Macroeconomic Effects of Demographic Change: The Role of Human Capital

Inaugural-Dissertation
zur Erlangung des Doktorgrades
des Fachbereichs Wirtschaftswissenschaften
der Johann Wolfgang Goethe-Universität
Frankfurt am Main

vorgelegt von

Christian Geppert

aus Stuttgart

Frankfurt am Main, 2015

This page was intentionally printed only with this statement.

Für Christine, Rainer und Lisa mit Roga und Hannah

This page was intentionally printed only with this statement.

Vorwort

Diese Dissertation ist über einen Zeitraum von rund fünf Jahren durch die Arbeit an vier verschiedenen Orten - Köln, Phoenix, Paris und Frankfurt - entstanden. Ihr liegt ein intensiver Lern- und Erfahrungsprozess zugrunde; angefangen bei Doktorandenkursen und zahlreichen Seminarvorträgen an der Universität zu Köln, über etliche Stunden des Selbststudiums bis hin zu eigener Lehrtätigkeit, Vorträgen und dem intellektuellen Austausch mit Gleichgesinnten auf einer Vielzahl von wissenschaftlichen Konferenzen. Es gibt einige Menschen, von denen ich dabei im Hinblick auf die Erstellung dieser Arbeit besonders profitiert habe und denen mein tief empfundener Dank gebührt.

Herauszuheben ist mein Doktorvater Alexander Ludwig, der mich während meiner gesamten Doktorandenzeit gefördert (und auch gefordert) hat, mir das Thema ”demographischer Wandel” näher gebracht und mit seinem schier unerschöpflichen Fachwissen mit Rat und Tat zur Seite gestanden hat. Seine engagierte und realitätsnahe Arbeitsweise ist mir stets ein Vorbild. Des Weiteren möchte ich gerne meinen Kollegen danken - allen voran Laura Kohlleppel, Désirée Rückert und dem 2012er-Jahrgang des Economics-PhD-Programms der Arizona State University, die mit mir ein Büro geteilt und mir fachlich und emotional durch manch schwierigen Tag geholfen haben.

Der größte Teil dieser Dissertation ist in meiner Zeit an der Universität zu Köln entstanden. Mein Dank gilt daher ausdrücklich dem Center for Macroeconomic Research an der Universität zu Köln, das mir durch optimale Arbeitsbedingungen den erfolgreichen Abschluss dieser Dissertation ermöglicht hat. Des Weiteren danke ich den Gutachtern dieser Arbeit für ihre hilfreichen Anmerkungen und ihre außerordentlich zeitnahen Bewertungen, die zum zügigen Abschluss des Promotionsverfahrens geführt haben.

Abgesehen von allen fachlichen Herausforderungen wäre diese Dissertation niemals zu einem guten Abschluss gekommen, wenn mir die emotionale Unterstützung meiner Familie und meiner Freunde nicht die Motivation geschenkt hätte, mich immer und immer wieder mit neuen Problemen auseinander zu setzen, die während eines solch langen und komplexen Projektes auftreten. Insbesondere meine Eltern, Christine und Rainer, meine Schwester Lisa mit Roga und Hannah, sowie Chanti, Benni, Pepe, Lena und Benni mit Mavie haben einen unermesslichen Anteil daran.

This page was intentionally printed only with this statement.

Contents

This page was intentionally printed only with this statement.

List of Figures

This page was intentionally printed only with this statement.

List of Tables

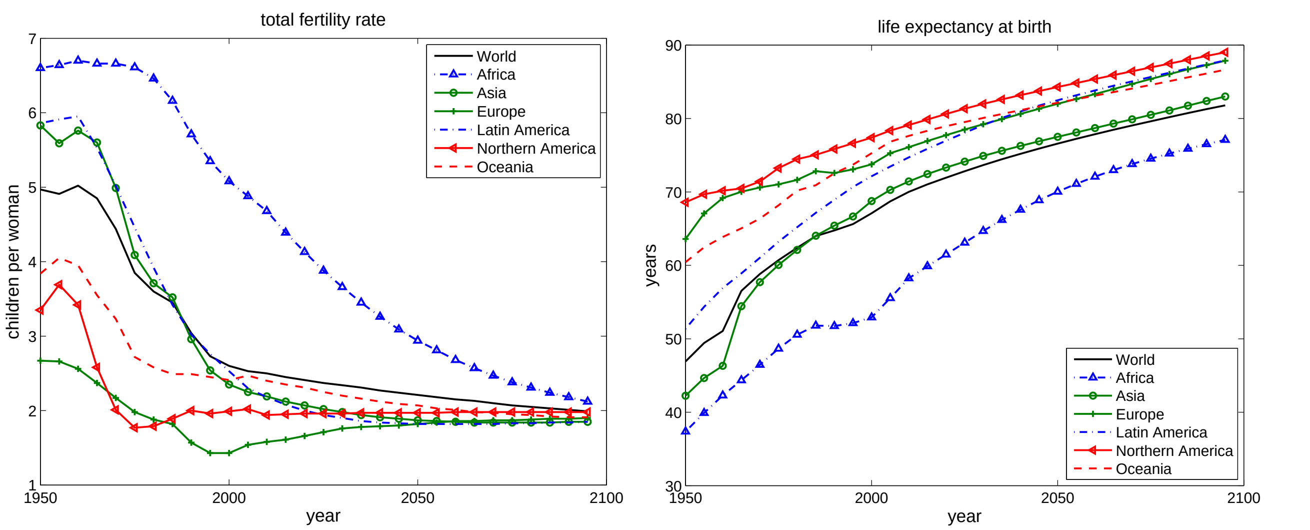
This page was intentionally printed only with this statement.

Introduction

"Skills are the currency for the 21st century – they will be decisive for economies, societies and the prospects of people." (OECD, 2013)

Demographic change belongs to the mega-trends of the 20th and the 21st century. While major industrialized countries like Japan, Germany, or Italy exhibit already strongly aging populations, emerging economies like China and India are going to follow the same process in the future. This process is driven by the two major forces; declining fertility rates and increasing life expectancy. Figure 0.1 depicts data and forecasts of demographic researchers of the United Nations for all six continents and the world (cf. United Nations, 2013). The left panel shows the total fertility rate which is measured as children per woman. Since the invention of

Figure 0.1: Demographic Data and Projections for the Continents and the World



Source: United Nations (2013), data and medium fertility prospects. *Notes*: The total fertility rate equals the number of children per woman over her reproductive period. Life expectancy measures the average years a newborn lives from the respective period onward.

the birth control pill in the 1960s, fertility has declined massively. While a woman had on average about 5 children back in 1950, the number dropped to less than 3 in 2000 and is expected to decline further to about 2 by 2100. The right panel shows life expectancy of an individual at birth. It has increased considerably over recent decades and is expected to increase further, from about 50 years back in 1950 to more than 80 years in 2100. Albeit differences in magnitude and timing, all continents show a similar pattern.

Both, academic literature and the political debate have discussed the implications of population aging in industrialized countries for many decades. Some of them speak of a *silver revolution*. For example, Siegel (1998) and Schieber and Shoven (1997) elaborate on an "asset market meltdown" (Poterba, 2001) when baby boomers retire and sell largely their assets in order to finance consumption. Others suspect a decay of public social security in the future, especially, in the case of pay-as-you-go financed pension systems. Among those are De Nardi, Imrohoroglu, and Sargent (1999), Storesletten (2000), Attanasio, Kitao, and Violante (2007), Börsch-Supan, Ludwig, and Winter (2006), Ludwig and Reiter (2010). One strand of the economic literature elaborates on the role of behavioral responses of households to the new market conditions which arise in the course of the demographic transition. From a macroeconomic perspective, the ongoing aging process gives rise to the relative scarcity of raw labor and the relative abundance of physical capital. Standard models suggest that this decreases asset returns and increases wages in general equilibrium. Meanwhile, Börsch-Supan, Ludwig, and Winter (2006), Lee and Mason (2010), Ludwig and Vogel (2010), and Ludwig, Schelkle, and Vogel (2012), among others, argue that the joint decline of interest rates and increase of wages offers individuals an incentive for more human capital accumulation and labor supply. This, in turn, dampens the labor scarcity in an economy and the associated price effects.

This thesis quantifies the price and welfare effects of demographic change using the example of Germany and the U.S. It accounts for human capital along the dimensions of tertiary education and on-the-job investments as well as for hours worked and reveals their relative importance.

The thesis consists of three self-contained chapters. Each of them makes use of a large-scale overlapping generations model in the tradition of Auerbach and Kotlikoff (1987). Chapters 1 and 2 take demographic forecasts as given and simulate the resulting economic dynamics for Germany and the U.S. respectively. While chapter 1 analyzes both, the inter- and the intra-generational dimension, chapter 2 focuses on the latter. Chapter 3 deviates from the first two chapters in its nature as it contributes to the literature of computational economics. More precisely, it develops a method which deals with the computational challenges of transitional dynamics in heterogeneous agent models with aggregate risk. The method can be applied to large-scale overlapping generations as well as other models with heterogeneous agents and is employed in chapter 2.

Chapter 1 investigates the impact of demographic change on the distributions of income, skills, and welfare in the German economy along the inter- and the intra-generational dimension.[1] It builds on an overlapping generations model in the tradition of Auerbach and Kotlikoff (1987) with households being heterogeneous in their (innate) ability for studying in college. The model accounts for capital-skill complementarity and several endogenous household choices. Households initially choose whether to receive tertiary education which splits them into *high-school* and *college* types, thereby, determining their degree of substitutability against capital in production. On a period-by-period basis, households decide on consumption as well as on the time they spend both, in the labor market and on skill formation on-the-job.

The major contribution of the chapter is to show quantitatively the impact of demographic change on different skill groups and to reveal how households will react rationally to the altered market conditions arising from demographic change. Furthermore, the chapter highlights the relevance of past skill-biased technological change for the future dynamics of the income and skill distributions.

Forecasts by United Nations (2013) under medium fertility assumptions predict that the working age-to-total population ratio defined as the population at age $20-64$ divided by the total population shows a severe drop of more than 10 percentage points until 2050. Even when accounting for a step-wise increase of the statutory retirement age to 67 as implemented by the German government in 2007 the drop in the working age-to-total population ratio remains substantial.[2]

The quantitative experiments reveal the following effects of demographic change comparing year 2010 to year 2050: 1) The skill premium declines by about 15 percentage points while the college educated share in the

[1]Chapter 1 is based on Geppert (2015b).

[2]In that case, the working age-to-total population ratio would decrease to about 53 percent by the year 2050 using medium fertility prospects.

workforce increases by about 3 percentage points. 2) The interest rate falls by 1 percentage point while the average wage increases by about 20% induced by a substitution of labor by capital and the aforementioned skill increase in the production. 3) The replacement rate falls massively by about 40 percentage points if the contribution rate to the public pension system is held fix.

Welfare effects of demographic change are substantial and vary between -3% and $+2\%$ of consumption in every period of lifetime depending on skill group and generation. All currently living generations lose. Despite the drop in the skill premium demographic change benefits skilled over unskilled households. This is mainly due to the co-incident decline of the interest rate which makes borrowing for education less costly. While less able households benefit strongly from equilibrium effects arising from a higher college share in the workforce, more able households benefit rather from higher idiosyncratic human capital investments on the job.

As a secondary result, the quantitative experiments show that past skill-biased technological change will depress strongly the future skill premium by additional 15 percentage points due to an ongoing increase in the relative supply of college workers. This causes strong welfare losses for college households of up to 8% of consumption in every period of lifetime. Note that the prediction of a strongly declining earnings premium and the associated welfare consequences could be turned around in case of ongoing skill-biased technological change in the future. However, this chapter remains agnostic with respect to the direction of technological change in the future in the sense that all future change in technology is assumed to be skill-neutral.

Chapter 2 contributes to the literature on the role of aging for asset pricing.[3] More precisely, it disentangles the effect of demographic change on returns to risk-free and risky assets in the U.S. and measures the net effect on their differential return, the equity premium. Therefore, it builds on an overlapping generations structure in the tradition of Auerbach and Kotlikoff (1987) with aggregate risk. The model accounts for idiosyncratic earnings risk and an endogenous portfolio choice. On a period-by-period basis, households decide on consumption as well as on the fraction of wealth which they want to save in stocks, bonds, and human capital.

The major contribution of the chapter is to quantify the effect of demographic change on asset returns in a model with a realistic periodicity of one year. Furthermore, it reveals the effect of increased human capital investments on financial market prices in the course of demographic change.

Forecasts of Human Mortality Database (2008) and United Nations (2007), predict a decline in the working age to population ratio by roughly 9 percentage points between 2010 and 2030. Our results show that the expected decrease of the average stock return until 2030 is in the order of magnitude of 0.16 percentage points. The decrease of the risk-free interest rate on bonds is slightly higher such that the equity premium increases by about 0.08 percentage points. These relatively mild changes in returns and the equity premium result from an interplay of three main effects. First, older households on average hold relatively fewer equity than younger households in the model as well as in the data (Ameriks and Zeldes, 2004).[4] Demographic change increases the size of the old population relative to the young which drives up the relative demand for bonds thereby increasing its relative price. Consequently, the equity premium tends to increase. The second effect is a portfolio adjustment effect isolated in Kuhle (2008) that works in the opposite direction: Ignoring the first effect, suppose that demographic change would lead to a decrease of the expected rates of return on both assets by the same amount (such that the ex-ante equity premium is constant). For a positive

[3] Chapter 2 is based on Geppert and Ludwig (2015) which is co-authored by Professor Alexander Ludwig.

[4] According to Ameriks and Zeldes (2004) life-cycle portfolio shares do not vary much with age conditional on participation in equity markets but participation decreases around the age of retirement.

equity premium, then, the percentage decrease of the risk-free rate of return is higher such that the investor increases her relative portfolio shares of equity. Consequently, the demand for bonds decreases. Hence, the equilibrium decrease of the equity premium is smaller than the first effect would postulate in isolation. Third, and most importantly, endogenous human capital adjustments have a large effect. As societies are aging, labor becomes a relatively scarce factor and households increase human capital investments. This increases productivity thereby decreasing the downward pressure on asset prices. If one instead holds the human capital shares constant, then the negative effects on asset returns are much larger. In that scenario, the average stock return decreases by about 0.70 percentage points until 2030 and the equity premium increases by about 0.27 percentage points.

A welfare analysis shows that the decline of asset returns and the co-incident increase of the human capital return benefits future generations relative to generations born in the past. Again, human capital adjustments reduce welfare consequences and their differences across generations considerably.

Chapter 3 develops a new method for computing transitional dynamics in heterogeneous agent models with aggregate risk.[5] Macroeconomic analyses use increasingly that kind of model in order to address questions which are related to both, inter- and intra-generational heterogeneity and, in particular, associated policy concerns. The method applies to the (stochastic) transition of such an economy if this transition is induced by exogenous deterministic dynamics such as, e.g., a fundamental tax reform or demographic change as assumed in the first two chapters. Assuming that households forecast the evolution of aggregate variables by applying an aggregate law of motion as suggested by Krusell and Smith (1997, 1998) such a transition induces time dependency of the aggregate law of motion. Using ideas from Judd (2002), the method parameterizes this dependency on time and is particularly easy to implement.

The major contribution of the new method is a particular small number of coefficients in the aggregate laws of motion to be determined. In the alternative standard brute force KS approach, in which one specifies a separate law of motion for each period in the transition, the coefficients of the laws of motion are identified solely by cross-sectional variation. Meanwhile, the coefficients of the time polynomials are identified by both, cross-sectional and time variation. Accordingly, the new method requires a much smaller number of simulations of the economy along the transition compared to the alternative brute force approach.

The illustration of the method uses an overlapping generations model with aggregate shocks in which a fundamental tax reform induces transitional dynamics. The quantitative experiment reveals a substantial reduction of total computing time by 45% compared to the brute force approach. Euler equation errors as well as errors from one-period-ahead and multi-periods-ahead predictions of the aggregate state variable are very low and similar in size to the brute force approach.

[5] Chapter 3 is based on Geppert (2015a).

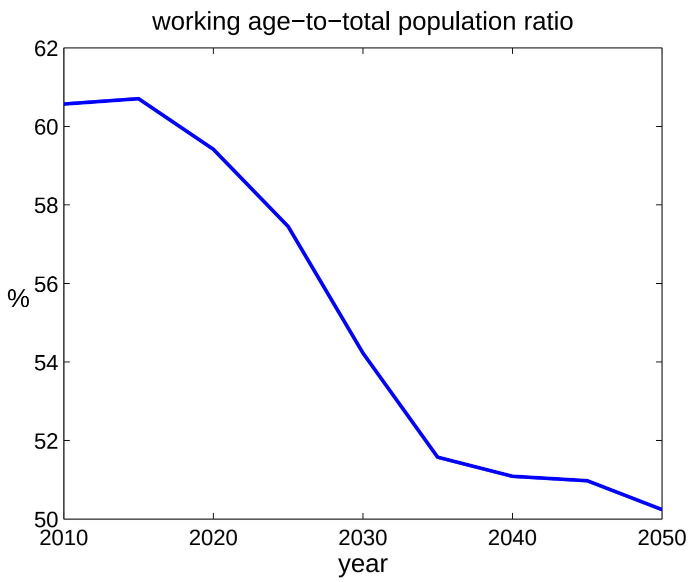
1

On the Distributional Implications of Demographic Change

1.1 Introduction

Like all other major economies, Germany faces severe population aging within the next decades. Figure 1.1 depicts the expected evolution of the working age-to-total population ratio by United Nations (2013) holding fix the retirement age at 65. The graph shows a severe drop of more than 10 percentage points until 2050. Even when accounting for a step-wise increase of the statutory retirement age to 67, as implemented by the German government in 2007, the drop in the working age-to-total population ratio remains substantial.[1]

Figure 1.1: Expected Demographic Change in Germany



Source: Own calculations based on United Nations (2013) using medium fertility prospects and holding the retirement age fix at 65. Note that taking account of the step-wise increase of the statutory retirement age to 67 as implemented by the German government in 2007 would lead to a working age-to-total population ratio in 2050 of about 53%.

These strong changes in the population structure will have important implications for the macroeconomic composition of capital and labor in the production as Ludwig, Schelkle, and Vogel (2012) remark. From a theoretical macroeconomic perspective, labor becomes ceteris paribus a scarce factor in an aging economy if labor market participation, education, or human capital formation do not increase strongly. In this case, labor is partially substituted by physical capital which, in turn, leads to increasing wages and decreasing interest

[1] In that case, the working age-to-total population ratio would decrease to about 53 percent by the year 2050 using medium fertility prospects.

rates. Furthermore, in the presence of capital-skill complementarity the abundance of physical capital benefits some workers more than others. Krusell et al. (2000) show that the latter is key for the explanation of the trend rise in the skill premium[2] in the U.S. over a course of thirty years. Hence, the aging process gives rise to an increase in *intra*-generational inequality in addition to the often discussed shift in *inter*-generational inequality. The latter arises from changes in the overall wage level, the interest rate, and the generosity of pay-as-you-go (PAYG) financed pension systems over time. However, the evolution of the aggregate of labor services[3] depends heavily on individual household behavior. Adjustments of the latter with respect to labor market participation (along the intensive and the extensive margin) and labor productivity might counteract or even overturn the labor scarcity along with the aforementioned resulting effects in equilibrium.

Against that background, this chapter investigates the impact of demographic change on the distributions of income, skills, and welfare in the German economy, along the inter- and the intra-generational dimension. Therefore, it builds on an overlapping generations structure in the tradition of Auerbach and Kotlikoff (1987) with households being heterogeneous in their (innate) ability for studying in college. The model accounts for capital-skill complementarity and several endogenous household choices. Households initially choose whether to receive tertiary education which splits them into *high-school* and *college* types thereby determining their degree of substitutability against capital in production. On a period-by-period basis, households decide on consumption as well as on the time they spend both, in the labor market and on skill formation on-the-job.

The major contribution of the chapter is to show quantitatively the impact of demographic change on different skill groups and to reveal how households can react to the altered market conditions arising from demographic change. Furthermore, the chapter highlights the relevance of past skill-biased technological change for the future dynamics of the income and skill distribution.

The quantitative experiments reveal the following effects of demographic change comparing year 2010 to year 2050: 1) The skill premium declines by about 15 percentage points while the college educated share in the workforce increases by about 3 percentage points. 2) The interest rate falls by 1 percentage point while the average wage increases by about 20% induced by a substitution of labor by capital and the aforementioned skill increase in the production. 3) The replacement rate falls massively by about 40 percentage points if the contribution rate to the public pension system is held fix.

Welfare effects of demographic change[4] are substantial and vary between -3% and $+2\%$ of consumption in every period of lifetime depending on skill group and generation. All currently living generations lose. Despite the drop in the skill premium, demographic change benefits skilled over unskilled households. This is mainly due to the co-incident decline of the interest rate which makes borrowing for education less costly. While less able households benefit strongly from equilibrium effects arising from a higher college share in the workforce, more able households rather benefit from higher idiosyncratic human capital investments on the job.

As a secondary result, the quantitative experiments show that past skill-biased technological change will depress the future skill premium by additional 15 percentage points due to an ongoing increase in the relative supply of college workers. This causes strong welfare losses for college households of up to 8% of consump-

[2] The authors define the *skill premium* as the ratio of wages paid to college workers to wages paid to non-college workers while this chapter refers to the corresponding ratio of earnings.

[3] Throughout this chapter, the terms “effective hours of labor supply”, “effective labor supply”, and “labor services” all refer to the productivity weighted hours of labor supply.

[4] Note that all welfare measures throughout the chapter exclude welfare gains from increasing survival probabilities over time which are exogenous in this model.

tion in every period of lifetime. Note that the prediction of a strongly declining earnings premium and the associated welfare consequences could be turned around in case of ongoing skill-biased technological change in the future. However, this chapter remains agnostic with respect to the direction of future technological change in the sense that all future change in technology is assumed to be skill-neutral.

After a brief literature review in the next section the theoretical model in use is described in section 1.3. Section 1.4 elaborates on the quantitative approach of the chapter and the calibration of the model. Results from simulations are shown in section 1.5. Finally, section 1.6 concludes.

1.2 Relation to the Literature

This chapter relates to three strands of the literature.

The first strand deals with the welfare consequences of demographic change. Here, the focus has been on the sustainability of PAYG financed public social security systems and the related inter-generational effects. This chapter is closest in relation to the part of that literature which highlights the importance of changes in household behavior in response to the altered economic conditions arising from demographic change. Among those De Nardi, Imrohoroglu, and Sargent (1999) and Ludwig, Schelkle, and Vogel (2012) are in a closed economy setting. The former paper raises the problem of excess burden due to distortionary government policies which try to maintain past welfare levels. The latter finds that equilibrium effects on wages and interest rates which arise in aging economies induce higher incentives for human capital formation. The authors show that those higher human capital investments mitigate the effects of demographic change on macroeconomic aggregates and prices, and reduce welfare losses of middle aged agents substantially. The results complement similar findings with respect to endogenous labor supply by Börsch-Supan, Ludwig, and Winter (2006). Based on those findings, this chapter adds to the literature on the welfare consequences of demographic change by accounting for the dimension of intra-cohort inequality and by investigating the role of tertiary education.

Based on the importance of interest rate dynamics for the welfare consequences of demographic change Börsch-Supan, Ludwig, and Winter (2006), Krüger and Ludwig (2007), and Attanasio, Kitao, and Violante (2007) extend the investigation to an open economy setting in which different regions of the world age at different paces. The authors show that capital flows evolve from more to less strongly aging regions of the world and evaluate the associated consequences for social security systems, a task that this chapter leaves for future research.

Within a second related strand of the literature, researchers claim that reductions in mortality rates have positive incentives for education and human capital formation from a theoretical point of view. Among those are De La Croix and Licandro (1999), Kalemli-Ozcan, Ryder, and Weil (2000), Boucekkine, Croix, and Licandro (2002), Boucekkine, Croix, and Licandro (2003), Lagerlöf (2003), Soares (2005), and Cervellati and Sunde (2005). Moreover, Cervellati and Sunde (2013) disprove Hazan (2009)'s claim that a necessary condition for positive incentives to arise is an increase in lifetime labor supply. These theoretical findings are confirmed by an empirical literature which suggests a positive causal effect of higher life expectancy on educational attainment, cf., e.g., Bleakley (2007), Jayachandran and Lleras-Muney (2009), and Oster, Shoulson, and Dorsey (2013).

The third strand of the literature is concerned with the past evolution of the skill premium and the wage distribution. Dustmann, Ludsteck, and Schönberg (2009) as well as Fuchs-Schündeln, Krueger, and Sommer (2010) elaborate on recent empirical trends in Germany and are important sources for the calibration of the model in this chapter. In a seminal paper, Katz and Murphy (1992) set up a simple supply and demand model of the labor market. They show that the model, together with some latent time trend in relative demand for skilled labor, is able to explain the evolution of the U.S. skill premium from 1963 to 1987. The results of Katz and Murphy (1992) have led many economists to search for the economic forces behind the measured time trend interpretable as latent skill-biased technological change. Among those, Krusell et al. (2000) show that a negative time trend in the price of capital equipment relative to the price of capital structures can explain the overall rise in the U.S. skill premium between 1963 and 1992. The result is based on the complementarity between capital equipment and skilled labor. However, in both papers the relative supply of aggregate effective labor hours by skilled versus unskilled households is key for the throughout explanation of the evolution of the skill premium. This is true in particular for the decline of the latter in the 1970s. While Krusell et al. (2000) neglect the efficiency part and account only for the relative supply of labor hours, Heckman, Lochner, and Taber (1998) do the opposite. In fact, they set up a model with endogenous productivity along two margins. First, households decide on tertiary education at the beginning of the life cycle and, second, they choose the time that they spend on on-the-job skill formation on a period-by-period basis.

This chapter adds to that strand of the literature by accounting for both, endogenous hours and productivity of labor supply, and by investigating their relative importance for the evolution of the skill premium in earnings. From a technical point of view the education decision is modeled based on Willis and Rosen (1979) and Keane and Wolpin (1997). The endogenous decisions on human capital and labor supply are in line with Becker (1967) and Ben-Porath (1967).

1.3 Model

1.3.1 Time and Demographics

Time is discrete and runs from $t = 0, 1, \ldots, \infty$. In every period, the economy is populated with $J + 1$ overlapping generations and the population structure[5] is time-dependent and exogenous. Households enter the economy at the age of $j = 0$, have the possibility to go to college[6] at ages $j = 0, ..., j_w - 1$, retire at the age of $j = j_r$, and live at most until turning $j = J + 1$ years. The population of age j in time period t is denoted by $N_{t,j}$ and the total population in time period t equals $N_t = \sum_{j=0}^{J} N_{t,j}$. Households face mortality risk represented by exogenous survival probabilities. $\varsigma_{t,j}$ is the probability of a household at age j and time t to survive until the next period.

1.3.2 Innate Ability and Endowments

A household enters the economically relevant time of life at age $j = 0$ being equipped with an idiosyncratic innate ability for tertiary education which is fully observable. The ability, indicated by superscript a, is represented exclusively by the amount of time per period, $\bar{\imath}$, which the household will have to spend on studying if it chooses to accomplish tertiary education. Technically speaking, upon entering the economy, a household draws from an ability distribution which is independent and identical across all newborn households in the

[5] I use the terms *demographic distribution* and *population structure* interchangeably throughout the chapter indicating the distribution of the population by age.

[6] I use the terms *tertiary education*, *formal education*, *schooling*, and *college* $(\mathcal{C})$ interchangeably throughout the chapter.

course of time:

$$\bar{i} \overset{iid}{\sim} \mathcal{D}(\mu, \sigma^2) \tag{1.1}$$

where $\mathcal{D}$ is some distribution with mean μ and variance σ^2. Moreover, a newborn household is endowed with a positive initial level of human capital, $h^a_{t,0} = h_0 > 0$, but neither physical capital, $k^a_{t,0} = k_0 = 0$, nor claims to the public pension system, $b^a_{t,0} = b_0 = 0$, for all t. Note that all endowments are time-independent and identical across households.

1.3.3 Optimal Education and Subsequent Choices

At the beginning of life, a household faces the decision of accomplishing tertiary education or not. In the following, I will speak of college ($\mathcal{C}$) and high-school ($\mathcal{H}$) households respectively. Attending college implies a time investment as was described in section 1.3.2.[7] In return, the household accumulates human capital and joins the tertiary educated labor force upon graduation. Trading off costs against benefits leads to the optimal educational choice given by

$$\mathcal{S}^a_{t,0} = arg \max_{\mathcal{S} \in \{\mathcal{H}, \mathcal{C}\}} \left\{ v^{a,\mathcal{S}}_{t,0}(k_0, h_0, b_0) \right\} \tag{1.2}$$

where $v^{a,\mathcal{S}}_{t,0}(\cdot)$ is the lifetime utility of a household with ability a from schooling $\mathcal{S}$ in period t at age $j = 0$.

In each single period, a household chooses consumption, c, hours spent on on-the-job human capital development[8], i, and hours supplied to the labor market, l, based on an utilitarian preference function, $u(c, 1 - i - l)$. $u(\cdot)$ fulfills standard assumptions, further specified in section 1.4.2, and features that a household gains utility from consumption and leisure, $1 - i - l$. Please see section 1.A.1 for a detailed derivation of the solution to the household problem.

1.3.4 Wealth Accumulation

Over the course of life, a household accumulates physical capital, human capital, and pension benefit entitlements as specified below.

The dynamic budget constraint is given by

$$c^a_{t,j} + k^a_{t+1,j+1} = e^a_{t,j} + (1 + r^K_t) \cdot k^a_{t,j} \tag{1.3}$$

where

$$e^a_{t,j} := \begin{cases} (1 - \tau_t) \cdot r^{\mathcal{S}}_t \cdot h^a_{t,j} \cdot l^a_{t,j} & if \quad j < j_r \\ \mathcal{P}_t(b^a_{t,j}) & else \end{cases}$$

are net earnings and pension benefits respectively. $r^{\mathcal{S}}$ is the return on labor services, $h \cdot l$, of an agent with education level $\mathcal{S}$. *Labor services* denote the human capital (productivity) weighted hours of labor supply. τ

[7] Note that tertiary education does not involve any pecuniary private costs other than foregone wages which comes close to the German university system.

[8] Note that currently enrolled college students are excluded from on-the-job human capital development by assumption such that their i equals $\bar{i}^a$ as described in section 1.3.2.

is the contribution rate to the pension system, and $\mathcal{P}(\cdot)$ is the pension benefit function. Note that the gross hourly wage of an agent with ability a and education level $\mathcal{S}$, which is the observable variable in the data, equals $r_t^{\mathcal{S}} \cdot h_{t,j}^a$.

Human capital represents the idiosyncratic labor productivity of a household. It accumulates according to the following law of motion:

$$h_{t+1,j+1}^a = \begin{cases} h_0 & if \quad j+1 < j_w \quad \wedge \quad \mathcal{S} = \mathcal{C} \\ (1+\bar{h}) \cdot h_0 & if \quad j+1 = j_w \quad \wedge \quad \mathcal{S} = \mathcal{C} \\ \varphi^{\mathcal{S}}(h_{t,j}^a, i_{t,j}^a) & else. \end{cases} \tag{1.4}$$

While studying in college, households remain at the initial human capital level, h_0. Upon graduation, they receive a fix markup, $\bar{h}$, on their human capital stock and join the tertiary educated labor force.[9] Households which are not currently enrolled in college are able to accumulate human capital by on-the-job time investments, i. The accumulation evolves according to the well-known production function $\varphi^{\mathcal{S}}(h, i)$ going back to Ben-Porath (1967). Note that $\varphi^{\mathcal{S}}(\cdot)$ differs by education type reflecting the difference in the evolution of labor productivity over the life cycle between educational groups observed in the data.[10]

Households collect benefit claims to the public pension system through their working life cycle:

$$b_{t+1,j+1}^a = \begin{cases} \vartheta_t(b_{t,j}^a, e_{t,j}^a) & if \quad j < j_r \\ b_{t,j}^a & else \end{cases} \tag{1.5}$$

where $\vartheta(\cdot)$ is an increasing function in both of its arguments which implies that new benefit claims are earnings related.

1.3.5 Production

Production takes place with a constant returns to scale production function which is based on Krusell et al. (2000). It features that labor services of different education types are not perfect substitutes:

$$\mathcal{F}_t(K_t, L_t^{\mathcal{H}}, L_t^{\mathcal{C}}) = \left\{ \alpha_2 \cdot \left[\Upsilon_t \cdot L_t^{\mathcal{H}} \right]^{\rho_2} + (1-\alpha_2) \cdot \left[\left(\alpha_1 \cdot (K_t)^{\rho_1} + (1-\alpha_1) \cdot (\Upsilon_t \cdot L_t^{\mathcal{C}})^{\rho_1} \right)^{\frac{1}{\rho_1}} \right]^{\rho_2} \right\}^{\frac{1}{\rho_2}} \tag{1.6}$$

where $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$, $\rho_1 < 1$, and $\rho_2 < 1$. K is the aggregate stock of physical capital. $L^{\mathcal{C}}$ and $L^{\mathcal{H}}$ denote the aggregate inputs of labor services by college graduates and high-school households respectively. Υ_t denotes the labor augmenting technology which improves at the exogenous fix rate g, i.e., $\Upsilon_{t+1} = (1+g) \cdot \Upsilon_t$ for all t. Note that this represents skill-neutral technological progress in the economy and captures trend growth in output per capita.[11] $1/(1-\rho_1)$ is the elasticity of substitution between college labor, $L_t^{\mathcal{C}}$, and capital, K_t, while $1/(1-\rho_2)$ is the elasticity of substitution between college labor (or capital)

[9] The jump in the human capital profile upon graduation indicates the newly achieved possibility to apply for jobs which require a formal tertiary degree and is in line with the approach of Kindermann (2014).

[10] Note that limited data availability inhibits a further break down into ability classes among those groups as was already pointed out by Kindermann (2014, p. 14).

[11] Note that growth is only balanced in the model in the case of a stationary demographic distribution.

and high school labor, $L_t^{\mathcal{H}}$, holding fix the relative price between college labor and capital. If ρ_1 is smaller than ρ_2 the economy features capital-skill complementarity (CSC).

Perfect competition among firms leads to standard first order conditions of the firm problem stating that prices equal marginal products minus depreciation:

$$r_t^K = \frac{\partial \mathcal{F}_t}{\partial K_t} - \delta^K, \; r_t^{\mathcal{H}} = \frac{\partial \mathcal{F}_t}{\partial L_t^{\mathcal{H}}}, \; r_t^{\mathcal{C}} = \frac{\partial \mathcal{F}_t}{\partial L_t^{\mathcal{C}}} \tag{1.7}$$

1.3.6 Government

The government plays a twofold role in this model.

First, it runs a pay-as-you-go (PAYG) pension system. In any period t, earnings of workers are taxed at the rate τ_t whereas households in retirement ($j \geq j_r$) receive a pension. The condition for a balanced budget of the PAYG system in every period is:

$$\tau_t \cdot \sum_{j=0}^{j_r-1} N_{t,j} \cdot \int r_t^{\mathcal{S}} \cdot h_{t,j}^a \cdot l_{t,j}^a \cdot \zeta(a) \, da = \sum_{j=j_r}^{J} N_{t,j} \cdot \int \mathcal{P}_t(b_{t,j}^a) \cdot \zeta(a) \, da \tag{1.8}$$

where $\zeta(a)$ denotes the fraction of the population with ability, a.

