

Review

Spontaneous Symmetry Breaking and Nambu–Goldstone Bosons in Quantum Many-Body Systems

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Abstract: Spontaneous symmetry breaking is a general principle that constitutes the underlying concept of a vast number of physical phenomena ranging from ferromagnetism and superconductivity in condensed matter physics to the Higgs mechanism in the standard model of elementary particles. I focus on manifestations of spontaneously broken symmetries in systems that are not Lorentz invariant, which include both nonrelativistic systems as well as relativistic systems at nonzero density, providing a self-contained review of the properties of spontaneously broken symmetries specific to such theories. Topics covered include: (i) Introduction to the mathematics of spontaneous symmetry breaking and the Goldstone theorem. (ii) Minimization of Higgs-type potentials for higher-dimensional representations. (iii) Counting rules for Nambu–Goldstone bosons and their dispersion relations. (iv) Construction of effective Lagrangians. Specific examples in both relativistic and nonrelativistic physics are worked out in detail.

Keywords: spontaneous symmetry breaking; Nambu–Goldstone bosons; effective field theory

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1. Introduction

Symmetry considerations play a key role in our understanding of Nature. Ever since the birth of science, the aesthetic beauty of symmetry has appealed to the minds of natural philosophers, and later physicists, seeking for the origin of the laws of Nature. With the advent of quantum mechanics in the twentieth century, symmetry techniques based on group theory became an indispensable tool of a theoretical physicist. While in classical physics symmetries are straightforwardly incorporated and directly connected to physical observables by means of the Noether theorem, the situation in the quantum theory is more subtle. A fundamental theorem due to Wigner (see Section 2.2 in [1]) states that invariance of observables under certain transformation implies the existence of a unitary operator on the Hilbert space of states. If the symmetry transformation is moreover compatible with the dynamics of the system, this unitary operator commutes with the Hamiltonian and gives rise to a characteristic multiplet structure in its spectrum. However, there are systems whose dynamics is invariant under a symmetry transformation, yet this symmetry is not manifest in the spectrum or physical observables. We speak of *spontaneous symmetry breaking* (SSB). This phenomenon is ubiquitous in condensed matter physics where it is responsible, for example, for the peculiar behavior of superconductors, superfluids, and ferromagnets. In high energy physics, it underlies much of our current understanding of the fundamental interactions and the origin of masses of elementary particles.

This paper provides a pedagogical review of the properties of spontaneously broken *continuous global internal symmetries* with special emphasis on systems lacking Lorentz invariance that is either intrinsically nonrelativistic systems or relativistic systems at nonzero density. (In the following, all these systems will for simplicity be referred to as nonrelativistic, unless explicitly indicated otherwise.) It aims to be a self-contained introduction into the subject, requiring just moderate knowledge of quantum field theory. Nevertheless, it will hopefully also offer some new material to the experts.

This paper is organized as follows. In Section 2, the notion of SSB is introduced and the mathematical subtleties associated with its implementation on the Hilbert space of a quantum system are discussed to some depth. A specific example, a free nonrelativistic field theory, is analyzed in detail in Section 3., illustrating explicitly the general material of the previous section, and at the same time showing some nontrivial features of SSB in nonrelativistic systems to be discussed in what follows. The technically easiest way to achieve SSB in an interacting field theory is to introduce an effective scalar field and adjust its phenomenological potential so that it has a symmetry-breaking minimum at the tree level. While this is rather trivial in the case of the Higgs mechanism in the standard model of elementary particles or the Ginzburg–Landau theory for ferromagnets, it becomes a difficult problem with the increasing dimension of the representation in which the scalar field transforms. Section 4. describes a general method to minimize Higgs potentials; the specific example of a spin-one color superconductor is worked out in detail. Section 5. presents, from the theoretical point of view, the most important part of the paper. I first formulate and prove the Goldstone theorem and point out its limitations in case of nonrelativistic systems, essentially following the classic review [2]. Then the issue of the counting rules for Nambu–Goldstone (NG) bosons is introduced, starting with the seminal contribution of Nielsen and Chadha [3], and then discussing the recent developments. In Section 6. the general results are elucidated using several explicit examples. Finally, Section 7. is devoted to the model-independent description

of SSB based on low-energy effective field theory, worked out in the general nonrelativistic case by Leutwyler [4]. Section 8. provides a summary and offers some comments on spontaneous breaking of spacetime symmetries, not covered by this paper.

What I have regretfully omitted from this paper is a thorough discussion of applications of SSB and detailed examples of its appearance in realistic interacting quantum field theories. Even though this would certainly help the reader close the gap between the general principles and their concrete applications in current literature, I had several reasons for this omission. First, the list of problems to discuss would strongly depend on personal taste, and even with a very limited selection would increase the volume of the paper inadequately. Second, I prefer to stay at an as general level as possible so that the paper may be found useful by a wider audience. I thus discuss the principles and the model-independent approach of effective field theory, and only use specific model examples to illustrate the general results. Finally, dealing with realistic field theories would almost inevitably mean using suitable approximation schemes. One then has to make extra effort to show that the obtained results are physical, and not just artifacts of the approximation. On the contrary, most of the examples presented in the text are, sometimes in a restricted sense, exactly solvable.

Since this is a review, most of its content is essentially compiled from existing literature. However, some of the presented material is new. This concerns in particular Section 6.: to the best of my knowledge this is the first time to give an explicit, and exactly solvable, example of a system exhibiting a NG boson whose energy is proportional to a *noninteger* power of momentum. The nonanalytic long-distance behavior of such a system clearly precludes any description based on conventional effective field theory. Furthermore, I have improved upon the proof of Theorem 4 on the NG boson counting in absence of charge density. Finally, as far as I know, even the identification of a free nonrelativistic particle as a type-II NG boson presented in Section 3. is novel. Throughout the text I use, for the sake of simplicity, the natural units in which $\hbar = c = 1$. Also, I use the Einstein summation convention and the Lorentz indices are contracted with the timelike Minkowski metric, $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

The literature dealing with various applications of the general principle of SSB is vast, and I therefore apologize to all authors not mentioned explicitly in the list of references for having omitted their valuable contributions. I have, at least, tried to include papers discussing to some extent the *general* properties of spontaneously broken symmetries, in particular those focusing on nonrelativistic systems. Besides papers covering specific subtopics to be discussed in the following sections, I would also like to point out several good reviews of the physics of Nambu–Goldstone bosons and their effective field theory description [5–9] as well as some specialized in the relativistic chiral perturbation theory [10–12].

2. Basic Properties of Spontaneously Broken Symmetries

Consider a quantum field theory whose action is invariant under a continuous group of symmetry transformations. By Noether’s theorem, this gives rise to the existence of a conserved current $j^\mu(x)$, satisfying the continuity equation, $\partial_\mu j^\mu(x) = 0$. In the standard, Wigner–Weyl realization of the symmetry, the associated integral charge Q serves as a generator of the symmetry transformations, which are implemented on the Hilbert space of states by unitary operators. The ground state is assumed to be a discrete, nondegenerate eigenstate of the Hamiltonian. Consequently, it bears a one-dimensional representation of the symmetry group, and therefore also is an eigenstate of the charge Q . The spectrum

of all eigenstates splits into multiplets of the symmetry, corresponding to irreducible representations of the symmetry group.

2.1. Ground state and finite symmetry transformations

A *spontaneously broken symmetry* can be naively characterized as such symmetry that the *ground state is not an eigenstate of its generator*. In order to give a more formal definition, let us follow the review [2]. Given the charge density $j^0(x)$, one introduces for an arbitrary finite space domain Ω the operator

$$Q_\Omega(t) = \int_\Omega d^3\mathbf{x} j^0(\mathbf{x}, t) \quad (1)$$

In the following I will use the same letter Ω to denote the domain and its volume; the precise meaning of the symbol will always be clear from the context. The *symmetry breaking condition* can be restated as the existence of a (not necessarily local) operator Φ such that

$$\lim_{\Omega \rightarrow \infty} \langle 0 | [Q_\Omega(t), \Phi] | 0 \rangle \neq 0 \quad (2)$$

where $|0\rangle$ is a translationally invariant ground state. This expectation value is known as the *order parameter*. Clearly, this formal definition immediately implies the previous one: if the vacuum were an eigenstate of the charge operator, the expectation value of this commutator would have to be zero.

It is customary to identify $Q(t) = \lim_{\Omega \rightarrow \infty} Q_\Omega(t)$ formally with the integral charge operator. However, strictly speaking, this operator does not exist [13]. Indeed, translational invariance of the vacuum would then imply translational invariance of the state $Q(t)|0\rangle$, and consequently

$$\langle 0 | Q(t) Q(t) | 0 \rangle = \int d^3\mathbf{x} \langle 0 | j^0(\mathbf{x}, t) Q(t) | 0 \rangle = \int d^3\mathbf{x} \langle 0 | j^0(\mathbf{0}, t) Q(t) | 0 \rangle \quad (3)$$

which diverges unless $Q(t)|0\rangle = 0$. This would, however, be in contradiction with the symmetry breaking condition (2).

The intuitive picture of spontaneous symmetry breaking, based on the observation that a symmetry transformation does not leave the ground state intact, suggests high degeneracy of equivalent ground states. Indeed, since the charge operator commutes with the Hamiltonian, so will a finite symmetry transformation generated by this operator. It will therefore transform the ground state into another state with the same energy. As long as the symmetry group is continuous, we will find *infinitely many degenerate ground states*. On account of the fact that they are all connected by symmetry transformations, they must be physically equivalent and any one of them can serve as a starting point for the construction of the spectrum of excited states. The issue of the choice of a ground state will be discussed in more detail in the next subsection.

However, mathematical implementation of these ideas is subtle. The finite volume charge operator $Q_\Omega(t)$ induces a “finite symmetry transformation,” $U_\Omega(\theta, t) = \exp[i\theta Q_\Omega(t)]$, which in turn gives rise to a “rotated ground state,” $|\theta, t\rangle_\Omega = U_\Omega^\dagger(\theta, t)|0\rangle$. However, very much like the limit $\lim_{\Omega \rightarrow \infty} Q_\Omega(t)$ does not exist, the operator $\exp[i\theta Q(t)]$ is not well defined either. In fact, it can be proved that

$$\lim_{\Omega \rightarrow \infty} \langle 0 | \theta, t \rangle_\Omega = \lim_{\Omega \rightarrow \infty} \langle 0 | \exp[-i\theta Q_\Omega(t)] | 0 \rangle = 0 \quad (4)$$

as will be shown later on explicit examples. It means that in the infinite volume (thermodynamic) limit, any two ground states, formally connected by a symmetry transformation, are actually orthogonal. The same conclusion holds for excited states constructed above these vacua. All these states therefore cannot be accommodated in a single separable Hilbert space, forming rather two distinct Hilbert spaces of their own. Any of these Hilbert spaces can, nevertheless, be taken as a basis for an equivalent description of the system, and the choice has no observable physical consequences.

Unlike the transformations of physical states, finite symmetry transformations of observables can be defined consistently in a theory which is sufficiently causal. Using the Baker–Campbell–Hausdorff formula one obtains for any operator A that

$$A_{\theta,t;\Omega} \equiv U_{\Omega}(\theta, t) A U_{\Omega}^{\dagger}(\theta, t) = A + i\theta [Q_{\Omega}(t), A] + \frac{1}{2}(i\theta)^2 [Q_{\Omega}(t), [Q_{\Omega}(t), A]] + \cdots \quad (5)$$

where $[Q_{\Omega}(t), A] = \int_{\Omega} d^3\mathbf{x} [j^0(\mathbf{x}, t), A]$

As long as the theory satisfies the microcausality condition, that is, the commutator of any two local operators separated by a spacelike interval vanishes, and as long as the operator A is localized in a finite domain of spacetime, there will be a region Ω_0 such that the charge density outside this region does not contribute to the commutator,

$$\int_{\mathbb{R}^3 \setminus \Omega_0} d^3\mathbf{x} [j^0(\mathbf{x}, t), A] = 0 \quad \text{whence} \quad \lim_{\Omega \rightarrow \infty} [Q_{\Omega}(t), A] = \int_{\Omega_0} d^3\mathbf{x} [j^0(\mathbf{x}, t), A] \quad (6)$$

The transformation $U_{\Omega}(\theta, t) A U_{\Omega}^{\dagger}(\theta, t)$ therefore has a well-defined limit as $\Omega \rightarrow \infty$. The expectation value of the rotated operator $A_{\theta,t;\Omega}$ in the vacuum $|0\rangle$ can then be interpreted as the expectation value of A in the rotated vacuum $|\theta, t\rangle_{\Omega}$. One should nevertheless keep in mind that in the limit of infinite volume the correct formal definition proceeds as above.

Let me recall at this point that in Lorentz invariant theories the microcausality condition is automatically guaranteed. This is, of course, not the case for nonrelativistic field theories where interaction is often modeled by a nonlocal instantaneous potential rather than by an exchange of a propagating mode. However, the above argument can still be applied provided the range of the interaction is short enough [14]. Typically, it is sufficient when the correlations decay exponentially at long distance. If they decrease just as some power of distance, the conditions for applicability of the general arguments concerning SSB have to be inspected case by case.

2.2. Explicit symmetry breaking and the choice of the ground state

In the preceding subsection it was demonstrated that SSB gives rise to multiple degenerate ground states in which the order parameter acquires different values. I also stressed that any of these degenerate vacua can be chosen as the physical ground state. A natural question then arises: why does the system actually choose one of the states with a definite value of the order parameter, and not their superposition in which the expectation value of the order parameter could be even made to vanish? The answer to this question is tightly connected to the presence of external perturbations and the thermodynamic limit. A thorough discussion of this issue is given in Section 19.1 of [15].

In finite volume the ground state of a quantum system is usually (though not necessarily always) nondegenerate. One still has a large number of states formally connected by symmetry transformations,

but their exact degeneracy is lifted by boundary conditions. (An exception is the frequently used periodic boundary condition that preserves translational invariance and thus inherits many of the properties of the infinite volume system. Nevertheless, this boundary condition seems somewhat artificial from a physical point of view.) The unique ground state is then given by a symmetric superposition of all the states, very much like the ground state of the quantum mechanical particle in a potential double well. The distance of the individual energy levels is essentially determined by the tunneling amplitude for the transition from one state with a definite value of the order parameter to another. In the infinite volume limit, this amplitude is exponentially suppressed and the ground states become perfectly degenerate. In principle it is indeed possible, though technically not so advantageous, to construct a quantum mechanical description of a spontaneously broken symmetry using a symmetric ground state. SSB is then manifested by long range correlations rather than nonzero vacuum expectation values [16].

In practice, the symmetry is almost never exact but is usually disturbed by small perturbations such as external fields. A typical example is the ferromagnet, where the role of the order parameter is played by the spontaneous magnetization. In principle, the magnetization can take any direction, but an arbitrarily weak external magnetic field will cause the magnetization to align with it. This phenomenon is general. As soon as the system volume is large enough, the energy difference induced by the external perturbation will be much larger than the tiny intrinsic splitting of the energy levels. The ground state will then be determined solely by the perturbation; this is called *vacuum alignment*. By choosing an appropriate perturbation one can select the corresponding vacuum. The key step is now that there is a basis in the space of degenerate ground states in which all observables become diagonal operators in the limit of infinite volume (see Section 19.1 in [15]). Therefore, it is the same basis of states in which the order parameter takes definite values, and one of which is selected by the external perturbation, regardless of what operator is used as the perturbation. This concludes the argument that the physical vacua are those with a definite value of the order parameter; their superpositions do not survive the infinite volume limit. One practical means of selecting the ground state thus is to switch on an external perturbation and *then* go to infinite volume. After this is done, the perturbation can be adiabatically switched off without disturbing the vacuum. It is symptomatic of SSB that the order of these two limits (infinite volume versus zero perturbation) cannot be reversed.

Before concluding the section, let me summarize the main features of SSB. First and most importantly, the symmetry is not realized by unitary operators on the Hilbert space, so it does not give rise to multiplets in the spectrum. A hallmark of SSB is the existence of an order parameter, that is, nonzero vacuum expectation value of an operator which transforms nontrivially under the symmetry group. There is a continuum of degenerate ground states, labeled by different values of the order parameter. Each of these ground states constitutes a basis of a distinct Hilbert space, all of them bearing unitarily inequivalent representations of the broken symmetry. This is the so-called *Nambu–Goldstone realization* of symmetry. From the phenomenological point of view, the most important consequence of SSB is the existence of soft modes in the spectrum, the Nambu–Goldstone bosons. Section 5. will be devoted to a detailed discussion of this phenomenon.

3. Example: Free Nonrelativistic Particle

As an illustration of the general abstract material of the previous section, let us consider the simplest example imaginable, the free nonrelativistic field theory (see also Chapter 2 of [17]). In order to demonstrate the variety of technical approaches to SSB, I will follow a different path: instead of defining the charge operator by an integral of the charge density over a finite space domain, the whole system will be quantized in finite volume Ω .

The dynamics of the system is governed by the Lagrangian density

$$\mathcal{L} = i\psi^\dagger \frac{\partial \psi}{\partial t} - \frac{1}{2m} \nabla \psi^\dagger \cdot \nabla \psi \quad (7)$$

Obviously, the action defined by this Lagrangian is invariant under the following three kinds of transformations,

$$\psi \xrightarrow{\#1} \psi + \theta, \quad \psi \xrightarrow{\#2} \psi + i\theta, \quad \psi \xrightarrow{\#3} e^{i\theta} \psi \quad (8)$$

with a real parameter θ . (Complex conjugated transformations for $\psi^\dagger$ are implied.) Note that the transformation #3, corresponding to the usual conservation of particle number, leaves the Lagrangian (7) itself invariant, while the transformations #1, #2 change it by a total derivative. Taking this into account, one infers from the Noether theorem the following list of conserved charge and current densities,

$$\begin{aligned} \varrho_1 &= i(\psi - \psi^\dagger), & \varrho_2 &= \psi + \psi^\dagger, & \varrho_3 &= \psi^\dagger \psi \\ \mathbf{j}_1 &= \frac{1}{2m} \nabla(\psi + \psi^\dagger), & \mathbf{j}_2 &= -\frac{i}{2m} \nabla(\psi - \psi^\dagger), & \mathbf{j}_3 &= \frac{1}{2im} (\psi^\dagger \nabla \psi - \psi \nabla \psi^\dagger) \end{aligned} \quad (9)$$

The three independent types of transformations listed in Equation (8) generate the symmetry group ISO(2), that is, the two-dimensional Euclidean group. Transformations #1, #2 correspond to translations in the plane defined by the real and imaginary part of ψ , while #3 yields the rotation in this plane. Demanding as usual that the Schrödinger picture fields $\psi(\mathbf{x}), \psi^\dagger(\mathbf{x})$ satisfy the canonical commutation relation,

$$[\psi(\mathbf{x}), \psi^\dagger(\mathbf{y})] = \delta^3(\mathbf{x} - \mathbf{y}) \quad (10)$$

one easily derives the algebra of the Noether charges in finite volume, $Q_a = \int_\Omega d^3\mathbf{x} \varrho_a(\mathbf{x})$,

$$[Q_3, Q_1] = -iQ_2, \quad [Q_3, Q_2] = +iQ_1, \quad [Q_1, Q_2] = 2i\Omega \quad (11)$$

While the first two relations are identical to their classical counterparts, the two translations at the classical level commute. The last commutator in (11) hence shows that after quantization, the algebra ISO(2) develops a *central charge*, and the representation of finite group transformations becomes *projective*.

