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Consumption and Wage Humps in a Life-Cycle Model with Education

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Non-Technical Summary

It has been documented that consumption expenditures of individuals are hump shaped over the life cycle with a peak around the age of 45-50 years. These findings contrast the classical, frictionless life-cycle consumption-savings models that predict either a monotonically increasing, monotonically decreasing, or flat consumption profile depending on parameter values. We present and explicitly solve a parsimonious and transparent model in which the optimal consumption of the individual can exhibit a hump-shaped life-cycle pattern. The key model feature generating the consumption hump is endogenous educational decisions. The agent decides how much time to spend on education. He benefits from education since it has a lasting positive effect on the wages he earns. On the other hand, by spending time on education the agent forgoes leisure, which leads to disutility. In addition, education might require a monetary investment by the agent. The endogenously determined price of leisure relative to consumption varies over life. Enjoying an extra hour of leisure means spending one hour less on education, which reduces wages in all future. The present value of the foregone wages is larger for young than for old agents. Therefore, the relative price of leisure decreases over life. The consumption bundle of perishable goods and leisure is optimally tilted towards more perishable consumption and less leisure early in life, but less perishable consumption and more leisure later in life. When we embed this mechanism in a setting where the return on savings exceeds the agent's subjective time preference rate, a hump-shaped pattern for perishable consumption emerges.

Our model is set up in continuous-time with an agent maximizing expected life-time Cobb-Douglas utility of the consumption of goods and leisure. At any point in time before retirement the agent has to decide upon his consumption of goods and the fraction of time spent on education. While time spent on education reduces leisure, it also increases wages in all future albeit with the impact being exponentially decaying over time. For parsimony (but not unlike the situation in European labor markets), the agent is assumed to work a constant fraction of time throughout life so his labor income is affected by education in the same way as the wage rate. After retirement, the agent receives a fixed income given by a fraction of labor income just before retirement and decides only on consumption. The individual invests any savings at a constant rate of return. We intentionally abstract from uncertainty to make the key mechanisms of the model as transparent as possible. We derive the optimal consumption rate and educational effort, as well as their dynamics, in closed form. We provide sufficient conditions for the presence of a hump in life-cycle consumption and investigate analytically the sensitivity of the timing of the hump with respect to central parameters of the model.

Our findings are illustrated for a set of realistic parameter values. With the benchmark parametrization and the agent being in the labor force from age 20 to 65 (and then being retired until age 90), the consumption increases at a modest rate until the age of 48 and then drops somewhat until retirement. This pattern is in line with empirical observations. The optimal time spent on education declines smoothly over time and, in turn, net leisure increases over time. In our benchmark parametrization the endogenously determined wage rate also exhibits a hump-shaped pattern, with the hump being located a few years later in life than the peak of consumption. However, our model is flexible in regards to the peak ages, and for alternative parameter constellations the wage peaks earlier than consumption.

There are also cases in which the wage is monotonically increasing over the entire working life (supported by some empirical studies) and still consumption is hump shaped. The observed hump-shaped life-cycle pattern in individuals' consumption cannot be explained by the classical consumption-savings model. We explicitly solve a model with utility of both consumption and leisure and with educational decisions affecting future wages. We show that optimal consumption is hump shaped and determine the peak age. The hump results from consumption and leisure being substitutes and from the implicit price of leisure being decreasing over time; more leisure means less education, which lowers future wages, and the present value of foregone wages decreases with age. Consumption is hump shaped whether the wage is hump shaped or increasing over life.

Consumption and Wage Humps in a Life-Cycle Model with Education

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The paper contains graphs in color, use color printer for best results.

Abstract

The observed hump-shaped life-cycle pattern in individuals' consumption cannot be explained by the classical consumption-savings model. We explicitly solve a model with utility of both consumption and leisure and with educational decisions affecting future wages. We show optimal consumption is hump shaped and determine the peak age. The hump results from consumption and leisure being substitutes and from the implicit price of leisure being decreasing over time; more leisure means less education, which lowers future wages, and the present value of foregone wages decreases with age. Consumption is hump shaped whether the wage is hump shaped or increasing over life.

Keywords: Education, leisure, consumption hump, wage hump

JEL subject codes: D11, D14, D91, I21, J24

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Consumption and Wage Humps in a Life-Cycle Model with Education

1 Introduction

The consumption expenditures of individuals have been documented to be hump shaped (inverted U-shaped) over the life cycle: they increase up to age 45-50 years and then decrease over the remaining life. These findings contrast the classical, frictionless life-cycle consumption-savings models that predict either a monotonically increasing, monotonically decreasing, or flat consumption profile depending on parameter values. This paper presents and explicitly solves a parsimonious and transparent model in which the optimal consumption of the individual can exhibit a hump-shaped life-cycle pattern.

The key model feature generating the consumption hump is endogenous educational decisions. Throughout life the agent decides how much time to spend on education. The agent benefits from education since it has a lasting positive impact on the wages he earns, which can finance higher future consumption. On the other hand, by spending time on education the agent enjoys less leisure, which negatively affects his utility. In addition, education might require a monetary investment by the agent. The endogenously determined price of leisure relative to consumption varies over life. Enjoying an extra hour of leisure means spending one hour less on education, which reduces wages in all future. The present value of the foregone wages is larger for young than for old agents, other things equal, hence the relative price of leisure decreases over life. Therefore, the consumption bundle of perishable goods and leisure is optimally tilted towards more perishable consumption and less leisure early in life, but less perishable consumption and more leisure later in life. When we embed this mechanism in a setting where the return on savings exceeds the agent's subjective time preference rate—so that optimal overall consumption is intrinsically increasing over life—a hump-shaped pattern for perishable consumption emerges.

More specifically, we set up a continuous-time model with an agent maximizing expected life-time Cobb-Douglas utility of the consumption of goods and leisure. At any point in time before retirement the agent has to decide upon his consumption of goods and the fraction of time spent on education. While time spent on education reduces leisure, it also increases wages in all future albeit with the impact being exponentially decaying over time. For parsimony (but not unlike the situation in European labor markets), the agent is assumed to work a constant fraction of time throughout life so his labor income is affected by education in the same way as the wage rate. After retirement, the agent receives a fixed income given by a fraction of labor income just before retirement and decides only on consumption. The individual invests any savings at a constant rate of return. We intentionally abstract from uncertainty to make the key mechanisms of the model as transparent as possible. We derive the optimal consumption rate and educational effort, as well as their dynamics, in closed form. We provide sufficient conditions for the presence of a hump in life-cycle consumption and investigate analytically the sensitivity of the timing of the hump with respect to central parameters of the model.

We illustrate our results for a set of realistic parameter values. With the benchmark parametrization and the agent being in the labor force from age 20 to 65 (and then being retired until age 90), the consumption increases at a modest rate until age 48 and then drops somewhat until retirement, a pattern in line with empirical observations. The optimal time spent on education declines smoothly over time and, consequently, net leisure increases over time. In our benchmark parametrization the endogenously determined wage rate also exhibits a hump-shaped pattern, with the hump being located a few years later in life than the peak of consumption. However, our model is flexible in regards to the peak ages, and for alternative parameter constellations the wage peaks earlier than consumption. There are also cases in which the wage is monotonically increasing over the entire working life (supported by some empirical studies, see below) and still consumption is hump shaped.

The remainder of the paper is organized as follows. Section 2 relates our paper to the existing literature. The model is formulated in Section 3, and Section 4 presents the closed-form solutions for the optimal decisions at any point in time. Section 5 illustrates the solution for benchmark parameter values, derives the life-cycle pattern in consumption, sufficient conditions for the presence of a consumption hump, and—if a hump occurs—the exact age at which consumption peaks. Section 6 discusses and illustrates how our results and, in particular, the peak age of consumption are affected by the values of key parameters in the model. Section 7 briefly reviews some related approaches in the literature to highlight which model specifications are able to produce a hump in life-cycle consumption. Finally, Section 8 concludes.

2 Related literature

Our paper connects two strands of literature on the decisions of economic agents over their life cycle: the literature on the consumption hump and the literature on how educational decisions affect wages and human capital.

The consumption hump literature was initiated by Thurow (1969) who first documented that observed individual consumption over life is hump shaped, a pattern that has since been confirmed using different data sources and periods, cf., e.g., Attanasio and Weber (1995), Attanasio, Banks, Meghir, and Weber (1999), Browning and Crossley (2001), Gourinchas and Parker (2002), and Fernández-Villaverde and Krueger (2007). The consumption hump is inconsistent with the classical, frictionless life-cycle consumption-saving theory of Ramsey (1928), Fisher (1930), Modigliani and Brumberg (1954), and Friedman (1957) (extended to uncertainty by Samuelson (1969) and Merton (1969, 1971)). This theory generates either monotonically increasing, monotonically decreasing, or flat consumption profiles over life depending on whether the individual's subjective time preference rate is smaller than, greater than, or equal to the (risk-adjusted expected) return on investments; see Section 7 for a review of both this model and some of the below-mentioned models.

Various papers have developed models that can generate a consumption hump. Thurow (1969) suggests that the hump results from imperfect credit markets (borrowing constraints) and recommends government intervention in the consumer loan market to allow for optimal adjustment of consumption. Heckman (1974) offers a neoclassical answer to Thurow (1969) by showing that a consumption hump may emerge from a model with perfect credit markets if consumers face an exogenously given age-dependent wage rate, are free to choose their hours of work, and obtain utility from both consumption and leisure. Under the reasonable assumptions that consumption and leisure are substitutes (e.g., Hokayem and Ziliak 2014) and the return on savings exceeds the time preference rate of the agent (as in mainstream asset pricing models), a consumption hump occurs in Heckman's model if the wage profile is hump-shaped, and then the consumption peaks later than the wage rate. For his arguments to work, it is crucial that the agent is free to choose his hours to work, however Carroll and Summers (1991) and Browning and Crossley (2001) conclude that realistic consumption humps can only be explained by unrealistically pronounced labor supply patterns.[1] Furthermore, the assumption of a hump-shaped wage profile is debatable. Some empirical studies do find a hump in wages, although with very different peak ages, cf. Hanoch and Honig (1985), Willis (1986), and Johnson and Neumark (1996). However, other studies cannot detect a decline in hourly wage rates up to retirement, whereas total wage income may still decline due to a reduction in hours worked, cf. Murphy and Welch (1990, 1992) and Casanova (2013). Although our model has fixed labor supply, it can generate a consumption hump whether or not the endogenously determined wage exhibits a hump, and if the wage is also hump shaped, the peak in consumption can occur before or after the peak in wages.

Other extensions of the canonical consumption-savings model that can produce a con-

[1] Bullard and Feigenbaum (2007) revitalize Heckman's explanation with endogenous labor supply by calibrating a general equilibrium overlapping generations model to US data.

sumption hump include uncertain exogenous income and precautionary savings (Nagatani 1972; Carroll 1997; Cocco, Gomes, and Maenhout 2005), variations in household size (Attanasio and Browning 1995; Browning and Ejrnæs 2009), mortality risk (Feigenbaum 2008; Hansen and İmrohoroğlu 2008), or consumer durables serving as collateral (Fernández-Villaverde and Krueger 2011). Since such models involve various frictions, the optimal consumption strategy is not available in closed form so that both the presence and the location of the hump are merely detected in the numerical solutions. In contrast, we formulate a model that offers a closed-form solution for optimal consumption, and therefore we can be explicit about both the conditions under which a consumption hump emerges and its precise location.

The literature on the role of education in the determination of wages and human capital was founded by Becker (1965), Ben-Porath (1967), and Mincer (1974), among others. Most papers in this field are, however, not concerned with the resultant impact on consumption. Ben-Porath (1967) maximizes the present value of future labor income in a model where (i) educational decisions are taken throughout life, (ii) wage increments are a strictly concave function of the time cost of education (hourly wage multiplied by hours spent on education), a feature known as Harrod neutrality, (iii) the wage rate depreciates over time unless educational efforts are sufficiently high, and (iv) labor supply is fixed. The maximization of the present value of future income is equivalent to utility maximization only if leisure does not enter the utility function. Heckman (1976b) reports monotonically increasing consumption in a model where utility depends on consumption and the monetary value of leisure (leisure time multiplied by wage rate), the wage dynamics are of the Ben-Porath (1967) type, and where both education, labor supply, and consumption are choice variables; however his analysis presumes an interior solution whose existence is questionable, cf. our discussion in Section 7. Heckman (1976a) and others report empirical findings inconsistent with the Harrod-neutral specification of wage dynamics. He further presents some theoretical results from a model without Harrod neutrality, but focuses on the special case where utility of consumption is additively separable from utility of leisure time. Also Ghez and Becker (1975) derive mainly qualitative results about consumption and earning humps in a model with endogenous education.

In contrast to the above-mentioned papers, we derive explicit expressions for the optimal consumption and educational decisions, as well as the resulting wage dynamics. This allows us to precisely characterize the life-cycle patterns in consumption and wages and, in particular, the presence and location of humps in these patterns. Our model focuses on how educational decisions affect the dynamics of wages and consumption with a fixed labor supply.[2] Our assumed wage dynamics are similar to the specifications discussed in

[2] This is also done in Ben-Porath (1967) and other classical models in the literature on investing in

Heckman (1976a) and is not Harrod neutral. We assume a Cobb-Douglas utility of consumption and leisure time, which is more in line with standard representations than the preferences considered by Heckman (1976a, 1976b).

