12th International Workshop on Termination (WST 2012)

WST 2012, February 19–23, 2012, Obergurgl, Austria

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Preface

This volume contains the proceedings of the *12th International Workshop on Termination (WST 2012)*, to be held February 19–23, 2012 in Obergurgl, Austria. The goal of the Workshop on Termination is to be a venue for presentation and discussion of all topics in and around termination. In this way, the workshop tries to bridge the gaps between different communities interested and active in research in and around termination. The *12th International Workshop on Termination* in Obergurgl continues the successful workshops held in St. Andrews (1993), La Bresse (1995), Ede (1997), Dagstuhl (1999), Utrecht (2001), Valencia (2003), Aachen (2004), Seattle (2006), Paris (2007), Leipzig (2009), and Edinburgh (2010).

The *12th International Workshop on Termination* did welcome contributions on all aspects of termination and complexity analysis. Contributions from the imperative, constraint, functional, and logic programming communities, and papers investigating applications of complexity or termination (for example in program transformation or theorem proving) were particularly welcome.

We did receive 18 submissions which all were accepted. Each paper was assigned two reviewers. In addition to these 18 contributed talks, WST 2012, hosts three invited talks by Alexander Krauss, Martin Hofmann, and Fausto Spoto.

I would like to take this opportunity to thank all those that helped to organise the current Workshop on Termination. First of let me thank the members of the program committee who provided invaluable assistance on the scientific end of the workshop. Moreover I'd like to thank the members of the organisation committee who did their best to guarantee that the actual event will be a success. Last, but not least, I'd like to thank our sponsors, namely the *Kurt Gödel Society*, and the *University of Innsbruck*, without whose assistance the workshop wouldn't haven taken place.

Innsbruck, February 17, 2012 Georg Moser

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Alexander Krauss

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This talk gives an overview on the structure of termination problems that arise in interactive theorem proving based on higher-order logic, specifically Isabelle/HOL.

I will explain what it means to justify a recursive definition in HOL, and discuss the extraction of termination proof obligations and some successful approaches to their automated proof.

Then I will focus on the higher-order case, which is not fully automated, as it requires some configuration by the user. I will discuss the state of the art and the issues that make further automation difficult.

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Amortized resource analysis

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Abstract of the Talk

Resource analysis aims at automatically determining and upper bound on the resource usage of a program as a function of its input size. Resources in this context can be runtime, heap- and stack size, number of occurrences of certain events, etc.

The amortized approach to resource analysis works by associating "credits" with elements of data structures and to "pay" for each consumption of resource from the credits currently available. In this way composite programs and programs with intermediate data structures can be analysed more conveniently than would otherwise be the case.

Amortization was introduced by Tarjan in the 70s in the context of manual complexity analyis of algorithms. More recently, it has been used for type-based automatic resource analysis.

The talk surveys the key concepts with simple examples and then moves on to survey some recent papers, notably the inference of multivariate polynomial resource bounds for functional programs and a type-based resource analysis of Java-like object-oriented programs. I will also try to say something about possible connections with term rewriting, in particular polynomial interpretations.

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Termination analysis with Julia: What are we missing?

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Abstract of the Talk

I will describe the structure and underlying theory of the termination analysis module of the Julia static analyzer. This tool is based on abstract interpretation and translates Java/Android code into CLP, whose termination is more easily proved. I will give some details about the implementation and the trade-offs between precision and efficiency. I will then present the results of analysis of a set of large programs and see concrete examples of where the tool does not prove termination. This will give us an idea of which actual problems are faced by a termination analyzer and how/if they can be solved in the future.

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On the Invariance of the Unitary Cost Model for Head Reduction[*]

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Abstract

The λ-calculus is a widely accepted computational model of higher-order functional programs, yet there is not any direct and universally accepted cost model for it. As a consequence, the computational difficulty of reducing λ-terms to their normal form is typically studied by reasoning on concrete implementation algorithms. Here, we show that when head reduction is the underlying dynamics, the unitary cost model is indeed invariant. This improves on known results, which only deal with weak (call-by-value or call-by-name) reduction. Invariance is proved by way of a *linear* calculus of explicit substitutions, which allows to nicely decompose any head reduction step in the λ-calculus into more elementary substitution steps, thus making the combinatorics of head-reduction easier to reason about. The technique is also a promising tool to attack what we see as the main open problem, namely understanding for which *normalizing* strategies the unitary cost model is invariant, if any.

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Keywords and phrases Lambda Calculus, Invariance, Cost Models

1 Introduction

Giving an estimate of the amount of time T needed to execute a program is a natural refinement of the termination problem, which only requires to decide whether T is either finite or infinite. The shift from termination to complexity analysis brings more informative outcomes at the price of an increased difficulty. In particular, complexity analysis depends much on the chosen computational model. Is it possible to express such estimates in a way which is independent from the specific machine the program is run on? An answer to this question can be given following computational complexity, which classifies functions based on the amount of time (or space) they consume when executed by *any* abstract device endowed with a *reasonable* cost model, depending on the size of input. When can a cost model be considered reasonable? The answer lies in the so-called invariance thesis [14]: any time cost model is reasonable if it is polynomially related to the (standard) one of Turing machines.

If programs are expressed as rewrite systems (e.g. as first-order TRSs), an abstract but effective way to execute programs, rewriting itself, is always available. As a consequence, a natural time cost model turns out to be *derivational complexity*, namely the (maximum) number of rewrite steps which can possibly be performed from the given term. A rewriting step, however, may not be an atomic operation, so derivational complexity is not by definition

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invariant. For first-order TRSs, however, derivational complexity has been recently shown to be an invariant cost model, by way of term graph rewriting [7, 5].

The case of λ-calculus is definitely more delicate: if β-reduction is weak, i.e., if it cannot take place in the scope of λ-abstractions, one can see λ-calculus as a TRS and get invariance by way of the already cited results [6], or by other means [12]. But if one needs to reduce "under lambdas" because the final term needs to be in normal form (e.g., when performing type checking in dependent type theories), no invariance results are known at the time of writing.

Here we give a partial solution to this problem, by showing that the unitary cost model is indeed invariant for the λ-calculus endowed with *head reduction*, in which reduction *can* take place in the scope of λ-abstractions, but *can only* be performed in head position. Our proof technique consists in implementing head reduction in a calculus of explicit substitutions.

Explicit substitutions were introduced to close the gap between the theory of λ-calculus and implementations [1]. Their rewriting theory has also been studied in depth, after Melliès showed the possibility of pathological behaviors [9]. Starting from graphical syntaxes, a new *at a distance* approach to explicit substitutions has recently been proposed [4]. The new formalisms are simpler than those of the earlier generation, and another thread of applications — to which this paper belongs — also started: new results on λ-calculus have been proved by means of explicit substitutions [4, 3].

Here, we make use of the *linear-substitution calculus* $\Lambda_{[\cdot]}$, a slight variation over a calculus of explicit substitutions introduced by Robin Milner [10]. The variation is inspired by the structural λ-calculus [4]. We study in detail the relation between λ-calculus head reduction and *linear head reduction* [8], the head reduction of $\Lambda_{[\cdot]}$, and prove that the latter can be at most quadratically longer than the former. This is proved without any termination assumption, by a detailed rewriting analysis.

To get the Invariance Theorem, however, other ingredients are required:

1. *The Subterm Property.* Linear head reduction has a property not enjoyed by head β-reduction: linear substitutions along a reduction $t \multimap^* u$ duplicates subterms of t only. It easily follows that $\multimap$-steps can be simulated by Turing machines in time polynomial in the size of t and the length of $\multimap^*$.
2. *Compact representations.* Explicit substitutions, decomposing β-reduction into more atomic steps, allow to take advantage of sharing and thus provide compact representations of terms, avoiding the exponential blowups of term size happening in plain λ-calculus. Is it reasonable to use these compact representations of λ-terms? We answer affirmatively, by exhibiting a dynamic programming algorithm for checking equality of terms with explicit substitutions modulo unfolding, and proving it to work in polynomial time in the size of the involved compact representations.
3. *Head simulation of Turing machines.* We also provide the simulation of Turing machines by λ-terms. We give a new encoding of Turing machines, since the known ones do not work with *head* β-reduction, and prove it induces a polynomial overhead.

We emphasize the result for head β-reduction, but our technical detour also proves invariance for linear head reduction. To our knowledge, we are the firsts to use the fine granularity of explicit substitutions for complexity analysis. Many calculi with bounded complexity (e.g. [13]) use `let`-constructs, an avatar of explicit substitutions, but they do not take advantage of the refined dynamics, as they always use big-steps substitution rules.

Arguably, the main contribution of this paper lies in the technique rather than in the invariance result. Indeed, the main open problem in this area, namely the invariance of the unitary cost model for any *normalizing* strategy remains open. But even if linear explicit substitutions cannot be *directly* applied to the problem, the authors strongly believe that this is anyway a promising direction, on which they are actively working at the time of writing.

2 Linear Explicit Substitutions and the Unitary Cost Model

First of all, we introduce the λ-calculus. Its terms are given by the grammar:

$$t, u, r \in \mathcal{T}_\lambda :: x \mid \mathcal{T}_\lambda\ \mathcal{T}_\lambda \mid \lambda x.\mathcal{T}_\lambda$$

and its reduction rule $\rightarrow_\beta$ is defined as the context closure of $(\lambda x.t)\ u \mapsto_\beta t\{x/u\}$. We will mainly work with head reduction, instead of full β-reduction. We define head reduction as follows. Let an *head context* $\hat{H}$ be defined by:

$$\hat{H} ::= [\cdot] \mid \hat{H}\ \mathcal{T}_\lambda \mid \lambda x.\hat{H}.$$

Then define *head reduction* $\rightarrow_{\mathtt{h}}$ as the closure by head contexts of $\mapsto_\beta$. Our definition of head reduction is slightly more liberal than the usual one, but none of its properties are lost.

The calculus of explicit substitutions we are going to use is a minor variation over a simple calculus introduced by Milner [10]. The grammar is standard:

$$t, u, r \in \mathcal{T} :: x \mid \mathcal{T}\ \mathcal{T} \mid \lambda x.\mathcal{T} \mid \mathcal{T}[x/\mathcal{T}].$$

The term $t[x/u]$ is an *explicit substitution* and binds x in t. Given a term t with explicit substitutions, its *unfolding* is the λ-term without explicit substitutions defined as follows:

$$x{\downarrow} := x{\downarrow} \qquad (t\ u){\downarrow} := t{\downarrow}\ u{\downarrow} \qquad (\lambda x.t){\downarrow} := \lambda x.t{\downarrow} \qquad (t[x/u]){\downarrow} := t{\downarrow}\{x/u{\downarrow}\}.$$

Head contexts are defined by the following grammar:

$$H ::= [\cdot] \mid H\ \mathcal{T} \mid \lambda x.H \mid H[x/\mathcal{T}].$$

We define *head linear reduction* $\multimap$ as $\multimap_{\mathtt{dB}} \cup \multimap_{\mathtt{ls}}$, where $\multimap_{\mathtt{dB}}$ and $\multimap_{\mathtt{ls}}$ are the closure by head contexts of:

$$(\lambda x.t)\mathtt{L}\ u \quad \mapsto_{\mathtt{dB}} \quad t[x/u]\mathtt{L} \qquad\qquad H[x][x/u] \quad \mapsto_{\mathtt{ls}} \quad H[u][x/u]$$

The key property of linear head reduction is the Subterm Property. A term u is a *box-subterm* of a term t if t has a subterm of the form $r\ u$ or of the form $r[x/u]$ for some r.

▶ **Lemma 1** (Subterm Property). *If $t \multimap^* u$ and r is a box-suterm of u, then r is a box-subterm of t.*

Linear head substitution steps duplicate sub-terms, but the Subterm Property guarantees that only sub-terms of the initial term t are duplicated, and thus each step can be implemented in time polynomial in the size of t, which is the size of the input, the fundamental parameter for complexity analysis. This is in sharp contrast with what happens in the λ-calculus, where the cost of a β-reduction step is not even polynomially related to the size of the initial term.

The subterm property does not only concern the cost of implementing reduction steps, but also the size of intermediate terms:

▶ **Corollary 2.** *There is a polynomial $p : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that if $t \multimap^k u$ then $|u| \leq p(k, |t|)$.*

From a rewriting analysis of head reduction and linear head reduction we get the following:

1. Any $\multimap$-reduction ρ projects via unfolding to a $\rightarrow_{\mathtt{h}}$-reduction $\rho\downarrow$ having as length exactly the number of $\multimap_{\mathtt{dB}}$ steps in ρ;
2. Any $\rightarrow_{\mathtt{h}}$-reduction ρ can be simulated by a $\multimap$-reduction having as many $\multimap_{\mathtt{dB}}$-steps as the the steps in ρ, followed by unfolding;

Moreover, by means of a simple measure and the subterm property we prove:

▶ **Theorem 3.** *Let $t \in \mathcal{T}_\lambda$. If $\rho : t \multimap^n u$ then $n = O(|\rho|^2_{\mathtt{dB}})$, where $|\rho|_{\mathtt{dB}}$ is the number of $\multimap_{\mathtt{dB}}$-steps in ρ.*

From the theorem and the previous two points there is a quadratic — and thus polynomial — relation between $\rightarrow_{\mathtt{h}}$-reductions and $\multimap$-reduction from a given term. Therefore, we get:

▶ **Corollary 4** (Invariance, Part I)**.** *There is a polynomial time algorithm that, given $t \in \mathcal{T}_\lambda$, computes a term u such that $u\downarrow = r$ if t has $\rightarrow_{\mathtt{h}}$-normal form r and diverges if u has no $\rightarrow_{\mathtt{h}}$-normal form. Moreover, the algorithm works in polynomial time on the derivation complexity of the input term.*

One may now wonder why a result like Corollary 4 cannot be generalized to, e.g., leftmost-outermost reduction, which is a normalizing strategy. Actually, linear explicit substitutions *can* be endowed with a notion of reduction by levels capable of simulating the leftmost-outermost strategy in the same sense as linear head-reduction simulates head-reduction here. And, noticeably, the subterm property *continues* to hold. What is not true anymore, however, is the quadratic bound we have proved in this section: in the leftmost-outermost strategy, one needs to perform *too many* substitutions not related to any β-redex. If one wants to generalize Corollary 4, in other words, one needs to further optimize the substitution process. But this is outside the scope of this paper.

One may also wonder whether explicit substitutions are nothing more than a way to *hide* the complexity of the problem under the carpet of compactness: what if we want to get the normal form in the *usual*, *explicit* form? Consider the sequence of λ-terms defined as follows, by induction on a natural number n (where u is the lambda term yxx): $t_0 = u$ and for every $n \in \mathbb{N}$, $t_{n+1} = (\lambda x.t_n)u$. t_n has size linear in n, and t_n rewrites to its normal form r_n in exactly n steps by head reduction strategy:

$$\begin{aligned}
&t_0 \equiv u \equiv r_0\\
&t_1 \rightarrow yuu \equiv yr_0yr_0 \equiv r_1\\
&t_2 \rightarrow (\lambda x.t_0)(yuu) \equiv (\lambda x.u)(r_1) \rightarrow yr_1r_1 \equiv r_2\\
&\quad\vdots
\end{aligned}$$

For every n, however, r_{n+1} contains two copies of r_n, hence the size of r_n is *exponential* in n. As a consequence, if we stick to the head strategy and if we insist on normal forms to be represented explicitly, without taking advantage of the shared representation provided by explicit substitutions, the number of head steps *is not* an invariant cost model: in a linear number of steps we reach an object which cannot even be written down in polynomial time.

This phenomenon is due to the λ-calculus being a very inefficient way to represent λ-terms. Explicit substitutions represent normal forms compactly, avoiding the exponential blow-up. We prove that this compact representation is reasonable in the following sense: even if computing the unfolding of a term $t \in \Lambda_{[\cdot]}$ takes exponential time, *comparing* the unfoldings of two terms $t, u \in \Lambda_{[\cdot]}$ for equality can be done in polynomial time (details in [2]). This way, linear explicit substitutions are proved to be a succint, acceptable, encoding of λ-terms in the sense of Papadimitriou [11]. The algorithm which compares the unfoldings is based on dynamic programming: for every subterm of t (resp. u) it computes its unfolding with respect to the substitutions in t (resp. u) and compare it with the unfoldings of the subterms of the other term. This can be done without really computing those unfoldings (which would require exponential space and time).

We address also the converse relation between Turing Machines and λ-calculus, by giving a new encoding of Turing Machines into the λ-calculus (details in [2]). The transitions of Turing Machines are simulated by head reduction in such a way that the running time of the machine is polynomially related to the length of the head reduction of the encoding term. The encoding is along the lines of existing representations of Turing Machines into λ-calculus, except that 1) natural numbers are represented via Scott numerals (instead of Church numerals), which are a better representation when evaluation is given by head reduction, and 2) The encoding is in continuation-passing style. The following theorem completes our invariance result:

▶ **Theorem 5** (Invariance, Part II). *Let Δ be an alphabet. If $f : \Delta^* \to \Delta^*$ is computed by a Turing machine $\mathcal{M}$ in time g, then there is a term $\mathcal{U}(\mathcal{M}, \Delta)$ such that for every $u \in \Delta^*$, $\mathcal{U}(\mathcal{M}, \Delta)\lceil u \rceil^{\Delta^*} \to_{\mathtt{h}}^{n} \lceil f(u) \rceil^{\Delta^*}$ where $n = \mathcal{O}(g(|u|) + |u|)$.*

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On a Correspondence between Predicative Recursion and Register Machines*

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Abstract

We present the small polynomial path order sPOP*. Based on sPOP*, we study a class of rewrite systems, dubbed systems of predicative recursion of degree d, such that for rewrite systems in this class we obtain that the runtime complexity lies in $O(n^d)$. We show that predicative recursive rewrite systems of degree d define functions computable on a register machine in time $O(n^d)$.

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Keywords and phrases Runtime Complexity, Polynomial Time Functions, Implicit Computational Complexity, Rewriting

1 Introduction

In [1] we propose the *small polynomial path order* (*sPOP** for short). The order sPOP* provides a characterisation of the class of *polynomial time computable function* (*polytime computable functions* for short) via term rewrite systems. Any polytime computable function gives rise to a rewrite system that is compatible with sPOP*. On the other hand any function defined by a rewrite system compatible with sPOP* is polytime computable. The proposed order embodies the principle of *predicative recursion* as proposed by Bellantoni and Cook [4]. Our result bridges the subject of (automated) complexity analysis of rewrite systems and the field of implicit computational complexity (*ICC* for short).

Based on sPOP*, one can delineate a class of rewrite systems, dubbed systems of *predicative recursion of degree d*, such that for rewrite systems in this class we obtain that the runtime complexity lies in $O(n^d)$. This is a tight characterisation in the sense that one can provide a family of systems of predicative recursion of depth d, such that their runtime complexity is bounded from below by $\Omega(n^d)$ [1]. In this note, we study the connection between functions f defined by predicative recursive term rewrite systems (TRSs) of degree d and register machines. We show that any such function can be computed by a register machine operating in time $O(n^d)$. This result further emphasises the fact that the runtime complexity of a TRS (cf. [7]) is an invariant cost model [3]. Our work was essentially motivated by Leivant's work on *predicative recurrence* [8] and Marion's *strict ramified primitive recursion* [10].

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Let $\mathcal{R}$ be a TRS and fix a (quasi)-precedence $\succcurlyeq := \succ \uplus \sim$ on the symbols of $\mathcal{R}$. We are assuming that the arguments of every function symbol are partitioned in to normal and safe ones. Notationally we write $f(t_1,\dots,t_k\,;t_{k+1},\dots,t_{k+l})$ with *normal* arguments $t_1,\dots,t_k$ separated from *safe* arguments $t_{k+1},\dots,t_{k+l}$ by a semicolon. We define the equivalence $\sim_{\mathsf{s}}$ on terms respecting this separation as follows: $s \sim_{\mathsf{s}} t$ holds if $s = t$ or $s = f(s_1,\dots,s_k\,;s_{k+1},\dots,s_{k+l})$ and $t = g(t_1,\dots,t_k\,;t_{k+1},\dots,t_{k+l})$ where $f \sim g$ and $s_i \sim_{\mathsf{s}} t_{\pi(i)}$ for all $i = 1,\dots,k+l$ such that the permutation π on $\{1,\dots,k+l\}$ maps normal to normal argument positions. We write $s \rhd_{\mathsf{n}} t$ if t is a proper subterm of s (modulo $\sim_{\mathsf{s}}$) at a normal argument position: $f(s_1,\dots,s_k\,;s_{k+1},\dots,s_{k+l}) \rhd_{\mathsf{n}} t$ if $s_i \trianglerighteq \cdot \sim_{\mathsf{s}} t$ and $i \in \{1,\dots,k\}$.

The following definition introduces small polynomial path orders $>_{\mathsf{spop}*}$. The order allows recursive definitions only on *recursive symbols* $\mathcal{D}_{\mathsf{rec}} \subseteq \mathcal{D}$. Symbols in $\mathcal{D} \setminus \mathcal{D}_{\mathsf{rec}}$ are called *compositional* and denoted by $\mathcal{D}_{\mathsf{comp}}$. To retain the separation under $\sim_{\mathsf{s}}$, we require $\sim \subseteq \mathcal{C}^2 \cup \mathcal{D}^2_{\mathsf{rec}} \cup \mathcal{D}^2_{\mathsf{comp}}$. We set $\geqslant_{\mathsf{spop}*} := \sim_{\mathsf{s}} \cup >_{\mathsf{spop}*}$ and also write $>_{\mathsf{spop}*}$ for the product extension of $>_{\mathsf{spop}*}$ to tuples $\vec{s} = \langle s_1,\dots,s_n\rangle$ and $\vec{t} = \langle t_1,\dots,t_n\rangle$: $\vec{s} \geqslant_{\mathsf{spop}*} \vec{t}$ holds if $s_i \geqslant_{\mathsf{spop}*} t_i$ for all $i = 1,\dots,n$ and $\vec{s} >_{\mathsf{spop}*} \vec{t}$ holds if additionally $s_{i_0} >_{\mathsf{spop}*} t_{i_0}$ for some $i_0 \in \{1,\dots,n\}$. We denote by $\mathcal{T}(\mathcal{F}^{\prec f},\mathcal{V})$ the set of terms build from variables and function symbols $\mathcal{F}^{\prec f} := \{g \mid f \succ g\}$.

▶ **Definition 1.1.** Let $s = f(s_1,\dots,s_k\,;s_{k+1},\dots,s_{k+l})$. Then $s >_{\mathsf{spop}*} t$ if either

1) $s_i \geqslant_{\mathsf{spop}*} t$ for some argument s_i of s.
2) $f \in \mathcal{D}$, $t = g(t_1,\dots,t_m\,;t_{m+1},\dots,t_{m+n})$ with $f \succ g$ and the following conditions hold: (i) $s \rhd_{\mathsf{n}} t_j$ for all normal arguments t_j of t, (ii) $s >_{\mathsf{spop}*} t_j$ for all safe arguments t_j of t, and (iii) $t_j \notin \mathcal{T}(\mathcal{F}^{\prec f},\mathcal{V})$ for at most one $j \in \{1,\dots,k+l\}$.
3) $f \in \mathcal{D}_{\mathsf{rec}}$, $t = g(t_1,\dots,t_k\,;t_{k+1},\dots,t_{k+l})$ with $f \sim g$ and the following conditions hold: (i) $\langle s_1,\dots,s_k\rangle >_{\mathsf{spop}*} \langle t_{\pi(1)},\dots,t_{\pi(k)}\rangle$ for some permutation π, (ii) $\langle s_{k+1},\dots,s_{k+l}\rangle \geqslant_{\mathsf{spop}*} \langle t_{\tau(k+1)},\dots,t_{\tau(k+l)}\rangle$ for some permutation τ.

The *depth of recursion* $\mathsf{rd}(f)$ is inductively defined in correspondence to the rank of f in $\succcurlyeq$, but only takes recursive symbols into account: Let $n = \max\ \{0\} \cup \{\mathsf{rd}(g) \mid f \succ g\}$. Then $\mathsf{rd}(f) := 1 + n$ if $f \in \mathcal{D}_{\mathsf{rec}}$ and otherwise $\mathsf{rd}(f) := n$. We say a constructor TRS $\mathcal{R}$ is *predicative recursive of degree* d if $\mathcal{R}$ is compatible with an instance $>_{\mathsf{spop}*}$ and the maximal depth of recursion of a function symbol in $\mathcal{R}$ is d.

▶ **Theorem 1.2** ([1]). *Let $\mathcal{R}$ be predicative recursive of degree d. Then the innermost runtime complexity of $\mathcal{R}$ lies in $O(n^d)$. Moreover, this bound is tight.*

As one anonymous reviewer points out, Theorem 1.2 also holds with respect to full rewriting, if $\mathcal{R}$ is in addition a non-duplicating overlay system [6].

2 Register Machines Compute Predicative TRSs

Let $\mathbb{W}$ denote the set of *words* over a *binary* alphabet. Fix a predicative constructor TRS $\mathcal{R}$ of degree d that *computes* functions over $\mathbb{W}$. We will now show that the functions computed by $\mathcal{R}$ can be realised on a *register machine* (*RM*) [5], operating in time asymptotic to n^d where n is the size of the input.

First we make precise the notion of computation on TRSs. We assume that the encoding of words $\mathbb{W}$ as terms makes use of dyadic successors s_0 and s_1 that append the corresponding character to its argument, as well as the constant ϵ to construct the empty word. Henceforth we set $\mathcal{C} := \{\mathsf{s}_0,\mathsf{s}_1,\epsilon\}$ and by the one-to-one correspondence between ground constructor terms and binary words $\mathbb{W}$ we allow ourselves to confuse these sets. Let f be a defined symbol in $\mathcal{R}$ of arity k. Then $\mathcal{R}$ computes the function $f : \mathbb{W}^k \to \mathbb{W}$ defined as $f(w_1, \dots, w_k) = w$ if $\mathsf{f}(w_1, \dots, w_k) \to^!_{\mathcal{R}} w$. This notion is well-defined if $\mathcal{R}$ is orthogonal (hence confluent) and completely defined, i.e., normal forms and constructor terms coincide.

A RM over $\mathbb{W}$ contains a finite set of registers $\mathsf{R} = \{x_1, \dots, x_n\}$ that store words over $\mathbb{W}$. We use the notion of RM from [5] adapted from $\mathbb{N}$ to binary words $\mathbb{W}$ and identify RMs with goto-programs over variables R that allow to (i) copy (the content of) one variable to another, (ii) appending $0, 1$, or removing the last bit of a variable, and (iii) that can perform conditional branches based on the last bit of a variable. A RM *computes* the function $f : \mathbb{W}^k \to \mathbb{W}$ with $k \leqslant n$ defined as follows: $f(w_1, \dots, w_k) = w$ if on initial assignment w_i to x_i for all $i = 1, \dots, k$ and ε to x_i for all $i = k+1, \dots, n$, the associated goto-program halts and the content of a dedicated output-variable x_o equals w. The complexity of an RM is given by the number of executed instructions as function in the sum of sizes of the input.

To simplify matters, we *normalise* right-hand sides of rewrite rules. Throughout the following, we denote by $\vec{u}, \vec{v}, \vec{w}$, possibly extended by subscripts, vectors of constructor terms. Let $\mathcal{R}_{\mathsf{n}}$ denote some fixpoint on $\mathcal{R}$ of following *normalisation* operator: if the TRS contains a rule $\mathsf{f}(\vec{u_f} \,; \vec{v_f}) \to \mathsf{g}(\vec{u_g} \,; \vec{t_1}, \mathsf{h}(\vec{u_h} \,; \vec{t_2}), \vec{t_3})$ where $\mathsf{f} \succ \mathsf{h}$, $\mathsf{h} \in \mathcal{D}$ and $\vec{t_1}, \vec{t_2}$ or $\vec{t_3}$ contain at least one defined symbol, replace the rule with $\mathsf{f}(\vec{u_f} \,; \vec{v_f}) \to \mathsf{g}'(\vec{u_f} \,; \vec{v_f}, \vec{t_1}, \vec{t_2}, \vec{t_3})$ and $\mathsf{g}'(\vec{u_f} \,; \vec{v_f}, \vec{x_1}, \vec{x_2}, \vec{x_3}) \to \mathsf{g}(\vec{u_g} \,; \vec{x_1}, \mathsf{h}(\vec{u_h} \,; \vec{x_2}), \vec{x_3})$. Here g' is a fresh composition symbol so that $\mathsf{f} \succ \mathsf{g}' \succ \mathsf{g}, \mathsf{h}$ and variables $\vec{x_1}, \vec{x_2}, \vec{x_3}$ do not occur elsewhere. Note that $\mathcal{R}_{\mathsf{n}}$ is well-defined as in each step the number of defined symbols in right-hand sides are decreasing.

▶ **Lemma 2.1.** *We have (i) $\to_{\mathcal{R}} \subseteq \to^+_{\mathcal{R}_{\mathsf{n}}}$ and (ii) $\mathcal{R}_{\mathsf{n}}$ is predicative recursive of degree d.*

By Property (i) it is easy to verify that any function computed by $\mathcal{R}$ is also computed by $\mathcal{R}_{\mathsf{n}}$. Property (ii) and the definition of $\mathcal{R}_{\mathsf{n}}$ allows the classification of each $\mathsf{f}(\vec{u_l} \,; \vec{v_l}) \to r \in \mathcal{R}_{\mathsf{n}}$ into one of the following forms.

- *Construction Rule*: r is a constructor term;
- *Recursion Rule*: $r = \mathsf{g}(\vec{u_g} \,; \vec{v_g}, \mathsf{f}'(\vec{u_r} \,; \vec{v_r}), \vec{w_g})$ where $\mathsf{f} \succ \mathsf{g}$ and $\mathsf{f} \sim \mathsf{f}'$;
- *Composition Rule*: $r = \mathsf{g}(\vec{u_g} \,; \vec{v_g}, \mathsf{h}(\vec{u_r} \,; \vec{v_r}), \vec{w_g})$ where $\mathsf{f} \succ \mathsf{g}, \mathsf{h}$.

In the latter two cases the context $\mathsf{g}(\vec{u_g} \,; \vec{v_g}, \Box, \vec{w_g})$ might also be missing. Note that for recursion rules, the sum of sizes of $\vec{u_l}$ is strictly greater than the sum of the sizes of $\vec{u_r}$.

▶ **Theorem 2.2.** *Let $\mathcal{R}$ be an orthogonal and completely defined predicative system of degree d. Every function f computed by $\mathcal{R}$ is computed by a register machine RM_f operating in time $\mathsf{O}(n^d)$, where n refers to the sum of the sizes of* normal *arguments.*

Proof. Consider a function f computed by $\mathcal{R}$, and let f be the corresponding defined symbol. We define a program P_{f} which, on *input variables* $\vec{I_f}$ initialised with $\vec{v}$, computes $f(\vec{v})$ in a dedicated *output variable* O_f, executing no more than $\mathsf{O}(n^{\mathsf{rd}(\mathsf{f})})$ instructions. The program P_{f} works by reduction according to the *normalised* TRS $\mathcal{R}_{\mathsf{n}}$. For this note that $\mathcal{R}_{\mathsf{n}}$ is orthogonal, hence Lemma 2.1 (1) gives that $\mathcal{R}_{\mathsf{n}}$ reduces $\mathsf{f}(\vec{v})$ to $f(\vec{v})$ independent on the evaluation strategy. The construction is by induction on the rank f in $\succcurlyeq$ (on the extended signature of $\mathcal{R}_{\mathsf{n}}$). We only consider the more involved inductive step. By induction hypothesis for each g below f in the precedence there exist a program P_{g} that compute the function defined by g operating in time $\mathsf{O}(n^{\mathsf{rd}(\mathsf{g})})$, where n is the sum of sizes of normal arguments to g.

Suppose the input variables $\vec{I_f}$ hold the arguments $\vec{v}$. Due to linearity, pattern matches can be hard-coded by looking at suffixes in $\vec{I_f}$ bounded in size by a constant. Consequently in a constant number of steps P_{f} can check which rules applies on $\mathsf{f}(\vec{v})$. First suppose $\mathsf{f} \in \mathcal{D}_{\mathsf{comp}}$, thus $\mathsf{f}(\vec{u})$ reduces either using a construction or composition rule.

The interesting case is when $\mathsf{f}(\vec{u}) \xrightarrow{\mathsf{i}}_{\mathcal{R}_{\mathsf{n}}} \mathsf{g}(\vec{v_1}, \mathsf{h}(\vec{w}), \vec{v_2})$ due to a composition rule. Since $\mathsf{f} \succ \mathsf{g}, \mathsf{h}$, induction hypothesis gives programs P_{g} and P_{h} that compute the functions defined by g and h respectively. The program P_{f} first stores the arguments to h in the dedicated input registers $\vec{I_h}$ and executes the code of P_{h}. Since $\vec{w}$ are constructor terms, initialisation of $\vec{I_h}$ requires only constant time similar to above. Further the sum of sizes of normal inputs in $\vec{u}$ and $\vec{w}$ differ only by a constant factor c_1, hence executing P_{h} takes time $\mathsf{O}((c_1 \cdot n)^{\mathsf{rd}(\mathsf{h})}) = \mathsf{O}(n^{\mathsf{rd}(\mathsf{h})})$. We repeat the procedure using a program P_{g} in time $\mathsf{O}(n^{\mathsf{rd}(\mathsf{g})})$. Here we employ that due to separation of safe and normal arguments, the complexity of computing the call of g does not depend on the result of $\mathsf{h}(\vec{w})$. Overall, employing $\mathsf{rd}(\mathsf{f}) \geqslant \mathsf{rd}(\mathsf{g}), \mathsf{rd}(\mathsf{h})$, the runtime is in $\mathsf{O}(n^{\mathsf{rd}(\mathsf{h})} + n^{\mathsf{rd}(\mathsf{g})}) \subseteq \mathsf{O}(n^{\mathsf{rd}(\mathsf{f})})$.

Now suppose $\mathsf{f} \in \mathcal{D}_{\mathsf{rec}}$ and thus $\mathsf{rd}(\mathsf{f}) \geqslant 1$. Consider an innermost reduction of f. Wlog

$$\mathsf{f}(\vec{v}) = \mathsf{f}_0(\vec{v_0}) \xrightarrow{\mathsf{i}}_{\mathcal{R}_n} \mathsf{g}_1(\vec{u_1}, \mathsf{f}_1(\vec{v_1}), \vec{w_1}) \xrightarrow{\mathsf{i}}_{\mathcal{R}} \mathsf{g}_1(\vec{u_1}, \mathsf{g}_2(\vec{u_2}, \dots, \mathsf{g}_{\mathsf{k}}(\vec{u_k}, \mathsf{f}_{\mathsf{k}}(\vec{v_k}), \vec{w_k}), \dots, \vec{w_2}), \vec{w_1})$$

where the first k applications follow from applying recursive rules, and $\mathsf{f}_{\mathsf{k}}(\vec{v_k})$ matches either a construction or composition rule. By definition the sum of sizes of normal arguments in the *recursion arguments* $\vec{v_0}, \dots, \vec{v_k}$ is strictly decreasing, and conclusively k is bounded by n. To compute $f(\vec{v})$, the program P_{f} evaluates the last term inside out, starting from $\mathsf{f}_{\mathsf{k}}(\vec{v_k})$. Since we have only a constant number of registers at our disposal, we cannot program the machine to memorise or recompute all recursion arguments $\vec{v_0}, \dots, \vec{v_k}$ in time linear in n. Instead, we employ per argument position of f an additional register and exploit the following one-to-one correspondence between arguments $\vec{v_{i+1}}$ and $\vec{v_i}$: $\vec{v_{i+1}}$ is obtained from $\vec{v_i}$ by flipping and chopping a constant number of bits according to the rewrite rule applied in step i. For $i = 0, \dots, n$, the machine performs this operation on the input registers storing $\vec{v_i}$, pushing the chopped bits onto the corresponding auxiliary registers in constant time. To recall the rule applied in step i, we associate each rule with a binary number of fixed size, and push this number on an additional register that we abuse as a *call stack*. Since $\vec{v_{i+1}}$ is obtained from $\vec{v_i}$ by executing a constant number of instructions, $\vec{v_k}$ is constructed in time $k = \mathsf{O}(n)$, allowing stepwise reconstruction of recursion arguments starting from $\vec{v_k}$.