As an aside let me remark that a projective representation of a symmetry group of a quantum system may arise either as a result of nontrivial global topology of the symmetry group, or as a consequence of a central charge in the Lie algebra as above. For semi-simple Lie algebras the central charges can always be removed by a proper redefinition of generators. In the case of the non-semi-simple Euclidean algebras ISO(N), the central charges vanish for all N but $N = 2$, which is exactly our case. Further details on the projective representations of symmetry groups may be found in [1] (Section 2.7) or [18] (Chapter 13).

3.1. Hilbert space and inequivalent ground states

In finite volume one can always define the (countable) basis of the physical Hilbert space using the Fock construction. In the following, we will implicitly assume periodic boundary conditions so that (discrete) translational invariance is preserved and the basis of one-particles states may be chosen as the eigenstates of the momentum operator. These one-particle states, labeled by the three-momentum $\mathbf{k}$, can thus be obtained from the Fock vacuum $|0\rangle$ by the action of creation operators $a_{\mathbf{k}}^\dagger$, $|\mathbf{k}\rangle \equiv a_{\mathbf{k}}^\dagger|0\rangle$. Let us suppose for the sake of generality that the one-particle states are normalized as $\langle \mathbf{k}'|\mathbf{k}\rangle = N_{\mathbf{k}}\delta_{\mathbf{k}\mathbf{k}'}$. Then the annihilation and creation operators satisfy the corresponding relation $[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = N_{\mathbf{k}}\delta_{\mathbf{k}\mathbf{k}'}$. The canonical commutator of the fields (10) subsequently fixes the normalization of their expansion in the annihilation and creation operators,

$$\psi(\mathbf{x}) = \sum_{\mathbf{k}} \frac{1}{\sqrt{N_{\mathbf{k}}\Omega}} e^{i\mathbf{k}\cdot\mathbf{x}} a_{\mathbf{k}} \quad (12)$$

Using this expansion of the fields, one finds an explicit representation of the symmetry generators,

$$Q_1 = i\sqrt{\frac{\Omega}{N_0}}(a_0 - a_0^\dagger), \quad Q_2 = \sqrt{\frac{\Omega}{N_0}}(a_0 + a_0^\dagger), \quad Q_3 = \sum_{\mathbf{k}} \frac{1}{N_{\mathbf{k}}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \quad (13)$$

From here we immediately see that $\|Q_{1,2}|0\rangle\|^2 = \Omega \rightarrow \infty$ in the infinite volume (thermodynamic) limit. This clearly indicates that the operators $Q_{1,2}$ are ill-defined on the Hilbert space built above the Fock vacuum in the limit that the space volume is sent to infinity.

The Hamiltonian of the system is in the second quantization expressed as

$$H = \sum_{\mathbf{k}} \frac{1}{N_{\mathbf{k}}} \frac{\mathbf{k}^2}{2m} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \quad (14)$$

Obviously, the Fock vacuum $|0\rangle$ represents one possible ground state with zero energy. However, we already know that other, equivalent, ground states may be obtained from $|0\rangle$ by applying symmetry transformations. Let us therefore denote

$$|z\rangle \equiv |\theta_1, \theta_2\rangle = e^{i(\theta_1 Q_1 + \theta_2 Q_2)} |0\rangle \quad \text{where} \quad z = \theta_1 + i\theta_2 \quad (15)$$

Substituting from (13) yields

$$|z\rangle = \exp \left[\sqrt{\frac{\Omega}{N_0}} (z a_0^\dagger - z^* a_0) \right] |0\rangle \quad (16)$$

and a simple manipulation shows that all these states are in fact coherent states, corresponding to the eigenvalue $z\sqrt{N_0\Omega}$ of the annihilation operator a_0 . Naturally, they are annihilated by all $a_{\mathbf{k}}$ with nonzero momentum $\mathbf{k}$. They have the same energy as the Fock vacuum since adding an arbitrary number of zero-momentum quanta does not change the total energy of the system. The magnitude of the scalar product of two such states is equal to

$$|\langle z'|z\rangle| = e^{-\Omega|z'-z|^2} \quad (17)$$

This demonstrates explicitly the general fact that in the infinite volume limit, any two ground states connected by a broken symmetry transformation become orthogonal.

3.2. Spontaneous symmetry breaking and the Nambu–Goldstone boson

Let us first analyze the pattern of symmetry breaking when the ground state is chosen as the Fock vacuum $|0\rangle$. Apparently, out of the three generators Q_a only the third one preserves the ground state, the other two are spontaneously broken. In the field space this corresponds to the fact that the full Euclidean group $\text{ISO}(2)$ is spontaneously broken to its rotation subgroup, $\text{SO}(2)$. The Goldstone theorem (to be discussed in detail in Section 5.) now asserts that for each broken generator there should be a state in the spectrum which couples to the associated current. In this case there is obviously only one such state, namely the one-particle state $|\mathbf{k}\rangle$, and the corresponding amplitudes read

$$\begin{aligned}\langle 0|\varrho_1(\mathbf{x})|\mathbf{k}\rangle &= i\sqrt{\frac{N_{\mathbf{k}}}{\Omega}}e^{i\mathbf{k}\cdot\mathbf{x}}, & \langle 0|\mathbf{j}_1(\mathbf{x})|\mathbf{k}\rangle &= \frac{i\mathbf{k}}{2m}\sqrt{\frac{N_{\mathbf{k}}}{\Omega}}e^{i\mathbf{k}\cdot\mathbf{x}} \\ \langle 0|\varrho_2(\mathbf{x})|\mathbf{k}\rangle &= \sqrt{\frac{N_{\mathbf{k}}}{\Omega}}e^{i\mathbf{k}\cdot\mathbf{x}}, & \langle 0|\mathbf{j}_2(\mathbf{x})|\mathbf{k}\rangle &= \frac{\mathbf{k}}{2m}\sqrt{\frac{N_{\mathbf{k}}}{\Omega}}e^{i\mathbf{k}\cdot\mathbf{x}}\end{aligned}\quad (18)$$

All these formulas are valid in the Schrödinger picture where the fields are time independent. However, transition to the Heisenberg picture is trivial in this noninteracting field theory.

Due to the dispersion relation, $E_{\mathbf{k}} = \mathbf{k}^2/2m$, the state $|\mathbf{k}\rangle$ represents a NG boson with momentum $\mathbf{k}$. Since its dispersion relation is quadratic, it is the simplest possible example of a type-II NG boson, to be discussed in depth later. Note that there is a *single NG boson* that couples to *two broken generators*. As will become clear in Section 5. this is only possible when the commutator of the two broken generators develops nonzero vacuum expectation value. A glance at Equation (11) shows that this is indeed the case. Moreover, our simple noninteracting field theory provides a rather nontrivial realization of this condition. The vacuum expectation value of the commutator $[Q_1, Q_2]$ is guaranteed by the existence of a central charge in the quantized theory, rather than by a nonzero charge density as one might naively expect based on the Lie algebraic structure of the symmetry.

What now if we choose as the ground state one of the coherent states $|z\rangle$ with nonzero z ? In field space this means shifting the ground state off the origin,

$$\langle z|\psi(\mathbf{x})|z\rangle = z \quad (19)$$

Since the new ground state $|z\rangle$ is (in finite volume) connected to the Fock vacuum $|0\rangle$ by a symmetry transformation, they are physically equivalent. The corresponding symmetry breaking patterns thus also have to be the same, just with a different basis of broken and unbroken generators. The geometric picture is clear: any expectation value of the field always breaks the two translations in the field space; it only preserves the rotations about the point $\langle z|\psi(\mathbf{x})|z\rangle = z$.

From a physical point of view, the states $|z\rangle$ are very interesting (see Section 2.3 in [17]). Note that the expectation of the particle number operator Q_3 is

$$\langle z|Q_3|z\rangle = \Omega|z|^2 \quad (20)$$

The operator Q_3 itself is apparently not well defined in the infinite volume limit. However, the density $\langle z|\varrho_3|z\rangle$ remains finite in this limit. The states $|z\rangle$ therefore describe a finite density free nonrelativistic many-body system with all particles occupying the lowest energy level, that is, a Bose–Einstein

condensate (this issue is discussed with much greater mathematical rigor in [19]). Different choices of z correspond to condensates with a different density and/or phase.

A cautious reader may now be confused by my previous statement, based on general considerations, that all ground states $|z\rangle$ are equivalent since they are connected by symmetry transformations. Yet, systems with different density do not look physically equivalent. The resolution of this seeming paradox is tightly bound with measurement theory [2]. Any device that can measure the system density must be able to distinguish states with different density, and therefore explicitly break those symmetry transformations that connect these states. On the other hand, if we only allow for a measuring apparatus that preserves the symmetry of the system, then the different ground states will really be indistinguishable.

Let us finally comment on the general technique that allows one to pick a single ground state out of the continuum of degenerate ones. One introduces an explicit symmetry breaking term in the Hamiltonian which splits the energy of the ground states. Then the infinite volume limit is performed and only afterwards the symmetry breaking term is removed. In our case such an explicit symmetry breaking term can be proportional to the particle number operator Q_3 . One replaces the Hamiltonian (14) with $H_\mu = H - \mu Q_3$. The parameter μ plays the role of a chemical potential. In view of Equation (20) the energy of the state $|z\rangle$ is then $\langle z|H_\mu|z\rangle = -\mu\Omega|z|^2$. In order that the system has a ground state, the chemical potential must be non-positive. For any strictly negative value of μ the Fock vacuum $|0\rangle$ becomes the single nondegenerate ground state, while the energy of all other coherent states goes to infinity in the thermodynamic limit. In other words, in presence of a Bose–Einstein condensate the chemical potential must be zero, as is of course well known. Obviously, any of the coherent states $|z\rangle$ can be selected as the preferred one by a proper choice of the explicit symmetry breaking term.

Even though this section was devoted to a free nonrelativistic particle, it is amusing to compare to the free *relativistic* massless complex scalar field theory. There, the action possesses the same symmetry group $\text{ISO}(2)$, while the spectrum contains two NG bosons with energy proportional to momentum: the quantum of the field and its antiparticle. The spectrum is therefore dramatically different even though the symmetry breaking pattern is the same. The reason is that in the relativistic case the shift generators Q_1 and Q_2 commute even after quantization since the contributions to the commutator from the particle and antiparticle sectors cancel each other. The algebra $\text{ISO}(2)$ thus does not develop a central charge and the implementation of broken symmetry as well as the spectrum of NG bosons is usual.

4. Achieving Spontaneous Symmetry Breaking: Minimization of Higgs Potentials

Demonstrating that a given physical system exhibits SSB requires two key steps. First, one has to identify a suitable order parameter. This is usually very difficult to do from first principles. The reason is that SSB is a nonperturbative phenomenon, so it cannot be achieved at any finite order of perturbation theory based on a ground state which preserves the symmetry. One then often relies on observations (such as in the case of ferromagnets where the order parameter, the spontaneous magnetization, is obvious) or physical insight (such as in the case of superconductors, where pair correlations are crucial).

The second step is the actual calculation of the order parameter. From the qualitative point of view, there are essentially two approaches. The first, physically more satisfactory but technically much more difficult, is to find a self-consistent (thus nonperturbative) symmetry-breaking solution to

the equations of motion, starting from whatever degrees of freedom are present in the theory. The NG bosons then typically appear as collective modes of these elementary degrees of freedom. This is the case, for instance, of the Heisenberg model of a ferromagnet to be discussed in Section 6., the celebrated Nambu–Jona-Lasinio model [20,21], or the technicolor theories in high energy physics. The most serious technical drawback of this approach is that one usually has to introduce an Ansatz for the symmetry-breaking solution and show in turn that it is consistent with the equation of motion. Thus one only gets such solutions that are put in by hand from the beginning.

The second, more phenomenological approach is to introduce an effective field with suitable transformation properties under the symmetry in question, which creates the NG bosons and whose ground state expectation value at the same time serves as the order parameter. This is done, for example, in the Higgs mechanism in the standard model of elementary particles or in the Ginzburg–Landau theory of phase transitions. While the minimization of a quartic potential with one complex doublet scalar field as in the standard model is straightforward, it can become a complicated task provided the order parameter transforms in a higher-dimensional representation of the symmetry group such as in some models of grand unification. It is therefore worthwhile to ask what can be said about the minima of a potential just from symmetry considerations. The investigation of this issue was initiated by Michel [22,23], and in the next subsection I will review some basic results. A heuristic method to minimize general Higgs-type potentials, developed by Kim [24], will then be presented and illustrated on an explicit nontrivial example.

4.1. Group action on the order parameter space

Let me introduce, in a non-rigorous manner following Section 6.2 of [25], some basic mathematical terms needed for the discussion of invariant potentials and their minima. Let G be a compact Lie group, acting smoothly on an infinitely differentiable manifold M . The points on this manifold may be thought of as values of the order parameter, and the action of a group element, $g \in G$, on a point $\phi \in M$ will be denoted simply as $g\phi$. For each point we also denote as H_ϕ the set of all group elements that leave ϕ invariant, that is, $H_\phi = \{g \in G \mid g\phi = \phi\}$. This set forms a subgroup of G and is called the *little (or isotropy) group* of ϕ . In physical terms, it consists exactly of those transformations left unbroken when the order parameter takes the value ϕ .

A very important notion that we will further use is that of the *orbit* $G(\phi)$ of a given point ϕ . It consists of all points of M which can be reached from ϕ by a (naturally, broken) symmetry transformation, $G(\phi) = \{g\phi \mid g \in G\}$. The relation defined by the condition that two points be connected by a group transformation is an equivalence, and the group orbits then define a decomposition of the manifold M into equivalence classes. Any potential V on the manifold which is invariant under the group action, $V(\phi) = V(g\phi)$ for all $\phi \in M$ and $g \in G$, may be thought of as a function on the orbits. The minimization problem for a given potential on M can therefore be reformulated as a minimization of a function on the space of orbits.

It is clear from the definition of the orbit that two points on the same orbit have isomorphic little groups. This is in fact an immediate consequence of the stronger statement that the two little groups are *conjugate*, $H_{g\phi} = gH_\phi g^{-1}$. In addition to the orbit $G(\phi)$ there may be other points on the manifold whose little group is conjugate to H_ϕ . For example, when M is a linear space such as in the Higgs

mechanism, multiplying ϕ by any (nonzero) number we obviously get a point with the same little group as ϕ . The set of all points with little groups conjugate to H_ϕ is called a *stratum*, $S(\phi)$. Intuitively, a stratum consists of all points of the same symmetry “class”: they have the same unbroken subgroup and the same symmetry-breaking pattern. In the phase diagram, a stratum would be associated with a particular phase. At a phase transition, the order parameter moves from one stratum to another, and the symmetry class changes.

We are now ready to formulate a fundamental theorem which constrains possible stationary points of group-invariant potentials [22]:

Theorem 1 (Michel) *Let G be a compact Lie group acting smoothly on the real manifold M and let $\phi \in M$. Then the orbit $G(\phi)$ is critical, that is, every smooth real G -invariant function on M is stationary on $G(\phi)$ if and only if $G(\phi)$ is isolated in its stratum, that is, there is a neighborhood U_ϕ of ϕ such that $U_\phi \cap S(\phi) = G(\phi)$.*

It is worth emphasizing that the theorem makes no particular assumption about the form of the invariant function, so it may be a Higgs-type quartic potential as well as the full quantum effective potential whose power expansion would contain terms of arbitrarily high orders in ϕ . In case M is a linear space Theorem 1 is actually not very useful. For every point $\phi \in M$ its stratum $S(\phi)$ also contains all points $\lambda\phi$ with nonzero λ , and the orbit $G(\phi)$ is therefore never isolated in its stratum unless $\phi = 0$. So in this case, the theorem simply asserts a rather obvious fact that the origin $\phi = 0$ is a stationary point of every G -invariant function on the space M .

On the other hand when the manifold M is not a linear space, Theorem 1 may have remarkably strong consequences. Let us consider a manifold M made of the values of the order parameter of fixed “length” and arbitrary “direction”, that is, constrain the linear space of the previous paragraph by the requirement $\|\phi\| = 1$, where $\|\cdot\|$ is a suitably defined norm. This is a situation one deals with in the low energy effective descriptions of spontaneously broken symmetries (such as the so-called nonlinear sigma model). As I will show in the next subsection, also the minimization of a quartic Higgs potential can under certain circumstances be reduced to minimization over directions of the order parameter, its norm being fixed. When classifying possible shapes of the order parameter by their symmetry properties, one often encounters the situation that the form of the order parameter is fixed up to a symmetry transformation (and trivial rescaling). These states are called *inert* in condensed matter physics (see Section 6.2.3 in [25]). By definition, their stratum consists of a single (hence isolated) orbit. The inert states are therefore found among stationary states of any G -invariant function on the manifold of order parameters of fixed norm. Explicit examples will be shown later.

Before concluding the general discussion let me stress that an invariant function can, of course, have other stationary states than those guaranteed by Theorem 1. Indeed, *any* state can be realized as a minimum of a suitably defined potential. To see this, note that a given group orbit can be uniquely specified by the values c_α of a set of group invariants, say $\mathcal{P}_\alpha(\phi)$. The desired potential then reads $f(\phi) = \sum_\alpha [\mathcal{P}_\alpha(\phi) - c_\alpha]^2$. Such potentials will, however, typically involve terms of high order. If we restrict ourselves to Higgs-type polynomials of fourth order, the list of allowed stationary states may become more narrow. Another point to emphasize is that Michel’s theorem 1 makes a claim about stationary states of a given function, while in physical applications one would usually like to know the

(absolute) minimum. To that end, Michel conjectured [23] that if the representation of the symmetry group G of the Higgs-type fourth order potential is irreducible on the real, its absolute minima are realized by orbits with maximal little groups. Even though this rule turned out not to hold in general (see [26,27] and references therein), it provides a useful guide for locating the minimum of Higgs potentials. A more rigorous and complete analysis of this problem may be found in [28].