Finally, we mention a few other papers addressing the impact of education on life-cycle earnings and decisions. Brown, Fang, and Gomes (2012) study the returns to education and find that individuals should be willing to pay 300 to 500 (200 to 250) thousand US dollars to obtain a college (high school) degree in order to benefit from the 32 to 42 percent (20 to 38 percent) increase in annual certainty-equivalent consumption. Cocco, Gomes, and Maenhout (2005) discuss how the optimal life-cycle consumption and investment decisions depend on the educational level of the agent. They do not endogenize the education decision, but impose a hump-shaped labor income profile and calibrate it to different education levels (no high school, high school, college). Another line of literature discusses the non-pecuniary or consumption value of education, see Schaafsma (1976), Lazear (1977), Heckman, Lochner, and Taber (1999), Carneiro, Hansen, and Heckman (2003), and Alstadsæter (2011), among others.

3 The model

We formulate a parsimonious continuous-time model that focuses on the educational and consumption-savings decisions in the active phase of an individual's life cycle. The individual enters the economy at time 0 and is assumed to be active in the labor market up to time T_0, where he retires and lives on until time $T > T_0$, where both T_0 and T are fixed. In the active phase the individual works a fraction $1 - \ell$ of time and earns wages at a rate of $w(t)(1 - \ell)$ per time period. His remaining time ℓ is split between time spent on education $\varepsilon(t)$ and (net) leisure $\ell - \varepsilon(t)$. For simplicity, we assume that labor supply—and thus ℓ—is fixed exogenously and constant over time.

The individual obtains utility from leisure (see details below) so education has an immediate utility cost. In addition, education has a monetary cost: the individual must pay $K\varepsilon(t)\,dt$ for education over the time span $[t, t + dt]$, where K is a non-negative constant.[3] The individual benefits from education via its effect on wages. We assume the wage is initially $w(0)$ and evolves according to

$$dw(t) = (\alpha\varepsilon(t) - \beta w(t))\ dt = w(t)\left(\frac{\alpha}{w(t)}\varepsilon(t) - \beta\right)\ dt, \tag{1}$$

human capital, cf. the survey by Weiss (1986). The assumption is obviously more appropriate if the agent is located in a country with tight working time regulations as seems realistic especially for some European countries; see, e.g., Causa (2009) who also provides a literature survey over cross-country studies on average hours worked.

[3] As will become clear from the results in the subsequent sections, the model can produce a consumption hump whether K is zero or strictly positive.

where α and β are non-negative constants.[4] The constant α captures the efficiency of education. This specification implies that education has a higher relative effect on wages when current wage is low, so there are decreasing returns to scale on education. The constant β is the relative depreciation rate of wages if the agent does not educate himself. With a positive β, the individual has to continue educating to increase or even maintain the wage at its current level. Hence, education has a lasting but decaying effect on wages. The values of the parameters ℓ, α, and β are characteristics of the individual.

The individual decides on the consumption rate $c(t)$ of the single physical good available in the economy. All wealth not consumed is invested in a risk-free financial asset offering a rate of return r per time period (continuously compounded). The individual enters the economy at time 0 with wealth $X(0)$. During the active phase the wealth $X(t)$ evolves according to

$$dX(t) = rX(t)\,dt + \left(w(t)(1-\ell) - c(t) - K\varepsilon(t)\right)\,dt, \quad t < T_0, \tag{2}$$

so that the wealth increment is the return on savings plus wages earned less consumption and the monetary costs of education.

In the retirement phase $[T_0, T]$, the individual does not educate and receives an income which is a constant fraction $\chi \geq 0$ of the full-time wage just before retirement, in line with the wide-spread final salary retirement plans.[5] The individual still earns interest on savings and decides on consumption. The wealth dynamics in retirement are thus

$$dX(t) = rX(t)\,dt + \left(\chi w(T_0) - c(t)\right)\,dt, \quad T_0 \leq t \leq T. \tag{3}$$

An important motivation for education in the active phase is to raise the final wage $w(T_0)$ and thus retirement income, so it is crucial to include the retirement phase in the model.

We assume a Cobb-Douglas utility function of consumption c and leisure $\ell - \varepsilon$,

$$u(c, \varepsilon) = \frac{1}{1-\gamma}\left(c^{\kappa}(\ell-\varepsilon)^{1-\kappa}\right)^{1-\gamma}, \tag{4}$$

in the active phase. Here $\kappa \in (0,1)$ captures the relative preference weight of consumption and leisure, and $1/\gamma$ is the elasticity of intertemporal substitution (EIS). In retirement,

[4] While a specification of the form (1) is crucial for obtaining a closed-form solution to the utility maximization problem, we expect that similar specifications also lead to an implicit price of leisure that is decreasing over life, which is the main driver of the consumption hump.

[5] The constant χ is related to the so-called replacement rate. To understand the meaning of χ, consider an example in which $1-\ell = 0.4$, i.e., the agent works 40% of his total time endowment. For a given wage, he thus receives $w(1-\ell)$ as labor income. If now $\chi = 0.2$, then his labor income decreases by 50% after retirement.

leisure is fixed so we assume a utility of consumption given by

$$U(c) = \frac{1}{1-\gamma} c^{1-\gamma}.$$

The indirect utility—i.e., the maximized life-time utility—is therefore

$$V(t,x,w) = \sup_{c,\varepsilon} \left\{ \int_t^{T_0} e^{-\rho(s-t)} u\left(c(s), \varepsilon(s)\right) \, ds + \int_{T_0}^{T} e^{-\rho(s-t)} U\left(c(s)\right) \, ds \right\}, \quad t \leq T_0, \tag{5}$$

where we maximize over consumption throughout the entire life and the fraction of time devoted to education up to retirement. The constant ρ is the agent's subjective time preference rate.

Note that, because of the assumed utility function, the agent always chooses strictly positive consumption c and leisure $\ell - \varepsilon$. One might further add the constraint that education ε stays non-negative, but then a closed-form solution is not available. We take an alternative approach here: we solve the problem without imposing the constraint and verify that the solution satisfies the constraint. For our benchmark parametrization optimal education is indeed positive throughout the active phase. Even late in the active phase the benefits from educating outweigh the monetary costs and the reduced leisure since education increases not only wages earned in the remaining active period but also the retirement income (as this is proportional to the final salary). For other combinations of parameters and state variables the constraint can be binding.

Throughout the paper we make the following assumption:

Assumption 1 *The parameters of the model satisfy*

$$\gamma > 1, \tag{6}$$

$$r > \rho \geq 0, \tag{7}$$

$$\frac{K}{\alpha} \leq \frac{\chi}{r}\left(1 - e^{-r(T-T_0)}\right) \leq \frac{1-\ell}{r+\beta}. \tag{8}$$

The condition (6) is standard in the consumption-portfolio choice literature and is backed by empirical studies (see, e.g., Meyer and Meyer 2005). In our two-good setting it implies that goods and leisure are substitutes, which finds strong empirical support, cf. Hokayem and Ziliak (2014). The condition (7) is valid, for example, in risk-free versions of consumption-based asset pricing models where the equilibrium interest rate equals the sum of the time preference rate (of the representative agent) and the product of the consumption growth rate and the reciprocal of the EIS (of the representative agent). By adding uncertainty to such models the risk-free interest rate is somewhat reduced due to precautionary savings. On the other hand the value of r in our model is better interpreted as the overall return on savings, which in general is higher than the risk-free rate due to the risk premium on risky investments. The condition (8) ensures that the utility maximization problem (5) has an interior solution and is satisfied for realistic parametrizations of the model.

4 The solution

The problem is formulated so that it captures the key aspects of education and consumption over the life cycle and still allows a closed-form solution. The following theorem presents the indirect utility and optimal decision rules before retirement which is our main focus (all proofs are in Appendix A).

Theorem 1 *The individual's indirect utility before retirement is*

$$V(t,x,w) = \frac{1}{1-\gamma} f(t)^{\gamma} \left(x + wg(t) + h(t)\right)^{1-\gamma}, \quad t \le T_0, \tag{9}$$

where the time-dependent functions f, g, and h are given by

$$\begin{aligned} f(t) &= \int_t^{T_0} e^{-\tilde{r}(s-t)} \delta(s)\, ds + \frac{1}{\tilde{r}} e^{-\tilde{r}(T_0-t)} \left(1 - e^{-\tilde{r}(T-T_0)}\right), \\ g(t) &= \frac{1-\ell}{r+\beta} \left(1 - e^{-(r+\beta)(T_0-t)}\right) + \frac{\chi}{r} e^{-(r+\beta)(T_0-t)} \left(1 - e^{-r(T-T_0)}\right), \\ h(t) &= \ell \int_t^{T_0} e^{-r(s-t)} \left(\alpha g(s) - K\right) ds, \end{aligned}$$

with

$$\tilde{r} = \frac{\rho}{\gamma} + \frac{\gamma-1}{\gamma} r = r - \frac{r-\rho}{\gamma}, \tag{10}$$

$$\delta(t) = \tilde{\kappa} \left(\alpha g(t) - K\right)^{\frac{(1-\kappa)(\gamma-1)}{\gamma}}, \tag{11}$$

$$\tilde{\kappa} = \kappa^{-\frac{\kappa(\gamma-1)}{\gamma}} (1-\kappa)^{-\frac{(1-\kappa)(\gamma-1)}{\gamma}}. \tag{12}$$

The optimal consumption and education policies are

$$c(t) = \kappa \delta(t) \frac{X(t) + g(t)w(t) + h(t)}{f(t)}, \tag{13}$$

$$\varepsilon(t) = \ell - (1-\kappa)\delta(t) \frac{X(t) + g(t)w(t) + h(t)}{f(t)\left(\alpha g(t) - K\right)}. \tag{14}$$

Furthermore, the function $g(t)$ is decreasing.

The right-most inequality in (8), which can be rewritten as $g(T_0) \leq (1-\ell)/(r+\beta)$, ensures that g is decreasing. The left-most inequality in (8) is equivalent to $\alpha g(T_0) > K$ and, since g is decreasing, this ensures $\alpha g(t) > K$ for all t, so that the relevant expressions in the theorem are well-defined. Intuitively, the left-most inequality means that the retirement phase is long enough for the benefits of education to extend into retirement.

The optimal choices of consumption and leisure are related via

$$c(t) = \frac{\kappa}{1-\kappa} \left(\alpha g(t) - K \right) \left(\ell - \varepsilon(t) \right), \tag{15}$$

which naturally depends on the parameter κ that captures the relative preference weights of consumption and leisure, but also depends on the term $\alpha g(t) - K$. In our setting an extra time unit of leisure means less education, which saves some monetary costs of education, but also leads to a lower life-long wage rate. The term $\alpha g(t) - K$ measures the resulting drop in human capital net of monetary costs of education, and therefore this term represents the implicit price of an extra unit of leisure. Since g is a decreasing function, the relative price of leisure declines over time. Consequently, the ratio of physical consumption to leisure also declines over time. We explore the life-cycle pattern in consumption in more detail in the next section.

We can interpret $wg(t)+h(t)$ as the individual's human capital net of costs of education. If the agent at time t decides to educate at the maximum rate in the remaining active phase, i.e., $\varepsilon(s) = \ell$ for $s \in [t, T_0]$, then (1) implies that the future wages he receives are

$$w(s) = w(t)e^{-\beta(s-t)} + \frac{\alpha\ell}{\beta} \left(1 - e^{-\beta(s-t)} \right),$$

and then the present value of all future income net of monetary educational costs is

$$\int_t^{T_0} e^{-r(s-t)} \left(w(s)[1-\ell] - K\ell \right) \, ds + \chi \int_{T_0}^{T} e^{-r(s-t)} w(T_0) \, ds = w(t)g(t) + h(t).$$

The agent's total wealth is the sum of the financial wealth $X(t)$ and the human capital $w(t)g(t) + h(t)$. Intuitively, the agent first calculates the value of the maximum possible future net wages, and then decides how much to spend on consumption goods and leisure. It is now clear that the form of the indirect utility is similar to well-known solutions of related life-cycle problems, including the models reviewed in Section 7 below. As in the related problems the function in front of the total wealth term in the indirect utility (f in our notation) depends on the relative price of the two goods, $\alpha g(t) - K$, as seen in the specification of the auxiliary function δ. Furthermore, the optimal consumption of the physical good and of leisure is a fraction of total wealth, where the fraction depends on age and the relative price of the two goods. The optimal total expenditures on consumption and leisure, $c(t)+(\alpha g(t)-K)(\ell-\varepsilon(t))$, constitute a fraction $\delta(t)/f(t)$ of total wealth, and the total consumption expenditures are then split according to the preference weights κ and $1-\kappa$.

5 Properties of the solution

This section investigates the properties of the solutions in order to illustrate and enhance our understanding of the economic forces at play.

5.1 Parameter values

We use the parameter values listed in Table 1 as the benchmark in our discussions. In Section 6 we discuss comparative statics.

[Table 1 about here.]

We have in mind an individual who is initially 20 years old, works until age 65, and then lives on for another 25 years. The values of the time preference rate and elasticity of intertemporal substitution are standard in the literature. The value of κ means that the individual splits the costs of consumption and leisure (foregone education and thus higher future wages) on a 60:40 basis. With $\ell = 0.6$, the individual works 40% of time and can spend 60% of time on either leisure or education.

The initial wage rate of $w = 4$ represents the annual labor income a worker would receive if he worked 100% of time, which with $1 - \ell = 0.4$ implies that the individual initially earns 1.6 monetary units per year. If, for concreteness, we think of a monetary unit as being 10,000 US dollars (USD), this constitutes an annual income of 16,000 USD. The wage depreciates by 5% per year if the individual does not educate. To understand the value of the parameter α, we first note that, using the benchmark parameters, the optimal decision is initially to spend around 33% of time on education, i.e. $\varepsilon_0 \approx 0.33$. With $\alpha = 1.5$ and $w = 4$, this implies that wages grow at a rate of around 12.4% per year less the 5% depreciation rate. The cost parameter $K = 1$ implies that the initially optimal educational effort costs 0.33 (corresponding to 3,300 USD) per year. The replacement rate of $\chi = 0.15$ means that, at retirement, the individual's income drops from 40% (equal to $1-\ell$) to 15% of the hypothetical maximal annual income, but then stays at this level for the rest of his life. The initial wealth is set to 3 (corresponding to 30,000 USD).