Recall that the sum of sizes of normal recursion arguments $\vec{v_i}$ $(i = 1, \dots, k)$ is decreasing and consequently bounded by n. Consider the evaluation of $\mathsf{f}_{\mathsf{k}}(\vec{v_k})$ that reduces by construction either using a composition or projection rule. In both cases we conclude that $\mathsf{f}_{\mathsf{k}}(\vec{v_k})$ is computed in time $\mathsf{O}(n^{\mathsf{rd}(\mathsf{f})-1})$ as in the case $\mathsf{f} \in \mathcal{D}_{\mathsf{comp}}$, employing $\mathsf{rd}(\mathsf{f}) > \mathsf{rd}(\mathsf{g})$ for all g such that $\mathsf{f} \succ \mathsf{g}$. The evaluation is then continued inside out exactly as in the case $\mathsf{f} \in \mathcal{D}_{\mathsf{comp}}$, recovering the arguments $\vec{v_i}$ from $\vec{v_{i+1}}$ after each step in constant time. Employing $\mathsf{rd}(\mathsf{f}) > \mathsf{rd}(\mathsf{g_i})$ we see that the application of $\mathsf{g_i}$ is bounded by $\mathsf{O}(n^{\mathsf{rd}(\mathsf{g})}) \subseteq \mathsf{O}(n^{\mathsf{rd}(\mathsf{f})-1})$. Overall, employing $k = \mathsf{O}(n)$, the procedure stops after executing at most $\mathsf{O}(n) + \mathsf{O}(n \cdot n^{\mathsf{rd}(\mathsf{f})-1}) = \mathsf{O}(n^{\mathsf{rd}(\mathsf{f})})$ instructions. This concludes the final case. ◀

3 Experimental Results

We have implemented sPOP* in the *Tyrolean Complexity Tool* $\mathsf{T_CT}$[1]. In Table 1 we contrast sPOP* to its predecessors *lightweight multiset path orders* (LMPO for short) [9] and *polynomial path orders* [2] (POP* for short)[2]. LMPO characterises the class of polytime computable functions, also by embodying the principle of predicative recursion. Since LMPO allows simultaneous recursion it fails at binding the runtime complexity polynomially. POP* characterises predicative recursive systems but cannot give a precise bound on the runtime

[1] $\mathsf{T_CT}$ is open source and available from `http://cl-informatik.uibk.ac.at/software/tct`.
[2] See `http://cl-informatik.uibk.ac.at/software/tct/experiments/wst2012` for full experimental evidence and explanation on the setup.

complexity. Finally we also included *multiset path orders* (MPO for short) in the table, as all mentioned orders are essentially syntactic restrictions of MPO.

Comparing LMPO and MPO, the experiments reveal that enforcing predicative recursion limits the power of our techniques by roughly one fourth on our testbed. Comparing POP* with sPOP* we see an increase in precision accompanied with only a minor decrease in power. Of the four systems that can be handled by POP* but not by sPOP*, two fail to be oriented because sPOP* weakens the multiset status to product status, and two fail because sPOP* enforces a more restrictive composition scheme.

bound	MPO	LMPO	POP*	sPOP*
$\mathsf{O}(1)$				9\0.06
$\mathsf{O}(n^1)$				32\0.07
$\mathsf{O}(n^2)$				38\0.09
$\mathsf{O}(n^3)$				39\0.20
$\mathsf{O}(n^k)$			43\0.05	39\0.20
yes	76\0.09	57\0.05	43\0.05	39\0.07
maybe	681\0.16	700\0.11	714\0.11	718\0.11

Figure 1 Number of oriented problems and average execution time in seconds.

4 Conclusion and Future Work

We have shown that predicative TRSs of recursion depth d can be computed by RMs operating in time $O(n^d)$. One question that remains open is the reverse direction on the correspondence between RMs and predicative TRSs. Using a pairing constructor for collecting the contents of the registers, the simulation of $\mathsf{O}(n^d)$ time-bounded RMs is straight forward to define using recursion up to depth d. Without such a constructor however, the proof gets significantly more involved. Still, we are sufficiently convinced of our argument to conjecture that also the reverse direction on the correspondence between RMs and predicative TRSs holds. More precisely, suppose f is computable by a RM in time $\mathsf{O}(n^d)$. Then there exists a predicative recursive TRS $\mathcal{R}$ of degree d that computes f. In future work we also want to investigate whether we can weaken the assumptions in Theorem 2.2 so that compatibility with sPOP* is no longer required.

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Higher-Order Interpretations and Program Complexity*

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Abstract

Polynomial interpretations and their generalizations like quasi-interpretations have been used in the setting of first-order functional languages to design criteria ensuring statically some complexity bounds on programs [3]. This fits in the area of implicit computational complexity, which aims at giving machine-free characterizations of complexity classes. Here we extend this approach to the higher-order setting. For that we consider the notion of simply typed term rewriting systems [8], we define higher-order polynomial interpretations (HOPI) for them and give a criterion based on HOPIs to ensure that a program can be executed in polynomial time. In order to obtain a criterion which is flexible enough to validate some interesting programs using higher-order primitives, we introduce a notion of polynomial quasi-interpretations, coupled with a simple termination criterion based on linear types and path-like orders.

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1 Introduction

The problem of statically analyzing the performance of programs can be attacked in many different ways. One of them consists in inferring *complexity* properties of programs early in development cycle, when the latter are still expressed in high-level programming languages, like functional or object oriented idioms. And in this scenario, results from an area known as *implicit* computational complexity (ICC in the following) can be useful: they consist in characterizations of complexity classes in terms of paradigmatic programming languages (λ-calculus, term rewriting systems, etc.) or logical systems (proof-nets, natural deduction, etc.), from which static analysis methodologies can be distilled. Examples are type systems, path-orderings and variations on the interpretation method. The challenge here is defining ICC systems which are not only simple, but also intensionally powerful: many natural programs among those with bounded complexity, should be recognized as such by the ICC system, i.e., are actually programs *of* the system.

One of the most fertile direction in ICC is indeed the one in which programs are term rewriting systems (TRS in the following) [3, 4], whose complexity can be kept under control by way of variations of the powerful techniques developed to check termination of TRS, namely path orderings [7, 5], dependency pairs and the interpretation method [6]. Many

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different complexity classes have been characterized this way, from polynomial time to polynomial space, to exponential time to logarithmic space. And remarkably, many of the introduced characterizations are intensionally very powerful, in particular when the interpretation method is relaxed and coupled with recursive path orderings, like in quasi-interpretations [4].

Here, we consider one of the simplest higher-order generalizations of TRSs, namely Yamada's simply-typed term rewriting systems (STTRSs in the following), we define a system of higher-order polynomial interpretations for them and prove that, following [3], this allows to exactly characterize, among others, the class of polynomial time computable functions. An extended version of this paper is available [2] which includes all proofs, together with a description of how the proposed approach can be adapted to quasi-interpretations, in the style of [4].

2 Simply-Typed Term Rewriting Systems

We recall here the definition of a STTRS, following [8, 1]. We will actually consider as *programs* a subclass of STTRSs, basically those where rules only deal with the particular case of a function symbol applied to a sequence of *patterns*. For first-order rewrite systems this corresponds to the notion of *constructor rewrite system*.

We consider a denumerable set of base types, which we call *data-types* and we shall denote as D or E. *Types* are defined by the following grammar:

$$A, B ::= D \;\big|\; A_1 \times \cdots \times A_n \to A.$$

A *functional type* is a type which contains an occurrence of $\to$. Some examples of base types are the type NAT of tally integers, and the type W_2 of binary words.

We denote by $\mathcal{F}$ the set of *function symbols* (or just *functions*), $\mathcal{C}$ that of *constructors* and $\mathcal{X}$ that of *variables*. Constructors $\mathbf{c} \in \mathcal{C}$ have a type of the form $D_1 \times \cdots \times D_n \to D$. Functions $\mathtt{f} \in \mathcal{F}$, on the other hand, can have any functional type. Variables $x \in \mathcal{X}$ can have any type. *Terms* are typed and defined by the following grammar:

$$t, t_i := x^A \;\big|\; \mathbf{c}^A \;\big|\; \mathtt{f}^A \;\big|\; (t^{A_1 \times \cdots \times A_n \to A}\; t_1^{A_1} \ldots t_n^{A_n})^A$$

where $x^A \in \mathcal{X}$, $\mathbf{c}^A \in \mathcal{C}$, $\mathtt{f}^A \in \mathcal{F}$. We denote by $\mathcal{T}$ the set of terms. Observe how application is primitive and is in general treated differently from other function symbols. This is what make STTRSs different from ordinary TRSs.

We define the size $|t|$ of a term t as the number of symbols (elements of $\mathcal{F} \cup \mathcal{C} \cup \mathcal{X}$) it contains. To simplify the writing of terms we will often elide their type. We will also write $(t\ \overline{s})$ for $(t\ s_1 \ldots s_n)$. Therefore any term t is of the form $(\ldots((\alpha\ \overline{s_1})\ \overline{s_2})\ldots\overline{s_k})$ where $k \geq 0$, $\alpha \in \mathcal{X} \cup \mathcal{C} \cup \mathcal{F}$. Moreover, we will use the following convention: any term t is of the form $(\ldots((s\ \overline{s_1})\ \overline{s_2})\ldots\overline{s_k})$ will be written $((s\ \overline{s_1}\ \ldots\overline{s_k}))$. A crucial class of terms are patterns, which in particular are used in defining rewriting rules. Formally, a *pattern* is a term generated by the following grammar:

$$p, p_i := x^A \;\big|\; (\mathbf{c}^{D_1 \times \ldots \times D_n \to D}\; p_1^{D_1} \ldots p_n^{D_n}).$$

$\mathcal{P}$ is the set of all patterns. Observe that if a pattern has a functional type then it must be a variable. We consider *rewriting rules* in the form $t \to s$ satisfying the following two constraints:

1. t and s are terms of the same type A, $FV(s) \subseteq FV(t)$, and any variable appears at most once in t.

2. t must have the form $((\mathtt{f}\ \overline{p_1} \dots \overline{p_k}))$ where each $\overline{p_i}$ for $i \in 1, \dots, k$ consists of patterns only. The rule is said to be a rule *defining* $\mathtt{f}$, while the total number of patterns in $\overline{p_1}, \dots, \overline{p_k}$ is the *arity* of the rule.

Now, a *simply-typed term rewriting system* is a set R of orthogonal rewriting rules such that for every function symbol $\mathtt{f}$, every rule defining $\mathtt{f}$ has the same arity, which is said to be the arity *of* $\mathtt{f}$.

We consider call-by-value reduction of STTRSs, i.e, only values will be passed as arguments to functions. Formally, we say that a term is a *value* if either:

1. it has type D and is in the form $(\mathbf{c}\ v_1 \dots v_n)$, where $v_1, \dots, v_n$ are themselves values.
2. it has functional type and is of the form $((\mathtt{f}, \overline{v_1}\ \dots\ \overline{v_n}))$, where the terms in $\overline{v_1}, \dots \overline{v_n}$ are themselves values *and* the total number of terms in $\overline{v_1}, \dots, \overline{v_n}$ is strictly smaller than the arity of $\mathtt{f}$.

We denote values as v, u and their set by $\mathcal{V}$.

3 Higher-Order Polynomial Interpretations

We want to demonstrate how first-order rewriting-based techniques for ICC can be adapted to the higher-order setting. Our goal is to devise criteria ensuring a complexity bound on programs of *first-order types* but using subprograms of *higher-order types*. A typical application will be to find out under which conditions a higher-order functional program such as *e.g.* `map`, `iteration` or `foldl`, fed with a (first-order) polynomial time program produces a polynomial time program.

As a first illustrative step we consider the approach based on polynomial interpretations from [3], which offers the advantage of simplicity. We thus build a theory of *higher-order polynomial interpretations* for STTRSs. It can be seen as a particular concrete instantiation of the methodology proposed in [8] for proving termination by interpretation.

Higher-order polynomials (HOPs) take the form of terms in a typed λ-calculus whose only base type is that of natural numbers. To each of those terms can be assigned a strictly monotonic function in a category $\mathbb{FSPOS}$ with products and functions. So, the whole process can be summarized by the following diagram:

$$\begin{tikzcd}\text{STTRSs} \arrow[r, "{[\cdot]}"] & \text{HOPs} \arrow[r, "{[\![\cdot]\!]}"] & \mathbb{FSPOS}\end{tikzcd}$$

3.1 Higher-Order Polynomials

Let us consider types built over a single base type $\mathbf{N}$:

$$A, B ::= \mathbf{N} \;\big|\; A \to A.$$

Let C_P be the following set of constants:

$$C_P = \{+ : \mathbf{N} \to \mathbf{N} \to \mathbf{N}, \times : \mathbf{N} \to \mathbf{N} \to \mathbf{N}\} \cup \{\overline{n} : \mathbf{N} \mid\ n \in \mathbb{N}^\star\}.$$

Observe that in C_P we have constants of type $\mathbf{N}$ only for *strictly* positive integers. We consider the following grammar of Church-typed terms

$$M := x^A \;\big|\; \mathbf{c}^A \mid (M^{A\to B} N^A)^B \mid (\lambda x^A.M^B)^{A \to B}$$

where $\mathbf{c}^A \in C_P$ and *in* $(\lambda x^A.M^B)$ *we require that* x *occurs free in* M. A *higher-order polynomial* (HOP) is a term of this grammar, which is in β-normal form. We use an infix notation for $+$ and $\times$. We assume given the usual set-theoretic interpretation of types and terms, denoted as $[\![A]\!]$ and $[\![M]\!]$: if M has type A and $FV(M) = \{x_1^{A_1}, \ldots, x_n^{A_n}\}$, then $[\![M]\!]$ is a map from $[\![A_1]\!] \times \ldots \times [\![A_n]\!]$ to $[\![A]\!]$. We denote by $\equiv$ the equivalence relation which identifies terms which denote the same function, e.g. we have: $\lambda x.(\overline{2} \times ((\overline{3} + x) + y)) \equiv \lambda x.(\overline{6} + (2 \times x + 2 \times y))$.

Noticeably, even if HOPs can be built using higher-order functions, the first order fragment only contains polynomials:

▶ **Lemma 1.** *If M is a HOP of type $\mathbf{N}$ and such that $FV(M) = \{y_1 : \mathbf{N}, \ldots, y_k : \mathbf{N}\}$, then the function $[\![M]\!]$ is a polynomial function.*

3.2 Semantic Interpretation.

Now, we consider a subcategory $\mathbb{FSPOS}$ of the category $\mathbb{SPOS}$ of strict partial orders as objects and *strictly monotonic* total functions as morphisms. Objects of $\mathbb{FSPOS}$ are the following:

- $\mathcal{N}$ is the domain of *strictly positive* integers, equipped with the natural strict order $\prec_{\mathcal{N}}$,
- 1 is the trivial order with one point;
- if σ, τ are objects, then $\sigma \times \tau$ is obtained by the product ordering,
- $\sigma \to \tau$ is the set of strictly monotonic total functions from σ to τ, equipped with the following strict order: $f \prec_{\sigma\to\tau} g$ if for any a of σ we have $f(a) \prec_\tau g(a)$.

We denote by $\preceq_\tau$ the reflexive closure of $\prec_\tau$. $\mathbb{FSPOS}$ is a subcategory of $\mathbb{SET}$ with all the necessary structure to interpret types. $[\![A]\!]_\prec$ denotes the semantics of A as an object of $\mathbb{FSPOS}$. We choose to set $[\![\mathbf{N}]\!]_\prec = \mathcal{N}$. Notice that any element of $e \in [\![A]\!]_\prec$ can be easily mapped onto an element $e\downarrow$ of $[\![A]\!]$. What about terms? Actually, $\mathbb{FSPOS}$ can again be shown to be sufficiently rich:

▶ **Proposition 2.** *Let M be a HOP of type A with free variables $x_1^{A_1}, \ldots, x_n^{A_n}$. Then for every $e \in [\![A_1 \times \ldots \times A_n]\!]_\prec$, there is exactly one $f \in [\![A]\!]_\prec$ such that $f\downarrow = [\![M]\!](e\downarrow)$. Moreover, this correspondence is strictly monotone and thus defines an element of $[\![A_1 \times \ldots \times A_n \to A]\!]_\prec$ which we denote as $[\![M]\!]_\prec$.*

3.3 Assignments and Polynomial Interpretations

To each variable x^A we associate a variable $\underline{x}^{\underline{A}}$ where $\underline{A}$ is obtained from A by replacing each occurrence of base type by the base type $\mathbf{N}$ and by curryfication. We will sometimes write x (resp. A) instead of $\underline{x}$ (resp. $\underline{A}$) when it is clear from the context.

An *assignment* $[\cdot]$ is a map from $\mathcal{C} \cup \mathcal{F}$ to HOPs such that if $f \in \mathcal{C} \cup \mathcal{F}$, $[f]$ is a closed HOP, of type $\underline{A_1}, \ldots, \underline{A_n} \to \underline{A}$. Now, for $t \in \mathcal{T}$ we define $[t]$ by induction on t:

- if $t \in \mathcal{X}$, then $[t]$ is $\underline{f}$;
- if $t \in \mathcal{C} \cup \mathcal{F}$, $[t]$ is already defined;
- otherwise, if $t = (t_0\, t_1 \ldots t_n)$ then $[t] \equiv (\ldots([t_0][t_1]) \ldots [t_n])$.

Observe that in practice, computing $[t]$ will in general require to do some β-reduction steps.

▶ **Lemma 3.** *If $s \to t$, then $[\![t]\!]_\prec \prec [\![s]\!]_\prec$.*

As a consequence, the interpretation of terms (of base type) is itself a bound on the length of reduction sequences:

▶ **Proposition 4.** *Let t be a closed term of base type D. Then $[t]$ has type $\mathbf{N}$ and any reduction sequence of t has length bounded by $[\![t]\!]_\prec$.*

3.4 A Complexity Criterion

Proving a STTRS to have an interpretation is not enough to guarantee its time complexity to be polynomial. To ensure that, we need to impose some constraints on the way constructors are interpreted.

We say that the assignment $[\,\cdot\,]$ is *additive* if any constructor $\mathbf{c}$ of type $D_1 \times \cdots \times D_n \to E$, where $n \geq 0$, is interpreted by a HOP $M_{\mathbf{c}}$ whose semantic interpretation $[\![M_{\mathbf{c}}]\!]_{\prec}$ is a polynomial function of the form:

$$P(y_1, \ldots, y_n) = \sum_{i=1}^{n} y_i + \gamma_{\mathbf{c}}, \text{ with } \gamma_{\mathbf{c}} \geq 1.$$

Additivity ensures that the interpretation of first-order values is proportional to their size:

▶ **Lemma 5.** *Let $[\,\cdot\,]$ be an additive assignment. Then there exists $\gamma \geq 1$ such that for any value v of type D, where D is a data type, we have $[\![v]\!]_{\prec} \leq \gamma \cdot |v|$.*

The base type W_n denotes the data-type of n-ary words, whose constructors are **empty** and $\mathbf{c}_1, \ldots, \mathbf{c}_n$. A function $f : (\{0,1\}^*)^m \to \{0,1\}^*$ is said to be *representable* by a STTRS R if there is a function symbol `f` of type $W_2^m \to W_2$ in R which computes f in the obvious way. Noticeably:

▶ **Theorem 6.** *The functions on binary words representable by STTRSs admitting an additive polynomial interpretation are exactly the polytime functions.*

Not many programs can be proved to be polytime by way of the criterion we have just introduced. This, however, can be partially solved by switching to quasi-interpretations, as described in [2].

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Recent Developments in the Matchbox Termination Prover

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Abstract

We report on recent and ongoing work on the Matchbox termination prover:

- a constraint compiler that transforms a Boolean-valued Haskell function into a Boolean satisfiability problem (SAT),
- a constraint solver for real and arctic matrix constraints that is using evolutionary optimization, and is running on massively parallel (graphics) hardware.

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1 Introduction

The program Matchbox [11] originally proved termination of string rewriting, using the method of match bounds [6]. The domain was extended to term rewriting, by adding proof methods of dependency pairs [1], matrix interpretations over the integers [4] and over arctic numbers [9].

These methods are typical instances of the following scheme: to automatically find a proof of termination, one solves constraint satisfaction problems. E.g., the precedence of function symbols, or the coefficients of polynomials and matrices, are constrained by the condition that the resulting path order, polynomial order, or matrix order, respectively, is compatible with a given rewriting system.

The constraint system could be solved by

- by domain-specific methods (e.g., Matchbox computes a certificate for match-boundedness by completion of automata),
- by generic search (exhaustively, randomly, or directed by some fitness function),
- by transformation to another constraint domain (e. g., Matchbox transforms integer and arctic polynomial inequalities to a Boolean satisfiability problem, and solves it with Minisat [3]).

In the present paper, we report on recent and ongoing work to

- extract a general framework for constraint programming by automatic transformation to SAT,
- and (independently) add a domain-specific solver for real and arctic matrix constraints that is using evolutionary optimization, and is running on massively parallel (graphics) hardware.

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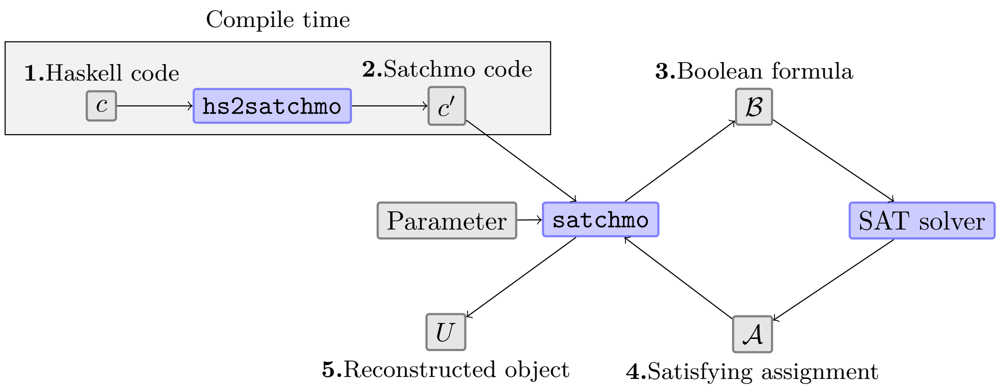
2 A Constraint Compiler

The idea behind constraint programming is to separate specification (encoded as constraint system) from implementation (the constraint solver). The obvious choice for the specification language is mathematical logic, equivalently, a pure (i.e. side-effect free) functional programming language like Haskell. A constraint system c can be seen as a function with type $c : U \to \texttt{Bool}$, where the solution is an object $s \in U$ such that $c(s)$ is True.

There are clever solvers for the case that $U' = \texttt{Bool}^*$, and c' is given by a formula in propositional logic. But typically, the application domain U is different. The translation from U to U' can be done manually by the programmer, or automatically, by some tool. The satchmo library (`http://hackage.haskell.org/package/satchmo`) used in Matchbox is an example for the "manual" approach. It is an embedded (in Haskell) domain specific language for the generation of SAT constraints. Several termination researchers built and published similar libraries for other host languages (Ocaml, Java). These interweave the generation of the boolean formula with the declaration of the constraint system in the host language. E.g., satchmo generates the formula as a side effect represented by a suitable State monad. Actually these are different processes and should be separated from each other.

Therefore Alexander Bau is building a constraint compiler that inputs source code of a Haskell function $c : U \to \texttt{Bool}$, as explained above, and produces a satchmo program. The domain U may use structured data types like tuples and lists, in addition to primitive types like booleans and integers. c may also depend on run-time parameters that are not known at compile time.

A prototypical use case is the search for a precedence that defines a lexicographical path order (LPO) that is compatible with a term rewriting system (TRS). In this case the constraint system consists of a Haskell implementation of LPO : TRS $\to$ Precedence $\to$ `Bool`. LPO applied to a TRS R and a precedence p returns `true`, iff lpo(p) is compatible with R. The first parameter (R) of LPO is known at run-time while the second (p) is not.



The constraint system is given as a program (a set of declarations) in a subset of Haskell. The constraint compiler performs a type-directed transformation, where the type system is an extension of the Damas-Milner type system [2] [10]. We additionally annotate each type constructor with a flag that indicates whether its value is known (as a parameter given at run-time) or unknown (and therefore has to be determined by the constraint solver).

We plan to extend the type system further, to take into account resource bounds [7]. E.g., we want to be statically certain that the size of the generated SAT constraint system is polynomially bounded in the size of the (known) input parameters.

3 Massively Parallel Constraint Solving

A constraint satisfaction problem can be converted into an optimization problem that is solved by evolutionary algorithms. For the domain of matrix interpretations, this approach was used by Dieter Hofbauer's termination prover *MultumNonMulta* (2006, 2007), see also [5]. We return to it now, since it allows for massive parallelisation.

In the context of numerical constraint solving by randomized, directed search, parallel processing is applicable because

- basic operations (on numbers) can be executed fast
- domain specific operations (matrix multiplications) can be sped up by parallelism (multiplication of n dimensional square matrices, using n^2 cores and n time)
- evolutionary search strategies can be sped up again, by treating several individuals in parallel (e.g., computing their fitness values)

General Purpose Graphical Processing Units (GPGPUs) provide massively parallel processing at affordable prices. Tobias Kalbitz and Maria Voigtländer are implementing matrix constraint solvers for CUDA capable graphics cards. CUDA (Compute Unified Device Architecture) [8] is a parallel programming model for NVIDIA's GPGPUs.

The following approach is used to find a strictly monotone matrix interpretation of dimension d that is compatible with a string rewriting system R over alphabet Σ (weakly compatible with each rule, and strictly compatible with at least one rule):

- A population consists of several individuals, each individual is a matrix interpretation, that is, a mapping $[\cdot] : \Sigma \to \mathbb{N}^{d\times d}$, where for each $a \in \Sigma$, the first column of $[a]$ is $(1, 0, \dots, 0)^T$, and the last row of $[a]$ is $(0, \dots, 0, 1)$. This condition ensures monotonicity.
- The *fitness* of an interpretation $[\cdot]$ is $\sum\{\max(0, [r]_{p,q} - [l]_{p,q})^2 \mid (l, r) \in R, 1 \le \{p, q\} \le d\}$, plus some very large penalty in case that $\neg\exists(l, r) \in R : [l]_{1,d} > [r]_{1,d}$. Lower fitness values are better, and value zero indicates that compatibility holds.
- An individual with fitness > 0 is changed by a *large* mutation: we randomly pick some $(l, r) \in R, 1 \le \{p, q\} \le d$ such that $[l]_{p,q} < [r]_{p,q}$, and we choose randomly a sequence of indices $p = p_0, p_1, \dots, p_n = q$ with $n = |l|$, and then increase each $[a_i]_{p_{i-1},p_i}$ by one, where $l = a_1 \dots a_n$. This ensures that $[l]_{p,q}$ increases.
- Next, this individual undergoes a series of *small* mutations where for any $a \in \Sigma, 1 \le i, j \le d$, the entry at $[a]_{i,j}$ is modified. We try sereval small mutations, until we find one that decreases fitness, and then repeat. The total number of small mutations is bounded.
- The resulting individual is placed back into the population, removing another individual of larger fitness.

▶ **Example 1.** With 1000 individuals, and 100 small steps after each large step, we find a compatible 5-dimensional interpretation for $a^2b^2 \to b^3a^3$ (Problem z001) with < 30.000 large steps with probability $> 50\%$. Of course, the total runtime is not bounded, as the evolution may go into a dead-end. So it is better to re-start than to wait.

Applying this idea to rational, and arctic, numbers, we meet the following challenges:

Real numbers are approximated by rational ("floating point" values), thus results of comparisons may be wrong. The solution is to introduce a "grid" for rounding input values, e.g. use only integer multiples of 1/2, or 1/10, say.

A fine grid implies a smooth objective function, and this may help evolutionary algorithms. On the other hand, a coarse grid reduces the search space, and may increase the chance that we find a solution by luck.

Note that we do not need a grid for arctic numbers, since we can use arctic integers.

On typical CUDA cards, a large number of compute cores is available (e.g., 512). They can only be used efficiently if the data that they process is stored in fast (thread-(block-)local) memory. The amount of such memory is severely limited (e.g., 16 kByte total, resulting in 300 byte per core)

CUDA cards are programmed in (a dialect of) C. This allows fine-grained control, but is highly impractical for large-scale programming. Therefore, we are isolating the low-level details in a C library, and provide it with an interface to Haskell, where we implement global flow of control. Still it is important that data stays on the card's memory, since transport to and from the host computer's memory is slow.

4 Future plans

We stress that the above is a report on ongoing work.

We plan to have an implementation ready for the termination competition in 2012. The code will be open-sourced.

Since the hardware of the competition platform does not include a GPGPU, we will run Matchbox/CUDA remotely.

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The recursive path and polynomial ordering[*]

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Abstract

In most termination tools two ingredients, namely recursive path orderings (RPO) and polynomial interpretation orderings (POLO), are used in a consecutive disjoint way to solve the final constraints generated from the termination problem.

We present a simple ordering that combines both RPO and POLO and defines a family of orderings that includes both, and extends them with the possibility of having, at the same time, an RPO-like treatment for some symbols and a POLO-like treatment for the others.

The ordering is extended to higher-order terms, providing an automatable use of polynomial interpretations in combination with beta-reduction.

1 Introduction

Term orderings have been extensively used in termination proofs of rewriting. They are used both in direct proofs of termination showing decreasingness of every rule or as ingredients for solving the constraints generated by other methods like the Dependency Pair approach [1] or the Monotonic Semantic Path Ordering [4].

The most widely used term orderings in automatic termination tools are the recursive path ordering (RPO) and the polynomial ordering (POLO). Almost all known tools implement these orderings. RPO and POLO are incomparable, so that they are used in a sequential way, first trying one method (maybe under some time limit) and, in case of failure, trying the other one afterwards.

As an alternative to this sequential application we propose a new ordering that combines both RPO and POLO. The new family of orderings, called RPOLO, includes strictly both RPO and POLO as well as the sequential combination of both. Our approach is based on splitting the set of symbols into those handled in an RPO-like way (called RPO-symbols) and those that are interpreted using a polynomial interpretation (called POLO-symbols). In this paper, only linear polynomial interpretations are considered. These interpretations are never applied to terms headed by an RPO-symbol. Instead, the term is interpreted as a new variable (labeled by the term). This is crucial to be able to extend the ordering to the higher-order case, since applying polynomial interpretations to beta-reduction is not easy. However, the introduction of different unrelated variables for every term makes us lose stability under substitutions and (weak) monotonicity. To avoid that, a context relating the variables is introduced, but then a new original proof of well-foundedness is needed.

Matrix interpretations [8, 7], have recently been adopted as the third alternative to define term orderings. As future work, it would be interesting to study if our results can be generalized to matrix interpretations and to more general interpretations fulfilling some required properties[1].

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Although the new ordering is strictly more powerful than its predecessors and thus examples that can be handled by RPOLO and neither by RPO nor by POLO can be cooked, in practice, there is no real gain when using RPOLO on the first-order examples coming from the Termination Problem Data Base.

Due to this, we show its practical usefulness by extending it, using the same techniques as for the higher-order recursive path ordering [10] (HORPO), to rewriting on simply typed higher-order terms union beta-reduction. The resulting ordering, called HORPOLO, can hence be used to prove termination of the so called Algebraic Functional Systems [9] (AFS), and provides an automatable termination method that allows the user to have polynomial interpretations on some symbols in a higher-order setting. Note that, although some polynomial interpretations for higher-order rewrite systems à la Nipkow where extensively studied in [11], it is unclear how to implement those techniques in an automatic tool.

Due to the space limitations we have not included here the definitions of the higher-order version of the ordering, but it is the natural extension following the same ideas applied to extend RPO to HORPO (for a full version of this work see [3]).

2 The recursive path and polynomial ordering (RPOLO)

Here we present the ordering for first-order terms. Let $\mathcal{F}$ be a signature split into two sets $\mathcal{F}_{POLO}$ and $\mathcal{F}_{RPO}$. We have a precedence $\succeq_{\mathcal{F}}$ on $\mathcal{F}_{RPO}$ and a polynomial interpretation I over the non-negative integers $\mathbb{Z}^+$ for the terms in $\mathcal{T}(\mathcal{F},\mathcal{X})$. Moreover, the interpretation I is defined by a linear interpretation f_I with coefficients in $\mathbb{Z}^+$ for every symbol f in $\mathcal{F}_{POLO}$ and a variable x_s for every term s with top symbol in $\mathcal{F}_{RPO}$:

$$I(s) = \begin{cases} f_I(I(s_1),\dots,I(s_n)) & \text{if } s = f(s_1,\dots,s_n) \text{ and } f \in \mathcal{F}_{POLO} \\ x_s & \text{otherwise} \end{cases}$$

In order to handle these introduced variables x_s, we define a context information to be used when comparing the interpretations. In what follows a *(polynomial) context* is a set of constraints of the form $x \geq E$ where x is a variable and E is a linear polynomial expression over $\mathbb{Z}^+$. Let us now show the way contexts are used when comparing polynomials.

▶ **Definition 1.** Let C be a context. The relation $\rightarrow_C$ on linear polynomial expressions over $\mathbb{Z}^+$ is defined by the rules $P + x \rightarrow_C P + E$ for every $x \geq E \in C$.

Let p and q be linear polynomial expressions over $\mathbb{Z}^+$. Then $p >_C q$ (resp. $p \geq_C q$) if there is some u such that $p \twoheadrightarrow^{=}_{C} u > q$ (resp. $p \twoheadrightarrow^{=}_{C} u \geq q$).

We use here (the reflexive closure of) a parallel rewriting step $\twoheadrightarrow^{=}_{C}$ instead of the transitive closure of $\rightarrow_C$ because it simplifies the proofs without losing any power.

The following three mutually recursive definitions introduce respectively the context $C(\mathcal{S})$ of a set of terms $\mathcal{S}$, the ordering $\succ_{RPOLO}$ and the compatible quasi-ordering $\sqsupseteq_{RPOLO}$.

▶ **Definition 2.** Let $\mathcal{S}$ be a set of terms u such that $top(u) \notin \mathcal{F}_{POLO}$. The context $C(\mathcal{S})$ is defined as the union of

1. $x_u \geq E+1$ for all $u \in \mathcal{S}$ and for all linear polynomial expressions E over $\mathbb{Z}^+$ and variables $\{x_{v_1},\dots,x_{v_n}\}$ such that $u \succ_{RPOLO} v_i$ for all $i \in \{1,\dots,n\}$.
2. $x_u \geq x_v$ for all $u \in \mathcal{S}$ and all v such that $u \sqsupseteq_{RPOLO} v$ and $top(v) \in \mathcal{F}_{RPO}$.

Note that $C(s)$ can be infinite. For this reason, in practice, when comparing a pair of terms s and t we only generate the part of $C(s)$ that is needed. This part is chosen by inspecting t.

▶ **Definition 3.** $s \sqsupseteq_{RPOLO} t$ iff

1. $s = t \in \mathcal{X}$, or
2. $s = f(s_1, \ldots, s_n)$ and
   a. $f \in \mathcal{F}_{POLO}$, $I(s) \geq_{C(s)} I(t)$ or
   b. $t = g(t_1, \ldots, t_n)$, $f, g \in \mathcal{F}_{RPO}$, $f =_{\mathcal{F}} g$ and
      i. $stat(f) = mul$ and $\{s_1, \ldots, s_n\}(\sqsupseteq_{RPOLO})_{mon}\{t_1, \ldots, t_n\}$, or
      ii. $stat(f) = lex$ and $\langle s_1, \ldots, s_n\rangle(\sqsupseteq_{RPOLO})_{mon}\langle t_1, \ldots, t_n\rangle$.

▶ **Definition 4.** $s = f(s_1, \ldots, s_n) \succ_{RPOLO} t$ iff

1. $f \in \mathcal{F}_{POLO}$ and $I(s) >_{C(s)} I(t)$, or
2. $f \in \mathcal{F}_{RPO}$, and
   a. $s_i \succeq_{RPOLO} t$ for some $i \in \{1, \ldots, n\}$, or
   b. $t = g(t_1, \ldots, t_m)$, $g \in \mathcal{F}_{POLO}$ and $s \succ_{RPOLO} u$ for all $u \in \mathcal{A}cc(t)$, or
   c. $t = g(t_1, \ldots, t_m)$, $g \in \mathcal{F}_{RPO}$ and
      i. $f \succ_{\mathcal{F}} g$ and $s \succ_{RPOLO} t_i$ for all $i \in \{1, \ldots, m\}$, or
      ii. $f =_{\mathcal{F}} g$, $stat(f) = mul$ and $\{s_1, \ldots, s_n\}(\succ_{RPOLO})_{mul}\{t_1, \ldots, t_m\}$, or
      iii. $f =_{\mathcal{F}} g$, $stat(f) = lex$, $\langle s_1, \ldots, s_n\rangle(\succ_{RPOLO})_{lex}\langle t_1, \ldots, t_m\rangle$ and $s \succ_{RPOLO} t_i$ for all $i \in \{1, \ldots, m\}$,

where $s \succeq_{RPOLO} t$ iff $s \succ_{RPOLO} t$ or $s \sqsupseteq_{RPOLO} t$ and $\mathcal{A}cc(s)$ is defined as $\{u \mid x_u \in \mathcal{V}ar(I(s))\}$ and, to ease the reading, $C(s)$ denotes $C(\mathcal{A}cc(s))$.

