The O(N=2) model in polar coordinates at nonzero temperature

Diplomarbeit

vorgelegt von

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Deutsche Zusammenfassung

In Kapitel 1 diskutieren wir einige wichtige Aspekte der Quantenchromodynamik (QCD) und führen das chirale Kondensat als Ordnungsparameter für den chiralen Phasenübergang ein. Der Schwerpunkt liegt dabei auf dem Konzept der Universalität und den Argumenten, weshalb das $O(4)$ Modell in die gleiche Universalitätsklasse fällt wie die effektive Theorie für den Ordnungsparameter der (masselosen) Zwei-Flavor QCD. Kapitel 2 erklärt den CJT-Formalismus auf pädagogische Weise und befasst sich mit der WKB-Methode. In Kapitel 3 und Kapitel 4 wird der CJT-Formalismus auf ein einfaches Z_2-symmetrisches Modell angewendet. In Kapitel 4 sind spontane Symmetriebrechung und der Tunneleffekt von Relevanz. Wie auch im Falle aller anderen Modelle, die innerhalb dieser Arbeit diskutiert werden, untersuchen wir das Verhalten bei endlicher Temperatur. Dies geschieht sowohl in 1+3 Dimensionen als auch in 1+0 Dimensionen. Im letzteren Fall ist es möglich, das effektive Potential am globalen Minimum (also den negativen Druck) mit dem Resultat aus der WKB-Näherung zu vergleichen. Unser Hauptinteresse gilt jedoch dem $O(2)$ Modell, wobei die Felder als Polarkoordinaten behandelt werden. Dieses Modell ist der erste Schritt in Richtung des $O(4)$ Modells in vierdimensionalen Polarkoordinaten. Obwohl im Prinzip autonom, sind alle Inhalte dieser Arbeit direkt mit Fragestellungen verbunden, die im Zuge der Untersuchung dieses Modells auftreten. In Kapitel 5 gehen wir direkt vom erzeugenden Funktional in kartesischen Koordinaten aus und wechseln zu Polarkoordinaten. Im folgenden sind wir mit der Frage beschäftigt, unter welchen Umständen es möglich ist, die gleichen Feynman-Regeln wie im Falle von kartesischen Koordinaten zu verwenden. Unter Annahme der gewohnten Feynman-Regeln wenden wir sodann den CJT-Formalismus auf das polare $O(2)$ Modell an. Ursprünglich war die Untersuchung in 1+0 Dimensionen dazu gedacht, den Wechsel zu Polarkoordinaten besser zu verstehen. Es stellte sich jedoch heraus, dass Infrarot-Divergenzen die Untersuchung erschweren. Dieses Problem erfordert besondere Aufmerksamkeit und motiviert die Untersuchung eines masselosen Feldes unter topologischen Zwangsbedingungen in Kapitel 8. In Kapitel 7 untersuchen wir das kartesische $O(2)$ Modell in 1+0 Dimensionen. Wir vergleichen das effektive Potential am globalen Minimum, berechnet innerhalb des CJT-Formalismus und mittels der WKB-Näherung. In Anhang B besprechen wir die Herleitung herkömmlicher thermischer Integrale in 1+0 und 1+3 Dimensionen, welche die Grundlage für unsere CJT-Rechnungen sowie für die Diskussion der Infrarot-Divergenzen bilden. In Kapitel 9 diskutieren wir den sogenannten Pfadintegral-Kollaps und schlagen eine Lösung für das Problem vor. In Kapitel 10 präsentieren wir unsere Schlussfolgerungen sowie einen Ausblick. Da wir unsere Darstellung im Rahmen einer Diplomarbeit so pädagogisch wie möglich halten wollten, haben wir uns entschieden, ausgiebigen Gebrauch von Anhängen zu machen. Die Anhänge A-H sind für Studierende gedacht, die mit gewissen Konzepten nicht vertraut sind. Wir verweisen innerhalb der Arbeit explizit auf diese Anhänge, um die Verbindung zwischen unserer Arbeit und dem zugrundeliegenden Hintergrund herzustellen.

Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbständig und ohne unerlaubte fremde Hilfe verfasst habe, und dass alle wörtlich oder sinngemäß aus Veröffentlichungen entnommenen Stellen dieser Arbeit unter Quellenangabe einzeln kenntlich gemacht sind.

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Contents

Preface

Outline

Chapter 1 contains the general background of our work. We briefly discuss important aspects of quantum chromodynamics (QCD) and introduce the concept of the chiral condensate as an order parameter for the chiral phase transition. Our focus is on the concept of universality and the arguments why the $O(4)$ model should fall into the same universality class as the effective Lagrangian for the order parameter of (massless) two-flavor QCD. Chapter 2 pedagogically explains the CJT formalism and is concerned with the WKB method. In chapter 3 the CJT formalism is then applied to a simple Z_2 symmetric toy model featuring a one-minimum classical potential. As for all other models we are concerned with in this thesis, we study the behavior at nonzero temperature. This is done in 1+3 dimensions as well as in 1+0 dimensions. In the latter case we are able to compare the effective potential at its global minimum (which is minus the pressure) with our result from the WKB approximation. In chapter 4 this program is also carried out for the toy model with a double-well classical potential, which allows for spontaneous symmetry breaking and tunneling. Our major interest however is in the $O(2)$ model with the fields treated as polar coordinates. This model can be regarded as the first step towards the $O(4)$ model in four-dimensional polar coordinates. Although in principle independent, all subjects discussed in this thesis are directly related to questions arising from the investigation of this particular model. In chapter 5 we start from the generating functional in cartesian coordinates and carry out the transition to polar coordinates. Then we are concerned with the question under which circumstances it is allowed to use the same Feynman rules in polar coordinates as in cartesian coordinates. This question turns out to be non-trivial. On the basis of the common Feynman rules we apply the CJT formalism in chapter 6 to the polar $O(2)$ model. The case of 1+0 dimensions was intended to be a toy model on the basis of which one could more easily explore the transition to polar coordinates. However, it turns out that we are faced with an additional complication in this case, the infrared divergence of thermal integrals. This problem requires special attention and motivates the explicit study of a massless field under topological constraints in chapter 8. In chapter 7 we investigate the cartesian $O(2)$ model in 1+0 dimensions. We compare

the effective potential at its global minimum calculated in the CJT formalism and via the WKB approximation. Appendix B reviews the derivation of standard thermal integrals in 1+0 and 1+3 dimensions and constitutes the basis for our CJT calculations and the discussion of infrared divergences. In chapter 9 we discuss the so-called path integral collapse and propose a solution of this problem. In chapter 10 we present our conclusions and an outlook. Since we were interested in organizing our work as pedagogical as possible within the narrow scope of a diploma thesis, we decided to make extensive use of appendices. Appendices A-H are intended for students who are not familiar with several important concepts we are concerned with. We will refer to them explicitly to establish the connection between our work and the general context in which it is settled.

Of central importance in the whole thesis is the concept of the *generating functional* and the *partition function*, respectively. In appendix A.1 we present the general context in which the partition function appears and its general definition within the operator formalism of second quantization. Alternatively, this definition can be rewritten via the path integral formalism. We restrict ourselves to scalar fields in this case. Furthermore, the understanding of the CJT formalism is based on knowledge about *n-point functions* (connected or disconnected, in the presence or in the absence of sources) and the context in which they arise. In appendix A.2 we give their definition taking account of the different modifications in which these quantities occur in this thesis, i.e., scalar field theory at zero or at nonzero temperature, respectively. From a didactic point of view, we believe that it is helpful if one can establish a relation between special cases and a general framework. Therefore, in appendix A.3 we want to keep an eye on the overall picture. We discuss the general concept of the generating functional for correlation functions, which also covers the partition function. We also briefly comment on the general concept of *Feynman rules* and we clarify the meaning of the terms *Green's function* and *propagator*.

Notation

Sometimes we abbreviate **Q**uantum **F**ield **T**heory as QFT and **S**tatistical **Q**uantum **F**ield **T**heory as SQFT.

In all vacuum expectation values, we assume the fields to be time ordered (Euclidean time ordered), so that we can omit the time order operator $\hat{T}$.

In analogy to "space-time", we occasionally use the term "space-inverse temperature" in SQFT. We refer to the field variables as *internal degrees of freedom*, whereas to space-time resp. space-inverse temperature as *external degrees of freedom*.

We speak of $1 + D$ dimensions, where D is the number of spatial dimensions and the 1 refers to time or, respectively, to temperature.

We use natural units where $\hbar = c = k_B = 1$. However, sometimes we write out $\hbar$ explicitly to indicate where it enters.

Ω denotes either spatial volume (to avoid confusion with the effective potential V) or the grandcanonical potential (which however appears only few times). In appendix G, spatial volume is exclusively denoted by V and the grandcanonical potential by Ω.

In the discretized version of path integrals, we use square brackets to divide the expression for the path integral into individual products. A product sign in such a bracket refers to the indices

in this bracket only.
At nonzero temperature we work in the imaginary-time formalism (Matsubara formalism) and use both, the Euclidean as well as the pseudo-Minkowskian notation. We explain our notation in great detail in section 2.2, appendix B and appendix E.
In the case of the path-integral representation for the partition function, we use the symbol $\oint$ to remind of the periodic boundary condition and the additional integration in the discretized version of the path integral.

Laß den Anfang mit dem Ende sich in
Eins zusammenzieh'n.
(Johann Wolfgang von Goethe)

Chapter 1

Introduction

Preliminaries

The focus of this diploma thesis is on the $O(N)$ model, which is an effective theory of QCD. Effective theories of QCD have their origin in the general theory of the strong interaction, Quantum chromodynamics (QCD), which describes quarks and gluons and the particles built out of the latter, the hadrons. Whereas appendix H summarizes the mathematical steps leading to the perturbative expansion of QCD (Perturbative QCD) and SQCD (Statistical Quantum chromodynamics, i.e., QCD at nonzero temperature), this introduction will present important facts on a less detailed level. The section on QCD, SQCD and lattice QCD reviews asymptotic freedom of QCD on a rather intuitive level in order to motivate the necessity of effective theories for the nonperturbative regime. We will also point out the reason for the different behavior of the renormalized coupling constant of QCD and QED (Quantum electrodynamics, i.e., the theory describing electrically charged particles and photons). To put it in a nutshell, QCD is too complicated to be solved. Instead, approaches for the nonperturbative regime have been developed, which can be divided into two main classes. On the one hand we have lattice QCD, on the other hand effective theories of QCD (as for example *Chiral Perturbation Theory* or the *Nambu-Jona-Lasinio model*). The latter treat hadrons as effective particles, whose inner structure is not evident.
Throughout this work, the notion of vacuum will play an important role. Let us give the definition in Quantum Field Theory (at nonzero temperature):

definition 1.1 (vacuum) *The vacuum, at a certain temperature T, is the state of lowest energy. If there exist more states of lowest energy, the system has accordingly many different vacua, which are said to be degenerate.*

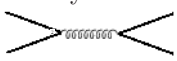
Note that the vacuum state changes with rising temperature. The explanation is that, the higher the temperature, the higher the probability for vacuum fluctuations, i.e., particle and anti-particle are produced and immediately annihilated. Also note that the vacuum state $|\Omega\rangle$ of a theory with interactions differs from the vacuum state of the free theory, which is usually denoted by $|0\rangle$. Although there might exist degenerate vacua, one has to choose one of them as the *real vacuum* if one wants to apply perturbation theory. In principle, all vacua could be equally

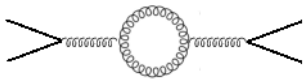
suited to describe physics. Similarly, gauge transformations leave physics invariant. However, in order to make quantitative predictions one has to fix the gauge. In our models, the vacua are identified with the global minima of the effective potential, a quantity which will be introduced later. Identifying the degrees of freedom with physical particles, the choice for the real vacuum is dictated by the quantum numbers of the vacuum ($J^{PC} = 0^{++}$, i.e., total spin J is zero, whereas parity P and C-parity C are positive).

QCD, SQCD and lattice QCD

Asymptotic freedom

The higher the energies one reaches in accelerators, the higher the square of the four-momentum transfer Q^2 in scattering processes, and therefore the smaller the length scale one is able to resolve. For instance, $\sqrt{Q^2}$ has to be large compared with $\hbar/R$ in order to see the constituents of a nucleon with radius R. Perturbative QCD only makes sense at very high energy scales (in the so-called *perturbative regime*), because for small momentum transfers or large distances (in the *nonperturbative regime*) the coupling constant in which we expand becomes large. Apart from the number of colors, N_c, and the number of flavors, N_f, the coupling constant is the only free parameter of the theory, i.e., has to be measured by experiment. However, before we can measure anything, QCD has to be renormalized. This leads from the bare coupling constant, g_S, to the renormalized coupling constant, $g_{S,R}(Q)$. The renormalized coupling constant can be measured in experiments and depends on the four-momentum transfer or, respectively, on the energy scale μ.

The so-called *Feynman rules* prescribe how to write a Feynman diagram in terms of an explicit analytical expression, and how to express quantities in terms of Feynman diagrams. The technique of Feynman diagrams can simply be regarded as a method how to handle perturbative expansions in a convenient way. Moreover, each diagram has a specified physical interpretation. The external lines can be interpreted as the observable ingoing and outgoing particles, whereas the internal lines can be imagined as virtual particles. Where lines meet in a vertex an interaction occurs. The internal (virtual) structure of a Feynman diagram determines a specific way how the process defined by the external lines can take place. For example, the diagram  can be regarded as a single gluon exchange (internal line) between two quarks (two incoming external lines and two outgoing external lines). An important quantity is the Lorentz-invariant amplitude $\mathcal{M}$, from which in turn all other quantities of interest (decay rates, cross sections, and so forth) could be derived. In the case of the physical process of two quarks interacting with each other, to lowest order in the coupling, $\mathcal{M}$ is given by the aforementioned diagram (translated with the Feynman rules of QCD). The principle how to construct $\mathcal{M}$ is easy to remember. First, we have to draw all diagrams with external lines describing the physical process under consideration, i.e., we have a diagram for each way the interaction might take place. $\mathcal{M}$ is then nothing else but the sum of all diagrams (each translated with the Feynman rules). Note that this picture coincides with the path integral approach. So to speak, all information about a process is included in its Lorentz-invariant amplitude, which is a superposition due to the Feynman rules of all possible ways the process can take place. Likewise, the expression for the

generating functional can be regarded as a sum over the weight of all possible paths the system is allowed to follow. For higher-order diagrams, the internal structure becomes more and more complicated. Some of them only yield negligible contributions to $\mathcal{M}$, however others are infinite and require renormalization. An important role is played by diagrams with gluon loops, as for example  . Each of them can be imagined as gluon exchange between quarks, in which the gluon gives rise to the production of virtual gluons (gluon loops) from the vacuum. Those graphs result from the gluonic self-interaction vertices and are responsible that (for $11N_c > 2N_f$) the measured renormalized coupling constant will decrease with increasing momentum transfer or smaller distance scale, as one can see from equation (1.3) below. This effect is known as *asymptotic freedom of QCD*.[1] In contrast to that, the renormalized coupling constant for QED increases with momentum transfer. This can be explained by the fact that photons do not interact with each other in contrast to gluons. Whereas the gauge-field contribution $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ in QED, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field-strength tensor, does not contain a self-interaction term for the gauge boson, this is the case in QCD. In general, a non-Abelian gauge symmetry with commutation relations $\left[t^a, t^b\right] = if^{abc}t^c$ for its generators yields an additional term in the field-strength tensor,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_S f^{abc} A^b_\mu A^c_\nu \ ,$$

allowing the gluons to interact with each other via a 3-gluon-vertex and a 4-gluon-vertex. Quarks interact with each other only via a 2-quark-1-gluon-vertex, i.e., their interaction is mediated by gluons. We will return to the discussion about the role of the gluonic self-interaction immediately after some general remarks on renormalization.

Consider a certain Lagrangian before renormalization. It involves bare parameters, usually bare masses and bare couplings. In the case of quantum field theories, one notices that some quantities of interest calculated from the bare Lagrangian turn out to be infinite. The idea of renormalization is to absorb these infinities, arising from divergent Feynman diagrams, into a redefinition of the parameters. Let us first state the result of the renormalization procedure in general: the Feynman rules have the same form before and after renormalization, only that one has to replace the bare parameters by the measurable renormalized ones. For example, the Feynman rules for QED have the same form before and after renormalization, but the bare coupling constant g_e and the bare mass m_e have to be replaced by the renormalized coupling constant $g_{e,R}$ and the renormalized mass $m_{e,R}$ respectively. Of course the explicit calculation of diagrams is different after renormalization, since the divergences have been absorbed. The rules how to treat the divergent terms in a diagram can be derived from the renormalization approach. Let us explain in principle how it works. We want to keep the discussion as abstract as possible. For a concrete example, QCD, we refer for instance to Ref. [2]. Consider a Lagrangian $\mathscr{L}$ involving several bare fields and bare parameters $\{p\}$. First, each of them is multiplied by its own *renormalization factor* (usually

[1] D.J.Gross, F.Wilczek and H.D.Politzer were able to show that Yang-Mills Theory is asymptotically free and can be used to describe the theory of the strong interaction. Also A.Zee searched for asymptotic freedom. Two months before Gross, Wilczek and Politzer published their work on asymptotic freedom, which was awarded with the Nobel prize in 2004, A.Zee conjectured on the basis of representation theory of Lie algebras that there are no asymptotically free quantum field theories in four dimensional space-time [1]. At that time he had not yet examined Yang-Mills Theory, which he, however, planned to analyze next.

denoted by the letter Z) to formally obtain the parameters $\{p_R\}$, which one calls renormalized parameters. Let us denote the Lagrangian one obtains by replacing in $\mathscr{L}$ the bare parameters and bare fields by the new ones as $\mathscr{L}_R$. Carrying out the renormalization, the renormalized parameters become functions of the four-momentum transfer Q, which will be explained later in this section in the context of cut-off regularization. As we already discussed, the dependence on Q, i.e., $\{p_R(Q)\}$, is equivalent to a dependence on the energy scale μ, i.e., $\{p_R(\mu)\}$. Before we can renormalize a theory, it needs to be regularized via a *regularization scheme*. The regularization scheme does not affect the results after the renormalization has been carried out. For example, in a cut-off regularization scheme one restricts the range of integration by introducing a cut-off, which makes divergent integrals (divergent Feynman diagrams) finite. In the momentum-space representation the cut-off is the upper limit for the absolute value of four-momentum transfer and will be denoted by M in the following. Another possibility is to change the space-time dimension, $d = 4 - \epsilon$, which has the same effect. This scheme is known as *dimensional regularization*. In the course of renormalization the original dimension will be recovered, i.e., $\epsilon \to 0$. So far we basically discussed the formal framework, now comes the nontrivial part, which we only want to sketch. The above mentioned renormalization factors can be calculated perturbatively in g_R and depend on ϵ. One can define an appropriate *counter term Lagrangian* $\mathscr{L}_C$, which consists of the same terms as $\mathscr{L}_R$, only that in front of each term there is some function of the renormalization factors. Let us denote the regularized version of $\mathscr{L}_R$ by $\mathscr{L}_R^\epsilon$ and the regularized version of $\mathscr{L}_C$ by $\mathscr{L}_C^\epsilon$. $\mathscr{L}_C$ is chosen such that

$$\mathscr{L}_R = \lim_{\epsilon \to 0} \left(\mathscr{L}_R^\epsilon + \mathscr{L}_C^\epsilon \right) \ , \tag{1.1}$$

i.e., such that the infinite diagrams arising from $\lim_{\epsilon \to 0} \mathscr{L}_R^\epsilon$ are exactly canceled by the diagrams from $\lim_{\epsilon \to 0} \mathscr{L}_C^\epsilon$. For dimensional reasons, performing the cancellation is inevitably accompanied by introducing a renormalization scale, which can be interpreted as the physical scale to which $\mathscr{L}_R$ applies. With the dependence on the physical scale the renormalization group comes into play, which is suited to encode the renormalization process in so-called renormalization group equations for the renormalized parameters. We will comment on the renormalization group in a separate paragraph.

Let us come back to the discussion of the gluonic self-interaction. In QED, the divergent diagrams containing electron-positron-loops enforce the introduction of a renormalized coupling constant. Although the diagrams with quark-antiquark loops in QCD have the same effect as those containing electron-positron loops within QED, the coupling constant for QCD does not increase with momentum transfer. Instead it decreases due to the diagrams with gluon loops. However, the way how the renormalized coupling is introduced is similar to QED, but one has to take into account the graphs arising from the gluonic self-interaction. Even though we used a special process (two quarks interacting with each other) to illustrate our discussion, also in general the gluonic self-interaction causes the asymptotic freedom of QCD. As already mentioned, the principal idea of renormalization is to absorb infinite contributions from diagrams into a redefinition of the coupling constant. For each individual process in QCD, one is able to show that exactly those diagrams (describing the process) which arise from the gluonic self-interaction assure asymptotic freedom, i.e., that the renormalized coupling constant (1.3), see below, decreases with higher momentum transfer. How to absorb infinite contributions from diagrams

into a redefinition of bare quantities can be understood representatively for a special case in QED, electron-muon scattering. David Griffiths explains in chapters 7.9 and 9.4 of his book [3], how to introduce a relation[2] between g_e, the cut-off M and $g_{e,R}$ as well as a formula for the dependence of $g_{e,R}$ on Q, such that divergent contributions are absorbed into the redefinition. In contrast to the aforementioned relation, the formula for the dependence on Q is in principle generally valid in QED at every order in g_e. Griffiths only neglects small contributions from certain diagrams. In this so-called "leading logarithm approximation" the dependence is given by

$$\alpha_{e,R}(Q) = \frac{\alpha_{e,R}(0)}{1 - (\alpha_{e,R}(0)/3\pi)\ln\left(Q^2/m_{e,R}^2\right)} \ , \quad \alpha_{e,R} = \frac{g_{e,R}^2}{4\pi} \ , \quad \alpha_{e,R}(0) = \alpha_e = \frac{1}{137} = \frac{g_e^2}{4\pi} \ . \tag{1.2}$$

However, there is a way how to obtain the formula for the momentum dependence of the renormalized coupling constant in a straightforward manner. So to speak, the intention of the discussion above was to point out how virtual quantum fluctuations lead to a screening of the bare coupling strength in the QED case, and to an antiscreening of the bare coupling strength in the QCD case. As already suggested above, the general derivation uses the renormalization group and is for example presented by Ryder in chapter 9.8 of his textbook [4]. We briefly discussed that in the course of renormalizing QCD the bare coupling constant g_S is replaced by $g_{S,R}(Q)$. The dependence on Q results from the renormalization procedure itself and is determined by the renormalization group equation $Q^2 \frac{\partial g_{S,R}(Q)}{\partial Q^2} = \beta\left(g_{S,R}(Q)\right)$, i.e., if $g_{S,R}$ is measured at a certain value for Q^2, the values for all other points are predicted by this equation. Namely

$$\alpha_{S,R}(Q) = \frac{\alpha_{S,R}(Q_0)}{1 + (\alpha_{S,R}(Q_0)/12\pi)(11N_c - 2N_f)\ln(Q^2/Q_0^2)} \ , \quad \alpha_{S,R} = \frac{g_{S,R}^2}{4\pi} \ . \tag{1.3}$$

Note that we have $\alpha_{S,R}(Q_0)$ instead of $\alpha_{S,R}(0)$, because $\alpha_{S,R}(0)$ would be large, whereas we have to refer to a point Q_0 at which a perturbative expansion is justified. Only for those values of Q^2 where a perturbative expansion in $g_{S,R}(Q)$ is justified (1.3) is meaningful. This is because the calculation is based on a perturbative expansion in the coupling constant, which breaks down for large values. This means that in the infrared regime, i.e., small Q^2, QCD in its most general form cannot be examined via perturbative renormalization group methods. $\alpha_{S,R}$ can be measured for different four-momentum transfers or, respectively, for different energy-scales on the basis of various processes, for which one is able to calculate certain observables depending on $\alpha_{S,R}$. For a detailed review see [5]. The right panel of fig.1.1 shows results for a measurement of $\alpha_{S,R}$ at different centre-of-mass energies $\sqrt{s}$ of a hadronic system which is produced in a decay $Z \to hadrons + \gamma$. We can observe that the smaller $\sqrt{s}$, the larger $\alpha_{S,R} = \frac{g_{S,R}^2}{4\pi}$.

Lattice QCD

Note that although effective theories of QCD apply to the nonperturbative regime, perturbative techniques are employed in these theories. Lattice QCD however, is a completely nonperturbative technique established by Wilson [6]. All quantities of interest can be derived from the generating functional, so the idea is to compute the generating functional numerically on a Euclidean

[2] This relation defines the renormalized coupling constant. So to speak, the divergence hiding in the cut-off M is absorbed into this definition. $M \to \infty$ corresponds to $\epsilon \to 0$ in dimensional regularization.

space-time lattice. Note that one generally works in Euclidean space-time because in that case no imaginary values have to be considered, which simplifies numerical calculations. If one wants to consider the system at nonzero temperature T one has to replace $L^3 \times \tau_{max}$ by $L^3 \times 1/T$. The generating functional is a path integral, which is defined as continuum limit of its discretized version. The continuous Euclidean space-time $L^3 \times \tau_{\max}$ is replaced by a discretized lattice, which consists of a finite number $N^3 \cdot N_\tau$ of points (also called *lattice sites*) $x^\mu = (a_\tau \cdot l, a \cdot i, a \cdot j, a \cdot k)$, where $a \equiv L/N$ is the distance between two lattice sites (also called *lattice spacing*) in a spatial direction, and $a_\tau \equiv \tau_{max}/N_\tau$ the lattice spacing in the Euclidean time direction. In order to obtain a discretized version of a path integral, the integrand is rewritten in discretized form, such that the continuum limit ($a_\tau \to 0$, $a \to 0$) yields the continuous version of the integrand. However, note that the discretized version is not unique.

Although the Lattice idea arises almost naturally from the definition of the path integral, the details are highly nontrivial. Particularly, one has to figure out how to handle infinities which would require renormalization in the continuum limit. Fortunately, the discrete lattice serves as a nonperturbative regularization scheme. The ultraviolet cut-off regulating ultraviolet divergences can be identified with the maximum momentum scale which is given by the inverse of the smallest length-scale inherent to the system, the inverse lattice spacing a^{-1}. The inverse of the largest length-scale, $(a \cdot N)^{-1}$, in turn determines the minimum momentum scale, i.e., the infrared cut-off. In principle, in the perturbative regime, usual perturbation theory can be applied using the lattice regularization. An advantage of lattice QCD is that we can start from the Euclidean version of the generating functional (H.12), since in its discretized version we have a finite number of gauge-field integrations. Hence, no ghost and gauge fixing terms have to be introduced. On the other hand, one has to worry about what to do with the Grassmann valued fermion fields. Fortunately, it is not necessary to do numerical calculations on the lattice with Grassmann variables. The integrations over the fermionic fields can be carried out, so that no Grassmann-valued fields remain. Another problem is that replacing derivatives by simple difference quotients costs much computation time. Apart from this, one should ensure that gauge invariance is respected by the Lattice formulation. Wilson accounted for both difficulties by associating gauge-fields with links connecting lattice sites. For details we refer to Ref. [7].

In practice, the Monte Carlo simulation, a numerical integration method based on probability, is employed. Hereby different algorithms can be used (e.g. Metropolis algorithm, Lagevin algorithm, Microcanonical algorithm) to improve the procedure. Nevertheless, these simulations have statistical errors in addition to the systematic errors arising from the lattice discretization. Recall that all thermodynamical quantities can be calculated from the Euclidean generating functional at nonzero temperature (the partition function). Of particular interest are the quantities related to the phase transition. A typical Lattice calculation, indicating a phase transition from a confined to a gas-like state at a critical temperature T_c, is shown on the left in figure 1.1.

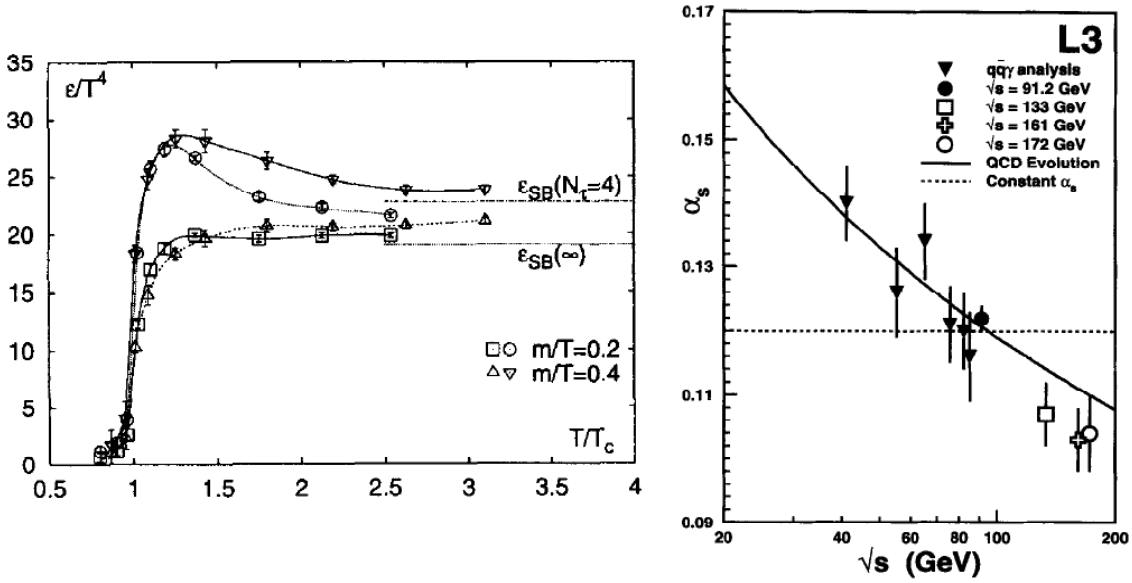


Figure 1.1: left: energy density against temperature, taken from [8]; right: the data points ▼ are experimental results for the running renormalized fine structure constant $\alpha_{S,R} = \frac{g^2_{S,R}}{4\pi}$ at different centre-of-mass energies $\sqrt{s}$ of a hadronic system which is produced in a decay $Z \rightarrow hadrons + \gamma$, taken from [9].

Condensates and chiral symmetry

We now turn to a nontrivial observation, which has not been yet completely understood. Let us first introduce an important quantity, the so-called *quark condensate*[3]. It can be defined as the correctly normalized 2-point function for $\overline{\psi}_p$ and ψ_q, where q and p are flavor indices. In the case of QCD, the *quark condensate in the absence of sources* is given by $\langle \overline{\psi}_p \psi_q \rangle^0$, where for the definition of $\langle\ \rangle^0$ we refer to appendix H, that is to (H.24). Absence of sources corresponds to vacuum. We have to distinguish the *quark condensate in the presence of sources*, $\langle \overline{\psi}_p \psi_q \rangle$, where we do not set the sources to zero. Note that the term "vacuum expectation value in the presence of sources" is somehow misleading, a better suited name would be "expectation value away from the vacuum". Note that in the case of SQCD we simply have to perform a Wick rotation, $t \longrightarrow \overline{t} = -i\tau$, and to impose periodic (antiperiodic) boundary conditions on the (fermionic) fields. The quark condensate, now depending on temperature, can be calculated, for example in lattice QCD, as outlined above. The interesting point is that it vanishes around a critical temperature, which corresponds to T_c as found in lattice calculations of the kind presented in the left of figure 1.1). Furthermore, the critical temperature is closely related to Spontaneous Symmetry Breaking (SSB) of chiral symmetry, which will be discussed in the following.
In the case of vanishing quark masses[4], $m = 0$, the QCD Lagrangian possesses, apart from the

[3]The quark condensate is an example of a fermionic condensate, which is similar to the Bose-Einstein condensate.

[4]As soon as $m \neq 0$, chiral symmetry is explicitly broken. More precisely, if all quark masses are equal, $m_{pq} = m\delta_{pq}$, there is only a $SU(N_f)^V$ symmetry left (omitting the vector subgroup $U(1)^V$, which does not affect chiral dynamics, and assuming the axial $U(1)^A$ symmetry is broken explicitly by instantons). Whereas, if

local $SU(N_c)$ gauge symmetry, a global symmetry, the so-called *chiral symmetry*, which is given by the chiral group $U(N_f)^r \times U(N_f)^l$. That is, the Lagrangian is invariant under transformations

$$\psi^{r,l} \longrightarrow U^{r,l}\psi^{r,l} \quad , \qquad U^{r,l} \equiv \exp\left(i \sum_{a=0}^{N_f^2-1} \alpha_{r,l}^a T_a \right) \in U(N_f) \ , \tag{1.4}$$

where $U^{r,l}$ acts on $\psi^{r,l}$, given by

$$\psi^{r,l} \equiv P^{r,l}\psi \quad , \quad P^{r,l} \equiv \frac{1+\gamma_5}{2} \quad , \quad \psi = \psi^r + \psi^l \ , \tag{1.5}$$

i.e., $\mathscr{L}(\psi^r, \psi^l) = \mathscr{L}(U^r\psi^r, U^l\psi^l)$. The T^a denote the generators of $U(N_f)$.[5]
Chiral symmetry is spontaneously broken by a nonvanishing quark condensate $\langle \overline{\psi}_p \psi_q \rangle^0 \neq 0$. Spontaneous symmetry breaking means that, whereas the Lagrangian has a certain symmetry (here $U(N_f)^r \times U(N_f)^l$), the vacuum has not. This implies that if the expectation value of an oberservable in the vacuum state (here $\langle \overline{\psi}_p \psi_q \rangle^0$) is not invariant under the same symmetry, it is spontaneously broken. Due to $\overline{\psi}\psi = \overline{\psi}^l\psi^r + \overline{\psi}^r\psi^l$ we can rewrite the quark condensate as $\langle \overline{\psi}_p \psi_q \rangle^0 = \langle \overline{\psi}_p^l \psi_q^r \rangle^0 + \langle \overline{\psi}_p^r \psi_q^l \rangle^0$. In the literature the term "chiral condensate" can refer to both, either to the quark condensate $\langle \overline{\psi}_p \psi_q \rangle^0$ or to $\langle \overline{\psi}_p^l \psi_q^r \rangle^0$. In the following however, we will call $\Phi_{pq} \sim \langle \overline{\psi}_p^l \psi_q^r \rangle$ *chiral condensate (in the presence of sources)* and $\Phi_{pq}^0 \sim \langle \overline{\psi}_p^l \psi_q^r \rangle^0$ *chiral condensate (in the absence of sources)* respectively. Obviously, a (non-)vanishing quark condensate is equivalent to a (non-)vanishing chiral condensate, so that both are equally suited to describe the chiral phase transition, which explains the confusion of names in the literature. Note that if we are in the chiral limit, $m = 0$, all quark flavors are equivalent, and therefore $\Phi_{pq}^0 = \varphi\delta_{pq}$. The chiral condensate still has a $U(N_f)^V$ symmetry[6], $\Phi^0 \longrightarrow U^V\Phi^0U^{V\dagger} = \Phi^0$, but is no longer invariant under axial transformations, $\Phi^0 \longrightarrow U^A\Phi^0U^{A\dagger} \neq \Phi^0$. Chiral symmetry is therefore spontaneously broken by a nonvanishing Φ^0.

Phase transitions

The order of a phase transition

Phase transitions can be divided into three categories. A phase transition can either be first-order, crossover or second-order, which is the border case between first-order and crossover. In this work, the aim of numerical calculations is to determine masses and condensate, as well as the so-called effective potential at nonzero temperature. From the evolution of the effective potential and the condensate with temperature, we are able to determine the order of the phase transition. The theory of phase transitions is discussed in a large amount of textbooks: some

$m_u \simeq m_d \ll m_i$ (where $i = s, c, t, b$), what remains is the approximate $SU(2)^V$ isospin symmetry.

[5] For $N_f = 2$ the generators are given by the unit matrix and the Pauli matrices.

[6] $U(N_f)^r \times U(N_f)^l$ is isomorphic to $U(N_f)^V \times U(N_f)^A$. Note that the group elements $U^V = \exp\left(i\alpha_V^a T_a\right)$ resp. $U^A = \exp\left(i\gamma_5\alpha_A^b T_b\right)$ do not act on independent "vector" resp. "axial" parts, as one might misinterpret the notation. Instead: $\psi \longrightarrow \exp\left(i\alpha_V^a T_a\right)\exp\left(i\gamma_5\alpha_A^b T_b\right)\psi$, where $\alpha_A^a \equiv (\alpha_r^a - \alpha_l^a)/2$ and $\alpha_V^a \equiv (\alpha_r^a + \alpha_l^a)/2$. Since $\overline{\psi}^l = \psi^{l\dagger}\gamma^0 \longrightarrow \psi^{l\dagger}U^{V\dagger}\gamma^0 = \overline{\psi}^l U^{V\dagger}$, we conclude that the matrix Φ^0 transforms as $\Phi^0 \longrightarrow U^V\Phi^0U^{V\dagger}$.

of them deal with classical thermodynamics, others with advanced topics in condensed matter or Statistical Quantum Field Theory. Since the relations between thermodynamic quantities are universally valid (for details see appendix A.1), also the classification of phase transitions used in SQFT is the same as in statistical mechanics. However, note that the thermodynamic potentials, namely the internal energy $U(S, V, \{Q_i\})$, the Helmholtz free energy $F(T, V, \{Q_i\})$, the enthalpy $H(S, p, \{Q_i\})$, the free enthalpy $G(T, p, \{Q_i\})$ and the grandcanonical potential $\Omega(T, V, \{\mu_i\})$, depend on conserved charges Q_i or on the associated chemical potentials μ_i. Whereas in the case of statistical mechanics the Q_i can be identified with particle numbers, the charges represent conserved quantum numbers in SQFT. Different *phases* are distinguished from each other by the criterion that certain observables (called *order parameters*) have different values across the phase transition. Density is an example for the distinction between gas, fluid and solid, whereas magnetization is one for the differentiation of para- and ferromagnetism. The classification of phase transitions goes back to Ehrenfest. The order n of a phase transition is defined as the lowest order partial derivative of a thermodynamic potential with respect to its natural variables, where at least one of the derivatives is discontinuous at the phase transition.

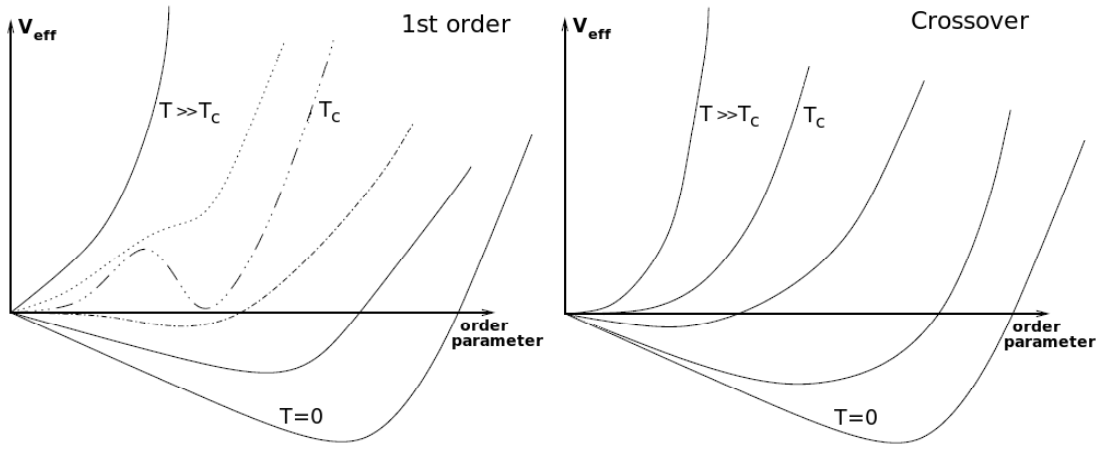


Figure 1.2: Compare with [10].

According to what one knows about phase transitions, it is safe to say that there is a rare feature, which clearly distinguishes second from first order: self-similarity. Consider the liquid-gas phase transition of boiling water. At low temperature and pressure it is of first order, but at very high temperature and pressure there is no distinction between gas and liquid. This means that the critical line ends in a critical point where the phase transition is of second order. Let us quote John Baez [11]: *Right at the critical point, something very cool happens: the system transforms in a simple way under scaling! What does this mean? Well, if you get some water right at the critical point, it looks "opalescent" like a moonstone. If you stare at it carefully, you'll see a bunch of liquid water droplets of all different sizes floating around in steam. However, if you look closely at any of these droplets, you see it's full of little droplets of liquid! It's like a random fractal: no matter how closely you look, you see the same thing. You can't tell if you're looking at water droplets in steam or bubbles of steam in water, and there is no distinguished length*

scale...at least until you get down to the scale of atoms, that is.
The so-called *correlation length* ξ is a measure of the range over which behavior in one part of the system influences behavior in another part. In the above example ξ can be regarded as the maximum size of droplets. Accordingly, if a system undergoes a second-order phase transition, its correlation length diverges at the critical point.
In the vacuum, for homogeneous systems (i.e., $\frac{\partial}{\partial V} \equiv \frac{1}{V}$) and for vanishing chemical potentials, the effective potential V_{eff} (which we introduce later in chapter 2.1) is related to thermodynamics via $V_{eff} = \frac{\Omega}{V} = \frac{F}{V} = -p$, where p is the pressure. T is a natural variable on which V_{eff} depends. As we will show in section 2.1, we are able to identify the order parameter (the condensate Φ) with the global minimum of the effective potential $V_{eff}(\phi)$. Note that in case of a first-order phase transition the entropy $S = -\left(\frac{\partial F}{\partial T}\right)_{V,\{Q_i\}}$ has a discontinuity at the critical temperature T_c. Accordingly, due to $V_{eff} \sim F$, we can conclude from the shape of the effective potential about T_c to the order of the phase transition. In the case of a first-order phase transition, $\Phi(T)$ has a discontinuity at T_c. Figure 1.2 shows examples for a 1st order and a crossover phase transition. The characteristic feature of a crossover is the continuous transition of the global minimum Φ from a nonzero value to zero. In the case of a first-order phase transition, we have a discontinuous jump from some nonzero value to zero.

After about a month of work I was
ordered to write up my results, as a result
of which I swore to myself that I would
choose a subject for research where it
would take at least five years before I had
anything worth writing about.
(Kenneth G. Wilson [12])

The renormalization group

The domain of applicability for the renormalization group is a wide one. The method in general allows to study a system at different distance scales and energy scales, respectively. All results derived from a renormalized theory depend on the energy scale μ at which the measurement takes place. Consider for example (1.3), which follows from a renormalization group study and describes how the renormalized QCD coupling constant depends on the scale. In this sense a renormalized theory can be compared to an object examined under a microscope.
Note that the term *renormalization* refers to the approach of how to handle divergences arising in quantum field theories. The method was developed initially for QED in the 1940's and was generalized later to other theories. The renormalization procedure is inevitably accompanied by the introduction of an energy scale μ at which the cancellation of the divergences is performed. The renormalized Lagrangian for this energy scale describes the system at μ. Quantum field theories are usually plagued with ultraviolet divergences arising at high energy scales, hence the renormalization group treats the ultraviolet behavior in this case. Regarding critical phenomena and universality, which will be discussed in the following paragraph, the renormalization group

is concerned with the infrared behavior. The general framework however, established by Wilson, is the same. The abstract formulation of the renormalization group merges two concepts, which were treated independent from each other until Wilson recognized their relation. The first one we already discussed: there is a set of transformations describing the dependence of the renormalized parameters on μ, arising naturally from the renormalization of quantum field theories. This concept constitutes the renormalization group in a field-theoretical context. The second concept was established by Kadanoff. Initially, one tried to explain the empirical phenomenon of universality with the help of mean-field theory. In the course of these studies, Kadanoff developed a prototype of the renormalization group, the so-called *block-spin renormalization group*. It originally referred to a simple model, the *Ising model*, which involves spin variables taking values ± 1 on a spatial lattice. This approach was suitable to describe the Ising model at large distance scales and was a first step towards explaining universality. The close relationship between critical phenomena and Quantum Field Theory can be understood from the fact that partition functions of classical statistical physics in four spatial dimensions have basically the same mathematical form as generating functionals of Euclidean Quantum Field Theory. By means of a perturbative expansion in the coupling constant(s) and ε, defined by $D = 4 - \varepsilon$, one is able to obtain the physical case of three spatial dimensions by setting ε to 1 at the end of the calculation, where one should not confuse this ε with the $\epsilon = 4 - d$ from dimensional regularization. This explains why it is possible to use renormalization group methods from Quantum Field Theory also in classical statistical physics, even though both theories have different physical content.

Let us now discuss the renormalization group in its abstract form, which is necessary to understand what universality has to do with the renormalization group. For further details we refer to [12], [13], [14] and [15].

Consider a system at length scale L_0, described by the Lagrangian $\mathscr{L}_0$ or, equivalently, by the Hamiltonian $\mathscr{H}_0$. Let the operator $\mathcal{R}_b$ describe how the Hamiltonian changes, when we increase or decrease the intrinsic length scale L_0 by a factor b, i.e., $L_0 \to L_0 \cdot b \equiv L_b$:

$$\mathcal{R}_b \mathscr{H}_0 = \mathscr{H}_b \ . \tag{1.6}$$

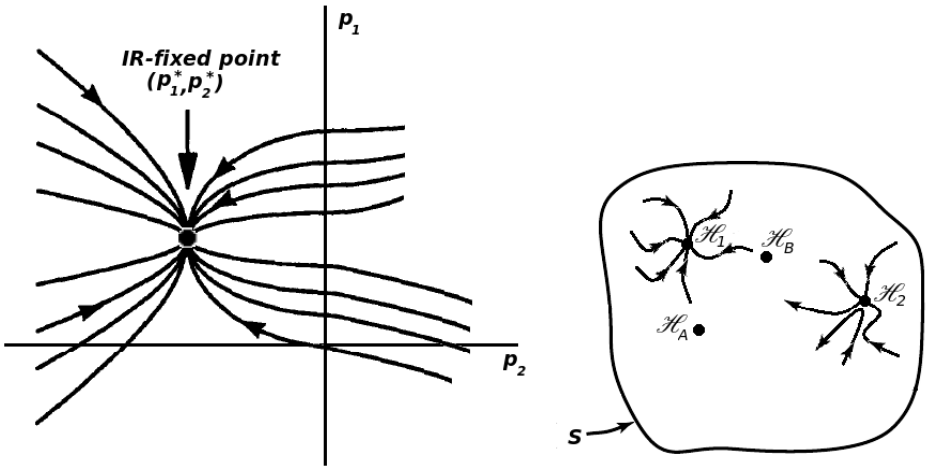
The Hamiltonian $\mathscr{H}_b$ characterizes the system at length scale L_b. Note that decreasing the length scale corresponds to increasing the energy scale. The set of all $\mathcal{R}_b$'s constitutes the so-called *renormalization group*:

$$\{\mathcal{R}_b \ , \ 0 < b < \infty\} \ . \tag{1.7}$$

Of course, it would be difficult to specify $\mathcal{R}_b$ for arbitrary b. Therefore the idea is to perform such a transformation in infinitely many infinitesimal steps:

$$\mathscr{H}_0 \longrightarrow \mathscr{H}_1 \longrightarrow \cdots \longrightarrow \mathscr{H}_{N-1} \longrightarrow \mathscr{H}_b \ , \quad N \to \infty \ . \tag{1.8}$$

Since infinitely many infinitesimal steps yield the same result as a single transformation, the operators $\mathcal{R}_b$ are indeed *group* elements of a continuously connected group. Because the Hamiltonian changes not much under an infinitesimal change of scale (the *how* is of course important), in principle one is able to take account of this change quantitatively to obtain a continuous *renormalization group flow* among Hamiltonians with different intrinsic length scales. Abstractly, one can consider a space S consisting of all kinds of Hamiltonians $\mathscr{H}$. In general there will be points



(a) Qualitative example of an infrared fixed point, parameter space, p_1-p_2 plane.

(b) Qualitative example of an infrared fixed point.

Figure 1.3: Renormalization group flow.

in S which are continuously connected with others via renormalization group transformations. Of course, regarding explicit calculations one has to parametrize the Hamiltonians in S. For the moment however, we want to keep the discussion general.
If the Hamiltonian $\mathscr{H}$ converges towards $\mathscr{H}^*$ when lowering the energy scale μ, the Hamiltonian $\mathscr{H}^*$ is called an *infrared fixed point Hamiltonian* or only *infrared fixed point*. Similarly, an *ultraviolet fixed point* is a fixed point when increasing the energy scale. A fixed point can either be stable or unstable. Consider the space S consisting of all possible Hamiltonians as depicted in figure 1.3(b). Changing the energy scale, a Hamiltonian $\mathscr{H}_A$ corresponding to a system at some scale will move along a *flow line*. If all flow lines in the neighborhood of a point $\mathscr{H}^*$ converge in this point, while all Hamiltonians in the neighborhood flow towards $\mathscr{H}^*$ when lowering the energy scale μ, then $\mathscr{H}^*$ is called a *infrared stable* fixed point. However, there are degrees of stability and instability, respectively. One can imagine cases where this defining condition is violated in certain directions, consider for instance $\mathscr{H}_2$ in figure 1.3(b). Whereas it is stable in some directions, it is not in others. One can classify the degrees of instability of a fixed point via the number of its so-called *relevant operators* (see below). More precisely, one defines an *n-unstable* fixed point as a fixed point with n relevant operators. Consider a Hamiltonian which is very close to $\mathscr{H}^*$, i.e.,

$$\mathscr{H} = \mathscr{H}^* + \Delta\mathscr{H} \ . \tag{1.9}$$

If one defines a linear operator L^* via

$$\frac{\partial}{\partial t}\Delta\mathscr{H} \equiv L^*(\Delta\mathscr{H}) + O(\Delta\mathscr{H}^2) \ , \quad t \equiv -\ln\mu \ , \tag{1.10}$$

one can expand $\Delta\mathscr{H}$ in terms of the eigenoperators $\mathcal{O}$ of L^*:

$$\Delta\mathscr{H} = \sum_n h_n \mathcal{O}_n \ . \tag{1.11}$$

The eigenoperators which correspond to eigenvalues l_n with positive real part are called *relevant operators*. Of course, the larger the space S one considers, the more complicated the flow structure or *renormalization group flow*. In principle there can exist arbitrarily many fixed points and arbitrarily complicated flow patterns. In practice one constrains the space of consideration to a sensible subspace of Hamiltonians.

Now consider a concrete Hamiltonian $\mathscr{H}$ or Lagrangian $\mathscr{L}$, such as for instance (1.42) or (1.21) below, instead of abstract points in some topological space. The abstract space S is now the set of all Lagrangians which are parametrized by $\mathscr{L}$. For example, the Lagrangians parametrized by (1.42) constitute a subspace of the set of Lagrangians parametrized by (1.31). Let the Lagrangian $\mathscr{L}$ contain a set of parameters $\{p\} = (p_1, p_2, ..., p_l)$. The renormalization group flow depicted in figure 1.3(b) of course implies a renormalization group flow in the parameter space, as illustrated in figure 1.3(a) for the p_1-p_2 plane. The renormalization group approach indeed implies a coupled system of continuous *renormalization group equations* for the set of parameters $\{p\}$:

$$\beta_i(\{p\}) = \mu \frac{\partial p_i}{\partial \mu} \ , \quad i = 1, ..., q \ , \tag{1.12}$$

$$\gamma_i(\{p\}) \sim \frac{\mu}{p_i} \frac{\partial p_i}{\partial \mu} \ , \quad i = q+1, ..., l \ , \tag{1.13}$$

where we divided the set of parameters into coupling constants, $p_1, \cdots, p_q$, and others, usually field renormalization factors and masses, $p_{q+1}, \cdots, p_l$. The letter β is usually reserved for renormalization group functions corresponding to coupling constants, and the system of equations for the β_i's can be solved separately. Each γ_i is usually called *anomalous dimension* for the parameter p_i. If an infrared fixed point $\mathscr{H}^*$ (or $\mathscr{L}^*$ respectively) exists, the set of parameters will flow into the corresponding infrared fixed point $\{p^*\}$ in parameter space when decreasing μ continuously. Hence, infrared fixed points are determined by $\beta_i \stackrel{!}{=} 0$ (and $\gamma_i \stackrel{!}{=} 0$). This is illustrated in figure 1.3(a) for the p_1-p_2 plane. If all parameters p_i^* are zero, one speaks of a *trivial* infrared fixed point. Inserting $\{p^*\}$ into the ansatz $\mathscr{L}$ yields the Lagrangian describing the system at the critical point.

In principle one can study the renormalization group flow for every renormalizable Lagrangian. The parameters determined by the renormalization group equations (1.12)-(1.13) at some scale μ are the physical (renormalized) parameters. Inserting them into the ansatz $\mathscr{L}$ from which one has derived the equations (1.12)-(1.13), one obtains the physical (renormalized) Lagrangian describing the system at the scale μ. The information how to treat divergences arising in quantum field theories is encoded in (1.12)-(1.13). It is important to understand that temperature T and energy scale are two different concepts. Consider for example a nucleus with a diameter of about $10^{-14}m$. To measure the diameter one has to use electrons of about $100MeV$, i.e., the energy scale is high. However, the temperature of a nucleus in its non-excited state (ground state) is always zero. For statistical systems, a flow pattern such as depicted in figure 1.3(a) looks different for different values of T. Consider again (1.8). We have not discussed how to construct such transformations. Loosely speaking, step by step one integrates out the short-range fluctuations in the partition function or, respectively, in the generating functional. Because the partition function depends on T, the flow pattern also does. In the context of critical behavior, it was Wilson's idea that the critical temperature T_c is determined by the appearance of an infrared fixed point. This becomes plausible from the picture we gave of a second-order phase transition. Right at T_c

the system becomes scale invariant, i.e., the Lagrangian which describes the system at T_c does not change if we further decrease μ. This corresponds to $\beta_i \stackrel{!}{=} 0$ (and $\gamma_i \stackrel{!}{=} 0$). Of course, there is a maximal value for μ up to which the system looks scale invariant. Beyond this maximal value the length scale is so small that one resolves the microscopical structure of the system. In fact, a whole "universality class" of microscopic structures is suited to generate the same picture at large enough length scales. In the context of critical behavior all Lagrangians which flow into the same infrared fixed point are therefore believed to belong to the same *universality class*. In this sense the parameter sets (p_1, p_2) shown in figure 1.3(a) define Lagrangians which belong to the same universality class. The original, phenomenological definition of the term *universality class* will be introduced in the following section.

Universality

In this thesis we consider systems in local thermal equilibrium and, accordingly, the static critical behavior. When we speak in the following of universality classes, we refer to static universality classes. Dynamical universality classes apply to systems in nonequilibrium systems, which we do not discuss.

It is an empirical fact that in case of second-order phase transitions the temperature dependence of many thermodynamic observables becomes very simple near the critical temperature T_c and depends only on general properties of the system. For example, near T_c, certain thermodynamic observables A often behave like

$$A(\epsilon) \sim \epsilon^p \ , \quad \epsilon \equiv \frac{T - T_c}{T_c} \ . \tag{1.14}$$

The dependence on T is "universally" characterized by the "critical exponent" p. There exist several relations involving critical exponents, characterizing the behavior of certain thermodynamic quantities around T_c. We want to state some important ones related to standard universality classes in the most common notation:

$$C \sim (-\epsilon)^{-\alpha} \ , \quad \text{for } T \stackrel{<}{\longrightarrow} T_c \ , \tag{1.15}$$

$$\chi_T \sim (-\epsilon)^{-\gamma} \ , \quad \text{for } T \stackrel{<}{\longrightarrow} T_c \ , \tag{1.16}$$

$$\xi \sim |\epsilon|^{-\nu} \ , \quad \text{for } T \approx T_c \ , \tag{1.17}$$

$$\phi \sim (-\epsilon)^{-\beta} \ , \quad \text{for } T \stackrel{>}{\longrightarrow} T_c \ . \tag{1.18}$$

where C is the specific heat, χ_T the transverse susceptibility, ξ the correlation length, and ϕ an order parameter for the phase transition. The critical exponents are related via

$$\alpha + 2\beta + \gamma = 2 \ , \tag{1.19}$$

$$\alpha = 2 - D\nu \ , \tag{1.20}$$

where D is the number of spatial degrees of freedom. For more differentiated presentations we refer to [15], [16], [17]. The astonishing observation is that the values of the critical exponents are equal for a large class of systems, which share certain general properties. Although the phenomenon itself was known earlier (see [18] for a review), Griffiths was the first to specify these properties. Griffiths basically looked at several simple (quantum) statistical mechanical models,

such as for example the Ising model or the Heisenberg model. These models have in common that they are defined on a D-dimensional spatial lattice. The comparison of theoretical estimates for critical exponents led Griffiths to the conclusion that these predominantly depend on 1) the lattice dimensionality, 2) the range of interaction and 3) symmetry properties of the order parameter for the phase transition [19]. This postulate is the prototype of the so-called "universality hypothesis" and refers exclusively to second-order phase transitions. Generally speaking it states that certain characteristics of a system (not necessarily defined on a lattice, although it is always possible to discretize space-time) near a second-order phase transition only depend on few properties[7]. Systems or quantities which possess the same critical exponents are said to be in the same *universality class*. Ten years later it became clear that at least four properties, in the following referred to as *essential, qualitative features (E.Q.F.'s)*, are necessary (*not* sufficient!) to characterize a universality class: 1) the spatial dimensionality D of the system, 2) the number of components of the order parameter, 3) the range and angular dependence of the interaction and 4) the symmetry properties of the order parameter. Meanwhile, the connection between critical phenomena and the renormalization group had been found, allowing an explanation of universality [20] (compare with the later review [12]). However, as Bruce points out, it was not possible so far to rigorously prove the universality hypothesis with the help of the renormalization group [21]. The above criteria 1)-4) were formulated in [21] by Bruce (also compare with the formulation by Amit in [22]) and have the character of inductive generalizations. Apart from a few topological considerations (see for instance [20]), only a restricted class of models was taken into account regarding their derivation. Nevertheless, their above presented abstract formulation has survived subsequent investigations of more advanced models, and nowadays it is believed that all systems in nature should belong to one of a comparatively small number of universality classes. According to [23] two specific Hamiltonians are enough to comprise these universality classes, namely the so-called Q-state Potts model and the n-vector model. The Ising model, the XY-model and the Heisenberg model are for example special cases of the n-vector model. Both models are defined on a lattice, however in principle field theory is included via the continuum limit. The further specification of the E.Q.F.'s in a general context is still ongoing work. In particular, consider a field-theoretical model, which in first place is defined in continuous space-time. We can discretize space-time to obtain its lattice version, but there is at least one problem relevant regarding universality: the continuous symmetry of the system is in general broken on the lattice. For example, one does not have a really satisfactory lattice formulation of QCD that is chirally symmetric in the chiral limit (vanishing quark masses) [7]. Therefore also lattice results regarding the justification of the $O(4)$-model as an effective theory have to be justified with precaution. Particularly condition 3) has not been further specified from a rigorous standpoint, although it is widely believed that in case of continuous symmetries only infinitesimal symmetry transformations of the order parameter play a role regarding universality. This point of view implies that only the Lie algebra (the group elements infinitesimally close to the identity) corresponding to the symmetry of the Lagrangian is important.

Using the renormalization group as a tool, it is possible to calculate critical exponents (see for example chapter 10 of [17]). Generally speaking, the critical exponents follow from the investigation of the renormalization group flow near the infrared fixed point which corresponds to the

[7]Note that in general the critical temperature is different for two models in the same universality class.

critical temperature T_c. In principle, one can now calculate critical exponents for different kinds of Lagrangians, in order to find out how their values are related to general properties of the Lagrangians. Moreover, one could analyze the topological structure of the renormalization group flow to find out to what extent the critical exponents depend on it. Hence, the renormalization group approach allows to check and to improve the universality hypothesis. Using the words of Bruce, *it is a conjecture largely substantiated by explicit calculation* that two systems fall into the same universality class, if they both approach the same infrared fixed point. Furthermore, it is suggested from such calculations that the above listed E.Q.F.'s predominantly determine the renormalization group flow. In order to decide on the basis of (1.12)-(1.13) whether two Lagrangians $\mathscr{L}_A$ and $\mathscr{L}_B$ fall into the same universality class, one has to analyze (1.12)-(1.13) for a Lagrangian $\mathscr{L}$ which completely parametrizes both $\mathscr{L}_A$ and $\mathscr{L}_B$. If exactly one stable infrared fixed point exists in the parameter space of $\mathscr{L}$, and $\mathscr{L}_A$ as well as $\mathscr{L}_B$ lie on a flow line leading towards this point, then both fall necessarily into the same universality class. It is possible to determine the stability properties of an infrared fixed point by looking at the eigenvalues of the stability matrix $\frac{\partial \beta_i}{\partial p_j}$. Again, these conclusions are restricted to those directions in the renormalization group flow which are parametrized by the Lagrangian under consideration. Recall that universality refers to second-order phase transitions. For a long time it has been taken for granted that the absence of a nontrivial infrared stable fixed point implies a first-order phase transition, and conversely that the existence of a nontrivial infrared stable fixed point implies a second-order phase transition. Meyer-Ortmanns objects that there may be regions in the coupling parameter space where flow lines do not lead to the infrared fixed point [24].

Effective models for the QCD order parameter

In principle, there is no strict regulation whether a model can be regarded as a reasonable candidate for an effective theory of QCD or not. Some models are proposed more or less ad hoc, on grounds of properties they have in common with QCD. The question is which properties of QCD are crucial for the results one intends to obtain. If one is interested in the behavior of the QCD chiral condensate in the region where a phase transition from spontaneously broken chiral symmetry to the restored phase takes place, the search for an effective Lagrangian for this order parameter is principally limited to candidates featuring spontaneous symmetry breaking. As already discussed, chiral symmetry is spontaneously broken by a nonvanishing chiral condensate Φ^0. It has been shown that certain pairs of hadrons (named *chiral partners*) should have the same mass, if both vacuum and Lagrangian are chirally symmetric. Being this not the case at low temperature, at high enough temperature and density one expects a restoration of chiral symmetry (i.e., the chiral condensate should vanish), taking place at the transition from hadronic matter to a gas-like state (quark-gluon plasma). In that way it is possible that the hot early universe is symmetric, whereas variety in nature emerges from cooling the system below the critical temperature. From Goldstone's theorem[8] we expect massless Goldstone bosons in the

[8]Goldstone's theorem states that if a continuous and global symmetry is spontaneously broken, there have to be massless particles in the spontaneously broken theory, exactly as many as there are broken generators. In the case of a spontaneously broken continuous gauge (i.e., local) symmetry one has to be careful. Higgs' theorem

spontaneously broken phase. However, note that there is an ongoing discussion concerning the question if the phase transition from confinement to deconfinement is really accompanied by a chiral phase transition and at which temperature they take place (see for instance [25]).
According to Wilson, there is a more rigorous point of view. In the early days of renormalization the cut-off regularization was more or less regarded as a formal approach. As already mentioned, the cut-off M is sent to infinity in the course of renormalization. Wilson's interpretation of the renormalization group equations allows for a physical interpretation of M. There is an upper limit for the energy scale μ of a renormalized theory, beyond which the renormalized Lagrangian is not suited to describe physics. As already discussed, this corresponds to an upper limit M for the absolute value of four-momentum transfer. In the case of theories which are part of the Standard Model, M is extremely large. This explains why it is possible to renormalize a theory in the limit $M \to \infty$. However, in the context of Wilson's approach M remains finite. Performing a Fourier decomposition of the fields, the partition function can be expressed as a path integral over the Fourier modes of the fields, which are functions of the four-momentum K. Wilson's idea was to integrate out the modes with $aM < \sqrt{K^2} < M$, where $0 < a < 1$. One obtains an effective partition function with an effective Lagrangian, which is suited to describe physics for absolute values of four-momentum transfers smaller than the effective cut-off $M_{eff} = aM$. Choosing a infinitesimally small, one can integrate out the fluctuations at high energy scales step by step, which finally leads to the continuous renormalization group equations.

Chiral linear sigma model

The *chirally invariant linear sigma model* for 2 flavors, $N_f = 2$, is an effective model for the QCD order parameter (i.e., the chiral condensate) of 2-flavor QCD. Assuming a very heavy gluon mass, it can be obtained from QCD in a straightforward manner. This approach, called "hadronization", was proposed by Kleinert [26]. It leads to the Lagrangian

$$\mathscr{L}_{\Phi'} = Tr(\partial_\mu \Phi'^\dagger)(\partial_\mu \Phi') + m^2 Tr \Phi'^\dagger \Phi' + \lambda_1 (Tr \Phi'^\dagger \Phi')^2 + \lambda_2 Tr(\Phi'^\dagger \Phi')^2 \ , \tag{1.21}$$

where Φ' is the matrix with components $\Phi'_{pq} \sim \langle \overline{\psi}^l_p \psi^r_q \rangle$. Note that Φ' differs from the later used Φ simply by a trivial factor of $\sqrt{2}$ due to notation.
Furthermore one identifies:

$$\Phi' = (\sigma + i\eta)t^0 + (\vec{a}_0 + i\vec{\pi})\vec{t} \, , \quad \Phi'^\dagger = (\sigma - i\eta)t^0 + (\vec{a}_0 - i\vec{\pi})\vec{t} \, , \tag{1.22}$$

where the t^i are the generators of $U(2)$, i.e., t^0 is half the 2×2 unit matrix, and the components of $\vec{t}$ are half the Pauli matrices. Hence, we obtain for example

$$Tr \Phi'^\dagger \Phi' = \frac{1}{2} \left(\sigma^2 + \eta^2 + \vec{a}_0^{\,2} + \vec{\pi}^2 \right) \ . \tag{1.23}$$

The fields σ, a_0^+, a_0^0, a_0^- are identified with the scalar mesons, whereas the fields η, π^+, π^0, π^- are identified with the pseudoscalar mesons.

says: if a gauge symmetry is spontaneously broken, the gauge-field becomes massive and "eats up" the would-be Goldstone boson.

Alternatively, one can derive the Lagrangian $\mathscr{L}_{\Phi'}$ in a less rigorous manner using a clever argument. Consider Φ' as a complex $N_f \times N_f$ matrix. As already discussed, the QCD Lagrangian for N_f flavors and vanishing quark masses is invariant under the chiral symmetry group

$$G_{ch} \equiv U(N_f)^r \times U(N_f)^l \ . \tag{1.24}$$

$\mathscr{L}_{\Phi'}$ is simply the most general renormalizable Lagrangian for a complex matrix field Φ' invariant under G_{ch} [27].
Due to the isomorphisms

$$U(N)^r \simeq U(1)^r \times [SU(N)/Z(N)]^r \ , \quad U(N)^l \simeq U(1)^l \times [SU(N)/Z(N)]^l \ , \tag{1.25}$$

$$U(1)^l \times U(1)^r \simeq U(1)^V \times U(1)^A \ , \tag{1.26}$$

we obtain the isomorphism

$$G_{ch} \simeq U(1)^V \times U(1)^A \times [SU(N_f)/Z(N_f)]^l \times [SU(N_f)/Z(N_f)]^r \ . \tag{1.27}$$

The vector subgroup $U(1)^V$ corresponds to quark number conservation. It should not affect the chiral phase transition, since it is not broken in the phase of spontaneously broken chiral symmetry. Since G_{ch} is a continuous symmetry, one believes that only its Lie algebra is important regarding universality. Therefore the two discrete $Z(N_f)$ symmetry groups can be omitted, so that we are left with the subgroup of G_{ch} which is relevant for the chiral phase transition:

$$G_f = U(1)^A \times SU(N_f)^l \times SU(N_f)^r \ . \tag{1.28}$$

Instantons explicitly break down the axial $U(1)_A$ symmetry to a discrete axial $Z_A(N_f)$ symmetry. Accordingly G_f boils down to

$$G_f^{'} \equiv Z(N_f)^A \times SU(N_f)^l \times SU(N_f)^r \ , \tag{1.29}$$

and accordingly, since $Z(N_f)^A$ is discrete, we are left with the symmetry group

$$G_r \equiv SU(N_f)^l \times SU(N_f)^r \tag{1.30}$$

relevant for the chiral phase transition in presence of the anomaly. According to [28], in four dimensions the most general renormalizable $U(N_f)^l \times U(N_f)^r$ Lagrangian for $N_f \geqslant 3$ is given by

$$\mathscr{L}_\Phi = \frac{1}{2} Tr(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + \frac{1}{2} m^2 Tr\Phi^\dagger\Phi + \frac{\pi^2}{3} g_1 (Tr\Phi^\dagger\Phi)^2 + \frac{\pi^2}{3} g_2 Tr(\Phi^\dagger\Phi)^2 \ , \quad (N_f \geqslant 3) \ , \tag{1.31}$$

whereas the most general renormalizable $SU(2)^l \times SU(2)^r$ invariant Lagrangian in the irreducible parametrization for Φ is given by

$$\mathscr{L}_P = \frac{1}{2} Tr(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + \frac{1}{2} m^2 Tr\Phi^\dagger\Phi + \frac{\pi^2}{3} g_1 (Tr\Phi^\dagger\Phi)^2 \ , \quad (N_f = 2) \ . \tag{1.32}$$

This is a crucial point. In presence of the anomaly the group $U(N_f)^l \times U(N_f)^r$ would be too large. One can use the $[\underline{2}, \overline{\underline{2}}]$ representation of $SU(2)^l \times SU(2)^r$ to restrict the model to G_r. In this case an irreducible parametrization of Φ is provided by

$$\Phi = (\sigma I_2 + i\vec{\pi} \cdot \vec{\tau}) \ , \tag{1.33}$$

where I_2 is the 2×2-unit matrix, $\vec{\tau}$ are the Pauli matrices, and σ as well as $\vec{\pi}$ are real-valued. In this case

$$\Phi^\dagger \Phi = (\sigma^2 + \vec{\pi}^2) I_2 \ ,$$

and therefore

$$(Tr\Phi^\dagger\Phi)^2 = Tr(\Phi^\dagger\Phi)^2 \ .$$

According to Paterson [28], the renormalization group equations in parameter space within the ε-expansion read

$$\beta_1 = -\varepsilon g_1 + \left[\frac{1}{3}(N_f^2 + 4) g_1^2 + \frac{4}{3} N_f g_1 g_2 + g_2^2 \right] \ , \tag{1.34}$$

$$\beta_2 = -\varepsilon g_2 + \left[2 g_1 g_2 + \frac{2}{3} N_f g_2^2 \right] \ . \tag{1.35}$$

Note that the factor $1/8\pi^2$ appears in Paterson's equations (3.14) and (3.15) simply because he uses a different notation. $\frac{\pi^2}{3} g_i$ in our notation corresponds to $\frac{1}{4!} u_i$ in Paterson's notation.
Paterson calculated the fixed points in parameter space for $\mathscr{L}_\Phi$ in case of $N_f \geqslant 3$ within the ε-expansion:

$$\vec{g}^*_{triv} \equiv (g_1^*, g_2^*) = (0, 0) \ , \tag{1.36}$$

$$\vec{g}^*_H \equiv (g_1^*, g_2^*) = (6\varepsilon/(2N_f^2 + 8), 0) \ . \tag{1.37}$$

The latter one, $\vec{g}^*_H$, is infrared unstable and coincides with the infrared fixed point which one encounters for the usual $O(M = 2N_f^2)$-model

$$\mathscr{L}_{O(M)} = \frac{1}{2} \sum_{i=1}^{M} \partial^\mu \phi_i \partial_\mu \phi_i + \frac{1}{4!} u \left(\sum_{i=1}^{M} \phi_i^2 \right)^2 \ . \tag{1.38}$$

The corresponding infrared fixed point for $\mathscr{L}_{O(M)}$ was calculated by Amit in [22] within the ε-expansion:

$$u^* = \frac{6}{M + 8} \varepsilon \ . \tag{1.39}$$

In the case of $\mathscr{L}_P$ for $N_f = 2$, however, Paterson argues that $\mathscr{L}_P$ is equivalent to $\mathscr{L}_{O(M)}$ with $M = 4$, i.e., the usual $O(4)$-model. He also points out that the phase transition in the $O(4)$-model is known to be of second order in $4 - \varepsilon$ ($\varepsilon > 0$) dimensions.
Note that in [29] Pisarski and Stein reproduce Paterson's results (1.34) and (1.35) as well as (1.36) and (1.37). They state that $\mathscr{L}_\Phi$ is the most general renormalizable Lagrangian also for $N_f \leq 2$ and that (1.37) becomes infrared stable when

$$N_f < \sqrt{2} \ . \tag{1.40}$$

In [27] Wilczek and Pisarski discuss the same system of equations, i.e., (1.34) and (1.35). We want to point out a typo. They state that the infrared fixed point is stable for $0 < N_f < \sqrt{2}$ with $O(2N_f)$ critical exponents, and they interpret the absence of infrared stable fixed points for

$N_f > \sqrt{3}$ in the sense that the phase transition is of first order. For $N_f = 1$ this is indeed not in contradiction with [22], since $O(2N_f^2) = O(2N_f)$ in this case. Note that $\mathscr{L}_\Phi$ as well as the $O(2)$-model have indeed an order parameter with two components (for $N_f = 1$ Φ is a complex number). Hence, the E.Q.F. which states that the number of components for the order parameter has to be equal for two models in the same universality class is fulfilled. In [27] the equations (5) correspond to (3) where Φ is a complex $N_f \times N_f$ matrix. In this case, according to Paterson, one has $O(2N_f^2)$ instead of $O(2N_f)$. The authors do not state explicitly[9] that $\mathscr{L}_P$ for $N_f = 2$ is the most general renormalizable, irreducible Lagrangian with $O(4)$ critical exponents. Also if there exists an infrared stable fixed point for $\mathscr{L}_\Phi$ in case of $N_f = 2$, the above mentioned E.Q.F. would not allow that it has $O(2N_f)$ critical exponents. Simply because for $N_f = 2$ $\mathscr{L}_\Phi$ has an order parameter with 8 components, since Φ is a complex 2×2-matrix. The 8 degrees of freedom correspond to the 4 scalar and the 4 pseudoscalar mesons. The $O(4)$-model instead has an order parameter with 4 components, where the 4 degrees of freedom are identified with the chiral partners σ and $\vec{\pi}$. Wilczek and Pisarski take account of this so-called $U(1)_A$-anomaly by adding the term

$$\mathscr{L}'_\Phi = c\left(det\Phi + det\Phi^\dagger\right) \tag{1.41}$$

to the Lagrangian, which ensures the explicit symmetry breaking. This implies that the system of equations (1.34) and (1.35) is no longer valid. The authors however do not write down the equations for this case, i.e., $\mathscr{L}_a \equiv \mathscr{L}_\Phi + \mathscr{L}'_\Phi$. They discuss the influence of the anomaly on a qualitative level. All their conclusions were summarized by Meyer-Ortmanns in [24]. Meyer-Ortmanns also emphasizes that these conclusions are based a) on the assumption that the absence of a stable infrared fixed point implies a first-order phase transition, and b) on the extrapolation of the ε-expansion to $\varepsilon = 1$.[10] Wilczek and Pisarski state that in presence of the anomaly, the case of two massless flavors is special. Considering a temperature dependent anomaly strength, the authors argue that since the η degree of freedom remains massive about the critical temperature, the phase transition can be of second order with $O(4)$ critical exponents, i.e., the $O(4)$ model can fall into the same universality class as $\mathscr{L}_\Phi + \mathscr{L}'_\Phi$. The argument itself is incomplete. Also the $\vec{a}_0$ has to remain massive in order to fulfill the above mentioned E.Q.F. Only if all these 4 particles remain massive, the order parameter undergoes long-range fluctuations in only 4 of its components. The massive 4 degrees of freedom could only undergo small fluctuations and are not important regarding universality.
Let us sum up. ***Assume*** QCD for vanishing quark masses and in presence of the $U(1)^A$-anomaly. ***Then***

1. the $O(4)$ model, i.e., $\mathscr{L}_{O(4)}$, falls into the same universality class as the Lagrangian describing the 2-flavor QCD chiral condensate,

2. the $O(2)$ model, i.e., $\mathscr{L}_{O(2)}$, falls into the same universality class as the Lagrangian describing the 1-flavor QCD chiral condensate,

if the QCD phase transition is of second order.