Finally, we set the return r on savings to 4%. In our parsimonious deterministic setting, r should be interpreted as the return on a well-diversified portfolio. Hence a savings return of 4% appears realistic.

5.2 Life-cycle patterns

[Figure 1 about here.]

Figure 1 illustrates the life-cycle patterns generated by the solution presented in Theorem 1. The fraction of time spent on education (green curve; left axis) is initially around 0.33 and then gradually declines over time. The earlier in life the individual educates, the longer he can enjoy the positive effects of education on future wages. The flip-side of time spent on education is net leisure, which therefore increases over life (red curve; left axis). As the benefits of educating decrease over life, the implicit costs of leisure decline, which naturally leads to increased net leisure. Even late in the active phase of life, additional education generally pays off since this will have a positive effect on the wage received in the remaining part of the active phase as well as the retirement income. The following theorem demonstrates that these observations are generally valid.

Theorem 2 *(a) The net leisure-time dynamics are given by*

$$d(\ell - \varepsilon)(t) = (\ell - \varepsilon)(t) \left(\frac{r - \rho}{\gamma} + \frac{1 + \kappa(\gamma - 1)}{\gamma} \alpha B(t) \right) dt,$$

where

$$B(t) = \frac{1 - \ell - (r + \beta) g(t)}{\alpha g(t) - K} \tag{16}$$

with $B(t) > 0$ and $B'(t) > 0$.

(b) Net leisure time is increasing over time, i.e., the time spent on education is decreasing over life.

(c) Optimal education is non-negative up to retirement if

$$(1 - \kappa)\tilde{\kappa} \frac{x + g(0)w + h(0)}{f(0)} e^{\frac{r-\rho}{\gamma} T_0} \left(\alpha g(T_0) - K \right)^{-\frac{1+\kappa(\gamma-1)}{\gamma}} \leq \ell. \tag{17}$$

Concerning the condition (17), note that $g(0)$, $h(0)$, and $f(0)$ implicitly depend on $g(T_0)$. If κ is sufficiently close to one, the condition is always satisfied. This is because in this case the agent puts a sufficiently high weight on consumption for him to educate, thus giving up leisure now for consumption in the future.

Figure 1 further shows that consumption is hump-shaped over life (blue curve; left axis): consumption increases moderately for the first 28 years or so (until age 48) and then declines until retirement. As shown in the following theorem this hump-shaped consumption pattern is typical in our setting.

Theorem 3 *(a) The consumption dynamics are given by*

$$dc(t) = c(t)\left(\frac{r-\rho}{\gamma} - \frac{(1-\kappa)(\gamma-1)\alpha}{\gamma}B(t)\right)dt. \tag{18}$$

(b) The growth rate of consumption is decreasing over life.

(c) There is at most one hump in consumption over the life cycle.

(d) There is a unique consumption hump if

$$B(0) < \frac{r-\rho}{\alpha(\gamma-1)(1-\kappa)} < B(T_0), \tag{19}$$

and then the hump occurs at time

$$t^* = T_0 + \frac{1}{r+\beta}\ln\left(\frac{\frac{1-\ell}{r+\beta} - C}{\frac{1-\ell}{r+\beta} - g(T_0)}\right), \tag{20}$$

where the constant C is defined as

$$C = \frac{(r-\rho)K/\alpha + (1-\kappa)(\gamma-1)(1-\ell)}{r-\rho+(1-\kappa)(\gamma-1)(r+\beta)}$$

and satisfies

$$g(T_0) \le C \le \frac{1-\ell}{r+\beta}. \tag{21}$$

Here (b) follows from (a) since $B'(t) > 0$ and $\gamma > 1$, and (c) follows from (b). Note that the argument of the log-function in (20) is between 0 and 1 because of the inequalities (21) and, consequently, $t^* < T_0$.

In the case without education, the growth rate of consumption equals the constant $(r-\rho)/\gamma$ which by assumption is positive. Therefore, consumption increases over the life-cycle as is well-known, cf., e.g., Gollier (2001, Ch. 15). In contrast, the endogenous education decision makes the consumption growth rate decreasing over time. If the consumption growth rate starts out positive and eventually becomes negative, the model produces a hump-shaped pattern in consumption over the life cycle.

The condition (19) is satisfied if the retirement phase is of intermediate length. Intuitively, if this phase becomes too long the agent will educate himself intensively until retirement and thus continue to substitute leisure by consumption so that consumption will never start to fall. If the retirement period becomes too short, theoretically there might be a hump, but the optimal education level can fall below zero and we get a corner solution before retirement. More formally, the argument is as follows. Since $\lim_{T_0\to\infty} g(t) = \frac{1-\ell}{r+\beta}$, the condition $B(0) < \frac{r-\rho}{\alpha(\gamma-1)(1-\kappa)}$ can be satisfied by fixing the length of the retirement period $\Delta = T - T_0$ and choosing T_0 large enough, since then $g(0) \to (1-\ell)/(r+\beta)$

implying $B(0) \searrow 0$. On the other hand, the condition $B(T_0) > \frac{r-\rho}{\alpha(\gamma-1)(1-\kappa)}$ is satisfied if and only if $C > g(T_0)$. Clearly, $g(T_0) < \chi/r$. So if $C > \chi/r$, the condition is satisfied for any value of $T - T_0$. If $C \leq \chi/r$, we can always find a small enough value of $T - T_0$ so that $C > g(T_0)$ (since $g(T_0)$ is close to zero for $T - T_0 \approx 0$). However, then the condition (17) for positive education might be violated and has to be checked separately.

Next, we consider the evolution of wages over the life cycle that follows from the optimal educational decisions and the assumed wage dynamics (1). In the benchmark case illustrated in Figure 1 the wage is hump-shaped over the working phase of life (purple curve; right axis): it increases over the first roughly 31 years (until age 51)—quite steeply in the early years—and then declines somewhat until retirement. The following theorem shows that the wage hump naturally materializes for suitable parameterizations of the model. However, there are also cases in which the wage dynamics do not exhibit a hump. For example, if the wage depreciation rate β is zero, even a small positive level of education leads to ever-increasing wages, but at the same time the consumption can still have the hump shape.

Theorem 4 *Assume that* (17) *is satisfied.*

(a) The wage w is increasing over time if $\beta = 0$.

(b) The equality $\alpha\varepsilon(t) = \beta w(t)$ holds at most for one $t \in [0, T]$. Suppose it holds for some $t \in [0, T]$ and let τ the unique time at which it holds. Then the wage rate $w(t)$ has a unique hump at τ: it increases over $[0, \tau]$, reaches its maximum at $t = \tau$, and then decreases over $t \in (\tau, T]$.

6 Comparative statics

We now study how key parameters of the model affect the solution and, in particular, the location of the humps in consumption and wages. First, consider the parameter α that controls the responsiveness of wages to education. Figure 2 shows the life-cycle patterns in consumption, wage, net leisure, and education for the benchmark value of $\alpha = 1.5$ and for a lower and a higher value of α. When the wage responds more positively to education, the same level of education leads to a higher human capital so that the agent can afford a higher overall level of consumption of both the physical good and leisure. In addition, the agent has stronger incentives to educate more and substitute some leisure by consumption. To put it differently, the price of leisure relative to the physical good increases when education is more effective. In Figure 2 the net effect of an increase in educational effectiveness is that the agent educates more and thus consumes less leisure, while the consumption of the physical good increases substantially.

[Figure 2 about here.]

For other values of state variables, increasing α leads to lower optimal educational effort. In such cases the income effect dominates the substitution effect. Formally,

$$\frac{\partial \varepsilon(t)}{\partial \alpha} = \kappa^{\frac{1}{\gamma}} \left(\frac{1-\kappa}{\kappa} \frac{1}{\alpha g(t) - K} \right)^{\frac{(1-\kappa)(1-\gamma)}{\gamma}+1} \times \left(\left[\frac{1+\kappa(\gamma-1)}{\gamma} \frac{g(t)}{\alpha g(t) - K} + \frac{f_\alpha(t)}{f(t)} \right] \frac{X(t) + g(t)w(t) + h(t)}{f(t)} - \frac{h_\alpha(t)}{f(t)} \right),$$

where α-subscripts indicate partial derivatives. Note that $h_\alpha(t)$ is positive. If the current wealth $X(t)$ and the current wage level $w(t)$ are low enough then, at least for some parameter values, education is decreasing in α. In such a case we see from (14) that optimal education is already near the limit ℓ, so marginal disutility of education (equivalently, marginal utility of net leisure) is large. Increasing the efficiency of education implies that the agent does not have to educate as much to obtain the same future wages. Hence, the agent lowers education today in order to increase net leisure. Conversely, if wealth and wage are large, the agent currently educates little, and by making education more efficient the willingness to educate increases. As long as education increases in α, the wage also increases. Formally, Eq. (1) implies that

$$w(t) = \mathrm{e}^{-\beta t} w_0 + \alpha \int_0^t \mathrm{e}^{\beta(s-t)} \varepsilon(s)\, ds$$

so that

$$\frac{\partial w(t)}{\partial \alpha} = \int_0^t e^{\beta(t-s)} \left(\varepsilon(s) + \alpha \frac{\partial \varepsilon(s)}{\partial \alpha} \right) ds$$

and if $\frac{\partial \varepsilon(s)}{\partial \alpha} > 0$, then $\frac{\partial w(t)}{\partial \alpha} > 0$. It seems that one cannot rule out $\frac{\partial w(t)}{\partial \alpha} < 0$ for low enough values of initial wealth and wage levels.

Figure 2 further shows that the hump-shaped pattern in consumption and wages over life is present for all three values of α considered. As α is increased, the maximum consumption is reached somewhat later and the age where the wage peaks increases more substantially. The former observation is generally valid since differentiation of the consumption hump time t^* in (20) yields

$$\frac{\partial t^*}{\partial \alpha} = -\frac{\partial C/\partial \alpha}{1-\ell-(r+\beta)C} = \frac{(r-\rho)K}{\alpha^2 \left[1-\ell-(r+\beta)C\right] \left[r-\rho+(1-\kappa)(\gamma-1)(r+\beta)\right]^2},$$

which is positive because of Assumption 1 and (21).

Next, Figure 3 illustrates the sensitivity of the results to the wage depreciation rate β. Other things equal, a higher β lowers future wages and thus the human capital, and therefore lowers the overall consumption of the physical good and leisure, whereas educational efforts increase. In addition, the relative price of leisure decreases, which causes a substitution from physical consumption to leisure with less education as a consequence. According to the graphs the net effect of a higher β is to slightly reduce education and increase leisure in the early years, but to increase education and decrease leisure in the later part of the active life. With a high β, fairly high educational efforts just before retirement are important to ensure a decent income throughout retirement. When β is increased, the consumption hump occurs later and the wage hump earlier in life. The latter observation is consistent with part (a) of Theorem 4 which concludes that the wage is increasing over the entire active life if β is decreased to zero.

[Figure 3 about here.]

Figure 4 depicts the sensitivity of life-cycle consumption and wages on the time preference rate ρ. As ρ increases, the agent increases consumption early in life and lowers consumption late in life to obtain a flatter life-cycle pattern, and the consumption hump occurs earlier in life. This follows formally from the fact that

$$\frac{\partial t^*}{\partial \rho} = \frac{\partial C/\partial \rho}{(r+\beta)C - (1-\ell)} = -\frac{(1-\kappa)(\gamma-1)\left[1-\ell-\frac{K}{\alpha}(r+\beta)\right]}{\left[1-\ell-(r+\beta)C\right]\left[r-\rho+(1-\kappa)(\gamma-1)(r+\beta)\right]^2}$$

is negative because of Assumption 1 and (21). Increasing ρ also leads to more leisure and thus less education and lower wages early in life, but later in life educational efforts and thus wages are increasing in ρ. The age at which the maximum wage is earned is increasing in ρ.

[Figure 4 about here.]

A higher interest rate leads, other things equal, to a lower present value of future income and thus a lower total wealth. Consequently, the agent reduces consumption as shown in the left panel of Figure 5. Because of the higher discount rate, the present value of the benefits from education declines so the agent educates less and increases net leisure. The lower educational efforts reduce wages, cf. the right panel of Figure 5, and thus human capital even further. An increase in the interest rate leads to a later hump in consumption but an earlier hump in wages.

[Figure 5 about here.]

Figure 6 reports the sensitivity of consumption and wages to the parameter κ that reflects the relative weight of perishable consumption in the utility function. Naturally, a higher κ leads to increased consumption of the physical good, as shown in the left panel, and lower consumption of leisure. Hence, the agent educates more which leads to higher future wages, as shown in the right panel, which reinforces the positive effect on perishable consumption. By increasing κ both the consumption hump and the wage hump occur later in life. The first observation follows formally from the fact that

$$\frac{\partial t^*}{\partial \kappa} = -\frac{\partial C/\partial \kappa}{1 - \ell - (r+\beta)C} = \frac{(\gamma - 1)(r - \rho)}{1 - \ell - (r+\beta)C} \frac{1 - \ell - (r+\beta)\frac{K}{\alpha}}{[r - \rho - (1-\kappa)(\gamma - 1)(r+\beta)]^2}$$

is positive because of Assumption 1 and (21).

[Figure 6 about here.]