4.2. Minimization of Higgs potentials

An elegant general method to minimize Higgs potentials was developed by Kim [24]. I will restrict to the simplest case where the order parameter transforms in an irreducible representation of the symmetry group. Then there is a single invariant quadratic term in the potential, proportional to $\sum_i \phi_i^2 \equiv \|\phi\|^2$. (We can without lack of generality assume that the components ϕ_i of the order parameter ϕ are real.) Assuming further reflection symmetry or any other constraint which rules out a cubic term, the most general potential up to fourth order in the field reads

$$V(\phi) = -\frac{1}{2}m^2 \|\phi\|^2 + \frac{1}{4} \sum_{\alpha} \lambda_{\alpha} \mathcal{P}_{\alpha}^{(4)}(\phi) \quad (21)$$

where $\mathcal{P}_{\alpha}^{(4)}$ is a set of algebraically independent fourth-order invariants in ϕ . (The mass term m^2 was written suggestively with a minus sign to ensure that a nontrivial minimum exists.) One of these invariants is always $\mathcal{P}_1^{(4)}(\phi) = (\|\phi\|^2)^2$. Factoring it out, one gets

$$V(\phi) = -\frac{1}{2}m^2 \|\phi\|^2 + \frac{1}{4} (\|\phi\|^2)^2 \left[\lambda_1 + \sum_{\alpha \neq 1} \lambda_{\alpha} \mathcal{A}_{\alpha}(\phi) \right] \quad (22)$$

where $\mathcal{A}_{\alpha}(\phi) = \mathcal{P}_{\alpha}^{(4)}(\phi) / \mathcal{P}_1^{(4)}(\phi)$. These quantities are dimensionless and represent the “angles”, that is, the orientation of the condensate in field space. The main observation now is that minimization of the potential with respect to ϕ is equivalent to successive minimization with respect to the norm $\|\phi\|$ and the angles $\mathcal{A}_{\alpha}$. For fixed values of the angles, the norm of the field and the value of the potential in the minimum are given by

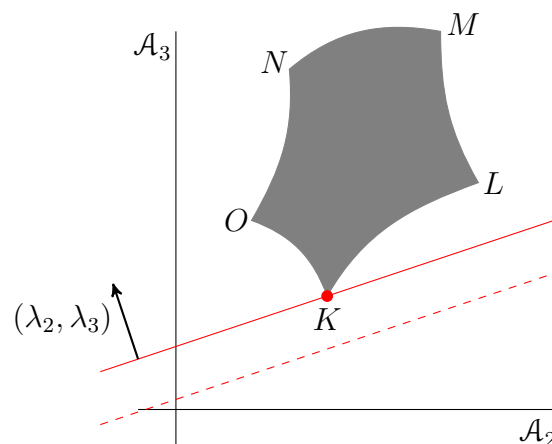
$$\|\phi\|_{\min}^2 = \frac{m^2}{\lambda_1 + \sum_{\alpha \neq 1} \lambda_{\alpha} \mathcal{A}_{\alpha}}, \quad V_{\min}(\phi) = -\frac{1}{4}m^2 \|\phi\|_{\min}^2 \quad (23)$$

In order to find the absolute minimum of the potential, one has to maximize $\|\phi\|_{\min}$, and hence minimize the expression $\Xi(\mathcal{A}_{\alpha}) \equiv \sum_{\alpha \neq 1} \lambda_{\alpha} \mathcal{A}_{\alpha}$ for given couplings λ_{α} . This may be done using an appealing geometric picture. Consider for simplicity the case of three independent quartic invariants $\mathcal{P}_{\alpha}^{(4)}(\phi)$, that is, two angles, $\mathcal{A}_2$ and $\mathcal{A}_3$. These cannot acquire arbitrary values, but rather span some domain in a two-dimensional plane which is indicated by the gray shaded area in Figure 1; it will be referred to as the *target space*. The shape of the target space is independent of the couplings λ_{α} , and is a sole characteristic of the algebraic structure of the symmetry group and its particular representation.

The set of constant $\Xi(\mathcal{A}_{\alpha})$ is in the $(\mathcal{A}_2, \mathcal{A}_3)$ plane represented by a straight line with the normal vector (λ_2, λ_3) . If we choose too low a value of Ξ , the line will not intersect the target space, that is, there is no ϕ that would yield the desired value of Ξ . This corresponds to the dashed red line. As we increase Ξ , the line shifts parallel until it touches for the first time the target space (solid line, point

K). This point determines the absolute minimum of $\Xi(\mathcal{A}_\alpha)$ and thus also the absolute minimum of the potential $V(\phi)$. It is then a matter of mere algebra to find out which shapes of the order parameter map onto the point K in the target space.

Figure 1. Geometric minimization of the quantity $\Xi(\mathcal{A}_\alpha) \equiv \sum_{\alpha \neq 1} \lambda_\alpha \mathcal{A}_\alpha$. The region of allowed values of $\mathcal{A}_\alpha$ is indicated by gray shading. The lines of constant $\Xi(\mathcal{A}_\alpha)$, having the common normal vector (λ_2, λ_3) , are in red. The red dot denotes the position of the absolute minimum of the potential.



This geometric picture can be conveniently used to scan the whole phase diagram as the couplings λ_α are varied. First, it is obvious that the shape of the order parameter in the minimum does not depend on λ_1 ; this merely affects its overall magnitude, and at the same time must be large enough to ensure the boundedness of the potential from below. Coming back to the example with three quartic invariants, the phase diagram can then be plotted in the (λ_2, λ_3) plane. For any values of the couplings, the ground state will always be represented by a point on the boundary of the target space. If the boundary is concave, as is the case of its edges adjacent to the point K , then by tuning the couplings continuously the ground state will reside at K , and then change abruptly to L or O . In other words, the system will undergo a first-order phase transition. If, on the other hand, the boundary has a convex segment, such as the edge between the M, N vertices, then tuning the couplings smoothly will result in a continuous change of the ground state. In the phase diagram, one will then observe an M -phase, an MN -phase (where the ground state travels from M to N and typically has a different symmetry structure), and finally an N -phase, the three of them separated by a sequence of two second-order transitions. An explicit example will be given in the next subsection.

When the number of quartic invariants differs from three, the situation is very similar. First of all, in the case of two invariants only the solution is trivial. The target space is one-dimensional and there are just two phases, corresponding to the minimum and maximum allowed values of $\mathcal{A}_2(\phi)$. This happens, for instance, in the Color–Flavor Locked phase of dense three-flavor quark matter [29]. For more quartic invariants the target space is multidimensional, and keeping track of all faces and edges of its boundary, which determine the phases appearing in the phase diagram, may become increasingly difficult. Yet the presented technique seems to be the most elegant way to accomplish this task. The same method can

also be used, with necessary complications, when the representation of the order parameter is reducible, or when other (cubic or higher-order) terms are present in the potential.

4.3. Example: spin-one color superconductor

As an explicit nontrivial example, I will now discuss the phases of a spin-one color superconductor. The details of all derivations may be found in [30], see also [31,32] for a partial discussion of the problem. A similar analysis for the case of d -wave pairing, that is, a theory of a traceless complex symmetric matrix with $SO(3) \times U(1)$ symmetry, was performed in [33]. Spin-one color superconductivity is a viable candidate phase for the ground state of cold dense quark matter at moderate densities (see [34] for a review). The most favored pairing pattern involves quarks of a single flavor, pairing in the color $SU(3)$ antitriplet and spin $SO(3)$ triplet channel. All we need to know for the purposes of the present paper is that the order parameter Δ is a complex 3×3 matrix, transforming as $\Delta \rightarrow U\Delta R$ where $U \in U(3)_L$ is unitary and $R \in SO(3)_R$ is orthogonal. (The overall phase transformations, complementing the color $SU(3)$, stem from conservation of baryon number.) Given the fact that a complex 3×3 matrix involves eighteen degrees of freedom, twelve of which can be transformed away, the order parameter is fully specified by a set of six independent real numbers. It can always be cast in the form

$$\Delta = \begin{pmatrix} \Delta_1 & ia_3 & -ia_2 \\ -ia_3 & \Delta_2 & ia_1 \\ ia_2 & -ia_1 & \Delta_3 \end{pmatrix} \quad (24)$$

This expression is very convenient for the analysis of the possible inequivalent structures of the order parameter, that is, the group orbits and strata. Focusing for simplicity on continuous symmetry transformations only, there are altogether eight nontrivial strata, summarized in Table 1. Three of them correspond to inert states: A, polar, and CSL (abbreviated from Color–Spin Locking). It is worth emphasizing that the isomorphy of little groups of two different values of the order parameter does not necessarily imply that they lie in the same stratum. Indeed, the cylindrical and ε , as well as the polar and A, phases do have isomorphic little groups.

To decide which of these phases occupy a part of the phase diagram one needs to write down the most general $U(3)_L \times SO(3)_R$ invariant potential of the Higgs type (21),

$$V(\Delta) = -\frac{1}{2}m^2\sqrt{\mathcal{P}_1^{(4)}(\Delta)} + \frac{1}{4}\sum_{\alpha=1}^3\lambda_\alpha\mathcal{P}_\alpha^{(4)}(\Delta) \quad (25)$$

where

$$\mathcal{P}_1^{(4)}(\Delta) = [\text{Tr}(\Delta\Delta^\dagger)]^2, \quad \mathcal{P}_2^{(4)}(\Delta) = \text{Tr}(\Delta\Delta^\dagger\Delta\Delta^\dagger), \quad \mathcal{P}_3^{(4)}(\Delta) = \text{Tr}[\Delta\Delta^T(\Delta\Delta^T)^\dagger] \quad (26)$$

The three independent quartic invariants satisfy the following inequalities which establish the shape of the target space:

$$\frac{\mathcal{P}_1^{(4)}}{3} \leq \mathcal{P}_2^{(4)} \leq \mathcal{P}_1^{(4)}, \quad 0 \leq \mathcal{P}_3^{(4)} \leq \mathcal{P}_2^{(4)}, \quad \frac{2}{3}\mathcal{P}_1^{(4)} \leq \mathcal{P}_2^{(4)} + \mathcal{P}_3^{(4)} \quad (27)$$

and finally

$$\sqrt{\mathcal{P}_1^{(4)}} \leq \sqrt{\mathcal{P}_3^{(4)}} + \sqrt{\mathcal{P}_2^{(4)} - \mathcal{P}_3^{(4)}} \quad (28)$$

which holds only for those values of Δ satisfying $\mathcal{P}_3^{(4)} \leq \mathcal{P}_1^{(4)}/9$. The conditions for the saturation of these inequalities determine which strata appear on the boundary of the target space, that is, what are the candidate ground states.

Table 1. Classification of different ground states of a spin-one color superconductor, based on the pattern of spontaneous breaking of *continuous* symmetries. First line: name of the phase; second line: unbroken continuous symmetry; third line: representative element of the stratum. Lower indices $_{L,R}$ denote subgroups of $U(3)_L$ and $SO(3)_R$, while $_v$ stands for a “diagonal” subgroup, mixing transformations from the two.

Oblate	Cylindrical	ε	A
$SO(2)_v$	$SO(2)_v \times U(1)_L$	$SO(2)_v \times U(1)_L$	$SU(2)_L \times SO(2)_v \times U(1)_L$
$\begin{pmatrix} \Delta_1 & +ia & 0 \\ -ia & \Delta_1 & 0 \\ 0 & 0 & \Delta_2 \end{pmatrix}$	$\begin{pmatrix} \Delta & +ia & 0 \\ -ia & \Delta & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \Delta_1 & +i\Delta_1 & 0 \\ -i\Delta_1 & \Delta_1 & 0 \\ 0 & 0 & \Delta_2 \end{pmatrix}$	$\begin{pmatrix} 1 & +i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
CSL	Polar	N_1	N_1
$SO(3)_v$	$SU(2)_L \times SO(2)_R \times U(1)_L$	$U(1)_L$	$SU(2)_L \times U(1)_L$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ z_1 & z_2 & z_3 \end{pmatrix}$

The target space of the spin-one color superconductor is plotted in the left panel of Figure 2. The three inert states are represented by single points, forming the corners of the target space. This is in agreement with the general discussion since their strata consist of a single orbit each. Accordingly, they occupy the majority of the phase diagram, shown in the right panel of Figure 2. However, a part of the phase diagram belongs to the non-inert ε phase, which corresponds to the AX segment of the boundary of the target space. This phase was missed in a previous analysis [31]. In contrast to the sketch in Figure 1, the remaining segments of the target space boundary are not strictly convex, but straight. Consequently, right at the phase transition between two adjacent phases, many more states can actually coexist than just the two. Thus, the A–polar line is occupied by all matrices of rank one, the polar–CSL line by all real matrices (up to a symmetry transformation), and the CSL–X segment by all matrices of the oblate type with $\Delta_2 = \sqrt{\Delta_1^2 + a^2}$.

In view of Michel’s conjecture mentioned at the end of Section 4.1, it is useful to investigate the little groups of the eight strata listed in Table 1. Their hierarchy is displayed in Figure 3. The three inert states all have maximal little groups, which is actually a general feature of the inert states. On the other hand, the little group of the ε phase is non-maximal, and its presence in the phase diagram thus provides a counterexample to Michel’s conjecture. Note that this does not follow from the mere fact that the ε state is noninert, as even such states can in principle have maximal little groups (see Section 6.2.4 in [25]). The ε and A phases are connected by a second-order phase transition; the fact that the little group of the ε phase is a subgroup of that of the A phase is in accord with the general Landau theory of phase transitions. At the other end of the hierarchy of symmetry breaking stand the oblate and N_1 states which

have no nontrivial (continuous) subgroups, hence representing the most general states preserving a part of the symmetry.

Figure 2. Left panel: target space of a spin-one color superconductor. The dashed lines are given by various bounding inequalities as described in the text. The AX segment of the boundary is occupied by the non-inert ε phase. Right panel: phase diagram of the spin-one color superconductor as a function of the quartic couplings. Thick solid and dashed lines stand for first and second order phase transitions, respectively.

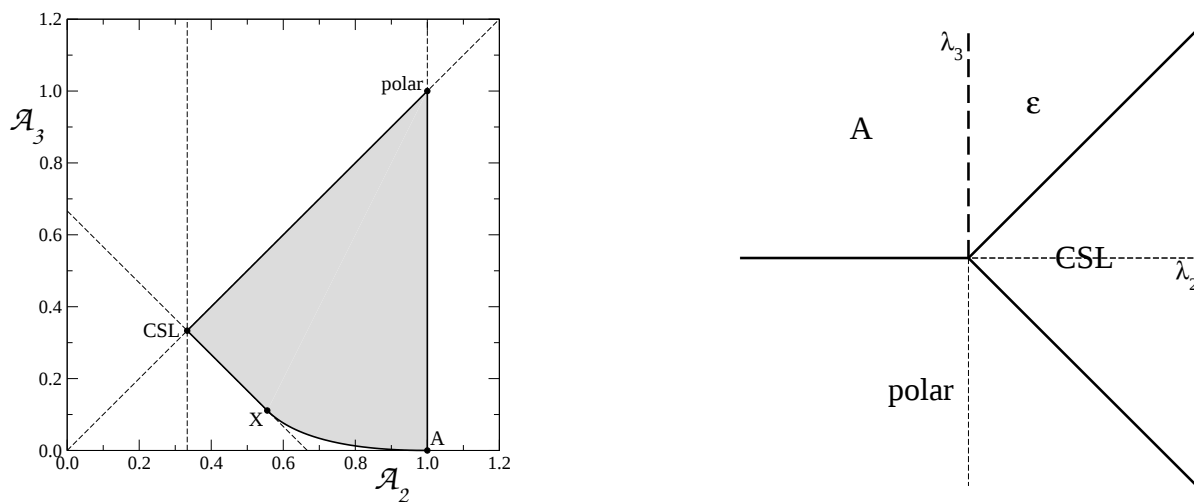


Figure 3. Hierarchy of the little groups in a spin-one color superconductor. Arrows indicate successive breaking into smaller and smaller subgroups. The relation $G_1 \rightarrow G_2$ means that the group G_2 is conjugate to a subgroup of G_1 , or in other words, the orientation of the order parameter in the two phases can be chosen so that $G_2 \subset G_1$.

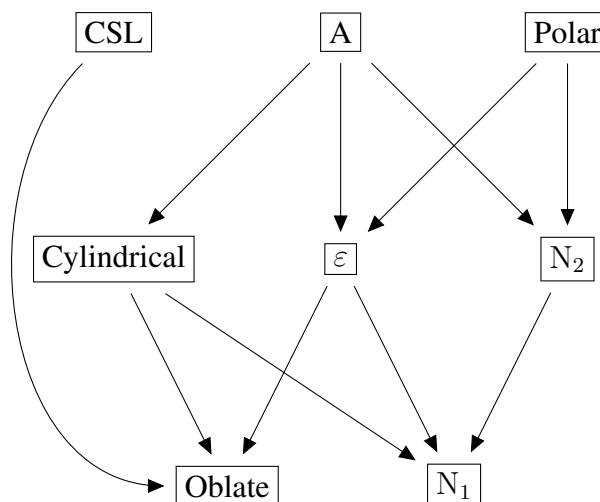
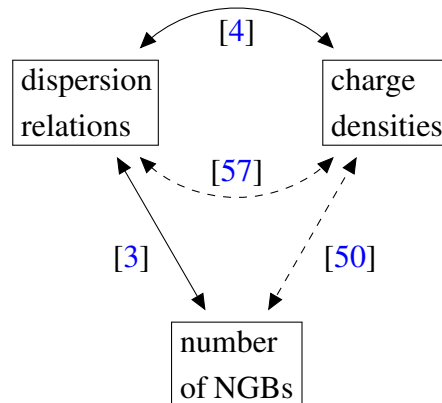


Figure 4. Characteristics of NG spectrum and their interconnections. Nielsen and Chadha [3] provided a general relation between the number of NG bosons and their dispersion relations. The connection of NG dispersion relations and the presence of charge densities was clarified by Leutwyler [4] using low-energy effective field theory. Partial results are indicated by dashed lines.



5. Goldstone Theorem and The Counting of Nambu–Goldstone Bosons

5.1. Goldstone theorem

One of the most striking consequences of SSB is the existence of soft modes in the spectrum whose energy vanishes in the long-wavelength limit. This is the celebrated Goldstone theorem [35,36] and the soft modes are usually referred to as the Nambu–Goldstone bosons. Since the aim of the paper is to discuss SSB in nonrelativistic systems, which involves a number of subtleties, I will now briefly review the standard proof of the Goldstone theorem in a version applicable to these systems.

From a physical point of view, the most important ingredient responsible for the presence of NG bosons is, apart from the symmetry breaking itself, the existence of a conserved charge. In Section 2.1, the integral charge $Q_\Omega(t)$ was defined in finite volume and it was shown that it does not exist in the limit $\Omega \rightarrow \infty$. Since SSB is based on the existence of the order parameter (2) which, being a commutator of the broken charge, is well defined even in infinite volume, let us inspect its time dependence. To that end, one takes the commutator $[\partial_\mu j^\mu(x), \Phi]$ and integrates it over the domain Ω . Using the continuity equation and the Gauss theorem one arrives at

$$\partial_0[Q_\Omega(t), \Phi] + \int_{\partial\Omega} [d\boldsymbol{\sigma} \cdot \mathbf{j}, \Phi] = 0 \quad (29)$$

The key technical assumption of the Goldstone theorem, apart from translational invariance of the vacuum, is the vanishing of the surface integral in the infinite volume limit. This, upon taking the vacuum expectation value, guarantees that the *order parameter is time independent*. The same remark as at the end of Section 2.1. applies here: in causal theories the vanishing of the surface term is guaranteed as long as the operator Φ is localized to a finite domain of spacetime. (In practice, it is often even strictly local.) In acausal theories such as some nonrelativistic models with instantaneous interaction, the surface integral tends to zero in the infinite volume limit when the interaction is of finite range or

decreases exponentially with distance. In case of long-range interactions, however, the disappearance of the surface term must be checked case by case.

