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Fitting Parsimonious Household-Portfolio Models to Data ⋆

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Abstract

US data and new stockholding data from fifteen European countries and China exhibit a common pattern: stockholding shares increase in household income and wealth. Yet, there is a multitude of numbers to match through models. Using a single utility function across households (parsimony), we suggest a strategy for fitting stockholding numbers, while replicating that saving rates increase in wealth, too. The key is introducing subsistence consumption to an Epstein-Zin-Weil utility function, creating endogenous risk-aversion differences across rich and poor. A closed-form solution for the model with insurable labor-income risk serves as calibration guide for numerical simulations with uninsurable labor-income risk.

Keywords: Epstein-Zin-Weil recursive preferences, subsistence consumption, household-portfolio shares, business equity, wealth inequality

JEL classification: G11, D91, D81, D14, D11, E21

1. Introduction

1.1 Data facts: the need for calibration guidance

In Figure 1, Panels A, B, and C show household-portfolio shares of stocks, plotted per household-income category. Across fifteen European-Union (EU) countries, China, and the US, portfolio shares of stocks are increasing in income.[1] This monotonic relationship between stock-portfolio shares and household income seems to be a robust cross-country pattern. A household-portfolio model should be able to generate such a pattern qualitatively. Yet, given the cross-country quantitative differences shown by Figure 1, models should be able to quantitatively replicate stockholding portfolio shares, too. Here we propose, (i) a common-across-households utility function that replicates the monotonic relationship between stock-portfolio shares and household incomes (we introduce subsistence consumption to Epstein-Zin-Weil (EZW) preferences),[2] and (ii) a closed-form solution that serves as calibration guide, by enabling minimum-distance data fitting at low computational cost.

1.2 The modeling and calibration challenge

In order to make a household-portfolio model replicate that stockholding shares are increasing in income/wealth, a promising idea is to create a monotonic relationship between income/wealth and risk aversion (RRA). One approach is to assume exogenously varying RRA coefficients, or varying rates of time preference, i.e., exogenous preference heterogene-

[1] For the US we use data before the subprime crisis of 2008, whereas for EU countries and China we use the only available databases from 2013, that also enable comparisons across EU countries. US household-portfolio data include direct stockholding, mutual funds, and retirement accounts. EU data include only direct stockholding and mutual funds, which is one reason that EU portfolio shares are lower compared to the US. Chinese data include direct stockholding, mutual funds and some other indirect stockholding data, such as Exchange Traded Funds (ETFs). In Online Data Appendices we provide details about our data sources, and database structure and quality.

[2] See Epstein and Zin (1989) and Weil (1989), and for the continuous-time version of recursive preferences that we use in this paper see Duffie ant Epstein, (1992a,b).

ity. Exogenous preference heterogeneity causes exogenous differences in saving rates. Yet, according to evidence, richer households have higher saving rates.[3] So, assuming exogenous preference heterogeneity creates another need: one should also connect exogenous preference parameters with initial conditions, i.e., with who is rich/poor in the data.[4] But models should be able to explain such saving-rate/wealth monotonic relationships instead of assuming them. In addition, assuming that poorer households have higher RRA coefficients creates a tension: on the one hand, the higher risk aversion of the poor could explain the low risky-asset-portfolio shares of the poor; on the other hand, saving rates may decline in income/wealth; higher RRA coefficients amplify precautionary motives that arise from uninsurable labor-income risk, driving the poor to saving more. Household portfolio models should deal with such challenging tensions.

1.3 One utility function for all households: dealing with heterogeneity parsimoniously

Using the same utility function with the same parameter values across all households serves two purposes. First, a single utility function might explain why the poor save less, instead of assuming it. Second, the parsimony of such an explanation can provide insights about the engines of wealth-distribution dynamics. Specifically, parsimonious explanations of qualitative and quantitative patterns in micro data can point at appropriate microfoundations for models which may promote our understanding of taxation and regulation issues regarding

[3] See, for example, Dynan et al. (2004).

[4] For example, Gomes and Michaelides (2005), assume heterogeneity in relative-risk aversion coefficients using Epstein-Zin-Weil preferences, in the context of a life-cycle model, starting from the income/wealth distribution of households with household heads aged 20, and using these initial conditions in order to produce endogenous wealth distributions that may match wealth-distribution data. Carroll (2000) assumes heterogeneous rates of time preference, which leads to differences in saving rates, in a general-equilibrium framework. These saving-rate differences in Carroll (2000) determine who is rich and poor in the long-run, because in the long run there is no dependence on initial conditions. Yet, for life-cycle models the same problem of having to assign different preference parameters to households with different initial conditions remains.

financial markets or the macroeconomy.[5]

As we demonstrate, introducing subsistence consumption to EZW preferences is capable of dealing with data challenges. Subsistence consumption is key to endogenously sorting out rich and poor households according to risk aversion, with the poor being more risk averse. So, the poor hold fewer risky-assets. At the same time, EZW preferences distinguish the elasticity of intertemporal substitution (IES) from RRA. Parameters affecting IES levels, which are also endogenous at the presence of subsistence consumption, give one more degree of freedom. This extra degree of freedom proves useful for replicating the lower saving rates of the poor. In particular, this separation between IES and RRA is important for breaking the tension between stockholding shares and saving rates that the higher risk aversion of the poor creates. The poorer take less risk, but their higher risk aversion is endogenous in our model, so their higher precautionary motives do not dominate and do not lead to higher saving rates. The higher risk aversion of the poor is also shaped by all their future plans: as the model shows, precautionary motives are dominated by optimal wealth-transition plans that poorer households make; they exit their subsistence concerns smoothly in the future through low, yet positively-valued, saving rates.

1.4 What numbers should one assign for subsistence consumption?

Econometric studies such as these of Atkeson and Ogaki (1996) or Donaldson and Pendakur (2006) do not reject the existence of subsistence consumption levels. Yet, issues of econometric model specification affect the robustness of subsistence estimates, making them rather unpopular among applied-theory researchers. Here we rely on estimates from surveys regarding living-standards comparisons across households and we claim that an adult needs

[5] For a review paper about doing quantitative macroeconomics with heterogeneous households see Heathcote, Storesletten, and Violante (2009).

an annual amount of about 3,000 US dollars in order to just survive.[6] Cross-country survey data in Koulovatianos et al. (2007, 2014) indicate annual subsistence costs per person in the order between 1,300-3,600 US dollars.

1.5 Developing calibration tools

We identify a closed-form solution for a version of the model with insurable labor-income risk and multiple risky assets in a continuous-time analysis with infinitely-lived households. This closed-form solution serves as a handy calibration tool, because it enables us to match data using minimum-distance-fitting techniques at low computational cost. We demonstrate the usefulness of this closed-form solution by simultaneously matching both the US stockholding shares of Panel C in Figure 1, and the business-equity shares of Panel D in Figure 1. In addition, using the closed-form solution as intuition guide, through sensitivity analysis, we are able to identify what parameter-value combinations guarantee satisfactory data fitting. The key ingredients for good data matching with two risky assets are: (i) returns of stocks and business equity should be weakly correlated and, (ii) household resources are expected to grow over time, so that poorer households can afford exiting subsistence concerns slowly by saving less and by taking less risk, while holding balanced portfolios.

The closed-form solution also serves as a guide for judging the computational efficiency of numerical algorithms in the absence of insurable labor-income risk. In addition, the calibrating parameters identified when labor-income risk is insurable are in the right ballpark for calibrating the uninsurable-labor-income risk model through trial and error (without using minimum-distance approaches). We also find that the parameters identified by the continuous-time model serve as a guide for calibrating the same model in discrete time. As

[6] Our calibration in this paper refers to US dollars in year 2007. For the survey evidence see Koulovatianos et al. (2007, 2014) who use data in six countries derived using the survey method first suggested by Koulovatianos et al. (2005), and our discussion in the calibration section.

discrete-time modeling is, perhaps, more popular to applied researchers, our study offers modeling pointers for applications and extensions to the research group that uses discrete-time analyses.

1.6 Some literature

Before initiating this research we have been motivated and guided by recent advances in the literature. A recent study by Wachter and Yogo (2010) has made a breakthrough, as it provided reasonable fit of theoretical household-portfolio shares to data. The key idea in Wachter and Yogo (2010) is that they distinguish between two categories of goods, basic goods and luxuries, so the rich invest more in risky assets because they are risking losses in mostly luxury consumption.[7] Similarly, Achury et al. (2012) introduced subsistence consumption into a simple Merton (1969, 1971) model with one type of goods, uncovering a similar mechanism to this of Wachter and Yogo (2010): the poor do not invest in risky assets because they are strongly averse to losing their subsistence consumption. Our study makes use of such building blocks, but pays attention to putting together as many pieces as possible analytically in order to study their interconnection. Specifically, our study provides analytical guidance on how a labor-income process affects portfolio choice and savings, extending Achury et al. (2012).

[7] This idea has been implicit in Browning and Crossley (2000).

2. Model

2.1 Observable budget-constraint characteristics

2.1.1 Income process

Time is continuous. At any instant $t \in [0, \infty)$ a household receives a labor income stream, $y(t)$, that evolves according to the geometric process

$$\frac{dy(t)}{y(t)} = \mu_y dt + \sigma_y dz_y(t) \ , \tag{1}$$

with $\sigma_y > 0$, $\mu_y \geq 0$, with $z_y(t)$ being a Brownian motion, and for a given $y(0) = y_0 > 0$.[8]

2.1.2 Asset returns

The household also possesses an initial stock of financial wealth, $a_0 \in \mathbb{R}$, and has the potential to invest this wealth in a risk-free asset with return r^f, and also in a set of $N \geq 1$ risky assets. The price of risky asset $i \in \{1, ..., N\}$, denoted by $p_i(t)$, is governed by the process

$$\frac{dp_i(t)}{p_i(t)} = R_i dt + \mathbf{e}_i \boldsymbol{\sigma} d\mathbf{z}^T(t) \ , \tag{4}$$

in which $\mathbf{z}(t) \equiv [z_1(t) \quad z_2(t) \quad \cdots \quad z_N(t)]$ is a row vector of Brownian motions with $z_i(t)$ being associated with asset $i \in \{1, ..., N\}$. The $N \times N$ matrix $\boldsymbol{\sigma}$ is derived from the de-

[8] Notice the equivalence between the continuous-time representation in (1) and its discrete-time permanent-income hypothesis counterpart in Carroll (1992, 1997). In particular, Carroll (1992, p. 65) uses a discrete-time stochastic framework in which income, Y_t, following his notation, is governed by $\ln(Y_t) = \ln(P_t) + \ln(V_t)$, $\ln(V_t) \sim N\left(0, \sigma_V^2\right)$, i.i.d. over time, with P_t denoting the permanent-labor-income component which obeys $\ln(P_{t+1}) = \ln(G) + \ln(P_t) + \ln(N_{t+1})$, and in which $\ln(N_t) \sim N\left(0, \sigma_N^2\right)$, i.i.d. over time. Combining these two equations leads to,

$$\ln(Y_{t+1}) - \ln(Y_t) = \ln(G) + \ln(\varepsilon_{t+1}) \ , \tag{2}$$

in which $\ln(\varepsilon_{t+1}) = \ln(N_{t+1}) + \ln(V_{t+1}) - \ln(V_t)$. Given the assumption that $\ln(N_t)$ and $\ln(V_t)$ are independent, which is stated in Carroll(1992, p. 70), it follows that $\ln(\varepsilon_{t+1}) \sim N\left(0, \sigma_N^2 + 2\sigma_V^2\right)$, i.i.d. over time. After applying Itô's Lemma on (1) and stochastically integrating over a time interval $[t, t + \Delta t]$ for all $t \geq 0$ and any $\Delta t \geq 0$, we obtain,

$$\ln[y(t + \Delta t)] - \ln[y(t)] = \left(\mu_y - \frac{\sigma_y^2}{2}\right) \Delta t + \sigma_y [z_y(t + \Delta t) - z_y(t)] \ . \tag{3}$$

Setting $\Delta t = 1$, $\mu_y - \sigma_y^2/2 = \ln(G)$, and $\sigma_y^2 = \sigma_N^2 + 2\sigma_V^2$, makes equations (3) and (2) to coincide.

composition of the covariance matrix, $\boldsymbol{\Sigma}$, which refers to risks of the N risky assets only. In particular, $\boldsymbol{\Sigma} = \boldsymbol{\sigma}\boldsymbol{\sigma}^T$. Finally, $\mathbf{e}_i$ is a $1 \times N$ vector in which the value 1 is in position $i \in \{1, ..., N\}$, while all other elements are zero.

2.1.3 Correlation between labor-income growth and asset returns

Labor income is correlated with risky asset $i \in \{1, ..., N\}$ through the correlation coefficient $\rho_{y,i}$. Specifically,

$$z_y(t) = \sqrt{1 - \rho_{y,1}^2 - ... - \rho_{y.N}^2} z_0(t) + \rho_{y,1} z_1(t) + ... + \rho_{y,N} z_N(t) \ , \tag{5}$$

in which $z_0(t)$ is also a Brownian motion. If $\rho_{y,1}^2 + ... + \rho_{y,N}^2 \neq 1$, then labor-income risk is uninsurable. If, instead, $\rho_{y,1}^2 + ... + \rho_{y,N}^2 = 1$, then labor risk can be eliminated by trading financial assets. Numerical analysis of portfolio choice with multiple risky assets and labor-income risk is a demanding task.[9] In addition, solving complex models numerically may mask some of its key mechanics. So, in order to facilitate the derivation of analytical results for many risky assets without the need to resort to numerical analysis, we use the restriction $\rho_{y,1}^2 + ... + \rho_{y,N}^2 = 1$.

The evolution of assets is governed by the budget constraint,

$$da(t) = \left\{ \left\{ \boldsymbol{\phi}(t) \mathbf{R}^T + \left[1 - \boldsymbol{\phi}(t) \mathbf{1}^T \right] r^f \right\} a(t) + y(t) - c(t) \right\} dt + a(t) \boldsymbol{\phi}(t) \boldsymbol{\sigma} d\mathbf{z}^T(t) \ , \tag{6}$$

in which $\mathbf{R} = [R_1 \ \cdots \ R_N]$ is a row vector containing all mean asset returns and $\boldsymbol{\phi}(t) = [\phi_1(t) \ \cdots \ \phi_N(t)]$ is a row vector containing the chosen fraction of financial wealth invested in risky asset i, for all $i \in \{1, ..., N\}$ at any time $t \geq 0$ ($\mathbf{A}^T$ denotes the transpose of any matrix $\mathbf{A}$). We do not impose any short-selling restrictions on $\boldsymbol{\phi}(t)$.

[9] Yet, our available toolkit for solving discrete-time dynamic portfolio choice problems with many assets and state variables has been recently advanced by Garlappi and Skoulakis (2010).

2.2 Preferences

The problem faced by a household is to maximize its lifetime expected utility subject to constraints (6) and (1). Our utility specification involves a small, yet influential step away from the continuous-time formulation and parameterization of recursive "Epstein-Zin" preferences, suggested by Duffie and Epstein (1992a,b). In particular, we use a subsistence-consumption level χ, defining expected utility as,

$$J(t) = E_t \left[\int_t^\infty f\left(c(\tau)\ ,\ J(\tau)\right) d\tau \right] , \tag{7}$$

in which $f(c, J)$ is a normalized aggregator of continuation utility, J, and current consumption, c, with

$$f(c, J) \equiv \rho(1-\gamma) \cdot J \cdot \frac{\left\{ \frac{c-\chi}{[(1-\gamma)J]^{\frac{1}{1-\gamma}}} \right\}^{1-\frac{1}{\eta}} - 1}{1 - \frac{1}{\eta}} , \tag{8}$$

and in which $\chi \geq 0$ and $\rho, \eta, \gamma > 0$. In Appendix A we show an intuitive result for the case with $\chi > 0$: if $\gamma = 1/\eta$, then expected utility converges to the case of time-separable preferences with hyperbolic-absolute-risk-aversion (HARA) momentary utility.[10] If $\chi = 0$ (standard formulation), then η denotes the household's elasticity of intertemporal substitution and γ is the coefficient of relative risk aversion. In Appendix A we show that the IES is equal to $\eta(1 - \chi/c)$ no matter if $\gamma \neq 1/\eta$ or not. So, in case $\chi > 0$, which is central to our analysis, parameter η sets the upper bound of IES (recall that $c \geq \chi$) and plays the role of the IES only asymptotically, as $c \to \infty$.

[10]Specifically, in Appendix A we show that $f(c, J)\,|_{\gamma=1/\eta}$ implies that continuation utility is

$$J(t) = \rho E_t \left\{ \int_t^\infty e^{-\rho(\tau-t)} \frac{[c(\tau) - \chi]^{1-\frac{1}{\eta}} - 1}{1 - \frac{1}{\eta}} d\tau \right\} . \tag{9}$$

Notice that Koo (1998) has provided theoretical analysis to a model that is similar to ours but he has restricted his attention to the consant-relative-risk aversion utility function given by (9) after setting $\chi = 0$. Other notable analyses with time0separable preferences are Duffie et al. (1997) and Henderson (2005).

2.3 (Closed-form) Solution

In equilibrium, continuation utility, $J^*(t)$, is a value function depending on the household's assets and labor income, so $J^*(t) = V(a(t), y(t))$ for all $t \geq 0$. With infinitely-lived households and constraints with time-invariant state-space representation, the optimization problem of the households falls in the category of stationary discounted dynamic programming. So, the time index is dropped from the Hamilton-Jacobi-Bellman equation (HJB) which is given by,

$$0 = \max_{c \geq \chi, \boldsymbol{\phi}} \left\{ f(c, V(a,y)) + \left\{ \left[\boldsymbol{\phi}\mathbf{R}^T + \left(1 - \boldsymbol{\phi}\mathbf{1}^T\right) r_f\right] a + y - c \right\} \cdot V_a(a,y) \right.$$
$$+ \frac{1}{2} a^2 \boldsymbol{\phi}\boldsymbol{\sigma}\boldsymbol{\sigma}^T\boldsymbol{\phi}^T \cdot V_{aa}(a,y) + \mu_y y \cdot V_y(a,y)$$
$$\left. + \frac{1}{2}(\sigma_y y)^2 \cdot V_{yy}(a,y) + \sigma_y a y \boldsymbol{\phi}\boldsymbol{\sigma}\boldsymbol{\rho}_y^T \cdot V_{ay}(a,y) \right\}, \tag{10}$$

in which V_x denotes the first partial derivative with respect to variable $x \in \{a, y\}$, V_{xx} is the second partial derivative with respect to x, the notation for the cross-derivative is obvious, and $\boldsymbol{\rho}_y = \begin{bmatrix} \rho_{y,1} & \cdots & \rho_{y,N} \end{bmatrix}$ is a row vector containing all correlation coefficients between each of asset returns and the income process. Finally, r_f denotes the return of investment in the risk-free asset.

The first-order conditions of the problem expressed by (10) are,

$$f_c(c, V(a,y)) = V_a(a,y), \tag{11}$$

$$\boldsymbol{\phi}^T = \left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(\mathbf{R}^T - r_f\mathbf{1}^T\right)\frac{V_a(a,y)}{-a \cdot V_{aa}(a,y)} - \sigma_y \frac{y}{a}\left(\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\right)^T \frac{V_{ay}(a,y)}{V_{aa}(a,y)}. \tag{12}$$

We make two technical assumptions that enable us to secure interiority of solutions and analytical tractability. The rationale behind these assumptions becomes more obvious after the statement of Proposition 1, so we provide intuition at a later point.

Assumption 1 *Initial conditions are restricted so that,*

$$a_0 + \frac{y_0}{r_y} > \frac{\chi}{r_f} \ ,$$

with

$$r_y \equiv r_f - \mu_y + \sigma_y \left(\mathbf{R} - r_f \mathbf{1}\right) \left(\boldsymbol{\rho}_y \boldsymbol{\sigma}^{-1}\right)^T$$

Assumption 2 *The parameter restriction,*

$$\frac{1}{\eta} > 1 - \frac{\rho}{r_f + \frac{\nu}{2\gamma}} \ ,$$

in which,

$$\nu \equiv \left(\mathbf{R} - r_f \mathbf{1}\right) \left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1} \left(\mathbf{R}^T - r_f \mathbf{1}^T\right) \ ,$$

holds.

Proposition 1 provides the formulas of the analytical solution to the model.

Proposition 1

If $\rho_{y,1}^2 + ... + \rho_{y,N}^2 = 1$*, short selling is allowed, and Assumptions 1 and 2 hold, the solution to the problem expressed by the HJB equation given by (10) is a decision rule for portfolio choice,*

$$\boldsymbol{\phi}^* = \boldsymbol{\Phi}(a, y) = \frac{1}{\gamma}\left(\mathbf{R} - r^f \mathbf{1}\right) \left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1} \left(1 - \frac{\frac{\chi}{r_f}}{a}\right) + \left[\frac{1}{\gamma}\left(\mathbf{R} - r_f \mathbf{1}\right) \left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1} - \sigma_y \boldsymbol{\rho}_y \boldsymbol{\sigma}^{-1}\right] \frac{\frac{y}{r_y}}{a} \ , \quad (13)$$

and a decision rule for consumption,

$$c^* = C(a, y) = \xi\left(a + \frac{y}{r_y} - \frac{\chi}{r_f}\right) + \chi \ , \quad (14)$$

in which

$$\xi = \rho\eta + (1-\eta) r_f - \frac{(\eta-1)}{2\gamma}\nu \ , \tag{15}$$

while the value function is given by,

$$V(a,y) = \rho^{-\eta\frac{1-\gamma}{1-\eta}} \xi^{\frac{1-\gamma}{1-\eta}} \frac{\left(a + \frac{y}{r_y} - \frac{\chi}{r_f}\right)^{1-\gamma}}{1-\gamma} \ .$$

Proof See Appendix B. □

The term $y(t)/r_y$ is the present value of expected lifetime labor earnings at time $t \geq 0$.[11] So, the sum $(a + y/r_y)$ equals the present value of total expected lifetime resources. The term χ/r_f is the present value of lifetime subsistence needs which uses the risk-free rate as its discount factor.[12] In light of these observations, the term $(a + y/r_y - \chi/r_f)$ equals the discretionary expected lifetime resources.

The decision rule of consumption, (14), is an affine function of discretionary resources, $(a + y/r_y - \chi/r_f)$, with its gradient, ξ, influenced by risk aversion, which is driven by parameter γ. In particular, if parameter η is lower than 1 (i.e., $IES = \eta(1 - \chi/c) < 1$), then a

[11]Since labor income is insurable, the effective discount factor, r_y, which is used to calculate the present value of expected lifetime labor earnings, involves three opportunity-cost ingredients. These ingredients are the risk-free rate, r_f, the trend of income, μ_y, and a term involving the excess returns and risks of other assets, $(\mathbf{R} - r_f\mathbf{1})(\boldsymbol{\sigma}^{-1})^T$. In addition, $r_y = r_f - \mu_y + \sigma_y(\mathbf{R} - r_f\mathbf{1})(\boldsymbol{\sigma}^{-1})^T \boldsymbol{\rho}_y^T$, takes into account the correlations of income with the risky assets, $\boldsymbol{\rho}_y$, and income volatility, σ_y. In particular, notice that $y(t) = y_0 \cdot e^{\mu_y t + \sigma_y z_y(t)}$ (see equation (1)) while equation (5) combined with the condition $\boldsymbol{\rho}_y\boldsymbol{\rho}_y^T = 1$ gives $z_y(t) = \boldsymbol{\rho}_y \cdot \mathbf{z}^T(t)$.