Concerning the parameter γ, the reciprocal of the elasticity of intertemporal substitution, the left panel of Figure 7 shows that a higher γ implies higher consumption early in life and lower consumption late in life. This entails an earlier consumption hump, which is confirmed by the observation that

$$\frac{\partial t^*}{\partial \gamma} = -\frac{\partial C/\partial \gamma}{1 - \ell - (r+\beta)C} = -\frac{(1 - \kappa)(r - \rho)}{1 - \ell - (r+\beta)C} \frac{1 - \ell - (r+\beta)\frac{K}{\alpha}}{[r - \rho - (1-\kappa)(1 - \gamma)(r+\beta)]^2}$$

is negative because of Assumption 1 and (21). The same holds for the consumption of net leisure, while education is lowered early in life and increased late in life. This produces the effect on wages seen in the right panel. In particular, a higher γ leads to a later wage hump.

[Figure 7 about here.]

Increasing the parameter K representing the monetary costs naturally generates lower education throughout life and thus both lower wages and lower consumption as illustrated in Figure 8. Optimal education is affected by the cost parameter K most strongly late in life, where the benefits from educating are smallest. For higher values of K both the consumption hump and the wage hump appear earlier in life. The first observation is substantiated by the fact that

$$\frac{\partial t^*}{\partial K} = -\frac{\partial C/\partial K}{1 - \ell - (r+\beta)C}$$

is negative because (21) holds and $\partial C/\partial K$ is clearly positive.

[Figure 8 about here.]

Figure 9 shows that an increase in initial financial wealth allows higher physical consumption throughout life. Also net leisure can be increased, which implies less education and thus lower wages and human capital. This dampens the effect of higher initial wealth on consumption. When increasing initial wealth, the age at which consumption is maximal is unaffected (t^* is independent of X_0), whereas the wage hump happens earlier in life.

[Figure 9 about here.]

Finally, Figure 10 shows that the parameter χ determining the ratio of retirement income to pre-retirement income only has a notable effect on consumption and wages in the last few years prior to retirement. Increasing χ boosts optimal education in the final active years and thus also wages and retirement income, which can finance an increase in consumption just before and in retirement. The consumption hump occurs later since

$$\frac{\partial t^*}{\partial \chi} = \frac{\partial g(T_0)/\partial \chi}{1 - \ell - (r+\beta)g(T_0)} = \frac{1 - e^{-r(T-T_0)}}{r\left(1 - \ell - (r+\beta)g(T_0)\right)}$$

is positive due to (8).

[Figure 10 about here.]

7 A review of related models

In many consumption-savings models with power utility of a single consumption good, the indirect utility turns out to be of the form $\frac{1}{1-\gamma} f(t)^\gamma \tilde{X}(t)^{1-\gamma}$ and the optimal consumption rate is

$$c(t) = \frac{\delta(t)}{f(t)} \tilde{X}(t), \tag{22}$$

where f and δ satisfy

$$f'(t) = \tilde{r} f(t) - \delta(t), \qquad \tilde{r} = r - \frac{r - \rho}{\gamma}.$$

Here $\tilde{X}(t)$ denotes the total wealth at time t, which is the sum of the financial wealth and, if present in the model, the human capital. Likewise, in models with Cobb-Douglas utility of two goods, the optimal consumption rate of the numeraire good is typically of the form

$$c(t) = \kappa \frac{\delta(t)}{f(t)} \tilde{X}(t), \tag{23}$$

where κ is the Cobb-Douglas preference weight of that good. In both cases, the wealth dynamics implied by the optimal decisions are of the form

$$d\tilde{X}(t) = r\tilde{X}(t)\,dt - \kappa^{-1}c(t)\,dt \tag{24}$$

where we let $\kappa = 1$ in the one-good case. The consumption dynamics in either case become

$$dc(t) = c(t)\mu_c(t)\,dt, \qquad \mu_c(t) = \frac{\delta'(t)}{\delta(t)} + \frac{r-\rho}{\gamma}. \tag{25}$$

The shape of the life-cycle pattern of consumption is therefore determined solely by the function $\delta(t)$ (or, equivalently, $f(t)$).

7.1 The canonical consumption-savings model

In the canonical, full certainty, frictionless consumption-savings model with an agent who maximizes the utility functional $\int_0^T e^{-\rho t}\frac{1}{1-\gamma}c(t)^{1-\gamma}\,dt$, receives no labor income, and obtains a rate of return r on savings, the optimal consumption-wealth ratio is

$$\frac{c(t)}{X(t)} = \tilde{r}\left(1 - e^{-\tilde{r}(T-t)}\right)^{-1}.$$

With $f(t) = 1 - e^{-\tilde{r}(T-t)}$, we have $f'(t) = -\tilde{r}e^{-\tilde{r}(T-t)}$ so that $\delta(t) = \tilde{r}f(t) - f'(t) = \tilde{r}$ and (22) holds. Since $\delta'(t) = 0$, we then have $\mu_c(t) = (r-\rho)/\gamma$, which shows that the age-pattern of consumption is either flat, increasing, or decreasing depending on whether the return on savings is equal to, greater than, or smaller than the subjective time preference rate.

7.2 Two perishable goods with time-dependent relative price

Suppose there are two perishable goods with $c(t)$ being the units consumed of good 1 and $q(t)$ the units consumed of good 2 per time period. Assume that good 1 is the numeraire and that the time t price of good 2 is $p(t)$, which is an exogenously given function of time. The agent maximizes the Cobb-Douglas utility functional $\int_0^T e^{-\rho t}\frac{1}{1-\gamma}\left(c(t)^\kappa q(t)^{1-\kappa}\right)^{1-\gamma}\,dt$. It can be shown that the ratio of optimal consumption of good 1 to wealth in this case is

$$\frac{c(t)}{X(t)} = \kappa p(t)^{\frac{(1-\kappa)(\gamma-1)}{\gamma}} f(t)^{-1},$$

where

$$f(t) = \int_t^T e^{-\tilde{r}(s-t)}p(s)^{\frac{(1-\kappa)(\gamma-1)}{\gamma}}\,ds.$$

Since $\delta(t) = \tilde{r} f(t) - f'(t) = p(t)^{\frac{(1-\kappa)(\gamma-1)}{\gamma}}$, we see that (23) holds. In this case

$$\frac{\delta'(t)}{\delta(t)} = \frac{(1-\kappa)(\gamma-1)}{\gamma} \frac{p'(t)}{p(t)},$$

so the growth rate of the consumption of good 1 is

$$\mu_c(t) = \frac{(1-\kappa)(\gamma-1)}{\gamma} \frac{p'(t)}{p(t)} + \frac{r-\rho}{\gamma}.$$

The life-cycle consumption pattern is determined by $p(t)$. Suppose $r > \rho$ and $\gamma > 1$. If $p(t)$ is first increasing, flat, or even weakly decreasing, and later sufficiently decreasing, then a consumption hump emerges.

7.3 Endogenous labor supply, but exogenous wages

Here we review the model of Heckman (1974). Complete proofs are provided in Appendix A.5. Heckman disregards educational decisions and assumes an exogenous life-cycle pattern of the wage rate, $w(t)$. This is equivalent to assuming that the dynamics of the wage rate are

$$dw(t) = w(t)\mu_w(t)\, dt$$

for some deterministic function μ_w. The agent can continuously adjust the fraction $L(t)$ of time spent on working, which thus generates income at the rate $L(t)w(t)$. The agent has Cobb-Douglas utility of consumption and leisure, $\frac{1}{1-\gamma}\left(c(t)^\kappa (1-L(t))^{1-\kappa}\right)^{1-\gamma}$. We can think of the agent as having the opportunity to work full time, which would generate a human capital of the form $w(t)g(t)$ for some age-dependent function $g(t)$ and thus a total wealth of $\tilde{X}(t) = X(t) + w(t)g(t)$. In a second step, the agent then decides to purchase leisure at the unit price of $w(t)$ corresponding to the wage foregone. It can be shown that the optimal consumption-wealth ratio is

$$\frac{c(t)}{\tilde{X}(t)} = \kappa\tilde{\kappa} \frac{w(t)^{\frac{(1-\kappa)(\gamma-1)}{\gamma}}}{f(t)} \tag{26}$$

with (we allow for a retirement phase exactly as in our model in Section 3)

$$f(t) = \tilde{\kappa} \int_t^{T_0} e^{-\tilde{r}(s-t)} w(s)^{\frac{(1-\kappa)(\gamma-1)}{\gamma}}\, ds + \frac{1}{\tilde{r}} e^{-\tilde{r}(T_0-t)} \left(1 - e^{-\tilde{r}(T-T_0)}\right).$$

Since

$$\delta(t) = \tilde{r} f(t) - f'(t) = \tilde{\kappa} w(t)^{\frac{(1-\kappa)(\gamma-1)}{\gamma}},$$

the consumption-wealth ratio in (26) is consistent with (23). The consumption growth rate is

$$\mu_c(t) = \frac{(1-\kappa)(\gamma-1)}{\gamma}\mu_w(t) + \frac{r-\rho}{\gamma}. \tag{27}$$

Note the equivalence with the case considered in Section 7.2; the wage enters the consumption growth rate only because it is the relative price of the second good, namely leisure.

Let us be more specific about the consumption hump in this model. A consumption hump may occur even though the wage rate is monotonic: if either

(i) $\gamma > 1$, $r < \rho$, $\mu_w(t) > 0$, and $\mu_w'(t) < 0$ for all $t \leq T_0$, or
(ii) $\gamma < 1$, $r > \rho$, $\mu_w(t) > 0$, and $\mu_w'(t) > 0$ for all $t \leq T_0$,

then $\mu_c(t)$ can be positive for small t and turn negative for large t. As we have argued below Assumption 1, the case $\gamma > 1$ and $r > \rho$ seems more realistic. For this case the Heckman (1974) model can produce a consumption hump if $\mu_w(t)$ starts out positive or even slightly negative and ends up being sufficiently negative. It is well-documented that wages generally increase in the earlier years in the job market so that the relevant situation is where $\mu_w(t)$ starts out positive and later turns (sufficiently) negative, i.e., the wage is hump shaped over life.

Suppose that the wage rate peaks at time $t_w \in (0, T_0)$ so that $\mu_w(t) > 0$ for $t < t_w$, $\mu_w(t_w) = 0$, $\mu_w(t) < 0$ for $t < t_w$, and μ_w is decreasing in t (wage is concave). Assuming $\gamma > 1$, we can see from (27) that the optimal consumption profile is hump shaped provided

$$-(1-\kappa)(\gamma-1)\mu_w(0) < r - \rho < -(1-\kappa)(\gamma-1)\mu_w(T_0),$$

since then the consumption growth rate is first positive and ends up negative at retirement. These inequalities may hold whether $r > \rho$, $r = \rho$, or $r < \rho$. If $r = \rho$, the consumption peaks at the same time as the wage rate peaks. If $r > \rho$ [respectively, $r < \rho$], a peak in consumption occurs later [earlier] than the wage rate. Some empirical studies report a hump in wages, but in that case it typically occurs after the consumption hump. Other studies report that wages are monotonically increasing over the working life, although flattening out up to retirement, and for this case the Heckman (1974) model cannot produce a consumption hump at all, whereas our model can.

The Heckman (1974) model also produces an optimal labor supply which, according to Carroll and Summers (1991) and Browning and Crossley (2001), varies more strongly with age than the typically observed labor supply.

7.4 Our model and a variation

Also in our model the consumption is of the form (23) and since

$$\delta'(t) = \delta(t)\alpha \frac{(1-\kappa)(\gamma-1)}{\gamma} \frac{g'(t)}{\alpha g(t) - K},$$

we obtain

$$\mu_c(t) = \frac{(1-\kappa)(\gamma-1)}{\gamma} \frac{\alpha g'(t)}{\alpha g(t) - K} + \frac{r-\rho}{\gamma},$$

which is equivalent to (18). As shown in part (b) of Theorem 3, the consumption growth rate is decreasing. Under the condition (19), the consumption growth rate starts out positive and later turns negative, which produces the hump.

Note that our model produces the same consumption pattern as a Heckman (1974) model with a wage rate exogenously specified to equal $\alpha g(t) - K$. In our model the price of leisure is not just the current wage rate, but the present value of all the future foregone earnings due to the reduction in education. This present value $\alpha g(t) - K$ can be hump shaped even though the wage rate is not.