Note that ordering keeps the flavor of the RPO definition, but adding some cases to handle the terms headed by polynomially interpreted symbols.

Now, we provide some examples of comparisons between terms that are included in our ordering and are neither included in RPO nor in POLO, i.e., using (linear) integer polynomial interpretations. In fact, since we consider constraints including both strict and non-strict literals, what we show is that they are included in the pair $(\succ_{RPOLO}, \succeq_{RPOLO})$.

▶ **Example 5.** Consider the following constraint consisting of three literals:

$$\begin{array}{lcl} H(f(g(g(x)),y),x) & > & H(f(g(y),x),f(g(y),x)) \\ H(x,g(y)) & \geq & H(y,x) \\ f(g(x),y) & \geq & f(y,x) \end{array}$$

The first literal cannot be proved by RPO since $f(g(g(x)),y)$ cannot be proved larger than $f(g(y),x)$ as no argument of the former is greater than $g(y)$. The constraints cannot be proved terminating by an integer polynomial interpretation either.

Let us prove it using RPOLO. We take $H \in \mathcal{F}_{RPO}$ with $stat(H) = mul$ and $f, g \in \mathcal{F}_{POLO}$ with $f_I(x,y) = x + y$ and $g_I(x) = x + 1$.

For the first literal, applying case 4.2(c)ii, we need to prove $\{f(g(g(x)),y),x\}(\succ_{RPOLO})_{mul}\{f(g(y),x),f(g(y),x)\}$, which holds since $f(g(g(x)),y) \succ_{RPOLO} f(g(y),x)$ by case 4.1 as $I(f(g(g(x)),y)) = x_x + x_y + 2 > x_x + x_y + 1 = I(f(g(y),x))$. The proof of the other two literals reuses part of the previous argument. ◀

Let us now show an example where we need symbols in $\mathcal{F}_{RPO}$ occurring below symbols that need to be in $\mathcal{F}_{POLO}$. Moreover, in this example a non-trivial use of the context is also necessary.

▶ **Example 6.** Consider the following constraint coming from a termination proof:

$$\begin{array}{lcl} f(0,x) & \geq & x \\ f(s(x),y) & \geq & s(f(x,f(x,y))) \\ H(s(f(s(x),y)),z) & > & H(s(z),s(f(x,y))) \end{array}$$

The third literal needs H and s to be in $\mathcal{F}_{POLO}$. To hint this fact, note that we cannot remove s and, in that case, no argument in $H(s(f(s(x),y)),z)$ can be greater than or equal to $s(z)$. On the other hand, since due to the third literal, s cannot be removed and needs a non-zero coefficient for its argument, there is no polynomial interpretation for f fulfilling the first two literals, i.e., f must be in $\mathcal{F}_{RPO}$.

Therefore, we take $H, s \in \mathcal{F}_{POLO}$ with $H_I(x,y) = x+y$ and $s_I(x) = x+1$, and $f \in \mathcal{F}_{RPO}$ with $stat(f) = lex$.

The first literal holds by case 4.2a. For the second one, $f(s(x),y) \succ_{RPOLO} s(f(x,f(x,y)))$ is proved by applying case 4.2b which requires $f(s(x),y) \succ_{RPOLO} f(x,f(x,y))$. We apply then case 4.2(c)iii, showing $s(x) \succ_{RPOLO} x$, by case 4.1, since $I(s(x)) = x_x + 1 > x_x = I(x)$, and $f(s(x),y) \succ_{RPOLO} x$ and $f(s(x),y) \succ_{RPOLO} f(x,y)$ for the arguments. The first one holds by applying cases 4.2a and 4.1 consecutively, and the second one by case 4.2(c)iii as before.

Finally, for the third literal we apply case 4.1, since

$$x_{f(s(x),y)} + x_z + 1 \rightarrow_{\{x_{f(s(x),y)} \geq x_{f(x,y)}+2\}} x_{f(x,y)} + 2 + x_z + 1 > x_z + x_{f(x,y)} + 2$$

Note that $x_{f(s(x),y)} \geq x_{f(x,y)} + 2$ belongs to the context of $H(s(f(s(x),y)),z)$ since we have $f(s(x),y) \succ_{RPOLO} s(f(x,y))$ and $I(s(f(x,y))) = x_{f(x,y)} + 1$. ◀

Let us mention that, although in the previous example we have used the context, in all non cooked examples we have tried the context is not used. However, the context is still necessary, since otherwise we can not prove neither stability under substitutions nor (weak) monotonicity.

HORPOLO has been implemented as base ordering in THOR-1.0 [2], a higher-order termination prover based on the monotonic higher-order semantic path ordering [6].

The implementation of HORPOLO is done by translating the ordering constraints $s > t$ and $s \geq t$ into problems in SAT modulo non-linear integer arithmetic (NIA) which is handled by the Barcelogic [2, 5] SMT-solver.

Just to hint on the power of the extension of the ordering to the higher-order case we provide an example that cannot be proved with other existing methods.

▶ **Example 7.** Let nat be a data type, $\mathcal{F} = \{s : [nat] \rightarrow nat, 0 : [] \rightarrow nat, dec : [nat \times nat] \rightarrow nat, grec : [nat \times nat \times nat \times (nat \rightarrow nat \rightarrow nat)] \rightarrow nat, + : [nat \times nat] \rightarrow nat, log2 : [nat \times nat] \rightarrow nat, sumlog : [nat] \rightarrow nat\}$ and $\mathcal{X} = \{x : nat, y : nat, u : nat, F : nat \rightarrow nat \rightarrow nat\}$.

Consider the following set of rules:

$$\begin{array}{lcl}
dec(0,x) & \rightarrow & 0 \\
dec(x,0) & \rightarrow & x \\
dec(s(x),s(y)) & \rightarrow & dec(x,y) \\
grec(0,d,u,F) & \rightarrow & u \\
grec(s(x),s(y),u,F) & \rightarrow & grec(dec(x,y),s(y),@(@(F,u),x),F)
\end{array}$$

[2] See `http://www.lsi.upc.edu/~albert/term.html`

$$\begin{array}{lcl}
0 + x & \rightarrow & x \\
s(x) + y & \rightarrow & s(x + y) \\
quad(0) & \rightarrow & 0 \\
quad(s(x)) & \rightarrow & s(s(s(s(quad(x))))) \\
sqr(x) & \rightarrow & sqrp(p(x, 0)) \\
sqrp(p(0, 0)) & \rightarrow & 0 \\
sqrp(p(s(s(x)), y)) & \rightarrow & sqrp(p(x, s(y))) \\
sqrp(p(0, s(y))) & \rightarrow & quad(sqrp(p(s(y), 0))) \\
sqrp(p(s(0), y)) & \rightarrow & quad(sqrp(p(y, 0))) + s(quad(y)) \\
sumsqr(x) & \rightarrow & grec(x, s(s(0)), 0, \lambda z_1 : nat.\lambda z_2 : nat.sqr(s(z_2)) + z_1)
\end{array}$$

The first rules define a tail recursive generalized form of the Gödel recursor where we can decrease in any given fixed amount at every recursive call. Using it, the rules compute the square root using the recurrence $x^2 = 4(x\,div\,2)^2$ when x is even and $x^2 = 4(x\,div\,2)^2 + 4(x\,div\,2) + 1$ when x is odd. Note that in the square definitions the even/odd checking is done along with the computation. To be able to handle this example we need to introduce the symbol p, which allows us to have $sqrp \in \mathcal{F}_{RPO}$ and $p \in \mathcal{F}_{POLO}$.

Some more examples as well as all details and proofs can be found in [3].

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Proving Termination of Java Bytecode with Cyclic Data*

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Abstract

In earlier work, we developed a technique to prove termination of Java Bytecode (JBC) automatically: first, JBC programs are automatically transformed to term rewrite systems (TRSs) and then, existing methods and tools are used to prove termination of the resulting TRSs. In this paper, we extend our technique in order to prove termination of algorithms on cyclic data such as cyclic lists or graphs automatically. We implemented our technique in the tool AProVE and performed extensive experiments to evaluate its practical applicability.

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Keywords and phrases Termination, Java Bytecode, Cyclic Data

1 Introduction

Most techniques for proving termination of imperative languages synthesize ranking functions (e.g., [6, 13]) and localize the termination test using Ramsey's theorem [10, 14]. For instance, such techniques are used in the tools Terminator [2, 7] and LoopFrog [9] to analyze termination of C programs. The heap is usually abstracted to integers using separation logic, cf. e.g. [11].

On the other hand, there also exist *transformational approaches* which automatically transform imperative programs to TRSs or to (constraint) logic programs. They allow to re-use the existing techniques and tools from term rewriting or logic programming also for termination of imperative programs. A tool to analyze C by a transformation to TRSs was presented in [8] and the tools Julia [15] and COSTA [1] prove termination of Java Bytecode (JBC) via a transformation to constraint logic programs. To deal with the heap, they use an abstraction to integers and represent objects by their *path length* (i.e., by the length of the maximal path obtained by following the fields of objects).

We presented a technique for termination of Java via a transformation to TRSs in [3, 4, 5, 12]. In contrast to other approaches for termination of imperative programs, we handle the heap by an abstraction to *terms*. In this paper, we extend our technique to also handle algorithms whose termination depends on the traversal or manipulation of cyclic objects. Up to now, transformational approaches could not deal with such programs. Our termination technique works in two steps: first, a JBC program is transformed into a *termination graph*, which is a finite representation of all possible program runs. This graph takes all sharing effects into account. In the second step, a TRS is generated from the graph.

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2 Handling Algorithms on Cyclic Data

```
public class L {
 int v;
 L n;
 static void visit(L x){
  int e = x.v;
  while (x.v == e) {
    x.v = e + 1;
    x = x.n; }}}
```

Figure 1 Java Program

We regard lists with a "next" field `n` where every element has an integer value `v`. The method `visit` in Fig. 1 stores the value of the first list element. Then it iterates over the list elements as long as they have the same value and "marks" them by modifying their value. The JBC for `visit` is shown in Fig. 2.

The algorithm terminates because it can distinguish already visited objects from unvisited ones by checking if the field `v` was changed. We recapitulate our representation of states in termination graphs in Sect. 2.1, explain the termination graph of `visit` in Sect. 2.2, and extend our approach in order to prove termination of algorithms like `visit` in Sect. 2.3.

2.1 Abstract States in Termination Graphs

```
00: aload_0         #load x
01: getfield v      #get v from x
04: istore_1        #store to e
05: aload_0         #load x
06: getfield v      #get v from x
09: iload_1         #load e
10: if_icmpne 28 #jump if x.v != e
13: aload_0         #load x
14: iload_1         #load e
15: iconst_1        #load 1
16: iadd            #add e and 1
17: putfield v      #store to x.v
20: aload_0         #load x
21: getfield n      #get n from x
24: astore_0        #store to x
25: goto 5
28: return
```

Figure 2 JBC for `visit`

In our termination graphs, we use abstract states to represent a possibly infinite number of non-abstract states. Such an abstract state is depicted in Fig. 3. The first three components of a state are written in the first line, separated by "|". The first component is the next instruction to evaluate. The second component represents the local variables as a list of references to the heap.[1] So "$\mathtt{x}:o_1$" indicates that the 0-th local variable `x` has the value o_1. The third component is the operand stack, which holds temporary results of JBC instructions. The empty stack is denoted by ε and "o_1, o_2" denotes a stack with top element o_1.

05 | $\mathtt{x}:o_1, \mathtt{e}:i_1$ | ε
o_1:L(?) $\quad i_1:\mathbb{Z} \quad o_1 \circlearrowleft$

Figure 3 State A

Below the first line, the heap is shown. It maps references to (abstract) values and contains annotations to specify sharing effects in parts of the heap that are not explicitly represented. We represent unknown integers by intervals, and abbreviate intervals such as $(-\infty, \infty)$ by $\mathbb{Z}$. If `Cl` is the name of a class, `Cl`(?) is an unknown object of type `Cl` (or a subtype) or `null`. Thus, "o_1:L(?)" means that at address o_1, we have an instance of `L` with unknown field values or that o_1 is `null`. More concrete objects are represented similarly, e.g., "o_2:L($\mathtt{v} = i_2, \mathtt{n} = o_3$)" describes some `L`-object at address o_2 whose field `v` contains the reference i_2 and whose field `n` contains o_3.

If one of our states contains the references o_1 and o_2, then the objects reachable from o_1 resp. o_2 are disjoint[2] and all these objects are tree-shaped (and thus acyclic), unless this is explicitly stated otherwise. Sharing can be represented in two ways in our states. Either, it is expressed directly (e.g., "o_1:L($\mathtt{v} = i_2, \mathtt{n} = o_1$)" implies that o_1 is cyclic) or *annotations* are

[1] To avoid a special treatment of integers, we also represent them using references to the heap. Furthermore, to ease readability, in examples we denote local variables by names instead of numbers.

[2] An exception are references to `null` or INTS, since in JBC, integers are primitive values where one cannot have any side effects. So if h is the heap of a state and $h(o_1) = h(o_2) \in$ INTS or $h(o_1) = h(o_2) =$ `null`, then one can always assume $o_1 = o_2$.

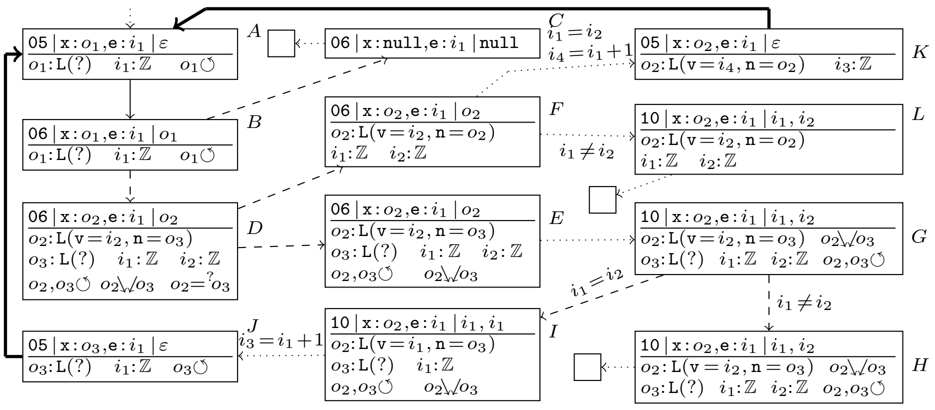

Figure 4 Termination Graph for `visit`

used to indicate (possible) sharing in parts of the heap that are not explicitly represented. For example, the *equality annotation* $o =^? o'$ means that the two references o and o' could actually be the same and the *joinability annotation* $o \curlyvee o'$ means that o and o' possibly have a common successor.

In our earlier papers [3, 12] we had another annotation to denote references that may point to non-tree-shaped objects. To maintain more information about possibly non-tree-shaped objects, we now introduce two new *shape annotations* instead. The *non-tree annotation* $o\diamondsuit$ means that o might have some successor that can be reached using two different cycle-free paths starting in o. The *cyclicity annotation* $o\circlearrowleft$ means that there could be cycles including o or reachable from o.

2.2 Constructing the Termination Graph

When calling `visit` for an arbitrary (possibly cyclic) list, one reaches state A from Fig. 4 after one loop iteration by symbolic evaluation and generalization. Now `aload_0` loads the value o_1 of `x` on the operand stack, yielding state B. To evaluate `getfield v`, we perform a *case analysis* (which we call *refinement*) and create successors C where o_1 is `null` and D where o_1 (now called o_2) is an actual instance of `L`. We copy the annotation $\circlearrowleft$ to its `n`-field o_3 and allow o_2 and o_3 to join. We also add $o_2 =^? o_3$, for the case where o_2 is a cyclic one-element list.

In C, we end with a `NullPointerException`. Before accessing o_2's fields in D, we have to resolve all possible equalities. Thus we refine D, obtaining E and F, corresponding to the cases where $o_2 \neq o_3$ and where $o_2 = o_3$. F needs no annotations anymore, as all reachable objects are completely represented in the state. In E we evaluate `getfield`, retrieving the value i_2 of the field `v`. Then we load `e`'s value i_1 on the operand stack, which yields G. To evaluate `if_icmpne`, we branch depending on the inequality of the top stack entries i_1 and i_2, resulting in H and I. We label the edges with the respective integer relations.

In I, we add 1 to i_1, creating i_3, which is written into the field `v` of o_2. Then, the field `n` of o_2 is retrieved, and the obtained reference o_3 is written into `x`, leading to J. As J is a renaming of A, it is an *instance* of A, meaning that A represents all non-abstract states represented by J. Therefore, we draw an *instance edge* (depicted by a thick arrow) from J to A. The states following F are analogous to the ones following E.

2.3 Proving Termination of Algorithms on Cyclic Data

To prove termination of algorithms like `visit`, the idea is to find a suitable *marking property* $M \subseteq$ Instances. So M is a set of objects that satisfy a certain property. We add an extra local variable with the name c_M to each state, counting the number of objects in the state that are in M. For each concrete state s, its value is the number of reachable objects of s which are in M. For an abstract state s representing some concrete state s', the value of c_M is an interval containing an upper bound for the number of objects with property M in s'. Then, we can analyze the termination graph for changes to this counter. In our example, we let M be the set of `L`-objects with `v` $= i_1$. Then in each loop iteration, the field `v` of some `L`-object is set to a value i_3 resp. i_4 which is *different* from i_1. Thus, the counter c_M decreases.

To detect a suitable marking property automatically, we restrict ourselves to properties "`Cl.f` $\bowtie i$", where `Cl` is a class, `f` a field in `Cl`, i a (possibly unknown) integer, and $\bowtie$ an integer relation. Then M is the set of all `Cl`-objects (including objects of subtypes of `Cl`) whose field `f` stands in relation $\bowtie$ to the value i.

The first step is to find some integer value that remains constant. In our example, we can easily infer this for i_1 automatically. The second step is to find `Cl`, `f`, and $\bowtie$ such that every cycle contains some state where $c_M > 0$, where M is the set of all objects satisfying `C.f` $\bowtie i$. We consider those states whose incoming edge is labeled with "$i \bowtie \ldots$" or "$\ldots \bowtie i$". In our example, I's incoming edge is labeled with "$i_1 = i_2$" and at the time of the comparison of i_1 and i_2 (i.e., in state G), i_2 was the value of o_2's field `v`, where o_2 is an `L`-object. This suggests the marking property "`L.v =` i_1". In I we thus know that $c_M > 0$. So the cycle $A, \ldots, E, \ldots A$ contains a state with $c_M > 0$ and one can automatically detect that the other cycle $A, \ldots, F, \ldots, A$ also has a similar state with $c_M > 0$.

In the third step, we add c_M as a new local variable with value i to all states. The edge from G to I is then labeled with "$i > 0$" which we inferred in the second step above (this label will be used in the resulting TRS). It remains to explain how to detect changes of the counter c_M. To this end, we use SMT solving. A counter for "`Cl.f` $\bowtie i$" can only change when a new object of type `Cl` (or a subtype) is created or when the field `Cl.f` is modified. So whenever "`new Cl'`" is called for some subtype `Cl'` of `Cl`, then we have to consider the default value d for the field `Cl.f`. If the underlying SMT solver can prove that $\neg d \bowtie i$ is a tautology, then c_M can remain unchanged. Otherwise, to ensure that c_M is an upper bound for the number of objects in M, c_M is incremented by 1. If a `putfield` replaces the value u in `Cl.f` by w, we have three cases:

(i) If $u \bowtie i \wedge \neg w \bowtie i$ is a tautology, then c_M may be decremented by 1.
(ii) If $u \bowtie i \leftrightarrow w \bowtie i$ is a tautology, then c_M remains the same.
(iii) In the remaining cases, we increment c_M by 1.

In our example, between I and J one writes i_3 to the field `v` of o_2. To find out how c_M changes from I to J, we create a formula containing all information on the edges in the path up to now. This results in $i_1 = i_2 \wedge i_3 = i_1 + 1$. We then check whether the information in the path implies $u \bowtie i \wedge \neg w \bowtie i$. In our example, the previous value u of o_2.`v` is i_1 and the new value w is i_3. Any SMT solver for integer arithmetic can easily prove that the resulting formula $i_1 = i_2 \wedge i_3 = i_1 + 1 \;\rightarrow\; i_1 = i_1 \wedge \neg i_3 = i_1$ is a tautology (i.e., its negation is unsatisfiable).

Thus, c_M is decremented by 1 in the step from I to J and hence, we label the edge from I to J with the relation "$i' = i - 1$" (where i' is the new value of c_M). Similarly, one can also easily prove that c_M decreases between F and K. We can now generate TRSs as in [4, 12]. The new counter results in an extra argument of the function symbols in the TRS. So for the cycle $A, \ldots, E, \ldots A$, after some "merging" of rewrite rules, we obtain the following TRS:

$$\mathsf{f}_A(\ldots, i) \to \mathsf{f}_I(\ldots, i) \quad | \; i > 0 \qquad \mathsf{f}_I(\ldots, i) \to \mathsf{f}_J(\ldots, i - 1) \qquad \mathsf{f}_J(\ldots, i') \to \mathsf{f}_A(\ldots, i')$$

For the other cycle $A, \ldots, F, \ldots A$ we obtain similar rules. Termination of the resulting TRS can easily be be shown automatically, which proves termination of the original method `visit`.

3 Experiments and Conclusion

We extended our earlier work [3, 4, 5, 12] on proving termination of JBC automatically to also handle methods whose termination behaviour depends on the cyclicity of the handled data. In the full version of the paper, we also describe additional extensions to handle further classes of such algorithms besides "marking algorithms" as in Sect. 2. We implemented our new approach in the termination tool AProVE and evaluated it on a large collection of JBC programs, including all methods of the classes `LinkedList` and `HashMap` from the `Collections` framework in the `java.util` package, which is part of the standard Java distribution. Our experiments demonstrate the practical applicability of our approach and they show that the new version of AProVE is significantly more powerful than its predecessor and than other tools for JBC termination analysis. For further details, we refer to the forthcoming full version of the paper.

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Proving Non-Termination for Java Bytecode*

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Abstract

Recently, we developed an approach for automated termination proofs of Java Bytecode (JBC), which is based on constructing and analyzing *termination graphs*. These graphs represent all possible program executions in a finite way. In this paper, we show that this approach can also be used to detect *non-termination*. We implemented our results in the termination prover AProVE and provide experimental evidence for the power of our approach.

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Keywords and phrases Non-Termination, Java Bytecode, SMT Solving

1 Introduction

Our approach is based on our earlier work to prove termination of JBC [3, 4, 11]. Here, a JBC program is first automatically transformed to a *termination graph* by symbolic evaluation. Afterwards, a term rewrite system is generated from the termination graph and existing techniques from term rewriting are used to prove termination of the rewrite system. As shown in the annual *International Termination Competition*,[1] our corresponding tool AProVE [9] is currently among the most powerful ones for automated termination proofs of Java programs.

Termination graphs finitely represent all runs through a program for a certain set of input values. In Sect. 2, we show that termination graphs can also be used to detect non-termination.

Methods to prove non-termination automatically have for example been studied for term rewriting (e.g., [8, 13]) and logic programming (e.g., [12]). We are only aware of two existing tools for automated non-termination analysis of Java: The tool Julia transforms JBC programs into constraint logic programs, which are then analyzed for non-termination [14] and under certain conditions, this allows to deduce non-termination of the original JBC program. The tool Invel [15] investigates non-termination of Java programs based on a combination of theorem proving and invariant generation using the KeY [2] system. In contrast to our approach, Invel and Julia only have limited support for proving non-termination of programs operating on the heap. Moreover, in contrast to our technique, neither Julia nor Invel return witnesses for non-termination. In Sect. 3 we compare the implementation of our approach in the tool AProVE with Julia and Invel. In our experiments, the non-termination analyzer in AProVE was substantially more powerful than the ones implemented in Julia and Invel.

* Supported by the DFG grant GI 274/5-3, the G.I.F. grant 966-116.6, and the DFG Research Training Group 1298 (*AlgoSyn*).

[1] See `http://www.termination-portal.org/wiki/Termination_Competition`

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Moreover, [10] presents a method for non-termination proofs of C programs. In contrast to our approach, [10] can deal with non-terminating recursion and integer overflows. On the other hand, [10] cannot detect *non-periodic* non-termination (where there is no fixed sequence of program positions that is repeated infinitely many times), whereas this is no problem for our approach.

2 Proving Non-Termination

```
static void nonLoop(
  int x, int y) {
   if (y >= 0) {
     while(x >= y) {
       int z = x - y;
       if (z > 0) {
         x--;
       } else {
         x = 2*x + 1;
         y++; }}}}
```

Figure 1 `nonLoop(x,y)`

Consider the method `nonLoop` in Fig. 1 which does not terminate if $\texttt{x} \geq \texttt{y} \geq 0$. For example, if $\texttt{x} = 2$, $\texttt{y} = 1$ at the beginning of the loop, then after one iteration we have $\texttt{x} = 1$, $\texttt{y} = 1$. In the next iterations, we obtain $\texttt{x} = 3$, $\texttt{y} = 2$; $\texttt{x} = 2$, $\texttt{y} = 2$; and $\texttt{x} = 5$, $\texttt{y} = 3$, etc. So this non-termination is *non-looping* and even *non-periodic* (since there is no fixed sequence of program positions that is repeated infinitely many times). Thus, non-termination cannot be proved by techniques like [10].

To prove non-termination of such methods, we first generate the *termination graph* automatically. Then we construct a formula that represents the loop condition and the computation on each path through the loop. Afterwards, we use an SMT solver to prove that the variable assignments after any run through the loop satisfy the loop condition again, and hence, the loop will be traversed again. If this proof succeeds, then we can conclude non-termination under the condition that at least one run through the loop is possible.

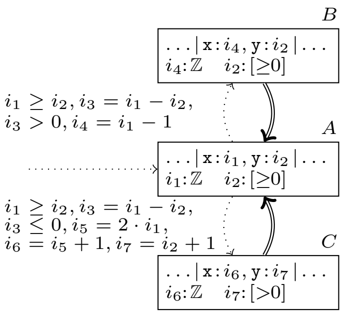
B
$\ldots|\texttt{x}:i_4,\texttt{y}:i_2|\ldots$
$i_4:\mathbb{Z}\quad i_2:[\geq 0]$
$i_1 \geq i_2, i_3 = i_1 - i_2,$
$i_3 > 0, i_4 = i_1 - 1$
A
$\ldots|\texttt{x}:i_1,\texttt{y}:i_2|\ldots$
$i_1:\mathbb{Z}\quad i_2:[\geq 0]$
$i_1 \geq i_2, i_3 = i_1 - i_2,$
$i_3 \leq 0, i_5 = 2 \cdot i_1,$
$i_6 = i_5 + 1, i_7 = i_2 + 1$
C
$\ldots|\texttt{x}:i_6,\texttt{y}:i_7|\ldots$
$i_6:\mathbb{Z}\quad i_7:[>0]$

Figure 2 Graph for `nonLoop`

Our approach is related to *abstract interpretation* [6], since the states in termination graphs are *abstract*, i.e., they represent a (possibly infinite) set of concrete system configurations of the program. For our symbolic evaluation, we *concretize* our states when needed for evaluation and *abstract* them again in order to "merge" similar states (this is needed to obtain a finite representation of all program runs). For our example program, it suffices to regard a simplified version of the termination graph, which is shown in Fig. 2. State A corresponds to the program position in the corresponding JBC program where one has just entered the body of the `while` loop. Thus, here we have two *local variables* `x` and `y` whose values are some integer numbers i_1 and i_2. Moreover, the state contains all information that we have about i_1 and i_2, i.e., i_2 is non-negative, whereas i_1 can be an arbitrary integer (we do not handle overflows).

By repeated symbolic evaluation and case analyses, the abstract program state A can be evaluated to the state B (if $i_3 = i_1 - i_2 > 0$) or to the state C (otherwise), representing the two possible paths through the loop. The states B and C are again at the same program position as A, i.e., at the beginning of the body of the `while` loop. Note that all concrete states that are represented by B or C are also represented by A, i.e., B and C are *instances* of A. Thus, we can draw so-called *instance edges* from both B and C to A, which concludes the construction of the termination graph. For more details on the construction of termination graphs (also for programs operating on the heap), we refer to the full version of the paper [5].

A node in a cycle with a predecessor outside of the cycle is called a *loop head node*. In Fig. 2, A is such a node. We consider all paths $p_1, \ldots, p_n$ from the loop head node back to itself (without traversing the loop head node in between), i.e., $p_1 = A, \ldots, B, A$ and $p_2 = A, \ldots, C, A$. Here, p_1 corresponds to the case where $\mathtt{x} \geq \mathtt{y}$ and $\mathtt{z} = \mathtt{x} - \mathtt{y} > 0$, whereas p_2 handles the case where $\mathtt{x} \geq \mathtt{y}$ and $\mathtt{z} = \mathtt{x} - \mathtt{y} \leq 0$. For each path p_j, we generate a *loop condition formula* φ_j (expressing the condition for entering this path) and a *loop body formula* ψ_j (expressing how the values of the variables are changed in this path).

Essentially, the formulas φ_j and ψ_j result from the constraints on the edges of the cycle. Here, φ_j contains those constraints that express *relations* between integers and ψ_j contains those constraints that express *operations* on integers. In our example, φ_1 is $i_1 \geq i_2 \wedge i_3 > 0$ and φ_2 is $i_1 \geq i_2 \wedge i_3 \leq 0$. Moreover, ψ_1 is $i_3 = i_1 - i_2 \wedge i_4 = i_1 - 1$ and ψ_2 is $i_3 = i_1 - i_2 \wedge i_5 = 2 \cdot i_1 \wedge i_6 = i_5 + 1 \wedge i_7 = i_2 + 1$. We use a labeling function ℓ^k where for any formula ξ, $\ell^k(\xi)$ results from ξ by labeling all variables with ^{k}. We use the labels $^1, \ldots, ^n$ for the paths through the loop and the label ^{r} for the **r**esulting variables (after having traversed the loop once). The question is whether after one run through the loop, one will leave the loop or whether one will stay within the loop. In other words, the question is whether the resulting variables after the first loop iteration violate all of the loop conditions.

$$\rho(p_1, \ldots, p_n) = \underbrace{\mu}_{\text{invariants}} \wedge \underbrace{(\bigvee\nolimits_{j=1}^{n} (\ell^j(\varphi_j) \wedge \ell^j(\psi_j) \wedge \iota_j))}_{\text{first run through the loop}} \quad \wedge \quad \underbrace{(\bigwedge\nolimits_{j=1}^{n} (\neg \ell^r(\varphi_j) \wedge \ell^r(\psi_j)))}_{\text{violation of loop conditions after first loop traversal}}$$

Here, μ is a set of obvious invariants that are known in the loop head node. So as we know "$i_2 \colon [\geq 0]$" in state A, μ is $i_2 \geq 0$ for our example. The formula ι_j connects the variables labeled with ^{j} to the unlabeled variables in μ and to the variables labeled with ^{r} in the formulas after the first loop traversal. So for every integer i in the loop head node, ι_j contains $i = i^j$. Moreover, if there is an instance edge from state s' to the loop head node s and the integer i' in s' corresponds to the integer i in s, then ι_j contains $i'^j = i^r$. For our example, ι_1 is $i_1 = i_1^1 \wedge i_2 = i_2^1 \wedge i_4^1 = i_1^r \wedge i_2^1 = i_2^r$.

Intuitively, satisfiability of the first two parts of $\rho(p_1, \ldots, p_n)$ corresponds to one successful run through the loop. The third part encodes that after the first loop iteration, none of the loop conditions holds anymore. Here, we do not only consider the negated loop conditions $\neg \ell^r(\varphi_j)$, but we also need $\ell^r(\psi_j)$, as φ_j can contain variables computed in the loop body. For example in the method `nonLoop`, $\ell^r(\varphi_1)$ contains $i_3^r > 0$. But to determine how i_3^r results from the "input arguments" i_1^r, i_2^r, one needs $\ell^r(\psi_1)$ which contains $i_3^r = i_1^r - i_2^r$.

The generated formula $\rho(p_1, \ldots, p_n)$ is an existentially quantified formula using non-linear integer arithmetic.[2] If an SMT solver proves unsatisfiability of this formula, we know that whenever a variable assignment satisfies a loop condition, then after one execution of the loop body, a loop condition is satisfied again (i.e., the loop runs forever). Note that we generalized the notion of "loop conditions", as we discover the conditions by symbolic evaluation of the loop. Consequently, we can also handle loop control constructs like `break` or `continue`.

So unsatisfiability of $\rho(p_1, \ldots, p_n)$ implies that the loop is non-terminating, provided that the loop condition can be satisfied at all. To check this, we use an SMT solver to find a model for $\sigma(p_1, \ldots, p_n) = \mu \wedge (\bigvee_{j=1}^{n} (\ell^j(\varphi_j) \wedge \ell^j(\psi_j) \wedge \iota_j))$. Moreover, to prove non-termination of a whole JBC method, one of course also has to prove that the non-terminating loop is reachable from the initial state of the method, cf. [5].

▶ **Theorem 1** (Proving Non-Termination). *Let s be a loop head node in a termination graph where the local variables of s only have integer values, and let $p_1, \ldots, p_n$ be all paths from s*

[2] As most programs do not contain non-linear expressions relevant for termination, $\rho(p_1, \ldots, p_n)$ is linear in most cases.

back to s. Let $\rho(p_1, \ldots, p_n)$ be unsatisfiable and let $\sigma(p_1, \ldots, p_n)$ be satisfiable by some model M. Let c be a concrete state represented by s, where every integer variable in c has been assigned the value given in M. Then c starts an infinite JBC evaluation.[3]

So from the model M of $\sigma(p_1, \ldots, p_n)$, we obtain an instance c of the loop head node where we replace unknown integers by the values in M. For our example, $i_1 = i_1^1 = i_3^1 = 1, i_2 = i_2^1 = i_2^r = i_4^1 = i_1^r = 0$ satisfies $\sigma(p_1, \ldots, p_n)$. From this, we can generate a witness state with $\mathtt{x} = 1$ and $\mathtt{y} = 0$ at the program position of the loop head node which indeed leads to non-termination.

Finally, we have to prove that this witness state is reachable from the initial state of the method `nonLoop`. To this end, we proceed step by step and automatically generate witnesses at preceding program positions by traversing the edges of the termination graph backwards and reversing the effects of the symbolic evaluation (the details of this witness generation are presented in [5]). If this succeeds and a witness state at the position of the initial state of the method could be generated, we present this witness to the user as a non-terminating counterexample. So in this way, we obtain a sound method for non-termination analysis of JBC, although the termination graph usually represents a superset of the executions that are possible in the original JBC program.

3 Evaluation and Conclusion

Based on termination graphs, we presented a technique to prove non-termination of JBC. While the approach presented in this paper fails on non-terminating programs that manipulate the heap or have sub-loops, the full version of the paper [5] also contains techniques to detect periodic non-termination of methods that operate on the heap and that may have sub-loops.

We implemented our approach in the termination tool AProVE [9], using the SMT solver Z3 [7] and evaluated it on a collection of 325 examples which contains all 268 JBC programs from the *Termination Problem Data Base* that is used in the annual *International Termination Competition*[4] and all 55 examples from [15] used to evaluate the Invel tool. For our evaluation, we compared the old version of AProVE (without support for non-termination), the new version AProVE-No, and Julia [14]. We were not able to obtain a running version of Invel, and thus we only compared to the results of Invel reported in [15].

	Invel Examples					Other Examples				
	Y	**N**	**F**	**T**	**R**	**Y**	**N**	**F**	**T**	**R**
AProVE-No	1	51	0	3	5	204	30	12	24	11
AProVE	1	0	5	49	54	204	0	27	39	15
Julia	1	0	54	0	2	166	22	82	0	4
Invel	0	42	13	0	?					

We used a time-out of 60 seconds for each example. "**Y**es" and "**N**o" indicate how often termination (resp. non-termination) could be proved, "**F**ail" states how often the tool failed in less than 1 minute, "**T**" indicates how many examples led to a **T**ime-out, and "**R**" gives the average **R**untime in seconds for each example. The experiments clearly show the power of our contributions, since AProVE-No is the most powerful tool for automated non-termination

[3] For the proof of the theorem, we refer to [5].
[4] We removed a controversial example whose termination depends on integer overflows.

proofs of Java resp. JBC programs. Moreover, the comparison between AProVE-No and AProVE indicates that the runtime for termination proofs did not increase due to the added non-termination techniques. To experiment with our implementation via a web interface and for further details, we refer to [1] and to the full version of the paper [5].

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Termination Analysis via Monotonicity Constraints over the Integers and SAT Solving[*]

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Abstract

We describe an approach for proving termination of programs abstracted to systems of monotonicity constraints in the integer domain. Monotonicity constraints are a non-trivial extension of the size-change termination method. In this setting, termination is PSPACE complete, hence we focus on a significant subset in NP, which we call MCNP, designed to be amenable to a SAT-based solution. We use ranking functions in terms of bounded differences between multisets of integers. Experiments with our approach as a back-end for termination analysis of Java Bytecode with AProVE and COSTA as front-ends reveal a good trade-off between precision and cost of analysis.