[12]To see why the discount factor of lifetime subsistence needs is the risk-free rate alone, consider the special case of a household with minimum assets, $\underline{a}$, such that $\underline{a} + y/r_y = \chi/r_f$, i.e. total expected lifetime resources equal subsistence needs (in slight violation of Assumption 1). In this special case, equation (13) implies that the household holds a portfolio of risky assets, $\boldsymbol{\phi}^* \cdot \underline{a} = -\sigma_y y/r_y \boldsymbol{\rho}_y \boldsymbol{\sigma}^{-1}$ which enables it to perfectly insure against labor-income risk. In this way, the equilibrium consumption profile of such a household is $c^*(t) = \chi$ for all $t \geq 0$. So, the ability to insure against labor-income risk enables the household to avoid consumption fluctuations and to meet the condition $c(t) \geq \chi$ with equality at all times. Since this special household does not have any opportunity left for fluctuations in total income through its savings behavior (its total income is equal to χ for all $t \geq 0$), its intertemporal opportunity cost is determined solely by the risk-free rate r_f.

higher level of γ reduces the propensity to consume, ξ, creating incentives for precautionary savings. Yet, the impact of an increase in risk aversion on the saving rate is not unambiguous. Risk aversion affects the optimal portfolio composition and hence the expected asset income of a household. In the following section we elaborate on the characteristics of the saving rate.

2.4 Characterizing the saving rate

The saving rate is a function of (a, y), it is denoted by $s(a, y)$, and is given by

$$s(a, y) = 1 - \frac{C(a, y)}{I(a, y)} ,$$

in which $C(a, y)$ is given by (14) and $I(a, y)$ is a household's total income, subject to its optimal portfolio-choice vector dictated by the decision rule $\mathbf{\Phi}(a, y)$ in (13). After some algebra we obtain,

$$s(a, y) = \frac{\left[\eta\left(r_f - \rho\right) + \frac{\eta+1}{2}\frac{\nu}{\gamma}\right]\left(a + \frac{y}{r_y} - \frac{\chi}{r_f}\right) - \mu_y \frac{y}{r_y}}{\left(\frac{\nu}{\gamma} + r_f\right)\left(a + \frac{y}{r_y} - \frac{\chi}{r_f}\right) + \chi - \mu_y \frac{y}{r_y}} . \tag{16}$$

Although equation (16) gives a closed form, it is still challenging to distinguish the dependence of the saving rate on total asset holdings, a, or on current income, y. One of the sources of complexity is the presence of subsistence consumption, χ. Yet, the introduction of χ in our model is crucial for our quantitative exploration.

2.4.1 How subsistence consumption pushes the saving rate towards being increasing in wealth

Households may save resources in order to be well above the level of lifetime subsistence needs, χ/r_f. A previous study indicates that the optimal transition of poorer households away from subsistence needs is slow, since the poor have lower saving rates (see Achury et al. (2012, Corollary 1, p. 113) which studies a model nested by our present framework for $\mu_y = 0$ and $\gamma = 1/\eta$).[13] With $\mu_y \neq 0$ the implied income trend affects incentives to save, since income growth exogenously shifts the resource constraint over time (unlike a which is endogenously determined). So, in order to examine how subsistence, χ, affects the dependence of the saving rates on a and y, we distinguish between two cases, $\mu_y = 0$ (nongrowing labor income) and $\mu_y \neq 0$ (growing labor income).

No expected income growth ($\mu_y = 0$) When $\mu_y = 0$, in Appendix C we show that a key parametric constraint in order to secure that the saving rate is strictly positive is,

$$s\left(a,y\right)|_{\left(\chi\geq 0,\mu_y=0\right)} > 0 \Leftrightarrow \frac{1}{\eta} > \frac{r_f + \frac{\nu}{2\gamma} - \rho}{r_f + \frac{\nu}{2\gamma}} > \frac{1}{\eta}\frac{-\frac{\nu}{2\gamma}}{r_f + \frac{\nu}{2\gamma}} \ . \tag{17}$$

In case $r_f + \nu/\left(2\gamma\right) > \rho$, Assumption 2 implies that the IES is smaller than an upper threshold determined by the rest of the model's parameters. This upper bound on the IES blocks the willingness to substitute consumption over time, preventing the possibility that households would seek corner solutions, and thus guaranteeing $c^*\left(t\right) > \chi$ for all $t \geq 0$.[14] The positive saving rate is also a result of a relatively low rate of time preference, ρ, which is another aspect taken care of by condition (17).

Non-zero expected income growth ($\mu_y \neq 0$) Focusing on the empirically plausible case of $\mu_y > 0$, after some algebra appearing in Appendix C, we identify a parametric restriction which guarantees that the saving rate is always increasing in wealth ($s_a\left(a,y\right) > 0$), even in the case of $\chi = 0$, given by,

$$s_a\left(a,y\right) > 0 \Leftrightarrow \chi\frac{\eta\left(r_f - \rho\right) + \frac{\eta+1}{2}\frac{\nu}{\gamma}}{\frac{\nu}{\gamma} + r_f} + \mu_y\xi\frac{\frac{y}{r_y}}{\left(\frac{\nu}{\gamma} + r_f\right)^2} > 0 \ . \tag{18}$$

[13] Achury et al. (2012) study a Merton (1969, 1971) model with additively-separable HARA preferences and no labor income.

[14] In this case of $r_f + \nu/\left(2\gamma\right) > \rho$, condition (17) is also automatically guaranteed, and a positive saving rate is guaranteed while $\mu_y = 0$.

The dependence of the saving rate on income, y, as initial condition (the sign of $s_y(a, y)$), is more cumbersome to understand. In Appendix C we provide some step-by-step qualitative analysis of (16), under the parametric restriction given by (18). Table 1 summarizes all results.

2.4.2 Key parametric constraints that make saving rates be increasing in income/wealth

In a strictly mathematical sense, the conclusions summarized by Table 1 refer to the closed-form solution case of insurable labor-income risk ($\boldsymbol{\rho}_y \boldsymbol{\rho}_y^T = 1$). Yet, Table 1 may serve as a calibration guide if labor-income risk is uninsurable ($\boldsymbol{\rho}_y \boldsymbol{\rho}_y^T \neq 1$). Our numerical simulation exercises below, indicate that Table 1 is helpful even if $\boldsymbol{\rho}_y \boldsymbol{\rho}_y^T \neq 1$. Specifically, the parametric restriction given by (18) enters our constraints in our minimum-distance search for best-fitting parameters to data under $\boldsymbol{\rho}_y \boldsymbol{\rho}_y^T = 1$ (insurability). When we introduce $\boldsymbol{\rho}_y \boldsymbol{\rho}_y^T \neq 1$ (uninsurability), most previous parameters remain unchanged.

sign of $s_a(a, y)$

	$\mu_y = 0$	$\mu_y > 0$
$\chi = 0$	0	+
$\chi > 0$	+	+

How the saving rate depends on wealth

sign of $s_y(a, y)$

	$\mu_y = 0$	$\mu_y > 0$
$\chi = 0$	0	$-$ if $a > 0$ and + if $a < 0$
$\chi > 0$	+	ambiguous

How the saving rate depends on income

Table 1 Dependence of the saving rate on wealth and income under restriction (18)

The way parametric restriction (18) relates to the results summarized by Table 1 can be verbally summarized as: *if household resources are expected to grow over time, then poorer households can afford exiting subsistence concerns slowly by saving at a lower rate, and they optimally choose to do so.* Finally, the parametric restriction given by (18) reveals the importance of the separation between parameters η and γ in the context of EZW preferences: *the presence of η gives us enough degrees of freedom to meet parametric restrictions (17) and (18) easily.* Specifically, if calibrating parameters overemphasize precautionary motives in a model, then the model may imply saving rates which are decreasing in wealth. Guided by (17) and (18) so as to guarantee that $s(a,y), s_a(a,y) > 0$, one can restore the empirically plausible pattern of saving rates which are increasing in wealth.

2.5 Characterizing portfolio composition in the case of two risky assets

The most complicated analytical aspect of determining the dependence of portfolio shares, $\boldsymbol{\phi}$, on total asset holdings, a, and income, y, is the role played by the covariance matrix of risky assets. In the case of two risky assets ($N = 2$), the covariance matrix is,

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_s^2 & \rho_{s,b}\sigma_s\sigma_b \\ \rho_{s,b}\sigma_s\sigma_b & \sigma_b^2 \end{bmatrix},$$

in which σ_i is the standard deviation of asset $i \in \{s, b\}$, with subscript "s" denoting "stocks" and subscript "b" denoting "business equity", while $\rho_{i,j}$ denotes the correlation coefficient between two risky assets $i, j \in \{s, b\}$. The stochastic structure of the problem with $N = 2$ involves three volatility parameters, σ_s, σ_b, and σ_y, and two correlation coefficients, $\rho_{s,b}$ and $\rho_{y,s}$, since correlation $\rho_{y,b}$ can be deduced from the labor-risk-insurability constraint $\rho_{y,s}^2 + \rho_{y,b}^2 = 1$.

In Appendix A we show that the solution based on (13) for $N = 2$ is,[15]

$$\phi_s^* = \frac{1}{\gamma}\cdot\frac{1}{1-\rho_{s,b}^2}\cdot\frac{\frac{R_s-r_f}{\sigma_s}-\rho_{s,b}\frac{R_b-r_f}{\sigma_b}}{\sigma_s}\cdot\left(1-\frac{\frac{\chi}{r_f}}{a}\right)$$
$$+\left[\frac{1}{\gamma}\cdot\frac{1}{1-\rho_{s,b}^2}\cdot\frac{\frac{R_s-r_f}{\sigma_s}-\rho_{s,b}\frac{R_b-r_f}{\sigma_b}}{\sigma_s}-\sigma_y\left(\frac{\rho_{y,s}}{\sigma_s}-\frac{\sqrt{1-\rho_{y,s}^2}\cdot\rho_{s,b}}{\sigma_s\sqrt{1-\rho_{s,b}^2}}\right)\right]\frac{\frac{y}{r_y}}{a}\,, \tag{20}$$

and

$$\phi_b^* = \frac{1}{\gamma}\cdot\frac{1}{1-\rho_{s,b}^2}\cdot\frac{\frac{R_b-r_f}{\sigma_b}-\rho_{s,b}\frac{R_s-r_f}{\sigma_s}}{\sigma_b}\cdot\left(1-\frac{\frac{\chi}{r_f}}{a}\right)$$
$$+\left[\frac{1}{\gamma}\cdot\frac{1}{1-\rho_{s,b}^2}\cdot\frac{\frac{R_b-r_f}{\sigma_b}-\rho_{s,b}\frac{R_s-r_f}{\sigma_s}}{\sigma_b}-\sigma_y\cdot\frac{\sqrt{1-\rho_{y,s}^2}}{\sigma_b\sqrt{1-\rho_{s,b}^2}}\right]\frac{\frac{y}{r_y}}{a}\,. \tag{21}$$

The first observation about equations (20) and (21) is that parameter η, which is tightly linked with the IES does not affect the composition of portfolios. On the contrary, an increase in the relative-risk aversion coefficient γ influences the optimal portfolio share of each risky asset. In particular, the comparison between the ratio of the two Sharpe ratios with the correlation coefficient between asset returns (i.e., how $\rho_{s,b}$ compares to $[(R_i - r_f)/\sigma_i]/[(R_j - r_f)/\sigma_j]$, $i, j \in \{s, b\}$ with $i \neq j$) determines whether an increase in γ leads to a decrease in both ϕ_s^* and ϕ_b^*, or in an increase in one of the two and in a reduction for the other.[16]

The dependence of ϕ_s^* and ϕ_b^*, on assets, a, and income, y, hinges upon a number of parameter combinations. If $\rho_{s,b} < [(R_i - r_f)/\sigma_i]/[(R_j - r_f)/\sigma_j]$, $i, j \in \{1, 2\}$ with $i \neq j$,

[15] In Appendix A we also show that the magnitude of the discount factor used to calculate the present value of lifetime labor income equals,

$$r_y = r_f - \mu_y + \sigma_y\left[\frac{R_s-r_f}{\sigma_s}\cdot\left(\rho_{y,s}-\frac{\sqrt{1-\rho_{y,s}^2}\cdot\rho_{s,b}}{\sqrt{1-\rho_{s,b}^2}}\right)+\frac{R_b-r_f}{\sigma_b}\cdot\frac{\rho_{y,b}}{\sqrt{1-\rho_{s,b}^2}}\right]. \tag{19}$$

Equation (19) reveals that apart from r_f, μ_y, and σ_y, a linear relationship between the Sharpe ratios weighted by expressions involving the correlation coefficients $\rho_{y,s}$ and $\rho_{s,b}$ plays a key role in determining the magnitude of r_y which critically affects the level of lifetime labor income y/r_y.

[16] Notice that since $\rho_{s,b} < 1$ it cannot be that an increase in γ causes both ϕ_s^* and ϕ_b^* to rise.

then the first term of (20) and (21) contributes to making ϕ_s^* and ϕ_b^* increasing in a, as long as $\chi > 0$. So, the presence of subsistence consumption, $\chi > 0$ contributes to having portfolio shares of risky assets that are increasing in wealth, in accordance with what the data say. Nevertheless, the second term introduces a separate role for the income/wealth ratio y/a in generating portfolio shares which are increasing in wealth. This role of y/a depends on a more complicated relationship among parameters related to asset returns, their covariance matrix, and the correlation of risky asset with labor income shocks. Yet, equations (20) and (21) provide a useful pointer towards a successful calibration exercise for the $N = 2$ case: two key ingredients in order to match that portfolio-shares are increasing in wealth or income in the data are, (a) a positive level of subsistence consumption, $\chi > 0$, and (b) a low correlation coefficient between the two risky assets, especially one that guarantees $\rho_{s,b} < [(R_i - r_f)/\sigma_i] / [(R_j - r_f)/\sigma_j]$, $i, j \in \{s, b\}$ with $i \neq j$. The simulation exercise demonstrates the quantitative importance of these two key ingredients.

3. A challenging calibration exercise: two risky assets

3.1 Benchmark calibration via minimum-distance fitting

Table 2 provides all calibrating parameters in the case of insurable labor-income risk ($\boldsymbol{\rho}_y \boldsymbol{\rho}_y^T = 1$). Setting labor-income risk, σ_y, equal to 8.21%, is within the ballpark of a standard parametrization motivated by micro data (see, for example, Gomes and Michaelides (2003 p. 736) for details). We also set the mean labor-income growth to 1.15%. Another standard parametrization is setting stock returns and their volatility close to their long-term values for R_s and σ_s of 7.56% and 21% (the corresponding values in Guvenen (2009) are 8% and 20%, while Gomes and Michaelides (2003) use 6% and 18%). Our calibration exercise worked better by giving the risk-free rate, r_f, the rather generous 3.56%, compared to the standard value close to 2% (see, for example, Gomes and Michaelides (2003) and Guvenen (2009)). While our implied equity premium is rather low (4%), it is not uncommon in the household-finance literature to consider such values. For example, an equity premium of 2.5% is within the range of values examined by Gomes and Michaelides (2003).

Preference Parameters

ρ	η	γ	χ
2.5%	0.08	4.78	2940[a]

Mean Returns

r_f	μ_y	R_s	R_b
3.56%	1.15%	7.56%	18%

Standard Deviations of Returns and Correlations

σ_y	σ_s	σ_b	ρ_{ys}	ρ_{yb}	ρ_{sb}
8.21%	21%	42.07%	48.93%	87.21%	1.74%

Table 2 Calibrating Parameters. [a]Annual subsistence cost per person in 2007 US Dollars.

Our preference parameters are close to the choices made by Achury et al. (2012), with the sole difference that the monthly subsistence consumption per person is USD 245 versus USD 230 in Achury et al. (2012). Nevertheless, the monthly amount of USD 245 is within the range of survey evidence about subsistence consumption reported by Koulovatianos et al. (2007, 2014), i.e. between USD 111 and 302.

After fixing the values of all parameters above, we performed minimum-distance-fitting of equations (20) and (21), using admissible ranges of all remaining parameter values, in order to best fit the model to the data.[17] The resulting fitted risky-asset shares of our

[17]We have also used the parametric restriction given by (18). Regarding the definition of after-tax incomes, income-tax calculations are based on data taken from the Federation of Tax Administrators at 444 N. Capital Street, Washington DC, projected from year 2003. See the Online Data Appendix for details on the tax rates and also Grant et al. (2010, Table 2).

benchmark calibration is given by Panels A and B in Figure 2. While the share of business equity seems imperfectly matched, simulated patterns of portfolio shares are both increasing in income/wealth. The span of simulated business-equity shares for all income/wealth categories is satisfactorily close to the span indicated by the data, showing promise for future work. Notably, we have excellent data fit for the stockholding portfolio data. The minimum-distance exercise implied a number of parameters for business equity that best match the data. Most interesting and robust are the implications that the mean and standard deviation of business-equity returns, R_b and σ_b, are 18% and 42.07%. The value $R_b = 18\%$ is not far from the average estimates in Moskowitz and Vissing-Jorgensen (2002, Table 4, p. 756). Regarding our model's implication that $\sigma_b = 42.07\%$, Moskowitz and Vissing-Jorgensen (2002, p. 765) mention: "*[...] the annual standard deviation of the smallest decile of public firm returns is 41.1 percent. A portfolio of even smaller private firms is likely to be as volatile.*" It can be difficult to estimate idiosyncratic risks borne by a household. Unobservable limitations in outside options, such as frictions in relocating business if other family incomes could increase by relocating, imperfect insurance from theft, etc., may justify that a value for σ_b in the order of 40% may still be low.

Regarding the correlation between labor income shocks and stock returns, ρ_{ys}, Gomes and Michaelides (2003 p. 736) suggest an educated value of 30%, but try higher values, too. Our implied value for ρ_{ys} is 48.93%, which immediately implies that $\rho_{yb} = \sqrt{1-(48.93\%)^2} = 87.21\%$, due to the parametric restriction $\rho_{y,s}^2 + \rho_{y,b}^2 = 1$, a key condition for obtaining closed-form solutions. The high correlation between business equity and family income may be plausible as a large fraction of households have family businesses and tend to employ family members or the owners themselves. Given equations (20) and (21) that we have derived above, we paid attention to the implied Sharpe ratios and concluded that an admissible and appropriate value for the crucial correlation between stock returns and business-equity returns, ρ_{sb}, is 1.74%.

In brief, our analysis suggests that data can be matched satisfactorily provided that business-equity returns are highly volatile and weakly correlated with stock returns. As idiosyncratic components of business-equity risk can be substantial, these implications seem plausible. Importantly, Panel C of Figure 2 shows that the model replicates saving rates across the rich and the poor within the ranges suggested by Dynan et al. (2004).

3.2 Sensitivity Analysis

Panels A and B of Figure 3 and Panel A of Figure 4 show the impact of changing the values of the subsistence parameter, χ, on risky-asset shares and the saving rate. A $12-18\%$ deviation above or below the benchmark value of USD 245 per month does not change fitted values of portfolio shares and saving rates substantially. Yet, a crucial exercise is to see the impact of discarding subsistence consumption from the model, through setting $\chi = 0$. Panels A and B of Figure 3 and Panel A of Figure 4 show that portfolio shares and saving rates become almost flat across the rich and the poor. The U-shaped part of the saving rate that arises in Panel A of Figure 4 is due to the cross-sectional pattern of the income-to-asset ratio, y/a, across the rich and the poor in the data, that is presented by Panel B of Figure 4. That y/a has an impact on the saving rate and portfolio shares is obvious from equations (16), (20), and (21).[18]

Another sensitivity-analysis exercise we preform focuses on changing the magnitude of the correlation coefficient between stock and business-equity returns, ρ_{sb}. Panels C and D of Figure 3 and Panel C of Figure 4 show that slight changes in this correlation coefficient

[18] In Appendix C, an algebraic manipulation of (16) makes the dependence of the saving rate on a/y more obvious (see equation (57) in Appendix C).

can have substantial impact on the portfolio shares and the saving rates. In other words, the benchmark value $\rho_{sb} = 1.74\%$ seems to be a sharp implication of the model: *returns of stocks and business equity should be weakly correlated, so as to ensure balanced portfolios across the rich and the poor.*

Finally, in Panel D of Figure 4 we vary the elasticity of intertemporal substitution, η, and observe the impact of these changes on the saving rate. We remind that η affects the saving rate alone and not the household portfolio shares at all, as (57), (20), and (21) reveal. Unsurprisingly, small changes in η affect the saving rates substantially. In our calibration, the shifts in the saving-rate pattern seem almost parallel, and also the implied saving-rate pattern under the constraint $\eta = 1/\gamma$ (time-separable preferences) is not quantitatively implausible. As the value of η leaves portfolio-shares unaffected (see equations (20) and (21)), it is notable that the assumption of Epstein-Zin-Weil recursive preferences is not crucial for the goodness of fit of our portfolio shares. Nevertheless, Epstein-Zin-Weil recursive preferences provide a valuable degree of freedom that may prove crucial for key extensions such as the introduction of tight liquidity constraints in a more descriptive and complicated version of the model.

4. Breaking the assumption that labor-income risk is insurable

The key assumption behind the uninsurability of labor-income risk is setting parameters so that,

$$\rho_{y,1}^2 + ... + \rho_{y,N}^2 = 1 \ . \tag{22}$$

If there is only one risky asset ($N = 1$), then (22) becomes $\rho_{y,1}^2 = 1$, i.e. labor-income risk is insurable if $\rho_{y,1} \in \{-1, 1\}$. So, in the case of one risky asset, letting $\rho_{y,1}$ depart from values -1 or 1, increases the degree of labor-income-risk uninsurability. Due to this one-dimensional representation/quantification of uninsurability, in this section we solve a single-risky-asset version of the model numerically, in order to clearly investigate whether uninsurability prevents our closed-form solution from providing calibration guidance.

4.1 Single-asset continuous-time model with uninsurable labor-income risk

In Online Appendix A we explain how we numerically solve the problem expressed by (10), under uninsurable labor-income risk ($\boldsymbol{\rho}_y \boldsymbol{\rho}_y^T \neq 1$), using Chebyshev-polynomial approximation and value-function iteration using the HJB equation. We have used our closed-form solution in order to check the validity and accuracy of our numerical approximations in the case of $\rho_{ys} = 1$, and we also tried two values for ρ_{ys}, namely 0.75, and 0.5.

parameter	$\rho_{ys} = 1$	$\rho_{ys} = 0.75$	$\rho_{ys} = 0.5$
ρ	3.67%	same	same
η	1.58	same	same
γ	3.78	4.85	5.5
χ	1829	2250	2700
r^f	3.85%	same	same
μ_y	2.06%	same	same
R_s	8.67%	same	same
σ_y	6.64%	same	8.5%
σ_s	21.2%	same	same

Table 3 Calibration for continuous-time model (Chebyshev approx.)

Our results are shown in Figure 5. In Panel A of Figure 5, we show the minimum-distance fit of the model to the data. Another hardly distinguishable curve in Panel A of Figure 5 is the fitted curve that has been obtained recursively, using Chebyshev-polynomial approximation.

