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Non-Trivial Fixed Points of the Scalar Field Theory

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Abstract

The phase structure of the scalar field theory with arbitrary powers of the gradient operator and a local non-analytic potential is investigated by the help of the RG in Euclidean space. The RG equation for the generating function of the derivative part of the action is derived. Infinitely many non-trivial fixed points of the RG transformations are found. The corresponding effective actions are unbounded from below and do probably not exhibit any particle content. Therefore they do not provide physically sensible theories.

1 Introduction

As the self-interacting scalar field with some continuous internal symmetry is a basic ingredient of the Standard Model and of GUT's, the phase structure of the one-component scalar field has also been of a permanent interest, in spite of the fact that it is only a toy model. It is well-known that scalar field theory exhibits a Gaussian fixed point [1]. For polynomial potentials that is a trivial IR fixed point corresponding to a free scalar field. The search for non-trivial fixed points in dimension $d = 4$ gave negative results (see [2] and the literature cited there). Perturbative renormalisability of scalar field theory was proven by means of the RG using the scaling property of the operators at the Gaussian fixed point [3, 4, 5]. A non-trivial fixed point was found in dimensions $2 < d < 4$, that however does not occur in dimension 4 [6]. A very interesting result has been obtained recently. Namely, there are relevant directions at the Gaussian fixed point for analytic potentials [7]. The existence of such relevant directions rises the question what is the low energy behaviour of the theories defined at high-energy scale by one of the relevant potentials in the neighbourhood of the Gaussian fixed point. Therefore a renewed search for non-trivial fixed points has acquired actuality once again.

More recently it was argued in [8] that the non-trivial directions were found in [7] as a consequence of the local potential approximation. As far as one is looking for the RG flow in the coarse-graining direction, any approximation neglecting irrelevant terms is justified and does not spoil the search for the IR fixed point along a given RG trajectory. Just the opposite is true if one tries to find UV fixed points with the help of the RG equations. Irrelevant terms negligible at a given scale become more and more important in the UV regime and reject the RG trajectory from the UV fixed point. Therefore tracing back a RG trajectory to the UV fixed point is practically impossible. How can one interpret the situation that a fixed point has a relevant direction and the model defined at some UV momentum scale lies close to this fixed point in this direction? Although it is not reasonable to ask for the UV fixed point, one can ask what is the IR effective theory for such a model. That is what we are doing in the present paper without making use of the local potential approximation.

In the present paper we show that the one-component scalar field theory has infinitely many non-trivial fixed points in any dimension $d > 2$ and determine the fixed point actions and the scaling operators in the neighbourhood of the non-trivial fixed points. It is crucial for finding the non-trivial fixed points the enlargement of the space of the scale dependent actions (Hamiltonians) considered by (a) including terms with arbitrary powers of the gradient operator, and (b) local but non-analytic potentials and wave function renormalization. Earlier investigations of the phase structure of the scalar field theory used more restricted parameter space. The models considered did not contain terms with higher than the second power of the gradient operator [6, 7], [11]-[23]. Below we show that the non-trivial fixed point actions contain high powers of the gradient operator. This may be the reason that they have been overseen before.

The phase structure of the scalar field theory is investigated with the help of the Wegner-Houghton equation [9]. As it is well-known, the approach of Wegner and Houghton enables one to carry out renormalization by integrating out the high-frequency modes Φ_q of the field $\Phi(x)$ step by step in infinitesimally thin momentum shells, $(k-\delta k, k)$. Making a quite general Ansatz with infinitely many terms for the action S_k at arbitrary scale k, the effect of the high frequency modes can be incorporated into the change of its couplings completely [9, 10, 24, 25]. In this procedure the contribution of each momentum shell to the action can be calculated exactly by making use of the small parameter $\delta k/k$. With the help of the Wegner-Houghton equation one can follow the evolution of the irrelevant coupling constants with the scale k, too. In the usual perturbative RG approach one only keeps track the evolution of the very limited number of terms included in the bare action and cannot notice if some of the irrelevant terms not included become relevant with decreasing scale k.

2 RG Equations

The action at the momentum scale k is assumed to have the form:

$$S_k = \int d^d x \left\{ G_k(\Phi, -\partial^2)\Phi + U_k(\Phi) \right\}, \tag{1}$$

where $G_k(\Phi, -\partial^2)$ is a local functional of the field $\Phi(x)$ and an analytic function of the derivatives:

$$G_k(\Phi, -\partial^2) = \sum_{n=1}^{\infty}\sum_{r=0}^{\infty} g_{nr}(k)\Phi^{r+1}(x)(-\partial)^{2n}. \tag{2}$$

Assuming that the system is enclosed in the finite volume V_d, we rewrite the bare potential,

$$\mathcal{U} \equiv \int d^d x U_k(\Phi) = \sum_{r=2}^{\infty} u_r(k) V_d^{-r} \sum_{q_1,\ldots,q_r}^{\leq k} \Phi_{q_1}\cdots\Phi_{q_r} V_d \delta_{q_1+\ldots+q_r} \tag{3}$$

and the derivative part,

$$\begin{aligned}\mathcal{G} &\equiv \int d^d x G_k(\Phi, -\partial^2)\Phi(x) \\ &= V_d^{-(r+2)} \sum_{n=1}^{\infty}\sum_{r=0}^{\infty} g_{nr}(k) \sum_{q_1,\ldots,q_{r+2}}^{\leq k} \left(q_1^2\right)^n \Phi_{q_1}\cdots\Phi_{q_{r+2}}\delta_{q_1+\ldots+q_{r+2}}.\end{aligned} \tag{4}$$