[9] Wilczek did that later in [30].

[10] In [31] Butti, Pelissetto and Vicari justify the extrapolation within a perturbative framework in which D is fixed to the physical value, i.e., $D = 3$. The β-functions are computed to six loops.

The $O(N)$ model

We already motivated the investigation of the $O(N)$ model. Now we want to focus on the model itself.
Let us begin the discussion of the $O(4)$ model with some important facts. First of all, let us recall the definition of the orthogonal group:

definition 1.2 (orthogonal matrix and symmetry group $O(N)$)

- *A matrix $Q \in \mathbb{R}^{N\times N}$ with orthonormal column vectors $(u_{*1}, ..., u_{*N})$ is said to be orthogonal.*
- *The set of all orthogonal $(N \times N)$-matrices Q is a representation of the orthogonal group $O(N)$.*

Group elements $Q \in O(N)$ have the following properties:

- $Q^{-1} = Q^T \quad , \quad |detQ| = 1 \quad ,$ where T denotes the transpose.
- The Euclidean product of vectors is invariant: $(Q\vec{u}) \cdot (Q\vec{v}) = \vec{u} \cdot \vec{v}$.
- The Euclidean norm is invariant: $|Q\vec{u}| = |u|$.

The group $O(N)$ is actually larger than necessary to cover all rotations in N-dimensional Euclidean space uniquely. The set of all group elements with determinant $+1$ are sufficient for this and form a group, the *special orthogonal group* $SO(N)$.

Regarding the discussion of the $O(4)$-model in the literature, there is a certain confusion concerning isomorphy. $SU(2) \times SU(2)$ is only homomorphic to $SO(4)$, but locally isomorphic because the generators are the same. Instead, an isomorphism is given by $SU(2) \times SU(2)/Z(2) \simeq SO(4)$ (see page 232 of [32]). Accordingly we have the isomorphism $Z(2) \times SU(2) \times SU(2) \simeq Z(2) \times SO(4) \times Z(2)$. Note that $Z(2) \times SO(4) \times Z(2)$ is *not* isomorphic to $O(4)$, since the centers of both groups are not isomorphic. Also $Z(2) \times SO(4)$ is not isomorphic to $O(4)$ due to the same reason. We only have local isomorphy of $SU(2) \times SU(2)$ and $O(4)$.
Note that there is a representation of $O(4)$-transformations acting on a four-component scalar vector $\vec{\phi}$ in the fourdimensional internal symmetry space. Because in the case of vanishing quark masses all components of the chiral condensate are equal, $\Phi^0_{pq} = \varphi\delta_{pq}$, it is isomorphic to a single scalar field. Accordingly, the absolute value of the $O(4)$-vector, $\phi \equiv |\vec{\phi}|$, is the most natural candidate for the chiral condensate. The most general renormalizable Lagrangian for $D = 3$ spatial dimensions, in case of an N-component scalar vector field $\vec{\phi}$, invariant under $O(N)$, is given by

$$\mathscr{L} = \frac{1}{2}\partial_\mu\vec{\phi}\partial^\mu\vec{\phi} + \mu^2\vec{\phi}^2 - \frac{\lambda}{N}\vec{\phi}^4. \tag{1.42}$$

To prove this, first note that in 1+3 dimensions $[\mathscr{L}] = MeV^4 \Rightarrow [\partial_\mu\vec{\phi}\partial^\mu\vec{\phi}] = MeV^4 \Rightarrow [\phi_i] = MeV$. Now consider all $O(N)$-symmetric terms, namely $a_1|\vec{\phi}|$, $a_2\vec{\phi}^2$, $a_3|\vec{\phi}|^3$, $a_4\vec{\phi}^4$, $a_5|\vec{\phi}|^5$, $(\cdots)$. The dimension of the couplings is obviously of the form $[a_i] = MeV^n$. Since $n \geq 0$ only for $i = 1, \ldots, 4$, all other terms are non-renormalizable. The terms involving the absolute value are non-renormalizable due to a more sophisticated argument. To show this, we use the fact that a

quadratically integrable function $f(x)$ can be rewritten as a polynomial in x which has an infinite number of terms. Consider the quadratically integrable function $f(x) = \sqrt{x}$ and the *Laguerre polynomials* $L_k(x) = e^x \frac{d^k}{dx^k}(x^k e^{-x})$. Because the *Laguerre functions* $u_k(x) = \frac{1}{k!} e^{-x/2} L_k(x)$ form a complete orthonormal system in $L^2(0,\infty)$, we can expand $\sqrt{x}$ as follows: $\sqrt{x} = \sum_{k=1}^{\infty} a_k u_k(x)$. The exponential functions appearing in $u_k(x)$ can be expanded into a usual Taylor series, so that we indeed end up with a polynomial. Accordingly $|\vec{\phi}| = \sqrt{\vec{\phi}^2} = \sqrt{\phi_1^2 + \dots \phi_N^2}$ can be rewritten as a polynomial in $\vec{\phi}^2$ with an infinite number of terms. Consequently $a_1|\vec{\phi}|$ and $a_3|\vec{\phi}|^3$ are non-renormalizable along the lines of our first argument.

As we assumed $N_f = 2$ flavors and vanishing quark masses, our effective model is valid for the lightest scalar and pseudoscalar mesons, which can be constructed from the light quark flavors u and d. Regarding the sigma particle and the three pions as chiral partners, this is a reasonable choice. One can add the most natural term to the Lagrangian which breaks the $O(N)$-symmetry explicitly and which leads to an analog of the so-called *Gell-Mann-Oakes-Renner relation*. This term is given by $\mathscr{L}_H \equiv H\phi_1$ and takes account of nonvanishing quark masses. However, we do not want to elaborate on this further.

As already mentioned, $\phi = |\vec{\phi}|$ plays the role of the chiral condensate. In the following chapters we denote by $\phi^0 \equiv \Phi$ the condensate in the absence of sources, i.e., at the global minimum. The extrema of the effective potential we denote by φ. If the Lagrangian (1.42) is an appropriate choice, we expect to observe spontaneous symmetry breaking in the $O(4)$-model, as in QCD. Whereas in QCD, Φ^0 is merely $SU(2)^V$- instead of $SU(2)^V \times SU(2)^A \times Z(2)^A$-symmetric in the spontaneously broken phase, we should find φ $O(4)$-symmetric resp. $O(3)$-symmetric ($O(3)$ is isomorphic to $SU(2)$) in the spontaneously restored resp. broken phase. This is indeed the case.

Finally, let us prove that the $O(4)$ model fulfills the E.Q.F.'s which are necessary to fall into the same universality as the effective Lagrangian for the QCD order parameter in case of 2 massless flavors. The E.Q.F.'s are fulfilled, which one can justify largely independent of the above discussion:

The **dimensionality of the system** is the same in both models. Both are defined in 1+D dimensional Euclidean space-time where $D = 3$ is the number of spatial degrees of freedom, i.e., we have $\varepsilon = 1$ in $D = 4 - \varepsilon$.

The **number of components of the order parameter** is the same since the η-particle and the $\vec{a}_0$ remain massive about T^* if one assumes that Φ is a complex 2×2 matrix, i.e., it has 8 components. Since long-range fluctuations of light particles dominate the behavior at the critical point, the 4 degrees of freedom corresponding to η and $\vec{a}_0$ can be neglected. Hence we are left with 4 components, as in case of the $O(4)$ order parameter $\vec{\phi}$. On the other hand, if one argues that in presence of the anomaly the relevant symmetry is $SU(2) \times SU(2)$ and a renormalizable Lagrangian is equivalent to that of the $O(4)$ model, this condition is trivially fulfilled.

The relevant **symmetry properties of the order parameter** are also the same. Note that the group $U(1)^V$ does not affect the chiral dynamics and that in presence of the anomaly the remaining continuous symmetry is locally isomorphic to $O(4)$. Note that one has to assume that only infinitesimal symmetry transformations of the order parameter are important regarding universality in a field-theoretical model.

Fortunately, lattice QCD for $N_f = 2$ indicates a second-order phase transition for QCD [33],

which is the basis for universality arguments[11]. Since a system which undergoes a second-order phase transition is scale invariant at the critical temperature T^*, one would expect the $O(4)$-model to become scale invariant at T^*. This would correspond to vanishing sigma and pion masses. However, approximations might ruin scale invariance.
Furthermore, one can derive that the phase transition in the $O(4)$-model is of second-order, and that the critical exponents are pretty close to those from lattice calculations [34].
In the following chapters, we focus on the implications arising from application of the CJT formalism. We investigate whether the Goldstone theorem is respected or not, and we examine what kind of phase transition can be observed, if there is one at all. Experimental parameters are the masses for the $f_0(600)$ (or σ) resonance, which is listed with mass $m_{f_0} = 400 - 1200\ MeV$ in the Particle Data Booklet [5], and the masses for the pions, listed with $m_{\pi^\pm} = 139.57018 \pm 0.00035 MeV$ and $m_{\pi^0} = 134.9766 \pm 0.0006 MeV$.

Further remarks on universality

Whereas quantities as the specific heat C or the transverse susceptibility χ_T can be directly measured, the order parameter is a theoretical quantity which has to be identified with a physical observable. In the case of the chiral $O(4)$ model for example we have $\phi \equiv |\vec{\phi}|$ which we identify with the chiral condensate. A ferromagnetic n-vector model involves the magnetization as order parameter, i.e., $\phi \equiv |\vec{M}| = M$. What about other relations like (1.15)-(1.18)? In the most general sense, two Lagrangians $\mathscr{L}_A$ and $\mathscr{L}_B$ can only fall into the same universality class if there exists a Lagrangian $\mathscr{L}$ which parametrizes both, i.e., such that $\mathscr{L}_A$ and $\mathscr{L}_B$ are special cases of $\mathscr{L}$. Otherwise there obviously would be no common reference system wherein critical exponents are defined. Consider for example a relation which is related to N-component scalar field theory, where $\vec{\phi} = (\phi_1, \cdots, \phi_N)$, in the absence of sources:

$$\mathcal{G}(\vec{x}) \sim \frac{1}{r^{D+\eta-2}} \ , \quad r \equiv |\vec{x}| \ . \tag{1.43}$$

This is a relation for the 2-point function in the absence of sources

$$\mathcal{G}(\vec{x} - \vec{y}) \equiv \langle \phi(\vec{x}) \phi(\vec{y}) \rangle |_{J=K=0} \ , \quad \text{(compare with definition A.11)} \ , \tag{1.44}$$

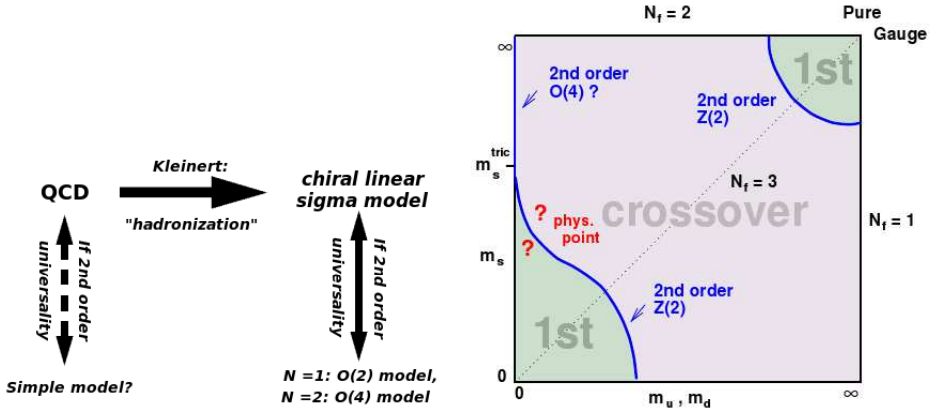
which involves a new critical exponent with specific values for each universality class in N-component scalar field theories. It turns out that it is related to the above critical exponents via

$$\eta = \frac{2D(\beta - 1 + \alpha/2) - 2\alpha + 4}{2 - \alpha} \ , \tag{1.45}$$

which again is a remarkable phenomenon [17]. On the other hand consider for instance QCD in the large-N_f limit, which was investigated by Gracey in [35]. Gracey derives critical exponents for the quark and the gluon fields, which obviously have no meaning in the Ising model for instance. Regarding these specific critical exponents both models have no common framework.

[11] For first-order phase transitions universality does not exist, because the correlation length remains finite (compare with p.520 of [14]).

In this sense every universality class can in principle have its own set of universal relations and critical exponents, such that different kinds of universality classes are not distinguished only by the value of some universal critical exponents. According to Gracey, at leading order in $1/N_f$, QCD should fall into the same universality class as the so-called *non-Abelian Thirring model*. The gluonic self-interaction diagrams, responsible for the increase of the renormalized coupling constant for small energy scales, are absent in the large-N_f limit, i.e., QCD at leading order in $1/N_f$ becomes similar to QED. In general, however, the philosophy is of course to keep the number of universality classes as small and elementary as possible.



(a) Brief summary.

(b) quark-mass diagram, taken from [36]

Figure 1.4: An overview.

Note that the chiral $O(4)$ model or the chiral linear sigma model are not parametrized by the QCD Lagrangian, as they are effective theories for the order parameter. However, via the above quoted technique of hadronization one is able to derive the chiral linear sigma model more or less directly from the QCD Lagrangian. Figure 1.4(a) overviews the models we discussed so far regarding QCD. We motivated at length the general background why to investigate the $O(4)$ and the $O(2)$ model. Although the $O(2)$ model should fall into the same universality class as one-flavor QCD, if the chiral phase transition is of second order, we are interested in this model mainly because it is the first step towards the $O(4)$ model in four-dimensional polar coordinates. Although physically not true, we also term the fields in the $O(2)$ model sigma and pion, having in mind the generalization to $O(4)$. The appropriate interpretation of the polar $O(2)$ model regarding one-flavor QCD is still ongoing work. Figure 1.4(b) summarizes considerations regarding the expected order of the QCD phase transition. For a detailed explanation we refer to Ref. [36]. We merely want to point out that along the critical lines (except for the $m_q = 0$ axes) where the phase transition is expected to be of second order, the phase transition should fall into the same universality class as the $Z(2)$ symmetric ($D = 3$-dimensional) Ising model. This would motivate the study of the system determined by the Lagrangian (3.1) with $m^2 < 0$ (compare with section 1.2. of [15] and page 295 of [22]). Within the scope of this thesis, however, we restrict ourselves to the model with a one-minimum potential as a mere toy model (see section 3.1).

Chapter 2

CJT formalism and WKB method

2.1 A resummation scheme: CJT formalism

Outline

In this section we want to explain the CJT formalism, which was established in 1974 by Cornwall, Jackiw and Tomboulis as a method to compute the effective action of a theory [37]. In principle the approach is not limited to scalar field theory. Already in the original work it is applied to an Abelian gauge theory of fermions and vector mesons. However, we restrict our discussion of the CJT formalism to scalar field theories, since we are not concerned with anything else in this thesis.
For those readers who are not familiar with the context in which the effective potential appears, we refer to appendix D which reviews the effective potential in scalar field theory with a single source J at nonzero temperature. The characteristic feature of the CJT formalism is to introduce a second, bilocal source K. Technically, the effective potential becomes a functional of the (connected) one-point function and the full connected two-point function in the presence of sources J and K. However, the case with physical meaning is $J = K = 0$. All physical quantities (the physical masses and the condensate) are defined at the global minimum. Note that for translationally invariant systems, which we assume in all our numerical simulations, the effective action is proportional to the effective potential[1]. In the first subsection, we will show that the common effective action based on a single source J is the same quantity as the effective action based on sources J and K, in case we set $K = 0$. Of course, the explicit expressions are different, since in the latter case the effective action explicitly depends on the connected two-point function. Nevertheless, the quantity itself remains the same. In the second subsection, we explain why the CJT formalism is a resummation scheme and what is actually resummed. In the third subsection we present the stationarity conditions which determine the condensate and the full connected propagator. The fourth subsection illustrates the role of the sources and establishes a connection with section 2.2.

[1] Compare with (D.28).

The effective action in the presence of sources J and K

Consider for simplicity a theory defined by the Minkowskian Lagrangian

$$\mathscr{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - U(\phi) \ , \tag{2.1}$$

where $U(\phi)$ is the classical potential. If we want to study the theory at nonzero temperature (SQFT), we have to use the Euclidean version of the Lagrangian (see appendices A and E), i.e.,

$$\mathscr{L}_E = \frac{1}{2}\left(\frac{\partial\phi}{\partial\tau'}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x'}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial y'}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial z'}\right)^2 + U\left(\phi\right) \ . \tag{2.2}$$

In this case the generating functional is given by

$$Z\left[J,K\right] = \mathcal{N}e^{W[J,K]} = \mathcal{N}\oint\mathcal{D}\phi e^{-S_E[\phi]+\phi J+\frac{1}{2}\phi K\phi} \ , \tag{2.3}$$

$$S_E\left(\phi\right) = \int d\tau'\int d^3\vec{x'}\left[\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau'}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x'}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial y'}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial z'}\right)^2 + U\left(\phi\right)\right] \equiv \int\limits_{X'}\left[T+U\right]$$

where $\phi = \phi(\tau', \vec{x'})$, $\phi J \equiv \int\limits_{X'}\phi\left(X'\right)J\left(X'\right)$ and $\phi K\phi \equiv \int\limits_{X',X''}\phi\left(X'\right)K\left(X',X''\right)\phi\left(X''\right)$. We use the symbol $\oint$ to remind of the periodic boundary condition and the additional integration in the discretized version of the path integral. For further details we refer to appendix A.

In this section we use primes to avoid confusion with the points appearing in the 1-point resp. in the 2-point function. However, in principle it does not play any role which symbol one chooses for internal integration variables.

According to definition (A.11), the thermal one-point function in the presence of sources, correctly normalized, is given by

$$\left\langle\phi\left(X\right)\right\rangle = \frac{\oint\mathcal{D}\phi\ e^{-S_E[\phi]+\phi J+\frac{1}{2}\phi K\phi}\phi\left(X\right)}{\oint\mathcal{D}\phi\ e^{-S_E[\phi]+\phi J+\frac{1}{2}\phi K\phi}} \quad , \tag{2.4}$$

and the thermal two-point function (synonymous: thermal propagator) in the presence of sources, correctly normalized, is given by

$$\left\langle\phi\left(X_1\right)\phi\left(X_2\right)\right\rangle = \frac{\oint\mathcal{D}\phi\ e^{-S_E[\phi]+\phi J+\frac{1}{2}\phi K\phi}\phi\left(X_1\right)\phi\left(X_2\right)}{\oint\mathcal{D}\phi\ e^{-S_E[\phi]+\phi J+\frac{1}{2}\phi K\phi}} \ . \tag{2.5}$$

From expression (2.3) we deduce

$$\frac{\delta W}{\delta J\left(X\right)} = \frac{1}{\oint\mathcal{D}\phi\ e^{-S_E[\phi]+\phi J+\frac{1}{2}\phi K\phi}}\oint\mathcal{D}\phi\ e^{-S_E[\phi]+\phi J+\frac{1}{2}\phi K\phi}\phi\left(X\right) = \left\langle\phi\left(X\right)\right\rangle \ , \tag{2.6}$$

$$\begin{aligned}\frac{\delta W}{\delta K\left(X_1,X_2\right)} &= \frac{1}{\oint\mathcal{D}\phi\ e^{-S_E[\phi]+\phi J+\frac{1}{2}\phi K\phi}}\oint\mathcal{D}\phi\ e^{-S_E[\phi]+\phi J+\frac{1}{2}\phi K\phi}\frac{1}{2}\phi\left(X_1\right)\phi\left(X_2\right) \\ &= \frac{1}{2}\left\langle\phi\left(X_1\right)\phi\left(X_2\right)\right\rangle \ .\end{aligned} \tag{2.7}$$

From the construction rules (D.24) we know how to express n-point functions in the presence of a source J in terms of the connected n-point functions in the presence of a source J. Cornwall, Jackiw and Tomboulis used this as a guide when they defined:

definition 2.1 (the connected thermal ... in the presence of sources)

- ***... one-point function ...*** *,* $G_K(X)$*,*

$$\frac{\delta W}{\delta J(X)} = \langle \phi(X) \rangle \equiv G_K(X) \ . \tag{2.8}$$

- ***... two-point function ...****,* $G_K(X_1, X_2)$*,*

$$\frac{\delta W}{\delta K(X_1,X_2)} = \frac{1}{2}\langle \phi(X_1)\,\phi(X_2) \rangle \equiv \frac{1}{2}\Big[G_K(X_1,X_2) + G_K(X_1)\,G_K(X_2) \Big] \ . \tag{2.9}$$

Using the above relations as definitions is based on the intuition that $G_K(X)$ resp. $G_K(X_1, X_2)$ play the same role in the theory containing the bilocal source as the functions $G(X)$ resp. $G(X_1, X_2)$ (compare with definition D.1) do in the theory using the conventional generating functional.
We want to emphasize an important fact:

- The definitions D.1 are, mathematically speaking, an embedding in the generalized definitions 2.1.
- **proof**: $G_K(X)|_{K=0} = G(X)$ as well as $G_K(X)|_{J=0,K=0} = \mathcal{G}(X)$. With that follows $\langle \phi(X_1)\,\phi(X_2) \rangle|_{K=0} = G_K(X_1,X_2)|_{K=0} + G(X_1)\,G(X_2)$. $\langle \phi(X_1)\,\phi(X_2) \rangle|_{K=0}$ is nothing else but $\langle \phi(X_1)\,\phi(X_2) \rangle_{\not K}$, the thermal propagator in the presence of a source J, which is defined for the conventional generating functional via (D.1). Hence, comparison with formula (D.25) leads to $G_K(X_1,X_2)|_{K=0} = G(X_1,X_2)$.

In complete analogy to the theory with a single source J, we define the effective action in the presence of the sources J and K as double Legendre transform. But let us first discuss the generalization of the double Legendre transform to functionals, which is not trivial. What plays the same role as the double Legendre transform for functions, in the case of functionals? Let us just motivate the result, using intuition instead of rigorous functional analysis. It is absolutely plausible that the double Legendre transform of $W[J,K]$ has the same form as in the case of functions, namely $W - \int\limits_X \frac{\delta W}{\delta J} J - \int\limits_X \int\limits_Y \frac{\delta W}{\delta K} K$, regarding the integrals as sums over variables with continuous indices. Plugging in the expressions from definition 2.1 yields $W - \int\limits_X G^{(1)} J - \int\limits_X \int\limits_Y \frac{1}{2} G_K^{(2)} K - \int\limits_X \int\limits_Y \frac{1}{2} G_K^{(1)} G_K^{(1)} K$. How to choose the arguments for $G_K^{(2)}$? Regarding $G_K(A,B)$ and $K(C,D)$ as matrices with continuous indices, the simplest operation leading to a functional with no explicit dependence on space-inverse temperature is the following: first a matrix multiplication $\sum_{X_2} G_K(A,X_2)\,K(X_2,C)$, then taking the trace of the result, i.e., $\sum_{X_1}\sum_{X_2} G_K(X_1,X_2)\,K(X_2,X_1)$. Hence, we have $\int\limits_{X_1}\int\limits_{X_2} G_K(X_1,X_2)\,K(X_2,X_1)$.
Now we can define the effective action in our theory with two sources[2]:

[2] If the reader wants to compare with formula (2.13) of the original work [37], we want to note that the authors abbreviate $\int\limits_{X_1}\int\limits_{X_2} G_K(X_1,X_2)\,K(X_2,X_1)$ as TrG_KK.

definition 2.2 (effective action)

$$\Gamma\left[G_K^{(1)}, G_K^{(2)}\right] \equiv W\left[J,K\right] - \int\limits_{X_1} J\left(X_1\right) G_K\left(X_1\right) - \frac{1}{2}\int\limits_{X_1}\int\limits_{X_2} G_K\left(X_1\right) K\left(X_1,X_2\right) G_K\left(X_2\right) - \frac{1}{2}\int\limits_{X_1}\int\limits_{X_2} G_K\left(X_1,X_2\right) K\left(X_2,X_1\right) \ .$$

Also the generalized definition 2.2 of the effective action covers the conventional effective action defined in (D.2) as a special case:

$$\Gamma\left[G^{(1)}, G^{(2)}\right] = W\left[J\right] - \int\limits_{X_1} J\left(X_1\right) G\left(X_1\right) = \Gamma\left[G^{(1)}\right] \tag{2.10}$$

proof: $\Gamma\left[G_K^{(1)}, G_K^{(2)}\right]|_{K=0} = \Gamma\left[G^{(1)}, G^{(2)}\right]$ and $W\left[J,K\right]|_{K=0} = W\left[J\right]$. With definition 2.2 this implies $\Gamma\left[G^{(1)}, G^{(2)}\right] = W\left[J\right] - \int\limits_{X_1} J\left(X_1\right) G\left(X_1\right)$. The right hand side is nothing else but $\Gamma\left[G^{(1)}\right]$.

This is a crucial point. The common effective action based on a single source J is the same quantity as the effective action based on sources J and K, in case we set $K = 0$. Although the explicit expressions are different, since in the latter case the effective action is explicitly dependent on the connected two-point function, the quantity itself remains the same. Hence, we found an alternative expression for the conventional effective action, which depends not only on the conventional $G\left(X\right)$ but also on the conventional $G\left(X_1,X_2\right)$.
From

$$\frac{\delta\Gamma}{\delta G_K\left(X\right)} = -J\left(X\right) - \int\limits_{X_2} K\left(X_1,X_2\right)\phi\left(X_2\right) \tag{2.11}$$

and

$$\frac{\delta\Gamma}{\delta G_K\left(X_1,X_2\right)} = -\frac{1}{2}K\left(X_1,X_2\right) \tag{2.12}$$

we see that $G\left(X\right)$ is determined (in consistence with the conventional theory) by

$$\frac{\delta\Gamma}{\delta G\left(X\right)} = -J\left(X\right) \quad , \tag{2.13}$$

whereas $G\left(X_1,X_2\right)$ has to fulfill

$$\frac{\delta\Gamma}{\delta G\left(X_1,X_2\right)} = 0 \ . \tag{2.14}$$

The effective potential in the CJT formalism

We now want to derive an explicit expression for the effective potential, which implies the resummed Dyson-Schwinger equation (D.23). Let us make the following ansatz for the effective

action:

$$\Gamma\left(G_K^{(1)}, G_K^{(2)}\right) \equiv -S_E\left(G_K^{(1)}\right) - \frac{1}{2} ln\Big(G_K^{(2)}\Big)^{-1} - \frac{1}{2}\left[D_K^{-1}\left(G_K^{(1)}\right) G_K^{(2)} - 1\right] + \Gamma_2\left(G_K^{(1)}, G_K^{(2)}\right) + C. \tag{2.15}$$

This ansatz is straightforward. The effective action is expected to be proportional to the classical action plus quantum corrections. This corresponds to the fact that the effective potential is the classical potential plus quantum corrections (compare with appendix D). The terms $-\frac{1}{2} ln\Big(G_K^{(2)}\Big)^{-1}$ and $-\frac{1}{2}\left[D_K^{-1}\left(G_K^{(1)}\right)\ G_K^{(2)} - 1\right]$ are suggested by the form of the Dyson-Schwinger equation. In the following we will determine Γ_2 such that the ansatz implies the resummed Dyson-Schwinger equation (D.23). The constant C can be chosen such that the effective potential has the correct $T = 0$ value.
From the ansatz follows immediately

$$\frac{\delta\Gamma\left(G_K^{(1)}, G_K^{(2)}\right)}{\delta G_K^{(2)}} = \frac{\delta\Gamma_2\left(G_K^{(1)}, G_K^{(2)}\right)}{\delta G_K^{(2)}} - \frac{1}{2}\frac{1}{\left(G_K^{(2)}\right)^{-1}}\left(-\left(G_K^{(2)}\right)^{-2}\right) - \frac{1}{2} D_K^{-1}\left(G_K^{(1)}\right) \ .$$

Together with equation (2.12) we have

$$-\frac{1}{2} K = \frac{1}{2}\left(G_K^{(2)}\right)^{-1} - \frac{1}{2} D_K^{-1} + \frac{\delta\Gamma_2\left(G_K^{(1)}, G_K^{(2)}\right)}{\delta G_K^{(2)}} \ ,$$

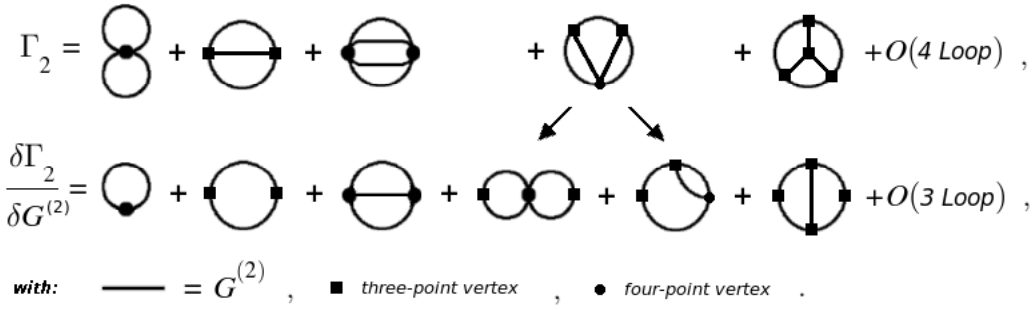
hence

$$0 = \frac{1}{2}\left(G^{(2)}\right)^{-1} - \frac{1}{2} D^{-1} + \frac{\delta\Gamma_2\left(G^{(1)}, G^{(2)}\right)}{\delta G^{(2)}} \ . \tag{2.16}$$

From comparison of equation (2.16) with (D.23) we finally determine

$$2\frac{\delta\Gamma_2\left(G^{(1)}, G^{(2)}\right)}{\delta G^{(2)}} = \Sigma' \quad , \quad D = \Delta \ .$$

Therefore one has to conclude that Γ_2 is the sum of all 2PI vacuum[3] graphs. We want to prove this for Γ_2 up to three-loop order in the case of a three-point and a four-point interaction vertex:



Up to prefactors, this is indeed Σ' up to two-loop order (see page 168). Three-point vertices can arise for example if one studies fluctuations around the vacuum (see section 2.2).
Note that the HTL approximation (see page 168) corresponds to taking into account only those

[3] *Vacuum* means no external lines.

diagrams (the double-bubble and the sunset diagram) in Γ_2 which yield Σ' to one-loop order. In the so-called *Hartree approximation* only the double-bubble diagram is taken into account, i.e., the sunset diagram in Γ_2 is neglected.
Let us now consider translationally invariant systems. For $G^{(1)}$ independent of space-inverse temperature (i.e., $\partial_\mu G^{(1)} \partial^\mu G^{(1)} = 0$), and for homogeneous systems (i.e., $\int d\tau d\mathbf{x}^3 \longrightarrow \Omega/T$) we can conclude from formulas (2.10) and (D.28):

$$V_{eff}\left[G^{(1)}, G^{(2)}\right] = -\frac{T}{\Omega}\Gamma\left[G^{(1)}, G^{(2)}\right]. \tag{2.17}$$

With this, we obtain[4]

result 2.1 (effective potential in the CJT formalism)

$$V_{eff} = U\left(G^{(1)}(k)\right) + \frac{1}{2}\int_k ln\Big(G^{(2)}(k)\Big)^{-1}$$
$$+\frac{1}{2}\int_k \left[D^{-1}\left(G^{(1)}, k\right)\ G^{(2)}(k)\right] + \Gamma_2\left(G^{(1)}(k), G^{(2)}(k)\right) + c\ .$$

Systems of equations

Conditions (2.13) and (2.14) play an important role. First consider $J = 0$, which says that we are at the vacuum (synonymous: ground state). In this context we want to mention that one refers to the expectation value in the absence of sources also as vacuum expectation value. Also note that the vacuum is different at different temperatures. This is why we speak of a thermal vacuum expectation value. In the vacuum, the full connected propagator at nonzero temperature, $\mathcal{G}(X_1, X_2)$, and the vacuum expectation value of the field at nonzero temperature, $\langle\phi(X)\rangle|_{J=0,K=0}$, are determined by the following system of equations:

$$\left.\frac{\delta\Gamma}{\delta G(X)}\right|_{J=0} = 0\ , \tag{2.18}$$

$$\left.\frac{\delta\Gamma}{\delta G(X_1, X_2)}\right|_{J=0} = 0\ . \tag{2.19}$$

For simplicity, let us consider translationally invariant systems, i.e., $G(X) \equiv G$ and $G(X_1, X_2) \equiv G(X_1 - X_2)$. In this case the effective action is proportional to the effective potential, so we are allowed to replace Γ by V_{eff} in the above conditions. Then (2.18) can be identified with the condition necessary for local extrema of the effective potential. In this thesis we refer to $\langle\phi(X)\rangle|_{K=0} \equiv \phi$ as "arbitrary point" ϕ. Accordingly the effective potential is a functional of ϕ and G, and we speak of the effective potential at an arbitrary point. In case we are at an extremum, we denote ϕ by φ and G by $\mathcal{G}$. The global minimum in turn we denote by Φ. In absence of sources the connected propagator is the same as the propagator. With this in mind (D.16) shall serve as a motivation why we are able to choose the ansatz

$$G \equiv \frac{1}{-k^2 + M^2} = \frac{1}{\omega_n^2 + \mathbf{k}^2 + M^2} \tag{2.20}$$

[4] $-S_E = -\int d\tau \int d^3\mathbf{x}\mathscr{L} = -\frac{\Omega}{T}\mathscr{L} = -\frac{\Omega}{T}U$, where in the last step we used $\partial_\mu\phi = 0$ for ϕ independent of space-inverse temperature. Then $-\frac{T}{\Omega}(-S_E) = U$.

for the full propagator in momentum space. M is the full mass, i.e., the physical mass.
Putting all together, the system of equations determining the extrema φ as well as the physical mass at the extrema is given by

result 2.2 (CJT system of equations at the extrema)

$$\frac{\delta V_{eff}(\phi, G)}{\delta \phi}|_{\phi=\varphi, G=\mathcal{G}} = 0 \ ,$$
$$\frac{\delta V_{eff}(\phi, G)}{\delta G}|_{\phi=\varphi, G=\mathcal{G}} = 0 \ .$$

Regarding the interpretation in our models, we refer to the global minimum Φ also as *condensate*. The effective potential at the global minimum, is related to the pressure p as follows:

$$V_{eff}\left[\varphi, \mathcal{G}\right] = -p \ . \tag{2.21}$$

In the now following proof, we use the symbol Ω for the grandcanonical potential (according to appendix G), whereas the symbol V refers to spatial volume (usually we use Ω instead). For homogeneous systems relation (G.6) is valid: $p = \frac{T}{V} \ln Z$. Together with formula (G.22) this yields $\Omega = -T \ln Z$. Z can be expressed within the path integral formulation and is related to W via $Z = e^W$, therefore $\Omega = -TW$. At the global minimum, definition 2.2 reduces to $\Gamma\left[\varphi, \mathcal{G}\right] = W$, where we have to set the sources to zero in W. Consequently, at the global minimum, this yields $\Omega = -T\Gamma\left[\varphi, \mathcal{G}\right]$, and with relation (2.17) we arrive at $\Omega = VV_{eff}$. So finally, we end up with $V_{eff} = \frac{\Omega}{V} = -p$. *q.e.d.*
However, one is not only interested in the global minimum, although all physical quantities (masses, thermodynamical quantities like the pressure, the condensate etc.) correspond to the vacuum. Consider an arbitrary point $\langle\phi(X)\rangle|_{K=0} \equiv \phi$. The nonvanishing source J deflects the expectation value for the field away from the vacuum. In theories featuring spontaneous symmetry breaking, also the effective potential at an arbitrary point ϕ is of interest.[5] In this case we are left with condition (2.14), since J is not specified any further, so that (2.13) is not a constraint. From (2.14) we obtain the condition which, together with the ansatz (2.20), determines the "mass at an arbitrary point ϕ":

result 2.3 (CJT gap equation at an arbitrary point)

$$\frac{\delta V_{eff}(\phi, G)}{\delta G} = 0 \ .$$

The role of the sources

Let us explain the role of the sources qualitatively. The generating functional contains all the physical information about the system, which is gained by a weighting of all possible paths $\phi\left(X'\right)$ the system may take. Such a path is illustrated qualitatively in fig.2.1, which requires further

[5] For exemplification consider a ferromagnet. From a heuristic point of view, the effective potential at an arbitrary point can be imagined as the free energy at a certain magnetization.

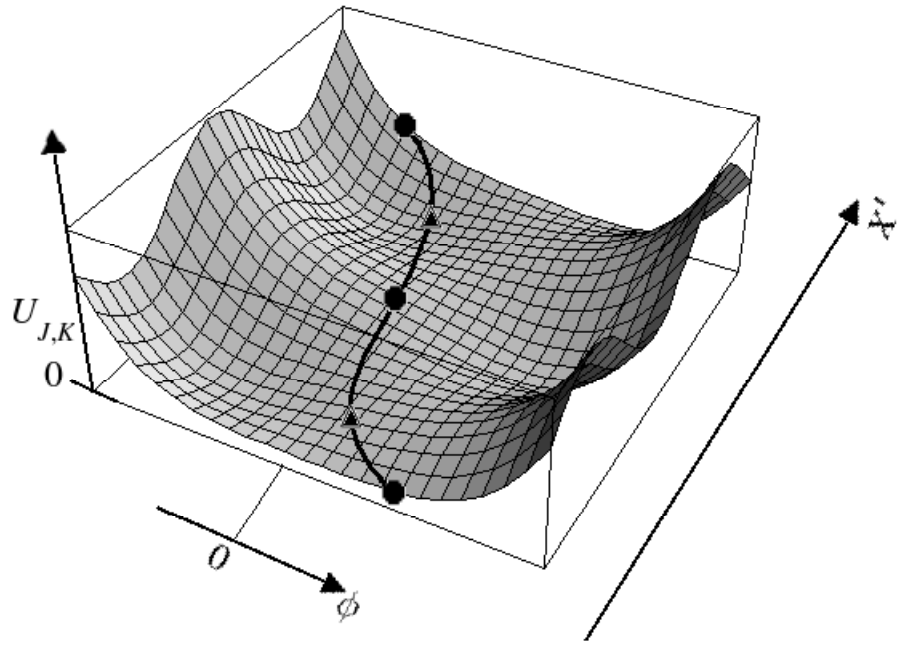


Figure 2.1: illustration of a path; the cutting plane in the front shows a potential that has the shape of U.

explanation. $\langle \phi(X) \rangle$ is a functional depending on the functions J and K. Instead of

$$-S_E \equiv -\int\limits_{X'} [T+U] \ ,$$

$$\text{we have} \int\limits_{X'} \left[-T-U+\phi(X')\,J(X') + \int\limits_{X''} \frac{1}{2}\phi(X')\,K(X',X'')\,\phi(X'') \right] \ .$$

So, instead of U, we have a "classical potential in the presence of sources"

$$U_{J,K} \equiv U[\phi(X')] - \phi(X')\,J(X') - \int\limits_{X''} \frac{1}{2}\phi(X')\,K(X',X'')\,\phi(X'') \equiv U[\phi(X')] + U_S[\phi(X')] \ .$$

This means, for each point X', the shape of the potential in which a path is considered, differs from that one of U in a characteristic way. For fixed X', the shape of U_S depends on how one chooses the functions J and K.

First consider (2.4) for vanishing sources, i.e., the vacuum expectation value for ϕ at nonzero temperature $\langle \phi(X) \rangle|_{J=0,K=0} = \frac{\int \mathcal{D}\phi \ e^{-S_E[\phi]}\phi(X)}{\int \mathcal{D}\phi \ e^{-S_E[\phi]}}$. Although it depends on temperature (the τ'-integration over the Lagrangian runs from 0 to $1/T$), only the classical potential enters the weighting: $U_{J,K}|_{J=0,K=0} = U$ at each point X'.

As already mentioned in the previous subsection, $\langle \phi(X) \rangle|_{J=0,K=0} \equiv \Phi$ is equal to the global minimum of the effective potential V_{eff} (given by result 2.1). It also became clear that $\langle \phi(X) \rangle|_{K=0}$ is simply an "arbitrary point ϕ" as depicted in figure 2.2. In the context of section 2.2, a non-vanishing source J effects that one does not fluctuate around the global minimum but around

an arbitrary point. The fact that the source J does not need to be specified further is mirrored in the fact that we consider an *arbitrary* point. For vanishing sources instead, one fluctuates around the global minimum. Of course also the bilocal source K does not need to be specified further. We showed that the introduction of K is simply a mathematical trick in order to make V_{eff} dependent on the connected two-point function.

2.2 Fluctuating around the vacuum

Throughout this work, the CJT formalism will be applied to a Lagrangian where the fields have undergone a shift. This is nothing but a redefinition of the fields, which is widely used in field theory and completely independent from the CJT formalism. Shifting the fields means choosing fixed points for the fields around which we fluctuate. If we choose the field configuration which corresponds to the vacuum (the global minimum), this corresponds to the picture that particles are excitations of the vacuum, since the fluctuations are related to particles.
Figure (2.2) shows the classical potential $U(\phi) = \pm\frac{m^2}{2}\phi^2 + \frac{\lambda}{N}\phi^4$, where $+/-$ corresponds to the left/right picture.

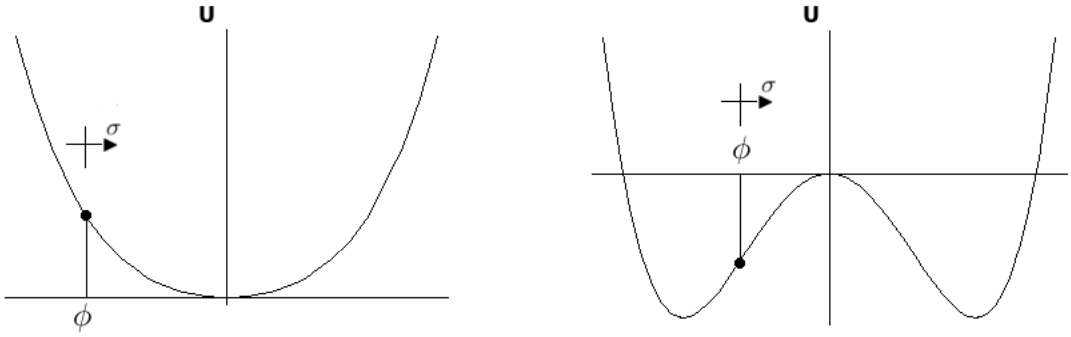


Figure 2.2: Classical potential for the Z_2 models, $H = 0$

A word on our notation first, in order to avoid confusion. As one usually learns QM and QFT before SQFT, one is usually more familiar with Minkowskian Lagrangians, as for example

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{N}\phi^4 \ . \tag{2.22}$$

From a Lagrangian one can read off the tree-level propagators and the vertex factors, which belong to the so-called *Feynman rules* for the theory. We will comment on the concept of Feynman rules in a general context briefly in appendix A.3. Here we are only concerned with technicalities. The Lagrangian (2.22) is given in position space. To obtain the Feynman rules and the tree-level quantities in momentum space, one should in principle first derive them in position space from (2.22) and then go over to momentum space via Fourier transformation. Since we are interested in the result only, we just want to give a mnemonic how to obtain the tree-level quantities in momentum space from (2.22). First note that $\partial_\mu\phi\partial^\mu\phi = \partial_\mu(\phi\partial^\mu\phi) - \phi\Box\phi$. Since $\partial_\mu(\phi\partial^\mu\phi)$

vanishes when we integrate to obtain the action,[6] we can neglect this term. Hence we are left with

$$\mathscr{L} = -\frac{1}{2}\phi\Box\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{N}\phi^4 \ . \tag{2.23}$$

Since $\Box$ goes over into $-k^2$ in momentum space, we can write down the following mnemonic[7]:

$$\text{“}\mathscr{L} = \frac{1}{2}k^2\phi^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{N}\phi^4 \text{ ”} \ . \tag{2.24}$$

Every Minkowskian Lagrangian has a corresponding Euclidean form which is used in SQFT. We discuss the transition from QFT to SQFT in appendix E. From result E.1 we read off the Euclidean version of the Lagrangian:

$$\mathscr{L}_E = \frac{1}{2}\partial_{\mu,E}\phi\partial^\mu_E\phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{N}\phi^4 \ , \tag{2.25}$$

where we indicate the use of the Euclidean metric by the subscript “E”. In complete analogy to the above we can neglect the term $\partial_{\mu,E}(\phi\partial^\mu_E\phi)$ in $\partial_{\mu,E}\phi\partial^\mu_E\phi = \partial_{\mu,E}(\phi\partial^\mu_E\phi) - \phi\Box_E\phi$, and we obtain

$$\mathscr{L}_E = -\frac{1}{2}\phi\Box_E\phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{N}\phi^4 \ . \tag{2.26}$$

Recall that the transition from the Minkowskian version to the Euclidean version is based on the Wick rotations

$$k^2 = k_0^2 - \mathbf{k}^2 \xrightarrow{k_0\to ip_4} = -p_4^2 - \mathbf{k}^2 \equiv -p^2 \ , \tag{2.27}$$

$$\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \xrightarrow{t\to -i\tau} -\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \equiv -\Box_E \ . \tag{2.28}$$

Since $\Box$ goes over to $-k^2$ in momentum space, the Euclidean d'Alembert operator, $\Box_E$, goes over into $-p^2$ in Euclidean momentum space. Accordingly the Euclidean version of our mnemonic (2.24) reads:

$$\text{“}\mathscr{L}_E = \frac{1}{2}p^2\phi^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{N}\phi^4 \text{ ”} \ . \tag{2.29}$$

In the case of QFT, the inverse tree-level propagator[8] is the prefactor in front of $\phi^2/2$ in our mnemonic (2.24):

$$D^{-1} = k^2 - m^2, \tag{2.30}$$

which is nothing but the inverse Feynman propagator in momentum space, if one compares with (E.19).

[6] The contribution can be rewritten as a 4-dimensional surface integral, which is zero, if we assume that ϕ vanishes on a surface lying at infinity.

[7] For instruction how to really write down the Lagrangian in momentum space we refer for example to [22].

[8] Again a remark on conventions: some textbooks prefer to use the symbol D for $k^2 - m^2$ instead of $\frac{1}{D}$.

Accordingly, the Euclidean tree-level propagator (which one uses in SQFT) is the prefactor in front of $\phi^2/2$ in (2.29):

$$D_E^{-1} = p^2 + m_t^2$$
$$m_t^2 = m^2,$$

which is the inverse Euclidean Feynman propagator in momentum space appearing in (E.21). Note that the term "tree-level" is justified. The tree-level mass m_t is equal to the bare mass (compare with appendix A), i.e., the mass without any loop corrections. The quartic term yields a 4-vertex with the vertex factor $-\frac{\lambda}{N}$.
We come to the same result if we read off the *pseudo-Minkowskian* tree-level propagator and the tree-level mass m_t from our mnemonic (2.24). The pseudo-Minkowskian tree-level propagator is now the *negative* prefactor in front of $\phi^2/2$:

$$D^{-1} = -k^2 + m_t^2 \tag{2.31}$$
$$m_t^2 = m^2 \ . \tag{2.32}$$

Note that (2.31) relates to Euclidean field theory, although the tree-level propagator D (and correspondingly the ansatz for the full connected propagator G which we will make later) is given in a pseudo-Minkowskian notation. The terms "Euclidean" and "pseudo-Minkowskian" only indicate the kind of notation one uses, and in fact, a quantity in Euclidean notation is the same as in pseudo-Minkowskian notation. The $-k^2$ actually stands for $-\left[(ip_4)^2 - \mathbf{k}^2\right] = p_4^2 + \mathbf{k}^2 = p^2$, i.e., one refers to the Minkowskian metric using ip_4 as zeroth component. We do not really like this notation, since the non-Euclidean zeroth component k_0 has to be *replaced* by ip_4 when changing to Euclidean field theory (compare with appendix E), i.e., $k_0 \rightarrow \overline{k}_0 = ip_4$, and is *not* equal to it, i.e., $k_0 \neq ip_4$. Nevertheless we will use this notation, since it is often employed in literature.
We now pick an arbitrary point ϕ, which is by definition independent of τ and $\vec{x}$ (we keep the same symbol ϕ) and fluctuate around it:

$$\phi\,(\tau, \vec{x}) = \phi + \sigma\,(\tau, \vec{x}) \ . \tag{2.33}$$

This yields, using $\partial_\mu(\phi+\sigma)\partial^\mu(\phi+\sigma) = \partial_\mu\sigma\partial^\mu\sigma$,

$$\text{“}\mathscr{L} = \frac{1}{2}k^2\sigma^2 - \frac{m^2}{2}(\phi+\sigma)^2 - \frac{\lambda}{N}(\phi+\sigma)^4 \text{ ”} \ .$$

Again, we are only interested in terms quadratic and quartic in σ (we do not need the cubic terms in the Hartree approximation). We read off the following tree-level mass and vertex factor:

$$D^{-1}(\phi) = -k^2 + m_t^2 \ , \tag{2.34}$$
$$m_t^2 \equiv m^2 + \frac{12\lambda}{N}\phi^2 \ , \tag{2.35}$$

$$-\tfrac{\lambda}{N}\sigma^4 \longrightarrow \quad \times\text{-}\frac{\lambda}{N} \ .$$

Note that the derivation of these quantities is absolutely independent from the CJT formalism.

We will simply use them in the CJT formalism.
Look at result 2.1. $G^{(1)}(k)$ is the connected thermal one-point function in the absence of source K, expressed in momentum space, i.e., $\langle\phi(k)\rangle|_{K=0}$. We can identify this expectation value with the arbitrary point ϕ, around which we fluctuate. For $J = 0$, this point is an extremum φ of the effective potential. The extremum we are interested in is the global minimum Φ.

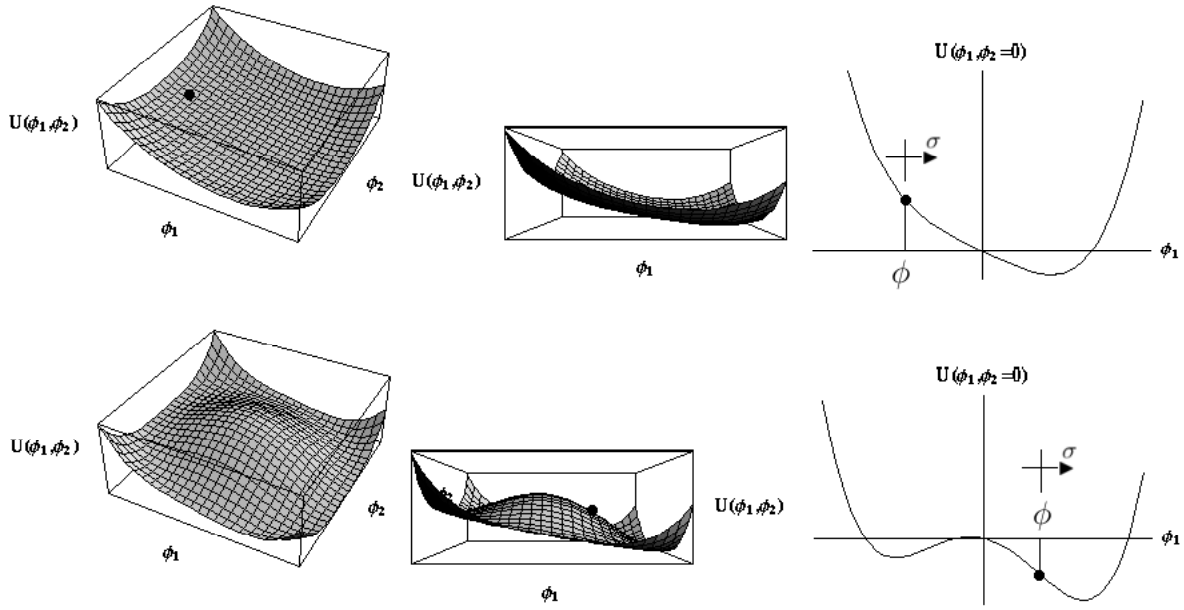


Figure 2.3: Top: $U(\phi_1, \phi_2) = +\frac{m^2}{2}\vec{\phi}^2 + \frac{\lambda}{N}\vec{\phi}^4 - H\phi_1$, $H \neq 0$, $m^2 > 0$. Bottom: Mexican hat potential, $U(\phi_1, \phi_2) = -\frac{m^2}{2}\vec{\phi}^2 + \frac{\lambda}{N}\vec{\phi}^4 - H\phi_1$, $H \neq 0$, $m^2 > 0$.

In case of more particles and different Lagrangians, we proceed in the same way. We will come back to this in the later chapters. For the rest of this section, we only want to discuss some general features of the potentials we use in our models. The classical potential $U(\phi_1, \phi_2) = +\frac{m^2}{2}\vec{\phi}^2 + \frac{\lambda}{N}\vec{\phi}^4 - H\phi_1$ has in principle the shape shown at the top of fig.2.3. Note that H is chosen relatively large, for $H = 0$ the potential would be symmetric with the minimum at zero. If we choose a minus sign in front of m^2, i.e., $U(\phi_1, \phi_2) = -\frac{m^2}{2}\vec{\phi}^2 + \frac{\lambda}{N}\vec{\phi}^4 - H\phi_1$, we obtain the well-known mexican hat potential, shown at bottom in fig.2.3. The effective potential V_{eff} in statistical field theory for $T = 0$ should be equal to the classical potential U (compare with appendix D). As the temperature grows, the shape of V_{eff} continuously changes, as one can see for example in fig.4.2.
In cartesian coordinates, the scalar field ϕ_1 corresponds to a particle, and ϕ_2 to another one. Particles are excitations of the vacuum. Having this in mind, it becomes plausible why we treat particles at temperature T mathematically as fluctuations around the global minimum of the effective potential. More precisely, the sigma particle is treated as a fluctuation in ϕ_1-direction, and the pion as a fluctuation in ϕ_2-direction. Independent from the direction of the fluctuations,

we choose the potential in a certain direction, as in fig.2.3 for example the ϕ_1-direction (as discussed on p.112, due to the negative parity of the pion this is the only choice for us). Now pick any point for $\phi \equiv \pm|\vec{\phi}| = \pm\sqrt{\phi_1^2 + \phi_2^2}$ you like, and allow fluctuations σ (pointing in ϕ_1-direction) and π (pointing in ϕ_2-direction, not shown in the figure because it is directed into the drawing plane) around that point.

2.3 Quantum Field Theory in 1+0 dimensions

2.3.1 Introductory remark

QFT resp. SQFT in 1+0 dimensions means spatial degrees of freedom do not exist and the generating functional E.1 resp. E.2 boils down to

$$Z = \int D\phi\,(t)\, e^{iS} \ , \tag{2.36}$$

$$\text{with } S = \int dt \left(\frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 - U\left(\phi\,(t)\right) \right) \ , \tag{2.37}$$

resp.

$$Z = \oint D\phi\,(\tau)\, e^{-S_E} \ , \tag{2.38}$$

$$\text{with } S_E = \int d\tau \left(\frac{1}{2} \left(\frac{\partial \phi}{\partial \tau} \right)^2 + U\left(\phi\,(\tau)\right) \right) \ , \tag{2.39}$$

where one has to bear in mind the periodic boundary condition $\phi(\tau) \stackrel{!}{=} \phi(\tau + \frac{1}{T})$.

If we replace $\phi(t)$ resp. $\phi(\tau)$ by $x(t)$ resp. $x(\tau)$, and the $\frac{1}{2}$ in front of the kinetic term by $\frac{m}{2}$, we obtain the generating functional for Quantum Mechanics resp. Statistical Quantum Mechanics. Therefore, the mathematical structure of (S)QFT in 1+0 dimensions and (S)QM is the same, and consequently we are able to use the WKB approximation to calculate the energy eigenvalues. These we can use with (A.1) to calculate Z, which is the same quantity as (2.38):

$$Z = Tr\left(e^{-\beta \hat{H}}\right) \stackrel{\{|E_n>\}}{=} \sum_n e^{-\beta E_n} \ . \tag{2.40}$$

First, note that $Z = \sum_n e^{-\beta E_n}$ is true only at the global minimum ($J = K = 0$). To calculate the generating functional in the presence of sources, we would have to include source terms in $\hat{H}$. Furthermore, note that in (S)QFT, $\hat{H}$ in formula (A.1) depends on ϕ.
From the proof of (2.21), we know how to calculate the effective potential at its global minimum:

$$V(\Phi) = -T \ln \left(\sum_n e^{-E_n/T} \right) \ , \tag{2.41}$$

where we used that there is no spatial volume in $1 + 0$ dimensions, i.e., $V = L^D = L^0 = 1$.

2.3.2 WKB method, $N = 1$

This section outlines the derivation of the WKB equation, from which the energy eigenvalues can be computed. For details we refer to Ref. [38].
The WKB method, introduced in 1926 by Wentzel, Kramers, and Brillouin, is an approach to find approximate solutions of the stationary Schrödinger equation

$$\frac{d^2u(x)}{dx^2} + k^2(x)u(x) = 0 \ , \quad k^2(x) = \frac{2m}{\hbar^2}(E - U(x)) \ . \tag{2.42}$$

According to the introductory remark, this quantum mechanical equation has its quantum field-theoretical analogon

$$\frac{d^2u(\phi)}{d\phi^2} + k^2(\phi)u(\phi) = 0 \ , \quad k^2(\phi) = \frac{2}{\hbar^2}(E - U(\phi)) \ , \tag{2.43}$$

where $U(\phi)$ is the classical potential of the system under consideration. Although $\hbar = 1$ in natural units, we keep it in the derivation of the WKB method, since the approach is basically an expansion in order of $\hbar$. Later we will set it to 1 again. In the following we review the derivation of the WKB equation in Quantum Mechanics, however in each step one is able to switch to the notation of Quantum Field Theory.
Starting from the ansatz

$$u(x) = Ce^{\frac{i}{\hbar}W(x)} \ , \tag{2.44}$$

one expands $W(x)$ in powers of $(i\hbar)$:

$$W(x) = \sum_{n=0}^{\infty}(i\hbar)^n W_n(x) = W_0(x) + i\hbar W_1(x) - \hbar^2 W_2(x) + O(\hbar^3) \ . \tag{2.45}$$

Insertion into (2.42) yields

$$0 = \hbar^0({W_0'}^2 - k^2\hbar^2) + i\hbar(2W_0'W_1' - W_0^{''}) + \hbar^2(-2W_0'W_2' - {W_1'}^2 + W_1^{''}) + O(\hbar^3) \ , \tag{2.46}$$

where we denote the derivative with respect to x by a prime.
Equation (2.46) is solved order by order in $\hbar$:

$$\hbar^0: \qquad {W_0'}^2(x) = k^2\hbar^2 \ \Rightarrow \ W_0 = \pm\hbar\int_{c_1}^{x} k(\rho)d\rho \ , \tag{2.47}$$

$$\hbar^0 \text{ and } \hbar^1: \qquad W_1' = \frac{W_0^{''}}{2W_0'} \ . \tag{2.48}$$

Using (2.47) we obtain

$$\frac{W_0^{''}}{2W_0'} = \frac{1}{2}\frac{\pm\hbar k'}{\pm\hbar k} = \left(\frac{1}{2}\ln k\right)' \ \Rightarrow \ W_1 = \ln\sqrt{k} \ , \tag{2.49}$$

$$\hbar^0 \text{ and } \hbar^1 \text{ and } \hbar^2: \qquad {W_1'}^2 + 2W_0'W_2' - W_1^{''} = 0 \ \Leftrightarrow \ W_2' = \frac{W_1^{''} - {W_1'}^2}{2W_0'} \ . \tag{2.50}$$

Using (2.47) and (2.49) we obtain

$$W_2 = \pm\frac{1}{4\hbar}\int_{c_2}^{x}\left(\frac{k''(\rho)}{k^2(\rho)} - \frac{3}{2}\frac{k'^2(\rho)}{k^3(\rho)}\right)d\rho\ . \tag{2.51}$$

Inserting the expressions for W_0 and W_1 into the ansatz (2.44), we obtain two special, linearly independent solutions:

$$u_\pm = C\cdot e^{-\ln\sqrt{k}}\cdot\exp\left(\pm i\int_{c_1}^{x}k(\rho)d\rho\right)\ , \tag{2.52}$$

where the WKB approximation consist of neglecting all terms of order $O(\hbar^2)$ in (2.45).
The general solution $\hat{u}$ is then given as a linear combination of u_+ and u_-:

result 2.4 (general solution, WKB approximation)

$$\hat{u} = c_+\frac{1}{\sqrt{k}}\exp\left(+i\int_{d_+}^{x}k(\rho)d\rho\right) + c_-\frac{1}{\sqrt{k}}\exp\left(-i\int_{d_-}^{x}k(\rho)d\rho\right)\ , \tag{2.53}$$

where the constants $c_\pm$ and $d_\pm$ have to be determined by the boundary conditions. Note that only two of the constants are independent because (2.42) is a differential equation of second order. One can show that the approximation is good if one is sufficiently far away from classical turning points of the potential $U(x)$. Result 2.4 is also valid in the classically forbidden region,

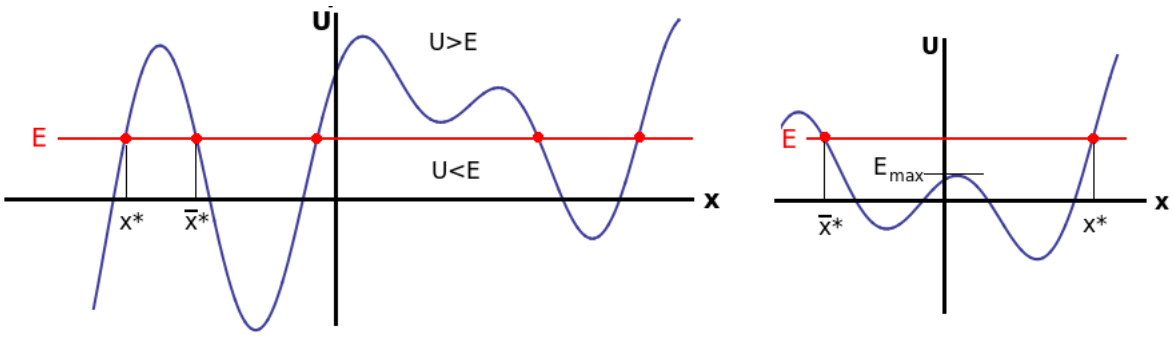


Figure 2.4: Illustration of classical turning points, the classically allowed region ($U < E$) and classically forbidden region ($U > E$). We distinguish between left turning points $\overline{x}^*$ and right turning points x^*. The right picture shows the special case of a bound state.

however, here k is imaginary:

$$E < U\ \Rightarrow\ k^2(x) = \frac{2m}{\hbar^2}\underbrace{(E-U)}_{<0}\ \Rightarrow\ k(x) = i\frac{\sqrt{2m}}{\hbar}\sqrt{|E-U|} \equiv i\kappa(x).$$

Thus, we can write the solution in the classically forbidden region as

$$\hat{u} = \frac{c^+/\sqrt{i}}{\sqrt{\kappa}}\exp\left(-\int_{d^+}^{x}\frac{\sqrt{2m}}{\hbar}\sqrt{|E-U|}\ d\rho\right) + \frac{c^-/\sqrt{i}}{\sqrt{\kappa}}\exp\left(\int_{d^-}^{x}\frac{\sqrt{2m}}{\hbar}\sqrt{|E-U|}\ d\rho\right)\ . \tag{2.54}$$

Remember that the WKB approximation does not hold at the turning points. However, one of the two independent parameters can be chosen such that the turning points are removable discontinuities, so that one parameter is left for normalization. This procedure is known as *Langer's method*. One obtains two expressions for $\hat{u}$ near each kind of turning point (left and right), one for the classically forbidden and one for the classically allowed region. In the following we only need the results for the classically allowed region:

$$\text{right turning point, } U < E: \ \hat{u}(x) = a\sqrt{\frac{6}{\pi}}\frac{1}{\sqrt{k}}\cos\left(\int_x^{x^*} k(\rho)d\rho - \frac{\pi}{4}\right) \ , \tag{2.55}$$

$$\text{left turning point, } U < E: \ \hat{u}(x) = \overline{a}\sqrt{\frac{6}{\pi}}\frac{1}{\sqrt{k}}\cos\left(\int_{\overline{x}^*}^{x} k(\rho)d\rho - \frac{\pi}{4}\right) \ , \tag{2.56}$$

which enable us to set up the WKB equation for a bound state as illustrated in the right picture of figure 2.4. From the condition that expressions (2.55) and (2.56) have to be equal, we obtain:

$$\pm\cos\left(\int_x^{x^*} k(\rho)d\rho - \frac{\pi}{4}\right) = \cos\left(\int_{\overline{x}^*}^{x} k(\rho)d\rho - \frac{\pi}{4}\right) \ . \tag{2.57}$$

This can be rewritten in a form known as WKB equation:

$$\int_{\overline{x}^*}^{x^*} k dx = \left(\frac{1}{2} + n\right)\pi \ . \tag{2.58}$$

From the introductory remark, we can conclude that the field-theoretical analogue for the WKB equation reads

result 2.5 (WKB equation)

$$\int_{\overline{\phi}^*}^{\phi^*} \sqrt{2\left(E - U(\phi)\right)}\, d\phi = (\frac{1}{2} + n)\pi \ .$$

To find the solutions E_n of this equation, the energy eigenvalues, we have to express ϕ^* and ϕ^* as functions of E and solve the integral equation numerically for each $n = 0, 1, ...$
Note that (2.58) as well as result 2.5 is only correct as long as $E > E_{max}$ (see figure 2.4). Otherwise there exists a probability for tunneling and the right-hand side of the WKB equation needs to be modified. For details we refer to Ref. [39] and to section 4.1.2.

2.3.3 Radial WKB method, $N = 2$

For two scalar field variables, the WKB method is a bit more difficult. We do not want to use the version for cartesian field coordinates, since in the case of potentials which only depend on the radial degree of freedom, $r = \sqrt{\phi_1^2 + \phi_2^2}$, i.e., $U(\phi_1, \phi_2) = U(r)$, there exists a radial WKB equation which is simpler. In later chapters, we will compare cartesian and polar CJT results

with those in this radial WKB approximation.
Starting from the stationary Schrödinger equation in polar coordinates,

$$\left(-\frac{\hbar^2}{2}\Delta + U(r)\right)\psi(r,\varphi) = E\psi(r,\varphi)\ , \quad \text{with } \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\ , \tag{2.59}$$

we make the separation ansatz $\psi \equiv R(r)Y_m(\varphi)$, where $R(r) \equiv \frac{u(r)}{\sqrt{r}}$ and $Y_m(\varphi) \equiv e^{im\varphi}$, to deduce the *stationary radial Schrödinger equation*:

$$\frac{\hbar^2}{2}\frac{d^2u(r)}{dr^2} + (E - U_m(r))\,u(r) = 0\ , \quad \text{with } U_m(r) \equiv U(r) + \frac{\hbar^2}{2}\frac{(m^2 - \frac{1}{4})}{r^2}. \tag{2.60}$$

Do not confuse the quantum number m with the bare mass m!
However, there is a problem with the equation in this form, which anticipates the derivation of a WKB equation. Simply because $U_{m=0} = U(r) - \frac{\hbar^2}{8r^2} \xrightarrow{r\to 0} -\infty$, which means that there is no global minimum, and above a certain critical energy E no left turning point. The solution is a conformal mapping $r = e^{\rho}$, as proposed in [40] and [41], which will be discussed in the following.
Application of the chain rule for the second derivative,

$$\frac{d^2u(r)}{dr^2} = \frac{d^2u(\rho)}{d\rho^2}\left(\frac{d\rho}{dr}\right)^2 + \frac{du(\rho)}{d\rho}\frac{d^2\rho}{dr^2}, \tag{2.61}$$

yields

$$\frac{d^2u(r)}{dr^2} = \frac{d^2u(\rho)}{d\rho^2}\frac{1}{r^2} - \frac{du(\rho)}{d\rho}\frac{1}{r^2}. \tag{2.62}$$

In order to deduce an equation of the same form as (2.60), we have to perform a second transformation:

$$u(\rho) = e^{\frac{\rho}{2}}\chi(\rho). \tag{2.63}$$

From (2.62) we obtain

$$\frac{d^2u(r)}{dr^2} = \frac{1}{r^2}e^{\frac{\rho}{2}}\left[\frac{d^2\chi}{d\rho^2} - \frac{1}{4}\chi\right], \tag{2.64}$$

which we insert together with $u(\rho) = e^{\frac{\rho}{2}}\chi(\rho)$ into equation (2.60), to obtain:

$$\frac{d^2\chi}{d\rho^2} + \frac{2}{\hbar^2}e^{2\rho}\left(E - U(\rho) - \frac{\hbar^2}{2}\frac{m^2}{r^2}\right)\chi(\rho) = 0. \tag{2.65}$$

The spurious $-\frac{1}{4}$ has disappeared, and we are able to derive the radial WKB equation, following the same steps which led to equation (2.5):

$$\int_{\overline{\rho}^*}^{\rho^*}\sqrt{\frac{2}{\hbar^2}e^{2\rho}\left(E - U(\rho) - \frac{\hbar^2}{2}\frac{m^2}{e^{2\rho}}\right)}\,d\rho = (\frac{1}{2} + n)\pi\ . \tag{2.66}$$

Having gained equation (2.66), we can transform back to r:

result 2.6 (radial WKB equation)

$$\int_{\overline{r}^*}^{r^*}\sqrt{\frac{2}{\hbar^2}\left(E - U(r) - \frac{\hbar^2}{2}\frac{m^2}{r^2}\right)}\,dr = (\frac{1}{2} + n)\pi.$$

Chapter 3

Z_2-symmetric one-minimum potential

In this chapter we study the system determined in 1+3 dimensions by the Lagrangian (3.1), and in 1+0 dimensions by the Lagrange function (3.9) respectively. Both are symmetric under the change of sign $\phi \longrightarrow -\phi$, which constitutes a Z_2 symmetry. We will examine the behavior of the system for nonzero temperature (compare with the comments in section 2.2).
Though we keep the variable N general, for numerical calculations we set $N = 1$. Note that this chapter (as well as the following ones) requires knowledge about thermal integrals, which we introduce in appendix B.