Second, the government taxes accidental bequests of physical capital by departed households at 100% and uses it for government consumption:

$$G_t = \sum_{j=1}^{J} (1 - \varsigma_{t-1,j-1}) \cdot N_{t-1,j-1} \cdot \int (1 + r_t^K) \cdot k_{t,j}^a \cdot \zeta(a) \, da \tag{1.9}$$

1.3.7 Equilibrium

Given the exogenous distributions of the population weights, $\{\{N_{t,j}\}_{j=0}^J\}_{t=0}^T$, and the survival rates, $\{\{\varsigma_{t,j}\}_{j=0}^J\}_{t=0}^T$, the exogenous time-constant ability distribution $\mathcal{D}$, as well as initial stocks of human capital, physical capital, and pension benefit entitlements, h_0, k_0, b_0, a competitive equilibrium consists of sequences of individual variables, $\{\{\{c_{t,j}^a, l_{t,j}^a, i_{t,j}^a, h_{t+1,j+1}^a, k_{t+1,j+1}^a, b_{t+1,j+1}^a\}_a\}_{j=0}^J\}_{t=0}^T$, sequences of aggregate variables, $\{L_t^{\mathcal{H}}, L_t^{\mathcal{C}}, K_{t+1}, Y_t, G_t, C_t, I_t^K\}_{t=0}^T$, government policies $\{\tau_t, \vartheta_t(\cdot), \mathcal{P}_t(\cdot)\}_{t=0}^T$, and prices, $\{r_t^{\mathcal{H}}, r_t^{\mathcal{C}}, r_t^K\}_{t=0}^T$, such that[12]

1. households behave optimally as according to equations (1.2) and (1.26–1.28),
2. firms behave optimally as according to the equations in (1.7),
3. factor markets clear:

$$K_{t+1} = \sum_{j=0}^{J} N_{t,j} \cdot \int k_{t+1,j+1}^a \cdot \zeta(a) \, da \tag{1.10}$$

$$L_t^{\mathcal{C}} = \sum_{j=0}^{j_r-1} N_{t,j} \cdot \int_{\underline{a}} l_{t,j}^a \cdot h_{t,j}^a \cdot \zeta(a) \, da \tag{1.11}$$

[12] The presentation of the equilibrium definition follows Ludwig, Schelkle, and Vogel (2012).

$$L_t^{\mathcal{H}} = \sum_{j=0}^{j_r-1} N_{t,j} \cdot \int^{\overline{a}} l_{t,j}^a \cdot h_{t,j}^a \cdot \zeta(a)\, da \tag{1.12}$$

where $\overline{a}$ indicates the marginal ability type being indifferent of going to college or not and $\zeta(a)$ is the probability density function of a,

4. the PAYG pension budget clears as according to (1.8),
5. accidental bequests finance government consumption as according to (1.9),
6. and the aggregate resource constraint holds:[13]

$$Y_t = C_t + I_t^K + G_t \tag{1.13}$$

1.4 Calibration and Numerical Solution

1.4.1 Solution Strategy

The solution of the model involves outer and inner loop iterations and follows the approach taken by Ludwig, Schelkle, and Vogel (2012). The outer loop solves for equilibrium by iterating on the aggregate capital stock, K, the aggregate college labor services, $L^{\mathcal{C}}$, the aggregate non-college labor services, $L^{\mathcal{H}}$, as well as the aggregate pension payments for all periods $t = 0, ..., T$. The inner loop solves for the household policy functions[14] in all periods $t = 0, ..., T$. In each outer loop iteration, household level variables are aggregated in order to update the aggregate stocks using a simple Gauss-Seidel algorithm as explained in Ludwig (2007).

The exogenous driving process[15] in the model is demographic change between 1950 and 2100 represented by both, a time-varying age structure of the population and increasing survival rates. Demographic data and projections are taken from United Nations (2013) under the medium fertility assumption.[16] Figure 1.2 shows the population age structure of the economy in the years 2010 and 2050. It reflects exemplarily the prediction of a strongly aging German adult population.

For computational reasons, the solution of the model begins in year 1750 ($t = 0$) in which an artificial initial steady state (in per efficient capita units) with the demographic structure of year 1950 is assumed.[17] I then compute the transitional dynamics to an artificial final steady state in year 2500 ($t = T$) with the fix demographic structure of year 2100.[18] According to data availability the calibration period runs from 1975 to 2010. The main period of projection is 2010 to 2050.

1.4.2 Calibration

The calibration of the model follows a two-step procedure. In the first step, model parameters are set exogenously in line with empirical evidence and the literature. In the second step, model parameters are set in order

[13] Please see section 1.A.2 for a detailed derivation.

[14] Please see section 1.B.1 in the appendix for details.

[15] Note that there is an additional exogenous driving force in the model, skill-biased technological change between 1975 and 2010, which serves calibration purposes. Please see section 1.4.2 for details.

[16] For a detailed description of data and estimates I refer to the author.

[17] The phase-in period with fix demographics until 1950 assures fully rational anticipation of changing market conditions arising from demographic change which takes place as of 1950.

[18] The phase-out period with fix demographics beyond 2100 assures that the transitional dynamics of the model, indeed, lead to the final steady state.

Figure 1.2: Demography: Data 2010 vs. Projections 2050

Source: Own calculations based on United Nations (2013), medium fertility prospects. *Notes:* The graphs show fractions of the adult population by 5-year age bins which begin with the indicated age. The adult population in the data corresponds to the total population in the model.

to match key moments in the data.

The period length in the model is five years. Accordingly, all references to a specific year or a specific age throughout the chapter stand for the respective 5-year period starting with the indicated year or age. Individuals are assumed to enter their economically relevant life at the age of 20 ($j = 0$), graduate potentially when turning 25 ($j_w = 1$), retire when turning 65 ($j_r = 9$), and die, the latest, when turning 100 ($J + 1 = 16$). Preferences over consumption and leisure follow Ludwig and Abiry (2015):[19]

$$u(c^a_{t,j}, 1 - i^a_{t,j} - l^a_{t,j}) := \begin{cases} \ln(c^a_{t,j}) + \gamma \frac{1}{1-\upsilon}\left((1 - i^a_{t,j} - l^a_{t,j})^{1-1/\upsilon} - 1\right) & if\ \theta = 1, \\ \frac{(c^a_{t,j})^{1-\theta}}{1-\theta} \cdot \left(1 + \gamma \frac{1-\theta}{1-\upsilon}\left((1 - i^a_{t,j} - l^a_{t,j})^{1-1/\upsilon} - 1\right)\right)^{\theta} & else, \end{cases} \tag{1.14}$$

where I restrict υ, without loss of generality, to the empirically relevant case of being smaller than unity. γ denotes the utility weight of leisure which is calibrate in order to match the average number of labor hours in the data. $1/\theta$ is the elasticity of inter-temporal substitution. The λ-constant Frisch elasticity of labor supply equals $\upsilon \cdot (1 - i^a_{t,j} - l^a_{t,j})/l^a_{t,j}$ and, thus, varies along with leisure and labor over the life cycle.[20] Note that from this it follows that labor supply elasticities differ also by skill group. As college households tend to supply more labor and consume less leisure than non-college households during most of their working lifetime they exhibit smaller labor supply elasticities. This is in line with the empirical literature on labor supply elasticities (cf. Browning, Hansen, and Heckman, 1999) without imposing preference heterogeneity across skill groups. υ is set to 0.21 thereby pinning down the average (hours weighted) elasticity in the model which equals 0.475 in year 2010. This is in line with the micro-evidence on labor supply elasticities (cf. Domeij and Flodén, 2006,

[19] Note that the considered preferences exhibit a jump at $\theta = 1$. This comes from the need of a homothetic utility function for computing consumption equivalent variation as in section 1.5. Therefore, I primarily consider the case $\theta > 1$ in the quantitative experiments.

[20] Please see Ludwig and Abiry (2015) and Ludwig, Schelkle, and Vogel (2012) for a discussion on implications arising from the life cycle variation of the Frisch elasticity.

for models without borrowing constraints).

On-the-job skill formation follows the well-known Ben-Porath (1967) human capital production function

$$\varphi^{\mathcal{S}}(h^a_{t,j}, i^a_{t,j}) := (1-\delta^h)\cdot h^a_{t,j} + \varpi^{\mathcal{S}}\cdot (h^a_{t,j}\cdot i^a_{t,j})^{\varrho}. \tag{1.15}$$

$0 < \varrho < 1$ governs the complementarity between old human capital and time investments in the accumulation of new human capital and δ^h is the depreciation rate of human capital. $\varpi^{\mathcal{S}}$ is calibrated in order to match the net earnings ratios between age groups in the data and differs by education type.[21]

Pension entitlements accumulate according to a purely earnings related scheme which comes close to the actual German public pension system:

$$\vartheta_t(b^a_{t,j}, r^{\mathcal{S}}_t\cdot h^a_{t,j}\cdot l^a_{t,j}) := b^a_{t,j} + r^{\mathcal{S}}_t\cdot h^a_{t,j}\cdot l^a_{t,j}/\bar{e}_t \tag{1.16}$$

where $\bar{e}_t := (r^{\mathcal{H}}_t\cdot L^{\mathcal{H}}_t + r^{\mathcal{C}}_t\cdot L^{\mathcal{C}}_t)/(\sum_{j=0}^{j_r-1} N_{t,j})$ are average earnings in period t. The contribution rate of the pension system is set exogenously as according to data from DRV (2014, p. 262) and held constant in future periods.[22] Pension benefits

$$\mathcal{P}_t(b^a_{t,j}) := \nu_t\cdot b^a_{t,j} \tag{1.17}$$

are payed proportional to the accumulated benefit stock. ν_t is the so-called *actual pension amount* payed to the household in period t for each point of accumulated benefit entitlements. It adjusts in every period so that the pension budget clears. Note that ν_t grows over time along with wages due to exogenous (skill-neutral) technological progress in Υ_t. This implies that the growth of the pension payment over the retirement spell of a household keeps track with wage growth which is (broadly) consistent with the German public pension scheme.

Time spent on studying in college (ability), $\bar{i}^a$, is assumed to be uniformly distributed. That distribution is then approximated by ten different ability groups of equal size[23] where

$$\bar{i}^a := \mu + \sigma\cdot[1, \frac{7}{9}, \frac{5}{9}, \frac{3}{9}, \frac{1}{9}, -\frac{1}{9}, -\frac{3}{9}, -\frac{5}{9}, -\frac{7}{9}, -1].$$

μ determines the average time per period spent in college and is calibrated such that the college educated share of the 25+ workforce in the model matches its counterpart in the data. σ governs the standard deviation of time per period spent in college and is set to 0.32 such that the model leads to empirically reasonable elasticities of the college decision as will be shown next. Therefore, I compute the percentage change of the college educated share to 1) a yearly college grant of US\$1000 and 2) a 1% increase in the skill premium. The change under 1) turns out to be 0.43 in 2010 which is in line with the micro elasticity for Germany estimated in Steiner and Wrohlich (2011)[24] and literature cited therein. Results from the quasi-experimental literature

[21] The model does not feature a progressive earnings tax system which implies that ratios between age groups of net earnings are identical to those of gross earnings by construction. I use data on net earnings as targets in order to match more closely the differences in the life cycle profiles of disposable income across educational groups.

[22]

	... − 1965	1970	1975	1980	1985	1990	1995	2000	2005	2010 − ...
τ	0.140	0.170	0.180	0.182	0.189	0.182	0.198	0.193	0.197	0.194

[23] Note that this does not hold necessarily for the two marginal ability groups around the cut-off value, $\underline{a}$. Their size is adjusted using interpolation techniques such that the tertiary educated share of the population observed in the data can be matched precisely. Hence, the selected number of groups matters for computational accuracy. Ten clusters showed to be sufficient as the use of 18 different ability types had only a minor impact on results.

[24] Steiner and Wrohlich (2011) estimate that the share of high-school graduates enrolling for college increases by 1.5 percentage points in response to a yearly college grant of €1000 while Autorengruppe Bildungsberichterstattung (2012) reports a high-

for the U.S. find higher numbers for an increase in college enrollment in the magnitude of 3 to 5 percentage points (cf., e.g., Kane, 2006 and Deming and Dynarski, 2009). However, the key driver for those high estimates are borrowing constraints which are absent in this model. Johnson (2013) and Findeisen and Sachs (2015) find a 2.4 and 4.1 percentage point increase in the college share, respectively, when abolishing borrowing constraints. Under 2), the experiment results in an increase of 0.60 percentage points in 2010 which is slightly higher than what is found in the literature (cf., e.g., Fredriksson, 1997, for Sweden). Section 1.5.1 shows that the model is able to match the past evolution of the college workforce share in the data. $\bar{\iota}$ turns out to lie in the interval $[0.34, 0.98]$.

Along with Heckman, Lochner, and Taber (1998), I assume exogenous skill-biased technological change in the calibration period, 1975 to 2010. It is imposed by a downward trend in the production function parameter α_2[25] and induces an upward trend in the skill return premium $r_t^{\mathcal{C}}/r_t^{\mathcal{H}}$. I set the constant yearly increase in $(1-\alpha_2)/\alpha_2$ in the period $1975-2010$ to 2%. While Heckman, Lochner, and Taber (1998) choose a value of 3.6% which is in line with the literature on the past evolution of the skill premium in the U.S. (cf., e.g., Katz and Murphy, 1992), the results in Dustmann, Ludsteck, and Schönberg (2009, p. 852) suggest a smaller value for Germany of about 60% of the U.S. value which comes close to the considered 2%. Section 1.5.1 shows that the model is able to trace jointly the past evolution of the skill premium and the college workforce share in the data.

Table 1.1 summarizes first and second stage parameters. Note that the elasticity of substitution between physical capital and high-school labor services is higher than the one between physical capital and college labor services which implies capital-skill complementarity. This is in line with empirical estimates and the selected values lie in the range considered in the literature (cf., e.g., Duffy, Papageorgiou, and Perez-Sebastian, 2004, Krusell et al., 2000, and Heckman, Lochner, and Taber, 1998).

1.5 Results

1.5.1 Cross-Sectional Profiles in 2010 and the Past Trend in the Skill Premium

Figure 1.3 shows resulting cross-sectional age profiles of the model economy in year 2010 by educational group. College households choose not to work besides studying and benefit from the markup in their human capital stock upon graduation which leads to a jump in their earnings profile (top right panel). Furthermore, college graduates are more efficient in developing human capital on-the-job. Together with higher relative working hours at old ages that leads to a steeper age-earnings profile of college households compared to high-school households, or, equivalently, to an increase in the earnings premium over the working life cycle (bottom right panel). Both is consistent with empirical evidence from OECD (2014a) and the German Socio-Economic Panel (cf. Kindermann, 2014). Note that the pension premium is at a lower level than the earnings premium albeit the absence of a re-distributive mechanism in the pension benefit function. This comes from the strong rise in the earnings premium over past decades (cf. figure 1.4) and shows that the pension premium serves as a lagged indicator for the earnings premium. Consumption (bottom left panel) and hours worked

school graduate share of 41% in the $30-35$ year old population.

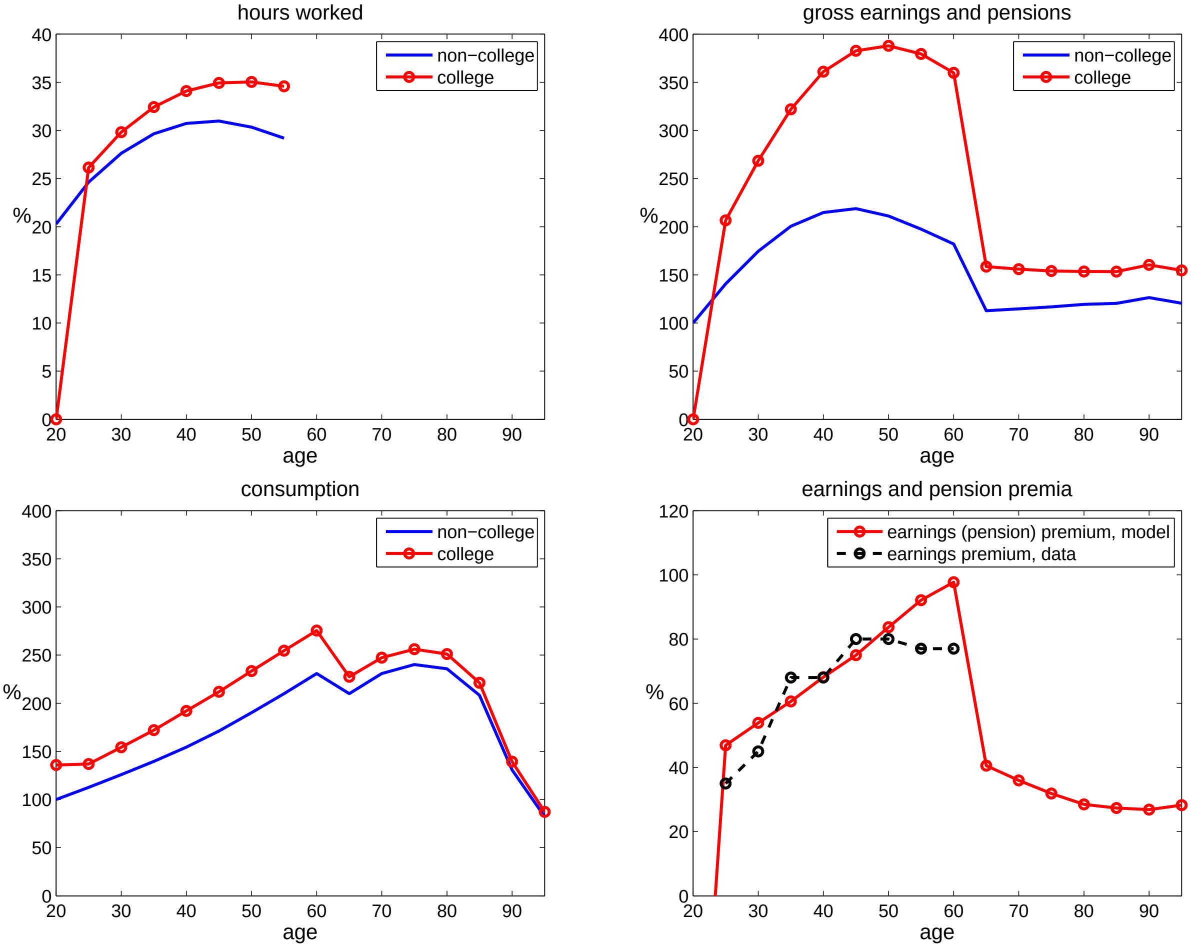
[25] Note that this implies an upward drift in the capital income share and the capital-output ratio during the calibration period. The upward drift is consistent with the findings of Piketty and Zucman (2014) for Germany who recently challenged the predominant view in the literature of the constancy of the aforementioned capital measures with a new data set.

Table 1.1: First and Second Stage Parameters

Parameter	Value	Target	Source, Comment
Firm sector			
Skill-neutral techn. progress: g	0.01	1^{st} stage	OECD (2014b): $g^{Y/N}$
Skill-biased techn. change: $g_{1975-2010}^{(1-\alpha_2)/\alpha_2}$	0.02	1^{st} stage	DLS, period: $1975-2010$
ES($L^{\mathcal{H}}, L^{\mathcal{C}}$)=ES($L^{\mathcal{H}}, K$): $1/(1-\rho_2)$	1.59	1^{st} stage	DPP
ES($K, L^{\mathcal{C}}$): $1/(1-\rho_1)$	0.64	1^{st} stage	DPP
Weight on L^H: α_2	0.34	$e^{\mathcal{C}}/e^{\mathcal{H}} = 1.71$	OECD (2014a)
Weight on K: α_1	0.51	$\frac{(r^{\mathcal{H}}L^{\mathcal{H}}+r^{\mathcal{C}}L^{\mathcal{C}})}{Y} = 0.65$	literature
Depreciation rate of K: δ^K	0.059	$I^K/Y = 0.18$	OECD (2014b)
Preferences			
EIS, $1/\theta$	0.5	1^{st} stage	literature, LSV
Leisure elasticity: υ	0.21	1^{st} stage	labor supply elast., see text
Time discount factor: β	0.999	$K/Y = 2.9$	literature
Weight of leisure: γ	0.34	aver. $l = 0.285$	OECD (2014b)
Endowment and ability			
Endowment: $\{h_0, k_0, b_0\}$	$\{1.0, 0.0, 0.0\}$	1^{st} stage	normalization
Std. time effort college: σ	0.32	1^{st} stage	$\mathcal{C}$ share elast., see text
Human capital accumulation			
Depreciation rate of h: δ^h	0.008	1^{st} stage	LSV
On-the-job h accum.: ϱ	0.65	1^{st} stage	BHH
Mean time effort college: μ	0.66	$\mathcal{C}$ share = 0.27	OECD (2014b)
h markup from college: $\bar{h}$	0.58	$r^{\mathcal{C}} = r^{\mathcal{H}}$	normalization
On-the-job h accum. $\mathcal{H}$: $\varpi^{\mathcal{H}}$	0.75	$e^{\mathcal{H}}_{55-60}/e^{\mathcal{H}}_{25} = 1.35$	OECD (2014a)
On-the-job h accum. $\mathcal{C}$: $\varpi^{\mathcal{C}}$	0.84	$e^{\mathcal{C}}_{55-60}/e^{\mathcal{C}}_{25} = 1.78$	OECD (2014a)
Pension system			
Contribution rate: τ	see text	1^{st} stage	German public pension sys.

Source: Baseline model and target data in the calibration period. The displayed values refer to year 2010 if not stated differently. *Notes:* The displayed values are converted to annualized rates where applicable. ES($L^{\mathcal{H}}, L^{\mathcal{C}}$) $\hat{=}$ elasticity of substitution between high school and college labor holding the relative price of college labor to capital fix. ES($K, L^{\mathcal{C}}$) $\hat{=}$ elasticity of substitution between college labor and physical capital. EIS $\hat{=}$ elasticity of inter-temporal substitution. BHH $\hat{=}$ Browning, Hansen, and Heckman (1999). DLS $\hat{=}$ Dustmann, Ludsteck, and Schönberg (2009). LSV $\hat{=}$ Ludwig, Schelkle, and Vogel (2012). DPP $\hat{=}$ Duffy, Papageorgiou, and Perez-Sebastian (2004). The average labor supply in the data refers to the number of hours per worker (incl. part time) divided by 5000. $e^{\mathcal{H}}$ and $e^{\mathcal{C}}$ are average net earnings by educational group. Note that the model does not feature a progressive earnings tax system which implies that ratios between educational groups of net earnings are identical to those of gross earnings by construction. I use data on net earnings as targets in order to match more closely the differences in disposable income across educational groups.

Figure 1.3: Cross-Sectional Age Profiles By Education Type in 2010

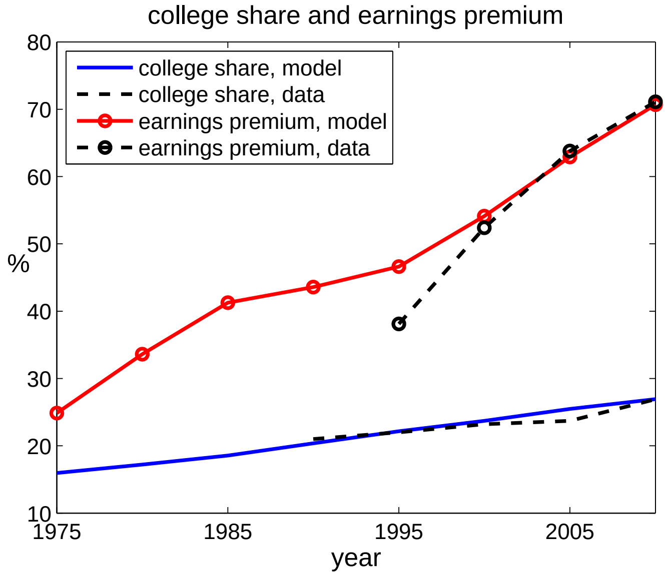


Source: Baseline model in year 2010: Selected average cross-sectional age profiles by education type. Net earnings premium data is taken from OECD (2014a). *Notes:* The top left panel shows hours worked as a percentage of total hours available to a household. The top right panel shows the sum of gross earnings and pensions by educational group normalized by the corresponding value of a 20 year old non-college household. The bottom left panel shows consumption by educational group normalized by the corresponding value of a 20 year old non-college household. The bottom right panel shows the net earnings and pension premium of a college household relative to a non-college household of the same age.

(top left panel) show the typical hump-shaped pattern with college households supplying more labor than high-school households. The closer to retirement, the larger is that difference. Note that the peak in the consumption profile is too late compared to the data which is a common problem in deterministic life cycle models.

Figure 1.4 reflects the good fit of the model to the data with respect to the evolution of the college share in the 25+ workforce and the average net earnings premium of college graduates. The rise in the two variables is a

Figure 1.4: Past Evolution of College Share and Earnings Premium



Source: Baseline model vs. data in the calibration period, 1975 − 2010. Data on the evolution of the college share in the 25+ workforce is taken from Statistisches Bundesamt (2014) and scaled to the target value of the college share in 2010 taken from OECD (2014b), cf. table 1.1. Net earnings premium data stems from OECD (2014a).

result of both, demographic change and a past trend in the direction of technological progress as observed in the data (cf. section 1.4.2). Note that the model slightly underestimates the increase in the earnings premium between 1995 and 2000. That is likely due to other factors being absent in the model which had affected the earnings premium at that time. For example, Dustmann, Ludsteck, and Schönberg (2009, pp. 859ff.) estimate that the decrease in the unionization rate of the labor force in the considered period can account for 28% of the increase in the 50-15 earnings gap while it is less important at the upper end of the earnings distribution.

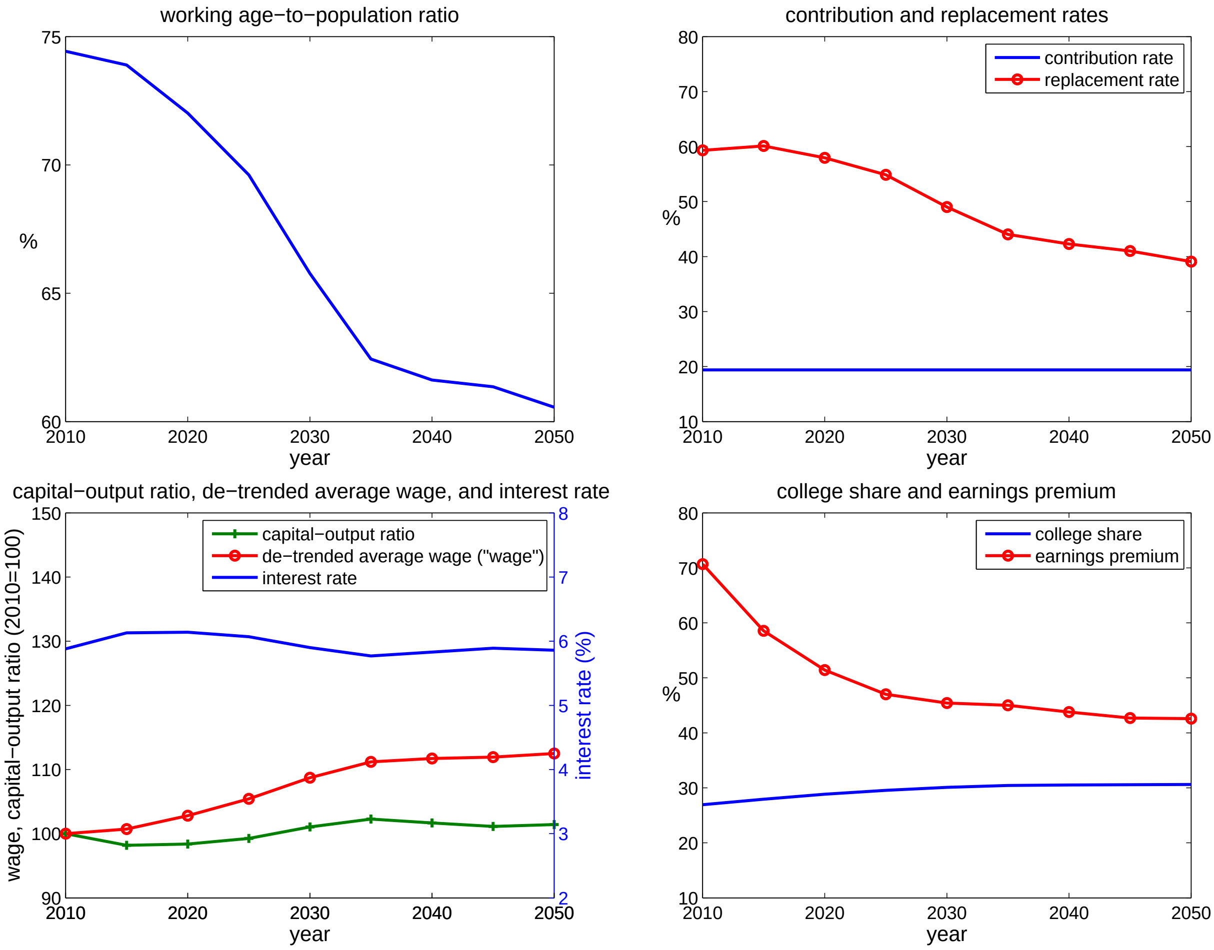
1.5.2 Aggregate Dynamics Beyond 2010

Let us now turn to the future dynamics in the considered model economy. Figure 1.5 shows the evolution of key aggregate measures in the main period of projection, i.e., year 2010 to year 2050.

The top-left panel depicts the evolution of the working age-to-population ratio (WAPR). This is the exogenous driving force in future periods of the model and a key measure for the demographic structure of the economy. It reflects the strong aging process which is expected to hit the German economy.

The top-right panel depicts the contribution and replacement rates of the public pension system and confirms the expected decline in the generosity of the pension system. Note that this scenario assumes that the government will hold the contribution rate fix in all future periods which results in a strong decline in the replacement rate of about 20 percentage points by 2050. This is within the range of estimates for pay-as-you-go

Figure 1.5: Aggregate Dynamics Beyond 2010



Source: Baseline model in the main period of projection, 2010 to 2050: Selected aggregate measures. *Notes:* The replacement rate in the top right panel refers to the average ratio of the pension payment to a 65 year old household to the sum of net earnings (adjusted by wage inflation) over its working life. The average gross hourly wage in the bottom left panel is shown net of the trend growth arising from exogenous skill-neutral technological change in Υ_t over time. Capital-output ratio and de-trended average wage are normalized to 100 in 2010. The bottom right panel shows the average net earnings premium of college graduates and the share of college graduates in the 25+ workforce in percent.

financed pension systems like the German one with defined contributions.

The bottom left panel shows an increase in the capital-output ratio with the associated increase in the detrended average gross wage[26] and the decline in the interest rate. It reveals the substitution of the scarce production factor, labor, by the one in abundance, capital, and is in line with the results of Ludwig, Schelkle, and Vogel (2012) for the US. However, while the rise in the capital-output ratio and the decline in the interest rate remain rather small the rise in the wage is more pronounced and equals 12.5% by 2050. This points to a second substitution effect taking place in the aging economy: Labor input of non-college households is substituted by labor input of college households. The re-composition of the labor force elevates the average wage because the latter households are more productive than the former and thus earn a higher wage.

The bottom right panel confirms the re-composition of the labor force by showing a rise in the college share of about 4 percentage points by 2050. Furthermore, it reveals that the rise in the college share is associated with a strong decline in the average earnings premium of college households. These developments are mainly driven by two effects.

The first effect arises from the scarcity of labor as an input in the production which follows ceteris paribus from the process of aging beyond 2010. Its scarcity makes labor relative to capital ceteris paribus more expensive and leads to a substitution of labor by capital. Due to the complementarity of capital and college labor input in the production the substitution of labor by capital is associated with a rise in the relative demand for college workers compared to non-college workers. This elevates ceteris paribus the earnings premium. However, in anticipation of this and other effects described below more households choose to go to college which, in turn, induces an increase in the relative supply of college labor and lowers ceteris paribus the earnings premium.

The second effect stems from skill-biased technological change in the calibration period due to which the share of newborn households selecting college has risen strongly in the period until 2010 (cf. figure 1.4). From this it follows that less educated retiring generations of workers are substituted gradually by better educated new generations of workers also beyond 2010. This implies a higher relative supply of college workers which lowers ceteris paribus the earnings premium.

As mentioned above, the bottom right panel shows a strong and steady overall decline in the earnings premium beyond 2010 and, thereby, reveals that the supply side effects overcompensate clearly the demand side effect. The decrease in the earnings premium occurs most strongly in the period 2010 to 2030 amounting to more than 20 percentage points. This is when the aforementioned second effect is at its peak. Note that it becomes optimal to pursue a college degree for a higher fraction of households despite the decline in the earnings premium. Reasons for that are the co-incident decline in the interest rate which makes borrowing for education less costly (and savings less attractive) as well as the increase in life expectancy. The latter prolongs the expected return-on-investment period of education (inter alia via an earnings related pension system) and induces a need for higher lifetime earnings. Education still shows to be the most efficient way of achieving it.

As an indicator for the development of overall economic inequality in the economy table 1.2 displays the change in the Gini coefficients of net total income (*income Gini*) and consumption (*consumption Gini*) from 2010 to 2050.[27] First, consider the top row. The number in the top left corner suggests that overall economic

[26] The average gross wage follows a trend in time arising from exogenous skill-neutral technological change in Υ_t. The trend is removed for the sake of meaningful comparison between different time periods.

[27] The model clearly underestimates the level of income inequality in the economy. However, note that income inequality in the data is driven by both, variation in initial conditions and differences in shocks over the course of life whereof the latter are absent in this model. Huggett, Ventura, and Yaron (2011) estimate for the U.S. that about two third of lifetime inequality stems from

Table 1.2: Measures of Economic Inequality: The Change in Ginis from 2010 to 2050

	net total income	consumption
$Gini(2050) - Gini(2010)$, baseline	$+6.1$	-3.4
$Gini(2050) - Gini(2010)$, fix demographics	-7.6	-0.2

Source: Baseline model in 2010 and 2050. *Notes:* The numbers show percentage point changes. The top row displays the total change in Gini coefficients while the bottom row shows the corresponding values if the Gini coefficients in 2050 are recomputed using the age distribution of the population in year 2010. Net total income is the sum of capital income and net earnings, respectively pension income of a household.

inequality rises strongly in the period 2010 to 2050 as the income Gini increases by 6.1 percentage points. However, the consumption Gini shows a decline of 3.4 percentage points over the same period indicating the opposite. Note that the change in a Gini coefficient can be induced by a change in the income (respectively consumption) distribution given the same age distribution of the population or by a change in the age distribution of the population itself. Here, both effects apply at the same time.

In order to isolate the first effect the bottom row of table 1.2 shows the change in Gini coefficients if the Gini coefficients in 2050 are recomputed using the age distribution of the population in year 2010. The number in the bottom left corner now indicates a strong drop in the income Gini. Recalling the dynamics displayed in figure 1.5 suggests that this stems primarily from receding income inequality between ability groups represented by the drop in the earnings premium. Hence, the overall increase in the income Gini (top row) stems primarily from a change in the population distribution, more precisely, a higher fraction of income poor (but asset rich) old households. Now, consider the number in the bottom right corner. It shows a rather stable consumption Gini which indicates rather stable lifetime income prospects of a household between 2010 and 2050 because the consumption decision depends on lifetime income rather than current temporary income. Again, the overall decrease in the consumption Gini (top row) stems primarily from a change in the population distribution, more precisely, a smaller fraction of consumption poor young households.

1.5.3 Welfare Effects Within and Across Generations

I now turn to the welfare effects arising from the dynamics described in section 1.5.2. Following the literature I measure welfare effects by consumption equivalent variation (cf. Ludwig, Schelkle, and Vogel, 2012, and the literature cited therein). A household's welfare is affected in two ways. The first effect arises from changes in survival probabilities which are exogenous in this model. The second effect stems from changing good and time allocations induced by changes in wages, interest rates and average pension amounts.

I want to isolate the second effect and, therefore, conduct the following auxiliary computation using the same (time- and age-dependent) survival probabilities throughout the entire welfare calculation. I compute a partial equilibrium version of the model given a particular exogenous vector of wages, interest rates, and average pension amounts for the entire transition which I will call *price vector* in the following. The price vector equals the corresponding vector from the equilibrium path of the baseline case up until year 2010 and keeps prices fixed from then onward. This represents an auxiliary world in which the future price changes arising from

variation in initial conditions. Assuming ad-hoc the same decomposition for Germany allows to compute the Gini of net total income which corresponds to the model. Using data from OECD (2014b) it amounts to $0.286 \cdot 2/3 = 0.191$ which is very close to the actual value in the model in 2010, 0.194.

demographic and past skill-biased technological change do not evolve.[28] Welfare effects are then measured as the percentage change in consumption in every period of lifetime that a household must be compensated with in order to be indifferent between the auxiliary world and the baseline case.[29]

In this model, total future price changes and their welfare effects are induced by the combination of demographic and past skill-biased technological change. In order to decompose total welfare effects I redo the entire welfare calculation described above under the assumption that no skill-biased technological change has taken place up until 2010. Note that, except for the mentioned change, parameters of the model are not re-calibrated in order to have a reasonable case of comparison.[30] Figure 1.6 presents resulting average welfare effects by educational class. The top row shows welfare effects which arise purely from demographic change while the bottom row displays the effects from the combination of demographic and past skill-biased technological change. The left panels depict welfare effects on all generations alive in year 2010 while the right panels display welfare effects on generations which turn 20 in the indicated year between 2010 and 2050.

Consider first the panels in the top row of figure 1.6 showing the average welfare effects from demographic change. Section 1.B.3 in the appendix presents the associated future dynamics of key aggregate measures in detail. The central findings can be summarized as follows: While 1) the replacement rate, 2) the interest rate, and 3) the earnings premium decline gradually over time, 4) the average wage level increases. Note that while effect 3) obviously benefits high-school households, effect 2) rather benefits college households because their advantage from smaller borrowing costs during the college years shows to dominate the loss from smaller capital income at wealthy old ages. Developments 1) and 4) affect college and high-school households similarly.