The Wegner-Houghton equation can be written as [9, 6, 24]:

$$k\partial_k S_k = \frac{k}{2\delta k}\sum_p{}' F_p K^{-1}_{p,-p} F_{-p} - \hbar \frac{k}{2\delta k}\sum_p{}' \left(\ln \frac{K_{p,p'}}{K_{p,-p}|_{\Phi=\Phi_c}} \right)_{p,-p}, \tag{5}$$

where Φ_c is either the vacuum expectation value Φ_0, or any constant field configuration, the sum $\sum_p{}'$ is taken over the momentum shell of the thickness δk at $p^2 = k^2$, i.e. $(1/V_d)\sum_p' \ldots = \delta k\ k^{d-1}\int d\omega (2\pi)^{-d}\ldots$ in the infinite volume limit, (with the infinitesimal solid angle $d\omega$ in the d dimensional momentum space) and $F_p = (\delta S_k/\delta\Phi_p)_{\Phi_p=0}$, $K_{p,p'} = (\delta^2 S_k/\delta\Phi_p\delta\Phi_{p'})_{\Phi_p=0}$, where the subscript denotes that the Fourier amplitudes of the modes in the momentum shell at k have to be set to zero. The first term on the r.h.s. of Eq. (5) is the tree level contribution occurring if $F_p \neq 0$. The second term on the r.h.s. of Eq. (5) is the one-loop contribution. The denominator in the argument of the logarithm on the r.h.s. of Eq. (5) ensures that the effective action S_k takes vanishing value for $\Phi \equiv \Phi_c$ for all values of k. Further on we shall work in the units $\hbar = 1$.

The action considered has the general structure $S_k(\Phi_q) = V_d\sigma_k(\phi_q)$ in terms of $\phi_q = \Phi_q/V_d$ and Eq. (5) takes the form:

$$k\partial_k\sigma_k(\phi_q) = \frac{k^d}{2(2\pi)^d}\int d\omega \left[V_d\ f_p k^{-1}_{p,-p} f_{-p} - \left(\ln \frac{k_{p,p'}}{k_{p,-p}|_{\phi=\phi_c}} \right)_{p,-p} \right] \tag{6}$$

with $f_p = \partial\sigma_k/\partial\phi_p|_{\phi_p=0}$, and $k_{p,p'} = \partial^2\sigma_k/\partial\phi_p\partial\phi_{p'}|_{\phi_p=0}$.

Our procedure of looking for the solutions of Eq. (6) is the following: we introduce the generating functions for the derivative part and the potential, and derive partial differential equations for them by an appropriate projection of the original equation (6) and its second functional derivative:

$$
\begin{aligned}
&k\partial_k \frac{\partial^2 \sigma_k}{\partial\phi_Q \partial\phi_{-Q}} \\
&= \frac{k^d}{2(2\pi)^d} \int d\omega \frac{\partial^2}{\partial\phi_Q \partial\phi_{-Q}} \left[V_d \, f_p k^{-1}_{p,-p} f_{-p} - \left(\ln \frac{k_{p,p'}}{k_{p,-p}|_{\phi=\phi_c}} \right)_{p,-p} \right]
\end{aligned} \tag{7}
$$

with $Q \neq p$. The projector $\mathcal{P}$ we use was introduced in [6] by the definition

$$
\mathcal{P}F = \left(\exp\left\{ x \frac{\partial}{\partial\phi_0} \right\} F \right)_{\phi\equiv 0} \tag{8}
$$

with the arbitrary functional F of the field. As shown in [6] the projection of any product of functionals equals to the product of the projections of those functionals.

Let us define now the generating function for the potential,

$$
V(x;k) = \sum_{r=2}^{\infty} u_r(k) x^r, \tag{9}
$$

and that for the derivative part of the action S_k,

$$
G(Q^2, x, k) = 2 \sum_{n=1}^{\infty} Q^{2n} \sum_{r=0}^{\infty} g_{nr}(k) x^r. \tag{10}
$$

Here $V(x,k)$ is the generating function introduced in [6].

Let us apply the projector $\mathcal{P}$ to both sides of Eqs. (6) and (7). Making use of the various steps of the projection described in the Appendix, we find the following coupled set of partial differential equations for the generating functions:

$$
k\partial_k V(x,k) = -k^d \alpha \ln \frac{\partial_x G(k^2,x,k) + \partial_x^2 V(x,k)}{[\partial_x G(k^2,x,k) + \partial_x^2 V(x,k)]_{x=x_c}}, \tag{11}
$$

$$
\begin{aligned}
&k\partial_k \partial_x G(Q^2,x,k) \\
&= -k^d \alpha \left\{ \frac{\partial_x^3 G(k^2,x,k)}{\partial_x G(k^2,x,k) + \partial_x^2 V(x,k)} \right. \\
&\quad \left. - \frac{\partial_x^2 G(Q^2,x,k) \, [\partial_x^2 G(k^2,x,k) + \partial_x^3 V(x,k)] + \left[\frac{1}{2}\partial_x^2 G(Q^2,x,k)\right]^2}{[\partial_x G(k^2,x,k) + \partial_x^2 V(x,k)]^2} \right\}.
\end{aligned} \tag{12}
$$

with $\alpha = \frac{1}{2}(2\pi)^{-d}\Omega_d$ (Ω_d the entire solid angle in the d dimensional momentum space). Due to the particular choice of the action the tree level terms do not occur in these equations. Therefore Eqs. (11) and (12) are safe in the limit $V_d \to \infty$.