3.1 1+3 dimensions

In this section we study the system determined by the Lagrangian

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{N}\phi^4 \ , \quad m^2 > 0 \ . \tag{3.1}$$

After having shifted the field, $\phi\,(\tau,\vec{x}) = \phi + \sigma\,(\tau,\vec{x})$, we obtain the pseudo-Minkowskian tree-level propagator and the vertex factor according to section 2.2:

$$D^{-1}(\phi) = -k^2 + m_t^2 \ , \tag{3.2}$$

$$m_t^2 \equiv m^2 + \frac{12\lambda}{N}\phi^2 \ , \tag{3.3}$$

$$-\tfrac{\lambda}{N}\sigma^4 \longrightarrow \quad \times\text{-}\frac{\lambda}{N} \ .$$

The effective potential in the CJT formalism is given by result 2.1:

$$V\,[\phi,G] = \frac{m^2}{2}\phi^2 + \frac{\lambda}{N}\phi^4 + \frac{1}{2}\int\limits_k \ln G^{-1} + \frac{1}{2}\int\limits_k \left(D^{-1}(\phi)G - 1\right) + V_2\,[\phi,G] \ , \tag{3.4}$$

$$\text{with } V_2 = -\left(-3\frac{\lambda}{N}[\int\limits_k G]^2\right) . \tag{3.5}$$

The mass at the arbitrary point ϕ is determined by result 2.3:

$$\frac{\delta V}{\delta G} = 0 \;\Leftrightarrow\; -\frac{1}{2}\frac{1}{G^{-1}}G^{-2} + \frac{1}{2}D^{-1} + 6\frac{\lambda}{N}\int\limits_k G = 0$$

$$\Leftrightarrow\; G^{-1} = D^{-1} + 12\frac{\lambda}{N}\int\limits_k G \;. \tag{3.6}$$

Since the tree-level propagator is given by $D^{-1}(\phi) = -k^2 + m_t^2$, and $12\frac{\lambda}{N}\int\limits_k G$ is independent of k, we are able to make the following ansatz for the connected thermal 2-point function in the presence of a source J, expressed in momentum space:

$$G \equiv \frac{1}{-k^2 + M^2} \;. \tag{3.7}$$

This leads to the so-called *gap equation for the mass* at an arbitrary point ϕ:

result 3.1 (gap equation)

$$M^2 = m^2 + 12\frac{\lambda}{N}\phi^2 + 12\frac{\lambda}{N}\int\limits_k G \;.$$

The condition which tells us that we are at an extremum φ is given by the first equation of result 2.2:

$$\frac{\delta V}{\delta \phi}\Big|_{\phi=\varphi;G=\mathcal{G}} = 0 \;\Leftrightarrow\; m^2\varphi + 4\frac{\lambda}{N}\varphi^3 + 12\frac{\lambda}{N}\varphi\int\limits_k \mathcal{G} = 0 \;. \tag{3.8}$$

Since $M^2 = m^2 + 12\frac{\lambda}{N}\varphi^2 + 12\frac{\lambda}{N}\int\limits_k \mathcal{G}$, substracting $0 = 3m^2 + 12\frac{\lambda}{N}\varphi^2 + 36\frac{\lambda}{N}\int_k \mathcal{G}$
yields $M^2 = -2m^2 - 24\frac{\lambda}{N}\int_k \mathcal{G} < 0$ $\lightning$, and we conclude that $\varphi(T) = 0$ always.
As a result, the system of equations determining M and ϕ at the extrema (here only one extremum exists, the global minimum) is given by:

result 3.2 (conditions determining mass and condensate at the global minimum)

$$\varphi(T) = 0 \;,$$
$$M^2 = m^2 + 12\frac{\lambda}{N}\int\limits_k G \;.$$

We will discuss the numerical solutions using $\int\limits_k G = \frac{1}{2\pi^2}\int\limits_0^\infty dk\; \frac{k^2}{\epsilon_k}\frac{1}{e^{\epsilon_k/T}-1}$, compare with (B.10), i.e., we drop the contribution Q_μ which would require renormalization.
At $T = 0$ result 3.2 yields

result 3.3 ($T = 0$)

$$M_0^2 = m^2 \;,$$
λ remains independent.

Since we dropped Q_μ, we also have to drop R_μ in result B.6. Using expression (B.12), we obtain $\int\limits_k \ln\left(G^{-1}\right) = 4\pi\frac{1}{(2\pi)^3}\int\limits_0^\infty dk k^2 2T \ln\left(1-e^{-\frac{\epsilon_k}{T}}\right)$.
Therefore the effective potential (3.4) reads

result 3.4 (effective potential)

$$V[\phi,G] = \frac{m^2}{2}\phi^2 + \frac{\lambda}{N}\phi^4 + \frac{T}{2\pi^2}\int\limits_0^\infty dk k^2 \ln\left(1-e^{-\frac{\epsilon_k}{T}}\right)$$
$$+\frac{1}{2}\left(m^2 + 12\frac{\lambda}{N}\phi^2 - M^2\right)\frac{1}{2\pi^2}\int\limits_0^\infty dk\ \frac{k^2}{\epsilon_k}\frac{1}{e^{\epsilon_k/T}-1} + 3\frac{\lambda}{N}\left[\frac{1}{2\pi^2}\int\limits_0^\infty dk\ \frac{k^2}{\epsilon_k}\frac{1}{e^{\epsilon_k/T}-1}\right]^2 ,$$

where we have to use result 3.1 to determine M for each point ϕ.
Figure 3.1 shows the numerical results of interest.

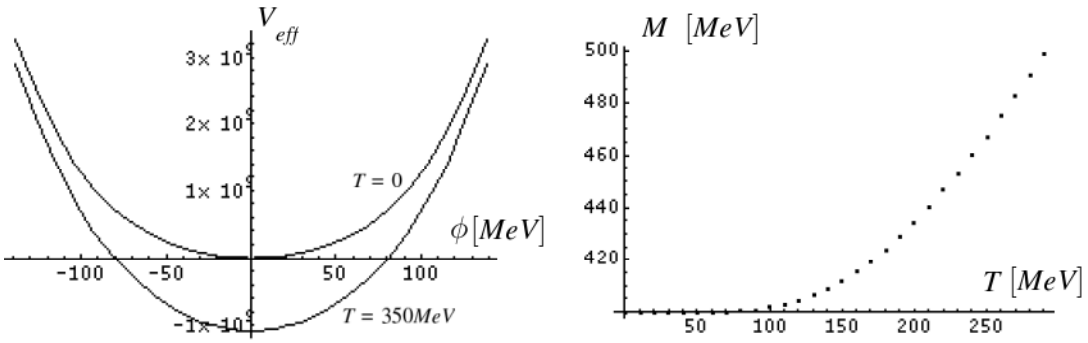


Figure 3.1: left: effective potential for $T = 0.01MeV$ (resp. classical potential) and $T = 350MeV$; right: mass at the global minimum.

3.2 1+0 dimensions

In 1+0 dimensions, the above Lagrangian becomes the Lagrange function

$$L = \frac{1}{2}\partial_0\phi\partial^0\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{N}\phi^4 , \quad m^2 > 0 . \tag{3.9}$$

3.2.1 CJT

Basically, the equations are the same as in section 3.1, provided that we replace $\int\limits_k G$ by $\int\limits_{k_0} G$ and $\int\limits_k \ln G^{-1}$ by $\int\limits_{k_0} \ln G^{-1}$. In 1+0 dimensions, the contributions corresponding to the terms Q_μ resp.

R_μ are finite. Therefore, we are not allowed to drop them. Hence:

$$\int_{k_0} G = \frac{1}{M}\left(\frac{1}{e^{M/T}-1}+\frac{1}{2}\right) \ , \tag{3.10}$$

$$\int_{k_0} \ln G^{-1} = M + 2T\ln\left(1-e^{-M/T}\right). \tag{3.11}$$

However, in 1+0 dimensions, we have to be careful with the dimension of our quantities, which are not the same as in 1+3 dimensions. Let us express all quantities in multiples of m. We know that

$$S_E = \int d\tau L_E \ , \quad L_E = \frac{1}{2}\frac{\partial\phi}{\partial\tau}\frac{\partial\phi}{\partial\tau} + \frac{m^2}{2}\phi^2 + \frac{\lambda}{N}\phi^4 \ , \quad [\tau] = MeV^{-1} \ , \quad [L_E] = MeV.$$

Defining $L_E = \widetilde{L}_E \cdot m$, where the twiddle means that $\widetilde{L}_E$ is dimensionless, we conclude:

$$\phi = \widetilde{\phi}\cdot m^{-\frac{1}{2}} \ , \quad \lambda = \widetilde{\lambda}\cdot m^3 \ , \quad \tau = \widetilde{\tau}\cdot m^{-1}. \tag{3.12}$$

Having expressed the above quantities in multiples of m, we now obtain the following results (with $M = \widetilde{M}\cdot m$ and $T = \widetilde{T}\cdot m$). For simplicity, the twiddle will be omitted.

$$D^{-1}(\phi) = -k^2 + m_t^2 \ , \tag{3.13}$$

$$m_t^2 \equiv 1 + \frac{12\lambda}{N}\phi^2 \ . \tag{3.14}$$

result 3.5 (gap equation)

$$M^2 = 1 + 12\frac{\lambda}{N}\phi^2 + 12\frac{\lambda}{N}\int_{k_0} G \ .$$

result 3.6 (conditions determining mass and condensate at the global minimum)

$$\varphi(T) = 0 \ ,$$
$$M^2 = 1 + 12\frac{\lambda}{N}\int_{k_0} G \ .$$

result 3.7 ($T = 0$)

$$M_0^2 = 1 + 12\frac{\lambda}{N}\frac{1}{2M_0} \ ,$$
λ remains independent.

result 3.8 (effective potential)

$$V[\phi, G] = \frac{1}{2}\phi^2 + \frac{\lambda}{N}\phi^4 + \frac{1}{2}\left(M + 2T\ln\left(1-e^{-M/T}\right)\right)$$
$$+\frac{1}{2}\left(1 + 12\frac{\lambda}{N}\phi^2 - M^2\right)\frac{1}{M}\left(\frac{1}{e^{M/T}-1}+\frac{1}{2}\right) + 3\frac{\lambda}{N}\left[\frac{1}{M}\left(\frac{1}{e^{M/T}-1}+\frac{1}{2}\right)\right]^2 \ .$$

3.2.2 WKB

Method

In 1+0 dimensions it is possible to calculate the effective potential at its global minimum in the WKB approximation. For this purpose we use the relation (2.41):

$$V(\Phi) = -T \ln \left(\sum_n e^{-E_n/T} \right) \ , \tag{3.15}$$

and the WKB equation (2.5)

$$\int_{\overline{\phi}^*}^{\phi^*} \sqrt{2\,(E - U(\phi)\,)}\, d\phi = (\frac{1}{2} + n)\pi. \tag{3.16}$$

Since the potential $U = \frac{1}{2}\phi^2 + \frac{\lambda}{N}\phi^4$ is symmetric, and only two turning points,

$$\phi^* = +\sqrt{-\frac{N}{4\lambda} + \sqrt{\frac{N^2}{16\lambda^2} + E\frac{N}{\lambda}}} \text{ and } \overline{\phi}^* = -\sqrt{-\frac{N}{4\lambda} + \sqrt{\frac{N^2}{16\lambda^2} + E\frac{N}{\lambda}}} \tag{3.17}$$

exist, the latter can be rewritten as

$$2\int_0^{\phi^*} \sqrt{2\,(E - U(\phi)\,)}\, d\phi = (\frac{1}{2} + n)\pi. \tag{3.18}$$

This equation can be solved numerically for E_n which are in multiples of m.

For $T = 0$, expression (3.15) contains only the lowest energy eigenvalue:

$$V(\Phi) = -T \ln \left[e^{-E_0/T} \left(1 + e^{\underbrace{(E_0 - E_1)}_{<0}/T} + \dots \right) \right] \xrightarrow{T \to 0} -T \ln e^{-E_0/T} = E_0 \ . \tag{3.19}$$

Analytic comparison between CJT and WKB for $\lambda = 0$

In the case of the quantum mechanical harmonic oscillator $L_h = \frac{1}{2}m_h\dot{x}^2 - \frac{1}{2}m_h\omega^2 x^2$, the WKB method is exact, i.e., it yields the correct eigenvalues $E_n = \hbar\omega(n + \frac{1}{2})$. From comparison of $U = \frac{m^2}{2}\phi^2$ and $U_h = \frac{1}{2}m_h\omega^2 x^2$, we conclude that U will lead to the eigenvalues

$$E_n = \hbar m(n + \frac{1}{2}). \tag{3.20}$$

Together with $M_0^2 \overset{\lambda=0}{=} m^2$ and the limit (3.19), we conclude:

$$V_0(\Phi) = \hbar\frac{1}{2}m = \hbar\frac{1}{2}M_0 \ , \tag{3.21}$$

where V_0 is short for $V(T=0)$.
The same follows from the CJT result 3.8 for $\lambda = 0$,

$$V\left[\Phi, \mathcal{G}\right] = \frac{1}{2}\left(M + 2T\ln\left(1 - e^{-M/T}\right)\right) + \frac{1}{2}\left(m^2 - m^2\right)\frac{1}{M}\left(\frac{1}{e^{M/T}-1} + \frac{1}{2}\right) \ , \tag{3.22}$$

which, for $T = 0$, yields $\frac{1}{2}M_0$, too. The $\hbar$ is missing simply because we worked with $\hbar = 1$ in the CJT formalism.

3.3 Numerical results

Figure 3.2 shows the effective potential for different temperatures (top: $\lambda = 10m^3$, bottom: $\lambda = 1 \cdot 10^3 m^3$).
In figure 3.4, we show the comparison of the effective potential at its global minimum, calculated within CJT and WKB. All quantities are expressed in multiples of m. To find out how many eigenvalues to consider we proceed as explained in the following. T_g denotes the temperature up to which the result is within the required accuracy, and we neglect eigenvalues above the energy E_g. We choose E_g such that $\ln e^{-E_0/T_g}$ equals $\ln\left(e^{-E_0/T_g} + e^{-E_g/T_g}\right)$ within five significant digits. Then we can expect that including the next larger eigenvalue would change the result $(-T\ln\left(\sum_n e^{-E_n/T}\right))$ at the utmost in the fifth significant digit. We also practically checked this approach by comparing to results with at least twice as many eigenvalues taken into account. For the comparison shown in figure 3.4 we chose $T_g = 50m$, which is high enough to reveal a difference between CJT and WKB. As explained above, it is safe to say that this difference does not stem from insufficient accuracy. For the required accuracy one needs the first 92 eigenvalues in case of $\lambda = 10m^3$, the first 27 eigenvalues in case of $\lambda = 1 \cdot 10^3 m^3$ and the first 8 eigenvalues in case of $\lambda = 1 \cdot 10^5 m^3$ respectively. We observe that the larger λ the larger the difference at $T = 0$. Although the difference between both curves (CJT resp. WKB) becomes larger with increasing temperature, the ratio goes to one. This becomes obvious if we shift by a constant, such that both curves coincide at $T = 0$ and start just below the T-axis. Figure 3.3 shows the ratio after the shift. It does not seem to depend significantly on λ.

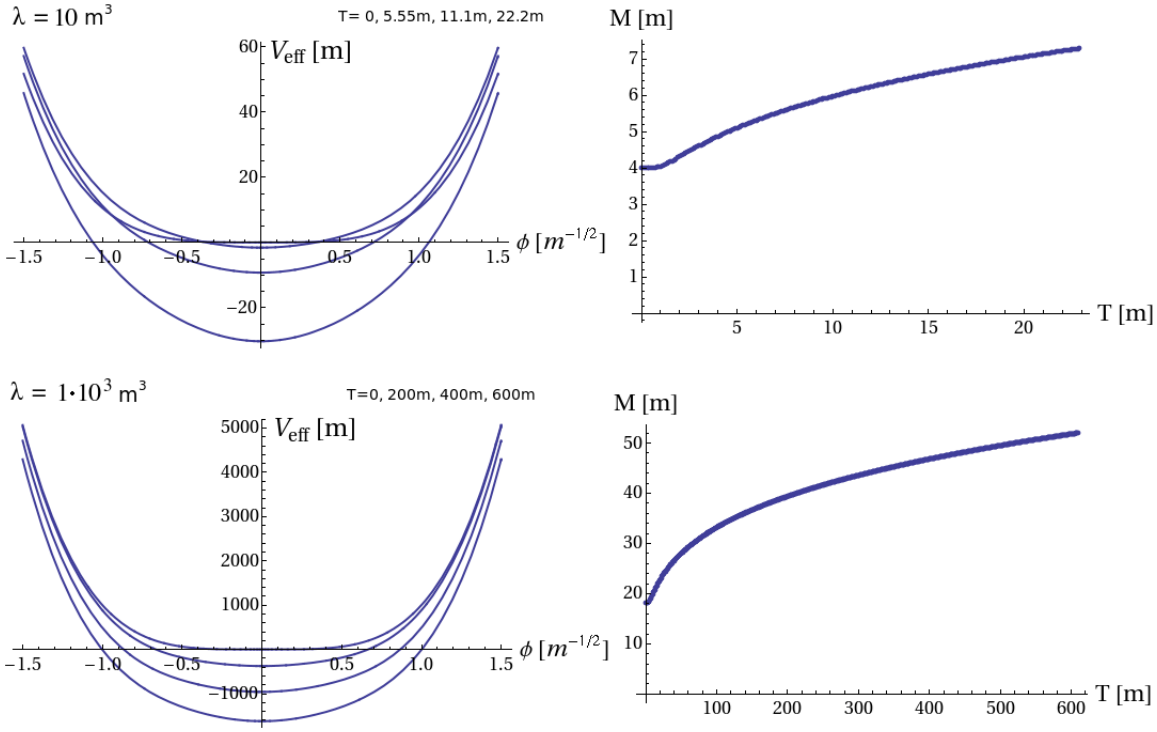


Figure 3.2: effective potential at ϕ, and mass at the global minimum.

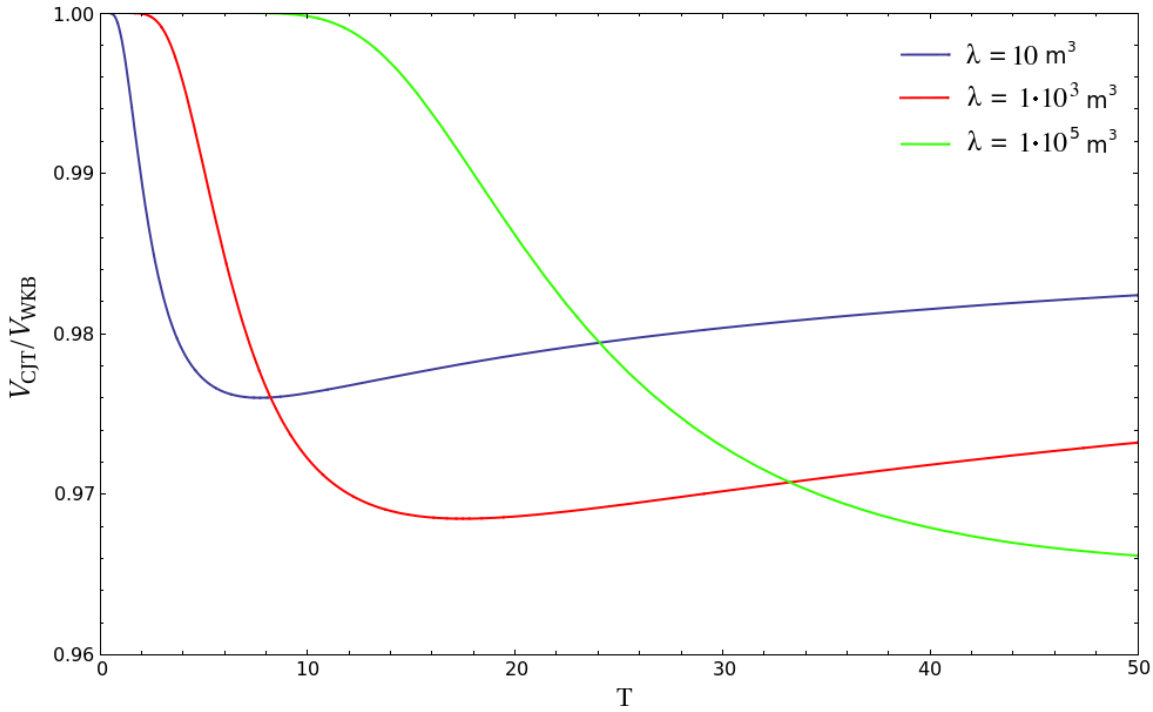


Figure 3.3: quantitative comparison between WKB and CJT.

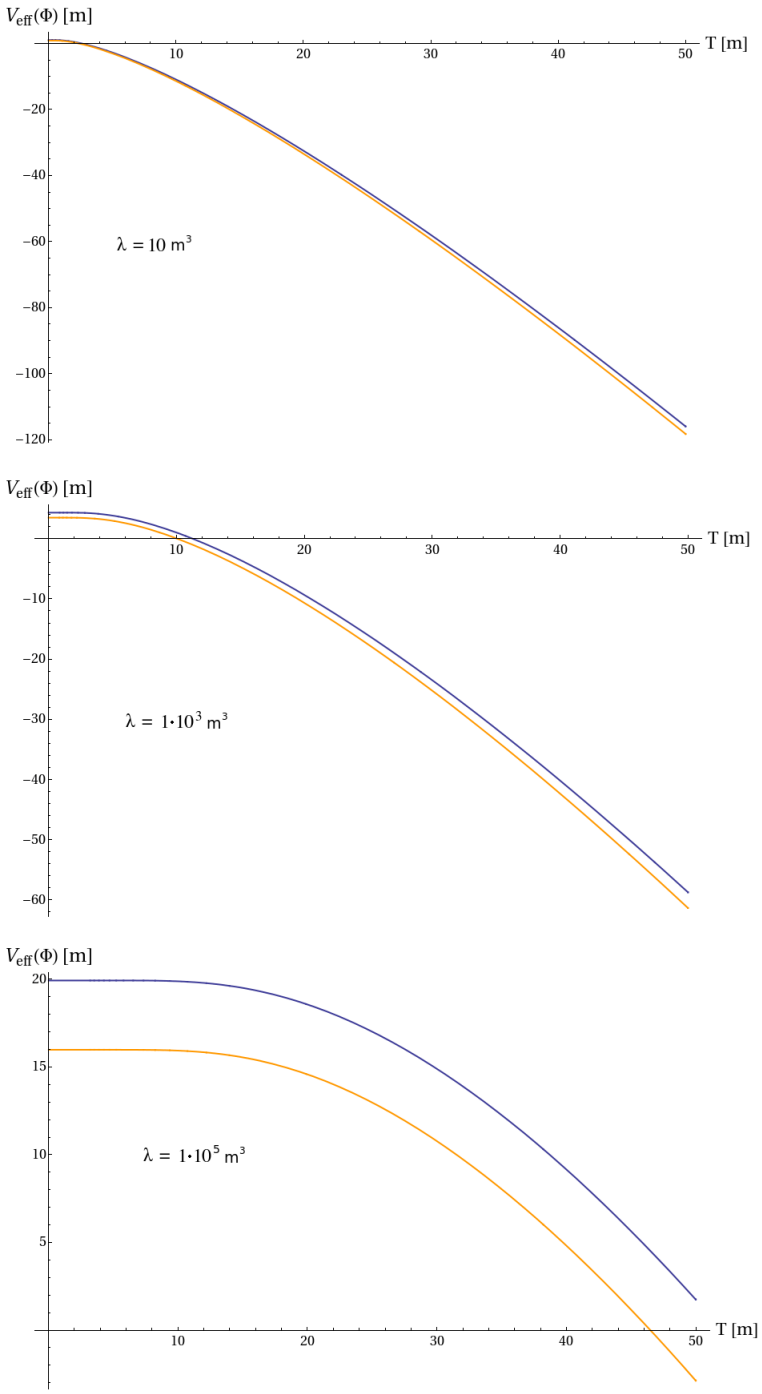


Figure 3.4: effective potential at its global minimum against temperature. blue: CJT, orange: WKB.

Chapter 4

Z_2-symmetric double-well potential

4.1 1+0 dimensions

In this chapter we want to restrict the discussion to 1+0 dimensions, since our major interest is the comparison between the CJT formalism and the WKB method. The system we are concerned with is determined by the Lagrange function

$$L = \frac{1}{2}\partial_0\phi\partial^0\phi + \frac{m^2}{2}\phi^2 - \frac{\lambda}{N}\phi^4 \ , \quad m^2 > 0 \ . \tag{4.1}$$

Again, we will study the behavior of the system at nonzero temperature.

4.1.1 CJT

A couple of equations we can borrow from section 3.2 by changing the sign in front of m^2. Let us list them explicitly.

result 4.1 (tree-level mass)

$$m_\sigma^2 = -m^2 + 12\frac{\lambda}{N}\phi^2 \ .$$

Therefore, in pseudo-Minkowskian notation, we have the inverse tree-level propagator

$$D_\sigma^{-1} = -k_0^2 - m^2 + 12\frac{\lambda}{N}\phi^2 \ . \tag{4.2}$$

result 4.2 (contribution from two-particle irreducible diagrams)

$$V_2 = 3\left(\frac{\lambda}{N}\right)\left[\int_{k_0} G_\sigma\left(k_0\right)\right]^2 \ .$$

result 4.3 (effective potential)

$$V[\phi] = -\frac{m^2}{2}\phi^2 + \frac{\lambda}{N}\phi^4 + \frac{1}{2}\int_{k_0} \ln G_\sigma^{-1}(k_0) + \frac{1}{2}\int_{k_0} \left[D_\sigma^{-1}(k_0)\, G_\sigma(k_0) - 1\right] + V_2 \ .$$

result 4.4 (gap equation for sigma particle)

$$M_\sigma^2 = -m^2 + 12\frac{\lambda}{N}\phi^2 + \frac{12\lambda}{N}\int_{k_0} G_\sigma \ .$$

However, in contrast to result 3.6, the extrema are nontrivial:

result 4.5 (stationarity condition for the field)

$$-m^2\varphi + 4\frac{\lambda}{N}\varphi^3 + 12\frac{\lambda}{N}\varphi\int_{k_0} \mathcal{G}_\sigma = 0 \ .$$

First, let us examine M_σ and ϕ at the extrema of the effective potential. We will find that there exists a critical value λ_{cr} for the coupling constant, from which on the only extremum is a global minimum at $\phi = 0$ for all temperatures. In this case, the effective potential for $T = 0$ does not reproduce the classical potential.
The system of equations given by results (4.4) and (4.5), with all quantities expressed in multiples of m, reads

$$M_\sigma^2 = -1 + 12\tfrac{\lambda}{N}\varphi^2 + \tfrac{12\lambda}{N}\int_{k_0} \mathcal{G}_\sigma \ , \tag{4.3}$$

$$-\varphi + 4\tfrac{\lambda}{N}\varphi^3 + 12\tfrac{\lambda}{N}\varphi\int_{k_0} \mathcal{G}_\sigma = 0 \ . \tag{4.4}$$

It can be rewritten, distinguishing between two cases:

$\underline{\varphi \neq 0}$

$$M_\sigma^2 = 2 - 24\tfrac{\lambda}{N}\int_{k_0} \mathcal{G}_\sigma \tag{4.5}$$

$$M_\sigma^2 = 8\tfrac{\lambda}{N}\varphi^2 \tag{4.6}$$

$\underline{\varphi = 0}$

$$M_\sigma^2 = -1 + 12\tfrac{\lambda}{N}\int_{k_0} \mathcal{G}_\sigma \tag{4.7}$$

In the following, we use result B.1 for $\int_{k_0} \mathcal{G}_\sigma$ (we will see that always $M_\sigma \neq 0$). At $T = 0$ this reduces to $\frac{1}{2M_\sigma}$.
Eliminating λ from equations (4.5) and (4.6), we obtain $M(T=0) \equiv M_0$ by choosing the positive solution:

$$M_0 = \frac{-3 + \sqrt{9 + 32f^4}}{4f^2} \quad , \quad (\varphi \neq 0) \tag{4.8}$$

with the notation $\varphi(T=0) \equiv f$.
In the same way, we can eliminate f and deduce, apart from a meaningless negative solution, two positive-real ones:

$$M_0^{>} = \frac{\sqrt[3]{2}\left(\sqrt[3]{6}N^2 + \left(\sqrt{729N^4\lambda^2 - 6N^6} - 27N^2\lambda\right)^{2/3}\right)}{3^{2/3}N\sqrt[3]{\sqrt{729N^4\lambda^2 - 6N^6} - 27N^2\lambda}} \quad , \quad (\varphi \neq 0) \tag{4.9}$$

$$M_0^{<} = \frac{(-6)^{2/3}N^2 - \sqrt[3]{-6}\left(\sqrt{729N^4\lambda^2 - 6N^6} - 27N^2\lambda\right)^{2/3}}{3N\sqrt[3]{\sqrt{729N^4\lambda^2 - 6N^6} - 27N^2\lambda}} \quad , \quad (\varphi \neq 0) \tag{4.10}$$

Likewise, we can express f by M_0:

$$f = \pm\sqrt{\frac{3M_0}{4-2M_0^2}} \quad , \quad (\varphi \neq 0) \tag{4.11}$$

Equation (4.7) has, apart from two meaningless complex solutions, a positive-real one:

$$M_0 = \frac{\sqrt[3]{27\lambda N^2 + \sqrt{3}\sqrt{N^6 + 243\lambda^2 N^4}}}{3^{2/3}N} - \frac{N}{\sqrt[3]{81\lambda N^2 + 3\sqrt{3}\sqrt{N^6 + 243\lambda^2 N^4}}} \quad , \quad (\varphi = 0) \tag{4.12}$$

Now we should check under which circumstances the two cases $\varphi \neq 0$ resp. $\varphi = 0$ hold. Therefore we assume that equation (4.5) holds true (i.e., $\varphi \neq 0$), $M > 0$, $T > 0$ and last but not least $\lambda > 0$. The *Mathematica*-function ***Reduce*** yields an equivalent system of conditions:

$$0 < M < \sqrt{2} \tag{4.13}$$

$$\wedge\ 0 < \lambda < \tfrac{1}{12}\left(2MN - M^3N\right) \tag{4.14}$$

$$\wedge\ T = \frac{M}{\ln\left(\frac{NM^3-2NM-12\lambda}{NM^3-2NM+12\lambda}\right)} = \frac{M}{\ln\left(1-\frac{24\lambda}{-2MN+M^3N+12\lambda}\right)} \ . \tag{4.15}$$

There is a little subtlety hidden in the above assumptions: we have not specified that there are upper and lower boundaries for M, therefore the assumption $M > 0$ (and consequently also the system of conditions we derived with ***Reduce***) includes too many solutions. Nevertheless, we can calculate the maximal value λ_{cr} for the coupling constant, which is possible within the assumptions. It is nothing else but the maximum of $\frac{1}{12}\left(2MN - M^3N\right)$ in the interval $0 < M < \sqrt{2}$:

$$\lambda_{cr} = \max_{0<M<\sqrt{2}}\left[\frac{1}{12}\left(2MN - M^3N\right)\right] \ .$$

In a next step, we feed ***Reduce*** with the condition $\frac{24\lambda}{-2MN+M^3N+12\lambda} < 0$, which is equivalent to $T > 0\ \wedge\ M > 0\ \wedge\ \lambda > 0$ as one can see from (4.15). ***Reduce*** offers the equivalent conditions

$$0 < \tfrac{4\lambda}{N} < \tfrac{4}{9}\sqrt{\tfrac{2}{3}}$$
$$\wedge\ M_{min} < M < M_{max} \ ,$$

where M_{min}/M_{max} is the smaller/larger positive solution of the equation $12\frac{\lambda}{N} - 2M_{min/max} + M_{min/max}^3 = 0$. One can directly see that this is nothing else but equation (4.5) for $T = 0$, if we assume $M > 0$. From this we easily conclude that $M_{max} = M_0^{>}$ and $M_{min} = M_0^{<}$. Additionally, we gained an explicit expression for λ_{cr}:

$$\lambda_{cr} = \frac{N}{9}\sqrt{\frac{2}{3}} \ . \tag{4.16}$$

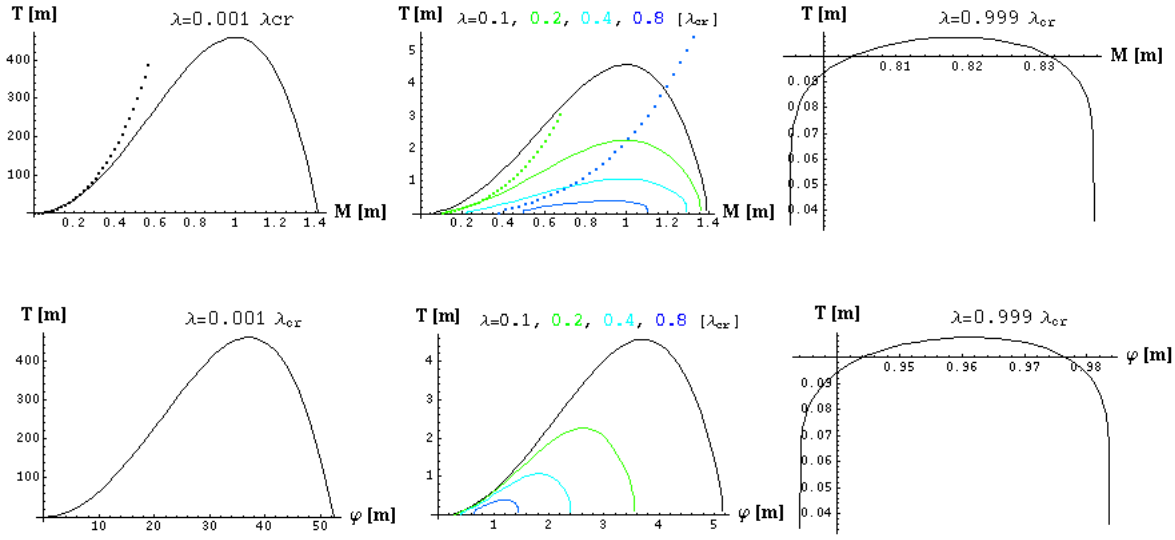


Figure 4.1: M and ϕ at the extrema of the effective potential (solid lines correspond to $\varphi \neq 0$, dotted lines to $\varphi = 0$).

Relation (4.15) allows us to plot $T(M)$ for $\varphi \neq 0$, and also $T(\varphi)$ for $\varphi \neq 0$ if we employ (4.6). One can observe from fig.4.1 what happens when λ is sent to λ_{cr}. The maximum (4.9) and minimum (4.10) approach each other more and more, until at λ_{cr} only that $M(T=0)$ is left, which follows from equation (4.7). Let us turn to the effective potential. Just for computational convenience, we add a constant to the effective potential (that does not change the physics), such that the minima of the classical potential touch the ϕ-axis. Remember, the symbol f stands for all extrema at zero temperature, whereas F refers to the right minimum at zero temperature alone. Then, with $F = \sqrt{\frac{Nm^2}{4\lambda}}$ and $U(F) = -\frac{\lambda}{N}F^4$, we find the shifted classical potential to be $\overline{U} = \frac{\lambda}{N}\left(\phi^2 - F^2\right)^2$.
Result 4.3, using $\overline{U}$, yields

$$V_{eff} = \frac{\lambda}{N}\left(\phi^2 - F^2\right)^2 + \frac{1}{2}\left(M + 2T\ln\left(1 - e^{-M/T}\right)\right)$$
$$+\frac{1}{2}\left(m_\sigma^2 - M^2\right)\frac{1}{M}\left(\frac{1}{e^{M/T}-1} + \frac{1}{2}\right) + \frac{3}{N}\lambda\left[\frac{1}{M}\left(\frac{1}{e^{M/T}-1} + \frac{1}{2}\right)\right]^2 \quad , \tag{4.17}$$

where we neglected the addend $T(1 - 2\ln(e-1))$, which would contribute a relatively small, ϕ-independent constant. Also note that for simplicity we keep the same symbol for the shifted effective potential.
When we want to plot the effective potential against ϕ at temperature T, we have to solve equation (4.3) at T for $M(\phi)$. Figure (4.2) shows the typical evolution for the case $\lambda < \lambda_{cr}$. We have a phase transition at temperature T^*, when the right minimum at $\phi = \varphi_R$ and the minimum at $\phi = 0$ are at the same height, i.e., the global minimum Φ changes from φ_R to zero. In accordance with fig.4.1, one can see in fig.4.3 that for λ_{cr} the minimum and the maximum are replaced by a saddle-point at $T = 0$. Figure (4.4) illustrates that even for $\lambda < \lambda_{cr}$ the classical potential is not reproduced for $T = 0$, due to $\frac{1}{M}\left(\frac{1}{e^{M/T}-1} + \frac{1}{2}\right) \xrightarrow{T\to 0} \frac{1}{2M}$ and

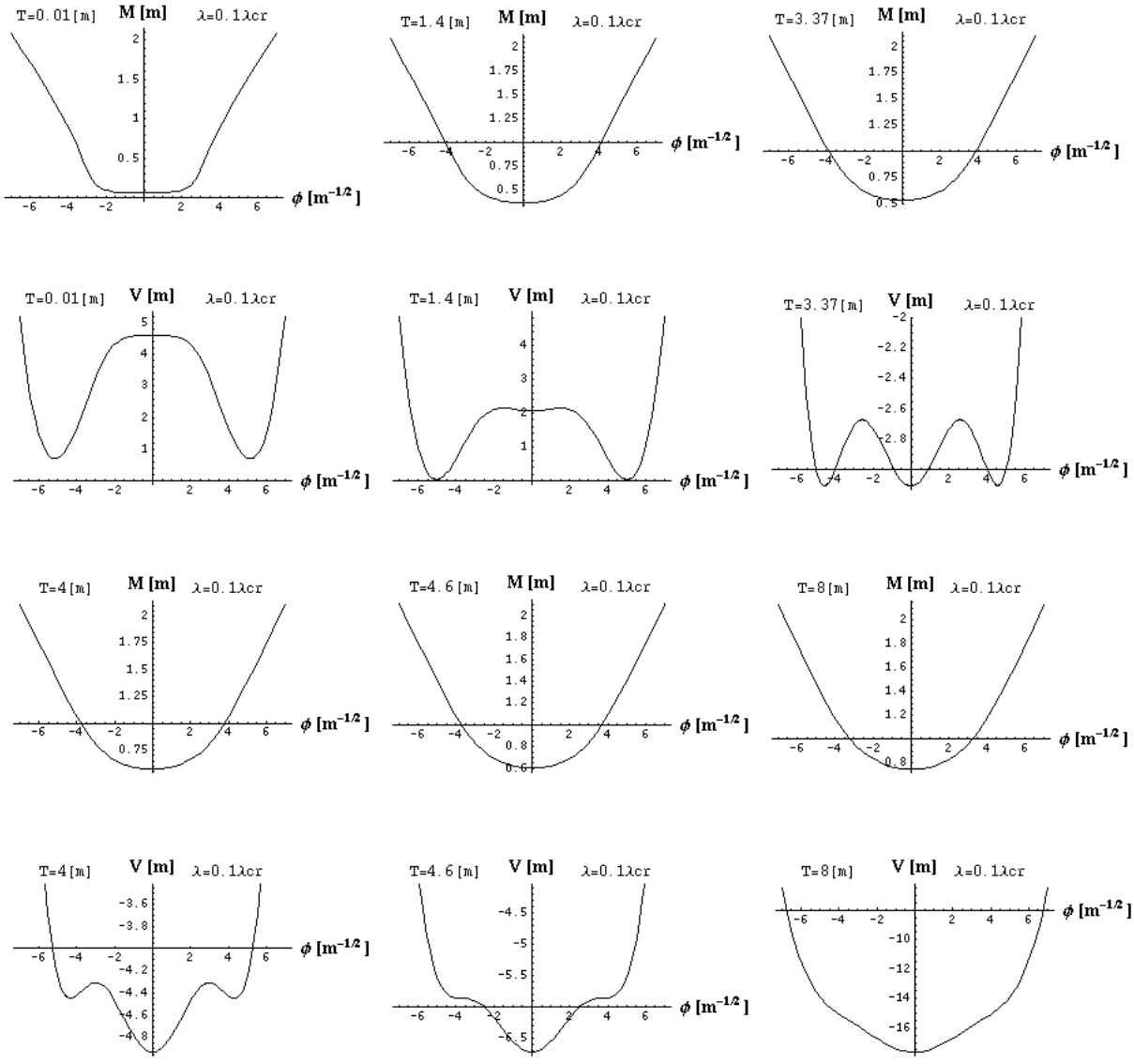


Figure 4.2: shifted effective potential for different temperatures, $\lambda = 0.1\lambda_{cr}$.

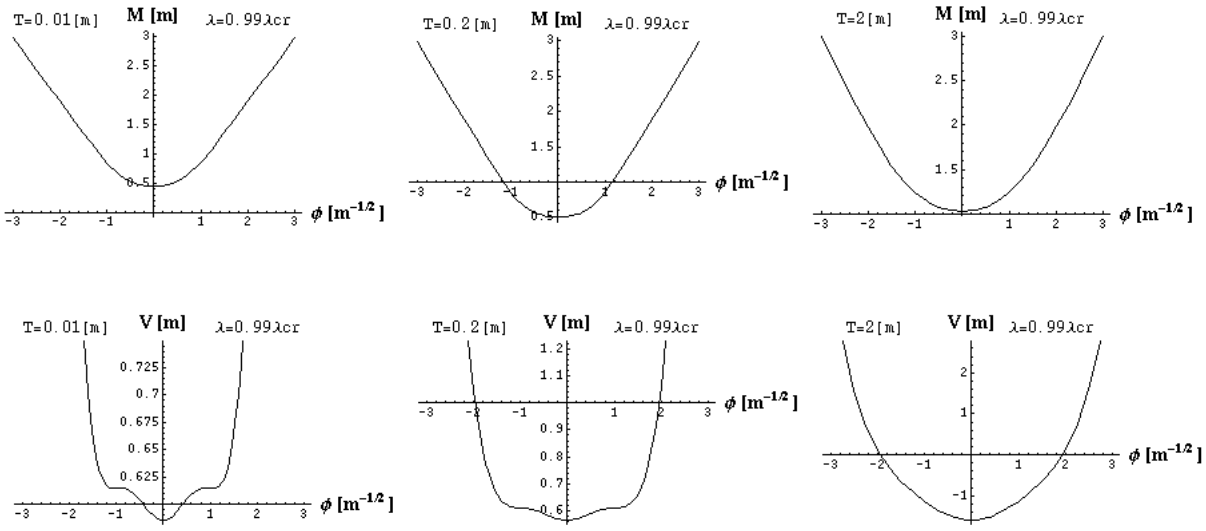


Figure 4.3: shifted effective potential for different temperatures, $\lambda = 0.99\lambda_{cr}$.

$\frac{1}{2}\left(M + 2T\ln\left(1 - e^{-M/T}\right)\right) \xrightarrow{T\to 0} \frac{M}{2}$. Even if $\lambda = 0$, although in this case the contribution from V_2 is zero and $M_0^2(\phi) = -m^2 + 12\frac{\lambda}{N}\phi^2 + 12\frac{\lambda}{N}\frac{1}{2M} \overset{\lambda=0}{=} m_\sigma^2$, the vacuum energy (synonymous: zero-point energy) $\frac{M}{2}$ is still left.

For $0 < \lambda < \lambda_{cr}$, a right minimum exists at $T = 0$, however it is global only until λ reaches the value λ^* at which $T^* = 0$. This means there is no T^*, i.e., no phase transition if $\lambda > \lambda^*$, because the global minimum is always zero. Figure (4.5) shows the critical temperature T^* vs. λ in multiples of λ_{cr}. It was numerically obtained by minimizing the difference between the effective potential at $\phi = \varphi_R$ and at $\phi = 0$. Of particular interest is the effective potential at the global minimum Φ, which is shown in fig.4.6 for various λ.

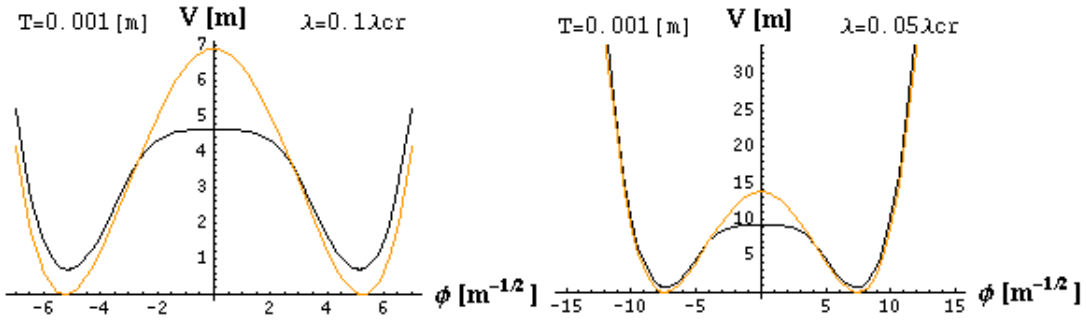


Figure 4.4: comparison between the classical potential (orange) and the effective potential for small T.

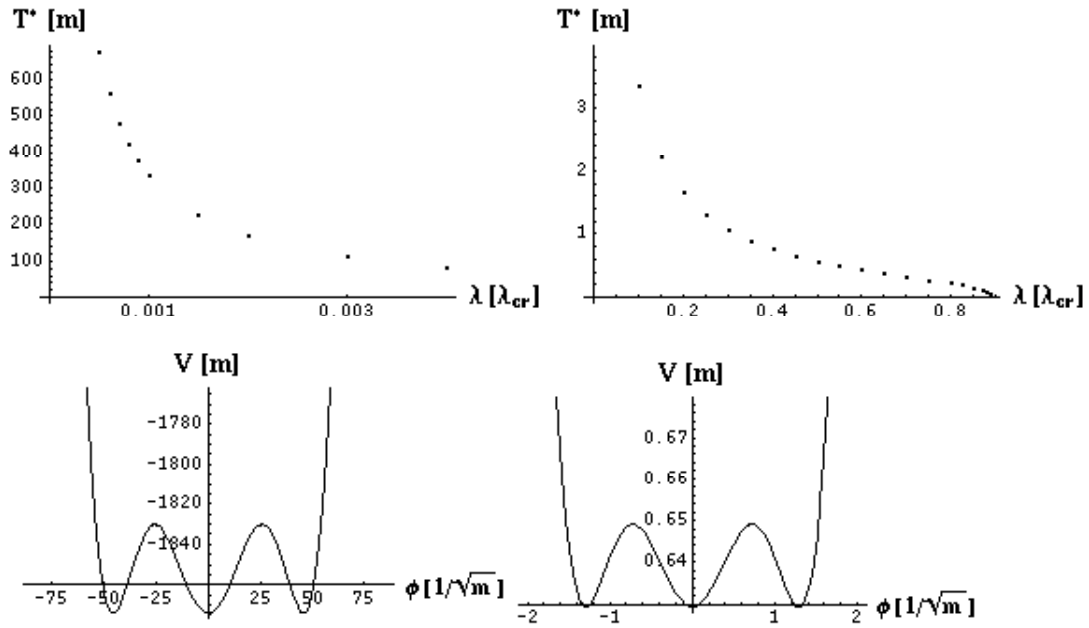


Figure 4.5: top-down, from left to right: critical temperature at which the phase transition occurs; effective potential for $\lambda = 0.001\lambda_{cr}$ at $T^* = 338.46m$; effective potential for $\lambda = 0.885\lambda_{cr} \simeq \lambda^*$ at $T = 0.01$.

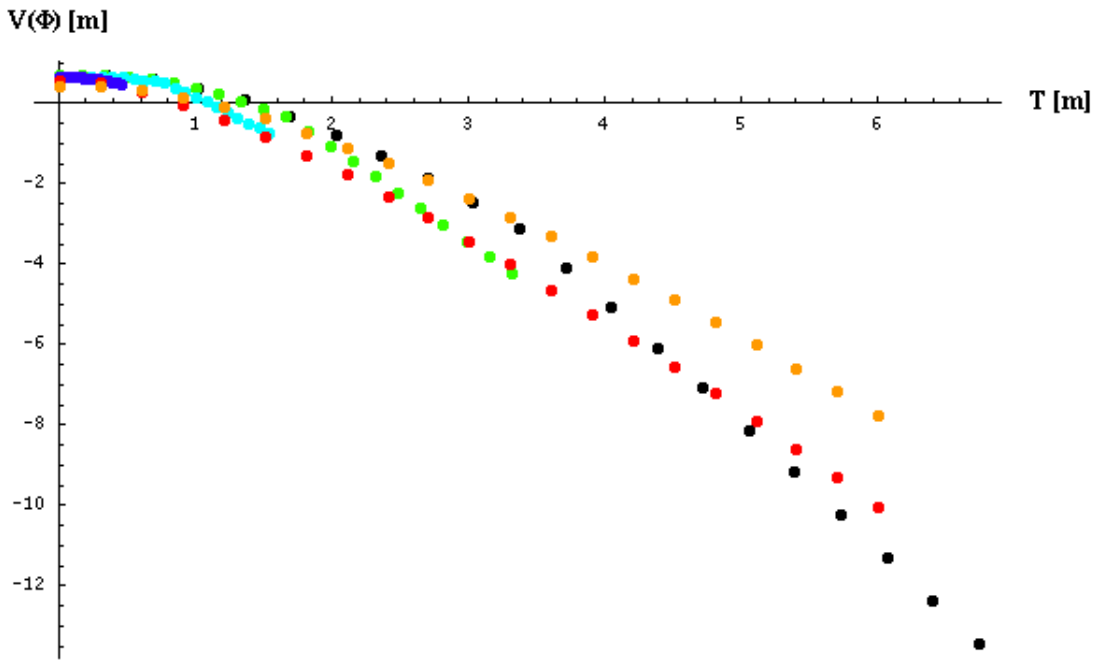


Figure 4.6: effective potential at its global minimum Φ; black: $\lambda = 0.1\lambda_{cr}$, green: $\lambda = 0.2\lambda_{cr}$, cyan: $\lambda = 0.4\lambda_{cr}$, dark blue: $\lambda = 0.8\lambda_{cr}$, red: $\lambda = \lambda_{cr}$, orange: $\lambda = 4\lambda_{cr}$; the curves for $\lambda < \lambda_{cr}$ each range from $T = 0.01$ to $T = 2T^*$.

4.1.2 WKB

As in the section before, we use the shifted classical potential $\overline{U} = \frac{\lambda}{N}(\phi^2 - F^2)^2$ for computational convenience. The minima of the potential are given by

$$f_{1/2} = \pm F = \pm\sqrt{\frac{Nm^2}{4\lambda}}, \tag{4.18}$$

and the turning points by

$$\phi_1 = +\sqrt{-\sqrt{\frac{NE}{\lambda}} + F^2}\ , \quad \phi_2 = +\sqrt{+\sqrt{\frac{NE}{\lambda}} + F^2}\ , \tag{4.19}$$

$$\phi_3 = -\sqrt{+\sqrt{\frac{NE}{\lambda}} + F^2}\ , \quad \phi_4 = -\sqrt{-\sqrt{\frac{NE}{\lambda}} + F^2}\ . \tag{4.20}$$

One now has to distinguish two cases (see figure 4.7). For $E < E_{max}$, where $E_{max} = \frac{\lambda}{N}F^4$,

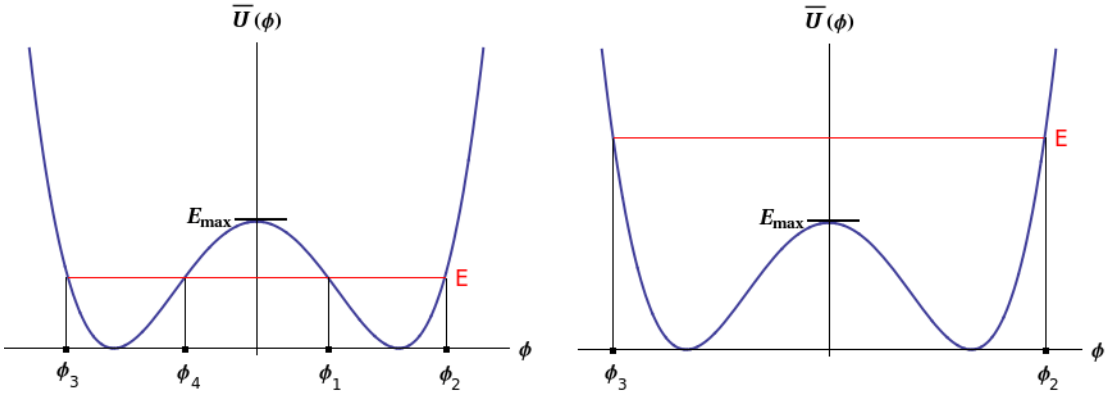


Figure 4.7: shifted classical potential.

we have two bound states and, as already mentioned in section 2.3.2, a certain probability for tunneling. Therefore one has to modify the WKB equation. According to [39] the eigenvalues are determined by the following modified WKB equations:

$$\int_{\phi_1(E)}^{\phi_2(E)} d\phi\ \sqrt{2\left[E - U(\phi)\right]} = \left(n + \frac{1}{2}\right)\pi + \frac{1}{2}e^{-L}\ , \tag{4.21}$$

$$\int_{\phi_1(E)}^{\phi_2(E)} d\phi\ \sqrt{2\left[E - U(\phi)\right]} = \left(n + \frac{1}{2}\right)\pi - \frac{1}{2}e^{-L}\ , \tag{4.22}$$

$$\text{with}\quad L = \int_{\phi_4(E)}^{\phi_1(E)} d\phi\ \sqrt{2\left[U(\phi) - E\right]}\ , \tag{4.23}$$

where we used the equality

$$\int_{\phi_1(E)}^{\phi_2(E)} d\phi \, \sqrt{2\left[E - U(\phi)\right]} = \int_{\phi_3(E)}^{\phi_4(E)} d\phi \, \sqrt{2\left[E - U(\phi)\right]} \, . \tag{4.24}$$

From (4.21) follow the eigenvalues E_n^+ and from (4.22) the eigenvalues E_n^-. These are the eigenvalues which lie below E_{max}.
For $E \geq E_{max}$ we have the ordinary WKB equation

$$\int_{\phi_3(E)}^{\phi_2(E)} d\phi \, \sqrt{2\left[E - U(\phi)\right]} = \left(n + \frac{1}{2}\right) \pi \, , \tag{4.25}$$

from which the eigenvalues E_n^{up} are determined, i.e., the eigenvalues which lie above E_{max}.
The sum in the partition function

$$Z = \sum_n e^{-E_n/T} \tag{4.26}$$

runs over all eigenvalues E_n^{up}, E_n^+ and E_n^-.
The effective potential at its global minimum is given by

$$V(\Phi) = -T \ln Z = -T \ln \left(\sum_n e^{-E_n/T} \right) . \tag{4.27}$$

In order to find out how many eigenvalues have to be included in the calculation, we proceed similarly to section 3.3. The comparison between the WKB method and the CJT formalism is shown in figures 4.8-4.10. We observe that the larger λ, the more both curves deviate from each other at $T = 0$, while their shape become more equal. For large λ and $T < T_g$, the curves are identical, up to an offset.

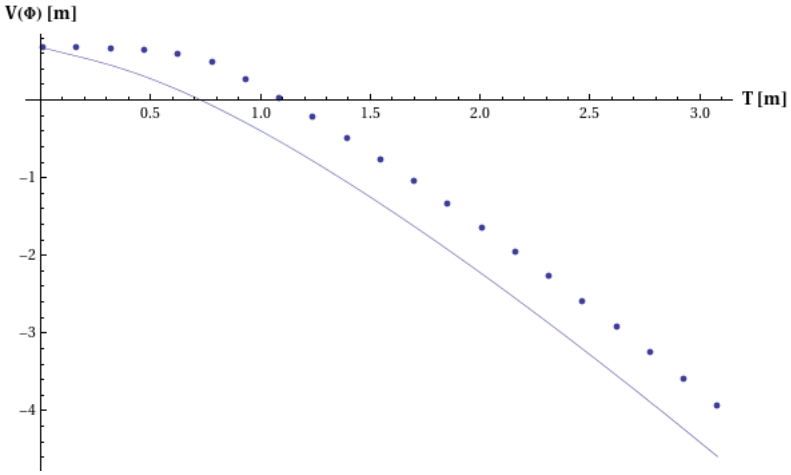


Figure 4.8: comparison of the effective potential at its global minimum for WKB (solid line) and CJT (dots), $\lambda = 0.4\lambda_{cr}$. Both curves are almost equal at $T = 0$.

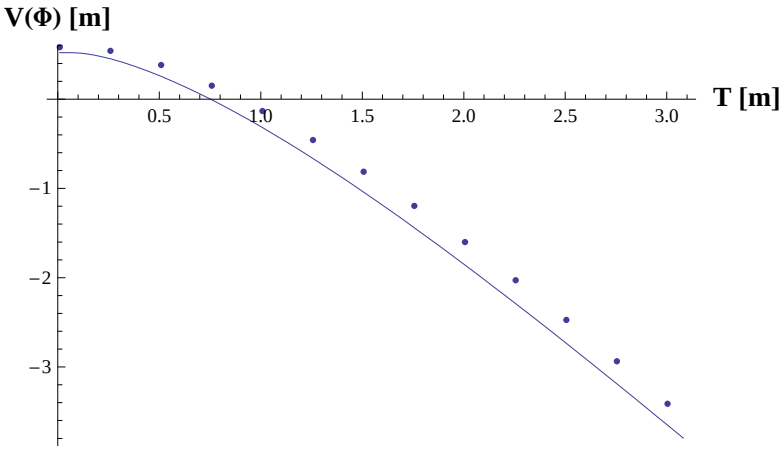


Figure 4.9: comparison of the effective potential at its global minimum for WKB (solid line) and CJT (dots), $\lambda = \lambda_{cr}$. $V_{WKB}(T=0) \simeq 0.521m$, $V_{CJT} \simeq 0.582m$.

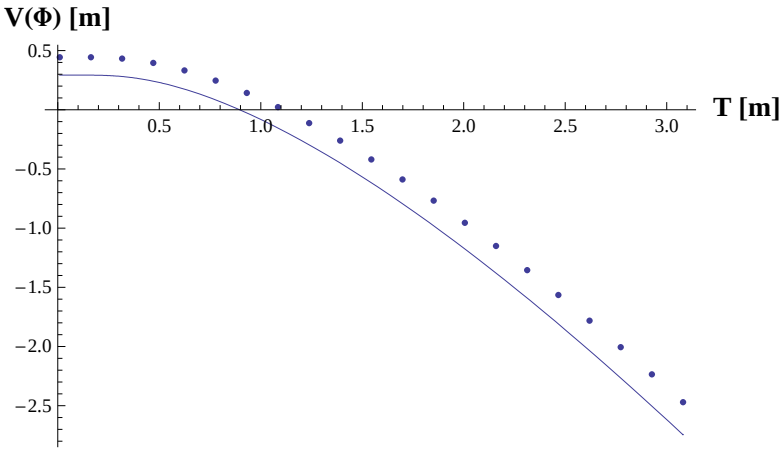


Figure 4.10: comparison of the effective potential at its global minimum for WKB (solid line) and CJT (dots), $\lambda = 4\lambda_{cr}$. $V_{WKB}(T=0) \simeq 0.293m$, $V_{CJT} \simeq 0.443m$.

Chapter 5

The generating functional in polar coordinates

So far we have discussed models for a single field ($N = 1$). The later chapters will proceed with $N = 2$, expressing the field variables (usually referred to as *internal degrees of freedom*) either in cartesian or in polar coordinates. In this chapter we refer to 1+0 dimensional SQFT. Except for section 5.2.3, we were able to generalize all our results presented in this chapter to 1+3 dimensions. For simplicity, and because the verification of certain marginal (but interesting) issues in 1+3 dimensions is still ongoing work, we will restrict ourselves to 1+0 dimensions.
Note that we use natural units where $\hbar = 1$.
For cartesian coordinates, the starting point is the generating functional (E.7). In the case we are not at the global minimum, source terms are present:

$$Z = \left[\prod_{n=1}^{N+1} \int_{-\infty}^{\infty} \frac{d\phi_{1,n}}{\sqrt{2\pi\hbar\varepsilon}}\right] \left[\prod_{n=1}^{N+1} \int_{-\infty}^{\infty} \frac{d\phi_{2,n}}{\sqrt{2\pi\hbar\varepsilon}}\right] e^{-\frac{1}{\hbar}A_E + sourceterms} \ , \tag{5.1}$$

$$\text{where } \phi_{1,a} = \phi_{1,b} \text{ and } \phi_{2,a} = \phi_{2,b} \ , \tag{5.2}$$

$$\text{with the Euclidean action } A_E = \varepsilon \sum_{n=1}^{N+1} \left[\frac{1}{2}\left(\frac{\vec{\phi}_n - \vec{\phi}_{n-1}}{\varepsilon}\right)^2 + U(\vec{\phi}_n, -i\tau_n)\right] \ , \text{ and with} \tag{5.3}$$

$$sourceterms = \varepsilon \sum_{n=1}^{N+1} \left[\phi_{1,n} J_{1,n} + \phi_{2,n} J_{2,n} + \sum_{m=1}^{N+1} \left[\frac{1}{2}\phi_{1,n} K_{1,nm} \phi_{1,m} + \frac{1}{2}\phi_{2,n} K_{2,nm} \phi_{2,m}\right]\right] \ . \tag{5.4}$$

Note that in the case of SQFT one has to bear in mind the periodic boundary condition

$$\vec{\phi}(\tau, \vec{x}) \stackrel{!}{=} \vec{\phi}(\tau + 1/T, \vec{x}) \ . \tag{5.5}$$

In the following, we set the sources to zero, which means we are at the global minimum.
According to chapter 8 of Kleinert's textbook [42], we have to change to polar coordinates in this sliced version, which is not equivalent to performing the coordinate transformation in the

continuous form of Z. The transformation is given by

$$\phi_{1,j} = r_j \cos\varphi_j \ , \quad \phi_{2,j} = r_j \sin\varphi_j \ , \tag{5.6}$$

with the Jacobian $\det F_j = \begin{vmatrix} \frac{\partial\phi_{1,j}}{\partial r_j} & \frac{\partial\phi_{1,j}}{\partial\varphi_j} \\ \frac{\partial\phi_{2,j}}{\partial r_j} & \frac{\partial\phi_{2,j}}{\partial\varphi_j} \end{vmatrix} = \begin{vmatrix} \cos\varphi_j & -r_j\sin\varphi_j \\ \sin\varphi_j & r_j\cos\varphi_j \end{vmatrix} = r_j \ .$

The kinetic term becomes

$$\exp\left(-\frac{1}{2}\sum_{j=1}^{N+1}\varepsilon\frac{\left(\vec{\phi}_j-\vec{\phi}_{j-1}\right)^2}{\varepsilon^2}\right) = \prod_{j=1}^{N+1} e^{-\frac{1}{2\varepsilon}\left(\vec{\phi}_j-\vec{\phi}_{j-1}\right)^2}$$
$$= \prod_{j=1}^{N+1} e^{-\frac{1}{2\varepsilon}[r_j^2+r_{j-1}^2-2r_jr_{j-1}\cos(\varphi_j-\varphi_{j-1})]} \ .$$

In the following, we assume a potential that depends on the radial degree of freedom only, for instance

$$U(\vec{\phi}_j) = \frac{m^2}{2}\vec{\phi}_j^{\,2} + \frac{\lambda}{N}\left(\vec{\phi}_j\cdot\vec{\phi}_j\right)^2 = \frac{m^2}{2}r_j^2 + \frac{\lambda}{N}r_j^4 \ . \tag{5.7}$$

With the above, the expression for the generating functional (5.1) merges into its representation in polar coordinates

$$Z = \left[\prod_{j=1}^{N+1}\int_0^\infty \frac{dr_jr_j}{\sqrt{2\pi\varepsilon}}\right]\left[\prod_{j=1}^{N+1}\int_0^{2\pi}\frac{d\varphi_j}{\sqrt{2\pi\varepsilon}}\right] e^{-A_E} \ , \tag{5.8}$$

$$\text{where } r_b = r_a \text{ and } \varphi_b = \varphi_a \ ,$$

$$\text{with } A_E[r_j,\varphi_j;\varepsilon] = \frac{1}{2}\sum_{j=1}^{N+1}\varepsilon\left(\frac{r_j^2+r_{j-1}^2-2r_jr_{j-1}\cos(\varphi_j-\varphi_{j-1})}{\varepsilon^2}\right) + \sum_{j=1}^{N+1}\varepsilon U(r_j) \ . \tag{5.9}$$

With

$$r_j^2+r_{j-1}^2-2r_jr_{j-1}\cos(\varphi_j-\varphi_{j-1}) = (r_j\cos\varphi_j - r_{j-1}\cos\varphi_{j-1})^2 + (r_j\sin\varphi_j - r_{j-1}\sin\varphi_{j-1})^2 \ ,$$

we obtain

$$\left[\prod_{j=1}^{N}\int_0^\infty \frac{dr_jr_j}{\sqrt{2\pi\varepsilon}}\right]\left[\prod_{j=1}^{N}\int_0^{2\pi}\frac{d\varphi_j}{\sqrt{2\pi\varepsilon}}\right] e^{-A_E} = \left[\prod_{j=1}^{N+1}\int_0^\infty dr_jr_j\right]\left[\prod_{j=1}^{N+1}\int_0^{2\pi}d\varphi_j\right]\times \tag{5.10}$$
$$\times\exp\left(-\frac{1}{2}\sum_{j=1}^{N+1}\varepsilon\left(\frac{(r_j\cos\varphi_j - r_{j-1}\cos\varphi_{j-1})^2}{\varepsilon^2} + \frac{(r_j\sin\varphi_j - r_{j-1}\sin\varphi_{j-1})^2}{\varepsilon^2}\right) - \sum_{j=1}^{N+1}\varepsilon U(r_j)\right).$$

With

$$\frac{(r_j\cos\varphi_j - r_{j-1}\cos\varphi_{j-1})^2}{\varepsilon^2} \xrightarrow{\varepsilon\to0} \left(\frac{\partial}{\partial\tau}(r\cos\varphi)|_{\tau=\tau_j}\right)^2 \ ,$$

$$\sum_{j=1}^{N+1}\varepsilon \xrightarrow{\varepsilon\to0} \int_0^{1/T} d\tau \ , \quad \text{where } 0\equiv\tau_0 \text{ and } 1/T\equiv\tau_{N+1} \ ,$$

$$\lim_{\varepsilon\to0} A_E \equiv S_E$$

and since $S_E = \int d\tau (L + U)$, we read off

$$L_E(r,\varphi) = \frac{1}{2}\frac{\partial}{\partial\tau}(r\cos\varphi)\frac{\partial}{\partial\tau}(r\cos\varphi) + \frac{1}{2}\frac{\partial}{\partial\tau}(r\sin\varphi)\frac{\partial}{\partial\tau}(r\sin\varphi) + U(r)\ , \tag{5.11}$$

which is consistent with applying the transformations (5.6) to $L_E(\phi_1, \phi_2)$. With

$$\begin{aligned}(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi)^2 &= \dot{r}^2\cos^2\varphi - 2\dot{r}\cos\varphi r\dot{\varphi}\sin\varphi + r^2\dot{\varphi}^2\sin^2\varphi,\\ (\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi)^2 &= \dot{r}^2\sin^2\varphi + 2\dot{r}\cos\varphi r\dot{\varphi}\sin\varphi + r^2\dot{\varphi}^2\cos^2\varphi,\end{aligned}$$

we arrive at

$$L_E(r,\varphi) = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\varphi}^2 + U(r)\ . \tag{5.12}$$

Hence, in the continuum limit $\varepsilon \to 0$, we can abbreviate (5.8) by

$$Z = \oint_0^\infty \mathcal{D}r r \oint_0^{2\pi} \mathcal{D}\varphi e^{-\int d\tau L_E(r,\varphi)}. \tag{5.13}$$

But beware! We are not allowed to apply the CJT formalism to $L_E(r,\varphi)$ a priori. We have to take care of three issues

- the Jacobian r which appears in the integration measure,
- the integration over r runs from 0 to ∞,
- the integration over φ runs from 0 to 2π.

We cannot assume in the first place that perturbation theory for such a path integral is the same as for the cartesian generating functional (5.1). In his textbook [43], Anthony Zee outlines the main steps which lead to the Feynman rules in the case of cartesian coordinates. He begins with a neat toy model (called "baby problem"), $Z = \int_{-\infty}^{\infty} dq e^{-\frac{1}{2}aq^2 - \frac{\lambda}{N}q^4 + Jq}$, which is suited to convince oneself that the Feynman rules for (5.13) differ from the well-known cartesian. Namely, their derivation is based on the Gaussian integral

$$\int_{-\infty}^{\infty} dq\ e^{-\frac{1}{2}aq^2 + Jq} = \left(\frac{2\pi}{a}\right)^{\frac{1}{2}} e^{\frac{J^2}{2a}}. \tag{5.14}$$

One possibility would be to derive the Feynman rules for (5.13), in a similar way presented by Zee. A much more elegant way is to do perturbation theory in general curvilinear coordinates, as Kleinert discusses it in chapter 10 of his textbook [42] for QM and SQM. Since Euclidean QM with a periodic boundary condition (Statistical Quantum Mechanics) has mathematically the same structure as SQFT in 1+0 dimensions (compare with appendix E), his results are of high interest for us. His discussion of SQM in chapter 10.11 has not been examined with respect to all implications for our work, yet. In this thesis, we focus on his considerations concerning the Jacobian, which we want to discuss in section 5.2.3. In section 10.6.1, Kleinert begins with the discussion with a relatively general case in Euclidean QM. This section is valid for path integrals at $T \geq 0$, in arbitrary curvilinear coordinates (the coordinate transformation only

needs to be holonomic), which contain the commonly used action ($U \neq 0$). In the following sections 10.7-10.10, Kleinert shows for the special case $T = 0$ together with $U = \omega^2 x^2$ that the Jacobian does not have an effect on perturbation theory. At each order in a perturbative expansion, the terms containing $\delta(0)$ are canceled by other terms. In sections 10.11.2/10.11.4 he proceeds with the proof for $T \neq 0$ together with $U = 0$ on the basis of holonomic coordinate transformations, whereas in sections 10.11.5/10.11.6 he broadens the scope to non-holonomic coordinate transformations. However, at the end of section 10.6.1, Kleinert predicts that the $\delta(0)$-terms will be canceled also in the general case, since this is what one expects due to consistency with $\delta(0) = 0$ in dimensional regularization (compare with section 5.2.2). So one of our three problems is solved. In chapter 9 we also want to present an alternative attempt to cope with the problem.
The problem of the integration intervals is still left. In section 5.3 we examine the integration over φ, whereas section 5.4 is dedicated to the integration over r. Both problems are solved. In the latter case, the solution is again dimensional regularization.