The closed-form solution confirms the accuracy of our numerical method. The first column of Table 3 shows all calibrating parameters that have been obtained through minimum-distance fitting. Breaking the labor-income-risk insurability assumption, we pursue solutions for other values of ρ_{ys}, namely $\rho_{ys} \in \{0.5, 0.75\}$. In Online Appendix C, we present evidence, using Panel-Study-of-Income-Dynamics (PSID) income data and Standard and Poors (S&P) stock-index data, showing that empirical estimates of ρ_{ys} vary between 32-51%.

Panel B of Figure 5 shows portfolio shares, ϕ, for calibrating parameters $\rho_{ys} \in \{0.5, 0.75\}$, if we use our best-fitting calibrating values in the case of $\rho_{ys} = 1$, presented in the first column of Table 3. These values for ϕ, are rather high compared to the data. So, we try different calibrating parameters, using a trial-and-error approach. The second and third columns of Table 3 show our calibration strategy. We gradually increase parameter γ, which is tightly linked with household risk-aversion, and we also increase subsistence consumption, χ, in order to achieve lower values for ϕ. In the case of $\rho_{ys} = 0.5$, which requires us to find ways to bring ϕ even further down (see Panel B of Figure 5), we also increase σ_y within bounds that are consistent with empirical evidence, in order to increase the background risk borne by households. The goodness of fit of these two calibration exercises is depicted by Panels C and D of Figure 5.

In brief, the closed-form solution has served as a useful calibration guide, allowing us to understand which parameter combinations can achieve satisfactory data fitting. Interestingly, trying the closed-form solution formula for $\rho_{ys} \in \{0.5, 0.75\}$, leads to the hardly distinguishable curves of Panels C and D in Figure 5. Although for $\rho_{ys} < 1$ the closed-form solution is mathematically incorrect, it seems that the closed-form formula is directly useful even under uninsurable labor-income risk. This numerical proximity between stockholding shares derived by the closed-form solution and the numerical solution in the case of uninsurable labor-income risk, may be sensitive to changes in parameter values. Yet, such an investigation is beyond the scope of the present study.

4.1.1 Borrowing constraints and risk of failing to meet subsistence needs

In all simulations we have imposed the constraints $c \geq \chi$, and $a \geq 0$, i.e., we have placed a borrowing limit. The borrowing limit is less relevant to the target group of stockholders that we are calibrating, since these households already have some financial assets, and the probability of a binding borrowing constraint is negligible in the calibration. The constraint $c \geq \chi$ is more likely to bind. The calibrating parameters indicated by the closed-form solution in the case of $\rho_{ys} = 1$, keep this likelihood low, with negligible impact on our simulations, even in the case of $\rho_{ys} = 0.5$, in which labor-uninsurability concerns are substantial. An intuitive explanation for the state described by $c < \chi$ is homelessness or failure to meet daily calorie-intake needs. Perhaps our target group of stockholders, conditional on their stockholding status, has initial conditions, a and y, that already give these households the opportunity to choose savings and risk-taking strategies that keep them well within an interior solution. Including non-stockholders with lower initial wealth is beyond the scope of our analysis, but an interesting future extension to pursue.

4.2 Extension to a Discrete-time analysis

For the discrete-time analogue to the continuous-time version of the model with one risky asset, the Bellman equation is,

$$V\left(a_t, y_t\right) = \max_{(c_t, \phi_t)} \frac{\left\{ (1-\beta)\left(c_t - \chi\right)^{1-\frac{1}{\eta}} + \beta \left\{(1-\gamma) E_t \left[V\left(R_{p,t+1} a_t + y_t - c_t\ ,\ y_{t+1}\right)\right]\right\}^{\frac{1-\frac{1}{\eta}}{1-\gamma}} \right\}^{\frac{1-\gamma}{1-\frac{1}{\eta}}}}{1-\gamma}, \tag{23}$$

in which,

$$R_{p,t+1} \equiv \left(R_{t+1} - r^f\right)\phi_t + r^f \ , \tag{24}$$

and with,

$$\ln\left(y_{t+1}\right) - \ln\left(y_t\right) = \mu_y + \varepsilon_{y,t+1} \ , \ \ \varepsilon_{y,t+1} \sim N\left(0, \sigma_y^2\right) \ , \tag{25}$$

$$\ln\left(P_{s,t+1}\right) - \ln\left(P_{s,t}\right) = R_s + \varepsilon_{s,t+1} \ , \ \ \varepsilon_{s,t+1} \sim N\left(0, \sigma_s^2\right) \ , \tag{26}$$

where $P_{s,t}$ denotes the stock price in period t, while,

$$\frac{Cov\left(\varepsilon_{s,t+1}, \varepsilon_{y,t+1}\right)}{\sigma_s \sigma_y} = \rho_{ys} \ , \tag{27}$$

R_t in equation (24) is given by,

$$R_t = e^{R_s + \varepsilon_{s,t}} \ , \tag{28}$$

y_t is given by,

$$y_t = e^{\mu_y + \varepsilon_{y,t}} \ , \tag{29}$$

and (a_0, y_0, ϕ_0) are given.

The computational algorithm is fully explained in Online Appendix B. In Panel A of Figure 6 we can see that, trying the best-fitting parameters of the continuous-time model presented in the first column of Tables 3 and 4, does not lead to a good match of the discrete-time model to the data.[19] In the second column of Table 4, we report the calibrating values that match the discrete-time model to the stockholding data in Panel A of Figure 6.[20] We have achieved this goodness of fit through a trial-and-error approach, and Panel B of Figure 6 shows that the parameter values used in the second column of Table 4 cannot match the data in the cases of $\rho_{ys} \in \{0.5, 0.75\}$.

[19]We have repeated these calibrating parameters in Table 4 in order to facilitate parameter comparisons with other calibration exercises, except for $\eta = 0.8$ instead of $\eta = 1.58$ in the continuous-time case. Parameter η is crucial for determining the level of the saving rate, and throughout we want to maintain strictly positive saving rates.

[20]The term "same" in Table 4 means "same parameter value as in the column on the left".

parameter	$\rho_{ys} = 1$ (cont. time)	$\rho_{ys} = 1$ (disc. time)	$\rho_{ys} = 0.75$	$\rho_{ys} = 0.5$
ρ	3.67%	4.67%	same	same
η	0.8	same	same	same
γ	3.78	5	5.75	7.95
χ	1829	3000	3200	3600
r^f	3.85%	3.5%	same	same
μ_y	2.06%	1.25%	same	same
R_s	8.67%	6.37%	same	7%
σ_y	6.64%	same	7%	7.4%
σ_s	21.2%	same	same	same

Table 4 Calibration for discrete-time model ("cont. time" refers to the calibration of the continuous-time model in Panel A of Figure 5, and "disc. time" refers to the calibration of the discrete-time model in Panel A of Figure 6)

The third and fourth column of Table 4 show the parameter values that lead to the fitted curves of Panels C and D of Figure 6. As in the case of the continuous-time model calibration appearing in Figure 5, the key ingredients of a good match of the model to the data as ρ_{ys} decreases, is an increase in γ (making households more risk-averse), an increase in χ (increasing the subsistence needs of households), and also increasing σ_y (introducing more background risk to the model). An important message of Table 4 and Figure 6 is that the guidance we have had from the continuous-time model allows us to get in the ballpark of best-matching parameters to data, so that we can achieve our calibration goals through a trial-and-error calibration approach.

5. Conclusion

New stockholding data from fifteen European countries and China reconfirm what we have already known from US data (SCF): stockholding shares increase in income and wealth. On the one hand, we should be able to robustly replicate this monotonic pattern qualitatively, using one utility function for all households (parsimony). On the other hand, we should be able to match stockholding numbers across the rich and the poor. At the same time, we should also be able to replicate that saving rates also increase with income/wealth, because saving rates are a major determinant of wealth-distribution dynamics. Here we have introduced subsistence consumption to an Epstein-Zin-Weil (EZW) utility function. Our approach has created endogenous risk-aversion differences across the rich and the poor, which explains that the poor take less household-portfolio risk. The ability of the EZW utility function to separate risk-aversion parameters from intertemporal-elasticity-of-substitution parameters, has allowed us to replicate that saving rates increase in income/wealth, too.

In order to quantitatively match stockholding-share data, we have identified a closed-form solution that is handy for minimum-distance data-fitting calibration. The closed-form solution is possible only if labor-income risk is insurable. Yet, we have demonstrated that calibrating parameters for this special case of insurable labor-income risk serve as a useful guide for numerical simulations with uninsurable labor-income risk as well. Our study points at a key takeout: introducing subsistence consumption to household-portfolio models seems promising for cracking central household-finance puzzles. Using the same utility function, it is worth investigating whether analytical results are possible for households with finite lives, in future work. Such an extension could shed light on whether life-cycle models can explain longitudinal-data facts.

6. Appendix A - Proofs regarding the structure of Epstein-Zin-Weil preferences with subsistence consumption

Proof that setting $\gamma = 1/\eta$ in equation (8) leads to time-separable preferences with HARA momentary utility

One can make a conjecture beforehand: we can use the transformation $\tilde{c} = c - \chi$; then $f(c, J) = \tilde{f}\left(\tilde{c}, \tilde{J}\right)$, in which $\tilde{f}\left(c, \tilde{J}\right)$ is the normalized aggregator function in the Epstein and Duffie (1992a,b) original formulation, with $\tilde{J}$ being its associated continuation utility (notice that $\tilde{f}\left(c, \tilde{J}\right) = f(c, J)|_{\chi=0}$); based on the identity $f(c, J) = \tilde{f}\left(\tilde{c}, \tilde{J}\right)$, one can use the well-known result that if $\gamma = 1/\eta$, then $\tilde{f}\left(c, \tilde{J}\right)|_{\gamma=1/\eta}$ implies that continuation utility is $\tilde{J}(t) = \rho \int_t^\infty e^{-\rho(s-t)} c(s)^{1-\gamma} / (1-\gamma)\, ds$; the conjecture to make is that $\tilde{f}\left(\tilde{c}, \tilde{J}\right)|_{\gamma=1/\eta}$ implies $\tilde{J}(t) = \rho \int_t^\infty e^{-\rho(s-t)} \tilde{c}(s)^{1-\gamma} / (1-\gamma)\, ds$, which is the desired result. Here we go through all steps of a formal proof in order to cross check whether the intuition and conjecture discussed above fail. In addition, throughout the proof, we use the expectations operator in order to cross check whether the result holds in the presence of parameter $\chi > 0$, when consumption is stochastic.

Equation (8) implies,

$$f(c, J)|_{\gamma=\frac{1}{\eta}} = \rho \frac{(c-\chi)^{1-\gamma}}{1-\gamma} - \rho J \ . \tag{30}$$

Let's use $J'(t)$ in order to denote the total derivative of J with respect to time evaluated at time t. Equation (7) implies that $E_t[J'(t)] = -E_t[f(c(t), J(t))]$, and after using equation (30) we obtain,

$$-E_t[J'(t)] = \rho E_t \left\{ \frac{[c(t)-\chi]^{1-\gamma}}{1-\gamma} \right\} - \rho E_t[J(t)] \ .$$

After multiplying both sides by $(1/\rho)\, e^{-\rho t}$ and after integrating the above equation with respect to time we obtain,

$$-\frac{1}{\rho}E_0\left[\int_0^T e^{-\rho t}J'(t)\,dt\right]=E_0\left\{\int_0^T e^{-\rho t}\frac{[c(t)-\chi]^{1-\gamma}}{1-\gamma}dt\right\}-E_0\left[\int_0^T e^{-\rho t}J(t)\,dt\right]\ ,\tag{31}$$

for some $T\geq 0$. After applying integration by parts we obtain,

$$\int_0^T e^{-\rho t}J(t)\,dt=-\frac{1}{\rho}\left[e^{-\rho T}J(T)-J(0)\right]+\frac{1}{\rho}\int_0^T e^{-\rho t}J'(t)\,dt\ ,$$

and substituting this last expression into (31) results in,

$$J(0)=E_0\left\{\rho\int_0^T e^{-\rho t}\frac{[c(t)-\chi]^{1-\gamma}}{1-\gamma}dt\right\}+e^{-\rho T}E_T[J(T)]\ .\tag{32}$$

Since the choice of T was arbitrary, equation (32) should hold for all $T\geq 0$. The requirement of having a well-defined expected-utility function for all $T\geq 0$, i.e.,

$$-\infty<E_T[J(T)]<\infty\qquad\text{for all } T\geq 0\ ,$$

implies that $\lim_{T\to\infty}e^{-\rho t}E_T[J(T)]=0$. So, equation (32) implies,

$$J(T)=E_T\left\{\rho\int_T^\infty e^{-\rho(t-T)}\frac{[c(t)-\chi]^{1-\gamma}}{1-\gamma}dt\right\}\ ,\ \text{for all } T\geq 0\ ,\tag{33}$$

which proves the statement that setting $\gamma=1/\eta$ in equation (8) leads to time-separable preferences with HARA momentary utility. □

Proof that the IES is equal to $\eta(1-\chi/c)$

We can consider two distinct time instants, t and $t+\Delta t$, for any $t\geq 0$, and $\Delta t>0$. Based on the definition of $J(t)$ given by (7), the IES at time t is,

$$IES(t)=-\lim_{\Delta t\to 0}\frac{d\ln\left[\frac{c(t+\Delta t)}{c(t)}\right]}{d\ln\left[\frac{\frac{\partial J(t)}{\partial c(t+\Delta t)}}{f_c(c(t),J(t))}\right]}\ .\tag{34}$$

With $\Delta t > 0$ it is,

$$J(t) = E_t\left[\int_t^{t+\Delta t_-} f(c(\tau), J(\tau))\, d\tau\right] + E_{t+\Delta t}\left[\int_{t+\Delta t}^{\infty} f(c(\tau), J(\tau))\, d\tau\right] ,$$

in which Δt_- is approaching Δt from below. Given the above equation and the definition of (7),

$$\lim_{\Delta t \to 0} \frac{\partial J(t)}{\partial c(t+\Delta t)} = \lim_{\Delta t \to 0} \left\{ E_t\left[\int_t^{t+\Delta t_-} f_J(c(\tau), J(\tau))\, d\tau\right] + 1 \right\} \cdot f_c(c(t+\Delta t), J(t+\Delta t)) \ , \tag{35}$$

in which the integral in the term $\lim_{\Delta t \to 0} \left\{ E_t\left[\int_t^{t+\Delta t_-} f_J(c(\tau), J(\tau))\, d\tau\right] + 1 \right\}$ is an acceptable approximation derived from $E_t\left[\int_t^{t+\Delta t_-} f_J(c(\tau), J(\tau)) \cdot \partial J(\tau) / \partial c(t+\Delta t)\, d\tau\right]$, given that $\Delta t \to 0$.

Combining (35) with (34) leads to,

$$IES(t) = \frac{-\lim\limits_{\Delta t \to 0} d \ln\left[\frac{c(t+\Delta t)}{c(t)}\right]}{d\left\{ \lim\limits_{\Delta t \to 0} \ln\left\{ E_t\left[\int_t^{t+\Delta t_-} f_J(c(\tau), J(\tau))\, d\tau\right] + 1 \right\} + \lim\limits_{\Delta t \to 0} \ln\left[\frac{f_c(c(t+\Delta t), J(t+\Delta t))}{f_c(c(t), J(t))}\right] \right\}} \ . \tag{36}$$

Since $\lim_{\Delta t \to 0} \{\ln[x(t+\Delta t)] - \ln[x(t)]\} = [\dot{x}(t)/x(t)]\, dt$ (in which $\dot{x}(t) \equiv dx(t)/dt$), equation (36) can be expressed as,

$$IES(t) = \frac{-d\left[\frac{\dot{c}(t)}{c(t)}\right]}{d\left\{ \lim\limits_{\Delta t \to 0} \frac{\ln\left\{ E_t\left[\int_t^{t+\Delta t_-} f_J(c(\tau), J(\tau)) d\tau\right] + 1 \right\}}{\Delta t} + \frac{d \ln[f_c(c(t), J(t))]}{dt} \right\}} \ . \tag{37}$$

The relationship between a discrete-time growth rate, g_d, with its continuous-time counterpart, g_c, is given by $g_c = \ln(1 + g_d)$. Since $\Delta t \to 0$ implies transition from a discrete-time approximation to continuous time, the term $\ln\left\{ E_t\left[\int_t^{t+\Delta t_-} f_J(c(\tau), J(\tau))\, d\tau\right] + 1 \right\} / \Delta t$ converges to $f_J(c(t), J(t))$, and can be substituted into (37) to give,

$$IES(t) = -\left\{ \frac{d\left\{ f_J(c(t), J(t)) + \frac{d \ln[f_c(c(t), J(t))]}{dt} \right\}}{d\left[\frac{\dot{c}(t)}{c(t)}\right]} \right\}^{-1} \ . \tag{38}$$

From (8) we obtain $f_c = \rho\left[(1-\gamma) J\right]^{(1/\eta-\gamma)/(1-\gamma)} \cdot (c-\chi)^{-1/\eta}$, which implies $d\ln(f_c)/dt = (1/\eta-\gamma)/(1-\gamma) \cdot \left(\dot{J}/J\right) - (1/\eta)\left[c/(c-\chi)\right] \cdot (\dot{c}/c)$ and becomes

$$\frac{d\ln\left[f_c\left(c(t), J(t)\right)\right]}{dt} = -\frac{\frac{1}{\eta}-\gamma}{1-\gamma} \cdot \frac{f\left(c(t), J(t)\right)}{J(t)} - \frac{1}{\eta} \cdot \frac{1}{1-\frac{\chi}{c(t)}} \cdot \frac{\dot{c}(t)}{c(t)}, \tag{39}$$

after noticing that (7) gives $\dot{J}(t) = -\, f\left(c(t), J(t)\right)$. After some algebra we obtain

$$f_J\left(c(t), J(t)\right) = \frac{\frac{1}{\eta}-\gamma}{1-\gamma} \cdot \frac{f\left(c(t), J(t)\right)}{J(t)}. \tag{40}$$

After substituting (39) and (40) into (38) we arrive at,

$$IES(t) = \eta \left\{ \frac{d\left[\frac{1}{1-\frac{\chi}{c(t)}} \cdot \frac{\dot{c}(t)}{c(t)}\right]}{d\left[\frac{\dot{c}(t)}{c(t)}\right]} \right\}^{-1}. \tag{41}$$

Assuming that the current consumption level, $c(t)$, is constant, and focusing only on the impact of the change in the growth rate of consumption on the change in the growth rate of the marginal utility of consumption, equation (41) implies $IES(t) = \eta\left[1-\chi/c(t)\right]$ which proves the statement. $\square$

7. Appendix B - Derivation of the closed-form solution

Proof of Proposition 1

We make a guess on the functional form of the value function, namely,

$$V(a, y) = b\frac{(a+\psi y-\omega)^{1-\gamma}}{1-\gamma}, \tag{42}$$

which implies,

$$V_a(a, y) = b\left(a+\psi y-\omega\right)^{-\gamma}, \tag{43}$$

and also

$$f_c\left(c, V\left(a, y\right)\right) = \rho b^{1-\frac{1-\frac{1}{\eta}}{1-\gamma}}\left(a + \psi y - \omega\right)^{\frac{1}{\eta}-\gamma}\left(c - \chi\right)^{-\frac{1}{\eta}} \ . \tag{44}$$

From (43), (44) and (11) it is,

$$c = \rho^{\eta} b^{-\eta\frac{1-\frac{1}{\eta}}{1-\gamma}}\left(a + \psi y - \omega\right) + \chi \ . \tag{45}$$

Similarly, calculating the appropriate partial derivatives and substituting them in (12), gives,

$$\boldsymbol{\phi}^T = \frac{1}{\gamma}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(\mathbf{R}^T - r_f\mathbf{1}^T\right)\left(1 + \psi\frac{y}{a} - \frac{\omega}{a}\right) - \sigma_y\psi\frac{y}{a}\left(\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\right)^T \ . \tag{46}$$

Substituting (45), (42), (8), (46), and all derivatives stemming from (42) into the HJB given by (10) results in,

$$\begin{aligned}
\rho b\frac{\left(a + \psi y - \omega\right)^{1-\gamma}}{1 - \frac{1}{\eta}} &= \frac{\rho^{\eta} b^{1-\eta\frac{1-\frac{1}{\eta}}{1-\gamma}}}{1 - \frac{1}{\eta}}\left(a + \psi y - \omega\right)^{1-\gamma} + \\
&+ b\left(a + \psi y - \omega\right)^{-\gamma}\left\{\frac{1}{\gamma}\left(\mathbf{R} - r_f\mathbf{1}\right)\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(\mathbf{R}^T - r_f\mathbf{1}^T\right)\left(a + \psi y - \omega\right) - \right. \\
&\left. - \sigma_y\psi y\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\left(\mathbf{R}^T - r_f\mathbf{1}^T\right) + r_f a + y - \chi - \rho^{\eta} b^{-\eta\frac{1-\frac{1}{\eta}}{1-\gamma}}\left(a + \psi y - \omega\right)\right\} - \\
&- \frac{\gamma}{2}a^2 b\left(a + \psi y - \omega\right)^{-\gamma-1}\left\{\frac{1}{\gamma}\left(\mathbf{R} - r_f\mathbf{1}\right)\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(1 + \psi\frac{y}{a} - \frac{\omega}{a}\right) - \sigma_y\frac{\psi y}{a}\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\right\} \times \\
&\times \boldsymbol{\sigma}\boldsymbol{\sigma}^T\left\{\frac{1}{\gamma}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(\mathbf{R}^T - r_f\mathbf{1}^T\right)\left(1 + \psi\frac{y}{a} - \frac{\omega}{a}\right) - \sigma_y\frac{\psi y}{a}\left(\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\right)^T\right\} + \\
&+ \psi b\left(a + \psi y - \omega\right)^{-\gamma}\mu_y y - \frac{\gamma}{2}b\psi^2\left(\sigma_y y\right)^2\left(a + \psi y - \omega\right)^{-\gamma-1} - \gamma\sigma_y a y b\psi\left(a + \psi y - \omega\right)^{-\gamma-1} \times \\
&\times \left\{\frac{1}{\gamma}\left(\mathbf{R} - r_f\mathbf{1}\right)\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(1 + \psi\frac{y}{a} - \frac{\omega}{a}\right) - \sigma_y\frac{\psi y}{a}\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\right\}\boldsymbol{\sigma}\boldsymbol{\rho}_y^T \ .
\end{aligned} \tag{47}$$

After some algebra, (47) leads to,

$$\begin{aligned}
\frac{\rho - \frac{1}{\eta}\rho^{\eta} b^{-\eta\frac{1-\frac{1}{\eta}}{1-\gamma}}}{1 - \frac{1}{\eta}} - \frac{1}{2\gamma}\left(\mathbf{R} - r_f\mathbf{1}\right)\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(\mathbf{R}^T - r_f\mathbf{1}^T\right) &= r^f\frac{a + \frac{1+\psi\left[\mu_y - \sigma_y\left(\mathbf{R} - r_f\mathbf{1}\right)\left(\boldsymbol{\rho}\boldsymbol{\sigma}^{-1}\right)^T\right]}{r_f}y - \frac{\chi}{r_f}}{a + \psi y - \omega} \\
&+ \frac{1}{2}\gamma\left(\frac{\sigma_y\psi y}{a + \psi y - \omega}\right)^2\left(\boldsymbol{\rho}_y\boldsymbol{\rho}_y^T - 1\right) \ .
\end{aligned} \tag{48}$$