Let us briefly consider a variation of our model that also allows a closed-form solution and, potentially, a consumption hump. This variation follows Heckman (1976b) and deviates from our main model in the specification of the wage dynamics and the preferences, and by disregarding direct monetary costs of education (i.e., $K = 0$). The wage dynamics are

$$dw(t) = (aw(t)\varepsilon(t) - \beta w(t))\, dt = w(t)\,(a\varepsilon(t) - \beta)\, dt$$

and are thus Harrod neutral in line with the original specification of Ben-Porath (1967). The utility is a Cobb-Douglas function of consumption and the monetary value of leisure instead of just leisure time. While Heckman (1976b) allows for endogenous labor supply (see discussion below), we consider the case with fixed labor supply $L = 1 - \ell$ so that the value function is defined as

$$\begin{aligned} V(t,x,w) = \sup_{c,\varepsilon} \Bigg\{ \int_t^{T_0} e^{-\rho(s-t)} \frac{1}{1-\gamma} \left(c(s)^\kappa \left\{ w(s)[1 - L - \varepsilon(s)] \right\}^{1-\kappa} \right)^{1-\gamma} ds \\ + \int_{T_0}^{T} e^{-\rho(s-t)} \frac{1}{1-\gamma} c(s)^{1-\gamma}\, ds \Bigg\}. \end{aligned} \tag{28}$$

The Bellman equation for the pre-retirement period is

$$
\begin{aligned}
0 = \sup_{c,\varepsilon} \Bigg\{ & V_t + V_x \left(rx + wL - c\right) + wV_w(a\varepsilon - \beta) \\
& + \frac{1}{1-\gamma} c^{\kappa(1-\gamma)} w^{(1-\kappa)(1-\gamma)} (1 - L - \varepsilon)^{(1-\kappa)(1-\gamma)} - \rho V \Bigg\}.
\end{aligned} \tag{29}
$$

The first-order conditions for c and ε are

$$
V_x = \kappa c^{\kappa(1-\gamma)-1} w^{(1-\kappa)(1-\gamma)} (1 - L - \varepsilon)^{(1-\kappa)(1-\gamma)}, \tag{30}
$$

$$
awV_w = (1-\kappa) c^{\kappa(1-\gamma)} w^{(1-\kappa)(1-\gamma)} (1 - L - \varepsilon)^{(1-\kappa)(1-\gamma)-1}, \tag{31}
$$

which imply

$$
c = \frac{\kappa}{1-\kappa} a \frac{wV_w}{V_x} (\ell - \varepsilon).
$$

It can be shown that

$$
V(t,x,w) = \frac{1}{1-\gamma} f(t)^{\gamma} \left(x + wg(t)\right)^{1-\gamma}
$$

solves the Bellman equation if f and g solve

$$
0 = f'(t) - \tilde{r} f(t) + \tilde{\kappa} \left(ag(t)\right)^{\frac{(1-\kappa)(\gamma-1)}{\gamma}}, \qquad 0 = g'(t) - (r + \beta - a\ell) g(t) + 1 - \ell.
$$

The optimal consumption rate becomes

$$
c(t) = \kappa \tilde{\kappa} \left(ag(t)\right)^{\frac{(1-\kappa)(\gamma-1)}{\gamma}} \frac{\tilde{X}(t)}{f(t)},
$$

where $\tilde{X}(t) = X(t) + w(t)g(t)$, This is consistent with (23) if we let

$$
\delta(t) = \tilde{\kappa} \left(ag(t)\right)^{\frac{(1-\kappa)(\gamma-1)}{\gamma}},
$$

which is similar to the expression (11) for δ in our model. It follows that

$$
\mu_c(t) = \frac{(1-\kappa)(\gamma-1)}{\gamma} \frac{g'(t)}{g(t)} + \frac{r-\rho}{\gamma}.
$$

Clearly, this model is also able to produce a consumption hump, but it builds on the non-standard assumption that utility depends on the monetary value of leisure instead of the leisure time.

7.5 Endogenous labor supply and education

A natural extension of the work mentioned above would be to allow the agent both to choose labor supply and educational efforts in addition to consumption. However, such models typically result in corner solutions where the agent will either work or educate, but not do both at the same time.

Consider for example the set-up of Heckman (1976b) described in the preceding subsection. If we add the labor supply $L(t)$ as a choice variable in (28), the corresponding first-order condition from the Bellman equation is

$$wV_x = (1-\kappa)c^{\kappa(1-\gamma)}w^{(1-\kappa)(1-\gamma)}(1-L-\varepsilon)^{(1-\kappa)(1-\gamma)-1}, \tag{32}$$

and combining that with (31) implies $V_x = aV_w$, which is necessary to obtain an interior solution. But both V_x and V_w generally depend on time, and $V_x = aV_w$ cannot be expected to hold except maybe for some carefully reverse-engineered model specifications. The derivative of the right-hand side of (29) with respect to L is

$$wV_x - (1-\kappa)c^{\kappa(1-\gamma)}w^{(1-\kappa)(1-\gamma)}(1-L-\varepsilon)^{(1-\kappa)(1-\gamma)-1},$$

whereas the derivative with respect to ε is

$$awV_w - (1-\kappa)c^{\kappa(1-\gamma)}w^{(1-\kappa)(1-\gamma)}(1-L-\varepsilon)^{(1-\kappa)(1-\gamma)-1}.$$

Hence, as long as $V_x > aV_w$, the agent wants to work more and educate less, and vice versa when $V_x < aV_w$. This leads to a corner solution where, at any given point in time, the agent either works or educates. An extra hour of work or education comes with the same direct utility cost through the foregone leisure, so the choice depends only on the added indirect utility via the extra current wage if working or the extra future wages if educating. The same conclusion holds if we allow endogenous labor supply in our main model.

Heckman (1976b) presents some discussion of the properties of the agent's optimal consumption, labor supply, and educational decisions and, among other things, concludes that consumption rises monotonically over the life cycle. However, his analysis presumes an interior solution (middle of his page S14) but in fact, as we have seen above, his setting leads to a corner solution. Williams (1979) considers a model of consumption, labor supply, and education with uncertainty about wage dynamics and investment returns. The uncertainty may result in an interior solution, at least in some states of the world, because then the agent balances the known benefits of an extra hour of work against the expectation and uncertainty about the benefits of an extra hour of education. Still corner solutions are optimal in some states of the world and, hence, an explicit solution does not seem to be available. Williams presents an approximate closed-form solution which in some states is violating the constraints and does not discuss the life-cycle pattern in consumption.

8 Conclusion

This paper has bridged two important strands of literature in economics: the literature on the consumption hump and the literature on how educational decisions affect wages and welfare. We have presented a simple and transparent model that offers closed-form solutions but still captures the key mechanisms and trade-offs. Our model shows that through the effect of educational decisions on future wages, the price of leisure relative to consumption of goods decreases over life, and this can twist the consumption pattern from being monotonically increasing in the corresponding model without education and leisure into the hump-shaped profile observed in the data.

At the same time our model provides an explanation of how educational efforts can shape the wage rates over the working life of an individual. Empirical studies disagree whether wages tend to be hump-shaped or monotonically increasing over life, but our model can produce either pattern—together with the consumption hump.

A Proofs

A.1 Proof of Theorem 1

In retirement. First we solve the utility maximization problem in retirement. Here, the indirect utility function is

$$J(t,x) = \sup_c \int_t^T e^{-\rho(s-t)} U(c(s))\, ds, \quad t \geq T_0,$$

and the associated Bellman equation is

$$0 = \sup_c \Big\{ J_t + J_x r x + J_x (w\chi - c) + U(c) - \rho J \Big\}, \qquad J(T,x) = 0,$$

where $w = w(T_0)$ is fixed. The first-order condition for consumption is given by $c = J_x^{-1/\gamma}$. We conjecture that

$$J(t,x) = \frac{1}{1-\gamma} f(t)^\gamma \left(x + wg(t)\right)^{1-\gamma}, \quad T_0 \leq t \leq T,$$

for suitable functions f and g. Substituting the conjecture and the first-order condition into the Bellman equation yields

$$\begin{aligned}\frac{\gamma}{1-\gamma} f' \frac{x+gw}{f} + g'w = \rho \frac{1}{1-\gamma} f \frac{x+gw}{f} - r\frac{(x+gw)}{f} f + rgw \\ - \left(w\chi - \frac{x+gw}{f} \right) - \frac{1}{1-\gamma} \frac{x+gw}{f}.\end{aligned}$$

Collecting all terms multiplying $(x+gw)/f$ and all remaining terms, we conclude that f and g satisfy the differential equations

$$f'(t) = \tilde{r} f(t) - 1, \qquad g'(t) = r g(t) - \chi,$$

where $\tilde{r}$ is given by (10), with terminal values $f(T) = g(T) = 0$. Solving these equations we find

$$f(t) = \frac{1}{\tilde{r}} \left(1 - e^{-\tilde{r}(T-t)}\right), \qquad g(t) = \frac{\chi}{r} \left(1 - e^{-r(T-t)}\right).$$

The optimal consumption rate in retirement is $c = (x+gw)/f$.

Before retirement. For $t \leq T_0$ the relevant Bellman equation is

$$
\begin{aligned}
0 = \sup_{c,\varepsilon} \Big\{ & V_t + V_x r x + V_x \left(w \left(1-\ell\right) - c - K\varepsilon \right) \\
& + V_w \left(\alpha\varepsilon - \beta w \right) + \frac{1}{1-\gamma} \left(c^{\kappa} (\ell - \varepsilon)^{1-\kappa} \right)^{1-\gamma} - \rho V \Big\},
\end{aligned} \tag{33}
$$

with the terminal condition

$$
V(T_0, x, w) = J(T_0, x) = \frac{1}{1-\gamma} f(T_0)^{\gamma} \left(x + w g(T_0) \right)^{1-\gamma}
$$

stemming from the solution of the problem after retirement. The first-order conditions imply

$$
V_x = \kappa c^{\kappa(1-\gamma)-1} \left(\ell - \varepsilon\right)^{(1-\kappa)(1-\gamma)}, \tag{34}
$$

$$
V_w \alpha - V_x K = (1-\kappa) c^{\kappa(1-\gamma)} \left(\ell - \varepsilon\right)^{(1-\kappa)(1-\gamma)-1}. \tag{35}
$$

Dividing the second condition by the first one yields a simple relation between consumption and net leisure time,

$$
\frac{c}{\ell - \varepsilon} = \frac{\kappa}{1-\kappa} \left(\alpha \frac{V_w}{V_x} - K \right). \tag{36}
$$

By applying this relation, Eq. (34) implies that optimal consumption is

$$
c = \kappa^{\frac{1}{\gamma}} \left(\frac{\kappa}{1-\kappa} \left[\alpha \frac{V_w}{V_x} - K \right] \right)^{-\frac{(1-\kappa)(1-\gamma)}{\gamma}} V_x^{-\frac{1}{\gamma}} = \kappa \tilde{\kappa} \left(\alpha \frac{V_w}{V_x} - K \right)^{-\frac{(1-\kappa)(1-\gamma)}{\gamma}} V_x^{-\frac{1}{\gamma}}, \tag{37}
$$

and then

$$
\ell - \varepsilon = (1-\kappa) \tilde{\kappa} \left(\alpha \frac{V_w}{V_x} - K \right)^{-\frac{(1-\kappa)(1-\gamma)}{\gamma} - 1} V_x^{-\frac{1}{\gamma}} \tag{38}
$$

follows. Substituting these expressions back into (33), we obtain after some reduction the ordinary differential equation

$$
0 = V_t + V_x \left[r x + w(1-\ell) - K\ell \right] + V_w (\alpha \ell - \beta w) + \frac{\gamma}{1-\gamma} \tilde{\kappa} \left(\alpha \frac{V_w}{V_x} - K \right)^{\frac{(1-\kappa)(\gamma-1)}{\gamma}} - \rho V. \tag{39}
$$

With our conjecture

$$
V(t, x, w) = \frac{1}{1-\gamma} f(t)^{\gamma} \left(x + w g(t) + h(t) \right)^{1-\gamma}, \quad t \leq T_0, \tag{40}
$$

where $h(T_0) = 0$, we have

$$V_t = \frac{\gamma}{1-\gamma} f' \left(\frac{x + gw + h}{f} \right)^{1-\gamma} + \left(\frac{x + gw + h}{f} \right)^{-\gamma} (wg' + h'),$$
$$V_x = \left(\frac{x + gw + h}{f} \right)^{-\gamma}, \qquad V_w = \left(\frac{x + gw + h}{f} \right)^{-\gamma} g$$

and, in particular, $V_w/V_x = g$. Substituting V_x and V_w into Eqs. (37)–(38), we obtain the expressions (13) and (14) for the optimal controls stated in the theorem. Substituting the conjecture into the reduced Bellman equation (39), we conclude that the following differential equations for f, g, and h must hold:

$$f'(t) = \tilde{r} f(t) - \delta(t), \tag{41}$$
$$g'(t) = (r + \beta)\, g(t) - (1 - \ell), \tag{42}$$
$$h'(t) = r h(t) - \ell\, (\alpha g(t) - K). \tag{43}$$

It is then straightforward to verify that the solutions for f, g, and h are as stated in the theorem. In fact standard integration rules yield

$$h(t) = \frac{\alpha\ell(1-\ell)}{\beta} \left[\mathcal{A}_r(T_0 - t) - \mathcal{A}_{r+\beta}(T_0 - t) + \frac{\chi\beta}{1-\ell} e^{-r(T_0 - t)} \mathcal{A}_r(T - T_0) \mathcal{A}_\beta(T_0 - t) \right]$$
$$- K\ell\mathcal{A}_r(T_0 - t), \tag{44}$$

where, for any k, we define

$$\mathcal{A}_k(\tau) = \frac{1}{k} \left(1 - e^{-k\tau} \right).$$

Straightforward differentiation yields

$$g'(t) = e^{-(r+\beta)(T_0 - t)} (r + \beta) \left(g(T_0) - \frac{1-\ell}{r+\beta} \right), \tag{45}$$

which is negative as long as the right-most inequality in (8) is satisfied, and thus g is decreasing.

To obtain an interior solution it is necessary that $V_w \alpha > V_x K$, which with our conjectured indirect utility function implies that $\alpha g(t) > K$ must hold for every $t \le T_0$. Since g is decreasing, we just need $\alpha g(T_0) > K$, which is the left-most inequality in (8).