5.1 Integrating out the angular degrees of freedom

If the classical potential U does not depend on φ, i.e., $U = U(r)$, we are able to perform the integrations over φ_n. With the help of the *modified Bessel functions*, $I_\nu(z)$, expression (5.8) can be rewritten as

$$\begin{aligned} Z = &\left[\prod_{n=1}^{N+1} \int_0^\infty \frac{dr_n r_n}{\sqrt{2\pi\varepsilon}}\right] \left[\prod_{n=1}^{N+1} \int_0^{2\pi} \frac{d\varphi_n}{\sqrt{2\pi\varepsilon}}\right] \left[\prod_{n=1}^{N+1} e^{-\varepsilon \frac{1}{2} \frac{r_n^2 + r_{n-1}^2}{\varepsilon^2}}\right] \times \\ &\times \underbrace{\left[\prod_{n=1}^{N+1} \exp\left(\varepsilon \frac{1}{2} \frac{2 r_n r_{n-1} \cos(\varphi_n - \varphi_{n-1})}{\varepsilon^2}\right)\right]}_{= \left[\prod_{n=1}^{N+1} \sum_{m_n=-\infty}^{\infty} I_{m_n}\left(\frac{r_n r_{n-1}}{\varepsilon}\right) e^{i m_n (\varphi_n - \varphi_{n-1})}\right]} \left[\prod_{n=1}^{N+1} e^{-\varepsilon U(r_n)}\right]. \end{aligned} \tag{5.15}$$

Let us begin with collecting all factors depending on φ_N and integrating this expression over φ_N:

$$\begin{aligned} &\int_0^{2\pi} d\varphi_N \left(\sum_{m_N=-\infty}^{\infty} I_{m_N}\left(\frac{r_N r_{N-1}}{\varepsilon}\right) e^{i m_N (\varphi_N - \varphi_{N-1})}\right) \left(\sum_{m_{N+1}=-\infty}^{\infty} I_{m_{N+1}}\left(\frac{r_{N+1} r_N}{\varepsilon}\right) e^{i m_{N+1} (\varphi_{N+1} - \varphi_N)}\right) \\ &= \sum_{m_N} \sum_{m_{N+1}} I_{m_N}\left(\frac{r_N r_{N-1}}{\varepsilon}\right) I_{m_{N+1}}\left(\frac{r_{N+1} r_N}{\varepsilon}\right) e^{i m_{N+1} \varphi_{N+1} - i m_N \varphi_{N-1}} \int_0^{2\pi} d\varphi_N e^{i \varphi_N (m_N - m_{N+1})} \\ &= 2\pi \sum_{m_{N+1}} I_{m_{N+1}}\left(\frac{r_N r_{N-1}}{\varepsilon}\right) I_{m_{N+1}}\left(\frac{r_{N+1} r_N}{\varepsilon}\right) e^{i m_{N+1} \varphi_{N+1} - i m_{N+1} \varphi_{N-1}} , \end{aligned} \tag{5.16}$$

where we used

$$\begin{aligned} \int_0^{2\pi} d\varphi_N e^{i \varphi_N (m_N - m_{N+1})} &= -\frac{i\left(-1 + e^{2\pi i (m_N - m_{N+1})}\right)}{m_N - m_{N+1}} = \begin{cases} 0 & \text{for } m_{N+1}, m_N \in \mathbb{Z} \text{ and } m_{N+1} \neq m_N \\ 2\pi & \text{for } m_{N+1} = m_N \end{cases} \\ &= 2\pi \delta_{m_{N+1}, m_N} . \end{aligned} \tag{5.17}$$

We continue with the integration over φ_{N-1}. We collect all factors depending on φ_{N-1} and perform the integration. With (5.16) we have:

$$\begin{aligned}
&\int_0^{2\pi} d\varphi_{N-1} \sum_{m_{N-1}} I_{m_{N-1}}\left(\frac{r_{N-1}r_{N-2}}{\varepsilon}\right) e^{im_{N-1}(\varphi_{N-1}-\varphi_{N-2})} \times \\
&\times 2\pi \sum_{m_{N+1}} I_{m_{N+1}}\left(\frac{r_N r_{N-1}}{\varepsilon}\right) I_{m_{N+1}}\left(\frac{r_{N+1}r_N}{\varepsilon}\right) e^{im_{N+1}\varphi_{N+1}-im_{N+1}\varphi_{N-1}} \\
= \sum_{m_{N+1}}\sum_{m_{N-1}} & I_{m_{N-1}}\left(\frac{r_{N-1}r_{N-2}}{\varepsilon}\right) I_{m_{N+1}}\left(\frac{r_N r_{N-1}}{\varepsilon}\right) I_{m_{N+1}}\left(\frac{r_{N+1}r_N}{\varepsilon}\right) e^{im_{N+1}\varphi_{N+1}-im_{N-1}\varphi_{N-2}} \times \\
&2\pi \underbrace{\int_0^{2\pi} d\varphi_{N-1} e^{i\varphi_{N-1}(m_{N-1}-m_{N+1})}}_{2\pi\delta_{m_{N-1},m_{N+1}}} \\
= \sum_{m_{N+1}} & I_{m_{N+1}}\left(\frac{r_{N-1}r_{N-2}}{\varepsilon}\right) I_{m_{N+1}}\left(\frac{r_N r_{N-1}}{\varepsilon}\right) I_{m_{N+1}}\left(\frac{r_{N+1}r_N}{\varepsilon}\right) e^{im_{N+1}\varphi_{N+1}-im_{N+1}\varphi_{N-2}} (2\pi)^2 \ .
\end{aligned} \tag{5.18}$$

It is obvious that we are able to proceed in the same manner, which leads to

$$\sum_{m_{N+1}} \left[\prod_{n=2}^{N+1} I_{m_{N+1}}\left(\frac{r_n r_{n-1}}{\varepsilon}\right)\right] e^{im_{N+1}\varphi_{N+1}-im_{N+1}\varphi_1} (2\pi)^{N-1} \tag{5.19}$$

after having performed the $\int_0^{2\pi} d\varphi_2$ -integration. With (5.19) the partition function reads:

$$\begin{aligned}
Z = &\left[\prod_{n=1}^{N+1} \int_0^\infty \frac{dr_n r_n}{\sqrt{2\pi\varepsilon}}\right] \int_0^{2\pi} \frac{d\varphi_1}{\sqrt{2\pi\varepsilon}} \int_0^{2\pi} \frac{d\varphi_{N+1}}{\sqrt{2\pi\varepsilon}} \left[\prod_{n=1}^{N+1} e^{-\varepsilon\frac{1}{2}\frac{r_n^2+r_{n-1}^2}{\varepsilon^2}}\right] \times \\
\times \frac{1}{(\sqrt{2\pi\varepsilon})^{N-1}} &\left(\sum_{m_{N+1}} \left[\prod_{n=2}^{N+1} I_{m_{N+1}}\left(\frac{r_n r_{n-1}}{\varepsilon}\right)\right] e^{im_{N+1}\varphi_{N+1}-im_{N+1}\varphi_1} (2\pi)^{N-1}\right) \times \\
&\left[\prod_{n=1}^{N+1} e^{-\varepsilon U(r_n,-i\tau_n)}\right] \left(\sum_{m_1} I_{m_1}\left(\frac{r_1 r_0}{\varepsilon}\right) e^{im_1(\varphi_1-\varphi_0)}\right) \ .
\end{aligned} \tag{5.20}$$

A last time we proceed as before:

$$\begin{aligned}
\sum_{m_{N+1}}\sum_{m_1} &\left[\prod_{n=2}^{N+1} I_{m_{N+1}}\left(\frac{r_n r_{n-1}}{\varepsilon}\right)\right] I_{m_1}\left(\frac{r_1 r_0}{\varepsilon}\right) e^{im_{N+1}\varphi_{N+1}-im_1\varphi_0} (2\pi)^{N-1} \int_0^{2\pi} d\varphi_1 e^{i\varphi_1(m_1-m_{N+1})} \\
&= \sum_{m_{N+1}} \left[\prod_{n=1}^{N+1} I_{m_{N+1}}\left(\frac{r_n r_{n-1}}{\varepsilon}\right)\right] e^{im_{N+1}(\varphi_{N+1}-\varphi_0)} (2\pi)^N \ ,
\end{aligned} \tag{5.21}$$

and therefore

$$Z=\left[\prod_{n=1}^{N+1}\int_0^\infty\frac{dr_nr_n}{\sqrt{2\pi\varepsilon}}\right]\int_0^{2\pi}\frac{d\varphi_{N+1}}{\sqrt{2\pi\varepsilon}}\left[\prod_{n=1}^{N+1}e^{-\varepsilon\frac{1}{2}\frac{r_n^2+r_{n-1}^2}{\varepsilon^2}}\right]\frac{1}{(\sqrt{2\pi\varepsilon})^N}\left[\prod_{n=1}^{N+1}e^{-\varepsilon U(r_n,-i\tau_n)}\right]\times$$
$$\times\left(\sum_m\left[\prod_{n=1}^{N+1}I_m\left(\frac{r_nr_{n-1}}{\varepsilon}\right)\right]e^{im(\varphi_{N+1}-\varphi_0)}(2\pi)^N\right). \tag{5.22}$$

Because of $\varphi_{N+1}=\varphi_0$, the integration over φ_{N+1} simply yields a factor 2π, so that we obtain

$$Z=\sum_{m=-\infty}^{\infty}(2\pi)^{N+1}\left(\frac{1}{\sqrt{2\pi\varepsilon}}\right)^{2N+2}\left[\prod_{n=1}^{N+1}\int_0^\infty dr_n\right]\left[\prod_{n=1}^{N+1}e^{-\varepsilon U(r_n,-i\tau_n)}\right]\times$$
$$\times\left[\prod_{n=1}^{N+1}I_m\left(\frac{r_nr_{n-1}}{\varepsilon}\right)e^{-\varepsilon\frac{1}{2}\frac{r_n^2+r_{n-1}^2}{\varepsilon^2}}r_n\right]. \tag{5.23}$$

The latter factor we want to rewrite as follows:

$$\left[\prod_{n=1}^{N+1}I_m\left(\frac{r_nr_{n-1}}{\varepsilon}\right)e^{-\varepsilon\frac{1}{2}\frac{r_n^2+r_{n-1}^2}{\varepsilon^2}}r_n\right]=\left[\prod_{n=1}^{N+1}I_m\left(\frac{r_nr_{n-1}}{\varepsilon}\right)e^{-\frac{r_nr_{n-1}}{\varepsilon}}r_n\right]\left[\prod_{n=1}^{N+1}e^{-\frac{1}{2\varepsilon}(r_n-r_{n-1})^2}\right],$$

due to $e^{-\varepsilon\frac{1}{2}\frac{r_n^2+r_{n-1}^2}{\varepsilon^2}}=e^{-\frac{1}{2\varepsilon}\left[(r_n-r_{n-1})^2+2r_nr_{n-1}\right]}$.
Since

$$\prod_{n=1}^{N+1}r_n=\sqrt{r_{N+1}}\sqrt{r_{N+1}r_N}\sqrt{r_Nr_{N-1}}\cdots\sqrt{r_2r_1}\sqrt{r_1}\overset{r_{N+1}=r_0}{=}\left[\prod_{n=1}^{N+1}\sqrt{r_nr_{n-1}}\right], \tag{5.24}$$

we have

$$\left[\prod_{n=1}^{N+1}I_m\left(\frac{r_nr_{n-1}}{\varepsilon}\right)e^{-\frac{r_nr_{n-1}}{\varepsilon}}r_n\right]=\left[\prod_{n=1}^{N+1}I_m\left(\frac{r_nr_{n-1}}{\varepsilon}\right)e^{-\frac{r_nr_{n-1}}{\varepsilon}}\sqrt{2\pi\frac{r_nr_{n-1}}{\varepsilon}}\right](\sqrt{\varepsilon})^{N+1}\frac{1}{(\sqrt{2\pi})^{N+1}},$$

and so we end up with

$$Z=\sum_{m=-\infty}^{\infty}\left[\prod_{n=1}^{N+1}\int_0^\infty\frac{dr_n}{\sqrt{2\pi\varepsilon}}\right]\left[\prod_{n=1}^{N+1}e^{-\varepsilon U(r_n,-i\tau_n)}\right]\left[\prod_{n=1}^{N+1}e^{-\frac{1}{2\varepsilon}(r_n-r_{n-1})^2}\right]\times$$
$$\times\left[\prod_{n=1}^{N+1}I_m\left(\frac{r_nr_{n-1}}{\varepsilon}\right)e^{-\frac{r_nr_{n-1}}{\varepsilon}}\sqrt{2\pi\frac{r_nr_{n-1}}{\varepsilon}}\right]. \tag{5.25}$$

We refer to $\widetilde{I}_m(z)\equiv I_m\left(\frac{r_nr_{n-1}}{\varepsilon}\right)e^{-z}\sqrt{2\pi z}$ as the *slightly modified Besselfunctions*, and so we write

result 5.1 (polar generating functional in case of $U=U(r)$)

$$Z=\sum_{m=-\infty}^{\infty}\left[\prod_{n=1}^{N+1}\int_0^\infty\frac{dr_n}{\sqrt{2\pi\varepsilon}}\right]\left[\prod_{n=1}^{N+1}e^{-\varepsilon U(r_n,-i\tau_n)}\right]\left[\prod_{n=1}^{N+1}e^{-\frac{1}{2\varepsilon}(r_n-r_{n-1})^2}\right]\left[\prod_{n=1}^{N+1}\widetilde{I}_m\left(\frac{r_nr_{n-1}}{\varepsilon}\right)\right].$$

5.2 Jacobian

5.2.1 Cut-off approach

Our first attempt was to handle the Jacobian by introducing a cut-off. Let us discuss this at the example of (5.13). We rewrite the Jacobian such that it can be included in the exponential:[1]

$$Z = \left[\prod_{j=1}^{N+1} \int_0^\infty \frac{dr_j e^{\ln r_j}}{\sqrt{2\pi\varepsilon}}\right] \oint_0^{2\pi} \mathcal{D}\varphi e^{-\int d\tau L_E(r,\varphi)} = \left[\prod_{j=1}^{N+1} \int_0^\infty \frac{dr_j}{\sqrt{2\pi\varepsilon}}\right] \oint_0^{2\pi} \mathcal{D}\varphi e^{-\int d\tau L_E(r,\varphi) + \sum_{j=1}^{N+1} \ln r_j} . \tag{5.26}$$

For simplicity, we use a rather suggestive notation, where we write only the parts of interest in a sliced form. Understood, one should write the whole expression either in the sliced or in the continuous version.
We can generate an effective Lagrange function, $\overline{L} \equiv L_E + L_\Lambda$, by rewriting

$$\sum_{j=1}^{N+1} \ln r_j = \frac{1}{\varepsilon}\varepsilon \sum_{j=1}^{N+1} \ln r_j \ ,$$

which becomes an integral in the continuum limit:

$$\frac{1}{\varepsilon}\varepsilon \sum_{j=1}^{N+1} \ln r_j \xrightarrow{\varepsilon\to 0} \left[\lim_{\varepsilon\to\infty} \frac{1}{\varepsilon}\right] \cdot \int_0^{1/T} d\tau \ln r \ .$$

We now introduce a cut-off by simply replacing $\left[\lim_{\varepsilon\to\infty} \frac{1}{\varepsilon}\right]$ by Λ:

$$Z = \oint_0^\infty \mathcal{D}r \oint_0^{2\pi} \mathcal{D}\varphi e^{-\int d\tau [L_E(r,\varphi) - \Lambda \ln r]} \ . \tag{5.27}$$

The above expression can be generalized to 1+3 dimensions:

$$Z = \oint_0^\infty \mathcal{D}r \oint_0^{2\pi} \mathcal{D}\varphi e^{-\int d\tau d^3\mathbf{x} [L_E(r,\varphi) - \Lambda_\tau \Lambda_\mathbf{x} \ln r]} \ . \tag{5.28}$$

We applied the CJT formalism to $\overline{L} = L_E - \Lambda_\tau \Lambda_\mathbf{x} \ln r$, with

$$L_E = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\varphi}^2 - \frac{m^2}{2}r^2 + \frac{\lambda}{2}r^4 - Hr\cos\varphi \ ,$$

and verified that the condensate, as well as the masses, behave more and more pathological, as we increase the value of $\Lambda \equiv \Lambda_\tau \Lambda_\mathbf{x}$. The results are shown in figures 5.1 and 5.2. For $\Lambda = 0$ the results for section 6.1.1 are reproduced. As Λ increases, the weak first-order phase transition, encountered in section 6.1.1, seems to become crossover for increasing Λ. However, the (physical) pion mass vanishes long before the condensate becomes zero. For larger Λ it becomes obvious that the condensate even increases before the pion mass vanishes at a certain temperature. These

[1] Note that one can easily make the argument of the logarithm dimensionless by inserting $(\frac{1}{r_0})^{N+1} r_0^{N+1}$, using the first factors in order to obtain $\ln \frac{r_j}{r_0}$, and absorbing the latter into a normalization constant.

numerical results show that the limit $\Lambda \to \infty$ is completely incapable of outweighing the infinity arising from the terms with φ in the denominator. Quite the contrary!
This kind of behavior is not a surprise, since $\Lambda = \delta^4(0) \xrightarrow{DR} 0$ in dimensional regularization (compare with result 5.2).

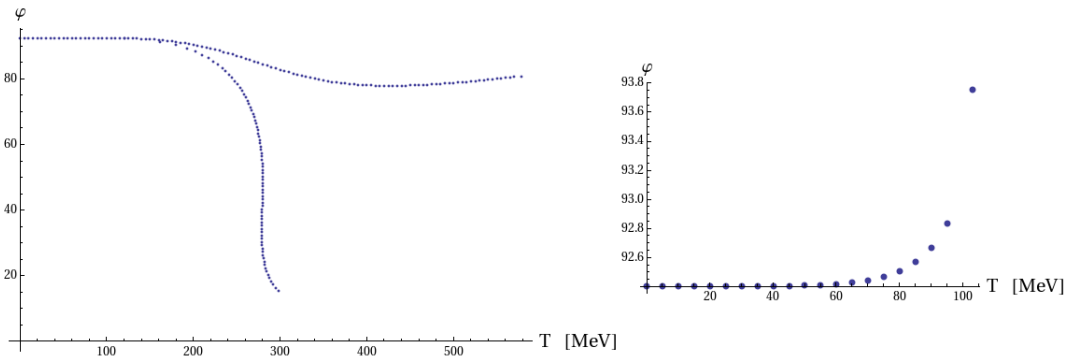


Figure 5.1: condensate; left: $\Lambda = 0$ and $\Lambda = 140MeV$, right: $\Lambda = 300MeV$.

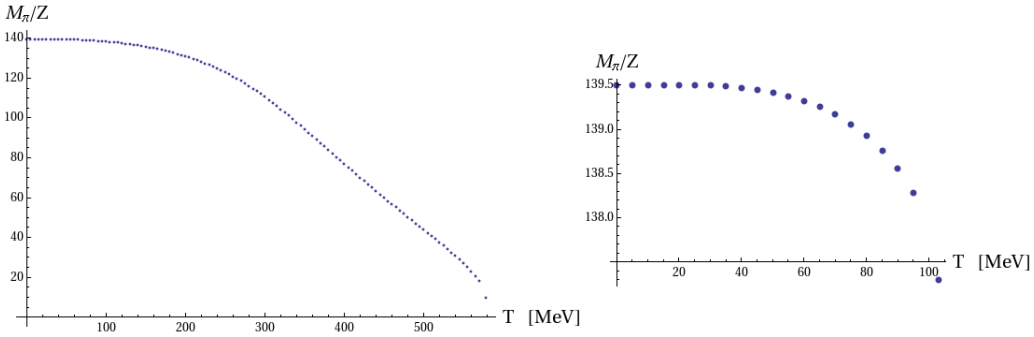


Figure 5.2: pion mass; left: $\Lambda = 140MeV$, right: $\Lambda = 300MeV$.

5.2.2 General solution: dimensional regularization

A comprehensible introduction into dimensional regularization was given by Leibbrandt [44]. In part II of his review, he discusses the technique of dimensional regularization for an ultraviolet divergent, Euclidean, four-dimensional integral of the form

$$I(q) = \int_{-\infty}^{\infty} \frac{dp_4}{2\pi} \int_{-\infty}^{\infty} \frac{d^3\mathbf{k}}{(2\pi)^3} J(p^2, p\cdot q) \ , \tag{5.29}$$

where $J(p^2, p\cdot q)$ is an arbitrary function of the Euclidean scalar products $p^2 = p_4^2 + \mathbf{k}^2$ and $p\cdot q = p_4 q_4 + \mathbf{k}\cdot\mathbf{q}$.
As the procedure is described in detail by Leibbrandt, we restrict ourselves to summarizing the technique. Usually the integral $I(q)$ will contain propagators and therefore the masses of the fields in our theory. In a first step, all propagators are to be replaced by

$$\frac{1}{p^2+M^2} = \int_0^{\infty} d\alpha \exp\left[-\alpha(p^2+M^2)\right] . \tag{5.30}$$

Secondly, one has to analytically continue the vectors p, q to 2ω-dimensional vectors, with complex ω. Accordingly the integrations have to be replaced by $\int_{-\infty}^{\infty} \frac{dp_4}{2\pi} \int_{-\infty}^{\infty} \frac{d^3\mathbf{k}}{(2\pi)^3} \to \int_{-\infty}^{\infty} \frac{d^{2\omega}p}{(2\pi)^{2\omega}}$.[2] Since we are only interested in the technique, we can regard the analytically continued expression just from a formal point of view.[3] With help of the formula

$$\int_{-\infty}^{\infty} \frac{d^{2\omega}p}{(2\pi)^{2\omega}} e^{-xp^2+2p\cdot b} = \begin{cases} \frac{(\pi/x)^{\omega}}{(2\pi)^{2\omega}} e^{b^2/x} & \text{for } M>0 \\ (4\pi x)^{-\omega} e^{b^2/x - xf(\omega)} & \text{for } M=0 \end{cases} , \tag{5.31}$$

where b is a 2ω-dimensional Euclidean vector, $x>0$ and $f(\omega)$ is not unique, but an arbitrary function satisfying certain conditions, we obtain an expression which merely contains an integration over α. Performing this integration usually leads to a result containing the Gamma function

$$\Gamma\left(g(\omega)\right) = \int_0^{\infty} dt\ t^{g(\omega)-1} e^{-t} \ , \quad \text{where } g \text{ is some function with } \ Re\left(g(\omega)\right) > 0 \ , \tag{5.32}$$

as this special function arises naturally from the computation of Gaussian integrals. As one wants ω to be defined on the whole complex plane, one analytically continues the definition (5.32) to

[2] This is done because the analytically continued expression $I_{2\omega}(q)$ is not divergent anymore. After the terms responsible for the divergence at $\omega = 2$ have been removed from $I_{2\omega}(q)$, so that we are left with $I'_{2\omega}(q)$, the limit $\lim_{\omega\to 2} I'_{2\omega}(q)$ will be taken, yielding the finite renormalized result.

[3] However, from a more rigorous point of view, we are able to understand the nature of this analytic continuation. $\int_{-\infty}^{\infty} \frac{d^4p}{(2\pi)^4} e^{-x(p_0^2+p_x^2+p_y^2+p_z^2)+2(p_0b_0+p_xb_x+p_yb_y+p_zb_z)} = \frac{1}{(2\pi)^4}\frac{\pi^2}{x^2} e^{b^2/x}$, which is obtained by carrying out the four integrations. This reveals the dependence of the result on the dimension r, which one should be able to prove in principal: $\int_{-\infty}^{\infty} \frac{d^{2r}p}{(2\pi)^{2r}} e^{-xp^2+2p\cdot b} = \frac{1}{(2\pi)^{2r}} \left(\frac{\pi}{x}\right)^r e^{b^2/x}$. The analytic continuation now consists of nothing more than defining $\int_{-\infty}^{\infty} \frac{d^{2\omega}p}{(2\pi)^{2\omega}} e^{-xp^2+2p\cdot b} \equiv \frac{1}{(2\pi)^{2\omega}} \left(\frac{\pi}{x}\right)^{\omega} e^{b^2/x}$, which is in this sense simply a suggestive abbreviation. All other analytically continued expressions can be understood in this sense, however we owe a rigorous proof.

the entire complex plane (except the points $\omega = 0, -1, -2, \cdots$):

$$\Gamma_W\left(g(\omega)\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+g(\omega))} + \int_1^{\infty} dt\ t^{g(\omega)-1} e^{-t}\ , \tag{5.33}$$

which is known as *Weierstrass's representation of the Gamma function*. For $Re\left(g(\omega)\right) > 0$ one recovers $\Gamma(g(\omega))$. After having replaced $\Gamma(g(\omega))$ by $\Gamma_W(g(\omega))$, one has to expand the result in a Laurent series about the point $\omega = 2$, which reveals the origin of the divergence. The divergent terms show up as addends with singularities at $\omega = 2$. Dropping these addends, one obtains the renormalized result for $I(q)$ by taking the limit $\omega \to 2$.

Although the above recipe is formulated such that it is most easily applied to I's appearing in perturbative calculations, it can be applied to any I of the above form, at least if the variables p_4 and $\mathbf{k}$ have the right dimension. As we will see, such integrals do not exclusively appear in Euclidean QFT, but also in SQFT.

We now come to the point of interest, which can be found in [44], too: $I \equiv \delta^4(0)$.

Let us first rewrite $\delta^4(0)$, such that it has the appropriate form to apply the above recipe.

$$\delta^4(0) = \frac{1}{(2\pi)^4} \int d^4p\ e^{i\ p\cdot 0} = \frac{1}{(2\pi)^4} \int d^4p\ 1 = \frac{1}{(2\pi)^4} \int d^4p\ \frac{p^2}{p^2}. \tag{5.34}$$

Using formula (5.30), we obtain

$$\delta^4(0) = \frac{1}{(2\pi)^4} \int d^4p\ p^2 \int_0^{\infty} d\alpha e^{-\alpha p^2}. \tag{5.35}$$

Analytically continuing (5.35) to a 2ω-dimensional vector space results

$$\delta^4(0) \to \frac{1}{(2\pi)^{2\omega}} \int d^{2\omega}p\ p^2 \int_0^{\infty} d\alpha e^{-\alpha p^2}. \tag{5.36}$$

Using a similar formula as (5.31) for $M = 0$, namely

$$\int \frac{d^{2\omega}p\ p^2}{(2\pi)^{2\omega}} e^{-xp^2} = (4\pi)^{-\omega} \left[\omega x^{-(1+\omega)} + x^{-\omega} f(\omega)\right] e^{-xf(\omega)}\ , \tag{5.37}$$

we obtain

$$\begin{aligned} \frac{1}{(2\pi)^{2\omega}} \int d^{2\omega}p\ p^2 \int_0^{\infty} d\alpha e^{-\alpha p^2} &= \int_0^{\infty} d\alpha \left((4\pi)^{-\omega} \left[\omega\alpha^{-(1+\omega)} + \alpha^{-\omega} f(\omega)\right] e^{-\alpha f(\omega)}\right) \\ &= \omega(4\pi)^{-\omega} \int_0^{\infty} d\alpha\ \alpha^{-(1+\omega)} e^{-\alpha f(\omega)} + f(\omega)(4\pi)^{-\omega} \int_0^{\infty} d\alpha\ \alpha^{-\omega} e^{-\alpha f(\omega)}. \end{aligned} \tag{5.38}$$

The integrals can be evaluated with the help of *Mathematica*:

$$\int_0^{\infty} d\alpha\ \alpha^{-(1+\omega)} e^{-\alpha f(\omega)} = \left(f(\omega)\right)^{\omega} \Gamma(-\omega)\ ,\quad \text{if } Re\left(f(\omega)\right) > 0 \text{ and } Re(\omega) < 0\ , \tag{5.39}$$

$$\int_0^{\infty} d\alpha\ \alpha^{-\omega} e^{-\alpha f(\omega)} = \left(f(\omega)\right)^{-1+\omega} \Gamma(1-\omega)\ ,\quad \text{if } Re\left(f(\omega)\right) > 0 \text{ and } Re(\omega) < 1\ . \tag{5.40}$$

Therefore, using the analytic continuation (5.33) of the Gamma function, we are able to write

$$\omega(4\pi)^{-\omega}\int_0^\infty d\alpha\ \alpha^{-(1+\omega)}e^{-\alpha f(\omega)} + f(\omega)(4\pi)^{-\omega}\int_0^\infty d\alpha\ \alpha^{-\omega}e^{-\alpha f(\omega)}$$
$$= (f(\omega))^{\omega}\,(4\pi)^{-\omega}\left[\omega\Gamma_W(-\omega) + \Gamma_W(1-\omega)\right]. \tag{5.41}$$

Finally, since $\lim_{\omega\to 2}\Gamma_W\left(g(\omega)\right) = \Gamma\left(g(2)\right)$, we end up with

$$\lim_{\omega\to 2}(f(\omega))^{\omega}\,(4\pi)^{-\omega}\left[\omega\Gamma_W(-\omega) + \Gamma_W(1-\omega)\right] = 0\ . \tag{5.42}$$

We have derived an important result, namely

result 5.2 (In dimensional regularization)

$$\delta^4(0) \xrightarrow{DR} 0\ .$$

Note that result 5.2 also holds true for the delta function at the origin in 1+0 dimensions. As Kleinert mentions in chapter 10 of his textbook [42], the vanishing of the delta function at the origin in dimensional regularization is consistent with *Veltman's rule*, which in turn is discussed in another of his books [17].

Although results should not depend on the regularization scheme used, we have to employ only one scheme in calculations, due to consistency. In appendix B, we already discussed how to renormalize the divergent contribution Q_μ in the CT-scheme. However, all numerical results presented in this thesis are for $Q_\mu = 0$, since the shape of the masses and the extrema φ, plotted against temperature T, for this case are not much different from those achieved by taking into account Q_μ in the CT-scheme. As one is able to perceive the shape of the effective potential just from $\varphi(T)$, we claim that setting $R_\mu = 0$ will not change its shape significantly, in comparison with taking into account R_μ in the CT-scheme. If we are able to show that $\delta^4(0) = 0$ also in the CT-scheme, the numerical results are significant. If not, one would be obliged to check, if accounting for Q_μ and R_μ has an important effect on the results. This is still ongoing work. Here, we just want to present our preliminary result for Q_μ in dimensional regularization (which needs to be checked).

According to [45], we are able to rewrite Q_μ as

$$Q_\mu = \int_{-\infty}^{\infty}\frac{d^4p}{(2\pi)^4}\frac{1}{p^2+M^2}\ , \quad \text{with } p=(p_0,\mathbf{p}) \text{ and } p^2 = p_0^2+\mathbf{p}^2\ , \tag{5.43}$$

due to $\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\frac{1}{p_0^2+x} = \frac{1}{2}\frac{1}{\sqrt{x}}$.

Instead of applying the CT-scheme, which was done in the aforementioned work, we employ the above recipe.

Using

$$\frac{1}{p_0^2+\mathbf{p}^2+M^2} = \int_0^\infty d\alpha\ e^{-\alpha(p_0^2+\mathbf{p}^2+M^2)}\ , \tag{5.44}$$

we obtain

$$Q_\mu = \int_0^\infty d\alpha \int_{-\infty}^{\infty} \frac{d^4p}{(2\pi)^4} e^{-\alpha(p_0^2+\mathbf{p}^2+M^2)}. \tag{5.45}$$

Analytically continuing p to a 2ω-dimensional vector, we obtain with help of formula (5.31) for $M > 0$:

$$Q_\mu \to \int_0^\infty d\alpha \int \frac{d^{2\omega}p}{(2\pi)^{2\omega}} e^{-\alpha(p^2+M^2)} = \int_0^\infty d\alpha \frac{1}{(2\pi)^{2\omega}} \left(\frac{\pi}{\alpha}\right)^\omega e^{-\alpha M^2}. \tag{5.46}$$

With the help of *Mathematica*, we deduce

$$\int_0^\infty d\alpha \frac{1}{(2\pi)^{2\omega}} \left(\frac{\pi}{\alpha}\right)^\omega e^{-\alpha M^2} = \frac{1}{(2\pi)^{2\omega}} (M^2)^{-1+\omega} \pi^\omega \Gamma(1-\omega) \ , \quad \text{if } Re(\omega) < 1 \ . \tag{5.47}$$

Using the analytic continuation (5.33) of the Gamma function, we are able to write

$$\int_0^\infty d\alpha \frac{1}{(2\pi)^{2\omega}} \left(\frac{\pi}{\alpha}\right)^\omega e^{-\alpha M^2} = \frac{1}{(2\pi)^{2\omega}} (M^2)^{-1+\omega} \pi^\omega \Gamma_W(1-\omega) \ . \tag{5.48}$$

Using the explicit form (5.33) and expanding the above expression in a Laurent series about $\omega = 2$ (again with the help of *Mathematica*), we isolate the term which is singular at $\omega = 2$: $\frac{M^2}{16\pi^2(\omega-2)}$. Taking the limit $\omega \to 2$, we end up with the result (which needs to be checked)

result 5.3 (In dimensional regularization)

$$Q_\mu \xrightarrow{DR} \frac{M^2\left(-1+\gamma+\ln\frac{M^2}{4\pi}\right)}{16\pi^2} \ ,$$

where $\gamma \simeq 0.577216$ is Euler's constant.

5.2.3 Diagrammatic expansion

As shown in the previous section, the Jacobian has no influence on the results. This is mirrored in the perturbative expansion of the generating functional. We want to show this on the basis of the toy model $S(\phi) = \frac{1}{2}\int d\tau \left[\dot\phi^2 + m^2\phi^2\right]$,[4] which was discussed by Kleinert and Chervyakov [46].

In this section, we will omit the subscript E, which indicates the Euclidean form.

$$Z = \oint_{-\infty}^{\infty} \mathcal{D}\phi e^{-S(\phi)} = \left[\prod_{n=1}^{N+1} \int_{-\infty}^{\infty} \frac{d\phi_n}{\sqrt{2\pi\varepsilon}}\right] e^{-A(\phi_n)}. \tag{5.49}$$

A coordinate transformation of the form

$$\phi = f(q) \ , \tag{5.50}$$

[4]Note that with the replacements $x \to \phi$, $\omega^2 \to m^2$ and $M \to 1$, this corresponds to the harmonic oscillator in Euclidean QM.

which is obviously holonomic, yields the generating functional

$$Z = \left[\prod_{n=1}^{N+1}\int \frac{dq_n}{\sqrt{2\pi\varepsilon}}\right]\left[\prod_{n=1}^{N+1}\frac{df(q_n)}{dq_n}\right] e^{-A(q_n)} = \oint \mathcal{D}q \frac{df(q)}{dq} e^{-S(q)}$$
$$= \left[\prod_{n=1}^{N+1}\int \frac{dq_n}{\sqrt{2\pi\varepsilon}}\right] e^{-A(q_n)+\frac{1}{\varepsilon}\epsilon\sum_{n=1}^{N}\ln\left[\frac{df(q_n)}{dq_n}\right]} \ . \tag{5.51}$$

Note that we can keep the argument of the logarithm dimensionless, since the latter expression is equal to

$$\left[\prod_{n=1}^{N+1}\int \frac{dq_n}{\sqrt{2\pi\varepsilon}}\frac{df(q_0)}{dq_0}\right] e^{-A(q_n)+\frac{1}{\varepsilon}\epsilon\sum_{n=1}^{N}\ln\left[\left(\frac{df(q_n)}{dq_n}\right)/\left(\frac{df(q_0)}{dq_0}\right)\right]} \ . \tag{5.52}$$

Let us abbreviate the Jacobian: $\frac{df(q_n)}{dq_n} \equiv J(q_n)$.
In the continuum limit $\varepsilon \to 0$ we obtain

$$Z = \oint \mathcal{D}q e^{-S(q)+\delta(0)\int d\tau \ln J(q)} \ . \tag{5.53}$$

We consider fluctuations around $q = 0$,

$$q \to 0 + \delta q \ , \tag{5.54}$$

but we will keep the symbol q for the fluctuation δq.
The physical meaning of the fluctuation is the same as in section 2.2. Our potential $U(\phi) = \frac{m^2}{2}\phi^2$ has its global minimum at $\phi = 0$. Assuming that $U(q)$ has its global minimum correspondingly at $q = 0$, our particle is described by a fluctuation about the vacuum.
We now want to make the coordinate transformation explicit:

$$\phi = f(q) \equiv q - \frac{g}{3}q^3 + \frac{g^2}{5}aq^5 - \cdots \tag{5.55}$$
$$\Rightarrow \quad \dot{\phi} = \dot{q} - gq^2\dot{q} + g^2aq^4\dot{q} - \cdots \ , \tag{5.56}$$

hence

$$\dot{\phi}^2 = \dot{q}^2 - gq^2\dot{q}^2 + g^2aq^4\dot{q}^2 - gq^2\dot{q}^2 + g^2q^4\dot{q}^2 + g^2aq^4\dot{q}^2 + O(g^3) \tag{5.57}$$

and

$$m^2\phi^2 = m^2\left(q^2 - g\frac{q^4}{3} + g^2a\frac{q^6}{5} - g\frac{q^4}{3} + g^2\frac{q^6}{9} + g^2a\frac{q^6}{5} + O(g^3)\right) \tag{5.58}$$

yield

$$S(q) = \frac{1}{2}\int d\tau \left(-2gq^2\dot{q}^2 - g\frac{2m^2}{3}q^4 + g^2(2a+1)\dot{q}^2q^4 + g^2m^2\left(\frac{2}{5}a + \frac{1}{9}\right)q^6 + \dot{q}^2 + m^2q^2 + O(g^3)\right). \tag{5.59}$$

With

$$J(q) = \frac{df(q)}{dq} = 1 - gq^2 + g^2aq^4 - \cdots \tag{5.60}$$

and

$$\ln\left(1-gq^2+g^2aq^4\right)\simeq -gq^2+g^2aq^4-\frac{1}{2}g^2q^4+O(g^3)\ , \tag{5.61}$$

we deduce

$$\begin{aligned} Z &= \oint \mathcal{D}q \exp\left\{-S(q)+\delta(0)\int d\tau \ln J(q)\right\} \\ &= \oint \mathcal{D}q \exp\left\{-S^0-S^{int}-S_J+O(g^3)\right\}\ , \end{aligned} \tag{5.62}$$

where we defined

$$S^0 \equiv \frac{1}{2}\int d\tau \left[\dot{q}^2+m^2q^2\right]\ , \tag{5.63}$$

$$S^{int} \equiv \frac{1}{2}\int d\tau \left[-2gq^2\dot{q}^2-g\frac{2m^2}{3}q^4+g^2(2a+1)\dot{q}^2q^4+g^2m^2\left(\frac{2}{5}a+\frac{1}{9}\right)q^6\right]\ , \tag{5.64}$$

$$S_J \equiv -\delta(0)\int d\tau \left[-gq^2+g^2(a-\frac{1}{2})q^4\right]. \tag{5.65}$$

Using another abbreviation, $S^{int}_{tot}\equiv S^{int}+S_J$, we rewrite the generating functional as

$$Z=\oint \mathcal{D}q e^{-S^0}e^{-S^{int}_{tot}}. \tag{5.66}$$

Taylor expanding the interaction part,

$$e^{-S^{int}_{tot}} = 1-S^{int}_{tot}+\frac{1}{2!}S^{int\,2}_{tot}-\frac{1}{3!}S^{int\,3}_{tot}+\cdots\ , \tag{5.67}$$

we obtain

$$Z=\oint \mathcal{D}q e^{-S^0}-\oint \mathcal{D}q e^{-S^0}S^{int}_{tot}+\frac{1}{2!}\oint \mathcal{D}q e^{-S^0}S^{int\,2}_{tot}-\cdots \tag{5.68}$$

With the abbreviations

$$\oint \mathcal{D}q e^{-S^0}\equiv Z_0\ , \tag{5.69}$$

$$\langle\cdots\rangle_0 \equiv \frac{1}{Z_0}\oint \mathcal{D}q e^{-S^0}(\cdots)\ , \tag{5.70}$$

we can write

$$Z=Z_0\left[1-\langle S^{int}_{tot}\rangle_0+\frac{1}{2!}\langle S^{int\,2}_{tot}\rangle_0-\cdots\right]. \tag{5.71}$$

Accordingly, the effective potential at its global minimum reads:

$$V=-T\ln Z=\underbrace{-T\ln Z_0}_{V_0}-T\left[-\langle S^{int}_{tot}\rangle_0+\frac{1}{2!}\langle S^{int\,2}_{tot}\rangle_0-\cdots\right]. \tag{5.72}$$

The expectation values appearing in (5.72) can be rewritten due to $\langle a+b\rangle_0=\langle a\rangle_0+\langle b\rangle_0$:

$$\begin{aligned} \langle S^{int}_{tot}\rangle_0 = \langle S^{int}+S_J\rangle_0 &= \int d\tau\langle -gq^2\dot{q}^2\rangle_0+\int d\tau\langle -g\frac{m^2}{3}q^4\rangle_0+\int d\tau\langle g^2(a+\frac{1}{2})\dot{q}^2q^4\rangle_0 \\ &+\int d\tau\langle g^2m^2\left(\frac{a}{5}+\frac{1}{18}\right)q^6\rangle_0+\int d\tau\langle g\delta(0)q^2\rangle_0+\int d\tau\langle -g^2\delta(0)\left(a-\frac{1}{2}\right)q^4\rangle_0+O(g^3)\ , \end{aligned} \tag{5.73}$$

$$\langle S_{tot}^{int\,2}\rangle_0 = g^2 \int d\tau \int d\tau' \Big[\langle q^2(\tau)\dot{q}^2(\tau)q^2(\tau')\dot{q}^2(\tau')\rangle_0 + \langle \frac{m^2}{3} q^2(\tau)\dot{q}^2(\tau)q^4(\tau')\rangle_0$$
$$+\langle -q^2(\tau)\dot{q}^2(\tau)\delta(0)q^2(\tau')\rangle_0 + \langle \frac{m^2}{3} q^4(\tau)q^2(\tau')\dot{q}^2(\tau')\rangle_0 + \langle \frac{m^4}{9} q^4(\tau)q^4(\tau')\rangle_0$$
$$+\langle -\frac{m^2}{3}\delta(0)q^4(\tau)q^2(\tau')\rangle_0 + \langle \delta(0)q^2(\tau)q^2(\tau')\dot{q}^2(\tau')\rangle_0$$
$$+\langle -\delta(0)\frac{m^2}{3} q^2(\tau)q^4(\tau')\rangle_0 + \langle \delta^2(0)q^2(\tau)q^2(\tau')\rangle_0\Big] \,. \tag{5.74}$$

To each of the expectation values, we can apply Wick's rule

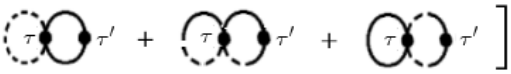
$$\langle a_1 a_2 \cdots a_r\rangle_0 = \sum_{all\ pairs} \langle \sqcup\sqcup\rangle_0 \cdots \langle \sqcup\sqcup\rangle_0 \quad \text{(for even r)}. \tag{5.75}$$

For example:

$$\langle q^2(\tau)\dot{q}^2(\tau)q^2(\tau')\rangle_0 = \langle q(\tau)q(\tau')\rangle_0\langle q(\tau)q(\tau')\rangle_0\langle \dot{q}(\tau)\dot{q}(\tau)\rangle_0$$
$$+\langle q(\tau)q(\tau')\rangle_0\langle \dot{q}(\tau)q(\tau')\rangle_0\langle q(\tau)\dot{q}(\tau)\rangle_0 + \langle \dot{q}(\tau)q(\tau')\rangle_0\langle \dot{q}(\tau)q(\tau')\rangle_0\langle q(\tau)q(\tau)\rangle_0 \,, \tag{5.76}$$

so that, up to combinatorial prefactors,

$$-g^2\delta(0)\int_0^{1/T} d\tau \int_0^{1/T} d\tau' \ \langle q^2(\tau)\dot{q}^2(\tau)q^2(\tau')\rangle_0 \quad =$$



where we used the abbreviations

$\langle\, q(\tau)q(\tau')\,\rangle_0 \equiv$

$\langle\, \dot{q}(\tau)\dot{q}(\tau')\,\rangle_0 \equiv$

$\langle\, \dot{q}(\tau)q(\tau')\,\rangle_0 \equiv$

for the propagators.
At $T = 0$, expression (F.12) is valid:

$$\langle q(\tau)q(\tau')\rangle_0 = \frac{1}{2m} e^{-m|\tau-\tau'|} \,, \tag{5.77}$$

from which we obtain

$$\langle \dot{q}(\tau)q(\tau')\rangle_0 = \partial_\tau \langle q(\tau)q(\tau')\rangle_0 = -\frac{1}{2}\epsilon(\tau-\tau')e^{-m|\tau-\tau'|} \,, \tag{5.78}$$

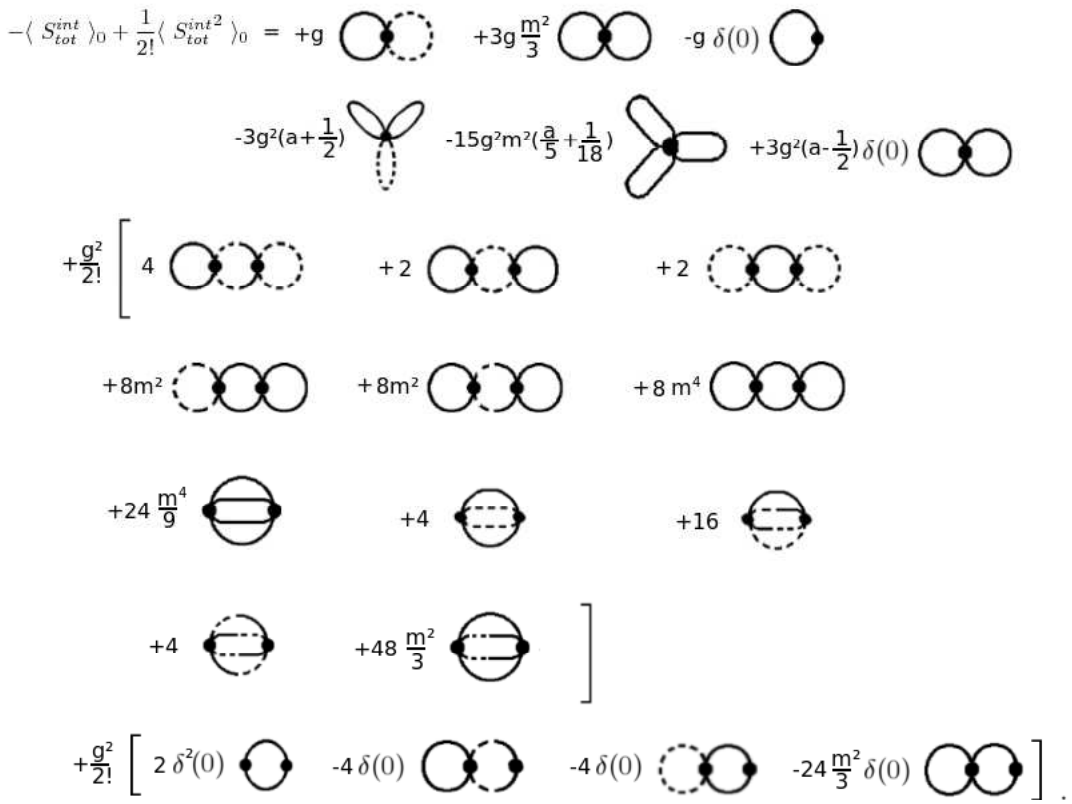
$$\text{with} \quad \epsilon(\tau-\tau') \equiv -1 + 2\int_{-\infty}^{\tau} d\tau'' \delta(\tau''-\tau') \tag{5.79}$$

and

$$\langle \dot{q}(\tau)\dot{q}(\tau')\rangle_0 = \partial_\tau\partial_{\tau'}\langle q(\tau)q(\tau')\rangle_0 = \delta(\tau-\tau') - \frac{m}{2}e^{-m|\tau-\tau'|}. \tag{5.80}$$

Although Kleinert does not mention it, the above expressions were derived for a path integral of the form (5.51), with the integration over q running from $-\infty$ to ∞. However, for an arbitrary

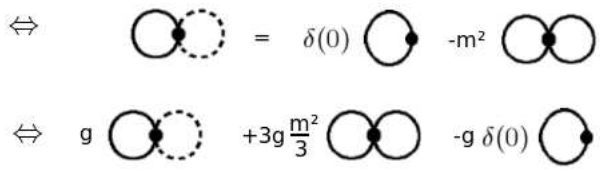
transformation $\phi = f(q)$, the integration interval is not necessarily of this kind (consider for example $\phi = \ln q$). However, we discuss in chapters 5.3 and 5.4 that we are able to extend the integration intervals in the case of polar coordinates from $[0, \infty]$ resp. $[0, 2\pi)$ to $[-\infty, \infty]$, at least in special cases. For each coordinate transformation this should be checked, when using the above expressions for the propagators.
Treating all expectation values in the same way as in the example above and taking care about the correct combinatorial prefactors, one ends up with



Note that diagrams containing factors $\langle \dot{q}(\tau) q(\tau) \rangle_0$ vanish, since $\langle \dot{q}(\tau) q(\tau) \rangle_0 = 0$ due to formula (5.78). The calculated loop diagrams can be found in section 10.8 of [42]. However, one does not need these explicit results for proving that, at each order in g, the graphs with $\delta(0)$ resp. $\delta^2(0)$ in front cancel only against terms from the other diagrams.
Let us examine this at order g:

$$\langle \dot{q}(\tau)\dot{q}(\tau') \rangle_0 = \delta(\tau - \tau') - \underbrace{\frac{m}{2} e^{-m|\tau-\tau'|}}_{= m^2 \langle q(\tau) q(\tau') \rangle_0}$$

$$\Rightarrow \int d\tau \langle q(\tau) q(\tau) \rangle_0 \langle \dot{q}(\tau) \dot{q}(\tau) \rangle_0 = \int d\tau \delta(0) \langle q(\tau) q(\tau) \rangle_0 - \int d\tau \; m^2 \langle q(\tau) q(\tau) \rangle_0 \langle q(\tau) q(\tau) \rangle_0$$



The cancellation takes place because of the $\delta(\tau - \tau')$ in $\langle \dot{q}(\tau)\dot{q}(\tau')\rangle_0$. So all diagrams containing this factor are involved.

5.3 Integration over angular degree of freedom

The aim of this section is to justify the extension of the integration interval from $[0, 2\pi)$ to $[-\infty, \infty]$. After some pre-discussion, section 5.3.2 presents the proof for potentials which exclusively depend on the radial degree of freedom (as is the case for our polar models in the chiral limit) and its generalization to 2π-periodic potentials $U(r, \varphi) = U(r, \varphi + 2\pi)$.

5.3.1 A faulty justification

Let us start with a very suggestive, but incorrect, line of arguments.
Since we may choose each interval of length 2π, which we denote by I, the calculation leading to formula (8.12), is the same for the intervals

$$\ldots,\ [-4\pi, -2\pi),\ [-2\pi, 0),\ [0, 2\pi),\ [2\pi, 4\pi),\ \ldots.$$

All lead to the same partition function:

$$\frac{1}{2}\left(-1 + \vartheta_3\left(0, e^{-\frac{2}{T}\frac{\pi^2}{d^2}}\right)\right) = \cdots = Z_{[-4\pi,-2\pi)} = Z_{[-2\pi,0)} = Z_{[0,2\pi)} = Z_{[2\pi,4\pi)} \cdots\ , \tag{5.81}$$

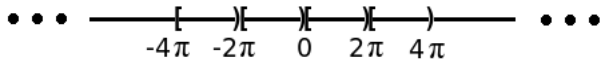
where ϑ_3 denotes the elliptic theta function. Each of the Z_I's can be expressed within the path integral formalism via

$$Z_I = \oint_I \mathcal{D}\varphi\ e^{-S_E}\ ,\quad \text{with } S_E = \int d\tau[\frac{1}{2}(\frac{\partial\varphi}{\partial\tau})^2]\ . \tag{5.82}$$

Therefore we have, where $\mathcal{N}$ is a countably infinite constant,

$$Z \equiv \sum_I Z_I = \oint_{-\infty}^{\infty} \mathcal{D}\varphi\ e^{-S_E} = \mathcal{N}\frac{1}{2}\left(-1 + \vartheta_3\left(0, e^{-\frac{2}{T}\frac{\pi^2}{d^2}}\right)\right), \tag{5.83}$$

which is illustrated by



This Z is the partition function (at the global minimum) for a scalar SQFT in 1+0 dimensions, describing a massless particle. $\mathcal{N}$, although countably infinite, simply plays the role of a normalization constant. Since it is related to the effective potential via $V_{eff} = -T \ln Z$, without appropriate normalization the effective potential is $-\infty$. Since the integration runs from $-\infty$ to ∞ the common Feynman rules can be applied to Z.

5.3.2 A proof for potentials independent of the angular degree of freedom and for 2π-periodic potentials

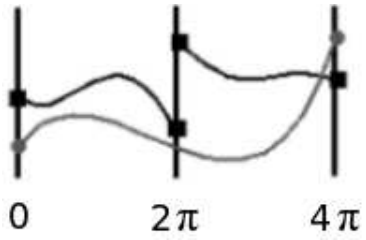
What is wrong in the above consideration? The crucial point is formula (5.83), which is not valid for path integrals. Consider for instance:

$$\underbrace{\left[\prod_i \int_0^{2\pi} d\varphi(\tau_i)\right]}_{=\left[\int_0^{2\pi} d\varphi(\tau_1)\right]\cdot\left[\int_0^{2\pi} d\varphi(\tau_2)\right]\cdots} e^{-S} + \underbrace{\left[\prod_i \int_{2\pi}^{4\pi} d\varphi(\tau_i)\right]}_{=\left[\int_{2\pi}^{4\pi} d\varphi(\tau_1)\right]\cdot\left[\int_{2\pi}^{4\pi} d\varphi(\tau_2)\right]\cdots} e^{-S} ,$$

which is *not* equal to

$$\underbrace{\left[\prod_i \left(\int_0^{2\pi} d\varphi(\tau_i) + \int_{2\pi}^{4\pi} d\varphi(\tau_i)\right)\right]}_{=\left(\left[\int_0^{2\pi} d\varphi(\tau_1)\right]+\left[\int_{2\pi}^{4\pi} d\varphi(\tau_1)\right]\right)\cdot\left(\left[\int_0^{2\pi} d\varphi(\tau_2)\right]+\left[\int_{2\pi}^{4\pi} d\varphi(\tau_2)\right]\right)\cdots} e^{-S} .$$

Decomposing the integration interval, as known from common integration, is therefore not possible for path integrals. The intuitive reason for this property is that for $[-\infty, \infty]$ the particle is allowed to take paths which were forbidden before.



Physical information, the constraint due to the infinite wall on the circle, has disappeared.

However, we are able to proceed in a different manner.
Actually, we just have to ask to what extent result 5.1 is altered, if we extend the range of the integrations from $\int_0^{2\pi}$ to $\int_{-\infty}^{\infty}$.
Because

$$\int_a^b d\varphi_N e^{i\varphi_N(m_N - m_{N+1})} = \int_0^{2\pi} d\varphi_N e^{i\varphi_N(m_N - m_{N+1})} \tag{5.84}$$

for each of the intervals $[a,b] = \cdots,\ [-4\pi,-2\pi],\ [-2\pi,0],\ [0,2\pi],\ [2\pi,4\pi],\ [4\pi,6\pi],\cdots$, we have

$$\int_{-\infty}^{\infty} d\varphi_N e^{i\varphi_N(m_N - m_{N+1})} = \begin{cases} 0 & \text{for } m_{N+1}, m_N \in \mathbb{Z} \text{ and } m_{N+1} \neq m_N \\ \mathcal{M}2\pi & \text{for } m_{N+1} = m_N \end{cases}$$
$$= \mathcal{M}2\pi\delta_{m_{N+1},m_N} \tag{5.85}$$

instead of (5.17). $\mathcal{M}$ is a countably infinite constant. Accordingly, we simply have $(\mathcal{M}2\pi)^N$ instead of $(2\pi)^N$ in equation (5.22):

$$Z = \left[\prod_{n=1}^{N+1}\int_0^\infty \frac{dr_n r_n}{\sqrt{2\pi\varepsilon}}\right]\int_{-\infty}^{\infty}\frac{d\varphi_{N+1}}{\sqrt{2\pi\varepsilon}}\left[\prod_{n=1}^{N+1} e^{-\varepsilon\frac{1}{2}\frac{r_n^2+r_{n-1}^2}{\varepsilon^2}}\right]\frac{1}{(\sqrt{2\pi\varepsilon})^N}\left[\prod_{n=1}^{N+1} e^{-\varepsilon U(r_n,-i\tau_n)}\right]\times$$
$$\times\left(\sum_m\left[\prod_{n=1}^{N+1} I_m\left(\frac{r_n r_{n-1}}{\varepsilon}\right)\right]e^{im(\varphi_{N+1}-\varphi_0)}(\mathcal{M}2\pi)^N\right).$$

The last integration over φ_{N+1} correspondingly yields a factor ∞ instead of 2π because of the constraint $\varphi_{N+1} = \varphi_0$. Therefore, the result 5.1 is simply altered by an overall factor $\frac{\infty}{\pi}\mathcal{M}^N$, which can be absorbed into a normalization constant. Hence

result 5.4 (intermediate result, proven in case of $U = U(r)$)

$$Z \sim \oint_0^\infty \mathcal{D}rr \oint_{-\infty}^{\infty}\mathcal{D}\varphi e^{-\int d\tau L_E(r,\varphi)}$$

is equivalent to expression (5.13).

Now consider a 2π-periodic potential $U(r,\varphi) = U(r,\varphi+2\pi)$ in (5.15). Again, one can pull the integrations over the φ_n in front of factors depending on φ_n. All these factors have the same form (note that $\varphi_{N+1} = \varphi_0$):

$$\int_0^{2\pi} d\varphi_n e^{i\varphi_n l - \varepsilon U(r_n,\varphi_n)}\ ,\ \text{where } l\in\mathbb{Z}.$$

For $U(r_n,\varphi_n+2\pi) = U(r_n,\varphi_n)$ we have (since $e^{i2\pi l}=1$)

$$\int_a^b d\varphi_n e^{i\varphi_n l-\varepsilon U(r_n,\varphi_n)} = \int_0^{2\pi} d\varphi_n e^{i\varphi_n l-\varepsilon U(r_n,\varphi_n)}$$

for each of the intervals $[a,b] = \cdots,\ [-4\pi,-2\pi],\ [-2\pi,0],\ [0,2\pi],\ [2\pi,4\pi],\ [4\pi,6\pi],\cdots$. Hence

result 5.5 (generalization)

$$Z \sim \oint_0^\infty \mathcal{D}rr \oint_{-\infty}^{\infty}\mathcal{D}\varphi e^{-\int d\tau L_E(r,\varphi)}$$

is equivalent to expression (5.13) in the case of 2π-periodic potentials $U(r,\varphi)$.

5.4 Integration over radial degree of freedom

In this section we want to examine if the integration over r can be extended from $[0,\infty]$ to $[-\infty,\infty]$. Note that, as already discussed, decomposing the integration interval, as known from ordinary integration, is not allowed in case of path integrals. The first idea is to proceed similarly to section 5.3.2, namely to examine to what extent Z changes when we change the integration

interval. We follow this idea in section 5.4.1. Unfortunately, this will not answer our question. The difficulty is that we are not able to perform the integrations over the r_n, hence the most simple basis for our considerations is expression (5.23):

$$Z = \sum_{m=-\infty}^{\infty} (2\pi)^{N+1} \left(\frac{1}{\sqrt{2\pi\varepsilon}}\right)^{2N+2} \left[\prod_{n=1}^{N+1} \int_0^\infty dr_n\right] \left[\prod_{n=1}^{N+1} e^{-\varepsilon U(r_n, -i\tau_n)}\right] \times$$
$$\times \left[\prod_{n=1}^{N+1} I_m\left(\frac{r_n r_{n-1}}{\varepsilon}\right) e^{-\varepsilon \frac{1}{2} \frac{r_n^2 + r_{n-1}^2}{\varepsilon^2}} r_n\right] .$$

We present the solution in section 5.4.2, which consists of a few lines due to dimensional regularization. For a special case, a free scalar field, we confirm this result in section 8.2.2 by examining the scalar field constrained to a half space and without constraint.

5.4.1 An impasse

We begin with writing out the r_{N+1}-integration over all factors in expression (5.23), which depend on r_{N+1}:

$$\int_0^\infty dr_{N+1} e^{-\varepsilon U(r_{N+1}, -i\tau_{N+1})} I_m\left(\frac{r_{N+1} r_N}{\varepsilon}\right) e^{-\varepsilon \frac{1}{2} \frac{r_{N+1}^2 + r_N^2}{\varepsilon^2}} r_{N+1} \ . \tag{5.86}$$

Due to

- $I_m\left(\frac{r_{N+1} r_N}{\varepsilon}\right) = I_{-m}\left(\frac{r_{N+1} r_N}{\varepsilon}\right)$,
- $I_m\left(\frac{r_{N+1} r_N}{\varepsilon}\right) = (-1)^m I_m\left(\frac{-r_{N+1} r_N}{\varepsilon}\right)$,
- $e^{-\varepsilon \frac{1}{2} \frac{r_{N+1}^2 + r_N^2}{\varepsilon^2}}$ is symmetric in r_{N+1} ,
- $r_{N+1} = -(-r_{N+1})$,
- the integrand vanishes at $r_{N+1} = 0$,

and if $e^{-\varepsilon U(r_{N+1}, -i\tau_{N+1})}$ is symmetric in r_{N+1} (as this is the case in the chiral limit), expression (5.86) is equal to

$$(-1)^{m+1} \int_{-\infty}^0 dr_{N+1} e^{-\varepsilon U(r_{N+1}, -i\tau_{N+1})} I_m\left(\frac{r_{N+1} r_N}{\varepsilon}\right) e^{-\varepsilon \frac{1}{2} \frac{r_{N+1}^2 + r_N^2}{\varepsilon^2}} r_{N+1} \ . \tag{5.87}$$

Each of the other integrations yields such a factor $(-1)^{m+1}$,

$$\int_0^\infty dr_N e^{-\varepsilon U(r_N, -i\tau_N)} I_m\left(\frac{r_N r_{N-1}}{\varepsilon}\right) e^{-\varepsilon \frac{1}{2} \frac{r_N^2 + r_{N-1}^2}{\varepsilon^2}} r_N \times$$
$$\times (-1)^{m+1} \int_{-\infty}^0 dr_{N+1} e^{-\varepsilon U(r_{N+1}, -i\tau_{N+1})} I_m\left(\frac{r_{N+1} r_N}{\varepsilon}\right) e^{-\varepsilon \frac{1}{2} \frac{r_{N+1}^2 + r_N^2}{\varepsilon^2}} r_{N+1}$$

and so forth.
We can split the sum into sums over even and odd m:

$$Z \equiv \sum_m Z_m = \sum_{m\ even} Z_m + \sum_{m\ odd} Z_m \ . \tag{5.88}$$

Let us denote

$$Z' \equiv \sum_{m=-\infty}^{\infty} (2\pi)^{N+1} \left(\frac{1}{\sqrt{2\pi\varepsilon}}\right)^{2N+2} \left[\prod_{n=1}^{N+1} \int_{-\infty}^{\infty} dr_n\right] \left[\prod_{n=1}^{N+1} e^{-\varepsilon U(r_n, -i\tau_n)}\right] \times$$
$$\times \left[\prod_{n=1}^{N+1} I_m\left(\frac{r_n r_{n-1}}{\varepsilon}\right) e^{-\varepsilon \frac{1}{2} \frac{r_n^2 + r_{n-1}^2}{\varepsilon^2}} r_n\right] .$$

Of course the same is possible with Z':

$$Z' \equiv \sum_m Z'_m = \sum_{m\ even} Z'_m + \sum_{m\ odd} Z'_m \ . \tag{5.89}$$

The sum over odd m is no problem, the prefactors from all integrations are $(-1)^{m+1} = 1$. Therefore:

$$\sum_{m\ odd} Z_m = \left(\frac{1}{2}\right)^{N+1} \sum_{m\ odd} Z'_m \ . \tag{5.90}$$

However, the even m spoil the proof. The prefactors are in this case $(-1)^{m+1} = -1$, which means

$$\sum_{m\ even} Z'_m = 0 \ , \tag{5.91}$$

and since $Z_m \geq 0$, the sum $\sum_{m\ even} Z_m$ does not vanish.
Note that we have not proven the opposite of what we intended to show. $Z \sim Z'$ does not contradict the above considerations, as we expect an infinite proportionality constant. Furthermore in the continuum limit also $\left(\frac{1}{2}\right)^{N+1}$ is infinite.

5.4.2 Dimensional regularization again

We now present the proof for potentials $U(r, \varphi)$.
We are able to rewrite the path integral (5.8) with the help of a modified Heaviside step function. Defining

$$\theta_{f(\varepsilon)}(x) \equiv \begin{cases} 1 & \text{for } x > 0 \\ f(\varepsilon) & \text{for } x \leq 0 \end{cases} \ , \quad \text{with any function satisfying } \lim_{\varepsilon \to 0} f(\varepsilon) = 0 \ , \tag{5.92}$$

we can write

$$Z = \lim_{\varepsilon \to 0} \left[\prod_{n=1}^{N+1} \int_{-\infty}^{\infty} \frac{dr_n r_n}{\sqrt{2\pi\varepsilon}}\ \theta_{f(\varepsilon)}(r_n)\right] \left[\prod_{n=1}^{N+1} \int_0^{2\pi} \frac{d\varphi_n}{\sqrt{2\pi\varepsilon}}\right] e^{-A_E} \tag{5.93}$$

$$= \lim_{\varepsilon \to 0} \left[\prod_{n=1}^{N+1} \int_{-\infty}^{\infty} \frac{dr_n r_n}{\sqrt{2\pi\varepsilon}}\right] \left[\prod_{n=1}^{N+1} \int_0^{2\pi} \frac{d\varphi_n}{\sqrt{2\pi\varepsilon}}\right] e^{-A_E + \frac{1}{\varepsilon}\varepsilon \sum_{n=1}^{N+1} \ln \theta_{f(\varepsilon)}(r_n)} \ . \tag{5.94}$$

The exponent becomes in the continuum limit

$$\lim_{\varepsilon\to 0}\left(-A_E+\frac{1}{\varepsilon}\varepsilon\sum_{n=1}^{N+1}\ln\theta_{f(\varepsilon)}(r_n)\right)=-S_E+\left[\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\right]\left[\lim_{\varepsilon\to 0}\varepsilon\sum_{n=1}^{N+1}\ln\theta_{f(\varepsilon)}(r_n)\right]. \tag{5.95}$$

For $r_n > 0$ we have

$$\left[\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\right]\left[\lim_{\varepsilon\to 0}\varepsilon\sum_{n=1}^{N+1}\ln\theta_{f(\varepsilon)}(r_n)\right]=\left[\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\right]\left[\lim_{\varepsilon\to 0}\varepsilon\sum_{n=1}^{N+1}\ln 1\right]=\delta(0)\cdot 0\ . \tag{5.96}$$

Since $\delta(0)=0$ in dimensional regularization, expression (5.96) vanishes.
Imagining the integrals $\int_{-\infty}^{\infty} dr_n$ as sums over addends with continuous indices r_n, we see from the representation (5.93) that the addends $r_n = 0$ vanish. Hence, we are left with $r_n < 0$ to check. In this case we have

$$\left[\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\right]\left[\lim_{\varepsilon\to 0}\varepsilon\sum_{n=1}^{N+1}\ln\theta_{f(\varepsilon)}(r_n)\right]=\left[\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\right]\left[\lim_{\varepsilon\to 0}\varepsilon\sum_{n=1}^{N+1}\ln f(\varepsilon)\right]. \tag{5.97}$$

Choosing $f(\varepsilon)\equiv e^{-1/\varepsilon}$, we obtain:

$$\left[\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\right]\left[\lim_{\varepsilon\to 0}\varepsilon\sum_{n=1}^{N+1}\left(-\frac{1}{\varepsilon}\right)\right]\xrightarrow{\varepsilon\to 0}\delta(0)\left[-\int_0^{1/T}\delta(0)d\tau\right]=-\frac{1}{T}\delta(0)^2\ , \tag{5.98}$$

which vanishes in dimensional regularization. Hence

$$Z=\oint_{-\infty}^{\infty}\mathcal{D}rr\oint_0^{2\pi}d\varphi e^{-S_E}\ , \tag{5.99}$$

and so we end up with the result

result 5.6 (In dimensional regularization)

$$Z=\oint_{-\infty}^{\infty}\mathcal{D}rr\oint_0^{2\pi}d\varphi e^{-S_E} \tag{5.100}$$

is equivalent to expression (5.13), where U can depend on r and φ.

Chapter 6

$O(2)$-symmetric double-well potential, polar coordinates

6.1 1+3 dimensions

Whereas the systems discussed in the previous sections were more or less toy models, the one described in this section has a physical meaning. We discuss an $O(2)$-symmetric linear sigma model in polar coordinates, regarding the fluctuation in radial direction as the sigma particle and the angular fluctuation as the pion.
Motivation for this model is the violation of Goldstone's theorem encountered in the chiral limit using cartesian coordinates [45] (see chapter 7.1). According to experience, polar coordinates are better suited for problems with rotational symmetry. In particular, there exists a similar problem with the Abelian Higgs theory. Tye and Vtorov-Karevsky [47] pointed out that the would-be Goldstone bosons have non-zero masses, if one expresses the Higgs field in cartesian coordinates, i.e., $\phi = (\phi_1, \phi_2)$. They were able to show (at least for $T = 0$) that this problem is cured when using polar coordinates instead, i.e., $\Phi = \phi_1 + i\phi_2 = \rho e^{i\chi}$.
For the free case, we showed in section 5.4 that $\int_0^\infty \mathcal{D}r \to \int_{-\infty}^\infty \mathcal{D}r$ is possible without effects on the Lagrangian to which we apply the Feynman rules. After having applied the Feynman rules, we restrict ourselves to $r > 0$ again.
For $\int_0^{2\pi} \mathcal{D}\varphi \to \int_{-\infty}^\infty \mathcal{D}\varphi$, the corresponding proof, given in section 5.3.2, is valid for potentials of the form $U(r)$ and for 2π-periodic potentials $U(r, \varphi)$. Hence, the proof covers the potential $U = -\frac{m^2}{2}r^2 + \frac{\lambda}{2}r^4 - Hr\cos\varphi$, which we will use in the following. After application of the Feynman rules, we only consider $\varphi \in [0, 2\pi)$.
According to section 5.2, the Jacobian of the coordinate transformation does not have any effect in dimensional regularization.[1]

[1] Correspondingly, if one wants to include the contributions to the thermal integrals requiring renormalization, one has to use dimensional regularization, too. This is still ongoing work, but we can already present the results neglecting these terms.

As in cartesian coordinates, we will plot the effective potential in ϕ_1-direction, i.e., $\phi_2 = 0$.

$$|\vec{\phi}| = \sqrt{\phi_1^2 + \phi_2^2} \overset{\phi_2=0}{=} |\phi_1| = r \ , \quad \phi_1 = \begin{cases} r & \text{for } \phi_1 > 0 \\ -r & \text{for } \phi_1 < 0 \end{cases} \ .$$

For simplicity, we will use the symbol ϕ instead of ϕ_1. At the global minimum, ϕ is the condensate Φ. At any extremum, we denote ϕ by φ instead.
Our system is determined by the Lagrangian

$$\mathscr{L} = \frac{1}{2}\partial_\mu \vec{\phi} \partial^\mu \vec{\phi} + \frac{m^2}{2}\vec{\phi}\vec{\phi} - \frac{\lambda}{2}(\vec{\phi}\vec{\phi})^2 + H\phi_1 \ . \tag{6.1}$$

We already found the Lagrangian for polar coordinates in the correct way, namely by changing to polar coordinates in the sliced version of the action, which lead to formula (5.12). For the sake of completeness, we will point out in the following that the same result follows from applying the coordinate transformation to the continuous Lagrangian 6.1.

$$\phi_1 = r\cos\varphi \ , \quad \phi_2 = r\sin\varphi \ , \tag{6.2}$$

$$\begin{aligned} \partial_\mu \vec{\phi}(\tau,\vec{x}) \partial^\mu \vec{\phi}(\tau,\vec{x}) &= \begin{pmatrix} \partial_\mu (r\cos\varphi) \\ \partial_\mu (r\sin\varphi) \end{pmatrix} \begin{pmatrix} \partial^\mu (r\cos\varphi) \\ \partial^\mu (r\sin\varphi) \end{pmatrix} \\ &= [(\partial_\mu r)\cos\varphi - (\partial_\mu\varphi) r\sin\varphi]\,[(\partial^\mu r)\cos\varphi - (\partial^\mu\varphi) r\sin\varphi] \\ &+ [(\partial_\mu r)\sin\varphi + (\partial_\mu\varphi) r\cos\varphi]\,[(\partial^\mu r)\sin\varphi + (\partial^\mu\varphi) r\cos\varphi] \\ &= \partial_\mu r \partial^\mu r \left(\cos^2\varphi + \sin^2\varphi\right) + r^2 \partial_\mu\varphi\partial^\mu\varphi \left(\sin^2\varphi + \cos^2\varphi\right) = \partial_\mu r\partial^\mu r + r^2\partial_\mu\varphi\partial^\mu\varphi \ , \end{aligned}$$

where we used $\partial_\mu \cos\varphi = \frac{\partial\cos\varphi}{\partial t} + \frac{\partial\cos\varphi}{\partial x} + \frac{\partial\cos\varphi}{\partial y} + \frac{\partial\cos\varphi}{\partial z} = -(\partial_\mu\varphi)\sin\varphi$,
and $\partial^\mu \cos\varphi = -(\partial^\mu\varphi)\sin\varphi$ respectively.

$$\Rightarrow \ \mathscr{L} = \frac{1}{2}\partial_\mu r\partial^\mu r + \frac{1}{2}r^2\partial_\mu\varphi\partial^\mu\varphi + \frac{m^2}{2}r^2 - \frac{\lambda}{2}r^4 + Hr\cos\varphi \ . \tag{6.3}$$

We now perform a shift, exactly as shown in figure 2.3. The radial fluctuation in ϕ_1-direction around $\phi = \begin{cases} r & \text{for } \phi_1 > 0 \\ -r & \text{for } \phi_1 < 0 \end{cases}$ is denoted by σ, since we identify it with the sigma particle. The angular degree of freedom corresponds to the pion. Since we deal with the potential in ϕ_1-direction only, we fluctuate around $\varphi = 0$. We normalize the angular fluctuation to ϕ, so that we describe the pion field by $\frac{\pi}{\phi}$. This yields the shifted Lagrangian

$$\mathscr{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\frac{(\sigma+\phi)^2}{\phi^2}\partial_\mu\pi\partial^\mu\pi + \frac{m^2}{2}(\sigma+\phi)^2 - \frac{\lambda}{2}(\sigma+\phi)^4 + H(\sigma+\phi)\cos\frac{\pi}{\phi} \ . \tag{6.4}$$

Since $\partial_\mu\sigma\partial^\mu\sigma = \partial_\mu(\sigma\partial^\mu\sigma) - \sigma\Box\sigma$, $\partial_\mu(\sigma\partial^\mu\sigma)$ vanishes in momentum space and $\Box$ becomes $-k^2$ in momentum space, we obtain in complete analogy to section 2.2 the following mnemonic:

$$\text{“}\mathscr{L} = \frac{1}{2}k^2\sigma^2 + \frac{1}{2}k^2\frac{(\sigma+\phi)^2}{\phi^2}\pi^2 + \frac{m^2}{2}(\sigma+\phi)^2 - \frac{\lambda}{2}(\sigma+\phi)^4 + H(\sigma+\phi)\cos\frac{\pi}{\phi} \ \text{”}. \tag{6.5}$$

Note that we will study the behavior of the system at nonzero temperature (compare the comments in section 2.2).
After removing the parentheses and expanding the cosine in a Taylor series around zero, $\cos\frac{\pi}{\phi} = 1 - \frac{1}{2}\frac{\pi^2}{\phi^2} + \frac{1}{4!}\frac{\pi^4}{\phi^4}$, the quadratic terms ($\sim \sigma^2$ resp. $\sim \pi^2$) offer the tree-level masses and the tree-level propagators. The other terms involving the fields σ resp. π, are interaction terms, from which we read off the vertex factors for each interaction vertex.
We have $-(-\frac{m^2}{2} + \frac{6\lambda}{2}\phi^2)\sigma^2$ and $-\left(-\frac{H}{2\phi}\pi^2\right)$. Hence, the tree-level masses, m_σ and m_π, are given by

result 6.1 (tree-level masses)

$$m_\sigma^2 = -m^2 + 6\lambda\phi^2 \ , \quad m_\pi^2 = \frac{H}{\phi} \ .$$

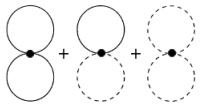
The inverse tree-level propagators in pseudo-Minkowskian notation read

result 6.2 (tree-level propagators)

$$D_\sigma^{-1}(k,\phi) = -k^2 + m_\sigma^2 = -k^2 - m^2 + 6\lambda\phi^2 \ , \tag{6.6}$$

$$D_\pi^{-1}(k,\phi) = -k^2 + \frac{H}{\phi} \ . \tag{6.7}$$

Using $V_{eff} = -\frac{T}{\Omega}\Gamma = V_\sigma + V_\pi = -\frac{T}{\Omega}(\Gamma_\sigma + \Gamma_\pi)$ with result 2.1 and

$\Gamma_2 =$ 

we obtain the polar effective potential in the CJT formalism:

result 6.3 (effective potential)

$$V_{eff} = -\frac{m^2}{2}\phi^2 + \frac{\lambda}{2}\phi^4 - H\phi + \frac{1}{2}\int\limits_k \ln G_\sigma^{-1} + \frac{1}{2}\int\limits_k \ln G_\pi^{-1} + \frac{1}{2}\int\limits_k \left[D_\sigma^{-1}G_\sigma - 1\right]$$
$$+\frac{1}{2}\int\limits_k \left[D_\pi^{-1}G_\pi - 1\right] + \frac{3}{2}\lambda\left[\int\limits_k G_\sigma\right]^2 - \frac{H}{8\phi^3}\left[\int\limits_k G_\pi\right]^2 - \frac{1}{2\phi^2}\left[\int\limits_k G_\sigma\right]\left[\int\limits_k k^2 G_\pi\right] . \tag{6.8}$$

Note that we use the notation of appendix B. $\frac{1}{-k^2+M^2} \to \frac{1}{p^2+M^2} \to \frac{1}{\omega_n^2+\mathbf{k}^2+M^2}$, but we denote the thermal integral $T\sum_{n=-\infty}^{\infty}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{\omega_n^2+\mathbf{k}^2+M^2}$ by $\int\limits_k \frac{1}{-k^2+M^2} \equiv T\sum_{n=-\infty}^{\infty}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{\omega_n^2+\mathbf{k}^2+M^2}$, which is usually found in literature. The symbol $\int\limits_k$ stands exclusively for the right-hand side expression, and *not* for a 4-dimensional integral. We hope that this way of notation avoids confusion, since Euclidean QFT is not exactly SQFT (compare with appendix B as well as appendices E and F).