3 Equations for the Dimensionless Coupling Constants

The equations for the dimensionless quantities can be obtained by rescaling the fields and the variable x as $\tilde{\Phi}_q = k^{(d+2)/2}\Phi_q$, and $\tilde{x} = k^{-(d-2)/2}x$ that corresponds to $\tilde{x}_\mu = kx_\mu$ and $\tilde{V}_d = k^d V_d$ in coordinate space and leaves the dimensionless expression $\Phi_q/(V_d x)$ unchanged. After this rescaling we obtain

$$V(x,k) = k^d \tilde{V}(\tilde{x},k), \qquad \partial_x^2 V(x,k) = k^d k^{-(d-2)} \partial_{\tilde{x}}^2 \tilde{V}(\tilde{x},k), \tag{13}$$

Requiring that $\mathcal{G}$ were dimensionless, we are lead to $g_{10} = \tilde{g}_{10}$ and

$$G(Q^2,x,k) = 2k^2 \sum_{n=1}^{\infty} \tilde{Q}^{2n} \sum_{r=0}^{\infty} \tilde{g}_{nr} \tilde{x}^{r+1} = k^2 \tilde{G}(\tilde{Q}^2,\tilde{x},k). \tag{14}$$

The RG equations for the dimensionless generating functions take then the following form:

$$\left(k\partial_k - \frac{d-2}{2}\tilde{x}\partial_{\tilde{x}} + d\right)\tilde{V}(\tilde{x},k) = -\alpha \ln \frac{\partial_{\tilde{x}}\tilde{G}(1,\tilde{x},k) + \partial_{\tilde{x}}^2\tilde{V}(\tilde{x},k)}{\left[\partial_{\tilde{x}}\tilde{G}(1,\tilde{x},k) + \partial_{\tilde{x}}^2\tilde{V}(\tilde{x},k)\right]_{\tilde{x}_c}}, \tag{15}$$

$$\begin{aligned}
&\left(k\partial_k - 2\tilde{Q}^2\partial_{\tilde{Q}^2} - \frac{d-2}{2}\tilde{x}\partial_{\tilde{x}} + 2\right)\partial_{\tilde{x}}\tilde{G}(\tilde{Q}^2,\tilde{x},k) \\
&= -\alpha\left\{\frac{\partial_{\tilde{x}}^3\tilde{G}(\tilde{Q}^2,\tilde{x},k)}{\partial_{\tilde{x}}\tilde{G}(1,\tilde{x},k) + \partial_{\tilde{x}}^2\tilde{V}(\tilde{x},k)}\right. \\
&\left. - \frac{\partial_{\tilde{x}}^2\tilde{G}(\tilde{Q}^2,\tilde{x},k)\left[\partial_{\tilde{x}}^2\tilde{G}(1,\tilde{x},k) + \partial_{\tilde{x}}^3\tilde{V}(\tilde{x},k)\right] + \left[\frac{1}{2}\partial_{\tilde{x}}^2\tilde{G}(\tilde{Q}^2,\tilde{x},k)\right]^2}{\left[\partial_{\tilde{x}}\tilde{G}(1,\tilde{x},k) + \partial_{\tilde{x}}^2\tilde{V}(\tilde{x},k)\right]^2}\right\}.
\end{aligned} \tag{16}$$

Now we generalize Eqs. (15) and (16) in two respects.

1. The above equations are valid, strictly speaking, only if the vacuum expectation value of the field is $\phi_0 = 0$. In the more general case $\phi_0 \neq 0$ we have to take into account that the generating functions V, and G depend on rather $x - x_0$ than on x, and x_0 has the dimensional scaling of x. Therefore the equations (15) and (16) must be modified replacing $\tilde{x}$ by $z = \tilde{x} - \tilde{x}_0$, and $x_c = x_0$ can be chosen.

   Due to the quadratic approximation of the action used by deriving the Wegner-Houghton equation [9], the minimum $x_0 \neq 0$ of the action must be sufficiently close to zero, otherwise the equation itself looses its validity.

2. Furthermore we assume the validity of Eqs. (15) and (16) also for generating functions V and G non-analytic in the variable z. In the case of potentials having a singularity at their (absolute) minimum at $z_c = 0$, we cannot keep the potential at a fixed value in its minimum. Therefore we choose an arbitrary point $z_c \neq 0$ for this purpose.

All over this paper we shall only deal with dimensions $d > 2$.

4 Fixed Point Solutions

The equations for the fixed points are obtained by setting zero the derivatives of the generating functions with respect of the scale k in Eqs. (15), (16). We shall seek the fixed point solutions $V^*(z)$, $G^*(\tilde{Q}^2, z)$ by making the Ansatz that the field dependent wave function renormalization can be separated in the derivative part, i.e. $\partial_z G^*(\tilde{Q}^2, z) = H^*(\tilde{Q}^2) h^*(z)$.

4.1 Gaussian fixed point

Assuming $V^* = const.$ and $h^* \equiv 1$, one easily finds the fixed point solution $H^*(Q^2) = H_0^* Q^2$, i.e. $G^*(Q^2, z) = H_0^* Q^2 z$ and the corresponding fixed point action

$$S^* = -\frac{1}{2} H_0^* \int d^d x \phi(x) \partial_x^2 \phi(x). \tag{17}$$

The choice $H_0^* = 1$ can be made without loss of generality (rescaling of the field).

4.2 Non-trivial fixed points

We recognize that the fixed point equation obtained from Eq. (15) has the solution $V^* = \frac{1}{2} C_V \ln z^2 + V_0^*$, $h^*(z) = z^{-2}$ with $C_V = 2\alpha/d$ and $V_0^* = (\alpha/d^2)\,(d - 2 - d \ln z_c^2)$. The logarithm on the r.h.s. of Eq. (15) is only well-defined for $b \equiv H^*(1) - C_V \neq 0$. Then we obtain a non-linear ordinary differential equation for the function $H^*(\tilde{Q}^2)$:

$$\tilde{Q}^2 \frac{dH^*(\tilde{Q}^2)}{d\tilde{Q}^2} = \kappa H^*(\tilde{Q}^2) - \nu \left[H^*(\tilde{Q}^2) \right]^2 \tag{18}$$

with

$$\kappa = \frac{1}{2} \left(d + \frac{2\alpha}{b} \right), \qquad \nu = \frac{\alpha}{2b^2}. \tag{19}$$