Keywords and phrases Termination Analysis, Monotonicity Constraints, SAT Encoding

1 Introduction

For termination analysis, we need a program abstraction that both captures the properties required to prove termination as often as possible and provides a decidable sufficient criterion for termination. Typically, such abstractions describe possible program steps by finitely many abstract transition rules. The abstraction considered in this paper is based on monotonicity-constraint systems (MCSs). The MCS abstraction is an extension of the SCT (size-change termination [4]) abstraction, which has been studied extensively during the last decade.[1] For SCT, an abstract transition rule is specified by a set of inequalities that show how the sizes of program data in the target state decrease compared to the source state. Size is measured by a well-founded base order. These inequalities are often represented by a *size-change graph.*

The size-change technique was conceived to deal with well-founded domains, where infinite descent is impossible. Termination is deduced by proving that any (hypothetical) infinite run would decrease some value monotonically and endlessly, in contradiction to well-foundedness. Extending this approach, a *monotonicity constraint* (MC) allows for any conjunction of order relations (strict and non-strict inequalities) involving any pair of variables from the source and target states. So in contrast to SCT, one may also have relations between two variables in the target state or two variables in the source state, which makes MCSs more expressive than size-change graphs. Another advantage of MCSs is that monotonicity constraints can imply termination under a different assumption—that the data are integers. Not being well-founded, integer data cannot be handled by SCT.

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[†] Part of this author's work was carried out while he was visiting DIKU, the University of Copenhagen.
[‡] Part of this author's work was carried out at LuFG Informatik 2, RWTH Aachen University.
[1] See `http://www2.mta.ac.il/~amirben/sct.html` for a summary and references.

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MCS and SCT both have the drawback that the decision problems for termination are PSPACE complete and a certificate can be of prohibitive complexity. [1] addresses this problem for SCT, identifying an NP complete subclass of SCT, called SCNP, which yields polynomial-size certificates. Moreover, [1] automates SCNP using a SAT solver. In practice, this method has good performance and power compared to a complete SCT decision procedure.

In this paper we tackle the similar termination problem for MCSs in the integer domain. The integer setting is more complicated than the well-founded setting, and termination is often proved by looking at *differences* of certain program values (which should be decreasing and lower-bounded). We use the following approach: (1) We associate two argument sets with each program point and define how to "subtract" them so that the difference can be used for ranking (generalizing the difference of two integers). (2) We introduce a concept of "ranking functions" which is less strict than typically used but still suffices for termination. After setting up the scenario in Sect. 2, Sect. 3 introduces *ranking structures* as termination witnesses and the class MCNP of terminating MCSs, which is in NP. Sect. 4 gives an empirical evaluation and concludes. For further details, please see also the full paper [2].

2 Monotonicity-Constraint Systems and Their Termination

Our method is programming-language independent. It works on an abstraction of the program provided by a front-end (assumed given). An abstract program is a transition system with states expressed in terms of a finite number of variables (*argument positions*).

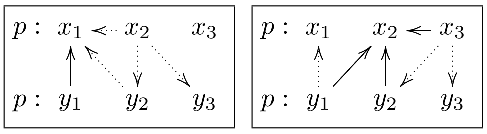
▶ **Definition 1** (monotonicity-constraint system, monotonicity constraint)**.** A *monotonicity-constraint system (MCS)* is an abstract program, represented by a directed multigraph called a *control-flow graph (CFG)*. The vertices are called *program points* and they have fixed numbers (arity) of *argument positions*. A *program state* is an association of a value from $\mathbb{Z}$ to each argument position of a program point p, denoted $p(x_1, \dots, x_n)$ and abbreviated $p(\bar{x})$. The set of all states is denoted St. The arcs of the CFG are associated with *transition rules* $p(\bar{x}) \coloneq \pi;\ q(\bar{y})$ specifying relations on program states. Here π is a *monotonicity constraint (MC)* on $V = \bar{x} \cup \bar{y}$, i.e., π is a conjunction of constraints $x \rhd y$ where $x, y \in V$, and $\rhd \in \{>, \geq\}$. We write $\pi \models x \rhd y$ if $x \rhd y$ is a consequence of π.

We often represent a MC as a directed graph (often denoted by the letter g), with vertices $\bar{x} \cup \bar{y}$, and two types of edges (x, y): strict and weak. If $\pi \models x > y$ then there is a strict edge from x to y and if $\pi \models x \geq y$ (but not $x > y$) then the edge is weak. Note that this paper has *two* kinds of graphs: those for transition rules, and the CFG induced by these rules.

▶ **Definition 2** (run, termination)**.** Let $\mathcal{G}$ be a transition system. A *run* of $\mathcal{G}$ is a sequence $p_0(\bar{x}_0) \xrightarrow{\pi_0} p_1(\bar{x}_1) \xrightarrow{\pi_1} p_2(\bar{x}_2) \dots$ of states labeled by constraints such that each labeled pair of states, $p_i(\bar{x}_i) \xrightarrow{\pi_i} p_{i+1}(\bar{x}_{i+1})$, corresponds to a transition rule $p_i(\bar{x}) \coloneq \pi_i;\ p_{i+1}(\bar{y})$ from $\mathcal{G}$ and such that π_i is satisfied. A transition system *terminates* if it has no infinite run.

▶ **Example 3.** This example presents a MCS in textual form as well as graphical form. This system is terminating, and later we shall prove this using our method. In the graphs, solid arrows stand for strict inequalities and dotted arrows stand for weak inequalities.

$$
\begin{array}{lll}
g_1 = & p(x_1, x_2, x_3) \ \coloneq & y_1 > x_1, y_2 \geq x_1, x_2 \geq y_2, x_2 \geq y_3, x_2 \geq x_1; \qquad\qquad p(y_1, y_2, y_3) \\
g_2 = & p(x_1, x_2, x_3) \ \coloneq & y_1 \geq x_1, y_1 > x_2, y_2 > x_2, x_3 \geq y_2, x_3 \geq y_3, x_3 > x_2;\ p(y_1, y_2, y_3)
\end{array}
$$



3 Ranking Structures for MCSs and MCNP

For a quasi-order $\succsim$, its *strict part* $x \succ y$ is $(x \succsim y) \wedge (y \not\succsim x)$, and it is *well-founded* if there is no infinite $\succ$-chain. A set is well-founded if it has a tacitly-understood well-founded order.

A *ranking function* maps program states into a well-founded set, such that every transition decreases the function's value. Generalizing, we introduce *ranking structures*, which are more flexible than ranking functions, and suffice for most practical termination proofs.

▶ **Definition 4** (anchor, intermittent ranking function)**.** Let $\mathcal{G}$ be a MCS with state space St. Let $(\mathcal{D}, \succsim)$ be a quasi-order and $\mathcal{D}_+$ a well-founded subset of $\mathcal{D}$. Consider a function $\Phi : St \to \mathcal{D}$. We say that $g \in \mathcal{G}$ is a Φ-*anchor* for $\mathcal{G}$ (or that g is *anchored* by Φ for $\mathcal{G}$) if for every run $p_0(\bar{x}_0) \xrightarrow{\pi_0} p_1(\bar{x}_1) \xrightarrow{\pi_1} \ldots \xrightarrow{\pi_{k-1}} p_k(\bar{x}_k) \xrightarrow{\pi_k} p_{k+1}(\bar{x}_{k+1})$ where both $p_0(\bar{x}_0) \xrightarrow{\pi_0} p_1(\bar{x}_1)$ and $p_k(\bar{x}_k) \xrightarrow{\pi_k} p_{k+1}(\bar{x}_{k+1})$ correspond to the transition rule g, we have $\Phi(p_i(\bar{x}_i)) \succsim \Phi(p_{i+1}(\bar{x}_{i+1}))$ for all $0 \le i \le k$, where at least one of these inequalities is strict; and $\Phi(p_i(\bar{x}_i)) \in \mathcal{D}_+$ for some $0 \le i \le k$. Such a function Φ is called an *intermittent ranking function (IRF).*

▶ **Example 5.** Consider the transition rules from Ex. 3 inducing the MCS $\mathcal{G}$. Let $\Phi_1(p(\bar{x})) = max(x_2, x_3) - x_1$. In any run built with g_1 and g_2, the value of Φ_1 is non-negative at least in every state followed by a transition by g_1. Moreover, a transition by g_1 decreases the value strictly and a transition by g_2 decreases it weakly. Hence, g_1 is anchored by Φ_1 for $\mathcal{G}$.

▶ **Definition 6** (ranking structure)**.** Consider $\mathcal{G}$ and $\mathcal{D}$ as in Def. 4. Let $\Phi_1, \ldots, \Phi_m : St \to \mathcal{D}$. Let $\mathcal{G}_1$ consist of all transition rules $g \in \mathcal{G}$ where Φ_1 anchors g for $\mathcal{G}$. For $2 \le i \le m$, let $\mathcal{G}_i$ consist of all transition rules $g \in \mathcal{G} \setminus (\mathcal{G}_1 \cup \ldots \cup \mathcal{G}_{i-1})$ where Φ_i anchors g in $\mathcal{G} \setminus (\mathcal{G}_1 \cup \ldots \cup \mathcal{G}_{i-1})$. We say that $\langle \Phi_1, \ldots, \Phi_m \rangle$ is a *ranking structure* for $\mathcal{G}$ if $\mathcal{G}_1 \cup \ldots \cup \mathcal{G}_m = \mathcal{G}$.

Note that by the above definition, for every $g \in \mathcal{G}$ there is a (unique) $\mathcal{G}_i$ with $g \in \mathcal{G}_i$.

▶ **Example 7.** For the program of Ex. 3, a ranking structure is $\langle \Phi_1, \Phi_2 \rangle$ with Φ_1 as in Ex. 5 and $\Phi_2(p(\bar{x})) = x_3 - x_2$. Here, we have $g_1 \in \mathcal{G}_1$ and $g_2 \in \mathcal{G}_2$.

▶ **Theorem 8.** *If there is a ranking structure for $\mathcal{G}$, then $\mathcal{G}$ terminates.*

The building blocks for our construction are two[2] quasi-orders on multisets of integers and a notion of *level mappings* from program states into pairs of multisets, whose *difference* (not set-theoretic difference; see Def. 13 below) will be used to rank the states. The difference will be itself a multiset, and we use the following relations to order such multisets.

▶ **Definition 9** (multiset types)**.** Let $\wp_n(\mathbb{Z})$ denote the set of multisets of integers of at most n elements, where n is fixed by context.[3] The *μ-ordered multiset type*, for $\mu \in \{ max, min \}$, is the quasi-ordered set $(\wp_n(\mathbb{Z}), \succsim^\mu)$ where:

1. *(max order)* $S \succsim^{max} T$ holds iff $max(S) \ge max(T)$, or T is empty; $S \succ^{max} T$ holds iff $max(S) > max(T)$, or T is empty while S is not.
2. *(min order)* $S \succsim^{min} T$ holds iff $min(S) \ge min(T)$, or S is empty; $S \succ^{min} T$ holds iff $min(S) > min(T)$, or S is empty while T is not.

▶ **Example 10.** For $S = \{10, 8, 5\}$, $T = \{9, 5\}$: $S \succ^{max} T$, $T \succsim^{min} S$.

▶ **Definition 11** (well-founded subset of multiset types)**.** For $\mu \in \{ max, min \}$, we define $(\wp_n(\mathbb{Z}), \succsim^\mu)_+$ as follows: For *min* (respectively *max*) order, the subset consists of the multisets whose minimum (resp. maximum) is non-negative.

▶ **Lemma 12.** *For all $\mu \in \{ max, min \}$, $(\wp_n(\mathbb{Z}), \succsim^\mu)$ is a total quasi-order, with $\succ^\mu$ its strict part; and $(\wp_n(\mathbb{Z}), \succsim^\mu)_+$ is well-founded.*

[2] The full version of this paper [2] additionally uses the *multiset order* and the *dual multiset order.*
[3] For monotonicity-constraint systems, n is the maximum arity of program points.

For MCs over the integers, we consider differences: in the simplest case, we have a "low variable" x that is non-descending and a "high variable" y that is non-ascending, so $y - x$ is non-ascending (and will decrease if x or y changes). If we also have a constraint like $y \geq x$, to bound the difference from below, we can use this for ranking a loop (we refer to this situation as "the Π"—due to the diagram on the right). In the more general case, we consider sets of variables. We will search for a similar Π situation involving a "low set" and a "high set". We next define how to form a difference of two sets so that one can follow the same strategy of "diminishing difference".

$$\begin{tikzcd} x & y \arrow[l, dotted] \arrow[d, dotted] \\ x' \arrow[u, dotted] & y' \end{tikzcd}$$

▶ **Definition 13** (multiset difference). Let L, H be non-empty multisets with types μ_L, μ_H respectively. For $\mu_L \in \{max, min\}$, we define $H - L = \{h - \mu_L(L) \mid h \in H\}$, and $H - L$ has the type of H. (Here, $\mu_L(L)$ signifies $min(L)$ or $max(L)$ depending on the value of μ_L). We write $H \mathrel{\dot{\trianglerighteq}} L$ if the difference belongs to the well-founded subset.

The following lemma provides the intuition for multiset difference as above.

▶ **Lemma 14.** *Let L, H be two multisets of types μ_L, μ_H, let μ_D be the type of $H - L$, and let L', H' be of the types of L, H respectively. Then $H \succsim^{\mu_H} H' \wedge L \precsim^{\mu_L} L' \Rightarrow H - L \succsim^{\mu_D} H' - L'$; $H \succ^{\mu_H} H' \wedge L \precsim^{\mu_L} L' \Rightarrow H - L \succ^{\mu_D} H' - L'$; $H \succsim^{\mu_H} H' \wedge L \prec^{\mu_L} L' \Rightarrow H - L \succ^{\mu_D} H' - L'$.*

Level mappings are functions that facilitate the construction of ranking structures.

▶ **Definition 15** (bi-multiset level mapping, or "level mapping" for short). Let $\mathcal{G}$ be a MCS. A *(bi-multiset) level mapping*, f_{μ_L,μ_H} maps each program state $p(\bar{x})$ to a pair of (possibly intersecting) multisets $p_f^{low}(\bar{x}) = \{ u_1, \ldots, u_l \} \subseteq \bar{x}$ and $p_f^{high}(\bar{x}) = \{ v_1, \ldots, v_k \} \subseteq \bar{x}$ with types indicated respectively by $\mu_L, \mu_H \in \{ max, min \}$. The selection of argument positions only depends on the program point p.

▶ **Example 16.** The following are the level mappings used (in Ex. 23) to prove termination of the program of Ex. 3. Here, each program point p is mapped to $\langle p_f^{low}(\bar{x}), p_f^{high}(\bar{x})\rangle$.

$$f^1_{min,max}(p(\bar{x})) = \langle\{ x_1 \}, \{ x_2, x_3 \}\rangle \qquad f^2_{min,max}(p(\bar{x})) = \langle\{ x_2 \}, \{ x_3 \}\rangle$$

Level mappings are applied to express the diminishing difference of their low and high sets. We also need to express a constraint relating the high and low sets, providing the horizontal bar of "the Π". A transition rule that has such a constraint is called *bounded.*

▶ **Definition 17** (bounded). Let $\mathcal{G}$ be a MCS, f a level mapping (for brevity we sometimes write f instead of f_{μ_L,μ_H}) and $g \in \mathcal{G}$. A transition rule $g = p(\bar{x}) \coloneq \pi; q(\bar{y})$ in $\mathcal{G}$ is called *bounded w.r.t. f* if $\pi \models p_f^{high} \mathrel{\dot{\trianglerighteq}} p_f^{low}$.

▶ **Definition 18** (orienting transition rules). Let f be a level mapping. (1) f *orients* transition rule $g = p(\bar{x}) \coloneq \pi; q(\bar{y})$ if $\pi \models p_f^{high}(\bar{x}) \succsim q_f^{high}(\bar{y})$ and $\pi \models p_f^{low}(\bar{x}) \precsim q_f^{low}(\bar{y})$; (2) f orients g *strictly* if, in addition, $\pi \models p_f^{high}(\bar{x}) \succ q_f^{high}(\bar{y})$ or $\pi \models p_f^{low}(\bar{x}) \prec q_f^{low}(\bar{y})$.

▶ **Example 19.** We refer to Ex. 3 and the level mapping $f^1_{min,max}$ from Ex. 16. Function $f^1_{min,max}$ orients all transition rules, where g_1 is bounded and oriented strictly w.r.t. $f^1_{min,max}$.

▶ **Corollary 20** (of Def. 18 and Lemma 14). *Let f be a level mapping and define $\Phi_f(p(\bar{x})) = p_f^{high}(\bar{x}) - p_f^{low}(\bar{x})$. If f orients $g = p(\bar{x}) \coloneq \pi; q(\bar{y})$, then $\pi \models \Phi_f(p(\bar{x})) \succsim \Phi_f(q(\bar{y}))$; and if f orients g strictly, then $\pi \models \Phi_f(p(\bar{x})) \succ \Phi_f(q(\bar{y}))$.*

The next theorem combines orientation and bounding to show how a level mapping induces anchors. We refer to cycles in the CFG also as "cycles in $\mathcal{G}$".

▶ **Theorem 21.** *Let $\mathcal{G}$ be a MCS and f a level mapping. Let $g = p(\bar{x}) :\!-\ \pi; q(\bar{y})$ be such that every cycle $\mathcal{C}$ including g satisfies these conditions: (1) all transitions in $\mathcal{C}$ are oriented by f, and at least one of them strictly; (2) at least one transition in $\mathcal{C}$ is bounded w.r.t. f. Then g is a Φ_f-anchor for $\mathcal{G}$, where $\Phi_f(p(\bar{x})) = p_f^{high}(\bar{x}) - p_f^{low}(\bar{x})$.*

▶ **Definition 22** (MCNP anchors and ranking functions)**.** Let $\mathcal{G}$ be a MCS and f a level mapping. We say that g is a MCNP-anchor for $\mathcal{G}$ w.r.t. f if f and g satisfy the conditions of Thm. 21. Φ_f is called a *MCNP (intermittent) ranking function (MCNP IRF).* A system of monotonicity constraints is in MCNP if it has a tuple of MCNP IRFs as a ranking structure.

It follows from Thm. 8 that if a MCS is in MCNP, then it terminates.

▶ **Example 23.** Consider again Ex. 3 and Ex. 16. Then, $\langle \Phi_{f^1}, \Phi_{f^2} \rangle$ is a ranking structure for $\mathcal{G}$. The facts in Ex. 19 imply that g_1 is an MCNP-anchor w.r.t. f^1. Moreover, f^2 is both strict and bounded on g_2.

One can now show that MCNP is in NP. Thus, for automation we use a SAT encoding to find termination proofs using an off-the-shelf SAT solver. We invoke a SAT solver iteratively to generate level-mappings and construct a ranking structure $\langle \Phi_1, \Phi_2, \ldots, \Phi_m \rangle$. Details on the algorithm and on the encoding can be found in the full version of this paper [2].

4 Experiments and Conclusion

We implemented a termination analyzer based on our SAT encoding for MCNP and tested it on several benchmark suites. For details on our experiments please see `http://aprove.informatik.rwth-aachen.de/eval/MCNP`. As part of our experiments, we applied our MCNP implementation on MCS abstractions of over 100 Java Bytecode programs which were obtained using the termination tools AProVE and COSTA. Our results show that MCNP is almost as powerful as the back-ends of AProVE and COSTA, and its performance is competitive, especially in comparison to the rewrite-based tool AProVE. Thus, it could be fruitful to use a combination of tools where the MCNP-analyzer is tried first and the rewrite-based analyzer is only applied for the remaining “hard” examples.

To conclude, we introduced a new approach to prove termination of monotonicity-constraint transition systems. The idea is to construct a ranking structure, of a novel kind, extending previous work in this area. For automation, we use a SAT-based approach, which we evaluated in extensive experiments. The results demonstrate the power of our approach and show that its integration into termination tools for Java Bytecode advances the state of the art of automated termination analysis. The full version of this paper has appeared in [2].

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Detecting Non-Looping Non-Termination*

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Abstract

We introduce a technique to prove non-termination of term rewrite systems automatically. In contrast to most previous approaches, our technique is also able to detect non-looping non-termination.

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1 Introduction

To prove constructively that a term rewrite system (TRS) is non-terminating, one has to provide a finite description of how to obtain an infinite rewrite sequence. The most common way to do so is in the form of loops. A loop is a finite sequence of rewrite steps, such that the start term of the sequence is embedded in the final term.

▶ **Definition 1** (Loops). A TRS $\mathcal{R}$ is called *looping*, if there is a term u, a context C, and a substitution μ such that $u \to_{\mathcal{R}}^{+} C[u\mu]$.

Since term rewriting is closed under substitutions and contexts, from any loop it is possible to construct an infinite rewrite sequence $u \to_{\mathcal{R}}^{n} C[u\mu] \to_{\mathcal{R}}^{n} C[C\mu[u\mu^2]] \to_{\mathcal{R}}^{n} \dots$ for some $n > 0$.

To detect loops automatically, one can start with a rule and then repeatedly *narrow* it using other rules (see e.g. [3, 4, 10, 12, 14, 15] for existing work on proving looping non-termination of TRSs). Narrowing is similar to rewriting, but instead of matching the left-hand side of a rule with a subterm, one uses unification. In this way, one constructs longer and longer rewrite sequences $u \to_{\mathcal{R}}^{+} v$. As soon as u *semi-unifies* with a subterm $v|_\pi$ of v (i.e., $u\delta_1\delta_2 = v|_\pi\delta_1$ for some substitutions δ_1 and δ_2), one has found a loop, since

$$u\delta_1 \to_{\mathcal{R}}^{+} v\delta_1 = v\delta_1[v|_\pi\delta_1]_\pi = v\delta_1[u\delta_1\delta_2]_\pi.$$

This approach is suitable for automation, since semi-unification is decidable and algorithms for semi-unification were presented in [6, 8], for example.

2 Non-Looping Non-Termination

While interesting classes and examples of non-looping TRSs were identified in earlier papers (e.g., [2, 13]), up to now virtually all methods to prove non-termination of TRSs automatically

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were restricted to looping non-termination.[1] A notable exception, although restricted to *string* rewrite systems (SRSs), is a technique and tool for non-termination proofs of non-looping SRSs given in [9]. This approach uses an abstract rewrite relation, where rules are not pairs of strings, but pairs of patterns of the form uv^nw. Here, u, v, and w are strings and n can be instantiated by a natural number.

Our goal is to extend this idea in order to obtain a powerful automated technique for non-termination of (possibly non-looping) *term* rewrite systems.

▶ **Example 2.** Let $\mathcal{R}$ be the TRS consisting of the following rules:

$$\begin{aligned} \mathsf{isNat}(\mathsf{0}) &\to \mathsf{true} && (1)\\ \mathsf{isNat}(\mathsf{s}(x)) &\to \mathsf{isNat}(x) && (2)\\ \mathsf{f}(\mathsf{true}, x) &\to \mathsf{f}(\mathsf{isNat}(x), \mathsf{s}(x)) && (3) \end{aligned}$$

This system is non-terminating, but not looping. To see this, note that any infinite rewrite sequence has the following form (up to contexts):

$$\begin{aligned} \mathsf{f}(\mathsf{true}, \mathsf{s}^n(\mathsf{0})) &\to_{\mathcal{R}} \mathsf{f}(\mathsf{isNat}(\mathsf{s}^n(\mathsf{0})), \mathsf{s}^{\mathsf{n+1}}(\mathsf{0}))\\ &\to_{\mathcal{R}}^{n+1} \mathsf{f}(\mathsf{true}, \mathsf{s}^{n+1}(\mathsf{0}))\\ &\to_{\mathcal{R}} \mathsf{f}(\mathsf{isNat}(\mathsf{s}^{n+1}(\mathsf{0})), \mathsf{s}^{n+2}(\mathsf{0}))\\ &\to_{\mathcal{R}}^{n+2} \mathsf{f}(\mathsf{true}, \mathsf{s}^{n+2}(\mathsf{0}))\\ &\ldots \end{aligned}$$

Since the number of steps required to reduce the isNat-terms to true increases in every iteration, this sequence cannot be represented as a loop. In other words, loops cannot capture non-periodic infinite rewrite sequences.

To represent such sequences, we extend the idea of [9] from strings to terms and define so-called *pattern rules* which are parameterized over the natural numbers. Instantiating the parameter results in a pair of terms u, v such that $u \to_{\mathcal{R}}^{+} v$. This allows us to capture certain rewrite sequences of arbitrary length by a finite representation.

For instance, the effect of repeated application of rule (2) on the same position can be captured by the pattern rule

$$\mathsf{isNat}(x)[x/\mathsf{s}(x)]^n \hookrightarrow \mathsf{isNat}(x) \qquad (4)$$

with the parameter n, where $[x/\mathsf{s}(x)]^n$ means that the substitution $[x/\mathsf{s}(x)]$ is applied n times. Then, for every natural number n, the term pair resulting from the instantiation is contained in $\to_{\mathcal{R}}^{n}$ (i.e., $\mathsf{isNat}(\mathsf{s}^n(x)) \to_{\mathcal{R}}^{n} \mathsf{isNat}(x)$).

To prove non-termination, we now proceed in a similar way as in existing techniques to find loops. More precisely, we extend the concept of *narrowing* from ordinary rules to pattern rules. In this way, we can generate new pattern rules that describe longer and longer rewrite sequences. Finally, we use a variant of *semi-unification* to check whether one of the newly obtained pattern rules directly leads to non-termination.

[1] For automated non-termination proofs of *programs*, the situation is similar, i.e., most of the existing automated approaches for non-termination also just detect loops. However for Java Bytecode, we recently presented an approach that can also detect certain forms of non-looping and non-periodic non-termination automatically, based on SMT solving [1]. But an adaption of that approach to term rewriting does not seem to be promising, since [1] can only handle non-periodic non-termination in cases where there are no sub-loops and where non-termination is due to operations on integers. Thus, this approach is not suitable for TRSs where one treats terms instead of integers and where sub-loops (i.e., recursively defined auxiliary functions like isNat) are common.

To illustrate the narrowing of pattern rules, we first narrow the pattern rule (4) with rule (1). In other words, the variable x in the pattern rule is instantiated by the term $\mathsf{0}$, and then the resulting right-hand side $\mathsf{isNat}(\mathsf{0})$ is rewritten using (1). This yields the new pattern rule

$$\mathsf{isNat}(x)[x/\mathsf{s}(x)]^n[x/\mathsf{0}] \quad\hookrightarrow\quad \mathsf{true}. \tag{5}$$

While we narrowed a pattern rule with an ordinary rule above, it is also possible to do the converse, i.e., one can narrow an ordinary rule with a pattern rule. To see this, we now narrow rule (3) with the pattern rule (5) which yields

$$\mathsf{f}(\mathsf{true},x)[x/\mathsf{s}(x)]^n[x/\mathsf{0}] \quad\hookrightarrow\quad \mathsf{f}(\mathsf{true},\mathsf{s}(x))[x/\mathsf{s}(x)]^n[x/\mathsf{0}]. \tag{6}$$

Moreover, one can even narrow pattern rules with pattern rules.

The following theorem extends the semi-unification criterion to pattern rules, in order to detect whether a pattern rule directly leads to non-termination. For a pattern rule $u\sigma^n\mu \hookrightarrow v\sigma^n\mu$, we do not only check whether the *base term* u on the left-hand side semi-unifies with the base term v on the right-hand side. In addition, one may also apply the pattern substitution σ arbitrary many times to u before performing the semi-unification.

▶ **Theorem 3** (Detecting Non-Termination of Pattern Rules). *Let $u\sigma^n\mu \to_{\mathcal{R}}^+ v\sigma^n\mu$ for all natural numbers n. If there are a position π of v, a natural number $k \in \mathbb{N}$, and two substitutions δ_1 and δ_2 such that $u\sigma^k\delta_1\delta_2 = v|_\pi\delta_1$ and such that both δ_1 and δ_2 commute*[2] *with both σ and μ, then $\mathcal{R}$ is non-terminating.*

Proof. We show that for all natural numbers n, the term $u\sigma^n\mu\delta_1$ rewrites to a term containing a subterm that is an instance of $u\sigma^{k+n}\mu\delta_1$. By repeating these rewrite steps on that subterm, we obtain an infinite rewrite sequence. Here, we denote the superterm relation by $\trianglerighteq$.

$$\begin{array}{lll}
 & u\sigma^n\mu\delta_1 & \\
\to_R^+ & v\sigma^n\mu\delta_1 & \text{by rewriting} \\
= & v\delta_1\sigma^n\mu & \text{since } \delta_1 \text{ commutes with both } \sigma \text{ and } \mu \\
\trianglerighteq & u\sigma^k\delta_1\delta_2\sigma^n\mu & \text{since } u\sigma^k\delta_1\delta_2 = v|_\pi\delta_1 \\
= & u\sigma^k\delta_1\sigma^n\mu\delta_2 & \text{since } \delta_2 \text{ commutes with both } \sigma \text{ and } \mu \\
= & u\sigma^{k+n}\mu\delta_1\delta_2 & \text{since } \delta_1 \text{ commutes with both } \sigma \text{ and } \mu
\end{array}$$

◀

In our example, the criterion of Thm. 3 can easily detect non-termination of the pattern rule (6), i.e., of $\mathsf{f}(\mathsf{true},x)\sigma^n\mu \hookrightarrow \mathsf{f}(\mathsf{true},\mathsf{s}(x))\sigma^n\mu$ where $\sigma = [x/\mathsf{s}(x)]$ and $\mu = [x/\mathsf{0}]$. Let $k = 1$. Then we have $u\sigma^k = v$, i.e., the term $\mathsf{f}(\mathsf{true},x)\sigma^k$ is equal to the base term $\mathsf{f}(\mathsf{true},\mathsf{s}(x))$ of the right-hand side of the pattern rule (6). Thus, we choose δ_1 and δ_2 to be the identity. Since then δ_1 and δ_2 trivially commute with σ and μ, non-termination of the original TRS in Ex. 2 follows from Thm. 3.

3 Conclusion

We introduced a new technique to prove non-termination of possibly non-looping TRSs automatically. The technique extends and subsumes previous approaches to detect loops which were based on narrowing and semi-unification. To this end, we adapted an idea of

[2] We say that two substitutions δ and σ *commute* if $\delta\sigma = \sigma\delta$.

[9] from string rewriting to term rewriting and introduced pattern rules which represent a whole class of rewrite sequences. Afterwards, we extended narrowing and semi-unification to pattern rules. The technical details and extensions of our approach will be presented in a forthcoming full version of the paper.

We implemented the resulting non-termination prover in the tool AProVE [5] and compared the new version AProVE-NL (for non-loop) with the previous version AProVE '11 and 3 other powerful tools for non-termination of TRSs (NTI [10], $\mathsf{T_TT_2}$ [7], VMTL [11]). We ran the tools on the 1438 TRSs of the *Termination Problem Data Base* (*TPDB*) used in the annual *International Termination Competition*.[3] In the table, we consider those 241 TRSs of the TPDB where at least one tool proved non-termination. Moreover, we also tested the tools on a selection of 58 typical non-looping non-terminating TRSs from different sources (*"nl"*). We used a time-out of 1 minute for each example. "**N**" indicates how often **N**on-termination was proved and "**R**" gives the average **R**untime in seconds for each example. Thus, AProVE-NL could solve 75.9 % of the non-looping examples without compromising its power on looping examples, whereas the other tools cannot handle non-looping non-termination. For further details on the evaluation, we also refer to the forthcoming full version of the paper.

	TPDB		*nl*	
	N	**R**	**N**	**R**
AProVE-NL	232	6.6	44	5.2
AProVE '11	228	6.6	0	60.0
NTI	214	7.3	0	60.0
$\mathsf{T_TT_2}$	208	9.1	0	60.0
VMTL	95	16.5	0	42.8

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Binomial Interpretations[*]

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Abstract

Polynomial interpretations can be used for proving termination of term rewrite systems. In this note, we contemplate *binomial* interpretations based on binomial coefficients, and show that they form a suitable basis for obtaining (weakly) monotone algebras. The main motivation is that this representation covers examples with negative coefficients like $f(x) = 2x^2 - x + 1$, and even some polynomials with rational coefficients like $f(x) = x(x-1)/2$ that map natural numbers to natural numbers.

1 Introduction

Using well-founded monotone algebras is a general and common method for proving termination of term rewrite systems. Many algebras have been suggested for this purpose. Here we are mainly interested in polynomial interpretations (introduced by Lankford, [6]). In [8] it is shown among other things that polynomial interpretations over the real numbers do not subsume polynomial interpretations over the natural numbers. Ultimately, the reason for this surprising result lies in the fact that there are polynomials that are non-negative for every natural number, but negative when evaluated for some real numbers. The example $f(x) = x(x-1)/2$ shows that there are polynomials with non-integer coefficients that nevertheless evaluate to integers at every integer argument. Binomial functions (see below) capture these polynomials precisely.

We are not the first to use binomial functions this way. Girard et al. [1] extend linear logic with resources bounded by *resource polynomials*, which are binomial functions with non-negative coefficients. In more recent work, Hofmann et al. [4, 3] use resource polynomials for amortized resource analysis of programs. The observation that binomial functions are closed under composition is much older. The earliest appearances that we are aware of originate in the study of nilpotent groups [2] and of recursively equivalent sets [7].

In the remainder of the paper, we exhibit some fundamental properties of binomial coefficients in Section 2, then sketch binomial interpretations in Section 3. In Section 4, we compare the power of binomial interpretations to standard polynomial interpretations.

2 Fundamentals

▶ **Definition 1.** For $n \in \mathbb{N}$ (where $\mathbb{N}$ is the set of non-negative integers), and x element of some ring, the *falling power*, $x^{\underline{n}}$, is defined as follows: (This notation is used by Knuth in [5])

$$x^{\underline{0}} = 1 \qquad\qquad x^{\underline{n}} = x \cdot (x-1)^{\underline{n-1}}$$

Falling powers are closely related to binomial coefficients. In fact, we can define binomial coefficients in terms of falling powers.

▶ **Definition 2.** For $k \in \mathbb{N}$, the *binomial coefficient* $\binom{x}{k}$ is defined as $\binom{x}{k} = \frac{x^{\underline{k}}}{k!}$.

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▶ Remark. Over some rings, the fraction $\frac{x^k}{k!}$ may not have a value for some x. For example, in the polynomial ring $\mathbb{Z}[x]$, $\binom{x}{2}$ does not exist. The fraction may also have several values (if the ring is not torsion-free). But we will only work over $\mathbb{Z}$ or $\mathbb{R}$, where this does not happen.

It is clear from the definition that the binomial coefficient $\binom{x}{k}$ is a polynomial in x of degree k with rational coefficients. It is well known that $\binom{x}{k} \in \mathbb{Z}$ whenever $x \in \mathbb{Z}$.

▶ **Lemma 3.** *Binomial coefficients satisfy a tremendous number of identities. We exhibit two of them. (Only (1) is used later, but the second one gives some insight into why binomial functions are closed under composition.)*

$$\binom{x+1}{k+1} - \binom{x}{k+1} = \binom{x}{k} \tag{1}$$

$$\binom{x+y}{k} = \sum_{i=0}^{k} \binom{x}{i}\binom{y}{k-i} \tag{2}$$

3 Binomial Interpretations

▶ **Definition 4.** A *monomial* over the variables V is a finite product $\prod_{v\in V'} \binom{v}{k_v}$ such that $V' \subseteq V$ and $0 < k_v \in \mathbb{N}$. We write 1 if the product is empty and v for $\binom{v}{1}$. A *binomial function* f over a domain D is a linear combination of monomials, $f(v_1,\dots,v_n) = \sum_{m\in M} a_m \cdot m$ where $a_m \in D$ and M is a finite set of monomials over the variables $V = \{v_1,\dots,v_n\}$. We can evaluate binomial functions in the obvious way, substituting values for the formal variables.

We define a difference operator on binomial functions, justified by the identity (1): We let $\Delta_v \left(\sum_{m\in M} a_m \cdot m\right) = \sum_{m\in M} a_m \cdot \Delta_v m$, where on monomials, $\Delta_v \prod_{w\in V} \binom{w}{k_w} = 0$ if $v \notin V$ or $k_v = 0$. Otherwise, $\Delta_v \prod_{w\in V} \binom{w}{k_w} = \binom{w}{k'_w}$ where $k'_v = k_v - 1$ and $k'_w = k_w$ for $w \neq v$. It is easy to see that $f(v_1,\dots,v_i+1,\dots,v_n) - f(v_1,\dots,v_i,\dots,v_n) = (\Delta_{v_i} f)(v_1,\dots,v_i,\dots,v_n)$ for all binomial functions f and variables v_i.