6.1.1 Explicitly broken symmetry, $H \neq 0$

Condition 2.3 yields

$$\frac{\delta V_{eff}}{\delta G_\sigma} = 0 \ , \quad \frac{\delta V_{eff}}{\delta G_\pi} = 0 \ . \tag{6.9}$$

Making the ansatz

$$G_\sigma^{-1} \equiv -k^2 + M_\sigma^2 \ , \tag{6.10}$$

$$G_\pi^{-1} \equiv -Z^2 k^2 + M_\pi^2 \ , \tag{6.11}$$

we deduce from (6.9) the gap equations for the masses at an arbitrary point ϕ, i.e., not necessarily at the extrema:

result 6.4 (At ϕ)

$$M_\sigma^2 = -m^2 + 6\lambda\phi^2 + 6\lambda \left[\int_k G_\sigma\right] - \frac{1}{\phi^2}\left[\int_k k^2 G_\pi\right] \ , \tag{6.12}$$

$$M_\pi^2 = \frac{H}{\phi} - \frac{H}{2\phi^3}\left[\int_k G_\pi\right] \ , \tag{6.13}$$

$$Z^2 = 1 + \frac{1}{\phi^2}\left[\int_k G_\sigma\right] \ . \tag{6.14}$$

At any extremum (where $J = 0$), result 2.2 implies

$$\frac{\delta V_{eff}}{\delta\phi}\Big|_{\phi=\varphi;G_\sigma=\mathcal{G}_\sigma;G_\pi=\mathcal{G}_\pi} = 0 \ . \tag{6.15}$$

This equation constitutes an additional equation, the stationarity condition for the field. Together with the gap equations at φ, we have a system of equations determining φ as well as M_σ, M_π and Z at the extrema:

result 6.5 (At the extrema)

$$m^2\varphi - 2\lambda\varphi^3 + H = 6\lambda\varphi\left[\int_k \mathcal{G}_\sigma\right] - \frac{H}{2\varphi^2}\left[\int_k \mathcal{G}_\pi\right] + \frac{3H}{8\varphi^4}\left[\int_k \mathcal{G}_\pi\right]^2 + \frac{1}{\varphi^3}\left[\int_k \mathcal{G}_\sigma\right]\left[\int_k k^2 G_\pi\right] \ , \tag{6.16}$$

$$M_\sigma^2 = -m^2 + 6\lambda\varphi^2 + 6\lambda\left[\int_k \mathcal{G}_\sigma\right] - \frac{1}{\varphi^2}\left[\int_k k^2\mathcal{G}_\pi\right] \ , \tag{6.17}$$

$$M_\pi^2 = \frac{H}{\varphi} - \frac{H}{2\varphi^3}\left[\int_k \mathcal{G}_\pi\right] \ , \tag{6.18}$$

$$Z^2 = 1 + \frac{1}{\varphi^2}\left[\int_k \mathcal{G}_\sigma\right] \ . \tag{6.19}$$

At $T = 0$ the system of equations 6.5 determines all constants:

result 6.6 (T=0)

$$m_\sigma(\varphi) = M_\sigma(T=0) \ , \tag{6.20}$$

$$m_\pi(\varphi) = M_\pi(T=0) \ , \tag{6.21}$$

$$H = m_\pi^2 \varphi \ , \tag{6.22}$$

$$\lambda = \frac{m_\sigma^2 - m_\pi^2}{4\varphi^2} \ , \tag{6.23}$$

$$m^2 = \frac{m_\sigma^2 - 3m_\pi^2}{2} \ . \tag{6.24}$$

At the global minimum Φ, we set φ equal the pion decay constant f_π. The masses at zero temperature are chosen according to the Particle Data Booklet.

6.1.2 Chiral limit, $H = 0$

In the case $H = 0$, equation (6.18) yields

$$M_\pi^2 = 0 \ . \tag{6.25}$$

$Z \neq 0$ according to equation (6.19), hence:

$$\int_k k^2 G_\pi \overset{M_\pi=0}{=} \frac{M_\pi^2}{Z^4} \frac{1}{(4\pi)^2} \Big[\underbrace{\frac{M_\pi^2}{Z^2} \ln \frac{M_\pi^2}{Z^2 \mu^2}}_{\xrightarrow[M_\pi \to 0]{} 0} - \frac{M_\pi^2}{Z^2} + \mu^2 \Big] + \frac{M_\pi^2}{Z^4} \frac{T^2}{12} = 0 \ . \tag{6.26}$$

One might be tempted to set $\int_k k^2 G_\pi$ to zero from the beginning, i.e., in V_{eff}. This would imply

$$Z^2 = 1 \ ,$$

because $\frac{1}{\phi^2} \int_k G_\sigma$ arises from $-\frac{1}{2\phi^2} \left[\int_k G_\sigma \right] \left[\int_k k^2 G_\pi \right]$. But it is not allowed. When deriving the gap equation for the pion one differentiates with respect to G_π *before* one concludes from the system of equations that $M_\pi = 0$.
Therefore we arrive at the results:

result 6.7 ($H = 0$, at extrema)

$$m^2 \varphi - 2\lambda \varphi^3 = 6\lambda \varphi \left[\int_k \mathcal{G}_\sigma \right] \ , \tag{6.27}$$

$$M_\sigma^2 = -m^2 + 6\lambda \varphi^2 + 6\lambda \left[\int_k \mathcal{G}_\sigma \right] \ , \tag{6.28}$$

$$Z^2 = 1 + \frac{1}{\varphi^2} \left[\int_k \mathcal{G}_\sigma \right] \ , \tag{6.29}$$

$$M_\pi^2 = 0 \ . \tag{6.30}$$

Note that equations (6.27) and (6.28) are independent of Z and M_π and can therefore be solved separately. At $T = 0$ one obtains:

result 6.8 ($H = 0$, $T = 0$)

$$H = m_\pi = 0 \ , \tag{6.31}$$

$$\lambda = \frac{m_\sigma^2}{4\varphi^2} \ , \tag{6.32}$$

$$m^2 = \frac{m_\sigma^2}{2} \ , \tag{6.33}$$

$$M_\sigma(T = 0) = m_\sigma \ . \tag{6.34}$$

At the global minimum Φ, we set φ equal the pion decay constant f_π. The sigma mass at zero temperature is chosen according to the Particle Data Booklet.

The masses at an arbitrary point ϕ, i.e., not necessarily at the extrema, are determined by the system of equations

result 6.9 ($H = 0$, **at** ϕ)

$$M_\sigma^2 = -m^2 + 6\lambda\phi^2 + 6\lambda \left[\int_k G_\sigma \right] \ , \tag{6.35}$$

$$Z^2 = 1 + \frac{1}{\phi^2} \left[\int_k G_\sigma \right] \ , \tag{6.36}$$

$$M_\pi^2 = 0 \ . \tag{6.37}$$

6.1.3 Results

We want to list the thermal integrals which were used for numerical calculations. Their derivation can be found in appendix B. In the numerical results presented in this section, we neglected the contributions from renormalization, i.e., $Q_\mu \equiv 0$ as well as $R_\mu \equiv 0$.

$$Q_T(M/Z) = \frac{1}{2\pi^2} \int_0^\infty dk \ \frac{k^2}{\sqrt{k^2 + (\frac{M}{Z})^2}} \frac{1}{e^{\sqrt{k^2+(\frac{M}{Z})^2}/T} - 1} \ , \tag{6.38}$$

$$\text{which in case of } M = 0 \text{ and } Z \neq 0 \text{ simplifies to } Q_T(0) = \frac{T^2}{12} \ . \tag{6.39}$$

$$\int_k G_\sigma = Q_\mu(M_\sigma) + Q_T(M_\sigma) \ , \tag{6.40}$$

$$\int_k G_\pi = \frac{1}{Z^2} \left[Q_\mu(\frac{M_\pi}{Z}) + Q_T(\frac{M_\pi}{Z}) \right] \ , \tag{6.41}$$

$$\int_k k^2 G_\pi = \frac{M_\pi^2}{Z^4} \left[Q_\mu(\frac{M_\pi}{Z}) + Q_T(\frac{M_\pi}{Z}) \right] \ . \tag{6.42}$$

For the effective potential we need in addition:

$$R_T(M/Z) = \frac{T}{\pi^2}\int_0^\infty dk k^2 \ln\left(1 - e^{-\frac{\sqrt{k^2+\frac{M^2}{Z^2}}}{T}}\right) , \tag{6.43}$$

$$\text{which in the case } M = 0 \text{ and } Z \neq 0 \text{ simplifies to } R_T(0) = -\frac{T^4\pi^2}{45} . \tag{6.44}$$

$$\begin{aligned}\frac{1}{2}\int_k \left[D_\sigma^{-1}G_\sigma - 1\right] &= \frac{1}{2}\int_k \left[(-k^2+m_\sigma^2)\frac{1}{-k^2+M_\sigma^2} - \frac{-k^2+M_\sigma^2}{-k^2+M_\sigma^2}\right] \\ &= \frac{1}{2}(m_\sigma^2 - M_\sigma^2)\int_k \frac{1}{-k^2+M_\sigma^2} = \frac{1}{2}(m_\sigma^2 - M_\sigma^2)\left[Q_\mu(M_\sigma) + Q_T(M_\sigma)\right] ,\end{aligned} \tag{6.45}$$

$$\begin{aligned}\frac{1}{2}\int_k \left[D_\pi^{-1}G_\pi - 1\right] &= \frac{1}{2}\int_k \left[(-k^2+m_\pi^2)\frac{1}{-Z^2k^2+M_\pi^2} - \frac{-Z^2k^2+M_\pi^2}{-Z^2k^2+M_\pi^2}\right] \\ &= \frac{1}{2}(m_\pi^2 - M_\pi^2)\frac{1}{Z^2}\left[Q_\mu\left(\frac{M_\pi}{Z}\right) + Q_T\left(\frac{M_\pi}{Z}\right)\right] + \frac{1}{2}(Z^2-1)\frac{M_\pi^2}{Z^4}\left[Q_\mu\left(\frac{M_\pi}{Z}\right) + Q_T\left(\frac{M_\pi}{Z}\right)\right] .\end{aligned} \tag{6.46}$$

$$\frac{1}{2}\int_k \ln G_\sigma^{-1} = \frac{1}{2}R_\mu(M_\sigma) + \frac{1}{2}R_T(M_\sigma) , \tag{6.47}$$

$$\frac{1}{2}\int_k \ln G_\pi^{-1} = \frac{1}{2}R_\mu\left(\frac{M_\pi}{Z}\right) + \frac{1}{2}R_T\left(\frac{M_\pi}{Z}\right) . \tag{6.48}$$

Our results are shown in figures 6.1-6.13. For a better understanding we will briefly comment on them in the following. Compare with our conclusions presented in chapter 10. Let us begin with the chiral limit $H = 0$. Below the temperature $T_< = 2f_\pi$ (which is determined analytically from $M_\sigma \equiv 0$) no solutions for $\varphi = 0$ exist. Below $T_<$ there is a region around $\phi = 0$ where the effective potential is not defined, until at $T_<$ a maximum at $\phi = 0$ arises. Above $T_<$ there are three extrema $\varphi \geq 0$: a minimum at $\phi = 0$, another minimum and a maximum. At the critical temperature $T^* = 2.787 f_\pi$ both minima are at the same level and the global minimum jumps from $\Phi \neq 0$ to $\Phi = 0$, i.e., a first-order phase transition takes place. Note that Goldstone's theorem is respected in the phase of spontaneously broken symmetry since the pion mass is always zero, however the pion and the sigma mass do not become degenerate above T^*. The easiest way to understand the case of explicit symmetry breaking ($H \neq 0$) is to look at the results for the effective potential first. Note that there is a region around the origin where the effective potential is not defined due to an imaginary solution for the sigma mass or, respectively, the pion mass. Due to the spurious terms with powers of φ in the denominator there is no extremum at the origin. At low temperature there is only a global minimum at $\phi > 0$. At a certain temperature a second minimum and a maximum occur. At $T^* = 279.617 MeV$ both are at the same level. Above a certain temperature only one minimum is left, which moves closer and closer to the origin. Above a certain temperature T_{max} no physical solutions exist. Figure 6.3 shows the extrema of the effective potential vs. temperature. The green points are solutions of the system

of equations. Therefore one would expect them to be saddle points of the effective potential. In fact they are not. The blue points exactly describe all the extrema of the effective potential (compare fig.6.3 with fig.6.9). Note that the global minimum (the condensate) never becomes zero, i.e., no phase transition occurs (compare the discussion in chapter 10).

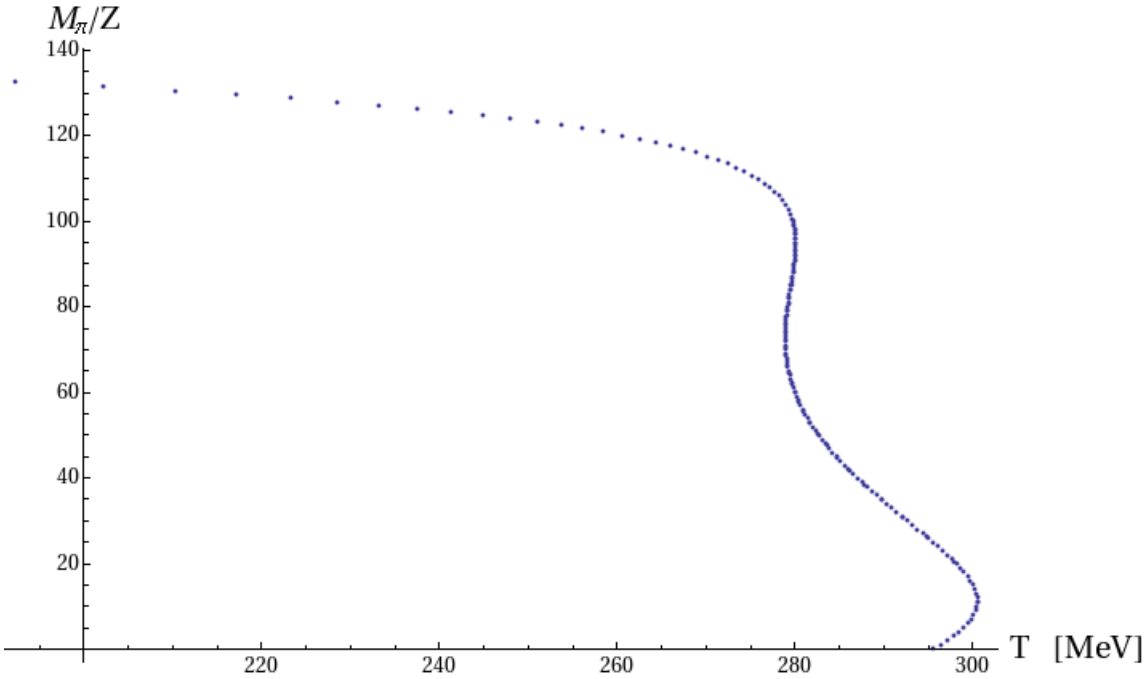


Figure 6.1: $H \neq 0$, physical pion mass at the extrema, positive ϕ_1-direction.

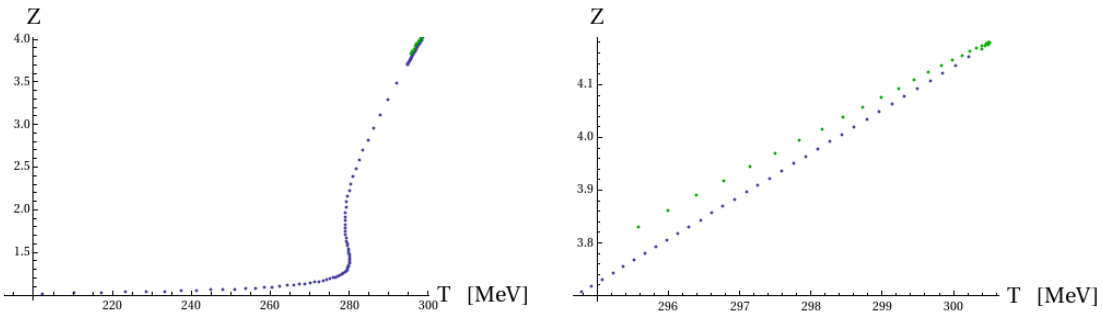


Figure 6.2: $H \neq 0$, wave renormalization factor at the extrema, positive ϕ_1-direction.

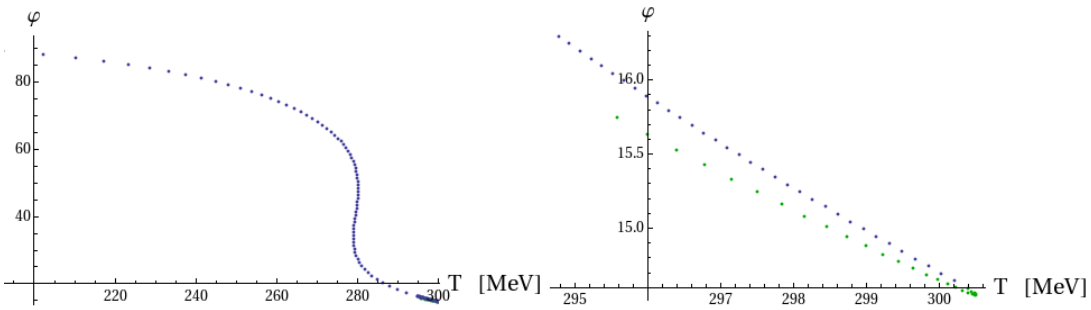


Figure 6.3: $H \neq 0$, φ, positive ϕ_1-direction.

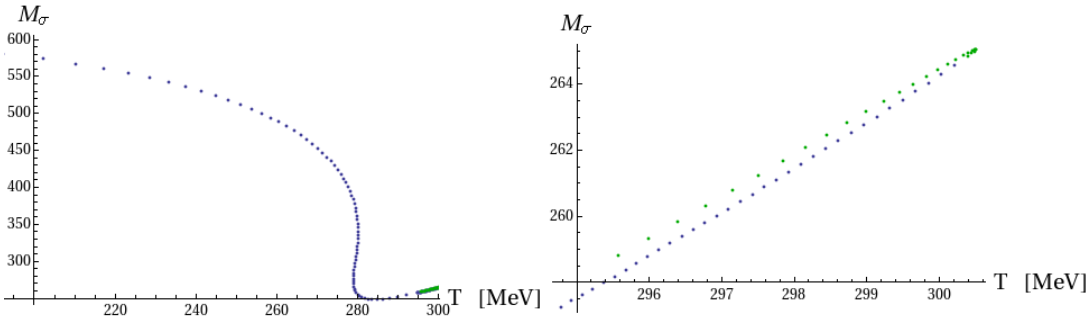


Figure 6.4: $H \neq 0$, sigma mass at the extrema, positive ϕ_1-direction.

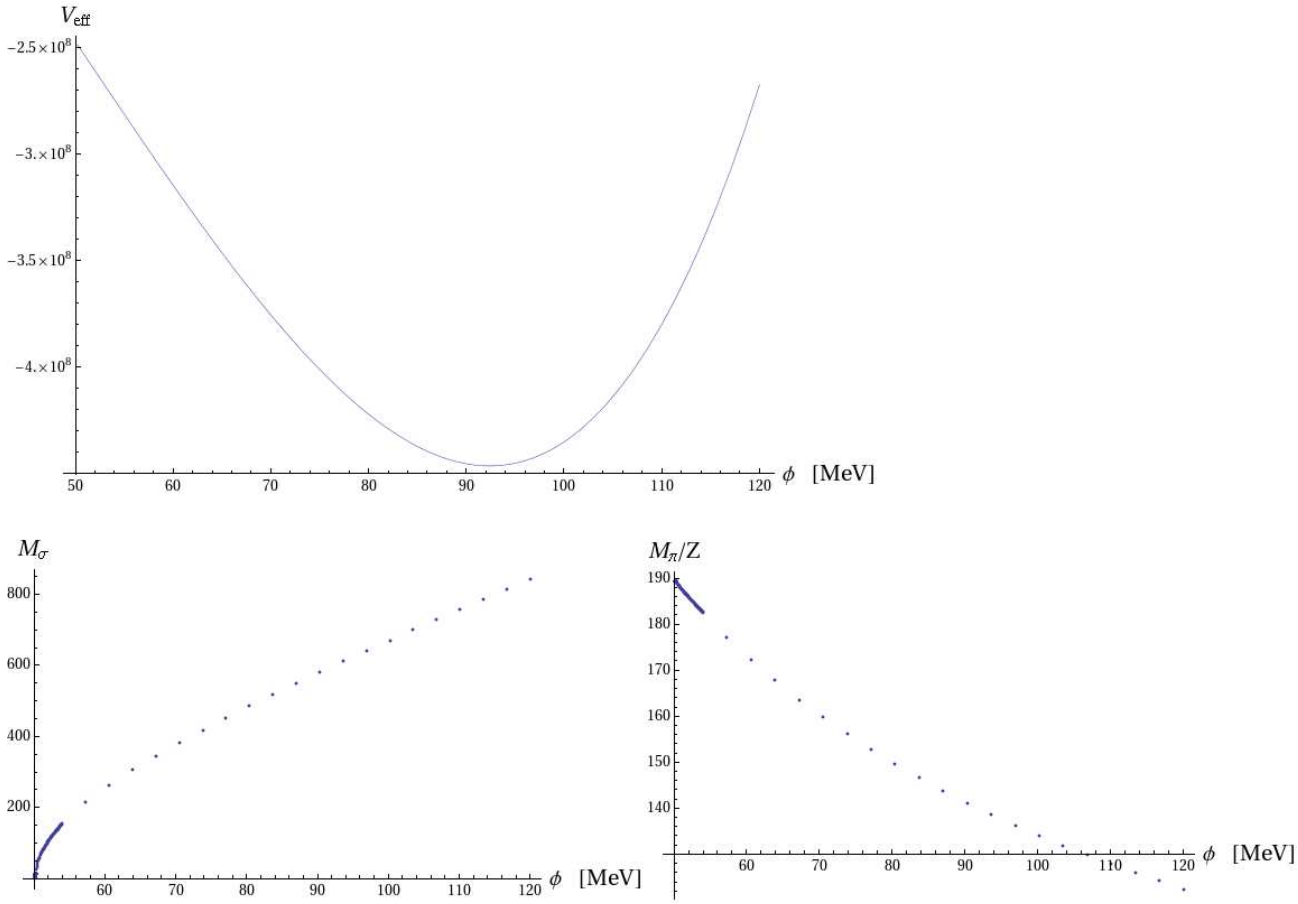


Figure 6.5: $H \neq 0$; $T = 0.1MeV$; effective potential, masses and wave renormalization factor at ϕ; positive ϕ_1-direction. We observe that the sigma mass becomes zero at a certain ϕ_{cr}. For $\phi < \phi_{cr}^{(r)}$ only imaginary solutions for the sigma mass exist. It shall be mentioned that at some $\phi_{cr}^{(l)} < 0$, the sigma mass obtain real values again.

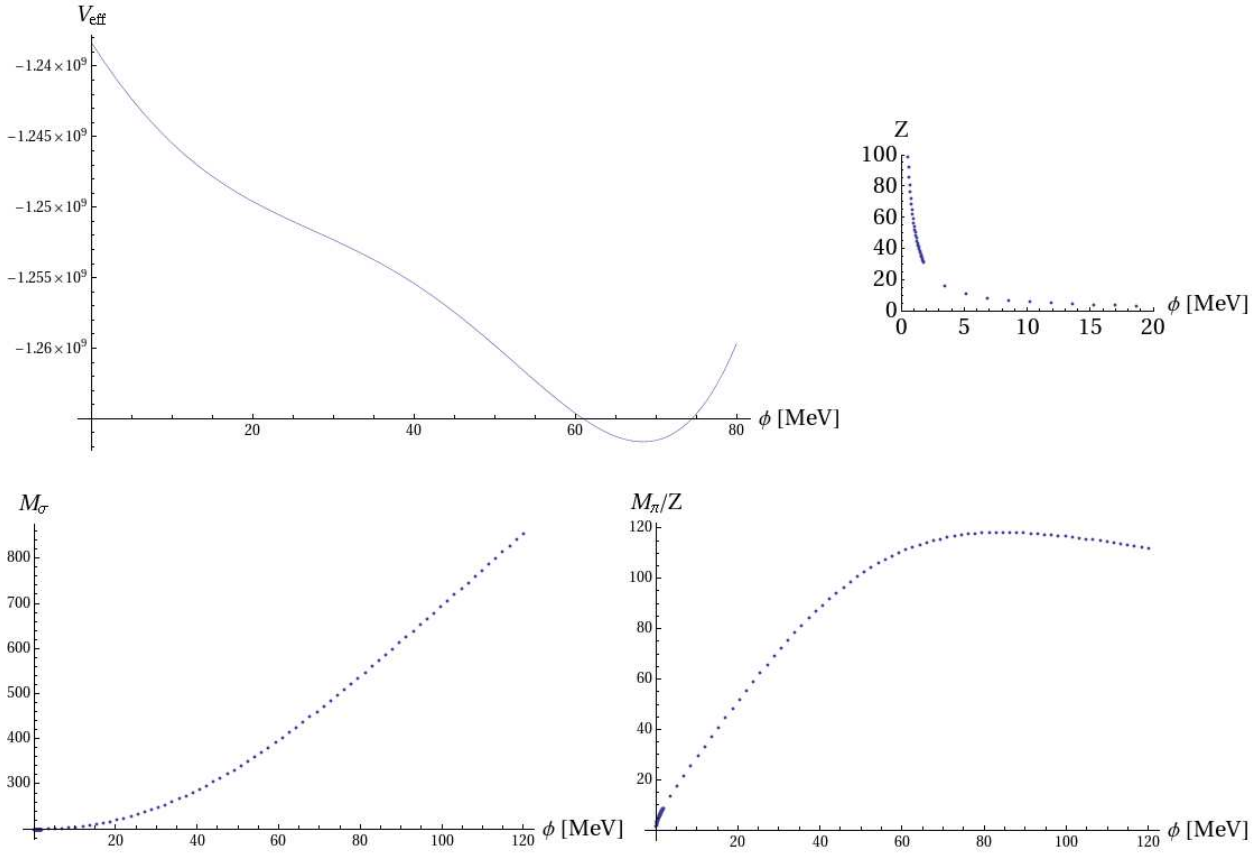


Figure 6.6: $H \neq 0$; $T = 270MeV$; effective potential, masses and wave renormalization factor at ϕ; positive ϕ_1-direction. The pion mass vanishes at some point ϕ_{cr}. Now it is the pion which takes imaginary values from that point on, until it becomes real again at some point on the negative ϕ-axis. This is already the case below $T = 270MeV$.

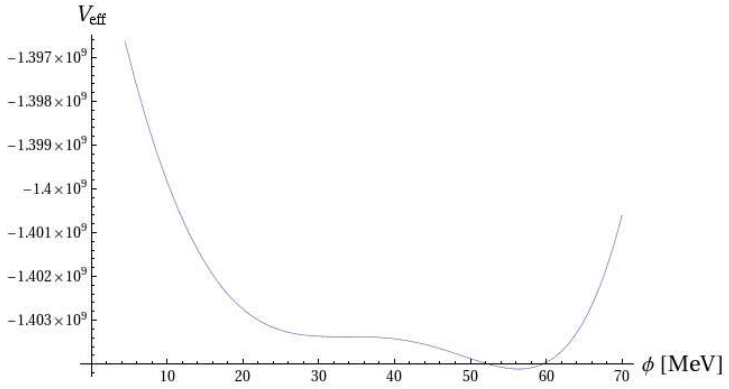


Figure 6.7: $H \neq 0$; $T = 278.91MeV$; effective potential at ϕ; positive ϕ_1-direction. Around this temperature, the second minimum evolves. We have a turning point. The masses and the wave renormalization factor do not differ much from those for $T = 270MeV$.

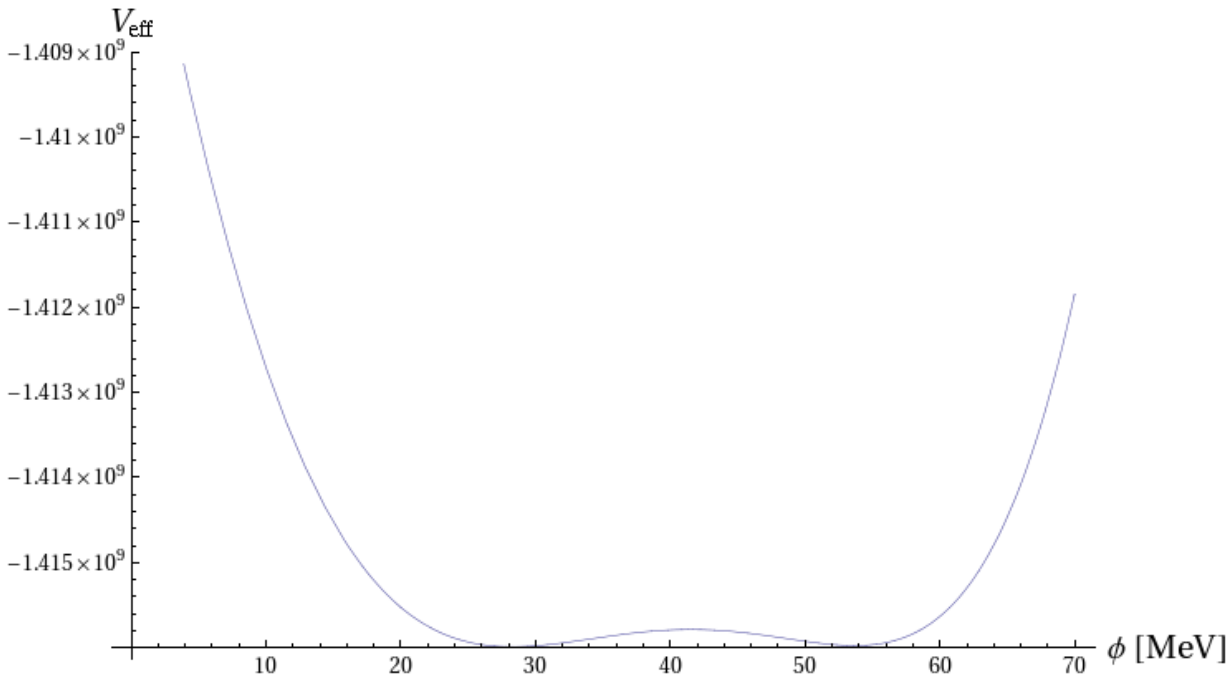


Figure 6.8: $H \neq 0$; We are at the critical temperature $T^* = 279.617MeV$; effective potential at ϕ; positive ϕ_1-direction. The masses and the wave renormalization factor do not differ much from those for $T = 270MeV$.

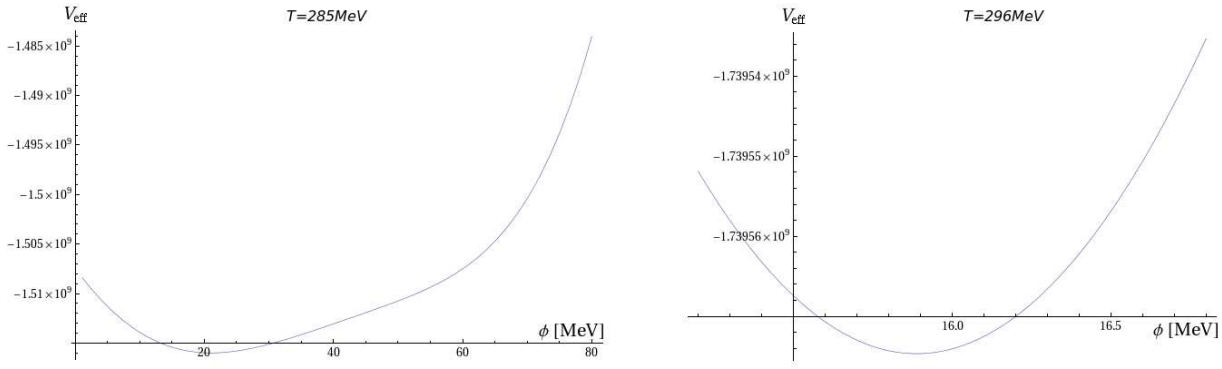


Figure 6.9: $H \neq 0$; left: $T = 285MeV$, right: $T = 296MeV$; effective potential at ϕ; positive ϕ_1-direction. The masses and the wave renormalization factor still look very similar to those for $T = 270MeV$.

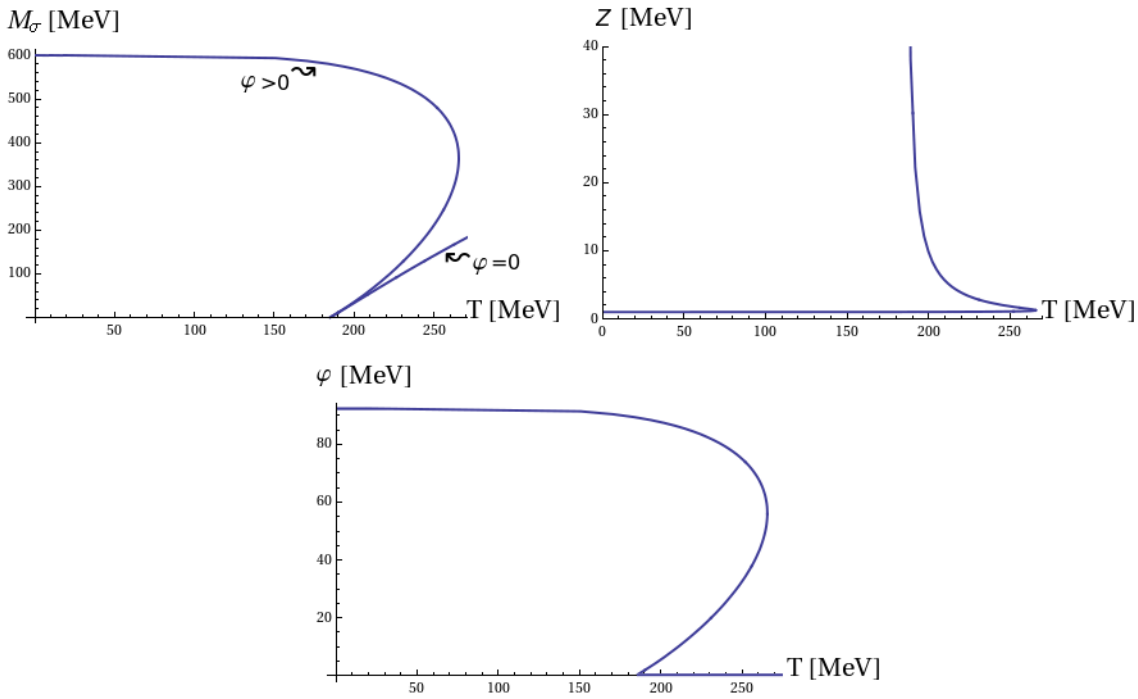


Figure 6.10: $H = 0$, extrema φ of the effective potential, sigma mass at the extrema, and the wave renormalization factor vs. temperature. Note that $Z = \infty$ for $\varphi = 0$.

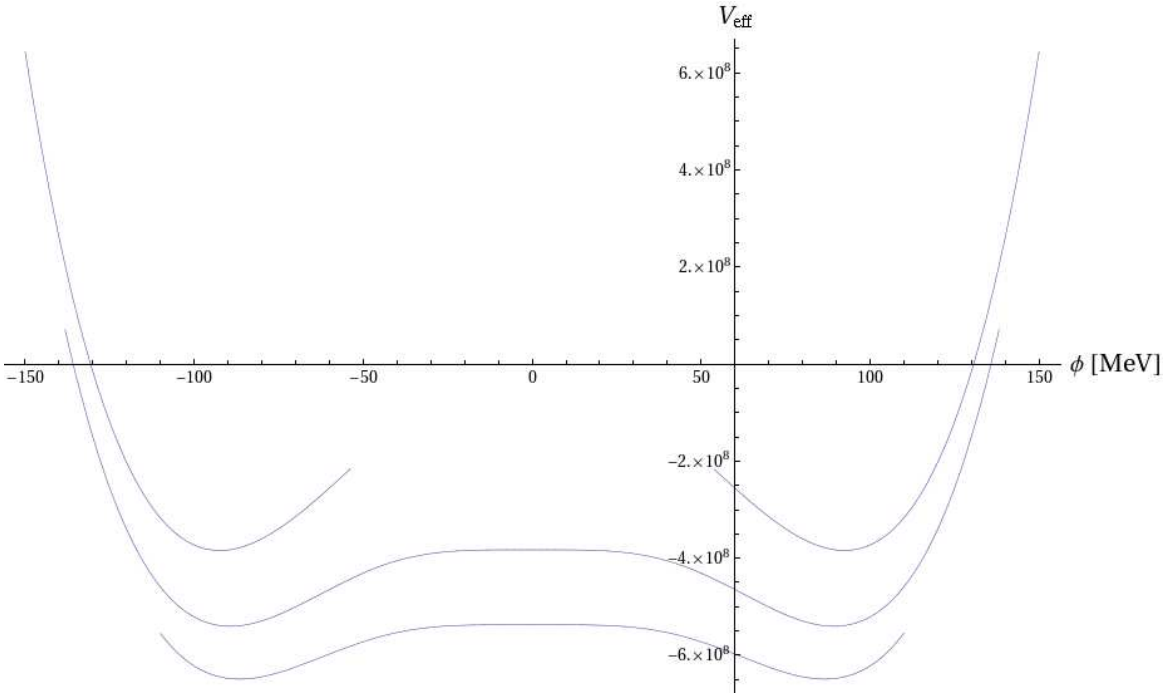


Figure 6.11: $H = 0$, effective potential at ϕ for the temperatures: $0.1MeV$, $0.999 \cdot 2f_\pi$, $2.25f_\pi$ (from above).

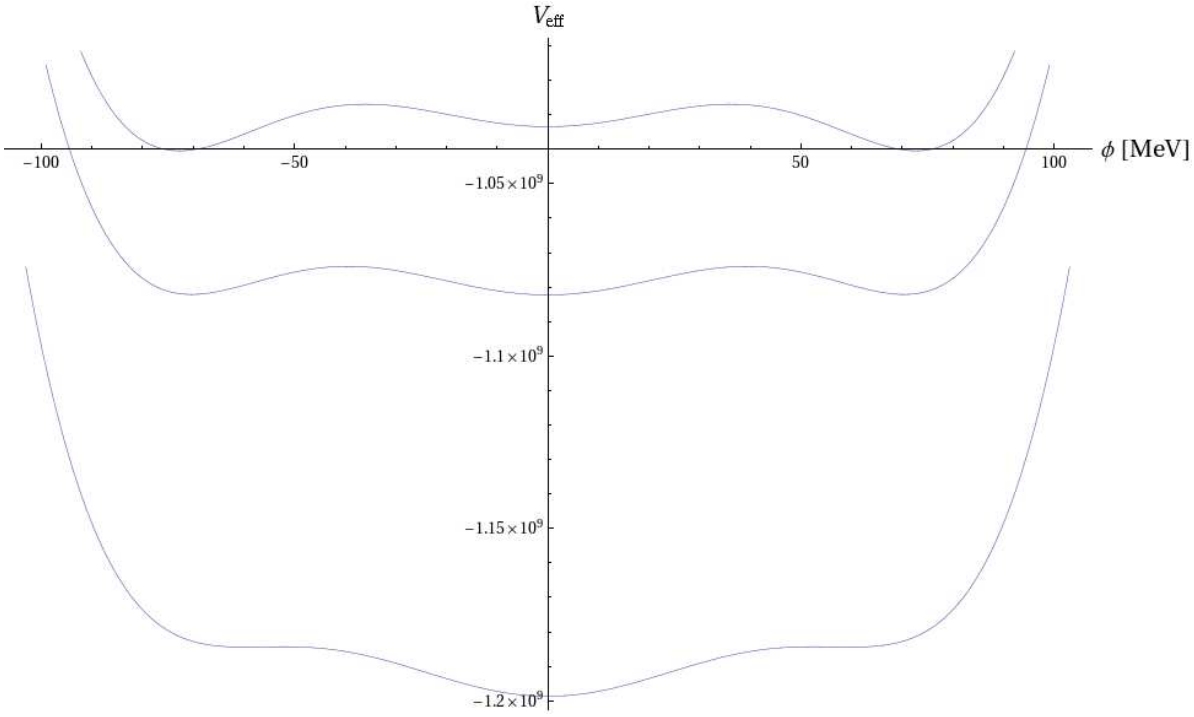


Figure 6.12: $H = 0$, effective potential at ϕ for the temperatures: $2.75f_\pi$, $T^* = 2.787f_\pi$, $2.87f_\pi$ (from above).

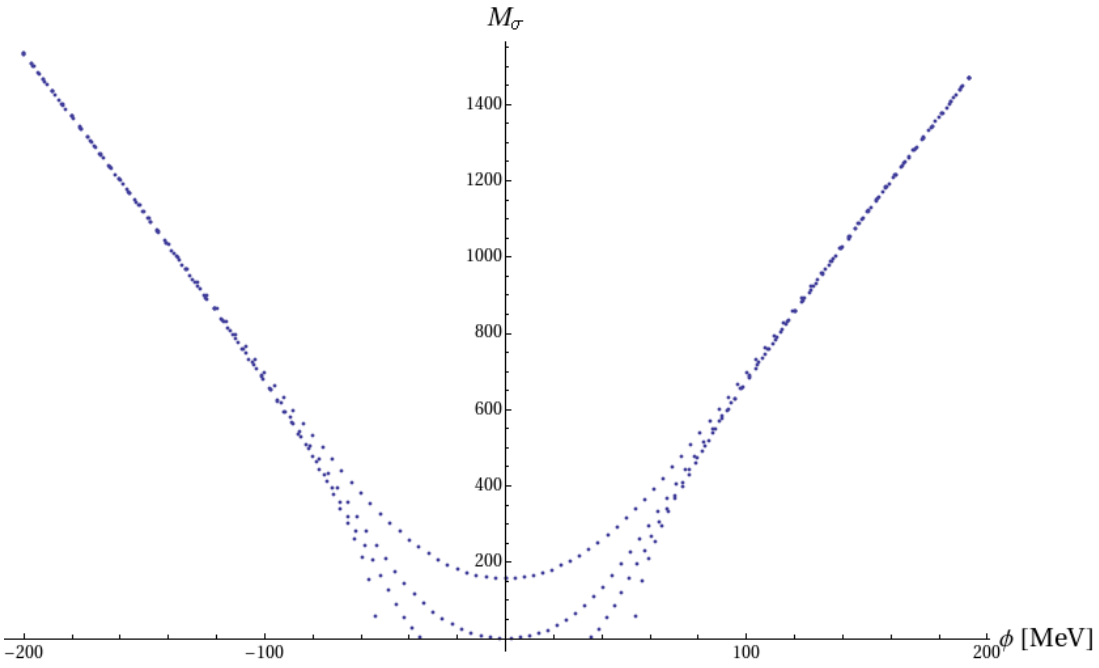


Figure 6.13: $H = 0$, sigma mass at ϕ for the temperatures: $0.1MeV$, $1.5f_\pi$, $0.999 \cdot 2f_\pi$, $T^* = 2.787f_\pi$ (from outer to inner).

6.2 1+0 dimensions

Usually radius r means the absolute value $|\vec{\phi}| = \sqrt{\phi_1^2 + \phi_2^2}$, nevertheless we want to distinguish between "left from origin" (negative direction) and "right from origin" (positive direction), which is necessary for non-symmetrical potentials. A convenient notation is therefore $r \equiv \pm|\vec{\phi}| \equiv \phi = \pm\sqrt{\phi_1^2 + \phi_2^2}$, where $+/-$ stands for the case in which $\vec{\phi}$ points into positive/negative direction. So we can use ϕ and r synonymously. We are interested in the radial ϕ_1-direction only, i.e., $r \equiv \pm|\vec{\phi}| \equiv \phi = \pm\sqrt{\phi_1^2}$, where the classical mexican-hat potential, known from fig.2.3, can have three different shapes (see figure 6.14), depending on the values of λ and H. Same procedure as

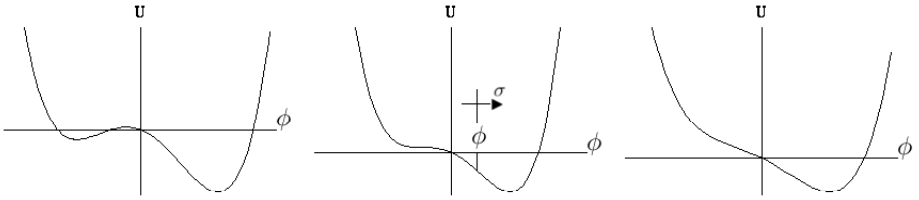


Figure 6.14: tree-level shapes of the classical potential.

usual: we choose an arbitrary point ϕ and fluctuate around it. The fluctuation in radial direction is denoted by σ. The angular fluctuation is directed orthogonal to the drawing plane, therefore not sketched in fig.6.14. Using the notation $\phi_1 = r\cos\varphi$, $\phi_2 = r\sin\varphi$, we fluctuate around $\varphi = 0$ (remember, we look at radial ϕ_1-direction). Normalizing the fluctuation by ϕ, we denote it by $\frac{\pi}{\phi}$. From the Lagrange function

$$L = \frac{1}{2}\dot{\sigma}^2 + \frac{1}{2}(\sigma+\phi)^2\frac{\dot{\pi}^2}{\phi^2} + \frac{m^2}{2}(\sigma+\phi)^2 - \frac{\lambda}{N}(\sigma+\phi)^4 + H(\sigma+\phi)\cos\frac{\pi}{\phi}$$

we read off, analogously to the previous sections, the tree-level masses and the vertex factors:

result 6.10 (tree-level masses)

$$m_\sigma^2 = -m^2 + 12\frac{\lambda}{N}\phi^2 \quad , \quad m_\pi^2 = \frac{H}{\phi} \ .$$

Therefore, in the pseudo-Minkowskian notation, we have the inverse tree-level propagators $D_\sigma^{-1} = -k_0^2 - m^2 + 12\frac{\lambda}{N}\phi^2$, for the sigma, and $D_\pi^{-1} = -k_0^2 + \frac{H}{\phi}$, for the pion.

result 6.11 (vertex factors)

$$-\tfrac{\lambda}{N}\sigma^4 \longrightarrow \quad \times\text{-}\frac{\lambda}{N}$$

$$+H\phi\tfrac{1}{4!}\tfrac{\pi^4}{\phi^4} \longrightarrow \quad \times + \frac{H}{4!\phi^3}$$

$$\tfrac{1}{2}\sigma^2\tfrac{1}{\phi^2}\underbrace{\partial_0\pi\partial^0\pi}_{=-k_0^2\pi^2} \longrightarrow \quad \times + \frac{1}{2\phi^2}$$

Employing result 6.11, we gain V_2:

result 6.12 (contribution from two-particle irreducible diagrams)

$$V_2 = -3\left(-\tfrac{\lambda}{N}\right)\left[\int\limits_{k_0} G_\sigma\left(k_0\right)\right]^2 - 3\left(+\tfrac{H}{4!\phi^3}\right)\left[\int\limits_{k_0} G_\pi\left(k_0\right)\right]^2 - \left(+\tfrac{1}{2\phi^2}\right)\left[\int\limits_{k_0} G_\sigma\left(k_0\right)\right]\left[\int\limits_{k_0} k_0^2 G_\pi\left(k_0\right)\right] \ .$$

The effective potential reads

result 6.13 (effective potential)

$$V\left[\phi\right] = -\tfrac{m^2}{2}\phi^2 + \tfrac{\lambda}{N}\phi^4 - H\phi\cos\left(0\right) + \tfrac{1}{2}\int\limits_{k_0}\ln G_\sigma^{-1}\left(k_0\right) + \tfrac{1}{2}\int\limits_{k_0}\ln G_\pi^{-1}\left(k_0\right) + \tfrac{1}{2}\int\limits_{k_0}\left[D_\sigma^{-1}\left(k_0\right)G_\sigma\left(k_0\right) - 1\right] + \tfrac{1}{2}\int\limits_{k_0}\left[D_\pi^{-1}\left(k_0\right)G_\pi\left(k_0\right) - 1\right] + V_2 \ .$$

(We have $\cos\left(0\right)$, because we fluctuate around 0).

The gap equation for the sigma follows from $\frac{\delta V}{\delta G_\sigma} = 0$. With the effective potential (6.13) we obtain:

$$\frac{1}{2}\ \frac{1}{G_\sigma^{-1}}\left(-1\right)G_\sigma^{-2} + \frac{1}{2}D_\sigma^{-1}\left(k_0,\phi\right) + \frac{3\lambda}{N}\ 2\ \left[\int\limits_{k_0} G_\sigma\right] - \frac{1}{2\phi^2}\left[\int\limits_{k_0} k_0^2 G_\pi\right] = 0 \ . \tag{6.49}$$

The gap equation for the pion follows from $\frac{\delta V}{\delta G_\pi} = 0$, so we have:

$$\frac{1}{2}\ \frac{1}{G_\pi^{-1}}\left(-1\right)G_\pi^{-2} + \frac{1}{2}D_\pi^{-1}\left(k_0,\phi\right) - \frac{3H}{4!\phi^3}\ 2\ \left[\int\limits_{k_0} G_\pi\right] - \frac{1}{2\phi^2}k_0^2\left[\int\limits_{k_0} G_\sigma\right] = 0 \ . \tag{6.50}$$

Making the Ansatz $\boxed{G_\sigma^{-1} = -k_0^2 + M_\sigma^2}$ resp. $\boxed{G_\pi^{-1} = -Z^2 k_0^2 + M_\pi^2}$ we are able to proceed with the calculation.
Equation (6.49) yields

result 6.14 (gap equation for sigma particle)

$$M_\sigma^2 = -m^2 + 12\frac{\lambda}{N}\phi^2 + \frac{12\lambda}{N}\int\limits_{k_0} G_\sigma - \frac{1}{\phi^2}\int\limits_{k_0} k_0^2 G_\pi \ ,$$

whereas equation (6.50) leads to

$$Z^2 k_0^2 - M_\pi^2 + \left(-k_0^2 + m_\pi^2\right) - \frac{12H}{4!\phi^3}\int\limits_{k_0} G_\pi - \frac{k_0^2}{\phi^2}\int\limits_{k_0} G_\sigma \ .$$

Separation of k_0-dependant and independent terms yields

result 6.15 (gap equations for pion)

$$y^2 = \frac{H}{\phi Z^2} - \frac{H}{2\phi^3}\frac{1}{Z^2}\int\limits_{k_0} G_\pi \ ,$$
$$Z^2 = 1 + \frac{1}{\phi^2}\int\limits_{k_0} G_\sigma \ ,$$

where $y \equiv M_\pi/Z$ denotes the physical pion mass.
Equations (6.49) and (6.50) hold at each point ϕ, also at the extrema φ. If we consider the equations at an extremum, we indicate this by writing $\mathcal{G}$ instead of G.
The stationarity condition for the field (which means that we are at an extremum), $\frac{\delta V}{\delta \phi}|_{\phi=\varphi, G=\mathcal{G}} = 0$, leads to:

result 6.16 (stationarity condition for the field)

$$-m^2\varphi + 4\frac{\lambda}{N}\varphi^3 - H + 12\frac{\lambda}{N}\varphi\int\limits_{k_0}\mathcal{G}_\sigma - \frac{H}{2\varphi^2}\int\limits_{k_0}\mathcal{G}_\pi$$
$$+\frac{9H}{4!}\varphi^{-4}\left(\int\limits_{k_0}\mathcal{G}_\pi\right)^2 + \varphi^{-3}\int\limits_{k_0}\mathcal{G}_\sigma\int\limits_{k_0}k_0^2\mathcal{G}_\pi = 0 \ .$$

Moreover, using the ansatz and the expressions for the tree-level propagators, the effective potential (result 6.13) can be rewritten:

$$V[\phi] = -\frac{m^2}{2}\phi^2 + \frac{\lambda}{N}\phi^4 - H\phi\cos(0) + \frac{1}{2}\left(m_\sigma^2 - M_\sigma^2\right)\int\limits_{k_0} G_\sigma(k_0) + \frac{1}{2}\left(Z^2-1\right)\int\limits_{k_0} k_0^2 G_\pi(k_0)$$
$$+\frac{1}{2}\left(m_\pi^2 - M_\pi^2\right)\int\limits_{k_0}\frac{1}{-Z^2k_0^2+M_\pi^2} + \frac{1}{2}\int\limits_{k_0}\ln G_\sigma^{-1}(k_0) + \frac{1}{2}\int\limits_{k_0}\ln G_\pi^{-1}(k_0) + V_2 \ . \qquad (6.51)$$

The case $H \neq 0$

If we want to calculate the effective potential (6.13) for arbitrary values ϕ at nonzero temperature, we first have to solve the non-linear system of equations (6.14,6.15) numerically for the masses M_σ,M_π and the wave renormalization factor Z at given temperature T and field ϕ.
At first, let us assume $M_\sigma > 0$ and $M_\pi > 0$. Using results B.1 and B.2, the thermal integrals are given by

$$\int\limits_{k_0} G_\sigma = \frac{1}{M}\left(\frac{1}{e^{M/T}-1} + \frac{1}{2}\right) \ ,$$
$$\int\limits_{k_0} G_\pi = \frac{1}{Z^2}\frac{1}{y}\left(\frac{1}{e^{y/T}-1} + \frac{1}{2}\right) \ ,$$
$$\int\limits_{k_0} k_0^2 G_\pi = \frac{M}{Z^3}\left(\frac{1}{e^{M/ZT}-1} + \frac{1}{2}\right) \ ,$$

where we used $\int\limits_{k_0}\frac{1}{-Z^2k_0^2+M_\pi^2} = \frac{1}{Z^2}\int\limits_{k_0}\frac{1}{-k_0^2+M_\pi^2/Z^2} = \frac{1}{Z^2}\frac{1}{y}\left(\frac{1}{e^{y/T}-1}+\frac{1}{2}\right)$ for the second one.
For practical calculations, we express all the quantities in multiples of m.
Numerical calculations show that we are confronted with the same problem as in 1+3 dimensions. If one calculates the effective potential at an arbitrary point ϕ, there is a region in which the effective potential is ill-defined due to imaginary solutions of the gap equations. Figure 6.15 shows the typical case where the pion mass $M_\pi(\phi)$ becomes imaginary at a certain point $\phi_{krit} > 0$.

The larger T the larger ϕ_{krit}. Since the condensate (i.e., the global minimum of the effective potential) decreases with increasing temperature, it reaches the ill-defined region at some critical value T_{max}, which rules out a phase transition. Furthermore, numerical calculations indicate that the smaller the explicit symmetry breaking the larger ϕ_{krit} and the smaller $M_\pi(\phi)/Z$. More precisely: $M_\pi(\phi)/Z \to 0$ for $H \to 0$. This has to do with the fact that results B.1 and B.4 diverge in the limit $M \to 0$. Hence, until one is able to understand the chiral limit, it does not make sense to investigate the case of explicitly broken symmetry any further.

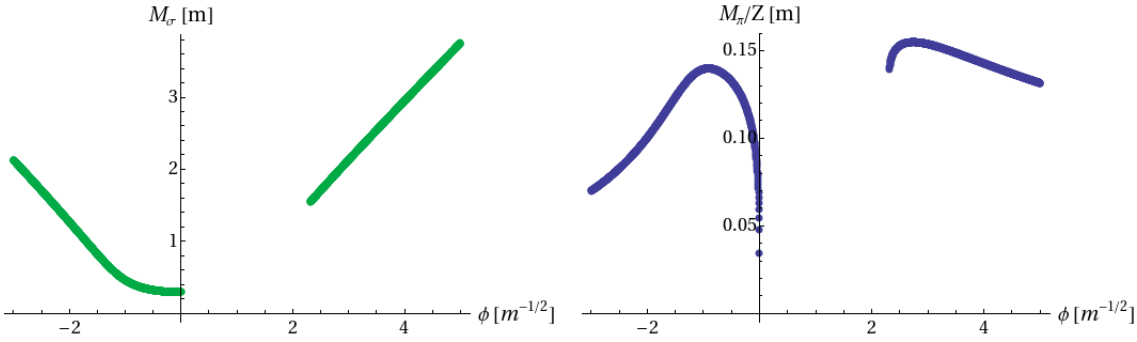


Figure 6.15: $T = 0.1m$, $\lambda = 0.1m^3$, $H = 0.1m^{\frac{3}{2}}$.

The case $H = 0$ (chiral limit)

From 6.15 we conclude that for $H = 0$, the physical pion mass y (as well as the pion mass M_π) is zero for all temperatures. $\int_{k_0} G_\sigma$ is positive definite, therefore $Z > 1$. With that, we can exclude that for given $\phi \neq 0$ the limit $H \to 0$ leads to a $y > 0$. Note that this is completely consistent with what we found out from numerical calculations for $H \to 0$.
Now that $y = 0$, we have to worry about what to do with the thermal integrals for the pion, because the results B.1 and B.4 diverge in the limit $M \to 0$. So far we were not able to resolve the problem completely, yet we present our intermediate results in section 8.2.3.

Chapter 7

$O(2)$-symmetric double-well potential, cartesian coordinates

7.1 1+3 dimensions

In this section we are concerned with the $O(2)$ model in cartesian coordinates, which is determined by the Lagrangian density

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \vec{\phi}\,(X)\,\partial^\mu \vec{\phi}\,(X) + m^2 \vec{\phi}\,(X)^2\right) - \frac{\lambda}{N}\left(\vec{\phi}\,(X)^2\right)^2 + H\phi_1 \ . \tag{7.1}$$

$\vec{\phi}\,(X) = \Big(\phi_1\,(X)\,,\ \phi_2\,(X)\Big)$ is a vector with real components.
Since we are interested in spontaneous symmetry breaking, we assume $m^2 > 0$.
The $O(4)$-model is of particular interest as we pointed out in great detail in chapter 1. The $O(N)$-model has been discussed by several authors. Therefore we just want to give a short summary of this theory for $N = 2$ at nonzero temperature. Everywhere in this section where an explicit N appears, it stands for $N = 2$. It makes comparison with the case $N > 2$ easier.
We choose a fixed (i.e., independent of X) arbitrary point $\Big(\phi_1,\ \phi_2\Big)$ in the $\phi_1\phi_2$-plane and fluctuate around it:

$$\mathscr{L} = \frac{1}{2}\Big(\partial_\mu\left[\phi_1+\sigma\right]\partial^\mu\left[\phi_1+\sigma\right] + \partial_\mu\left[\phi_2+\pi\right]\partial^\mu\left[\phi_2+\pi\right] + m^2\left[\left(\phi_1+\sigma\right)^2 + \left(\phi_2+\pi\right)^2\right]\Big) - \frac{\lambda}{N}\Big(\left(\phi_1+\sigma\right)^2 + \left(\phi_2+\pi\right)^2\Big)^2 + H\left(\phi_1+\sigma\right)$$

$$\Leftrightarrow \qquad \mathscr{L} = \frac{1}{2}\left(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\pi\partial^\mu\pi + m^2\left[{\phi_1}^2 + 2\sigma\phi_1 + \sigma^2 + {\phi_2}^2 + 2\pi\phi_2 + \pi^2\right]\right) - \frac{\lambda}{N}\left({\phi_1}^2 + {\phi_2}^2 + 2\phi_1\sigma + 2\phi_2\pi + \sigma^2 + \pi^2\right)^2 + H\left(\phi_1+\sigma\right) \ .$$

The only informations we need from the above are the prefactors of the quadratic and the quartic term in σ resp. π:

$$
\begin{aligned}
&- \left(-\frac{m^2}{2} + 6\frac{\lambda}{N}{\phi_1}^2 + 2{\phi_2}^2 \right) \sigma^2 \ , \\
&- \left(-\frac{m^2}{2} + 6\frac{\lambda}{N}{\phi_2}^2 + 2{\phi_1}^2 \right) \pi^2 \ , \\
&- \frac{\lambda}{N}\sigma^4 \ , \\
&- \frac{\lambda}{N}\pi^4 \ .
\end{aligned}
$$

The quadratic terms specify the tree-level propagators. In pseudo-Minkowskian notation they are given by:

$$
\begin{aligned}
{D_\sigma}^{-1} &= -k^2 - m^2 + 12\frac{\lambda}{N}{\phi_1}^2 + 4\frac{\lambda}{N}{\phi_2}^2 \ , \\
{D_\pi}^{-1} &= -k^2 - m^2 + 12\frac{\lambda}{N}{\phi_2}^2 + 4\frac{\lambda}{N}{\phi_1}^2 \ .
\end{aligned}
$$

The quartic terms will be needed for the vertex factors when we calculate Feynman diagrams.

Imagine how the mexican hat potential (see fig.2.3) would look like for $H = 0$. There is an infinite number of states (ϕ_1, ϕ_2) for which the potential is minimal. All of them lie on a circle. If the particles described by our theory would both have positive parity, the vacuum would be the circle as a whole. Transforming $\vec{\phi}$ under $U(1)$ means rotating on a circle in the (ϕ_1, ϕ_2)-pane. Consequently the vacuum would be $U(1)$-invariant, such as the Lagrangian, and we would have no spontaneous symmetry breaking for such a theory. However, we want to describe a σ-particle and a pion with our theory. Pions have negative parity, because of that there can be no pions in the vacuum (compare with definition 1.1 of the vacuum). If we describe the σ-particle by ϕ_1 and the pion by ϕ_2, this corresponds to the condition $\phi_2|_{vac} \equiv \varphi_2 = 0$, i.e., ϕ_2 at the vacuum is zero. One can see later that this has to be the case at nonzero temperature, too. Consequently we have to choose $(\varphi_1, 0)$ as vacuum. Together with the origin, this determines the ϕ_1-direction ($\phi_2 = 0$) as the direction of our interest. By this constraint, the vacuum is no longer $U(1)$-invariant: a rotation leads to a state with negative parity. That means our $U(1)$-symmetry is spontaneously broken.
For the case where the $U(1)$-symmetry is explicitly broken by the $H\phi_1$-term, the minimum on the right is lower than that one on the left, so it is a global minimum. One can show that it has positive parity i.e $\varphi_2 = 0$, too. This is easy to see only at tree-level: the conditions $\frac{\partial U}{\partial \phi_{1,2}} \stackrel{!}{=} 0$ lead to the equations $-2m^2 + 4\left({\phi_1}^2 + {\phi_2}^2\right) = 0$ and $-2m^2 + 4\left({\phi_1}^2 + {\phi_2}^2\right) - \frac{H}{\phi_1}$, which cannot be fullfilled for $\phi_1 \neq 0$, $\phi_2 \neq 0$, $H \neq 0$.

The above expressions for the tree-level propagators reduce to:

result 7.1 (tree-level propagators)

$$
{D_\sigma}^{-1} = -k^2 - m^2 + 12\frac{\lambda}{N}\phi^2 \ , \tag{7.2}
$$

$$
{D_\pi}^{-1} = -k^2 - m^2 + 4\frac{\lambda}{N}\phi^2 \ . \tag{7.3}
$$

Making the following ansatz for the inverse connected thermal 2-point functions in the presence of a source J:

$$G_\pi^{-1} \equiv -k^2 + M_\pi^2 \;\text{ resp. }\; G_\sigma^{-1} \equiv -k^2 + M_\sigma^2 \;, \tag{7.4}$$

the stationarity condition for the field leads to equation (7.5) and the gap equations (at the extrema) to equations (7.6) and (7.7):

result 7.2 (system of equations at the extrema)

$$H = \varphi \left[-m^2 + \frac{4\lambda}{2}\varphi^2 + \frac{12\lambda}{2} \int_k \mathcal{G}_\sigma + \frac{4\lambda}{2} \int_k \mathcal{G}_\pi \right] \;, \tag{7.5}$$

$$M_\sigma{}^2 = -m^2 + \frac{4\lambda}{2} \left[3\varphi^2 + 3 \int_k \mathcal{G}_\sigma + \int_k \mathcal{G}_\pi \right] \;, \tag{7.6}$$

$$M_\pi{}^2 = -m^2 + \frac{4\lambda}{2} \left[\varphi^2 + \int_k \mathcal{G}_\sigma + 3 \int_k \mathcal{G}_\pi \right] \;. \tag{7.7}$$

The solution to these equations can be found in [45]. One observes that in the chiral limit, $H = m_\pi = 0$, Goldstone's theorem is violated (except for large N) due to the nonvanishing mass of the pions in the phase of spontaneously broken symmetry, since the system of equations (result 7.2) implies for $H = 0$, if $\varphi \neq 0$:

$$M_\sigma^2 = \sqrt{\frac{8\lambda}{N}}\varphi \neq 0 \;, \tag{7.8}$$

$$M_\pi^2 = \frac{8\lambda}{N} \left[\int_k \mathcal{G}_\pi - \int_k \mathcal{G}_\sigma \right] , \tag{7.9}$$

which means that $M_\pi = 0$ is not a solution.

A solution to the problem was proposed by Nemoto, Naito, and Oka [48]. They improved the above procedure by an $O(N)$-symmetric ansatz for G_π^{-1} and G_σ^{-1}:

$$G_\pi^{-1} \equiv -k^2 + M_\pi^2 \;\text{ with }\; M_\pi^2 = -m^2 + \frac{X(\phi^2)}{N}\phi^2 \;, \tag{7.10}$$

$$G_\sigma^{-1} \equiv -k^2 + M_\sigma^2 \;\text{ with }\; M_\sigma^2 = -m^2 + \frac{X(\phi^2) + Y(\phi^2)}{N}\phi^2, \tag{7.11}$$

where M_π and M_σ are regarded as unphysical quantities, determined from the system of equations (result 7.2). The physical masses at an arbitrary point (σ, π), $\overline{M}_\pi$ and $\overline{M}_\sigma$, are given by the definitions

$$\overline{M}_\pi^2 \equiv \frac{d^2 V_{eff}}{d\pi^2} \;\text{ and }\; \overline{M}_\sigma^2 \equiv \frac{d^2 V_{eff}}{d\sigma^2}, \tag{7.12}$$

which are usually restricted to tree-level, whereas the physical mass is defined as the pole in the propagator.

Since the effective potential V_{eff} depends on M_π and M_σ, the physical masses will depend on them, too. According to Nemoto, Naito, and Oka, Goldstone's theorem is respected even for finite values of N.

7.2 1+0 dimensions

7.2.1 Condensate and masses at the global minimum

Replacing $\int\limits_k G$ by $\int\limits_{k_0} G$ and $\int\limits_k \ln G^{-1}$ by $\int\limits_{k_0} \ln G^{-1}$, where

$$\int\limits_{k_0} G = \frac{1}{M}\left(\frac{1}{e^{M/T}-1}+\frac{1}{2}\right) , \tag{7.13}$$

$$\int\limits_{k_0} \ln G^{-1} = M + 2T\ln\left(1-e^{-M/T}\right), \tag{7.14}$$

and expressing all quantities in multiples of m, i.e.,

$$\phi = \widetilde{\phi}\cdot m^{-\frac{1}{2}} \ , \quad \lambda = \widetilde{\lambda}\cdot m^{3} \ , \quad \tau = \widetilde{\tau}\cdot m^{-1} \ , \quad H = \widetilde{H}\cdot m^{\frac{3}{2}} \ , \quad M = \widetilde{M}\cdot m \ , \quad T = \widetilde{T}\cdot m \ , \tag{7.15}$$

we obtain from (7.1) resp. (7.2):

result 7.3 (tree-level propagators)

$$D_\sigma{}^{-1} = -k_0^2 - 1 + 12\frac{\lambda}{N}\phi^2 \ , \tag{7.16}$$

$$D_\pi{}^{-1} = -k_0^2 - 1 + 4\frac{\lambda}{N}\phi^2 \ . \tag{7.17}$$

result 7.4 (system of equations at the extrema)

$$H = \varphi\left[-1+\frac{4\lambda}{N}\varphi^2 + \frac{12\lambda}{N}\int\limits_{k_0}\mathcal{G}_\sigma + \frac{4\lambda(N-1)}{N}\int\limits_{k_0}\mathcal{G}_\pi\right] \ , \tag{7.18}$$

$$M_\sigma{}^2 = -1+\frac{4\lambda}{N}\left[3\varphi^2 + 3\int\limits_{k_0}\mathcal{G}_\sigma + (N-1)\int\limits_{k_0}\mathcal{G}_\pi\right] \ , \tag{7.19}$$

$$M_\pi{}^2 = -1+\frac{4\lambda}{N}\left[\varphi^2 + \int\limits_{k_0}\mathcal{G}_\sigma + (N+1)\int\limits_{k_0}\mathcal{G}_\pi\right] \ . \tag{7.20}$$

where the twiddle has been omitted for simplicity.
We have to distinguish between two cases, $\varphi = 0$ and $\varphi \neq 0$. Note that for $H \neq 0$ we always have $\varphi \neq 0$ due to equation (7.18), whereas $H = 0$ allows $\varphi \neq 0$ as well as $\varphi = 0$ (however it will turn out that the case $\varphi \neq 0$ only exists for $\lambda < \lambda_{max}$).

I) $\varphi = 0$

result 7.5 (system of equations at the extrema, $\varphi = 0$)

$$H = 0 \ , \tag{7.21}$$

$$M_\sigma{}^2 = -1 + \frac{4\lambda}{N}\left[3\int_{k_0}\mathcal{G}_\sigma + (N-1)\int_{k_0}\mathcal{G}_\pi\right] \ , \tag{7.22}$$

$$M_\pi{}^2 = -1 + \frac{4\lambda}{N}\left[\int_{k_0}\mathcal{G}_\sigma + (N+1)\int_{k_0}\mathcal{G}_\pi\right] \ . \tag{7.23}$$

Let us assume $M_\sigma \neq 0$ and $M_\pi \neq 0$ (which turns out to be true later). At $T = 0$, the gap equations read

$$M_\sigma^{0\,2} = -1 + \frac{4\lambda}{N}\left(\frac{3}{2M_\sigma^0} + \frac{N-1}{2M_\pi^0}\right) \ , \tag{7.24}$$

$$M_\pi^{0\,2} = -1 + \frac{4\lambda}{N}\left(\frac{1}{2M_\sigma^0} + \frac{N+1}{2M_\pi^0}\right) \ , \tag{7.25}$$

from which we conclude

$$M_\pi^{0\,2} + \frac{4\lambda}{M_\sigma^0 N} = M_\sigma^{0\,2} + \frac{4\lambda}{M_\pi^0 N}$$

$$\Rightarrow \quad M_\pi^0 = M_\sigma^0 \text{ is solution for arbitrary } \lambda. \tag{7.26}$$

Equations (7.24),(7.25), and (7.26) yield

$$M_{\pi/\sigma}^0{}^2 = -1 + \frac{4\lambda}{N}\left(\frac{2+N}{2M_{\pi/\sigma}^0}\right) \ . \tag{7.27}$$

Since $M_{\pi/\sigma}^0 > 0$, we find

result 7.6 (Masses at $T = 0$, $\varphi = 0$)

$$M_{\pi/\sigma}^0 = \frac{-3^{2/3}N^2 + 3^{\frac{1}{3}}\left(9N^2(2+N)\lambda + \sqrt{3N^6 + 81N^4(2+N)^2\lambda^2}\right)^{\frac{2}{3}}}{3N\left(9N^2(2+N)\lambda + \sqrt{3N^6 + 81N^4(2+N)^2\lambda^2}\right)^{\frac{1}{3}}} \ .$$

Result 7.6 can be used as initial value, when solving the system of equations numerically.