Since we have assumed that $\rho_{y,1}^2+...+\rho_{y,N}^2 = 1$, $\boldsymbol{\rho}_y\boldsymbol{\rho}_y^T = 1$, and the last term of the right-hand side in (48) vanishes. Moreover, we set

$$\omega = \chi / r_f \tag{49}$$

and

$$\psi = \frac{1+\psi\left[\mu_y - \sigma_y\left(\mathbf{R}-r_f\mathbf{1}\right)\left(\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\right)^T\right]}{r_f}\ , \tag{50}$$

which gives

$$\psi = 1/r_y\ . \tag{51}$$

After substituting (50) into (48) we obtain

$$\frac{\rho - \frac{1}{\eta}\rho^\eta b^{-\eta\frac{1-\frac{1}{\eta}}{1-\gamma}}}{1-\frac{1}{\eta}} - \frac{1}{2\gamma}\left(\mathbf{R}-r_f\mathbf{1}\right)\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(\mathbf{R}^T - r_f\mathbf{1}^T\right) = r_f\ . \tag{52}$$

Solving (52) for $\rho^\eta b^{-\eta(1-1/\eta)/(1-\gamma)}$ gives,

$$\rho^\eta b^{-\eta\frac{1-\frac{1}{\eta}}{1-\gamma}} = \xi\ , \tag{53}$$

in which ξ is given by equation (15). Moreover, substituting (51) and (49) into (46) leads to (13). Substituting formulas (51) and (49) in (42) reveals that Assumption 1 is both necessary and sufficient in order that $V(a,y)$ be well-defined. From (15) the requirement that $\xi > 0$ is equivalent to the condition given by Assumption 2 in order to guarantee that, under Assumption 1 and equation (14), $c \geq \chi$ for all (a,y), completing the proof. $\square$

8. Appendix C - Characterization of the saving rate

Proofs of all relationships appearing in Table 1

Equation (16) implies,

$$s\left(a,y\right)|_{\left(\chi>0,\mu_{y}=0\right)}=\frac{\eta\left(r_{f}-\rho\right)+\frac{\eta+1}{2}\frac{\nu}{\gamma}}{\frac{\nu}{\gamma}+r_{f}+\frac{\chi}{a+\frac{y}{r_{y}}-\frac{\chi}{r_{f}}}}\ , \tag{54}$$

which, in turn, implies a positive dependence of the saving rate on both wealth and income ($s_{a}\left(a,y\right)|_{\left(\chi>0,\mu_{y}=0\right)}$, $s_{y}\left(a,y\right)|_{\left(\chi>0,\mu_{y}=0\right)}>0$, if and only if $s\left(a,y\right)|_{\left(\chi>0,\mu_{y}=0\right)}>0$).[21] On the contrary, setting $\chi=0$, equation (16) implies,

$$s\left(a,y\right)|_{\left(\chi=0,\mu_{y}=0\right)}=\frac{\eta\left(r_{f}-\rho\right)+\frac{\eta+1}{2}\frac{\nu}{\gamma}}{\frac{\nu}{\gamma}+r_{f}}\ , \tag{55}$$

i.e., saving rates are the same across the rich and the poor. In order that the saving rate in both (54) and (55) be strictly positive, parameters should be such that the numerator in both formulas is strictly positive. After some algebra, we find that $s\left(a,y\right)|_{\left(\chi\geq0,\mu_{y}=0\right)}>0$ if and only if,

$$1-\frac{\rho}{r_{f}+\frac{\nu}{2\gamma}}>\frac{1}{\eta}\frac{-\frac{\nu}{2\gamma}}{r_{f}+\frac{\nu}{2\gamma}}\ . \tag{56}$$

Combining (56) with Assumption 2 leads to (17).

Focusing on the empirically plausible case of $\mu_{y}>0$, equation (16) implies, after some algebra,

$$s\left(a,y\right)=1-\frac{\xi+\frac{\chi}{r_{f}}\frac{\eta\left(r_{f}-\rho\right)+\left(\eta-1\right)\frac{\nu}{2\gamma}}{a+\frac{y}{r_{y}}}}{\frac{\nu}{\gamma}+r_{f}-\frac{\chi}{r_{f}}\frac{\nu}{\gamma}\frac{1}{a+\frac{y}{r_{y}}}-\mu_{y}\frac{1}{1+r_{y}\frac{a}{y}}}\ . \tag{57}$$

[21] This monotonicity result is in accordance with findings in Achury et al. (2012, Proposition 3 and Corollary 1). The Achury et al. (2012) model can be nested in our analysis if we set $\mu_{y}=0$. If $\mu_{y}=0$, our assumption of full labor-income insurability ($\boldsymbol{\rho}_{y}\boldsymbol{\rho}_{y}^{T}=1$) makes labor income a trendless noise which can be fully absorbed by a and fully incorporated into future household asset holdings, a, which are endogenously accumulated. Yet, even within the special case of $\mu_{y}=0$, Achury et al. (2012) study a more special case for us here, this with $\gamma=1/\eta$. Equation (54) shows that, for $\mu_{y}=0$, the monotonicity of the saving rate in Achury et al. (2012, Proposition 3 and Corollary 1) can be generalized for $\gamma\neq1/\eta$.

Equation (57) is indicative of the importance of setting parameter $\chi > 0$. By setting $\chi = 0$, (57) implies,

$$s(a, y)\,|_{\chi=0} = 1 - \frac{\xi}{\frac{\nu}{\gamma} + r_f - \mu_y \frac{1}{1 + r_y \frac{a}{y}}} \ ,$$

In this case of homothetic preferences it is easy to verify the monotonicity of $s_a(a, y)$ with respect to income, namely,

$$s_y(a, y)\,|_{\left(\chi=0,\mu_y>0\right)} < 0 \quad \text{if } a > 0 \ , \qquad s_y(a, y)\,|_{\left(\chi=0,\mu_y>0\right)} > 0 \quad \text{if } a < 0 \ . \tag{58}$$

The negative dependence of the saving rate on income when $a > 0$ in (58) reflects a dominant wealth effect on consumption. Since income grows exogenously at rate $\mu_y > 0$, higher future-consumption levels can be achieved without further sacrifices, i.e. without a higher saving rate. That both current and future consumption are normal goods corroborates this intuition. For indebted households ($a < 0$), an increase in labor income reduces the relative cost of servicing the current debt.

A ceteris-paribus increase in a implies an increase in the ratio a/y, which further implies a comparative disadvantage for the resource that grows without making sacrifices (i.e., y if $\mu_y > 0$). This comparative disadvantage is captured by the role of the ratio a/y in equation (57). From (16), after some algebra, we can verify (18).Noticing that $\eta(r_f - \rho) + (\eta + 1)\nu/(2\gamma) > 0$ is implied by condition (17), the equivalence given by (18) implies,

$$s_a(a, y)\,|_{\left(\chi\geq 0,\mu_y>0\right)} > 0 \ . \tag{59}$$

So, our conclusions regarding a saving rate which is increasing in a if $\chi > 0$, drawn by equation (54) above for the case of $\mu_y = 0$ are reconfirmed and strengthened. The positive dependence of the saving rate on a implied by (59) is a key takeout of our analytical investigation. Equation (57) makes clear that the sign of $s_y(a, y)\,|_{\left(\chi>0,\mu_y>0\right)}$ is ambiguous, necessitating calibration and numerical investigation. Finally, the less empirically plausible case of $\mu_y < 0$ implies ambiguous monotonicity of the saving rate with respect to a and y in most cases, which would require numerical verification.

9. Appendix D - Characterization of portfolio shares in the case of two risky assets

Proof of equations (20), (21), and (19)

The decomposition of matrix $\mathbf{\Sigma}$ is

$$\mathbf{\Sigma} = \boldsymbol{\sigma}\boldsymbol{\sigma}^T = \begin{bmatrix} \sigma_s & 0 \\ \rho_{s,b}\sigma_b & \sigma_b\sqrt{1-\rho_{s,b}^2} \end{bmatrix} \cdot \begin{bmatrix} \sigma_s & \rho_{s,b}\sigma_b \\ 0 & \sigma_b\sqrt{1-\rho_{s,b}^2} \end{bmatrix}, \tag{60}$$

with

$$\boldsymbol{\sigma}^{-1} = \frac{1}{\sigma_s\sigma_b\sqrt{1-\rho_{s,b}^2}} \begin{bmatrix} \sigma_b\sqrt{1-\rho_{s,b}^2} & 0 \\ -\rho_{s,b}\sigma_b & \sigma_s \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma_s} & 0 \\ \frac{-\rho_{s,b}}{\sigma_s\sqrt{1-\rho_{s,b}^2}} & \frac{1}{\sigma_b\sqrt{1-\rho_{s,b}^2}} \end{bmatrix}, \tag{61}$$

so,

$$\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1} = \begin{bmatrix} \rho_{y,s} & \rho_{y,b} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sigma_s} & 0 \\ \frac{-\rho_{s,b}}{\sigma_s\sqrt{1-\rho_{s,b}^2}} & \frac{1}{\sigma_b\sqrt{1-\rho_{s,b}^2}} \end{bmatrix}$$

or,

$$\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1} = \begin{bmatrix} \frac{\rho_{y,s}}{\sigma_s} - \frac{\rho_{y,b}\rho_{s,b}}{\sigma_s\sqrt{1-\rho_{s,b}^2}} & \frac{\rho_{y,b}}{\sigma_b\sqrt{1-\rho_{s,b}^2}} \end{bmatrix}. \tag{62}$$

Notice that since,

$$\mathbf{\Sigma}^{-1} = \left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1} = \frac{1}{\sigma_s^2\sigma_b^2\left(1-\rho_{s,b}^2\right)} \begin{bmatrix} \sigma_b^2 & -\rho_{s,b}\sigma_s\sigma_b \\ -\rho_{s,b}\sigma_s\sigma_b & \sigma_s^2 \end{bmatrix}, \tag{63}$$

after some algebra, the term $(1/\gamma)\left(\mathbf{R} - r^f\mathbf{1}\right)\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}$ in (13) is,

$$\frac{1}{\gamma}\left(\mathbf{R} - r^f\mathbf{1}\right)\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1} = \frac{1}{\gamma} \cdot \frac{1}{1-\rho_{s,b}^2} \begin{bmatrix} \frac{\frac{R_s - r_f}{\sigma_s} - \rho_{s,b}\frac{R_b - r_f}{\sigma_b}}{\sigma_s} & \frac{\frac{R_b - r_f}{\sigma_b} - \rho_{s,b}\frac{R_s - r_f}{\sigma_s}}{\sigma_b} \end{bmatrix}. \tag{64}$$

After combining (64) and (62) with (13), and after imposing the constraint $\rho_{y,s}^2 + \rho_{y,b}^2 = 1$, we obtain equations (20) and (21). Finally, since $r_y = r_f - \mu_y + \sigma_y \left(\mathbf{R} - r_f \mathbf{1}\right) \left(\boldsymbol{\rho}_y \boldsymbol{\sigma}^{-1}\right)^T$, after combining equation (62) with the constraint $\rho_{y,s}^2 + \rho_{y,b}^2 = 1$, we obtain equation (19). $\square$

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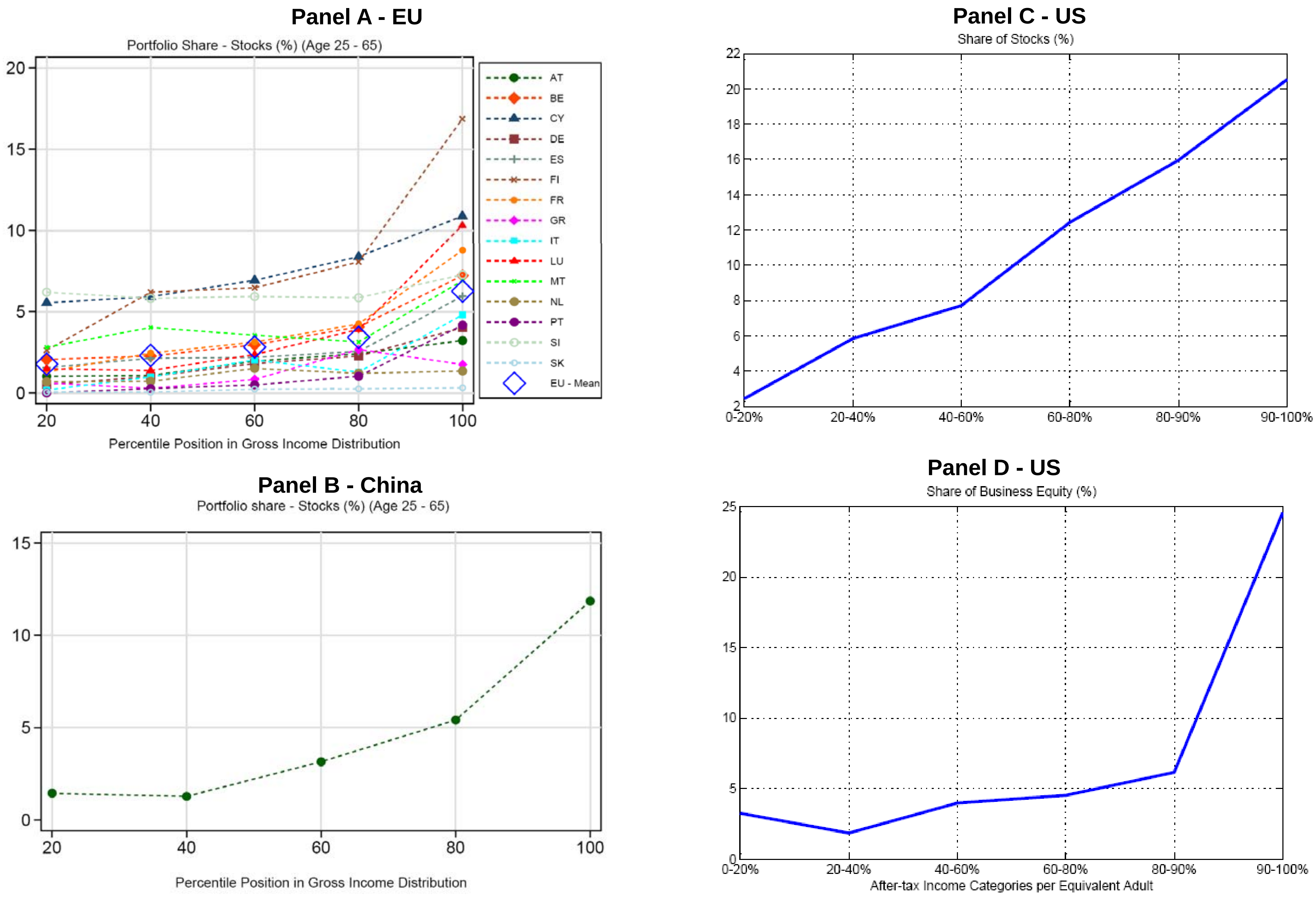

Figure 1 Sources: for the EU data, European Central Bank (ECB) Household Finance and Consumption Survey, 1st Wave-2013; for China, China Household Finance Survey, 1st Wave-2013; for the US, Survey of Consumer Finances (SCF-2007). In EU and China, Portfolio share = Total Stock Holding / Total Financial Assets), in the US, (Portfolio share = Total Stock Holding / Total Assets). 'AT' = Austria, 'BE' = Belgium, 'CY' = Cyprus, 'DE' = Germany, 'ES' = Spain, 'FI' = Finland, 'FR' = France, 'GR' = Greece, 'IT' = Italy, 'LU' = Luxembourg, 'MT' = Malta, 'NL' = Netherlands, 'PT' = Portugal, 'SI' = Slovenia, 'SK' = Slovakia.

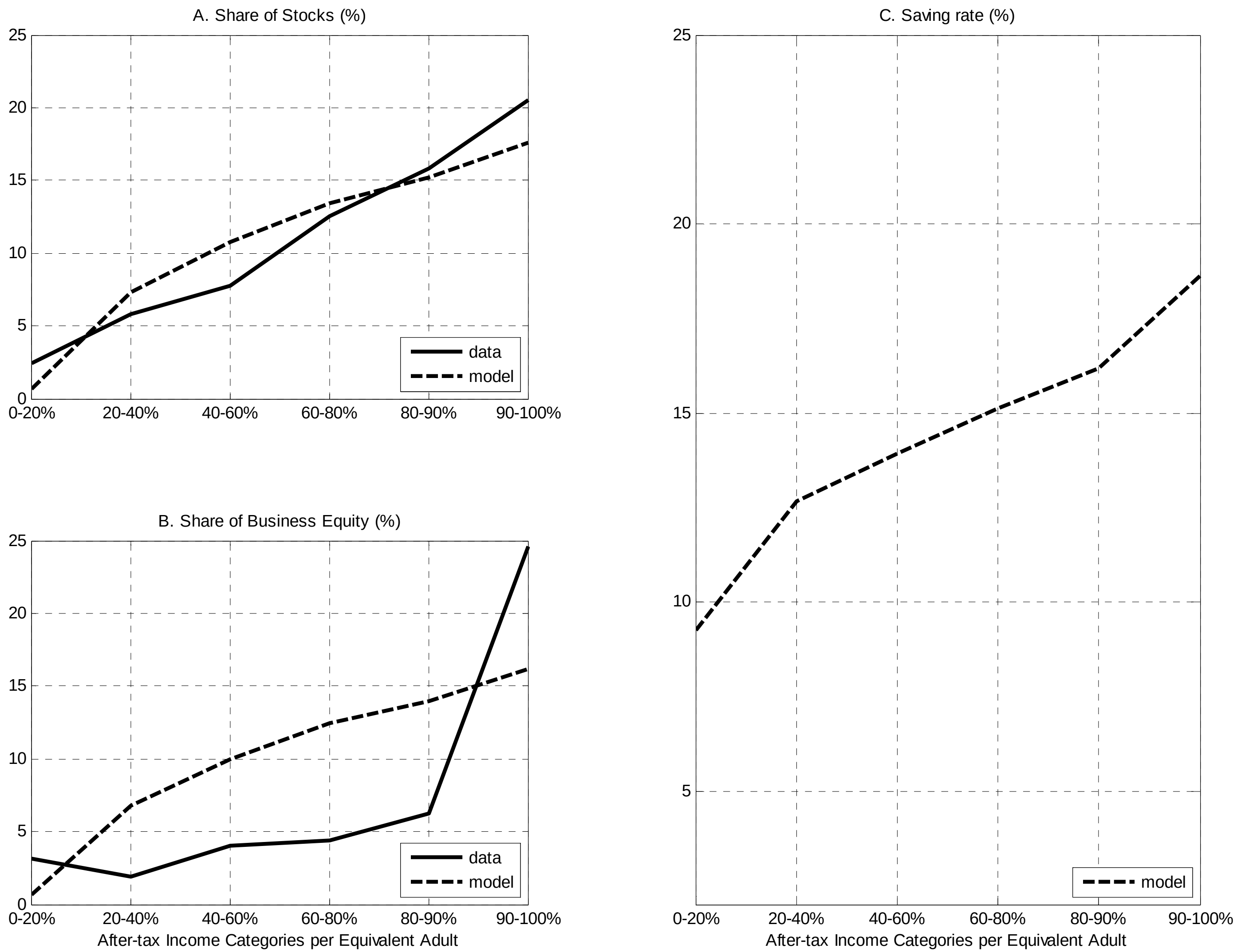

Figure 2 Benchmark calibration of the model, using the closed-form solution. Income is in 2007 USD.

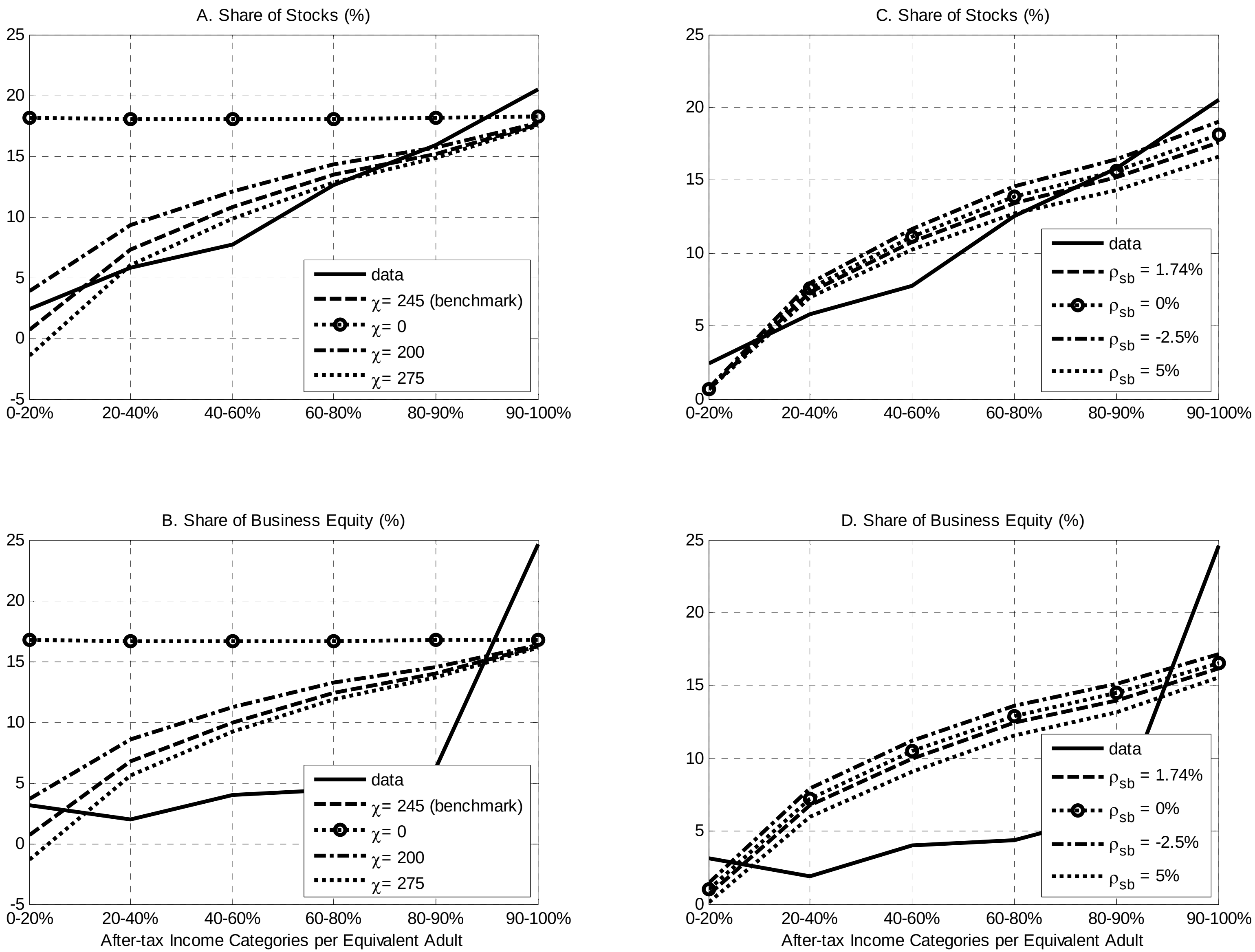

Figure 3 Sensitivity analysis of $\mathbf{\Phi}(a, y)$ by varying subsistence consumption (χ), and the correlation between stock and business-equity returns (ρ_{sb})