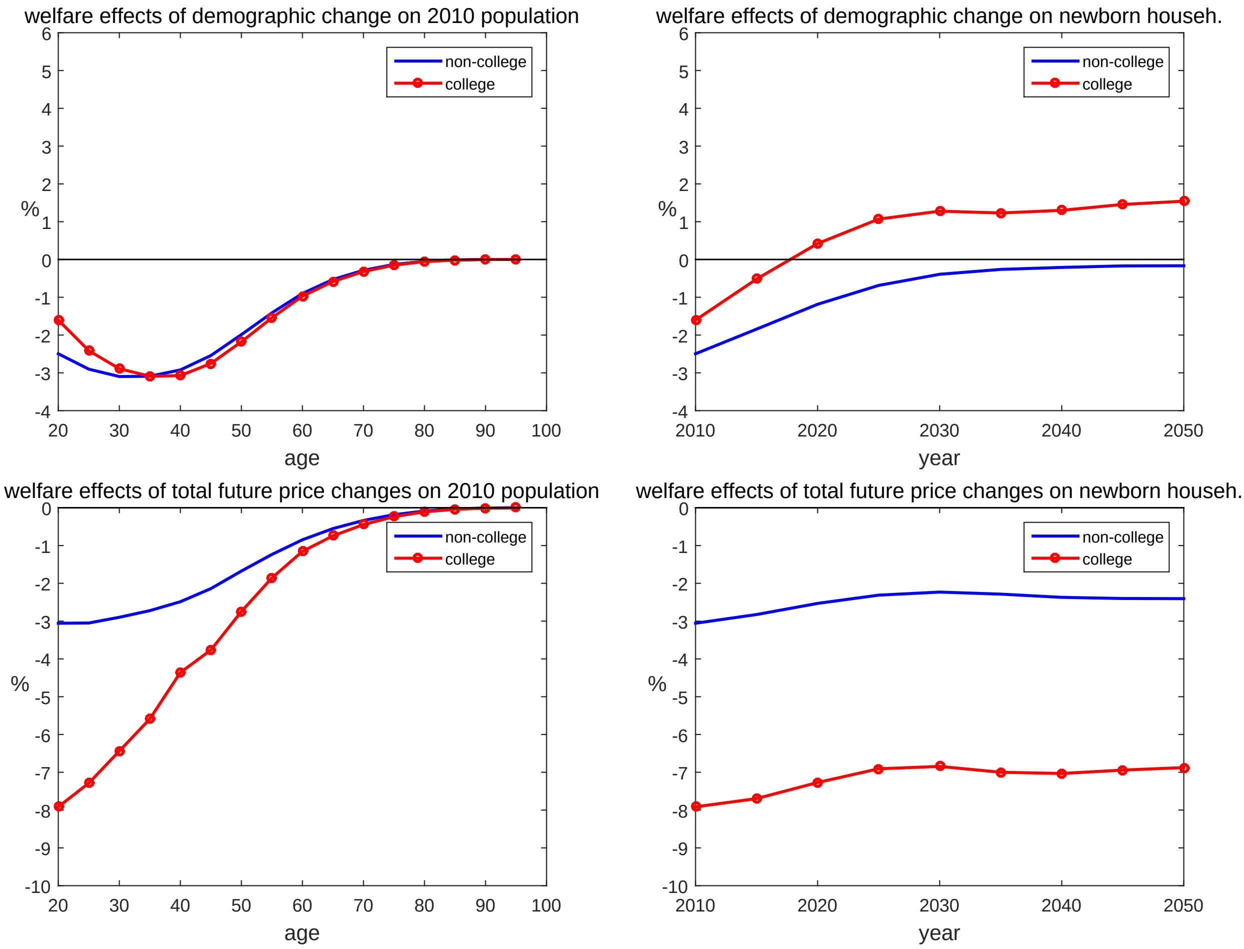
The top left panel shows the net welfare effects on all generations alive in 2010 and features an u-shaped pattern for both education types. This was already found by Ludwig, Schelkle, and Vogel (2012) in an overlapping generations model without differentiation by education. The top right panel displays welfare effects of newborn generations in years 2010 to 2050 and shows that welfare losses from demographic change on newborn households vanish in the course of time and turn even positive for both educational groups at some point in the future. Taken both panels together, one can make the following two major observations: First, the later in time a household is born, the less it suffers from demographic change irrespective of education. This is due to the developments described above, in particular, rising wages. However, that first observation does not apply to households which are more than about 35 years old in 2010. The reason is that they benefit less from rising wages due to a shorter expected remaining working life and from decreasing interest rates for financing education whereof the latter only applies to college households. Furthermore, welfare effects run off with increasing age per definition because the household's expected remaining lifetime decreases. As a net effect, welfare losses are severest for the generation aged $30-40$ in 2010 and amount to about 3% of consumption in every period of lifetime. This holds for college as well as for high-school households. Second, the later in time a household is born, the more it benefits from going to college. This shows that effect 2) of dropping interest rates becomes more and more important over time while the opposite holds true for effect

[28] Note that household choices are fully rational given the exogenous price vector while the procedure does not involve the general equilibrium as described in section 1.3.7.

[29] Based on the homotheticity of the value function, consumption equivalent variation can be measured as $cev_t := (\frac{v_{t,0}}{v_{t,0}^A})^{\frac{1}{1-\theta}} - 1$, where $v_{t,0}$ and $v_{t,0}^A$ are the lifetime values of the generation born in period t in the baseline case and the auxiliary world respectively.

[30] I.e., the parameter values of table 1.1 still apply except for $g_{1975-2010}^{(1-\alpha_2)/\alpha_2}$ which equals 0.

Figure 1.6: Welfare Effects of Demographic Change (Top Panels) and Total Future Price Changes (Bottom Panels) on Current and Future Populations



Source: Baseline model in the main period of projection, 2010 to 2050. *Notes:* Welfare effects measured as consumption equivalent variation. Negative values indicate welfare losses. Top row: Welfare effects from demographic change. Bottom row: Welfare effects from future price effects induced by both, demographic and past skill-biased technological change. Left panels: Welfare effects on all generations alive in year 2010. Right panels: Welfare effects on generations turning 20 between 2010 and 2050.

3), the decline in the earnings premium.

Let us now move on to the bottom panels. Here, welfare effects arise from total future price changes which are induced by the combination of demographic change and past skill-biased technological change. Comparing the graphs in the bottom panels to the respective graphs in the top panels reveals the central welfare effect arising from past skill-biased technological change: A strong negative effect for college households. This holds for all considered generations. Why is this the case? Note that the baseline economy matches the past evolution of the college share in the workforce which is characterized by a stronger upward trend (cf. figure 1.4) than in the case of demographic change only. From this it follows that, the increase in the college share in the future, in particular until 2040, is also more severe for the baseline economy. This is due to a larger difference between the college shares of newborn households and substituted retiring households in that period. The stronger future increase of the college share has two consequences. First, the decline in the earnings premium is larger in the baseline economy, and, second, the decline in the interest rate is smaller. Both induces negative welfare effects on college households.

1.5.4 The Role of Changes in Household Behavior

This section investigates the quantitative importance of changes in household behavior in response to altering market conditions arising from demographic change[31] along three dimensions: tertiary education, working hours, and human capital investments. Correspondingly, I run three experiments, each of them re-computes the general equilibrium path of the model economy and evaluates the associated welfare effects (following the approach described in section 1.5.3) under exactly one restriction: From period 2010 onward, the choice of tertiary education, working hours, or human capital investments respectively is restricted as specified in more detail below.[32]

Figure 1.7 shows the welfare effects of demographic change on newborn households in the respective model variant by education. As implications are similar for current generations they are omitted for the sake of brevity, here.

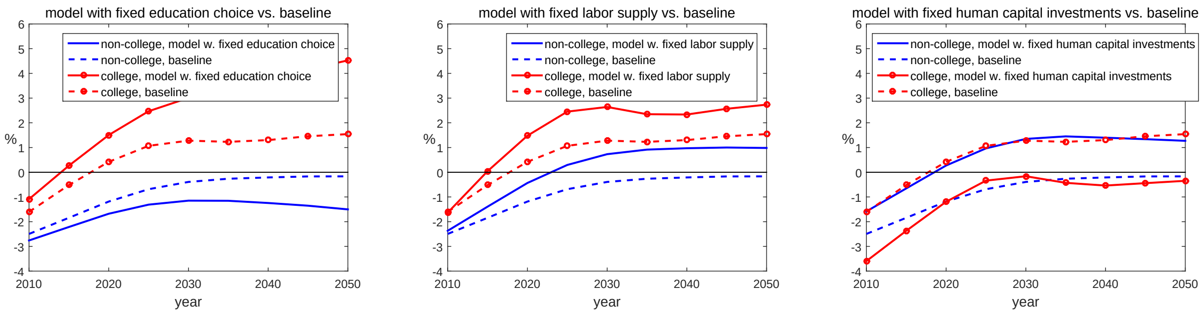
The left panel depicts the welfare effects for the model variant in which the share of newborn households selecting tertiary education is held fix artificially from 2010 onward. This dampens the supply side effect on the labor market which was described in detail in section 1.5.2. It leads to a smaller drop in the earnings premium with positive (negative) welfare implications for college (high-school) households. Note that welfare advantages of college households from restricting the college choice become larger in time because the aforementioned dampening of the supply side effect applies to every period past 2010 and thus accumulates.

The panel in the center shows welfare effects of demographic change if hours worked are held fix artificially from 2010 onward. This implies that the age-profiles of hours worked by ability group remain time-constant as of 2010. The graphs show that the increase of working hours is an important adjustment channel for households to the altered market conditions arising from demographic change. In particular, it enables households to benefit more strongly from rising wages (per hour). Note that the restriction of adjustments along the working hours margin has similar welfare costs for both education types.

[31] Here, I focus on the case without past skill-biased technological change in order to answer more precisely the question how well households can react to challenges from demographic change. However, at least qualitatively, results are similar to the baseline case.

[32] Note that each experiment assures the general equilibrium under an additional constraint in the household problem.

Figure 1.7: The Role of Changes in Household Behavior: Welfare Effects of Demographic Change in Different Model Variants



Source: Variants of the model with restricted household choices (solid lines) vs. baseline model (dashed lines, cf. figure 1.6): Welfare effects of demographic change on newborn households, i.e. generations turning 20 in the main period of projection, 2010 to 2050. *Notes:* Welfare effects measured as consumption equivalent variation. Negative values indicate welfare losses of demographic change.

The right panel contains the corresponding graphs for the model variant in which the hours spent on human capital formation are held fix artificially from 2010 onward. This implies that the age-profile of human capital by ability group remains constant as of 2010. College households exhibit a higher productivity in human capital formation on-the-job than high-school households. The picture shows clearly that this turns into a true asset in the demographic transition. It enables college households to benefit from higher returns to human capital investments at a lower cost than high-school households.

To sum up, upward adjustments of hours worked affect all households similarly, increased human capital investments rather benefit college households, and the opposite of the latter is induced by more tertiary education. While increased human capital investments directly elevate the idiosyncratic productivity and in turn the idiosyncratic wage of a household, the increase in the tertiary educated share in the work force depresses the market-wide return premium to college labor input. Hence, the latter acts through an equilibrium effect.

1.6 Concluding Remarks

This chapter investigates the effects of demographic change on the distributions of income, skills, and welfare in the German economy. The dynamic overlapping generations model features heterogeneity along both, the inter- and the intra-generational dimension whereof the latter is given by an idiosyncratic innate ability for tertiary education. Besides consumption, the model features three choices which can help households to overcome the altered economic conditions arising from demographic change: tertiary education, on-the-job skill formation, and hours worked.

The quantitative experiments reveal the following effects of demographic change comparing year 2010 to year 2050: 1) The skill premium declines by about 15 percentage points while the college educated share of the workforce increases by about 3 percentage points. 2) The interest rate falls by 1 percentage point while the average wage increases by about 20% induced by a substitution of labor by capital and the aforementioned skill increase in the production. 3) The replacement rate falls massively by about 40 percentage points if the contribution rate to the public pension system is held fix.

Welfare effects of demographic change are substantial and vary between -3% and $+2\%$ of consumption in every period of lifetime depending on skill group and generation. All currently living generations lose. Despite the drop in the skill premium, demographic change benefits skilled over unskilled households. This is mainly due to the co-incident decline of the interest rate which makes borrowing for education less costly. While less able households benefit strongly from equilibrium effects arising from a higher college share in the workforce, more able households benefit rather from higher idiosyncratic human capital investments on the job.

As a secondary result, the quantitative experiments show that past skill-biased technological change will depress strongly the future skill premium by additional 15 percentage points due to an ongoing increase in the relative supply of college workers. This causes strong welfare losses for college households of up to 8% of consumption in every period of lifetime. Note that the prediction of a strongly declining earnings premium and the associated welfare consequences could be turned around in case of ongoing skill-biased technological change in the future. However, this chapter remains agnostic with respect to the direction of future technological change in the sense that all future change in technology is assumed to be skill-neutral.

The investigation shows that it is important to consider the skill-composition of aggregate labor input in the context of aging economies and thereby extents the results of Ludwig, Schelkle, and Vogel (2012) by heterogeneity in formal skills. There are a couple of interesting extensions that future research may approach. First, in this model, a household trades off opportunity costs against pecuniary benefits when deciding on tertiary education. The latter does not involve any direct pecuniary costs, neither at the private nor at the public level. In reality, the German university system is mostly financed by public means. Aging economies are likely to fall short of financial means as the tax base erodes as Keuschnigg, Davoine, and Schuster (2013) argue. A link of tax and educational system would thus be an interesting extension. Furthermore, Winter (2014) states that both, parental transfers and borrowing constraints are important determinants of the college decision in the U.S. Kindermann (2014) argues that the former is also true for Germany. In a dynastic world, in which the ability of children is linked to the ability of their parents parental transfers might keep lower ability household out of college, as they would be disadvantaged along two dimensions. However, declining interest rates in the course of demographic change would then act as a counter-argument as they would harm rich and more able households more than the poor and less able. The evolution of interest rates and the capital structure of the economy drives the college decision and is also important for the welfare consequences across skill groups. It is thus important to shed more light on the distributional consequences across skill groups in economic settings with an international capital market in the tradition of Krüger and Ludwig (2007). This is another task that I leave for future research.

Appendix 1.A Theoretical Appendix

1.A.1 Recursive Household Problem

I hereafter define recursively the household problem. Households take returns as given and maximize their lifetime utility over the choice of consumption, hours supplied to the labor market and hours spent on skill accumulation. Note that the latter choice is restricted while a household is enrolled in college as described in section 1.3.3.

In order to derive the solution of the household problem I apply a transformation which assures that all variables are trend-stationary. This is consistent with the computational implementation of the solution of the household problem and does not alter results. Therefore, I divide variables which exhibit a trend growth arising from the exogenous technological progress, $\Upsilon_{t+1} = (1+g)\cdot\Upsilon_t$, by the technology level: $\tilde{k}^a_{t,j} := k^a_{t,j}/\Upsilon_t$, $\tilde{c}^a_{t,j} := c^a_{t,j}/\Upsilon_t$, $\tilde{r}^{\mathcal{S}}_t := r^{\mathcal{S}}_t/\Upsilon_t$, $\tilde{\mathcal{P}}_t(b^a_{t,j}) := \mathcal{P}_t(b^a_{t,j})/\Upsilon_t$, $\tilde{\vartheta}_t(b^a_{t,j}, r^{\mathcal{S}}_t \cdot h^a_{t,j} \cdot l^a_{t,j}) := \vartheta_t(b^a_{t,j}, r^{\mathcal{S}}_t \cdot h^a_{t,j} \cdot l^a_{t,j})/\Upsilon_t$.[33] Other variables are already trend-stationary and do not need to be transformed. In the following, I drop the indexes t and j for the sake of simplicity and indicate next period's variables by the symbol $'$, irrespective of whether they are only time dependent or age and time dependent.
Given the preferences of section 1.4.2 the de-trended household problem reads as:

$$v^a(\tilde{k}^a, h^a, b^a) = \max_{\tilde{c}^a, l^a, i^a, \tilde{k}^{a\prime}, h^{a\prime}, b^{a\prime}} \{u(\tilde{c}^a, i^a, l^a) + \hat{\beta}\cdot v^{a\prime}(\tilde{k}^{a\prime}, h^{a\prime}, b^{a\prime})\} \tag{1.18}$$

subject to

$$\begin{aligned}
\tilde{k}^{a\prime} &= \begin{cases} \frac{1}{1+g}\cdot\left((1+r^K)\cdot\tilde{k}^a + (1-\tau)\cdot\tilde{r}^{\mathcal{S}}\cdot h^a\cdot l^a - \tilde{c}^a\right) & if \quad j < j_r \\ \frac{1}{1+g}\cdot\left((1+r^K)\cdot\tilde{k}^a + \tilde{\mathcal{P}}(b^a) - \tilde{c}^a\right) & else \end{cases}\\
b^{a\prime} &= \begin{cases} \tilde{\vartheta}(b^a, \tilde{r}^{\mathcal{S}}\cdot h^a\cdot l^a) & if \quad j < j_r - 1 \\ b^a & else \end{cases}\\
h^{a\prime} &= \begin{cases} h_0 & if \quad j < j_w - 1 \ \wedge\ \mathcal{S} = \mathcal{C} \\ (1+\bar{h})\cdot h_0 & if \quad j = j_w - 1 \ \wedge\ \mathcal{S} = \mathcal{C} \\ \varphi^{\mathcal{S}}(h^a, i^a) & else \end{cases}\\
h^a_0 &= h_0 > 0,\ \tilde{k}^a_0 = \tilde{k}_0 = 0,\ b^a_0 = b_0 = 0\\
c^a&, l^a, i^a \geq 0\\
i^a &= \bar{i}^a \quad if \quad j < j_w - 1 \ \wedge\ \mathcal{S} = \mathcal{C}
\end{aligned} \tag{1.19}$$

where $\hat{\beta} := \beta\cdot\varsigma\cdot(1+g)^{1-\theta}$. Note that the transformation of the utility function in period t, age j follows $u(c, 1-i-l) = u(\tilde{c}, 1-i-l)\cdot\Upsilon^{1-\theta}$. That transformation also applies to the value function.

In the following, I derive the first order conditions (FOCs) of the de-trended household problem given initial conditions and the educational choice. Note that the constraint $i^a = 0$ never binds due to the production

[33] Note that the values of trend-stationary variables do not depend on the time period in a steady state of the economy in which the demographic age distribution of the economy is assumed to be fix over time.

function of human capital. λ is the Lagrange multiplier associated with the constraint $l^a \geq 0$.

$$0 = u_{\tilde{c}} + \hat{\beta} \cdot v'_{\tilde{k}'} \cdot (-\frac{1}{1+g}) \quad if \quad j < J \tag{1.20}$$

$$0 = u_i + \hat{\beta} \cdot v'_{h'} \cdot \varphi_i^{\mathcal{S}} = 0 \quad if \quad j < j_r - 1 \wedge not(j < j_w \wedge \mathcal{S} = \mathcal{C}) \tag{1.21}$$

$$0 = u_l + \hat{\beta} \cdot \left(v'_{\tilde{k}'} \cdot \frac{1}{1+g} \cdot (1-\tau) \cdot \tilde{r}^{\mathcal{S}} \cdot h + v'_{b'} \cdot \tilde{\vartheta}_l \right) + \lambda \quad if \quad j < j_r \tag{1.22}$$

where $u_{\tilde{c}} := \partial u(\tilde{c}, i, l)/\partial \tilde{c}$, $u_i := \partial u(\tilde{c}, i, l)/\partial i$, $u_l := \partial u(\tilde{c}, i, l)/\partial l$,
$\varphi_i^{\mathcal{S}} := \partial \varphi^{\mathcal{S}}(h, i)/\partial i$, $\tilde{\vartheta}_l := \partial \tilde{\vartheta}(b, \tilde{r}^{\mathcal{S}} \cdot h \cdot l)/\partial l$, $v'_{\tilde{k}'} = \partial v'(\tilde{k}', h', b')/\partial \tilde{k}'$, $v'_{h'} = \partial v'(\tilde{k}', h', b')/\partial h'$, and $v'_{b'} = \partial v'(\tilde{k}', h', b')/\partial b'$.

Next, I derive the first partial derivatives of the value function with respect to the state variables $\tilde{k}$, h, and b using the envelope theorem:

$$v_{\tilde{k}} = \begin{cases} \hat{\beta} \cdot v'_{\tilde{k}'} \cdot \frac{1}{1+g} \cdot (1 + r^K) & if \quad j < J \\ u_{\tilde{c}} \cdot (1 + r^K) & else \end{cases} \tag{1.23}$$

$$v_h = \begin{cases} \hat{\beta} \cdot \left(v'_{\tilde{k}'} \cdot \frac{1}{1+g} \cdot (1-\tau) \cdot \tilde{r}^{\mathcal{S}} \cdot l + v'_{h'} \cdot \varphi_h^{\mathcal{S}} + v'_{b'} \cdot \tilde{\vartheta}_h \right) & if \quad j < j_r \\ 0 & else \end{cases} \tag{1.24}$$

$$v_b = \begin{cases} \hat{\beta} \cdot v'_{b'} \cdot \tilde{\vartheta}_b & if \quad j < j_r \\ \hat{\beta} \cdot \left(v'_{\tilde{k}'} \cdot \frac{1}{1+g} \cdot \tilde{\mathcal{P}}_b + v'_{b'} \right) & if \quad j_r \leq j < J \\ u_{\tilde{c}} \cdot \tilde{\mathcal{P}}_b & else \end{cases} \tag{1.25}$$

where $\varphi_h^{\mathcal{S}} := \partial \varphi^{\mathcal{S}}(h, i)/\partial h$, $\tilde{\vartheta}_h := \partial \tilde{\vartheta}(b, \tilde{r}^{\mathcal{S}} \cdot h \cdot l)/\partial h$, $\tilde{\vartheta}_b := \partial \tilde{\vartheta}(b, \tilde{r}^{\mathcal{S}} \cdot h \cdot l)/\partial b$, and $\tilde{\mathcal{P}}_b := \partial \tilde{\mathcal{P}}(b)/\partial b$.

Using these equations yields the following FOCs:

$$u_{\tilde{c}} = \frac{1}{1+g} \cdot \hat{\beta} \cdot (1 + r^{K\prime}) \cdot u'_{\tilde{c}'} \qquad if \quad j < J \tag{1.26}$$

$$-u_i = \hat{\beta} \cdot v'_{h'} \cdot \varphi_i^{\mathcal{S}} \quad if \quad j < j_r - 1 \wedge not(j < j_w \wedge \mathcal{S} = \mathcal{C}) \tag{1.27}$$

$$-u_l = \left((1-\tau) \cdot \tilde{r}^{\mathcal{S}} \cdot h + \frac{v'_{b'}}{v'_{\tilde{k}'}} \cdot (1+g) \cdot \tilde{\vartheta}_l \right) \cdot u_{\tilde{c}} + \lambda \quad if \quad j < j_r \tag{1.28}$$

1.A.2 Derivation of the Aggregate Resource Constraint

The individual budget constraints write as

$$k_{t+1,j+1}^a = (1 + r_t^K) \cdot k_{t,j}^a + (1 - \tau_t) \cdot r_t^{\mathcal{S}} \cdot h_{t,j}^a \cdot l_{t,j}^a + \mathcal{P}_t(b_{t,j}^a) - c_{t,j}^a \; \forall \, t, j.$$

I take the population weighted sum of the individual budget constraints in order to derive the aggregate resource constraint in period t:

$$\sum_{j=0}^{J} N_{t,j} \cdot \int k_{t+1,j+1}^a \cdot \zeta(a) \, da$$
$$= \sum_{j=0}^{J} N_{t,j} \cdot \int \left((1 + r_t^K) \cdot k_{t,j}^a + (1 - \tau_t) \cdot r_t^{\mathcal{S}} \cdot h_{t,j}^a \cdot l_{t,j}^a + \mathcal{P}_t(b_{t,j}^a) - c_{t,j}^a \right) \cdot \zeta(a) \, da$$

$$
\begin{aligned}
&\sum_{j=0}^{J} N_{t,j} \cdot \int k^a_{t+1,j+1} \cdot \zeta(a)\, da \\
&= \sum_{j=1}^{J} \varsigma_{t-1,j-1} \cdot N_{t-1,j-1} \cdot \int (1+r^K_t) \cdot k^a_{t,j} \cdot \zeta(a)\, da + N_{t,0} \cdot \int (1+r^K_t) \cdot k^a_{t,0} \cdot \zeta(a)\, da \\
&+ \sum_{j=0}^{J} N_{t,j} \cdot \int (1-\tau_t) \cdot r^{\mathcal{S}}_t \cdot h^a_{t,j} \cdot l^a_{t,j} \cdot \zeta(a)\, da + \sum_{j=0}^{J} N_{t,j} \cdot \int \mathcal{P}_t(b^a_{t,j}) \cdot \zeta(a)\, da \\
&- \sum_{j=0}^{J} N_{t,j} \cdot \int c^a_{t,j} \cdot \zeta(a)\, da + \sum_{j=}^{J} (1-\varsigma_{t-1,j-1}) \cdot N_{t-1,j-1} \cdot \int (1+r^K_t) \cdot k^a_{t,j} \cdot \zeta(a)\, da \\
&- \sum_{j=1}^{J} (1-\varsigma_{t-1,j-1}) \cdot N_{t-1,j-1} \cdot \int (1+r^K_t) \cdot k^a_{t,j} \cdot \zeta(a)\, da \\
&\sum_{j=0}^{J} N_{t,j} \cdot \int k^a_{t+1,j+1} \cdot \zeta(a)\, da \\
&= \sum_{j=1}^{J} N_{t-1,j-1} \cdot \int k^a_{t,j} \cdot \zeta(a)\, da + r^K_t \cdot \sum_{j=1}^{J} N_{t-1,j-1} \cdot \int k^a_{t,j} \cdot \zeta(a)\, da \\
&+ \sum_{j=0}^{j_r-1} N_{t,j} \cdot \int r^{\mathcal{S}}_t \cdot h^a_{t,j} \cdot l^a_{t,j} \cdot \zeta(a)\, da - \tau_t \cdot \sum_{j=0}^{j_r-1} N_{t,j} \cdot \int r^{\mathcal{S}}_t \cdot h^a_{t,j} \cdot l^a_{t,j} \cdot \zeta(a)\, da \\
&+ \sum_{j=j_r}^{J} N_{t,j} \cdot \int \mathcal{P}_t(b^a_{t,j}) \cdot \zeta(a)\, da - \sum_{j=0}^{J} N_{t,j} \cdot \int c^a_{t,j} \cdot \zeta(a)\, da \\
&- \sum_{j=1}^{J} (1-\varsigma_{t-1,j-1}) \cdot N_{t-1,j-1} \cdot \int (1+r^K_t) \cdot k^a_{t,j} \cdot \zeta(a)\, da
\end{aligned}
$$

Using the equilibrium conditions (1.8–1.12) as well as the following two conditions

- zero profits due to constant returns to scale production:

$$
\begin{aligned}
&\mathcal{F}_t(K_t, L^{\mathcal{H}}_t, L^{\mathcal{C}}_t) - (r^K_t + \delta^K) \cdot K_t - r^{\mathcal{H}}_t \cdot L^{\mathcal{H}}_t - r^{\mathcal{C}}_t \cdot L^{\mathcal{C}}_t = 0 \\
\Leftrightarrow\ & Y_t = (r^K_t + \delta^K) \cdot K_t + r^{\mathcal{H}}_t \cdot L^{\mathcal{H}}_t + r^{\mathcal{C}}_t \cdot L^{\mathcal{C}}_t
\end{aligned}
$$

- accumulation of the aggregate capital stock:

$$
K_{t+1} = (1-\delta^K) \cdot K_t + I^K_t
$$

leads to the aggregate resource constraint holding in equilibrium:

$$
\begin{aligned}
K_{t+1} =& K_t + r^K_t \cdot K_t + r^{\mathcal{H}}_t \cdot L^{\mathcal{H}}_t + r^{\mathcal{C}}_t \cdot L^{\mathcal{C}}_t - C_t - G_t \\
\Leftrightarrow\ \ Y_t =& C_t + I^K_t + G_t
\end{aligned}
\tag{1.29}
$$

Appendix 1.B Computational Appendix

1.B.1 Computational Implementation

The numerical solution is implemented in Fortran 90 using routines which are partly based on Press et al. (1996). The determination of the equilibrium path involves outer (aggregate model) and inner (household problem) loop iterations. Furthermore, a very outer loop serves for calibration purposes. At all stages, I apply an error tolerance level of at least $1 \cdot 10^{-4}$. Section 1.B.2 contains details on the numerical solution of the household problem.

1.B.2 Solving the Household Problem

I solve the household problem for the policy functions $\{\tilde{c}^{a,\mathcal{S}}_{t,j}, i^{a,\mathcal{S}}_{t,j}, l^{a,\mathcal{S}}_{t,j}\}_{t,j,a,\mathcal{S}}$, i.e., de-trended consumption, hours spent on skill development, and hours supplied to the labor market for all combinations of ability type, schooling type, age, and time period. In addition, I solve for the optimal schooling decision at age $j = 0$ for all ability types in all time periods. In the following, I omit the superscripts a and $\mathcal{S}$ for convenience where applicable and indicate next period's variables by $'$ irrespective of whether it is age dependent, time dependent, or both.

I apply a backward shooting method using the equations of the household problem, its first order conditions, and the first derivatives of the value function as derived in section 1.A.1.

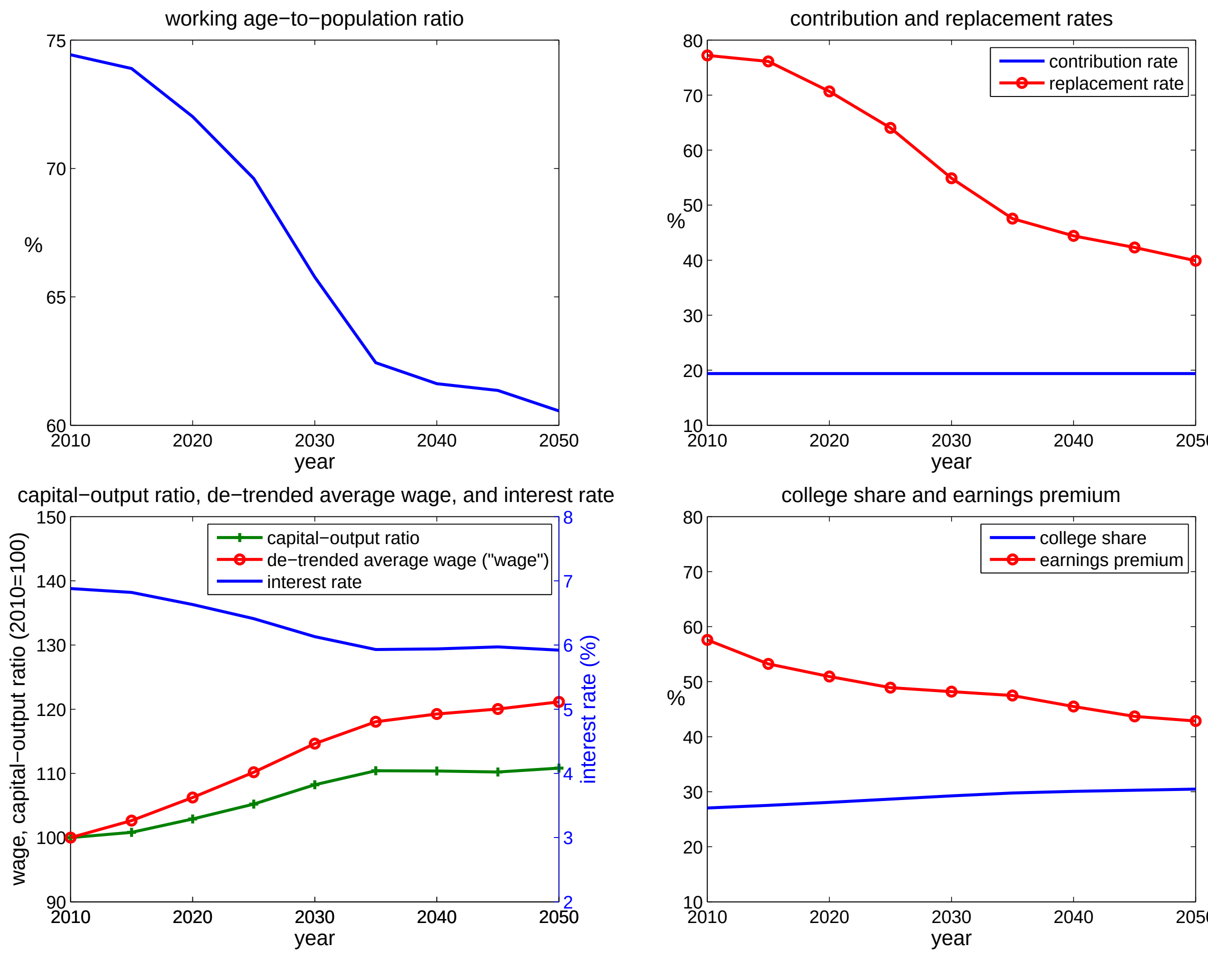
1. Guess $(\tilde{k}_J, h_J, b_J)$.
2. Start at age $j = J$ where $v'_{\tilde{k}'} = v'_{h'} = v'_{b'} = v' = \tilde{k}' = 0$ $\tilde{k}$, h, and b are given , and the household chooses $l = i = 0$. Use (1.18) for determining $\tilde{c}$. Compute v from (1.18) and $v_{\tilde{k}}, v_h, v_b$ according to (1.23–1.25).
3. Go backwards in age for $j = J-1, J-2, ..., j_r$. Set $i = l = 0$. Given $u'_{\tilde{c}'}$ determine $\tilde{c}$ from (1.26), compute $\tilde{k}$, h, and b from (1.18). Compute $v_{\tilde{k}}, v_h, v_b$ according to (1.23–1.25) and v from (1.18).
4. Go backwards in age for $j = j_r - 1, j_r - 2, ..., 0$ and proceed as described below in order to determine $\tilde{c}$, i, and l. In the following cases, choices are restricted: At $j = j_r - 1$ set $i = 0$ as any time spent on skill development does not pay off in retirement. At $j = 0, ..., j_w - 1 \wedge \mathcal{S} = \mathcal{C}$, set $i = \bar{i}$ as the household is currently enrolled in college.
    a) Determine $\tilde{c}, i, l$:
        i. Guess i.
        ii. Compute h from (1.18).
        iii. Compute $u_{\tilde{c}}$ from (1.26) and $u_l/u_{\tilde{c}}$ from (1.28).
        iv. Compute l from $u_{\tilde{c}}$ and $u_l/u_{\tilde{c}}$. If $l < 0$ set $l = 0$.
        v. Compute $\tilde{c}$ using the preference function $u(\cdot)$.
        vi. Compute $\hat{i}$ from (1.27).
        vii. If $\|\hat{i} - i\| > \epsilon$ go to step 4(a)i. and update the guess of i.[34]
    b) Compute $v, \tilde{k}, h$, and b from equation (1.18) and $v_{\tilde{k}}, v_h, v_b$ according to (1.23–1.25).
5. If $\|(\tilde{k}_0, h_0, b_0) - (0, 1, 0)\| > \epsilon$ go to step 2 and update the guess of $(\tilde{k}_J, h_J, b_J)$.
6. At the beginning of period $j = 0$, each household faces the decision on tertiary education. Given all policy rules resolved according to steps 2.–4., given its ability type, a, a household chooses formal schooling according to equation (1.2).

[34] ϵ denotes the error tolerance level.

1.B.3 Aggregate Dynamics Beyond 2010 Without Past Skill-Biased Technological Change

Figure 1.B.1 shows the evolution of key aggregate measures in the main period of projection, from year 2010 to year 2050 in the auxiliary economy without past skill-biased technological change. The central findings can

Figure 1.B.1: Aggregate Dynamics Beyond 2010 Without Past Skill-Biased Technological Change



Source: Auxiliary model without past skill-biased technological change in the main period of projection, 2010 to 2050: Selected aggregate measures. *Notes:* The replacement rate in the top right panel refers to the average ratio of the pension payment to a 65 year old household to the sum of net earnings (adjusted by wage inflation) over its working life. The average gross hourly wage in the bottom left panel is shown net of the trend growth arising from exogenous skill-neutral technological change in Υ_t over time. Capital-output ratio and de-trended average wage are normalized to 100 in 2010. The bottom right panel shows the average net earnings premium of college graduates and the share of college graduates in the 25+ workforce in percent.

be summarized as follows: While 1) the replacement rate, 2) the interest rate, and 3) the earnings premium decline gradually over time, 4) the average wage level increases.

As an indicator for the development of overall economic inequality in the economy table 1.B.1 displays the change in the Gini coefficients of net total income and consumption from 2010 to 2050. The resulting numbers are similar to the results with past skill-biased technological change in the main text. However, the change in the income Gini shows to be about 2 to 3 percentage points more positive than in the case with past skill-biased technological case. This holds for both computed values (top and bottom row) and mirrors the development of the earnings premium which exhibits a smaller drop in the case without past skill-biased technological

Table 1.B.1: Measures of Economic Inequality: The Change in Ginis from 2010 to 2050 Without Past Skill-Biased Technological Change

	net total income	consumption
$Gini(2050) - Gini(2010)$, baseline	$+7.9$	-3.6
$Gini(2050) - Gini(2010)$, fix demographics	-4.6	$+2.2$

Source: Auxiliary model without past skill-biased technological change. *Notes:* The numbers show percentage point changes. The top row displays the total change in Gini coefficients while the bottom row shows the corresponding values if the Gini coefficients in 2050 are recomputed using the age distribution of the population in year 2010. Net total income is the sum of capital income and net earnings, respectively pension income of a household.

change as it is discussed in the main text.