It is known that binomial functions over $\mathbb{N}$ are closed under addition, multiplication and composition [1]. In practice, the best way to compute the results of these operations appears to be to use the identity $(\Delta_v^n \binom{v}{k})(0) = \delta_{n,k}$, where $\delta_{n,k} = 1$ if $n = k$ and $\delta_{n,k} = 0$ otherwise. Once the degree d of a unary binomial function $f(x)$ is known, one can compute its coefficients from $f(0), f(1), \dots, f(d)$. This can be extended to multiple variables by treating a binomial function in V with $v \in V$ as a unary function in v with coefficients that are binomial functions over $V \setminus \{v\}$. The difference operator also plays a crucial role in showing that all integer-valued (over the integers) polynomials can be expressed by binomial functions, as follows. Let $f(v)$ be an integer-valued polynomial of degree $d > 0$. Then $(\Delta_v f)(v) = f(v+1) - f(v)$ is an integer-valued polynomial of degree $d-1$. The function f can be reconstructed from $\Delta_v f$ and $f(0)$, and ultimately from the values $f_i = (\Delta_v^i f)(0)$ for $0 \leqslant i \leqslant d$, and we have just seen that these define a binomial function. In fact, $f(v) = \sum_{i=0}^{d} f_i \binom{v}{i}$.

▶ **Definition 5.** Let $\mathcal{F}$ be a signature where each $f \in \mathcal{F}$ has an arity $\mathsf{ari}(f)$. Furthermore let $V = \{v_1, v_2, \dots\}$ be a countable set of variables. A *binomial $\mathcal{F}$-algebra $\mathcal{A}$ over a domain D* assigns to each $f \in \mathcal{F}$ an interpretation $f_{\mathcal{A}}$ that is a binomial function over D with variables $\{v_1, \dots v_{\mathsf{ari}(f)}\}$. A binomial $\mathcal{F}$-algebra induces an $\mathcal{F}$-algebra with carrier D by evaluating the binomial functions.

To use a binomial $\mathcal{F}$-algebra for proving termination of a TRS $\mathcal{R}$, it has to induce a well-founded monotone algebra that is compatible with $\mathcal{R}$.

▶ **Theorem 6.** *Let $\mathcal{A}$ be a $\mathcal{F}$-algebra over $\mathbb{N}$. Then $\mathcal{A}$ is a well-founded monotone algebra, provided that for all $f \in \mathcal{F}$, $f_{\mathcal{A}}(v_1, \dots, v_n) = \sum_{m \in M} a_m \cdot m$ implies $a_{v_i} \geqslant 1$ for $1 \leqslant i \leqslant n$. Furthermore, for any binomial function $f(v_1, \dots, v_n) = \sum_{m \in M} a_m \cdot m$ over $\mathbb{N}$, we have $f(v_1, \dots, v_n) > 0$ for all possible values of $v_i \in \mathbb{N}$, if, and only if, $a_1 \geqslant 0$.*

Proof. Note that over $\mathbb{N}$, all binomial functions are weakly monotone and nowhere negative. The theorem follows easily from that observation. ◀

4 Comparison to Polynomial Interpretations

We will show below that neither polynomial interpretations over $\mathbb{R}$ nor over $\mathbb{N}$ subsume binomial interpretations. Note that *linear* binomial interpretations are identical to linear polynomial interpretations over the integers—the increased power requires higher degree polynomials. Using the method by Neurauter and Middeldorp [8], which can force weakly compatible polynomial interpretations to be linear with non-integer coefficients, it is clear that binomial interpretations do not subsume polynomial interpretations over $\mathbb{Q}$. On the other hand, if negative coefficients are allowed, binomial interpretations subsume polynomial interpretations over $\mathbb{N}$ with integer coefficients, by way of the identity

$$x^k = \sum_{i=0}^{k} i! \left\{ {n \atop i} \right\} \binom{x}{i}$$

where $\left\{ {n \atop i} \right\}$ denotes Stirling numbers of the second kind, which are non-negative integers. The same relation allows us to transform polynomial interpretations with non-negative coefficients to binomial interpretations with non-negative coefficients.

We adapt an example from [8] to show that binomial interpretations are not subsumed by polynomial interpretations over $\mathbb{R}$ or $\mathbb{N}$. Let $\mathcal{R}$ be the following TRS.

$$\begin{aligned}
\mathsf{s}(0) &\to \mathsf{f}(0) && (1) & \mathsf{s}(\mathsf{f}(\mathsf{s}(x))) &\to \mathsf{h}(\mathsf{f}(x), \mathsf{g}(x)) && (6)\\
\mathsf{s}^2(0) &\to \mathsf{f}(\mathsf{s}(0)) && (2) & \mathsf{f}(\mathsf{g}(\mathsf{s}(x))) &\to \mathsf{g}(\mathsf{g}(\mathsf{f}(\mathsf{s}(x)))) && (7)\\
\mathsf{g}(x) &\to \mathsf{h}(x, \mathsf{h}(x, x)) && (3) & \mathsf{h}(\mathsf{s}^2(x), \mathsf{h}(x, x)) &\to \mathsf{g}(x) && (8)\\
\mathsf{s}(x) &\to \mathsf{h}(0, x) && (4) & \mathsf{s}(x) &\to \mathsf{h}(x, 0) && (9)\\
\mathsf{g}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{s}(\mathsf{g}(x))) && (5) & \mathsf{h}(\mathsf{f}(x), \mathsf{s}(\mathsf{g}(x))) &\to \mathsf{f}(\mathsf{s}(x)) && (10)
\end{aligned}$$

▶ **Theorem 7.** *Termination of the TRS $\mathcal{R}$ can be shown by a binomial interpretation.*

Proof. We let $[0] = 0$, $[\mathsf{s}](x) = x+1$, $[\mathsf{f}](x) = 3\binom{x}{2}+x$, $[\mathsf{g}](x) = 3x+1$, $[\mathsf{h}](x, y) = x+y$. These are strictly monotone functions on $\mathbb{N}$. For compatibility with $\mathcal{R}$, we obtain the following constraints.

$$\begin{aligned}
1 &> 0 && (1) & 3\tbinom{x}{2} + 4x + 2 &> 3\tbinom{x}{2} + 4x + 1 && (6)\\
2 &> 1 && (2) & 27\tbinom{x}{2} + 48x + 22 &> 27\tbinom{x}{2} + 36x + 13 && (7)\\
3x + 1 &> 3x && (3) & 3x + 2 &> 3x + 1 && (8)\\
x + 1 &> x && (4) & x + 1 &> x && (9)\\
3x + 4 &> 3x + 3 && (5) & 3\tbinom{x}{2} + 4x + 2 &> \tbinom{x}{2} + 4x + 1 && (10)
\end{aligned}$$

Since these constraints are all satisfied, we conclude that $\mathcal{R}$ is terminating. ◀

▶ **Theorem 8.** *Termination of $\mathcal{R}$ cannot be shown using polynomial interpretations over $\mathbb{N}$ or $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geqslant 0\}$.*

Proof. The argument follows that from [8]. We will first show that regardless of the domain $\mathbb{N}$ or $\mathbb{R}^+$, a polynomial interpretation that is compatible with $\mathcal{R}$ must assign $[\mathsf{f}]$ a quadratic polynomial with leading coefficient $\frac{3}{2s_0}$ for some $s_0 \in \mathbb{N}$, ruling out $\mathbb{N}$ as a domain. The second part of the proof is devoted to showing that $[\mathsf{s}](x) = x + \delta$, where δ is the parameter defining the well-founded order $<_\delta$ on $\mathbb{R}^+$. Using this fact we will conclude that $[\mathsf{f}](x) < 0$ for some $x \in \mathbb{R}$, establishing the claim for the domain $\mathbb{R}^+$.

For the first part, we can treat both domains $\mathbb{N}$ and $\mathbb{R}^+$ simultaneously, as follows. When working over $\mathbb{R}^+$, we use $>_\delta$ as well-founded order and $\geqslant$ as compatible quasi-order to obtain a well-founded monotone algebra, where $a >_\delta b$ iff $a \geqslant b + \delta$ and $\delta > 0$ is a fixed real numgber. Over $\mathbb{N}$, the well-founded order and quasi-order are $>$ and $\geqslant$, respectively. If we let $\delta = 1$ over $\mathbb{N}$, then $>_\delta = >$, and the two definitions of the orders coincide.

Assume that we are given polynomials $[\mathsf{0}] = z$, $[\mathsf{s}] = s$, $[\mathsf{f}] = f$, $[\mathsf{g}] = g$ and $[\mathsf{h}] = h$ with coefficients in $\mathbb{R}$ ($\mathbb{Z}$) for domain $\mathbb{R}^+$ ($\mathbb{N}$). Furthermore let these polynomials be strictly monotone with respect to $>_\delta$ over the domain and compatible with $\mathcal{R}$. To establish compatibility with the rules, we evaluate both sides of all rules and compare the resulting polynomials. First consider rules (7) and (5), and compare the degrees of both sides: We have $\deg(f)\deg(g) \geqslant \deg(g)^2\deg(f)$ and $\deg(g)\deg(s) \geqslant \deg(s)^2\deg(g)$, from which we conclude that $\deg(g) = \deg(s) = 1$ (note that because of strict monotonicity, none of the polynomials can be constant). So $g(x) = g_1x + g_0$ and $s(x) = s_1x + s_0$ for some $g_1, s_1 \geqslant 1$ and $g_0, s_0 \geqslant 0$. Furthermore by comparing the leading coefficients of (5), namely $g_1s_1 \geqslant s_1^2g_1$ we see that $s_1 = 1$. Next we find constraints on h. To that end, consider (3). Since the left-hand side evaluates to a linear polynomial, so must the right-hand side. Therefore, we may assume that $h(x, y) = h_xx + h_yy + h_0$ where $h_x, h_y \geqslant 1$ and $h_0 \geqslant 0$. By comparing leading coefficients of (4) and (9) we find that $s_1 \geqslant h_x$ and $s_1 \geqslant h_y$, i.e., $h_x = h_y = 1$. Using these values, we can find a lower bound on s_0 from the compatibility of (9), namely $s_0 \geqslant z_0 + h_0 + \delta$, which implies $s_0 \geqslant \delta > 0$. Finally we find a bound on the degree of f. Using (10) we conclude that $x + 2s_0 + h_0 \geqslant f(x + s_0) - f(x)$ Because $s_0 \geqslant \delta > 0$, the degree of $f(x + s_0) - f(x)$ is one less than that of $f(x)$, and since the left-hand side is linear, f can at most be quadratic. To summarize, we can express z, s, f, g and h as follows.

$$z = z_0 \qquad s(x) = x + s_0 \qquad f(x) = f_2x^2 + f_1x + f_0$$
$$g(x) = g_1x + g_0 \qquad h(x, y) = x + y + h_0$$

We also know that $z_0, f_0, g_0, h_0 \geqslant 0$ and $s_0 \geqslant \delta$. Next we compare the leading coefficients in (3,8). For (3), we get $g_1 \geqslant 3$, while for (8), $3 \geqslant g_1$. Therefore, $g_1 = 3$.

Now let us determine f_2. From compatibility of (6) we find that $f_2x^2 + (2f_2s_0 + f_1)x + O(1) >_\delta f_2x^2 + (f_1 + g_1)x + O(1)$, where $O(1)$ stands for a constant term not containing x. From this we conclude that $2f_2s_0 \geqslant g_1$. Similarly from compatibility (10) we have $f_2x^2 + (f_1 + g_1)x + O(1) >_\delta f_2x^2 + (2f_2s_0 + f_1)x + O(1)$, which implies $g_1 \geqslant 2f_2s_0$. Therefore, $f_2 = \frac{g_1}{2s_0} = \frac{3}{2s_0}$. In particular, no polynomial interpretation over $\mathbb{N}$ can exist, because s_0 and f_2 cannot both be integers.

Therefore, from now on, we assume that we are given a polynomial interpretation over $\mathbb{R}$. Our next step will be to determine s_0. We already know that $s_0 \geqslant \delta$. By strict monotonicity, we must have $f(\delta) - f(0) \geqslant \delta$, which is equivalent to $f_2\delta + f_1 \geqslant 1$. Now consider (2). We have $z_0 + 2s_0 - \delta \geqslant f_2(z_0 + s_0)^2 + f_1(z_0 + s_0) + f_0 \geqslant f_2s_0(z_0 + s_0) + (1 - f_2\delta)(z_0 + s_0)$. Therefore, $s_0 - \delta \geqslant f_2(z_0 + s_0)(s_0 - \delta) \geqslant f_2s_0(s_0 - \delta) = \frac{3}{2}(s_0 - \delta)$, which implies $\delta \geqslant s_0$, from which we conclude that $s_0 = \delta$. Using (4), this implies $z_0 + h_0 \leqslant 0$, i.e., $z_0 = h_0 = 0$. Then, from (1), we conclude that $f_0 = 0$. Finally, we consider (2) once more. Compatibility now implies $2\delta - \delta \geqslant \frac{3}{2}\delta + f_1\delta$, or $-\frac{1}{2} \geqslant f_1$. This, however, leads to a contradiction, since

$f(\frac{\delta}{6}) = \frac{3}{2\delta} \cdot \frac{\delta^2}{6^2} + f_1 \frac{\delta}{6} \leqslant \frac{1}{24}\delta - \frac{1}{12}\delta < 0$ lies outside the domain $\mathbb{R}^+$. ◀

5 Conclusion

We have described an extension of polynomial interpretations with integer coefficients using binomial coefficients. These binomial interpretations arise naturally as a characterization of integer-valued polynomials with integer arguments and rational coefficients. We have also shown that binomial interpretations are not subsumed by polynomial interpretations over the real numbers.

As future work, we plan to incorporate binomial interpretations into $\mathsf{T_TT_2}$.

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On Modularity of Termination Properties of Rewriting under Strategies

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Abstract

The modularity of termination and confluence properties of term rewriting systems has been extensively studied, for disjoint unions and other more types of combinations. However, for rewriting under strategies the theory is less well explored. Here we extend the modularity analysis of termination properties systematically to (variants of) innermost and outermost rewriting. It turns out — as expected — that in essence innermost rewriting behaves nicely w.r.t. modularity of termination properties, whereas this is not at all the case for outermost rewriting, at least not without further assumptions.

Keywords and phrases Modularity, Preservation under Signature Extensions, Termination Properties, Rewriting under Strategies

1 Introduction and Overview

Whereas most known modularity results refer to unrestricted rewriting, cf. e.g. [2]–[16], in applications and programming language contexts one very often has restrictions imposed on the evaluation mechanism like (position-based) strategies. For instance, *innermost* rewriting closely corresponds to *eager* evaluation and *call-by-value* whereas *outermost* rewriting is close to *lazy* evaluation and *call-by-name*. Here we will study the modularity behaviour of normalization and termination of (different versions of) innermost and outermost rewriting. It will turn out that in this regard innermost rewriting has nice properties (which is not very surprising) whereas outermost rewriting is highly non-modular.

We will entirely focus here on the case of *disjoint* unions, cf. e.g. [14, 13]. Most results easily extend to slightly more general combinations like (at most) *constructor sharing* or *composable* systems, cf. e.g.[10]. The interference of the usual modularity analysis, taking into account the *layered* structure of *mixed* terms with strategy-based restrictions of rewriting steps is in general (highly) non-trivial, especially for the case of outermost rewriting.

The remainder of this extended abstract is structured as follows. In Section 2 we will very briefly recall some notions and notations. In the main Section 3 we first review what is known and then study the modularity of weak and strong termination properties of (variants of) innermost and outermost rewriting. As a next step we analyze under which additional assumptions modularity can be recovered (for outermost rewriting). Then, motivated by the negative results, we study a very special case of modularity, namely preservation under signature extensions. Finally, we briefly discuss directions for further research.

Due to lack of space we omit any proofs. Yet, for some negative results we give concrete counterexamples.

2 Preliminaries

We assume familiarity with the basics of term rewriting and of modularity in term rewriting (cf. e.g. [2], [4], [11]).

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We will deal with *(term) rewriting systems* (TRS) $\mathcal{R}^{\mathcal{F}} = (\mathcal{F}, R)$ consisting of (a signature $\mathcal{F}$ and) rules $l \to r$ over some signature $\mathcal{F}$ and set of variables $\mathcal{V}$. The rules $l \to r$ satisfy two conditions: The left-hand side l must not be a variable, and every variable appearing in r also appears in l. The *rewrite relation* induced by a TRS $\mathcal{R}$ is denoted by $\to_{\mathcal{R}}$ or $\to$ if $\mathcal{R}$ is clear from the context or irrelevant. We sometimes use notations like $s \to_q t$ or $s \to_{>\epsilon} t$ to indicate that the position of the redex contraction is q or is strictly below the root, repectively.

Innermost rewriting $\to_i$ is defined as follows, slightly abusing notation: $s \to_i t$ if t is obtained from s by contracting an *innermost redex*, i.e., a subterm of s which is reduced at the root such that all its proper subterms are in normal form. *Leftmost innermost rewriting* is defined by $s \to_{li} t$ if $s \to_i t$ such that in this step a leftmost innermost redex is contracted. *(Maximal) parallel innermost rewriting* $\to_{pi}$ is given by $s \to_{pi} t$ if $s = C[s_1, \ldots, s_n]_{p_1,\ldots,p_n} \to_i^* C[t_1, \ldots, t_n]_{p_1,\ldots,p_n} = t$ such that s_{p_i}, $1 \le i \le n$ are all the innermost redexes of s and $s_i \to_{i,\epsilon} t_i$ for all i. Analogously, the relations *outermost rewriting* $\to_o$, *leftmost outermost rewriting* $\to_{lo}$ and *(maximal) parallel outermost rewriting* $\to_{po}$ are defined. Note that whereas a parallel innermost step can always be sequentialized into a sequence of ordinary innermost steps, the analogous property does not hold in general for parallel outermost rewriting.

An orthogonal TRS is *left-normal* if in every rule $l \to r$ the constant and function symbols in the left-hand side precede (in the linear term notation) the variables.

Two TRSs $\mathcal{R}_1^{\mathcal{F}_1}$ and $\mathcal{R}_2^{\mathcal{F}_2}$ are *disjoint* if $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ (which then also implies $R_1 \cap R_1 = \emptyset$).

Finally, a *modular reduction* step $s \rightsquigarrow t$ means normalization of s in one system (i.e., reduction in one system to normal form w.r.t. that system), cf. [9].

3 Modularity of Termination Properties of Rewriting under Strategies

We will consider innermost and outermost rewriting as well as variants thereof, namely *leftmost* and *(maximal) parallel* versions of both, as well as weak termination and termination, also known as weak normalization (WN) and strong normalization (SN), respectively.

First let us recall in Table 1 the main basic results that are known concerning modularity of WN and SN, without mentioning the many advanced results about (non-)modularity of termination of standard rewriting.

property	is modular?	reason/reference
SN	$-$	[14, 13]
WN	$+$	[15, 16], [5], [3], [9]
SN($\rightsquigarrow$)	$+$	[9]
SIN	$+$	[6]
WIN	$+$	[6]

Table 1 Some known modularity results for termination properties of standard rewriting

From this table it is clear that — apart from termination properties of general rewriting — only (modularity of termination of) innermost rewriting has been studied to some extent, but outermost rewriting not at all, to the best of our knowledge. In the sequel we will investigate modularity of both termination (SN) and weak termination (WN) for standard, leftmost and (maximal) parallel innermost as well as outermost rewriting. For brevity we use the following abbreviations: WIN = WN($\to_i$), WLIN = WN($\to_{li}$), WPIN = WN($\to_{pi}$), SIN = SN($\to_i$), SLIN = SN($\to_{li}$), SPIN = SN($\to_{pi}$).

For a better understanding of the following tables let us mention that normalization and termination, respectively, of innermost rewriting remain invariant, when the variants $\rightarrow_{li}$ or $\rightarrow_{pi}$ are used instead of $\rightarrow_i$.

▶ Fact 3.1 (*selection invariance* for innermost rewriting). WIN $\Longleftrightarrow$ WLIN $\Longleftrightarrow$ WPIN and SIN $\Longleftrightarrow$ SLIN $\Longleftrightarrow$ SPIN.

This property, called *selection invariance* in [8] (cf. also [7]), seems to be 'folklore knowledge' in rewriting. A formal proof of a particular case (namely the equivalence of SLIN and SIN) is given in [8, Theorem]. Table 2 below shows which of the termination properties of innermost and outermost rewriting are modular, and which are not in general.

Table 3 then exhibits which of the negative results even hold for orthogonal TRSs (and which positive results hold for orthogonal systems). Observe that in left-normal normalizing TRSs leftmost-outermost reduction is normalizing, hence terminating (cf. e.g. [4]. Furthermore, in orthogonal systems we clearly have WLON $\Longleftrightarrow$ SLON.

property	is modular?	reason/reference
SIN,SLIN,SPIN	+	Table 1, Fact 3.1
WIN,WLIN,WPIN	+	Table 1, Fact 3.1
SON	−	Table 4
WON	−	Table 4
SLON	−	Table 4
WLON	−	Table 4
SPON	−	Table 4
WPON	−	Table 4

Table 2 Modularity of innermost and outermost termination properties

property	is modular?
SON	−
WON, WPON, SPON[1]	+
WLON, SLON	− (but holds for left-normal TRSs)

Table 3 Modularity of outermost termination properties for orthogonal TRSs

Further easy positive results are possible by requiring non-collapsingness of the TRSs involved, which we do not detail here. Another question is, whether the negative results of Table 2 turn into positive ones, at least for the very special case of signature extensions. But as shown below in Table 4, this is only the case for left-linear TRSs. Observe that the positive preservation results in Table 4 for left-linear TRSs crucially rely on the property of left-linear systems that in a term $s = s[l\sigma]$, the redex $s|_p = l\sigma$ (for some rule $l \to r$) is still a redex after reducing in the 'substitution part' of $l\sigma$.

In the following we present a few counterexamples supporting some of the previous negative claims.

[1] Note for left-normal orthogonal TRSs any outermost rewriting strategy is well-known to be normalizing, not only parallel outermost. Thus, in this case the properties WON, WPON and SONcoincide.

property	is preserved under signature extensions?
SON	– (but holds for left-linear TRSs)
WON	– (but holds for left-linear TRSs)
SLON	– (but holds for left-linear TRSs)
WLON	– (but holds for left-linear TRSs)
SPON	– (but holds for left-linear TRSs)
WPON	– (but holds for left-linear TRSs)

Table 4 Preservation of outermost termination properties under signature extensions

▶ **Example 1** (counterexample to preservation of SON, WON, SLON, WLON under signature extensions)**.** Consider the TRS $\mathcal{R}$ over the signature $\mathcal{F} = \{f_1, f_2, g, c, d_1, d_2\}$:

$$\begin{aligned} g(f_1(x,y,y)) &\to g(f_2(x,x,y)) & g(f_2(*,x,y)) &\to c \\ g(f_1(y,x,y)) &\to g(f_2(x,x,y)) & g(f_2(x,*,y)) &\to c \\ f_2(x,x,y) &\to f_1(x,x,y) & g(f_2(x,y,*)) &\to c \\ d_1 &\to d_2 & g(f_2(x,x,x)) &\to c \end{aligned}$$

Here, the "$*$"-pattern notation in 3 of the rules is to be interpreted as follows: For $l \to r$ of shape $C[*] \to r$, the rule stands for the whole family of rules $C[*] \to r$ where $*$ is sucessively replaced by all most general f-patterns, for all $f \in \mathcal{F}$, i.e., by $f(x_1, \ldots, x_{ar(f)})$, such that the x_i are distinct fresh (w.r.t. $C[.]$) variables. With some effort one can show that $\mathcal{R}$ is SON, hence also WON, SLON and WLON. However, if we add a fresh unary function symbol H, all these properties get lost. To wit, consider $s = g(f_1(H(d_1), H(d_1), H(d_2)))$ which initiates the (only) outermost derivations (the contracted outermost redexes are underlined)

$$\begin{aligned} s \quad &= \quad g(f_1(\underline{H(d_1)}, H(d_1), H(d_2))) \to_o \underline{g(f_1(H(d_2), H(d_1), H(d_2)))} \\ \to_o \quad & \quad g(\underline{f_2(H(d_1), H(d_1), H(d_2))}) \to_o s \to \cdots \quad \text{and} \end{aligned}$$

$$\begin{aligned} s \quad &= \quad g(f_1(H(d_1), \underline{H(d_1)}, H(d_2))) \to_o \underline{g(f_1(H(d_1), H(d_2), H(d_2)))} \\ \to_o \quad & \quad g(\underline{f_2(H(d_1), H(d_1), H(d_2))}) \to_o s \to \cdots \end{aligned}$$

▶ **Example 2** (counterexample to to preservation of SPON, WPON under signature extensions)**.** Consider the TRS $\mathcal{R}$ over the signature $\mathcal{F} = \{f_1, f_2, g_1, g_2, c, d_1, d_2\}$:

$$\begin{aligned} g_1(f_1(x,y),x) &\to g_2(f_1(x,y),x) & g_2(f_1(*,x),y) &\to c \\ f_1(x,y) &\to f_2(x,y) & g_2(f_1(x,*),y) &\to c \\ g_2(f_2(x,y),y) &\to g_1(f_1(x,y),x) & g_2(f_1(x,y),*) &\to c \\ d_1 &\to d_2 & g_2(f_1(x,x),x) &\to c \end{aligned}$$

Again, with some effort one can show that $\mathcal{R}$ is SPON, hence also WPON. However, if we add the fresh unary function symbol H, these properties get lost. To wit, consider $s = g_2(f_1(H(d_1), H(d_2)), H(d_1))$ which initiates the (only) parallel outermost derivation (the contracted parallel outermost redexes are underlined)

$$\begin{aligned} s \quad &= \quad g_2(\underline{f_1(H(d_1), H(d_2))}, H(\underline{d_1})) \to_{po} \underline{g_2(f_2(H(d_1), H(d_2)), H(d_2))} \\ \to_{po} \quad & \quad \underline{g_1(f_1(H(d_1), H(d_2)), H(d_1))} \to_{po} s \to \cdots . \end{aligned}$$

Concerning future work, it is quite natural to ask how the situation looks like for strategies other than innermost and outermost and for restrictions of rewriting like *context-sensitivity* or *forbidden patterns*. Furthermore more general combinations of TRSs like *constructor sharing* or *composable* ([10]) ones are of interest, too. Another line of research is to take into account typing, i.e. to ask whether imposing a type discipline may facilitate the verification of termination properties of rewriting under strategies, cf. e.g. [17, 1]. On a more technical level it appears interesting to investigate relationships to other settings and approaches where non-left-linearity causes major problems, e.g., in (automatically) proving outermost termination and in dependency pair based termination proofs where signature extensions play a major role ([12]).

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Termination Graphs Revisited*

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Abstract

We revisit termination graphs from the viewpoint of runtime complexity. Suitably generalising the construction proposed in the literature, we define an alternative representation of Jinja Bytecode (JBC) executions as *computation graphs*. We show that the transformation from JBC programs to computation graphs is *sound*, i.e., an infinite execution gives rise to an infinite path in the computation graph. Moreover, we establish that the transformation is *complexity preserving*.

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1 Introduction

In [9, 4, 3] termination of Jinja Bytecode (JBC for short) programs is studied. To this extent the execution of a JBC program P is represented in a finite graph, a so-called *termination graph*. Based on this graph, integer term rewrite systems $\mathcal{R}$ (cf. [5]) are defined, such that termination of $\mathcal{R}$ yields termination of P. That is, the proposed transformation from JBC to rewrite systems is *non-termination preserving*.

In this note we revisit termination graphs from the viewpoint of runtime complexity. Suitably generalising the earlier construction, we propose an alternative representation of JBC executions in graph form as *computation graphs* G. The nodes of the computation graph are abstract states, representing sets of states of the Jinja Virtual Machine (JVM). The edges represent symbolic evaluations together with refinements and abstraction steps.

We show that the transformation from JBC programs to computation graphs is non-termination preserving, that is, any infinite evaluation of P gives rise to the existence of infinite paths in G. Moreover, we establish that the transformation is *complexity preserving*. For this we measure the runtime complexity of P as a function that relates the maximal length of evaluations to the size of the initial state. (Note that this measure overestimates the size of the input to P only by a constant factor.) Moreover, the computation complexity maps the maximal length of a path in G to the size of the initial abstract state. We show that the runtime complexity of P is asymptotically bounded in the computation complexity of G. Disregarding the viewpoint of complexity, this paper provides a simplification and clarification of the concepts proposed in [9, 4, 3] and thus may be of independent interest.

In the following we give a brief overview about the Jinja source language, cf. [7] for details. Values are either *Boolean*, *integers*, *references*, the null reference (denoted as `null`), or the dummy value (denoted as `unit`). We usually refer to (non-null) references as *addresses*. The dummy value `unit` is used for the evaluation of assignments (see [7]) and also used in the JVM to allocate uninitialised local variables.

A Jinja *program* consists of a set of *class declarations*. Each class is identified by a *class name* and further consists of the name of its direct *superclass*, *field declarations* and *method*

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declarations. A field declaration is a pair of *field name* and *field type*. A method declaration consists of the *method name*, a list of *parameter types*, the *result type* and the *method body*. The method body consists of a list of *parameter identifiers* and an *expression*. The definition for JBC programs is almost identical to the definition of Jinja programs. The sole exception is that the JBC method body is represented by a triple (mxs, mxl, ins), where mxs denotes the maximal size of the operand stack, mxl the number of registers (not including the `this` pointer and the parameters) and ins a sequence of JBC instructions. We consider Jinja programs and JBC programs to be well-formed [7]. Further, we presuppose normal evaluation, that is, no exceptions are raised and demand that all data structures are non-cyclic.

JBC is executable on the JVM. A state of the JVM is represented by a pair of *heap* and *frames*. The heap represents the global memory of the program and associates *addresses* to *objects*. A frame represents the execution environment of a method and is a quintuple (stk, loc, cn, mn, pc) such that: stk denotes the *operand stack*, loc denotes the *registers*, cn denotes the *class name*, mn denotes the *method name*, and pc is the *program counter*. Both, operand stack and registers store values. For each frame the number of registers is fixed and the maximum size of the operand stack is computed during compilation. Thus the operand stack can be conceived as an array. See [7] for further details.

2 Abstract States

We extend Jinja by abstract variables `Class` for each class considered. Further, $\texttt{Bool} := \{\texttt{true}, \texttt{false}\}$ denotes an *abstract Boolean value* and any interval $I \subseteq [-\infty, \infty]$ denotes an *abstract integer value*. We write `Int` instead of I, if the concrete interval is not relevant and we identify the interval $[z, z]$, where $z \in \mathbb{Z}$ with the integer z. An *abstract value* is either a Jinja value, or an abstract Boolean or integer value. Furthermore, we make use of an infinite supply of *abstract locations* $\zeta_0, \zeta_1, \zeta_2, \dots$ In the following $\preceq$ denotes the subclass relation.

An *abstract state* is a triple consisting of the heap, the list of frames, and a set of annotations. A *heap* is a mapping from *addresses* to *objects*, where an object is either an abstract variable or pair (cn, ft): cn denotes the *class name* and ft denotes the field table, i.e., a mapping from $(cn', fieldid)$ to abstract values, where $cn \preceq cn'$. Let obj be an object. We define the projections cl and ft as follows: (i) $\mathsf{cl}(obj) := cn$, if $obj = (cn, ft)$, and $\mathsf{cl}(obj) := \texttt{Class}$, if obj is an abstract variable of type `Class`. (ii) $\mathsf{ft}((cn, ft)) := ft$. Registers and operand stack of a frame, now store abstract values. Furthermore, we define *annotations* of addresses in a state s, denoted as iu. Annotations are pairs $p \neq q$ of addresses, where $p, q \in heap$ and $p \neq q$. The intuition of iu is to express that for $p \neq q \in iu$, we disallow sharing of these addresses in concrete states. An abstract state which does not contain abstract variables and where addresses cannot be shared further is a *concrete* (or *Jinja*) state. We define a bijection ϕ that associates every non-address value in $heap$ with an abstract location. We define the graph Φ as function of $heap$:

$$\Phi(heap) := \{(\zeta, val) \mid \exists \text{ address } a\text{: } \mathsf{rg}(\mathsf{ft}(heap(a))) = val,\ val \text{ not an address, } \zeta \text{ fresh}\} \ .$$

Finally, we set $\phi(\zeta) := val$ if $(\zeta, val) \in \Phi(heap)$.

▶ **Definition 1.** We represent $heap$ as a directed graph $H = (V_H, Succ_H, L_H, E_H)$, where the nodes, the successor relation and the labeling function are defined as follows: (i) $V_H := \mathsf{dom}(heap) \cup \mathsf{dom}(\phi)$ (ii) $Succ_H(u) := [ft^*((C_1, id_1)), \dots, ft^*((C_k, id_k))]$, if u is an address, $\mathsf{ft}(heap(u)) = ft$ and $\mathsf{dom}(ft) = \{(C_1, id_1), \dots, (C_k, id_k)\}$, otherwise $Succ_H(u) := []$. Here $ft^*((C, id)) := ft((C, id))$, if $ft((C, id))$ is an address and $ft^*((C, id)) := \phi^{-1}(ft((C, id)))$ otherwise. (iii) $L_H(u) := \mathsf{cl}(heap(u))$, if u is an address and $L_H(u) := \phi(u)$ otherwise. (iv)

$E_H(u \to v) := (C, id)$, if u is an address, $\mathsf{ft}(heap(u)) = ft$ and $ft((C, id)) = v$. Otherwise we set $E_H(u \to v) := \epsilon$.

We call a value val reachable from an address a in $heap$, if there exists a path from a to val in the heap graph of $heap$. For a given state $s = (heap, frms, iu)$ with top-frame $frm = (stk, loc, cn, mn, pc)$, we are only concerned with the part of the heap that is reachable from frm. Let $heap \restriction frm$ denote the restriction of $heap$ to all nodes reachable from $\{stk(i) \mid i \in \{1, \dots, m\} \cup \{loc(i) \mid i \in \{1, \dots, n\}\}$. We generalise the bijection ϕ so that also non-address values in the registers and on the operand stack are represented. For that we define the graph Φ as a function of stk, loc, and $heap$ in the natural way. Finally, we set $\phi(\zeta) := val$ if $(\zeta, val) \in \Phi(stk, loc, heap)$.

▶ **Definition 2.** Let $s = (heap, frms, iu)$ be a state and let $frm = (stk, loc, cn, mn, pc)$ be the top-frame. Let $\mathsf{dom}(stk) = \{1, \dots, m\}$ and let $\mathsf{dom}(loc) = \{1, \dots, n\}$. Recall that stk can be conceived as array. We use os_i (l_i) to denote index i of the stack (register i). Moreover suppose H denotes $heap \restriction frm$. We define the *state graph of s* as 5-triple $S = (V_S, Succ_S, L_S, E_S, iu)$, where the first four components denote a directed graph with edge labels and iu denotes a set of annotations. The nodes, the successor relation, and the labeling function of the directed graph are defined as follows: (i) $V_S := \{1, \dots, m{+}n\} \cup V_H \cup \mathsf{dom}(\phi)$ (ii) $Succ_S(u) := [stk^*(u)]$, if $u \in \{1, \dots, m\}$. Otherwise, if $u \in \{m + 1, \dots, n\}$, then $Succ_S(u) := [loc^*(u - m)]$. Finally, if $u \in V_H$, then $Succ_S(u) := Succ_H(u)$. Here $stk^*(u)$ and $loc^*(u)$ is defined like ft^* as introduced in Definition 1. (iii) $L_S(u) := os_u$, if $u \in \{1, \dots, m\}$ and $L_S(u) := l_{u-m}$, if $u \in \{m + 1, \dots, n\}$. Otherwise, $L_S(u) := \mathsf{cl}(heap(u))$, if u is an address. Finally, $L_S(u) := \phi(u)$. (iv) $E_S(u \to v) := E_H(u \to v)$, if $u, v \in H$. Otherwise, we set $E_S(u \to v) := \epsilon$.

We often confuse a state s and its representation as a state graph and addresses v with its corresponding object $heap(v)$. We define a binary relation $\sqsubseteq$ on abstract values. Let v, w be values. Then $v \sqsubseteq w$ if (i) $v \in \{\mathtt{null}, \mathtt{unit}\}$ and either $v = w$ or $w \in \{\mathtt{null}, \mathtt{Class}, \mathtt{Bool}, \mathtt{Int}\}$, or (ii) $v, w \subseteq [-\infty, \infty]$ and $v \subseteq w$, (iii) $v, w \in \{\mathtt{true}, \mathtt{false}, \mathtt{Bool}\}$ and either $v = w$ or $w = \mathtt{Bool}$, or (iv) v, w are class names or abstract class variables and $v \preceq w$. Based on the definition of $\sqsubseteq$ we introduce the following variant of graph morphism, called *state morphism*.