With these remarks in mind, I will from now on simply assume that the surface term in Equation (29) vanishes. This makes sure that the order parameter is time independent as $\Omega \rightarrow \infty$. It is given by the spatial integral of $\langle 0|[j^0(\mathbf{x}, t), \Phi]|0\rangle$. Inserting partition of unity in terms of eigenstates of the Hamiltonian and using translational invariance of the vacuum, this becomes

$$\langle 0|[j^0(\mathbf{x}, t), \Phi]|0\rangle = \sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3} [e^{-ik \cdot x} \langle 0|j^0(0)|n_{\mathbf{k}}\rangle \langle n_{\mathbf{k}}|\Phi|0\rangle - e^{+ik \cdot x} \langle 0|\Phi|n_{-\mathbf{k}}\rangle \langle n_{-\mathbf{k}}|j^0(0)|0\rangle] \quad (30)$$

The summation runs over discrete labels distinguishing different excitation branches in the spectrum as well as continuous internal variables of the multiparticle states. The states are normalized by $\langle n_{\mathbf{k}}|m_{\mathbf{q}}\rangle = (2\pi)^3 \delta_{mn} \delta^3(\mathbf{k} - \mathbf{q})$. Integral over total three-momentum of the modes is indicated explicitly. Integrating now over the domain Ω yields

$$\langle 0|[Q_{\Omega}(t), \Phi]|0\rangle = \sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3} [e^{-iE_n t} \varphi_{\Omega}(\mathbf{k}) \langle 0|j^0(0)|n_{\mathbf{k}}\rangle \langle n_{\mathbf{k}}|\Phi|0\rangle - e^{+iE_n t} \varphi_{\Omega}(-\mathbf{k}) \langle 0|\Phi|n_{-\mathbf{k}}\rangle \langle n_{-\mathbf{k}}|j^0(0)|0\rangle] \quad (31)$$

where $\varphi_{\Omega}(\mathbf{k}) = \int_{\Omega} d^3\mathbf{x} e^{ik \cdot \mathbf{x}}$. In a large volume, this function is strongly peaked around $\mathbf{k} = \mathbf{0}$ and thus only states with low momentum will contribute to the right hand side. At the same time we know that as $\Omega \rightarrow \infty$ the left hand side becomes time independent. This is only possible when the energy of the contributing states vanishes in the limit of zero momentum. Finally, we use the broken symmetry assumption which states that the left hand side actually is nonzero, so that there must be at least one state that couples to both the broken current $j^0(x)$ and the interpolating field Φ . The most general formulation of the Goldstone theorem therefore is:

Theorem 2 (Goldstone) *Spontaneous breaking of a continuous global internal symmetry implies the existence of a mode in the spectrum such that $\lim_{\mathbf{k} \rightarrow 0} E_{\mathbf{k}} = 0$.*

Several remarks to the theorem, its technical assumptions and derivation are in order. First of all, the theorem guarantees existence of a NG mode in the spectrum. In the most general formulation it does not tell us *how many* NG bosons there are. In Lorentz invariant theories there turns out to be exactly one NG boson for each broken generator. (Also, due to the strong constraints on the form of the dispersion relation, a NG boson is then simply a massless particle.) Perhaps this is the reason why, somewhat unfortunately, many textbooks focused on relativistic quantum field theory and its applications to particle physics overlook the issue of NG boson counting. Here, it will be discussed in detail in the following subsection for the case of internal symmetries. It is also quite nontrivial when spacetime symmetries are spontaneously broken. For more details, the reader is referred to [37].

Second, the Goldstone theorem provides information about the low energy behavior of the NG boson. Since it constitutes a whole excitation branch in the spectrum, it will (but does not necessarily need to) presumably also exist at high momentum. However, in this range its behavior is determined by details of short distance physics, hence it is nonuniversal and cannot be predicted solely from the broken symmetry. Examples are the acoustic phonon in crystalline solids whose dispersion relation is linear

at low momentum, but gradually flattens until the group velocity becomes zero at the edge of the Brillouin zone, or the Bogolyubov–Anderson mode in Bose–Einstein condensates whose dispersion at high momentum becomes that of a free boson, essentially insensitive to the presence of the condensate.

Third, as was repeatedly stressed above, a sufficient technical condition for the integral charge Q to be time independent, and therefore for the Goldstone theorem to hold, is the causality which is inherent in all Lorentz invariant theories. In such a case the consequences of the Goldstone theorem cannot be escaped. Yet it does not mean that the predicted gapless NG boson can actually be observed, for it may appear in the unphysical sector of the Hilbert space. A distinguished class of systems where this may happen are gauge theories. In continuum, they are ill-defined unless the gauge is fixed appropriately. The remaining global symmetry can then be broken spontaneously. In covariant gauges that preserve Lorentz invariance and hence causality, a NG boson appears, but it is in the unphysical part of the Hilbert space, as mentioned above [38–40]. On the other hand, in noncovariant (such as Coulomb) gauges, the presence of a long-range interaction invalidates the Goldstone theorem. This is well known to happen, for instance, in electric superconductors where the soft mode acquires nonzero energy even in the long-wavelength limit. In the lattice formulation where there is no need for gauge fixing, spontaneous breaking of the gauge symmetry is ruled out by Elitzur’s theorem [41].

Fourth, the assumption of unbroken translational invariance of the vacuum is natural. Without translational invariance, there is no conserved momentum to label the excitations so the whole quasiparticle picture loses its meaning. Note that for the sake of the low-energy physics that the Goldstone theorem is concerned with, only discrete translational invariance like in crystalline solids is in principle sufficient. The values of momentum are then restricted to a finite range, but this still allows for well-defined low-momentum excitations.

Finally, it should be made clear that the Goldstone theorem speaks of a *limit* of the energy as momentum goes to zero. This is ensured by the derivation above where integration is first performed over a finite space domain Ω . The function $\varphi_\Omega(\mathbf{k})$ is thus smeared around the origin and only in the limit $\Omega \rightarrow \infty$ goes to $(2\pi)^3\delta^3(\mathbf{k})$. This distinction is very important since in principle there can be isolated states with zero momentum and energy which do not represent NG bosons, but still give a spurious contribution to the right hand side of Equation (31). Thanks to the fact that the integrand is smeared by the factor $\varphi_\Omega(\mathbf{k})$, these states fortunately do not affect the value of the momentum integral. In finite volume, such states may be, for example, the other, degenerate ground states. One can then show explicitly [14] that their contribution is suppressed when the limit of infinite volume is taken.

Concerning the observable consequences of SSB, I have focused exclusively on the presence of massless particles in the spectrum so far. However, the Goldstone theorem has in fact much broader consequences. The reason is that we can consider not just the (quasi-)local field Φ , but in general an n -point Green’s function constructed from such fields as the interpolating field that gives rise to an order parameter. An immediate corollary of the Goldstone theorem then is that if two Green’s functions, connected by a symmetry transformation, are different then there must be a NG mode in the spectrum. For example, if the masses of two particles, created by fields that lie in the same multiplet of the symmetry, are different then the spectrum exhibits a NG state [42]. This makes it clear that a spontaneously broken symmetry by no means resembles symmetry that is broken explicitly. Indeed,

the former still gives rise to *exact* constraints on the Green's functions of the theory, conveniently summarized in terms of a set of Ward identities.

5.2. Goldstone boson counting: Dispersion relations

As already suggested in the previous subsection, it is in general a complicated task to determine the precise number of NG bosons and its connection to the number of broken symmetry generators. In order to appreciate how frequently one deals with nontrivial realizations of broken symmetry, let me list several examples from nonrelativistic as well as relativistic physics. The most profound example is that of ferromagnets versus antiferromagnets. In both the $SU(2)$ symmetry of spin rotations is spontaneously broken by mutual spin alignment to the $U(1)$ subgroup of rotations about the direction of total magnetic moment (staggered, in the case of antiferromagnets). This means that two generators are spontaneously broken. While an antiferromagnet possesses two NG modes (spin waves or *magnons*), both with a linear dispersion relation at low momentum, there is just one NG mode in a ferromagnet, its dispersion relation being quadratic. Similarly, in the so-called canted phase of ferromagnets, the $SU(2)$ symmetry is completely broken, and there is one NG mode of either of the above mentioned types [43].

The same phenomenon was predicted to occur in multicomponent Bose–Einstein condensates of alkalic atomic gases [44,45]. In particular with three spin polarizations the global symmetry in question is $SO(3) \times U(1)$, corresponding to rotational invariance and conservation of particle number. There are two phases, roughly analogous to the polar and A phases of superfluid ^3He or the spin-one color superconductor analyzed in Section 4.3., leaving unbroken $SO(2)$ and $U(1)$ subgroups which are isomorphic, but realized differently due to a different structure of the ground state. (The respective values of the order parameter lie in different strata.) The three broken generators give rise to three NG bosons with linear dispersions in the polar phase, and to one linear and one quadratic NG mode in the A phase. Similar conclusions have recently been reported for spin-two Bose–Einstein condensates [46].

In a three-component Fermi gas [47–49], the global $SU(3) \times U(1)$ symmetry is spontaneously broken in presence of Cooper pairs built from fundamental fermions. Due to Pauli principle, these pairs transform as an antisymmetric tensor (antitriplet) of $SU(3)$, leaving an $SU(2) \times U(1)'$ symmetry unbroken. The five broken generators lead to one linear and two quadratic NG bosons. One should perhaps add that while in (anti)ferromagnets the dispersion relation of magnons can be probed directly by means of neutron scattering, similar direct measurements for superfluid atomic gases are beyond the reach of current experiments.

In relativistic physics, the issue of NG boson counting has been discussed primarily in the context of dense nuclear and quark matter. For instance, a condensate of neutral kaons can accompany the Color–Flavor Locked (CFL) phase of three-flavor quark matter at very high densities. Such a system exhibits a global $SU(2) \times U(1)$ symmetry stemming from isospin and strangeness conservation. The condensate leaves just its $U(1)'$ subgroup intact, thus breaking spontaneously three generators. These give rise to one linear and one quadratic NG boson [50–52]. The same behavior as in condensed matter ferromagnets can be observed in nuclear matter [53]. Yet other examples can be found among the various color superconducting phases of dense quark matter. In a two-flavor color superconductor, known as the 2SC phase, the symmetry as well as its breaking pattern are the same as in the above mentioned three-component Fermi gas [54]. However, in this case the symmetry is gauged, and as will become clear

later, this result may thus be just an artifact of the model treatment based on a global color invariance. On the other hand, in a spin-one color superconductor, one can find spin waves and the “abnormal” NG bosons with quadratic dispersion relation are then physical [55].

All the preceding examples suggest that there is a deeper connection between the number of NG bosons and their dispersion relations. Quite generally, when there are some modes with quadratic dispersion relation, the total number of NG bosons seems less than the number of broken generators, in contrast to the naive expectation based on experience with Lorentz invariant systems. Incidentally, a great deal of understanding of this problem was achieved long before many of the above listed examples were even known. More than thirty years ago, Nielsen and Chadha [3] formulated the following theorem:

Theorem 3 (Nielsen and Chadha) *Assume that translational invariance is not completely broken spontaneously and that there are no long-range interactions. Then the energy of a NG boson is analytic in momentum. Denoting NG modes whose energy is proportional to an odd power of momentum as type-I, and those whose energy is proportional to an even power of momentum as type-II, the number of type-I NG bosons plus twice the number of type-II NG bosons is greater than or equal to the number of broken symmetry generators.*

The requirement on the absence of long-range forces was technically formulated as a commutativity condition: for any two local operators $A(x)$ and $B(0)$ there is a real positive number τ such that

$$|\langle 0|[A(\mathbf{x}, t), B(0)]|0\rangle| \rightarrow e^{-\tau|\mathbf{x}|} \quad \text{as} \quad |\mathbf{x}| \rightarrow \infty \quad (32)$$

It is this assumption which is responsible for the analyticity of the Fourier transform of the commutator (30), and in turn for the analyticity of the dispersion relation. Obviously it is a stronger assumption than is necessary for the Goldstone theorem itself. Indeed, in Section 6. I will show examples of systems with long-range forces where the Goldstone theorem is still valid, but the NG dispersion relation is not analytic.

The actual proof of the claim about the number of NG bosons is algebraic and somewhat involved, so I will not repeat details and instead refer the reader to the original paper [3]. It is notable that the statement of the Nielsen–Chadha theorem is rather general. First, it does not specify the power of momentum to which energy is proportional. Second, it only gives an inequality for the number of NG bosons. On the other hand, I am not aware of any example of a NG boson whose energy would be proportional to a higher power of momentum than two. Also, the question of the possible general saturation of the inequality for the number of NG bosons is yet to be understood. Nevertheless, note that the counting may be obscured by the presence of gapless non-NG modes, as recognized recently in the context of spin-two Bose–Einstein condensates [46].

5.3. Goldstone boson counting: Charge densities

What I have not mentioned before in the list of examples exhibiting type-II NG bosons was that they are all accompanied by nonvanishing density of some of the conserved charges. Indeed, this is exactly the sought property that distinguishes ferromagnets and antiferromagnets: the ferromagnetic ground state features nonzero net spin density. The connection of the NG boson counting and the presence of

charge density in the ground state is rather general and a number of (partial) results exists in this respect. I will start with a theorem due to Schäfer *et al.* [50]:

Theorem 4 (Schäfer *et al.*) *If $\langle 0|[Q_a, Q_b]|0\rangle = 0$ for all pairs of broken generators Q_a, Q_b , then the number of NG bosons is at least equal to the number of broken generators.*

Note that in [50] the theorem was formulated as a strict equality. However, this does not follow from the proof presented there. In general it is difficult to place an upper bound on the number of NG modes; see also the remark at the end of the previous subsection in this respect. The original version of the proof was based on the identification of vectors $Q_a|0\rangle$ with the NG states. This may be troublesome because as explained in Section 2, these vectors are not well defined in the infinite volume limit. Also, it is not clear *a priori* that the number of linearly independent vectors $Q_a|0\rangle$ is equal to the number of NG states in the spectrum. Below I present a slightly more rigorous modification of the proof which avoids this step.

Take $j_a^0(x)$ as the broken charge density in Equation (30) and $j_b^0(0)$ as the interpolating field for the NG mode. Integrating over the space in the infinite volume limit then yields

$$\langle 0|[Q_a, j_b^0(0)]|0\rangle = \sum_n [\langle 0|j_a^0(0)|n_0\rangle\langle n_0|j_b^0(0)|0\rangle - \langle 0|j_b^0(0)|n_0\rangle\langle n_0|j_a^0(0)|0\rangle] \quad (33)$$

Denoting $M_{an} = \langle 0|j_a^0(0)|n_0\rangle$, this equals $(MM^\dagger)_{ab} - (MM^\dagger)_{ba}$. The assumption of the theorem is equivalent to the requirement that $MM^\dagger$ be symmetric, and hence real (it is automatically Hermitian). It may therefore be diagonalized by a real orthogonal transformation, $MM^\dagger \rightarrow RMM^\dagger R^T$, which is equivalent to $M \rightarrow RM$, that is, to a change of basis of the broken generators. (It is essential that the transformation matrix R is real, for otherwise $R_{ab}Q_b$ would not necessarily be a generator of the symmetry.) Assume now that the number of NG states, n_{NG} , is smaller than the number of broken currents, n_{BC} . The rank of the $n_{\text{BC}} \times n_{\text{NG}}$ matrix M then cannot be larger than n_{NG} , and so cannot the rank of the $n_{\text{BC}} \times n_{\text{BC}}$ diagonal matrix $MM^\dagger$. The basis of broken generators may be ordered so that the first n_{NG} diagonal elements of $MM^\dagger$ are nonzero while the remaining $n_{\text{BC}} - n_{\text{NG}}$ ones are zero. Since $MM^\dagger$ is just the matrix of scalar products of rows of M , this means that the last $n_{\text{BC}} - n_{\text{NG}}$ rows are zero. Consequently, the generators $Q_{n_{\text{NG}}+1}, \dots, Q_{n_{\text{BC}}}$ have zero matrix elements with all NG states $|n_0\rangle$, which is in contradiction with the assumption that they are broken. The theorem is thus proved.

A short precaution is appropriate regarding the interpretation of Theorem 4. Since the symmetry generators furnish a Lie algebra, it is tempting to conclude that nonzero density of some of the conserved charges is a necessary condition for the number of NG bosons to be smaller than the number of broken generators [50,56]. This is indeed the case in all systems listed in the previous subsection. However, the otherwise trivial example of a free nonrelativistic particle (see Section 3.) teaches us that it is possible to achieve nonzero vacuum expectation value of a commutator of two generators even if the expectations of all generators themselves vanish. The resolution is that the Lie algebra of symmetry generators picks a central charge upon quantization of the theory. One therefore cannot generally conclude that nonzero charge density is necessary in order to have fewer NG bosons than broken generators, unless additional assumptions are made such as that the Lie algebra of symmetry generators is semi-simple, in which case there are no nontrivial central charges (see Section 2.7 in [1]).

Looking back at Equation (33), one observes that in case that the left hand side is actually nonzero, there must be a NG mode that couples to both broken generators Q_a, Q_b [56]. This indicates at a very elementary level that the opposite to Theorem 4 should also hold: nonzero vacuum expectation value of a commutator of two broken generators implies that the number of NG bosons is smaller than the number of broken generators. However, it would be desirable to prove this statement at a similar level of rigor as Theorem 4 itself.

While the Nielsen–Chadha theorem 3 relates the number of NG bosons to their dispersion relations, Theorem 4 of Schäfer *et al.* provides a (partial) connection of the number of NG bosons to charge density. (Keeping in mind the example of Section 3. I use the term “charge density” here as a synonym for “expectation value of a commutator of two broken generators.”) However, there is also a rather straightforward connection between the latter two, that is, NG dispersion relations and charge density. This was shown by Leutwyler [4] in the framework of low-energy effective field theory. A general argument based just on basic symmetry assumptions and analyticity is presented below. For a summary of all these relations, see Figure 4.

Consider now two broken generators, Q_a and Q_b , such that their commutator has a nonzero vacuum expectation value. I will demonstrate that, quite generally, this implies existence of one NG boson with nonlinear (typically quadratic) dispersion relation at low momentum. For the sake of simplicity I will assume rotational invariance. This may seem rather restrictive, note however that even the low-energy behavior of many solid state systems such as ferromagnets or superconductors is, to leading order, described by a rotationally invariant effective field theory. Keeping this in mind, the transition amplitude for the annihilation of the NG boson by the current operator can be parameterized as

$$\langle 0 | j_a^\mu(0) | n_{\mathbf{k}} \rangle = i k_{\text{on}}^\mu F_{an}(|\mathbf{k}|) + i \delta^{\mu 0} G_{an}(|\mathbf{k}|) \quad (34)$$

where k_{on}^μ is the on-shell wave vector of the one-particle state, that is, k_{on}^0 is given by the dispersion relation, $k_{\text{on}}^0 = E_{n,\mathbf{k}}$. Using current conservation, this leads to the following equation for the NG dispersion relation,

$$(E_{n,\mathbf{k}}^2 - \mathbf{k}^2) F_{an} + E_{n,\mathbf{k}} G_{an} = 0 \quad (35)$$

A few special cases are worth mentioning. When $G_{an} = 0$ the standard Lorentz invariant dispersion relation is recovered. When the ratio G_{an}/F_{an} is small of order $\mathcal{O}(|\mathbf{k}|)$ in the limit $\mathbf{k} \rightarrow 0$, the dispersion relation comes out linear with phase velocity differing from one. Finally, when G_{an}/F_{an} has a nonzero limit as $\mathbf{k} \rightarrow 0$, the dispersion relation takes the low-momentum form

$$E_{n,\mathbf{k}} = \mathbf{k}^2 \frac{F_{an}}{G_{an}} \quad (36)$$

This is the case of most interest since it turns out to be implied by the presence of nonzero charge density.