A.2 Proof of Theorem 2

(a) First define total wealth as

$$\tilde{X}(t) = X(t) + g(t)w(t) + h(t).$$

Differentiation leads to

$$
\begin{aligned}
d\tilde{X}(t) &= dX(t) + g(t)\,dw(t) + w(t)\,dg(t) + dh(t) \\
&= rX(t)\,dt + (w(t)[1-\ell] - c(t) - K\varepsilon(t))\,dt + g(t)\,(\alpha\varepsilon(t) - \beta w(t))\,dt \\
&\quad + w(t)\,([r+\beta]g(t) - [1-\ell])\,dt + (rh(t) - \ell[\alpha g(t) - K])\,dt \\
&= r\,(X(t) + g(t)w(t) + h(t))\,dt - c(t)\,dt - (\alpha g(t) - K)(\ell - \varepsilon(t))\,dt \\
&= r\tilde{X}(t)\,dt - c(t)\,dt - (\alpha g(t) - K)(\ell - \varepsilon(t))\,dt \\
&= r\tilde{X}(t)\,dt - \frac{\delta(t)}{f(t)}\tilde{X}(t)\,dt
\end{aligned}
$$

and also

$$
\begin{aligned}
d\left(\frac{\tilde{X}(t)}{f(t)}\right) &= -\frac{f'(t)}{f(t)^2}\tilde{X}(t)\,dt + \frac{1}{f(t)}\,d\tilde{X}(t) \\
&= -\frac{\tilde{r}f(t) - \delta(t)}{f(t)^2}\tilde{X}(t)\,dt + \frac{1}{f(t)}\left(r\tilde{X}(t)\,dt - \frac{\delta(t)}{f(t)}\tilde{X}(t)\,dt\right) \\
&= (r - \tilde{r})\frac{\tilde{X}(t)}{f(t)}\,dt,
\end{aligned} \tag{46}
$$

where we have used (41). From (11) and (14), optimal net leisure is

$$
\begin{aligned}
(\ell - \varepsilon)(t) &= (1-\kappa)\frac{\delta(t)}{\alpha g(t) - K}\frac{\tilde{X}(t)}{f(t)} \\
&= (1-\kappa)\tilde{\kappa}\,(\alpha g(t) - K)^{\frac{(1-\kappa)(\gamma-1)}{\gamma} - 1}\frac{\tilde{X}(t)}{f(t)}.
\end{aligned}
$$

By applying (42) and (46) we obtain

$$
\begin{aligned}
d(\ell - \varepsilon)(t) &= -\frac{1 + \kappa(\gamma - 1)}{\gamma}\frac{\alpha g'(t)}{\alpha g(t) - K}(\ell - \varepsilon)(t)\,dt + (r - \tilde{r})(\ell - \varepsilon)(t)\,dt \\
&= (\ell - \varepsilon)(t)\left\{-\frac{1 + \kappa(\gamma - 1)}{\gamma}\alpha\frac{(r+\beta)g(t) - (1-\ell)}{\alpha g(t) - K} + r - \tilde{r}\right\}dt \\
&= (\ell - \varepsilon)(t)\left\{\frac{r - \rho}{\gamma} + \frac{1 + \kappa(\gamma - 1)}{\gamma}\alpha B(t)\right\}dt,
\end{aligned}
$$

where $B(t)$ is defined in (16) and we use (10).

To show that $B(t)$ is positive, notice that by (45)

$$
(r+\beta)g(t) - (1-\ell) = g'(t) = e^{-(r+\beta)(T_0 - t)}\left[\chi\frac{r+\beta}{r}\left(1 - e^{-r(T - T_0)}\right) - (1-\ell)\right] < 0,
$$

due to (8). Finally, $\alpha g(t) - K > 0$ since $\alpha g(T_0) > K$ by assumption and $g'(t) < 0$. This produces the desired result. Differentiation gives

$$B'(t) = \frac{\alpha g'(t)^2 - (r+\beta)g'(t)[\alpha g(t) - K]}{(\alpha g(t) - K)^2} > 0$$

as claimed.

(b) This follows from (a) since we assume $r > \rho$ and $\gamma > 1$.

(c) Non-negative education is equivalent to $\ell - \varepsilon \leq \ell$. Since $(\ell - \varepsilon)(t)$ is increasing, it is sufficient to check that $(\ell - \varepsilon)(T_0) \leq \ell$. Let

$$A(t) = \frac{r-\rho}{\gamma} + \frac{1+\kappa(\gamma-1)}{\gamma}\alpha B(t),$$

so that $d(\ell-\varepsilon)(t) = (\ell-\varepsilon)(t)A(t)\,dt$ and thus $(\ell-\varepsilon)(T_0) = (\ell-\varepsilon)_0 \exp\{\int_0^{T_0} A(s)\,ds\}$. Since

$$\int \alpha \frac{(r+\beta)\,g(t) - (1-\ell)}{\alpha g(t) - K}\,dt = \int \alpha \frac{g'(t)}{\alpha g(t) - K}\,dt = \ln(\alpha g(t) - K)$$

we get

$$\begin{aligned}
(\ell-\varepsilon)(T_0) &= (\ell-\varepsilon)_0 e^{\frac{r-\rho}{\gamma}T_0} \left(\frac{\alpha g(T_0) - K}{\alpha g(0) - K}\right)^{-\frac{1+\kappa(\gamma-1)}{\gamma}} \\
&= (1-\kappa)\tilde{\kappa}\,(\alpha g(0) - K)^{-\frac{1+\kappa(\gamma-1)}{\gamma}} \frac{x + g(0)w + h(0)}{f(0)} \\
&\quad \times e^{\frac{r-\rho}{\gamma}T_0} \left(\frac{\alpha g(T_0) - K}{\alpha g(0) - K}\right)^{-\frac{1+\kappa(\gamma-1)}{\gamma}},
\end{aligned}$$

which should be smaller than ℓ. Therefore, we obtain the condition (17).

A.3 Proof of Theorem 3

(a) From (13), optimal consumption is

$$c(t) = \kappa\tilde{\kappa}\,(\alpha g(t) - K)^{\frac{(1-\kappa)(\gamma-1)}{\gamma}} \frac{\tilde{X}(t)}{f(t)}.$$

By applying (42) and (46) we obtain

$$\begin{aligned}
dc(t) &= \frac{(1-\kappa)(\gamma-1)}{\gamma} \frac{\alpha g'(t)}{\alpha g(t) - K} c(t)\,dt + (r - \tilde{r})c(t)\,dt \\
&= c(t)\left\{\frac{r-\rho}{\gamma} - \frac{(1-\kappa)(\gamma-1)}{\gamma}\alpha B(t)\right\}\,dt.
\end{aligned}$$

(b) This claim follows from (a) since $B'(t) > 0$ and $\gamma > 1$.

(c) This claim follows directly from (b).

(d) First note that

$$g(t) = \frac{1-\ell}{r+\beta}\left(1 - e^{-(r+\beta)(T_0-t)}\right) + e^{-(r+\beta)(T_0-t)} g(T_0),$$

and if we substitute this into the expression (16) for $B(t)$, simplify, and multiply numerator and denominator by $\exp\{(r+\beta)(T_0-t)\}$, we can rewrite $B(t)$ as

$$B(t) = \frac{1-\ell-(r+\beta)g(T_0)}{\left(\frac{\alpha(1-\ell)}{r+\beta} - K\right) e^{(r+\beta)(T_0-t)} + \alpha\left(g(T_0) - \frac{1-\ell}{r+\beta}\right)}.$$

When the inequalities in (19) are satisfied, the consumption drift is positive at $t = 0$ and negative at $t = T_0$. Since B is continuous and increasing, a time point t^* exists where $B(t^*) = \frac{r-\rho}{\alpha(\gamma-1)(1-\kappa)}$. Applying the above expression for $B(t)$, we find that t^* is given by (20). The consumption drift is zero at t^*. In combination with (b), it follows that there is a consumption hump at t^*.

The claim $C \leq (1-\ell)/(r+\beta)$ is equivalent to

$$(1-\ell)\left(r-\rho+(1-\kappa)(\gamma-1)(r+\beta)\right) \geq (r+\beta)\left((r-\rho)\frac{K}{\alpha} + (1-\kappa)(\gamma-1)(1-\ell)\right),$$

which reduces to the inequality $1-\ell \geq (r+\beta)K/\alpha$ that holds due to the assumption (8). Finally, the right-most inequality in the assumption (19) is equivalent to

$$\frac{r-\rho}{\alpha(1-\gamma)(1-\kappa)} > \frac{(r+\beta)g(T_0)-(1-\ell)}{\alpha g(T_0) - K},$$

which again is equivalent to

$$g(T_0) < \frac{(r-\rho)\frac{K}{\alpha} - (1-\kappa)(1-\gamma)(1-\ell)}{r-\rho-(1-\kappa)(1-\gamma)(r+\beta)} = C$$

as was to be shown.

A.4 Proof of Theorem 4

(a) The claim is obvious since $\alpha \geq 0$ and, due to (17), $\varepsilon(t) \geq 0$.

(b) Assume $\alpha\varepsilon(t) = \beta w(t)$ holds for at least one $t \in [0, T]$. First define

$$\tau \triangleq \min\left\{t \in [0,T] \,:\, \alpha\varepsilon(t) = \beta w(t)\right\}.$$

Note that

$$\frac{d^2 w(t)}{dt^2} = \alpha\frac{d\varepsilon(t)}{dt} - \beta\frac{dw(t)}{dt}.$$

At any point where $\alpha\varepsilon(t) = \beta w(t)$ we have $dw(t)/dt = 0$ and thus $d^2w(t)/dt^2 < 0$ since education is decreasing over time according to Theorem 2. Hence, any local extremum of the wage rate is a local maximum. It follows that the wage growth rate is positive on the interval $[0, \tau]$ and negative on the interval $[\tau, T]$. Furthermore, τ must be the only time point at which the wage growth rate is zero, which implies that the minimum in the definition of τ can be omitted and that the first statement in the claim holds.

A.5 The model of Heckman (1974) considered in Section 7.3

The dynamics of the wage rate are

$$dw(t) = w(t)\mu_w(t)\,dt$$

for some deterministic function μ_w. The agent can continuously adjust the fraction $L(t)$ of time spent on working, which thus generates income at the rate $L(t)w(t)$. Before retirement, the wealth dynamics are therefore

$$dX(t) = rX(t)\,dt + \left(L(t)w(t) - c(t)\right)\,dt,$$

and as before we assume a retirement phase with the wealth dynamics (3) but focus on behavior before retirement. The indirect utility is now defined as

$$\begin{aligned} V(t,x) = \sup_{c,L} \Bigg\{ & \int_t^{T_0} e^{-\rho(s-t)} \frac{1}{1-\gamma} \left(c(s)^{\kappa}(1-L(s))^{1-\kappa}\right)^{1-\gamma}\,ds \\ & + \int_{T_0}^{T} e^{-\rho(s-t)} \frac{1}{1-\gamma} c(s)^{1-\gamma}\,ds \Bigg\}, \quad t \le T_0. \end{aligned} \tag{47}$$

Theorem 5 *The indirect utility can be represented as*

$$V(t,x) = \frac{1}{1-\gamma} f(t)^{\gamma} \left(x + w(t)g(t)\right)^{1-\gamma}, \quad t \le T_0,$$

where the functions f and g are given by

$$\begin{aligned} f(t) &= \tilde{\kappa} \int_t^{T_0} e^{-\tilde{r}(s-t)} w(s)^{\frac{(1-\kappa)(\gamma-1)}{\gamma}}\,ds + \frac{1}{\tilde{r}} e^{-\tilde{r}(T_0-t)} \left(1 - e^{-\tilde{r}(T-T_0)}\right), \\ g(t) &= \int_t^{T_0} e^{-r(s-t)+\int_t^s \mu_w(u)\,du}\,ds + \frac{\chi}{r} e^{-r(T_0-t)+\int_t^{T_0} \mu_w(u)\,du} \left(1 - e^{-r(T-T_0)}\right), \end{aligned}$$

and $\tilde{r}$ and $\tilde{\kappa}$ are given by (10) and (12). The optimal consumption and labor supply are

$$c(t) = \kappa\tilde{\kappa}w(t)^{\frac{(1-\kappa)(\gamma-1)}{\gamma}} \frac{X(t)+g(t)w(t)}{f(t)}, \tag{48}$$

$$L(t) = 1-(1-\kappa)\tilde{\kappa}w(t)^{-\frac{1+\kappa(\gamma-1)}{\gamma}} \frac{X(t)+g(t)w(t)}{f(t)}. \tag{49}$$

The dynamics of optimal consumption are

$$dc(t) = c(t)\mu_c(t)\,dt, \qquad \mu_c(t) = \frac{(1-\kappa)(\gamma-1)}{\gamma}\mu_w(t) + \frac{r-\rho}{\gamma}. \tag{50}$$

Proof: In retirement the problem is identical to that in our main model so we have

$$J(t,x) = \frac{1}{1-\gamma} f(t)^{\gamma} \left(x + w(T_0)g(t)\right)^{1-\gamma}, \quad T_0 \le t \le T,$$
$$f(t) = \frac{1}{\tilde{r}}\left(1-e^{-\tilde{r}(T-t)}\right), \qquad g(t) = \frac{\chi}{r}\left(1-e^{-r(T-t)}\right)$$

with $\tilde{r} = r - \frac{r-\rho}{\gamma}$.