▶ **Definition 3.** Let S, T be state graphs. A *state morphism* from T to S (denoted $m \colon T \to S$) is a function $m \colon V_T \to V_S$ such that (i) for all $u \in T$, $L_T(u) \sqsupseteq L_S(m(u))$, (ii) for all $u \in T$, $m^*(Succ_T(u)) = Succ_S(m(u))$, and (iii) for all $u \xrightarrow{\ell} v \in T$ and $m(u) \xrightarrow{\ell'} m(v) \in S$, $\ell = \ell'$.

If no confusion can arise we refer to a state morphism simply as *morphism*. It is easy to see that the composition $m_1 \circ m_2$ of two morphisms m_1, m_2 is again a morphism.

▶ **Definition 4.** Let $s = (heap, frms, iu)$ and $t = (heap', frms', iu')$ be states. Then t is an *instance* of s (denoted as $t \sqsubseteq s$) if the following conditions hold: (i) all corresponding program counters in the frame lists $frms$ and $frms'$ coincide, (ii) there exists a morphism $m \colon s \to t$, and (iii) for all $p \neq q \in iu$ we have that $m(p) \neq m(q)$ and $iu' \subseteq m^*(iu)$. If t an instance of s, then we call s an *abstraction* of t, denoted as $s \sqsupseteq t$.

It is an easy consequence of the composability of morphism that the instance relation $\sqsubseteq$ is transitive. Let $s = (heap, frms, iu)$ be a state and let p, q denote distinct addresses in $heap$ such that $p \neq q \notin iu$. Then we say p and q are *unifiable* (denoted as $p \overset{?}{=} q$) if there exists a state t and a morphism $m \colon s \to t$, such that $m(p) = m(q)$.

▶ **Definition 5.** Let s be a state and let $S = (V_S, Succ_S, L_S, E_S, iu)$ be its state graph. The *size* of s, denoted $|s|$, is defined as follows: $\sum_{l \in L_S} |l|$, where $|l|$ is $\mathsf{abs}(l)$ if $l \in \mathbb{Z}$, otherwise 1. (As usual $\mathsf{abs}(l)$ denotes the absolut value of the integer l.)

3 Computation Graph

The operational semantics of the JVM yields the single-step execution of each JBC command [7]. Based on these instructions, and actually mimicking them quite closely, we define how abstract states are symbolically evaluated. We write $\mathit{instr\_of}(C, M)$ to denote the *instruction list* of method M in class C of the considered JBC program P. Furthermore, we set $\mathit{method}(C, M) = (D, Ts, T, \mathit{mbody})$ to denote that method M with type signature $M\colon Ts \to T$ is defined in the superclass D of C and its body is $\mathit{mbody} = (\mathit{mxs}, \mathit{mxl}, \mathit{ins})$. See [7] for a suitable implementation of the functions *instr_of* and *method*.

Let $s = (\mathit{heap}, \mathit{frms}, \mathit{iu})$ be an abstract state with top-frame $\mathit{frm} = (\mathit{stk}, \mathit{loc}, \mathit{cn}, \mathit{mn}, \mathit{pc})$. Suppose $\mathit{instr} = \mathit{instr\_of}(C, M)(\mathit{pc})$, i.e., the current instruction. By case distinction on *instr*, one defines the *symbolic evaluation* of P. In Definition 6 we have worked out the cases for `Putfield` and `Invoke`. The other cases are left to the reader.

▶ **Definition 6.** Consider a `Putfield` F C instruction. Let v be a value and r be an address such that $\mathit{heap}(r) = (D, \mathit{ft})$. We set $\mathit{ft}' := \mathit{ft}\{(C, F) \mapsto v\}$ to denote the updating of field F in ft. Suppose there exists no address $p \in \mathit{heap}$ such that $r \overset{?}{=} p$. Then we define the following step:

$$\frac{(\mathit{heap}, (v :: r :: \mathit{stk}, \mathit{loc}, \mathit{cn}, \mathit{mn}, \mathit{pc}) :: \mathit{frms}, \mathit{iu})}{(\mathit{heap}\{r \mapsto (D, \mathit{ft}')\}, (\mathit{stk}, \mathit{loc}, \mathit{cn}, \mathit{mn}, \mathit{pc} + 1) :: \mathit{frms}, \mathit{iu})}\ .$$

Now, consider an instruction `Invoke` M n. Suppose r denotes the address of the calling object and $\mathsf{cl}(\mathit{heap}(r)) = C$ and $\mathit{method}(C, M) = (D, Ts, T, (\mathit{mxs}, \mathit{mxl}, \mathit{ins}))$. We set:

$$\frac{(\mathit{heap}, (p_{n-1} :: \cdots :: p_0 :: r :: \mathit{stk}, \mathit{loc}, \mathit{cn}, \mathit{mn}, \mathit{pc}) :: \mathit{frms}, \mathit{iu})}{(\mathit{heap}, \mathit{frm}' :: (\mathit{stk}, \mathit{loc}, \mathit{cn}, \mathit{mn}, \mathit{pc}) :: \mathit{frms}, \mathit{iu})}\ ,$$

where $\mathit{loc}' := [r, p_0, \dots, p_{n-1}] \mathbin{@} \mathit{units}$ and $\mathit{frm}' = ([], \mathit{loc}', D, M, 0)$. Here *units* denotes an array of `unit`-values of size *mxl*.

In addition to symbolic evaluations, we define refinement steps on abstract states s if the information given in s is not concrete enough to execute a given instruction. Following [4] we make use of *case distinction*, *class instance*, *sharing*, and *unsharing*. We will restrict to an informal explanation of these refinement steps. *Case distinction* refines abstract Boolean or integer values if a symbolic instruction is otherwise not possible as the state is underspecified. The refinement *class instance* either replaces an abstract class variable by the null-pointer or refines the information about the class. Finally, the refinement steps *sharing*, and *unsharing* either explicitly share unifiable addresses p, q by identifying them, or the annotation $p \neq q$ is added to mark that these references must not be shared. We arrive at the definition of a computation graph.

▶ **Definition 7.** A *computation graph* $G = (V_G, E_G)$ is a directed graph, where V_G are abstract states and $s \to t \in E_G$ if either (i) t is obtained from s by a symbolic evaluation, (ii) t is obtained by a refinement step, or (iii) $s \sqsubseteq t$ holds.

Let G be a computation graph. We write $G\colon s \rightharpoonup_G t$ to indicate that state t is directly reachable in G from s. If s is reachable from t in G we write $G\colon s \rightharpoonup^*_G t$. Let s and t be concrete states. Then we denote by $P\colon s \xrightarrow{\mathsf{jvm}}_1 t$ the one-step transition relation of the JVM. If there exists a (normal) evaluation of s to t, we write $P\colon s \xrightarrow{\mathsf{jvm}} t$. The next lemma states that any single-step execution on the JVM can be simulated by at least one step in the computation graph.

▶ **Lemma 8.** *Let s be a state and let s' be a concrete state such that $s' \sqsubseteq s$. Then $P\colon s' \xrightarrow{\mathsf{jvm}}_1 t'$ implies the existence of a state t such that $t' \sqsubseteq t$ and $G\colon s \xrightarrow{+}_G t$.*

We define the *runtime* of a JVM for a given normal evaluation $P\colon s \xrightarrow{\mathsf{jvm}} t$ as the number of single-step executions. The *computation length* denotes the maximal length of a path in the computation graph G such that $G\colon s \xrightarrow{*}_G t$. Recall Definition 5, defining the size of some state s. We define the *runtime complexity* with respect to P as follows: $\mathsf{rcj}(n) := \max\{m \mid P\colon s \xrightarrow{\mathsf{jvm}} t$ holds such that the runtime is m and $|s| \leqslant n\}$. Similarly we set: $\mathsf{cc}(n) := \max\{m \mid G\colon s \xrightarrow{*}_G t$ holds such that the computation length is m and $|s| \leqslant n\}$.

▶ **Theorem 9.** *Let s' and t' be concrete states. Suppose $P\colon s' \xrightarrow{\mathsf{jvm}} t'$. Then there exists an abstraction s of s' and a computation $G\colon s \xrightarrow{*}_G t$ such that $t' \sqsubseteq t$. Furthermore $\mathsf{rcj} \in O(\mathsf{cc})$.*

4 Conclusion

In this note we propose computation graphs as suitable representation of the execution of a JVM. We show that this representation is complexity preserving. In future work it needs to be clarified whether our result on complexity preservation still holds, if the *cycles* of the computation graph are considered separately [9]. Furthermore, the notion of (innermost) runtime complexity for integer rewrite systems need to be clarified (cf. [6, 2] for the standard definition of runtime complexity of a rewrite system). Finally, methods for runtime complexity need to be adapted to integer rewrite systems (cf. [1, 8, 6] for examples of existing techniques).

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Matrix Interpretations for Polynomial Derivational Complexity

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Abstract

Matrix interpretations can be used to bound the derivational complexity of term rewrite systems. In general, the obtained bounds are exponential. In this paper we use joint spectral radius theory in order to completely characterize matrix interpretations that induce polynomial upper bounds on the derivational complexity of (compatible) rewrite systems.

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1 Introduction

This paper is concerned with automated complexity analysis of term rewrite systems. Given a terminating rewrite system, the aim is to obtain information about the maximal length of rewrite sequences in terms of the size of the initial term. This is known as derivational complexity. Matrix interpretations [3] are a popular method for automatically proving termination of rewrite systems. They can readily be used to establish upper bounds on the derivational complexity of compatible rewrite systems. However, in general, matrix interpretations induce exponential (rather than polynomial) upper bounds. In order to obtain polynomial upper bounds, the matrices used in a matrix interpretation must satisfy certain (additional) restrictions, the study of which is the central concern of [8, 9, 11]. So what are the conditions for polynomial boundedness of a matrix interpretation? In the literature, two different approaches have emerged. On the one hand, there is the automata-based approach of [11], where matrices are viewed as weighted (word) automata computing a weight function, which is required to be polynomially bounded. The result is a complete characterization (i.e., necessary and sufficient conditions) of polynomially bounded matrix interpretations over $\mathbb{N}$. On the other hand, there is the algebraic approach pursued in [9] (originating from [8]) that can handle matrix interpretations over $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ but only provides sufficient conditions for polynomial boundedness. In what follows, we use joint spectral radius theory [5, 4] to extend the latter to a complete characterization of polynomially bounded matrix interpretations over $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$.

2 Preliminaries

We assume familiarity with the basics of term rewriting [1, 10]. Let $\mathcal{V}$ denote a countably infinite set of variables and $\mathcal{F}$ a fixed-arity signature. The set of *terms* over $\mathcal{F}$ and $\mathcal{V}$ is denoted by $\mathcal{T}(\mathcal{F}, \mathcal{V})$. The *size* $|t|$ of a term t is defined as the number of function symbols and

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variables occurring in it. A *term rewrite system* (*TRS* for short) $\mathcal{R}$ over $\mathcal{T}(\mathcal{F},\mathcal{V})$ is a *finite* set of rewrite rules $\ell \to r$ such that $\ell \notin \mathcal{V}$ and $\mathcal{V}\mathsf{ar}(\ell) \supseteq \mathcal{V}\mathsf{ar}(r)$. The smallest rewrite relation that contains $\mathcal{R}$ is denoted by $\to_{\mathcal{R}}$. As usual, $\to_{\mathcal{R}}^{+}$ ($\to_{\mathcal{R}}^{*}$) denotes the transitive (and reflexive) closure of $\to_{\mathcal{R}}$ and $\to_{\mathcal{R}}^{n}$ its n-th iterate. For a terminating TRS $\mathcal{R}$, the *derivation height* of a term t with respect to $\mathcal{R}$ is defined as $\mathsf{dh}(t,\to_{\mathcal{R}}) = \max\{n \mid t \to_{\mathcal{R}}^{n} u \text{ for some term } u\}$. The *derivational complexity* of $\mathcal{R}$ is the function $\mathsf{dc}_{\mathcal{R}}\colon \mathbb{N}\setminus\{0\} \to \mathbb{N}, k \mapsto \max\{\mathsf{dh}(t,\to_{\mathcal{R}}) \mid |t| \leqslant k\}$.

An $\mathcal{F}$-*algebra* $\mathcal{A}$ consists of a carrier set A and a collection of interpretations $f_{\mathcal{A}}\colon A^k \to A$ for each k-ary function symbol $f \in \mathcal{F}$. By $[\alpha]_{\mathcal{A}}(\cdot)$ we denote the usual evaluation function of $\mathcal{A}$ according to an assignment α which maps variables to values in A. An $\mathcal{F}$-algebra together with a well-founded order $>$ on A is called a *(well-founded) monotone algebra* if for each function symbol $f \in \mathcal{F}$ the interpretation function $f_{\mathcal{A}}$ is monotone with respect to $>$ in all arguments. Any monotone algebra $(\mathcal{A},>)$ (or just $\mathcal{A}$ if $>$ is clear from the context) induces a well-founded order on terms: $s >_{\mathcal{A}} t$ if and only if $[\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t)$ for all assignments α. A TRS $\mathcal{R}$ and a monotone algebra $\mathcal{A}$ are *compatible* if $\ell >_{\mathcal{A}} r$ for all $\ell \to r \in \mathcal{R}$.

Let R be a commutative ring (e.g., $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$). The ring of all n-*dimensional square matrices* over R is denoted by $R^{n\times n}$. The set of all non-negative n-dimensional square matrices of $\mathbb{Z}^{n\times n}$ ($\mathbb{R}^{n\times n}$) is denoted by $\mathbb{N}^{n\times n}$ ($\mathbb{R}_0^{n\times n}$), and we write A^{T} for the *transpose* of a matrix (vector) A. With any matrix $A \in \mathbb{R}_0^{n\times n}$ we associate a directed (weighted) graph $G(A)$ on n vertices, numbered from 1 to n, such that there is a directed edge (of weight A_{ij}) in $G(A)$ from i to j if and only if $A_{ij} \neq 0$. In this situation, A is said to be the *adjacency matrix* of the graph $G(A)$. The *weight of a path* in $G(A)$ is the product of the weights of its edges. With a (non-empty) finite set of matrices $S \subseteq \mathbb{R}_0^{n\times n}$ we associate the directed (weighted) graph $G(S) := G(M)$, where M denotes the component-wise maximum of the matrices in S, i.e., $M_{ij} = \max\{A_{ij} \mid A \in S\}$ for all $1 \leqslant i,j \leqslant n$. Following [5], we define a directed graph $G^k(S)$ for $k \geqslant 2$ on n^k vertices representing ordered tuples of vertices of $G(S)$, such that there is an edge from vertex $(i_1,\dots,i_k)$ to $(j_1,\dots,j_k)$ if and only if there is a matrix $A \in S$ with $A_{i_\ell j_\ell} > 0$ for all $\ell = 1,\dots,k$.

For *matrix interpretations* (over $\mathbb{R}$), we fix a *dimension* $n \in \mathbb{N}\setminus\{0\}$ and use the set $\mathbb{R}_0^n$ as the carrier of an algebra $\mathcal{M}$, together with the order $>_\delta$ on $\mathbb{R}_0^n$ defined as $(x_1,x_2,\dots,x_n)^{\mathrm{T}} >_\delta (y_1,y_2,\dots,y_n)^{\mathrm{T}}$ if $x_1 >_{\mathbb{R},\delta} y_1$ and $x_i \geqslant_{\mathbb{R}} y_i$ for $2 \leqslant i \leqslant n$. Here $x >_{\mathbb{R},\delta} y$ if and only if $x \geqslant_{\mathbb{R}} y + \delta$. Each k-ary function symbol f is interpreted by a linear function of the following shape: $f_{\mathcal{M}}(\vec{v}_1,\dots,\vec{v}_k) = F_1\vec{v}_1 + \cdots + F_k\vec{v}_k + \vec{f}$ where $\vec{v}_1,\dots,\vec{v}_k$ are (column) vectors of variables, $F_1,\dots,F_k \in \mathbb{R}_0^{n\times n}$ and $\vec{f}$ is a vector in $\mathbb{R}_0^n$. To ensure monotonicity, it suffices that the top left entry $(F_i)_{11}$ of each matrix F_i is at least one. Then it is easy to see that $(\mathcal{M},>_\delta)$ forms a well-founded monotone algebra for any $\delta > 0$. We obtain matrix interpretations over $\mathbb{Q}$ by restricting to the carrier $\mathbb{Q}_0^n$. Similarly, matrix interpretations over $\mathbb{N}$ operate on the carrier $\mathbb{N}^n$ and use $\delta = 1$. We denote by $S_{\mathcal{M}}$ the set of matrices occurring in (the interpretation functions of) $\mathcal{M}$. We set $S_{\mathcal{M}} = \{0\}$ in the pathological case when $\mathcal{M}$ contains no matrices. Further, we denote by $S_{\mathcal{M}}^k = \{A_1\cdots A_k \mid A_i \in S_{\mathcal{M}}, 1 \leqslant i \leqslant k\}$ the set of all products of length k of matrices taken from $S_{\mathcal{M}}$. For $k = 0$, this yields the singleton set $S_{\mathcal{M}}^0 = \{I\}$ containing only the identity matrix. Finally, $S_{\mathcal{M}}^*$ denotes the (matrix) monoid generated by $S_{\mathcal{M}}$, i.e., $S_{\mathcal{M}}^* = \bigcup_{k=0}^{\infty} S_{\mathcal{M}}^k$. We often drop the subscript $\mathcal{M}$ if it is clear from the context.

Let t be an arbitrary term and α_0 the assignment that maps every variable to 0. In the sequel, we abbreviate $[\alpha_0]_{\mathcal{M}}(t)$ by $[t]_{\mathcal{M}}$ (or just $[t]$ if $\mathcal{M}$ is clear from the context), and we write $[t]_j$ ($1 \leqslant j \leqslant n$) for the j-th component of $[t]$.

3 Growth of Matrix Interpretations

Following [11], we define the notion of growth of a matrix interpretation as follows.

▶ **Definition 3.1.** Let $\mathcal{M}$ be a matrix interpretation. The *growth function* of $\mathcal{M}$ is defined as $\mathsf{growth}_{\mathcal{M}}(k) = \max\{[t]_1 \mid t \text{ is a term and } |t| \leqslant k\}$.

According to [9], for a TRS $\mathcal{R}$ and a compatible matrix interpretation $\mathcal{M}$, we have $\mathsf{dh}(t, \rightarrow_{\mathcal{R}}) \leqslant \frac{1}{\delta} \cdot [t]_1$, and therefore $\mathsf{dc}_{\mathcal{R}}(k) \leqslant \frac{1}{\delta} \cdot \mathsf{growth}_{\mathcal{M}}(k)$. As the growth of $\mathcal{M}$ is at most exponential (in the worst case), the derivational complexity of the TRSs one can handle in this way can at most be exponential. This was shown in [3] for matrix interpretations over $\mathbb{N}$, but the result obviously extends to matrix interpretations over $\mathbb{Q}$ and $\mathbb{R}$. In order to establish polynomial derivational complexity, the matrices occurring in $\mathcal{M}$ must satisfy certain additional properties that guarantee polynomial boundedness of $\mathsf{growth}_{\mathcal{M}}(k)$.

4 Algebraic Methods for Bounding Polynomial Growth

In this section we study an algebraic approach to characterize polynomial growth of matrix interpretations (over $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$). We employ the following definition [7, 8, 9].

▶ **Definition 4.1.** A matrix interpretation $\mathcal{M}$ is *polynomially bounded (with degree $d \in \mathbb{N}$)* if the growth of the entries of all matrix products in $S^*_{\mathcal{M}}$ is polynomial (with degree d) in the length of such products, i.e., $\max\{M_{ij} \mid M \in S^k_{\mathcal{M}}\} \in O(k^d)$ for all $1 \leqslant i, j \leqslant n$, where n is the dimension of $\mathcal{M}$.

Obviously, the condition given in Definition 4.1 is sufficient for polynomial boundedness of $\mathsf{growth}_{\mathcal{M}}(k)$. Moreover, as shown in [7], it is also necessary in the following sense. If $\mathsf{growth}_{\mathcal{M}}(k)$ is polynomially bounded and $\mathcal{M}$ is compatible with a TRS $\mathcal{R}$, then there exists a matrix interpretation $\mathcal{N}$ compatible with $\mathcal{R}$ such that $\mathsf{growth}_{\mathcal{N}}(k) = \mathsf{growth}_{\mathcal{M}}(k)$ and the entries of all matrix products in $S^*_{\mathcal{N}}$ are of polynomial growth (i.e., $\mathcal{N}$ conforms to Definition 4.1). The proof given in [7] leverages the connection between matrix interpretations and weighted word automata. This is possible since matrix interpretations correspond to a rather restricted form of tree automata, called *path-separated* [6]. The idea is to transform a matrix interpretation into the corresponding automaton, trim this automaton by removing useless states and then transform the resulting automaton back into a (compatible) matrix interpretation. Thus, the interpretation $\mathcal{N}$ can be obtained from $\mathcal{M}$ by simply dropping some rows and columns (the ones whose indices correspond to the useless states) in the matrices and vectors occurring in the interpretation functions of $\mathcal{M}$.

The relationship between polynomially bounded matrix interpretations and the derivational complexity of compatible TRSs is as follows (cf. [9, 7]).

▶ **Lemma 4.2.** *Let $\mathcal{R}$ be a TRS and $\mathcal{M}$ a compatible matrix interpretation. If $\mathcal{M}$ is polynomially bounded with degree d, then $\mathsf{dc}_{\mathcal{R}}(k) \in O(k^{d+1})$.* ◀

In the sequel, we employ joint spectral radius theory [5, 4], a branch of mathematics dedicated to studying the growth rate of products of matrices taken from a set, to obtain a complete characterization of polynomially bounded matrix interpretations (over $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$). All matrix norms $\|\cdot\|$ are assumed to be submultiplicative, i.e., $\|AB\| \leqslant \|A\| \cdot \|B\|$.

▶ **Definition 4.3.** Let $S \subseteq \mathbb{R}^{n \times n}$ be a finite set of real square matrices, and let $\|\cdot\|$ denote a matrix norm. The growth function growth_S associated with S is defined as $\mathsf{growth}_S(k, \|\cdot\|) = \max\{\, \|A_1 \cdots A_k\| \mid A_i \in S, 1 \leqslant i \leqslant k \,\}$.

Using the (submultiplicative) matrix norm $\|\cdot\|_1$ given by the sum of the absolute values of all matrix entries, we observe that a matrix interpretation $\mathcal{M}$ is polynomially bounded (with degree d) if and only if $\mathsf{growth}_{S_\mathcal{M}}(k, \|\cdot\|_1)$ is polynomial in k (with degree d). The asymptotic behaviour of $\mathsf{growth}_{S_\mathcal{M}}(k, \|\cdot\|_1)$ can be characterized by the joint spectral radius of $S_\mathcal{M}$.

▶ **Definition 4.4.** Let $S \subseteq \mathbb{R}^{n\times n}$ be finite, and let $\|\cdot\|$ denote a matrix norm. The *joint spectral radius* $\rho(S)$ of S is defined as $\rho(S) = \lim_{k\to\infty} \max\{\, \|A_1 \cdots A_k\|^{1/k} \mid A_i \in S, 1 \leqslant i \leqslant k \,\}$.

It is well-known that this limit always exists and that it does not depend on the chosen norm, which follows from the equivalence of all norms in $\mathbb{R}^n$. Because of this and due to the fact that we are only interested in the asymptotic behaviour of $\mathsf{growth}_S(k, \|\cdot\|)$, from now on we simply write $\mathsf{growth}_S(k)$. The following theorem (due to [2]) provides a characterization of polynomial boundedness of $\mathsf{growth}_S(k)$ by the joint spectral radius of S.

▶ **Theorem 4.5** ([2, Theorem 1.2])**.** *Let $S \subseteq \mathbb{R}^{n\times n}$ be a finite set of matrices. Then $\mathsf{growth}_S(k) \in O(k^d)$ for some $d \in \mathbb{N}$ if and only if $\rho(S) \leqslant 1$. In particular, $d \leqslant n-1$.* ◀

Hence, polynomial boundedness of $\mathsf{growth}_S(k)$ is decidable if $\rho(S) \leqslant 1$ is decidable. But it is well-known that the latter is undecidable in general, even if S consists of finitely many non-negative rational matrices (cf. [5, Theorem 2.6]). However, in case S is a finite set of non-negative integer matrices, then $\rho(S) \leqslant 1$ is decidable. In particular, there exists a polynomial-time algorithm that decides it (cf. [5, Theorem 3.1]). This algorithm is based on the following lemma.

▶ **Lemma 4.6** ([5, Lemma 3.3])**.** *Let $S \subseteq \mathbb{R}_0^{n\times n}$ be a finite set of non-negative real square matrices. Then there is a product $A \in S^*$ such that $A_{ii} > 1$ for some $i \in \{1, \ldots, n\}$ if and only if $\rho(S) > 1$.* ◀

According to [5], for $S \subseteq \mathbb{N}^{n\times n}$, the existence of such a product can be characterized in terms of the graphs $G(S)$ and $G^2(S)$ one can associate with S. More precisely, there is a product $A \in S^*$ with $A_{ii} > 1$ if and only if

1. there is a cycle in $G(S)$ containing at least one edge of weight $w > 1$, or
2. there is a cycle in $G^2(S)$ containing at least one vertex (i, i) and at least one vertex (p, q) with $p \neq q$.

Hence, we have $\rho(S) \leqslant 1$ if and only if neither of the two conditions holds, which can be checked in polynomial time according to [5]. Furthermore, as already mentioned in [5, Chapter 3], this graph-theoretic characterization does not only hold for non-negative integer matrices, but for any set of matrices such that all matrix entries are either zero or at least one (because then all paths in $G(S)$ have weight at least one).

▶ **Lemma 4.7.** *Let $S \subseteq \mathbb{R}_0^{n\times n}$ be a finite set of matrices where all matrix entries are either zero or at least one. Then $\rho(S) \leqslant 1$ is decidable in polynomial time.* ◀

So, in the situation of Lemma 4.7, polynomial boundedness of $\mathsf{growth}_S(k)$ is decidable in polynomial time. In addition, the exact degree of growth can be computed in polynomial time (cf. [5, Theorem 3.3, Lemma 3.2 and Proposition 3.3]).

▶ **Theorem 4.8.** *Let $S \subseteq \mathbb{R}_0^{n\times n}$ be a finite set of matrices such that $\rho(S) \leqslant 1$ and all matrix entries are either zero or at least one, and let $d \geqslant 0$ be the largest integer possessing the following property: there exist d different pairs of indices $(i_1, j_1), \ldots, (i_d, j_d)$ such that for every pair (i_s, j_s) the indices i_s, j_s are different and there is a product $A \in S^*$ for which $A_{i_s i_s}, A_{i_s j_s}, A_{j_s j_s} \geqslant 1$, and for each $1 \leqslant s \leqslant d-1$, there exists $B \in S^*$ with $B_{j_s i_{s+1}} \geqslant 1$. Then $\mathsf{growth}_S(k) \in \Theta(k^d)$ if $d \geqslant 1$ and $\mathsf{growth}_S(k) \in O(k^d)$ if $d = 0$. Moreover, the* growth rate *d is computable in polynomial time and $d \leqslant n-1$.* ◀

Next we elaborate on the ramifications of joint spectral radius theory on complexity analysis of TRSs via polynomially bounded matrix interpretations. To begin with, we note that a matrix interpretation $\mathcal{M}$ is polynomially bounded if and only if $\rho(S_{\mathcal{M}}) \leqslant 1$. This follows directly from Theorem 4.5. Due to the relationship between polynomially bounded matrix interpretations and the derivational complexity of compatible TRSs expressed in Lemma 4.2, we immediately obtain the following result, which holds for matrix interpretations over $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$.

▶ **Theorem 4.9.** *Let $\mathcal{R}$ be a TRS and $\mathcal{M}$ a compatible matrix interpretation of dimension n. If $\rho(S_{\mathcal{M}}) \leqslant 1$, then $\mathsf{dc}_{\mathcal{R}}(k) \in O(k^n)$.* ◀

As this theorem assumes the worst-case growth rate for $\mathsf{growth}_{S_{\mathcal{M}}}(k)$, the inferred degree of the polynomial bound may generally be too high (and unnecessarily so). Yet with the help of Theorem 4.8, from which we obtain the exact growth rate, Theorem 4.9 can be strengthened (in this respect), at the expense of having to restrict the set of permissible matrices.

▶ **Theorem 4.10.** *Let $\mathcal{R}$ be a TRS and $\mathcal{M}$ a compatible matrix interpretation of dimension n where all matrix entries are either zero or at least one. If $\rho(S_{\mathcal{M}}) \leqslant 1$, then $\mathsf{dc}_{\mathcal{R}}(k) \in O(k^{d+1})$, where d refers to the growth rate obtained from Theorem 4.8.* ◀

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Encoding Induction in Correctness Proofs of Program Transformations as a Termination Problem*

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Abstract

The diagram-based method to prove correctness of program transformations consists of computing complete set of (forking and commuting) diagrams, acting on sequences of standard reductions and program transformations. In many cases, the only missing step for proving correctness of a program transformation is to show the termination of the rearrangement of the sequences. Therefore we encode complete sets of diagrams as term rewriting systems and use an automated tool to show termination, which provides a further step in the automation of the inductive step in correctness proofs.

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1 Introduction

The motivation for this work is derived from proving correctness of program transformations in program calculi, in particular in extended lambda calculi that model core-languages of variants of Haskell.

In our setting a *program calculus* is a tuple $(\mathcal{E}, \mathcal{C}, \xrightarrow{sr}, \mathcal{A})$ where $\mathcal{E}$ is the set of *expressions*, $\mathcal{C}$ is the set of *contexts*, i.e. usually $\mathcal{C}$ consists of all expressions of $\mathcal{E}$ where one subexpression is replaced by the context hole, $\xrightarrow{sr} \subseteq \mathcal{E} \times \mathcal{E}$ is a small-step reduction relation (called **s**tandard **r**eduction) which defines the operational semantics of the program calculus and $\mathcal{A} \subseteq \mathcal{E}$ is a set of *answers*, which are usually $\xrightarrow{sr}$-irreducible. The evaluation of a program expression $e \in \mathcal{E}$ is a sequence of standard reduction steps to an answer $a \in \mathcal{A}$, i.e. $e \xrightarrow{sr,*} a$, where $\xrightarrow{sr,*}$ denotes the reflexive-transitive closure of $\xrightarrow{sr}$. If such an evaluation exists, then we write $e\Downarrow$ and say e *converges*, otherwise we write $e\Uparrow$ and say e *diverges*. The semantics is the *contextual equivalence* of expressions: $e \sim_c e' :\iff e \leq_c e' \wedge e' \leq_c e$, where $e \leq_c e' :\iff \forall C \in \mathcal{C} : C[e]\Downarrow \implies C[e']\Downarrow$.

A program transformation $\xrightarrow{T} \subseteq (\mathcal{E} \times \mathcal{E})$ is a binary relation on expressions. It is called *correct* if for all e, e' with $e \xrightarrow{T} e'$ the equivalence $e \sim_c e'$ holds. Usually a *context-closure* T' of the program transformation T is considered (w.r.t. all contexts, or a restricted class of contexts, if a context lemma is available), such that proving $e \xrightarrow{T'} e'$ implies $e\Downarrow \iff e'\Downarrow$ suffices to conclude that T is a correct program transformation. In the following we do not distinguish between T and its context-closure T', and assume that a program transformation is always closed by an appropriate class of contexts such that the correctness proof is reduced to show

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equivalence of convergence for all $e \xrightarrow{T} e'$. Moreover, with $\xleftarrow{T}$ denoting the inverse of $\xrightarrow{T}$ the correctness of $\xrightarrow{T}$ holds, if $\xrightarrow{T}$ as well as $\xleftarrow{T}$ are convergence preserving (where $\xrightarrow{T}$ is convergence preserving if $e \xrightarrow{T} e' \implies (e\Downarrow \implies e'\Downarrow)$).

In the application below, we will use different program transformations $T_1, \dots, T_k$ in which case we set $T = \bigcup_{i=1}^{k} T_i$. In general, there are also different kinds of standard reductions, which are used in the concrete proofs, hence we extend the standard reduction to $\xrightarrow{sr} \subseteq \mathcal{E} \times \mathcal{E} \times L$ where L is a set of labels. Sometimes we indicate the label l by writing $\xrightarrow{sr,l}$. We say the program transformation $\xrightarrow{T}$ is *answer-preserving*, if $a \in \mathcal{A}$ and $a \xrightarrow{T} e$ implies $e \in \mathcal{A}$; and *weakly answer-preserving*, if $a \in \mathcal{A}$ and $a \xrightarrow{T} e$ implies $e\Downarrow$.

The diagram-based proof method operates on abstract reduction sequences (ARS), which are strings consisting only of the standard reductions with their labels, and the program transformations, but the expressions are ignored with the exception of an abstract symbol A for an answer. A *forking diagram* is a rewriting rule $L \rightsquigarrow R$ on ARSs. The semantics of a diagram $L \rightsquigarrow R$ is that the reduction sequence L can be transformed (or rewritten) into the reduction sequence R. We also allow diagrams that speak about transitive closures of reductions. We are only interested in ARSs that are a mix of $\xleftarrow{sr}$ and $\xrightarrow{T}$-reductions, perhaps labeled, together with an answer token A to the left. The idea of the diagrams is that they transform reduction sequences into evaluations. In general, this rewriting is non-deterministic, which is the price for abstracting away the term structure. *Completeness* of a set $DF(\xrightarrow{T})$ of forking diagrams for transformation $\xrightarrow{T}$ means that every ARS $A \xleftarrow{sr,+} \xrightarrow{T}$ is modifiable by a diagram. For $\xleftarrow{T}$ we call the diagrams in $DF(\xleftarrow{T})$ *commuting diagrams*.

Usually, forking diagrams are of the form $\xleftarrow{sr,l_n} \dots \xleftarrow{sr,l_1} \xrightarrow{T_k} \rightsquigarrow \xrightarrow{T_1} \dots \xrightarrow{T_m} \xleftarrow{sr,l_{n'}} \dots \xleftarrow{sr,l_1}$ where labels l_i may also be omitted and where also the meta-symbols $+$ and $*$ may occur for the transitive/transitive-reflexive closure of a standard reduction or transformation.

We also need another form of diagrams, the *answer diagrams*, $DA(\xrightarrow{T})$, which are called *complete* (for transformation $\xrightarrow{T}$), if every ARS $A \xrightarrow{T,+}$ is modifiable by a diagram in DA. In the case of answer-preservation, these extra diagrams are simply $A \xrightarrow{T} \rightsquigarrow A$, and for a weakly answer-preserving transformation, the diagrams are of the form $A \xrightarrow{T} \rightsquigarrow A \xleftarrow{sr,l_n} \dots \xleftarrow{sr,l_1}$, where usually only a subset of the labels occur as l_i which may ease the termination proof.

In applications to calculi, the computation of the diagram sets for a given program transformation is done by analyzing the syntax of expressions and the syntax of rules and by covering all possibilities, where usually labels are heavily used at $\xleftarrow{sr}$, depending on the kind of reduction rules, and often, several program transformations occur in the diagram set. This computation may be done by hand, but there is also a proposal for automating this in an expressive core calculus of Haskell, see [3, 4].

The diagram based method to show correctness of a program transformation $\xrightarrow{T}$ is performed by the steps:

1. Show (weak) answer-preservation of $\xrightarrow{T}$ and compute the DA-diagrams.
2. Compute complete sets of forking-diagrams for $\xrightarrow{T}$.
3. Show that every reduction sequence $a \xleftarrow{sr,*} e \xrightarrow{T} e'$ where $a \in \mathcal{A}$ can be transformed using the diagrams from steps 1 and 2 into $a' \xleftarrow{sr,*} e'$, where $a' \in \mathcal{A}$. This is usually done by an induction on the application of diagrams.
4. Do the same by performing steps (1), (2), (3) for the inverse relation $\xleftarrow{T}$.

Since answer-preservation implies weak answer-preservation, we show the next theorem only for the weak case.