It has the analytic solution:

$$H^*(\tilde{Q}^2) = \frac{\kappa}{\nu} \tilde{Q}^{2\kappa} \left[C + \tilde{Q}^{2\kappa} \right]^{-1} \tag{20}$$

with

$$\frac{\kappa}{\nu} = 2C_V \frac{2\kappa}{d} \left[\frac{2\kappa}{d} - 1 \right]^{-2}, \qquad C = \left(3 - \frac{2\kappa}{d} \right) \left(\frac{2\kappa}{d} - 1 \right)^{-1} \tag{21}$$

for positive integer $\kappa \neq d/2,\ 3d/2$. The non-trivial fixed point solution is then given by

$$\begin{aligned} \tilde{G}^*(\tilde{Q}^2, z) &= -2C_V \left(1 - \frac{d}{2\kappa} \right)^{-1} z^{-1} \tilde{Q}^{2\kappa} \left[3 - \frac{2\kappa}{d} + \left(\frac{2\kappa}{d} - 1 \right) \tilde{Q}^{2\kappa} \right]^{-1}, \\ V^*(z) &= \frac{1}{2} C_V \ln z^2 + V_0^*. \end{aligned} \tag{22}$$

The powers $\kappa = d/2$ and $3d/2$ must be excluded due to $H^*(1) \to \infty$. Analyticity and $H^*(0) = 0$ are only satisfied for $0 < \kappa$ integers. Therefore there are infinitely many non-trivial fixed points characterized by the positive integers $\kappa \neq d/2,\ 3d/2$. Eq. (20) is non-linear, therefore the linear combinations of the solutions corresponding to various values of κ are not solutions.

It is worthwhile noticing that $\partial_z \tilde{G}^* = \tilde{Q}^2$ solves Eq. (16) trivially. Then Eq. (15) leads to the fixed point equation found in [6]. As discussed there and also in [14] (for $d = 3$), a non-trivial IR fixed point solution of Eq. (15) exists for $2 < d < 4$ and, very probably, this fixed point does not occur for dimension $d = 4$. It is the constancy of the wave function renormalization the basic assumption for obtaining such a solution.

5 Linearized RG Transformation

As to the next we investigate the linearized RG transformations in the neighbourhood of the fixed points in order to determine the scaling operators. Let us write for the generating functions $V(z,k) = V^*(z) + \delta V(z,k)$, and $G(\tilde{Q}^2, z, k) = G^*(\tilde{Q}^2, z) + \delta G(\tilde{Q}^2, z, k)$ in the neighbourhood of a given fixed point. In order to find the scaling operators at each of the fixed points we shall seek the eigensolutions of the operator $k\partial_k$ with the eigenvalues $-\lambda$ satisfying Eqs. (15) and (16) in their linearized form. We make the separation Ansatz for the solution:

$$\delta V(z,k) = (k/\Lambda)^{-\lambda}\, \varphi(z), \qquad \partial_z \delta G(Q^2, z, k) = (k/\Lambda)^{-\lambda}\, \phi(Q^2)\psi(z) \tag{23}$$

with $\phi(0) = 0$. The positive, vanishing and negative eigenvalues λ correspond to relevant, marginal, and irrelevant directions, respectively, at the given fixed point in the parameter space.

5.1 Linearized RG equations at the Gaussian fixed point

The linearized version of Eq. (15) at the Gaussian fixed point is given by:

$$\begin{aligned} &\left(k\partial_k - \frac{d-2}{2} z\partial_z + d\right) \delta V(z,k) \\ &= -\alpha \left[\delta\partial_z G(1,z,k) - \delta\partial_z G(1,0,k) + \partial_z^2 \delta V(z,k) - \partial_z^2 \delta V(z,k)\Big|_{z=0}\right], \end{aligned} \tag{24}$$

$$\left(k\partial_k + \alpha\partial_z^2 - \frac{d-2}{2} z\partial_z - 2\tilde{Q}^2 \partial_{\tilde{Q}^2} + 2\right) \partial_z \delta G(\tilde{Q}^2, z, k) = 0. \tag{25}$$

For the eigensolutions (23) we obtain the following equations from Eq. (25)

$$\left(\alpha\partial_z^2 - \frac{d-2}{2} z\partial_z - \lambda + 2 - C\right) \psi(z) = 0, \tag{26}$$

$$\left(2\tilde{Q}^2 \partial_{\tilde{Q}^2} - C\right) \phi(\tilde{Q}^2) = 0 \tag{27}$$

with the constant C. For $C = 2n$ with $n = 1, 2, \ldots$ we obtain a complete set of analytic solutions of Eq. (27) satisfying $\phi(0) = 0$:

$$\phi = \phi_0 (\tilde{Q}^2)^n \tag{28}$$

with the constant ϕ_0. Let us write $\psi(z) = \gamma Z(y)$, $z = \beta y$ and define the constants β and γ as $\gamma = 4/(d-2)$, $\beta^2 = \alpha\gamma$. Then we obtain the following ordinary differential equation of second order for the function $Z(y)$ from Eq. (26):

$$\left[\partial_y^2 - 2y\partial_y + (2 - \lambda - 2n)\gamma\right] Z(y) = 0. \tag{29}$$

There is a complete set of analytic solutions characterized by $(2-\lambda-2n)\gamma = 2n'$ with $n' = 1, 2, \ldots$, namely the Hermite polynomials $Z(y) = H_{n'}(y)$. Thus we find that the eigenvalues

$$\lambda_{nn'} = 2 - 2n - n'\frac{d-2}{2} \tag{30}$$

are given by the integers n and n' and the corresponding eigensolutions are:

$$\partial_z \delta G \;=\; \left(\frac{k}{\Lambda}\right)^{-\lambda_{nn'}} \frac{4}{d-2}\phi_0 (\tilde{Q}^2)^n H_{n'}(z/\beta). \tag{31}$$