2

Risky Human Capital, Aging, the Equity Premium, and Welfare

2.1 Introduction

As the population in all major industrialized countries the population in the U.S. is aging reducing the fraction of the population in working age. This process is driven by falling mortality rates and declining birth rates, which substantially reduces population growth rates. Based on Human Mortality Database (2008) and United Nations (2007), figure 2.1 compresses the stylized facts on demographic change by displaying the predicted time paths of two key demographic indicators for the US. The blue solid line in the figure (left scale) is the predicted working age-to-population ratio – here defined as the number of the working age population of age 20-64 to the total adult population of age 20-110 – and the red line with dots (right scale) is the corresponding time path of the old-age dependency ratio – here defined as the number of the population of age 65 and older as a fraction of the working age population.[1] While the working age-to-population ratio is projected to decrease by roughly 9 percentage points between 2010 and 2030, which we take as the base years of comparison throughout the chapter, the old-age dependency ratio increases by about 15 percentage points.[2] These projected developments will make raw labor a scarce factor relative to physical capital with ensuing decreases of the rate of return to capital.

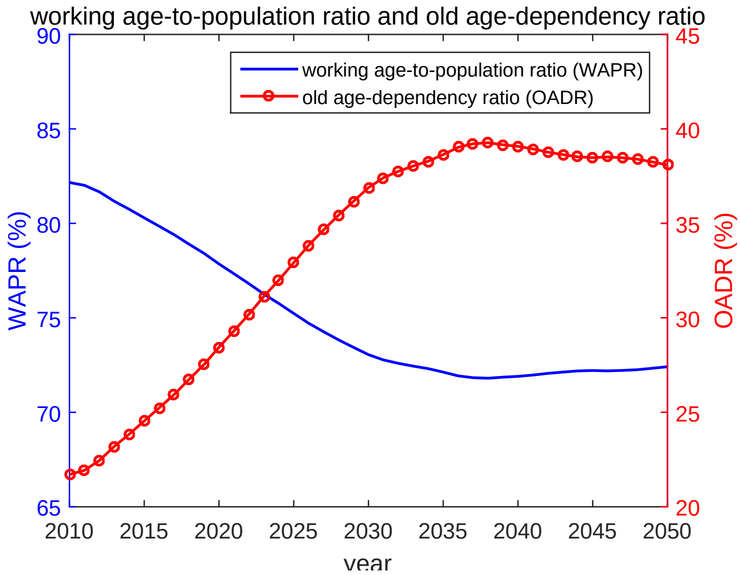
What will be the financial market consequences of these demographic developments? No consensus has been reached in the academic literature on this prominent question posed by Abel (2001, 2003), Poterba (2001) and several others. Despite significant effects of demographic change on the rate of return to capital, it has recently been argued that the size of these effects seems too small such that the catchphrase "asset market meltdown" (Poterba, 2001) is not justified in the context of population aging, cf., e.g., Börsch-Supan, Ludwig, and Winter (2006) and Krüger and Ludwig (2007). Quite in contrast, there is little agreement on the qualitative as well as the quantitative effects of demographic change on the differential returns between risky and risk-free assets Bakshi and Chen (1994), Brooks (2004), Börsch-Supan, Ludwig, and Sommer (2003), Geanakoplos, Magill, and Quinzii (2004), and Kuhle (2008). While Brooks (2004) reports substantial increases in the equity premium, the approximate calculations in Börsch-Supan, Ludwig, and Sommer (2003) rather suggest a small increase. Geanakoplos, Magill, and Quinzii (2004) conclude that "the equity premium is smaller when the population of savers is older" which the authors interpret as a contradiction to the findings of Bakshi and Chen (1994) and Brooks (2004).

The contribution of this chapter is to develop a multi-generation overlapping generations (OLG) model in the tradition of Auerbach and Kotlikoff (1987) in order to provide a quantitative assessment of the effects of

[1] The age bounds in these definitions correspond to the definitions used in the macroeconomic simulation model, cf. section 2.2.

[2] The choice of year 2030 as a base year of comparison is motivated by the insight that demographic developments somewhat flatten out after 2030, cf. figure 2.1, and because demographic projections are inherently more uncertain after a horizon of about 30 years.

Figure 2.1: Expected Demographic Change in the U.S.



Source: Own calculations based on United Nations (2007) and Human Mortality Database (2008). *Notes:* The working age-to-population ratio is here defined as the number of the working age population of age 20-64 to the total adult population of age 20-110. The old-age dependency ratio is here defined as the number of the population of age 65 and older as a fraction of the working age population.

demographic change on the equity premium. Any serious attempt to quantify these effects should be based on simulation models with a periodicity of one to at most five years. Models that run at a lower frequency implicitly impose restrictions on household's ability to adjust their portfolio which may severely bias the predictions. The periodicity of our model is therefore annual and we calibrate the model to the projected trends of U.S. demography in the coming decades.

However, the annual frequency of our model also, in principal, implies tremendous computational costs. To avoid these we adopt the risky human capital framework developed in Krebs (2003) and Krebs and Wilson (2004) in an overlapping generations setup. By assuming that the technology of human capital production is linear, our setup considerably simplifies the numerical solution of the model's household sector conditional on the law of motion of the aggregate state of the economy, also see Merton (1969) and Samuelson (1969).

Yet, human capital in this model does not only serve a computational purpose. First, it enables us to account for the considerable evidence that individual households face a substantial amount of uninsurable idiosyncratic labor income risk which, in a model with aggregate risk, generates additional precautionary savings. Our analysis therefore also relates to the literature on the importance of background risk for asset pricing, see, e.g., Storesletten, Telmer, and Yaron (2007). Second, it has recently been shown that human capital investments are an important adjustment channel to demographic change, see Ludwig, Schelkle, and Vogel (2012) and the literature cited there. The argument is that demographic change leads to a relative shortage of raw labor and abundance of physical capital which tends to increase the relative return on human capital leading to an increase of educational investment.

The above discussion illustrates how the model developed in this chapter *kills three birds with one stone*. The *stone* is the idea to use the linear human capital technology in an OLG application and thereby to simplify solution of the household model. The three *birds* are, first, preserving computational tractability despite a large model with complex transitional dynamics, second, idiosyncratic risk and thereby meaningful asset pricing, and third, human capital as an important channel to adjust to the consequences of demographic change.

On the aggregate side we follow the literature (e.g., Gomes and Michaelides, 2008 and Storesletten, Telmer, and Yaron, 2007) and compute an approximate rational expectations equilibrium by applying a variant of the methodology developed in Krusell and Smith (1997, 1998). The Krusell-Smith methodology is modified in order to account for the fact that demographic change enters the model through a time-varying exogenous process. As the periodicity of the model is annual, transitions are rather smooth and the Krusell-Smith method is robust and accurate.

Our results show that the expected decrease of the average stock return until 2030 is in the order of magnitude of 0.16 percentage points. The decrease of the risk-free interest rate on bonds is slightly higher such that the equity premium increases by about 0.08 percentage points. These relatively mild changes in returns and the equity premium result from an interplay of three main effects. First, older households on average hold relatively fewer equity than younger households in the model as well as in the data (cf. Ameriks and Zeldes, 2004).[3] Demographic change increases the size of the old population relative to the young which drives up the relative demand for bonds thereby increasing its relative price. Consequently, the equity premium tends to increase. The second effect is a portfolio adjustment effect isolated in Kuhle (2008) that works in the opposite direction: Ignoring the first effect, suppose that demographic change would lead to a decrease of the expected rates of return on both assets by the same amount (such that the ex-ante equity premium is constant). For a positive equity premium, then, the percentage decrease of the risk-free rate of return is higher such that the investor increases her relative portfolio shares of equity and the demand for bonds decreases. Hence, the equilibrium increase of the equity premium is smaller than the first effect would postulate in isolation. Third, and most importantly, endogenous human capital adjustments have a large effect. As societies are aging, labor becomes a relatively scarce factor and households increase human capital investments. This increases productivity thereby decreasing the downward pressure on asset prices. If we instead hold the human capital shares constant, then, the negative effects on asset returns are much larger. In that scenario, the average stock return decreases by about 0.70 percentage points until 2030 and the equity premium increases by about 0.27 percentage points.

A welfare analysis shows that the decline of asset returns and the co-incident increase of the human capital return benefits future generations relative to generations born in the past. Again, human capital adjustments reduce welfare consequences and their differences across generations considerably.

The remainder of this chapter is structured as follows. Section 2.2 presents the large scale overlapping generations model. Section 2.3 discusses calibration of the current version and the numerical solution. Section 2.4 presents simulation results. Finally, section 2.5 concludes.

2.2 Model

We extend the classical Diamond (1965) economy to a multi-period setup as in Auerbach and Kotlikoff (1987) with idiosyncratic and aggregate risk. On the household side, the novelty in this chapter is to assume that labor income is a choice variable of households rather than being exogenously given. This feature is implemented by adopting the human capital framework developed in Krebs (2003) and Krebs and Wilson (2004) in an overlapping generations setup. In each period, a household of a given age chooses to invest a fraction of

[3] According to Ameriks and Zeldes (2004) life-cycle portfolio shares do not vary much with age conditional on participation in equity markets but participation decreases around the age of retirement.

her overall wealth in human capital, respectively financial capital. As for the fraction of wealth invested in financial assets, the household solves a standard portfolio allocation problem by choosing how much to invest into risky physical capital and risk-free bonds. Consequently, there are three assets in the economy: risky human capital, risky physical capital and risk-free bonds. In this setup, once portfolio allocation decisions are made, household consumption and savings policies are linear functions of wealth, cf. Merton (1969) and Samuelson (1969). Therefore, conditional on expectations on the evolution of aggregate prices, the household problem is easy to solve. This feature of the model is particularly useful because it enables us to solve a large-scale OLG model with rather complex dynamics without incurring tremendous computational costs. On the firm side, the model is standard.

2.2.1 Risk and Time

Time is discrete and runs from $t = 0, \ldots, \infty$. Aggregate uncertainty is represented by an event tree. The economy starts with some fixed event λ_0, and each node of the tree is a history of exogenous shocks $\lambda^t = (\lambda_0, \lambda_1, \ldots, \lambda_t)$. The shocks are assumed to follow a Markov chain with finite support L and strictly positive transition matrix Π. For notational convenience, we will only index variables by time thereby suppressing the dependency of variables on λ^t but it is understood that all choice variables are functions of history.

2.2.2 Production

Production takes place with a standard Cobb-Douglas production function with total output at time t given by

$$Y_t = z_t \cdot K_t^{\alpha} \cdot (\Upsilon_t \cdot H_t)^{1-\alpha} \tag{2.1}$$

where K_t is the aggregate stock of physical capital, H_t is the aggregate stock of human capital, and z_t is a stochastic shock to total factor productivity. Υ_t is a human capital augmenting productivity parameter which grows at the exogenous constant rate g which captures the observed trend in GDP in the data.

Profit maximization of firms leads to the standard first order conditions stating that marginal products equal returns minus depreciation rates:

$$r_t^K = \alpha \cdot z_t \cdot \left(\frac{K_t}{\Upsilon_t \cdot H_t}\right)^{\alpha-1} - \delta_t^K \tag{2.2a}$$

$$r_t^H = \Upsilon_t \cdot (1-\alpha) \cdot z_t \cdot \left(\frac{K_t}{\Upsilon_t \cdot H_t}\right)^{\alpha}. \tag{2.2b}$$

Note that r^H grows along with Υ over time while r^K is trend-stationary. Following Krüger and Kübler (2006), Storesletten, Telmer, and Yaron (2007), Gomes and Michaelides (2008) and others we assume that the depreciation rate of physical capital, δ^K, is stochastic.

2.2.3 Demographics

The economy is populated with $J + 1$ overlapping generations and the underlying population dynamics is the exogenous driving force of the model. Households enter the model at the age of 20 ($j = 0$) and live at most until turning 101 ($j = J + 1 = 81$). Population of age j in time period t is given recursively as

$$N_{t,j} = \begin{cases} N_{t-1,j-1} \cdot \varsigma_{t-1,j-1} & \text{for } j = 1, \dots, J \\ \sum_{l=0}^{j_f} f_{t-20,l} \cdot N_{t-20,l} & \text{for } j = 0 \end{cases} \tag{2.3}$$

where $\varsigma_{t,j}$ and $f_{t,j}$ denote time and age-specific survival rates and fertility rates respectively. j_f is the age of menopause. Processes governing mortality and fertility are assumed to be non-stochastic.

2.2.4 Preferences

We take Epstein-Zin preferences. Let θ be the coefficient of relative risk-aversion and ξ denote the elasticity of inter-temporal substitution.

$$u_{t,j} = \left[c_{t,j}^{\frac{1-\theta}{\gamma}} + \beta \cdot \left(\mathbb{E}_{t,j}[\varsigma_{t,j} \cdot u_{t+1,j+1}^{1-\theta}] \right)^{\frac{1}{\gamma}} \right]^{\frac{\gamma}{1-\theta}} \tag{2.4}$$

where $\gamma := (1-\theta)/(1-1/\xi)$. $0 < \beta < 1$ is the standard discount factor. For $\theta = 1/\xi$ we have $\gamma = 1$ and are back at standard CRRA preferences. β is the raw time discount factor and $c_{t,j}$ is consumption at time t, age j. $\mathbb{E}_{t,j}$ is the expectations operator and expectations are taken with respect to idiosyncratic shocks to human capital and aggregate shocks to productivity and physical capital depreciation conditional on information at time t, age j. As $\varsigma_{t,J}$ equals 0 for all t, equation (2.4) implies that $u_J = c_J$.

2.2.5 Endowments

When entering the economy at age $j = 0$, households are endowed with an initial level of human capital, $h_{t,0} = h_0$ for all t and financial wealth $k_{t,0}$ which is set to zero for all t for convenience. Summing financial assets and human capital makes up households' total wealth. Each period, households choose to invest a fraction of their total wealth in financial assets and in human capital respectively. Let $i_{t,j}^h$ denote the amount of wealth invested in human capital.

Human capital earns a gross rate of return of r_t^H which is the marginal product of human capital. The term $r_t^H \cdot h_{t,j}$ can be understood as gross earnings of a household at age j in period t.
We assume that human capital depreciates at the individual level by the age-specific deterministic rate δ_j^h. The age-profile of $\{\delta_j^h\}_{j=1}^{j_r}$ enables us to calibrate the model such that it mimics decreasing returns to human capital accumulation as assumed elsewhere in the literature (e.g. Huggett, Ventura, and Yaron, 2011). We assume the following functional form

$$\delta_j^h = -\chi_0 + \exp(\chi_1 \cdot j), \quad \chi_0 > 0, \chi_1 \geq 0, \tag{2.5}$$

which is monotonically increasing in j such that $1 - \chi_0 \leq \delta_j^h \leq \delta_{j+1}^h$ for all j. χ_1 is the rate at which the household's human capital depreciation accelerates when getting older.
After the return to human capital is paid the household is hit by an additive idiosyncratic shock to its human capital holdings:

$$\eta_t \sim \mathcal{D}(0, \sigma^2(\lambda_t)) \tag{2.6}$$

where $\mathcal{D}$ is some distribution with mean zero further specified in the quantitative section 2.3. Although the shock is idiosyncratic, it depends on the current state of the economy, λ_t, because, as further discussed below, the variance of the idiosyncratic human capital shock, σ_t^2, depends on the current state of the economy. Collecting all these elements, the human capital accumulation equation in period t, age j, is given by

$$h_{t+1,j+1} = h_{t,j} \cdot (1 - \delta_j^h + \eta_t) + \tilde{i}_{t,j}^h, \qquad h_{t,j} \geq 0 \;\; \forall\, t, j, \tag{2.7}$$

where $\tilde{i}_{t,j}^h := i_{t,j}^h / \Upsilon_t$.[4] Note that all variables in (2.7) are trend-stationary, i.e., they do not exhibit exogenous growth along with human capital productivity in production Υ_t[5].

The household faces a portfolio decision between risky and one period ahead risk-free financial assets, which we denote in the following by *risky equity* and *risk-free bonds* respectively.[6] Let $\alpha_{t,j}^s$ be the fraction of financial assets invested in risky equity in period t, age j. The dynamic financial asset accumulation equation in period t, age j, is then given by

$$k_{t+1,j+1} = k_{t,j} \cdot (1 + r_t^f + \alpha_{t,j}^s \cdot (r_t^s - r_t^f)) + h_{t,j} \cdot r_t^H - i_{t,j}^h - c_{t,j} \tag{2.8}$$

2.2.6 Recursive Household Problem

We hereafter define recursively the household problem conditional on a law of motion of the aggregate state of the economy. We present a de-trended version of the household problem derived in section 2.A.1 of the appendix. The symbol ˜ indicates that a variable has been transformed, e.g. $\tilde{k} := k/\Upsilon$, thereby removing the trend from (exogenous) labor-augmenting technological progress. It is convenient to express next period's values with symbol $'$, irrespective of whether they are only time-dependent or both, age- and time-dependent. The states of the household problem are the exogenous states j, t, and λ, the endogenous idiosyncratic state of (de-trended) cash-on-hand, $\tilde{x}$, further specified below as well as the distribution of (de-trended) wealth, Ω, which is the endogenous aggregate state of the economy. The associated law of motion is $\Omega' = \Phi(\Omega, \lambda, \lambda', N')$. The existence of aggregate shocks implies that Ω evolves stochastically over time. Notice that λ' is a determinant of Ω' because it specifies $r^{K\prime}$ and $r^{H\prime}$. A change in demography, N', induces a transition of the economy from an initial stationary equilibrium to another. The (de-trended) household problem at age j in period t is then given by

$$v(\tilde{x}, \lambda, \Omega) = \max_{\tilde{c}, \tilde{x}', \widehat{\alpha}^{s\prime}, \widehat{\alpha}^{h\prime}} \{\tilde{c}^{\frac{1-\theta}{\gamma}} + \widehat{\beta} \cdot (\mathbb{E}[v'(\tilde{x}', \lambda', \Omega')^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}} \tag{2.9}$$

$$\text{s.t. } \tilde{x}' = \frac{1}{1+g} \cdot (\tilde{x} - \tilde{c}) \cdot (1 + \widehat{r}'), \; \tilde{x}_0 > 0 \text{ given} \tag{2.10}$$

$$\Omega' = \Phi(\Omega, \lambda, \lambda', N'), \; N' \text{ given}$$

$$\pi(\lambda' \mid \lambda), \; \lambda_0 \text{ given}$$

$$\eta \sim \mathcal{D}(0, \sigma^2(\lambda)).$$

where $\tilde{x} := (\tilde{k} + h/(1+g)) \cdot (1 + \widehat{r})$, $\widehat{r}' := r^{f\prime} + \widehat{\alpha}^{s\prime} \cdot (r^{s\prime} - r^{f\prime}) + \widehat{\alpha}^{h\prime} \cdot (\widehat{r}^{h\prime} - r^{f\prime})$, $\widehat{r}^{h\prime} := (1+g) \cdot (\tilde{r}^{H\prime} + 1 - \delta^{h\prime} + \eta') - 1$, $\widehat{\alpha}^{s\prime} := \alpha^{s\prime} \cdot \tilde{k}' / (\tilde{k}' + h'/(1+g))$, $\widehat{\alpha}^{h\prime} := h'/(1+g)/(\tilde{k}' + h'/(1+g))$, and $\widehat{\beta} := \beta \cdot \varsigma^{\frac{1}{\gamma}} \cdot (1+g)^{\frac{1-\theta}{\gamma}}$.

[4] We assume that costs for human capital investment, i_t^h, grow with the same rate as Υ_t.

[5] As the return to human capital r_t^H already exhibits a trend growth along with Υ_t, human capital must be trend stationary in order to assure that gross human capital earnings, $h_{t,j} \cdot r_t^H$, grow at the same rate as Υ_t over time.

[6] The assumption of risk-free bonds is not innocent and relies on the assumption that debtors (which can be households and firms) always repay their debts. This is especially not subject to neither the aggregate state of the economy nor, in case of a household, the idiosyncratic state of the debtor.

The expectation $\mathbb{E}$ above is taken with respect to the realization of tomorrow's aggregate state λ' conditional on state λ today and the realization of tomorrow's idiosyncratic shock, η'. Note that λ' determines the shock to technology, z', the shock to physical capital depreciation, $\delta^{K\prime}$, and the variance of the idiosyncratic human capital depreciation shock, $\sigma^{2\prime}$.

Using results derived in Samuelson (1969) we can next state the following properties of the optimal policy functions.

Proposition 1. *Denote by $\widehat{\alpha}^{s*\prime}$ and $\widehat{\alpha}^{h*\prime}$ the optimal portfolio decisions that are the solutions to*

$$\mathbb{E}[(m')^{1-\theta-\gamma}(1+\widehat{r}')^{-\theta}(r^{s\prime}-r^{f\prime})] = 0 \tag{2.11}$$

$$\mathbb{E}[(m')^{1-\theta-\gamma}(1+\widehat{r}')^{-\theta}(\widehat{r}^{h\prime}-r^{f\prime})] = 0 \tag{2.12}$$

where m denotes the marginal propensity to consume out of (de-trended) cash-on-hand, $\tilde{x}$. Then the optimal (de-trended) consumption function is linear in cash-on-hand,

$$\tilde{c} = m \cdot \tilde{x}. \tag{2.13}$$

The marginal propensity to consume out of cash-on-hand is given by

$$m := \frac{[\beta^{\gamma}\cdot\varsigma\cdot\wp]^{\frac{1}{1-\theta-\gamma}}}{1+[\beta^{\gamma}\cdot\varsigma\cdot\wp]^{\frac{1}{1-\theta-\gamma}}} \text{, where} \quad \wp := \mathbb{E}[(m')^{1-\theta-\gamma}\cdot(1+\widehat{r}')^{1-\theta}].$$

Proof. Please see section 2.A.1 in the appendix. □

Note that portfolio decisions do not dependent on and consumption is linear in current cash-on-hand. These features are due to the homotheticity of preferences and are particularly useful in the numerical solution of the simulation model. They transfer the results in Krebs (2003) and Krebs and Wilson (2004) which were derived in an infinite horizon model to a (finite) life cycle household problem. Furthermore, it implies that we do not need to break down the wealth distribution into idiosyncratic characteristics other than age which we impose in the remainder of this chapter for the sake of easier presentation.

2.2.7 Government

The government taxes accidentally bequeathed financial wealth of departed households and uses it for government consumption. For simplicity we assume that the tax rate on bequests is 100%. Note that the government receives bequeathed wealth including associated interests in period $t+1$.[7]

$$\tilde{G}_{t+1} = \frac{1}{1+g}\cdot\sum_{j=0}^{J} N_{t,j}\cdot(1-\varsigma_{t,j})\cdot\tilde{x}_{t,j}\cdot(1-m_{t,j}) \\ \cdot(1+r^f_{t+1}+\alpha^s_{t+1,j+1}\cdot(r^s_{t+1}-r^f_{t+1})) \tag{2.14}$$

[7] Again, an aggregation over age only suffices due to the independence of policy functions of $\tilde{x}$ as explained at the end of section 2.2.6.

2.2.8 Equilibrium

Equilibrium in the economy is defined recursively and presented in de-trended form, cf. section 2.2.6. It requires market clearing in all periods while optimal decisions and aggregation conditions have to hold. In the following, $'$ ($_{\prime}$) indicates next (last) period's variables while we make the dependency on age, j, explicit.

A recursive competitive equilibrium is a value function $v(j, \tilde{x}, \lambda, \Omega)$ and policy functions, $\widehat{\alpha}^{s\prime}(j, \lambda, \lambda', \Omega')$, $\widehat{\alpha}^{h\prime}(j, \lambda, \lambda', \Omega')$, $m(j, \lambda, \lambda', \Omega')$, for the household, policy functions for the firm, $\tilde{K}(\lambda, \Omega)$, $H(\lambda, \Omega)$, pricing functions $r^s(\lambda, \Omega)$, $r^H(\lambda, \Omega)$, $r^{f\prime}(\lambda, \Omega)$, the demographic distribution, N, the wealth distribution, Ω, and its associated (aggregate) law of motion, $\Phi(\Omega, \lambda, \lambda', N')$, such that for all (λ, Ω)

1. $v(\cdot)$, $\tilde{x}(j, \lambda, \Omega)$, $\tilde{c}(j, \tilde{x}, \lambda, \Omega)$, $\widehat{\alpha}^{s\prime}(\cdot)$, $\widehat{\alpha}^{h\prime}(\cdot)$, $m(\cdot)$ are measurable, $v(\cdot)$ satisfies the household's recursive problem, and $\widehat{\alpha}^{s\prime}(\cdot)$, $\widehat{\alpha}^{h\prime}(\cdot)$, $m(\cdot)$ are the associated policy functions following from the conditions in proposition 1, given $\mathbb{E}[r^{s\prime}(\lambda', \Phi(\Omega, \lambda, \lambda', N'))]$, $\mathbb{E}[\widehat{r}^{h\prime}(j+1, \eta', \lambda', \Phi(\Omega, \lambda, \lambda', N'))]$, $r^{f\prime}(\cdot)$ and $\tilde{x}(\cdot)$,
2. firms behave optimally as according to equations (2.2),
3. government consumption financed by accidental bequests fulfills equation (2.14),
4. market clearing on bond, stock, human capital, and final good markets as according to equations (2.15), (2.16), (2.17), and (2.18) respectively:

$$\tilde{K}'(\cdot) \cdot \frac{\ell}{1+\ell} = \frac{1}{1+g} \cdot \sum_{j=0}^{J} N(j) \cdot \tilde{x}(j, \cdot) \cdot (1 - m(j, \cdot)) \cdot (1 - \widehat{\alpha}^{s\prime}(j, \cdot) - \widehat{\alpha}^{h\prime}(j, \cdot)) \tag{2.15}$$

$$\tilde{K}'(\cdot) \cdot \frac{1}{1+\ell} = \frac{1}{1+g} \cdot \sum_{j=0}^{J} N(j) \cdot \tilde{x}(j, \cdot) \cdot (1 - m(j, \cdot)) \cdot \widehat{\alpha}^{s\prime}(j, \cdot) \tag{2.16}$$

$$H(\cdot) = \sum_{j=0}^{J} N(j) \cdot \frac{\tilde{x}(j, \cdot)}{1 + \widehat{r}(j, \cdot)} \cdot \widehat{\alpha}^{h}(j, \cdot) \tag{2.17}$$

$$\tilde{Y}(\lambda, \Omega) = \tilde{C}(\lambda, \Omega) + \tilde{G}(\lambda, \Omega) + \tilde{I}^K(\lambda, \Omega) + I^h(\lambda, \Omega) \tag{2.18}$$

   where $\ell := \frac{B}{S}$ is the leverage ratio of bonds over stocks of the representative firm which is exogenous and fix by assumption, cf. section 2.3. The bond price $q^f(\lambda, \Omega) := (1 + r^{f\prime}(\lambda, \Omega))^{-1}$ is determined such that it clears the bond market in period t. (2.18) is the aggregate resource constraint which is derived in section 2.A.2 of the appendix,
5. the aggregate law of motion Φ satisfies

$$\Omega' = \Phi(\Omega, \lambda, \lambda', N'). \tag{2.19}$$

   It is generated by the exogenous population dynamics, the exogenous stochastic processes and the endogenous asset accumulation decisions as captured by the policy functions,
6. the initial wealth distribution, Ω_0;
7. the transition matrices for the exogenous processes.

Definition 1. *A stationary recursive competitive equilibrium is a special case of the equilibrium described above. It is characterized by time-constant individual policy functions* $m(\cdot)$, $\widehat{\alpha}^{s\prime}(\cdot)$, $\widehat{\alpha}^{h\prime}(\cdot)$, *and a time-constant aggregate law of motion* $\Phi(\cdot)$. *This requires a time-constant demographic distribution,* N.

2.3 Calibration and Numerical Solution

In terms of expectations, we solve an approximate rational expectations equilibrium by applying a variant of the method of Krusell and Smith(1997, 1998) as further described in appendix 2.B.1.

Calibration of the model is in part by reference to other studies and in part by informal matching of moments procedures. The period length is one year. Table 2.1 summarizes structural model parameters where target values refer to year 2010. The additional parameters governing stochastic and demographic processes are only described in the text.

Table 2.1: First and Second Stage Parameters

Parameter	Value	Target	Target Source, Comment
Firm sector			
Capital share: α	0.36	1^{st} stage	wage share (NIPA)
Technological progress: g	0.018	1^{st} stage	TFP growth (NIPA)
Leverage ratio: ℓ	0.67	1^{st} stage	RZ
Mean depreciation rate K: δ_0^K	0.086	$r^f = 0.013$	PST, Shiller (2015)
Households			
Life cycle: $j = \{0, j_r, J\}$	$\{0, 45, 80\}$	1^{st} stage	biological age: $\{20, 65, 100\}$
Elasticity inter-temp. substit., ξ	1.5	1^{st} stage	Bansal and Yaron (2004)
Endowment: $\{h_0, k_0\}$	$\{1.0, 0.0\}$	1^{st} stage	normalization
Time discount factor: β	0.936	$K/Y = 2.65$	NIPA
Relative risk aversion: θ	8.4	$r^s - r^f = 0.062$	PST
Depreciation rate h: $\{\chi_0, \chi_1\}$	$\{0.976, 0.0007\}$	$\{r^H \cdot h_j\}_{j=20}^{64}$	PSID

Source: Baseline model: The target year is 2010. *Notes*: We target the average of the post-Second World War risk-free rates of PST and Shiller (2015). RZ $\hat{=}$ Rajan and Zingales (1995). PST $\hat{=}$ Piazzesi, Schneider, and Tuzel (2007). NIPA $\hat{=}$ National Income and Product Accounts. TFP $\hat{=}$ Total factor productivity.

The time- and age-specific demographic data for the population dynamics in (2.3) are based on Human Mortality Database (2008) and the United Nations' population projections in United Nations (2007).

We assume that aggregate risk is driven by a four state Markov chain with support $L = \{\lambda_1, \ldots, \lambda_4\}$ and transition matrix $\Pi = (\pi_{ik})$. Each aggregate state maps into a combination of low or high technology shocks and low or high physical capital depreciation. Precisely, we assume that

$$z_t = z(\lambda_t) = \begin{cases} z_0(1+\bar{z}) \text{ for } \lambda \in \lambda_1, \lambda_2 \\ z_0(1-\bar{z}) \text{ for } \lambda \in \lambda_3, \lambda_4 \end{cases}, \delta_t^k = \delta^k(\lambda_t) = \begin{cases} \delta_0^k + \overline{\delta}^k \text{ for } \lambda \in \lambda_1, \lambda_3 \\ \delta_0^k - \overline{\delta}^k \text{ for } \lambda \in \lambda_2, \lambda_4. \end{cases} \tag{2.20}$$

One feature specific to the model is that the endogenous fluctuations generated by the financial savings and human capital accumulation channels are higher than in the standard model with exogenous labor income. Therefore, the auto-correlation of the exogenous technology shock process, ρ^z, and the probability of a high (low) depreciation state conditional on being in a low (high) technology state, ρ^δ, must be lower than in the standard model. We assume $\rho^\delta = 0.6$ and $\rho^z = 0.7$ which comes close to Gomes and Michaelides (2008) who use 0.5 and 0.67 respectively. $\bar{z}$ is set to 0.02 which results in a standard deviation of GPD growth of 4% and a standard deviation of consumption growth of 3.8%. This is slightly higher than the 3% measured usually in the data. The standard deviation of the shock to the depreciation rate of physical capital is set to $\overline{\delta}^k = 0.1$ such that the model matches the standard deviation of the stock return in the data of about 16.7% (cf., e.g., Shiller, 2015).

The value of the capital share parameter, $\alpha = 0.36$, is based on an estimation of the aggregate production function for the US (cf. Krüger and Ludwig, 2007) and lies in the usual range considered in the literature. The value of the mean depreciation rate of physical capital, $\delta_0^K = 0.086$, lies at the upper end of the range of empirical estimates and leads to a risk-free interest rate of 1.3%. We assume that the representative firm keeps an exogenous fixed leverage ratio, $\ell := \frac{B}{S}$, which is set to the empirically observed value, 0.67 (cf. Rajan and Zingales, 1995). Thereby, corporate bonds are in positive net supply.

The value of households' raw time discount factor, $\beta = 0.936$, is at the lower range of values considered in the literature. It yields the in NIPA data observed capital-output ratio of 2.65. The elasticity of inter-temporal substitution, ξ, equals 1.5. It lies in the range considered in the asset pricing literature (cf. the discussion in Bansal and Yaron, 2004, pp. 1492-93) and results in a hump-shaped consumption profile which is in line with the data, cf. Fernández-Villaverde and Krüger (2006). While being mostly flat between 45 and 65 the peak lies at around the age of 55. The value of the coefficient of relative risk aversion, $\theta = 8.4$, must be considered high relative to the literature. However, Mehra and Prescott (1985) argue that the upper bound of reasonable values of the parameter of risk aversion is 10. With this value, the model is able to generate an empirically observed equity premium of about 6.2%.

Due to the homotheticity of preferences, the initial level of human capital h_0 is irrelevant and we normalize human capital by setting $h_0 = 1$. We calibrate the human capital depreciation rate, δ^h, by setting the corresponding parameters, χ_0 and χ_1, such that the model matches observed wage profiles based on PSID data provided by Huggett, Ventura, and Yaron (2011).[8] Idiosyncratic shocks to human capital, η, are uncorrelated but the variance of η depends on the current state of the economy which has been documented in the data and used in the asset pricing literature (cf. Storesletten, Telmer, and Yaron, 2004 and Constantinides and Duffie, 1996 respectively). We follow the approach of Storesletten, Telmer, and Yaron (2007) and set the standard deviation σ_t to

$$\sigma_t = \sigma(\lambda) = \begin{cases} 0.2 & \text{for } \lambda \in (\lambda_1, \lambda_2) \\ 0.1 & \text{for } \lambda \in (\lambda_3, \lambda_4) \end{cases} \tag{2.21}$$

which is within the range considered in Krebs and Wilson (2004).

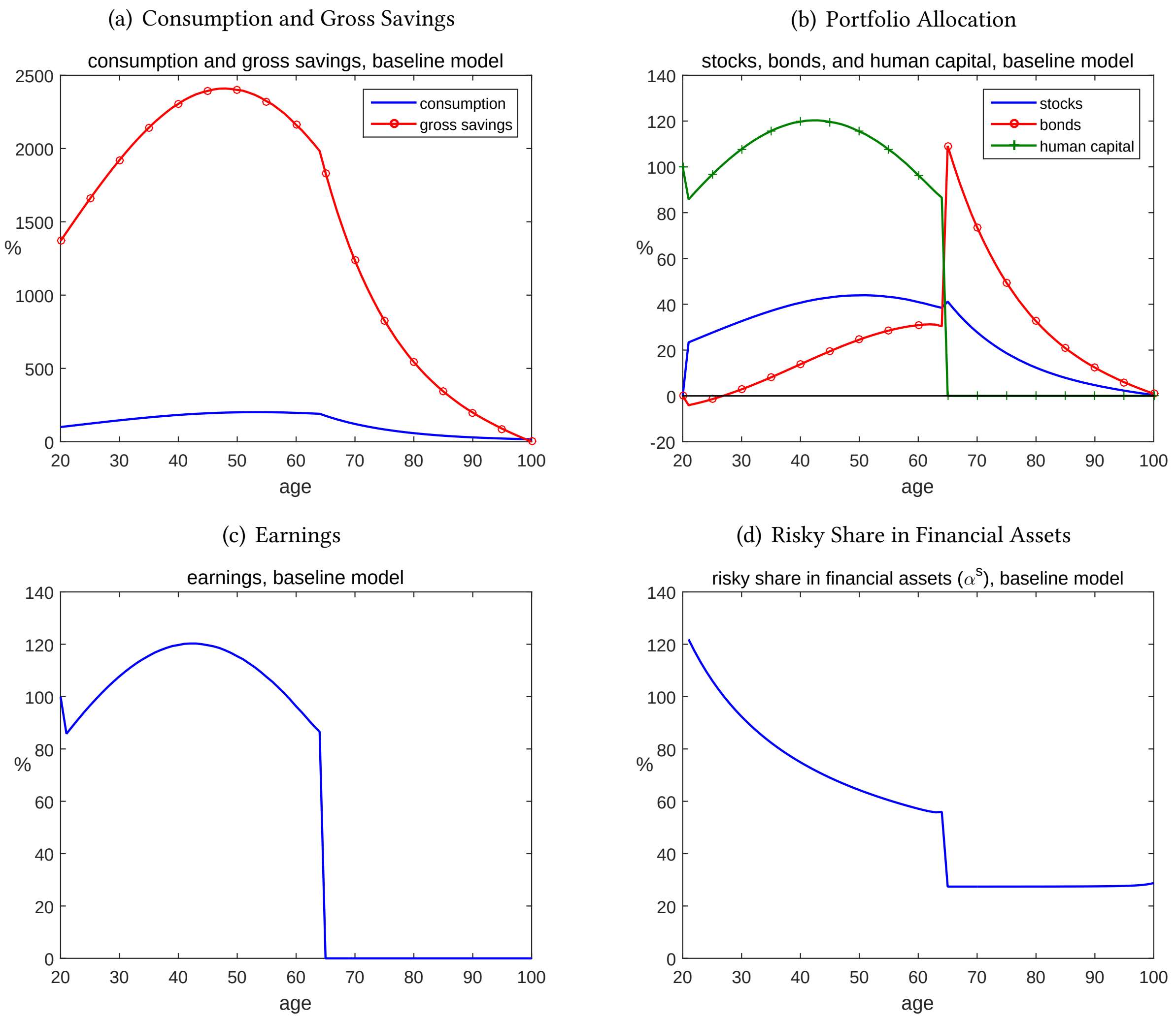
2.4 Results

2.4.1 Cross-Sectional Profiles in 2010

Figure 2.2 shows resulting key cross-sectional age profiles of the model economy in year 2010. The top left panel shows consumption and gross savings by age. Consumption is hump-shaped as in the data (cf., e.g., Fernández-Villaverde and Krüger, 2006) and remains at its maximum level between 48 and 64. Gross savings exhibit the typical saving-dis-saving pattern as in standard life-cycle models. The top right panel depicts the portfolio allocation of households by age. It shows the following pattern. Households enter their economically relevant lifetime with zero financial assets but positive human capital. Subsequently, the latter follows a hump-shaped pattern over the working life which results in a corresponding pattern in the age-earnings profile (bottom left panel). This is a target in the calibration. The pattern of financial asset holdings shows in the bottom right panel. It depicts the share of risky assets in the financial portfolio and follows a declining pattern over the working life cycle. Note that this stems from the co-incident decrease of human wealth which

[8] We thank Mark Huggett for sending us the data.