1
$$\begin{tikzcd}
\cdot \arrow[r, "iS{,}seq"] \arrow[d, "n{,}a"'] & \cdot \arrow[d, dashed, "n{,}a"] \\
\cdot \arrow[r, dashed, "iS{,}seq"'] & \cdot
\end{tikzcd}$$

2
$$\begin{tikzcd}
\cdot \arrow[r, "iS{,}seq"] \arrow[d, "n{,}a"'] & \cdot \arrow[dl, dashed, "n{,}a"] \\
\cdot &
\end{tikzcd}$$

3
$$\begin{tikzcd}
\cdot \arrow[r, "iS{,}seq"] \arrow[d, "n{,}a"'] & \cdot \arrow[ddl, dashed, "n{,}a"] \\
\cdot \arrow[d, "n{,}seq"'] & \\
\cdot &
\end{tikzcd}$$

4
$$\begin{tikzcd}
\cdot \arrow[rr, "iS{,}seq"] \arrow[d, "n{,}cp"'] & & \cdot \arrow[d, dashed, "n{,}cp"] \\
\cdot \arrow[r, dashed, "iS{,}seq"'] & \cdot \arrow[r, dashed, "iS{,}seq"'] & \cdot
\end{tikzcd}$$

(a) Forking diagrams for the transformation $(i\mathcal{S}, seq)$

1	$iSseq(n(a,x)) \to n(a, iSseq(x))$ $iSseq(n(seq,x)) \to n(seq, iSseq(x))$ $iSseq(n(cp,x)) \to n(cp, iSseq(x))$	**3**	$iSseq(n(a,n(seq,x))) \to n(a,x)$ $iSseq(n(seq,n(seq,x))) \to n(seq,x)$ $iSseq(n(cp,n(seq,x))) \to n(cp,x)$
2	$iSseq(n(a,x)) \to n(a,x)$ $iSseq(n(seq,x)) \to n(seq,x)$ $iSseq(n(cp,x)) \to n(cp,x)$	**4**	$iSseq(n(cp,x)) \to n(cp, iSseq(iSseq(x)))$
		Answer diagram	$iSseq(w) \to w$

(b) TRS encoding of forking- and answer-diagrams for the transformation $(i\mathcal{S}, seq)$

Figure 1 Diagrams and their TRS encoding for the transformation $(i\mathcal{S}, seq)$

▶ Proposition 1.1. Let $\xrightarrow{T}$ be weakly answer preserving, and let $DF(\xrightarrow{T})$ and $DA(\xrightarrow{T})$ be the complete sets of forking and answer diagrams, respectively, for $\xrightarrow{T}$. Then termination of $DF(\xrightarrow{T}) \cup DA(\xrightarrow{T})$ implies that $\xrightarrow{T} \subseteq \leq_c$.

Proof. Starting with $e\Downarrow$ and $e \xrightarrow{T} e'$, this corresponds to an ARS of the form $A \xleftarrow{sr,l_n} \dots \xleftarrow{sr,l_1} \xrightarrow{T}$. Completeness of $DF(\xrightarrow{T})$ and $DA(\xrightarrow{T})$ guarantees that an ARS in normal-form is of the form $A \xleftarrow{sr,l'_m} \dots \xleftarrow{sr,l'_1}$, which shows $e'\Downarrow$. Since $\xrightarrow{T}$ is assumed to be closed for context application, this implies $e \leq_c e'$. Since this holds for all $e \xrightarrow{T} e'$, we have shown $\xrightarrow{T} \subseteq \leq_c$. ◀

▶ **Theorem 1.2.** *If the assumptions of Proposition 1.1 hold for $\xrightarrow{T}$ as well as for $\xleftarrow{T}$ – including complete sets $DF(\xrightarrow{T}), DF(\xleftarrow{T}), DA(\xrightarrow{T})$, and $DA(\xleftarrow{T})$ – then $\xrightarrow{T} \subseteq \sim_c$, which means that T is a correct program transformation.*

Based on the description above and Theorem 1.2, we encode reduction sequences as terms, and complete sets of diagrams as term rewriting systems on the sequences. As we will demonstrate for encoding some of our diagrams including transitive closure we require *conditional integer term rewriting systems* (ITRS). However, these can also be treated by the automated termination prover AProVE [1, 2]. Hence we can use the AProVE system to show termination of TRSs / conditional ITRSs, which provides a further step in the automation of the inductive step in correctness proofs.

2 Encodings of Reductions and Sets of Diagrams

We give some examples for the encoding of complete sets of diagrams into (I)TRSs. The diagrams are taken from [5] for an extended call-by-need lambda calculus with a standard reduction called *normal order reduction*, denoted as $\xrightarrow{n}$, and expressions considered as answers are called *weak head normal forms* (WHNFs).

We first consider the transformation *seq*. Figure 1a shows the forking diagrams $DF(\xrightarrow{i\mathcal{S},seq})$ for the transformation $(i\mathcal{S}, seq)$, which is the context-closure of *seq*. The label a signifies an arbitrary reduction label. The solid lines in the diagrams represent the left hand sides and the dashed lines the right hand sides in the diagram rules of the form $L \rightsquigarrow R$. The diagrams can also be represented in their flat form, e.g. the flat form of the first diagram is $\xleftarrow{n,a}\xrightarrow{i\mathcal{S},seq} \rightsquigarrow \xrightarrow{i\mathcal{S},seq}\xleftarrow{n,a}$. Figure 1b shows the TRS encoding of the forking- and answer-diagrams for the transformation $(i\mathcal{S}, seq)$, where x is a variable, and all other symbols are function symbols. We highlight on some properties of the encoding:

- The special answer token (i.e. WHNF-token) is represented as the constant w.
- The abstract reduction sequences in the graphical diagrams are encoded from *right to left*, i.e. the flat diagram $\xleftarrow{n,a}\xrightarrow{i\mathcal{S},seq} \rightsquigarrow \xrightarrow{i\mathcal{S},seq}\xleftarrow{n,a}$ is represented as the rewrite rule $iSseq(n(a,x)) \to n(a, iSseq(x))$. This is done to express the fact that diagrams turn reduction sequences into evaluations. E.g. the sequence $w \xleftarrow{n,a}\xleftarrow{n,a}\xleftarrow{n,a}\xrightarrow{i\mathcal{S},seq}$ is represented by the term $iSseq(n(a,n(a,n(a,w))))$ and can be turned into the evaluation $w \xleftarrow{n,a}\xleftarrow{n,a}\xleftarrow{n,a}$ (either by repeated application of the first diagram and a closing application of the answer-diagram or by a single application of the second diagram).
- The labels of normal order reductions and transformations are encoded differently: Labels of transformations are encoded directly into function symbols (like *iSseq*) whereas labels of normal order reductions are encoded as parameters of function applications, e.g. in the term $n(a,x)$ the constant a denotes the label of the normal order reduction. Here a is a constant that represents arbitrary reduction labels (that are not *seq* or *cp*) whereas the constants *seq* and *cp* denote those specific labels (this is also the reason why we need three rewrite rules per diagram in the present example). The different encoding of names has mainly technical reasons: The automatic proofs using AProVE are in some cases only possible with the described encoding.

Since *seq* is answer-preserving the TRS encoding of $DA(\xrightarrow{i\mathcal{S},seq})$ consists of the single diagram $iSseq(w) \to w$. For the *seq* transformation the termination of the TRS encoded complete diagram set could be automatically shown.

Figure 2a gives another example of a complete set of forking diagrams $DF(\xrightarrow{i\mathcal{S},llet})$ for the transformation $(i\mathcal{S}, llet)$, which is answer-preserving. In the diagrams $\xrightarrow{n,lll^+}$ represents a (non-empty) sequence of *lll*-reductions i.e. the transitive closure of those reductions. These symbols require a special treatment in the encoding into TRS, since they represent an infinite set of diagrams. If the symbol $\xrightarrow{n,lll^+}$ occurs on the left hand side of a diagram, this means that any given (non-empty) reduction sequence of *lll*-reductions can be matched. In the encoding this symbol is represented by the function symbol *nlllPlusL* and there are additional rules which allow to contract a given sequence of *lll*-reductions into the symbol $\xrightarrow{n,lll^+}$ (see Figure 2c). If a symbol $\xrightarrow{n,lll^+}$ occurs on the right hand side of a diagram, then a naive approach would be to add rules $\xrightarrow{n,lll^+} \rightsquigarrow \xrightarrow{n,lll}$ and $\xrightarrow{n,lll^+} \rightsquigarrow \xrightarrow{n,lll^+}\xrightarrow{n,lll}$. However, this approach does not work, since it introduces nontermination in the corresponding TRS. Hence, we use integer term rewrite systems for the encoding, which allow to rewrite the symbol $\xrightarrow{n,lll^+}$ into a sequence of $\xrightarrow{n,lll}$-reductions of arbitrary but *fixed* length. In the encoding we use the function symbol *nlllPlusR* for the occurrence of $\xrightarrow{n,lll^+}$ on the right hand side. For diagrams 3 and 4, an integer variable k is introduced by the rewriting rule which is like guessing a natural number. Additionally we add ITRS-rules to rewrite the symbol into a sequence of k $\xrightarrow{n,lll}$-reductions (see Figure 2d).

Termination of $DF(\xrightarrow{i\mathcal{S},llet}) \cup DA(\xrightarrow{i\mathcal{S},llet})$ can be automatically checked using AProVE.

1
$$\begin{tikzcd}
\cdot \arrow[r, "i\mathcal{S}{,}llet"] \arrow[d, "n{,}a"'] & \cdot \arrow[d, dashed, "n{,}a"'] \\
\cdot \arrow[r, dashed, "i\mathcal{S}{,}llet"] & \cdot
\end{tikzcd}$$

2
$$\begin{tikzcd}
\cdot \arrow[r, "i\mathcal{S}{,}llet"] \arrow[d, "n{,}a"'] & \cdot \arrow[dl, dashed, "n{,}a"] \\
\cdot &
\end{tikzcd}$$

3
$$\begin{tikzcd}
\cdot \arrow[r, "i\mathcal{S}{,}llet"] \arrow[d, "n{,}lll^+"'] & \cdot \arrow[dl, dashed, "n{,}lll^+"] \\
\cdot &
\end{tikzcd}$$

4
$$\begin{tikzcd}
\cdot \arrow[r, "i\mathcal{S}{,}llet"] \arrow[d, "n{,}lll^+"'] & \cdot \arrow[d, dashed, "n{,}lll^+"'] \\
\cdot \arrow[r, dashed, "i\mathcal{S}{,}llet"] & \cdot
\end{tikzcd}$$

5
$$\begin{tikzcd}
\cdot \arrow[r, "i\mathcal{S}{,}llet"] \arrow[d, "n{,}a"'] & \cdot \arrow[ddl, dashed, "n{,}a"] \\
\cdot \arrow[d, "n{,}llet"'] & \\
\cdot &
\end{tikzcd}$$

(a) Forking diagrams for the transformation $(i\mathcal{S}, llet)$

3 $iSllet(nlllPlusL(x)) \to nlllPlusR(k, x)$	**4** $iSllet(nlllPlusL(x)) \to nlllPlusR(k, iSllet(x))$

(b) ITRS encoding of the third and fourth forking diagram for the transformation $(i\mathcal{S}, llet)$

$$\boxed{\begin{aligned} n(lll, nlllPlusL(x)) &\to nlllPlusL(x) \\ n(lll, x) &\to nlllPlusL(x) \end{aligned}}$$

$$\boxed{\begin{aligned} nlllPlusR(0, x) &\to x \\ nlllPlusR(k, x) &\to nlllPlusR(k-1, n(lll, x)) \text{ if } k > 0 \end{aligned}}$$

(c) Contracting $\xrightarrow{n,lll}$-sequences into $\xrightarrow{n,lll^+}$ **(d)** Expansion of $\xrightarrow{n,lll^+}$ into k $\xrightarrow{n,lll}$-reductions

Figure 2 Diagrams and ITRS encoding for the transformation $(i\mathcal{S}, llet)$

Conclusion We tested the complete sets of (forking as well as commuting) diagrams of several program transformations from [5] and they could all be shown terminating with the above method using AProVE as a tool for automatic termination proofs. While the encoding of most of the diagrams from [5] was in general rather straightforward, there are also cases, where additional knowledge (beyond the mere information of the diagram) has to be employed in the encoding, or where the automatic proof can only be found for a particular syntactic variant. An increasing set of (I)TRS-encoded diagrams and the corresponding termination proofs in AProVE can be found on the website:
`http://www.ki.informatik.uni-frankfurt.de/research/dfg-diagram/auto-induct/`.

Future work is to connect the automated termination prover with the diagram calculator of [3, 4] and thus to complete the tool for automated correctness proofs of program transformations. Another direction is to check more sets of diagrams which probably requires to develop more sophisticated encoding techniques.

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A Relative Dependency Pair Framework*

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1 Introduction

Relative rewriting and the dependency pair framework (DP framework) are two strongly related termination methods. In both formalisms we consider whether the combination of two TRSs allows an infinite derivation:

- Relative termination of $\mathcal{R}/\mathcal{S}$ can be defined as strong normalization of $\to_{\mathcal{R}} \cdot \to_{\mathcal{S}}^*$.
- Finiteness of a DP problem $(\mathcal{P}, \mathcal{R})$ can be defined as strong normalization of $\xrightarrow{\epsilon}_{\mathcal{P}} \cdot \to_{\mathcal{R}}^*$ where $\xrightarrow{\epsilon}$ allows steps only at the top. Moreover, *minimality* can be incorporated by requiring that all terms are terminating w.r.t. $\mathcal{R}$.

The above definitions have two orthogonal distinctions of rules. In both formalisms there are *strict* and *weak* rules: $\mathcal{P}$ and $\mathcal{R}$ are the strict rules of $(\mathcal{P}, \mathcal{R})$ and $\mathcal{R}/\mathcal{S}$, respectively, while $\mathcal{R}$ and $\mathcal{S}$ are the respective weak rules. In the DP framework, there is the additional distinction between rules that may only be applied at the top ($\mathcal{P}$) and those that can be applied at arbitrary positions ($\mathcal{R}$).

Note that the restriction to top rewriting is an important advantage for proving termination in the DP framework. It allows to use non-monotone orders for orienting the strict rules. Furthermore, if minimality is considered, we can use termination techniques (e.g., usable rules or the subterm criterion) that are not available for relative rewriting.

However, also relative rewriting has some advantages which are currently not available in the DP framework: Geser showed that it is always possible to split a relative termination problem into two parts [4]. Relative termination of $(\mathcal{R}_s \cup \mathcal{R}_w)/(\mathcal{S}_s \cup \mathcal{S}_w)$ can be shown by relative termination of both $(\mathcal{R}_s \cup \mathcal{S}_s)/(\mathcal{R}_w \cup \mathcal{S}_w)$ and $\mathcal{R}_w/\mathcal{S}_w$. Hence, it is possible to show (in a first relative termination proof) that the strict rules $\mathcal{R}_s \cup \mathcal{S}_s$ cannot occur infinitely often and afterwards continue (in a second relative termination proof) with the problem $\mathcal{R}_w/\mathcal{S}_w$. A major advantage of this approach is that in the first proof we can apply arbitrary techniques which may increase the size of the TRSs drastically (e.g., semantic labeling [11]), or which may even be incomplete (e.g., string reversal in combination with innermost rewriting, where by reversing the rules we have to forget about the strategy). As long as relative termination of $(\mathcal{R}_s \cup \mathcal{S}_s)/(\mathcal{R}_w \cup \mathcal{S}_w)$ could be proven, we can afterwards continue independently with the problem $\mathcal{R}_w/\mathcal{S}_w$.

Such a split is currently not possible in the DP framework since there are no top weak rules and also no strict rules which can be applied everywhere.

In this paper we generalize the DP framework to a relative DP framework, where such a split is possible. To this end, we consider DP problems of the form $(\mathcal{P}, \mathcal{P}_w, \mathcal{R}, \mathcal{R}_w)$, where we have strict and weak, top and non-top rules. (This kind of DP problems were first suggested by Jörg Endrullis at the *Workshop on the Certification of Termination Proofs* in 2007 and

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are already used in his termination tool Jambox [3]. Unfortunately the suggestion did not get much attention back then and we are not aware of any publications on this topic.) In this way, problems that occur in combination with semantic labeling and dependency pairs—which can otherwise be solved by using a dedicated semantics for DP problems [9]—can easily be avoided. Furthermore, the new framework is more general than [9] since it also solves some problems that occur when using other termination techniques like uncurrying [6, 8].

2 A Relative Dependency Pair Framework

We assume familiarity with term rewriting [2] and the DP framework [5].

▶ **Definition 1.** A *relative dependency pair problem* $(\mathcal{P}, \mathcal{P}_w, \mathcal{R}, \mathcal{R}_w)$ is a quadruple of TRSs with *pairs* $\mathcal{P} \cup \mathcal{P}_w$ (where pairs from $\mathcal{P}$ are called *strict* and those of $\mathcal{P}_w$ *weak*) and *rules* $\mathcal{R} \cup \mathcal{R}_w$ (where rules of $\mathcal{R}$ are called *strict* and those of $\mathcal{R}_w$ *weak*).

For relative DPPs the notion of chains and finiteness is adapted in the following way.

▶ **Definition 2.** An infinite sequence of pairs $s_1 \to t_1$, $s_2 \to t_2$, ... forms a $(\mathcal{P}, \mathcal{P}_w, \mathcal{R}, \mathcal{R}_w)$-*chain* if there exists a corresponding sequence of substitutions σ_1, σ_2, ... such that

$$s_i \to t_i \in \mathcal{P} \cup \mathcal{P}_w \text{ for all } i \tag{1}$$

$$t_i\sigma_i \to^*_{\mathcal{R}\cup\mathcal{R}_w} s_{i+1}\sigma_{i+1} \text{ for all } i \tag{2}$$

$$s_i \to t_i \in \mathcal{P} \text{ or } t_i\sigma_i \to^*_{\mathcal{R}\cup\mathcal{R}_w} \cdot \to_{\mathcal{R}} \cdot \to^*_{\mathcal{R}\cup\mathcal{R}_w} s_{i+1}\sigma_{i+1} \text{ for infinitely many } i \tag{3}$$

For minimal chains, we additionally require

$$\mathsf{SN}_{\mathcal{R}\cup\mathcal{R}_w}(t_i\sigma_i) \text{ for all } i \tag{4}$$

A relative DPP $(\mathcal{P}, \mathcal{P}_w, \mathcal{R}, \mathcal{R}_w)$ is *finite*, iff there is no minimal infinite $(\mathcal{P}, \mathcal{P}_w, \mathcal{R}, \mathcal{R}_w)$-chain.

Hence, a (minimal) $(\mathcal{P}, \mathcal{P}_w, \mathcal{R}, \mathcal{R}_w)$-chain is like a (minimal) $(\mathcal{P} \cup \mathcal{P}_w, \mathcal{R} \cup \mathcal{R}_w)$-chain—as defined in [1]—with the additional demand that there are infinitely many strict steps using $\mathcal{P}$ or $\mathcal{R}$. It is easy to see that $(\mathcal{P}, \mathcal{R})$-chains can be expressed in the new framework.

▶ **Lemma 3.** *The DP problem $(\mathcal{P}, \mathcal{R})$ is finite iff there exists a minimal $(\mathcal{P}, \mathcal{R})$-chain iff there exists a minimal $(\mathcal{P}, \emptyset, \emptyset, \mathcal{R})$-chain iff the relative DPP $(\mathcal{P}, \emptyset, \emptyset, \mathcal{R})$ is finite.*

Note that in contrast to DPPs $(\mathcal{P}, \mathcal{R})$, for relative DPPs, $\mathcal{P} = \emptyset$ does not imply finiteness of $(\mathcal{P}, \mathcal{P}_w, \mathcal{R}, \mathcal{R}_w)$.

▶ **Example 4.** The relative DPP $(\emptyset, \{\mathsf{F}(\mathsf{a}) \to \mathsf{F}(\mathsf{b})\}, \{\mathsf{b} \to \mathsf{a}\}, \emptyset)$ is not finite.

However, a sufficient criterion for finiteness is that there are either no pairs, or that there are neither strict pairs nor strict rules.

▶ **Lemma 5** (Trivially finite relative DPPs). *If $\mathcal{P} \cup \mathcal{P}_w = \emptyset$ or $\mathcal{P} \cup \mathcal{R} = \emptyset$ then $(\mathcal{P}, \mathcal{P}_w, \mathcal{R}, \mathcal{R}_w)$ is finite.*

3 Processors in the Relative Dependency Pair Framework

Processors and soundness of processors in the relative DP framework are defined as in the DP framework, but operate on relative DPPs instead of DPPs (a processor is sound if finiteness of all resulting relative DPPs implies finiteness of the given relative DPP).

Note that most processors can easily be adapted to the new framework where most often it suffices to treat the relative DPP $(\mathcal{P}, \mathcal{P}_w, \mathcal{R}, \mathcal{R}_w)$ as the DPP $(\mathcal{P} \cup \mathcal{P}_w, \mathcal{R} \cup \mathcal{R}_w)$.

However, when starting with the initial relative DPP $(DP(\mathcal{R}), \emptyset, \emptyset, \mathcal{R})$ it is questionable whether we ever reach relative DPPs containing weak pairs or strict rules. If this is not the case, then our generalization would be useless. Therefore, in the following we give evidence that the additional flexibility is beneficial.

Easy examples are semantic labeling and uncurrying. Both techniques are transformational techniques where each original step is transformed into one main transformed step together with some auxiliary steps. For the auxiliary steps one uses auxiliary pairs and rules (the decreasing rules and the uncurrying rules, respectively). If there are auxiliary pairs $\mathcal{P}_{aux}$, then in the DP framework, $\mathcal{P}_{aux}$ can only be added as strict pairs, whereas in the relative DP framework, we can add $\mathcal{P}_{aux}$ to the weak pairs, and hence we do not have to delete all pairs of $\mathcal{P}_{aux}$ anymore for proving finiteness.

As another example, we consider top-uncurrying of [8, Def. 19], where some rules $\mathcal{R}$ are turned into pairs. Again, in the DP framework this would turn the weak rules $\mathcal{R}$ into strict pairs, which in fact would demand that we prove termination of $\mathcal{R}$ twice: Once via the original DPs for $\mathcal{R}$, and a second time after the weak rules of $\mathcal{R}$ have been converted into strict pairs. For example, in [8, Ex. 21] termination of the minus-rules is proven twice. This is no longer required in the relative DP framework where one can just turn the weak rules $\mathcal{R}$ into weak pairs $\mathcal{R}$.

Finally, in the relative DP framework we can apply the split technique known from relative rewriting.

▶ **Definition 6** (Split processor). The relative DPP $(\mathcal{P}_s^1 \cup \mathcal{P}_w^1, \mathcal{P}_s^2 \cup \mathcal{P}_w^2, \mathcal{R}_s^1 \cup \mathcal{R}_w^1, \mathcal{R}_s^2 \cup \mathcal{R}_w^2)$ is finite if both $(\mathcal{P}_s^1 \cup \mathcal{P}_s^2, \mathcal{P}_w^1 \cup \mathcal{P}_w^2, \mathcal{R}_s^1 \cup \mathcal{R}_s^2, \mathcal{R}_w^1 \cup \mathcal{R}_w^2)$ and $(\mathcal{P}_w^1, \mathcal{P}_w^2, \mathcal{R}_w^2, \mathcal{R}_w^2)$ are finite.

A more instructive way of putting the above definition for termination tool authors that are used to standard DP problems is as follows. Start from the relative DPP $(\mathcal{P}, \emptyset, \emptyset, \mathcal{R})$. Identify pairs $\mathcal{P}'$ and rules $\mathcal{R}'$ that should be deleted. Then use the split processor to obtain the two relative DPPs $(\mathcal{P}', \mathcal{P} \setminus \mathcal{P}', \mathcal{R}', \mathcal{R} \setminus \mathcal{R}')$ and $(\mathcal{P} \setminus \mathcal{P}', \emptyset, \emptyset, \mathcal{R} \setminus \mathcal{R}')$.

Clearly, the split processor can be used to obtain relative DPPs with strict rules and weak pairs, but the question is how to apply it. We give two possibilities.

Semantic labeling is often used in a way, that after labeling one tries to remove all labeled variants of some rules $\mathcal{R}_s$ and pairs $\mathcal{P}_s$, and afterwards removes the labels again to continue on a smaller unlabeled problem.

▶ **Example 7.** Consider a DP problem $p_1 = (\{1, 2\}, \{3\})$. After applying semantic labeling, all pairs and rules occur in labeled variants $1.x$, $2.x$, and $3.x$, so the resulting DP problem might look like $(\{1.a, 1.b, 2.a, 2.b\}, \{3.a, 3.b, 3.c\})$. Applying standard techniques to remove pairs and rules one might get stuck at $p_2 = (\{2.a, 2.b\}, \{3.a, 3.c\})$. Although p_1 contains less rules than p_2, p_2 is somehow simpler since all rules $1.x$ have been removed. And indeed, after applying unlabeling on p_2 the resulting DP problem $p_3 = (\{2\}, \{3\})$ is smaller than p_1.

Since the removal of labels is problematic for soundness, a special semantics was developed in [9]. This is no longer required in the relative DP framework. After $\mathcal{R}_s$ and $\mathcal{P}_s$ have been identified, one just applies the split processor to transform $(\mathcal{P}, \emptyset, \emptyset, \mathcal{R})$ into $(\mathcal{P}_s, \mathcal{P} \setminus \mathcal{P}_s, \mathcal{R}_s, \mathcal{R} \setminus \mathcal{R}_s)$ and $(\mathcal{P} \setminus \mathcal{P}_s, \emptyset, \emptyset, \mathcal{R} \setminus \mathcal{R}_s)$. The proof that all labeled variants of rules in $\mathcal{R}_s$ and pairs in $\mathcal{P}_s$ can be dropped, proves finiteness of the first problem, and one can continue on the latter problem without having to apply unlabeling.

▶ **Example 8.** Using split, we can restructure the proof of Example 7 without using unlabeling: We know that in the end, we only get rid of pair 1. Hence, we apply split on p_1 to obtain p_3 and $p_4 = (\{1\}, \{2\}, \emptyset, \{3\})$. Thus, we get the same remaining problem p_3 if we can prove finiteness of p_4. But this can be done by replaying the proof steps in Example 7. Applying the same labeling as before, we obtain $p_5 = (\{1.a, 1.b\}, \{2.a, 2.b\}, \emptyset, \{3.a, 3.b, 3.c\})$. Removing pairs and rules as before, we simplify p_5 to $p_6 = (\emptyset, \{2.a, 2.b\}, \emptyset, \{3.a, 3.c\})$ and this relative DP problem is trivially finite by Lemma 5.

Note that using [9] it was only possible to revert the labeling, but not to revert other techniques like the closure under flat contexts which is used in combination with root-labeling [7]. However, using the split processor this is also easily possible, since one just has to apply the split processor before applying the closure under flat contexts.

A further advantage of the relative DP framework in comparison to [9] can be seen in the combination of semantic labeling with the dependency graph processor.

▶ **Example 9.** Consider a DP problem $p_1 = (\{1, 2\}, \{3, 4\})$ which is transformed into $(\{1.a, 1.b, 2.a, 2.b\}, \{3.a, 3.b, 4.a, 4.b\})$ using semantic labeling. Applying the dependency graph and reduction pairs yields two remaining DP problems $p_2 = (\{2.a\}, \{4.a\})$ and $p_3 = (\{2.b\}, \{3.a, 4.b\})$. Using unlabeling we have to prove finiteness of the two remaining problems $p_4 = (\{2\}, \{4\})$ and $p_5 = (\{2\}, \{3, 4\})$. Note that finiteness of p_5 does not imply finiteness of p_4, so one indeed has to perform two proofs.

However, when using the split processor, only p_5 remains: we observe from p_2 and p_3 that only pair 1 could be removed. So, we start to split p_1 into p_5 and $p_6 = (\{1\}, \{2\}, \emptyset, \{3, 4\})$. Labeling p_6 yields $(\{1.a, 1.b\}, \{2.a, 2.b\}, \emptyset, \{3.a, 3.b, 4.a, 4.b\})$ which is simplified to the two problems $(\emptyset, \{2.a\}, \emptyset, \{4.a\})$ and $(\emptyset, \{2.b\}, \emptyset, \{3.a, 4.b\})$ with the same techniques as before. Both problems are trivially finite by Lemma 5.

Other Techniques may also take advantage of the split processor. For example, the dependency pair transformation of narrowing [1, 5] is not complete in the innermost case but might help to remove some pairs and rules. If it turns out that after some narrowing steps some original pairs and rules can be removed, then one can just insert a split processor before narrowing has been performed. In this way one has obtained progress in proving finiteness and in the remaining system the potential incomplete narrowing steps have not been applied. In other words, the split processor allows to apply incomplete techniques without losing overall completeness.

4 Conclusions and Future Work

We presented the relative DP framework which generalizes the existing DP framework by allowing weak pairs and strict rules. It forms the basis of our proof checker CeTA (since version 2.0) [10] where we additionally integrated innermost rewriting (in the form of $\mathcal{Q}$-restricted rewriting) [5]. One of the main features of the new framework is the possibility to split a DP problem into two DP problems which can be treated independently. Examples to illustrate the new features are provided in the IsaFoR-repository (e.g., `div_uncurry.proof.xml` uses weak pairs for uncurrying, and in `secret_07_trs_4_top.proof.xml` the split processor is used to avoid unlabeling).

It is an obvious question, whether the relative DP framework can be used to characterize relative termination. In a preliminary version we answered this question positively by presenting a theorem that $\mathcal{R}/\mathcal{S}$ is relative terminating iff there is no infinite $(DP(\mathcal{R}), DP(\mathcal{S}), \mathcal{R}, \mathcal{S})$-chain. However, it was detected that the corresponding proof contained a gap (it was the only proof that we did not formalize in Isabelle/HOL) and that the whole theorem did not hold (by means of a counterexample).

An interesting direction for future work is to unify termination (via relative DP problems) with relative termination. One reason is that this would allow to reduce the formalization effort, since results for termination are expected to be corollaries carrying over from relative termination.

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Automated Runtime Complexity Analysis for Prolog by Term Rewriting*

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Abstract

For term rewrite systems (TRSs), a huge number of automated termination analysis techniques have been developed during the last decades, and by automated transformations of Prolog programs to TRSs, these techniques can also be used to prove termination of Prolog programs. Very recently, techniques for automated termination analysis of TRSs have been adapted to prove asymptotic upper bounds for the runtime complexity of TRSs automatically. In this paper, we present ongoing work to transform Prolog programs automatically into TRSs in such a way that the resulting TRSs have at least the same asymptotic upper complexity bound. Thus, techniques for complexity analysis of TRSs can be applied to prove upper complexity bounds for Prolog programs.

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Keywords and phrases Prolog, Complexity, Analysis, Term Rewriting, Automated Reasoning

1 Introduction

Automated complexity analysis of term rewrite systems has recently gained a lot of attention (see e.g., [2, 3, 5, 7, 9, 10, 17, 18]). In particular, there are also new complexity categories for TRSs at the annual *International Termination Competition* [16]. The reason why these complexity categories are integrated in the termination competition is that the techniques used to analyze asymptotic complexity of TRSs were adapted from techniques used for automated termination analysis of TRSs.

Moreover, techniques for termination analysis of TRSs were used to analyze termination of logic programs by transforming such programs into TRSs in a non-termination preserving way (see e.g., [11, 12]). In fact, this transformational approach for termination analysis of logic programs turned out to be more powerful than techniques to analyze termination of logic programs directly.

While there already exists some work on direct complexity analysis for logic programs (e.g., [6]), these approaches can only handle restricted classes of definite logic programs and logic programs with linear arithmetic. Our goal is to achieve better results for automated complexity analysis of logic programs by a transformational approach similar to the ones

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for termination analysis. However, just using the existing transformations to term rewriting does not work, because they are not complexity-preserving. This is mainly due to the fact that backtracking in the logic program is replaced by non-deterministic choice in the TRS.

Instead, we propose a new transformation based on a graph representing all possible executions of a given logic program. This is similar to our approach for termination analysis of Prolog in [13, 14] which goes beyond definite logic programs. In this way, the transformation is also applicable to Prolog programs using built-in predicates like cuts. In contrast to previous transformations which could only be used for termination analysis, our new transformation can also be used for complexity analysis. We briefly introduce some notations and the considered operational semantics of Prolog programs in Section 2. Then we explain very shortly the graph construction from [13, 14] in Section 3. Afterwards, we propose a method to obtain TRSs from such graphs which have at least the same complexity as the original Prolog program in Section 4. We conclude in Section 5.

2 Preliminaries

For the basics of term rewriting, see, e.g., [4]. For a TRS $\mathcal{R}$ with defined symbols $\Sigma_d = \{root(\ell) \mid \ell \to r \in \mathcal{R}\}$, a term $\mathsf{f}(t_1, \dots, t_n)$ is called *basic* if $\mathsf{f} \in \Sigma_d$ and $t_1, \dots, t_n$ do not contain symbols from Σ_d. The *innermost runtime complexity function* $irc_\mathcal{R}$ [7] maps any $n \in \mathbb{N}$ to the length of the longest sequence of $\xrightarrow{i}_\mathcal{R}$-steps starting with a basic term t with $|t| \leq n$. Here, "$\xrightarrow{i}_\mathcal{R}$" is the innermost rewrite relation.

See, e.g., [1] for the basics on logic programming. For Prolog programs, we consider the operational semantics defined in [15] which is equivalent to the semantics defined in the ISO standard [8] for Prolog. As in [8, 15], we do not distinguish between predicate and function symbols. A *query* is a sequence of terms, where $\square$ denotes the empty query. A *clause* is a pair $h \leftarrow B$ where the *head* h is a term and the *body* B is a query. If B is empty, then one writes just "h" instead of "$h \leftarrow \square$". A *Prolog program* $\mathcal{P}$ is a finite sequence of clauses. We often denote the application of a *substitution* σ by $t\sigma$ instead of $\sigma(t)$. A substitution σ is the *most general unifier* (*mgu*) of s and t iff $s\sigma = t\sigma$ and, whenever $s\gamma = t\gamma$ for some other unifier γ, there is a δ with $X\gamma = X\sigma\delta$ for all $X \in \mathcal{V}(s) \cup \mathcal{V}(t)$. If s and t have no *mgu* σ, we write $mgu(s,t) = fail$. $Slice(\mathcal{P}, t)$ are all clauses for t's predicate, i.e., $Slice(\mathcal{P}, p(t_1, ..., t_n)) = \{c \mid c = \text{“}p(s_1, ..., s_n) \leftarrow B\text{”} \in \mathcal{P}\}$.

To describe the semantics of Prolog, we use *states*. A *state* has the form $\langle G_1 \mid \dots \mid G_n \rangle$ where $G_1 \mid \dots \mid G_n$ is a sequence of goals. Essentially, a *goal* is just a *query*, i.e., a sequence of terms. In addition, a goal can also be labeled by a clause c, where the goal $(t_1, \dots, t_k)^c$ means that the next resolution step has to be performed using the clause c. The *initial state* for a query $(t_1, \dots, t_k)$ is $\langle (t_1, \dots, t_k) \rangle$, i.e., this state contains just a single goal.[1] Then the operational semantics can be defined by a set of inference rules on these states, cf. [15]. We show the four inference rules for the core part of Prolog by which definite logic programs can be defined in Figure 1.

So we define the *runtime complexity function* of a Prolog program w.r.t. a query as the function that maps the term size of the query[2] to the length of the maximal derivation that is possible with these inference rules when starting in the initial state for the query. As shown in [15], this is equivalent to the complexity according to the ISO standard for Prolog

[1] We omit answer substitutions for simplicity, since they do not contribute to the complexity.

[2] More precisely, we only measure the size of the *input arguments* of the query, i.e., of those arguments which are guaranteed to be ground.

$$\frac{\square \mid S}{S}\ (\textsc{Suc}) \qquad \frac{(t,Q) \mid S}{(t,Q)^{c_1} \mid \cdots \mid (t,Q)^{c_a} \mid S}\ (\textsc{Case}) \ \text{ if } \mathit{Slice}_{\mathcal{P}}(t) = (c_1,\ldots,c_a)$$

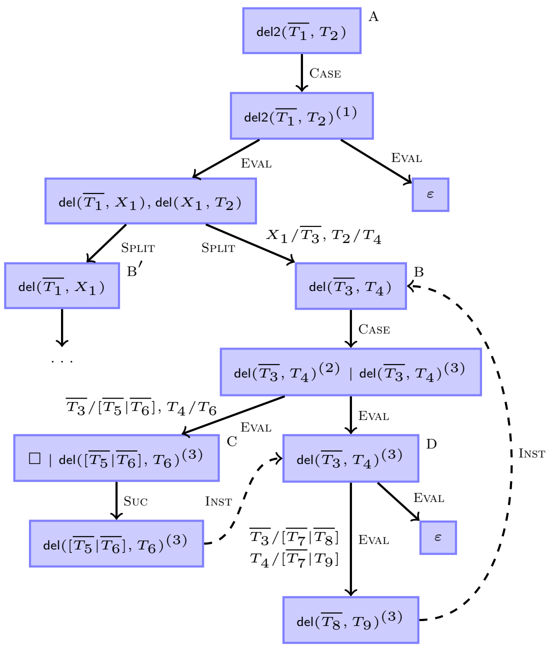
$$\frac{(t,Q)^{h \,:\!-\, B} \mid S}{(B\sigma, Q\sigma) \mid S}\ (\textsc{Eval}) \ \text{ if } \sigma = mgu(t,h) \qquad \frac{(t,Q)^{h \,:\!-\, B} \mid S}{S}\ (\textsc{Backtrack}) \ \text{ if } mgu(t,h) = \mathit{fail}$$

Figure 1 Inference Rules for the Subset of Definite Logic Programs

[8] (when using the asymptotic number of unification attempts as a complexity measure and when attempting to find all solutions for a query). This complexity measure is also used in the previous approach [6] for direct complexity analysis of logic programs. Note that by using appropriate cuts, our approach can also easily be adapted in order to analyze the complexity of finding only the first solution. The goal of our approach is to generate a TRS $\mathcal{R}$ such that $irc_{\mathcal{R}}$ is (asymptotically) an upper bound to the runtime complexity function of the Prolog program.