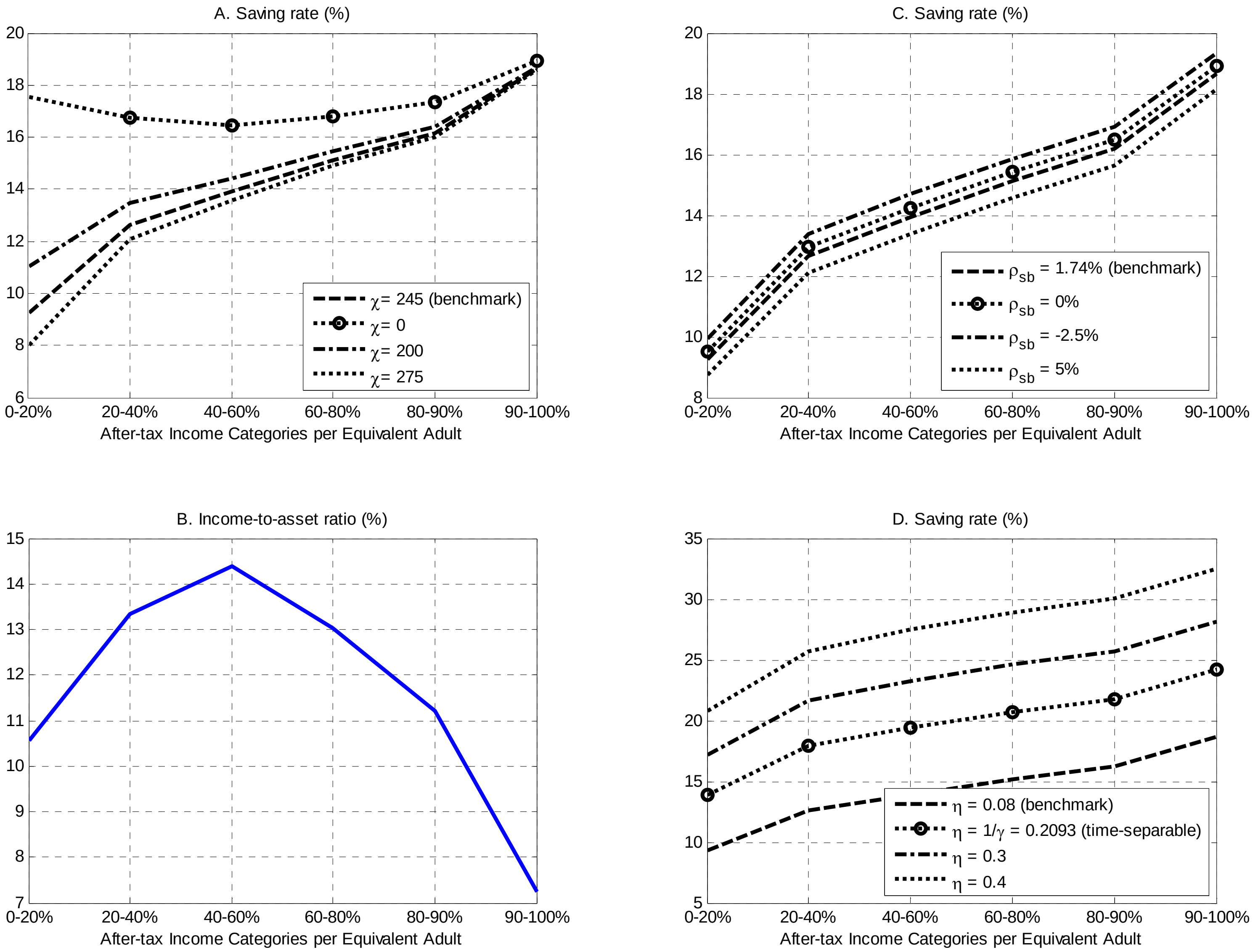

Figure 4 Sensitivity analysis of the saving rate, $s(a, y)$, by varying subsistence consumption (χ), the correlation between stock and business-equity returns (ρ_{sb}), and parameter η.

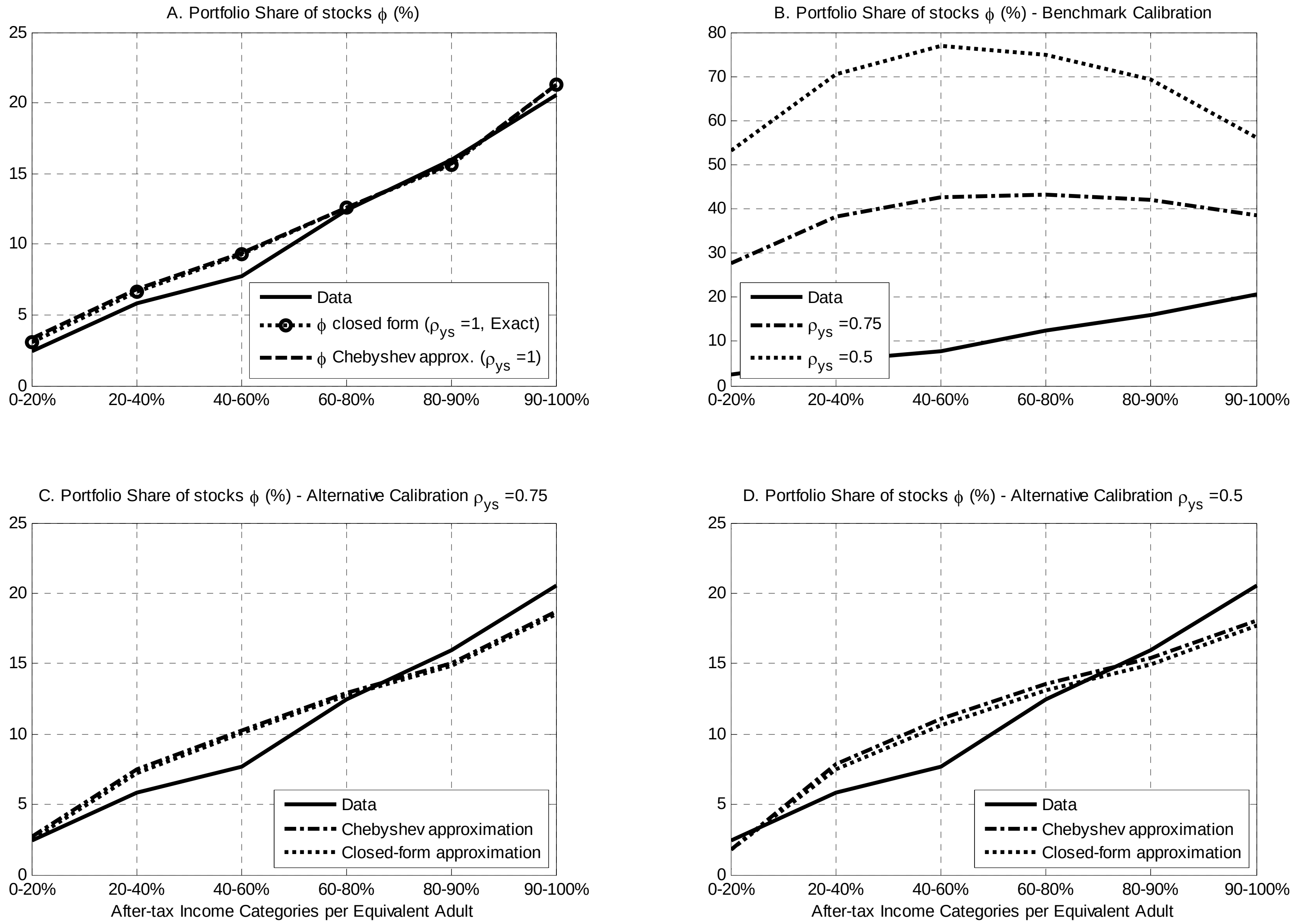

Figure 5 Sensitivity analysis of stockholding shares by varying the correlation between stock returns and income growth (ρ_{ys}) in the continuous-time simulated version of the model with a single risky asset.

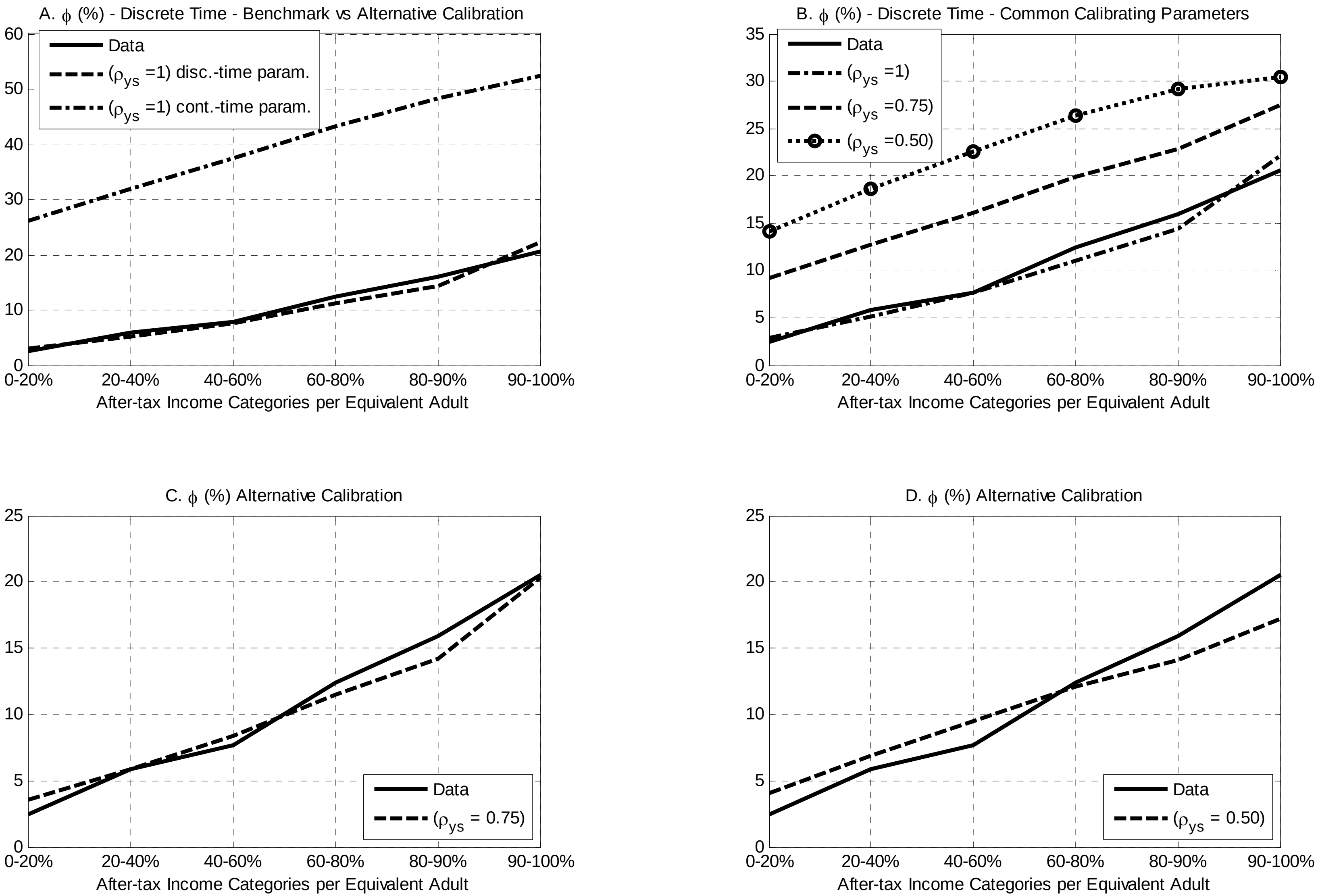

Figure 6 Discrete-time numerical simulations, using exponential projection on the model with a single risky asset. Sensitivity analysis of stockholding shares by varying the correlation between stock returns and income growth (ρ_{ys}).

Online Appendices

for

Fitting Parsimonious Household-Portfolio Models to Data

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1. Appendix A – Simulating the continuous-time model

1.1 Algebraic manipulations

The first-order conditions of the problem expressed by equation (10) in the main body of the paper are,

$$f_c\left(c, V\left(a, y\right)\right) = V_a\left(a, y\right) \ , \tag{1}$$

$$\boldsymbol{\phi}^T = \left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(\mathbf{R}^T - r_f\mathbf{1}^T\right)\frac{V_a\left(a, y\right)}{-a \cdot V_{aa}\left(a, y\right)} - \sigma_y\frac{y}{a}\left(\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\right)^T\frac{V_{ay}\left(a, y\right)}{V_{aa}\left(a, y\right)} \ . \tag{2}$$

Based on equation (8) in the paper,

$$f_c\left(c, V\right) = \rho\left[\left(1-\gamma\right)V\right]^{1-\frac{1-\frac{1}{\eta}}{1-\gamma}}\left(c-\chi\right)^{-\frac{1}{\eta}} \ . \tag{3}$$

In order to make notation somewhat easier to follow, set,

$$\theta \equiv \frac{1-\frac{1}{\eta}}{1-\gamma} \ . \tag{4}$$

Combining (4) with (3) we obtain,

$$f_c\left(c, V\right) = \rho\left[\left(1-\gamma\right)V\right]^{1-\theta}\left(c-\chi\right)^{\theta(1-\gamma)-1} \ . \tag{5}$$

Combining (5) with (1) gives,

$$c^* = C\left(a, y\right) = \chi + \left\{\rho^{-1}V_a^* \cdot \left[\left(1-\gamma\right)V^*\right]^{\theta-1}\right\}^{\frac{1}{\theta(1-\gamma)-1}} \ , \tag{6}$$

in which V^* is the fixed point of the Hamilton-Jacobi-Bellman (HJB) equation given by equation (10) in the paper. From equation (8) in the paper,

$$f\left(c^*, V^*\right) = \frac{\rho}{\theta\left(1-\gamma\right)}\left(c^*-\chi\right)^{\theta(1-\gamma)}\left[\left(1-\gamma\right)V^*\right]^{1-\theta} - \frac{\rho}{\theta}V^* \ . \tag{7}$$

Finally, (2) implies,

$$\left(\boldsymbol{\phi}^*\right)^T = \left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(\mathbf{R}^T - r_f\mathbf{1}^T\right)\frac{V_a^*}{-a \cdot V_{aa}^*} - \sigma_y\frac{y}{a}\left(\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\right)^T\frac{V_{ay}^*}{V_{aa}^*} \ . \tag{8}$$

The max operator on the right-hand side of the HJB equation which is given by (10) in the paper, can be discarded at the fixed point, V^*. Using equations (6), (8), and (7), we can incorporate the optimizers $\boldsymbol{\phi}^*$and c^* in the HJB equation, in order to obtain,

$$
\begin{aligned}
1 = \Bigg\{ & \frac{\rho}{\theta\left(1-\gamma\right)}\left(c^*-\chi\right)^{\theta(1-\gamma)}\left[\left(1-\gamma\right)V^*\right]^{1-\theta} + \left\{\left[\boldsymbol{\phi}^*\mathbf{R}^T + \left(1-\boldsymbol{\phi}^*\mathbf{1}^T\right)r_f\right]a + y - c^*\right\}\cdot V_a^* \\
& + \frac{1}{2}a^2\boldsymbol{\phi}^*\boldsymbol{\sigma}\boldsymbol{\sigma}^T\left(\boldsymbol{\phi}^*\right)^T\cdot V_{aa}^* + \mu_y y\cdot V_y^* \\
& + \frac{1}{2}\left(\sigma_y y\right)^2\cdot V_{yy}^* + \sigma_y a y \boldsymbol{\phi}\boldsymbol{\sigma}\boldsymbol{\rho}_y^T\cdot V_{ay}^*\Bigg\} \Big/ \left(\frac{\rho}{\theta}V^*\right) . \qquad (9)
\end{aligned}
$$

Equation (9) is a second-order (bivariate) partial differential equation, which we solve numerically. Yet, the numerical solution of partial differential equations can be challenging in terms of numerical accuracy, rounding problems, or error-accumulation problems. For example, in a slightly alternative version of equation (9), the term $(\rho/\theta)\,V^*$ would be on the left-hand side of (9); in the present version of (9) we have divided both sides of that alternative version by the term $(\rho/\theta)\,V^*$; we have done so, because, for successful calibrating parameter values of the model, the numerical values of V^* are often small numbers in the order of 10^{-15}; such small-valued functions V^* usually neglect convergence criteria, and a resolution of this problem is to normalize the HJB equation, as we did in (9).

1.2 Chebyshev-polynomial approximation

The Chebyshev-approximated function we use has the form,

$$V\left(a,y\right) \simeq \sum_{i=0}^{\nu_1-1}\sum_{j=0}^{\nu_1-1}\theta_{ij}T_i\left(X\left(a\right)\right)T_j\left(X\left(y\right)\right) \ , \qquad (10)$$

in which $T_j\left(x\right)$ is the Chebyshev polynomial of degree $j \in \{0, 1, ...\}$, given by,

$$T_j\left(x\right) = \cos\left(j\cdot\arccos\left(x\right)\right) \ , \qquad (11)$$

with,

$$T_j'(x) = \frac{\partial \cos(j \cdot \arccos(x))}{\partial x} = j\frac{\sin(j \arccos(x))}{\sqrt{1-x^2}} \ , \tag{12}$$

$$T_j''(x) = \frac{\partial^2 \cos(j \cdot \arccos(x))}{\partial x^2} = \frac{1}{1-x^2}\left[x \cdot \frac{j \cdot \sin(j \arccos(x))}{\sqrt{1-x^2}} - j^2 \cdot \cos(j \cdot \arccos(x))\right] \ ,$$

and based on formulas (11) and (12) we have the concise formula for the second derivative,

$$T_j''(x) = \frac{1}{1-x^2}\left[x \cdot T_j'(x) - j^2 \cdot T_j(x)\right] \ . \tag{13}$$

Regarding functions $X(a)$ and $X(y)$ in (10), notice that the domain of $T_j(x)$ is $[-1,1]$. Thanks to linearity properties of vector spaces it is straightforward to implement the Chebyshev projection method to values $a \in [\underline{a}, \bar{a}]$ and $y \in [\underline{y}, \bar{y}]$ through the linear transformation,

$$X(z) = \frac{2}{\bar{z}-\underline{z}} \cdot z - \frac{\bar{z}+\underline{z}}{\bar{z}-\underline{z}} \ , \quad z \in \{a, y\} \ , \tag{14}$$

in which $\underline{a}$ and $\underline{y}$ are the smallest values of the grids for a and y, while $\bar{a}$ and $\bar{y}$ are the largest values of a and y.

1.2.1 Forming the endogenous Chebyshev grids

Chebyshev polynomials can avoid accumulating rounding errors as the polynomial degree of the approximating function increases. While using state-space grids, this ability stems from the "discrete-orthogonality properties" of Chebyshev polynomials. These properties hold at specific gridpoints on the interval $[-1,1]$, at values $\bar{x}_k$, such that $T_n(x_k) = 0$, $k \in \{1, ..., n\}$, with,[1]

$$\bar{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right), \quad k \in \{1, ..., n\} \ . \tag{15}$$

Using m gridpoints for each dimension, a and y, the $m \times 1$ vector which is computed by (15) is denoted by $\bar{\mathbf{x}}$, and it is called the "Chebyshev nodes". In order to project the gridpoints

[1] This error-minimizing property of gridpoints $\{x_k\}_{k=1}^{n}$ with $T_n(x_k) = 0$ can be proved formally. See, for example, Judd (1992) and further references therein.

given by $\bar{\mathbf{x}}$ back onto variables a and y, use the inverse transformation of (14), in order to create the corresponding $m \times 1$ vectors, $\mathbf{a}_{grid} = \bar{\mathbf{a}}$, and $\mathbf{y}_{grid} = \bar{\mathbf{y}}$, namely,

$$\bar{\mathbf{a}} = A(\bar{\mathbf{x}}) = \frac{(\bar{\mathbf{x}} + 1)(\bar{a} - \underline{a})}{2} + \underline{a} \ , \tag{16}$$

and

$$\bar{\mathbf{y}} = Y(\bar{\mathbf{x}}) = \frac{(\bar{\mathbf{x}} + 1)(\bar{y} - \underline{y})}{2} + \underline{y} \ . \tag{17}$$

1.2.2 Best-fitting the two-dimensional Chebyshev polynomial to a known function

Let's assume that we have an $m_a \times 1$ grid for a, $\bar{\mathbf{a}}$, calculated using (16), and an $m_y \times 1$ grid for y, $\bar{\mathbf{y}}$, calculated using (17). Any known function, $V(a, y)$, can map the grid of Chebyshev nodes (discretized domain) to an $m_a \times m_y$ matrix, $\bar{\mathbf{V}}$, defined as,

$$\bar{\mathbf{V}} = [\bar{v}_{k,\ell}] = [V(\bar{a}_k, \bar{y}_\ell)] = [V(A(\bar{x}_{a,k}), Y(\bar{x}_{y,\ell}))] \ , \quad k \in \{1, ..., m_a\}, \ell \in \{1, ..., m_y\}, \tag{18}$$

in which $A(\cdot)$ and $Y(\cdot)$ are given by (16) and (17). Let's also assume that the Chebyshev polynomial degree for dimension a is ν_a, and ν_y for dimension y. In order to achieve a best Chebyshev polynomial fitting of the functional form given by (10) on the elements of matrix $\bar{\mathbf{V}}$, we minimize least-squares residuals. The formulas for the optimal Chebyshev-approximation estimator $\hat{\theta}_{i,j}$ are given by (see, for example, Heer and Maußner, 2005, Ch. 8, p. 441),

$$\hat{\theta}_{0,0} = \frac{1}{m_a m_y} \sum_{k=1}^{m_a} \sum_{\ell=1}^{m_y} \bar{v}_{k,\ell} \tag{19}$$

$$\hat{\theta}_{i,0} = \frac{2}{m_a m_y} \sum_{k=1}^{m_a} \sum_{\ell=1}^{m_y} \bar{v}_{k,\ell} T_i(\bar{x}_{a,k}) \tag{20}$$

$$\hat{\theta}_{0,j} = \frac{2}{m_a m_y} \sum_{k=1}^{m_a} \sum_{\ell=1}^{m_y} \bar{v}_{k,\ell} T_j(\bar{x}_{y,\ell}) \tag{21}$$

$$\hat{\theta}_{i,j} = \frac{4}{m_a m_y} \sum_{k=1}^{m_a} \sum_{\ell=1}^{m_y} \bar{v}_{k,\ell} T_i(\bar{x}_{a,k}) T_j(\bar{x}_{y,\ell}) \ , \tag{22}$$

for $i \in \{1, ..., \nu_a - 1\}$ and $j \in \{1, ..., \nu_y - 1\}$. For convenience, we can summarize the optimal-fitting conditions given by equations (19) through (22) using some particular matrix arrays.