Figure 2.2: Cross-Sectional Profiles in 2010



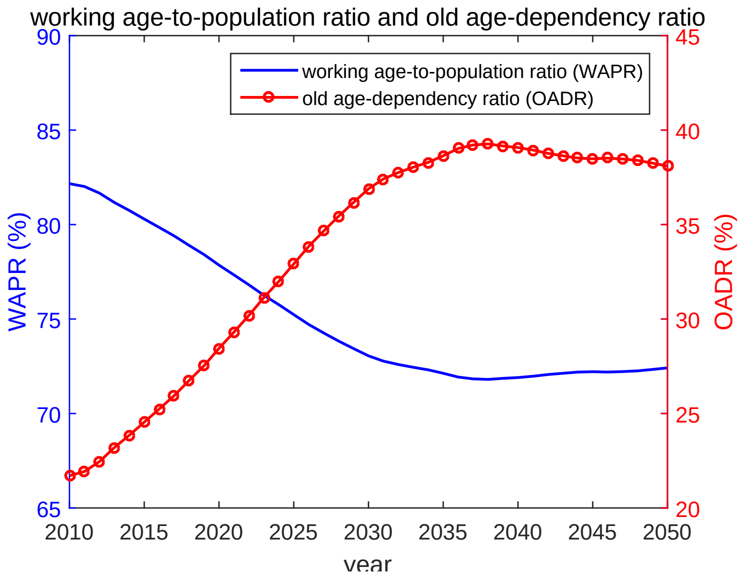
Source: Baseline model in year 2010: Selected average cross-sectional age profiles. *Notes:* The top left panel shows consumption and savings as a percentage of consumption at age 20. The top right panel plots stocks, bonds, and human capital as a percentage of (initial) human capital at age 20. The bottom left panel shows earnings as a percentage of earnings at age 20. The bottom right panel shows the risky share of the financial portfolio in percent.

is defined as the present value of expected remaining lifetime earnings. Despite its riskiness, human wealth shows to resemble rather the holdings of one period ahead risk-free bonds than of stocks. Consequently, households re-allocate their portfolio toward bonds when human wealth decreases (cf., e.g., Campbell and Viceira, 2002, ch. 6 et seqq.). As human wealth equals zero as of the retirement of a household the risky financial portfolio share remains approximately constant over the retirement spell.

2.4.2 Macroeconomic Aggregates and Asset Returns

In the following, we show resulting time paths of key variables in the model induced by the demographic transition. These are macroeconomic aggregates and the returns to the different kinds of assets as well as the resulting equity premium. In order to reveal the role of human capital in the demographic transition we show results of two variants of the model for the main period of projection, i.e., 2010 to 2050. Left sub-figures depict results for the baseline case while right sub-figures belong to an auxiliary variant of the model in which all human capital shares in total wealth are held fix at the level of 1960. By the latter, we approximate a model

Figure 2.3: Working Age-to-Population Ratio and Old-Age Dependency Ratio



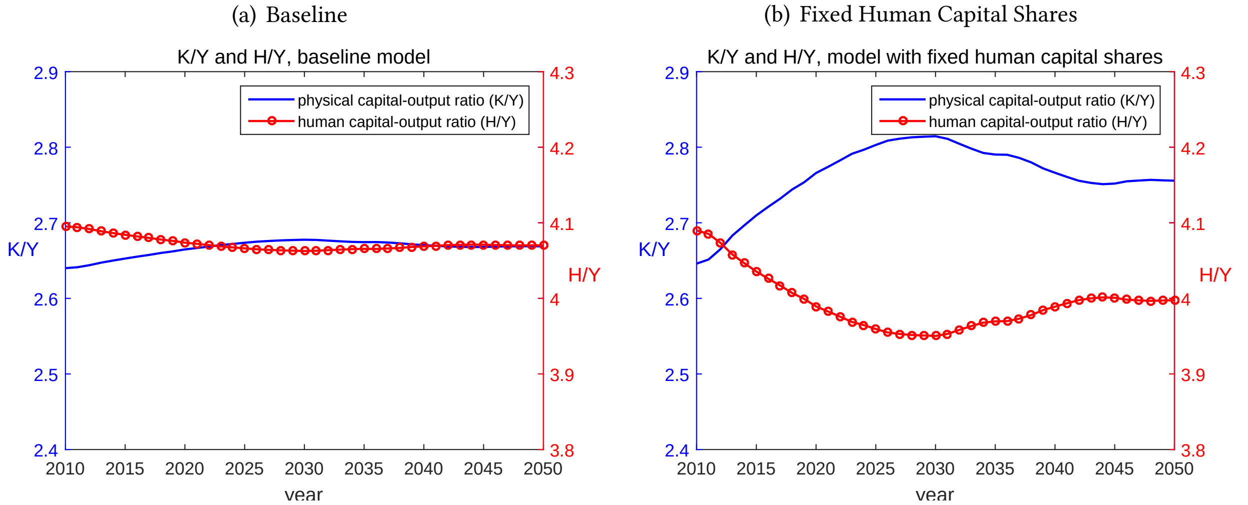
Source: Own calculations based on United Nations (2007) and Human Mortality Database (2008). *Notes:* The working age-to-population ratio is here defined as the number of the working age population of age 20-64 to the total adult population of age 20-110. The old-age dependency ratio is here defined as the number of the population of age 65 and older as a fraction of the working age population.

without human capital adjustments and are able to show their mitigating effect for dynamics of aggregate measures and asset returns in the demographic transition.

Figure 2.3 summarizes the demographic transition of the U.S. economy which is the exogenous driving force in the model. It reflects the aging process in the U.S. economy by depicting the evolution of its working age-to-population ratio. However, note that the demographic structure of the model is much richer than that summary statistic featuring the entire distribution of the population and its survival probabilities over age. The most severe change in the age structure of the economy is expected to evolve until around 2030. Subsequently, there is almost no further change in the working age-to-population ratio until 2050. However, although this is not captured by the summary statistics, the age distribution still changes albeit with a lower degree of severity.

The blue solid lines in figure 2.4 and the red lines with dots show the resulting paths of human capital-output ratio, H/Y, and physical capital-output ratio, K/Y, respectively. Conventional analyses suggest that

Figure 2.4: Macroeconomic Aggregates: Physical and Human Capital



Source: Baseline model (panel (a)) and auxiliary model with fixed human capital shares (panel (b)) in the main period of projection.
Notes: Human capital shares in the auxiliary model are held fix at the level of the initial stationary equilibrium in 1960.

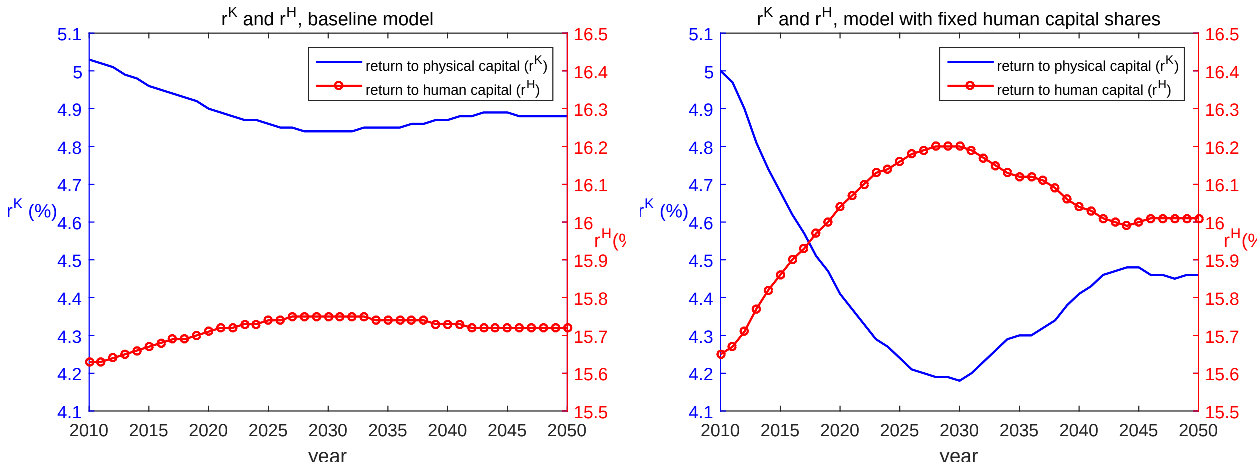
aging induces a relative shortage of labor and a relative abundance of physical capital in the economy. This corresponds to an increase in the physical capital-output ratio and a decrease in the human capital-output ratio. Both applies in the baseline case depicted by the left panel of figure 2.4. It implies that endogenous human capital adjustments do not offset the shortage of labor arising from the aging process. However, the right panel reveals that labor scarcity is much more pronounced if human capital shares remained constant. This underscores the importance of endogenous individual human capital adjustments for the macroeconomic composition of capital and labor in the demographic transition. Note that dynamics are less pronounced as of around 2030 which re-emerges in subsequent figures. As explained above, demographic shifts are less severe as of 2030 leading to smaller movements in aggregates and prices.

What does the change in the aggregate measures imply for the returns to physical and human capital? Figure 2.5 plots the corresponding time paths. It shows that rates of return to physical capital decline which is the mirror image of the increasing capital-output ratios in figure 2.4. Again, this is consistent with conventional analyses. Correspondingly, human capital returns follow an increasing pattern which reflects the augmenting relative scarcity of labor in the demographic transition. Note that the effects are quantitatively small in the baseline model but sizable when we hold human capital shares constant. This confirms the findings of Ludwig, Schelkle, and Vogel (2012). In the latter case the change in the risky interest rate amounts to a substantial one percentage point decrease until 2030.

Let us now turn to the key concern of this chapter, i.e., how the effect of demographic change on physical asset returns differs by the risk nature of assets. While returns to equity exhibit business cycle risk, returns to bonds are risk-free for a time horizon of one period. Figure 2.6 plots the corresponding time paths resulting from the model simulation. Both returns decline over time which corresponds to the result of a declining return to physical capital described above. Again effects are much larger in the model with constant human

Figure 2.5: Risky Asset Returns

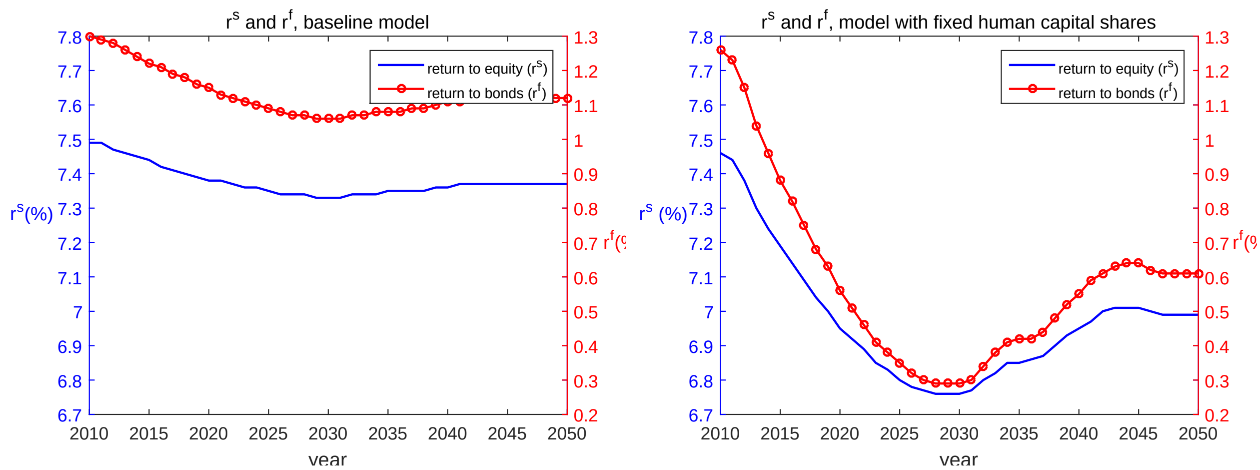
(a) Baseline (b) Fixed Human Capital Shares



Source: Baseline model (panel (a)) and auxiliary model with fixed human capital shares (panel (b)) in the main period of projection. *Notes:* All values are in percent. Human capital shares in the auxiliary model are held fix at the level of the initial stationary equilibrium in 1960.

Figure 2.6: Return to Equity and Risk-Free Rate

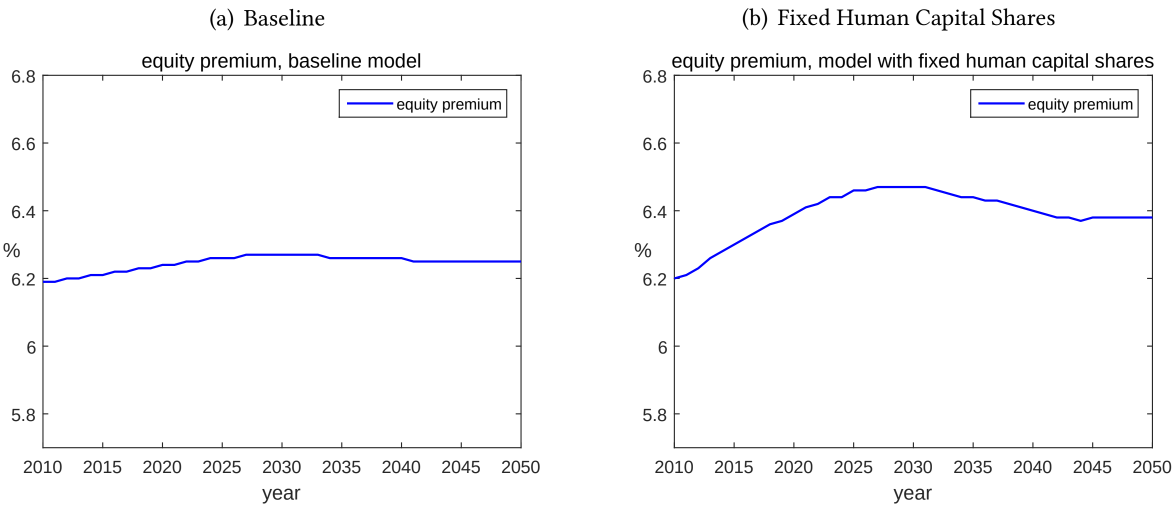
(a) Baseline (b) Fixed Human Capital Shares



Source: Baseline model (panel (a)) and auxiliary model with fixed human capital shares (panel (b)) in the main period of projection. *Notes:* All values are in percent. Human capital shares in the auxiliary model are held fix at the level of the initial stationary equilibrium in 1960.

capital shares. The development shows that we can expect low returns for (many) decades to come, irrespective of the risk nature of the asset. Moreover, figure 2.7 reveals the relative size of those declines by plotting the evolution of the equity premium, defined as the difference between the return to equity and the return to bonds. While effects in the baseline case are rather negligible, the equity premium increases by about 30

Figure 2.7: Equity Premium



Source: Baseline model (panel (a)) and auxiliary model with fixed human capital shares (panel (b)) in the main period of projection. *Notes:* The lines show the equity premium defined as the differential return between equity and bonds. Human capital shares in the auxiliary model are held fix at the level of the initial stationary equilibrium in 1960.

basis points until 2030 in the model with fixed human capital shares. This comes from the fact that households hold relatively more bonds in their financial portfolio as was shown in panel (d) of figure 2.2.[9] In the course of aging, that implies a higher relative demand for bonds in the economy which drives down the return to bonds more strongly than the return to equity. The comparison of the two panels shows that human capital adjustments work as an opposing force to that boosting effect of aging on the equity premium. Higher human capital investments, in particular, by young households imply higher earnings along their working life, enforced by the co-incident increase in the return to human capital. This implies higher human wealth of a household. As was discussed in section 2.4.1 and shown elsewhere in the literature (cf., e.g., Campbell and Viceira, 2002, ch. 6 et seqq.) rising human wealth elevates the risky share in the financial portfolio of a household as long as the (positive) correlation of their returns is not too high.[10]

Tables 2.2 and 2.3 summarize the results described above by reporting the corresponding numbers for the years 2010, 2030, and 2050.

2.4.3 Consequences for Welfare

What are the welfare consequences of the price dynamics described in the previous section? How do they differ across generations? In order to answer these questions, we follow Davila et al. (2012), Harenberg and Ludwig (2015), and others, and measure welfare by ex-ante expected utility at the beginning of a household's

[9]In fact, this includes a counteracting portfolio adjustment effect isolated in Kuhle (2008) which arises if the absolute return level drops which is the case here. Please see the explanation at the end of the introductory section 2.1.
[10]The correlation in the model equals about 0.1.

Table 2.2: Summary of Baseline Results

	WAPR	K/Y	r^f	r^s	$\{\hat{r}_j^h\}_{avrg.}$	EP
2010 (in %)	82.00	265.00	1.30	7.49	11.64	6.19
2030 (in %)	73.00	268.00	1.06	7.33	11.81	6.27
2050 (in %)	72.00	267.00	1.12	7.37	11.80	6.25
$\Delta_{\{2030-2010\}}$ *(in %p)*	-9.00	+3.00	-0.24	-0.16	+0.17	+0.08
$\Delta_{\{2050-2010\}}$ *(in %p)*	-10.00	+2.00	-0.18	-0.12	+0.16	+0.06

Source: Baseline model in the main period of projection. *Notes:* $WAPR \widehat{=}$ working age-to-population ratio. $\{\hat{r}_j^h\}_{avrg.} \widehat{=}$ average return to human capital of all agents alive. $EP := r^s - r^f$. The top three lines show the values of the considered variables for the year 2010, 2030, and 2050 in percent. The bottom two rows show the percentage point ($\%p$) change of the considered variables from 2010 to 2030 and 2010 to 2050.

Table 2.3: Effect of Endogenous Human Capital

	WAPR	k/y	r^f	r^s	$\{\hat{r}_j^h\}_{avrg.}$	EP
Baseline						
$\Delta_{\{2030-2010\}}$ (in %p)	-9.00	+3.00	-0.24	-0.16	+0.17	+0.08
$\Delta_{\{2050-2010\}}$ (in %p)	-10.00	+2.00	-0.18	-0.12	+0.16	+0.06
Holding human capital shares constant						
$\Delta_{\{2030-2010\}}$ (in %p)	-9.00	+16.00	-0.97	-0.70	+0.60	+0.27
$\Delta_{\{2050-2010\}}$ (in %p)	-10.00	+11.00	-0.65	-0.47	+0.42	+0.18

Source: Baseline model (top two rows) and auxiliary model with fixed human capital shares (bottom two rows) in the main period of projection. *Notes:* $WAPR \widehat{=}$ working age-to-population ratio. $\{\hat{r}_j^h\}_{avrg.} \widehat{=}$ average return to human capital of all agents alive. $EP := r^s - r^f$. The numbers show the percentage point ($\%p$) change of the considered variables from 2010 to 2030 and 2010 to 2050.

life, $\mathbb{E}[v_{\cdot,0}]$. All households of a given cohort are ex-ante identical and turn heterogeneous along the life cycle due to idiosyncratic shocks to human capital. The welfare concept of ex-ante expected utility is the natural objective of a social planner who is behind the veil of ignorance (cf. Davila et al., 2012, p. 2439). From this it follows that it provides also the natural perspective on the consequences that arise from the exogenous force in this model.

Demographic change affects a household's welfare in two ways. The first effect arises from changes in survival probabilities which are exogenous in this model. The second effect stems from changing good allocations induced by changes in wages and asset returns. We want to isolate the second effect and, therefore, conduct the following auxiliary computation. We compute welfare of households which face on the individual level the time- and age-dependent survival probabilities of the demographic transition while living in the aggregate environment of the initial stationary equilibrium in which no demographic change takes place.[11] Given that, households decide fully rational as in the baseline case. We then measure welfare effects of demographic change in terms of consumption equivalent variation, i.e., how much compensation in percent of consumption a household must receive in all periods of lifetime in order to be indifferent between the worlds with and without demographic change on the aggregate level.[12] Based on the homotheticity of the value function, consumption equivalent variation in period t can be measured as

$$cev_t := \frac{\mathbb{E}[v_{t,0}]}{\mathbb{E}[v^A_{t,0}]} - 1 \tag{2.22}$$

where $v^A_{\cdot,0}$ is the lifetime value of a newborn household in the auxiliary world without demographic change on the aggregate level.

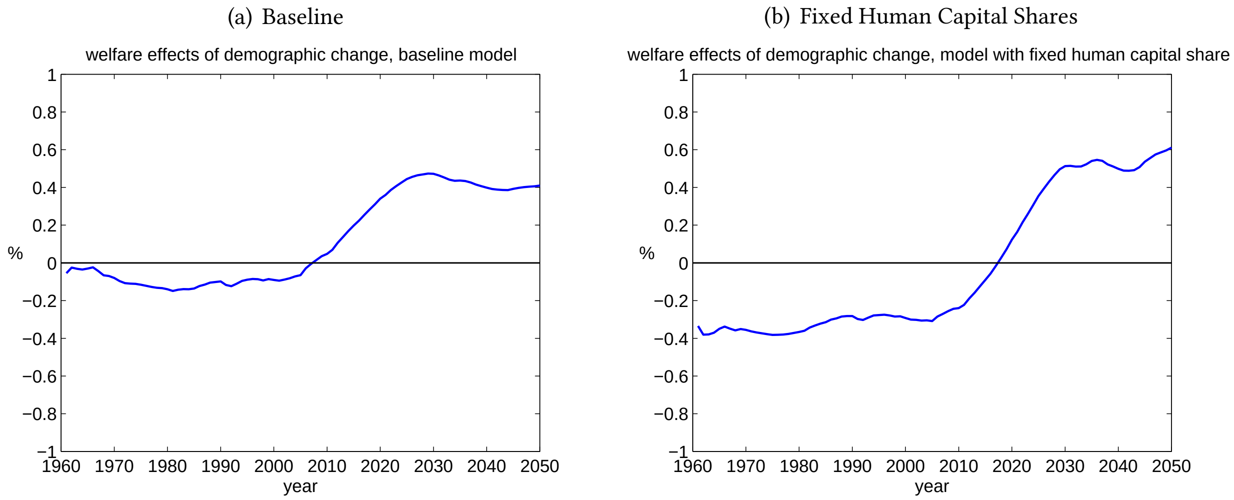
Figure 2.8 shows resulting welfare effects of demographic change for all generations born as of 1960. Again, the *birth* of a generation occurs when individuals turn 20. As before, the left sub-figure depicts the result for the baseline case while the right sub-figure belongs to the auxiliary variant of the model in which all human capital shares in total wealth are held fix at the level of 1960.

The left panel shows that welfare effects of demographic change differ considerably across generations. While generations born after 2005 benefit from the price effects induced by demographic change early generations, i.e., those born still in the 20^{th} century, lose. This can be traced back to the timing of the dynamics of the returns to physical and human capital which were described in the previous section. Note that assets play a changing role along the life cycle. While young households rather own little assets and hold negative bonds for financing human capital accumulation old households rely on assets as the only source of income. Consequently, early generations lose from the decline in the physical capital return as major return changes evolve in the period 2010 to 2030. At the same time, those generations benefit rather little from rising returns to human capital in the aforementioned period. This is because they have spent a significant part of their working life in a period with comparatively low returns to human capital. As a result, early generations, in particular the baby boomers, suffer from welfare losses due to demographic change. On the contrary, future generations benefit from demographic change as the aforementioned developments hit them at another point in the life cycle. A declining risk-free rate at young ages and increasing returns to human capital, in particular, at middle ages when human capital is at a high level lead to their welfare gains. Moreover, asset returns start

[11] This implies that both, the coefficients in the aggregate law of motion and the age distribution of the economy equal the corresponding values of the initial stationary equilibrium.

[12] We simulate the two model variants using both, identical initial conditions and the identical 50000 time series of aggregate shock realizations.

Figure 2.8: Welfare Effects of Demographic change



Source: Welfare evaluation in the baseline model (panel (a)) and in the auxiliary model with fixed human capital shares (panel (b)) in the main period of projection. *Notes:* Welfare effects of the generation born in the indicated year measured as consumption equivalent variation. *Birth* of a generation occurs when individuals turn 20. Negative values indicate welfare losses from demographic change. Human capital shares in the auxiliary model are held fix at the level of the initial stationary equilibrium in 1960.

to rise slightly again as of 2030 which affects their capital income in retirement positively.

The right panel of figure 2.8 shows the corresponding welfare effects for the auxiliary model variant with fixed human capital shares. Also in this auxiliary model, future generations benefit from demographic change while early generations lose. The comparison to the baseline model in the left panel shows that human capital adjustments mitigate the welfare effects. This mirrors the mitigating effect of human capital on return dynamics which was discussed above. Note that the mitigating effect works in both directions. While early generations suffer from stronger welfare losses if human capital is fixed, future generations benefit from higher welfare gains. This is an effect already observed by Ludwig, Schelkle, and Vogel (2012). It shows that, when investing into human capital, each household faces a trade-off between changes in individual productivity on the one hand and induced movements in equilibrium market returns on the other. Apparently, there is a positive net effect for early generations while the opposite is true for future generations. Look exemplarily at the generations born in the 1960*s*. They are already retired by the year 2010 when human capital returns start to rise significantly. Consequently, their individual human capital stocks are irrelevant in that period and, moreover, they are not affected by changes in returns to human capital. However, they finance consumption from savings and capital income, and suffer from declining asset returns. Hence, they are heavily interested in a high human capital stock of working generations in that period. Again, to state it clearly, the increase in the equity premium in the auxiliary model with fixed human capital shares observed in figure 2.7(b) induces positive welfare effects for young compared to old households in that period. This is due to the declining pattern in the risky financial portfolio share over age as implied by figure 2.2(d).

2.5 Conclusion

We ask the question to what extent demographic change of the upcoming decades will affect financial market prices and in particular the differential return between equity and bonds in the U.S. We develop a multi-generation OLG model featuring a linear human capital technology killing three birds with one stone. The three birds are, first, preserving computational tractability despite a large model with complex transitional dynamics, second, meaningful asset pricing by accounting for idiosyncratic risk, and, third, human capital as an important adjustment channel to the consequences of demographic change as shown elsewhere in the literature.

Our results show that the expected decrease of the average stock return until 2030 is in the order of magnitude of 0.16 percentage points. The decrease of the risk-free interest rate on bonds is slightly higher such that the equity premium increases by about 0.08 percentage points. These relatively mild decreases of returns and corresponding small increase of the equity premium results from an interplay of three main effects. First, older households on average hold relatively fewer equity than younger households in the model as well as in the data Ameriks and Zeldes (2004).[13] Demographic change increases the size of the old population relative to the young which drives up the relative demand for bonds thereby increasing its relative price. Consequently, the equity premium tends to increase. The second effect is a portfolio adjustment effect isolated in Kuhle (2008) that works in the opposite direction: Ignoring the first effect, suppose that demographic change would lead to a decrease of the expected rates of return on both assets by the same amount (such that the ex-ante equity premium is constant). For a positive equity premium, then, the percentage decrease of the risk-free rate of return is higher such that the investor increases her relative portfolio shares of equity. Consequently, the demand for bonds decreases. Hence, the equilibrium decrease of the equity premium is smaller than the first effect would postulate in isolation. Third, and most importantly, endogenous human capital adjustments have a large effect. As societies are aging, labor becomes a relatively scarce factor and households increase human capital investments. This increases productivity thereby decreasing the downward pressure on asset prices. If we instead hold the human capital shares constant, then the negative effects on asset returns are much larger. In that scenario, the average stock return decreases by about 0.70 percentage points until 2030 and the equity premium increases by about 0.27 percentage points.

A welfare analysis shows that the decline of asset returns and the co-incident increase of the human capital return benefits future generations relative to generations born in the past. Welfare effects show to be mitigated and homogenized across generations compared to a model with exogenous human capital.

Note that the level of welfare effects in this model is relatively low compared to what was found in the literature (cf., e.g., Ludwig, Schelkle, and Vogel, 2012). This has two major reasons. First, the absence of a pay-as-you-go pension system reduces inter-generational income shifts considerably compared to both, literature and the data. This leads to an overstating of relative welfare gains of future generations. Second, this model assumes a relatively high time discount rate, $1/\beta - 1$. Accordingly, welfare effects that evolve late in life are strongly discounted reducing the overall welfare effect of demographic change. This affects, in particular, welfare of generations born in 20^{th} century when demographic change was less pronounced.[14]

There are various additional interesting dimensions for future research. In this model there are no direct costs of market participation, neither at the aggregate nor at the idiosyncratic level. In particular, adjustment costs at the aggregate level as studied by, e.g., Abel (2003) which endogenize the price of capital might insert an interesting dimension to the general equilibrium effects of demographic change in a model of endogenous portfolio choice. Moreover, the correlation structure between human capital returns and risky asset returns

[13] According to Ameriks and Zeldes (2004) life-cycle portfolio shares do not vary much with age conditional on participation in equity markets but participation decreases around the age of retirement.

[14] The latter result is affected by the assumption of a stationary demographic characterization of the economy until 1960 which results from the lack of adequate data on much earlier periods. However, fertility rates start to deteriorate, considerably, with the invention of the birth control pill, primarily by the mid of 1960s.

is an important determinant of the financial portfolio decision along the life cycle and, in turn, the return effects of demographic change. Accordingly, idiosyncratic entrepreneurial risk as studied by, e.g., Angeletos and Calvet (2006) and Angeletos (2007) would be an interesting extension to study.

Appendix 2.A Theoretical Appendix

2.A.1 Solution of the Household Problem

We hereafter define recursively the household problem conditional on a law of motion of the aggregate state of the economy. It is convenient to express next period's values with symbol $'$, irrespective of whether they are only time-dependent or both, age- and time-dependent. The states of the household problem are the exogenous states j, t, and λ, the endogenous idiosyncratic states k and h, as well as the endogenous aggregate state of the economy, Γ, which is the distribution of physical capital, k, and human capital, h, among households. The associated law of motion is $\Gamma' = \Phi(\Gamma, \lambda, \lambda', N')$. The household problem at age j in period t is then given by

$$
\begin{aligned}
v(k,h,\lambda;\Gamma) = \max_{c,\alpha^{s\prime},i^h,k'} & \left\{ c^{\frac{1-\theta}{\gamma}} + \beta \cdot \left(\mathbb{E}[\varsigma \cdot v(k',h';\lambda';\Gamma')^{1-\theta}] \right)^{\frac{1}{\gamma}} \right\}^{\frac{\gamma}{1-\theta}} \\
\text{subject to} & \qquad (2.23) \\
& k' = k \cdot (1 + r^f + \alpha^s \cdot (r^s - r^f)) + h \cdot r^H - i^h - c \\
& h' = h \cdot (1 - \delta^h + \eta) + \tilde{i}^h \\
& h_0 > 0,\ k_0 = 0 \text{ given} \\
& \Gamma' = \Phi(\Gamma, \lambda, \lambda', N'),\ N' \text{ given} \\
& \pi(\lambda' \mid \lambda),\ \lambda_0 \text{ given.} \\
& \eta \sim \mathcal{D}(0, \sigma^2(\lambda)).
\end{aligned}
$$

The expectation $\mathbb{E}$ above is taken with respect to the realization of tomorrow's aggregate state λ' conditional on state λ today and the realization of tomorrow's idiosyncratic shock, η'. Note that λ' determines the shock to technology, z', the shock to physical capital depreciation, $\delta^{k\prime}$, and the variance of the idiosyncratic human capital depreciation shock, $\sigma^{2\prime}$.

In the following, we transform the household problem and derive the first-order conditions of its solution. We start with de-trending the accumulation equation for financial assets, (2.8), leading to:

$$
\tilde{k}' = \frac{1}{1+g} \cdot \left(\tilde{k} \cdot (1 + r^f + \alpha^s \cdot (r^s - r^f)) + h \cdot \tilde{r}^H - \tilde{i}^h - \tilde{c} \right)
$$

where, e.g., $\tilde{r}^H = r^H / \Upsilon$.

Let's combine the de-trended accumulation equation for financial assets with the human capital production technology given by equation (2.7).[15] Thereby we get:

$$
\begin{aligned}
\tilde{k}' &= \frac{1}{1+g} \cdot \Big(\tilde{k} \cdot (1 + r^f + \alpha^s \cdot (r^s - r^f)) + h \cdot \tilde{r}^H \\
&\quad + h \cdot (1 - \delta^h + \eta) - h' - \tilde{c} \Big) \\
\tilde{k}' + \frac{1}{1+g} \cdot h' &= \frac{1}{1+g} \cdot \Big((\tilde{k} + \frac{h}{1+g}) \cdot (1 + r^f) + \tilde{k} \cdot \alpha^s \cdot (r^s - r^f) \\
&\quad + \frac{h}{1+g} \cdot ((1+g) \cdot (\tilde{r}^H + 1 - \delta^h + \eta) - (1 + r^f)) - \tilde{c} \Big)
\end{aligned}
$$

[15] Again, human capital must be trend stationary in order to assure that gross human capital earnings, $h_{t,j} \cdot r_t^H$, grow at the same rate as Υ_t over time.

Define $\tilde{w} := \tilde{k} + \frac{1}{1+g} \cdot h$ as total (de-trended) wealth, we get:

$$\tilde{w}' = \frac{1}{1+g} \cdot (\tilde{w} \cdot (1+\widehat{r}) - \tilde{c})$$

where $\widehat{r} := r^f + \widehat{\alpha}^s \cdot (r^s - r^f) + \widehat{\alpha}^h \cdot (\widehat{r}^h - r^f)$ is the transformed net return on the total portfolio in period t, age j, and $\widehat{\alpha}^s := \tilde{k} \cdot \alpha^s / \tilde{w}$, $\widehat{\alpha}^h := \frac{h}{1+g} / \tilde{w}$, and $\widehat{r}^h := (1+g) \cdot (\tilde{r}^H + 1 - \delta^h + \eta) - 1$.
Let $\tilde{x} := \tilde{w} \cdot (1+\widehat{r})$ be total resources, or, alternatively, "cash-on-hand" (Deaton, 1991). It follows that

$$\tilde{x}' = \frac{1}{1+g} \cdot (\tilde{x} - \tilde{c}) \cdot (1 + \widehat{r}').$$

Next, we transform the utility function into a de-trended version

$$\begin{aligned} u &= [c^{\frac{1-\theta}{\gamma}} + \beta \cdot (\mathbb{E}[\varsigma \cdot u'^{1-\theta}])^{\frac{1}{\gamma}}]^{\frac{\gamma}{1-\theta}} \\ \Upsilon \cdot \tilde{u} &= [\Upsilon^{\frac{1-\theta}{\gamma}} \cdot \tilde{c}^{\frac{1-\theta}{\gamma}} + \beta \cdot (\mathbb{E}[\varsigma \cdot \Upsilon'^{1-\theta} \cdot \tilde{u}'^{1-\theta}])^{\frac{1}{\gamma}}]^{\frac{\gamma}{1-\theta}} \\ \tilde{u} &= [\tilde{c}^{\frac{1-\theta}{\gamma}} + \widehat{\beta} \cdot (\mathbb{E}[\tilde{u}'^{1-\theta}])^{\frac{1}{\gamma}}]^{\frac{\gamma}{1-\theta}} \text{ where } \widehat{\beta} = \beta \cdot \varsigma^{\frac{1}{\gamma}} \cdot (1+g)^{\frac{1-\theta}{\gamma}}, \end{aligned}$$

and finally state the de-trended household problem in period t, age j as:

$$\begin{aligned} v(\tilde{x}, \lambda, \Omega) &= \max_{\tilde{c}, \tilde{x}', \widehat{\alpha}^{s\prime}, \widehat{\alpha}^{h\prime}} \{\tilde{c}^{\frac{1-\theta}{\gamma}} + \widehat{\beta} \cdot (\mathbb{E}[v'(\tilde{x}', \lambda', \Omega')^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}} \\ \text{s.t. } \tilde{x}' &= \frac{1}{1+g} \cdot (\tilde{x} - \tilde{c}) \cdot (1 + \widehat{r}'),\ \tilde{x}_0 > 0 \text{ given} \\ \Omega' &= \Phi(\Omega, \lambda, \lambda', N'),\ N' \text{ given} \\ &\pi(\lambda' \mid \lambda),\ \lambda_0 \text{ given} \\ &\eta \sim \mathcal{D}(0, \sigma^2(\lambda)). \end{aligned}$$

Note that Ω is the distribution of (de-trended) wealth, $\tilde{w}$, among households with associated law of motion $\Phi(\cdot)$. Ω follows directly from the transformation of Γ.