Inserting the solution (31) in Eq. (24) and introducing $\bar{\varphi}(z) = \varphi(z) - C_2 = \gamma f(y)$ with $C_2 = C_1(d - \lambda_{nn'})^{-1}$, and $C_1 = \alpha\left[\phi_0\gamma H_{n'}(0) + \partial_z^2\varphi\big|_{z=0}\right]$, we find

$$\left[\partial_y^2 - 2y\partial_y + (d - \lambda_{nn'})\gamma\right] f(y) \;=\; -\alpha\phi_0\gamma H_{n'}(y). \tag{32}$$

By making use of the differential equation of the Hermite polynomials we find the solution $f(y) = f_0 H_{n'}(y)$ with the constant f_0 determined via

$$d - 2 + 2n \;=\; -\frac{\alpha\phi_0}{f_0}. \tag{33}$$

The constant C_2 takes the value $C_2 = -\gamma f_0 H_{n'}(0)$. Then we can write the eigensolutions corresponding to the eigenvalue $\lambda_{nn'}$ with n, $n' = 1, 2, \ldots$ as

$$\delta V \;=\; \left(\frac{k}{\Lambda}\right)^{-\lambda_{nn'}} f_0 \frac{4}{d-2}\left[H_{n'}(z/\beta) - H_{n'}(0)\right], \tag{34}$$

$$\partial_z \delta G \;=\; \left(\frac{k}{\Lambda}\right)^{-\lambda_{nn'}} \frac{-4f_0}{\alpha}\left(1 + \frac{2n}{d-2}\right)(\tilde{Q}^2)^n H_{n'}(z/\beta). \tag{35}$$

All these eigensolutions are irrelevant as $\lambda_{nn'} < 0$ for the values taken by n and n'.

It must be considered separately the case of the field independent wave function renormalization $\psi(z) = const.$, corresponding formally to $n' = 0$. There is no need then to introduce the constant C. We can write $\psi(z) \equiv 1$ without loss of generality. The equations (24) and (25) decouple and the eigensolutions for the kinetic part and for the potential are completely independent. We find the complete set of solutions $\partial_z \delta G = (k/\Lambda)^{-\lambda_n}\phi_0(\tilde{Q}^2)^n$ belonging to the eigenvalues $\lambda_n = 2 - 2n$. The solution $n = 1$ is marginal, the higher order derivative terms with $n > 1$ are irrelevant. By introducing $\bar{\varphi} = \varphi - C_0/(d-\lambda)$ with $C_0 = \alpha\, \partial_z^2 \varphi\big|_{z=0}$ and writing $\bar{\varphi}(z) = \gamma f(y)$ with $z = \beta y$, we obtain the following equation for the potential:

$$\left[\partial_y^2 - 2y\partial_y + (d-\lambda)\gamma\right] f(y) = 0. \tag{36}$$

Let us now introduce the new variable $u = y^2$ and define $\bar{f}(u) = f(y)$, then we find the confluent hypergeometric equation for the function $\bar{f}(u)$:

$$\left[u\partial_u^2 + \left(\frac{1}{2} - u\right)\partial_u - (a-1)\right] \bar{f}(u) = 0 \tag{37}$$

with $a - 1 = \frac{1}{4}(\lambda - d)\gamma$.

It has two independent solutions that can be chosen as

$$\bar{f}_1 = G\left(a-1, \frac{1}{2}, u\right), \qquad \bar{f}_2 = e^u G\left(\frac{3}{2} - a, \frac{1}{2}, -u\right); \tag{38}$$

or

$$\bar{f}_1' = F\left(a-1, \frac{1}{2}, u\right), \qquad \bar{f}_2' = u^{1/2} F\left(a - \frac{1}{2}, \frac{3}{2}, u\right) \tag{39}$$

with the confluent hypergeometric (Kummer) function $F(a, b, x)$ and the function $G(a, b, x) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} F(a, b, x) + \frac{\Gamma(b-1)}{\Gamma(a)} x^{1-b} F(a-b+1, 2-b, x)$. For $a = 1-(K/2)$ with $K = 1, 2, \ldots$ the solutions $\bar{f}_1$ are the Hermite polynomials as found in [6]: $f(y) = 2^{-K} H_K(y)$ corresponding to the eigenvalues $\lambda_K = d - (K/2)(d-2)$. So we recover the well-known classification of the polynomial interactions $\varphi(z) \sim [H_K(z/\beta) - H_K(0)]$ at the Gaussian fixed point. The quadratic potential ($K = 2$) is relevant. The quartic potential ($K = 4$) is relevant, marginal, and irrelevant for $2 < d < 4$, $d = 4$, and $d > 4$, resp., higher order terms in the potential are irrelevant for $d = 4$. It was shown in Ref. [6] that the non-linear terms of the RG equation for the potential render the quartic potential also irrelevant for $d = 4$.

It has been observed more recently [7] that the non-polynomial eigenpotentials

$$\varphi_a(z) = \gamma\left[F\left(a-1, \frac{1}{2}, \frac{z^2}{\beta^2}\right) - 1\right] \tag{40}$$

corresponding to the solution $\bar{f}_1' = F(a-1, \frac{1}{2}, y^2)$ for $a_c \equiv -2/(d-2) < a < 0$ i.e. $\lambda_a = 2 + (d-2)a > 0$ have a minimum. These potentials are relevant and behave asymptotically as $z^{2a-3}\exp\{\beta^{-2}z^2\}$.