Consider the matrices,

$$\mathbf{T}_a\left(X\left(\bar{\mathbf{a}}\right)\right) = \mathbf{T}_a\left(\bar{\mathbf{x}}_a\right) = \begin{bmatrix} T_0\left(\bar{x}_{a,1}\right) & T_1\left(\bar{x}_{a,1}\right) & \cdots & T_{\nu_a-1}\left(\bar{x}_{a,1}\right) \\ T_0\left(\bar{x}_{a,2}\right) & T_1\left(\bar{x}_{a,2}\right) & \cdots & T_{\nu_a-1}\left(\bar{x}_{a,2}\right) \\ \vdots & \vdots & \ddots & \vdots \\ T_0\left(\bar{x}_{a,m_a}\right) & T_1\left(\bar{x}_{a,m_a}\right) & \cdots & T_{\nu_a-1}\left(\bar{x}_{a,m_a}\right) \end{bmatrix},$$

and

$$\mathbf{T}_y\left(X\left(\bar{\mathbf{y}}\right)\right) = \mathbf{T}_y\left(\bar{\mathbf{x}}_y\right) = \begin{bmatrix} T_0\left(\bar{x}_{y,1}\right) & T_1\left(\bar{x}_{y,1}\right) & \cdots & T_{\nu_y-1}\left(\bar{x}_{y,1}\right) \\ T_0\left(\bar{x}_{y,2}\right) & T_1\left(\bar{x}_{y,2}\right) & \cdots & T_{\nu_y-1}\left(\bar{x}_{y,2}\right) \\ \vdots & \vdots & \ddots & \vdots \\ T_0\left(\bar{x}_{y,m_y}\right) & T_1\left(\bar{x}_{y,m_y}\right) & \cdots & T_{\nu_y-1}\left(\bar{x}_{y,m_y}\right) \end{bmatrix}.$$

Notice that $\mathbf{T}_a\left(\bar{\mathbf{x}}_a\right)$ is of size $m_a \times \nu_a$, while $\mathbf{T}_y\left(\bar{\mathbf{x}}_y\right)$ is an $m_y \times \nu_y$ matrix. Consider also the two matrices,

$$\mathbf{I}_{m_a} = \begin{bmatrix} \frac{1}{m_a} & 0 & \cdots & 0 \\ 0 & \frac{2}{m_a} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{2}{m_a} \end{bmatrix},$$

and

$$\mathbf{I}_{m_y} = \begin{bmatrix} \frac{1}{m_y} & 0 & \cdots & 0 \\ 0 & \frac{2}{m_y} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{2}{m_y} \end{bmatrix},$$

with $\mathbf{I}_{m_a}$ being of size $\nu_a \times \nu_a$, and with $\mathbf{I}_{m_y}$ being of size $\nu_y \times \nu_y$.

The $\nu_a \times \nu_y$ matrix $\hat{\mathbf{\Theta}}$ that contains all Chebyshev coefficients $\hat{\theta}_{i,j}$ for $i \in \{0, ..., \nu_a - 1\}$ and $j \in \{0, ..., \nu_y - 1\}$, as these are given by the optimal-fitting conditions (19) through (22), are summarized by,

$$\hat{\mathbf{\Theta}} \equiv \arg\min_{\mathbf{\Theta}} \sum_{k=1}^{m_a} \sum_{\ell=1}^{m_y} \left[\mathbf{T}_a\left(\bar{\mathbf{x}}_{a,k}\right) \cdot \Theta \cdot \mathbf{T}_y\left(\bar{\mathbf{x}}_{y,\ell}\right)^{\mathsf{T}} - \bar{\mathbf{V}}_{k,\ell} \right]^2 = \mathbf{I}_{m_a} \cdot \mathbf{T}_a\left(\bar{\mathbf{x}}_a\right)^{\mathsf{T}} \cdot \bar{\mathbf{V}} \cdot \mathbf{T}_y\left(\bar{\mathbf{x}}_y\right) \cdot \mathbf{I}_{m_y} \ , \tag{23}$$

in which $\mathbf{T}_a\left(\bar{\mathbf{x}}_{a,k}\right)$ and $\mathbf{T}_y\left(\bar{\mathbf{x}}_{y,l}\right)$ are the k-th and ℓ-th row of matrices $\mathbf{T}_a\left(\bar{\mathbf{x}}_a\right)$ and $\mathbf{T}_y\left(\bar{\mathbf{x}}_y\right)$.

Finally, notice the matrix array,

$$\bar{\mathbf{V}} = \mathbf{T}_a\left(\bar{\mathbf{x}}_a\right) \cdot \hat{\mathbf{\Theta}} \cdot \mathbf{T}_y\left(\bar{\mathbf{x}}_y\right)^{\mathsf{T}} \ , \tag{24}$$

which is easy to verify from the expression given by (23) and the Chebyshev discrete-orthogonality conditions, which imply,

$$\mathbf{T}_a\left(\bar{\mathbf{x}}_a\right) \cdot \mathbf{I}_{m_a} \cdot \mathbf{T}_a\left(\bar{\mathbf{x}}_a\right)^{\mathsf{T}} = \mathbf{I}_{(m_a \times m_a)} \quad \text{and} \quad \mathbf{T}_y\left(\bar{\mathbf{x}}_y\right) \cdot \mathbf{I}_{m_y} \cdot \mathbf{T}_y\left(\bar{\mathbf{x}}_y\right)^{\mathsf{T}} = \mathbf{I}_{(m_y \times m_y)} \ ,$$

and in which $\mathbf{I}_{(m_a \times m_a)}$ and $\mathbf{I}_{(m_y \times m_y)}$ are identiy matrices of size $m_a \times m_a$ and $m_y \times m_y$.

1.2.3 Computing all partial derivatives efficiently, and dealing with the small values of the indirect utility function

Let

$$\mathbf{A} \equiv \mathbf{T}_a\left(X\left(\bar{\mathbf{a}}\right)\right) \ , \text{and} \ \ \mathbf{Y} \equiv \mathbf{T}_y\left(X\left(\bar{\mathbf{y}}\right)\right) \ . \tag{25}$$

Let also,

$$\mathbf{A}_1 \equiv \frac{\partial \mathbf{T}_a\left(X\left(\bar{\mathbf{a}}\right)\right)}{\partial a} = \frac{2}{\bar{a} - \underline{a}} \mathbf{T}'_a\left(X\left(\bar{\mathbf{a}}\right)\right) \ , \tag{26}$$

with,

$$
\mathbf{T}_a'\left(X\left(\bar{\mathbf{a}}\right)\right)=\mathbf{T}_a'\left(\bar{\mathbf{x}}_a\right)=\begin{bmatrix} T_0'\left(\bar{x}_{a,1}\right) & T_1'\left(\bar{x}_{a,1}\right) & \cdots & T_{\nu_a-1}'\left(\bar{x}_{a,1}\right) \\ T_0'\left(\bar{x}_{a,2}\right) & T_1'\left(\bar{x}_{a,2}\right) & \cdots & T_{\nu_a-1}'\left(\bar{x}_{a,2}\right) \\ \vdots & \vdots & \ddots & \vdots \\ T_0'\left(\bar{x}_{a,m_a}\right) & T_1'\left(\bar{x}_{a,m_a}\right) & \cdots & T_{\nu_a-1}'\left(\bar{x}_{a,m_a}\right) \end{bmatrix}, \tag{27}
$$

in which $T_j'(x)$ is computed using (12). Notice that the term $2/\left(\bar{a}-\underline{a}\right)$ is the result of applying the chain rule of differentiation on $\mathbf{T}_a\left(X\left(a\right)\right)$, in which $X\left(a\right)$ is given by (14). Similarly,

$$
\mathbf{A}_2 \equiv \frac{\partial^2 \mathbf{T}_a\left(X\left(\bar{\mathbf{a}}\right)\right)}{\partial a^2}=\left(\frac{2}{\bar{a}-\underline{a}}\right)^2 \mathbf{T}_a''\left(X\left(\bar{\mathbf{a}}\right)\right)\ , \tag{28}
$$

with,

$$
\mathbf{T}_a''\left(X\left(\bar{\mathbf{a}}\right)\right)=\mathbf{T}_a''\left(\bar{\mathbf{x}}_a\right)=\begin{bmatrix} T_0''\left(\bar{x}_{a,1}\right) & T_1''\left(\bar{x}_{a,1}\right) & \cdots & T_{\nu_a-1}''\left(\bar{x}_{a,1}\right) \\ T_0''\left(\bar{x}_{a,2}\right) & T_1''\left(\bar{x}_{a,2}\right) & \cdots & T_{\nu_a-1}''\left(\bar{x}_{a,2}\right) \\ \vdots & \vdots & \ddots & \vdots \\ T_0''\left(\bar{x}_{a,m_a}\right) & T_1''\left(\bar{x}_{a,m_a}\right) & \cdots & T_{\nu_a-1}''\left(\bar{x}_{a,m_a}\right) \end{bmatrix}, \tag{29}
$$

in which $T_j''(x)$ is computed using (13). We also produce matrices $\mathbf{Y}_1$ and $\mathbf{Y}_2$, in accordance with formulas (26), (27), (28), and (29).

For reasonable calibrating parameters, the numerical values of $V\left(a,y\right)$ are often small numbers in the order of 10^{-15}. The problem is that such small-valued functions circumvent loops with tight convergence criteria. In order to deal with this problem, we normalize $V\left(a,y\right)$, through the transformation,

$$
V\left(a,y\right) \equiv \frac{\left[\tilde{V}\left(a,y\right)\right]^{1-\gamma}}{1-\gamma}\ . \tag{30}
$$

Using (24), for any estimator $\hat{\mathbf{\Theta}}^{(n)}$, during the n-th iteration of a recursive process, we approximate the value of $\tilde{V}(a,y)$ by,

$$\tilde{V}^{(n)}(a,y) \simeq \mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^{T} \ . \tag{31}$$

According to (30) and (31),

$$V^{(n)}(a,y) \simeq \frac{\left[\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^{T}\right]^{1-\gamma}}{1-\gamma} \ . \tag{32}$$

The transformation given by (32) allows us to achieve Chebyshev-polynomial coefficients (contained in in matrix $\hat{\mathbf{\Theta}}^{(n)}$) with values large enough for implementing a recursive numerical method that searches for a fixed point for matrix $\hat{\mathbf{\Theta}}^{(n)}$.

Using (32), the partial derivatives $V_a^{(n)}$ and $V_y^{(n)}$ are given by,

$$V_a^{(n)}(a,y) \simeq \left(\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^{T}\right)^{-\gamma}\mathbf{A}_1\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^{T} \ , \tag{33}$$

and

$$V_y^{(n)}(a,y) \simeq \left(\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^{T}\right)^{-\gamma}\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}_1^{T} \ . \tag{34}$$

Using (33) and (34), we obtain,

$$V_{aa}^{(n)}(a,y) \simeq \left(\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^{T}\right)^{-\gamma}\left[-\gamma\left(\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^{T}\right)^{-1}\left(\mathbf{A}_1\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^{T}\right)^{2}+\mathbf{A}_2\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^{T}\right] \ , \tag{35}$$

$$V_{yy}^{(n)}(a,y) \simeq \left(\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^{T}\right)^{-\gamma}\left[-\gamma\left(\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^{T}\right)^{-1}\left(\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}_1^{T}\right)^{2}+\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}_2^{T}\right] \ , \tag{36}$$

and

$$V_{ay}^{(n)}(a,y) \simeq \left(\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^{T}\right)^{-\gamma}\left[-\gamma\left(\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^{T}\right)^{-1}\left(\mathbf{A}_1\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}^{T}\right)\left(\mathbf{A}\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}_1^{T}\right)+\mathbf{A}_1\hat{\mathbf{\Theta}}^{(n)}\mathbf{Y}_1^{T}\right] \ . \tag{37}$$

1.2.4 Matrix array for computing all functions of the HJB equation using nonlinear regression techniques

Matrices described by equations (30) through (37) use the matrix array,

$$V(a,y) \simeq \underbrace{V_{matrix}}_{m_a \times m_y} \equiv \begin{bmatrix} V(a_1,y_1) & V(a_1,y_2) & \cdots & V\left(a_1,y_{m_y}\right) \\ V(a_2,y_1) & V(a_2,y_2) & \cdots & V\left(a_2,y_{m_y}\right) \\ \vdots & \vdots & \ddots & \vdots \\ V(a_{m_a},y_1) & V(a_{m_a},y_2) & \cdots & V\left(a_{m_a},y_{m_y}\right) \end{bmatrix}. \tag{38}$$

For V_{matrix} in (38) we use the $(m_a \cdot m_y) \times 1$ vector array,

$$V_{vector\_array} = (m_a \cdot m_y) \times 1 \left\{ \begin{bmatrix} V(a_1,y_1) \\ V(a_2,y_1) \\ \vdots \\ V(a_{m_a},y_1) \\ ---- \\ V(a_1,y_2) \\ V(a_2,y_2) \\ \vdots \\ V(a_{m_a},y_2) \\ ---- \\ \vdots \\ ---- \\ V\left(a_1,y_{m_y}\right) \\ V\left(a_2,y_{m_y}\right) \\ \vdots \\ V\left(a_{m_a},y_{m_y}\right) \end{bmatrix} \right. \tag{39}$$

The array in (39) can be achieved by matching two $(m_a \cdot m_y) \times 1$ vectors,

$$\bar{\mathbf{a}}_{grid\_long} = \mathbf{1}_{(m_y \times 1)} \otimes \bar{\mathbf{a}} \ , \tag{40}$$

which corresponds to m_y stacked vectors $\bar{\mathbf{a}}$, and

$$\bar{\mathbf{y}}_{grid\_long} = \bar{\mathbf{y}} \otimes \mathbf{1}_{(m_a \times 1)} \ , \tag{41}$$

which is m_y stacked vectors of size $m_a \times 1$, with each $m_a \times 1$ vector having m_a identical elements, m_a times each element of $\bar{\mathbf{y}}$, stacked in the order of elements of $\bar{\mathbf{y}}$.

Using the vector array in (39), we express all matrices described by equations (30) through (37), using the Matlab command "`reshape`", and we use all partial derivatives in the same $(m_a \cdot m_y) \times 1$ vector array in order to express c^* and $\boldsymbol{\phi}^*$ according to equations (6) and (8).

1.3 Ensuring that consumption is above subsistence and treatment of borrowing constraints

The functional form of utility that we use satisfies an Inada condition as $c \to \chi$, which is obvious from (3). This is the reason that equation (6) holds. The RHS of (6) has a simple interpretation: as long as V^* is well-defined, it is guaranteed that $c > \chi$. Notice that the first-order condition given by (1) holds even if there is a borrowing constraint $a \geq b$. The presence of a borrowing constraint, $a \geq b$, does not affect (2) either. In order to implement $a \geq b$, all we need to do is to ensure that the deterministic part of the budget constraint is nonnegative when $a = b$, i.e.

$$\left[\boldsymbol{\phi}^* \mathbf{R}^T + \left(1 - \boldsymbol{\phi}^* \mathbf{1}^T\right) r_f\right] b + y - c^* \geq 0 \ . \tag{42}$$

Inserting (42) into (9) is achieved by the modified version of (9),

$$1 = \left\{ \frac{\rho}{\theta\left(1-\gamma\right)} \left(c^* - \chi\right)^{\theta(1-\gamma)} \left[\left(1-\gamma\right) V^*\right]^{1-\theta} + \right.$$

$$
+\max\left\{\left[\boldsymbol{\phi}^{*}\mathbf{R}^{T}+\left(1-\boldsymbol{\phi}^{*}\mathbf{1}^{T}\right)r_{f}\right]a+y-c^{*}\ ,\ 0\right\}\Big|_{a=b}\cdot V_{a}^{*}
$$

$$
+\frac{1}{2}a^{2}\boldsymbol{\phi}^{*}\boldsymbol{\sigma}\boldsymbol{\sigma}^{T}\left(\boldsymbol{\phi}^{*}\right)^{T}\cdot V_{aa}^{*}+\mu_{y}y\cdot V_{y}^{*}
$$

$$
\left.+\frac{1}{2}\left(\sigma_{y}y\right)^{2}\cdot V_{yy}^{*}+\sigma_{y}ay\boldsymbol{\phi}\boldsymbol{\sigma}\boldsymbol{\rho}_{y}^{T}\cdot V_{ay}^{*}\right\}/\left(\frac{\rho}{\theta}V^{*}\right)\ , \tag{43}
$$

using an indicator function in order to implement the conditionality operator $(\cdot)|_{a=b}$. The presence of the term $\max\left\{\left[\boldsymbol{\phi}^{*}\mathbf{R}^{T}+\left(1-\boldsymbol{\phi}^{*}\mathbf{1}^{T}\right)r_{f}\right]a+y-c^{*}\ ,\ 0\right\}\Big|_{a=b}\cdot V_{a}^{*}$ in (43) has not affected our results, as we had strictly positive saving rates in all our calibration exercises. Notably, our borrowing constraint is $b=\underline{a}$. As we have averaged across income groups of stockholders, $\underline{a}$ is well above 0, it amounts to USD $85,520$, which is a rather tight borrowing constraint. Yet, at this level of wealth ($\underline{a}$), and for all gridpoints for y, households chose interior solutions.

1.4 The recursive algorithm

Using the HJB equation (equation (9)), we perform iterations on $\boldsymbol{\Theta}$, using the Matlab command "`nlinfit`" which is designed in order to solve nonlinear minimum least-squares econometric models. The inputs of "`nlinfit`" are a (nonlinear econometric) model, a matrix of regressors, and a vector of model parameters that need to be estimated. In order to match the input structure of the "`nlinfit`" Matlab procedure, we compute all the above $(m_a \cdot m_y) \times 1$ vectors corresponding to equations (30) through (37) and also to (6) and (8), and we use equation (9) in order to produce a Matlab m-file "`HJB.m`" with inputs $\bar{\mathbf{a}}_{grid\_long}$, $\bar{\mathbf{y}}_{grid\_long}$ and $\boldsymbol{\Theta}_{vector} \equiv$`reshape`$(\boldsymbol{\Theta}, \nu_a \cdot \nu_y, 1)$, which is an $(\nu_a \cdot \nu_y) \times 1$ vector resulting from stacking all columns of $\boldsymbol{\Theta}$. This "`HJB.m`" function defines the model to be estimated, and we also create an $(m_a m_y) \times 2$ matrix with columns consisting of vectors $\bar{\mathbf{a}}_{grid\_long}$ and $\bar{\mathbf{y}}_{grid\_long}$, which is the regressor matrix.

1.5 The importance of a good first guess

The initial guess is the $\hat{\mathbf{\Theta}}_{vector}^{(0)}$ which corresponds to the closed-form solution given by Proposition 1 in the paper, for the special case $\boldsymbol{\rho}_y \boldsymbol{\rho}_y^T = 1$. When $\boldsymbol{\rho}_y \boldsymbol{\rho}_y^T = 1$ holds, the performance of the algorithm is satisfactory, since $\hat{\mathbf{\Theta}}_{vector}^{(0)} = \hat{\mathbf{\Theta}}_{vector}^{*}$ in one iteration.

We perform iterations for the version of the model with a single risky asset, for cases in which $\rho_{ys} = 1$, and also for cases in which $\rho_{ys} < 1$. We compute the decision rules of the model for $\rho_{ys} \in \{0.5, 0.75\}$, taking gradual steps down from $\rho_{ys} = 1$ to $\rho_{ys} = 0.75$, and then from $\rho_{ys} = 0.75$ to $\rho_{ys} = 0.5$. In each case, we use the solution found in the previous step as a first guess in the "`nlinfit`" Matlab procedure, finding that this strategy is stable and efficient. Typically, setting $\nu_a = \nu_y = 5$, and $m_a = m_y = 20$, performs satisfactorily well, producing all results plotted in Figure 5 of the paper in about 13.5 seconds on a state-of-the art laptop.

1.6 Dealing with the nonlinear relationship between assets and income across income categories in the data

Panel C of Figure 4 shows that, after ranking households according to their after-tax adult-equivalent income, a and y are linked through a nonlinear relationship in the data. This nonlinear relationship is not reflected by the two grids, $\bar{\mathbf{a}}$ and $\bar{\mathbf{y}}$. This failure of reflecting the nonlinear relationship occurs because grids $\bar{\mathbf{a}}$ and $\bar{\mathbf{y}}$ should be consistent with Chebyshev nodes, in order to ensure that discrete-orthogonality conditions hold accurately. Discrete-orthogonality conditions are a necessary requirement for good performance of the Chebyshev approximation. The fact that grids $\bar{\mathbf{a}}$ and $\bar{\mathbf{y}}$ do not reflect the nonlinear relationship in the data means that we cannot directly select matrix elements from the resulting matrix

$$\bar{\mathbf{\Phi}} = \left[\bar{\phi}_{k,\ell}^{*}\right] = \left[\Phi\left(\bar{a}_k, \bar{y}_\ell\right)\right] = \left[\Phi\left(A\left(\bar{x}_{a,k}\right), Y\left(\bar{x}_{y,\ell}\right)\right)\right] \ , \quad k \in \{1, ..., m_a\} \, , \, \ell \in \{1, ..., m_y\} \, ,$$

of the code in order to report them in Figure 5. In order to deal with this issue, we first interpolate the a_{data} and y_{data} data observations that correspond to the six income categories in panel C of Figure 4 in order to capture the nonlinear relationship in that figure, say

$$y_{data} = g\left(a_{data}\right) \ ,$$

using the "`spline`"-interpolation option of Matlab's "`interp1`" routine; specifically, we produce an $m_a \times 1$ vector, called $\mathbf{y}^{nl}$, that uses $\bar{\mathbf{a}}$ as the interpolation domain, so,

$$\mathbf{y}^{nl} = g\left(\bar{\mathbf{a}}\right) \ . \tag{44}$$

In order to produce Figure 5, the goal is to report portfolio shares which are consistent with

$$\phi^* = \Phi\left(\bar{\mathbf{a}}, g\left(\bar{\mathbf{a}}\right)\right) \ .$$

So, for all $k \in \{1, ..., m_a\}$, fix an $\bar{a}_k$ gridpoint and define the function,

$$\tilde{\phi}_k\left(y\right) \equiv \Phi\left(\bar{a}_k, y\right) \ ,$$

using the "`spline`"-interpolation option of Matlab's "`interp1`" routine, using gridpoints $\bar{\mathbf{y}}$ that correspond to the k-th row of matrix $\bar{\mathbf{\Phi}}$ as the domain, and the $m_y \times 1$ vector $\left[\Phi\left(\bar{a}_k, \bar{\mathbf{y}}\right)\right]^T$ as the image of function $\tilde{\phi}_k\left(y\right)$. So,

$$\phi_k^* \equiv \Phi\left(\bar{a}_k, g\left(\bar{a}_k\right)\right) = \left\{\Phi\left(\bar{a}_k, y\right) \ | \ \tilde{\phi}_k^{-1}\left(y\right) = y_k^{nl}\right\} \ ,$$

in which y_k^{nl} corresponds to the k-th element of vector $\mathbf{y}^{nl}$, defined by (44), fills in a new $m_a \times 1$ vector, $\boldsymbol{\phi}^*$. Vector $\boldsymbol{\phi}^*$ contains the values that we report in Figure 5, after interpolating the pair $\left(\mathbf{y}^{nl}, \boldsymbol{\phi}^*\right)$ and projecting this interpolation on the 6×1 vector $\mathbf{y}_{data}$, using the "`spline`"-interpolation option of Matlab's "`interp1`" routine.