In what follows, we prove that the optimal household policy functions are given by proposition 1.

Proof. We guess that $v = m^l \cdot \tilde{x}$ where l is some parameter to be determined below and m is the marginal propensity to consume out of $\tilde{x}$ and show below that this is indeed true. From the guess it follows that

$$\begin{aligned} v &= \max_{\tilde{c}, \tilde{x}', \widehat{\alpha}^{s\prime}, \widehat{\alpha}^{h\prime}} \{\tilde{c}^{\frac{1-\theta}{\gamma}} + \widehat{\beta} \cdot (\mathbb{E}[(m'^l \cdot \tilde{x}')^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}} \text{ s.t. } \tilde{x}' = \frac{1}{1+g} \cdot (\tilde{x} - \tilde{c}) \cdot (1 + \widehat{r}') \\ v &= \max_{\tilde{c}, \widehat{\alpha}^{s\prime}, \widehat{\alpha}^{h\prime}} \{\tilde{c}^{\frac{1-\theta}{\gamma}} + \widehat{\beta} \cdot (\mathbb{E}[m'^{l \cdot (1-\theta)} \cdot (\frac{1}{1+g} \cdot (\tilde{x} - \tilde{c}) \cdot (1 + \widehat{r}'))^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}} \\ v &= \max_{\tilde{c}, \widehat{\alpha}^{s\prime}, \widehat{\alpha}^{h\prime}} \{\tilde{c}^{\frac{1-\theta}{\gamma}} + (\frac{1}{1+g} \cdot (\tilde{x} - \tilde{c}))^{\frac{1-\theta}{\gamma}} \cdot \widehat{\beta} \cdot (\mathbb{E}[(m'^l \cdot (1 + \widehat{r}'))^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}} \end{aligned}$$

Next, we compute the first-order conditions (FOCs) with respect to $\tilde{c}, \widehat{\alpha}^{s\prime}, \widehat{\alpha}^{h\prime}$:

- FOC with respect to consumption:

$$0 = \frac{\gamma}{1-\theta} \cdot \{\tilde{c}^{\frac{1-\theta}{\gamma}} + (\frac{1}{1+g} \cdot (\tilde{x} - \tilde{c}))^{\frac{1-\theta}{\gamma}} \cdot \widehat{\beta} \cdot (\mathbb{E}[(m'^l \cdot (1 + \widehat{r}'))^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}-1}$$

$$
\begin{aligned}
&\cdot \{\frac{1-\theta}{\gamma} \cdot \tilde{c}^{\frac{1-\theta-\gamma}{\gamma}} - \frac{1-\theta}{\gamma \cdot (1+g)} \cdot (\frac{1}{1+g} \cdot (\tilde{x}-\tilde{c}))^{\frac{1-\theta-\gamma}{\gamma}} \cdot \widehat{\beta} \\
&\cdot (\mathbb{E}[(m'^{l} \cdot (1+\widehat{r}'))^{1-\theta}])^{\frac{1}{\gamma}}\} \\
\tilde{c} =& (\tilde{x}-\tilde{c}) \cdot (\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}} \cdot \widehat{\beta}^{\frac{\gamma}{1-\theta-\gamma}} \cdot (\mathbb{E}[(m'^{l} \cdot (1+\widehat{r}'))^{1-\theta}])^{\frac{1}{1-\theta-\gamma}}
\end{aligned}
$$

Defining $n := \widehat{\beta}^{\frac{\gamma}{1-\theta-\gamma}} \cdot (\mathbb{E}[(m'^{l} \cdot (1+\widehat{r}'))^{1-\theta}])^{\frac{1}{1-\theta-\gamma}}$, $o := (\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}}$, and $m := \frac{o \cdot n}{1+o \cdot n}$, we get

$$
\tilde{c} = m \cdot \tilde{x}.
$$

- FOC with respect to stock portfolio share:

$$
\begin{aligned}
0 =& \frac{\gamma}{1-\theta} \cdot \{\tilde{c}^{\frac{1-\theta}{\gamma}} + (\frac{1}{1+g} \cdot (\tilde{x}-\tilde{c}))^{\frac{1-\theta}{\gamma}} \cdot \widehat{\beta} \cdot (\mathbb{E}[(m'^{l} \cdot (1+\widehat{r}'))^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}-1} \\
&\cdot (\frac{1}{1+g} \cdot (\tilde{x}-\tilde{c}))^{\frac{1-\theta}{\gamma}} \cdot \widehat{\beta} \cdot \frac{1}{\gamma} \cdot (\mathbb{E}[(m'^{l} \cdot (1+\widehat{r}'))^{1-\theta}])^{\frac{1}{\gamma}-1} \\
&\cdot \mathbb{E}[m'^{l \cdot (1-\theta)} \cdot (1-\theta) \cdot (1+\widehat{r}')^{-\theta} \cdot (r^{s\prime} - r^{f\prime})] \\
0 =& \mathbb{E}[m'^{l \cdot (1-\theta)} \cdot (1+\widehat{r}')^{-\theta} \cdot (r^{s\prime} - r^{f\prime})]
\end{aligned}
$$

- FOC with respect to human capital portfolio share:

$$
\begin{aligned}
0 =& \frac{\gamma}{1-\theta} \cdot \{\tilde{c}^{\frac{1-\theta}{\gamma}} + (\frac{1}{1+g} \cdot (\tilde{x}-\tilde{c}))^{\frac{1-\theta}{\gamma}} \cdot \widehat{\beta} \cdot (\mathbb{E}[(m'^{l} \cdot (1+\widehat{r}'))^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}-1} \\
&\cdot (\frac{1}{1+g} \cdot (\tilde{x}-\tilde{c}))^{\frac{1-\theta}{\gamma}} \cdot \widehat{\beta} \cdot \frac{1}{\gamma} \cdot (\mathbb{E}[(m'^{l} \cdot (1+\widehat{r}'))^{1-\theta}])^{\frac{1}{\gamma}-1} \\
&\cdot \mathbb{E}[m'^{l \cdot (1-\theta)} \cdot (1-\theta) \cdot (1+\widehat{r}')^{-\theta} \cdot (\widehat{r}^{h\prime} - r^{f\prime})] \\
0 =& \mathbb{E}[m'^{l \cdot (1-\theta)} \cdot (1+\widehat{r}')^{-\theta} \cdot (\widehat{r}^{h\prime} - r^{f\prime})]
\end{aligned}
$$

What is left is to show that indeed $v = m^{l} \cdot \tilde{x}$. Using $\tilde{c} = m \cdot \tilde{x}$, $n = \widehat{\beta}^{\frac{\gamma}{1-\theta-\gamma}} \cdot (\mathbb{E}[(m'^{l} \cdot (1+\widehat{r}'))^{1-\theta}])^{\frac{1}{1-\theta-\gamma}}$, $m = \frac{o \cdot n}{1+o \cdot n}$, and $o = (\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}}$ in u we get:

$$
\begin{aligned}
v =& \left\{ (m \cdot \tilde{x})^{\frac{1-\theta}{\gamma}} + \left(\frac{1}{1+g} \cdot (\tilde{x} - m \cdot \tilde{x})\right)^{\frac{1-\theta}{\gamma}} \cdot n^{\frac{1-\theta-\gamma}{\gamma}} \right\}^{\frac{\gamma}{1-\theta}} \\
=& \tilde{x} \cdot \left\{ m^{\frac{1-\theta}{\gamma}} + (1-m)^{\frac{1-\theta}{\gamma}} \cdot (\frac{1}{1+g})^{\frac{1-\theta}{\gamma}} \cdot n^{\frac{1-\theta-\gamma}{\gamma}} \right\}^{\frac{\gamma}{1-\theta}} \\
=& \tilde{x} \cdot \left\{ (\frac{o \cdot n}{1+o \cdot n})^{\frac{1-\theta}{\gamma}} + (\frac{1}{1+o \cdot n})^{\frac{1-\theta}{\gamma}} \cdot o^{\frac{1-\theta-\gamma}{\gamma}} \cdot n^{\frac{1-\theta-\gamma}{\gamma}} \right\}^{\frac{\gamma}{1-\theta}} \\
=& \tilde{x} \cdot \left\{ \frac{(o \cdot n)^{\frac{1-\theta}{\gamma}} + (o \cdot n)^{\frac{1-\theta-\gamma}{\gamma}}}{(1+o \cdot n)^{\frac{1-\theta}{\gamma}}} \right\}^{\frac{\gamma}{1-\theta}} = \tilde{x} \cdot \left\{ \frac{(o \cdot n + 1) \cdot (o \cdot n)^{\frac{1-\theta-\gamma}{\gamma}}}{(1+o \cdot n)^{\frac{1-\theta}{\gamma}}} \right\}^{\frac{\gamma}{1-\theta}} \\
=& \tilde{x} \cdot \left\{ \frac{(o \cdot n)^{\frac{1-\theta-\gamma}{\gamma}}}{(1+o \cdot n)^{\frac{1-\theta-\gamma}{\gamma}}} \right\}^{\frac{\gamma}{1-\theta}} = \tilde{x} \cdot m^{\frac{1-\theta-\gamma}{1-\theta}}
\end{aligned}
$$

Hence, $v = m^{l} \cdot \tilde{x}$ where $l = \frac{1-\theta-\gamma}{1-\theta}$. Plugging this into the FOCs yields:

$$
\tilde{c} = m \cdot \tilde{x}
$$

$$0 = \mathbb{E}[(m')^{1-\theta-\gamma} \cdot (1+\widehat{r}')^{-\theta} \cdot (r^{s\prime} - r^{f\prime})]$$
$$0 = \mathbb{E}[(m')^{1-\theta-\gamma} \cdot (1+\widehat{r}')^{-\theta} \cdot (\widehat{r}^{h\prime} - r^{f\prime})]$$

Defining $\wp := \mathbb{E}[(m'^{\frac{1-\theta-\gamma}{1-\theta}} \cdot (1+\widehat{r}'))^{1-\theta}]$, the marginal propensity to consume equals:

$$m = \frac{o \cdot n}{1 + o \cdot n} = \frac{(\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}} \cdot \widehat{\beta}^{\frac{\gamma}{1-\theta-\gamma}} \cdot \wp^{\frac{1}{1-\theta-\gamma}}}{1 + (\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}} \cdot \widehat{\beta}^{\frac{\gamma}{1-\theta-\gamma}} \cdot \wp^{\frac{1}{1-\theta-\gamma}}}$$
$$= \frac{(\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}} \cdot (\beta \cdot \varsigma^{\frac{1}{\gamma}} \cdot (1+g)^{\frac{1-\theta}{\gamma}})^{\frac{\gamma}{1-\theta-\gamma}} \cdot \wp^{\frac{1}{1-\theta-\gamma}}}{1 + (\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}} \cdot (\beta \cdot \varsigma^{\frac{1}{\gamma}} \cdot (1+g)^{\frac{1-\theta}{\gamma}})^{\frac{\gamma}{1-\theta-\gamma}} \cdot \wp^{\frac{1}{1-\theta-\gamma}}}$$
$$= \frac{(\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}} \cdot (\beta \cdot \varsigma^{\frac{1}{\gamma}})^{\frac{\gamma}{1-\theta-\gamma}} \cdot (1+g)^{\frac{1-\theta}{1-\theta-\gamma}} \cdot \wp^{\frac{1}{1-\theta-\gamma}}}{1 + (\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}} \cdot (\beta \cdot \varsigma^{\frac{1}{\gamma}})^{\frac{\gamma}{1-\theta-\gamma}} \cdot (1+g)^{\frac{1-\theta}{1-\theta-\gamma}} \cdot \wp^{\frac{1}{1-\theta-\gamma}}}$$
$$= \frac{(\beta \cdot \varsigma^{\frac{1}{\gamma}})^{\frac{\gamma}{1-\theta-\gamma}} \cdot \wp^{\frac{1}{1-\theta-\gamma}}}{1 + (\beta \cdot \varsigma^{\frac{1}{\gamma}})^{\frac{\gamma}{1-\theta-\gamma}} \cdot \wp^{\frac{1}{1-\theta-\gamma}}} = \frac{(\beta^{\gamma} \cdot \varsigma \cdot \wp)^{\frac{1}{1-\theta-\gamma}}}{1 + (\beta^{\gamma} \cdot \varsigma \cdot \wp)^{\frac{1}{1-\theta-\gamma}}}$$

□

2.A.2 Derivation of the Aggregate Resource Constraint

Deriving the aggregate resource constraint, first we take the population weighted sums of the (de-trended) individual budget constraints and the individual human capital accumulation constraints in period t (cf. equations (2.8) and (2.7)) and add them up. Note that it is understood that we sum over all individuals of each age bin characterized by the idiosyncratic mean zero-shock η without making this explicit. We then get

$$\begin{aligned}
&(1+g) \cdot \sum_{j=0}^{J} N_{t,j} \cdot \tilde{k}_{t+1,j+1} + \sum_{j=0}^{J} N_{t,j} \cdot h_{t+1,j+1} \\
&\quad = \sum_{j=0}^{J} N_{t,j} \cdot \tilde{k}_{t,j} \cdot (1 + r_t^f + \alpha_{t,j}^s \cdot (r_t^s - r_t^f)) + \sum_{j=0}^{J} N_{t,j} \cdot h_{t,j} \cdot \tilde{r}_t^H \\
&\quad\quad - \sum_{j=0}^{J} N_{t,j} \cdot \tilde{c}_{t,j} + \sum_{j=0}^{J} N_{t,j} \cdot h_{t,j} \cdot (1 - \delta_j^h) \\
&(1+g) \cdot \sum_{j=0}^{J} N_{t,j} \cdot \tilde{k}_{t+1,j+1} + \sum_{j=0}^{J} N_{t,j} \cdot \varsigma_{t,j} \cdot h_{t+1,j+1} + \sum_{j=0}^{J} N_{t,j} \cdot (1 - \varsigma_{t,j}) \cdot h_{t+1,j+1} \\
&+ N_{t+1,0} \cdot h_{t+1,0} - N_{t+1,0} \cdot h_{t+1,0} \\
&\quad = \sum_{j=1}^{J} N_{t-1,j-1} \cdot \varsigma_{t-1,j-1} \cdot \tilde{k}_{t,j} \cdot (1 + r_t^f + \alpha_{t,j}^s \cdot (r_t^s - r_t^f)) \\
&\quad\quad + \sum_{j=1}^{J} N_{t-1,j-1} \cdot (1 - \varsigma_{t-1,j-1}) \cdot \tilde{k}_{t,j} \cdot (1 + r_t^f + \alpha_{t,j}^s \cdot (r_t^s - r_t^f))
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{j=1}^{J} N_{t-1,j-1}\cdot(1-\varsigma_{t-1,j-1})\cdot\tilde{k}_{t,j}\cdot(1+r^f_t+\alpha^s_{t,j}\cdot(r^s_t-r^f_t))\\
&+\sum_{j=0}^{J} N_{t,j}\cdot h_{t,j}\cdot(1+\tilde{r}^H_t-\delta^h_j)-\sum_{j=0}^{J} N_{t,j}\cdot\tilde{c}_{t,j}\\
(1+g)\cdot&\sum_{j=0}^{J} N_{t,j}\cdot\tilde{k}_{t+1,j+1}+\sum_{j=0}^{J} N_{t+1,j}\cdot h_{t+1,j}+\sum_{j=0}^{J} N_{t,j}\cdot(1-\varsigma_{t,j})\cdot h_{t+1,j+1}\\
=&\sum_{j=1}^{J} N_{t-1,j-1}\cdot\tilde{k}_{t,j}+\sum_{j=1}^{J} N_{t-1,j-1}\cdot\tilde{k}_{t,j}\cdot(r^f_t+\alpha^s_{t,j}\cdot(r^s_t-r^f_t))\\
&-\sum_{j=1}^{J} N_{t-1,j-1}\cdot(1-\varsigma_{t-1,j-1})\cdot\tilde{k}_{t,j}\cdot(1+r^f_t+\alpha^s_{t,j}\cdot(r^s_t-r^f_t))\\
&+\sum_{j=0}^{J} N_{t,j}\cdot h_{t,j}\cdot(1+\tilde{r}^H_t-\delta^h_j)-\sum_{j=0}^{J} N_{t,j}\cdot\tilde{c}_{t,j}+N_{t+1,0}\cdot h_{t+1,0}\\
(1+g)\cdot&\tilde{K}_{t+1}+H_{t+1}\\
=&\tilde{K}_t+\tilde{K}_t\cdot r^K_t+H_t+H_t\cdot\tilde{r}^H_t-\tilde{G}_t-\tilde{C}_t\\
&-\sum_{j=0}^{J} N_{t,j}\cdot(1-\varsigma_{t,j})\cdot h_{t+1,j+1}+N_{t+1,0}\cdot h_{t+1,0}-\sum_{j=0}^{J} N_{t,j}\cdot h_{t,j}\cdot\delta^h_j
\end{aligned}
$$

or, finally,

$$
\tilde{Y}_t=\tilde{C}_t+\tilde{G}_t+\tilde{I}^K_t+I^h_t \tag{2.24}
$$

where we used equilibrium conditions summarized in 2.2.8 as well as $K_{t+1}=K_t\cdot(1-\delta^K_t)+I^K_t$ and $I^H_{t+1}=H_t+I^H_t$. Note that I^h_t is the aggregate of gross human capital investments defined as: $I^h_t := I^H_t-N_{t+1,0}\cdot h_{t+1,0}+\sum_{j=0}^{J}N_{t,j}\cdot(1-\varsigma_{t,j})\cdot h_{t+1,j+1}+\sum_{j=0}^{J}N_{t,j}\cdot h_{t,j}\cdot\delta^h_j$.

Appendix 2.B Computational Appendix

Numerical computations are implemented in Fortran 90 using routines which are partly based on Press et al. (1996). If not otherwise stated the convergence criterion of a root finding algorithm is set to 10^{-6} and the weight on resulting variables in the updating step of a Gauss-Seidel algorithm (cf., e.g., Ludwig, 2007) to 10%.

2.B.1 Numerical Solution

We solve an approximate rational expectations equilibrium by adapting the computational method developed in Krusell and Smith (1997, 1998) to the case of transitional dynamics with time-varying aggregate laws of motion due to a time-varying demographic distribution, N. Therefore, we follow the approach in chapter 3.

The solution of the model begins in year 1960 ($t=0$) in which we assume a fix demographic distribution leading to an artificial initial stationary equilibrium. We redo the exercise in year 2500 ($t=T$) with a fix demographic distribution of year 2100. Aggregate laws of motion (ALOM) in those stationary equilibria are assumed to be linear functions of a small number of moments of the endogenous aggregate state. They are specified as

$$\ln \kappa' = \phi_{t,\lambda}^{\kappa,0} + \phi_{t,\lambda}^{\kappa,1} \cdot \ln \tilde{K} + \phi_{t,\lambda}^{\kappa,2} \cdot \ln \kappa + \phi_{t,\lambda}^{\kappa,3} \cdot \Im \tag{2.25a}$$

$$\Im' = \phi_{t,\lambda'}^{\Im,0} + \phi_{t,\lambda'}^{\Im,1} \cdot \ln \tilde{K} + \phi_{t,\lambda'}^{\Im,2} \cdot \ln \kappa' + \phi_{t,\lambda'}^{\Im,3} \cdot \Im \tag{2.25b}$$

where $\kappa := \tilde{K}/H$ is the (de-trended) capital-human capital ratio, $\Im := E[r^{s\prime}] - r^{f\prime}$ is the ex-ante equity premium, and $'$ indicates the next simulation step. Note that the ALOMs depend on both, time and the exogenous aggregate state. We determine the coefficients in the ALOMs by Monte Carlo simulations, further described below.[16]

In the main step of the procedure, we compute the (stochastic) transition of the economy from the initial to the final stationary equilibrium which is induced by the exogenous deterministic dynamics of the demographic distribution between 1960 and 2100. The standard *brute force* approach would be to assume a separate law of motion (2.25) for each time period in the transition.[17] Instead, we follow the approach in chapter 3 and specify parameterized laws of motion for the transition by multiplying the coefficients of the stationary equilibria with time polynomials using ideas from Judd (2002):

$$\begin{aligned} \ln \kappa_{t+1} = \exp(-\nu^{\kappa} \cdot t) \cdot & \sum_{l=0}^{3} \phi_{0,\lambda}^{\kappa,l} \cdot P_{\lambda}^{\kappa,l}(t) \cdot y_t^l \\ & + (1 - \exp(-\nu^{\kappa} \cdot t)) \cdot \sum_{l=0}^{3} \phi_{T,\lambda}^{\kappa,l} \cdot y_t^l \end{aligned} \tag{2.26a}$$

$$\text{where } y_t^l \in \{1, \ln \tilde{K}_t, \ln \kappa_t, \Im_t\};$$

$$\begin{aligned} \ln \Im_{t+1} = \exp(-\nu^{\Im} \cdot t) \cdot & \sum_{l=0}^{3} \phi_{0,\lambda'}^{\Im,l} \cdot P_{\lambda'}^{\Im,l}(t) \cdot y_t^l \\ & + \left(1 - \exp(-\nu^{\Im} \cdot t)\right) \cdot \sum_{l=0}^{3} \phi_{T,\lambda'}^{\Im,l} \cdot y_t^l \end{aligned} \tag{2.26b}$$

$$\text{where } y_t^l \in \{1, \ln \tilde{K}_t, \ln \kappa_{t+1}, \Im_t\}.$$

Here, ν^i is the coefficient that determines the speed of convergence of the law of motion of variable $i \in \{\kappa, \Im\}$ to the corresponding law of motion in the final stationary equilibrium of the economy. $P_{\lambda}^{i,l}(t)$ for all $i \in \{\kappa, \Im\}$, $l \in \{0, 1, 2, 3\}$, and $\lambda \in L$, are flexible global time polynomials of Chebyshev's first kind writing as:

$$P_{\lambda}^{i,l}(t) = \sum_{q=0}^{n_q} \psi_{\lambda}^{i,l,q} \cdot \mathcal{T}^q(t) \tag{2.27}$$

where $\mathcal{T}^0(t) = 1,\ \mathcal{T}^1(t) = t,\ \mathcal{T}^q(t) = 2 \cdot t \cdot \mathcal{T}^{q-1}(t) - \mathcal{T}^{q-2}(t)$ for all $q \geq 2$, and n_q is the order of the polynomial. We determine the coefficients in the time polynomials by Monte Carlo simulations, further described next.

[16]Note that, in a narrow definition, the ALOMs consist only of (2.25a). $\Im$ does not belong to the aggregate state variables because it can be derived contemporaneously from Ω. We follow Krusell and Smith (1997), Storesletten, Telmer, and Yaron (2007), Harenberg and Ludwig (2015), and others and assume a law of motion for $\Im$ in order to avoid solving for the bond market equilibrium in all future states of the world in the determination of policy functions. Basically, the latter would be feasible but causing tremendous computational costs.

[17]The advantage of this approach compared to the brute force approach is discussed at the end of this section.

In analogy to Krusell and Smith (1997, p. 404) and Gomes and Michaelides (2008), we employ the following algorithm for the determination of the coefficient vector Φ.[18] Note that Φ contains the coefficients of the ALOM (2.25) in t in the case of solving for the stationary equilibrium in period t (i.e., $t_1 = t_2 = t$). Meanwhile Φ contains the coefficients of the time polynomials in (2.26) given by (2.27) in the case of the transition (i.e., $t_1 = 1$ and $t_2 = T - 1$):

1. Build grids for κ, $\tilde{K}$, and $\Im$.[19]
2. Draw M^A series of M^B aggregate shock realizations where $M^A = 1$ in the case of a stationary equilibrium while $M^B = T - 1$ in case of the transition.
3. Iterate on the vector of coefficients Φ until convergence (fixed point iteration).
   a) Choose an initial guess for Φ.[20]
   b) Solve the household problem (2.9) for policy functions $(m, \widehat{\alpha}^h, \widehat{\alpha}^s)$ from $t = t_2, ..., t_1$ at all (λ, j) and $(\kappa, \tilde{K}, \Im)$ of the respective grid. Therefore, use (2.25) respectively (2.26) and the exogenous law of motion for λ in order to make expectations on $(\kappa', \Im')$ and to determine $(r^{s\prime}, r^{f\prime}, r^{h\prime})$. Store the policy functions.
   c) Simulate the economy $M^A \cdot M^B$ times.
      i. Determine λ_{m^A, m^B}.
      ii. If $m^B = 1$ choose initial $(\kappa, \tilde{K}, \Im, r^f)$[21] otherwise use $(\hat{\kappa}', \hat{\tilde{K}}', \hat{\Im}, \hat{r^{f\prime}})_{m^B-1}$ of the previous iteration step.
      iii. Iterate on $\Im$ until the bond market clears.
         A. Interpolate on the policy functions with respect to $(\kappa, \tilde{K}, \Im)$.
         B. Determine r^s and r^H using (2.2).
         C. Aggregate and determine $(\hat{\kappa}', \hat{\tilde{K}}')$ as well as the aggregate excess demand on the bond market.
      iv. Store $(\hat{\kappa}', \hat{\tilde{K}}', \hat{\Im})$.
   d) Discard the first M^D of M^B periods whereof $M^D = 0$ in case of the transition. Determine $\hat{\Phi}$ by running regressions using $\{\{\hat{\kappa}, \hat{\tilde{K}}, \hat{\Im}\}_{m^B=M^D+1}^{M^B}\}_{m^A=1}^{M^A}$ together with (2.25) respectively (2.26).
   e) Update the coefficient vector according to $\Phi^{new} = \vartheta \cdot \Phi^{old} + (1 - \vartheta) \cdot \hat{\Phi}$ where $0 < \vartheta < 1$ is an arbitrary adjustment factor.

We select second order time polynomials in (2.27) and use $M^B = 82500$ and $M^D = 7500$ for the stationary equilibria as well as $M^A = 5000$ for the transition. Note that while the coefficients of the laws of motion of the alternative brute force approach would be identified solely by cross-sectional variation, those of the time polynomials are identified also by time variation. Accordingly, this method requires a much smaller number

[18] Note that, for the sake of simplicity, we use the same symbol, Φ, for denoting the actual ALOM and its approximating coefficient vector.

[19] We build grids around the solution of the Mean Shock Equilibrium (MSE) which assumes aggregate uncertainty to realize at its unconditional mean while otherwise fully accounting for the stochastic feature of the model. Please see section 2.B.2 for more detailed information.

[20] We use the solutions of the MSE in order to choose the initial coefficient guesses. For the initialization of the transition we run non-linear regressions of (2.26) given the mean shock path, $\{\Omega^{\mathcal{M}}\}_{t1}^{t2}$. This yields rates of convergence speed, ν^{κ} and $\nu^{\Im}$, and initial guesses for the coefficients in (2.27).

[21] We use the corresponding MSE values in case of a stationary equilibrium respectively a random realization of the initial stationary equilibrium in case of the transition.

of simulations M^A compared to the brute force approach.

After convergence, the Euler equation errors of households are small with a maximum error of 0.005 while R^2 of all regressions are higher than 0.985.

2.B.2 Mean Shock Equilibrium

As an initialization step, we solve for a degenerate path of the economy where the realizations of all aggregate shocks are at their respective means. We accordingly set $\lambda = (z, \delta^K)$ to $\bar{\lambda} = (\bar{z}, \bar{\delta^K}) = (\mathbb{E}[z], \mathbb{E}[\delta^K])$. We assume that households accurately solve their forecasting problem for each realization of the aggregate state. This means that we approximate the above approximate law of motion as

$$(\kappa', \Im') = \hat{\hat{\Phi}}(t; \kappa, \tilde{K}, \Im, \bar{\lambda}, \bar{\lambda}') \tag{2.28}$$

Observe that in the two stationary equilibria of our model, that is in periods $t = 0$ and $t = T$, respectively, we have the fixed point relation

$$(\kappa', \Im') = \hat{\hat{\Phi}}(t; \kappa, \tilde{K}, \Im, \bar{\lambda}, \bar{\lambda}') = (\kappa, \Im) \tag{2.29}$$

With these assumptions, we can solve the mean shock path by standard Gauss-Seidel iterations as, e.g., described in Ludwig (2007). That is, we first solve for the steady state equilibria in periods $t = 0$ and $t = T$, respectively, and then compute the transitional dynamics between those steady states. While the numerical methods are the same as in the solution to a deterministic economy, the actual behavior of households fully takes into account the stochastic nature of the model. The fixed-point computed in this auxiliary equilibrium gives $\kappa_t^{\mathcal{M}}$, $\tilde{K}_t^{\mathcal{M}}$, and $\Im_t^{\mathcal{M}}$ as aggregate moments and cross-sectional distributions of agents as induced by the mean shock path. We denote these distributions by $\Omega_t^{\mathcal{M}}$.

The employed algorithm of the MSE determination iterates on the vector of aggregate (state) variables $\{\kappa, \Im\}_{t_1}^{t_2}$ until convergence (fixed point iteration) as follows. Again, $t_1 = t_2 = t$ in the case of a stationary equilibrium while $t_1 = 1$ and $t_2 = T - 1$ in the case of the transition:

1. Choose an initial guess for $\{\kappa, \Im\}_{t_1}^{t_2}$.
2. Solve the household problem (2.9) for policy functions $(m, \widehat{\alpha}^h, \widehat{\alpha}^s)$ from $t = t_2, ..., t_1$ for all (λ, j) of the respective grid.[22]
3. Iterate forward in time for $t = t_1, ..., t_2$ and iterate in each t on $\Im$ until the bond market in t clears:
    a) Choose an initial guess for $\Im$.
    b) Update the policy functions $(m, \widehat{\alpha}^h, \widehat{\alpha}^s)$ at t for all (λ, j) of the respective grid.
    c) Aggregate over all households in the mean shock state by interpolating on the policy functions and returns. Determine $\hat{\kappa}'$ and the aggregate excess demand on the bond market.
4. Update $\{\kappa, \Im\}_{t_1}^{t_2}$ according to $\{\kappa, \Im\}_{t_1}^{t_2 \, new} = \vartheta \cdot \{\kappa, \Im\}_{t_1}^{t_2 \, old} + (1 - \vartheta) \cdot \{\hat{\kappa}, \hat{\Im}\}_{t_1}^{t_2}$ where $0 < \vartheta < 1$ is an arbitrary adjustment factor.

[22] This step is not executed in case of a stationary equilibrium as step 3b solves for policy functions, anyway.

3

Computing Transitional Dynamics in Heterogeneous Agent Models with Aggregate Risk by Parameterized Laws of Motion

3.1 Introduction

Macroeconomic analyses use increasingly heterogeneous agent models with aggregate risk. I develop a solution method to compute the (stochastic) transitions of such economies if these transitions are induced by exogenous deterministic dynamics such as, e.g., a fundamental tax reform. Assuming that households forecast the evolution of aggregate variables by applying an aggregate law of motion as suggested by Krusell and Smith (1997, 1998) such a transition induces time-dependency of the aggregate law of motion. I here suggest a straightforward way to parameterize this dependency on time. My method is particularly easy to implement and combines ideas from Krusell and Smith (1997, 1998) with those of Judd (2002). The implementation of the method involves the following key steps:

First, I assume an initial and a final stationary state in periods $t = 0$ and $t = T$ respectively in each of which the exogenous deterministic process is time-constant. I solve for the aggregate laws of motion in those two stationary equilibria separately by applying the standard Krusell and Smith (1997, 1998) methodology. In the main step of the procedure, I specify parameterized laws of motion for the transition phase, $t = 1, ..., T - 1$, by multiplying all coefficients in the stationary equilibria with time polynomials. Thereby, I generalize ideas developed for the parameterization of deterministic time paths by Judd (2002) to the parameterization of laws of motion. I determine the values of the polynomial coefficients of these augmented laws of motion by simple regressions using the endogenous aggregate state variables of M^{TR} stochastic simulations of the model along the transition.

The major advantage of the new method is a particular small number of coefficients to be determined, depending on the degree of the employed polynomial. In the alternative standard brute force Krusell and Smith (1997, 1998) approach, in which each period in the transition features a separate law of motion, the coefficients of the laws of motion are identified solely by cross-sectional variation. Meanwhile, the coefficients of the time polynomials are identified by both, cross-sectional and time variation. Accordingly, the new method requires a much smaller number of simulations of the economy along the transition than the alternative brute force approach.

I illustrate the method using an overlapping generations (OLG) model with aggregate shocks in which a fundamental tax reform induces transitional dynamics. The set-up is extremely simple and features analytical solutions of the household problem. This is particularly useful when evaluating the accuracy of the method as all errors stem from approximations at the aggregate level of the model. However, these simplifications are without loss of generality of the method and, as I further discuss, the extension of the approach to more general models is straightforward. The quantitative experiment reveals a substantial reduction of total computing time by about 45% compared to the brute force approach. Euler equation errors as well as errors from one-period ahead and multi-periods ahead predictions of the aggregate state variable are very low and similar in size to the brute force approach.

There are various techniques to solve heterogeneous agents models with aggregate uncertainty. A good overview is given in Algan et al. (2014). Some of them, like Preston and Roca (2007), use only perturbation techniques while others, like Den Haan (1997) and Den Haan and Rendahl (2010), focus on projection methods. Meanwhile, Den Haan (1996), Krusell and Smith (1998), Algan, Allais, and Den Haan (2008), Reiter (2009), and Reiter (2010) merge several techniques. Den Haan (2010b) compares the different algorithms in terms of accuracy and computing time. The most prominent approach stems from Krusell and Smith (1997, 1998). The basic idea is to summarize the cross-sectional distribution of capital by a finite number of its moments and to set up a linear law of motion in these moments. Parameters of that function are determined by Monte Carlo Simulations of the model. The aforementioned literature has in common that it focuses on stationary economies. The most simple way to extend the Krusell and Smith (1997, 1998) method to economies under transition would be to set up a time-independent linear law of motion of the type described above. However, approximation errors are likely to rocket. Another straightforward option is to set up a time-dependent law of motion, for each period in the transition separately. This is what I call the *brute force* approach described above. Evidently, approximation errors should be low but the numerical implementation is tremendous for more complex models. On the contrary, this chapter introduces time-dependency of the aggregate law of motion in a parametric and potentially non-linear form. It mirrors the approach taken by Judd (2002) in a dynamic deterministic infinite horizon economy. He proposes to make use of the special structure of many equilibrium time paths by assuming that the dynamic path of endogenous variables is some explicit function of time.

The newly developed method in this chapter does not require other assumptions than the Krusell and Smith (1997, 1998) method to work. In this sense it is neither more nor less general than that. Both assume that the cross-sectional distribution of, e.g., wealth can be approximated by a small number of its moments. Applicability of the method has to be evaluated on the basis of resulting approximation errors in the respective case.

The remainder of the analysis proceeds as follows. Section 3.2 describes the details of the solution method. Section 3.3 presents the model used for illustrative purposes while section 3.4 contains the details on its quantitaive application. Results are presented in section 3.5. Finally, section 3.6 concludes.

3.2 Solution Method

3.2.1 Class of Models

I develop a solution procedure for a particular class of heterogeneous agent models with aggregate shocks in which transitional dynamics are induced by exogenous deterministic processes, e.g., by a fundamental tax reform, deterministic trends in productivity or demographic change. Those deterministic processes might come as fully anticipated shocks at some period sufficiently far in the future or as surprise shocks, i.e., zero probability events as considered in numerous studies on the effects of fundamental tax reforms. Heterogeneity may come in the form of intra-generational heterogeneity, e.g., due to idiosyncratic income shocks, inter-generational heterogeneity in age (as in standard OLG models) or a combination of both forms. Time in these models is discrete and runs from $t = 0, \ldots, \infty$. The computational solution of the model economy is only up to some period $T < \infty$ where a stationary equilibrium is assumed and verified. Initialization of the model economy in period 0 is assumed to be given by a stationary equilibrium. Aggregate uncertainty is represented by an event tree. The economy starts with some fixed event z_0, and each node of the tree is a history of exogenous shocks $z^t = (z_0, z_1, \ldots, z_t)$. The shocks are assumed to follow a Markov chain with finite support Z with n_z elements, strictly positive transition matrix Π, and marginal distribution, $\mathcal{Z}$. z denotes the corresponding exogenous state variable. Throughout, I also assume that quasi-aggregation applies so that it is possible to describe the aggregate dynamics of the economy by means of a small number of aggregate state variables.

Equilibrium in this class of models is defined as a sequence of factor prices and allocations such that markets clear while optimal decisions and aggregation conditions have to hold in all periods, given the exogenous deterministic and stochastic processes.