In order to decide which of the possibilities outlined above is actually realized, one needs more information about the dynamics of the system. To that end, consider the time-ordered current–current correlation function, $D_{ab}^{\mu\nu}(x - y) = -i \langle 0 | T \{ j_a^\mu(x) j_b^\nu(y) | 0 \rangle$. Let us also assume the following form of the equal-time commutator of two charge density operators, $[j_a^0(\mathbf{x}, t), j_b^0(\mathbf{y}, t)] = i \delta^3(\mathbf{x} - \mathbf{y}) C_{ab}(\mathbf{x}, t)$. By this general notation the possibility of a nontrivial central charge is taken into account. Taking the

divergence of the correlation function and employing the Källén–Lehmann spectral representation, one arrives at the following Ward identity in momentum space,

$$i\langle 0|C_{ab}(0)|0\rangle = k_\mu D_{ab}^{\mu 0}(k^0, \mathbf{k}) = \sum_n \left[\frac{(k_\mu k_{\text{on}}^\mu F_{an} + k^0 G_{an})(k_{\text{on}}^0 F_{bn}^* + G_{bn}^*)}{k^0 - E_{n,\mathbf{k}} + i\varepsilon} - \frac{(k_{\text{on}}^0 F_{bn} + G_{bn})(k_\mu \tilde{k}_{\text{on}}^\mu F_{an}^* + k^0 G_{an}^*)}{k^0 + E_{n,\mathbf{k}} - i\varepsilon} \right] \quad (37)$$

where $\tilde{k}^\mu = (k^0, -\mathbf{k})$. The index ν was set to zero since only then the left-hand side can have nonzero vacuum expectation value due to the assumed rotational invariance. In order that the particle and antiparticle poles on the right-hand side are canceled, they must appear at energies satisfying Equation (35), which is just another derivation of the NG dispersion relation. Upon cancelation of the poles, the right-hand side of Equation (37) takes the form

$$2i \operatorname{Im} [E_{n,\mathbf{k}}^2 F_{an} F_{bn}^* + E_{n,\mathbf{k}} (F_{an} G_{bn}^* + F_{bn}^* G_{an}) + G_{an} G_{bn}^*] \quad (38)$$

Taking now the limit of zero momentum, the broken symmetry condition expressed in terms of the nonvanishing expectation value of $C_{ab}(0)$ implies a “density rule”,

$$\operatorname{Im} G_{an} G_{bn}^* = \frac{1}{2} \langle 0|C_{ab}(0)|0\rangle \quad (39)$$

In this expression, G_{an} stands for the zero-momentum limit of the amplitude function defined by Equation (34).

This is the desired result. It means that when the commutator of Q_a and Q_b has nonzero vacuum expectation value, the amplitudes G_a, G_b have finite limits as $\mathbf{k} \rightarrow 0$. According to Equation (36) the dispersion relation of the NG boson that couples to both broken generators is then nonlinear. In fact, in all cases I am aware of it is quadratic. However, this does not strictly speaking follow from the general argument presented here.

5.4. Linear sigma model

The above arguments connecting the number of NG bosons, their dispersion relations and charge densities are general, yet not as conclusive as one might desire. This last part of the section will therefore be devoted to a discussion that is less general, but allows much stronger statements. For many purposes it is often sufficient to introduce an effective scalar field whose vacuum expectation value plays the role of an order parameter. This was actually the underlying motivation behind Section 4, where the general problem of minimization of potential for such a field was considered. Let me first recall one of the standard proofs of the Goldstone theorem that applies to Lorentz invariant theories [36].

Consider a set of real scalar fields ϕ_i , transforming linearly in a real representation of a symmetry group with generators $(T_a)_{ij}$, supposed to be pure imaginary and Hermitian. This means in practice that the symmetry group is realized by orthogonal transformations, which can always be ensured as long as the group is compact (see Chapter 7 of [18]). The restriction to real fields is not essential for every complex field can be decomposed into two real ones, and I do not make any assumptions about (ir)reducibility of the group representation.

For linearly realized symmetry, the quantum effective potential $V_{\text{eff}}(\phi)$ of the theory is invariant under the same transformations as the classical Lagrangian (see Section 16.4 in [15]). The condition of invariance therefore reads

$$\frac{\partial V_{\text{eff}}}{\partial \phi_i}(T_a)_{ij}\phi_j = 0 \quad (40)$$

for all generators T_a . Differentiating with respect to ϕ and setting it to its vacuum expectation value φ , satisfying $\partial V_{\text{eff}}(\varphi)/\partial \phi_i = 0$, one obtains

$$\frac{\partial^2 V_{\text{eff}}(\varphi)}{\partial \phi_i \partial \phi_k}(T_a)_{ij}\varphi_j = 0 \quad (41)$$

The matrix of second partial derivatives is the mass matrix of the theory and as long as Lorentz invariance is preserved, it completely determines the spectrum of the system. This is the Goldstone theorem at work: spontaneous breaking of the generator T_a by definition means that it does not annihilate the vacuum, that is, $(T_a)_{ij}\varphi_j \neq 0$. But then the real vector $\chi_{ai} = i(T_a)_{ij}\varphi_j$ is a zero mode of the mass matrix, hence corresponding to a NG boson.

How many NG bosons are there? It seems almost obvious that there is one for each broken generator, but let us be a bit more precise at this point. The vectors χ_a give rise to a matrix of scalar products, $h_{ab} = \chi_a^T \chi_b = \varphi^T T_a T_b \varphi$. This matrix is real, symmetric and positive semi-definite, and may be diagonalized by an orthogonal transformation, $h \rightarrow R h R^T$. This in turn corresponds to the change of basis of the Lie algebra of generators, $T_a \rightarrow R_{ab} T_b$. After the transformation, the nonzero vectors χ_a , equal in number to the number of broken generators, form an orthogonal system. Since they are all zero modes of the mass matrix, it is now obvious that there is exactly one NG mode for each broken generator.

In order to see what difference lack of Lorentz invariance brings in, I will now switch to a class of theories with “minimal” breaking of Lorentz symmetry, induced by chemical potential. These describe many-body systems with relativistic dynamics. Lorentz invariance is broken explicitly by medium effects. The main message will be that *the number of zero modes of the mass matrix is still equal to the number of broken generators, but in general no longer coincides with the number of NG modes*.

Consider a general model for a complex field ϕ , defined by the Lagrangian [56]

$$\mathcal{L} = D_\mu \phi^\dagger D^\mu \phi - V(\phi) \quad (42)$$

where $V(\phi)$ is a renormalizable potential with terms up to fourth order in the field ϕ . The covariant derivative reads $D_\mu \phi = (\partial_\mu - i A_\mu) \phi$ and makes the Lagrangian formally gauge invariant (see Section 2.4 in [58]). In the end the background gauge field is set to $A_\mu = (\sum_a \mu_a T_a, \mathbf{0})$, where μ_a are chemical potentials associated with a set of mutually commuting conserved charges represented by matrices T_a . Upon expanding the covariant derivatives, the Lagrangian becomes

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - 2 \text{Im} \phi^\dagger A^\mu \partial_\mu \phi - \tilde{V}(\phi) \quad (43)$$

with the modified potential $\tilde{V}(\phi) = V(\phi) - \phi^\dagger A^\mu \partial_\mu \phi$. The chemical potential thus gives a negative contribution to the mass matrix, and when it exceeds certain critical value, the perturbative vacuum $\varphi = 0$ will no longer be stable and the field will condense. As soon as the field develops nonzero

expectation value, some of the symmetry generators become spontaneously broken. Keeping in mind that the zero modes of the mass matrix are $iT_a\varphi$, one reparameterizes the field as

$$\phi(x) = \varphi + H(x) + i\Pi(x)\varphi \quad (44)$$

Here $\Pi(x)$ is a linear combination of broken generators and represents the NG fields, while $H(x)$ stands for the massive, “Higgs” modes. After some manipulation, the bilinear part of the Lagrangian which governs the spectrum at tree level, becomes

$$\begin{aligned} \mathcal{L}_{\text{bilin}} = & \partial_\mu H^\dagger \partial^\mu H - \tilde{V}_{\text{bilin}}(H) - 2 \operatorname{Im} H^\dagger A^\mu \partial_\mu H + \\ & + \varphi^\dagger \partial_\mu \Pi \partial^\mu \Pi \varphi - 4 \operatorname{Re} H^\dagger A^\mu \partial_\mu \Pi \varphi - \operatorname{Im} \varphi^\dagger A^\mu [\Pi, \partial_\mu \Pi] \varphi \end{aligned} \quad (45)$$

where $\tilde{V}_{\text{bilin}}$ is the bilinear part of the modified potential, which only depends on the Higgs field H . This is the manifestation of the NG nature of the field Π : it drops from the mass part of the Lagrangian.

The Noether currents in the model are $j_a^\mu = -i(D^\mu \phi^\dagger T_a \phi - \phi^\dagger T_a D^\mu \phi)$, and the charge densities in the ground state read $j_a^0 = 2\varphi^\dagger A^0 T_a \varphi$. Consequently, the last term in the Lagrangian (45) is proportional to the density of a commutator of two generators. Not surprisingly, it plays a crucial role in the structure of the NG spectrum, in accordance with Theorem 4. In order to understand the spectrum of the system governed by the Lagrangian (45), it is desirable to simplify the problem as much as possible; it looks like one has to deal with a very complicated mixing of all the modes. Fortunately, it turns out that when the fields H, Π are properly decomposed into irreducible representations of the unbroken subgroup, the whole Lagrangian splits into sectors in which just two modes can mix with each other at a time (see [56] for details). A key role in the argument is played by the statement, whose validity is not restricted to the framework of the linear sigma model, formulated here as a simple theorem:

Theorem 5 *It is possible to choose a basis of the symmetry generators in such a way that only mutually commuting charges have nonzero vacuum expectation value.*

Unfortunately, I am not aware of a completely general proof of this claim. However, for the wide class of systems whose symmetry group is the special unitary group and its products, the proof is simple. Denote the expectations of the charge operators as $q_a = \langle 0 | Q_a | 0 \rangle$. Forming a linear combination with the generator matrices T_a one gets a Hermitian matrix $q_a T_a$. Under the unitary group transformation U on the Hilbert space, the charge operators transform in the adjoint representation, $Q_a \rightarrow U Q_a U^\dagger$. This in turn translates into the transformation rule for the charge expectations q_a . However, under the adjoint action of the unitary group, the matrix $q_a T_a$ can always be diagonalized. In such a diagonal representation, only mutually commuting charges have nonzero expectation values, as was to be proved.

The search for charges with nonzero expectation value is greatly facilitated by the fact that they must transform as singlets under the unbroken symmetry. The candidates can therefore be found by a decomposition of the Lie algebra of generators into irreducible representations of the unbroken subgroup. Once all charges that have nonzero vacuum expectation value have been identified, they can be completed to form the Cartan subalgebra of the Lie algebra. The standard root decomposition of Lie algebras (see Chaps. 6 and 8 of [59]) then guarantees that the remaining generators split up into pairs whose commutators lie in the Cartan subalgebra. Only such pairs can be mixed by the last term in the

Lagrangian (45). The task to determine the NG spectrum thus boils down to the problem of mixing of two fields, captured by the Lagrangian

$$\mathcal{L}_{\text{bilin}} = \frac{1}{2}(\partial_\mu \pi)^2 + \frac{1}{2}(\partial_\mu H)^2 - \frac{1}{2}f^2(\mu)H^2 - g(\mu)H\partial_0\pi \quad (46)$$

As the notation suggests, π is the NG field while H is a Higgs field as long as its mass function $f(\mu)$ is nonzero. The coefficient $g(\mu)$ accounts for the last two terms in Equation (45), which have just a single time derivative. The spectrum of this Lagrangian consists of two modes with squared energies at low momentum given by the following formulas,

$$E_{1,\mathbf{k}}^2 = f^2(\mu) + g^2(\mu) + \dots, \quad E_{2,\mathbf{k}}^2 = \frac{f^2(\mu)}{f^2(\mu) + g^2(\mu)}\mathbf{k}^2 + \frac{g^4(\mu)}{[f^2(\mu) + g^2(\mu)]^3}\mathbf{k}^4 + \dots \quad (47)$$

If $f(\mu) \neq 0$, the Lagrangian indeed describes a massive state and a type-I NG boson with linear dispersion relation at low momentum. On the other hand, if $f(\mu) = 0$, that is, if both π and H have no mass term, the spectrum contains a massive mode and only one NG boson of type-II with quadratic dispersion relation at low momentum, $E_{2,\mathbf{k}} = \mathbf{k}^2/|g(\mu)|$. This is a sheer consequence of the term in the Lagrangian with a single time derivative, and in turn of the nonzero expectation value of a commutator of two broken generators. One can thus conclude with the strong statement that *there is one type-II NG boson with quadratic dispersion relation for each pair of generators whose commutator has nonzero vacuum expectation value*. Explicit examples, further elucidating this result, will be analyzed in the next section.

6. Further Examples

In order to shed some light on the general arguments given in the preceding section, I will now work out in detail some examples of systems featuring type-II NG bosons. Interestingly enough, these examples turn out to be nontrivial and thus hopefully contribute to deeper understanding of the underlying physics. I will start with a nonrelativistic, completely solvable toy model.

6.1. Nonrelativistic Boulware–Gilbert model

Following [2], consider a nonrelativistic version of the model introduced in [38], defined in finite volume Ω by the Hamiltonian

$$\mathcal{H} = \frac{1}{2} \int_\Omega d^3\mathbf{x} [\pi^2 + (\nabla\phi)^2] + \frac{1}{2} \int_\Omega d^3\mathbf{x} d^3\mathbf{y} \pi(\mathbf{x})V(\mathbf{x} - \mathbf{y})\pi(\mathbf{y}) \quad (48)$$

The canonical coordinate and momentum satisfy the usual commutation relation, $[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y})$. Since the Hamiltonian is bilinear, this is essentially a free field theory.

The theory can be solved by switching to Fourier transformed variables, $\phi_{\mathbf{k}} = \int_\Omega d^3\mathbf{x} \phi(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}$, and analogously for $\pi_{\mathbf{k}}$ and $V_{\mathbf{k}}$. The Hamiltonian thus becomes

$$\mathcal{H} = \frac{1}{2\Omega} \sum_{\mathbf{k}} [\mathbf{k}^2 \phi_{\mathbf{k}} \phi_{-\mathbf{k}} + (1 + V_{\mathbf{k}}) \pi_{\mathbf{k}} \pi_{-\mathbf{k}}] \quad (49)$$

From here it is already straightforward to derive the Lagrangian,

$$\mathcal{L} = \frac{1}{2\Omega} \sum_{\mathbf{k}} \left(\frac{\dot{\phi}_{\mathbf{k}} \dot{\phi}_{-\mathbf{k}}}{1 + V_{\mathbf{k}}} - \mathbf{k}^2 \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \right) \quad (50)$$

Inverting the kernel of this bilinear Lagrangian one obtains the time-ordered Green's function (propagator) of the theory,

$$D(k_0, \mathbf{k}) = \frac{1 + V_{\mathbf{k}}}{k_0^2 - E_{\mathbf{k}}^2 + i\varepsilon} \quad (51)$$

where $E_{\mathbf{k}}^2 = \mathbf{k}^2(1 + V_{\mathbf{k}})$ is the dispersion relation of the one-particle excitation of the model.

Apparently, when the potential $V(\mathbf{x})$ falls off sufficiently fast at large distance so that a finite limit $\lim_{\mathbf{k} \rightarrow 0} V_{\mathbf{k}}$ exists, the theory has a gapless excitation. This is a NG boson associated with the invariance of the Lagrangian (50) under the constant shift, $\phi(\mathbf{x}) \rightarrow \phi(\mathbf{x}) + \theta$. The associated current is $\mathbf{j} = -\nabla \phi$ and charge density $\varrho = \pi$. The integral charge then is $Q_{\Omega} = \int_{\Omega} d^3\mathbf{x} \varrho(\mathbf{x}) = \pi_0$, which indeed commutes with the Hamiltonian.

Let us now have a closer look at the NG dispersion relation in the case of a long-range interaction. Assume that the potential is radially symmetric, regular at the origin and drops as $1/r^{\alpha}$ at large distance r , considering the following approximation,

$$V(r) \approx \begin{cases} V_0 & \text{for } r \ll r_0 \\ \gamma/r^{\alpha} & \text{for } r \gg r_0 \end{cases} \quad (52)$$

The momentum integration in

$$V_{\mathbf{k}} = \frac{4\pi}{k} \int_0^{\infty} dr r V(r) \sin kr = \frac{4\pi}{k^3} \int_0^{\infty} dx x V(x/k) \sin x \quad (53)$$

where $k = |\mathbf{k}|$, can then be split into three regions, $0 < x < kr_0$, $kr_0 < x < 1$, and $1 < x < \infty$, which are well separated provided the momentum is small enough. In the integration over the first region we approximate the potential by V_0 and the sine by its argument, getting $\frac{4}{3}\pi V_0 r_0^3$. In the second region we approximate the potential by γ/r^{α} and the sine by its argument, which yields $4\pi\gamma (k^{\alpha-3} - r_0^{3-\alpha}) / (3 - \alpha)$. In the last region we approximate just the potential, leading to the integral $4\pi\gamma k^{\alpha-3} \int_1^{\infty} dx \sin x / x^{\alpha-1}$. We do not need to evaluate this integral explicitly, it is enough to know that it converges as long as $\alpha > 1$. Putting all the pieces together, we arrive at the asymptotic scaling of the energy at low momentum,

$$E_{\mathbf{k}} \sim \begin{cases} k & \text{for } 3 \leq \alpha \\ k^{(\alpha-1)/2} & \text{for } 1 < \alpha \leq 3 \end{cases} \quad (54)$$

For $\alpha \leq 1$ the Fourier transform of the potential does not exist and the theory is ill-defined in three spatial dimensions.

The conclusion is that for $\alpha > 3$ the range of the interaction is short enough to ensure applicability of the Nielsen–Chadha theorem 3; the energy is analytic in momentum. On the other hand, for $1 < \alpha \leq 3$ the interaction is long-ranged, but there is still a NG boson. It is, however, a very nontrivial one. The nonanalytic structure of the dispersion relation clearly precludes a low-energy description in terms of an effective Lagrangian. Expansion in powers of momentum is impossible in such a case.

Another interesting special case is the screened Coulomb interaction, $V(r) = \gamma e^{-\mu r}/4\pi r$ [2]. Recalling that $V_{\mathbf{k}} = \gamma/(\mathbf{k}^2 + \mu^2)$, one observes that the system can be stable even with an attractive interaction, provided it is not too strong so that $V_0 \geq -1$. In the limiting case, $\gamma = -\mu^2$, we find $E_{\mathbf{k}} = \mathbf{k}^2/\sqrt{\mathbf{k}^2 + \mu^2}$. Energy is quadratic in momentum and the NG boson is type-II! Does it mean that we have found an explicit example of a system exhibiting a strict inequality in the Nielsen–Chadha theorem 3? Not really, because there are in fact *two* broken conserved charges. The reason is that for $V_0 = -1$, the operator ϕ_0 also commutes with the Hamiltonian (49). There are thus two conserved charges, π_0 and ϕ_0 , both being spontaneously broken. Moreover, their commutator is proportional to the unit operator, and hence obviously has nonzero vacuum expectation value in accord with the general arguments of the previous section. Yet another nontrivial fact is that the new conserved charge is not of the Noether type; it is not associated with a symmetry of the Lagrangian. This emphasizes the fact that the Goldstone theorem assumes just the existence of a conserved charge, and it is not important where this charge comes from.