2. Appendix B – Simulating the discrete-time model

2.1 Statement of the Problem

The household solves,

$$V(a_t, y_t) = \max_{(c_t, \phi_t)} \frac{\left\{ (1-\beta)(c_t - \chi)^{1-\frac{1}{\eta}} + \beta \left\{ (1-\gamma) E_t \left[V\left(R_{p,t+1} a_t + y_t - c_t \ , \ y_{t+1}\right) \right] \right\}^{\frac{1-\frac{1}{\eta}}{1-\gamma}} \right\}^{\frac{1-\gamma}{1-\frac{1}{\eta}}}}{1-\gamma} , \tag{45}$$

in which,

$$R_{p,t+1} \equiv \left(R_{t+1} - r^f\right) \phi_t + r^f \ , \tag{46}$$

and with,

$$\ln(y_{t+1}) - \ln(y_t) = \mu_y + \varepsilon_{y,t+1} \ , \ \ \varepsilon_{y,t+1} \sim N\left(0, \sigma_y^2\right) \ , \tag{47}$$

$$\ln(P_{s,t+1}) - \ln(P_{s,t}) = R_s + \varepsilon_{s,t+1} \ , \ \ \varepsilon_{s,t+1} \sim N\left(0, \sigma_s^2\right) \ , \tag{48}$$

where $P_{s,t}$ denotes the stock price in period t, while,

$$\frac{Cov\left(\varepsilon_{s,t+1}, \varepsilon_{y,t+1}\right)}{\sigma_s \sigma_y} = \rho_{ys} \ , \tag{49}$$

R_t in equation (46) is given by,

$$R_t = e^{R_s + \varepsilon_{s,t}} \ , \tag{50}$$

y_t is given by,

$$y_t = e^{\mu_y + \varepsilon_{y,t}} \ , \tag{51}$$

and (a_0, y_0, ϕ_0) are given.

The problem stated by (45) is the discrete-time analogue to the continuous-time version of the model with one risky asset (here we focus on stock-market portfolio holdings). Notice that in this case of a single risky asset, the condition for insurability (diversifiability) of labor-income risk becomes $\rho_{ys}^2 = 1$, so labor-income risk is uninsurable if $\rho_{ys} \notin \{-1, 1\}$. Let,

$$\theta \equiv \frac{1 - \frac{1}{\eta}}{1 - \gamma} ,$$

which transforms (45) into,

$$V(a_t, y_t) = \max_{(c_t, \phi_{t+1})} \frac{\left\{(1-\beta)(c_t - \chi)^{(1-\gamma)\theta} + \beta\left\{(1-\gamma) E_t\left[V\left(R_{p,t+1} a_t + y_t - c_t \; , \; y_{t+1}\right)\right]\right\}^{\theta}\right\}^{\frac{1}{\theta}}}{1-\gamma} . \tag{52}$$

2.2 Necessary conditions

Applying the envelope theorem on (52),

$$V_a(a_t, y_t) = \frac{\partial\left[RHS \text{ of eq. } (52)\right]}{\partial a_t} , \tag{53}$$

while the first-order conditions of (52) with respect to c_t give,

$$(1-\beta)(c_t - \chi)^{(1-\gamma)\theta - 1}\left\{(1-\beta)(c_t - \chi)^{(1-\gamma)\theta} + \beta\left\{(1-\gamma) E_t\left[V(a_{t+1}, y_{t+1})\right]\right\}^{\theta}\right\}^{\frac{1}{\theta}-1} =$$

$$= \frac{\partial\left[RHS \text{ of eq. } (52)\right]}{\partial a_t} . \tag{54}$$

In equilibrium, the optimal sequence $\left\{\left(c_t^*, \phi_{t+1}^*, x_{t+1}^*\right)\right\}_{t=0}^{\infty}$ satisfies the Bellman equation given by (52), so, after discarding the max operator, (52) gives,

$$\left\{(1-\beta)(c_t^* - \chi)^{(1-\gamma)\theta} + \beta\left\{(1-\gamma) E_t\left[V\left(a_{t+1}^*, y_{t+1}\right)\right]\right\}^{\theta}\right\} = \left[(1-\gamma) V(a_t, y_t)\right]^{\theta} . \tag{55}$$

Combining equations (55) and (54) we obtain,

$$(1-\beta)(c_t^* - \chi)^{(1-\gamma)\theta - 1}\left[(1-\gamma) V(a_t, y_t)\right]^{1-\theta} = \frac{\partial\left[RHS \text{ of eq. } (52)\right]}{\partial a_t} . \tag{56}$$

So, combining (53) with (56) we obtain,

$$c_t^* = C(a_t, y_t) = \chi + \left\{\frac{1}{1-\beta} V_a(a_t, y_t)\left[(1-\gamma) V(a_t, y_t)\right]^{\theta - 1}\right\}^{\frac{1}{(1-\gamma)\theta - 1}} . \tag{57}$$

Equation (57) is crucial for solving the model numerically using value-function iteration. Equation (57) states that, once we have a guess for the value function, $V(a,y)$, we immediately have a closed-form solution for the decision rule, $C(a,y)$, which depends only on $V(a,y)$and $V_a(a,y)$. So, if we use a projection method for approximating $V(a,y)$, then we can immediately incorporate the formula given by (57) into the RHS of the Bellman equation. Most importantly, equation (57) helps in the direct computation of portfolio shares, directly from the first-order condition with respect to ϕ.

The first-order condition with respect to ϕ implies,

$$E_t\left[V_a\left(R_{p,t+1}a_t + y_t - c_t\ ,\ y_{t+1}\right)\left(R_{t+1} - r^f\right)\right] = 0\ ,$$

the detailed version of which is,

$$\underbrace{E_t\left\{V_a\left(\left[\left(R_{t+1} - r^f\right)\phi_t + r^f\right]a_t + y_t - C(a_t,y_t)\ \ ,\ y_{t+1}\right)\left(R_{t+1} - r^f\right)\right\}}_{\substack{\shortparallel \\ h(\phi_t,a_t,y_t)}} = 0\ . \qquad (58)$$

So, based on (58), the decision rule for the portfolio share $\phi_t^* = \Phi(a_t,y_t)$, is the implicit function that solves,

$$h(\Phi(a,y), a, y) = 0\ . \qquad (59)$$

2.3 Algorithm: Value-Function Iteration

2.3.1 Overview

We use an initial guess on the value function V defined by (52), $V^{(0)}$. Then we utilize the contraction-mapping property of the Bellman equation described by the recursion,

$$V^{(j+1)}(a_t,y_t) = \max_{(c_t,\phi_{t+1})} \frac{\left\{(1-\beta)(c_t-\chi)^{(1-\gamma)\theta} + \beta\left\{(1-\gamma)E_t\left[V^{(j)}\left(R_{p,t+1}a_t + y_t - c_t\ ,\ y_{t+1}\right)\right]\right\}^{\theta}\right\}^{\frac{1}{\theta}}}{1-\gamma}\ . \qquad (60)$$

in order to generate a Cauchy sequence $\left\{V^{(j)}\right\}_{j=0}^{\infty}$ with $V^{(j)} \to V^*$, which is a typical value-function iteration approach. The key issue in value-function iteration approaches is how one numerically implements the max operator on the right-hand side (RHS) of the Bellman equation. In order to perform maximization on the RHS of (60), we solve the first-order conditions given by (57) and (58), in each step of the recursive procedure, which relies on (the typically incorrect) value function $V^{(j)}$. For deriving the decision rule for consumption, $C^{(j)}(a, y)$ which is conditional upon the value function $V^{(j)}(a, y)$, equation (57) provides an explicit formula,

$$C^{(j)}(a_t, y_t) = \chi + \left\{ \frac{1}{1-\beta} V_a^{(j)}(a_t, y_t) \left[(1-\gamma) V^{(j)}(a_t, y_t) \right]^{\theta - 1} \right\}^{\frac{1}{(1-\gamma)\theta - 1}} . \tag{61}$$

The formula $C^{(j)}(a_t, y_t)$ can be substituted directly into the RHS of (60), but we do have an analytical expression for the decision rule $\Phi^{(j)}(a_t, y_t)$. In order to compute $\Phi^{(j)}(a_t, y_t) \equiv \left\{\phi \mid h^{(j)}(\phi, a, y) = 0\right\}$, we need to numerically solve,

$$h^{(j)}(\phi, a, y) = 0 \ , \tag{62}$$

in which,

$$h^{(j)}(\phi_t, a_t, y_t) \equiv E_t \left\{ V_a^{(j)} \left(\left[\left(R_{t+1} - r^f \right) \phi_t + r^f \right] a_t + y_t - C^{(j)}(a_t, y_t) \ , \ y_{t+1} \right) \left(R_{t+1} - r^f \right) \right\} . \tag{63}$$

Both in (63), and in RHS of (60), there is an expectations operator, $E_t(\cdot)$, that needs to be computed. This computation of the expectations operator is discussed in a separate subsection below.

Another technical necessity in (63) is how to compute $V_a^{(j)}(a, y)$, the partial derivative of the value function. In order to achieve this derivative computation, we employ a simple exponential-projection method which approximates functions using,

$$f(a, y) \simeq \hat{f}(a, y) \equiv e^{\sum_{i=0}^{\nu} \sum_{j=0}^{\nu} \xi_{ij} [\ln(a)]^i [\ln(y)]^j} . \tag{64}$$

An advantage of this $\hat{f}(x)$ approximation given by (64), is that we can take explicit derivatives, namely,

$$f_a(a,y) \simeq \hat{f}_a(a,y) = \hat{f}(a,y) \sum_{i=0}^{\nu} \sum_{j=0}^{\nu} i \cdot \xi_{ij} \frac{[\ln(a)]^{i-1}}{a} [\ln(y)]^j \ . \tag{65}$$

For values of parameter $\gamma > 1$, the mapping $m(\cdot) = (\cdot)^{(1-\gamma)} / (1-\gamma)$, which is applied on the RHS of (60), is known to give negative values. This property, of having negative values for the the RHS of (60), is inherited by the value function on the LHS of (60) as well. Yet, the exponential-projection technique we suggest in (64), can only match positive values. In order to tackle this problem, we use the transformation,

$$V(a,y) = \frac{\left[\tilde{V}(a,y)\right]^{1-\gamma}}{1-\gamma} \Leftrightarrow \tilde{V}(a,y) = [(1-\gamma)V(a,y)]^{\frac{1}{1-\gamma}} \ . \tag{66}$$

A consequence of the transformation given by (66) is,

$$V_a(a,y) = \left[\tilde{V}(a,y)\right]^{-\gamma} \tilde{V}_a(a,y) \ . \tag{67}$$

So, we create a Matlab m-file, named "`Vtilde.m`", which implements the exponential approximation

$$\tilde{V}(a,y) \simeq e^{\sum_{i=0}^{\nu} \sum_{j=0}^{\nu} \xi_{ij} [\ln(a)]^i [\ln(y)]^j} \ , \tag{68}$$

on any grid for the state variables, a and y.

Using this projection approach, we take a first guess on the value function, $\tilde{V}^{(0)}$, and we obtain an estimate of the vector $\left\{ \left\{ \xi_{i,k}^{(0)} \right\}_{i=0}^{\nu} \right\}_{k=0}^{\nu}$ through the "`nlinfit`" command in Matlab. Our first guess, $\tilde{V}^{(0)}$, uses the calibrating parameters that we have found in continuous time, and the continuous-time functional form for the value function, $V(a,y)$ for the special case in which $\rho_{ys} = 1$.

Using the recursive procedure described above, through (60) we generate a sequence of coefficients $\left\{\left\{\left\{\xi_{i,k}^{(j)}\right\}_{i=0}^{\nu}\right\}_{k=0}^{\nu}\right\}_{j=0}^{\infty}$, with $\lim_{j\to\infty}\left\{\left\{\xi_{i,k}^{(j)}\right\}_{i=0}^{\nu}\right\}_{k=0}^{\nu}=\left\{\left\{\xi_{i,k}^{*}\right\}_{i=0}^{\nu}\right\}_{k=0}^{\nu}$, in which

$$V^{*}(a,y)\simeq\frac{e^{(1-\gamma)\sum_{i=0}^{\nu}\sum_{j=0}^{\nu}\xi_{ij}^{*}[\ln(a)]^{i}[\ln(y)]^{j}}}{1-\gamma}\ ,$$

in which $V^{*}(a,y)$ solves (52).

2.3.2 Approximating the joint density for the stochastic process for the interest rate and the labor-income growth

In equations (47), (48), and (49) above, we have mentioned that the model's two shocks ε_s and ε_y are distributed so that,

$$\varepsilon_s\sim N\left(0,\sigma_s^2\right),\ \varepsilon_y\sim N\left(0,\sigma_y^2\right)\ \text{, and}\ \frac{Cov\left(\varepsilon_s,\varepsilon_y\right)}{\sigma_s\sigma_y}=\rho_{ys}\ . \tag{69}$$

We want to compute a joint-probability matrix in order to describe the joint density of shocks,

$$s_{shock}\equiv R_s+\varepsilon_s,\ \text{ and }\ y_{shock}\equiv\mu_y+\varepsilon_y\ , \tag{70}$$

based on the stochastic structure given by (69). The joint density of (s_{shock},y_{shock}) is this of a bivariate normal with,

$$\phi\left(s_{shock},y_{shock}\right)=\frac{1}{2\pi\sigma_s\sigma_y\sqrt{1-\rho^2}}\times$$
$$\times\exp\left\{-\frac{1}{2(1-\rho_{ys}^2)}\left[\frac{(s_{shock}-R_s)^2}{\sigma_s^2}+\frac{(y_{shock}-\mu_y)^2}{\sigma_y^2}-\frac{2\rho_{ys}(s_{shock}-R_s)(y_{shock}-\mu_y)}{\sigma_s\sigma_y}\right]\right\}\ . \tag{71}$$

After some algebraic manipulations, it can be proved that, s_{shock} conditional upon y_{shock} is also normally distributed with,

$$s_{shock}|_{y_{shock}}\sim N\left(R_s+\frac{\sigma_s}{\sigma_y}\rho_{ys}(y_{shock}-\mu_y),(1-\rho_{ys}^2)\sigma_s^2\right)\ ,$$

so,

$$\phi\left(s_{shock}\ ,\ y_{shock}\right)=\phi\left(s_{shock}\mid y_{shock}\right)\cdot\phi\left(y_{shock}\right)\ , \tag{72}$$

in which

$$\phi\left(y_{shock}\right)=\frac{1}{\sigma_y\sqrt{2\pi}}\exp\left[-\frac{\left(y_{shock}-\mu_y\right)^2}{2\sigma_y^2}\right]\ ,$$

since $y_{shock}\sim N\left(\mu_y,\sigma_y^2\right)$. In order to calculate $\phi\left(s_{shock}\mid y_{shock}\right)$ and $\phi\left(y_{shock}\right)$, we use the fact that,

$$\frac{s_{shock}|_{y_{shock}}-\left[R_s+\frac{\sigma_s}{\sigma_y}\rho_{ys}(y_{shock}-\mu_y)\right]}{\sigma_s\sqrt{1-\rho_{ys}^2}}\sim N\left(0,1\right)\ \text{, and}\ \frac{y_{shock}-\mu_y}{\sigma_y}\sim N\left(0,1\right)\ ,$$

and we then use (72) in order to compute $\phi\left(s_{shock}\ ,\ y_{shock}\right)$ in matrix form. So, if the grid for s_{shock} is an $m_s\times 1$ vector and the grid for y_{shock} is an $m_y\times 1$ vector, then let the joint-probability matrix

$$\underbrace{M_{sy}}_{m_s\times m_y}\equiv\left[M_{sy,k\ell}\right]=\left[\phi\left(s_{shock,k}\ ,\ y_{shock,\ell}\right)\right]\ ,\quad k\in\{1,...,m_s\}\,,\ell\in\{1,...,m_y\}\ . \tag{73}$$

For specifying the grids for s_{shock} and y_{shock}, we split the continuum into equispaced intervals, and then we proceed to calculating the probabilities associated with the midpoint of each interval, using Matlab's built-in calculator for the normal density (the command "`normcdf`", which calculates cumulative probabilities for a standard normal).

Because the support of normally distributed variables is $(-\infty,\infty)$, we need to choose an upper and lower level of the support for s_{shock} and y_{shock}. For a standard normal notice that, in Matlab, "`normcdf(-3)=0.0013`", "`normcdf(-10)=7.6199e-24`", "`normcdf(-12)=1.7765e-33`", with the latter being a negligible number. In order to avoid accumulating errors (numbers such as 10^{-33} tend to create this error-accumulation problem), for the lowest gridpoint of s_{shock} (same for y_{shock}) called "$r_{\min}$", we use

$$s_{shock\_\min}=R_s+\sigma_s\cdot(-10)\ ,$$

and for the largest gridpoint we use

$$s_{shock\_\max} = R_s + \sigma_s \cdot (+10) \ ,$$

in which -10 is a calibrating parameter related to the standard normal, ensuring that the suppport of s_{shock} does not have probability kinks at its endpoints, or that there is no error-accumulation problem (after plotting both the joint density function of $(s_{shock} \ , \ y_{shock})$, and individual density functions, we have concluded that the value "`normcdf(-10)=7.6199e-24`" works best. The Matlab m-file "`mkprobmatrix2normals.m`" produces matrix M_{sy}, after inserting the vector $\left(s_{shock}, y_{shock}, R_s, \mu_y, \sigma_s, \sigma_y, \rho_{ys}\right)$ as this m-file's input.

2.4 Computing the portfolio share that satisfies the first-order conditions: applying the expectations operator

First, we choose grids for a and y calculated in accordance with the nonlinear relationship between a and y in the data (see Panel C in Figure 4 and the expression $y_{data} = g\left(a_{data}\right)$ given by (44)). So, we generate two $n \times 1$ vectors, a_{grid} and y_{grid}, that satisfy $y_{grid} = g\left(a_{grid}\right)$. Consider that we are at the j-th iteration of the value-function iteration method, using $V^{(j)}$ for all calculations. At this stage we want to compute the function $h^{(j)}\left(\phi_t, a_t, y_t\right)$ based on (58), and a concern is how to apply the expectations operator in that function. Using a loop, for each $i \in \{1, ..., n\}$, we express function $h^{(j)}\left(\phi_t, a_t, y_t\right)$ in equation (58) as,

$$\sum_{k=1}^{m_s}\sum_{\ell=1}^{m_y} M_{sy,k\ell} \left\{ V_a^{(j)} \left(\left[\left(\underbrace{e^{s_{shock,k}}}_{\overset{\shortparallel}{R_{t+1}}} - r^f \right) \phi_t + r^f \right] \underbrace{a_{grid,i}}_{\overset{\shortparallel}{a_t}} + \underbrace{y_{grid,i}}_{\overset{\shortparallel}{y_t}} \right.\right.$$
$$\left.\left. - \ C \left(\underbrace{a_{grid,i}}_{\overset{\shortparallel}{a_t}} \ , \ \underbrace{y_{grid,i}}_{\overset{\shortparallel}{y_t}} \right) \ , \ \underbrace{y_{grid,i} \cdot e^{y_{shock,\ell}}}_{\overset{\shortparallel}{y_{t+1}}} \right) \left(\underbrace{e^{s_{shock,k}}}_{\overset{\shortparallel}{R_{t+1}}} - r^f \right) \right\} = 0 \ , \qquad (74)$$

in which $V_a^{(j)}(\cdot)$ is given by (67) for a given vector of coefficients $\left\{\left\{\xi_{i,k}^{(j)}\right\}_{i=0}^{\nu}\right\}_{k=0}^{\nu}$. The expression given by (74) defines a function $h^{(j)}(\phi, a_{grid,i}, y_{grid,i})$, for each $i \in \{1, ..., n\}$. We use Matlab's "`fsolve`" routine in order to solve the nonlinear equation $h^{(j)}(\phi, a_{grid,i}, y_{grid,i}) = 0$, so,

$$\phi^{(j)}(a_{grid,i}, y_{grid,i}) = \left\{\phi \mid h^{(j)}(\phi, a_{grid,i}, y_{grid,i}) = 0\right\} \quad \text{for all } i \in \{1, ..., n\} \ . \tag{75}$$

2.5 Performing value-function iteration

Here we use the Bellman equation given by (60) in order to perform value function iteration. We use (75) and (61) in order to incorporate $\phi^{(j)}(a_t, y_t)$ and $C^{(j)}(a_t, y_t)$ into the RHS of (60). One difficulty is the computation of the expectations term on the RHS of (60). We use,

$$E_t\left[V^{(j)}(R_{p,t+1}a_t + y_t - c_t \ , \ y_{t+1})\right] = \sum_{k=1}^{m_s}\sum_{\ell=1}^{m_y} M_{sy,k\ell}\left\{V^{(j)}\left(\left(\left[\left(\underbrace{e^{s_{shock,k}}}_{\substack{\shortparallel \\ R_{t+1}}} - r^f\right)\phi_t + r^f\right]\underbrace{a_{grid,i}}_{\substack{\shortparallel \\ a_t}}\right.\right.\right.$$

$$\left.\left.\left. + \underbrace{y_{grid,i}}_{\substack{\shortparallel \\ y_t}} - C\left(\underbrace{a_{grid,i}}_{\substack{\shortparallel \\ a_t}} \ , \ \underbrace{y_{grid,i}}_{\substack{\shortparallel \\ y_t}}\right) \ , \ \underbrace{y_{grid,i} \cdot e^{y_{shock,\ell}}}_{\substack{\shortparallel \\ y_{t+1}}}\right)\left(\underbrace{e^{s_{shock,k}}}_{\substack{\shortparallel \\ R_{t+1}}} - r^f\right)\right\} . \tag{76}$$

Because the curvature of the value function is more profound at low income/wealth levels, we adjust the grids for a and y so that they are more dense at low income/wealth levels. This strategy allows us to obtain efficient approximations even with 35 gridpoints for a_{grid} and y_{grid} in total (e.g., raising the number of gridpoints to 150 does not make an essential difference). Convergence in value function and/or coefficients $\left\{\left\{\xi_{i,k}\right\}_{i=0}^{\nu}\right\}_{k=0}^{\nu}$, is usually achieved in about 2 minutes for each model parameterization in Figure 6. Producing all graphs in Figure 6 takes about 12 minutes on a state-of-the art laptop.

2.6 Ensuring that consumption is above subsistence and treatment of borrowing constraints

The utility function we use satisfies an Inada condition as $c \to \chi$, which is obvious from (54). The RHS of (57) has the interpretation that, as long as V^* is well-defined, $c > \chi$ is guaranteed. In order to implement a borrowing constraint of the form $a_{t+1} \geq b$ we modify (60) as,

$$V^{(j+1)}(a_t, y_t) =$$

$$\max_{(c_t, \phi_{t+1})} \frac{\left\{(1-\beta)(c_t-\chi)^{(1-\gamma)\theta} + \beta\left\{(1-\gamma)E_t\left[V^{(j)}\left(\max\left\{R_{p,t+1}a_t + y_t - c_t, b\right\}\big|_{a_t=b}\ ,\ y_{t+1}\right)\right]\right\}^{\theta}\right\}^{\frac{1}{\theta}}}{1-\gamma}. \tag{77}$$

using an indicator function in order to implement the conditionality operator $(\cdot)|_{a_t=b}$. As in our continuous-time analysis, the presence of the borrowing constraint has not affected our results. For our borrowing constraint $b = \underline{a}$, at this level of wealth ($\underline{a}$), and for all gridpoints for y, households chose interior solutions.