II) $\varphi \neq 0$

Equations (7.19) and (7.18) and, respectively, (7.20) and (7.19) and, respectively, (7.19) and (7.20) result in

result 7.7 (system of equations at the extrema, $\varphi \neq 0$)

$$M_\sigma^2 = \frac{8\lambda}{N}\varphi^2 + \frac{H}{\varphi} \ , \tag{7.28}$$

$$M_\pi^2 = \frac{H}{\varphi} - \frac{8\lambda}{N}\int\limits_{k_0} \mathcal{G}_\sigma + \frac{8\lambda}{N}\int\limits_{k_0} \mathcal{G}_\pi \ , \tag{7.29}$$

$$M_\sigma^2 - 3M_\pi^2 = 2 - \frac{8\lambda}{N}(N+2)\int\limits_{k_0} \mathcal{G}_\pi \ . \tag{7.30}$$

At $T = 0$, denoting $\varphi(T = 0)$ by f, equations (7.28) resp. (7.29) read

$$M_\sigma^{0\,2} = \frac{8\lambda}{N}f^2 + \frac{H}{f} \ , \tag{7.31}$$

$$M_\pi^{0\,2} = \frac{H}{f} + \frac{8\lambda}{N}\left(\frac{1}{2M_\pi^0} - \frac{1}{2M_\sigma^0}\right), \tag{7.32}$$

which in turn yield

$$M_\pi^{0\,2} = M_\sigma^{0\,2} + \frac{8\lambda}{N}\left(\frac{1}{2M_\pi^0} - \frac{1}{2M_\sigma^0} - f^2\right) \tag{7.33}$$

$$\Leftrightarrow \quad \lambda = \frac{N(M_\pi^{0\,2} - M_\sigma^{0\,2})}{8\left(\frac{1}{2M_\pi^0} - \frac{1}{2M_\sigma^0} - f^2\right)}$$

$$\Leftrightarrow \quad f = \pm\sqrt{\frac{N}{8\lambda}(M_\sigma^{0\,2} - M_\pi^{0\,2}) + \frac{1}{2M_\pi^0} - \frac{1}{2M_\sigma^0}} \ ,$$

whereas equation (7.30) reads

$$M_\sigma^{0\,2} - 3M_\pi^{0\,2} = 2 - \frac{4\lambda}{N}\left(\frac{N}{M_\pi^0} + \frac{2}{M_\pi^0}\right) \ . \tag{7.34}$$

Since N, λ, and H are given, we are in principle able to determine M_π^0,M_σ^0, and f from relations (7.31),(7.32) and (7.34) numerically, which in turn serve as starting values when solving the system of equations (7.7) numerically.

Figures (7.1) and (7.2) show the results in presence of the explicit symmetry breaking term, i.e., $H \neq 0$ (where we always have $\varphi \neq 0$). Solving the system of equations (7.7) numerically for small λ, we observe that the masses $M_\pi(T)$ and $M_\sigma(T)$ become more and more equal with larger λ. Now we know from equations (7.28) and (7.29) that, as soon as $M_\pi = M_\sigma$, we have $\frac{8\lambda}{N}\varphi^2 = 0$, i.e., $\phi = 0$ $\lightning$. Moreover, the pion mass cannot become larger than that for the sigma, because $M_\pi > M_\sigma$ (which implies $\int\limits_{k_0} \mathcal{G}_\pi < \int\limits_{k_0} \mathcal{G}_\sigma$) would result in $M_\pi^2 < \frac{H}{\varphi}$, due to relation (7.29). Together with relation (7.28), which says $M_\sigma^2 > \frac{H}{\varphi}$, we end up with the contradiction $M_\sigma > M_\pi$ $\lightning$. Therefore, we conclude (since there are no solutions in the case $\varphi = 0$):

There is a value λ_c for the coupling constant λ, above which no solutions exist for the case $H \neq 0$. It can be calculated numerically: in the case $N = 2$ and $H = 0.1$ we find $\lambda_c \simeq 0.24m^3$.

However, this is not the only critical value for λ. There is another one in the case $H = 0$. Consider the system of equations (7.7) in the chiral limit $H = 0$. For $T = 0$, equations (7.29) and, respectively, (7.30) yield

$$M_\pi^{0\,2} = \frac{8\lambda}{N}\left(\frac{1}{2M_\pi^0} - \frac{1}{2M_\sigma^0}\right) \tag{7.35}$$

$$\text{and} \quad M_\sigma^{0\,2} - 3M_\pi^{0\,2} = 2 - \frac{8\lambda}{N}(N+2)\frac{1}{2M_\pi^0}\ , \ \text{respectively.} \tag{7.36}$$

For given λ and N, this system of equations can be solved numerically for M_π^0 and M_σ^0. We find that only for $\lambda < \lambda_{max}$ real solutions exist. That means:

Presuming $H = 0$, there is a value λ_{max} for the coupling constant λ, above which no solutions for $\varphi \neq 0$ exist, i.e., we have solutions for $\varphi = 0$ only. It can be calculated numerically: for $N = 2$ we find $\lambda_{max} \simeq 0.115m^3$.

The numerical solutions for the case $H = 0$ are shown in figure 7.3.

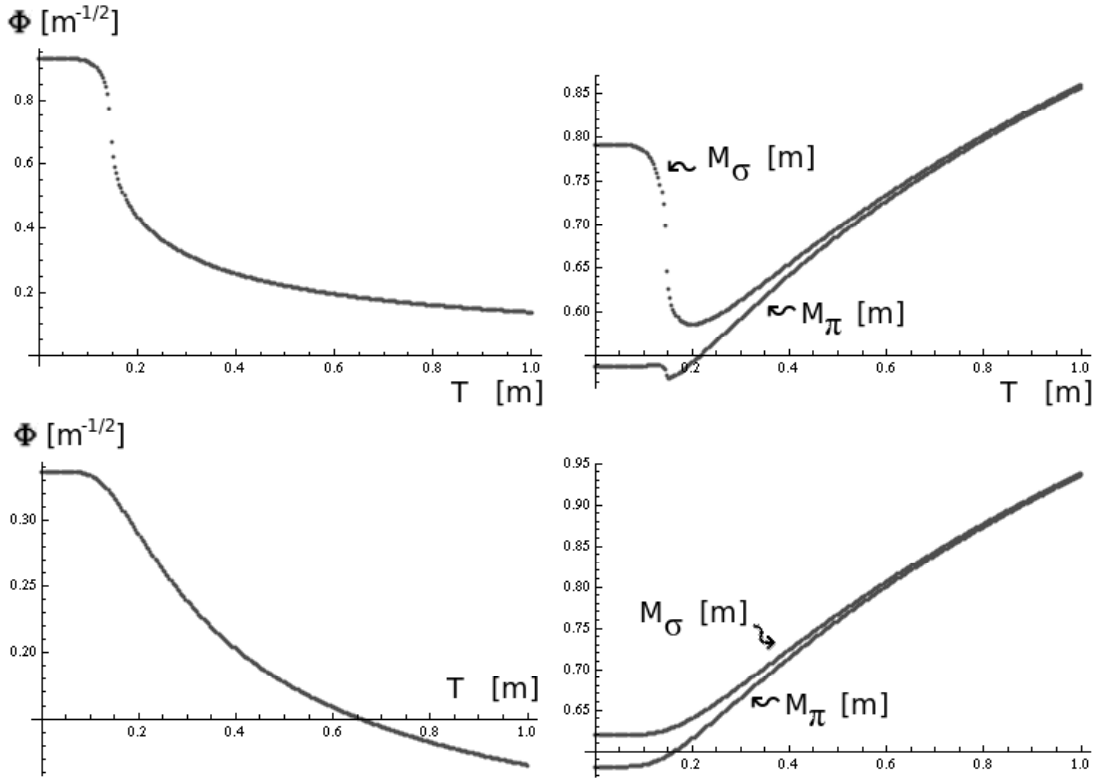


Figure 7.1: $H = 0.1m^{\frac{3}{2}}$. Top: condensate and masses at the global minimum for $\lambda = 0.15m^3$; Bottom: condensate and masses at the global minimum for $\lambda = 0.191m^3$.

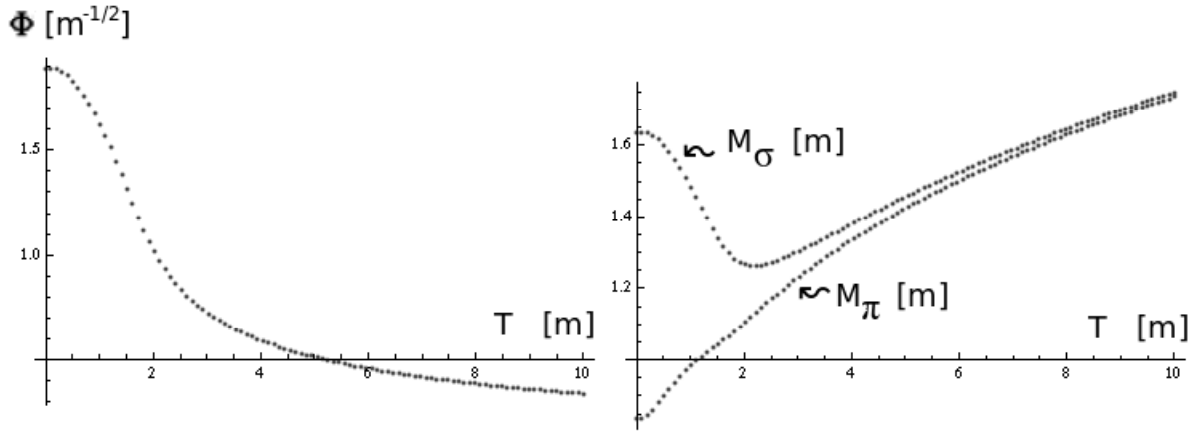


Figure 7.2: Illustration of the influence of H. Here we show $H = 1m^{\frac{3}{2}}$. Condensate and masses at the global minimum for $\lambda = 0.15m^3$.

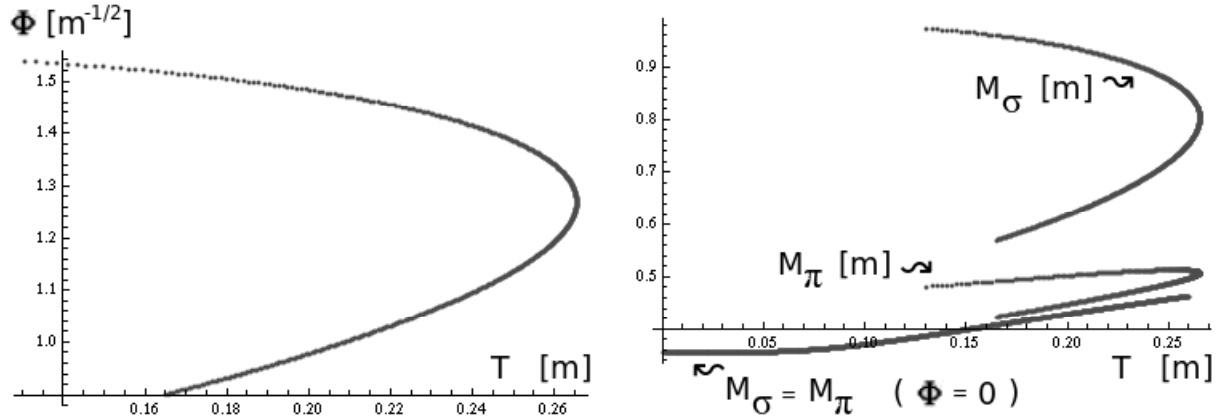


Figure 7.3: $H = 0$. Condensate and masses at the global minimum for $\lambda = 0.1m^3$. Since $0.1m^3 < \lambda_{max}$, we have solutions for $\varphi = 0$ and $\varphi \neq 0$. As one can see, the pion and the sigma mass become degenerate (i.e., become equal) for $\varphi = 0$.

7.2.2 Effective potential

As known from the sections before, we need the masses at a point ϕ if we want to calculate the effective potential. These are determined by relations (7.19) and (7.20), replacing φ by ϕ:

result 7.8 (gap equations for the sigma and the pion mass)

$$M_\sigma{}^2 = -1 + \frac{4\lambda}{N}\left[3\phi^2 + 3\int_{k_0}\mathcal{G}_\sigma + (N-1)\int_{k_0}\mathcal{G}_\pi\right] , \tag{7.37}$$

$$M_\pi{}^2 = -1 + \frac{4\lambda}{N}\left[\phi^2 + \int_{k_0}\mathcal{G}_\sigma + (N+1)\int_{k_0}\mathcal{G}_\pi\right] . \tag{7.38}$$

The effective potential is given by

result 7.9 (effective potential)

$$\begin{aligned} V(\phi) = U(\phi) &+ \frac{1}{2}\left(M_\sigma + 2T\ln\left(1-e^{-M_\sigma/T}\right)\right) + \frac{(N-1)}{2}\left(M_\pi + 2T\ln\left(1-e^{-M_\pi/T}\right)\right) \\ &+ \frac{1}{2}\left(m_\sigma^2 - M_\sigma^2\right)\int_{k_0} G_\sigma + \frac{(N-1)}{2}\left(m_\pi^2 - M_\pi^2\right)\int_{k_0} G_\pi \\ &+ 3\frac{\lambda}{N}\left[\int_{k_0} G_\sigma\right]^2 + (N+1)(N-1)\frac{\lambda}{N}\left[\int_{k_0} G_\pi\right]^2 + 2(N-1)\frac{\lambda}{N}\left[\int_{k_0} G_\sigma\right]\left[\int_{k_0} G_\pi\right] . \end{aligned}$$

Figure (7.4) shows the results in the **chiral limit** $H = 0$ for $\lambda = 10m^3$. To calculate the effective

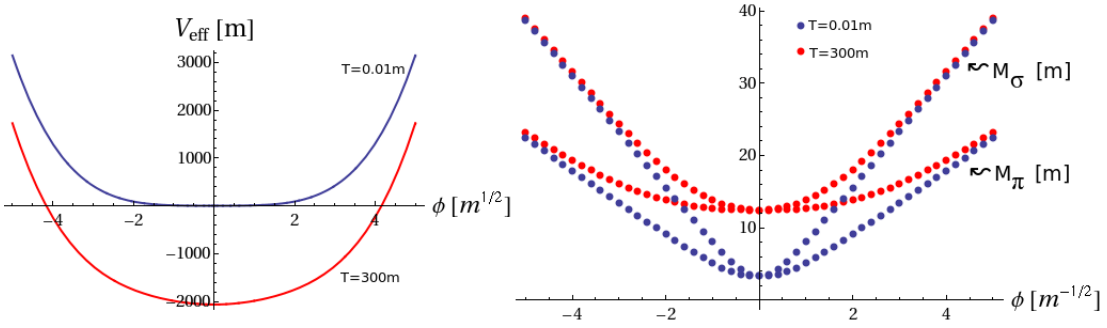


Figure 7.4: $H = 0$, effective potential and masses for temperature $T = 0.01m$ resp. $T = 300m$. Since $\lambda = 10m^3 > \lambda_{max}$, the global minimum is always at $\phi = 0$, also for $T = 0$, and therefore the effective potential at $T = 0$ does not coincide with $U(T = 0)$.

potential at its global minimum in the **chiral limit** $H = 0$, we simply have to set ϕ to zero in result (7.9). As already mentioned, we can determine it alternatively via the radial WKB method. From the radial WKB equation (2.6), we obtain the eigenvalues E_{mn}. These we use in

$$V(\Phi) = -T\ln\left(\sum_{m,n} e^{-E_{mn}/T}\right) , \tag{7.39}$$

which follows from

$$Z[J=K=0] = Tr\left(e^{-\beta H(J\hat{=}K=0)}\right) \overset{\{|E_{mn}>\}}{=} \sum_{m,n} e^{-\beta E_{mn}} \quad \text{and} \quad V(\Phi) = -T \ln Z[J=K=0] \ .$$

We begin with the calculation of the lowest eigenvalue, which we obtain for $m = n = 0$. From this one, we can determine what is the upper energy bound, above which eigenvalues can be neglected, up to a given temperature. For $\lambda = 10m^3$, $N = 2$, $H = 0$ and a maximal temperature of $T_g = 5m$, we can stop at $m = 11$, where the lowest energy eigenvalue is $E_{m=11,n=0} \simeq 52.588m$. Figure (7.5) shows the comparison between CJT and radial WKB in the **chiral limit**.

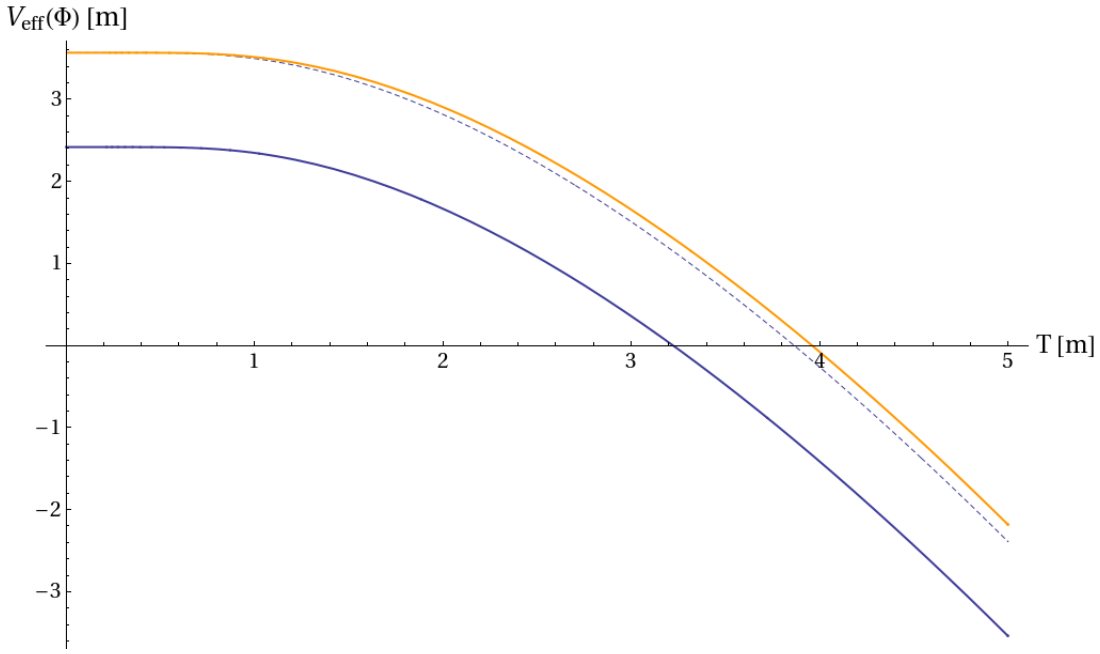


Figure 7.5: comparison of radial WKB and CJT, $\lambda = 10m^3$, $N = 2$, $H = 0$. orange: WKB, blue: CJT, dashed line: blue line shifted by a constant. We cannot compare with CJT in polar coordinates, yet. For an explanation see page 110.

Zweifel und Überzeugung sind heimlich alte Freunde. Nachts im Traum erwischt man sie schon mal beim Würfeln.
(Peter Horton)

Chapter 8

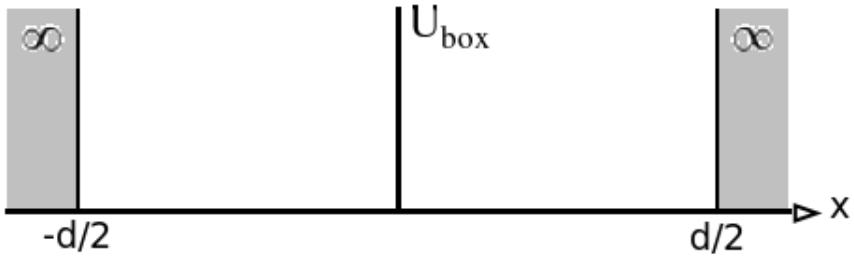
Critical remarks and the problem of infrared divergences

8.1 Free point particle...

8.1.1 ... in a box

Let us examine a simple special case, to understand the origin for the infrared divergence, namely a free point particle in a box. The classical potential is given by

$$U_{box} = \begin{cases} 0 & \text{for } x \in [-\frac{d}{2}, \frac{d}{2}] \\ \infty & \text{otherwise} \end{cases} \tag{8.1}$$



Let us begin with **Quantum Mechanics** (**QM**), for which the Lagrange function reads

$$L = \frac{1}{2}m\dot{x}^2 + U_{box} \ . \tag{8.2}$$

In chapter 6 of his textbook [42], Kleinert derives the formula (6.63) for the time-evolution amplitude:

$$\langle x_b t_b | x_a t_a \rangle_{box} = \frac{2}{d}\sum_{\nu=1}^{\infty} \sin k_\nu x_b \sin k_\nu x_a e^{-i\frac{k_\nu^2}{2m}(t_b - t_a)} \ , \tag{8.3}$$

where $k_\nu = \frac{2\pi\nu}{d}$.
From the time-evolution amplitude we calculate:

$$Tr\left(e^{-i(t_b-t_a)\hat{H}/\hbar}\right) = \int_{-d/2}^{d/2} \langle x_a t_b | x_a t_a \rangle_{box} dx_a$$

$$= \frac{2}{d}\sum_{\nu=1}^{\infty} \underbrace{\int_{-d/2}^{d/2} \left(\sin^2 \frac{2\pi\nu}{d} x_a\right) dx_a}_{=d/2} e^{-i\frac{(2\pi\nu)^2}{2md^2}(t_b-t_a)} = \sum_{\nu=1}^{\infty} e^{-i\left(\frac{2\pi}{d}\right)^2 \frac{\nu^2}{2m}(t_b-t_a)} \ . \tag{8.4}$$

The transition to **Statistical Quantum Field Theory (SQFT) in 1+0 dimensions** is straightforward:
the replacements $-it_b + it_a \to -\tau_b + \tau_a = -1/T$, $x(t) \to \phi(\tau)$ and $m \to 1$ yield the partition function

$$Z_{box} = \sum_{\nu=1}^{\infty} e^{-\left(\frac{2\pi}{d}\right)^2 \frac{\nu^2}{2T}} = \frac{1}{2}\left(-1 + \vartheta_3\left(0, e^{-\frac{2\pi^2}{Td^2}}\right)\right) \ , \tag{8.5}$$

where ϑ_3 denotes the elliptic theta function. Alternatively, we can express Z_{box} as a path integral. In the case $d \to \infty$, i.e., $[-\frac{d}{2}, \frac{d}{2}] \to [-\infty, \infty]$, the path integral is given by

$$\lim_{d\to\infty} Z_{box} = \lim_{d\to\infty} \oint_{-d/2}^{d/2} \mathcal{D}\phi e^{-\int d\tau \left(\frac{1}{2}\dot{\phi}^2 + U_{box}\right)} \ , \tag{8.6}$$

from which it becomes obvious that $d \to \infty$ corresponds to the limit $M \to 0$ in 1+0 dimensional SQFT. Do not confuse m (which denotes the mass of a quantum mechanical particle) with the mass of a particle described via Quantum Field Theory, which we denote by M.
The effective potential for (8.6) is given in the CJT formalism by

$$V_{eff} = \frac{1}{2}\int_{k_0} \ln G^{-1} + \frac{1}{2}\int_{k_0} [D^{-1}G - 1] = \frac{1}{2}\int_{k_0} \ln G^{-1} + const \ , \tag{8.7}$$

$$\text{with } \int_{k_0} \ln G^{-1} = \lim_{M\to 0}\left[M + 2T \ln\left(1 - e^{-\frac{M}{T}}\right)\right] = -\infty \ , \tag{8.8}$$

where we used result B.4 and

$$0 \stackrel{!}{=} \frac{\delta V_{eff}}{\delta G} = -\frac{1}{2}G^{-1} + \frac{1}{2}D^{-1} \quad \Rightarrow \quad D = G \ . \tag{8.9}$$

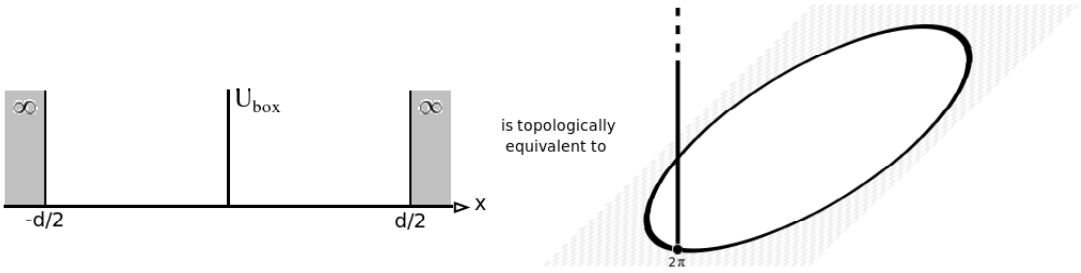
Note that

$$V_{eff} = -T \ln\left(\lim_{d\to\infty} Z_{box}\right) = -T \lim_{d\to\infty} \ln\left[\frac{1}{2}\left(-1 + \vartheta_3\left(0, e^{-\frac{2\pi^2}{Td^2}}\right)\right)\right] = -\infty \tag{8.10}$$

is consistent with (8.7).

8.1.2 ...on a circle with an infinite wall

Our potential U_{box} can be deformed to a circle of circumference d, with an infinite wall at d:



The angle φ lies in a finite interval: $\varphi \in [0, d)$.
Again, we begin with **Quantum Mechanics** (**QM**). According to topology:

$$\int_0^d \langle \varphi_a t_b | \varphi_a t_a \rangle_{wall} d\varphi_a = \int_0^d \langle \varphi_a t_b | \varphi_a t_a \rangle_{box} d\varphi_a$$

$$= \frac{2}{d} \sum_{\nu=1}^{\infty} \underbrace{\int_0^d \left(\sin^2 \frac{2\pi\nu}{d} \varphi_a \right) d\varphi_a}_{=d/2} e^{-i\frac{(2\pi\nu)^2}{2md^2}(t_b - t_a)} = \sum_{\nu=1}^{\infty} e^{-i\left(\frac{2\pi}{d}\right)^2 \frac{\nu^2}{2m}(t_b - t_a)} \ . \tag{8.11}$$

Therefore we obtain the partition function

$$Z_{wall} = \sum_{\nu=1}^{\infty} e^{-\left(\frac{2\pi}{d}\right)^2 \frac{\nu^2}{2T}} = \frac{1}{2} \left(-1 + \vartheta_3 \left(0, e^{-\frac{2}{T}\frac{\pi^2}{d^2}} \right) \right) = Z_{box} \tag{8.12}$$

in the case of **SQFT in 1+0 dimensions**. ϑ_3 denotes the elliptic theta function.

Due to the infinite wall at d, the particle moves in the interval $[0, d)$. Starting at 0, it hits the wall at d, which it cannot cross. Therefore values $\varphi + nd$ (where $n = 1, 2, \dots$) are not allowed. As well forbidden are values $\varphi - nd$, since it cannot pass through the wall from the other side either. The constraint $\varphi \in [0, 2\pi)$ is known from the definition of polar coordinates. In order to have a unique mapping from cartesian to polar coordinates, φ is limited to an interval of length 2π, for instance $[0, 2\pi)$. The interval is open, because $\varphi = 0$ and $\varphi = 2\pi$ are identical points. Although one is tended to link polar coordinates to the circle with an infinite wall, this is not correct. The point 2π is missing in polar coordinates, but the periodicity remains. Therefore we have to consider a third case.

8.1.3 ...on a circle with periodicity

Consider a free particle on a circle without a wall. $\varphi \in [0, 2\pi)$, but values $\varphi + 2\pi n$ are allowed too where φ and $\varphi + 2\pi n$ are indistinguishable.

We can use Kleinert's [42] formula (6.24), which refers to **QM**, to calculate

$$\int_0^{2\pi}\langle\varphi_a t_b|\varphi_a t_a\rangle_{cir}d\varphi_a=\sum_{l=-\infty}^{\infty}\int_0^{2\pi}\frac{1}{\sqrt{2\pi i(t_b-t_a)/m}}e^{\frac{i}{2}m\frac{(\varphi_a-\varphi_a+2\pi l)^2}{t_b-t_a}}d\varphi_a$$

$$=2\pi\sum_{l=-\infty}^{\infty}\frac{1}{\sqrt{2\pi i(t_b-t_a)/m}}e^{\frac{i}{2}m\frac{(2\pi l)^2}{t_b-t_a}}=\sqrt{\frac{2\pi m}{i(t_b-t_a)}}\vartheta_3\left(0,e^{-\frac{2\pi^2 m}{i(t_b-t_a)}}\right), \tag{8.13}$$

which yields the partition function in **SQFT in 1+0 dimensions**:

$$Z_{cir}=\sqrt{2\pi T}\vartheta_3\left(0,e^{-2\pi^2 T}\right). \tag{8.14}$$

Alternatively we could use Kleinert's formula (6.23) to calculate

$$\int_0^{2\pi}\langle\varphi_a t_b|\varphi_a t_a\rangle_{cir}d\varphi_a=\frac{1}{2\pi}\sum_{l=-\infty}^{\infty}\int_0^{2\pi}\exp\left[il(\varphi_a-\varphi_a)-i\frac{l^2}{2m}(t_b-t_a)\right] \tag{8.15}$$

$$=\sum_{l=-\infty}^{\infty}e^{-\frac{l^2}{2m}i(t_b-t_a)}=\vartheta_3\left(0,e^{-\frac{i(t_b-t_a)}{2m}}\right) \tag{8.16}$$

and

$$Z_{cir}=\vartheta_3\left(0,e^{-\frac{1}{2T}}\right)\ , \tag{8.17}$$

respectively. The expressions (8.14) and (8.17) are equal due to[1]

$$\vartheta_3(z,q)=\sqrt{\frac{i\pi}{-i\ln q}}e^{\frac{z^2}{\ln q}}\left(1+2\sum_{k=1}^{\infty}e^{\frac{k^2\pi^2}{\ln q}}\cosh\left(\frac{2k\pi z}{\ln q}\right)\right) \tag{8.18}$$

$$\Rightarrow\ \vartheta_3\left(0,e^{-\frac{1}{a}}\right)=\sqrt{\pi a}\left(1+2\sum_{k=1}^{\infty}e^{-ak^2\pi^2}\right)=\sqrt{\pi a}\vartheta_3\left(0,e^{-\pi^2 a}\right)\ . \tag{8.19}$$

Note that result 5.4 is not in contradiction with the fact that (8.17) is finite, since it only refers to polar coordinates i.e to the Lagrange function (5.12), which is different from $L_E=\frac{1}{2}\dot{\varphi}^2$.

8.2 Problems with $M=0$

8.2.1 An unsolved question

Let us turn to a free point particle on a circle with periodicity, as described in section 8.1.3. In their textbooks [42, 49], Kleinert resp. Chaichian derive the time-evolution amplitude in this case. They start with the bra-ket definition $\langle\varphi_b t_b|\varphi_a t_a\rangle_{cir}$, rewrite it in the standard way, involving the time-evolution operator and after some manipulations they derive the expression

$$\langle\varphi_b t_b|\varphi_a t_a\rangle_{cir}=\left[\prod_{j=1}^{N}\int_0^{2\pi}d\varphi_j\right]\left[\prod_{j=1}^{N+1}\sum_{m_n=-\infty}^{\infty}\frac{1}{2\pi}\right]\exp\left(i\sum_{n=1}^{N+1}\left[m_n(\varphi_n-\varphi_{n-1})-\frac{\epsilon}{\hbar}H(\hbar_n,\varphi_n)\right]\right), \tag{8.20}$$

[1] Compare with *wolfram.com, EllipticTheta[3,z,q], Series representations, Other*

which plays the role of a path integral in phase-space representation. The integration over momentum has turned into a discrete sum, which has its origin in the periodicity $\varphi = \varphi + 2\pi l$. However, there exists a form in which the sum is turned into a continuous integration over an auxiliary variable. Apart from a Kronecker delta and a sum over l, this form - (8.21) - has the same shape as the ordinary path integral for the particle on a line in phase-space representation. Kleinert presents it in formula (6.16) resp. Chaichian in formula (2.4.44):

$$\langle \varphi_b t_b | \varphi_a t_a \rangle_{cir} = \sum_{l=-\infty}^{\infty} \left[\prod_{j=1}^{N} \int_{-\infty}^{\infty} d\varphi_j \right] \left[\prod_{j=1}^{N+1} \int_{-\infty}^{\infty} \frac{dp_j}{2\pi\hbar} \right] \times$$
$$\times \exp\left(\frac{i}{\hbar} \sum_{j=1}^{N+1} \left[p_j(\varphi_j + 2\pi l \delta_{j,N+1} - \varphi_{j-1}) - \epsilon H(p_j, \varphi_j) \right] \right), \tag{8.21}$$

which we can rewrite as configuration-space path integral (compare with appendix C):

$$\langle \varphi_b t_b | \varphi_a t_a \rangle_{cir} = \sum_{l=-\infty}^{\infty} \frac{1}{\sqrt{2\pi\hbar i\epsilon/m}} \left[\prod_{j=1}^{N} \int_{-\infty}^{\infty} \frac{d\varphi_j}{\sqrt{2\pi\hbar i\epsilon/m}} \right] \times$$
$$\times \exp\left(\frac{i}{\hbar} \epsilon \sum_{j=1}^{N+1} \left[\frac{m}{2} \left(\frac{\varphi_j + 2\pi l \delta_{j,N+1} - \varphi_{j-1}}{\epsilon} \right)^2 - U(\varphi_j, t_j) \right] \right) . \tag{8.22}$$

Staring at relation (8.22), we may ask, how far away

$$\int_{-\infty}^{\infty} \langle \varphi_a t_b | \varphi_a t_a \rangle_{line} , \tag{8.23}$$

$$\text{where } \langle \varphi_a t_b | \varphi_a t_a \rangle_{line} \equiv \frac{1}{\sqrt{2\pi\hbar i\epsilon/m}} \left[\prod_{j=1}^{N} \int_{-\infty}^{\infty} \frac{d\varphi_j}{\sqrt{2\pi\hbar i\epsilon/m}} \right] \times$$
$$\times \exp\left(\frac{i}{\hbar} \epsilon \sum_{j=1}^{N+1} \left[\frac{m}{2} \left(\frac{\varphi_j - \varphi_{j-1}}{\epsilon} \right)^2 - U(\varphi_j, t_j) \right] \right) \text{ has exactly} \tag{8.24}$$

the same form as the time-evolution amplitude

for a particle on a line,

is from the expression

$$\int_{0}^{2\pi} \langle \varphi_a t_b | \varphi_a t_a \rangle_{cir} . \tag{8.25}$$

We can check that for the free case, where the time-evolution amplitudes are given by Chaichians formulas (2.4.48) and (2.4.49):

$$\langle \varphi_b t_b | \varphi_a t_a \rangle_{line} = \frac{1}{\sqrt{2\pi\hbar i(t_b - t_a)/m}} \exp\left(-\frac{m}{2\hbar} \frac{(\varphi_b - \varphi_a)^2}{i(t_b - t_a)} \right) , \tag{8.26}$$

$$\langle \varphi_b t_b | \varphi_a t_a \rangle_{circle} = \sum_{l=-\infty}^{\infty} \frac{1}{\sqrt{2\pi\hbar i(t_b - t_a)/m}} \exp\left(-\frac{m}{2\hbar} \frac{(\varphi_b + 2\pi l - \varphi_a)^2}{i(t_b - t_a)} \right) . \tag{8.27}$$

From this we calculate (denoting $t \equiv t_b - t_a$)

$$\int_{-\infty}^{\infty} \langle \varphi_a t_b | \varphi_a t_a \rangle_{line} d\varphi_a = 2\infty \frac{1}{\sqrt{2\pi\hbar i t}} \equiv \mathcal{N} \frac{1}{\sqrt{2\pi\hbar i t}} \ , \tag{8.28}$$

$$\int_{0}^{2\pi} \langle \varphi_a t_b | \varphi_a t_a \rangle_{circle} d\varphi_a = 2\pi \sum_{l=-\infty}^{\infty} \frac{1}{\sqrt{2\pi\hbar i t}} \exp\left(-\frac{m}{2\hbar} \frac{(2\pi l)^2}{it} \right) \ . \tag{8.29}$$

Via $it \to \tau = 1/T$, $\varphi(t) \to \varphi(\tau)$ (the angle becomes a field) and $m \to 1$, we obtain the partition functions in **SQFT in 1+0 dimensions**

$$Z_{ask} = \mathcal{N} \frac{1}{\sqrt{2\pi \frac{1}{T}}} \ , \tag{8.30}$$

$$Z_{cir} = \sqrt{2\pi T} \vartheta_3 \left(0, e^{-2\pi^2 T} \right) = \vartheta_3 \left(0, e^{-\frac{1}{2T}} \right) \ , \tag{8.31}$$

where we used the formula[2]

$$\frac{\vartheta_3 \left(0, e^{-\pi x} \right)}{\vartheta_3 \left(0, e^{-\frac{\pi}{x}} \right)} = \frac{1}{\sqrt{x}} \tag{8.32}$$

to rewrite the elliptic theta function ϑ_3. Note that we have set $\hbar \equiv 1$. Z_{cir} is the correct partition function for the field constrained to a circle with periodicity, and we ask how far away Z_{ask} is from Z_{cir}.
Note that in the case $T \to \infty$ we obtain

$$\lim_{T\to\infty} Z_{cir} = \lim_{T\to\infty} \sqrt{2\pi T} \ , \tag{8.33}$$

due to $\vartheta_3 \left(0, 0 \right) = 1$. Hence, if we choose the normalization constant as $\mathcal{N} \equiv 2\pi$, both partition functions are equal in this limit:

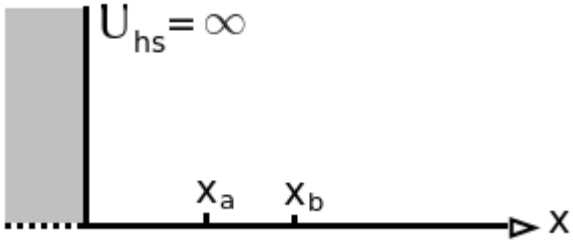
$$\lim_{T\to\infty} Z_{ask}|_{\mathcal{N}\equiv 2\pi} = \lim_{T\to\infty} Z_{cir} \ . \tag{8.34}$$

Although this is an interesting observation, for nonzero T both partition functions are different from each other. This is surprising because, according to result 5.4, one would guess that both partition functions should be equal if one chooses a certain finite value for $\mathcal{N}$. However, the fact that this is not the case is not in contradiction with result 5.4 so far. Our explanation is that starting directly from the generating functional for a periodic field, which takes values on a finite interval, is in principle different than going over to polar coordinates starting from cartesian coordinates. This becomes obvious by looking at (5.12) and (5.13). Furthermore, the case of $M = 0$ is exceptional due to the infrared divergence of the thermal integrals for $M \to 0$ (for the discussion we refer to section 8.2.3). We believe that the solution to this question requires further investigation of these divergences, as well as knowledge about the Feynman rules for path integrals with topological constraints.

[2]Compare with *wolfram.com/JacobiThetaFunctions.html*

8.2.2 A problem and its solution

In this section we want to discuss the analogon to the question raised in section 8.2.1 in case of the radial degree of freedom.
We can regard (5.13) as a path integral on a space with the topological constraint $r_n \in [0, \infty]$. Note that we can restrict the interval to $r_n \in (0, \infty]$, as the integrand vanishes for $r_n = 0$.
Now consider a free particle on a half space in **QM**.



$$U_{hs} = \begin{cases} \infty & \text{for } x = 0 \\ 0 & \text{otherwise} \end{cases}$$
$$x > 0$$

According to Kleinert (formula (6.46) of [42]) or Chaichian (formula (2.4.31) of [49]), in this case the time-evolution amplitude is given by

$$\langle x_b t_b | x_a t_a \rangle_{hs} = \frac{1}{\sqrt{2\pi i\hbar(t_b - t_a)/m}} \left[\exp\left(-\frac{m}{2\hbar} \frac{(x_b - x_a)^2}{i(t_b - t_a)} \right) - \exp\left(-\frac{m}{2\hbar} \frac{(x_b + x_a)^2}{i(t_b - t_a)} \right) \right] . \quad (8.35)$$

From (8.35) we obtain the partition function in **SQM**:

$$Z_{hs} = \left(\int_{(0)}^{\infty} \langle x\ t_b | x\ t_a \rangle_{hs}\ dx \right)_{t_b - t_a \to -i\hbar/T} . \quad (8.36)$$

With

$$\int_{(0)}^{\infty} \langle x\ t_b | x\ t_a \rangle_{hs}\ dx = \sqrt{\frac{m}{2\pi i\hbar(t_b - t_a)}} \int_{(0)}^{\infty} \left[1 - e^{\frac{i}{\hbar} 2m \frac{x^2}{(t_b - t_a)}} \right] dx ,$$

we receive

$$Z_{hs} = \sqrt{\frac{mT}{2\pi\hbar^2}} \left(\infty - \sqrt{\frac{\hbar^2\pi}{8mT}} \right) = \infty \sqrt{\frac{mT}{2\pi\hbar^2}} - \frac{1}{4\sqrt{m}} . \quad (8.37)$$

Let us compare this result with the partition function of a free particle without constraints (i.e., on the line), which is given by

$$Z_0 = \left(\int_{(0)}^{\infty} \langle x\ t_b | x\ t_a \rangle_0\ dx \right)_{t_b - t_a \to -i\hbar/T} = \int_{-\infty}^{\infty} \sqrt{\frac{mT}{2\pi\hbar^2}}\, dx = \infty \sqrt{\frac{mT}{2\pi\hbar^2}} , \quad (8.38)$$

where we inserted the time-transition amplitude for the free particle on the line,

$$\langle x_b t_b | x_a t_a \rangle_0 = \frac{1}{\sqrt{2\pi i\hbar(t_b - t_a)/m}} \exp\left(-\frac{m}{2\hbar} \frac{(x_b - x_a)^2}{i(t_b - t_a)} \right) , \quad (8.39)$$

which is the same as (8.26), we simply took account of the notation by changing φ to x.
Since

$$\frac{1}{4\sqrt{m}} \ll \infty \sqrt{\frac{mT}{2\pi\hbar^2}} \ , \tag{8.40}$$

we can identify Z_{hs} with Z_0. Since $(0,\infty]$ is homeomorphic to the real numbers $\mathbb{R}$, our conclusion is at first sight not in contradiction with topology.
Setting $m = 1$ in (8.37) and (8.38), we obtain the partition function for massless scalar **SQFT in 1+0 dimensions** where the field is constrained to the interval $(0,\infty]$ and, respectively, where the field is not constrained. Let us summarize our conclusion in the context of massless scalar **SQFT in 1+0 dimensions**:
If we want to extend the integration interval in the constrained partition function

$$Z_{hs} = \oint_{(0)}^{\infty} \mathcal{D}\phi e^{-\int_0^{1/T} d\tau \ \frac{1}{2}\dot{\phi}^2} \tag{8.41}$$

from $(0,\infty]$ to $[-\infty,\infty]$, we have to add a topological correction term $S_{top,E}$ to the Euclidean action $S_{0,E} = \int_0^{1/T} d\tau \ \frac{1}{2}\dot{\phi}^2$ (compare formula (2.4.30) of [49]):

$$Z_{hs} = \oint_{-\infty}^{\infty} \mathcal{D}\phi e^{-S_{0,E}-S_{top,E}} = \oint_{-\infty}^{\infty} \mathcal{D}\phi e^{-\int_0^{1/T} d\tau \ \frac{1}{2}\dot{\phi}^2 \ - \int_0^{1/T} d\tau \ \left[-i\pi \frac{\partial}{\partial\tau}\theta(-\phi)\right]} \ , \tag{8.42}$$

where θ is the Heaviside step function. Due to (8.40), we conclude that $S_{top,E}$ can be neglected. This conclusion can be applied to result 5.1, which reads for $U = 0$:

$$Z = \sum_{m=-\infty}^{\infty} \left[\prod_{n=1}^{N+1} \int_{(0)}^{\infty} \frac{dr_n}{\sqrt{2\pi\varepsilon}}\right] \left[\prod_{n=1}^{N+1} e^{-\frac{1}{2\varepsilon}(r_n - r_{n-1})^2}\right] \exp\left(\delta(0)\varepsilon \sum_{n=1}^{N+1} \ln \widetilde{I}_m\left(\frac{r_n r_{n-1}}{\varepsilon}\right)\right) \ ,$$

where we used that it is possible to take the zero out of the integration interval. As discussed in section 5.2, the delta function vanishes in dimensional regularization, $\delta(0) = 0$, so that all terms are independent of m, and we can do the sum:

$$Z \sim \left[\prod_{n=1}^{N+1} \int_{(0)}^{\infty} \frac{dr_n}{\sqrt{2\pi\varepsilon}}\right] e^{-\varepsilon \sum_{n=1}^{N+1} \frac{1}{2} \frac{(r_n - r_{n-1})^2}{\varepsilon^2}} \ , \tag{8.43}$$

with an infinite proportionality constant. As we have shown above, it is possible to extend the range of integration to the full real line, since the topological correction term can be neglected:

$$Z \sim \left[\prod_{n=1}^{N+1} \int_{-\infty}^{\infty} \frac{dr_n}{\sqrt{2\pi\varepsilon}}\right] e^{-\varepsilon \sum_{n=1}^{N+1} \frac{1}{2} \frac{(r_n - r_{n-1})^2}{\varepsilon^2}} \ . \tag{8.44}$$

This affirms result 5.6.

8.2.3 Speculations about how to cope with $M = 0$ in 1+0 dimensions

The question is how to deal with

$$\int_{k_0} \frac{1}{-Z^2 k_0^2} = T \sum_{n=-\infty}^{\infty} \frac{1}{Z^2 (2\pi n)^2 T^2} \ , \tag{8.45}$$

$$\int_{k_0} \frac{k_0^2}{-Z^2 k_0^2} = \frac{1}{Z^2} T \sum_{n=-\infty}^{\infty} \frac{-\omega_n^2}{\omega_n^2} \tag{8.46}$$

and

$$\int_{k_0} \ln\left(-Z^2 k_0^2\right) = T \sum_{n=-\infty}^{\infty} \ln\left(Z^2 \left(2\pi n\right)^2 T^2\right) \ . \tag{8.47}$$

Section 8.1.1 shows that for $M \to 0$ the infrared divergence of result B.4 is real and does not require special handling. However, this does not exclude that $\int\limits_{k_0} \ln G^{-1}$ is finite for $M = 0$. Note that the derivation of results B.1 and B.4 is only valid for $M \neq 0$. Even in 1+3 dimensions the problem of handling massless fields is problematic in certain cases. For a discussion of inconsistencies arising in the extension of the method of dimensional regularization to massless fields, we refer to Ref. [44]. Regarding the chiral limit for polar coordinates in 1+0 dimensions (see page 110) we need to know whether such a finite expression exists or not, and how it looks like. Therefore consider the partition function

$$Z = \oint_{-\infty}^{\infty} \mathcal{D}\phi e^{-\int d\tau \frac{1}{2}\dot{\phi}^2} \ .$$

The effective potential for Z is given in the CJT formalism by

$$V_{eff} = \frac{1}{2} \int_{k_0} \ln G^{-1} + \frac{1}{2} \int_{k_0} [D^{-1} G - 1] = \frac{1}{2} \int_{k_0} \ln G^{-1} + const \ , \tag{8.48}$$

where we used

$$0 \stackrel{!}{=} \frac{\delta V_{eff}}{\delta G} = -\frac{1}{2} G^{-1} + \frac{1}{2} D^{-1} \quad \Rightarrow \quad D = G \ .$$

On the other hand we know from (8.30) that Z can be written as (we simply have to rename φ by ϕ)

$$Z = \mathcal{N} \sqrt{\frac{T}{2\pi}} \ ,$$

from which follows

$$V_{eff} = -T \ln Z = -T \ln\left(\mathcal{N} \sqrt{\frac{T}{2\pi}}\right) \ . \tag{8.49}$$

Comparing (8.48) and (8.49), the only possibility for $\int\limits_{k_0} \ln G^{-1}$ to be finite is to choose a non-infinite normalization constant $\mathcal{N} \equiv \mathcal{N}_0$. The question is then, if there is a reasonable procedure

how to obtain $-T \ln\left(\mathcal{N}_0 \sqrt{T/2\pi}\right)$ from $\frac{1}{2} \int\limits_{k_0} \ln G^{-1} = \frac{1}{2} T \sum_{n=-\infty}^{\infty} \ln\left[(2\pi n)^2 T^2\right]$.
Note that if one wants to derive results B.1 and B.4, one is forced to drop addends which are independent of M. Hence, one could argue that one should proceed in the same way in the case of $M = 0$. This means dropping all addends which do not depend on M, i.e., setting (8.45),(8.46), and (8.47) equal zero. One might ask why the massless pions occurring in the chiral limit for polar coordinates in 1+0 dimensions (see page 110) would not contribute to the effective potential, which is irritating since the lightest particles should dominate. One could insist that in 1+0 dimensions one should not identify massless modes with observable particles.

8.3 Remarks on the use of Veltman's rule

When using Veltman's rule one has to be careful. It would be a mistake to assume that one could get rid of everything in front of the exponential in a generating functional. Consider a general generating functional with an integrand of the form $A \exp\left(-\int_X S\right)$. If we want to argue that A has no influence in perturbation theory, due to the trick $A = \exp(\ln A)$ and $\delta(0) \equiv 0$ in dimensional regularization, according to our understanding one has to take care of two important points. First, one should ensure that $\delta(0) \int_X \ln A$ written in the *discretized* version (lattice version) vanishes in the continuum limit $\varepsilon \to 0$. This is what we did for example in case of section 5.4.2, where the role of A is played by a modified Heavyside step function. Second, one should assure oneself that dimensional regularization is applicable. To give an example, dimensional regularization is problematic in gauge theories involving chiral transformations [44, 50]. Consider for example $SU(2)$ gauge theory. In 1981 Weiss discussed the $SU(2)$ gauge theory without fermions and the influence of the invariant measure. The invariant measure of a group, also known as *Haar measure* or *Hurwitz measure*, appears in the generating functional of a gauge-field theory. Due to gauge invariance and the hence resulting redundant information when integrating over gauge-equivalent gauge-fields, the invariant measure comes into play. In the above notation, this measure plays the role of A. Using the trick $A = \exp(\ln A)$, Weiss came to the conclusion that the contribution of A is canceled by terms regarded as gauge artifacts [51]. In 1995, Sailer, Schäfer and Greiner argued that this cancellation does not take place in second order of the coupling [52]. However, in 1997, Borisenko and Boháčik relativized this drastic statement by reviewing the controversial subject. They conclude that at least in dimensional regularization, the invariant measure does not contribute to the generating functional and can be omitted from the very beginning [53]. However, we want to point out a dilemma: chiral symmetry is not restored at high temperatures within Chiral Perturbation Theory in the mean-field approximation, if the invariant measure is neglected [54].
We suspect that, one day, one might be able to prove that ghost contributions cancel with gauge-fixing contributions by the help of dimensional regularization, which would correspond to the fact that a gauge transformation should not change physics. However, the method of dimensional regularization has to be developed further for this purpose, as revealed for example by the difficulties with chiral fermions.

Chapter 9

The path integral collapse

The path integral in polar coordinates, or more precisely the radial path integral, kept people occupied for a long time. And still, there seem to exist different views towards the subject. Just recently (November 2007) a paper was published by Jackiw [55], which assumes that the resolution of the problem was given by Edwards and Gulyaev [56]. However, Kleinert already criticized the use of the asymptotic expansion for the slightly modified Bessel function in the edition of his textbook published in 1993 [57]. According to Steiner [58], Langguth and Inomata were the first to realize that the asymptotic expansion used by Edwards and Gulyaev is problematic. They recognized that the expansion is applicable only if $Re[z] > 0$, which is not fullfilled for the Minkowskian path integral. They solved the problem by an analytic continuation of the particle mass $m \to m + i\eta$. In turn, Steiner recognized that one simply has to use the Euclidean form of the path integral, as the Wick rotation $t \to -i\tau$ yields $z \in \mathbb{R}$ as well. However, according to Steiner, Grosche [59], and Kleinert, there is another problem left, which was ignored by Langguth and Inomata, which is known as *path integral collapse*. According to these authors, the path integral collapse is a consequence of misusing the asymptotic expansion (9.2).
In the following, we want to put a more optimistic view of the problem up to discussion. Our motivation was the question: Why should the correct centrifugal barrier come out of the asymptotic expansion, if one is not allowed to use it? Note that the discussion on the basis of the transition amplitude is completely analogous to that for the generating functional (compare with appendix A).
We should mention that the statements in this appendix should be regarded as interim results which are still under examination, not the least because (9.2) resp. (9.15) only show asymptotic behavior.

Usual asymptotic expansion

The starting point is result 5.1:

$$Z = \sum_{m=-\infty}^{\infty} \left[\prod_{n=1}^{N+1} \int_0^{\infty} \frac{dr_n}{\sqrt{2\pi\varepsilon}} \right] \left[\prod_{n=1}^{N+1} e^{-\varepsilon U(r_n, -i\tau_n)} \right] \left[\prod_{n=1}^{N+1} e^{-\frac{1}{2\varepsilon}(r_n - r_{n-1})^2} \right] \left[\prod_{n=1}^{N+1} \widetilde{I}_m \left(\frac{r_n r_{n-1}}{\varepsilon} \right) \right] .$$

We can easily switch to its Minkowskian version by the inverse Wick rotation $\tau \to it$, i.e., $\varepsilon \to i\epsilon$:

$$Z = \sum_{m=-\infty}^{\infty} \left[\prod_{n=1}^{N+1} \int_0^{\infty} \frac{dr_n}{\sqrt{2\pi i\epsilon}} \right] \left[\prod_{n=1}^{N+1} e^{-i\epsilon U(r_n,t_n)} \right] \left[\prod_{n=1}^{N+1} e^{i\frac{1}{2\epsilon}(r_n - r_{n-1})^2} \right] \left[\prod_{n=1}^{N+1} \widetilde{I}_m \left(\frac{r_n r_{n-1}}{i\epsilon} \right) \right] . \tag{9.1}$$

This is the version encountered in QFT[1] in the first place. In SQFT (resp. SQM) on the other hand, we would naturally start with the Euclidean version. Nevertheless, using the Euclidean version instead of the Minkowskian does not necessarily mean considering a statistical theory. Using (9.1) in QFT (resp. QM) is nothing but a mathematical trick. After having obtained a certain result, one is usually allowed to apply the inverse Wick rotation to the result, in order to obtain the quantum-field theoretical outcome. However, we have to be careful with this statement in our case, since the asymptotic expansion breaks down because of $Re[\frac{r_n r_{n-1}}{i\epsilon}] = 0$. As we are interested in the Euclidean path integral only, we do not have to investigate this question further.

The crucial point is the asymptotic expansion of the slightly modified Bessel functions (see formula 9.7.1 of [60]):

$$\widetilde{I}_m(z) \equiv I_m(z)\, e^{-z}\sqrt{2\pi z} \sim 1 - \frac{m^2 - \frac{1}{4}}{2z} + \frac{(m^2 - \frac{1}{4})(m^2 - \frac{9}{4})}{2!4z^2} - \cdots \tag{9.2}$$

$$\text{has asymptotic behavior for } |arg(z)| < \frac{1}{2}\pi \quad (\Leftrightarrow \ Re(z) > 0) \quad \text{and large } |z|. \tag{9.3}$$

As one can see, we cannot apply (9.2) to $\widetilde{I}_m\left(\frac{r_n r_{n-1}}{i\epsilon}\right)$, since $Re\left[\frac{r_n r_{n-1}}{i\epsilon}\right] = 0$. For $\widetilde{I}_m\left(\frac{r_n r_{n-1}}{\varepsilon}\right)$ in contrast, this is possible, if $|r_n r_{n-1}/\varepsilon|$ is large. Since $\varepsilon \to 0$, this is fulfilled for $r_j \neq 0 \ \forall\ j$. As the zero is part of the integration interval $[0, \infty]$, Grosche, Kleinert, and Steiner conclude that the asymptotic expansion breaks down, because there exist paths with $r_j = 0$.

Although this is true, looking at (5.15),

$$\begin{aligned} Z = &\left[\prod_{n=1}^{N+1} \int_0^{\infty} \frac{dr_n r_n}{\sqrt{2\pi\varepsilon}} \right] \left[\prod_{n=1}^{N+1} \int_0^{2\pi} \frac{d\varphi_n}{\sqrt{2\pi\varepsilon}} \right] \left[\prod_{n=1}^{N+1} e^{-\varepsilon\frac{1}{2}\frac{r_n^2 + r_{n-1}^2}{\varepsilon^2}} \right] \times \\ &\times \left[\prod_{n=1}^{N+1} \exp\left(\varepsilon \frac{1}{2} \frac{2 r_n r_{n-1} \cos(\varphi_n - \varphi_{n-1})}{\varepsilon^2} \right) \right] \left[\prod_{n=1}^{N+1} e^{-\varepsilon U(r_n)} \right] , \end{aligned}$$

reveals that we are able to take the zero out of the integration interval, i.e., $[0, \infty] \to (0, \infty]$. To make this clear, let us write the above expression in the abstract form

$$Z = \int_0^{\infty} \frac{dr_0 r_0}{\sqrt{2\pi\varepsilon}} \cdots \int_0^{\infty} \frac{dr_j r_j}{\sqrt{2\pi\varepsilon}} \cdots \int_0^{\infty} \frac{dr_{N+1} r_{N+1}}{\sqrt{2\pi\varepsilon}} F(r_1, \cdots, r_j, \cdots, r_{N+1}, \varepsilon). \tag{9.4}$$

Due to the improper integral

$$\int_a^{\infty} f(x)\, dx \equiv \lim_{b\to\infty} \int_a^{b} f(x)\, dx \ , \tag{9.5}$$

[1] or QM respectively, if we replace $1 \to m$, $\phi \to x$.

and[2]

$$\lim_{r_j \to 0} F(r_1, \cdots, r_j, \cdots, r_{N+1}, \varepsilon) \text{ is finite for finite } r_{k \neq j} \text{ and } \varepsilon \neq 0 \ ,$$

we are able to change the integration interval to $(0, \infty]$. Since (9.5) is only a limit, one exact zero, $r_j = 0$, is enough for the integrand to vanish, due to the r_j in (9.4).
Now, having eliminated 0 from the integration intervals, we are allowed to use the asymptotic expansion (9.2). Proceeding as usual,

$$\widetilde{I}_m(z) \equiv I_m(z)\, e^{-z}\sqrt{2\pi z} \sim 1 - \frac{m^2 - \frac{1}{4}}{2z} + \cdots \simeq e^{-\frac{m^2-\frac{1}{4}}{2z}} + \cdots \ , \tag{9.6}$$

we obtain the criticized version of the radial path integral:

$$Z = \sum_{m=-\infty}^{\infty} \left[\prod_{n=1}^{N+1} \int_{(0)}^{\infty} \frac{dr_n}{\sqrt{2\pi\varepsilon}} \right] \left[\prod_{n=1}^{N+1} e^{-\varepsilon U(r_n, -i\tau_n)} \right] \left[\prod_{n=1}^{N+1} e^{-\frac{1}{2\varepsilon}(r_n - r_{n-1})^2} \right] \left[\prod_{n=1}^{N+1} e^{-\frac{1}{2}\varepsilon \frac{m^2-\frac{1}{4}}{r_n r_{n-1}}} \right]$$

$$\Leftrightarrow$$

$$Z = \sum_{m=-\infty}^{\infty} \left[\prod_{n=1}^{N+1} \int_{(0)}^{\infty} \frac{dr_n}{\sqrt{2\pi\varepsilon}} \right] \exp\left\{ -\varepsilon \sum_{n=1}^{N+1} \left[\frac{1}{2} \frac{(r_n - r_{n-1})^2}{\varepsilon^2} + U(r_n, -i\tau_n) + \frac{1}{2} \frac{m^2 - \frac{1}{4}}{r_n r_{n-1}} \right] \right\} \ , \tag{9.7}$$

which becomes

$$Z = \sum_{m=-\infty}^{\infty} \oint_{(0)}^{\infty} \mathcal{D}r \ \exp\left\{ -\int_{\tau_a}^{\tau_b} d\tau \left[\frac{1}{2}\dot{r}^2 + U(r, -i\tau) + \frac{1}{2} \frac{m^2 - \frac{1}{4}}{r^2} \right] \right\} \equiv \sum_m Z_m \tag{9.8}$$

in the continuum limit $\varepsilon \to 0$. In the case of SQFT, we have $\tau_a = 0$ and $\tau_b = 1/T$.
The fact that $Z_{m=0} = \infty$ (due to $+\frac{1}{2}\frac{1/4}{r^2} \to \infty$, for $r \to 0$) is known as *path integral collapse*.
Note that expression (9.8) is not in conflict with

$$Z = \sum_{m=-\infty}^{\infty} \left[\prod_{n=1}^{N+1} \int_{(0)}^{\infty} \frac{dr_n}{\sqrt{2\pi\varepsilon}} \right] \left[\prod_{n=1}^{N+1} e^{-\varepsilon U(r_n, -i\tau_n)} \right] \left[\prod_{n=1}^{N+1} e^{-\frac{1}{2\varepsilon}(r_n - r_{n-1})^2} \right] \left[\prod_{n=1}^{N+1} \widetilde{I}_m\left(\frac{r_n r_{n-1}}{\varepsilon} \right) \right] \ , \tag{9.9}$$

which we want to explain in the following.
Because of

$$\widetilde{I}_m(0) = I_m(0)\sqrt{2\pi \cdot 0}\ e^{-0} = 0 \ , \tag{9.10}$$

$$\lim_{z \to \infty} \widetilde{I}_m(z) = 1 \tag{9.11}$$

and

$$\lim_{\varepsilon \to 0} \frac{r_n r_{n-1}}{\varepsilon} = \infty \quad \forall r_n, r_{n-1} \in (0, \infty] \ , \tag{9.12}$$

[2] assuming that $U(0)$ is not $-\infty$.

we are able to rewrite (9.9) as follows:

$$Z = \sum_{m=-\infty}^{\infty} \left[\prod_{n=1}^{N+1} \int_0^{\infty} \frac{dr_n}{\sqrt{2\pi\varepsilon}} \right] \left[\prod_{n=1}^{N+1} e^{-\varepsilon U(r_n,-i\tau_n) - \frac{1}{2\varepsilon}(r_n - r_{n-1})^2} \right]$$
$$= \mathcal{N} \left[\prod_{n=1}^{N+1} \int_0^{\infty} \frac{dr_n}{\sqrt{2\pi\varepsilon}} \right] \left[\prod_{n=1}^{N+1} e^{-\varepsilon U(r_n,-i\tau_n) - \frac{1}{2\varepsilon}(r_n - r_{n-1})^2} \right] , \tag{9.13}$$

where $\mathcal{N}$ is a countably infinite constant. Since both, (9.8) as well as (9.13), are infinite, there is no contradiction so far.

Another asymptotic expansion

Last but not least, we want to present an alternative approach, which is consistent with (9.13). The difficulties with $m = 0$ are circumvented by using an alternative asymptotic series expansion instead of (9.2), namely:[3]

$$I_m(z) \sim \frac{e^z + e^{i\pi m - z}}{\sqrt{2\pi z}} , \quad \frac{-\pi}{2} < arg[z] \leq \frac{\pi}{2} , \quad |z| \to \infty. \tag{9.14}$$

Using (9.10) and

$$\frac{e^z + e^{i\pi m - z}}{\sqrt{2\pi z}} \cdot \sqrt{2\pi z}\ e^{-z} = 1 + i e^{i\pi m} e^{-2z} = 1 + i(-1)^m e^{-2z} ,$$

we obtain

$$\widetilde{I}_m(z) \begin{cases} = 0 & \text{for } z = 0 \\ \sim 1 + i(-1)^m e^{-2z} & \text{for } \frac{-\pi}{2} < arg[z] \leq \frac{\pi}{2} , \ |z| \to \infty \ . \end{cases} \tag{9.15}$$

Since $z = \frac{r_n r_{n-1}}{\varepsilon} \in \mathbb{R}$ in our case, we are allowed to use (9.15). Note that the larger $z \in \mathbb{R}$, the smaller the spurious imaginary part. The starting point is again result 5.1. Let us consider the product of the slightly modified Bessel functions in this expression. Because of

$$\lim_{\varepsilon \to 0} \frac{r_n r_{n-1}}{\varepsilon} = \begin{cases} 0 & \text{if } \ r_n = 0 \ \vee r_{n-1} = 0 \\ \infty & \text{if } \ r_n \neq 0 \ \wedge \ r_{n-1} \neq 0 \ , \end{cases} \tag{9.16}$$

we conclude that either the whole product vanishes, namely if at least one of the points $r_0, r_1, \cdots, r_{N+1}$ is zero, or we are allowed to apply the case $|z| \to \infty$ in (9.15) to each of the $N+1$ factors:

$$\left[\prod_{n=1}^{N+1} \widetilde{I}_m \left(\frac{r_n r_{n-1}}{\varepsilon} \right) \right] \overset{\varepsilon \to 0}{=}$$

$$\begin{cases} 0 & \text{if at least one point zero} \\ \left[1 + i(-1)^m e^{-2\frac{r_1 r_0}{\varepsilon}} \right] \cdots \left[1 + i(-1)^m e^{-2\frac{r_{N+1} r_N}{\varepsilon}} \right] & \text{else} \end{cases} .$$

With

$$\lim_{\varepsilon \to 0} \left[1 + i(-1)^m e^{-2\frac{r_n r_{n-1}}{\varepsilon}} \right] = 1 \quad \text{for } r_n \neq 0 \ \wedge \ r_{n-1} \neq 0 , \tag{9.17}$$

[3] Compare with http://functions.wolfram.com/Bessel-TypeFunctions/BesselI/06/02/02/02/

we finally conclude

$$\lim_{\varepsilon\to 0}\left[\prod_{n=1}^{N+1}\widetilde{I}_m\left(\frac{r_n r_{n-1}}{\varepsilon}\right)\right]=\begin{cases}0 & \text{if at least one point zero}\\ 1 & \text{else}\end{cases}\quad .$$

Accordingly, the integrand in result 5.1 either vanishes, namely for paths $(r_0, r_1, \cdots, r_{N+1})$ where at least one of the points is zero (see fig.9.1), or the integrand is simply

$$\left[\prod_{n=1}^{N+1} e^{-\varepsilon U(r_n,-i\tau_n)}\right]\left[\prod_{n=1}^{N+1} e^{-\frac{1}{2\varepsilon}(r_n-r_{n-1})^2}\right]\ ,$$

since the slightly modified Bessel functions yield a factor of 1 in the continuum limit. So again, we end up exactly with (9.13).

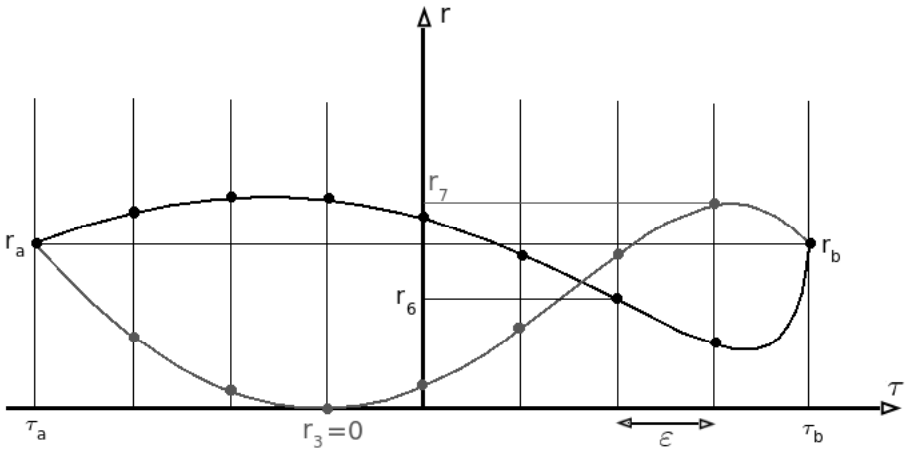


Figure 9.1: Illustration of two paths in the discussed path integral. The path which touches the τ-axis at one of its points does not contribute to the path integral.

Wer einmal nicht mehr irrt, der hat auch zu arbeiten aufgehört.

(Max Planck)

Chapter 10

Conclusions and Outlook

Conclusions

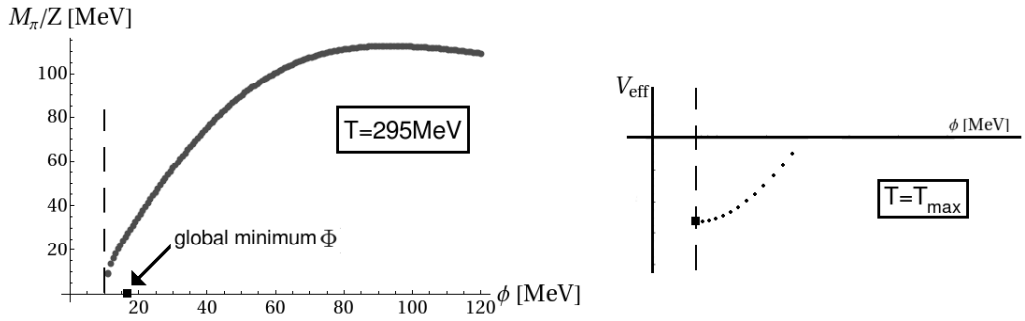
In this thesis, we studied several models at nonzero temperature. In particular, we examined a Z_2-symmetric toy model and the $O(2)$ model as well as questions arising from these studies. Our CJT calculations are in Hartree approximation, i.e., only double-bubble diagrams are taken into account in V_2.

In the case of our Z_2-symmetric toy model with the one-minimum classical potential no phase transition occurs, both in 1+3 dimensions as well as in 1+0 dimensions. In the latter case, we compared the effective potential at its global minimum (the negative pressure) calculated in the CJT formalism and via the WKB method. Whereas the difference between the WKB and the CJT result increases continuously with temperature T, the ratio is nearly one and gets even closer to one at high temperature. The offset at $T = 0$ gets larger with increasing coupling constant λ, which we can trace back to the terms in the thermal integrals which would require renormalization in 1+3 dimensions. In 1+0 dimensions they are finite and cannot be dropped, as the analytic comparison between CJT and WKB for the harmonic oscillator-like potential shows. This is also verified in further numerical comparison between WKB and CJT. We analyzed a large range in the coupling, from very small to very high, and we ensured that we took enough eigenvalues into account in the WKB method.

In case of the double-well classical potential, we encounter a first-order phase transition in 1+0 dimensions. It is an interesting observation that the CJT tree-level potential differs from the classical potential the larger the coupling λ. The terms in the thermal integrals which would require renormalization in 1+3 dimensions are responsible for this. Whereas for example the coupling λ in case of the $O(2)$ model in 1+3 dimensions can be fixed by identifying the tree-level masses with the observed vacuum masses of the sigma and the pion respectively, in 1+0 dimensions it is a free parameter. We find that above the critical value $\lambda_{cr} = \frac{N}{9}\sqrt{\frac{2}{3}}$ the tree-level potential (the effective potential for $T \to 0$) is completely deformed to a one-minimum potential without other extrema. Accordingly, the critical temperature at which the phase transition takes place decreases until it becomes zero at about $\lambda_{cr} \simeq 0.885\lambda_{cr}$. We calculated the effective potential at its global minimum for different values of λ. The comparison between CJT and WKB is complicated by tunneling. Although we took into account the tunneling in our WKB

calculation, we used an approximation which is problematic if there exist energy eigenvalues near E_{max}. However, in the case where all eigenvalues are far enough away from E_{max}, the agreement between WKB and CJT is just as good as in the case of the one-minimum potential.
Regarding the cartesian 1+0 dimensional $O(2)$ model in case of the double-well potential, in contrast to 1+3 dimensions we can vary λ and H independently from each other. In the case of explicitly broken symmetry, $H \neq 0$, we find that no solutions exist at all above some critical value for λ. In the chiral limit, $H = 0$, we still have solutions, but we observe a behavior similar to the case of the Z_2-symmetric double-well potential. Above another critical value for the coupling, the tree-level potential is deformed to a one-minimum potential without other extrema, and no phase transition can occur. Although this behavior seems to be pathologic at first look, the effective potential at its global minimum calculated via CJT is again in good agreement with the result from radial WKB. The behavior of the global minimum (i.e., the condensate) strongly depends on the choice for the values of H and λ. Although we only found crossover phase transitions so far, we believe that for other choices we should be able to obtain a second-order phase transition. In accordance with the results in 1+3 dimensions [45], the masses of the sigma and the pion become closer and closer for decreasing condensate. In the chiral limit, $H = 0$, the phase transition is of first order and the masses are degenerate at the critical temperature.
Note that we performed our CJT calculations in the Hartree approximation, which is nearly a HTL approximation (up to neglected sunset diagrams in V_2), i.e., nearly a complete one-loop calculation. The loop expansion is equivalent to an expansion in powers of $\hbar$, and accordingly the WKB approximation corresponds to one-loop order. From our comparison of the effective potential at its global minimum in the CJT formalism and the WKB approximation, we conclude that not only the HTL approximation is meaningful in the high-temperature limit but also the Hartree approximation.
Our major interest was in the $O(2)$ model with the fields treated as polar coordinates. In 1+3 dimensions we dropped the terms in the thermal integrals which would require renormalization, since the results for the cartesian case suggest that these terms do not influence the results significantly. We applied the CJT formalism in 1+3 dimensions as well as in 1+0 dimensions using the same Feynman rules as known from cartesian coordinates and neglecting the Jacobian from the beginning. Let us first summarize the results for 1+3 dimensions. We calculated the pion mass, the sigma mass, the chiral condensate and the effective potential at the global minimum (in the vacuum) as well as the "masses" and the effective potential away from the global minimum, i.e., at an arbitrary value for the "order parameter in the presence of a source J", $|\vec{\phi}|$, in radial ϕ_1-direction. Although the results for $H \neq 0$ look like a weak first-order phase transition at first glance, they do not describe a phase transition at all, since the condensate does not vanish. Instead, there is a maximal temperature T_{max} above which no physical solutions exist. To be more precise, near the would-be critical temperature, there is a region about the origin where the pion "mass" becomes imaginary. As the global minimum of the effective potential (the condensate) decreases, it hits this region at T_{max}. This situation is illustrated in figure 10.1, and we trace the reason back to the terms

$$\frac{1}{2}\frac{\sigma^2}{\phi^2}\partial_\mu\pi\partial^\mu\pi \quad \text{and} \quad H\sigma\cos\frac{\pi}{\phi} \tag{10.1}$$



(a) The global minimum approaches the ill-defined region.