Before retirement the Bellman equation is

$$0 = \sup_{c,L}\left\{V_t + V_x(rx - c + Lw) + \frac{1}{1-\gamma}\left(c^{\kappa}(1-L)^{1-\kappa}\right)^{1-\gamma} - \rho V\right\}. \tag{51}$$

The first-order conditions are

$$V_x = \kappa c^{\kappa(1-\gamma)-1}(1-L)^{(1-\kappa)(1-\gamma)},$$
$$wV_x = (1-\kappa)c^{\kappa(1-\gamma)}(1-L)^{(1-\kappa)(1-\gamma)-1}.$$

Dividing the second condition by the first, we get

$$c = \frac{\kappa}{1-\kappa}w(1-L).$$

Substituting this back into one of the first-order conditions, we obtain

$$c = \kappa^{\frac{1}{\gamma}}\left(\frac{\kappa}{1-\kappa}\right)^{1-\frac{1+\kappa(\gamma-1)}{\gamma}} w^{1-\frac{1+\kappa(\gamma-1)}{\gamma}} V_x^{-\frac{1}{\gamma}} = \kappa\tilde{\kappa}w^{\frac{(1-\kappa)(1-\gamma)}{\gamma}} V_x^{-\frac{1}{\gamma}} \tag{52}$$

and thus

$$1-L = (1-\kappa)\tilde{\kappa}w^{-\frac{1+\kappa(\gamma-1)}{\gamma}} V_x^{-\frac{1}{\gamma}}. \tag{53}$$

After substitution of these optimal controls and subsequent simplifications, the Bellman equation reads

$$0 = V_t + V_x(rx + w) + \frac{\gamma}{1-\gamma}\tilde{\kappa} w^{\frac{(1-\kappa)(\gamma-1)}{\gamma}} V_x^{1-\frac{1}{\gamma}} - \rho V. \tag{54}$$

The conjecture

$$V(t,x) = \frac{1}{1-\gamma} f(t)^{\gamma} \left(x + w(t)g(t)\right)^{1-\gamma}, \quad t \leq T_0,$$

implies

$$\begin{aligned} V_t &= \frac{\gamma}{1-\gamma} f'(t) \left(\frac{x + w(t)g(t)}{f(t)}\right)^{1-\gamma} + \left(\frac{x + w(t)g(t)}{f(t)}\right)^{-\gamma} \left(w'(t)g(t) + w(t)g'(t)\right), \\ V_x &= \left(\frac{x + w(t)g(t)}{f(t)}\right)^{-\gamma}. \end{aligned} \tag{55}$$

Next, we insert these expressions into (54), replace the term $rx+w$ by $r(x+wg)+w(1-rg)$, and collect terms involving $(x+wg)^{1-\gamma}$ as well as the other terms. This leads to the conclusion that $f(t)$ and $g(t)$ must satisfy the differential equations

$$0 = f'(t) - \tilde{r} f(t) + \tilde{\kappa} w(t)^{\frac{(1-\kappa)(\gamma-1)}{\gamma}}, \qquad 0 = g'(t) + \left(\mu_w(t) - r\right) g(t) + 1.$$

The values $f(T_0)$, $g(T_0)$ from the solution after retirement serve as terminal conditions. The solutions are as stated in the theorem. Furthermore, the expressions for the optimal controls presented in the theorem follow from substitution of (55) into (52) and (53).

With the optimal controls, the wealth dynamics become

$$dX(t) = rX(t)\,dt + w(t)\,dt - \tilde{\kappa} w(t)^{\frac{(1-\kappa)(\gamma-1)}{\gamma}} \frac{X(t) + w(t)g(t)}{f(t)}\,dt,$$

and therefore

$$\begin{aligned} d\left(X(t) + w(t)g(t)\right) &= dX(t) + w(t)\left(g'(t) + \mu_w(t)\right) g(t)\,dt \\ &= dX(t) + w(t)\left(rg(t) - 1\right)\,dt \\ &= \left(X(t) + w(t)g(t)\right)\left(r - f(t)^{-1}\tilde{\kappa} w(t)^{\frac{(1-\kappa)(\gamma-1)}{\gamma}}\right)\,dt, \end{aligned}$$

from which it follows that

$$
\begin{aligned}
d\left(\frac{X(t)+w(t)g(t)}{f(t)}\right) &= \frac{1}{f(t)} d\left(X(t)+w(t)g(t)\right) - \frac{f'(t)}{f(t)^2}\left(X(t)+w(t)g(t)\right)\, dt \\
&= \frac{X(t)+w(t)g(t)}{f(t)}\left(r-\tilde{r}\right)\, dt \\
&= \frac{X(t)+w(t)g(t)}{f(t)} \frac{r-\rho}{\gamma}\, dt.
\end{aligned}
$$

Combining this with Eq. (48) for the optimal consumption rate, we conclude that

$$
\begin{aligned}
dc(t) &= c(t)\frac{(1-\kappa)(\gamma-1)}{\gamma}\frac{w'(t)}{w(t)}\, dt + c(t)\frac{r-\rho}{\gamma}\, dt \\
&= c(t)\left[\frac{(1-\kappa)(\gamma-1)}{\gamma}\mu_w(t) + \frac{r-\rho}{\gamma}\right] dt \\
&= c(t)\left[\mu_w(t) + \frac{1}{\gamma}\big(r-\rho-[1+\kappa(\gamma-1)]\mu_w(t)\big)\right] dt
\end{aligned}
$$

as claimed. □

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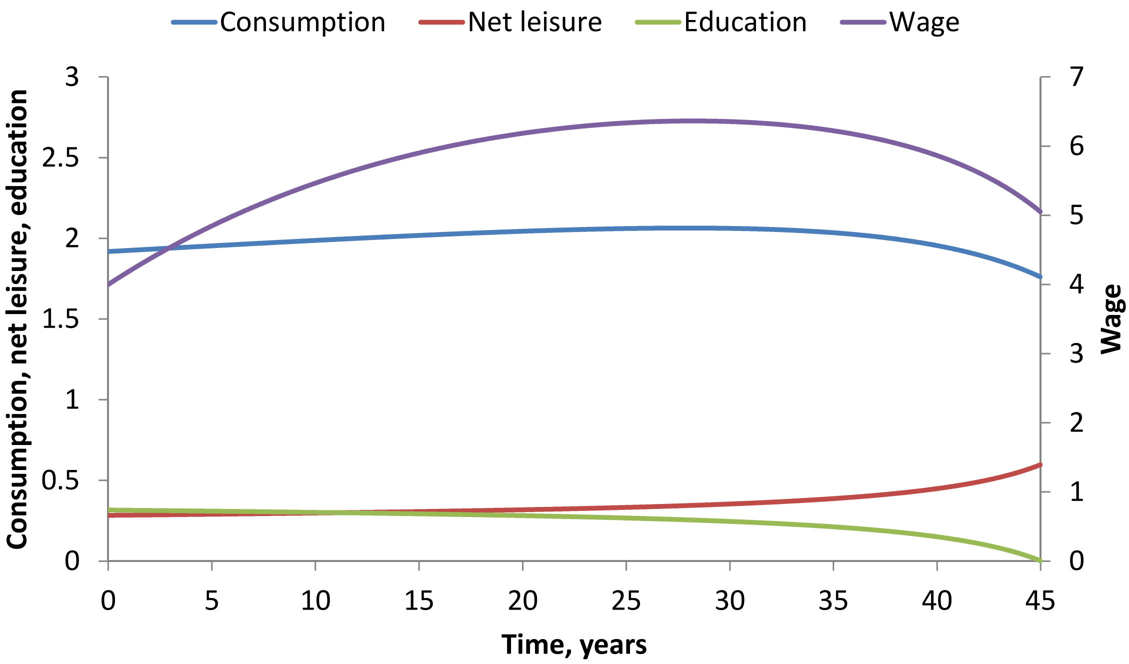

Figure 1: Life-cycle patterns for benchmark parameter values.
The figure depicts how the optimal consumption, education, and net leisure from Theorem 1 as well as the derived wage rate vary over the life of the agent. The benchmark parameter values listed in Table 1 are used. Note that consumption, net leisure, and education are read off the left axis, whereas the wage rate is read off the right axis.

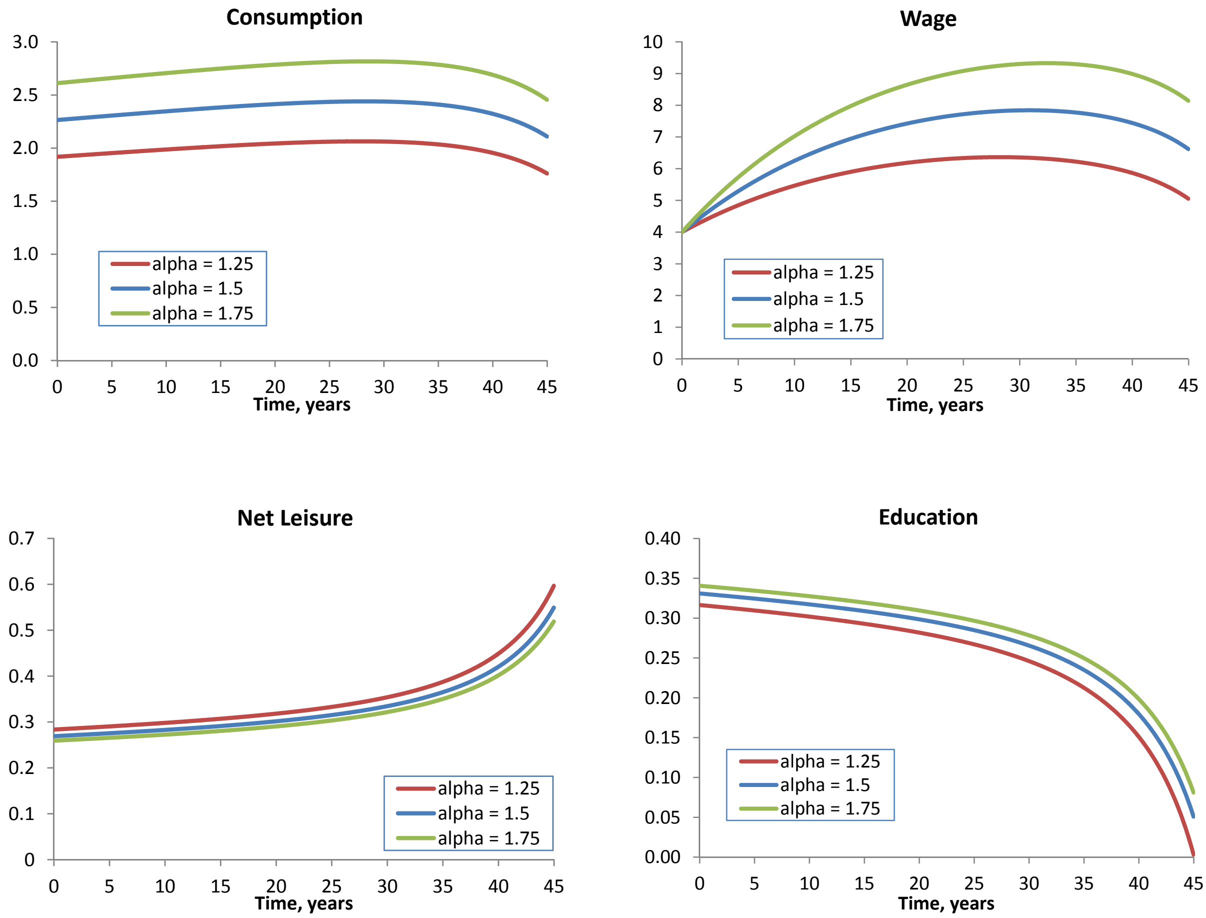

Figure 2: Sensitivity of results to educational efficiency parameter α. The four panels show how consumption, wages, net leisure, and education vary over time for three different values of α. For all other parameters the benchmark values listed in Table 1 are used.

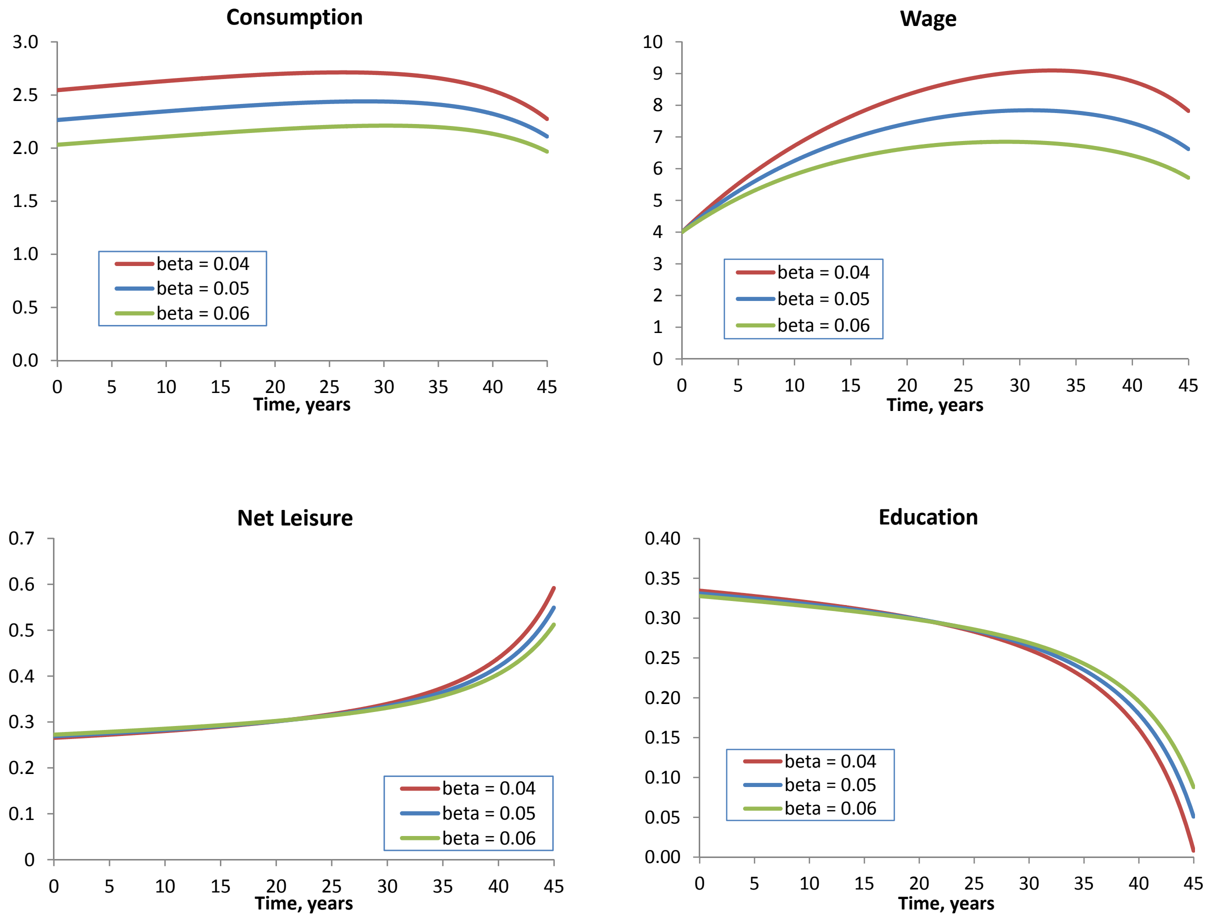

Figure 3: Sensitivity of results to wage depreciation parameter β.
The four panels show how consumption, wages, net leisure, and education vary over time for three different values of β. For all other parameters the benchmark values listed in Table 1 are used.