3. Calculating the correlation coefficient between risky-asset returns and labor-income growth

3.1 Labor-income dynamics: PSID 1970-2009

We use data from the Panel Study of Income Dynamics (PSID) between 1970 - 2009 in order to estimate the labor-income growth component that cannot be explained by household-demographic characteristics such as age, marital status, household composition, and some other (perhaps unobservable) family characteristics, such as cultural background, peer effects, etc. This labor-income growth component is our data proxy for variable y_{shock}, as defined by (47) and (70). The main estimation procedure follows Cocco, Gomes and Maenhout (2005). Cocco, Gomes and Maenhout (2005) restrict their sample to households headed by males. Unlike them, we keep households with both males and females as a household head, since we focus on explaining stockholding data from the Survey of Consumer Finances (SCF), in which we have not distinguished the gender of household heads. To single out the retirement behavior, which is abstract away from our model, we keep a subsample by eliminating retirees, nonrespondents and students.

Our definition of labor income is relatively inclusive in terms of fiscal transfers and government benefits, in order to focus on the pure absence of self-insuring potential against labor-income risk. We define labor income as total reported labor income plus unemployment compensation, workers compensation, social security, supplemental social security, other welfare, child support, and total transfers (mainly help from relatives). These calculations have been made for both the head of household and if a spouse is present we drop zero-income observations. We also deflate labor income using the Consumer Price Index, with 1992 as the base year.

We regress the logarithm of labor income on dummies for age, family fixed effects, marital status, and household composition. Using fixed-effect estimation, the econometric-model specification is,

$$y_{i,t} = \alpha + \mu_i + \mathbf{X}_{i,t}\boldsymbol{\beta} + \varepsilon_{i,t} \ , \quad y_{i,t} \equiv \ln(Y_{i,t}) \ , \tag{78}$$

in which $\mathbf{X}_{i,t}$ is the set of control variables. In order to explore the error structure further, we generate the residual from the above fitted model (78),

$$\widehat{\varepsilon_{i,t}} = y_{i,t} - \widehat{y_{i,t}} \ . \tag{79}$$

Combining (78) and (79), we formulate the cross-sectional mean of the unexplained part of the labor-income growth rate $\overline{\Delta\widehat{y_t}}$, as

$$y_{shock} \equiv \overline{\Delta\widehat{y_t}} = \frac{\sum_{i=1}^{N} \Delta\widehat{y_{i,t}}}{N} = \frac{\sum_{i=1}^{N} \widehat{\varepsilon_{i,t}} - \sum_{i=1}^{N} \widehat{\varepsilon_{i,t-1}}}{N} \ , \tag{80}$$

which y_{shock} is the labor-income-shock concept that we use in the theoretical model.

3.2 Risky-asset returns

For generating the time series of risky-asset returns, we use the Standard and Poor's (S&P) stock-market index from 1970 to 2009, and calculate S&P-index returns as annual averages. The formula of the variable proxying s_{shock} in our theoretical model is,

$$s_{shock} \equiv \frac{\text{S\&P Index}_t}{\text{S\&P Index}_{t-1}} - 1 \ . \tag{81}$$

3.3 Correlation coefficient between risky-asset returns and labor-income growth

Table A.1 gives the correlation coefficient between y_{shock} and s_{shock}.

	Full Sample	**College Graduates**
Sample Period	1970 - 2009	1970 - 2009
$corr(y_{shock}, s_{shock})$	31.89%	50.78%

Table A.1

For the full sample, the correlation coefficient is about 32%. Because stockholders tend to have higher educational level, we also focus on college graduates by restricting the PSID-sample to college graduates, finding a higher number, which is about 51%.

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Online Data Appendix A: European Household Finance and Consumption Survey 2013

for

Fitting Parsimonious Household-Portfolio Models to Data

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1. Data

The Eurosystem Household Finance and Consumption Survey (HFCS) is a joint project of all central banks of the Eurosystem. HFCS includes detailed household-level data on various aspects of household balance sheets and related economic and demographic variables, including income, pensions, employment, gifts and measures of consumption.

HFCS provides country-representative data, which have been collected in 15 euro area members for a sample of more than 62,000 households. These 15 countries are Belgium, Germany, Greece, Spain, France, Italy, Cyprus, Luxembourg, Malta, Netherlands, Austria, Portugal, Slovenia, Slovakia, and Finland.

For each country we consider only household heads between age 25 and 65 years old, which retained 42,553 households from the original sample. In addition, we also dropped households with zero income (215 observations).

The HFCS survey uses a multiple stochastic imputation strategy to recover the missing value or the non-responding households. It provides five imputed values (replicates) for every missing value corresponding to a variable.[1] We calculate the multiple imputed mean and standard deviation of our targeted variables (gross income and portfolio share on stocks) in Table 1. In Table 2, we calculate the mean of portfolio share on stocks for all Eurosystem countries, classifying by income category across the income distribution.

2. Definition of Variables

1. Stock Equity (direct and indirect stockholding excluding any pension accounts.)

    - Publicly Traded Stocks.

[1] A detailed description of the imputation procedure applied in the HFCS is given in chapter 6 of the Eurosystem Household Finance and Consumption Survey methodological report for the first wave. (https://www.ecb.europa.eu/pub/pdf/other)

- Mutual Funds: it includes funds predominately in equity, bonds, money market instruments, real estate, hedge funds and other fund types. The share of stock holding is adjusted conditional on fund types.[2]

2. Total Financial Assets: it includes deposits (sight accounts, saving accounts), investments in mutual funds, bonds, investments held in non-self-employment private businesses, publicly traded shares, managed investment accounts, money owed to households as private loans, other financial assets (options, futures, index certificates, precious metals, oil and gas leases, future proceeds from a lawsuit or estate that is being settled, royalties or any other), private pension plans and whole life insurance policies. However, current value of public and occupational pension plans is not included.

3. Total Income: it is measured as gross income and is defined as the sum of labor and non-labor income for all household members. Labor income is collected for all household members aged 16 and older, other income sources are collected at the household level. In some countries, as gross income is not well known by respondents it is computed from the net income given by the respondent. Specifically, the measure for gross income includes the following components: employee income, self-employment income, income from pensions, regular social transfers, regular private transfers, income from real estate property (income received from renting a property or land after deducting costs such as mortgage interest repayments, minor repairs, maintenance, insurance and other charges), income from financial investments (interest and dividends received from publicly traded companies and the amount of interest from assets such as bank accounts, certificates of deposit, bonds, publicly traded shares etc. received during

[2] Note: stockholding from any public and occupational pension plans or individual retirement accounts are not included in our calculation.

the income reference period less expenses incurred), income from private business and partnerships and other non-specified sources of income.[3]

4. Weight: weights are assigned in order to normalize the sample to representative-sampling standards. [4]

5. Income Percentiles: they are generated from the variable "total income".

3. Portfolio Share of Stockholding

We define the portfolio share of stockholding for income group j of country k as,

$$\phi_j^k = \frac{\sum_{n=1}^{N_j^k} \frac{\text{Stock}_{i,j,k}}{\text{Total Financial Assets}_{i,j,k}}}{N_j^k}$$

where N_j^k is the amount of households within income group j of country k.

REFERENCES

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[3] See section 9.2.4 of the Methodological Report in details on the collection of income variables in various countries.

[4] All statistics in this document are calculated using the household weight provided. Within each country, the sum of estimation weights equals the total number of households in the country, so that the sum of weights in the whole dataset equals the total number of households in the 15 countries participating in the 1st wave of the survey.

Table 1: Portfolio Share on Stocks by Income Percentile

Country Code	Income Percentiles	Gross Income (EUR)	Gross Income(s.e.)	Portfolio Share on Stocks	Portfolio Share on Stocks(s.e.)
AT	20	12,976.54	256	1.02%	0.0022
AT	40	24,967.37	404	1.08%	0.0028
AT	60	37,117.68	632	1.96%	0.0071
AT	80	53,357.09	1874	2.41%	0.0045
AT	100	109,326.63	15025	3.22%	0.0047
BE	20	9,281.07	54	2.06%	0.0003
BE	40	24,812.93	253	2.27%	0.0011
BE	60	40,320.70	354	2.96%	0.0020
BE	80	62,584.55	277	4.06%	0.0021
BE	100	138,496.66	3590	7.22%	0.0041
CY	20	12,187.64	152	5.55%	0.0081
CY	40	26,616.23	237	5.92%	0.0124
CY	60	37,687.29	372	6.93%	0.0077
CY	80	54,505.03	739	8.38%	0.0113
CY	100	115,287.60	3738	10.89%	0.0152
DE	20	11,943.90	226	0.53%	0.0013
DE	40	26,100.98	215	1.01%	0.0016
DE	60	39,488.18	192	1.81%	0.0012
DE	80	56,594.20	123	2.27%	0.0011
DE	100	115,300.16	632	4.02%	0.0011
ES	20	10,789.32	40	1.56%	0.0005
ES	40	20,483.62	25	2.14%	0.0011
ES	60	28,516.08	61	2.19%	0.0023
ES	80	39,324.07	176	2.55%	0.0011
ES	100	79,661.73	367	5.96%	0.0021
FI	20	15,486.37	0	2.63%	0.0000
FI	40	30,490.31	0	6.19%	0.0000
FI	60	44,594.16	0	6.47%	0.0000
FI	80	61,658.27	0	8.06%	0.0000
FI	100	105,894.09	0	16.87%	0.0000
FR	20	13,363.59	0	1.30%	0.0002
FR	40	23,818.85	0	2.45%	0.0003
FR	60	33,383.88	0	3.13%	0.0011
FR	80	45,058.05	0	4.24%	0.0004
FR	100	87,867.95	0	8.78%	0.0008
GR	20	9,343.90	53	0.62%	0.0009
GR	40	18,067.57	56	0.28%	0.0028
GR	60	25,897.98	68	0.84%	0.0043
GR	80	36,438.48	67	2.67%	0.0058
GR	100	68,567.06	758	1.75%	0.0040
IT	20	10,963.57	0	0.18%	0.0000
IT	40	21,951.38	0	0.99%	0.0000
IT	60	31,572.36	0	2.03%	0.0000
IT	80	44,861.28	0	1.29%	0.0000
IT	100	84,829.13	0	4.81%	0.0000
LU	20	23,090.13	330	1.50%	0.0071
LU	40	45,705.58	529	1.39%	0.0070
LU	60	68,371.82	227	2.36%	0.0036

LU	80	99,800.81	633	3.91%	0.0067
LU	100	210,510.78	1895	10.32%	0.0055
MT	20	7,749.43	70	2.81%	0.0040
MT	40	14,410.10	32	4.03%	0.0027
MT	60	21,843.21	96	3.55%	0.0029
MT	80	32,611.60	91	3.13%	0.0025
MT	100	55,681.82	329	6.91%	0.0012
NL	20	15,827.96	638	0.69%	0.0024
NL	40	32,431.63	698	0.72%	0.0061
NL	60	43,275.19	617	1.51%	0.0080
NL	80	57,456.25	604	1.21%	0.0040
NL	100	91,322.43	1316	1.35%	0.0008
PT	20	5,634.04	93	0.01%	0.0001
PT	40	11,801.17	66	0.28%	0.0003
PT	60	16,916.69	92	0.50%	0.0005
PT	80	24,892.49	142	1.03%	0.0004
PT	100	55,466.72	233	4.17%	0.0014
SI	20	2,976.53	121	6.20%	0.0009
SI	40	12,617.03	175	5.82%	0.0042
SI	60	22,103.94	146	5.93%	0.0063
SI	80	31,954.13	573	5.86%	0.0263
SI	100	60,898.29	957	7.27%	0.0187
SK	20	5,215.67	69	0.05%	0.0002
SK	40	9,139.51	77	0.05%	0.0003
SK	60	12,591.09	26	0.23%	0.0004
SK	80	16,646.84	56	0.25%	0.0009
SK	100	30,152.51	256	0.32%	0.0002

Note: s.e. stands for the multiple imputed standard errors.

Source: European Household Finance and Consumption Survey 2013

Table 2: Portfolio Share on Stocks by Income Percentile (EU mean)

Region	Income Percentiles	Portfolio Share on Stocks
Euro System Countries	20	1.779%
	40	2.308%
	60	2.828%
	80	3.422%
	100	6.258%

Source: European Household Finance and Consumption Survey 2013

Online Data Appendix B: Chinese Household Finance Survey – 1^{st} Wave 2013

for

Fitting Parsimonious Household-Portfolio Models to Data

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1. Data

The China Household Finance Survey (CHFS) is conducted by the survey and research center for China Household Finance, which is based at Southwestern University of Finance and Economics. This survey is the only nationally representative survey in China that has detailed information about household finance and assets, including housing, business assets, financial assets, and other household assets. In addition, the survey also provides information about income and expenditures, social and commercial insurance, and more.

We use the 1st survey that was conducted in summer 2011 with a sample size of 8,438 households and 29,500 individuals,which covers 21 provinces (including the autonomous regions) and 4 Municipalities (Beijing, Shanghai, Tianjin and Chongqing). This survey employs a stratified 3-stage probability proportion to size (PPS) random sample design, which is necessary to ensure that the survey is nationally representative[1] .

We consider only household heads between age 25 years old and 65 years old, which retained 6,952 households. In addition, we have dropped households with zero income (226 observations).

2. Definition of Variables

1. Stock Equity (direct and indirect stockholding excluding any pension account)

    - Publicly Traded Stocks.
    - Non Publicly Traded Stocks.
    - Mutual Funds: it includes funds predominatly in equity, bonds, money market instruments, also includes mixed stratergy funds and other types.

[1] Details about the sampling design could refer http://www.chfsdata.org/detail-14,15.html.

- Financial Products (categorized as Wealth Management Products)

2. Total Financial Assets: it comprises total balance of demand deposits, total balance of time deposits, stocks (public traded and non-public traded), bonds, mutual funds, derivatives, warrants, other financial derivatives, financial products, foreign currency assets, gold, cash at home and other type of liquid assets.

3. Total Income: it includes income from all sources (salary, interest, dividend, compensations, transfers etc).

4. Weight: weights are assigned in order to normalize the sample to representative-sampling standards, the weight variable in the data is "swgt".

5. Income Percentiles: they are generated from variable "total income".

3. Portfolio Share of Stockholding

We define the portfolio share of stockholding for income group j as,

$$\phi_j = \frac{\sum_{n=1}^{N_j} \frac{\text{Stock}_{i,j}}{\text{Total Financial Assets}_{i,j}}}{N_j}$$

where N_j is the amount of households within income group j. Table 1 shows the detailed information of portfolio share on stocks across the income distribution, together with inforamtion on total asset holding and total financial-asset holding.

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Table 1: Portfolio Share on Stocks by Income Percentile

Country Code	Income Percentiles	Portfolio Share on Stocks (%)	Gross Income (CNY)	Total Assets (CNY)	Total Financial Assets (CNY)
China	20	1.443	4,220.57	396,191.75	23,482.43
	40	1.273	16,367.13	275,096.63	16,263.76
	60	3.150	30,631.76	440,228.28	27,103.34
	80	5.414	52,124.18	594,769.38	37,879.68
	100	11.860	208,030.73	1,810,124.50	167,894.25

Source: Chinese Household Finance Survey 1st Wave 2013

Online Data Appendix C: US Survey of Consumer Finances (SCF) 2007

for

Fitting Parsimonious Household-Portfolio Models to Data

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1 Description of Variables (Source: Survey of Consumer Finances (SCF) 2007)

1. `Stock Equity (Direct and Indirect Stockholding)`:

   (a) **Direct stockholding**
   - Publicly Traded Stocks.

   (b) **Stockholding through mutual funds**
   - Saving and Money Market Accounts.
   - Mutual Funds.
   - Annuities, Trusts and Managed Investment Accounts.

   (c) **Stockholding through Retirement Accounts**
   - IRA/KEOGH Accounts.
   - Past Pension Accounts.
   - Current Benefits and Future Benefits from Pensions.

2. `Business Equity`:
   - Actively Managed Business.
   - Non-Actively Managed Business.

3. `Total Assets`: Assets of all categories covered in the SCF 2007 database (stocks, business equity, bonds, saving and checking accounts, retirement accounts, life insurance, primary residence, and other residential real estate, nonresidential real estate, vehicles, artwork, jewelry, etc.).

4. `Total Income`: Income from all sources (salary, interest, dividend, compensations, transfers etc).

5. `Weight`: Weights are assigned in order to normalize the sample to representative-sampling standards (see the section "Analysis Weights" in the "Codebook for the 2007 Survey of Consumer Finances").[1]

6. `Income Percentiles`: Benchmark value from Bucks et al. (2009a, Table A.2, p. A53).

[1] The "Codebook for the 2007 Survey of Consumer Finances" is downloadable from `http://federalreserve.gov/econresdata/scf/scf_2007documentation.htm`

7. `Equivalence Scales`: The equivalence scale is $\sqrt{n}$ in which n is the number of household members. This equivalence-scale measure approximates the standard OECD equivalance scales.

Table 1: **Income Percentiles**

Percentile	Total Labor Income
20	20,600
40	36,500
60	59,600
80	98,200
90	140,900

Notes: Full sample in 2007 USD. Data in the survey is in 2006 USD, which is adjusted according to the CPI-U table (U.S. Department of Labor Bureau of Labor Statistics, Consumer Price Index). The 2006-2007 average to average change is 2.84%.

2 Matching Data with Descriptive Statistics in the SCF 2007 Chartbook

To show that our database is constructed in a reliable way, we compare key statistics with those reported in the SCF2007 Chartbook. Our robustness checks are:

- **Matching median values of key variables in the SCF 2007 chartbook**:
  The reason for choosing medians instead of means in order to perform a robustness check is that median values capture more information regarding a variable's distribution. In addition, mean values can be substantially affected by outliers. Indeed, our database matches median values in the SCF2007 chartbook.

- **Matching median values of each income group in the SCF 2007 chartbook**:
  Our database generated should match the income benchmark in small differences by income quintile or decile, which is a more demanding task. Our results are listed in the following tables demonstrate that the matching is satisfactory.

Table 2: **Median Values of Key Variables**

Variables	SCF2007 Chartbook	Our Data
Total Asset	221.5	221.9
Total Income	47.3	46.5
Stock Equity	35.0	34.8
Business Equity	100.5	80.6

Notes: Full sample. Values in thousands of 2007 US dollars.

Table 3: **Median Values of Pre-Tax Family Income for All Families, Classified by Income Percentile**

Income Percentile	SCF2007 Chartbook	Our Data
Less than 20%	12.3	12.3
20%-39.9%	28.8	28.8
40%-59.9%	47.3	47.1
60%-79.9%	75.1	74.9
80%-89.9%	114.0	114.8
90%-100%	206.9	209.0

Notes: Full sample, in thousands of 2007 US dollars. Data in the survey are in 2006 US dollars. We adjusted them according to the CPI-U table (U.S. Department of Labor Bureau of Labor Statistics, Consumer Price Index). 2006-2007 Average to Average change is 2.84% .

Table 4: **Median Values of Total Assets for Families with Positive Asset Holdings, Classified by Income Percentile**

Income Percentile	SCF2007 Chartbook	Our Data
Less than 20%	23.5	26.1
20%-39.9%	84.9	90.1
40%-59.9%	183.5	182.2
60%-79.9%	343.1	345.6
80%-89.9%	567.5	561.2
90%-100%	1358.4	1355.5

Notes: Full sample, in thousands of 2007 US dollars.

Table 5: **Median Values of Different Asset Types for Families with Positive Asset Holdings by Income Percentile**

	SCF2007 Chartbook		Our Data	
Income Percentile	**Stocks**	**Business**	**Stocks**	**Business**
Less than 20%	6.5	50.0	7.3	43.6
20%-39.9%	8.4	19.5	10.0	20.4
40%-59.9%	17.7	30.8	17.4	33.8
60%-79.9%	34.2	55.1	33.8	41.6
80%-89.9%	62.0	72.1	59.1	70.6
90%-100%	219.6	379.5	223.8	381.0

Notes: Full sample, in thousands of 2007 US dollars.

3 Portfolio Shares of Risky Assets

Portfolio shares of risky assets are calculated by income groups. For each income group we have the formula,

$$SHARE_i = \frac{\sum_k \frac{\sum_n SHARE_{obs(n)}}{N}}{K} \ ,$$

in which n is the observation number, k is the imputation number and i is the risky-asset type. Final results are shown in the following tables. SCF weights are not shown in the above formula but have been included in the calculation. The comparison between Tables 6 and 7 justifies why we did not restrict the full sample into a particular age range such as household heads aged between 25-59 years old. Demographic or life-cycle biases seem to play a rather mild role, so we have chosen to utilize the entirety of the infomation provided by the SCF 2007 database in our calibration exercises.

Table 6: **Portfolio Share on Risky Assets by Income Percentile (Full Sample per Equivalent Adult)**

	Risky Assets (%)		General Information			Tax Information	
Income Percentile	**Stocks**	**Business**	**Total Income**	**Total Assets**	**Income/Asset (%)**	**Effective Marginal Tax Rate**	**After-tax Income**
Generated Data							
Less than 20%	2.44	3.24	9.03	85.52	10.56	-1.83%	9.19
20%-39.9%	5.84	1.84	19.42	139.82	13.89	2.78%	18.88
40%-59.9%	7.72	3.97	32.20	210.93	15.27	6.47%	30.11
60%-79.9%	12.44	4.51	49.84	327.22	15.23	14.28%	42.72
80%-89.9%	15.96	6.14	74.61	511.32	14.60	22.63%	57.73
90%-100%	20.53	24.55	252.12	2452.22	10.28	29.27%	178.33

Notes: Full sample, in thousands of 2007 US dollars.

Table 7: **Portfolio Share on Risky Assets by Income Percentile (Age group 25-59 per Equivalent Adult)**

	Risky Assets (%)		General Information			Tax Information	
Income Percentile	**Stocks**	**Business**	**Total Income**	**Total Assets**	**Income/Asset (%)**	**Effective Marginal Tax Rate**	**After-tax Income**
Generated Data							
Less than 20%	2.78	3.35	10.15	73.74	13.76	-1.83%	10.33
20%-39.9%	5.05	3.35	22.93	99.79	23.00	2.78%	22.30
40%-59.9%	8.32	3.73	37.60	188.48	19.95	6.47%	35.17
60%-79.9%	12.90	4.74	54.49	303.89	17.93	14.28%	46.71
80%-89.9%	14.89	7.26	78.69	475.10	16.57	22.63%	60.88
90%-100%	18.52	25.31	242.57	1908.87	12.71	29.27%	171.57

Notes: Age group 25-59, in thousands of 2007 US dollars.

References

[1] Bucks, Brian K., Arthur B. Kennickell, Traci L. Mach and Kevin B. Moore (2009a): "Changes in U.S. Family Finances from 2004 to 2007: Evidence from the Survey of Consumer Finances," Survey of Consumer Finances, Board of Governors of the Federal Reserve System.

[2] Bucks, Brian K., Arthur B. Kennickell, Traci L. Mach and Kevin B. Moore (2009b): "2007 SCF Chartbook," Survey of Consumer Finances, Board of Governors of the Federal Reserve System.

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