3.2.2 Parameterized Laws of Motion

I start by considering the specification of aggregate laws of motion as in Krusell and Smith (1997, 1998). Assume that, in the initial stationary equilibrium the state-z dependent law of motion for the endogenous aggregate state variable y_i, $i = 1, \ldots, n_i$, is given by

$$\ln y_{i,t+1} = \phi_{i,i,0}(0, z) + \sum_{l=1}^{n_l} \sum_{p=1}^{n_p} \phi_{i,l,p}(0, z) \cdot \ln y_{l,t+1-p}. \tag{3.1}$$

This law of motion is the actual aggregate law of motion of the economy up to period 0. In this notation, the various sub-indices and arguments read as follows: Coefficient $\phi_{i,l,p}(0, z)$ is the effect of variable l on variable i in period 0 and state z at lag p. It is the respective constant for $p = 0$ and $l = i$. Notice that I do not consider higher order terms or interaction terms at each time lag p in the above specification but such generalizations are of course possible. Also notice that most applications of the standard Krusell-Smith method restrict $n_p = 1$.

Correspondingly, the law of motion in the final stationary equilibrium reached after T periods is assumed to be given by

$$\ln y_{i,t+1} = \phi_{i,i,0}(T, z) + \sum_{p=1}^{n_p} \sum_{l=1}^{n_l} \phi_{i,l,p}(T, z) \cdot \ln y_{l,t+1-p}. \tag{3.2}$$

Along the transition in the periods $t = 1, 2, ..., T - 1$, I borrow ideas from Judd (2002) and consider parameterized laws of motion using a trans-log function:

$$\begin{aligned} \ln y_{i,t+1} = & \exp(-\nu_i \cdot t) \cdot \left(\phi_{i,i,0}(0, z) \cdot P_{i,i,0}(t, z) + \sum_{p=1}^{n_p} \sum_{l=1}^{n_l} \phi_{i,l,p}(0, z) \cdot P_{i,l,p}(t, z) \cdot \ln y_{l,t+1-p} \right) \\ & + (1 - \exp(-\nu_i \cdot t)) \cdot \left(\phi_{i,i,0}(T, z) + \sum_{p=1}^{n_p} \sum_{l=1}^{n_l} \phi_{i,l,p}(T, z) \cdot \ln y_{l,t+1-p} \right) \end{aligned} \tag{3.3}$$

Here, ν_i is the coefficient that determines the speed of convergence of the law of motion of variable i to the law of motion in the final stationary equilibrium of the economy. $P_{i,l,p}(t, z)$ are flexible global time polynomials of Chebyshev's first kind writing as:

$$P_{i,l,p}(t,z) = \sum_{q=0}^{n_q} \psi_{i,l,p,q}(z) \cdot \mathcal{T}_q(t) \tag{3.4}$$

where $\mathcal{T}_0(t) = 1,\ \mathcal{T}_1(t) = t,\ \mathcal{T}_q(t) = 2 \cdot t \cdot \mathcal{T}_{q-1}(t) - \mathcal{T}_{q-2}(t)\ \forall q \geq 2$, and n_q is the order of the polynomial.

3.2.3 Implementation Steps

The solution procedure targets at determining the coefficients in (3.1) – (3.4) using Monte Carlo methods and quasi-aggregation.

As an initializing step it involves the solution of the equilibrium of a degenerate version of the model which I refer to as the *mean shock path*. More precisely, the equilibrium results from a degenerate stochastic shock process where all shocks, $\{z_t\}_{t=0}^T$, are set to their respective unconditional means. The solution for this path is by standard procedures and easy to implement. It does not require the solution of policy functions on grids for the aggregate endogenous state variables because their evolution is assumed to be known to the agents. However, of course it requires the solution of the policy functions on the exogenous states t, z, and j. This means that I use the policy functions of a stochastic economy but consider a degenerate sequencing of events for the actual shocks which allows me to solve for an initial and a final steady state and the transition in-between. In the case that the unconditional mean of z, $\mathbb{E}[z]$, is not an element of Z this requires interpolation of policies on z in the aggregation step. This applies to standard finite state Markov processes with n_z even. Notice that, using the mean shock path instead of the deterministic path of the economy means that I fully account for the effects of risk on the agents' decisions. For example, I account for the effects of precautionary savings on the aggregate capital stock which shifts its entire time profile upward relative to the deterministic model version.

The following list provides details on the full solution procedure:

1. *Mean Shock Path*:

   Solve for the initial and final steady states and the transitional dynamics between them of an economy with a degenerate shock process where all shocks are set to their respective unconditional means given by $\mathbb{E}[z] = Z \circ \mathcal{Z}$, where $\circ$ denotes the operator symbol for the Hadamard product. I refer to the resulting time path of the endogenous aggregate (state) variables as $\{y_{i,t}^{\mathcal{M}}, y_{l,t}^{\mathcal{M}}\}_{t=0}^T$ for all $i = 1, ..., n_i$ and $l = 1, ..., n_l$. Solution of this path is by the assumption that the aggregate laws of motion along are exactly known to the households.

2. *Time Dependent Grids*:

   Determine time specific grids, $\mathcal{G}_{l,t}$ for all variables y_l, $i = 1, ..., n_l$, by specifying appropriate bounds around $\{y_{l,t}^{\mathcal{M}}\}_{t=0}^T$.

3. *Stationary Equilibria*:

   Apply a standard Krusell and Smith (1997, 1998) method to determine the coefficients, $\phi_{i,l,p}(0,z)$ and $\phi_{i,l,p}(T,z)$, for all i, l, p, z, of the aggregate laws of motion in the initial and the final stationary equilibrium respectively. I do so by M^{SS} stochastic simulations of each stationary equilibrium and by fixed point iterations to pin down the coefficients using standard first- or second-order methods.

4. *Rates of Convergence Speed*:

Given $\phi_{i,l,p}(0,z)$ and $\phi_{i,l,p}(T,z)$ for all i,l,p,z, determine the rates of convergence speed, ν_i, in the laws of motion (3.3), for all i, by non-linear regressions using the mean shock path, $\{y_{i,t}^{\mathcal{M}}, y_{l,t}^{\mathcal{M}}\}_{t=0}^{T}$ for all i and l.[1] Note that this also yields coefficient estimates of the time polynomials, $\psi_{i,l,p,q}(z)$ for all i,l,p,q,z, which I use as an initial guess of the next solution step.

5. *Transitional Dynamics*:

   This is the key step of the procedure. Given $\phi_{i,l,p}(0,z)$, $\phi_{i,l,p}(T,z)$, and ν_i for all i,l,p,z, solve for the coefficients, $\psi_{i,l,p,q}(z)$ for all i,l,p,q,z, of the time polynomials in equation (3.4) by the following procedure:

   a) Given $\psi_{i,l,p,q}(z)$ for all i,l,p,q,z, solve the policy functions at all points in the grids $\mathcal{G}_{l,t}$ for all l,t.
   b) Simulate the economy M^{TR} times along the transition of T-1 periods and obtain $\{y_{i,t}\}_{t=1}^{T}$ for all i and $\{y_{l,t}\}_{t=0}^{T-1}$ for all l.
   c) Regress $\{y_{i,t}\}_{t=1}^{T}$ on $\{y_{l,t}\}_{t=0}^{T-1}$ according to equation (3.3) and obtain $\hat{\psi}_{i,l,p,q}(z)$ for all i,l,p,q,z.
   d) If $\parallel \psi_{i,l,p,q}(z) - \hat{\psi}_{i,l,p,q}(z) \parallel < \epsilon$, where ϵ is some convergence criterion, stop. Else, update $\psi_{i,l,p,q}(z)$ in step 5a using standard first- or second-order methods for fix point procedures.

6. *Error Evaluation*:

   Given the policy functions at all grid points determined in the precedent step evaluate the approximation errors in M^{EE} stochastic simulations along the transition. Compute Euler equation errors as well as prediction errors of the aggregate variables $y_{i,t}$ for all periods t and report the maximum errors.[2]

3.2.4 Comparison to Brute Force Approach

The most obvious alternative approach to assuming parameterized laws of motion in time given by equation (3.3) is a brute force Krusell-Smith approach. This is an extrapolation of the Krusell-Smith method from stationary equilibria to the transition phase in the sense that each single period $t \in (1,2,...,T-1)$ of the transition is characterized by a separate aggregate law of motion (cf. (3.1) and (3.2)). In the following, I discuss both, the properties of and the conditions for parameterized time laws to be particularly useful in comparison to the brute force approach. Henceforth, I call the former approach *PTLM* and the latter *BFKS*.

Implementation of BFKS follows the same steps[3] as PTLM (cf. section 3.2.3) with the fundamental difference being the number of coefficients to be determined by step 5 of the procedure. BFKS involves the determination of one coefficient ($\phi_{i,l,p}(t,z)$) for each i,l,p,z, **and t**, amounting to $(n_i \times n_l \times n_p \times n_z \times T-1)$ coefficients, while PTLM implies one coefficient ($\psi_{i,l,p,q}(z)$) for each i,l,p,z, **and q**, i.e. $(n_i \times n_l \times n_p \times n_z \times n_q)$ coefficients. Note that, in general, $(n_i \times n_l \times n_p \times n_z \times T-1) \gg (n_i \times n_l \times n_p \times n_z \times n_q)$ due to the large number of periods in the transition[4], $T-1$, compared to the rather small order, n_q, of the polynomials.

[1] In the quantitative experiment of section 3.4, I use a scaled version of the mean shock path exploiting information from the stationary equilibria determined in step 3 of the procedure. Please see appendix 3.B.1 for details.

[2] A detailed description of the error evaluation will be given in section 3.4.2.

[3] Note that while step 4 is not strictly necessary in the BFKS procedure it implicitly determines the coefficients of the aggregate laws of motion of all time periods in the transition. As computational costs of step 4 are negligible and for the sake of valid comparison to PTLM I use those coefficients as an initial guess in step 5 of the BFKS procedure.

[4] The number of periods in the transition, $T-1$ is set exogenously such that convergence of the coefficients to the final stationary equilibrium is assured. It does not vary by method for the sake of valid comparison.

There are two sources of potential differences in computing time across the two alternative approaches: 1) The number of simulations, M^{TR}, used in step 5b and, 2), the number of iteration steps until convergence in the coefficients in step 5d in the previous section. As I will show for an illustrative example in section 3.5, the main advantage of PTLM compared to BFKS lies in source 1) making it eminently useful in applications where simulation of the model is particularly computationally costly. The smaller number of simulations needed is possible due to the aforementioned smaller number of coefficients to be determined by PTLM. Imposing identical degrees of freedom in the regression step of either method, a smaller number of coefficients affords a smaller number of total observations which translates into a smaller M^{TR}.

3.3 Model

This section develops the model economy used for illustration and evaluation of the PTLM method in section 3.5. I consider a particularly simple setup with inter-generational heterogeneity only in which an analytical solution for policy functions exists. Thereby, errors of approximation of the rational expectations solution arise solely from the aggregate level by parameterized laws of motion and interpolation which is not related to the specific details of the economic model.

3.3.1 Exogenous Shock Process

There is only one shock in the economy, a productivity shock z that follows a time invariant Markov chain with transition probabilities $\pi(z'|z)$. I consider two realizations of the shock, $z \in \{z_l, z_h\}$.

3.3.2 Demographics

Households enter the model at $j = 0$ and live (without mortality risk) until some maximum age J. Hence, the economy is populated with $J + 1$ overlapping generations. All generations have equal size and the total population is constant in time, i.e., $N_0 = N_1 = \ldots = N_J$ for all t.

3.3.3 Production

Production takes place with a non-standard production function where capital is the only input:

$$Y_t = z_t \cdot K_t^{\alpha}. \tag{3.5}$$

K_t is the aggregate stock of physical capital, $0 < \alpha < 1$ governs decreasing marginal productivity of capital. Assuming that capital depreciates at the rate δ^K, profits are given by

$$\Xi_t = Y_t - (r_t^K + \delta^K) \cdot K_t = z_t \cdot K_t^{\alpha} - (r_t^K + \delta^K) \cdot K_t \tag{3.6}$$

and profit maximization implies that

$$r_t^K = \alpha \cdot z_t \cdot K_t^{\alpha-1} - \delta^K. \tag{3.7}$$

Equilibrium profits are accordingly given by

$$\Xi_t = (1 - \alpha) \cdot z_t \cdot K_t^{\alpha}. \tag{3.8}$$

3.3.4 Preferences

The life-time utility function of households is given by

$$\mathbb{E}_t \sum_{j=0}^{J} \beta^j \cdot u(c_{t+j,j}) \tag{3.9}$$

where $\mathbb{E}$ is the expectations operator and expectations are taken with respect to the technology shock z. β is the raw time discount factor and $c_{t,j}$ is consumption at time t, age j.

The per period utility function is CRRA, hence

$$u(c_{t,j}) = \begin{cases} \frac{1}{1-\theta} \cdot c_{t,j}^{1-\theta} & \text{if } \theta \neq 1 \\ \ln(c_{t,j}) & \text{if } \theta = 1 \end{cases} \tag{3.10}$$

where θ is the coefficient of relative risk aversion.

3.3.5 Endowments

When entering the economy at age $j = 0$ households are endowed with no initial assets, i.e., $k_{t,0} = 0$ for all periods t, but, unlike all other generations, they receive a transfer from the government which is financed by taxing profits at a confiscatory rate[5],

$$e_{t,0} = \frac{\Xi_t}{N_0}. \tag{3.11}$$

The dynamic budget constraint and the respective transversality condition are given by

$$k_{t+1,j+1} = k_{t,j} \cdot R_t + e_{t,j} - c_{t,j} \tag{3.12}$$

$$k_{t,J+1} = 0 \tag{3.13}$$

, where $R_t := (1 + r_t^K \cdot (1 - \tau_t^K))$ is the gross after tax interest rate and τ_t^K denotes the capital income tax rate. Note that $e_{t,j} = 0$ for all $j \in (1, 2, ..., J)$ and all t.

3.3.6 Recursive Household Problem

I define the household problem recursively conditional on a law of motion of the aggregate state of the economy. Rather than using $k_{t,j}$ as the individual state variable, it is convenient to solve the household problem in terms of total resources available. Let $x_{t,j} = k_{t,j} \cdot R_{t,j} + e_{t,j}$ be total resources, or, alternatively, "cash-on-hand" (Deaton, 1991). Observe that then[6]

$$x_{t+1,j+1} = (x_{t,j} - c_{t,j}) \cdot R_{t+1,j+1}. \tag{3.14}$$

I express next period's values with symbol $'$, irrespective of whether they are only time dependent or both, age and time dependent. The states of the household problem are the exogenous states t, j and z, the endogenous cash-on-hand of the household, x, as well as the endogenous aggregate state of the economy, Ω, with

[5] Please note, that $e_{t,0}$ is not part of the capital stock, K_t.

[6] Please note that this expression holds for all $j \in (0, 1, ..., J)$ as $e_{t,j} = 0$ for all $j \in (1, 2, ..., J)$ in all periods t.

associated law of motion $\Omega' = \Phi(\Omega, z, z')$. The household problem in period t, age j is then given by

$$v(x; z, t, j; \Omega) = \max_{c,x'} \left\{ u(c) + \beta \cdot \mathbb{E}[v(x'; z', t+1, j+1; \Omega')] \right\} \tag{3.15}$$

subject to

$$\begin{aligned} x' &= (x - c) \cdot R' = (x - c) \cdot (1 + r^{K\prime} \cdot (1 - \tau^{K\prime})) \\ \Omega' &= \Phi(\Omega, z, z'). \end{aligned}$$

The expectation $\mathbb{E}$ above is taken with respect to the realization of tomorrow's productivity shock, z'.

Using standard results, cf., e.g., Samuelson (1969), I can next state the following property of the optimal consumption policy functions:

Proposition 2. *The optimal age-dependent consumption function is linear in cash-on-hand,*

$$c = m \cdot x,$$

whereby the marginal propensity m to consume out of cash-on-hand x is given by

$$m := \frac{(\beta \cdot \wp)^{-\frac{1}{\theta}}}{1 + (\beta \cdot \wp)^{-\frac{1}{\theta}}} \quad \text{, where} \quad \wp := \mathbb{E}[m'^{-\theta} \cdot R'^{1-\theta}].$$

Proof. See section 3.A.1 in the appendix. □

The linearity of consumption in cash-on-hand results from the homotheticity of preferences and greatly simplifies the numerical solution of the model economy.

3.3.7 Government

The role of the government in this model is twofold. First, the government taxes profits at a confiscatory rate and redistributes them to newborn households in a lump-sum fashion, cf. equation (3.11). Second, the government taxes capital income thereby financing government consumption,

$$G_t = \tau_t^K \cdot r_t^K \cdot \sum_{j=0}^{J} k_{t,j} \cdot N_j \tag{3.16}$$

which is otherwise neutral.

3.3.8 Equilibrium

Equilibrium in the economy is defined recursively and requires market clearing in all periods, while optimal decisions and aggregation conditions have to hold. Individual households, at the beginning of period t are indexed by their age j and their cash-on-hand holdings x. As consumption policies are linear in x it suffices to keep track of the beginning of period distribution of wealth over age, Ω. The existence of aggregate shocks implies that Ω evolves stochastically over time. I use Φ to denote the law of motion of Ω which is given by

$$\Omega' = \Phi(\Omega, z, z') \tag{3.17}$$

. Notice that z' is a determinant of Ω' because it specifies $r^{K\prime}$ and Ξ'. A change in the tax rate, $\tau^{K\prime}$, induces a transition of the economy from an initial stationary equilibrium to another. In analogy to models without aggregate risk, the aggregate law of motion is time-dependent. Therefore, the recursive equilibrium of the economy is defined as follows:

A recursive competitive equilibrium is a value function, $v(j, x, z, \Omega)$, and a policy function, $m(j, z, z', \Omega')$, for the household, a policy function, $K(z, \Omega)$, and pricing functions, $r(z, \Omega)$ and $\Xi(z, \Omega)$, for the firm, policy functions, τ^K and $G(z, \Omega)$, for the government, the demographic distribution, N, the wealth distribution, Ω, and its associated (aggregate) law of motion, $\Phi(\Omega, z, z')$, such that for all (z, Ω)

1. $v(\cdot)$, $x(j, z, \Omega)$, $c(j, x, z, \Omega)$, $m(\cdot)$ are measurable, $v(\cdot)$ satisfies a household's recursive problem, and $m(\cdot)$ is the associated policy function, given
   $\mathbb{E}[r'(z', \Phi(\Omega, z, z'))]$, $\mathbb{E}[\Xi'(z', \Phi(\Omega, z, z'))]$, and $x(\cdot)$, following from

   $$m = \frac{(\beta \cdot \wp)^{-\frac{1}{\theta}}}{1 + (\beta \cdot \wp)^{-\frac{1}{\theta}}}$$

   , where $\wp := \mathbb{E}[m'^{-\theta} \cdot R'^{1-\theta}]$ and $R' := 1 + r^{K\prime} \cdot (1 - \tau^{K\prime})$. Further

   $$c = m \cdot x;$$

2. $K(\cdot)$ satisfies, given z,

   $$r^K = \alpha \cdot z \cdot K^{\alpha-1} - \delta^K \tag{3.18}$$

   $$\Xi = (1 - \alpha) \cdot z \cdot K^{\alpha}; \tag{3.19}$$

3. $G(\cdot)$ satisfies, given τ^K,

   $$G = \tau^K \cdot r^K \cdot \sum_{j=0}^{J} k_j \cdot N_j; \tag{3.20}$$

4. aggregation over all households yields

   $$K' = \sum_{j=0}^{J} N_j \cdot (1 - m_j) \cdot x_j \tag{3.21}$$

   $$C = \sum_{j=0}^{J} N_j \cdot m_j \cdot x_j \tag{3.22}$$

5. the aggregate resource constraint,

   $$C + G + I = Y \tag{3.23}$$

   holds;

6. the aggregate law of motion, Φ, satisfies

   $$\Omega' = \Phi(\Omega, z, z') \tag{3.24}$$

. It is generated by the exogenous processes for technology and the tax rate as well as the endogenous asset accumulation decisions as captured by policy functions;

7. the initial endowment of newborn households equals the transfer from the government;

$$e_0 = \frac{\Xi}{N_0}, \tag{3.25}$$

8. the transition matrix of the exogenous technology process (time invariant Markov chain) consists of the probabilities $\pi(z'|z)\ \forall z, z' \in \{z_l, z_h\}$.

Definition 2. *A stationary recursive competitive equilibrium is a special case of the equilibrium described above in which policy functions, $m(\cdot)$, $K(\cdot)$, $G(\cdot)$, τ^K, and the aggregate law of motion, Φ, are constant in time.*

3.4 Quantitative Application

3.4.1 Illustrative Experiment and Laws of Motion

The economy starts in an initial stationary equilibrium with a zero capital income tax rate, $\tau_0^K = 0$. This is the equilibrium that the economy is in up to and including period 0. I consider the following experiment through which I induce a trivial transition of the economy: The economy is hit by a surprise shock in period 1 which is characterized by an exogenous increase of the capital income tax rate to $\tau_t^K = 0.25$ for all periods $t = 1, \ldots, T$.[7]

I follow Krusell and Smith (1997) and many others and take the capital stock as the only relevant endogenous state variable at the aggregate level of the considered economy. The state z-dependent laws of motion in the initial and the final stationary equilibrium respectively are given by

$$\ln K_{t+1} = \phi_0(0, z) + \phi_1(0, z) \cdot \ln K_t \tag{3.26a}$$

$$\ln K_{t+1} = \phi_0(T, z) + \phi_1(T, z) \cdot \ln K_t. \tag{3.26b}$$

During the transition, I consider the state z-dependent parameterized time law of motion, cf. equation (3.3), and choose order $n_q = 0$ for the Chebyshev polynomials of equation (3.4) leading to

$$\begin{aligned} \ln K_{t+1} = {} & \exp(-\nu \cdot t) \cdot (\phi_0(0, z) \cdot \psi_0(z) + \phi_1(0, z) \cdot \psi_1(z) \cdot \ln K_t) \\ & + (1 - \exp(-\nu \cdot t)) \cdot (\phi_0(T, z) + \phi_1(T, z) \cdot \ln K_t) \,. \end{aligned} \tag{3.27}$$

Given ν I only have to estimate four parameters in the main step of the solution procedure (cf. section 3.2.3): $\psi_p(z)$ for all $p \in (0, 1)$, $z \in (z_l, z_h)$. This is the major advantage compared to the BFKS approach in which each period of the transition has a separate law of motion of the form (3.26) amounting to a total number of 1600 coefficients (cf. section 3.2.4).

I evaluate the accuracy of the PTLM method by running an error evaluation as explained in section 3.4.2 and comparing errors to the BFKS method. Note that throughout, I solve households' policy functions on a grid for the physical capital stock with n_k nodes, and interpolate between the solutions at those nodes in the

[7] As an alternative experiment, one may assume a fully anticipated shock in the sense that the tax change takes place in some period $t_0 > 1$ which is sufficiently far in the future so that anticipation effects do not move the economy away from the initial law of motion already in period 1.

aggregation step of the respective procedure. Section 3.B.1 in the appendix presents the details on all solution steps involved. Resulting transitional dynamics, corresponding computing time and approximation errors are discussed in detail in section 3.5.

3.4.2 Error Evaluation

I evaluate the quality of the approximation of the aggregate law of motion by reporting two types of errors. Let $m = 1, \dots, M^{EE}$ be the stochastic simulations of the economy. In each simulation, let $\{K_{t,m}\}_{t=1}^{T}$ be the simulated time path, $K_{1,m}$ given. Furthermore, let $\hat{K}_{t,m}^{j}$ denote the j-periods ahead prediction of the aggregate capital stock in simulation m starting predictions from $K_{t,m}$ in period t. I do so by iterating on the aggregate laws of motion of the respective method under evaluation. As a first measure, I compute the maximum prediction errors across simulations at forecasting horizons 1 and J periods, PE_t^1 and PE_t^J, for each t given by

$$PE_t^i = \max \left\{PE_{t,m}^i\right\}_{m=1}^{M^{EE}} = \max \left\{ \max \left\{ \left\| \frac{\hat{K}_{t,m}^j}{K_{t,m}^j} - 1 \right\| \right\}_{j=1}^{i} \right\}_{m=1}^{M^{EE}} \tag{3.28}$$

where i denotes the prediction horizon. Thereby, I follow the call of Den Haan (2010a) for evaluating the prediction power of laws of motion for more than one period ahead. Note that I choose the forecasting horizon J as this is the maximum remaining lifetime of all households alive in period t. Observe further, that these prediction errors are independent of potential errors in the solution of policy functions due to the selected structure of the individual household problem. Along with prediction errors, I report the R^2 of regressions which is the standard measure in the literature containing similar information to PE_t^1.

As a second measure, I compute standard Euler equation errors. Those are also solely caused by the approximation of the aggregate law of motion because, again, the household problem exhibits analytical solutions. Let $c_{t,j,m}$ be period t consumption of a household of age j in simulation m. The maximum Euler equation error in period t is

$$EE_t = \max \left\{EE_{t,m}\right\}_{m=1}^{M^{EE}} = \max \left\{ \max \left\{ee_{t,j,m}\right\}_{j=0}^{J-1} \right\}_{m=1}^{M^{EE}} \tag{3.29}$$

where j denotes the age of the household and

$$ee_{t,j,m} = \left\| 1 - \beta \cdot \frac{\mathbb{E}\left[(1 + r_{t+1,m}^K) \cdot (1 - \tau_{t+1}^K) \cdot u_c(c_{t+1,j+1,m})\right]}{u_c(c_{t,j,m})} \right\|. \tag{3.30}$$

For each t, j, and m, I directly observe $K_{t,m}$ as well as the denominator of the fraction in (3.30). Meanwhile, I compute the corresponding numerator by applying the aggregate law of motion and computing the expectation over all possible future states of technology, $z_{t+1,m}$, conditional on being in state $z_{t,m}$.

3.4.3 Calibration

Parameters of the model economy are summarized in table 3.1. The only parameter that requires commenting on is the elasticity of production with respect to capital, α. It is set such that the rate of return to capital in the initial stationary equilibrium equals 7%.

Table 3.1: Parameters

Parameter	Value
Household Sector	
Discount factor, β	0.9
Coefficient of relative risk aversion, θ	2.0
Technology	
Capital share, α	0.672
Support of Markov process, $\{z_l, z_h\}$	$\{0.99, 1.01\}$
Transition matrix, Π	$\{0.9, 0.1; 0.1, 0.9\}$
Time and Maximum Age	
Time horizon, T	401
Maximum model age, J	60
Government	
Capital income tax rate before reform, τ_0^K	0.0
Capital income tax rate after reform, τ_t^K $\forall t = 1, 2, ..., T$	0.25

Source: Model economy. *Notes:* α is set such that the rate of return to capital in the initial stationary equilibrium equals 7%. All other parameter values are chosen exogenously.

3.5 Results

This section illustrates the properties of the solution approach (PTLM) by presenting results of the experiment described in section 3.4.1. I show the advantages and disadvantages of PTLM by comparing its computing time and approximation errors to the according measures when applying the alternative brute force approach (BFKS) as described in section 3.2.4. For both methods, I employ a simple Gauss-Seidel algorithm described in, e.g., Ludwig (2007) when iterating on the coefficients of the respective laws of motion in the main step 5 of the procedure (cf. section 3.2.3) which showed to be quicker than second-order methods. For the sake of a valid comparison of computing times I choose consistent starting values across methods. Note that the description of results focuses on the transition phase while figures also depict values from initial and final stationary equilibria for comparative purposes. Results of the preparatory steps 1 to 4 of the solution procedure which are identical across methods and all computational details can be found in section 3.B.1.

Recall from sections 3.2.4 and 3.4.1 that the main potential advantage of PTLM over BFKS is the smaller number of simulated time series needed for the determination of the coefficients in the respective aggregate laws of motion. The number of coefficients to be determined in the BFKS method equals 1600. Therefore, I set the number of simulated time paths to $M^{TR} = 50,000$ which is at the upper end of typical values considered in the literature.[8] Imposing the same degrees of freedom in the regression step of the PTLM method would require only $M^{TR} = 50,000 \cdot 4/1600 = 125$ simulated time paths for the determination of its four coefficients. However, such a small number of simulations would neither assure the unbiasedness of initial conditions[9] nor of the stochastic technology process. Therefore, I set M^{TR} to $5,000$ when applying the PTLM method. This leaves me with a reduction of simulated time series in the amount of 90% compared to the BFKS approach.

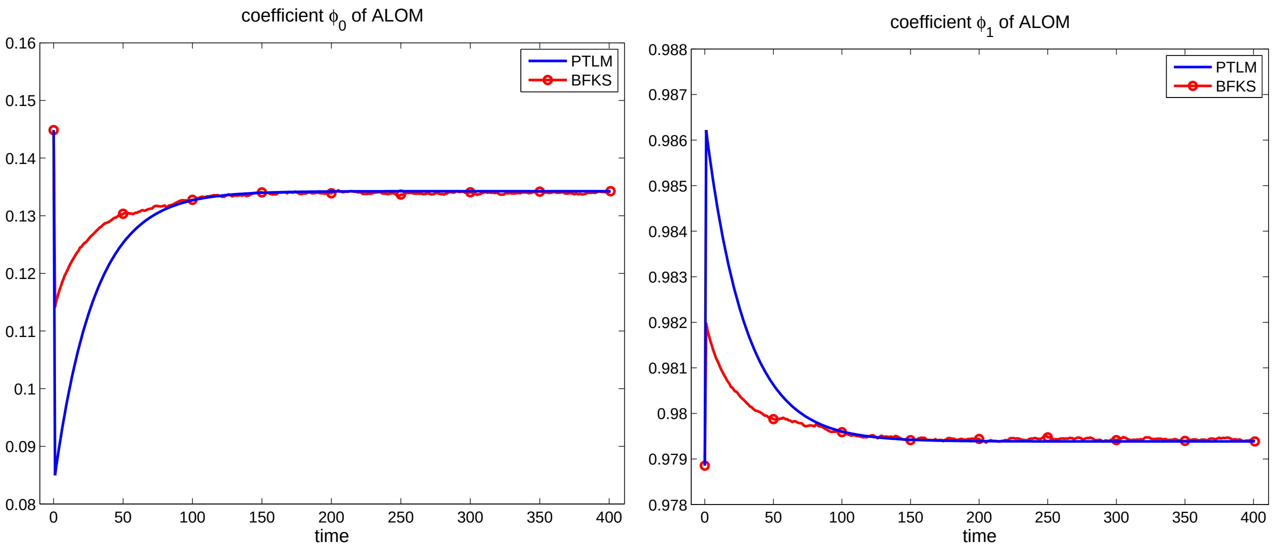
[8] E.g., Krusell and Smith (1998), Gomes and Michaelides (2008), Storesletten, Telmer, and Yaron (2007), and Harenberg and Ludwig (2015) consider $10,000$, $5,500$, $20,000$, and $36,000$ simulations respectively when computing stationary equilibria. Santos and Peralta-Alva (2005) elaborate on laws of large numbers for Markovian stochastic processes.

[9] Initial conditions are given by the combination of the initial capital stock, K_1, wealth distribution, $\{k_{1,j}\}_{j=0}^{J}$, and technology state, z_1.

3.5.1 Aggregate Laws of Motion

Figure 3.1 shows resulting coefficients of the aggregate law of motion (ALOM) of the respective method exemplary in the low productivity state, z_l.[10] More precisely, the left panel of figure 3.1 depicts $\phi_0(t,z)$

Figure 3.1: ALOM Coefficients, $\phi_0(t,z_l)$ and $\phi_1(t,z_l)$, under PTLM and BFKS



Source: Solution of the model economy under PTLM (blue solid lines) and BFKS (red lines with dots). *Notes:* The figure shows resulting coefficients of the aggregate law of motion exemplary in the low productivity state, z_l. Graphs contain the values from initial and final stationary equilibria. They show a jump on impact ($t = 1$) and a smooth transition to the final stationary equilibrium while effects differ quantitatively by method.

while the right panel shows $\phi_1(t,z)$ for all $t \in (0,...,T)$, cf. equation (3.26). Note that the construction of $\phi_p(t,z)$, $p \in (0,1), t \in (1,...,T-1)$ for the PTLM method follows directly from equation (3.27).[11] While exhibiting quantitative differences both methods show the same qualitative pattern, in that the coefficients jump on impact (period $t = 1$) and transit smoothly to the values of the final stationary equilibrium thereafter.

3.5.2 Computing Time

Next, I analyze the computing times of the two methods under consideration. The center column of table 3.2 shows computing times for the key step 5 of the solution procedure (cf. sections 3.2.3 and 3.B.1). This is the only source of a potential difference in total computing time across methods. The numbers show that PTLM is the much quicker method consuming only about 10% of the time that BFKS requires. This is almost perfectly identical to the ratio of simulated time series under the two methods which confirms the proposition of section 3.2.4. Those time advantages in the key step of the solution procedure translate also into a smaller total computing time of PTLM including all identical solution steps across methods (cf. sections 3.2.3 and 3.B.1). This is shown in the right column of figure 3.2. Naturally, the total time advantage of PTLM is smaller than its counterpart in the key solution step only. In the relatively simple model under consideration total

[10] Coefficients in the high technology state, z_h, show similar dynamics and approximation degrees. They are not shown here for the sake of brevity.

[11] Equation (3.27) rewrites as $\ln K_{t+1} = \phi_0(t,z) + \phi_1(t,z) \cdot \ln K_t$ where

$$\phi_p(t,z) = \exp(-\nu \cdot t) \cdot \phi_p(0,z) \cdot \psi_p(z) + (1 - \exp(-\nu \cdot t)) \cdot \phi_p(T,z) \qquad (3.31)$$

for all $t \in (1,...,T-1), p \in (0,1), z \in (z_l, z_h)$.

Table 3.2: Computing Time

	only differing key step 5	*total*
PTLM	233.2	2452.3
BFKS	2236.5	4455.6
PTLM/BFKS	10.4 %	55.0 %

Source: Solution of model economy under PTLM and BFKS. *Notes:* Computing time is reported in seconds. The center column shows computing time of the key step 5 of the solution procedure while excluding computing time for all remaining solution steps which are identical across methods. The right column shows the total computing time under the respective method. *PTLM* uses $M^{TR} = 5,000$ stochastic time series while *BFKS* uses $M^{TR} = 50,000$ stochastic time series. This is the main source of time advantage.

time advantage amounts to a substantial level of 45%. This number might even increase either in the case of a more complex transition which requires the computation of a longer transition period or in the case of a more complex model. Consequently, the weight of step 5 in the total computing time would rise boosting the time advantage of PTLM over BFKS. However, more complex transitions or models might also afford the application of higher order polynomials in PTLM in order to keep approximation errors low. This increases the number of simulations in PTLM and depresses its time advantage over BFKS. Hence, the effect of the complexity of the model and the transition under consideration on the total time advantage of PTLM remains indeterminate.

Computational tractability plays a key role in macroeconomic analyses using heterogeneous agent models. This suggests that the PTLM approach potentially shifts out the feasibility bound of computing transitional dynamics in heterogeneous agent models with aggregate risk using Monte Carlo methods. The application in chapter 2 gives rise to this hope.