6.2. Heisenberg ferromagnet

The next example of SSB is already quite realistic and represents a model for isotropic ferromagnets [14]. The degrees of freedom are spin- $\frac{1}{2}$ particles fixed to nodes of a crystal lattice, represented by operators $\mathbf{s}_i$. (The Roman subscript labels the lattice sites.) The Hamiltonian of the model reads

$$\mathcal{H} = -\frac{1}{2} \sum_{ij} J_{ij} \mathbf{s}_i \cdot \mathbf{s}_j \quad (55)$$

It is invariant under simultaneous rotations of all the spins, forming the group SU(2). As long as all the couplings J_{ij} are positive, the ground state of the Hamiltonian is obviously formed by all spins aligned in the same direction. Formally, one rewrites $\mathbf{s}_i \cdot \mathbf{s}_j = \frac{1}{2}(\mathbf{s}_i + \mathbf{s}_j)^2 - \frac{3}{4}$. With all spins aligned, each pair resides in a state with total spin one so that the ground state energy is $-\frac{1}{8} \sum_{ij} J_{ij}$.

Any state of the i th particle may be labeled by a direction $\mathbf{n}$ as $|i, \mathbf{n}\rangle$, meaning that it is an eigenstate of the projection $\mathbf{n} \cdot \mathbf{s}_i$ with the eigenvalue $1/2$. The degenerate ground states of the Hamiltonian (55) are then $|0_{\mathbf{n}}\rangle = \prod_{i=1}^N |i, \mathbf{n}\rangle$, where N is the total number of lattice sites. Using an explicit expression for the eigenvectors of spin- $\frac{1}{2}$ operators, one finds the scalar product of two states associated with different directions, $|\langle \mathbf{n}_1 | \mathbf{n}_2 \rangle| = \cos(\theta_{\mathbf{n}_1, \mathbf{n}_2}/2)$, where $\theta_{\mathbf{n}_1, \mathbf{n}_2}$ is the angle between the two directions. The scalar product of two ground states of the Heisenberg model therefore is

$$|\langle 0_{\mathbf{n}_1} | 0_{\mathbf{n}_2} \rangle| = \left(\cos \frac{\theta_{\mathbf{n}_1, \mathbf{n}_2}}{2} \right)^N \quad (56)$$

and it apparently goes to zero as $N \rightarrow \infty$ unless $\mathbf{n}_1$ and $\mathbf{n}_2$ are identical. This is in agreement with the discussion in Section 2, where it was pointed out as one of the characteristic features of SSB.

To construct the whole Hilbert space above the ground state $|0_{\mathbf{n}}\rangle$, one can conveniently use the formalism of creation and annihilation operators. To that end, recall that the two-dimensional space of spin $\frac{1}{2}$ may be viewed as the Fock space of the fermionic oscillator. One defines annihilation operators $a_i(\mathbf{n})$ and creation operators $a_i^\dagger(\mathbf{n})$ so that $a_i(\mathbf{n})|i, \mathbf{n}\rangle = 0$ and $\{a(\mathbf{n}), a^\dagger(\mathbf{n})\} = 1$. These are nothing else than the lowering and raising operators familiar from the theory of angular momentum. In addition to

their defining relations above, they satisfy $[a_i(\mathbf{n}), a_i^\dagger(\mathbf{n})] = 2\mathbf{n} \cdot \mathbf{s}_i$. Note that, in this setting, annihilation and creation operators at different lattice sites *commute* rather than anticommute as usual. The change of sign induced by the interchange of two distinguishable fermions is, however, merely a convention.

The Hilbert space of the Heisenberg ferromagnet is set up as a Fock space above the vacuum $|0_{\mathbf{n}}\rangle$. In the ground state all spins point in the direction $\mathbf{n}$, while the excited states are obtained by the action of the creation operators $a_i^\dagger(\mathbf{n})$ that flip the spin at the i th lattice site to the opposite direction. The countable basis of the space Hilbert space contains all vectors of the form $a_{i_1}^\dagger(\mathbf{n})a_{i_2}^\dagger(\mathbf{n}) \cdots |0_{\mathbf{n}}\rangle$ where a *finite* number of spins are flipped. It is now obvious that in the infinite volume limit, all basis vectors from the Hilbert space corresponding to a direction $\mathbf{n}_1$ are orthogonal to all basis vectors from the Hilbert space characterized by a different direction $\mathbf{n}_2$. This means that these two spaces are completely orthogonal.

The symmetry group SU(2) is generated by the operator of the total spin, $\mathbf{S} = \sum_i \mathbf{s}_i$. Only rotations about the direction $\mathbf{n}$ of the ground state, that is, the spontaneous magnetization, are unbroken. The unbroken group U(1) is generated by the spin projection $\mathbf{n} \cdot \mathbf{S}$. The two remaining generators of the symmetry group are spontaneously broken. They are obviously ill-defined operators, as are the induced finite transformations, for they take any state out of the above constructed separable Hilbert space.

In order to identify the NG boson state(s), one rewrites the Hamiltonian (55) in terms of the annihilation and creation operators. Since the orientation of the ground state is now fixed, the argument ($\mathbf{n}$) of these operators will be for simplicity omitted. First observe that

$$\mathbf{s}_i \cdot \mathbf{s}_j = -\frac{1}{2}(a_i^\dagger - a_j^\dagger)(a_i - a_j) + a_i^\dagger a_i a_j^\dagger a_j + \frac{1}{4} \quad (57)$$

The Hamiltonian preserves the “particle number,” that is, the number of flipped spins generated by the operator $\sum_i a_i^\dagger a_i$. This is of course, up to irrelevant constants, nothing but the $\mathbf{n}$ -projected component of the total spin, which is not spontaneously broken and thus can be used to label physical states. Let us restrict our attention to the “one-particle” space, spanned on the basis $|i\rangle = a_i^\dagger |0_{\mathbf{n}}\rangle$. The physical reason behind this restriction is that the sought NG boson turns out to be the spin wave, a traveling perturbation induced by flipping a single spin.

On the one-particle space, the second term on the right hand side of Equation (57) gives zero unless $i = j$. The one-particle Hamiltonian thus reads, up to an irrelevant constant,

$$H_{1P} = \frac{1}{4} \sum_{ij} J_{ij} (a_i^\dagger - a_j^\dagger)(a_i - a_j) \quad (58)$$

and acts on the basis states as

$$H_{1P}|i\rangle = \frac{1}{2} \sum_j J_{ij} (|i\rangle - |j\rangle) \quad (59)$$

It is worth emphasizing that so far we have nowhere used the discrete translational invariance implied by the symmetries of the crystal lattice. In fact, the indices i, j have been just labels distinguishing different degrees of freedom. However, as we already know, translational invariance is needed in order to have well defined quasiparticle excitations carrying conserved momentum. The stationary states can then be sought as the common eigenstates of the discrete translation operators, that is, the plane waves, $|\mathbf{k}\rangle = \sum_i e^{i\mathbf{k} \cdot \mathbf{x}_i} |i\rangle$. Assuming finally that the couplings depend just on the distance of the lattice sites,

$J_{ij} = J(|\mathbf{x}_i - \mathbf{x}_j|)$, one finds by direct substitution that the plane wave $|\mathbf{k}\rangle$ indeed is an eigenstate of the one-particle Hamiltonian (59), with the energy given by

$$E_{\mathbf{k}} = \frac{1}{2}(J_0 - J_{\mathbf{k}}) \quad (60)$$

where $J_{\mathbf{k}}$ is the discrete Fourier transform, $J_{\mathbf{k}} = \sum_i J(\mathbf{x}_i) e^{-i\mathbf{k} \cdot \mathbf{x}_i}$. As a concrete example, the nearest neighbor interaction of strength J on a square lattice of spacing ℓ would give the dispersion $E_{\mathbf{k}} = J(3 - \cos k_x \ell - \cos k_y \ell - \cos k_z \ell) = 2J \left(\sin^2 \frac{k_x \ell}{2} + \sin^2 \frac{k_y \ell}{2} + \sin^2 \frac{k_z \ell}{2} \right)$. The dispersion relation of the NG boson is quadratic at low momentum, hence it is of type-II. In fact, this follows directly from the Nielsen–Chadha theorem 3 since there are two broken generators and only one NG state. In this case one can easily see that acting with the two broken generators on the ground state $|0_n\rangle$ formally produces states that differ just by a phase factor.

At the end, let us again look more closely at the NG dispersion relation (60) and assume a long-range potential as in Equation (52). Since we are only interested in the low-momentum behavior, the discrete Fourier transform of the interaction can be replaced with the continuous radial integral (53) with an additional prefactor $1/\ell^3$ taking into account the volume of the elementary lattice cell. In order that the spectrum be well defined at all, the Fourier transform at zero momentum must exist, which is true only for $\alpha > 3$. By an analysis similar to that for the Boulware–Gilbert model one finds that

$$E_{\mathbf{k}} \sim \begin{cases} k^2 & \text{for } 5 \leq \alpha \\ k^{\alpha-3} & \text{for } 3 < \alpha \leq 5 \end{cases} \quad (61)$$

In conclusion, for $\alpha \geq 5$ the interaction drops sufficiently rapidly so that the Nielsen–Chadha theorem applies and the dispersion relation is analytic, concretely quadratic since the two broken generators demand a type-II NG boson. For $3 < \alpha \leq 5$ the NG boson still exists, but its energy is once again nonanalytic in momentum. Remember that even though we have not solved the Heisenberg model (55) completely, Equation (60) still represents the *exact* energy of exact eigenstates of the Hamiltonian, and therefore this conclusion about the NG energy is not just an artifact of some approximation. Finally, note that for $\alpha \geq 5$ the Fourier transform at low momentum may be evaluated by Taylor expansion up to order $\mathbf{k}^2$, resulting in the dispersion relation

$$E_{\mathbf{k}} = \frac{\pi \mathbf{k}^2}{3\ell^3} \int_0^\infty dr r^4 J(r) \quad (62)$$

6.3. Linear sigma model

For a final example let me get back to the linear sigma model. Unlike the previous two examples where the NG state and its dispersion relation were constructed exactly, here just the classical approximation will be used. As a trade-off one is able to gain deeper insight in the nature of type-II NG bosons.

For illustration purpose, I will restrict to the model with $SU(2) \times U(1)$ symmetry, defined by the Lagrangian [50,51]

$$\mathcal{L} = D_\mu \phi^\dagger D^\mu \phi - M^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 \quad (63)$$

Here ϕ is a complex doublet of scalar fields and the covariant derivative includes chemical potential μ associated with the $U(1)$ factor of the symmetry group, $D_\nu \phi = (\partial_\nu - i\delta_{\nu 0} \mu) \phi$. For $\mu > M$ the

perturbative Fock vacuum becomes unstable with respect to fluctuations of the field ϕ , and the field develops nonzero vacuum expectation value, φ . At tree level, $\varphi^\dagger \varphi \equiv v^2 = (\mu^2 - M^2)/(2\lambda)$. The four Noether currents of the theory have the form

$$j_\nu^a = -2 \operatorname{Im} \phi^\dagger \tau^a \partial_\nu \phi + 2\mu \delta_{\nu 0} \phi^\dagger \tau^a \phi, \quad j_\nu = -2 \operatorname{Im} \phi^\dagger \partial_\nu \phi + 2\mu \delta_{\nu 0} \phi^\dagger \phi \quad (64)$$

where τ^a are the Pauli matrices. Choosing the orientation of the ground state so that v is real and positive and resides solely in the lower component of φ , the only unbroken generator will be $\tau^+ = \frac{1}{2}(\mathbb{1} + \tau^3)$. The broken generators can be conveniently chosen as $\tau^- = \frac{1}{2}(\mathbb{1} - \tau^3)$ and $\tau^{1,2}$. Then, only the generator τ^- has nonzero expectation value in the ground state. Therefore, the commutator $[\tau^1, \tau^2]$ also has nonzero expectation value and we expect one type-II NG boson which couples to both j_ν^1 and j_ν^2 . In addition, there should be one type-I NG boson coupled to τ^- .

Remember from Section 5.4. that the NG fields can be identified as $iT_a \varphi$, using the broken generators T_a . This means that the upper component of ϕ , ϕ_1 , represents the type-II NG boson while the imaginary part of the lower component, ϕ_2 , represents the type-I NG boson. The real part of ϕ_2 corresponds to fluctuations of the magnitude of the condensate, and is expected to be massive. Moreover, ϕ_1 carries unit charge of the unbroken $U(1)'$ symmetry generated by τ^+ while ϕ_2 does not. Therefore, these two components do not mix and their bilinear Lagrangians and propagators can be evaluated separately.

One thus finds that ϕ_1 excites modes with dispersion relations $E_{\mp, \mathbf{k}} = \sqrt{\mathbf{k}^2 + \mu^2} \mp \mu$. The upper sign corresponds to the type-II NG boson; indeed, at low momentum the energy is approximately $E_{-, \mathbf{k}} \approx \mathbf{k}^2/2\mu$. The lower sign describes a mode with mass 2μ . As predicted in Section 5.4. the presence of nonzero vacuum expectation value of the commutator of two broken generators turns the two associated modes into one massive state and one NG boson of type-II. Similarly, the dispersion relations of the two modes in ϕ_2 are found to be

$$E_{\mp, \mathbf{k}}^2 = \mathbf{k}^2 + 3\mu^2 - M^2 \mp \sqrt{4\mu^2 \mathbf{k}^2 + (3\mu^2 - M^2)^2} \quad (65)$$

Again, the upper sign refers to the NG boson, whose dispersion relation at low momentum this time is $E_{-, \mathbf{k}} \approx |\mathbf{k}| \sqrt{(\mu^2 - M^2)/(3\mu^2 - M^2)}$.

Apart from the dispersion relations, the knowledge of the propagators allows us to determine the couplings of the modes to the fields from the residua of the respective poles using the Källén–Lehmann spectral representation [60]. Denoting the type-I and type-II boson states as $|\pi(\mathbf{k})\rangle$ and $|G(\mathbf{k})\rangle$, one obtains

$$\begin{aligned} \langle 0 | \phi_2(0) | \pi(\mathbf{k}) \rangle &= \frac{1}{\sqrt{2E_{-, \mathbf{k}}}} \frac{1}{\sqrt{E_{+, \mathbf{k}}^2 - E_{-, \mathbf{k}}^2}} [-(E_{-, \mathbf{k}} - \mu)^2 + \mathbf{k}^2 + 2\mu^2 - M^2] \\ \langle 0 | \phi_2^\dagger(0) | \pi(\mathbf{k}) \rangle &= -\frac{1}{\sqrt{2E_{-, \mathbf{k}}}} \frac{1}{\sqrt{E_{+, \mathbf{k}}^2 - E_{-, \mathbf{k}}^2}} [-(E_{-, \mathbf{k}} + \mu)^2 + \mathbf{k}^2 + 2\mu^2 - M^2] \end{aligned} \quad (66)$$

and

$$\langle 0 | \phi_1(0) | G(\mathbf{k}) \rangle = \frac{1}{\sqrt{2\sqrt{\mathbf{k}^2 + \mu^2}}} \quad (67)$$

and finally $\langle 0|\phi_1^\dagger(0)|G(\mathbf{k})\rangle = 0$ thanks to the unbroken $U(1)'$ charge. These expressions allow one to evaluate, with the help of Equation (64), the couplings of the NG bosons to the broken currents [57],

$$\begin{aligned}\langle 0|j_1^\nu(0)|G(\mathbf{k})\rangle &= -i\langle 0|j_2^\nu(0)|G(\mathbf{k})\rangle = v(k_{\text{on}}^\nu + 2\mu\delta^{\nu 0})\langle 0|\phi_1(0)|G(\mathbf{k})\rangle \\ \langle 0|j_-^\nu(0)|\pi(\mathbf{k})\rangle &= v\left[(k_{\text{on}}^\nu + 2\mu\delta^{\nu 0})\langle 0|\phi_2(0)|\pi(\mathbf{k})\rangle - (k_{\text{on}}^\nu - 2\mu\delta^{\nu 0})\langle 0|\phi_2^\dagger(0)|\pi(\mathbf{k})\rangle\right]\end{aligned}\quad (68)$$

With all these formulas at hand one readily checks all general formulas derived in Section 5.3. such as the density rule (39).

In order to gain further insight into the nature of the type-II NG boson, let us investigate the solution of the classical equation of motion. The bilinear part of the Lagrangian (63) containing just the field ϕ_1 is

$$\mathcal{L}_{\text{bilin},1} = 2i\mu\phi_1^\dagger\partial_0\phi_1 + \partial_\mu\phi_1^\dagger\partial^\mu\phi_1 \quad (69)$$

up to a total derivative. This has plane wave solutions of the type $\phi_1(x) = ae^{-ik\cdot x}$ with $k_0 = \sqrt{\mathbf{k}^2 + \mu^2} - \mu$, which is exactly the dispersion relation of the type-II NG boson in the model. Substituting into Equation (64) one arrives at the expressions for the currents carried by this plane wave,

$$\begin{aligned}j_1^\nu &= -2v(k_{\text{on}}^\nu + 2\delta^{\nu 0}\mu)\text{Im}\phi_1, & j_2^\nu &= -2v(k_{\text{on}}^\nu + 2\delta^{\nu 0}\mu)\text{Re}\phi_1 \\ j_+^\nu &= 2(k_{\text{on}}^\nu + \delta^{\nu 0}\mu)|\phi_1|^2, & j_-^\nu &= 2\delta^{\nu 0}\mu v^2\end{aligned}\quad (70)$$

The second line shows that the plane wave is associated with a uniform current of the unbroken charge, τ^+ . This corresponds to the observation made above that the field ϕ_1 , and hence its quantum $|G(\mathbf{k})\rangle$, carries unit charge of the unbroken group $U(1)'$. Also, there is uniform background density of the τ_- charge, induced by the condensate v . It is independent of the plane wave since the associated current does not couple to $|G(\mathbf{k})\rangle$.

The first line of Equation (70) reveals that the isospin vector of charge densities rotates in the (1, 2) plane perpendicular to the direction of the condensate. In other words, the type-II NG boson is a circularly polarized isospin wave. This is, of course, also in accord with the fact that it carries the unbroken charge. One may then naturally ask where the plane wave with opposite polarization is. In this model, this “antiparticle” of the type-II NG boson is heavy; its mass is 2μ .

Interestingly, the answer to this question is quite different in a nonrelativistic ferromagnet, analyzed in the previous subsection. There, too, the magnon is a spin wave which carries (minus) one unit of the unbroken charge, that is, projection of spin into the direction of total magnetization. However, in this case, the spin wave with opposite circular polarization does not even exist. The explanation is twofold. First, quantum mechanically, the ground state is the state with the highest possible spin, and the magnon corresponds to a propagating perturbation caused by flipping of one of the spins. The spin wave polarized in the opposite way would have to have total spin one unit higher than the ground state, which is not possible. Classically, imagine the magnon as a single test spin propagating on a background of other spins. The Heisenberg Hamiltonian (55) shows that this background acts on the test spin as a homogeneous magnetic field. The evolution of a spin in a magnetic field is well known as the Larmor precession. Its sense is fixed by the magnetic moment of the spin; rotation of the spin in the opposite direction is not possible.