(b) The situation at T_{max}.

Figure 10.1: Why no physical solutions exist above T_{max}.

appearing in the Lagrangian (6.4). As the explicit CJT calculation verifies, the divergence arising from $\phi \to 0$ is pathological. In the chiral limit however, the pion mass is always zero and all spurious terms in the system of equations vanish. We obtained a first-order phase transition at $T^* = 2.787 f_\pi$. Goldstone's theorem is respected in the phase of spontaneously broken symmetry since the pion mass is always zero, however the pion and the sigma mass do not become degenerate above T^*.

We thought that an analysis of the polar $O(2)$ model in 1+0 dimensions could help to understand the origin of the spurious behavior, since in 1+0 dimensions the thermal integrals become simple algebraic expressions. Indeed, for $H \neq 0$ the same problem occurs as in 1+3 dimensions. In 1+0 dimensions it makes no sense to identify the tree-level masses with the physical masses for the sigma and the pion respectively, hence H and λ are not fixed. We studied the limit $H \to 0$ and found that the "pion" mass has to be zero in the chiral limit. In 1+0 dimensions we are then faced with an additional problem, namely the infrared divergence ($M \to 0$) of the algebraic expressions for the thermal integrals. To learn more about the nature of the infrared divergence, we examined massless scalar field theory, for which one can calculate the partition function and the effective potential analytically, i.e., independent of thermal integrals. We considered the case where the field variable underlies no constraints and runs from $-\infty$ to ∞, which corresponds to a cartesian coordinate, and we also investigated the influence of topological constraints corresponding to a radial coordinate and a periodically constrained variable (like the angular variable in polar coordinates). Indeed, in accordance with the CJT formalism, the effective potential in absence of constraints (cartesian coordinates) is $-\infty$ in the limit $M \to 0$. In the case of a periodically constrained variable however, the effective potential is finite. Comparing the analytical results for the partition functions for the periodically constrained variable and the unconstrained cartesian one, we found that they are equal up to an overall countably infinite constant. Note that the derivation of the infrared-divergent algebraic expressions is only valid for $M \neq 0$, and one cannot exclude that the limit $M \to 0$ is different from $M = 0$. A massless particle is something special, it has to move with speed of light and might be treated in a special way. We came to the conclusion that this particular question can only be clarified, if we derive the Feynman rules for a periodically constrained coordinate, or for polar coordinates respectively.

Since we do not know the Feynman rules for polar coordinates, we were concerned with the ques-

tion if the use of the common Feynman rules for cartesian coordinates can be justified. Although the last word is not spoken yet, we argued why this should be possible. In case of the angular variable, we showed that extending the range of integration from $[0, 2\pi)$ to $[-\infty, \infty]$ simply yields a countably infinite constant in front of the partition function, which can be absorbed into the normalization. Regarding the radial variable, we justified the extension from $[0, \infty]$ to $[-\infty, \infty]$ in dimensional regularization. We also investigated if the Jacobian of the coordinate transformation from cartesian to polar coordinates has an influence on the results. Following the discussion of Kleinert, we convinced ourselves that it does not in dimensional regularization. Due to Veltman's rule it can indeed be omitted from the beginning. Additionally, we made sure that changing to polar coordinates in the discretized version of the path integral yields the same Lagrange function as if we directly change to polar coordinates in the continuous version of the Lagrange function. We carried out the calculations explicitly in 1+0 dimensions, however, we also checked that the conclusions are valid also in 1+3 dimensions. For a toy model in 1+0 dimensions one can confirm the observation that the Jacobian does not affect the results via expanding the effective potential at its global minimum perturbatively, treating $\delta(0)$ as finite. This was done by Kleinert and Chervyakov and was reviewed in our discussion.
The fact that the polar generating functional requires renormalization, due to the infinite contribution $\sim \delta(0)$ of the Jacobian, allows for a new understanding of the path integral collapse (see chapter 9). The asymptotic expansion of the slightly modified Bessel functions, which yields the centrifugal barrier, therefore remains plausible. The observation that the generating functional is infinite, due to $m = 0$ (what is called path integral collapse), can be explained by the fact that the generating functional requires renormalization. Furthermore, we argued that the use of another asymptotic expansion yields the same result for the generating functional as dimensional regularization, up to a countably infinite constant.

Outlook

So far, we discussed 1+3 dimensional models neglecting terms in the thermal integrals which require renormalization. In cartesian coordinates they do not alter the results significantly. In polar coordinates, however, they might have an influence. Technically, the problem is that the pion mass becomes zero before the condensate can become zero. Since the omitted terms would have prefactors with powers of the condensate in the denominator, this might strongly affect the behavior of the equations in the limit $\varphi \to 0$. Due to consistency, one should perform the renormalization procedure in dimensional regularization. An alternative explanation for the spurious behavior would be that one cannot use the cartesian Feynman rules, although we offered a justification why this should be possible. Whereas the extension of the integration interval for a cyclic variable (polar angle) alone would be questionable, since a circle is not topologically equivalent to a straight line, the simultaneous extension of the integration intervals for the radial and the angular variable are at first sight not in contradiction with topology. Polar coordinates and cartesian coordinates are equally suited to describe a two-dimensional space. However, further verification of our conclusions would be as complicated as the search for the polar Feynman rules, which is a task for the future.
It is also of interest to generalize the present study to 4-dimensional polar coordinates, which

would correspond to the polar $O(4)$ model. The polar $O(4)$ model has the advantage that 3 degrees of freedom can be identified with the three pions, π^0 and $\pi^\pm$, and the remaining one with their chiral partner, the sigma particle. In this context we want to mention a subtle issue. It is believed that only the local isomorphy of symmetry groups is important regarding universality, although this has not been rigorously proven. As we pointed out, $SU(2) \times SU(2)$ is locally isomorphic to $O(4)$, but an exact isomorphism is given by $SU(2) \times SU(2)/Z(2) \simeq SO(4)$. Hence it would be even more reliable to consider a quaternionic prescription of $SO(4)$ (since $SU(2)$ can be represented by unit norm quaternions, there has to be a quaternionic representation of $SO(4)$, too).

In this thesis we were concerned with effective theories for the chiral condensate. Figure 1.4(a) recapitulates how these models come into play. However, we believe that there should be an accessible way to study the properties of QCD near a critical point referring to a second-order phase transition. A system at the critical temperature of a second-order phase transition becomes very simple: the correlation length diverges and the system becomes self-similar (scale-invariant). This is mirrored in a very simple power-law behavior for thermodynamic quantities and gives rise to the phenomenon of universality. Accordingly, one should expect that the complicated theory of QCD should drastically simplify at the critical point. More precisely, the effective Lagrangian describing QCD at the critical temperature should be scale invariant. In the classical limit massless QCD is scale invariant, however, at the quantum level loop corrections spoil scale invariance, which is known as *scale anomaly*. Similarly, massless ϕ^4 theory is scale-invariant at the classical level but not at the quantum level. The corresponding anomaly in the Ward identity is the origin of the nontrivial critical exponents as they appear in the $O(N)$ model [17]. The running coupling constant of QCD at finite temperature depends both on the scale and on temperature. According to [61], we speculate that a perturbative renormalization group study in the context of critical behavior might be possible, if the critical temperature is large enough.

Willst du dich am Ganzen erquicken,
So mußt du das Ganze im Kleinsten
erblicken.
(Johann Wolfgang von Goethe)

Appendix A

Basics

Introductory remark

As already mentioned in the preface of our thesis, in this appendix we give some important definitions. Furthermore it shall serve as a guide how to embed these rather special definitions into the general framework constituted by literature. To be more precise, we are concerned with the following concepts: *generating functionals*, *partition functions*, *expectation values*, *n-point functions*, *propagators*, and *Green's functions*. While these quantities belong to the standard repertoire of theoretical physics, it is hard to review all the modifications in which they appear. However, we believe that the facts presented in the following are crucial for a broad understanding of these concepts. For further details we refer to literature in general, which should be accessible with the help of appendix A.3. Nonetheless, we want to give some specific recommendations. For Quantum Mechanics and Quantum Mechanics at zero as well as at nonzero temperature (Quantum Field Theory in 1+0 dimensions and Quantum Field Theory in 1+0 dimensions at zero temperature as well as at nonzero temperature) we refer to [42] or [57]. These books also provide an excellent introduction to the concept of *path integrals*. Regarding scalar Quantum Field Theory in 1+3 dimensions at zero temperature, we recommend [62], and for the discussion of the ϕ^4-theory at nonzero temperature [17]. However, note that in general our notation differs from the above mentioned publications. For the path-integral representation of the classical partition function, which we do not discuss explicitly in the following, we refer to chapter 1 of [63].

A.1 Partition function

The concept of the *partition function* has its origin in Statistical Mechanics, and also in general it usually refers to theories at nonzero temperature. The term *generating functional* however is used both in theories at zero temperature and in theories at nonzero temperature. Basically, all information about a system is encoded in its partition function or in its generating functional, respectively. In general, the generating functional is a functional of sources, and functional differentiation with respect to these sources yields the (disconnected) n-point functions. Similarly,

the logarithm of the generating functional generates the connected n-point functions. Appendix D covers the thermal n-point functions as they appear in perturbative ϕ^4-theory at nonzero temperature. In the case of vanishing sources, the thermal generating functional (i.e., the partition function in the presence of sources) equals the partition function. In the limit $T \to 0$ the thermal generating functional is proportional to the Euclidean non-thermal generating functional.
The theory which provides the justification of thermodynamics on a microscopical level is called *statistical physics*. In the following we will discuss theories at nonzero temperature in thermal equilibrium. Consider any kind of system, statistical mechanical, quantum statistical or quantum-field statistical. The most general case of consideration is a system Σ_{GC} where energy E and conserved charges Q_i (conserved quantum numbers as for example baryon number) can be exchanged with the surrounding medium Σ. Fixed quantities in the system Σ_{GC} are only the volume V, the chemical potentials μ_i and the temperature T, i.e., we assume that the system Σ is a so-called *heat bath* for our system Σ_{GC}. In statistical physics the system Σ_{GC} is described within the theoretical concept of the *grand canonical ensemble*. All thermodynamic quantities can be determined from the so-called *grand canonical partition function* $\mathcal{Z}(T, V, \{\mu_i\})$. Of course, these "microscopic" definitions for thermodynamic quantities imply relations between them, which in turn can be used to derive measuring methods for them. These relations are the same in all of the three statistical ensembles (*microcanonical ensemble*, *canonical ensemble*, *grand canonical ensemble*). The most important ones we want to list in appendix G. In case of macroscopic, asymptotically large systems (i.e., sufficiently many particles of each species), there is another important issue. In this case also the microscopic definitions which are of course different in the three ensembles, are consistent with each other. One simply has to identify the grand canonical average for the charges, $\langle Q_i \rangle$, with the fixed values Q_i in the canonical ensemble. However, we can only refer to [16] for the explicit proof in the case of Statistical Mechanics where the only charge is the number of particles, i.e., $Q \equiv N$.
Any theory at nonzero temperature is described by the *grand canonical partition function*

definition A.1 (grand canonical partition function)

$$\mathcal{Z} = Tre^{-\beta(\hat{H}-\mu_i\hat{Q}_i)} \ , \quad \beta \equiv \frac{1}{T} \ ,$$

where $\hat{H}$ is the Hamilton operator of the system.
The *grand canonical thermal expectation value* of any physical observable $\hat{A}$ is given by

definition A.2 (grand canonical thermal expectation value)

$$\langle \hat{A} \rangle = \frac{1}{\mathcal{Z}} Tr\left(e^{-\beta(\hat{H}-\mu_i\hat{Q}_i)}\hat{A}\right) \ .$$

For $\mu_i = 0$ one can use the canonical ensemble instead. The *partition function* is given by

definition A.3 (partition function)

$$Z = Tre^{-\beta\hat{H}} \ .$$

We can perform the trace in the basis $\{|E_n\rangle\}$ constituted by the energy eigenstates $|E_n\rangle$:

$$Z = Tre^{-\beta\hat{H}} = \sum_n e^{-\beta E_n} \ . \tag{A.1}$$

The *thermal expectation value* is given by

definition A.4 (thermal expectation value)

$$\langle \hat{A} \rangle = \frac{1}{Z} Tr \left(e^{-\beta \hat{H}} \hat{A} \right) \ .$$

Note that definitions A.1-A.4 refer to the vacuum because no source terms occur. Also note that for $\hbar \to 0$ these definitions boil down to those of classical statistical physics.
The trace can be computed in any basis of orthonormal eigenstates. Alternatively one can use the path integral formalism to rewrite these quantities. In the following we will restrict ourselves to scalar fields. For the transition from Quantum Mechanics to Quantum Field Theory to Statistical Quantum Field Theory see appendix E. Note that we work in natural units where $\hbar \equiv 1$. For simplicity, we will first discuss a single scalar field in 1+0 dimensions, since the generalization to M scalar fields, treated as cartesian coordinates, is then not very complicated. Also the transition to 1+3 dimensions is straightforward, so we do not state it explicitly. Here we use the letter M instead of N to avoid confusion with the lattice label N.
Let us begin with introducing the *Euclidean time-evolution operator* from Euclidean time $\tau_a \in \mathbb{R}$ to $\tau_b \in \mathbb{R}$:

definition A.5 (Euclidean time-evolution operator)

$$\hat{U}_E(\tau_b, \tau_a) = e^{-(\tau_b - \tau_a)\hat{H}} \ .$$

Single scalar field, 1+0 dimensions

The *Euclidean time-evolution amplitude* (or *Euclidean kernel*)

definition A.6 (Euclidean time-evolution amplitude)

$$(\phi_b, \tau_b | \phi_a, \tau_a) \equiv \langle \phi_b | \hat{U}_E(\tau_b, \tau_a) | \phi_a \rangle$$

is called amplitude, because $|(\phi_b, \tau_b|\phi_a, \tau_a)|^2$ is the probability that the field has the value ϕ_a at Euclidean time τ_a and the value ϕ_b at Euclidean time τ_b. For each of the variables ϕ_a,ϕ_b,τ_a and τ_b we can choose every value we like. So far this is simply Euclidean field theory which is equivalent to the usual Minkowskian version. The additional features which come into play at nonzero temperature are the following:

- one has to identify $\tau_b \equiv \frac{1}{T}$ and $\tau_a \equiv 0$,
- one has to impose the periodic boundary condition $\phi(\tau) \stackrel{!}{=} \phi(\tau + \frac{1}{T})$.

Identifying $\tau_b \equiv \frac{1}{T}$ and $\tau_a \equiv 0$, we obtain

definition A.7 (nonzero temperature evolution operator)

$$\hat{U}_E(\beta, 0) = e^{-\hat{H}/T} \ ,$$

definition A.8 (thermal evolution amplitude)

$$(\phi_b, \beta|\phi_a, 0) \equiv \langle\phi_b|\hat{U}_E(\beta, 0)|\phi_a\rangle \ .$$

From definitions A.3 and A.7 we obtain

$$Z = Tr\hat{U}_E(\beta, 0) = \sum_a \langle\phi_a|e^{-\beta\hat{H}}|\phi_a\rangle = \int d\phi_a \ (\phi_a, \beta|\phi_a, 0) \ , \tag{A.2}$$

where $\{|\phi_a\rangle\}$ constitutes a complete orthonormal system.

result A.1 (alternative expression for the partition function)

$$Z = \int d\phi_a \ (\phi_a, \beta|\phi_a, 0) \ .$$

Definition A.6 can be rewritten as a path integral:

result A.2 (Euclidean time-evolution amplitude, lattice version)

$$(\phi_b, \tau_b|\phi_a, \tau_a) \simeq \frac{1}{\sqrt{2\pi\varepsilon}} \left[\prod_{n=1}^{N} \int \frac{d\phi_n}{\sqrt{2\pi\varepsilon}}\right] e^{-A_E} \ ,$$

$$with \ A_E = \varepsilon \sum_{n=1}^{N+1} \left[\frac{1}{2}\left(\frac{\phi_n - \phi_{n-1}}{\varepsilon}\right)^2 + U(\phi_n, -i\tau_n)\right] \ ,$$

$$\phi_{N+1} \equiv \phi_b \ , \quad \phi_0 \equiv \phi_a \ , \quad \tau_{N+1} \equiv \tau_b \ and \ \tau_0 \equiv \tau_a \ .$$

The above expression is called the *discretized*, or *lattice*, *version*. In the continuum limit, $N \to \infty$ and $\varepsilon \to 0$, this defines a so-called *path integral*, which one denotes by

$$(\phi_b, \tau_b|\phi_a, \tau_a) = \int \mathcal{D}\phi \ e^{-S_E} \ , \tag{A.3}$$

where $S_E = \lim_{\varepsilon\to 0} A_E$ is the Euclidean action

$$S_E = \int d\tau \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 + U(\phi, -i\tau)\right] \ . \tag{A.4}$$

The symbol U denotes the classical potential of the system. The *function* U is the same in Minkowskian and Euclidean field theory. We only have to bear in mind that we performed a Wick rotation $t \to -i\tau$, which we indicate suggestively by $U(\phi, -i\tau)$.
In case of the partition function Z, one has to set $\phi_{N+1} = \phi_0$, and there is an extra integration over $\phi_{N+1} = \phi_0$. This corresponds to taking the trace. Furthermore, one has to set $\tau_{N+1} = 1/T$ and $\tau_0 = 0$ as well as to impose the periodic boundary condition $\phi(\tau) \stackrel{!}{=} \phi(\tau + \beta)$. Hence, from result A.2 we obtain

result A.3 (partition function, lattice version)

$$Z \simeq \left[\prod_{n=1}^{N+1} \int \frac{d\phi_n}{\sqrt{2\pi\varepsilon}}\right] e^{-A_E} \ ,$$

$$where \ \phi_{N+1} = \phi_0 \ .$$

In the continuum limit, $N \to \infty$ and $\varepsilon \to 0$, this defines the path-integral representation of the partition function:

$$Z = \oint \mathcal{D}\phi e^{-S_E} \ . \tag{A.5}$$

Note that in comparison with the generating functional in QFT (see for example [62]), there is the constraint $\phi_{N+1} = \phi_0$ and the additional integration over ϕ_{N+1} in the discretized version A.3. We use the symbol $\oint$ to remind of this fact.

M scalar fields, 1+0 dimensions

Let us generalize the above discussion to M scalar fields, ϕ_i, which are the cartesian components of the vector $\vec{\phi} = (\phi_1, \cdots, \phi_M)$.

definition A.9 (Euclidean time-evolution amplitude)

$$(\vec{\phi}_b, \tau_b | \vec{\phi}_a, \tau_a) \equiv \langle \phi_{1,b} \ , \cdots, \phi_{M,b} | \hat{U}_E(\tau_b, \tau_a) | \phi_{1,a} \ , \cdots, \phi_{M,a} \rangle \ .$$

result A.4 (alternative expression for the partition function)

$$Z = \int d\phi_{1,a} \cdots \int d\phi_{M,a} \ (\vec{\phi}_a, \beta | \vec{\phi}_a, 0) \ .$$

result A.5 (Euclidean time-evolution amplitude, lattice version)

$$(\vec{\phi}_b, \tau_b | \vec{\phi}_a, \tau_a) \simeq \left(\frac{1}{\sqrt{2\pi\varepsilon}} \right)^M \left[\prod_{n=1}^{N} \int \frac{d\phi_{1,n}}{\sqrt{2\pi\varepsilon}} \right] \cdots \left[\prod_{n=1}^{N} \int \frac{d\phi_{M,n}}{\sqrt{2\pi\varepsilon}} \right] e^{-A_E} \ ,$$

$$with \ A_E = \varepsilon \sum_{n=1}^{N+1} \left[\frac{1}{2} \left(\frac{\vec{\phi}_n - \vec{\phi}_{n-1}}{\varepsilon} \right)^2 + U(\vec{\phi}_n, -i\tau_n) \right] \ ,$$

$$\vec{\phi}_{N+1} \equiv \vec{\phi}_b \ , \quad \vec{\phi}_0 \equiv \vec{\phi}_a \ , \quad \tau_{N+1} \equiv \tau_b \ and \ \tau_0 \equiv \tau_a \ .$$

result A.6 (partition function, lattice version)

$$Z \simeq \left[\prod_{n=1}^{N+1} \int \frac{d\phi_{1,n}}{\sqrt{2\pi\varepsilon}} \right] \cdots \left[\prod_{n=1}^{N+1} \int \frac{d\phi_{M,n}}{\sqrt{2\pi\varepsilon}} \right] e^{-A_E} \ ,$$

$$where \ \vec{\phi}_{N+1} = \vec{\phi}_0 \ .$$

In the continuum limit, $N \to \infty$ and $\varepsilon \to 0$, this defines the path-integral representation of the partition function:

$$Z = \oint \mathcal{D}\vec{\phi} e^{-S_E} \ . \tag{A.6}$$

A.2 N-point functions

definition A.10 (correctly normalized n-point function...) *syn.: correctly normalized vacuum expectation value for* $\phi(X_1)...\phi(X_n)...$

- *...in the presence of sources:*

$$\langle \phi(X_1)\cdots\phi(X_n)\rangle^{(T=0)} = \frac{\int \mathcal{D}\phi\ e^{iS[\phi]+\phi J+\frac{1}{2}\phi K\phi}\phi(X_1)\ \ldots\ \phi(X_n)}{\int \mathcal{D}\phi\ e^{iS[\phi]+\phi J+\frac{1}{2}\phi K\phi}}$$

- *...(in the absence of sources):*

$$\langle \phi(X_1)\cdots\phi(X_n)\rangle^{(T=0)}|_{J=0,K=0} = \left.\frac{\int \mathcal{D}\phi\ e^{iS[\phi]+\phi J+\frac{1}{2}\phi K\phi}\phi(X_1)\ \ldots\ \phi(X_n)}{\int \mathcal{D}\phi\ e^{iS[\phi]+\phi J+\frac{1}{2}\phi K\phi}}\right|_{J=0,K=0}$$

where $\phi = \phi(t,\vec{x})$, $S[\phi] = \int dt \int d^3\vec{x}\ \mathscr{L}(\phi)$, $\phi J \equiv i\int dt \int d^3\vec{x}\ J(t,\vec{x})\phi(t,\vec{x})$
and $\phi K\phi \equiv i\int dt \int d^3\vec{x}\ \ i\int dt' \int d^3\vec{x'}\ \phi(t,\vec{x})K(t,\vec{x},t',\vec{x'})\phi(t',\vec{x'})$.

definition A.11 (correctly normalized thermal n-point function...) *syn.: correctly normalized thermal vacuum expectation value for* $\phi(X_1)...\phi(X_n)...$

- *...in the presence of sources:*

$$\langle \phi(X_1)\cdots\phi(X_n)\rangle = \frac{\oint \mathcal{D}\phi\ e^{-S_E[\phi]+\phi J+\frac{1}{2}\phi K\phi}\phi(X_1)\ \ldots\ \phi(X_n)}{\oint \mathcal{D}\phi\ e^{-S_E[\phi]+\phi J+\frac{1}{2}\phi K\phi}}$$

- *...(in the absence of sources):*

$$\langle \phi(X_1)\cdots\phi(X_n)\rangle|_{J=0,K=0} = \left.\frac{\oint \mathcal{D}\phi\ e^{-S_E[\phi]+\phi J+\frac{1}{2}\phi K\phi}\phi(X_1)\ \ldots\ \phi(X_n)}{\oint \mathcal{D}\phi\ e^{-S_E[\phi]+\phi J+\frac{1}{2}\phi K\phi}}\right|_{J=0,K=0}$$

where $\phi = \phi(\tau,\vec{x})$, $S_E[\phi] = \int d\tau \int d^3\vec{x}\ \mathscr{L}_E(\phi)$, $\phi J \equiv \int d\tau \int d^3\vec{x}\ J(\tau,\vec{x})\phi(\tau,\vec{x})$
and $\phi K\phi \equiv \int d\tau \int d^3\vec{x} \int d\tau' \int d^3\vec{x'}\ \phi(\tau,\vec{x})K(\tau,\vec{x},\tau',\vec{x'})\phi(\tau',\vec{x'})$.

Appendix D covers the thermal n-point functions as they appear in perturbative ϕ^4-theory at nonzero temperature. Connected thermal n-point functions we determine via definitions D.1 and 2.1.

A.3 The big picture

In the following we refer to textbooks in general. We believe that a major difficulty lies in establishing the connection between definitions given in different publications. Often quantities occur in different modifications, special cases or different contexts, which complicates a uniform nomenclature, although they are in principle closely related or even equal in some cases. Examples are the generating functional and the partition function, as well as the 2-point function, the Green's function, and the propagator. In the following we want to present some crucial facts, of which we believe that they are essential for the understanding of these concepts.

definition A.12 (Meaning of the term generating functional)

If the functional $F[g(x)]$ of the function $g(x)$ can be expanded into a functional power series

$$F[g(x)] = \sum_n F_n[g(x)] \ , \quad F_n[g(x)] = (n!)^{-1/2} \int f_n(x_1, ..., x_n) g(x_1) \cdots g(x_n) dx_1 \cdots dx_n \ ,$$

then $F[g(x)]$ is called a generating functional for the functions $f_n(x_1, ..., x_n)$.

Generating functionals as a method were established in 1928 by Fock [64]. Step by step, the concept of the generating functional captured a central role in Quantum Field Theory. In Quantum Field Theory one calls $g(x)$ a *source* and $f_n(x_1, ..., x_n)$ *n-point correlation functions* or *n-point functions*. Note that the definition A.12 (taken from [65]) covers the expressions (D.6), (D.17) and (D.26). However, it can be generalized to more sources. Consider for example two sources $J_A(x)$ and $J_B(y)$, which correspond to two different species of fields or particles, respectively.

If the functional $F[J_A(x), J_B(y)]$ of the functions $J_A(x)$ and $J_B(y)$ can be expanded

into a functional power series $\quad F[J_A(x), J_B(y)] = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F_{n,m}[J_A(x), J_B(y)] \ ,$

$$F_{n,m}[J_A(x), J_B(y)] \sim \int f_{n,m}(x_1, ..., x_n; y_1, ..., y_m) \times$$

$$\times J_A(x_1) \cdots J_A(x_n) J_B(y_1) \cdots J_B(y_m) dx_1 \cdots dx_n dy_1 \cdots dy_m \ ,$$

then $F[J_A(x), J_B(y)]$ is called a generating functional for the functions $f_{n,m}(x_1, ..., x_n; y_1, ..., y_m)$.

Note that $f_{0,m} \equiv f_{0,m}(y_1, ..., y_m)$ and $f_{n,0} \equiv f_{n,0}(x_1, ..., x_n)$.

Within the literature there is no uniform nomenclature, usually one names the functions $f_{n,m}$ according to the corresponding particles. So to speak they are "*m-n-point functions*". Obviously, this concept can be easily generalized to an arbitrary number of sources to define the generating functional for "*$l-m-n-(\cdots)$-point functions*". As a concrete example consider the *2-Fermion-2Boson-function* within Yukawa theory (see page 499 of [62]):

$$G^{(4)}_{FB}(x_1, x_2; y; x) \sim \left. \frac{\delta Z[\eta, \overline{\eta}, J]}{\delta J(x_1) \delta J(x_2) \delta \eta(x) \delta \overline{\eta}(y)} \right|_{J=\eta=\overline{\eta}=0} \ .$$

Furthermore, the concept could be generalized to sources which depend on more than one space-time point. Consider for example an additional bilocal source $K(y, z)$:

If the functional $F[J(x), K(y, z)]$ of the functions $J(x)$ and $K(y, z)$ can be expanded

into a functional power series $\quad F[J(x), K(y, z)] = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} F_{n,2k}[J(x), K(y, z)] \ ,$

$$F_{n,2k}[J(x), K(y, z)] \sim \int f_{n,2k}(x_1, ..., x_n; y_1, ..., y_k, z_1, ..., z_k) \times$$

$$\times J(x_1) \cdots J(x_n) K(y_1, z_1) \cdots K(y_k, z_k) dx_1 \cdots dx_n dy_1 \cdots dy_k dz_1 \cdots dz_k \ ,$$

then $F[J(x), K(y, z)]$ is called a generating functional

for the functions $f_{n,2k}(x_1, ..., x_n; y_1, ..., y_k, z_1, ..., z_k)$.

Note that $f_{0,2k} \equiv f_{0,2k}(y_1, ..., y_k, z_1, ..., z_k)$ and $f_{n,0} \equiv f_{n,0}(x_1, ..., x_n)$.

Again this can be generalized to an arbitrary number of sources of all kinds (such as $L(x,y,z)$ etc.).
All these kinds of correlation functions can be obtained from their generating functional via functional differentiation with respect to sources. Note that a theory can be studied either at zero or nonzero temperature. For further details see appendix E. Generically speaking, the generating functional, usually denoted by $\mathcal{Z}$ (or by Z for vanishing chemical potentials or zero temperature), is a functional of sources and has the form

$$\mathcal{Z}[sources] \sim \int \mathcal{D}\sqcup_1 \cdots \int \mathcal{D}\sqcup_n \; e^{A+sourceterms+other} \;, \tag{A.7}$$

where the symbols $\sqcup_i$ are replacement characters, A is the action of the theory (up to a constant prefactor) and *other* stands for contributions arising for example from gauge invariance (ghost and gaugefixing terms) or from the Jacobian of a coordinate transformation. Note that in the case of nonzero temperature we should write $\oint$ (which throughout the rest of this work reminds of the periodic boundary condition) instead of $\int$ in order to be consistent with our usual notation. For simplicity we use only the sign $\int$ in this section. One can split $A + other$ into a *free* part $A_0^{all} \equiv A_0 + other_0$, which contains all terms quadratic resp. bilinear in the fields, and the *interaction* part $A_I^{all} \equiv A_I + other_I$, which covers the rest. Factorizing,

$$e^{A+sourceterms+other} = e^{A_0^{all}+sourceterms+A_I^{all}} = e^{A_0^{all}+sourceterms} \cdot e^{A_I^{all}} \;, \tag{A.8}$$

expanding the interaction part into a Taylor series,

$$e^{A_I^{all}} = 1 + A_I^{all} + \frac{1}{2!}\left(A_I^{all}\right)^2 + \cdots \;, \tag{A.9}$$

and abbreviating

$$\mathcal{Z}_0[sources] \equiv \int \mathcal{D}\sqcup_1 \cdots \int \mathcal{D}\sqcup_n \; e^{A_0^{all}+sourceterms} \tag{A.10}$$

$$\langle \cdots \rangle_0^{sources} \equiv \frac{1}{\mathcal{Z}_0[sources]} \int \mathcal{D}\sqcup_1 \cdots \int \mathcal{D}\sqcup_n \; e^{A_0^{all}+sourceterms}(\cdots) \;, \tag{A.11}$$

we obtain

$$\mathcal{Z}[sources] = \mathcal{Z}_0[sources] \underbrace{\left[1 + \langle A_I^{all}\rangle_0^{sources} + \frac{1}{2!}\langle (A_I^{all})^2\rangle_0^{sources} + \cdots\right]}_{\equiv \mathcal{Z}_I[sources]} \;. \tag{A.12}$$

In any theory, one is allowed to expand $\mathcal{Z}[sources]$ perturbatively, if the interaction is sufficiently weak. Compare for example (D.8) or (H.14).
For every theory there exists a set of so-called *Feynman rules*, which prescribe a) how to write a Feynman diagram in terms of an explicit analytical expression, and b) how to express quantities in terms of Feynman diagrams. Whereas the rules themselves are usually well-arranged, their derivation is of course complicated. Regarding the technical question how to use Feynman rules, we recommend Griffith's pedagogical discussion for a toy model [3], from which more advanced Feynman rules become easier to comprehend.
Now we are prepared for an important remark:

- For any theory $\mathcal{Z}_I[sources]$ is the sum of all Feynman diagrams (with their combinatorial prefactors) that can be constructed from the Feynman rules.
- For any theory $\mathcal{Z}_I[0]$ is the sum of all vacuum Feynman diagrams (with their combinatorial prefactors) that can be constructed from the Feynman rules. A vacuum diagram is a diagram without external legs.
- For any theory $\ln \mathcal{Z}_I[0]$ is the sum of all connected vacuum Feynman diagrams (with their combinatorial prefactors) that can be constructed from the Feynman rules. Note that $\ln \mathcal{Z}_I[sources]$ can, however, include disconnected diagrams.

In most cases one is interested in quantities in the vacuum and in vacuum expectation values of observables. In principle all can be derived from $\mathcal{Z}[0]$. Note that $\mathcal{Z}[0] = \mathcal{Z}_0[0]\mathcal{Z}_I[0]$, where $\mathcal{Z}_0[0]$ is a constant. In general $\mathcal{Z}[0]$ is a series of infinitely many diagrams, and divergent diagrams can occur. Accordingly, this will be the case also for quantities one derives from $\mathcal{Z}[0]$. *Resummation* techniques are methods how to handle series of infinitely many diagrams, whereas *renormalization* is an approach how to handle divergent diagrams. These procedures yield *resummed* and *renormalized* quantities, respectively.
Our final remark refers to the (thermal) n-point functions generated by $\mathcal{Z}$, which can be defined either in the path-integral formulation, or alternatively within the operator formalism of second quantization as vacuum expectation values of (Euclidean) time-ordered products of field operators. Up to a constant prefactor both definitions are equal. A special case is the (thermal) free 2-point function which is also called (thermal) Green's function or (thermal) propagator. Note that this is misleading, since the (thermal) 2-point function is not in every theory a Green's function in the mathematical sense. Mathematically speaking, a Green's function $Gr(x, y)$ of a linear differential operator L is any solution of the equation

$$L\, Gr(x, y) = \delta(x - y)\ , \tag{A.13}$$

In scalar field theory for example, the free (thermal) 2-point function indeed is proportional to a special Green's function for the (thermal) Klein-Gordon operator, the so-called free (thermal) Feynman propagator. For further details, see appendix F and [62]. The (thermal) Feynman propagator is defined by (E.19) and (E.24), respectively.

Appendix B

Thermal integrals

The Feynman rules for statistical theories, i.e., theories at nonzero temperature (see for example $\lambda\phi^4$-theory in 1+3 dimensions in momentum space: e.g. p.35 in Kapusta's textbook [66]) contain so called "thermal integrals". Using the imaginary-time formalism they look as follows:

for theories in 1+3 dimensions

$$\int\limits_k f(k) \equiv T \sum_{n=-\infty}^{\infty} \int \frac{d^3\mathbf{k}}{(2\pi)^3} f(2\pi i n T, \mathbf{k}) \ , \tag{B.1}$$

and for theories in 1+0 dimensions

$$\int\limits_{k_0} f(k_0) \equiv T \sum_{n=-\infty}^{\infty} f(2\pi i n T) \ . \tag{B.2}$$

Note that the integral sign $\int\limits_k$ on the left-hand side is nothing more than an abbreviation. We will explain the meaning of the right side in the following.

When calculating thermal integrals, we have to consider a subtlety, namely the periodic boundary condition, which turns the integration over the Euclidean zeroth component into a sum. Simply carrying out the Wick rotation $t \to \overline{t} = -i\tau$ resp. $k_0 \to \overline{k}_0 = ip_4$ is not enough. This is only the first step on the way to a thermal integral. The integrals, appearing within the perturbative expansion in QFT, are rotated in Euclidean QFT:

$$i \int\limits_{-\infty}^{\infty} \frac{dk_0}{2\pi} \int\limits_{-\infty}^{\infty} \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{k^2 - M^2} \xrightarrow{k_0 \to i p_4} \int\limits_{-\infty}^{\infty} \frac{dp_4}{2\pi} \int\limits_{-\infty}^{\infty} \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{p^2 + M^2} \ , \tag{B.3}$$

where we have the Minkowskian scalar product $k^2 = k_0^2 - \mathbf{k}^2$ and the Euclidean scalar product $p^2 = p_4^2 + \mathbf{k}^2$.

The transition to SQFT is made by imposing a periodic boundary condition on the propagator, which yields

$$\int\limits_{-\infty}^{\infty} \frac{dp_4}{2\pi} \int\limits_{-\infty}^{\infty} \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{p^2 + M^2} \xrightarrow{p_4 \to \omega_n} T \sum_{n=-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{\omega_n^2 + \mathbf{k}^2 + M^2} \ , \tag{B.4}$$

where $\omega_n = 2\pi n T \in \mathbb{R}$ denote the Matsubara frequencies for bosons.

B.1 Thermal integrals in 1+0 dimensions

B.1.1 Thermal integral over propagator

Let us first compute the thermal integral over the propagator $G(k_0) = \frac{1}{-k_0^2+M^2}$:

$$\int\limits_{k_0} \frac{1}{-k_0^2+M^2} \overset{k_0=i\omega_n}{=} T \sum_{n=-\infty}^{\infty} \frac{1}{\omega_n^2+M^2} \ . \tag{B.5}$$

A short reminder from complex function theory: if $f_1(z)$ has a pole of first order at $z=a$, while $f_2(z)$ is holomorphic at a, then follows $Res_a(f_1 f_2) = f_2(a)\, Res_a f_1$. In our case $f_1(k_0) = \coth\frac{k_0}{2T}$ has poles of first order in $k_0 = i\omega_n$, therefore $Res f_1(k_0) = 2T$. $f_2(k_0) = \frac{1}{-k_0^2+M^2}$ has no poles on imaginary axis $\underline{\text{if } M \neq 0}$. Therefore, using the residue theorem:

$$T \sum_{n=-\infty}^{\infty} \frac{1}{\omega_n^2+M^2} = T\frac{1}{2\pi i} \int\limits_C dz\ \frac{1}{-z^2+M^2}\frac{1}{2T}\coth\frac{z}{2T} \ , \tag{B.6}$$

where C is a contour in the complex plane, which encloses all of the poles. Note that for application of the residue theorem, we analytically continued k_0 to the complex plane in the above formula, i.e., we write z. The simplest way to choose our contour is the following:

$$\begin{aligned}
&T\frac{1}{2\pi i}\int\limits_C dz\ \frac{1}{-z^2+M^2}\frac{1}{2T}\coth\frac{z}{2T} \\
&= \frac{1}{2}\left[\frac{1}{2\pi i}\int\limits_{i\infty-\epsilon}^{-i\infty-\epsilon} dz\ \frac{1}{-z^2+M^2}\coth\frac{z}{2T} \quad + \quad \frac{1}{2\pi i}\int\limits_{-i\infty+\epsilon}^{i\infty+\epsilon} dz\ \frac{1}{-z^2+M^2}\coth\frac{z}{2T}\right] \\
&= \underbrace{\frac{1}{2\pi i}\int\limits_{i\infty-\epsilon}^{-i\infty-\epsilon} dz\ \frac{1}{-z^2+M^2}\left(-\frac{1}{2}-\frac{1}{e^{-z/T}-1}\right)}_{=-\int\limits_{-i\infty+\epsilon}^{i\infty+\epsilon} dz\ \frac{1}{-(-z)^2+M^2}\left(-\frac{1}{2}-\frac{1}{e^{z/T}-1}\right)} \quad + \quad \frac{1}{2\pi i}\int\limits_{-i\infty+\epsilon}^{i\infty+\epsilon} dz\ \frac{1}{-z^2+M^2}\left(\frac{1}{2}+\frac{1}{e^{z/T}-1}\right) \ ,
\end{aligned}$$

where we used $\coth\frac{z}{2T} = \pm 1 \pm \frac{2}{e^{z/T}-1}$ in the last step,

$$\begin{aligned}
&= \frac{1}{2\pi i}\int\limits_{-i\infty+\epsilon}^{i\infty+\epsilon} dz\ \frac{1}{2}\left(\frac{1}{-z^2+M^2}+\frac{1}{-(-z^2)+M^2}\right) \\
&+\frac{1}{2\pi i}\int\limits_{-i\infty+\epsilon}^{i\infty+\epsilon} dz\ \frac{1}{e^{z/T}-1}\left(\frac{1}{-z^2+M^2}+\frac{1}{-(-z^2)+M^2}\right) \ .
\end{aligned}$$

Now we use that $-z^2+M^2 > 0$ for all imaginary values of z, thus we can set $\epsilon = 0$ in the first integral:

$$= \underbrace{\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} dz\ \frac{1}{-z^2+M^2}}_{\equiv Q_\mu^0} \quad + \quad \underbrace{\frac{1}{2\pi i}\int\limits_{-i\infty+\epsilon}^{i\infty+\epsilon} dz\ \frac{1}{e^{z/T}-1}\frac{2}{-z^2+M^2}}_{\equiv Q_T^0} \ .$$

$$Q_\mu^0 \overset{x\equiv iz}{=} \frac{1}{2\pi i}\int_\infty^{-\infty}\frac{dx}{i}\;\frac{1}{-(-ix)^2+M^2} = \int_{-\infty}^{\infty}\frac{dx}{2\pi}\;\frac{1}{x^2+M^2} \overset{MM}{=} \frac{1}{2M}\;.$$

We again apply the residue theorem to Q_T^0 . The contour D encloses the only pole of the integrand in the right half-plane, which lies at $z = M$ and is of first order. It goes straight upwards from $-i\infty+\epsilon$ to $i\infty+\epsilon$, back to $-i\infty+\epsilon$ over a half-circle with radius $|z|\to\infty$. The integral over the half-circle contributes zero, because the integrand goes to zero for $|z|\to\infty$.

$$Q_T^0 = \frac{1}{2\pi i}\int_D dz\;\frac{1}{e^{z/T}-1}\frac{2}{-z^2+M^2} = \frac{1}{2\pi i}(-2\pi i)\,Res_{z=M}\left(\frac{1}{e^{z/T}-1}\frac{2}{-z^2+M^2}\right)\;.$$

Note that the minus sign comes in, because we have winding number -1 for closing clockwise.

$$\begin{aligned} Res_{z=M}\left(\frac{1}{e^{z/T}-1}\frac{2}{-z^2+M^2}\right) &= \lim_{z\to M}\Big[(z-M)\,\frac{1}{e^{z/T}-1}\frac{2}{-z^2+M^2}\Big] \\ &= \lim_{z\to M}\frac{z-M}{\left(e^{z/T}-1\right)\left(\frac{-z^2+M^2}{2}\right)}\;. \end{aligned}$$

We can use L'Hospital's rule to calculate the limit:

$$= \lim_{z\to M}\frac{1}{\frac{1}{T}\,e^{z/T}\left(\frac{-z^2+M^2}{2}\right)-z\left(e^{z/T}-1\right)} = \frac{1}{M-M\;e^{M/T}}\;.$$

So we conclude

result B.1 ($M\neq 0$)

$$\int_{k_0}\frac{1}{-k_0^2+M^2} = \frac{1}{2M}-\frac{1}{M}\left(\frac{1}{1-e^{M/T}}\right) = \frac{1}{M}\left(\frac{1}{e^{M/T}-1}+\frac{1}{2}\right)\;.$$

One can deduce B.1 alternatively by carrying out a simple sum:

$$\int_{k_0}\frac{1}{-k_0^2+M^2} \overset{k_0=i\omega_n}{=} T\sum_{n=-\infty}^{\infty}\frac{1}{\omega_n^2+M^2} \overset{MM}{=} \frac{1}{2M}\coth\frac{M}{2T} = \frac{1}{M}\left(\frac{1}{e^{M/T}-1}+\frac{1}{2}\right)\;.$$

However the detailed derivation above shows the correspondence between Q_μ^0 resp. Q_T^0 and Q_μ resp. Q_T, which will be defined in section B.2. This is also where we remind of the fact that Q_T (the analogue to Q_T^0) is $\frac{T^2}{12}$ for $M=0$. So far, we only discussed the case $M\neq 0$. It is interesting that the result B.1 goes to infinity for $M\to 0$. More precisely, not only Q_μ^0 but Q_T^0 as well.

B.1.2 Thermal integral over propagator times k_0^2

$$\begin{aligned} \int_{k_0}\frac{k_0^2}{-Z^2k_0^2+M^2} &= \frac{1}{Z^2}\int_{k_0}\frac{k_0^2}{-k_0^2+M^2/Z^2} = \frac{1}{Z^2}T\sum_{n=-\infty}^{\infty}\frac{-\omega_n^2}{\omega_n^2+M^2/Z^2} \\ &= \underbrace{\frac{1}{Z^2}T\sum_{n=-\infty}^{\infty}-1}_{divergent\;\;part\;\;(drop)} + \underbrace{\frac{1}{Z^2}T\sum_{n=-\infty}^{\infty}\frac{M^2/Z^2}{(2\pi nT)^2+M^2/Z^2}}_{\overset{MM}{=}\frac{M^2}{Z^2}\frac{Z}{2TM}\coth(M/2TZ)} \\ &\overset{drop}{=} \frac{M}{Z^3}\frac{1}{2}\coth(M/2TZ) = \frac{M}{Z^3}\left(\frac{1}{e^{M/ZT}-1}+\frac{1}{2}\right)\;. \end{aligned}$$

result B.2 ($M \neq 0$)

$$\int\limits_{k_0} \frac{k_0^2}{-Z^2k_0^2+M^2} = \frac{M}{Z^3}\left(\frac{1}{e^{M/ZT}-1}+\frac{1}{2}\right) .$$

For $M \to 0$ this yields:

result B.3 ($M \to 0$)

$$\int\limits_{k_0} \frac{k_0^2}{-Z^2k_0^2+M^2} \xrightarrow{M\to 0} \frac{T}{Z^2} .$$

B.1.3 Thermal integral over logarithmic inverse propagator

We now want to compute

$$\int\limits_{k_0} \ln\left(-Z^2k_0^2+M^2\right) \overset{k_0=i\omega_n}{=} T\sum_{n=-\infty}^{\infty} \ln\left(Z^2\left(2\pi n\right)^2T^2+M^2\right) . \tag{B.7}$$

The sum

$$v \equiv \sum_{n=-\infty}^{\infty} \ln\left(Z^2(2\pi n)^2T^2+M^2\right)$$

is clearly divergent, so that we will have to drop infinite unphysical terms. More precisely, we will drop those addends which are independent of M. For this purpose we first take the partial derivative with respect to M:

$$\frac{\partial v}{\partial M} = \sum_{n=-\infty}^{\infty} \frac{2M}{Z^2(2\pi n)^2T^2+M^2} = \frac{2M}{M^2} + 2\sum_{n=1}^{\infty} \frac{2M}{Z^2(2\pi n)^2T^2+M^2}$$
$$= \frac{2}{M} + 4\frac{1}{Z2\pi T}\underbrace{\sum_{n=1}^{\infty} \frac{M/(Z2\pi T)}{n^2+M^2/(Z2\pi T)^2}}_{=-\frac{1}{2}\frac{Z2\pi T}{M}+\frac{1}{2}\pi\coth\frac{M}{Z2T}} = \frac{1}{ZT}\coth\frac{M}{Z2T} = 2\frac{1}{ZT}\left(\frac{1}{2}+\frac{1}{e^{\frac{M}{ZT}}-1}\right) .$$

Now, having performed the Matsubara sum, we integrate to receive back v up to a constant which may depend on T and Z:

$$v + const(Z,T) = \int \frac{\partial v}{\partial M} dM = \int \frac{2}{ZT}\left(\frac{1}{2}+\frac{1}{e^{\frac{M}{ZT}}-1}\right) dM$$
$$= \frac{2}{ZT}\left(-\frac{M}{2}+ZT\ln\left(-1+e^{\frac{M}{ZT}}\right)\right) = \frac{2}{ZT}\left(\frac{M}{2}-ZT\ln e^{\frac{M}{ZT}}-ZT\ln\frac{1}{e^{\frac{M}{ZT}}-1}\right)$$
$$= \frac{2}{ZT}\left(\frac{M}{2}+ZT\ln\left(1-e^{-\frac{M}{ZT}}\right)\right) .$$

Dropping $T \cdot const(Z,T)$ we obtain:

result B.4 ($M \neq 0$)

$$\int\limits_{k_0} \ln\left(-Z^2k_0^2+M^2\right) = \frac{M}{Z}+2T\ln\left(1-e^{-\frac{M}{TZ}}\right) .$$

Note that there is a good reason why we are able to drop addends which are independent of M. To derive the corresponding thermal integral in 1+3 dimensions, namely result B.6, we will have to replace M^2 by $\mathbf{k}^2 + M^2$ and integrate over momentum $\mathbf{k}$. Hence, addends independent of M correspond to addends independent of M and $\mathbf{k}$ in 1+3 dimensions. The momentum-integration makes these addends divergent. Following Dolan and Jackiw [67], one is allowed to drop such divergent terms, even though they depend on temperature (explicitly and/or implicitly). Intuitively, since these addends do not play any role in 1+3 dimensions, the same should be true in 1+0 dimensions. Let us verify this statement by comparing the effective potential at its global minimum calculated via WKB and via CJT in case of the harmonic oscillator-like potential $U = \frac{1}{2}\phi^2$, i.e., the model from section 3.2 at $\lambda = 0$:
From result 3.6 we know that the CJT formalism yields $M(T) = 1$. Note that we work in units of m, i.e., $m \equiv 1$. Accordingly, the effective potential at its global minimum, using result B.4, is given by

$$V_{CJT}(T) = \frac{1}{2} \int\limits_{k_0} \ln G^{-1} = \frac{1}{2} \left(1 + 2T \ln \left(1 - e^{-1/T} \right) \right) \ . \tag{B.8}$$

We now insert the above potential and the turning points $\phi^*_{1/2} = \pm\sqrt{2E}$ into the WKB equation:

$$2 \int\limits_0^{\sqrt{2E}} \sqrt{2(E - \frac{1}{2}\phi^2)}\, d\phi = \left(\frac{1}{2} + n \right) \pi \ .$$

Performing the integral yields the eigenvalues

$$E_n = \frac{1}{2} + n \ ,$$

which in turn we insert into expression (3.15) to calculate the effective potential at its global minimum via WKB:

$$V_{WKB}(T) = -T \ln \left(\sum_{n=0}^{\infty} e^{-(1/2+n)/T} \right) = -T \ln \left(\frac{1}{2} \mathrm{csch} \frac{1}{2T} \right)$$
$$= -T \ln \left(\frac{1}{2 \sinh \frac{1}{2T}} \right) = T \ln \left(e^{\frac{1}{2T}} - e^{-\frac{1}{2T}} \right) = \frac{1}{2} + T \ln \left(1 - e^{-1/T} \right) \ ,$$

which equals $V_{CJT}(T)$. As a matter of fact, there should be no additional temperature-dependent addend in (B.8), as the expression for $\int\limits_{k_0} \ln G^{-1}$ does not explicitly depend on λ.
As a last argument, we compute the Matsubara sum in an alternative way by using the formula [66]

$$\ln \left(a^2 + b^2 \right) = \int\limits_1^{a^2} \frac{1}{\theta + b^2}\, d\theta + \ln(1 + b^2) = \int\limits_1^{a} \frac{2\theta}{\theta^2 + b^2}\, d\theta + \ln(1 + b^2) \ . \tag{B.9}$$

As one will see, different addends (which again may depend explicitly or implicitly on T or Z) will be dropped and nevertheless we will end up with the same result B.4.

We can proceed as follows:

$$\sum_{n=-\infty}^{\infty} \ln\left(Z^2\left(2\pi n\right)^2 T^2 + M^2\right) = \underbrace{\sum_n \ln\left(Z^2 T^2\right)}_{drop} + \sum_n \ln\left((2\pi n)^2 + \frac{M^2}{Z^2T^2}\right)$$

$$= \int_1^{\frac{M^2}{Z^2T^2}} \underbrace{\sum_n \frac{1}{\theta + (2\pi n)^2}}_{=\frac{\coth(\sqrt{\theta}/2)}{2\sqrt{\theta}}} d\theta \; + \underbrace{\sum_n \ln\left(1 + (2\pi n)^2\right)}_{drop} = 2\ln\left[\operatorname{csch}\left(\frac{1}{2}\right)\sinh\left(\frac{M}{2TZ}\right)\right]$$

$$= \underbrace{1 - 2\ln(e-1)}_{drop} - \frac{M}{TZ} + 2\ln\left(e^{\frac{M}{TZ}} - 1\right) = \frac{M}{TZ} - 2\ln e^{\frac{M}{TZ}} + 2\ln\left(e^{\frac{M}{TZ}} - 1\right) = \frac{M}{TZ} + 2\ln\left(1 - e^{\frac{M}{TZ}}\right),$$

or slightly differently:

$$\sum_{n=-\infty}^{\infty} \ln\left(Z^2\left(2\pi n\right)^2 T^2 + M^2\right) = \underbrace{\sum_{n=-\infty}^{\infty} \ln T^2}_{drop} + \sum_{n=-\infty}^{\infty} \int_1^{M^2/T^2} \frac{d\theta}{\theta + \left(2\pi n\right)^2 Z^2} + \underbrace{\sum_{n=-\infty}^{\infty} \ln\left(1 + \left(2\pi n\right)^2 Z^2\right)}_{drop}$$

$$= \int_1^{M^2/T^2} d\theta \underbrace{\sum_{n=-\infty}^{\infty} \frac{1}{\theta + \left(2\pi n\right)^2 Z^2}}_{\frac{\coth\left(\sqrt{\theta}/2Z\right)}{2Z\sqrt{\theta}} = \frac{1 + \frac{2}{e^{\sqrt{\theta}/Z} - 1}}{2Z\sqrt{\theta}}} = \int_1^{M^2/T^2} d\theta \, \frac{1}{2Z\sqrt{\theta}}\left(1 + \frac{2}{e^{\sqrt{\theta}/Z} - 1}\right) \overset{u \equiv \sqrt{\theta}}{=} \int_1^{M/T} du \, \frac{1}{Z}\left(1 + \frac{2}{e^{u/Z} - 1}\right)$$

$$= \frac{1}{Z} - \frac{M}{ZT} - 2\ln\left(e^{1/Z} - 1\right) + 2\ln\left(e^{\frac{M}{TZ}} - 1\right) = \underbrace{\frac{1}{Z} - 2\ln\left(e^{1/Z} - 1\right)}_{drop} + \frac{M}{ZT} - 2\frac{M}{ZT} - 2\ln\left(\frac{1}{e^{\frac{M}{TZ}} - 1}\right)$$

$$= \frac{M}{ZT} + 2\ln\left(\frac{e^{\frac{M}{TZ}} - 1}{e^{\frac{M}{TZ}}}\right) = \frac{M}{ZT} + 2\ln\left(1 - e^{-\frac{M}{TZ}}\right).$$

B.2 Thermal integrals in 1+3 dimensions

Now $k = (k_0, \mathbf{k})$, of course $\mathbf{k} \in \mathbb{R}^3$, $k^2 = k_0^2 - \mathbf{k}^2$ with the imaginary zeroth component $k_0 = i\omega_n$.

B.2.1 Thermal integral over propagator

Again let us begin with the thermal integral over the propagator $G(k) = \frac{1}{-k^2 + M^2}$. From result B.1 obviously follows:

$$\int_{k_0} \frac{1}{-k_0^2 + \mathbf{k^2} + \mathbf{M^2}} = \frac{1}{\epsilon_k(M)}\left(\frac{1}{e^{\epsilon_k(M)/T} - 1} + \frac{1}{2}\right)$$

with $\epsilon_k(M) \equiv \sqrt{\mathbf{k^2} + \mathbf{M^2}}$.
Then, due to $\int_k \frac{1}{-k^2 + M^2} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int_{k_0} \frac{1}{-k_0^2 + \mathbf{k^2} + \mathbf{M^2}}$, we have:

result B.5 (thermal integral over propagator in 1+3 dimensions)

$$\int_k \frac{1}{-k^2+M^2} = \underbrace{\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{\epsilon_k(M)}\,\frac{1}{e^{\epsilon_k(M)/T}-1}}_{\equiv Q_T(M)} + \underbrace{\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\epsilon_k}}_{\equiv Q_\mu(M)}$$

where the integrations over spatial momenta $\mathbf{k}$ run from $-\infty$ to ∞ . Q_T can be expressed as an integral over solid angle using $\int d^3\mathbf{k} = \int_{\mathbf{0}}^{\mathbf{4\pi}} \mathbf{d\Omega} \int_{\mathbf{0}}^{\infty} \mathbf{k^2}$ (where $k = |\mathbf{k}|$) :

$$Q_T(M) = \frac{1}{2\pi^2}\int_0^\infty dk\,\frac{k^2}{\sqrt{k^2+M^2}}\frac{1}{e^{\sqrt{k^2+M^2}/T}-1}\ . \tag{B.10}$$

Note that for $M=0$ this integral gives the well-known result $Q_T(M=0) \overset{MM}{=} \frac{T^2}{12}$.
However, Q_μ is divergent and requires renormalization. In the CT renormalization scheme, counter terms are introduced to substract the UV divergences (compare with [45]) and one obtains:

$$Q_\mu(M) = \frac{1}{(4\pi)^2}\left[M^2\ln\frac{M^2}{\mu^2} - M^2 + \mu^2\right]\ . \tag{B.11}$$

The renormalization scale μ is an additional parameter and will be chosen later.

B.2.2 Thermal integral over logarithmic inverse propagator

From result B.4 and $\int_k \ln G^{-1} = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\int_{k_0}\ln G^{-1}$ directly follows :

result B.6 ($M \neq 0$)

$$\begin{aligned}\int_k \ln G^{-1} &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\frac{\sqrt{M^2+Z^2\mathbf{k}^2}}{Z} + 2T\ln\left(1-e^{-\frac{\sqrt{M^2+Z^2\mathbf{k}^2}}{TZ}}\right)\right]\\ &= \underbrace{\int \frac{d^3\mathbf{k}}{(2\pi)^3}\epsilon_k(M/Z)}_{\equiv R_\mu} + \underbrace{\int \frac{d^3\mathbf{k}}{(2\pi)^3}2T\ln\left(1-e^{-\frac{\epsilon_k(M/Z)}{T}}\right)}_{\equiv R_T}\end{aligned}$$

where $\epsilon_k(M/Z) \equiv \sqrt{\mathbf{k}^2+\frac{M^2}{Z^2}}$.
R_T can be expressed as an integral over solid angle:

$$R_T = \int_0^{4\pi} d\Omega_2 \int_0^\infty \frac{dk k^2}{(2\pi)^3} 2T\ln\left(1-e^{-\frac{\sqrt{k^2+\frac{M^2}{Z^2}}}{T}}\right) = 4\pi\frac{1}{(2\pi)^3}\int_0^\infty dk k^2 2T\ln\left(1-e^{-\frac{\sqrt{k^2+\frac{M^2}{Z^2}}}{T}}\right)\ . \tag{B.12}$$

In the case $M=0$ and $Z\neq 0$ this simplifies to

$$R_T = -\frac{T^4\pi^2}{45}\ . \tag{B.13}$$

Appendix C

Path integrals in phase space and configuration space

Let us consider **Quantum Mechanics** with a Hamilton function of the form $H(p,x,t) = T(p,t) + U(x,t) = \frac{p^2}{2m} + U(x,t)$. We want to show in the following that the path-integral representation of the *time-evolution amplitude*[1] in configuration space,

$$(x_b,t_b|x_a,t_a) = \frac{1}{\sqrt{2\pi\hbar i\epsilon/m}} \left[\prod_{n=1}^{N} \int_{-\infty}^{\infty} \frac{dx_n}{\sqrt{2\pi\hbar i\epsilon/m}}\right] e^{\frac{i}{\hbar}A} \tag{C.1}$$

$$\text{with } A \equiv \epsilon \sum_{n=1}^{N+1} \left[\frac{m}{2}\left(\frac{x_n - x_{n-1}}{\epsilon}\right)^2 - U(x_n,t_n)\right] , \tag{C.2}$$

is equal to its representation in phase space,

$$(x_b,t_b|x_a,t_a) = \left[\prod_{n=1}^{N} \int_{-\infty}^{\infty} dx_n\right] \left[\prod_{n=1}^{N+1} \int_{-\infty}^{\infty} \frac{dp_n}{2\pi\hbar}\right] e^{\frac{i}{\hbar}\overline{A}} \tag{C.3}$$

$$\text{with } \overline{A} = \sum_{n=1}^{N+1} \left[p_n(x_n - x_{n-1}) - \epsilon H(p_n,x_n,t_n)\right] . \tag{C.4}$$

The proof goes as follows:

$$\begin{aligned}\overline{A} = \sum_{n=1}^{N+1} \big[& \frac{\epsilon}{2m} 2m p_n \frac{(x_n - x_{n-1})}{\epsilon} - \frac{\epsilon}{2m} p_n^2 \\ & - \frac{\epsilon}{2m}\frac{m^2}{\epsilon^2}(x_n - x_{n-1})^2 + \frac{\epsilon}{2m}\frac{m^2}{\epsilon^2}(x_n - x_{n-1})^2 - \epsilon U(x_n,t_n)\big] \\ = \sum_{n=1}^{N+1} \big[& -\frac{\epsilon}{2m}\left(p_n - \frac{m}{\epsilon}(x_n - x_{n-1})\right)^2 + \epsilon\frac{m}{2}\left(\frac{x_n - x_{n-1}}{\epsilon}\right)^2 - \epsilon U(x_n,t_n)\big] .\end{aligned} \tag{C.5}$$

[1] The quantity is called amplitude, because $|(x_b,t_b|x_a,t_a)|^2$ is the probability that a particle at position x_a and time t_a, can be found at time t_b at position x_b. For each of the variables x_a,x_b,t_a and t_b, we can choose every value we like.

The Fresnel integral formula,

$$\int_{-\infty}^{\infty} \frac{dq}{\sqrt{2\pi}} e^{-i\frac{a}{2}q^2} = \frac{1}{\sqrt{|a|}} \begin{cases} \times\sqrt{i} & \text{if } a<0 \\ \times\frac{1}{\sqrt{i}} & \text{if } a>0 \end{cases} , \tag{C.6}$$

can be rewritten with $q \equiv (p_n - m\frac{x_n - x_{n-1}}{\epsilon})$ and $\frac{dq}{dp_n} = 1$:

$$\frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \frac{dp_n}{\sqrt{2\pi}} e^{-i\frac{(\frac{\epsilon}{\hbar m})}{2}(p_n - m\frac{x_n - x_{n-1}}{\epsilon})^2} = \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{\sqrt{|\frac{\epsilon}{\hbar m}|}} \frac{1}{\sqrt{i}} = \frac{1}{\sqrt{2\pi\hbar i\epsilon/m}} . \tag{C.7}$$

Inserting (C.5) into (C.3) yields:

$$\left[\prod_{n=1}^{N}\int_{-\infty}^{\infty} dx_n\right] \underbrace{\left[\prod_{n=1}^{N+1}\int_{-\infty}^{\infty} \frac{dp_n}{2\pi\hbar} e^{-i\frac{(\frac{\epsilon}{\hbar m})}{2}(p_n - m\frac{x_n - x_{n-1}}{\epsilon})^2}\right]}_{=\left(\frac{1}{\sqrt{2\pi\hbar i\epsilon/m}}\right)^{N+1}} e^{\frac{i}{\hbar}\sum_{n=1}^{N+1}[\epsilon\frac{m}{2}\left(\frac{x_n - x_{n-1}}{\epsilon}\right)^2 - \epsilon U(x_n, t_n)]} . \tag{C.8}$$

q.e.d.

The Euclidean time amplitude in **Euclidean Quantum Field Theory in 1+0 dimensions** follows immediately from the above via Wick rotation. Compare appendix E for details. Due to $x \to \phi$ and $m \to 1$, the momentum $p = m\dot{x}$ has to be replaced by the conjugate field, $p \to \dot{\phi} \equiv \pi$. Hence:
the Euclidean time-evolution amplitude is given by

$$(\phi_b, \tau_b|\phi_a, \tau_a) = \frac{1}{\sqrt{2\pi\hbar\varepsilon}} \left[\prod_{n=1}^{N}\int_{-\infty}^{\infty} \frac{d\phi_n}{\sqrt{2\pi\hbar\varepsilon}}\right] e^{-\frac{1}{\hbar}A_E} \tag{C.9}$$

$$\text{with } A_E \equiv \varepsilon \sum_{n=1}^{N+1} \left[\frac{1}{2}\left(\frac{\phi_n - \phi_{n-1}}{\varepsilon}\right)^2 + U(\phi_n, -i\tau_n)\right] , \tag{C.10}$$

as well as by

$$(\phi_b, \tau_b|\phi_a, \tau_a) = \left[\prod_{n=1}^{N}\int_{-\infty}^{\infty} d\phi_n\right] \left[\prod_{n=1}^{N+1}\int_{-\infty}^{\infty} \frac{d\pi_n}{2\pi\hbar}\right] e^{-\frac{1}{\hbar}\overline{A}_E} \tag{C.11}$$

$$\text{with } \overline{A}_E = \sum_{n=1}^{N+1} \left[-i\pi_n(\phi_n - \phi_{n-1}) + \varepsilon H(\pi_n, \phi_n, -i\tau_n)\right] . \tag{C.12}$$

In **Statistical Quantum Field Theory in 1+0 dimensions**, the partition function in momentum-space representation is then given by

$$Z = \int_{-\infty}^{\infty} (\phi_a, \frac{1}{T}|\phi_a, 0) d\phi_a = \left[\prod_{n=1}^{N+1}\int_{-\infty}^{\infty} d\phi_n\right] \left[\prod_{n=1}^{N+1}\int_{-\infty}^{\infty} \frac{d\pi_n}{2\pi\hbar}\right] e^{-\frac{1}{\hbar}\overline{A}_E} , \tag{C.13}$$

$$\text{with the periodic boundary condition } \phi(\tau) \stackrel{!}{=} \phi(\tau + 1/T) .$$

Appendix D

The effective potential in theories with a single source J

Preliminary remarks

In this appendix we define the effective potential for scalar ϕ^4-theory with a single source J at nonzero temperature. We review the context in which the effective potential appears, which allows for an adequate understanding of this widely used quantity. We introduce the effective potential in three consecutive steps. In the first step we clarify how the generating functional Z is calculated perturbatively with the help of n-point functions. In a second step we introduce the advantageous concept of connected n-point functions, which in turn allow for a perturbative expansion of $\ln Z$ containing only connected Feynman diagrams. We discuss relations between n-point functions, which are essential for the comprehension of the CJT formalism. Finally, in a third step, we present yet another concept which simplifies perturbative calculations. Working with the Legendre transform of $\ln Z$ and irreducible n-point functions, one gets by with 1-Particle-Irreducible diagrams. We end up with the conclusions that the effective potential is effectively the classical potential of the system plus quantum corrections, and that its global minimum is determined by the vacuum expectation value of the field.
We refer to the definition A.11 for $K = 0$, whereas we denote the correctly normalized thermal n-point function...

- ...in the presence of a single source J by:

$$\langle \phi(X_1) \cdots \phi(X_n) \rangle_{\not K} \equiv \frac{\oint \mathcal{D}\phi \; e^{-S_E[\phi]+\phi J} \phi(X_1) \; \ldots \; \phi(X_n)}{\oint \mathcal{D}\phi \; e^{-S_E[\phi]+\phi J}} \; , \tag{D.1}$$

- ...(in the absence of source) by:

$$\langle \phi(X_1) \cdots \phi(X_n) \rangle_{\not K} |_{J=0} \equiv \left. \frac{\oint \mathcal{D}\phi \; e^{-S_E[\phi]+\phi J} \phi(X_1) \; \ldots \; \phi(X_n)}{\oint \mathcal{D}\phi \; e^{-S_E[\phi]+\phi J}} \right|_{J=0} \; . \tag{D.2}$$

Although this notation is not very elegant, we think it is convenient regarding the discussion in chapter 2.

Introducing the effective potential

First step

$$\frac{\delta Z\left[J\right]}{\delta J\left(X\right)}=\mathcal{N}\oint\mathcal{D}\phi\ \phi\left(X\right)\ \exp\left(-S_{E}\left[\phi\right]+\int\limits_{X_{1}}\phi\left(X_{1}\right)\ J\left(X_{1}\right)\right)=Z\left[J\right]\left\langle\phi\left(X\right)\right\rangle_{\not K}\,, \tag{D.3}$$

$$\begin{aligned}\frac{\delta Z\left[J\right]}{\delta J\left(X_{1}\right)\delta J\left(X_{2}\right)}&=\mathcal{N}\oint\mathcal{D}\phi\ \phi\left(X_{1}\right)\phi\left(X_{2}\right)\ \exp\left(-S_{E}\left[\phi\right]+\int\limits_{X}\phi\left(X\right)\ J\left(X\right)\right)\\&=Z\left[J\right]\left\langle\phi\left(X_{1}\right)\phi\left(X_{2}\right)\right\rangle_{\not K}\,.\end{aligned} \tag{D.4}$$

In case of the thermal n-point function (in absence of a source), this generalizes with the normalization convention $Z\left[0\right]\equiv1$ to

$$\left\langle\phi\left(X_{1}\right)\cdots\phi\left(X_{n}\right)\right\rangle_{\not K}|_{J=0}=\left.\frac{\delta Z\left[J\right]}{\delta J\left(X_{1}\right)\ \ldots\ \delta J\left(X_{n}\right)}\right|_{J=0}\,. \tag{D.5}$$

The right-hand side expression provides us with the coefficients of a Taylor expansion generalized to the continuous case, so that the generating functional can be constructed from the n-point functions via

$$Z\left[J\right]=\sum_{n=0}^{\infty}\int\limits_{X_{1}}\ \ldots\ \int\limits_{X_{n}}\frac{1}{n!}\ \left\langle\phi\left(X_{1}\right)\cdots\phi\left(X_{n}\right)\right\rangle_{\not K}|_{J=0}\ J\left(X_{1}\right)\ \ldots\ J\left(X_{n}\right)\ . \tag{D.6}$$

Perturbation theory for a generating functional of the form $Z\left[J\right]$ can be found in common textbooks (compare with Greiner's and Reinhardt's textbook [62], note, however, that this treatment is at zero temperature, whereas we are at nonzero temperature). The perturbation series for the generating functional

$$\begin{aligned}Z\left[J\right]&=\mathcal{N}\oint\mathcal{D}\phi\exp\left(-S_{E}\left[\phi\right]+\int\limits_{X_{1}}\phi\left(X_{1}\right)J\left(X_{1}\right)\right)\\&=\mathcal{N}e^{\int U_{I}\left(\frac{\delta}{\delta J\left(X\right)}\right)}\underbrace{\oint\mathcal{D}\phi\exp\left(-S_{E,0}\left[\phi\right]+\int\limits_{X_{1}}\phi\left(X_{1}\right)J\left(X_{1}\right)\right)}_{\equiv Z_{0}\left[J\right]}\end{aligned} \tag{D.7}$$

reads, using the normalization condition $Z\left[0\right]\equiv1$:

$$\begin{aligned}Z\left[J\right]&=Z_{0}\left[J\right]\frac{1+\lambda u_{1}\left[J\right]+\lambda^{2}u_{2}\left[J\right]+...}{1+\lambda u_{1}\left[0\right]+\lambda^{2}u_{2}\left[0\right]+...}\\&=1+\lambda\left(u_{1}\left[J\right]-u_{1}\left[0\right]\right)+\lambda^{2}\left(u_{2}\left[J\right]-u_{2}\left[0\right]\right)+\lambda^{2}u_{1}\left[0\right]\left(u_{1}\left[0\right]-u_{1}\left[J\right]\right)\ +\ ...\end{aligned} \tag{D.8}$$

$$\text{with}\quad u_{1}\left[J\right]=Z_{0}^{-1}\left[J\right]\ \left[\int\limits_{X}U_{I}\left(\frac{\delta}{\delta J\left(X\right)}\right)\right]\ Z_{0}\left[J\right]\quad, \tag{D.9}$$

$$\text{and}\quad u_{2}\left[J\right]=Z_{0}^{-1}\left[J\right]\ \left[\int\limits_{X}U_{I}\left(\frac{\delta}{\delta J\left(X\right)}\right)\right]\ \left[\int\limits_{Y}U_{I}\left(\frac{\delta}{\delta J\left(Y\right)}\right)\right]\ Z_{0}\left[J\right]\ . \tag{D.10}$$

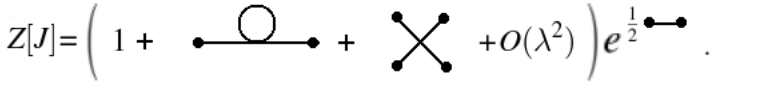
The generating functional for the free theory, $Z_0[J]$, can be expressed analytically as

$$Z_0[J] = \exp\left(\frac{1}{2}\int_{X'}\int_{X} J(X')\,\Delta(X'-X)\,J(X)\right) . \tag{D.11}$$

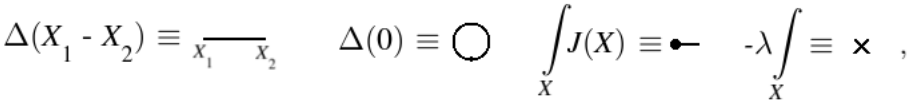
For $U = \frac{m^2}{2}\phi^2 + \frac{\lambda}{N}\phi^4 = \frac{m^2}{2}\phi^2 + U_I$, we obtain (up to prefactors), performing four functional derivatives with respect to $J(Y)$,

$$\frac{\delta^4}{\delta J(Y)^4} Z_0[J] = Z_0[J]\left[\Delta(0)^2 + \Delta(0)\left(\int_X \Delta(Y-X)\,J(X)\right)^2 + \left(\int_X \Delta(Y-X)\,J(X)\right)^4\right], \tag{D.12}$$

and further



We introduced a few simple translation rules (which are called the Feynman rules in position space, and can be recognized by just looking at the example we translated) and the following elementary abbreviations:



where $\Delta(X)$ stands for the thermal Feynman propagator in position space [1]

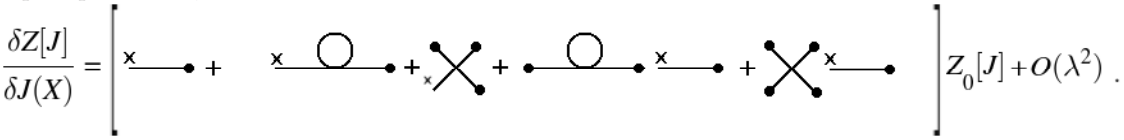
$$\Delta(X) = T\sum_{n=-\infty}^{\infty}\int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{-i(\omega_n\tau - \mathbf{kx})} \underbrace{\frac{1}{\omega_n^2 + \mathbf{k^2} + \mathbf{m^2}}}_{\equiv\Delta(\omega_n,\mathbf{k})} . \tag{D.13}$$

m is called *bare* mass, because in the free case ($U = \frac{m^2}{2}$, i.e., no interaction) we have

$$\langle\phi(X_1)\,\phi(X_2)\rangle_{\not K}|_{J=0} \sim \Delta(X_1 - X_2) , \tag{D.14}$$

whereas interaction terms lead to a *dressed* (synonymous: *full*) propagator containing the *dressed* mass, as we will see in the following.