The various eigenvalues λ correspond to different directions in the parameter space around the Gaussian fixed point. For $k \to 0$ and $\lambda < 0$ the renormalization trajectory flows in the fixed point, representing a trivial IR fixed point for $d = 4$. On the other hand, the trajectories for models with $a_c < a < 0$ move away from the fixed point. It represents an UV fixed point in this case, the corresponding models exhibit asymptotic freedom [7], but it is not yet clear how these models behave in the IR. As the equations for the potential and the derivative part of the action decouple completely, the couplings of the potential terms are independent of the couplings of the derivative terms.

5.2 Linearized RG transformation at the non-trivial fixed points

As we have shown above, there are infinitely many fixed points characterized by the positive integer powers $\kappa \neq d/2$ of the operator ∂^2. Restricting ourselves to the solutions satisfying $\delta G(0,z,k) = 0$, the linearized RG equations take the following form:

$$\begin{aligned}
&\left(k\partial_k - \frac{d-2}{2} z\partial_z + d\right) \delta V(z,k) \\
&= -\frac{\alpha}{b}\left[z^2\partial_z \delta G(1,z,k) + z^2\partial_z^2 \delta V(z,k) - z_c^2 \partial_z \delta G(1,z_c,k) - z_c^2\partial_z^2 \delta V(z_c,k)\right],
\end{aligned} \tag{41}$$

$$\begin{aligned}
&\left(k\partial_k - 2\tilde{Q}^2\partial_{\tilde{Q}^2} - \frac{d-2}{2} z\partial_z + 2\right)\partial_z \delta G(\tilde{Q}^2, z.k) \\
&= -\frac{\alpha}{b}\left\{\left[z^2\partial_z^2 + 2z\partial_z + \frac{1}{b}H^*(\tilde{Q}^2) z\partial_z\right]\partial_z \delta G(\tilde{Q}^2,z,k)\right. \\
&\left. + \frac{2}{b}H^*(\tilde{Q}^2)\left[1 + z\partial_z + \frac{1}{b}H^*(\tilde{Q}^2)\right]\left[\partial_z \delta G(1,z.k) + \partial_z^2 \delta V(z.k)\right]\right\}.
\end{aligned} \tag{42}$$

Let us look once again for the eigensolutions corresponding to the eigenvalue $-\lambda$ of the operator $k\partial_k$ in the form given by Eq. (23), assuming $\phi(0) = 0$ and making the Ansatz

$$\phi(1)\psi(z) = -\partial_z^2 \varphi(z). \tag{43}$$

Then we find the following coupled set of partial differential equations:

$$\left(-\frac{d-2}{2} z\partial_z - \lambda + d\right)\varphi(z) = 0, \tag{44}$$

$$\begin{aligned}
&\left(-2\tilde{Q}^2\partial_{\tilde{Q}^2} - \frac{d-2}{2} z\partial_z - \lambda + 2\right)\phi(\tilde{Q}^2)\psi(z) \\
&= -\frac{\alpha}{b}\left[z^2\partial_z^2 + 2z\partial_z + \frac{1}{b}H^*(\tilde{Q}^2) z\partial_z\right]\phi(\tilde{Q}^2)\psi(z).
\end{aligned} \tag{45}$$

Eq. (44) has the solution:

$$\varphi(z) = \varphi_0 z^{2s}, \qquad s = \frac{d-\lambda}{d-2}. \tag{46}$$

Then the wave function renormalization $\psi(z)$ is given by

$$\psi(z) = -\frac{\varphi_0}{\phi(1)} 2s(2s-1) z^{2s-2}. \tag{47}$$

Finally we obtain from Eq. (45) the equation

$$\tilde{Q}^2\partial_{\tilde{Q}^2} \ln \phi(\tilde{Q}^2) = \rho + \frac{\alpha}{b^2}(s-1) H^*(\tilde{Q}^2) \tag{48}$$

with

$$\rho \;=\; \frac{1}{2}\left[\frac{\alpha}{b}(2s-2)(2s-1)-\frac{d-2}{2}(2s-2)-\lambda+2\right]. \tag{49}$$

Introducing the new variable $\xi = \tilde{Q}^{2\kappa}$, one can easily integrate this equation and find its solution:

$$\phi(\tilde{Q}^2) \sim \tilde{Q}^{2\rho}\left[1+C^{-1}\tilde{Q}^{2\kappa}\right]^{C_Q} \tag{50}$$

with $C_Q = 2\kappa(2-\lambda)/(d-2)$. The analytic solutions satisfying $\phi(0) = 0$ are those with

$$\rho = C_Q\left[2-\frac{d}{2}+\frac{d}{2\kappa}\left(C_Q\frac{\kappa-1}{\kappa}-1\right)\right] > 0, \tag{51}$$

$C_Q > 0$, and both ρ and C_Q integers.

One finds for

1. d odd:

$$\begin{aligned} C_Q &= 2\kappa n, \qquad n = 1, 2, \ldots \\ \rho &= \kappa n(4-d) + nd\left[2n(\kappa-1)-1\right], \\ \lambda_n &= 2-n(d-2); \end{aligned} \tag{52}$$

2. d even:

$$\begin{aligned} C_Q &= \kappa n, \qquad n = 1, 2, \ldots \\ 2\rho &= n\kappa(4-d) + nd\left[n(\kappa-1)-1\right], \\ \lambda_n &= 20n\left((d/2)-1\right). \end{aligned} \tag{53}$$

Correspondingly

1. there are a relevant ($n = 1$), a marginal ($n = 2$), and infinitely many irrelevant ($n > 2$) scaling operators for dimensions $d = 3$ and $d = 4$;
2. there are a marginal ($n = 1$) and infinitely many irrelevant ($n > 1$) scaling operators for $d = 6$;
3. for all the other dimensions d all scaling operators are irrelevant.