7. Low-Energy Effective Field Theory for NG Bosons

Effective field theory (EFT) is a powerful approach to problems involving several energy scales that was developed in the full generality in particle physics [61], but has grown into an indispensable tool in essentially all branches of physics that use the methods of quantum field theory. The basic assumption is that the physical degrees of freedom split into groups well separated in energy. In case one is interested only in low-energy, or long-distance, observables, EFT may be used to describe the system in a way which involves just the low-energy degrees of freedom. The effects of the microscopic, short-distance dynamics are incorporated in the values of coupling constants of the effective theory. For more details see the review papers [5–8].

There are fortunate cases where the low-energy EFT can be derived directly from the underlying microscopic theory by “integrating out” the heavy, high-energy degrees of freedom. A profound example is the Euler–Heisenberg Lagrangian for the electromagnetic field, which can be inferred from quantum electrodynamics by eliminating electrons or any other massive charged particles present. In such cases EFT still provides a very convenient tool which dispenses with the degrees of freedom irrelevant for the description of low-energy physics. However, the true power of EFT lies in its successful application to systems where the high-energy degrees of freedom cannot be simply integrated out of the microscopic theory, either because of its nonperturbative nature, or because the low-energy modes are not even present in the microscopic theory (such as the hadrons that are not the fundamental degrees of freedom in quantum chromodynamics). The utility of EFT then relies on the observation that field theory is merely a convenient way to incorporate the general principles such as unitarity and cluster decomposition but contains no further dynamical assumptions, and it can therefore reproduce the predictions at low energy of *any* microscopic theory as long as it involves the appropriate low-energy degrees of freedom and symmetries, see e.g. [9] or Section 19.5 in [15].

A distinguished class of systems featuring separation of scales is represented by those with SSB. The NG bosons, guaranteed by the Goldstone theorem, are then the low-energy degrees of freedom. In the most frequent case that there are no other gapless states in the spectrum, they are in fact the only degrees of freedom in the low-energy EFT. A systematic approximation scheme for calculations of low-energy observables is then provided by expansion in powers of momentum, or derivatives. (This requires absence of long-range interactions, as shown in Sections 6.1. and 6.2. on explicit examples with non-analytic low-energy behavior that makes momentum expansion impossible.) For a detailed explanation how to set up a consistent power counting scheme, see [7,9]. The standard procedure to construct invariant Lagrangians for NG bosons [62,63] is briefly reviewed in the following subsection. After then I will point out some specific issues concerning its application to nonrelativistic systems.

7.1. Coset construction of effective Lagrangians

As was shown already in Section 5. NG states are created from the vacuum by the action of the broken symmetry generators. Formally, a *global* broken symmetry transformation, which merely moves the vacuum into an equivalent ground state, may be viewed as a zero-momentum NG boson. The finite-momentum NG state can then be excited by a *local* infinitesimal broken symmetry transformation. This corresponds to the intuitive picture of NG modes as small local fluctuations of the order parameter.

Denoting the vacuum expectation value of the order parameter field $\phi(x)$ as φ , all possible values of the order parameter that can be reached from φ by a symmetry transformation span a manifold, known as the *coset space* of the broken symmetry, G/H . (Strictly speaking, the coset space may in general consist of several orbits of the symmetry group G . For the construction of the effective Lagrangian, only the orbit containing the vacuum φ is important.) The elements of the coset space may be thought of as the sets $gH = \{gh \mid h \in H\}$ with arbitrary $g \in G$. Thanks to the group properties, two sets g_1H and g_2H are either identical or disjoint, and G/H is the space of these classes.

While the ground state is represented by a single point in the coset space, a general NG field configuration can be viewed as a map from the spacetime to the coset space, $\phi(x) : \mathbb{R}^4 \rightarrow G/H$. The task to construct the most general G -invariant Lagrangian for the NG bosons is equivalent to the geometric problem of constructing a G -invariant function on G/H , given the action of the group on the order parameter.

Let the symmetry group G be compact and semi-simple. Then one can choose a basis T_a of its Lie algebra such that $\text{Tr}(T_a T_b) = \delta_{ab}$ (see Chapter 1 of [18]). Splitting the generators T_a into the unbroken ones, V_α , generating the unbroken subgroup H , and the broken ones, A_i , it follows that $\text{Tr}(V_\alpha V_\beta) = \delta_{\alpha\beta}$, $\text{Tr}(A_i A_j) = \delta_{ij}$, and $\text{Tr}(V_\alpha A_i) = 0$. Furthermore, the structure constants f_{abc} of the Lie algebra of G are fully antisymmetric and the broken generators span a representation of the unbroken group, hence the commutation relations

$$[V_\alpha, V_\beta] = if_{\alpha\beta\gamma} V_\gamma, \quad [V_\alpha, A_i] = if_{\alpha ij} A_j, \quad [A_i, A_j] = if_{ij\alpha} V_\alpha + if_{ijk} A_k \quad (71)$$

Any element $g \in G$ can be, at least in the neighborhood of unity, decomposed as $g = e^{i\pi^i A_i} e^{iv^\alpha V_\alpha}$. Since $e^{iv^\alpha V_\alpha}$ leaves the ground state intact, the points of the coset space can be parameterized as $\phi(\pi) = U(\pi)\varphi$, where $U(\pi) = e^{i\pi^i A_i}$. Thus, the NG fields π^i serve as coordinates on G/H . The action of the symmetry group on G/H is defined by left multiplication, $\phi(\pi) \xrightarrow{g} g\phi(\pi) = \phi(\tilde{\pi})$. Decomposing $gU(\pi)$ as $U(\tilde{\pi})h(g, \pi)$, where $h(g, \pi) = e^{iv^\alpha(g, \pi)V_\alpha}$ belongs to the unbroken subgroup H , one arrives at the transformation rule

$$U(\pi) \xrightarrow{g} U(\tilde{\pi}(g, \pi)) = gU(\pi)h^\dagger(g, \pi) \quad (72)$$

This defines a nonlinear realization of the group G on G/H . In order that the group structure be preserved, the function $h : G \times G/H \rightarrow H$ must satisfy basic constraints such as $h(\mathbb{1}, \pi) = \mathbb{1}$ and the associativity, $h(g'g, \pi) = h(g', \tilde{\pi}(g, \pi))h(g, \pi)$. Also, for transformations from the unbroken subgroup it is independent of π , $h(h, \pi) = h$, so that *the unbroken group acts on G/H linearly*, $\tilde{\pi}^i A_i = h(\pi^i A_i)h^\dagger$.

The effective Lagrangian is most easily constructed using the *Maurer–Cartan form*,

$$\omega(\pi) = -iU^\dagger(\pi) dU(\pi) \quad (73)$$

Employing the very useful formula,

$$e^{-A} de^A = \int_0^1 dt e^{-tA} (dA) e^{tA} \quad (74)$$

one easily realizes that the Maurer–Cartan form lies in the Lie algebra of G . Under the group action (72) it transforms as

$$\omega \xrightarrow{g} \tilde{\omega}(g, \omega) = h\omega h^\dagger - ih dh^\dagger \quad (75)$$

Since $h dh^\dagger$ lies in the Lie algebra of H , it is useful to decompose the Maurer–Cartan form into projections on the spaces of unbroken and broken generators, $\omega = \omega_\parallel + \omega_\perp$. The “longitudinal” part transforms as a gauge field, $\omega_\parallel \xrightarrow{g} h\omega_\parallel h^\dagger - ih dh^\dagger$, while the “transverse” part transforms covariantly $\omega_\perp \rightarrow h\omega_\perp h^\dagger$.

Should it be necessary to include other, non-NG modes in the effective theory, one proceeds as follows. Taking advantage of the fact that the unbroken subgroup H is realized linearly even on the NG bosons, consider a set of “matter” fields ψ that transform in the representation $\mathcal{R}$ of H , $\psi \xrightarrow{h} h_{\mathcal{R}}\psi$. This is promoted to a nonlinear realization of the full group G , defined by

$$\psi \xrightarrow{g} h_{\mathcal{R}}(g, \pi)\psi \quad (76)$$

Any Lagrangian invariant under H can be made invariant under the full group G using this procedure. Note that since the transformation matrix $h_{\mathcal{R}}(g, \pi)$ depends on the NG fields π and thus implicitly on the spacetime coordinates, the transformation becomes local. In order that terms containing derivatives of ψ be still G -invariant, one uses the longitudinal component of the Maurer–Cartan form, $\omega_\parallel = \omega_{\parallel\mu} dx^\mu$, to promote derivatives to covariant ones, $\partial_\mu \psi \rightarrow (\partial_\mu + i\omega_{\parallel\mu})\psi$. The most general G -invariant Lagrangian can then be constructed [9] from the objects $\omega_\perp$, ψ and their covariant derivatives. For instance, the expression

$$\mathcal{L}_{\text{bilin}} = \text{Tr}(f_\pi^2 \omega_\perp \omega_\perp) \quad (77)$$

obviously includes the kinetic term for the NG bosons, as can be seen with the help of Equation (74). The f_π^2 is a Hermitian, positive definite matrix that must commute with all matrices from H in order that $\mathcal{L}_{\text{bilin}}$ be G -invariant. If the broken generators transform irreducibly under H , it is proportional to the unit matrix by Schur’s lemma, hence providing merely an overall normalization of the Lagrangian. In general it contains one arbitrary parameter per each irreducible representation, thus giving an independent normalization of the kinetic term for each multiplet of NG bosons.

One often meets the situation that the symmetry in question is local rather than global. This can happen both, when the system interacts with a dynamical gauge (such as electromagnetic) field and when external sources are coupled to the conserved currents of the theory [64]. Consider a gauge field $\mathcal{A}$ transforming under local G transformations as usual as $\mathcal{A} \xrightarrow{g} g\mathcal{A}g^\dagger + ig dg^\dagger$. It is straightforward to see that the Maurer–Cartan form, defined now by $i\omega = U^\dagger(\pi)\mathcal{D}U(\pi) = U^\dagger(\pi)(d - i\mathcal{A})U(\pi)$, satisfies the same rule (75) as in the case of a global symmetry.

This concludes the brief review of the coset construction of effective Lagrangians. It is useful to note that while the choice of the coset parameterization as $U(\pi) = e^{i\pi^i A_i}$ is very convenient in particular because it renders the action of the unbroken subgroup linear, the construction is not limited to this choice. In fact, the key elements are only the geometry of the coset space and its symmetry. The choice of coordinates π^i has no observable effects; one speaks of *reparameterization invariance* of physical observables [62].

7.2. Geometric interpretation

The kinetic term (77) can be evaluated more or less explicitly. Using the orthogonality of the generators, the Maurer–Cartan form can be expanded in V_α, A_i . Denoting $\omega_\perp = \omega_{\perp i} A_i$, one finds

$$\omega_{\perp i} = \Sigma_{ij} d\pi^j, \quad \Sigma_{ij} = \int_0^1 dt \operatorname{Tr} \left(A_i e^{-it\pi^k A_k} A_j e^{+it\pi^k A_k} \right) \quad (78)$$

The kinetic term acquires the form $g_{ij} d\pi^i d\pi^j$ with $g_{ij} = \Sigma_{ki} \Sigma_{lj} \operatorname{Tr}(f_\pi^2 A_k A_l)$. (I deliberately use differentials instead of spacetime derivatives in order to emphasize the geometric nature of the quantities, and to avoid having to treat space and time derivatives separately.) The matrix g_{ij} can be interpreted as a *metric* on the coset space. The construction of the most general term in the effective Lagrangian with two derivatives therefore reduces to the geometric problem of finding all G-invariant metrics on G/H [4,64].

In order to see what are the coordinate transformations that leave the metric g_{ij} invariant, one must explicitly evaluate the change of the fields π^i under a G transformation. Writing Equation (72) as

$$e^{-i\pi^i A_i} e^{i\tilde{\pi}^i A_i} e^{iv^\alpha V_\alpha} = e^{-i\pi^i A_i} g e^{i\pi^i A_i} \quad (79)$$

one sets $g = e^{i\theta^a T_a}$ and expands both sides to first order in the infinitesimal transformation parameters θ^a . (No assumption on the smallness of π^i is made though.) Denoting further

$$R_{\alpha i} = \int_0^1 dt \operatorname{Tr} \left(V_\alpha e^{-it\pi^k A_k} A_i e^{+it\pi^k A_k} \right) \quad (80)$$

$e^{-i\pi^k A_k} V_\alpha e^{i\pi^k A_k} = V_\beta P_{\beta\alpha}^{(VV)} + A_i P_{i\alpha}^{(AV)}$, and $e^{-i\pi^k A_k} A_i e^{i\pi^k A_k} = V_\alpha P_{\alpha i}^{(VA)} + A_j P_{ji}^{(AA)}$, Equation (79) reduces to

$$(V_\alpha R_{\alpha j} + A_i \Sigma_{ij}) \delta\pi^j + V_\alpha v^\alpha = \left[V_\alpha P_{\alpha\beta}^{(VV)} + A_i P_{i\beta}^{(AV)} \right] \theta^\beta + \left[V_\alpha P_{\alpha j}^{(VA)} + A_i P_{ij}^{(AA)} \right] \theta^j \quad (81)$$

Comparing coefficients at A_i , a formula for the shift of NG fields, $\delta\pi^i$, follows,

$$\delta\pi^i = (\Sigma^{-1})^{ij} \left[P_{j\beta}^{(AV)} \theta^\beta + P_{jk}^{(AA)} \theta^k \right] \equiv \xi_a^i \theta^a \quad (82)$$

The set of objects ξ_a^i that depend just on the coset geometry define the *Killing vectors* on G/H. They represent infinitesimal motions on G/H induced by the symmetry group transformations. A similar expression for the parameters v^α of the compensating transformation $h(g, \pi)$ is implied.

The above formulas were derived for infinitesimal transformation parameters θ^a , but hold for arbitrary values of the NG fields π^i . In practice, the vacuum is usually set at $\pi^i = 0$ and for calculation of low-energy observables such as scattering amplitudes, one gets by with a power expansion in π^i . All expressions of this subsection can then be evaluated explicitly in terms of the structure constants of the Lie algebra of symmetry generators. To first order in the NG fields one finds, for instance,

$$\Sigma_{ij} = \delta_{ij} - \frac{1}{2} f_{ijk} \pi^k, \quad \xi_\alpha^i = f_{j\alpha}^i \pi^j, \quad \xi_j^i = \delta_j^i - \frac{1}{2} f_{jk}^i \pi^k \quad (83)$$

This essentially confirms the picture that underlies all discussions of effective Lagrangians for NG bosons: unbroken symmetry is realized linearly on the NG fields whereas broken transformations are

equivalent, to lowest order, to a mere shift. This makes sure that the NG fields correspond to gapless modes in the spectrum.

It is worthwhile to inquire how the Maurer–Cartan form changes in presence of a gauge field. According to our discussion above, it picks an additional contribution,

$$-U^\dagger(\pi)\mathcal{A}U(\pi) = -\left[V_\beta P_{\beta\alpha}^{(VV)} + A_i P_{i\alpha}^{(AV)}\right]\mathcal{A}^\alpha - \left[V_\alpha P_{\alpha i}^{(VA)} + A_j P_{ji}^{(AA)}\right]\mathcal{A}^i \quad (84)$$

One then immediately finds that the transverse component of the Maurer–Cartan form that determines the kinetic term of the NG bosons, modifies according to $\omega_{\perp i} = \Sigma_{ij} d\pi^j \rightarrow \Sigma_{ij} \mathcal{D}\pi^j$, where $\mathcal{D}\pi^i = d\pi^i - \xi_a^i \mathcal{A}^a$. This shows that even in this very general setting where the symmetry group is realized nonlinearly, the infinitesimal shift of the fields under group motion serves to construct their gauge-covariant derivative.

In the above derivation, the metric g_{ij} was derived from the Maurer–Cartan form and its G-invariance followed from the G-invariance of the Lagrangian. It is instructive to reverse the argument and use an invariant metric on the coset space to construct Equation (77) in a purely geometric way. To this end, recall that we deal with compact symmetry groups that can be represented by unitary matrices. The space of complex matrices possesses a natural distance function, invariant under both left and right multiplication by unitary matrices, defined by $d(x, y) = \|x - y\|$, where $\|x\| = \sqrt{\text{Tr}(x^\dagger x)}$. (I use the term *distance* instead of metric to distinguish this object from the metric *tensor*, which roughly speaking measures the distance on the tangent space at unity.) This induces a distance function on the coset G/H , manifestly invariant under the action (72) of G ,

$$d_{G/H}(x, y) = \min_{h \in H} d(xh, y) \quad (85)$$

It is easy to show that this prescription indeed yields a well defined distance. First, the unbroken subgroup H is compact so that the minimum in Equation (85) exists. Furthermore, $d_{G/H}$ is independent of the choice of the representative elements x, y for the cosets xH, yH and it is zero if and only if x and y lie in the same coset. Finally, it is straightforward to prove the triangle inequality as a consequence of that for the distance d .

To determine the metric *tensor* following from Equation (85), one takes the points x, y infinitesimally far apart, $x = U(\pi)$ and $y = U(\pi + d\pi)$, whence

$$d_{G/H}(x, y) = \min_{h \in H} \|U(\pi + d\pi) - U(\pi)h\| = \min_{h \in H} \|U^\dagger(\pi)U(\pi + d\pi) - h\| = \min_{h \in H} \|\mathbf{i}\omega(\pi) + \mathbf{1} - h\| \quad (86)$$

The minimum will be realized for h infinitesimally close to unity, hence $\mathbf{1} - h$ will lie in the Lie algebra of H , and affect only the longitudinal part of the Maurer–Cartan form, $\omega_{\parallel}(\pi)$. Clearly, the minimum is achieved when $\mathbf{1} - h$ exactly cancels $\omega_{\parallel}$ so that

$$d_{G/H}(x, y) = \|\omega_{\perp}\| = \sqrt{\text{Tr}(\omega_{\perp}\omega_{\perp})} \quad (87)$$

Consequently, the bilinear Lagrangian (77) is directly related to a G-invariant metric on the coset space G/H . The only difference is the factor f_π^2 present in Equation (77). Mathematically, this comes from the fact that Equation (87) represents an intrinsic metric of the coset space, invariant under both left and right G transformations, while Equation (77) is by construction invariant under left G multiplication, but only under H acting from the right [64].

7.3. Nonrelativistic effective Lagrangians

In the preceding two subsections a general method how to build effective Lagrangians was presented. The construction is based on an implicit *assumption* that the Lagrangian is invariant under the symmetry group of the system. Global symmetry of the theory can be conveniently formulated in terms of invariance of the generating functional of Green's functions of the conserved currents, $j_a^\mu(x)$, with respect to *local* transformations of external sources, $\mathcal{A}_\mu^a(x)$, coupled to these currents [65,66]. This implies a set of Ward identities that the theory must satisfy in order to preserve the symmetry. In Lorentz invariant theories without quantum anomalies, the most general solution of the Ward identities can indeed always be obtained from a Lagrangian invariant under the symmetry group [64]. On the other hand, in systems that lack Lorentz invariance (“nonrelativistic” systems) this conclusion is no longer valid. The most general solution of the Ward identities to the lowest order of the derivative expansion was given by Leutwyler [4] who showed that in general it is just the action, not the Lagrangian, what is invariant.