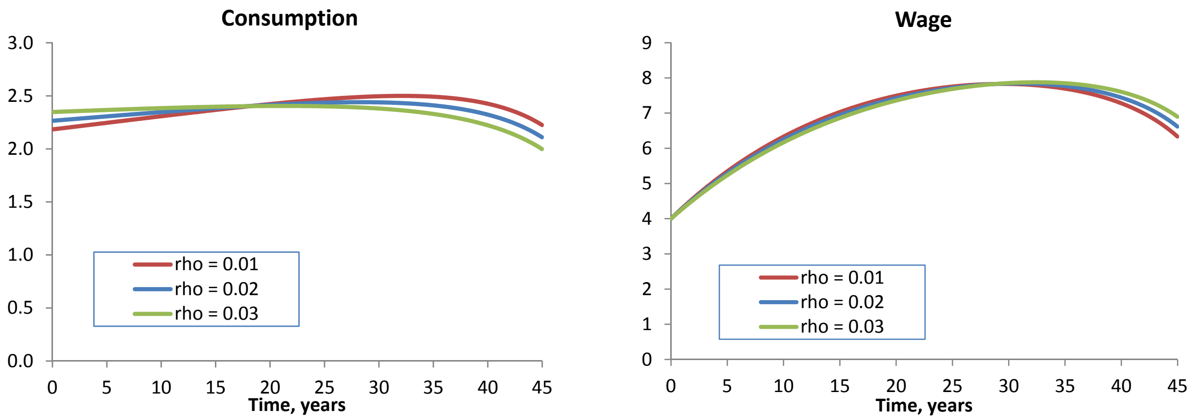

Figure 4: Sensitivity of results to time preference rate ρ.
The two panels show how consumption and wages, net leisure, and education vary over time for three different values of ρ. For all other parameters the benchmark values listed in Table 1 are used.

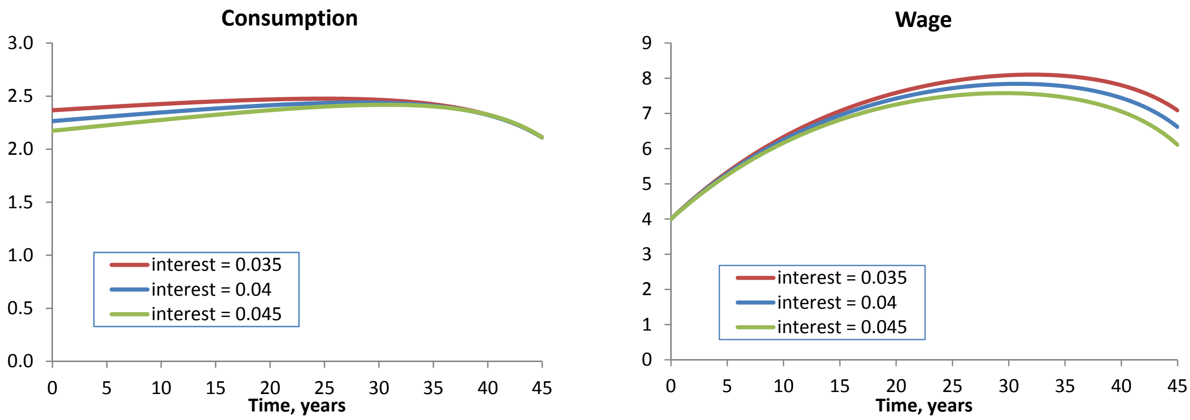

Figure 5: Sensitivity of results to interest rate r.
The two panels show how consumption and wages, net leisure, and education vary over time for three different values of r. For all other parameters the benchmark values listed in Table 1 are used.

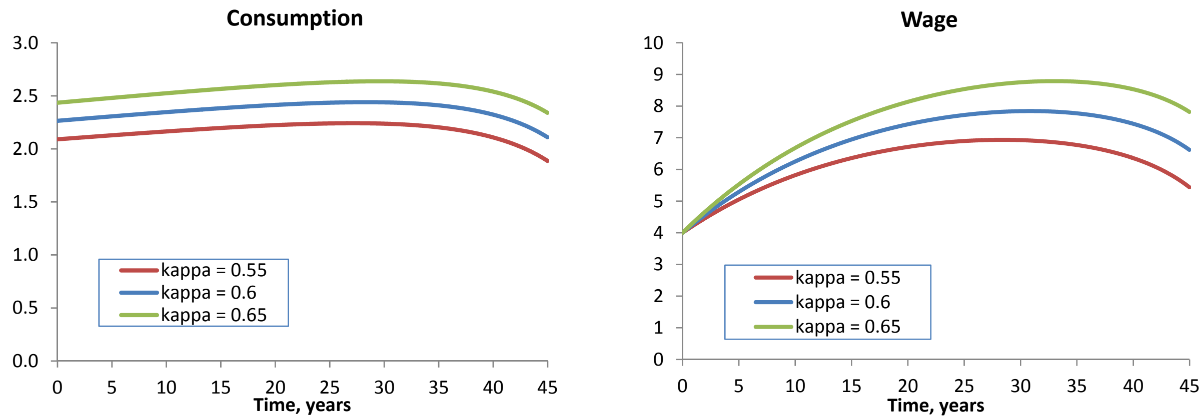

Figure 6: Sensitivity of results to consumption-leisure weight κ.
The two panels show how consumption and wages, net leisure, and education vary over time for three different values of κ. For all other parameters the benchmark values listed in Table 1 are used.

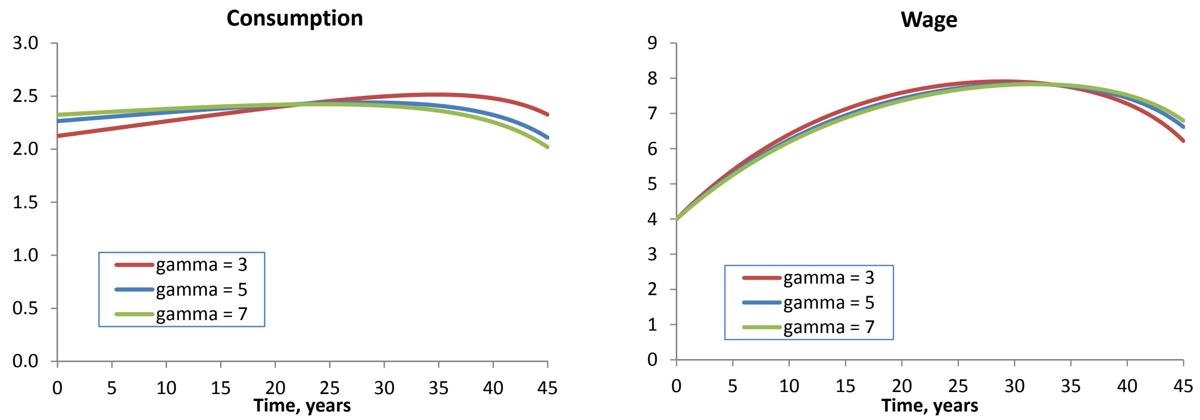

Figure 7: Sensitivity of results to EIS parameter γ.
The two panels show how consumption and wages, net leisure, and education vary over time for three different values of γ. For all other parameters the benchmark values listed in Table 1 are used.

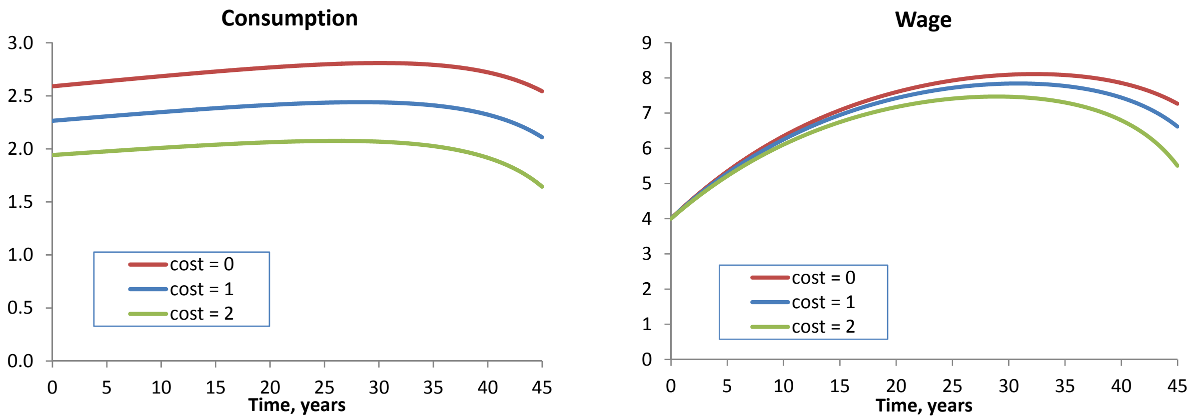

Figure 8: Sensitivity of results to educational cost parameter K.
The two panels show how consumption and wages, net leisure, and education vary over time for three different values of K. For all other parameters the benchmark values listed in Table 1 are used.

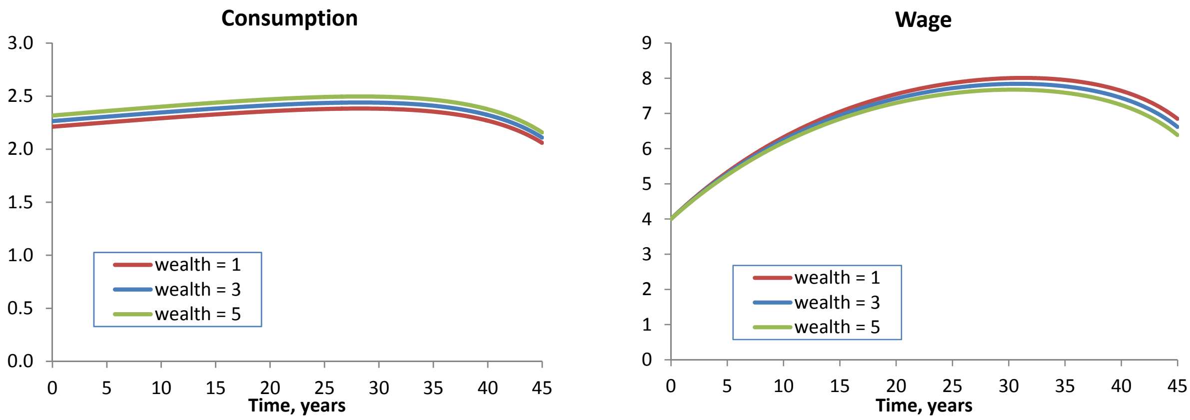

Figure 9: Sensitivity of results to the initial wealth level x.
The two panels show how consumption and wages, net leisure, and education vary over time for three different values of x. For all other parameters the benchmark values listed in Table 1 are used.

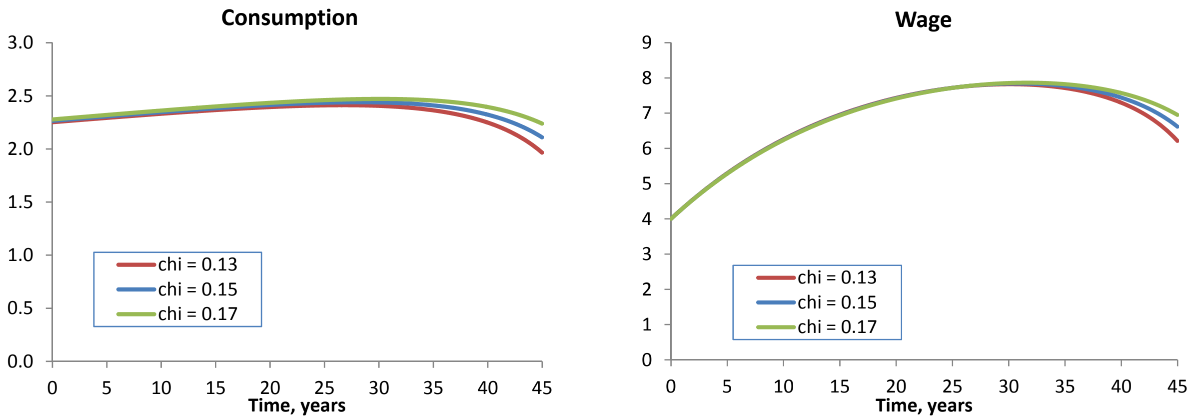

Figure 10: Sensitivity of results to the replacement rate χ.
The two panels show how consumption and wages, net leisure, and education vary over time for three different values of χ. For all other parameters the benchmark values listed in Table 1 are used.

Parameter	Description	Benchmark
T_0	Years until retirement	45
$T - T_0$	Years in retirement	25
ρ	Time preference rate	0.02
γ	Reciprocal of EIS	5
κ	Preference weight	0.6
ℓ	Maximal net leisure	0.6
w	Initial wage level	4
β	Depreciation rate of wages	0.05
α	Efficiency of education	1.5
K	Monetary cost of education	1
χ	Replacement rate	0.15
x	Initial wealth	3
r	Return on savings	0.04

Table 1: Benchmark values of parameters. EIS is short for elasticity of intertemporal substitution.

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