For the dimension $d = 4$ there are no scaling operators analytic in $\tilde{Q}^2$ at the fixed point with $\kappa = 1$. The other fixed points with $\kappa > 1$ exhibit a single relevant scaling operator and can be considered as corresponding to critical theories. To each of those fixed points belongs a critical surface positioned perpendicularly to the relevant direction at the given fixed point and the critical surfaces separate different phases.

Let us make a few important remarks.

(i) The higher order derivative terms lead to non-localized interactions. Causality is, however, not violated due to the analytic dependence of the Lagrangian on the gradient operator [31]. Unitarity depends on whether the real energy eigenvalue states are all of positive norm [32]. It is, however, not a necessary requirement for an effective theory to be unitary.

(*ii*) The fixed point actions belonging to all of the non-trivial fixed points are unbounded from below due to the logarithmic potential $V^*(z)$ and rather probably none of them possesses any particle excitation.
(*iii*) The sign of the derivative term of the fixed point action depends on κ. Let us consider the case of dimension $d = 4$. For $\kappa = 1$ and $\kappa > 6$ all Fourier modes give a positive contribution to the derivative part, whereas their contributions are negative for $2 < \kappa < 6$. The possibility of the existence of such fixed points has already been argued in [30]. For such fixed point theories the vacuum corresponds to a periodic field configuration with the wave length $2\pi/k$, but with vanishing amplitude due to the logarithmic potential.
(*iv*) It might happen that the highly non-linear RG equations have other fixed point solutions, not satisfying the separation Ansatz, but making more physical sense.

6 Conclusions

We have found infinitely many non-trivial fixed points for the one-component scalar field theory in the enlarged space of actions including derivative interactions and non-analytic potentials. The fixed point actions found are however not really physical, they are unbounded from below and do not support particle excitations. If the RG trajectories starting at the Gaussian fixed point along a relevant direction flow towards them, the corresponding theories are not sensical. In that case the only reasonable models are those being trivial.

Appendix: Derivation of the RG Equations

The first and second derivatives of the action defined via Eqs. (1), (2), and (3) take the form:

$$\begin{aligned}\frac{\partial\sigma_k}{\partial\phi_Q} &= \sum_{n=1}^{\infty}\sum_{r=0}^{\infty} g_{nr} \sum_{q_1,\dots,q_{r-1}}^{\le k} \left[\left(Q^2\right)^n + \left(q_1^2\right)^n (r+1)\right] \phi_{q_1}\cdots\phi_{q_{r+1}}\delta_{q_1+\dots+q_{r+1}+Q} \\ &\quad + \sum_{r=2}^{\infty} u_r \sum_{q_1,\dots,q_{r-1}}^{\le k} r\phi_{q_1}\cdots\phi_{q_{r+1}}\delta_{q_1+\dots+q_{r+1}+Q}, \end{aligned} \tag{54}$$

$$\begin{aligned}\frac{\partial^2\sigma_k}{\partial\phi_Q\partial\phi_{Q'}} &= \sum_{n=1}^{\infty}\sum_{r=0}^{\infty} g_{nr} \sum_{q_1,\dots,q_r}^{\le k} \left[\left(Q^2\right)^n + \left((Q')^2\right)^n + \left(q_1^2\right)^n r\right](r+1)\phi_{q_1}\cdots \\ &\quad \cdots\phi_{q_r}\delta_{q_1+\dots+q_r+Q+Q'} \\ &\quad + \sum_{r=2}^{\infty} u_r \sum_{q_1,\dots,q_{r-2}}^{\le k} r(r-1)\phi_{q_1}\cdots\phi_{q_{r-2}}\delta_{q_1+\dots+q_{r-2}}. \end{aligned} \tag{55}$$

If we take the momenta $Q = p$, $Q' = p'$ from the momentum shell at $|p| = |p'| = k$, we obtain f_p and $k_{p,p'}$ resp. from Eqs. (54) and (55) by excluding the momenta of the shell from the sums, i.e. by changing the upper limit of the sums over the momenta from $\leq k$ to $< k$. Furthermore we find

$$
\begin{aligned}
&\frac{\partial k_{p,p'}}{\partial \phi_Q} \\
&= \sum_{n=1}^{\infty}\sum_{r=0}^{\infty} g_{nr} \sum_{q_1,\ldots,q_{r-1}}^{<k} \left[2k^{2n} + \left(Q^2\right)^n + \left(q_1^2\right)^n (r-1)\right] r(r+1)\phi_{q_1}\cdots \\
&\quad \cdots \phi_{q_{r-1}} \delta_{q_1+\ldots+q_{r-1}+p+p'+Q} \\
&\quad + \sum_{r=2}^{\infty} u_r \sum_{q_1,\ldots,q_{r-3}}^{<k} (r-2)(r-1)r\phi_{q_1}\cdots\phi_{q_{r-3}}\delta_{q_1+\ldots+q_{r-3}+p+p'+Q}, \qquad (56)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial^2 k_{p,p'}}{\partial \phi_Q \partial \phi_{Q'}} \\
&= \sum_{n=1}^{\infty}\sum_{r=0}^{\infty} g_{nr} \sum_{q_1,\ldots,q_{r-2}}^{<k} \left[2k^{2n} + \left(Q^2\right)^n + \left((Q')^2\right)^n + \left(q_1^2\right)^n (r-2)\right] \cdot \\
&\quad \cdot (r-1)r(r+1)\phi_{q_1}\cdots\phi_{q_{r-2}}\delta_{q_1+\ldots+q_{r-2}+p+p'+Q+Q'} \\
&\quad + \sum_{r=2}^{\infty} u_r \sum_{q_1,\ldots,q_{r-4}}^{<k} (r-3)(r-2)(r-1)r\phi_{q_1}\cdots\phi_{q_{r-4}}\delta_{q_1+\ldots+q_{r-4}+p+p'+Q+Q'}.
\end{aligned}
\qquad (57)
$$