The Lagrangian of a low-energy EFT in principle contains an infinite number of terms. The predictive power of EFT lies in the fact that at any desired accuracy of an observable to be calculated, only a finite number of terms contribute. The Lagrangian is systematically expanded in powers of space (s) and time (t) derivatives, $\mathcal{L} = \sum_{s,t} \mathcal{L}^{(s,t)}$. In order to achieve a consistent power counting scheme, it is convenient to assign the external source $\mathcal{A}_\mu^a$ the same order as the derivative ∂_μ . However, the scaling of spatial and temporal derivatives may in general differ. For the sake of simplicity, I will assume that the system is invariant under the continuous rotation group $SO(3)$. This is certainly plausible for fluids, and even for some crystalline (such the plain cubic) structures, where the anisotropy induced by the lattice only appears at higher orders in the derivative expansion. The first few terms in the effective Lagrangian then read,

$$\mathcal{L}^{(0,1)} = C_i(\pi)\dot{\pi}^i + \varrho_a(\pi)\mathcal{A}_0^a, \quad \mathcal{L}^{(0,2)} = \frac{1}{2}\bar{g}_{ij}(\pi)\mathcal{D}_0\pi^i\mathcal{D}_0\pi^j, \quad \mathcal{L}^{(2,0)} = -\frac{1}{2}g_{ij}(\pi)\mathcal{D}_k\pi^i\mathcal{D}_k\pi^j \quad (88)$$

where the dot denotes a time derivative, and the covariant derivative, $\mathcal{D}_\mu\pi^i = \partial_\mu\pi^i - \xi_a^i\mathcal{A}_\mu^a$, was introduced below Equation (84). Both $g_{ij}(\pi)$ and $\bar{g}_{ij}(\pi)$ are G-invariant metrics on the coset space. It follows from the above discussion that if the NG fields transform irreducibly under the unbroken symmetry, they must be identical functions on G/H up to a trivial scale factor. Both $\mathcal{L}^{(0,2)}$ and $\mathcal{L}^{(2,0)}$ are obviously invariant under simultaneous gauge transformations of the NG fields and the external sources.

It would be tempting to conclude that the coefficient functions $C_i(\pi)$ and $\varrho_a(\pi)$ are related by $\varrho_a = -C_i\xi_a^i$ so that $\mathcal{L}^{(0,1)} = C_i(\pi)\mathcal{D}_0\pi^i$. However, this does not follow from general symmetry considerations. Instead, the Ward identities encoded in the gauge invariance of the generating functional imply a set of constraints for all coefficients $C_i(\pi)$, $\varrho_a(\pi)$, $g_{ij}(\pi)$, $\bar{g}_{ij}(\pi)$, $\xi_a^i(\pi)$ [4]. The functions $\xi_a^i(\pi)$, defined by Equation (82), are Killing vectors of *both* invariant metrics, $g_{ij}(\pi)$ and $\bar{g}_{ij}(\pi)$, that is, they obey the Killing equation $\nabla_i\xi_{ja} + \nabla_j\xi_{ia} = 0$. The Riemannian covariant derivative ∇_i is defined with the help of the Christoffel symbol, $\nabla_i\xi_{ja} = \partial_i\xi_{ja} - \Gamma_{ij}^k\xi_{ka}$. (Following [4] I use the notation $\partial_i \equiv \partial/\partial\pi^i$, not to be mixed up with a derivative with respect to the space coordinate.) Note that using the two metrics to lower the index in ξ_a^i , one obtains two *different* objects, ξ_{ia} and $\bar{\xi}_{ia}$, that satisfy Killing equations with respective Christoffel symbols.

Since the Killing vectors ξ_a^i generate the symmetry transformation (82) on the coset space, they must reproduce the structure of the Lie algebra of the symmetry generators, $d_a \xi_b^i - d_b \xi_a^i = f_{ab}^c \xi_c^i$, where $d_a \equiv \xi_a^i \partial_i$. Finally, the coefficients $C_i(\pi)$ and $\varrho_a(\pi)$ satisfy the identities

$$d_a \varrho_b = f_{ab}^c \varrho_c, \quad \xi_a^i (\partial_i C_j - \partial_j C_i) = \partial_j \varrho_a \quad (89)$$

While the former asserts that $\varrho_a(\pi)$ transforms in the adjoint representation of the symmetry group and fixes its value at any point of G/H in terms of that at the origin ($\pi = 0$), the latter determines the function $C_i(\pi)$ up to a gradient. This ambiguity is intrinsic to EFT and has no observable consequences. Indeed, changing $C_i(\pi)$ by a gradient merely modifies the Lagrangian by a total time derivative, hence leaving the action intact.

Several remarks are in order here. First, $\mathcal{L}^{(0,1)}$ is in general not gauge invariant: under the transformation (82) it changes by a total derivative, $\delta \mathcal{L}^{(0,1)} = \frac{d}{dt} [\theta^a (C_i \xi_a^i + \varrho_a)]$. Second, evaluating the action at its extremum in order to establish the leading contribution to the generating functional, one finds by differentiation with respect to $\mathcal{A}_0^a$ that $\varrho_a(\pi)$ at the origin determines the density of the conserved charge, $\langle 0 | j_a^0(x) | 0 \rangle = \varrho_a(0)$. When all charge densities are zero, it follows from Equation (89) that both $\varrho_a(\pi)$ and $C_i(\pi)$ vanish identically and the lowest order Lagrangian is essentially the same as in Lorentz invariant theories, up to a possible redefinition of the phase velocity of the NG bosons. The energy of the NG bosons is linear in momentum so that one assigns space and time derivatives the same degree in the power expansion, and the leading order Lagrangian reads $\mathcal{L} = \mathcal{L}^{(0,2)} + \mathcal{L}^{(2,0)}$. Also, when the symmetry is Abelian Equation (89) guarantees that $\varrho_a(\pi) = \varrho_a(0)$ on the whole coset space, and in turn $C_i(\pi) = 0$. Since constant $\varrho_a(\pi)$ does not contribute to the dynamics of NG bosons, merely generating background charge density, the equation of motion and dispersion relation of the NG bosons will be the same as above.

On the other hand, nonzero density of a non-Abelian charge leads to nonzero $C_i(\pi)$. The Lagrangian is gauge invariant only up to a total derivative, and the term with the single time derivative leads to a quadratic dispersion relation of the NG boson. One must then count a time derivative as two space derivatives so that the lowest order Lagrangian is $\mathcal{L} = \mathcal{L}^{(0,1)} + \mathcal{L}^{(2,0)}$. This confirms the observation that density of a non-Abelian charge implies a type-II NG boson in the spectrum, made in Section 5. on a general ground.

It is instructive to illustrate the construction of effective Lagrangians on an explicit example. The most convenient one is provided by the (anti)ferromagnet. Both these systems possess an $SU(2)$ symmetry group of spin rotations, which is spontaneously broken by an ordered ground state to its $U(1)$ subgroup. As already mentioned in Section 5.2. in a ferromagnet all spins sitting on the crystal lattice are aligned in the same direction and there is a net total magnetization. Accordingly, the two broken generators give rise to one type-II NG boson with quadratic dispersion relation. On the contrary, in an antiferromagnet antiparallel alignment of neighboring spins is favored and the ground state, though ordered, does not develop nonzero spin density. There are two NG bosons with linear dispersion relations, corresponding to two linearly polarized spin waves.

The coset space is $SU(2)/U(1) \simeq SO(3)/SO(2)$ in both cases and it is equivalent to a two-sphere, S^2 . It can therefore be most conveniently represented by a unit vector, $\mathbf{n}$. The symmetry group $SU(2)$ acts on $\mathbf{n}$ through rotations, which can be parameterized by a vector $\mathbf{g}$ whose direction and magnitude represent

the axis and angle of the rotation, respectively. An infinitesimal symmetry transformation therefore reads $\delta \mathbf{n} = \mathbf{n} \times \mathbf{g}$, and the covariant derivative in presence of an external source $\mathcal{A}_\mu$ consequently $\mathcal{D}_\mu \mathbf{n} = \partial_\mu \mathbf{n} + \mathcal{A}_\mu \times \mathbf{n}$. Since all charge densities are zero in the antiferromagnet, the term $\mathcal{L}^{(0,1)}$ is missing in the Lagrangian. The leading order Lagrangian thus is

$$\mathcal{L}_{\text{antiferro}} = \frac{1}{2} F_t^2 \mathcal{D}_0 \mathbf{n} \cdot \mathcal{D}_0 \mathbf{n} - \frac{1}{2} F_s^2 \mathcal{D}_k \mathbf{n} \cdot \mathcal{D}_k \mathbf{n} \quad (90)$$

Obviously, for an arbitrary set of coordinates π^1, π^2 on the coset, the invariant metrics will be $g_{ij} = F_s^2 \partial_i \mathbf{n} \cdot \partial_j \mathbf{n}$ and $\bar{g}_{ij} = F_t^2 \partial_i \mathbf{n} \cdot \partial_j \mathbf{n}$. From the Lagrangian one can also extract the Killing vectors, $\mathbf{h}_i = -F_s^2 \mathbf{n} \times \partial_i \mathbf{n}$. Indices are raised with the inverse metric, $g^{ij} = \varepsilon^{ik} \varepsilon^{jl} \partial_k \mathbf{n} \cdot \partial_l \mathbf{n} / F_s^2 \|\partial_1 \mathbf{n} \times \partial_2 \mathbf{n}\|^2$.

The case of the ferromagnet is more subtle. The ground state has nonzero charge density and accordingly the term $C_i(\pi)$ appears in the Lagrangian. First observe that the zero component of the external field, $\mathcal{A}_0^a$, couples to the Noether charge density, which is proportional to the total magnetization. Hence $\mathcal{A}_0^a$ is interpreted as an external magnetic field. The direction of the magnetization in the ground state is aligned with it so that the vector of couplings $\varrho_a(\pi)$ is simply proportional to the order parameter $\mathbf{n}(\pi)$ up to a constant, $\varrho = \alpha \mathbf{n}$. By explicitly solving the constraint (89), the function $C_i(\pi)$ can be shown to have a topological nature, related to the Brouwer degree of the map $\mathbf{n}(\pi)$ [4]. Constructing a path $\tilde{\pi}^i(\pi, \lambda)$, $0 \leq \lambda \leq 1$ such that $\tilde{\pi}^i(\pi, 0) = 0$ and $\tilde{\pi}^i(\pi, 1) = \pi^i$, and setting $\mathbf{n} = \mathbf{n}(\tilde{\pi}(\pi, \lambda))$, one has

$$C_i(\pi) = \alpha \int_0^1 d\lambda (\partial_i \mathbf{n} \times \partial_\lambda \mathbf{n}) \cdot \mathbf{n} \quad (91)$$

The integral is independent of the choice of the path, connecting the origin with the point π^i . This concludes the construction of the leading order effective Lagrangian for the ferromagnet. Details can be found in [4].

One comment concerning the difference between the antiferromagnet and the ferromagnet is worthwhile here. Due to the presence of a term with a single time derivative, the effective Lagrangian of the ferromagnet is sometimes claimed to break time reversal invariance. However, this is somewhat misleading. What breaks the time reversal is the ferromagnetic *ground state*, thanks to the nonzero charge density, that is, a zero component of a conserved current. On the other hand, any microscopic description of a ferromagnet will be time reversal invariant. Indeed, for example the Hamiltonian (55) can describe both the ferromagnetic and the antiferromagnetic state, depending on the sign of the coupling constants J_{ij} . Both systems therefore must have the same symmetry. Since one of the basic requirements on the low-energy EFT is that it should reproduce the symmetry of the underlying theory, the effective Lagrangian of the ferromagnet must actually inherit this time reversal invariance. What really distinguishes ferromagnets from antiferromagnets is the way time reversal is implemented on the coset space, which is natural because this implementation does depend on the structure of the ground state. The transformation of the NG fields under time reversal in the antiferromagnet is such that it prohibits the term with a single time derivative, whereas in the ferromagnet it is allowed [67].

7.4. Applications of effective field theory

Effective field theory has become an invaluable tool of a theoretical physicist in all branches of physics where the methods of quantum field theory are used. It often facilitates calculations that would otherwise

be very complicated, or hard to carry out at all. It would be hopeless to even try to give a representative list of all its applications. I will therefore just mentioned briefly a few explicit examples that are related to other material presented in this paper. The application of EFT to ferromagnets was worked out in [4,67–69], and in a similar spirit also for antiferromagnets [70–73]. The utility of EFT of course goes beyond the mere derivation of the equation of motion and dispersion relation of spin waves. One can calculate other observables as well such as the cross section for scattering of magnons, or of slow neutrons off the magnons [9]. The use of EFT as well as other techniques for the nonrelativistic weakly interacting Bose gas and Bose–Einstein condensation is reviewed in [74]. See also [75] for the calculation of the damping rate of the corresponding NG boson and the construction of the effective Lagrangian based on Galilean invariance. Subtle features associated with a consistent definition of thermodynamics in various approximation schemes for Bose condensed systems are discussed in detail in [76]. Extension of Galilean invariance in nonrelativistic physics to general coordinate invariance was worked out in [77] and applied to strongly attractive Fermi gas at unitarity.

Among the varied applications of EFT, I would like to discuss in a little more detail the technique to construct the effective action for relativistic superfluids from the equation of state, developed by Son [78]. It is based on the observation that in a relativistic many-body system at zero temperature, the chemical potential is the sole source of Lorentz violation. Lorentz invariance can therefore be restored when the chemical potential is treated as a zero component of a background gauge field, very much like in Section 5.4. where the linear sigma model was investigated.

Consider a relativistic system with Abelian U(1) symmetry that acts on the order parameter ϕ as a phase transformation: $\phi \rightarrow \phi e^{i\theta}$. Assigning a chemical potential μ to the Noether charge of this symmetry and denoting as in Section 5.4. $A_\mu = (\mu, \mathbf{0})$, the quantum effective action of the system, $\Gamma_{\text{eff}}[A_\mu, \phi]$, will be invariant under a simultaneous *local* transformation of ϕ and A_μ , $A_\mu \rightarrow A_\mu + \partial_\mu \theta$. As soon as the order parameter develops nonzero vacuum expectation value, φ , it can be conveniently parameterized as $\phi(x) = e^{i\pi(x)}[\varphi + H(x)]$. The phase transformation of ϕ is equivalent to a shift of the NG field, $\pi \rightarrow \pi + \theta$.

At low energy the modes created by the “Higgs” field cannot be excited. In order to obtain an effective action for the NG boson only, one “integrates out” the Higgs mode by minimizing the effective action with respect to the modulus $|\phi|$. The leading term in the derivative expansion of the resulting effective action, $\Gamma_{\text{eff}}[A_\mu, \pi]$, depends just on A_μ and $\partial_\mu \pi$. The underlying gauge invariance makes sure that they appear in the effective Lagrangian only in the combination $\mathcal{D}_\mu \pi = \partial_\mu \pi - A_\mu$. At the minimum of the effective action, the effective Lagrangian is equal to minus the energy density of the ground state in the background field A_μ (see Section 16.3 in [15]). Recalling that the energy density is equal to minus the thermodynamic pressure, $p(\mu)$, and observing that $\mu = \sqrt{A^\mu A_\mu} = \sqrt{\mathcal{D}_\mu \pi \mathcal{D}^\mu \pi}$ for a constant field π , the full dependence on $\mathcal{D}_\mu \pi$ can be restored with the knowledge of the function $p(\mu)$, that is, the equation of state,

$$\Gamma_{\text{eff}}[\mu, \pi] = \int d^4x p(\sqrt{\mathcal{D}_\mu \pi \mathcal{D}^\mu \pi}) = \int d^4x p\left(\sqrt{(\partial_0 \pi - \mu)^2 - (\nabla \pi)^2}\right) \quad (92)$$

One therefore arrives at the conclusion that to leading order in the derivative expansion, the full quantum effective action of the superfluid NG boson is completely determined by the equation of state. Since the equation of state is usually straightforward to calculate perturbatively, this is a rare example of a system where the low-energy EFT can be derived explicitly from the underlying microscopic theory.

The effective Lagrangian can be used to derive transport coefficients and other observables associated with low-energy processes in which only the NG bosons can be excited [79].

8. Conclusions

In the present paper, I have provided a self-contained introduction to spontaneous symmetry breaking and the properties of the associated NG bosons. I discussed to some detail the mathematical subtleties, connected with the implementation of a broken symmetry on the Hilbert space of states. The focus of the paper was on nontrivial features of SSB specific to nonrelativistic systems. These include unconventional dispersion relations of NG bosons as well as improved rules for their counting. The general results were demonstrated on several, mostly exactly solvable, examples.

Due to limited space, several issues were touched only superficially, or not at all. The first of them is the possibility of explicit symmetry breaking. If weak enough, spontaneous breaking of such symmetry will give rise to a pseudo-NG boson. Its energy at zero momentum is not strictly zero, but it is still small compared to the characteristic energy scale of the system. However, the counting of NG bosons then becomes blurred, since it may not be possible to distinguish an “approximately linear” dispersion relation from an “approximately quadratic” one. Only when the explicit symmetry breaking is weak, one can still distinguish type-I and type-II NG bosons [57].

A special case of explicit symmetry breaking is represented by a wide class of theories where a part of the Lagrangian, responsible for the NG boson spectrum at the tree level or in a similar approximation, has a higher symmetry than the rest of it. This typically happens in relativistic scalar theories coupled to gauge fields [80,81], but it can also occur in some condensed matter systems such as superfluid ^3He [82,83]. In the appropriate approximation, the system then exhibits more gapless states than would correspond to the symmetry of the full theory, some of them stemming from the extended symmetry of the part of the Lagrangian. These spurious NG bosons receive a gap once quantum corrections are taken into account. This mechanism can be responsible for the presence of naturally light states in the spectrum.

Second, in this paper I have considered exclusively continuous internal symmetries. While spontaneous breaking of a discrete symmetry does not give rise to NG bosons, spontaneous breaking of spacetime symmetries is subtle. First of all, the NG field configurations can be imagined as small local fluctuations of the order parameter. Since the local versions of different spacetime transformations may coincide, there are typically fewer NG bosons than broken generators [37]. Examples of spontaneous breaking of spacetime symmetries include homogeneous but anisotropic states such as in superfluid ^3He (see [25] for an extensive review), spin-one color superconductors [32], or even imbalanced spin-zero color superconductors [84], and in Bose–Einstein condensates of relativistic vector fields [85,86], as well as inhomogeneous states such as crystalline solids [87] or superconductors with inhomogeneous pairing [88]. Accordingly, the behavior of NG bosons may be highly nontrivial. For example, in helical ferromagnets [89,90] the local magnetization field forms a spiral structure. Along the axis of the helix, the average magnetization is zero and the NG boson is type-I. In the transverse directions, the NG boson feels the uniform magnetization and is type-II like in ordinary ferromagnets.

My main motivation for writing this paper was to provide a review on spontaneous symmetry breaking that would be general enough to cover both, relativistic many-body systems as well as intrinsically

nonrelativistic systems, and thus to bridge the gap between the communities. If a reader with expertise in high energy physics discovers that there is more to SSB and NG bosons than usually presented in the textbooks on particle physics, and if a reader with background in condensed matter or atomic physics finds here a general framework for the variety of fascinating phenomena he or she is familiar with, my goal will be achieved.

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