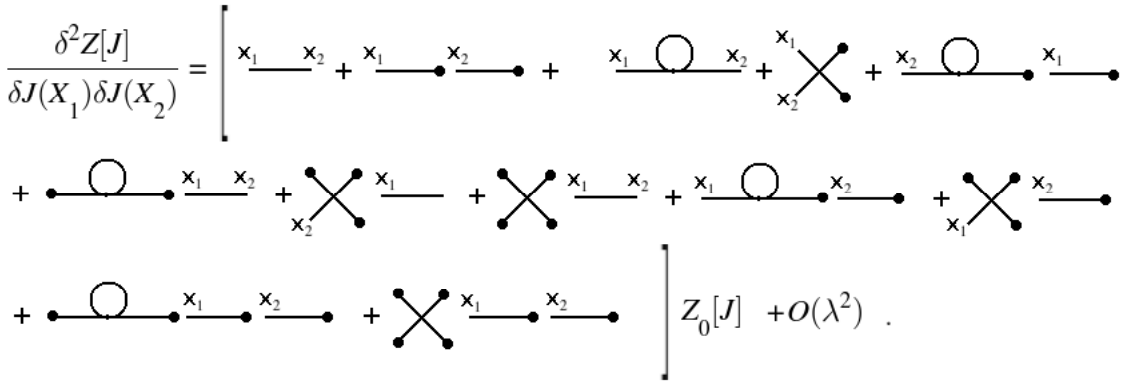
Up to prefactors, we have:



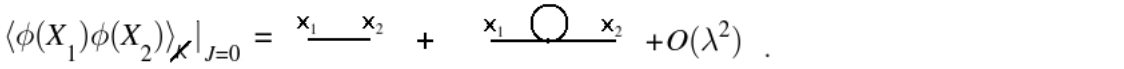
Note that, together with expression (D.3), one can see that $\langle\phi(X)\rangle_{\not K}$ will include disconnected Feynman graphs.

One more functional derivative yields:

[1] The (thermal) Feynman propagator is the most important representative from the collection of functions (called the *Greens functions* for the Klein-Gordon operator), which solve the (thermal) inhomogeneous Klein-Gordon equation with the four-dimensional delta function as inhomogeneity.



With $Z_0[J] = 1$ and the normalization convention $Z[J] \equiv 1$, only two diagrams survive for $J = 0$. So we obtain from expression (D.4) the thermal propagator to order g (up to prefactors):



We can observe that, to order λ, the interaction term $\frac{\lambda}{N}\phi^4$ changes the formula (D.14) to the above one. One can bring it into the following form:

$$\langle\phi(X_1)\,\phi(X_2)\rangle_{\not K}|_{J=0} \sim T\sum_{n=-\infty}^{\infty}\int\frac{d^3\mathbf{k}}{(2\pi)^3}e^{-i(\omega_n\tau-\mathbf{kx})}\frac{1}{\omega_n^2+\mathbf{k^2}+\mathbf{m^2}+\bigcirc}\ . \tag{D.15}$$

M in $M^2 \equiv m^2 + \bigcirc$ (up to a prefactor in front of $\bigcirc$), is called the *full* (dressed) mass to order λ.

The dressed (to order λ) thermal Feynman propagator in momentum space is obtained via inverse Fourier transformation. One reads it off directly from the above expression:

$$\Delta(\omega_n,\mathbf{k}) = \frac{1}{\omega_n^2+\mathbf{k}^2+M^2}\ . \tag{D.16}$$

However, one has to recognize that the Fourier transformation involves the discrete Matsubara frequencies. This is why there appears a sum instead of an integral, which is the correct form of the Fourier transformation on a finite interval with periodic constraint.

Second step

It is absolutely equivalent to work with $W \equiv \ln Z$ instead of Z. All quantities of our theory can be constructed from Z by certain prescriptions and therefore equivalently for W. The Taylor expansion for W is given by

$$W[J] = \sum_{n=0}^{\infty}\int_{X_1}\ ...\ \int_{X_n}\frac{1}{n!}\ \frac{\delta W[J]}{\delta J(X_1)\ ...\ \delta J(X_n)}|_{J=0}\ J(X_1)\ ...\ J(X_n)\ . \tag{D.17}$$

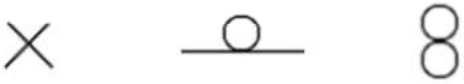
It can be shown that the graphical representation of the functions $\frac{\delta W[J]}{\delta J(X_1)\ ...\ \delta J(X_n)}|_{J=0}$, only possesses connected diagrams.

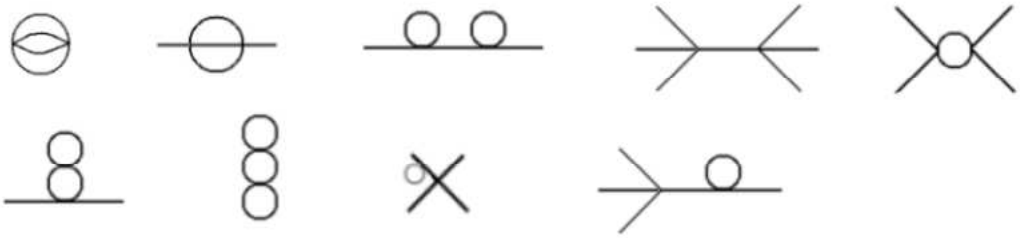
definition D.1 (connected thermal n-point functions)
The coefficients $\frac{\delta W[J]}{\delta J(X_1)\ ...\ \delta J(X_n)}|_{J=0}$ *we name* ***connected thermal n-point functions*** *(in the absence of a source), and use the symbol* $\mathcal{G}(X_1,...,X_n)$ *resp.* $\mathcal{G}^{(n)}$ *for them.*
Furthermore, we refer to $\frac{\delta W[J]}{\delta J(X_1)\ ...\ \delta J(X_n)}$ *as the* ***connected thermal n-point functions in the presence of a source*** J [a]*, using the symbol* $G(X_1,...,X_n)$ *resp.* $G^{(n)}$.

[a]Note that the connected n-point functions in the presence of a source may contain disconnected graphs. Only after setting the source to zero, those diagrams vanish

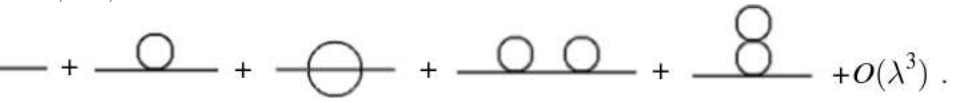
Let us draw all the possible connected diagrams to order λ^2:

$O(\lambda^0)$:

$O(\lambda)$: 

$O(\lambda^2)$: 

The connected 2-point function $\mathcal{G}^{(2)}$ is given by the sum of all possible connected graphs with 2 external points, i.e.,

$\mathcal{G}^{(2)} =$  $+O(\lambda^3)$.

Together with Δ, it is possible to construct $\mathcal{G}^{(2)}$ with the help of the so-called **proper self energy** Σ which is defined (up to prefactors) as the sum of all truncated[2] 1PI[3] diagrams (which had two external lines before truncation), using the following recipe (which follows from combinatorics):

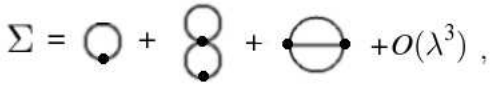
$$\begin{aligned}&\mathcal{G}(X,Y) = \Delta(X,Y) + \Delta(X,A)\ \Sigma(A,B)\ \Delta(B,Y)\\ &+\Delta(X,A)\ \Sigma(A,B)\ \Delta(B,C)\ \Sigma(C,D)\ \Delta(D,Y)\ +...\ .\end{aligned} \tag{D.18}$$

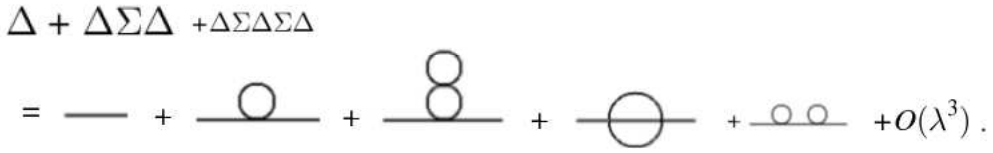
Equation (D.18) can be manipulated further:

$$\begin{aligned}\mathcal{G}^{(2)} &= \Delta(1+\Sigma\Delta+\Sigma\Delta\Sigma\Delta) = \Delta(1-\Sigma\Delta)^{-1}\\ &\iff \left(\mathcal{G}^{(2)}\right)^{-1} = \Delta^{-1}-\Sigma\ .\end{aligned} \tag{D.19}$$

[2]the truncated diagram is gained simply by removing the external lines
[3]1PI diagrams are those connected diagrams, which cannot be disconnected by cutting one internal line.

To order λ^2 we have:





Moreover, there exists a *resummed* Dyson-Schwinger equation (see page 256 of [68]). Instead of equation (D.18) one obtains the following relation between $G^{(2)}$ (the connected thermal 2-point function in the presence of a source J) and Σ' (the *resummed* self-energy):

$$G\left(X,Y\right)=\Delta\left(X,Y\right)+\Delta\left(X,A\right)\ \Sigma'\left(A,B\right)\ G\left(B,Y\right)\ . \tag{D.20}$$

This is obviously a self-consistent equation. G on the right-hand side can be iterated by reinserting the right-hand side:

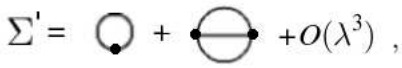
$$G=\Delta+\Delta\Sigma\left(\Delta+\Delta\Sigma G\right)=\Delta+\Delta\Sigma\Delta+\Delta\Sigma\Delta\Sigma G=\ \cdots\ . \tag{D.21}$$

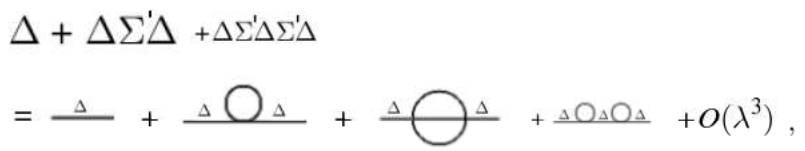
Hence we end up with

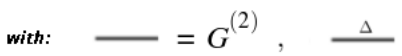
$$\begin{aligned}G\left(X,Y\right)&=\Delta\left(X,Y\right)+\Delta\left(X,A\right)\ \Sigma'\left(A,B\right)\ \Delta\left(B,Y\right)\\&+\Delta\left(X,A\right)\ \Sigma'\left(A,B\right)\ \Delta\left(B,C\right)\ \Sigma'\left(C,D\right)\ \Delta\left(D,Y\right)\ +...\end{aligned} \tag{D.22}$$

$$\Longleftrightarrow\ \left(G^{(2)}\right)^{-1}=\Delta^{-1}-\Sigma'\ . \tag{D.23}$$

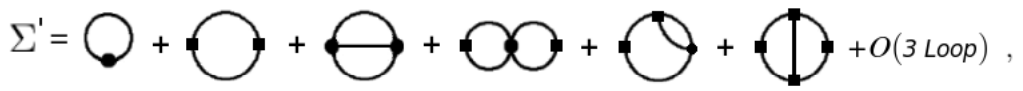
To order λ^2 we have:

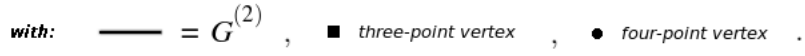






One can pick an arbitrary point ϕ, which is by definition independent of τ and $\vec{x}$ (we keep the same symbol ϕ) and fluctuate around it, i.e. $\phi(\tau,\vec{x})=\phi+\sigma(\tau,\vec{x})$. The ϕ^4-interaction term yields a three-point and a four-point interaction vertex for the fluctuation σ. In that case the resummed self-energy (up to two-loop order) is given by:





In principle one would have to sum an infinite set of diagrams, which is practically impossible. However, any truncation of the series at some given order represents a well-defined approximation. Taking only 1-loop diagrams in Σ' into account is known as *Hard-Thermal Loop (HTL)* approximation (in the high-temperature limit).
Apart from that it is possible to derive a rule how to construct the n-point functions in the presence of a source from the connected n-point functions in the presence of a source. Here, we give the formulas for the one-point and the two-point function:

$$\langle \phi(X) \rangle_{\not K} = G(X) \ , \tag{D.24}$$

$$\langle \phi(X_1)\, \phi(X_2) \rangle_{\not K} = G(X_1, X_2) + G(X_1)\, G(X_2) \ . \tag{D.25}$$

Third step

As well as with $W[J] \equiv \ln Z(J)$, one can work with its Legendre transform Γ (in the functional sense), assuming that the source J can be expressed as a functional of $\langle \phi(X) \rangle_{\not K}$, and has an explicit dependence on X. It contains the same amount of information as W and is therefore completely equivalent.

definition D.2 (effective action)

$$\Gamma\left[\langle \phi(X) \rangle_{\not K}\right] \equiv W[J] - \int\limits_X J(X)\, \langle \phi(X) \rangle_{\not K} \ .$$

The Taylor expansion reads

$$\Gamma\left[\langle \phi(X) \rangle_{\not K}\right] = \sum_{n=0}^{\infty} \int\limits_{X_1} \ldots \int\limits_{X_n} \frac{1}{n!}\, \frac{\delta \Gamma\left[\langle \phi(X) \rangle_{\not K}\right]}{\delta \langle \phi(X_1) \rangle_{\not K} \ldots \delta \langle \phi(X_n) \rangle_{\not K}} \Big|_{J=0} \langle \phi(X_1) \rangle_{\not K} \ldots \langle \phi(X_n) \rangle_{\not K} \ . \tag{D.26}$$

It is possible to prove that $\Gamma^{(n)} \equiv \frac{\delta \Gamma\left[\langle \phi(X) \rangle_{\not K}\right]}{\delta \langle \phi(X_1) \rangle_{\not K} \ldots \delta \langle \phi(X_n) \rangle_{\not K}} \big|_{J=0}$, called **irreducible n-point vertex function** (in the absence of a source), is (up to prefactors) equal to the sum of all possible 1PI diagrams with n external points that can be constructed. The figure on p.167 shows all connected graphs, and we see that $\Gamma^{(2)}$ is given by

Altogether, we found that it is possible to construct a quantity $\Gamma\left[\langle \phi(X) \rangle_{\not K}\right]$ (from which all other quantities of interest can be constructed) via formula (D.26), working with 1PI diagrams only, which is much more convenient.
Definition (D.2) directly results in

$$\frac{\delta \Gamma\left[\langle \phi \rangle_{\not K}\right]}{\delta \langle \phi(X) \rangle_{\not K}} = \frac{\delta W}{\delta \langle \phi(X) \rangle_{\not K}} - \int\limits_Y \frac{\delta J(Y)}{\delta \langle \phi(X) \rangle_{\not K}} \langle \phi(Y) \rangle_{\not K} - J(X) = -J(X) \ , \tag{D.27}$$

due to

$$\frac{\delta W}{\delta \langle \phi(X) \rangle_{\not K}} = \int\limits_Y \frac{\delta W}{\delta J(Y)} \frac{\delta J(Y)}{\delta \langle \phi(X) \rangle_{\not K}} = \int\limits_Y \langle \phi(Y) \rangle_{\not K} \frac{\delta J(Y)}{\delta \langle \phi(X) \rangle_{\not K}} \ .$$

We would like to find a function which is (among other properties) characterized by the property that, for $J = 0$, it is globally minimized at $\langle \phi(X) \rangle_{\not K}|_{J=0}$. Expanding all of the $\langle \phi \rangle_{\not K}$'s into a Taylor series around a common point, Γ can always be brought into the form

$$\Gamma[\langle \phi \rangle_{\not K}] = \int\limits_X \left(-V_{eff}(\langle \phi \rangle_{\not K}) + \frac{1}{2} \partial_\mu \langle \phi \rangle_{\not K} \partial^\mu \langle \phi \rangle_{\not K} F(\langle \phi \rangle_{\not K}) + \text{higher order derivatives} \right) . \quad \text{(D.28)}$$

Indeed, for $\langle \phi \rangle_{\not K}$ independent of space-inverse temperature, we conclude from equations (D.27) and (D.28) that the so-called **effective potential** V_{eff} is suitable, because

$$(i) \quad \langle \phi \rangle_{\not K}|_{J=0} \text{ fulfills the necessary condition } \frac{d\, V_{eff}(\langle \phi \rangle_{\not K}|_{J=0})}{d\, \langle \phi \rangle_{\not K}|_{J=0}} = 0 \ , \quad \text{(D.29)}$$

and (as more elaborated calculation would show[4])

(ii) Neglecting all diagrams with loops (we speak of **tree-level**) yields $V_{eff} = U$.

We know that an expansion in loops corresponds to an expansion in powers of $\hbar$, so that tree-level means nothing but the classical limit $\hbar \longrightarrow 0$. But the vacuum for the classical limit is the global minimum of the classical potential U. So all in all, we have to regard V_{eff} as the potential of the system with quantum corrections (which entirely arise from diagrams with loops), and we have to conclude:

result D.1 (calculation rule)

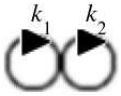
$\langle \phi \rangle_{\not K}|_{J=0}$ is the global minimum of V_{eff}.

Remark: Feynman rules in momentum space

So far, the Feynman diagrams introduced in this appendix are abbreviations which can be translated via the Feynman rules in position space. However, if one is interested in explicitly calculating a diagram resp. the mathematical expression it stands for, this would require an explicit expression for the Feynman propagator in position space, given by formula (D.13). Unfortunately, even carrying out just the Matsubara sum, yields a lengthy result containing hypergeometric functions. Therefore, one works in momentum space. Let us consider for example the double-bubble diagram  . Translated via the Feynman rules in position space, it is equal to

$$\begin{aligned} -\lambda \int\limits_X \Delta(0)^2 &= -\lambda \int\limits_0^{1/T} \int d^3\mathbf{x} \left[T \sum_{n=-\infty}^{\infty} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{\omega_n^2 + \mathbf{k}^2 + m^2} \right]^2 \\ &= -\lambda \frac{\Omega}{T} \left[T \sum_{n=-\infty}^{\infty} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{\omega_n^2 + \mathbf{k}^2 + m^2} \right]^2 . \end{aligned}$$

[4]  are obviously the only 1PI diagrams at tree-level, therefore only $n = 2, 4$ in the sum in definition (D.2) contribute nonvanishing addends.

Let us now draw a new kind of diagram, 

, by simply assigning a momentum to each line (the direction of the arrow is arbitrary, it depends on how one chooses the sign for each of the momenta). The Feynman rules how to translate such a diagram, can be found for example in the textbook of Kapusta chapter 3. A factor of $T\sum_{n=-\infty}^{\infty}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{\omega_n^2+\mathbf{k}^2+m^2}$ for each internal line, a factor of $-\lambda$ at each vertex and an overall factor of Ω/T yields likewise

$$-\lambda\frac{\Omega}{T}\left[T\sum_{n=-\infty}^{\infty}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{\omega_n^2+\mathbf{k}^2+m^2}\right]^2 .$$

We calculated the same quantity, using the Feynman rules in momentum space. Taking any connected diagram or sum of connected diagrams, one can either translate them using the position space Feynman rules, or alternatively by applying the momentum space Feynman rules. Both results will be equal.

Appendix E

From Quantum Mechanics to Statistical Quantum Field Theory

The generating functional for N particles (with positions $x_1, ..., x_N$) is given in Quantum Mechanics by

$$Z = \int Dx_1(t) ... \int Dx_N(t) \exp\left(i \int_{t_a}^{t_b} dt \left(\sum_{n=1}^{N} \frac{1}{2} m \left[\frac{dx_n}{dt} \right]^2 - U(x_1, ...x_N) \right) + sourceterms \right) . \tag{E.1}$$

Note that (E.1) refers to zero temperature. We denote the generating functional by the letter Z, because, up to a constant prefactor, it is equal to the partition function A.3 for $T \to 0$ and vanishing sources.
In a heuristic sense, Zee motivates in his textbook the definition of the generating functional for a scalar Quantum Field Theory:

definition E.1 (generating functional, QFT)

$$Z = \int D\phi(t, \mathbf{x}) \, e^{iS + sourceterms} ,$$

$$with \; S = \int dt \int d^3\mathbf{x} \left(\frac{1}{2} \partial_\mu \phi(t, \mathbf{x}) \, \partial^\mu \phi(t, \mathbf{x}) - U(\phi(t, \mathbf{x})) \right) ,$$

$\partial_\mu \phi(t, \mathbf{x}) \, \partial^\mu \phi(t, \mathbf{x})$ *is understood using the Minkowski metric* $g^{\mu\nu} = (+1, -1, -1, -1)$,

$$i.e., \; \partial_\mu \phi(t, \mathbf{x}) \, \partial^\mu \phi(t, \mathbf{x}) = (\frac{d\phi}{dt})^2 - (\frac{d\phi}{dx})^2 - (\frac{d\phi}{dy})^2 - (\frac{d\phi}{dz})^2 .$$

As one can learn from Zee's textbook [43], what was the mass m in Quantum Mechanics has been absorbed into the field ϕ by redefining $\phi \longrightarrow \frac{\phi}{\sqrt{\rho}}$, where ρ is the mass density. Mass in Quantum Field Theory is generated by appropriate mass terms in U. Whereas expression (E.1) is the generating functional for a non-relativistic quantized theory, definition E.1 describes a relativistic and quantized theory. We now want to *replace* the real time t by negative imaginary

time $\overline{t}$. This is called a *Wick rotation* from real time to negative imaginary time:

$$t \longrightarrow \overline{t} \equiv -i\tau \ , \qquad \text{where } t \in \mathbb{R},\ \tau \in \mathbb{R} \ . \tag{E.2}$$

We obtain the so-called *Euclidean* version of the generating functional:

result E.1 (Euclidean generating functional,QFT)

$$Z = \int D\phi\,(\tau, \mathbf{x})\, e^{-S_E + sourceterms} \ ,$$

$$\text{where } S_E = \int_{\tau_a}^{\tau_b} d\tau \int d^3\mathbf{x} \left(\frac{1}{2} \partial_{\mu,E}\phi\,(\tau, \mathbf{x})\, \partial_E^{\mu}\phi\,(\tau, \mathbf{x}) + U\left(\phi\,(\tau, \mathbf{x})\right) \right) \ ,$$

$\partial_{\mu,E}\phi\,(\tau, \mathbf{x})\, \partial_E^{\mu}\phi\,(\tau, \mathbf{x})$ *is understood using the Euclidean metric*

$$\delta^{\mu\nu} = diag\,(+1, +1, +1, +1), \ \textit{i.e.,}\ \partial_{\mu,E}\phi\,(\tau, \mathbf{x})\, \partial_E^{\mu}\phi\,(\tau, \mathbf{x}) = (\frac{d\phi}{d\tau})^2 + (\frac{d\phi}{dx})^2 + (\frac{d\phi}{dy})^2 + (\frac{d\phi}{dz})^2 \ .$$

Actually, in result E.1 one should write "Z_E" instead of "Z" to distinguish the quantity from Z in definition E.1, which is not the same. Since in the literature the letter Z is used for different kinds of partition functions and generating functionals anyway, we do not introduce a subscript. Let us recall the steps we performed. First, we replaced t by $\overline{t}$. Then we did the simple coordinate transformation $\overline{t} = -i\tau$, i.e.,

$$\frac{d\overline{t}}{d\tau} = -i \iff i d\overline{t} = d\tau \ , \quad \frac{d\phi}{d\overline{t}} = \frac{d\phi}{-i d\tau} \implies \left(\frac{d\phi}{d\overline{t}}\right)^2 = -\left(\frac{d\phi}{d\tau}\right)^2 \ ,$$

which resulted E.1. Note that the Wick rotation known from Quantum Mechanics is just a formal trick. After the Wick rotation, further computations are easier to handle, but the results for physical quantities would be imaginary, i.e., one has to rotate back after having performed the complicated steps. In this sense the Euclidean version of the generating functional is equivalent to the Minkowskian one. In contrast, in the *imaginary-time formalism* (or *Matsubara formalism*), SQFT is based on the Euclidean generating functional where the following additional features come into play:

- one has to identify $\tau_b \equiv \frac{1}{T}$ and $\tau_a \equiv 0$,
- one has to impose the periodic boundary condition $\phi(\tau, \vec{x}) \stackrel{!}{=} \phi(\tau + \frac{1}{T}, \vec{x})$.

The idea that these constraints yield a theory at nonzero temperature was established by Matsubara [69]. For further details see [70]. The generating functional for our scalar Quantum Field Theory at nonzero temperature is therefore:

definition E.2 (generating functional, SQFT)

$$Z = \oint D\phi\,(\tau,\mathbf{x})\, e^{-S_E + sourceterms}\ ,$$

$$\text{where } S_E = \int_0^{1/T} d\tau \int d^3\mathbf{x} \left(\frac{1}{2} \partial_{\mu,E}\phi\,(\tau,\mathbf{x})\, \partial_E^{\mu}\phi\,(\tau,\mathbf{x}) + U\left(\phi\,(\tau,\mathbf{x})\right) \right)\ ,$$

$\partial_{\mu,E}\phi\,(\tau,\mathbf{x})\,\partial_E^{\mu}\phi\,(\tau,\mathbf{x})$ is understood using the the Euclidean metric

$$\delta^{\mu\nu} = diag\,(+1,+1,+1,+1)\,,\ \textit{i.e.,}\ \partial_{\mu,E}\phi\,(\tau,\mathbf{x})\,\partial_E^{\mu}\phi\,(\tau,\mathbf{x}) = (\frac{d\phi}{d\tau})^2 + (\frac{d\phi}{dx})^2 + (\frac{d\phi}{dy})^2 + (\frac{d\phi}{dz})^2\ .$$

One has to bear in mind the periodic boundary condition $\phi(\tau,\vec{x}) \stackrel{!}{=} \phi(\tau + \frac{1}{T},\vec{x})$.

For clarification, let us carry out the transition from Quantum Mechanics to 1+0 dimensional SQFT within the discretized version of the path integral. Without loss of generality let us consider the quantum-mechanical time-transition amplitude

$$\langle x_{1,b} x_{2,b} t_b | x_{1,a} x_{2,a} t_a \rangle = \frac{1}{2\pi\hbar i\epsilon/m} \left[\prod_{n=1}^{N} \int \frac{dx_{1,n}}{\sqrt{2\pi\hbar i\epsilon/m}} \right] \left[\prod_{n=1}^{N} \int \frac{dx_{2,n}}{\sqrt{2\pi\hbar i\epsilon/m}} \right] e^{\frac{i}{\hbar}A}\ , \tag{E.3}$$

$$\text{with } A \equiv \epsilon \sum_{n=1}^{N+1} \left[\frac{m}{2} \left(\frac{\vec{x}_n - \vec{x}_{n-1}}{\epsilon} \right)^2 - U(\vec{x}_n, t_n) \right]\ . \tag{E.4}$$

Performing a Wick rotation $t \to \overline{t} = -i\tau$ resp. $\epsilon = t_n - t_{n-1} \to -i(\tau_n - \tau_{n-1}) \equiv -i\varepsilon$, carrying out the replacements $m \to 1$, $\vec{x} \to \vec{\phi}$ and identifying $\tau_a \equiv 0$, $\tau_b \equiv \beta$, we obtain the corresponding expression for the thermal evolution amplitude in **SQFT in 1+0 dimensions**:

$$\langle \phi_{1,b} \phi_{2,b} \beta | \phi_{1,a} \phi_{2,a} 0 \rangle = \frac{1}{2\pi\hbar\varepsilon} \left[\prod_{n=1}^{N} \int \frac{d\phi_{1,n}}{\sqrt{2\pi\hbar\varepsilon}} \right] \left[\prod_{n=1}^{N} \int \frac{d\phi_{2,n}}{\sqrt{2\pi\hbar\varepsilon}} \right] \times$$
$$\times \exp\left(\frac{\varepsilon}{\hbar} \sum_{n=1}^{N+1} \left[\frac{1}{2} \left(\frac{\vec{\phi}_n - \vec{\phi}_{n-1}}{-i\varepsilon} \right)^2 - U(\vec{\phi}_n, -i\tau_n) \right] \right),$$

or shorter:

$$\langle \phi_{1,b} \phi_{2,b} \beta | \phi_{1,a} \phi_{2,a} 0 \rangle = \frac{1}{2\pi\hbar\varepsilon} \left[\prod_{n=1}^{N} \int \frac{d\phi_{1,n}}{\sqrt{2\pi\hbar\varepsilon}} \right] \left[\prod_{n=1}^{N} \int \frac{d\phi_{2,n}}{\sqrt{2\pi\hbar\varepsilon}} \right] e^{-\frac{1}{\hbar}A_E} \tag{E.5}$$

$$\text{with } A_E = \varepsilon \sum_{n=1}^{N+1} \left[\frac{1}{2} \left(\frac{\vec{\phi}_n - \vec{\phi}_{n-1}}{\varepsilon} \right)^2 + U(\vec{\phi}_n, -i\tau_n) \right]. \tag{E.6}$$

According to result A.4, the partition function Z is obtained by setting $\phi_{1,a} = \phi_{1,b}$ as well as $\phi_{2,a} = \phi_{2,b}$, and then integrating over $\phi_{1,a} = \phi_{1,b}$ as well as over $\phi_{2,a} = \phi_{2,b}$:

$$Z = \left[\prod_{n=1}^{N+1} \int \frac{d\phi_{1,n}}{\sqrt{2\pi\hbar\varepsilon}}\right] \left[\prod_{n=1}^{N+1} \int \frac{d\phi_{2,n}}{\sqrt{2\pi\hbar\varepsilon}}\right] e^{-\frac{1}{\hbar}A_E} \ , \tag{E.7}$$

$$\text{where } \phi_{1,a} = \phi_{1,b} \text{ and } \phi_{2,a} = \phi_{2,b} \ , \tag{E.8}$$

$$\text{with the Euclidean action } A_E = \varepsilon \sum_{n=1}^{N+1} \left[\frac{1}{2}\left(\frac{\vec{\phi}_n - \vec{\phi}_{n-1}}{\varepsilon}\right)^2 + U(\vec{\phi}_n, -i\tau_n)\right] . \tag{E.9}$$

Moreover, let us specify the periodic boundary condition on the Greens function for the harmonic oscillator in SQM and for free SQFT in 1+0 dimensions with a mass term $m^2\phi^2$ respectively. The proportionality of the Greens function and the 2-point function for these cases is shown in appendix F. The generating functional for the **harmonic oscillator** in **QM** is given by

$$Z[J] = \int_{-\infty}^{\infty} \mathcal{D}x \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} dt \left[\frac{M}{2}(\dot{x}^2 - \omega^2 x^2) + x(t)J(t)\right]\right\} . \tag{E.10}$$

As shown for example in chapter 3 of Kleinert's textbook [42], the Greens function $Gr(t,t')$ for this case is defined as the general solution of

$$\left(-\frac{\partial^2}{\partial t^2} - \omega^2\right) Gr(t,t') \equiv \delta(t-t') \quad , \quad \text{with } t,t' \in [t_a, t_b] \ . \tag{E.11}$$

Performing a Wick-rotation $t \to \overline{t} = -i\tau$,
we obtain

the generating functional for the **harmonic oscillator** in **Euclidean QM**:

$$Z[J] = \int_{-\infty}^{\infty} \mathcal{D}x \exp\left\{-\frac{1}{\hbar}\int_{\tau_a}^{\tau_b} d\tau \left[\frac{M}{2}(\dot{x}^2 + \omega^2 x^2) - x(\tau)J(\tau)\right]\right\} \ , \tag{E.12}$$

as well as the differential equation determining the Euclidean Greens function $Gr_E(\tau,\tau')$:

$$\left(-\frac{\partial^2}{\partial \tau^2} + \omega^2\right) Gr_E(\tau,\tau') \equiv \delta(\tau-\tau') \quad , \quad \text{with } \tau,\tau' \in [\tau_a, \tau_b] \ . \tag{E.13}$$

Imposing periodic boundary conditions,
i.e., $Gr_E^P(\tau,\tau') \equiv Gr_E^P(\tau-\tau') = Gr_E^P\left(\tau - \tau' + (\tau_b - \tau_a)\right)$,
and setting $\tau_a \equiv 0$, $\tau_b \equiv \frac{1}{T}$,
we obtain

the generating functional for the **harmonic oscillator** in **SQM**:

$$Z[J] = \oint_{-\infty}^{\infty} \mathcal{D}x \exp\left\{-\frac{1}{\hbar}\int_{0}^{1/T} d\tau \left[\frac{M}{2}(\dot{x}^2 + \omega^2 x^2) - x(\tau)J(\tau)\right]\right\} \ , \tag{E.14}$$

as well as the differential equation which defines the periodic Euclidean (i.e., quantum statistical) Greens function $Gr_E^P(\tau-\tau')$:

$$\left(-\frac{\partial^2}{\partial\tau^2}+\omega^2\right)Gr_E^P(\tau-\tau')\equiv\delta(\tau-\tau')\quad,\quad\text{with }\tau-\tau'\in[0,\frac{1}{T})\ . \tag{E.15}$$

Finally, making the replacements

$$M\to 1,\ \omega^2\to m^2,\ x\to\phi\ ,$$

we end up with

the generating functional for **free SQFT in 1+0 dimensions with a mass term $\mathbf{m^2\phi^2}$**:

$$Z[J]=\oint_{-\infty}^{\infty}\mathcal{D}\phi\exp\left\{-\frac{1}{\hbar}\int_0^{1/T}d\tau\left[\frac{1}{2}\dot{\phi}^2+\frac{m^2}{2}\phi^2-\phi(\tau)J(\tau)\right]\right\}\ , \tag{E.16}$$

and again the differential equation defining the quantum field statistical Greens function $Gr_S(\tau,\tau')$:

$$\left(-\frac{\partial^2}{\partial\tau^2}+m^2\right)Gr_S(\tau-\tau')\equiv\delta(\tau-\tau')\quad,\quad\text{with }\tau-\tau'\in[0,\frac{1}{T})\ . \tag{E.17}$$

At the end of this appendix, let us discuss the Wick rotation in momentum space. In QFT, the *Minkowskian* position-space representation is equivalent to a *Minkowskian* momentum-space representation which is based on the Fourier transformation. Accordingly, the *Euclidean* position-space representation of SQFT is equivalent to a *Euclidean* momentum-space representation based on the Fourier transformation. The Wick rotation in momentum space,

$$k_0\longrightarrow\overline{k}_0\equiv ip_4\ , \tag{E.18}$$
$$\text{where } k_0\in\mathbb{R},\ p_4\in\mathbb{R}\ ,$$

is the analogue of the transformation (E.2).
Consider for example the Feynman propagator $\Delta_F(X)$ as defined in **QFT**:

$$\Delta_F(X)\equiv\int\frac{d^4k}{(2\pi)^4}e^{-i\mathbf{kx}}\underbrace{\frac{1}{k^2-m^2}}_{\equiv\Delta_F(k)}\ , \tag{E.19}$$

which is, up to a prefactor, equal to the free 2-point function in the absence of a source:

$$\Delta_F(X-Y)\equiv\frac{1}{i}\langle\phi(X)\,\phi(Y)\rangle_{free}^{(T=0)}|_{J=K=0}\ , \tag{E.20}$$

where $\langle\ \rangle_{free}^{(T=0)}|_{J=K=0}$ refers to definition A.10 with the subscript *free* indicating that the action (which enters the definition of the expectation value) is that for the free theory (i.e., $U_I=0$) . $\Delta_F(k)$ is the momentum-space representation of $\Delta_F(X)$.
Let us perform the Wick rotations (E.2) and (E.18):

$$\underbrace{\int dk_0\int\frac{d^3\mathbf{k}}{(2\pi)^4}e^{-ik_0t+i\mathbf{kx}}\frac{1}{k_0^2-\mathbf{k}^2-m^2}}_{=\Delta_F(X)}\longrightarrow-i\underbrace{\int\frac{dp_4}{2\pi}\int\frac{d^3\mathbf{k}}{(2\pi)^3}e^{-i(p_4\tau-\mathbf{kx})}\overbrace{\frac{1}{p_4^2+\mathbf{k}^2+m^2}}^{\equiv\Delta_{F,E}(p_4,\mathbf{k})}}_{\equiv\Delta_{F,E}(X)}\ . \tag{E.21}$$

$\Delta_{F,E}(X)$ is the *Euclidean Feynman propagator* in position space. $\Delta_{F,E}(p_4, \mathbf{k})$ is the *Euclidean Feynman propagator* in momentum space. From the Euclidean Feynman propagator we obtain the Feynman propagator at nonzero temperature, i.e., in **SQFT**, as follows. Due to the constraint of periodicity

$$\phi(1/T, \mathbf{x}) \stackrel{!}{=} \phi(0, \mathbf{x}) \ \forall \ \mathbf{x} \ , \tag{E.22}$$

which one has to impose at nonzero temperature, we have

$$p_4 \stackrel{!}{=} 2\pi n T \equiv \omega_n \tag{E.23}$$

for our bosonic scalar field. Imposing the constraint (E.23) on the Euclidean Feynman propagator, we obtain the *thermal Feynman propagator* in position space, denoted by $\Delta(X)$ in this thesis, and in momentum space, denoted by $\Delta(\omega_n, \mathbf{k})$:

$$\Delta(X) = T \sum_{n=-\infty}^{\infty} \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{-i(\omega_n \tau - \mathbf{k}\mathbf{x})} \overbrace{\frac{1}{\omega_n^2 + \mathbf{k}^2 + m^2}}^{\equiv \Delta(\omega_n, \mathbf{k})} . \tag{E.24}$$

Appendix F

Proportionality of free 2-point function and Greens function

In this appendix we refer to the definition A.11 for $K = 0$ and $X = \tau$, whereas we denote the correctly normalized thermal n-point function...

- ...in the presence of a single source J by:

$$\langle \phi(\tau_1) \cdots \phi(\tau_n) \rangle_{\not K} \equiv \frac{\oint \mathcal{D}\phi \; e^{-\frac{1}{\hbar} S_E[\phi] + \frac{1}{\hbar}\phi J} \phi(\tau_1) \; ... \; \phi(\tau_n)}{\oint \mathcal{D}\phi \; e^{-\frac{1}{\hbar} S_E[\phi] + \frac{1}{\hbar}\phi J}} \; , \tag{F.1}$$

- ...(in the absence of a source) by:

$$\langle \phi(\tau_1) \cdots \phi(\tau_n) \rangle_{\not K}|_{J=0} \equiv \left. \frac{\oint \mathcal{D}\phi \; e^{-\frac{1}{\hbar} S_E[\phi] + \frac{1}{\hbar}\phi J} \phi(\tau_1) \; ... \; \phi(\tau_n)}{\oint \mathcal{D}\phi \; e^{-\frac{1}{\hbar} S_E[\phi] + \frac{1}{\hbar}\phi J}} \right|_{J=0} . \tag{F.2}$$

For consistency, we use the subscript $\not K$ to indicate the absence of the source K in the definition. Of course, the absence of the source is equivalent to setting the source to zero, indicated by $|_{K=0}$, however we want to be consistent with the notation we use in appendix D. Note that we write out $\hbar = 1$ explicitly.

F.0.3 Proving the proportionality

In the following we want to show that, for $S_E = \int_0^{1/T} d\tau \left[\frac{1}{2}\dot{\phi}^2 + \frac{m^2}{2}\phi^2 \right]$, the free[1] thermal propagator (in the absence of a source), $\langle \phi(\tau_1)\phi(\tau_2) \rangle_{\not K}|_{J=0}$, is, up to a prefactor, equal to the Greens-function $Gr_S(\tau_1 - \tau_2)$ (see (E.17) for its definition):

result F.1 (free SQFT in 1+0 dimensions with a mass term $m^2\phi^2$)

$$\langle \phi(\tau_1)\phi(\tau_2) \rangle_{\not K}|_{J=0} = \hbar Gr_S(\tau_1 - \tau_2) \; .$$

[1] free, because we refer to the free action.

The proof goes as follows:
Using

$$\int_0^{1/T} d\tau \frac{1}{2}\dot{\phi}\dot{\phi} = \left[\frac{1}{2}\phi\dot{\phi}\right]_0^{1/T} - \int_0^{1/T} \frac{1}{2}\phi\ddot{\phi}d\tau \qquad \text{(integration by parts)} \ , \tag{F.3}$$

$$\left[\frac{1}{2}\phi\dot{\phi}\right]_0^{1/T} = 0 \qquad \text{due to } \phi(0) = \phi(1/T) \ , \tag{F.4}$$

and

$$\begin{aligned} &\int_0^{1/T} d\tau_2 \frac{1}{2}\phi(\tau_1)\left(-\frac{\partial^2}{\partial\tau_1^2} + m^2\right)\delta(\tau_1-\tau_2)\phi(\tau_2) \\ &= \int_0^{1/T} d\tau_2 \left[-\frac{1}{2}\phi(\tau_1)\frac{\partial^2}{\partial\tau_1^2}\delta(\tau_1-\tau_2)\phi(\tau_2)\right] + \int_0^{1/T} d\tau_2 \left[\frac{1}{2}m^2\phi(\tau_1)\phi(\tau_2)\delta(\tau_1-\tau_2)\right] \\ &= -\frac{1}{2}\phi(\tau_1)\frac{\partial^2}{\partial\tau_1^2}\phi(\tau_1) + \frac{1}{2}m^2\phi(\tau_1)^2 \ , \end{aligned} \tag{F.5}$$

we are able to rewrite the generating functional as follows:

$$Z[J] = \oint_{-\infty}^{\infty} \mathcal{D}\phi \exp\Big(-\frac{1}{\hbar}\int_0^{1/T} d\tau_1 [\underbrace{\int_0^{1/T} d\tau_2 \frac{1}{2}\phi(\tau_1)\left(-\frac{\partial^2}{\partial\tau_1^2} + m^2\right)\delta(\tau_1-\tau_2)\phi(\tau_2)}_{\equiv\frac{1}{2}\phi D\phi} - \phi(\tau_1)J(\tau_1)]\Big), \tag{F.6}$$

with the definition $D(\tau_1-\tau_2) \equiv \left(-\frac{\partial^2}{\partial\tau_1^2} + m^2\right)\delta(\tau_1-\tau_2)$. Comparison with definition (E.17) yields $D^{-1}(\tau_1-\tau_2) = Gr_S(\tau_1-\tau_2)$.
Treating $D(\tau_1,\tau_2)$ as matrix with continuous indices, and completing the square by introducing $\phi' \equiv \phi - D^{-1}J$, one is able to rewrite Z in the form[2]

$$Z[J] = \oint_{-\infty}^{\infty} \mathcal{D}\phi' e^{-\frac{1}{\hbar}\int_0^{1/T} d\tau_1 \frac{1}{2}\phi' D\phi'} \underbrace{\exp\left\{\frac{1}{2\hbar}\int_0^{1/T} d\tau_1 \int_0^{1/T} d\tau_2 \ J(\tau_1)D^{-1}(\tau_1-\tau_2)J(\tau_2)\right\}}_{\text{independent of } \phi'} \ . \tag{F.7}$$

As the second exponent is independent of ϕ', one is able to perform the path integral, which leads to

$$Z[J] = \frac{1}{2\sinh\frac{m}{2T}} \exp\left\{\frac{1}{2\hbar}\int_0^{1/T} d\tau_1 \int_0^{1/T} d\tau_2 \ J(\tau_1)D^{-1}(\tau_1-\tau_2)J(\tau_2)\right\}. \tag{F.8}$$

[2]Compare with page 241 of [42].

Since

$$\langle\phi(\tau_1)\cdots\phi(\tau_n)\rangle_{\not K} = \frac{1}{Z[J]}\hbar\frac{\delta}{\delta J(\tau_1)}\cdots\hbar\frac{\delta}{\delta J(\tau_n)}Z[J]\ , \tag{F.9}$$

which follows directly from expressions (E.16) and (F.1), we have:

$$\begin{aligned}
&\langle\phi(\tau_1)\phi(\tau_2)\rangle_{\not K}|_{J=0} = \\
&= 2\sinh\left(\frac{m}{2T}\right)\hbar^2\left[\frac{\delta}{\delta J(\tau_1)}\frac{\delta}{\delta J(\tau_2)}\frac{1}{2\sinh\frac{m}{2T}}\exp\left\{\frac{1}{2\hbar}\int_0^{1/T}d\tau_1\int_0^{1/T}d\tau_2\ J(\tau_1)D^{-1}(\tau_1-\tau_2)J(\tau_2)\right\}\right]_{J=0} \\
&= \hbar^2\left[\frac{1}{2\hbar}2D^{-1}(\tau_1-\tau_2)\exp\left\{\frac{1}{2\hbar}\int_0^{1/T}d\tau_1\int_0^{1/T}d\tau_2\ J(\tau_1)D^{-1}(\tau_1-\tau_2)J(\tau_2)\right\}\right]_{J=0} \\
&= \hbar D^{-1}(\tau_1-\tau_2) = \hbar G r_S(\tau_1-\tau_2)\ .
\end{aligned} \tag{F.10}$$

q.e.d.

F.0.4 Explicit expression for the free thermal propagator

According to [42], p.242, $\langle\phi(\tau_1)\phi(\tau_2)\rangle_{\not K}|_{J=0} = \hbar G r_S(\tau_1-\tau_2)$ is given by

$$\langle\phi(\tau_1)\phi(\tau_2)\rangle_{\not K}|_{J=0} = T\sum_{n=-\infty}^{\infty}\frac{1}{\omega_n^2+m^2}e^{-i\omega_n(\tau_1-\tau_2)} = \frac{1}{2m}\frac{\cosh\left[m(|\tau_1-\tau_2|-\frac{1}{2T})\right]}{\sinh\left(\frac{m}{2T}\right)} \tag{F.11}$$

$$\text{with } |\tau_1-\tau_2|\in[0,\hbar/T)\ ,\quad \omega_n = \hbar 2\pi n T\ ,\quad (n=0,\pm1,\pm2,\cdots)\ ,$$

where m is the same as in the free action.
In the zero-temperature limit, the above expression becomes:

$$\langle\phi(\tau_1)\phi(\tau_2)\rangle_{\not K}|_{J=0} \overset{T=0}{=} \hbar\int\frac{d\omega_n}{2\pi}\frac{1}{\omega_n^2+m^2}e^{-i\omega_n(\tau_1-\tau_2)} = \frac{1}{2m}e^{-m|\tau_1-\tau_2|}\ . \tag{F.12}$$

Appendix G

Important thermodynamic relations

In this appendix we want to list some important thermodynamic relations. For further details regarding Statistical Mechanics and Statistical Quantum Mechanics we recommend the textbook of Nolting [16]. Note that we work in natural units where $k_B \equiv 1$.
Let us begin with an important concept. *Thermal equilibrium* means that the thermodynamic quantities do not explicitly depend on space and time. *Local thermal equilibrium* means that although quantities are varying in space and time, this takes place so slowly that for any point one can assume thermodynamic equilibrium.
Whereas the microscopic definitions for thermodynamic quantities in statistical physics depend on the ensemble, the relations between thermodynamic quantities[1],
conserved charges: Q_i, chemical potentials: μ_i,
spatial volume: V, temperature: T, entropy: S, pressure: p,
Helmholtz free energy: F, Gibbs free energy: G, internal energy: E or U, grand canonical potential: Ω,
are universally valid (compare with the discussion in section A.1).
We want to give the microscopic definitions for some thermodynamic quantities in the grand canonical ensemble:

$$p = T \left(\frac{\partial}{\partial V} \ln \mathcal{Z} \right)_{T,\{\mu_i\}} , \tag{G.1}$$

$$\Omega = -T \ln \mathcal{Z} , \tag{G.2}$$

$$U = - \left(\frac{\partial \ln \mathcal{Z}}{\partial \beta} \right)_{\{\mu_i\}} + \sum_i \frac{\mu_i}{\beta} \left(\frac{\partial \ln \mathcal{Z}}{\partial \mu_i} \right)_{\beta} , \tag{G.3}$$

$$S = \ln \mathcal{Z} + \beta U - \beta \sum_i \mu_i Q_i , \tag{G.4}$$

$$F = \Omega + \sum_i \mu_i Q_i . \tag{G.5}$$

[1] We list only the commonly used. Note that one can of course define other thermodynamic quantities, such as for example enthalpy.

For $\mu_i = 0$ and homogeneous systems (i.e., $\frac{\partial}{\partial V} \equiv \frac{1}{V}$), we obtain from definitions A.1, A.3, (G.1) and (G.5):

$$p \overset{\mu_i=0}{=} \frac{T}{V} \ln Z = -\frac{\Omega}{V} = -\frac{F}{V} \ . \tag{G.6}$$

Finally, we want to list the most important relations between thermodynamic quantities. Depending on the system, it is convenient to work either with E, F, G or Ω.

$$E = E(S, V, \{Q_i\}) \ , \tag{G.7}$$

$$\text{1st law of thermodynamics: } \ dE = TdS - pdV + \sum_i \mu_i dQ_i \ , \tag{G.8}$$

$$T = \left(\frac{\partial E}{\partial S}\right)_{V,\{Q_i\}} \ , \tag{G.9}$$

$$p = -\left(\frac{\partial E}{\partial V}\right)_{S,\{Q_i\}} \ , \tag{G.10}$$

$$\mu_j = \left(\frac{\partial E}{\partial Q_j}\right)_{S,V,\{Q_i\}_{i\neq j}} \ . \tag{G.11}$$

$$F = E - TS, \ F = F(T, V, \{Q_i\}) \ , \tag{G.12}$$

$$\text{1st law of thermodynamics: } \ dF = -SdT - pdV + \sum_i \mu_i dQ_i \ , \tag{G.13}$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,\{Q_i\}} \ , \tag{G.14}$$

$$p = -\left(\frac{\partial F}{\partial V}\right)_{T,\{Q_i\}} \ , \tag{G.15}$$

$$\mu_j = \left(\frac{\partial F}{\partial Q_j}\right)_{T,V,\{Q_i\}_{i\neq j}} \ . \tag{G.16}$$

$$G = E - TS + pV, \ G = G(T, p, \{Q_i\}) \ , \tag{G.17}$$

$$\text{1st law of thermodynamics: } \ dG = -SdT + Vdp + \sum_i \mu_i dQ_i \ , \tag{G.18}$$

$$S = -\left(\frac{\partial G}{\partial T}\right)_{p,\{Q_i\}} \ , \tag{G.19}$$

$$V = \left(\frac{\partial G}{\partial p}\right)_{T,\{Q_i\}} \ , \tag{G.20}$$

$$\mu_j = \left(\frac{\partial G}{\partial Q_j}\right)_{T,p} \ . \tag{G.21}$$

$$\Omega = -pV, \ \Omega = \Omega(T, V, \{\mu_i\}) \ , \tag{G.22}$$

$$\text{1st law of thermodynamics: } \ d\Omega = -SdT - pdV - \sum_i Q_i d\mu_i \ , \tag{G.23}$$

$$S = -\left(\frac{\partial \Omega}{\partial T}\right)_{V,\{\mu_i\}} \ , \tag{G.24}$$

$$p = -\left(\frac{\partial \Omega}{\partial V}\right)_{T,\{\mu_i\}} = -\frac{\Omega}{V} \ , \tag{G.25}$$

$$Q_j = -\left(\frac{\partial \Omega}{\partial \mu_j}\right)_{T,V,\{\mu_i\}_{i\neq j}} \ . \tag{G.26}$$

Appendix H

QCD and SQCD

H.1 QCD

Consider a non-Abelian group G, whose elements can be represented as $N \times N$-matrices $U = \exp\left(-it^a\theta^a\right)$ $(a = 1, ..., n)$ belonging to the fundamental representation of G. Then consider a "fermion field" $\psi(X)$ with N components $\psi_i(X)$, on which U can act, and n "gauge-fields" A^a_μ, labeled by $a = 1, ..., n$, with 4 components labeled by Lorentz indices $\mu = 0, ..., 3$. Then the general form of a Lagrangian invariant under *non-Abelian local gauge transformations*

$$\psi_i \longrightarrow U_{ij}\psi_j \ , \tag{H.1}$$

$$t^a A^a_\mu = U\left(t^a A^a_\mu - \frac{i}{g}U^{-1}\partial_\mu U\right)U^{-1} \ , \tag{H.2}$$

is given by

$$\mathscr{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \overline{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi \ , \tag{H.3}$$

$$\text{with } D_\mu \equiv \partial_\mu - igt^a A^a_\mu \longrightarrow U D_\mu U^{-1} \ , \tag{H.4}$$

$$\text{and } F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \ , \tag{H.5}$$

$$\text{where } f^{abc} \text{ are the structure constants of G.} \tag{H.6}$$

We speak of a *non-Abelian local gauge theory* (syn. *local Yang-Mills theory*).
For $G = SU(N_c)$, we obtain the Lagrangian used in QCD, for N_c colors. For $N_c = 3$ the generators t^a would be half the Gell-Mann matrices, i.e., $t^a = \lambda_a/2$. ψ is a N_c-tuple with N_f-tuples as components with 4-tuples as components. For example for $N_c = 3$ and $N_f = 3$:

$$\psi = \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} u_r \\ d_r \\ s_r \\ u_g \\ d_g \\ s_g \\ u_b \\ d_b \\ s_b \end{pmatrix} = \begin{pmatrix} u_{r,0} \\ u_{r,1} \\ u_{r,2} \\ u_{r,3} \\ d_{r,0} \\ d_{r,1} \\ d_{r,2} \\ d_{r,3} \\ s_{r,0} \\ s_{r,1} \\ s_{r,2} \\ s_{r,3} \\ u_{g,0} \\ u_{g,1} \\ u_{g,2} \\ u_{g,3} \\ d_{g,0} \\ d_{g,1} \\ d_{g,2} \\ d_{g,3} \\ s_{g,0} \\ s_{g,1} \\ s_{g,2} \\ s_{g,3} \\ u_{b,0} \\ u_{b,1} \\ u_{b,2} \\ u_{b,3} \\ d_{b,0} \\ d_{b,1} \\ d_{b,2} \\ d_{b,3} \\ s_{b,0} \\ s_{b,1} \\ s_{b,2} \\ s_{b,3} \end{pmatrix} \tag{H.7}$$

Note that the matrices U act on ψ represented as N_c-tuple. Together with the transformation law (H.2), the invariance of the Lagrangian under non-Abelian local gauge transformations can be seen using the compact notation (H.3).

$\mathscr{L}$ can be divided into a free part and an interaction part, i.e.,

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_I \ . \tag{H.8}$$

Although one always works with the compact notation, writing it out explicitly clears up its meaning:

$$\begin{aligned}\mathscr{L}_0 = &\sum_{q=1}^{N_f}\sum_{\mu=0}^{3}\sum_{c=1}^{N_c}\overline{\psi}_{q\mu c}\left[i\sum_{\nu=0}^{3}(\gamma^\nu\partial_\nu) - m_{qq'}\delta_{qq'}\delta_{\mu\mu'}\delta_{cc'}\right]\psi_{q'\mu'c'} \\ &-\frac{1}{4}\sum_{a=1}^{N_c^2-1}\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu)\ , \end{aligned} \tag{H.9}$$

$$\begin{aligned}\mathscr{L}_I = g\sum_{q\mu\mu'cc'}\overline{\psi}_{q\mu c}\Big[&\left(\gamma^0\right)_{\mu\mu'}\left\{\left(t^1\right)_{cc'}A_0^1 + \left(t^2\right)_{cc'}A_0^2 + \cdots + \left(t^8\right)_{cc'}A_0^8\right\} \\ &+\left(\gamma^1\right)_{\mu\mu'}\left\{\left(t^1\right)_{cc'}A_1^1 + \left(t^2\right)_{cc'}A_1^2 + \cdots + \left(t^8\right)_{cc'}A_1^8\right\} \\ &+\left(\gamma^2\right)_{\mu\mu'}\left\{\left(t^1\right)_{cc'}A_2^1 + \left(t^2\right)_{cc'}A_2^2 + \cdots + \left(t^8\right)_{cc'}A_2^8\right\} \\ &+\left(\gamma^3\right)_{\mu\mu'}\left\{\left(t^1\right)_{cc'}A_3^1 + \left(t^2\right)_{cc'}A_3^2 + \cdots + \left(t^8\right)_{cc'}A_3^8\right\}\Big]\psi_{q\mu'c'} \\ -g\sum_{a,b,c=1}^{N_c^2-1}f_{abc}\partial_\mu A_\nu^a A_b^\mu A_c^\nu &- \frac{g^2}{4}\sum_{a,b,c,d,e}^{N_c^2-1}f_{abc}f_{cde}A_a^\mu A_b^\nu A_\mu^c A_\nu^d\ . \end{aligned} \tag{H.10}$$

where

$$\sum_{q\mu\mu'cc'} \equiv \sum_{q=1}^{N_f}\sum_{\mu=0}^{3}\sum_{\mu'=0}^{3}\sum_{c=1}^{N_c}\sum_{c'=1}^{N_c}\ .$$

The Lagrangian alone simply describes a classical field theory, "chromodynamics". Within the path integral formalism, the theory can be quantized (referring to second quantization) by introduction of the following adequate generating functional [2]:

$$\begin{aligned}&Z\left[J,\xi,\overline{\xi},\eta,\overline{\eta}\right] = \\ &\Big[\prod_{\mu,a,t,\mathbf{x}}\int dA_\mu^a(t,\mathbf{x})\Big]\Big[\prod_{a,t,\mathbf{x}}\int d\chi^a(t,\mathbf{x})\Big]\Big[\prod_{a,t,\mathbf{x}}\int d\overline{\chi}^a(t,\mathbf{x})\Big]\Big[\prod_{q,c,\mu,t,\mathbf{x}}\int d\psi(t,\mathbf{x})\Big]\Big[\prod_{q,c,\mu,t,\mathbf{x}}\int d\overline{\psi}(t,\mathbf{x})\Big] \\ &\times\exp\left(i\int dt\int d^3\mathbf{x}\left(\mathscr{L}' + A_\mu^a J^{a\mu} + \overline{\chi}^a\xi^a + \overline{\xi}^a\chi^a + \overline{\psi}_i\eta_i + \overline{\eta}_i\psi_i\right)\right)\ , \end{aligned} \tag{H.11}$$

where $\mathscr{L}' = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \overline{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi + \mathscr{L}_{GF} + \mathscr{L}_{FP}$. The so-called *gauge fixing part*, $\mathscr{L}_{GF} = -\frac{1}{2\alpha}\left(\partial^\mu A_\mu^a\right)^2$, and the *Fadeev-Popov part* $\mathscr{L}_{FP} = \left(\partial^\mu\overline{\chi}^a\right)D_\mu^{ab}\chi^b$ came into play, because the naive guess

$$\begin{aligned}Z\left[J,\eta,\overline{\eta}\right] = &\Big[\prod_{\mu,a,t,\mathbf{x}}\int dA_\mu^a(t,\mathbf{x})\Big]\Big[\prod_{q,c,\mu,t,\mathbf{x}}\int d\psi(t,\mathbf{x})\Big]\Big[\prod_{q,c,\mu,t,\mathbf{x}}\int d\overline{\psi}(t,\mathbf{x})\Big] \\ &\times\exp\left(i\int dt\int d^3\mathbf{x}\left(\mathscr{L} + A_\mu^a J^{a\mu} + \overline{\psi}_i\eta_i + \overline{\eta}_i\psi_i\right)\right) \end{aligned} \tag{H.12}$$

yields infinities, resulting from not having fixed the gauge. In this case, too much paths are included in the path integral over A_μ^a. Infinitely many paths are related to others by unphysical

gauge transformations. The same physical information is taken into account redundantly. Getting rid of the unphysical part by fixing the gauge leads to $\mathscr{L}_{GF}$ and $\mathscr{L}_{FP}$. In QED, the same problem arises, however the Fadeev-Popov part can be integrated out, in contrast to QCD, where the ghost fields couple to the gauge-fields.
For the same reasons as in (D.7), it is possible to separate the interaction part of the potential, U_I, from the free term, and to replace the sources in U_I by functional derivatives:

$$Z\left[J,\xi,\overline{\xi},\eta,\overline{\eta}\right]=\exp\left(i\int dt\int d^3\mathbf{x}\;U_I\left(\frac{\delta}{i\delta J^{a\mu}},\frac{\delta}{i\delta\overline{\xi}^a},\frac{\delta}{i\delta(-\xi^a)},\frac{\delta}{i\delta\overline{\eta}},\frac{\delta}{i\delta(-\eta)}\right)\right)Z_0\left[J,\xi,\overline{\xi},\eta,\overline{\eta}\right]\;. \tag{H.13}$$

The expansion (D.8) can be performed in the same way:

$$Z\left[J,\xi,\overline{\xi},\eta,\overline{\eta}\right]=1+\lambda\left(u_1\left[J,\xi,\overline{\xi},\eta,\overline{\eta}\right]-u_1\left[0,0,0,0,0\right]\right)+\lambda^2\left(u_2\left[J,\xi,\overline{\xi},\eta,\overline{\eta}\right]-u_2\left[0,0,0,0,0\right]\right)+$$
$$\lambda^2u_1\left[0,0,0,0,0\right]\left(u_1\left[0,0,0,0,0\right]-u_1\left[J,\xi,\overline{\xi},\eta,\overline{\eta}\right]\right)\;+\;...\;, \tag{H.14}$$

$$\text{with}\quad u_1\left[J,\xi,\overline{\xi},\eta,\overline{\eta}\right]=Z_0^{-1}\left[J,\xi,\overline{\xi},\eta,\overline{\eta}\right]\;\left[\int\limits_X U_I\left(\frac{\delta}{i\delta J^{a\mu}},\frac{\delta}{i\delta\overline{\xi}^a},\frac{\delta}{i\delta(-\xi^a)},\frac{\delta}{i\delta\overline{\eta}},\frac{\delta}{i\delta(-\eta)}\right)\right]$$
$$\times\;Z_0\left[J,\xi,\overline{\xi},\eta,\overline{\eta}\right]\;, \tag{H.15}$$

$$\text{and}\quad u_2\left[J,\xi,\overline{\xi},\eta,\overline{\eta}\right]=Z_0^{-1}\left[J,\xi,\overline{\xi},\eta,\overline{\eta}\right]\;\left[\int\limits_X U_I\left(\frac{\delta}{i\delta J^{a\mu}},\frac{\delta}{i\delta\overline{\xi}^a},\frac{\delta}{i\delta(-\xi^a)},\frac{\delta}{i\delta\overline{\eta}},\frac{\delta}{i\delta(-\eta)}\right)\right]$$
$$\times\;\left[\int\limits_Y U_I\left(\frac{\delta}{i\delta J^{a\mu}},\frac{\delta}{i\delta\overline{\xi}^a},\frac{\delta}{i\delta(-\xi^a)},\frac{\delta}{i\delta\overline{\eta}},\frac{\delta}{i\delta(-\eta)}\right)\right]\;Z_0\left[J,\xi,\overline{\xi},\eta,\overline{\eta}\right]\;. \tag{H.16}$$

The generating functional for free fields, Z_0, can be expressed analytically:

$$Z_0\left[J,\xi,\overline{\xi},\eta,\overline{\eta}\right]=Z_0^G\left[J\right]Z_0^{FP}\left[\xi,\overline{\xi}\right]Z_0^F\left[\eta,\overline{\eta}\right]\;, \tag{H.17}$$

$$Z_0^G\left[J\right]=\exp\left(\frac{i}{2}\int d^4xd^4yJ^{a\mu}\left(x\right)D_{\mu\nu}^{ab}\left(x-y\right)J^{b\nu}\left(y\right)\right)\;, \tag{H.18}$$

$$Z_0^{FP}\left[\xi,\overline{\xi}\right]=\exp\left(i\int d^4xd^4y\;\overline{\xi}^a\left(x\right)D^{ab}\left(x-y\right)\xi^b\left(y\right)\right)\;, \tag{H.19}$$

$$Z_0^F\left[\eta,\overline{\eta}\right]=\exp\left(i\int d^4xd^4y\;\overline{\eta}\left(x\right)S\left(x-y\right)\eta\left(y\right)\right)\;, \tag{H.20}$$

with the bare propagators for the gluon, Fadeev-Popov ghost and quark (their analogue in statistical scalar QFT is (D.13))

$$D_{\mu\nu}^{ab}\left(x\right)\delta^{ab}\int\frac{d^4k}{\left(2\pi\right)^4}\frac{e^{-i\mathbf{kx}}}{k^2+i\epsilon}\left(g_{\mu\nu}-(1-\alpha)\frac{k_\mu k_\nu}{k^2}\right), \tag{H.21}$$

$$D^{ab}\left(x\right)=\delta^{ab}\int\frac{d^4k}{\left(2\pi\right)^4}\frac{-1}{k^2+i\epsilon}e^{-i\mathbf{kx}}, \tag{H.22}$$

$$S\left(x\right)=\int\frac{d^4p}{\left(2\pi\right)^4}\frac{1}{m-\gamma^\mu p_\mu}e^{-i\mathbf{px}}\;. \tag{H.23}$$

Let us abbreviate

$$Z_{\sqcup_1 \ldots \sqcup_n}\left[J,\xi,\overline{\xi},\eta,\overline{\eta}\right] \equiv$$
$$\left[\prod_{\mu,a,t,\mathbf{x}}\int dA^a_\mu(t,\mathbf{x})\right]\left[\prod_{a,t,\mathbf{x}}\int d\chi^a(t,\mathbf{x})\right]\left[\prod_{a,t,\mathbf{x}}\int d\overline{\chi}^a(t,\mathbf{x})\right]\left[\prod_{q,c,\mu,t,\mathbf{x}}\int d\psi(t,\mathbf{x})\right]\left[\prod_{q,c,\mu,t,\mathbf{x}}\int d\overline{\psi}(t,\mathbf{x})\right]$$
$$\times \sqcup_1 \cdots \sqcup_n \ \times \exp\left(i\int dt\int d^3\mathbf{x}\left(\mathscr{L}' + A^a_\mu J^{a\mu} + \overline{\chi}^a\xi^a + \overline{\xi}^a\chi^a + \overline{\psi}_i\eta_i + \overline{\eta}_i\psi_i\right)\right) \ ,$$

where the symbols $\sqcup_i$ are replacement characters.
In this notation, the correctly normalized n-point functions (in the absence of sources) are defined via

$$\langle \sqcup_1 \ \ldots \ \sqcup_n\rangle^0 \equiv \left.\frac{Z_{\sqcup_1 \ldots \sqcup_n}\left[J,\xi,\overline{\xi},\eta,\overline{\eta}\right]}{Z\left[J,\xi,\overline{\xi},\eta,\overline{\eta}\right]}\right|_0 \ , \tag{H.24}$$

where $\big|_0$ stands for $\big|_{J=\xi=\overline{\xi}=\eta=\overline{\eta}=0}$ and each of the symbols $\sqcup_i$ can be replaced by either A^a_μ, χ^a, $\overline{\chi}^a$, ψ or $\overline{\psi}$.
Multiple functional differentiation yields

$$\frac{\delta^n Z\left[J,\xi,\overline{\xi},\eta,\overline{\eta}\right]}{\delta\cup_1 \ \ldots \ \delta\cup_n} =$$
$$\left[\prod_{\mu,a,t,\mathbf{x}}\int dA^a_\mu(t,\mathbf{x})\right]\left[\prod_{a,t,\mathbf{x}}\int d\chi^a(t,\mathbf{x})\right]\left[\prod_{a,t,\mathbf{x}}\int d\overline{\chi}^a(t,\mathbf{x})\right]\left[\prod_{q,c,\mu,t,\mathbf{x}}\int d\psi(t,\mathbf{x})\right]\left[\prod_{q,c,\mu,t,\mathbf{x}}\int d\overline{\psi}(t,\mathbf{x})\right]$$
$$\times \ (i\sqcup_1) \ \ldots \ (i\sqcup_n) \ \times \exp\left(i\int dt\int d^3\mathbf{x}\left(\mathscr{L}' + A^a_\mu J^{a\mu} + \overline{\chi}^a\xi^a + \overline{\xi}^a\chi^a + \overline{\psi}_i\eta_i + \overline{\eta}_i\psi_i\right)\right) \ , \tag{H.25}$$

where each of the symbols $\sqcup_i$ can be replaced by either A^a_μ, χ^a, $\overline{\chi}^a$, ψ or $\overline{\psi}$. Then the symbol $\cup_i$ has to be replaced by the corresponding source, whereas in case of η and ξ there is an extra minus sign (arising from the anticommutation of Grassmann numbers).
Choosing $Z\left[0,0,0,0,0\right] \equiv 1$, we conclude from comparison of (H.24) and (H.25):

$$i^n\langle \sqcup_1 \ \ldots \ \sqcup_n\rangle^0 = \left.\frac{\delta^n Z\left[J,\xi,\overline{\xi},\eta,\overline{\eta}\right]}{\delta\cup_1 \ \ldots \ \delta\cup_n}\right|_0 \ . \tag{H.26}$$

For example, the 1-gluon-2quark 3-point function (in the absence of sources) is given by

$$\langle A^a_\mu\overline{\psi}_i\psi_j\rangle^0 = \frac{1}{i^3}\left.\frac{\delta^3 Z\left[J,\xi,\overline{\xi},\eta,\overline{\eta}\right]}{\delta J^{a\mu} \ \delta(-\eta_i) \ \delta\overline{\eta}_j}\right|_0 \ . \tag{H.27}$$

Plugging the expressions (H.17-H.20) into the expansion (H.14), it is possible to calculate Z in the same way, as we did on p.165 for ϕ^4 theory. In analogy to page 165, one can introduce diagrams and translation rules (the Feynman rules for QCD), which enables us to express every single contribution to Z diagrammatically.

H.2 SQCD

The generalization to nonzero temperature has to be performed in complete analogy to appendix E. That means, section H.1 with the replacements

$$\int_x f(x) \longrightarrow \int_X f(X) \equiv \int_0^{1/T} d\tau \int d^3\mathbf{x}\, f(\tau, \mathbf{x}) \ , \tag{H.28}$$

$$iS \longrightarrow -S_E \ , \tag{H.29}$$

$$\begin{aligned} & D^{ab}_{\mu\nu}(x)\,\delta^{ab} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-i\mathbf{kx}}}{k^2+i\epsilon} \left(g_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2} \right) \\ \longrightarrow\ & -iT\delta^{ab} \sum_{n=-\infty}^{\infty} \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i(\omega_n\tau - \mathbf{kx})} \frac{1}{\omega_n^2 + \mathbf{k}^2 - i\epsilon} \left(g_{\mu\nu} - (1-\alpha)\frac{\overline{k}_\mu \overline{k}_\nu}{(-\omega_n^2 - \mathbf{k}^2)} \right) \ , \end{aligned} \tag{H.30}$$

$$\begin{aligned} & D^{ab}(x) = \delta^{ab} \int \frac{d^4k}{(2\pi)^4} \frac{-1}{k^2+i\epsilon} e^{-i\mathbf{kx}} \\ \longrightarrow\ & -iT\delta^{ab} \sum_{n=-\infty}^{\infty} \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i(\omega_n\tau - \mathbf{kx})} \frac{-1}{\omega_n^2 + \mathbf{k}^2 - i\epsilon} \ , \end{aligned} \tag{H.31}$$

$$\begin{aligned} & S(x) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{m - \gamma^\mu p_\mu} e^{-i\mathbf{px}} \\ \longrightarrow\ & iT\delta^{ab} \sum_{n=-\infty}^{\infty} \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i(\omega_n\tau - \mathbf{px})} \frac{1}{m - \gamma^\mu \overline{p}_\mu} \ , \end{aligned} \tag{H.32}$$

$$\text{with } \overline{k}_\mu \equiv (i\omega_n, -\mathbf{k}) \ , \tag{H.33}$$

yields SQCD, if one imposes periodic (antiperiodic) boundary conditions on the (fermionic) fields. Note that in case of the fermionic fields ω_n refers to the fermionic Matsubara frequencies $(2n+1)\pi T$.

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"When I was in high school, I came across a popular article, written by the famous mathematician Andrey Nikolaevich Kolmogorov. He suggested some litmus tests for mathematical talent. Something like this: you must be able to imagine a section of a cube by a plane passing through its center perpendicularly to the diagonal. I tried and failed. Next, he said, there are algebraic abilities - he gave some expressions and the problem was to expand it in simple factors. Again - no luck. So I decided that I didn't have the abilities to be a mathematician and attempted to study physics. I tried to read some popular books and also some freshman physics textbooks. The result was not good - I was bored by the slow pace there and the more detailed their explanations were getting, the less I understood. And then one day I bought in a second-hand book store a book "Mechanics" by Landau and Lifshitz. That was the moment of epiphany. When I started reading it, I got very excited. The beauty of the "least action principle", the arguments based on symmetry, the unifying view of the world, the fantastic intellectual intensity of this book left permanent impression on me. I realized that whether or not I have abilities - this is my science and I will study it no matter what."

(Alexander M. Polyakov [71])

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