By means of the generating functions (9) and (10) we obtain:

$$
\begin{aligned}
\mathcal{P}\sigma_k &= V(x;k), && (58)\\
\mathcal{P}\frac{\partial^2 \sigma_k}{\partial\phi_Q \partial\phi_{-Q}} &= \partial_x G(Q^2,x;k) + \partial_x^2 V(x;k), && (59)\\
\mathcal{P}k_{p,p'} &= \left[\partial_x G(k^2,x;k) + \partial_x^2 V(x;k)\right]\delta_{p+p'} \equiv \mathcal{A}\delta_{p+p'}, && (60)\\
\mathcal{P}\frac{\partial k_{p,p'}}{\partial \phi_Q} &= \left[\partial_x^2 G(k^2,x;k) + \frac{1}{2}\partial_x^2 G(Q^2,x;k) + \partial_x^3 V(x;k)\right]\delta_{p+p'+Q} \\
&\equiv \mathcal{B}\delta_{p+p'+Q}, && (61)\\
\mathcal{P}\frac{\partial^2 k_{p,p'}}{\partial\phi_Q \partial\phi_{Q'}} &= \left[\partial_x^3 G(k^2,x;k) + \frac{1}{2}\partial_x^3 G(Q^2,x;k)\right. \\
&\quad \left. + \frac{1}{2}\partial_x^3 G((Q')^2,x;k) + \partial_x^4 V(x;k)\right]\delta_{p+p'+Q+Q'} \\
&\equiv \mathcal{C}\delta_{p+p'+Q+Q'}, && (62)\\
\mathcal{P}(\ln k)_{p,p'} &= \delta_{p+p'}\ln\left[\partial_x G(k^2,x;k) + \partial_x^2 V(x;k)\right] && (63)
\end{aligned}
$$

Applying the projector $\mathcal{P}$ on the r.h.s. of Eq. (6), the tree level term vanishes due to $\mathcal{P}f_p = 0$, as observed in [6]. Due to that the projection of the second derivative of the tree level term in Eq. (7) also takes the simpler form:

$$
\mathcal{P}\frac{\partial^2}{\partial\phi_Q \partial\phi_{-Q}}\left(f_p k^{-1}_{p.p'} f_{p'}\right) = \mathcal{P}\frac{\partial f_p}{\partial \phi_{-Q}}\mathcal{P}k^{-1}_{p,p'}\mathcal{P}\frac{\partial f_{p'}}{\partial\phi_Q} + \mathcal{P}\frac{\partial f_p}{\partial\phi_Q}\mathcal{P}k^{-1}_{p,p'}\mathcal{P}\frac{\partial f_{p'}}{\partial\phi_{-Q}}. \qquad (64)
$$

However we find that $\mathcal{P}\left(\partial f_p/\partial\phi_Q\right) = 0$ for any $Q \neq -p$, and therefore the tree level term vanishes after acting with the projector $\mathcal{P}$ on the r.h.s. of Eq. (7).

Making use of the matrix identity $\partial(kk^{-1})_{p,p'}/\partial\phi_Q = 0$, we find

$$\frac{\partial k^{-1}_{p,p'}}{\partial\phi_Q} = -\left(k^{-1}\frac{\partial k}{\partial\phi_Q}k^{-1}\right)_{p,p'} \tag{65}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial\phi_Q\partial\phi_{-Q}}(\ln k)_{p,p'} &= \frac{1}{2}\left[k^{-1}\frac{\partial^2 k}{\partial\phi_Q\partial\phi_{-Q}} + \frac{\partial^2 k}{\partial\phi_Q\partial\phi_{-Q}}k^{-1}\right.\\ &\left.-k^{-1}\frac{\partial k}{\partial\phi_{-Q}}k^{-1}\frac{\partial k}{\partial\phi_Q} - \frac{\partial k}{\partial\phi_Q}k^{-1}\frac{\partial k}{\partial\phi_{-Q}}k^{-1}\right]_{p,p'}\end{aligned} \tag{66}$$

and then obtain

$$\mathcal{P}\frac{\partial^2}{\partial\phi_Q\partial\phi_{-Q}}(\ln k)_{p,p'} = \left(\frac{\mathcal{C}}{\mathcal{A}} - \frac{\mathcal{B}^2}{\mathcal{A}^2}\right)\delta_{p+p'}. \tag{67}$$

Let us apply now the projector $\mathcal{P}$ on both sides of Eqs. (6) and (7):

$$k\partial_k V(x,k) = -\alpha k^d \ln\frac{\mathcal{A}}{\mathcal{A}_{x_c}}, \tag{68}$$

$$k\partial_k\left[\partial_x G(Q^2,x,k) + \partial_x^2 V(x,k)\right] = -\alpha k^d\left[\frac{\mathcal{C}}{\mathcal{A}} - \frac{\mathcal{B}^2}{\mathcal{A}^2}\right]. \tag{69}$$

As far as $G(Q^2,x,k)$ is assumed being analytic in the variable Q^2, Eqs. (68) and (69) are consistent if and only if the equation obtained from Eq. (69) by setting $Q^2 = 0$,

$$k\partial_k\partial_x^2 V = -\alpha k^d\left[\frac{\mathcal{C}}{\mathcal{A}} - \frac{\mathcal{B}^2}{\mathcal{A}^2}\right]_{Q^2=0}, \tag{70}$$

is the consequence of Eq. (68). Indeed, Eq. (70) can be obtained from Eq. (68) by taking the second partial derivative of its both sides with respect of the variable x.

Subtracting Eq. (70) from Eq. (69) we obtain the coupled set of partial differential equations (11) and (12) for the generating functions.

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