3 Graph Construction

By adapting the inference rules to *classes* of queries, one obtains derivation trees instead of derivation sequences, because now the rules operate on abstract states which represent sets of concrete states. Thus, one abstract state may represent states where a unification succeeds and also states where the same unification attempt fails. For these abstract states, we use abstract variables representing fixed, but arbitrary terms. Moreover, one can constrain the terms represented by the abstract variables to be only ground terms (depicted by overlining the abstract variable). However, to obtain finite graphs instead of infinite trees, one needs an inference rule which can refer back to already existing states. Such Inst edges can be drawn in our derivation graph if the current state represents a subset of those concrete states that are represented by the earlier already existing state (i.e., the current state is an *instance* of the earlier state). Moreover, we also need a Split inference rule which splits up states in order to find such instances. In our example, the sub-graph below node B′ is analogous to the graph below node B. See [13, 14] for more details, further inference rules, and more explanation on the graph construction.



A
$\mathsf{del2}(\overline{T_1}, T_2)$
Case
$\mathsf{del2}(\overline{T_1}, T_2)^{(1)}$
Eval
Eval
ε
$\mathsf{del}(\overline{T_1}, X_1), \mathsf{del}(X_1, T_2)$
Split
Split
$X_1/\overline{T_3}, T_2/T_4$
B′
$\mathsf{del}(\overline{T_1}, X_1)$
...
B
$\mathsf{del}(\overline{T_3}, T_4)$
Case
$\mathsf{del}(\overline{T_3}, T_4)^{(2)} \mid \mathsf{del}(\overline{T_3}, T_4)^{(3)}$
$\overline{T_3}/[\overline{T_5}|\overline{T_6}], T_4/T_6$
Eval
Eval
C
$\square \mid \mathsf{del}([\overline{T_5}|\overline{T_6}], T_6)^{(3)}$
D
$\mathsf{del}(\overline{T_3}, T_4)^{(3)}$
Inst
Suc
Inst
Eval
$\mathsf{del}([\overline{T_5}|\overline{T_6}], T_6)^{(3)}$
$\overline{T_3}/[\overline{T_7}|\overline{T_8}]$
$T_4/[\overline{T_7}|T_9]$
Eval
ε
$\mathsf{del}(\overline{T_8}, T_9)^{(3)}$

Figure 2 Example Graph

Let us consider an example program deleting two arbitrary elements from a list and the corresponding derivation graph in Figure 2 when calling the program with queries of the form `del2(`t_1`,`t_2`)`, where t_1 is ground.

```
(1) del2(XS, YS)        :- del(XS, ZS), del(ZS, YS).
(2) del([X|XS], XS).
(3) del([X|XS], [X|YS]) :- del(XS, YS).
```

This program has quadratic complexity due to backtracking. However, if one encodes the clauses directly as rules in a TRS, the backtracking is lost and we generate a TRS with linear complexity. To obtain a transformation which over-approximates the complexity of the original program (i.e., where the innermost runtime complexity of the resulting TRS is an upper bound for the complexity of the Prolog program), we encode the *paths* of the graph representation of the program, which explicitly represents the backtracking possibilities.

4 Synthesizing TRSs

To obtain TRSs from the graph, we consider the different successors of SPLIT nodes separately. So in our example, we first generate a TRS from the graph that results from removing the sub-graph with root node B and then we generate a TRS from the graph that (essentially) results from removing the sub-graph with root node B′. Afterwards the runtime complexities of these two TRSs are multiplied in order to obtain an upper bound for the runtime complexity of the original logic program. The reason is that due to backtracking, every solution for the query in the leftmost SPLIT child triggers an evaluation of the query in the rightmost SPLIT child.

To generate TRSs from graphs, we encode each state s by two fresh function symbols f_s^{in} and f_s^{out}. The arguments of f_s^{in} are the (ground) input arguments of s and for f_s^{out}, we use the output arguments of s that are guaranteed to be ground after evaluating the query in s. (To detect those arguments, we use a groundness analysis for the logic program.)

We generate rewrite rules for the paths that start in the initial node of the graph or in a successor state of an INST or SPLIT node and that end in a SUC node or in a successor of an INST or SPLIT node. Moreover, except for the first and last node of the path, the path may not traverse successors of INST or SPLIT nodes. In our example we obtain a TRS $\mathcal{R}_1$ for the graph where we disregard the subgraph with root node B and a TRS $\mathcal{R}_2$ for the graph where we disregard the subgraph with root node B′. So in $\mathcal{R}_2$ we obtain rules for the paths from A to B (Rule (1)), B to C (Rule (2)), B to D directly (Rule (3)), B to D via C (Rule (4)), and D to B (Rule (5)). We always apply all substitutions along the path to the left-hand side of the resulting rewrite rule. See [12] for more details on how to encode clauses as rules in a TRS. Moreover, the rule from A to B has to take into account that the input argument T_3 of B is the result of evaluating B′. Thus, when computing the complexity of $\mathcal{R}_2$, we can also use the rules of $\mathcal{R}_1$ for *evaluating* terms, but these evaluations do not contribute to the complexity of $\mathcal{R}_2$. Such notions of complexity have already been used in existing frameworks for complexity analysis of TRSs [7, 10, 18]. For space reasons, here we only present $\mathcal{R}_2$. The TRS $\mathcal{R}_1$ is analogous to $\mathcal{R}_2$ except for $\mathcal{R}_1$'s first rule which is $f_{\mathrm{A}}^{in}(T_1) \to f_{\mathrm{B'}}^{in}(T_1)$.

$$\begin{array}{rcrcrcll}
f_{\mathrm{A}}^{in}(T_1) & \to & f_{\mathrm{B}}^{in}(T_3) & \mid & f_{\mathrm{A}}^{in}(T_1) & \to^* & f_{\mathrm{B'}}^{out}(T_3) & (1)\\
f_{\mathrm{B}}^{in}([T_5 \mid T_6]) & \to & f_{\mathrm{B}}^{out}(T_6) & & & & & (2)\\
f_{\mathrm{B}}^{in}(T_3) & \to & f_{\mathrm{B}}^{out}(T_4) & \mid & f_{\mathrm{D}}^{in}(T_3) & \to^* & f_{\mathrm{D}}^{out}(T_4) & (3)\\
f_{\mathrm{B}}^{in}([T_5 \mid T_6]) & \to & f_{\mathrm{B}}^{out}(T_6) & \mid & f_{\mathrm{D}}^{in}([T_5 \mid T_6]) & \to^* & f_{\mathrm{D}}^{out}(T_6) & (4)\\
f_{\mathrm{D}}^{in}([T_7 \mid T_8]) & \to & f_{\mathrm{D}}^{out}([T_7 \mid T_9]) & \mid & f_{\mathrm{B}}^{in}(T_8) & \to^* & f_{\mathrm{B}}^{out}(T_9) & (5)
\end{array}$$

To analyze these conditional TRSs for complexity, they are first transformed into unconditional TRSs using a standard transformation, cf. e.g., [11]. In our example, the resulting complexities of $\mathcal{R}_1$ and $\mathcal{R}_2$ are both linear. Since their complexities have to be multiplied due to backtracking, this yields a quadratic upper bound for our example logic program.

5 Conclusion

We proposed a new method for automated complexity analysis of Prolog programs based on automated complexity analysis of term rewriting. First experiments with a prototype implementation have shown promising results. The next steps for this approach are formal definitions and proofs of correctness, a complete implementation, and a thorough experimental evaluation.

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A Report on the Certification Problem Format*

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1 Introduction

Tools that perform automated deductions are available in several areas. There are SAT solver, SMT solver, automated theorem provers for first-order logic (FTP), termination analyzer, complexity analyzer, etc. In most areas, the community was able to agree on a *common* input format, like the DIMACS-, SMT-LIBv2-, TPTP-, or TPDB-format. Such a format is beneficial for several reasons. For example, users can easily try several tools on a problem, and it is possible to compare tools by running experiments on large databases of problems.

One problem when using tools for automated deduction is that they are complex pieces of software, which may contain bugs. These bugs may be harmless or they can lead to wrong answers. To this end, certification of the generated answers becomes an important task.

Of course, to certify an answer, the result of an automated deduction tool must not be just a simple yes/no-answer, but it must provide an accompanying sufficiently detailed proof which validates the answer. For satisfiability proofs and nontermination proofs, this is often simple by given the satisfying assignment or a looping derivation; however, it can become more complex for FTP where a model may be infinite, and it may also be hard to represent complex nonterminating derivations in a finite way. In contrast, for proofs of unsatisfiability or termination, often proof are compositional and consist of several basic proof steps.

We discuss some differences of these compositional proofs in order to illustrate the special demands that arise for a format for termination proofs of term rewrite systems (TRSs).

- *Complexity of basic proof steps*: A proof of unsatisfiability for SAT can be performed in various frameworks (natural deduction, resolution, DPLL), which all have very simple inference rules. Also for FTP, the basic proof steps are rather easy (natural deduction, resolution, superposition, basic step in completion procedure). In contrast, basic proof steps in SMT solvers can be complex (apply decision procedures for supported theories) and also for termination proofs of TRSs a single proof step can be complex. For example, for removing rules via matrix interpretations [6] one has to compute matrix multiplications; and for a single application of the dependency graph processor [1], one has to approximate an undecidable problem.
- *Number of basic proof steps in a compositional proof*: In comparison to SAT, SMT, and FTP, the number of proof steps in a termination proof is rather low. Consequently, termination proofs are usually small.
- *Frameworks*: In some frameworks the inference rules are mostly fixed (e.g., natural deduction or resolution). Nevertheless, efficiently finding proofs in these frameworks requires lots of research, e.g., by developing strategies how to apply the inference rules. In the DP framework for proving termination, the set of techniques is not at all fixed. Often, the power of termination tools is increased by the invention of new ways to prove termination, e.g., by inventing new reduction pairs, new transformations, etc.
- *Determinism of basic proof steps* Several proof steps are completely determined, like the rules of natural deduction or a resolution step. But there are also basic proof steps that

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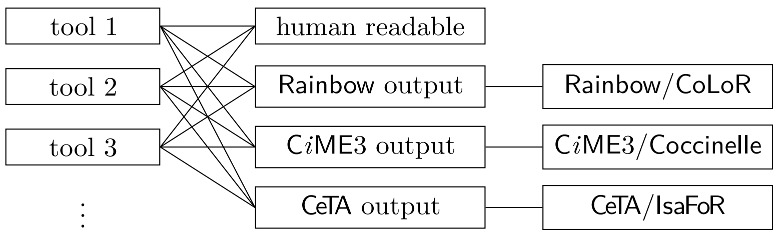

Figure 1 Certification of termination proofs before CPF

need further information to determine the result. For example, from one conflict in DPLL one can learn different conflict clauses; and for an application of the dependency graph processor, the result depends on the used approximation.

To summarize, termination proofs are usually small, but each basic proof step is complex. Moreover, the set of applied termination techniques is constantly growing.

In this report, we shortly present the certification problem format (CPF), a format developed to represent termination proofs. It has four major benefits. First, it is easy for termination tools to generate CPF files; second, it is easy to add new techniques to CPF; third, it provides enough information for certification; finally, it is a *common* proof format that is supported by several tools and certifiers.

All details on CPF and several example proofs are freely available at the following URL.

`http://cl-informatik.uibk.ac.at/software/cpf/`

2 The Certification Problem Format

Before any certifiers for termination proofs have been developed, each termination tool for TRSs provided proofs in a human-readable HTML or plain text file. From these files it was hard to extract the relevant proof steps since parameters of termination techniques are mixed with human readable explanations. Moreover, the output was not standardized at all, but every tool produced proofs in its own output format.

Hence, when the first certifiers for termination proofs have been developed (Rainbow/CoLoR [2], C*i*ME3/Coccinelle [4], and later CeTA/IsaFoR [19]), each of the certifiers demanded a proof written in their own format as input. Hence, to support certifiable proofs using all certifiers, a termination tool had to write several proof routines output, as illustrated in Figure 1.

To reduce the number of required proof outputs for termination tools, the three groups of the certifiers decided to develop *one* proof format that should be supported by all certifiers, namely CPF. Therefore, for generation of certifiable proofs, termination tools now only have to support CPF output. As CPF was also developed with several feedbacks from various termination tools it is widely accepted in the community and it is currently used as the only format during the termination competition for certified categories.

CPF is an XML-format. Choosing XML instead of ASCII was possible as termination proofs are rather small. So, the additional size-overhead of XML documents does not play such a crucial role as it might have played for (large) unsatisfiability proofs for SAT or SMT.

Using XML has several advantages: it is easy to generate, since often programming languages directly offer libraries for XML processing; even before certifiers can check the generated proofs, one can use standard XML programs to check whether a CPF file respects the required XML structure; and finally, it was easy to write a pretty printer to obtain

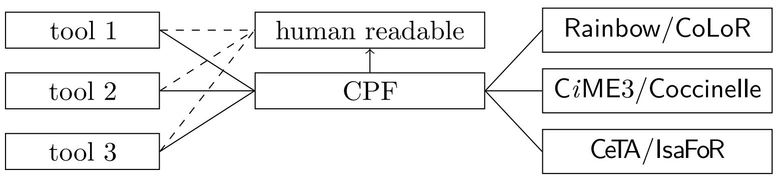

Figure 2 Certification of termination proofs using CPF

human readable proofs from CPF files. This pretty printer is written as XSL transformation (cpfHTML.xsl), so that a browser directly renders CPF proofs in a human readable way. Since this pretty printer is freely available, in principle it is no longer required for termination tool authors to write their own human readable proof output: an export to CPF completely suffices. A problem might occur if the tool uses some techniques that are not yet covered by CPF, but then it is still easily possible to extend or modify the existing pretty printer.

Note that CPF also allows to represent partial proofs: The fact, that CPF does not support all known (and in the future) developed termination techniques is reflected by allowing assumptions and by allowing intermediate results as input.

So, after the invention of CPF, the workflow and required proof export routines for certification have changed from Figure 1 to Figure 2.

3 Design decisions

In order to gain a wide acceptance for both certifiers and termination tools, representative members of the whole community have been integrated in the design process of CPF.

One major decision was that CPF should provide enough information for all three certifiers. Currently, there are some elements in CPF that are completely ignored by some certifier, which in turn are essential for another certifier.

In order to keep the burden on termination tools low, after the required amount of information has been identified, usually no further informations are required in CPF. One exception is that the proofs must be sufficiently detailed to guarantee determinism.

▶ **Example 1.** One standard technique to prove termination of a TRS $\mathcal{R}$ is to remove rules by using reduction orders [13, 10]. If the reduction order $\succ$ is provided, then usually the result is clear: it is the the remaining TRS $\mathcal{R} \setminus \succ$. So in principle, in CPF it should be sufficient to provide $\succ$. However, since there are several variants of reduction orders and since some reduction orders like polynomial orders are undecidable, it is unclear how $\succ$ is exactly defined or how it is approximated. To be more concrete, if a polynomial interpretation over the naturals is provided such that the left-hand side ℓ evaluates to $p_\ell = x^2 + 1$ and the right-hand side r to $p_r = x$, then some approximations can only detect $\ell \succsim r$ whereas a finer analysis delivers $\ell \succ r$. To avoid such problems in CPF, for rule removal it is required that the remaining system $\mathcal{R} \setminus \succ$ is also explicitly stated.

An alternative way to achieve determinism is to explicitly demand that in the proof the exact variant or approximation of the reduction pair is provided, so that the certifier can recompute the identical result. However, this alternative has the disadvantage that every variant or approximation has to be exactly specified and even worse, a certifier has to provide algorithms to compute all variants of reduction pairs that are used in termination tools. In contrast, with the current solution the certifiers can just implement one (powerful) variant / approximation of a reduction pair. Then during certification it must just be ensured that all removed rules are indeed strictly decreasing (and the remaining TRS is weakly decreasing).

Note that determinism of each proof step is also important for an early detection of errors. Otherwise, it might happen that a difference in the internal proof state of the termination tool and the state in the certifier remains undetected for several proof steps. And then errors are reported in proof steps which are perfectly okay.

▶ **Example 2.** Let $\mathcal{R} = \{\mathsf{f}(\mathsf{s}(x)) \to \mathsf{f}(\mathsf{t}(x)), \mathsf{g}(x) \to \mathsf{g}(\mathsf{s}(x))\}$. Consider a wrong proof where first the polynomial order with $\mathcal{P}ol(h(x)) = x$ for all $h \in \{\mathsf{f}, \mathsf{g}, \mathsf{s}, \mathsf{t}\}$ is used to remove the rule $\mathsf{g}(x) \to \mathsf{g}(\mathsf{s}(x))$; second, the only dependency pair $\mathsf{f}^\sharp(\mathsf{s}(x)) \to \mathsf{f}^\sharp(\mathsf{t}(x))$ is generated; finally, termination is proven since the dependency graph contains no edges.

If during the certification one just applies the same techniques without checking the intermediate results, then one first applies rule removal without removing any rule; second, one computes the dependency pairs including $\mathsf{g}^\sharp(x) \to \mathsf{g}^\sharp(\mathsf{s}(x))$; and finally, the error is reported that the dependency graph is not empty. Hence, the error in the first step is not detected, but in the final step—although the final step in the termination tool is sound.

Minor design decisions had to be made for all supported techniques,[1] e.g., the exact names and the exact representation of the relevant parameters, etc. For these decisions, usually the person who wanted to add a new technique to CPF was asked to provide a proposal. This proposal was then integrated into a development version of CPF and put under discussion on the CPF mailing list. Comments during the discussion were integrated in the proposal, and after the discussion has stopped, the modified proposal was then integrated into the official CPF version.

4 Problems and Future of CPF

Very recently, other classes than termination proofs were added to CPF, namely confluence proofs, completion proofs (is a TRS convergent and equivalent to an equational theory?), and equational proofs. Especially for completion proofs, the size of CPF files has grown tremendously. In experiments, example proofs of over 400 megabytes have been generated.[2] For these large proofs, both the completion tool and the certifier spend most of their time for proof export or parsing of XML documents.

Hence, action is required to counter the size-overhead of XML documents. Possibilities would include indexing of terms and rules. Moreover, for rule removal techniques, one might change CPF in such a way that the removed rules have to be provided instead of the remaining rules.[3] The latter change would also allow to represent the rule removal techniques for termination and relative termination in the same way, which in turn would allow to merge the proof techniques for termination and relative termination.

[1] Currently CPF supports several classes of reduction pairs (in alphabetical order): argument filters [1], matrix orders [6], polynomial orders over several carriers [13, 12, 15], recursive path orders [5], and SCNP reduction pairs [3]. Moreover, the techniques of dependency graph [1], dependency pairs [1, 8], dependency pair transformations [1, 8], loops, matchbounds [7], root labeling [16], rule removal [13, 10], semantic labeling [21], size-change termination [14, 18], string reversal, subterm criterion [10], switching to innermost termination [9], uncurrying [11, 17], and usable rules[1, 20, 8] are supported.

[2] We mention a completion proof for an example with 4 equations and 11 rules in the completed TRS. In this proof, only to show that all rules in the TRS can be derived from the equations, $\approx 90,000$ reductions have been performed, and the accumulated terms in these derivations consists of over 5 million function symbols and variables. Since symbols are strings and since there is the XML-overhead, in total, one obtains a 406 MB file (converting all symbols to integers still results in a 266 MB file). CeTA spend only 1 % of its time for checking the proof, and 99 % for parsing.

[3] If the remaining system has to specified, several steps of removing a single rule require quadratic size, whereas if one specifies the removed rules, then the size of the overall proof is linear.

However, these changes would be non-conservative changes which requires adaptations of the proof generating tools and the certifiers. Therefore, we believe that it should be discussed thoroughly by the community whether such changes should be made. Everyone is invited to contribute in the discussion.

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Automating Ordinal Interpretations

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Abstract

In this note we study weakly monotone interpretations for direct proofs of termination which is sound if the interpretation functions are "simple". This is e.g. the case for standard addition and multiplication of ordinal numbers. We compare the power of such interpretations to polynomial interpretations over the natural numbers and report on preliminary experimental results.

1 Introduction

Polynomial interpretations [9] are a well-established termination technique. By now powerful techniques are known for their automation [1,5]. Recently it has been shown that allowing different domains, (e.g., $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$) results in incomparable termination criteria [11,14]. Matrix interpretations consider linear interpretations over vectors or matrices of numbers (in $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$) and have been shown to be powerful in theory and practice [2,6,4,18]. However, for other extensions (e.g., elementary functions [10,12] or interpretations into ordinal numbers [16]) practical implementations remain an open problem.

In this note we revisit polynomial interpretations using ordinals as carrier [16]. Based on recent results [17], we present an implementation for string rewrite systems (with interpretation functions of a special shape), which is the first one to our knowledge. Our efforts could be seen as a first step towards automatically proving the battle of Hercules and Hydra [16] terminating. However—for the encoding of the battle from [3]—Moser [13, Section 7] anticipates that an extension of polynomial interpretations into ordinal domains is not sufficient. In the remainder of this introductory section we recall preliminaries.

Ordinals: We assume basic knowledge of ordinals [8]. By O we denote the set of ordinal numbers strictly less than ϵ_0. Every ordinal $\alpha \in \mathsf{O}$ has a unique representation in Cantor Normal Form (CNF): $\alpha = \sum_{1 \leqslant i \leqslant n} \omega^{\alpha_i} \cdot a_i$, where $a_1, \ldots, a_n \in \mathbb{N} \setminus \{0\}$ and $\alpha_1, \ldots, \alpha_n \in \mathsf{O}$ are also in CNF, with $\alpha > \alpha_1 > \cdots > \alpha_n$. We denote standard addition and multiplication on O (and hence also on $\mathbb{N}$) by $+$ and $\cdot$. We furthermore drop $\cdot$ whenever convenient.

Term Rewriting: We assume familiarity with term rewriting and termination [15]. Let $>$ be a relation and $\geqslant$ its reflexive closure. A function f is monotone if $a > b$ implies $f(\ldots, a, \ldots) > f(\ldots, b, \ldots)$ and weakly monotone if $a > b$ implies $f(\ldots, a, \ldots) \geqslant f(\ldots, b, \ldots)$. A function f is simple if $f(\ldots, a, \ldots) \geqslant a$.

An $\mathcal{F}$-*algebra* $\mathcal{A}$ consists of a carrier set A and an interpretation function $f_{\mathcal{A}} \colon A^k \to A$ for each k-ary function symbol $f \in \mathcal{F}$. By $[\alpha]_{\mathcal{A}}(\cdot)$ we denote the usual evaluation function of $\mathcal{A}$ according to an assignment α which maps variables to values in A. An $\mathcal{F}$-algebra together with a well-founded order $>$ on A is called a (well-founded) $\mathcal{F}$-algebra $(\mathcal{A}, >)$. Often we denote $(\mathcal{A}, >)$ by $\mathcal{A}$ if $>$ is clear from the context. The order $>$ induces a well-founded order on terms: $s >_{\mathcal{A}} t$ if and only if $[\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t)$ for all assignments α. A TRS $\mathcal{R}$ and an algebra $\mathcal{A}$ are *compatible* if $\ell >_{\mathcal{A}} r$ for all $\ell \to r \in \mathcal{R}$. A well-founded algebra $(\mathcal{A}, >)$ is a *monotone (weakly monotone / simple) algebra* if for every function symbol $f \in \mathcal{F}$ the interpretation function $f_{\mathcal{A}}$ is monotone (weakly monotone / simple) in all arguments. By $\mathcal{O}$ ($\mathcal{N}$) we denote well-founded algebras with the carrier O ($\mathbb{N}$) and the standard order $>$.

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For (direct) termination proofs one typically exploits the following theorem.

▶ **Theorem 1.1.** *A TRS is terminating if and only if it is compatible with a well-founded monotone algebra.* ◀

However, monotonicity can be replaced by weak monotonicity, provided the interpretation functions are simple. This result is less known.

▶ **Theorem 1.2** ([19, Proposition 12])**.** *A TRS is terminating if it is compatible with a well-founded weakly monotone simple algebra.* ◀

2 Ordinal Interpretations

Although standard addition, multiplication and exponentiation on ordinals are in general only weakly monotone, Theorem 1.2 nevertheless constitutes a way to use interpretations into the ordinals with these operations in termination proofs.

The next example shows that (fairly small) ordinals add power to linear interpretations.

▶ **Example 2.1.** Consider the SRS $\mathcal{R}$ consisting of the rule $\mathsf{a}(\mathsf{b}(x)) \to \mathsf{b}(\mathsf{a}(\mathsf{a}(x)))$. The linear ordinal interpretation

$$\mathsf{a}_{\mathcal{O}}(x) = x + 1 \qquad\qquad \mathsf{b}_{\mathcal{O}}(x) = x + \omega$$

is simple and proves termination of $\mathcal{R}$ since $x + \omega + 1 >_{\mathcal{O}} x + 1 + 1 + \omega = x + \omega$. Linear interpretations with coefficients in $\mathbb{N}$ are not sufficient. Assuming abstract interpretations $\mathsf{a}_{\mathcal{N}}(x) = a_1 x + a_0$ and $\mathsf{b}_{\mathcal{N}}(x) = b_1 x + b_0$, we obtain the constraints

$$a_1 b_1 \geqslant b_1 a_1 a_1 \qquad\qquad a_1 b_0 + a_0 > b_1 a_1 a_0 + b_1 a_0 + b_0$$

Since $\mathsf{a}_{\mathcal{N}}$ and $\mathsf{b}_{\mathcal{N}}$ must be simple (or monotone) $a_1, b_1 \geqslant 1$. From the first constraint we conclude $a_1 = 1$, which makes the second one unsatisfiable.

In the rest of this note we consider ordinal interpretations (for SRSs) of the following shape

$$f_{\mathcal{O}}(x) = x \cdot f' + \omega^d \cdot f_d + \cdots + \omega^1 \cdot f_1 + f_0 \tag{1}$$

where $f', f_d, \ldots, f_0 \in \mathbb{N}$. Interpretations of the shape (1) will be called *linear ordinal interpretations (of degree d)*. Note that interpretations of the shape (1) are weakly monotone and simple if $f' \geqslant 1$. To show the power of linear ordinal interpretations (with respect to the derivational complexity) we define the parametrized SRS $\mathcal{R}_m$.

▶ **Definition 2.2.** For any $m \in \mathbb{N}$ the SRS $\mathcal{R}_m$ consists of the rules

$$\mathsf{a}_i(\mathsf{a}_{i+1}(x)) \to \mathsf{a}_{i+1}(\mathsf{a}_i(\mathsf{a}_i(x))) \qquad\qquad \mathsf{a}_{i+1}(x) \to x$$

for each $0 \leqslant i < m$.

Note that $\mathcal{R}_0$ is empty. We have the following properties.

▶ **Lemma 2.3.** *For any $\mathcal{R}_m$ and $i \leqslant m$ we have $\mathsf{a}_i(\mathsf{a}_{i+1}^n(x)) \to^{2^n-1} \mathsf{a}_{i+1}^n(\mathsf{a}_i^{2^n}(x))$.*

Proof. By induction on n. In the base case $n = 0$ and $\mathsf{a}_i(x) \to^0 \mathsf{a}_i(x)$. In the step case

$$\begin{aligned}\mathsf{a}_i(\mathsf{a}_{i+1}^{n+1}(x)) &\to \mathsf{a}_{i+1}(\mathsf{a}_i(\mathsf{a}_i(\mathsf{a}_{i+1}^n(x)))) \to^{2^n-1} \mathsf{a}_{i+1}(\mathsf{a}_i(\mathsf{a}_{i+1}^n(\mathsf{a}_i^{2^n}(x))))\\ &\to^{2^n-1} \mathsf{a}_{i+1}(\mathsf{a}_{i+1}^n(\mathsf{a}_i^{2^n}(\mathsf{a}_i^{2^n}(x)))) = \mathsf{a}_{i+1}^{n+1}(\mathsf{a}_i^{2^{n+1}}(x))\end{aligned}$$

◀

shape	yes	time (avg.)	timeout (60s)
linear interpretations	19	0.8	1
linear ordinal interpretations (degree 1)	40	2.5	1
linear ordinal interpretations (degree 2)	40	3.8	6
linear ordinal interpretations (degree 3)	38	2.1	21
$\sum$	40	–	–

Table 1 Evaluation on 720 SRSs of TPDB 7.0.2

▶ **Lemma 2.4.** *We have* $\mathsf{a}_0(\mathsf{a}_1(\cdots(\mathsf{a}_{m-1}(\mathsf{a}_m^n(x))))) \to_{\mathcal{R}_m}^* \mathsf{a}_0^{2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}}(x)$ *where the tower of* 2*'s has height* m.

Proof. By induction on m. In the base case $m = 0$ and the claim trivially holds. In the step case we have

$$\mathsf{a}_0(\cdots(\mathsf{a}_m(\mathsf{a}_{m+1}^n(x)))) \to^{2^n-1} \mathsf{a}_0(\cdots(\mathsf{a}_{m+1}^n(\mathsf{a}_m^{2^n}(x)))) \to^n \mathsf{a}_0(\cdots(\mathsf{a}_m^{2^n}(x))) \to^* \mathsf{a}_0^{2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}}(x)$$

where Lemma 2.3 is applied in the first step and the induction hypothesis in the last step. ◀

As a consequence of Lemma 2.4 we get that $\mathsf{dc}_{\mathcal{R}_m}(n) = \Omega(2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}})$ where the tower of 2's has height m.

▶ **Lemma 2.5.** *For every* $\mathcal{R}_m$ *with* $m \in \mathbb{N}$ *there exists a compatible linear ordinal interpretation of degree* m *but not of degree* $m - 1$.

Proof. To show the first item we take $(\mathsf{a}_i)_{\mathcal{O}}(x) = x + \omega^i$. Then $\mathsf{a}_i(\mathsf{a}_{i+1}(x)) >_{\mathcal{O}} \mathsf{a}_{i+1}(\mathsf{a}_i(\mathsf{a}_i(x)))$ because of $x + \omega^{i+1} + \omega^i > x + \omega^i + \omega^i + \omega^{i+1} = x + \omega^{i+1}$ and $\mathsf{a}_{i+1}(x) >_{\mathcal{O}} x$ because of $x + \omega^{i+1} > x$ for all $x \in \mathsf{O}$. The second item follows from the claim that for any linear ordinal interpretation compatible with $\mathcal{R}_m$ we have that at least ω^i occurs in $(\mathsf{a}_i)_{\mathcal{O}}(x)$. The claim is proved by induction on m. ◀

From Lemma 2.5 we infer that allowing larger degrees increases the power of linear ordinal interpretations and in connection with Lemma 2.4 it shows that linear ordinal interpretations can prove SRSs terminating whose derivational complexity is multiple exponential.

3 Implementation and Evaluation

We implemented linear ordinal interpretations for SRSs of the shape (1). As illustration, we abstractly encode the rule $\mathsf{a}(\mathsf{b}(x)) \to \mathsf{b}(\mathsf{a}(\mathsf{a}(x)))$ with $d = 1$. For the left-hand side we get

$$x \cdot b' \cdot a' + \omega^1 \cdot b_1 \cdot a' + b_0 \cdot a' + \omega^1 \cdot a_1 + a_0$$

which can be written in the canonical form

$$x \cdot b' \cdot a' + \omega^1 \cdot (b_1 \cdot a' + a_1) + (a_1 > 0\,?\,0 : b_0 \cdot a') + a_0$$

where the $(\cdot\,?\,\cdot : \cdot)$ operator implements if-then-else, i.e., if a_1 is greater than zero then the summand $b_0 \cdot a'$ vanishes. To determine whether

$$x \cdot l' + \omega^1 \cdot l_1 + l_0 \geqslant x \cdot r' + \omega^1 \cdot r_1 + r_0$$

for all values of x, we use the criterion $l' \geqslant r' \wedge (l_1 > r_1 \vee (l_1 = r_1 \wedge l_0 \geqslant r_0))$. Finally, $f' \geqslant 1$ ensures that the interpretation $f_{\mathcal{O}}$ is simple while the interpretation functions are then weakly monotone for free. Hence the search for suitable coefficients can be encoded in non-linear integer arithmetic.

The results[1] are given in Table 1 where 4 bits are used to represent the coefficients $f_0, \ldots, f_n, f'$ and 8 bits are allowed for intermediate calculations. The column labeled "yes" indicates how many systems the given method could show terminating. Times are given in seconds.

4 Conclusion

We conclude this note with a short discussion on the relationship of linear ordinal interpretations with matrix interpretations [4]. In contrast to the latter the induced ordering is still total which makes it valuable for ordered completion. Secondly as Lemmata 2.4 and 2.5 show interpretations of the shape (1) allow to prove termination of SRSs whose derivational complexity is beyond exponential while matrix interpretations are restricted to an exponential upper bound.

Concerning future work we want to investigate if and how Theorem 1.2 could make automated termination and complexity tools more powerful.

For matrix interpretations over $\mathbb{N}$ (as defined in [4]) the answer is that Theorem 1.2 does not increase the power of the method. The reason is that the condition for a function to be simple ($M_{ii} \geqslant 1$ for all $1 \leqslant i \leqslant d$ where d is the dimension of the matrices) is a stronger requirement than monotonicity demanding $M_{11} \geqslant 1$.

However, if one considers matrix interpretations over O then additional termination proofs can be obtained (note that any linear ordinal interpretation corresponds to a matrix interpretation over O).

Another natural question is whether Theorem 1.2 helps arctic interpretations. Because of monotonicity requirements, direct proofs with arctic matrices are currently limited to dummy systems (SRSs augmented with constants).

Finally we recall that Theorem 1.2 allows direct proofs with polynomial interpretations augmented with "max". This has already been observed in [19, example on p. 13] but seems to have been forgotten. A similar statement holds for quasi-periodic functions [20].

As future work we want to consider linear ordinal interpretations for TRSs. The problem for TRSs is that for comparisons of polynomials the absolute positiveness approach [7] might not apply. To see this note that $f_1 \geqslant g_1$ and $f_2 \geqslant g_2$ does not imply $x \cdot f_1 + y \cdot f_2 \geqslant y \cdot g_2 + x \cdot g_1$ for all values of x and y if $f_1, f_2, g_1, g_2 \in \mathbb{N}$ and $x, y \in \mathsf{O}$. To also cope with such cases we propose a combination of *standard* and *natural* operations on ordinals, as illustrated in the following example, where $\oplus$ denotes natural addition on O.

▶ **Example 4.1** (Adapted from [17, Example 17]). Consider the TRS $\mathcal{R}$ consisting of the single rule $\mathsf{s}(\mathsf{f}(x, y)) \to \mathsf{f}(\mathsf{s}(y), \mathsf{s}(\mathsf{s}(x)))$. The weakly monotone interpretation $\mathsf{f}_{\mathcal{O}}(x, y) = (x \oplus y) + \omega$ and $\mathsf{s}_{\mathcal{O}}(x) = x + 1$ is simple and induces a strict decrease between left- and right-hand side:

$$(x \oplus y) + \omega + 1 >_{\mathcal{O}} ((y+1) \oplus (x+2)) + \omega = (x \oplus y) + 3 + \omega = (x \oplus y) + \omega$$

[1] Details are available from `http://colo6-c703.uibk.ac.at/ttt2/tpoly/`.

Hence $\mathcal{R}$ can be oriented by a linear ordinal interpretation. Again, linear interpretations with coefficients in $\mathbb{N}$ are not sufficient. Assuming abstract interpretations $\mathsf{f}_{\mathcal{N}}(x, y) = f_1 x + f_2 y + f_0$ and $\mathsf{s}_{\mathcal{N}}(x) = s_1 x + s_0$, we get the constraints

$$s_1 f_1 \geqslant f_2 s_1 s_1 \qquad s_1 f_2 \geqslant f_1 s_1 \qquad s_1 f_0 + s_0 > f_1 s_0 + f_2(s_0 + s_1 s_0) + f_0$$

Since $\mathsf{s}_{\mathcal{N}}$ and $\mathsf{f}_{\mathcal{N}}$ must be simple (or monotone) $s_1, f_1, f_2 \geqslant 1$. From the first two constraints we conclude $s_1 = 1$, such that the third simplifies to $f_0 + s_0 > f_0 + (f_1 + 2f_2)s_0$. This contradicts f_1 and f_2 being positive.

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