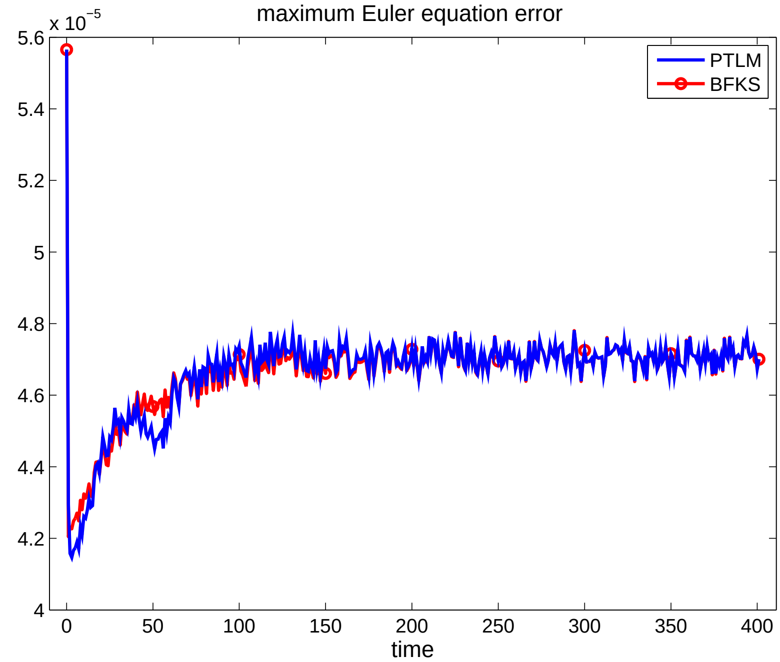
3.5.3 Approximation Errors

Finally, I evaluate the errors arising from approximating the aggregate law of motion by the respective method based on the evaluation measures presented in section 3.4.2. Therefore, I simulate the economy $M^{EE} = 50,000$ times for each of the two methods. Figure 3.2 shows resulting Euler equation errors. The graphs reveal that Euler equation errors are very low throughout and do not exceed 0.005% in the transition of the model economy for either method. This points to a very good level of approximation by both methods. The error level drops on impact ($t = 1$) and follows a smooth transition to the level of the final stationary equilibrium thereafter. Note that Euler equation errors are almost identical across methods.[12]

A similar conclusion can be drawn from the evaluation of prediction errors both, with a forecasting horizon of 1 and up to J periods ahead. Errors are defined as relative deviations of the predicted capital stocks from their realized counterparts, cf. equation (3.28). The left panel of figure 3.3 reveals that the maximum error of 1-period ahead predictions during the transition is very small and does not exceed 0.025% for either method. This coincides with very high values of the minimum R^2 measure of the respective regressions equaling 0.999999 (PTLM) and 0.999972 (BFKS). Meanwhile, prediction errors with a forecasting horizon of up to J periods are one order of magnitude higher. This is shown in the right panel of figure 3.3 and concerns in particular the first 150 periods in the transition. While quantitative differences across methods are still

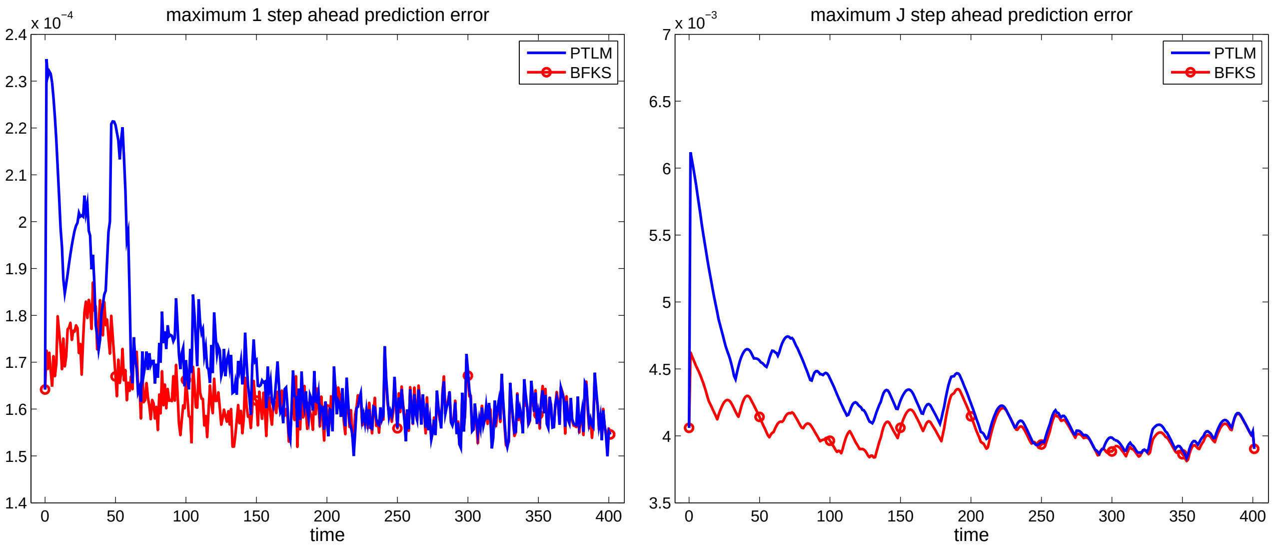
[12] I use the same initial conditions and the same stochastic time series of technology shocks for both methods for the sake of valid comparison across methods.

Figure 3.2: Maximum Euler Equation Error



Source: Evaluation of approximation errors under PTLM (blue solid line) and BFKS (red line with dots). *Notes:* Graphs show the maximum Euler equation errors of all agents alive in the respective period and contain the values from initial and final stationary equilibria. Both lines exhibit a drop in the error level on impact ($t = 1$) and a mostly smooth transition to the error level of the final stationary equilibrium thereafter. The absolute error level is very low and errors differ just negligibly across methods. Errors are evaluated by simulating $M^{EE} = 50,000$ time series each.

Figure 3.3: Maximum Errors of 1- and up to J-Periods Ahead Predictions



Source: Evaluation of approximation errors under PTLM (blue solid lines) and BFKS (red lines with dots). *Notes:* Graphs show the maximum errors of predicting the aggregate capital stock 1 (left panel) and J (right panel) periods ahead in the respective period, cf. equation (3.28). Errors are defined as relative deviations of the predicted capital stocks from their realized counterparts. Graphs contain the values from initial and final stationary equilibria. They show a higher error level close to the policy change ($t = 1$) and a relatively smooth transition to the error level of the final stationary equilibrium thereafter. The absolute error level is low. BFKS appears to outperform PTLM by up to 0.007% and 0.15% for 1- and up to J-periods ahead predictions respectively. Errors are evaluated by simulating $M^{EE} = 50,000$ time series each.

small BFKS appears to outperform PTLM in particular in the periods close to the exogenous policy change ($t = 1$). However, note that error levels of PTLM are throughout acceptable[13] and the advantage of BFKS with respect to the approximation precision does not exceed 0.007% and 0.15% for 1- and up to J-periods ahead predictions. Table 3.3 summarizes approximation errors.

Table 3.3: Summary of Approximation Errors

	maximum/minimum		average	
	PTLM	*BFKS*	*PTLM*	*BFKS*
EE	4.8E-05	4.8E-05	1.6E-05	1.6E-05
PE^1	2.3E-04	1.9E-04	4.7E-05	4.5E-04
PE^J	6.1E-03	4.6E-03	6.0E-04	5.7E-04
R^2	0.999999	0.999972	0.999999	0.999980

Source: Evaluation of approximation errors under PTLM and BFKS. *Notes:* EE is the maximum Euler equation error of all agents alive, cf. (3.29). PE^1 is the prediction error with forecasting horizon 1. PE^J is the maximum error of predicting up to J periods ahead, cf. (3.28). R^2 is the minimum coefficient of determination of all regressions. The center column shows the maximum (for EE, PE^1, PE^J) respectively minimum (for R^2) values of all simulations and time periods in the transition excluding values from initial and final stationary equilibria (the latter can be found in section 3.B.1). The right column depicts the corresponding average values over all simulations. Numbers indicate a very good level of approximation by both methods. BFKS appears to outperform PTLM slightly, in particular with respect to predictions of up to J periods ahead. Errors are evaluated by simulating $M^{EE} = 50,000$ time series each.

3.6 Conclusion

I develop a procedure to solve for transitional dynamics in models with aggregate shocks when these transitional dynamics are driven by exogenous deterministic processes. My method is easy to implement in that it combines well-established tools for solving heterogeneous agent models. The key element of the approach is a flexible parameterization of the coefficients in the laws of motion of aggregate endogenous state variables. Identification of the underlying parameters is by regression using realizations from stochastic simulations of the model along the transition. This central step of the procedure requires a much smaller number of stochastic simulations than an alternative brute force approach because I exploit time variation in addition to variation across simulations.

Using an illustrative example, I demonstrate that the parameterization of the coefficients of the aggregate laws of motion by time polynomials of Chebyshev's first type lead to very small prediction errors. The errors are similar in size to a brute force approach while outperforming the latter in terms of computing time by substantial 45%. In the example, I deliberately develop a very simple economic model with inter-generational heterogeneity only in order to focus on the determination of the coefficients in the parameterized laws of motion as the key element of the procedure. However, the application of my method to more general models is straightforward as long as the mean shock path of the economy is easy to determine. Models that combine idiosyncratic shocks as in Aiyagari (1994) with aggregate risk have to be solved using standard procedures that aggregate over cross-sectional measures along the path with mean shock realizations. Furthermore, I used the aggregate capital stock as the only relevant endogenous aggregate state variable. Chapter 2, considers an application of the method using two aggregate state variables and second order polynomials in a model

[13]Compare Den Haan (2010a) for a notion on acceptable error ranges.

of deterministic demographic change. It shows that the time advantage of the method is key and renders computations feasible.

Appendix 3.A Theoretical Appendix

3.A.1 Solution of the Household Problem

In what follows, I solve for the household policy function.

Proposition 3. *The optimal age-dependent consumption function is linear in cash-on-hand,*

$$c = m \cdot x,$$

whereby the marginal propensity m to consume out of cash-on-hand x is given by

$$m := \frac{(\beta \cdot \wp)^{-\frac{1}{\theta}}}{1 + (\beta \cdot \wp)^{-\frac{1}{\theta}}} \quad \text{, where} \quad \wp := \mathbb{E}[m'^{-\theta} \cdot R'^{1-\theta}].$$

Proof. I guess that $v = m^{-\theta} \cdot x^{1-\theta}/(1-\theta)$ where m is the marginal propensity to consume out of x and show below that this is indeed true. From the guess it follows that

$$v \quad = \quad \max_c \left\{ \frac{c^{1-\theta}}{1-\theta} + \frac{(x-c)^{1-\theta} \cdot \beta \cdot \mathbb{E}[(m')^{-\theta} \cdot R'^{1-\theta}]}{1-\theta} \right\}$$

The first order condition with respect to c is

$$c^{-\theta} - (x-c)^{-\theta} \cdot \beta \cdot \mathbb{E}[(m')^{-\theta} \cdot R'^{1-\theta}] = 0.$$

Defining $n := \beta \cdot \mathbb{E}[(m')^{-\theta} \cdot R'^{1-\theta}]$ I get $c^{-\theta} = (x-c)^{-\theta} \cdot n$, or equivalently

$$c \quad = \quad m \cdot x \ \text{ where } m = \frac{n^{-\frac{1}{\theta}}}{1 + n^{-\frac{1}{\theta}}}.$$

It is left to show that indeed $v = m^{-\theta} \cdot x^{1-\theta}/(1-\theta)$. Using $c = m \cdot x$ and $n = \beta \cdot \mathbb{E}[(m')^{-\theta} \cdot R'^{1-\theta}]$ in v I get:

$$\begin{aligned}
v \quad &= \quad \frac{(m \cdot x)^{1-\theta}}{1-\theta} + (x - m \cdot x)^{1-\theta} \cdot \frac{n}{1-\theta} \\
&= \quad \frac{x^{1-\theta}}{1-\theta} \cdot m^{1-\theta} + \frac{x^{1-\theta}}{1-\theta} \cdot (1-m)^{1-\theta} \cdot n \\
&= \quad \frac{x^{1-\theta}}{1-\theta} \cdot \left\{ m^{1-\theta} + \left((1-m) \cdot n^{\frac{1}{1-\theta}} \right)^{1-\theta} \right\} \\
&= \quad \frac{x^{1-\theta}}{1-\theta} \cdot \left\{ \left(\frac{n^{-\frac{1}{\theta}}}{1 + n^{-\frac{1}{\theta}}} \right)^{1-\theta} + \left(\frac{1}{1 + n^{-\frac{1}{\theta}}} \cdot n^{\frac{1}{1-\theta}} \right)^{1-\theta} \right\} \\
&= \quad \frac{x^{1-\theta}}{1-\theta} \cdot \left\{ \left(\frac{n^{-\frac{1}{\theta}} + n^{\frac{1}{1-\theta}}}{1 + n^{-\frac{1}{\theta}}} \right)^{1-\theta} \right\} = \frac{x^{1-\theta}}{1-\theta} \cdot \left\{ \frac{n^{\frac{\theta-1}{\theta}} + n}{\left(1 + n^{-\frac{1}{\theta}}\right)^{1-\theta}} \right\}
\end{aligned}$$

$$= \frac{x^{1-\theta}}{1-\theta} \cdot \left\{ n \cdot \frac{1+n^{-\frac{1}{\theta}}}{\left(1+n^{-\frac{1}{\theta}}\right)^{1-\theta}} \right\} = \frac{x^{1-\theta}}{1-\theta} \cdot \left\{ \frac{n}{\left(1+n^{-\frac{1}{\theta}}\right)^{-\theta}} \right\}$$

$$= \frac{x^{1-\theta}}{1-\theta} \cdot \left\{ \frac{n^{-\frac{1}{\theta}}}{1+n^{-\frac{1}{\theta}}} \right\}^{-\theta} = \frac{x^{1-\theta}}{1-\theta} \cdot m^{-\theta}$$

□

Appendix 3.B Computational Appendix

Numerical computations are implemented in Fortran 90 using routines which are partly based on Press et al. (1996). All computing times refer to an Intel Xeon CPU E5-2620 v2 @ 2.10 GHz not parallelizing the code. If not otherwise stated the convergence criterion of a root finding algorithm is set to 10^{-6} and the weight on resulting variables in the updating step of a Gauss-Seidel algorithm (cf., e.g., Ludwig, 2007) is set to 99%.

3.B.1 Solution Steps

I present the implementation steps of my procedure solving the model of section 3.3 in an order corresponding to section 3.2.3.

1. *Mean Shock Path*:

   Taking the exogenous variation of capital taxes as given, first, I solve for the mean shock path of the aggregate capital stock in the economy. To do so, I set the realization of the productivity shock to $\mathbb{E}[z] = 1$ in all t. Second, I solve for the initial steady state in $t = 0$ and the final steady state in $t = T$ by iterating on K_0, respectively K_T until convergence. Third, I solve for the transition by iterating on $\{K_t\}_{t=1}^{T-1}$ until convergence given K_0 and K_T. This is done by applying fixed point iterations using a standard Gauss-Seidel method. Throughout, the solution of the household problem is analytical taking expectations on z and given that the aggregate law of motion along the transition is exactly known. The resulting mean shock time path of the aggregate capital stock, $\{K_t^{\mathcal{M}}\}_{t=0}^{T}$, is shown in figure 3.B.1.

2. *Time Dependent Grids*:

   Next, I specify time dependent grids for the aggregate capital stock, $\mathcal{G}_t^K$. I do so by choosing $n_K = 11$ equally spaced grid points in the range $\{(1-s^K) \cdot K_t^{\mathcal{M}}, (1+s^K) \cdot K_t^{\mathcal{M}}\}$ for all t setting $s^K = 0.5$.

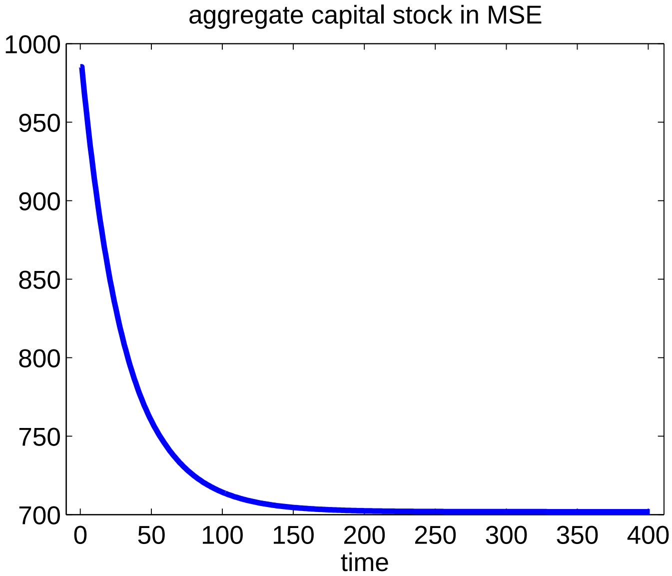
3. *Stationary Equilibria*:

   I apply a standard Krusell and Smith (1997, 1998) method to determine the coefficients, $\phi_p(0, z)$ and $\phi_p(T, z)$ for all $p \in (0, 1)$, $z \in (z_l, z_h)$, of the aggregate laws of motion in the initial and the final stationary equilibrium respectively, cf. (3.26). I do so by $M^{SS} = 55,000$ stochastic simulations of each stationary equilibrium given $K_0 = K_t^{\mathcal{M}}$ as initial value while discarding the first $5,000$ simulations. This involves fixed point iterations applying a Gauss-Seidel algorithm to minimize $\parallel \phi_p(t, z) - \hat{\phi}_p(t, z) \parallel$ for all $p \in (0, 1)$, $z \in (z_l, z_h)$, $t \in (0, T)$. $\hat{\phi}_p(t, z)$ denote the resulting coefficients from regressions which are non-linear functions of $\phi_p(t, z)$.

   Resulting coefficients and approximation errors are shown in table 3.B.1. Euler equation errors are very small and prediction errors are lower than 1% even for the multi-period ahead forecasts.

4. *Rate of Convergence Speed*:

Figure 3.B.1: Mean Shock Path of Aggregate Capital Stock



Source: Solution of the auxiliary mean shock path.

Table 3.B.1: Initial and Final Law of Motion

	$t = 0$	$t = T$
Coefficients		
$\phi_0(t, z_l)$	0.1450	0.1323
$\phi_1(t, z_l)$	0.9788	0.9797
$\phi_0(t, z_h)$	0.1513	0.1394
$\phi_1(t, z_h)$	0.9782	0.9789
Errors		
EE_t	5.6E-05	4.7E-05
PE_t^1	1.6E-04	1.5E-04
PE_t^J	4.1E-03	3.9E-03
R_t^2	0.999973	0.999976

Source: Solution of the model economy in the initial and the final stationary equilibria. *Notes:* Coefficients and approximation errors are all based on $M^{SS} = M^{EE} = 55,000$ stochastic simulations of which the respective first $5,000$ are discarded. Errors are computed in analogy to section 3.4.2. EE_t is the maximum Euler equation error of all simulations and agents alive in period t, cf. (3.29). PE_t^1 is the maximum prediction error with forecasting horizon 1 of all simulations in period t. PE_t^J is the maximum error of predicting up to J periods ahead of all simulations in period t, cf. (3.28). R_t^2 is the minimum coefficient of determination of regressions of all simulations in period t.

Given $\phi_p(0,z)$ and $\phi_p(T,z)$ for all p, z, I determine[14] the rate of convergence speed, ν, in equation (3.27) to equal 0.035. This results from a non-linear regression using a scaled version of the mean shock path, $\{K_t^{\mathcal{M}}\}_{t=0}^{T}$, where I exploit information from the stationary equilibria.[15]

Note that this also yields coefficient estimates of the time polynomials, $\psi_p(z)$ for all p, z, which are used as the initial guess in the next step of the procedure.[16]

5. *Transitional Dynamics*:

   This is the key step of the procedure. Given $\phi_p(0,z)$ and $\phi_p(T,z)$ for all p, z as well as ν, solve for the coefficients of the time polynomials, $\psi_p(z)$ for all $p \in (0,1), z \in (z_l, z_h)$, in equation (3.27) by the following procedure:

   a) Given $\psi_p(z)$ for all p, z, solve the policy functions at all points in the grids $\mathcal{G}_{K,t}$ for all $t = 1, ..., T-1$.
   b) Simulate the economy M^{TR} times along the transition of T-1 periods given a random initial value, K_1, from the simulation of the initial stationary equilibrium under step 3 and obtain $\{K_t\}_{t=1}^{T}$.
   c) Regress $\{K_t\}_{t=2}^{T}$ on the right-hand-side of equation (3.27) using $\{K_t\}_{t=1}^{T-1}$ and obtain $\hat{\psi}_p(z)$ for all p, z.
   d) If $\| \psi_p(z) - \hat{\psi}_p(z) \| < \epsilon$, where ϵ is some convergence criterion, stop. Else, update $\psi_p(z)$ in step 5a.

6. *Error Evaluation*:

   Given the policy functions at all grid points determined in the precedent step I evaluate approximation errors in M^{EE} stochastic simulations along the transition according to section 3.4.2.

Resulting transitional dynamics as well as corresponding approximation errors and computing times are presented in section 3.5.

[14] I apply a derivative-free Powell algorithm based on Press et al. (1996) which turned out to be more robust than alternative gradient based methods.

[15] I multiply the mean shock path by scaling factors $\{\jmath_t\}_{t=0}^{T}$. The scaling factors in the two stationary equilibria equal $\jmath_0 := \mathbb{E}[K_0]/K_0^{\mathcal{M}}$ and $\jmath_T := \mathbb{E}[K_T]/K_T^{\mathcal{M}}$ where $\mathbb{E}[K_0]$ and $\mathbb{E}[K_T]$ denote the average capital stocks from simulations in periods 0 and T respectively. Finally, I interpolate linearly between $\jmath_0$ and $\jmath_T$ yielding $\{\jmath_t\}_{t=1}^{T-1}$.

[16] For the sake of consistency, I use the implied resulting values of $\phi_p(t,z)$ for all $t \in (1,...,T-1), p \in (0,1), z \in (z_l, z_h)$ as an initial guess in the BFKS approach which I compare my method to. Note that equation (3.27) rewrites as $\ln K_{t+1} = \phi_0(t,z) + \phi_1(t,z) \cdot \ln K_t$ where

$$\phi_p(t,z) := \phi_p(0,z) \cdot \psi_p(z) \cdot \exp(-\nu \cdot t) + \phi_p(T,z) \cdot (1 - \exp(-\nu \cdot t)$$

for all $t \in (1,...,T-1), p \in (0,1), z \in (z_l, z_h)$.

This page was intentionally printed only with this statement.

Bibliography

Abel, A. B. (2001). “The Effects of Investing Social Security Funds in the Stock Market when Fixed Costs Prevent Some Households from Holding Stocks”. *American Economic Review* 91, pp. 128–148.

——— (2003). “The Effects of a Baby Boom on Stock Prices and Capital Accumulation in the Presence of Social Security”. *Econometrica* 71, pp. 551–578.

Aiyagari, S. R. (1994). “Uninsured Idiosyncratic Risk and Aggregate Saving”. *Quarterly Journal of Economics* 109, pp. 659–684.

Algan, Y., O. Allais, and W. J. Den Haan (2008). “Solving Heterogeneous-Agent Models with Parameterized Cross-Sectional Distributions”. *Journal of Economic Dynamics and Control* 32, pp. 875–908.

Algan, Y. et al. (2014). “Chapter 6 - Solving and Simulating Models with Heterogeneous Agents and Aggregate Uncertainty”. In: *Handbook of Computational Economics Vol. 3*. Ed. by K. Schmedders and K. L. Judd. Amsterdam: Elsevier, pp. 277–324.

Ameriks, J. and S. P. Zeldes (2004). “How Do Household Portfolio Shares Vary with Age?” Mimeo.

Angeletos, G.-M. (2007). “Uninsured Idiosyncratic Investment Risk and Aggregate Saving”. *Review of Economic Dynamics* 10, pp. 1–30.

Angeletos, G.-M. and L.-E. Calvet (2006). “Idiosyncratic Production Risk, Growth and the Business Cycle”. *Journal of Monetary Economics* 53, pp. 1095–1115.

Attanasio, O., S. Kitao, and G. L. Violante (2007). “Global Demographic Trends and Social Security Reform”. *Journal of Monetary Economics* 54(1), pp. 144–198.

Auerbach, A. J. and L. J. Kotlikoff (1987). *Dynamic Fiscal Policy*. Cambridge, MA: Cambridge University Press.

Autorengruppe Bildungsberichterstattung (2012). *Bildung in Deutschland 2012*. Bielefeld: W. Bertelsmann Verlag.

Bakshi, G. S. and Z Chen (1994). “Baby Boom, Population Aging and Capital Markets”. *Journal of Business* 67, pp. 165–202.

Bansal, R. and A. Yaron (2004). “Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles”. 59(4), pp. 1481–1509.

Becker, G. S. (1967). *Human Capital and the Personal Distribution of Income: An Analytical Approach*. Woytinsky Lecture: University of Michigan.

Ben-Porath, Y. (1967). “The Production of Human Capital and the Life Cycle of Earnings”. *Journal of Political Economy* 75(4), pp. 352–65.

Bleakley, H. (2007). “Desease and Development: Evidence from Hookworm Eradication in the American South”. *Quarterly Journal of Economics* 122(1), pp. 73–117.

Börsch-Supan, A., A. Ludwig, and M. Sommer (2003). *Demographie und Kapitalmärkte - Die Auswirkungen der Bevölkerungsalterung auf Aktien-, Renten- und Immobilienvermögen*. Köln: DIA.

Börsch-Supan, A., A. Ludwig, and J. Winter (2006). “Aging, Pension Reform and Capital Flows: A Multi-Country Simulation Model”. *Economica* 73, pp. 625–658.

Boucekkine, R., D. de la Croix, and O. Licandro (2002). “Vintage Human Capital, Demographic Trends, and Endogenous Growth”. *Journal of Economic Theory* 104(2), pp. 340–375.

——— (2003). “Early Mortality Declines at the Dawn of Modern Grwth”. *Journal of Economics* 105, pp. 401–418.

Brooks, R. (2004). “The Equity Premium and the Baby Boom”. *Econometric Society 2004 North American Winter Meetings* 155, pp. 1–25.

Browning, M., L. P. Hansen, and J. J. Heckman (1999). “Micro Data and General Equilibrium Models”. In: *Handbook of Macroeconomics*. Ed. by J. B. Taylor and W. M. Elsevier, pp. 543–633.

Campbell, J. Y. and L. M. Viceira (2002). *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors.* Oxford University Press.

Cervellati, M. and U. Sunde (2005). “Human Capital, Life Expectancy, and the Processs of Development”. *American Economic Review* 95(5), pp. 1653–1672.

——— (2013). “Life Expectancy, Schooling, and Lifetime Labor Supply: Theory and Evidence Revisited”. *Econometrica* 81(5), pp. 2055–2086.

Constantinides, G. M. and D Duffie (1996). “Asset Pricing with Heterogeneous Consumers”. *Journal of Political Economy* 104, pp. 219–240.

Davila, J. et al. (2012). “Constrainded Efficiency in the Neoclassical Growth Model with Uninsurable Idiosyncratic Shocks”. *Econometrica* 80(6), pp. 2431–2467.

De La Croix, D. and O. Licandro (1999). “Life Expectancy and Endogenous Growth”. *Economics Letters* 65, pp. 255–263.

De Nardi, M., S. Imrohoroglu, and T. J. Sargent (1999). “Projected U.S. Demographics and Social Security”. *Review of Economic Dynamics* 2(3), pp. 575–615.

Deaton, A. (1991). “Saving and Liquidity Constraints”. *Econometrica* 59(5), pp. 1221–1248.

Deming, D and S Dynarski (2009). “Into College, Out of Poverty? Policies to Increase the Post-Secondary Attainment of the Poor”. *NBER Working Paper* 15387.

Den Haan, W. J. (1996). “Heterogeneity, Aggregate Uncertainty and the Short-Term Interest Rate”. *Journal of Business and Economic Statistics* 14, pp. 399–411.

——— (1997). “Solving Dynamic Models with Aggregate Shocks and Heterogeneous Agents”. *Macroeconomic Dynamics* 1, pp. 355–386.

——— (2010a). “Assessing the Accuracy of the Aggregate Law of Motion in Models with Heterogeneous Agents”. *Journal of Economic Dynamics and Control* 34, pp. 79–99.

——— (2010b). “Comparison of Solutions to the Incomplete Markets Model with Aggregate Uncertainty”. *Journal of Economic Dynamics and Control* 34, pp. 4–27.

Den Haan, W. J. and P. Rendahl (2010). “Solving the Incomplete Markets Model with Aggregate Uncertainty Using Explicit Aggregation”. *Journal of Economic Dynamics and Control* 34, pp. 69–78.

Diamond, P. A. (1965). “National Debt in a Neoclassical Growth Model”. *American Economic Review* 55, pp. 1126–1150.

Domeij, D. and M. Flodén (2006). “The Labor-Supply Elasticity and Borrowing Constraints: Why Estimates are Biased”. *Review of Economic Dynamics* 9, pp. 242–262.

DRV (2014). *Rentenversicherung in Zeitreihen, Oktober 2014.* Deutsche Rentenversicherung Bund.

Duffy, J., C. Papageorgiou, and F. Perez-Sebastian (2004). “Capital-Skill Complementarity? Eidence from a Panel of Countries”. *The Review of Economics and Statistics* 86(1), pp. 327–344.

Dustmann, C., J. Ludsteck, and U. Schönberg (2009). “Revisiting the German Wage Structure”. *Quarterly Journal of Economics* 124, pp. 843–881.

Fernández-Villaverde, J. and D. Krüger (2006). “Consumption over the Life Cycle: Facts from Consumer Expenditure Survey Data”. *The Review of Economics and Statistics* 89(3), pp. 552–565.

Findeisen, S. and D. Sachs (2015). “Designing Efficient College and Tax Policies”. Mimeo.

Fredriksson, P. (1997). “Economic Incentives and the Demand for Higher Education”. *The Scandinavian Journal of Economics* 99(1), pp. 129–142.

Fuchs-Schündeln, N., D. Krueger, and M. Sommer (2010). "Inequality trends for Germany in the last two decades: A tale of two countries". *Review of Economic Dynamics* 13, pp. 103–132.

Geanakoplos, J., M. Magill, and M. Quinzii (2004). "Demography and the Long-Run Predictability of the Stock Market". *Brookings Papers on Economic Activity* 1, pp. 241–307.

Geppert, C. (2015a). "Computing Transitional Dynamics in Heterogenous Agent Models with Aggregate Risk by Parameterized Laws of Motion". *Mimeo.*

——— (2015b). "On the Distributional Implications of Demographic Change". *Mimeo.*

Geppert, C. and A. Ludwig (2015). "Risky Human Capital, Aging, the Equity Premium, and Welfare". *Mimeo.*

Gomes, F. and A. Michaelides (2008). "Asset Pricing with Limited Risk Sharing and Heterogeneous Agents". *The Review of Financial Studies* 21(1), pp. 415–449.

Harenberg, D. and A. Ludwig (2015). "Social Security and the Interactions between Aggregate and Idiosyncratic Risk". *Mimeo.*

Hazan, M. (2009). "Longevity and Lifetime Labor Supply: Evidence and Implications". *Econometrica* 77(6), pp. 1829–1863.

Heckman, J. J., L. Lochner, and C. Taber (1998). "Explaining Rising Wage Inequality: Explorations with a Dynamic General Equilibrium Model of Labor Earnings with Heterogeneous Agents". *Review of Economic Dynamics* 1, pp. 1–58.

Huggett, M., G. Ventura, and A. Yaron (2011). "Sources of Lifetime Inequality". *American Economic Review* 101, pp. 2923–2954.

Human Mortality Database (2008). "No Title". University of California, Berkeley (USA), and Max Planck Institutefor Demographic Research (Germany) [accessed at http://www.mortality.org/ on April 12, 2010].

Jayachandran, S. and A. Lleras-Muney (2009). "Life Expectancy and Human Capital Investments: Evidence from Maternal Mortality Declines". *Quarterly Journal of Economics* 124(1), pp. 349–398.

Johnson, M. T. (2013). "Borrowing Constraints, College Enrollment, and Delayed Entry". *Journal of Labor Economics* 31(4), pp. 669–725.

Judd, K. L. (2002). "The Parametric Path Method: An Alternative to Fair-Taylor and L-B-J for Solving Perfect Foresight Models". *Journal of Economic Dynamics and Control* 26, pp. 1557–1583.

Kalemli-Ozcan, S, H Ryder, and D Weil (2000). "Mortality Decline, Human Capital Investment, and Economic Growth". *Journal of Development Economics* 62, pp. 1–23.

Kane, T. J. (2006). "Public Interventions in Post-Secondary Education". In: *Handbook of the Economics of Education.* Ed. by E. Hanushek and F. Welch. Elsevier, pp. 1369–1401.

Katz, L. F. and K. M. Murphy (1992). "Changes in Relative Wages, 1963-1987: Supply and Demand Factors". *Quarterly Journal of Economics* 107(1), pp. 35–78.

Keane, M. and K Wolpin (1997). "The Career Decisions of Young Men". *Journal of Political Economy* 105, pp. 473–522.

Keuschnigg, C., T. Davoine, and P. Schuster (2013). *Aging, Informality and Public Finances in Poland and Slovakia - A General Equilibrium Approach, Background Paper for the ECA Old-Age Insurance World Bank Report.* Tech. rep. Vienna: Institute for Advanced Studies (IHS).

Kindermann, F. (2014). "Earnings Related Pension Schemes and Human Capital Formation". *Journal of Pension Economics and Finance* 14(1), pp. 19–54.

Krebs, T. (2003). "Human Capital Risk and Economic Growth". *Quarterly Journal of Economics* 118, pp. 709–745.

Krebs, T. and B. Wilson (2004). "Asset Returns in an Endogenous Growth Model with Incomplete Markets". *Journal of Economic Dynamics and Control* 28, pp. 817–839.

Krüger, D. and F. Kübler (2006). "Pareto Improving Social Security Reform When Financial Markets are Incomplete!?" *American Economic Review* 96(3), pp. 737–755.

Krüger, D. and A. Ludwig (2007). "On the Consequences of Demographic Change for Rates of Return to Capital and the Distribution of Wealth and Welfare". *Journal of Monetary Economics* 54, pp. 49–87.

Krusell, P. and A. A. Smith (1997). "Income and Wealth Heterogeneity, Portfolio Choice, and Equilibrium Asset Returns". *Macroeconomic Dynamics* 1, pp. 387–422.

——— (1998). "Income and Wealth Heterogeneity and the Macroeconomy". *Journal of Political Economy* 106, pp. 867–896.

Krusell, P. et al. (2000). "Capital-Skill Complementarity and Inequality: A Macroeconomic Analysis". *Econometrica* 68 (5), pp. 1029–1053.

Kuhle, W. (2008). "Demography and Equity Premium". *MEA Discussion Paper* 157-08.

Lagerlöf, N.-P. (2003). "From Malthus to Modern Growth: Can Epidemics Explain the Three Regimes?" *International Economic Review* 44(2), pp. 755–777.

Lee, R. D. and A. Mason (2010). "Some Macroeconomic Aspects of Global Population Aging". *Demography* 47 (1S), pp. 151–172.

Ludwig, A. (2007). "The Gauss-Seidel-Quasi-Newton Method: A Hybrid Algorithm for Solving Dynamic Economic Models". *Journal of Economic Dynamics and Control* 31(5), pp. 1610–1632.

Ludwig, A. and R. Abiry (2015). "A New Utility Function for Macroeconomists". Mimeo.

Ludwig, A. and M. Reiter (2010). "Sharing Demographic Risk - Who is Afraid of the Baby Bust?" *American Economic Journal: Economic Policy* 2(4), pp. 83–118.

Ludwig, A., T. Schelkle, and E. Vogel (2012). "Demographic Change, Human Capital and Welfare". *Review of Economic Dynamics* 15, pp. 94–107.

Ludwig, A. and E. Vogel (2010). "Mortality, Fertility, Education and Capital Accumulation in a Simple OLG Economy". *Journal of Population Economics* 23, pp. 703–735.

Mehra, R and E Prescott (1985). "The Equity Premium: A Puzzle". *Journal of Monetary Economics* 15, pp. 145–161.

Merton, R. C. (1969). "Litetime Portfolio Selection under Uncertainty: The Continuous-Time Case". *The Review of Economics and Statistics* 51, pp. 247–257.

OECD (2013). *Secretary-General's Report to Ministers 2013.* Paris: OECD Publishing.

——— (2014a). "Indicator A6 What are the earnings advantages from education?" In: *Education at a Glance 2014: OECD Indicators.* Paris: OECD Publishing.

——— (2014b). *OECD.Stat (database).* [accessed at http://stats.oecd.org on October 15, 2014]. Paris: OECD Publishing.

Oster, E., I. Shoulson, and E. R. Dorsey (2013). "Limited Life Expectancy, Human Capital and Health Investments". *American Economic Review* 103(5), pp. 1977–2002.

Piazzesi, M., M. Schneider, and S. Tuzel (2007). "Housing, Consumption, and Asset Pricing". *Journal of Financial Economics* 83(3), pp. 531–569.

Piketty, T. and G. Zucman (2014). "Capital Is Back: Wealth-Income Ratios in Rich Countries 1700–2010". *Quarterly Journal of Economics* 129(3), pp. 1255–1310.

Poterba, J. M. (2001). "Demographic Structure and Assets Returns". *The Review of Economics and Statistics* 83, pp. 565–584.

Press, W. H. et al. (1996). *Numerical Recipes in Fortran 90. The Art of Parallel Scientific Computing*. New York: Cambridge University Press.

Preston, B and M Roca (2007). "Incomplete Markets, Heterogeneity and Macroeconomic Dynamics". *NBER Working Paper* 13260.

Rajan, R. G. and L. Zingales (1995). "What Do We Know about Capital Structure? Some Evidence from International Data". *The Journal of Finance* 50(5), pp. 1421–1460.

Reiter, M. (2009). "Solving Heterogenous-Agent Models by Projection and Perturbation". *Journal of Economic Dynamics and Control* 33, pp. 649–665.

——— (2010). "Solving the Incomplete Markets Economy with Aggregate Uncertainty by Backward Induction". *Journal of Economic Dynamics and Control* 34, pp. 28–35.

Samuelson, P. A. (1969). "Lifetime Portfolio Selection by Dynamic Stochastic Programming". *The Review of Economics and Statistics* 51(3), pp. 239–246.

Santos, M. S. and A. Peralta-Alva (2005). "Accuracy of Simulations for Stochastic Dynamic Models". *Econometrica* 73(6), pp. 1939–1976.

Schieber, S. and J. Shoven (1997). "The Consequences of Aging on Private Pension Fund Saving and Asset Markets". In: *Public Policy Toward Pensions*. Ed. by S. Schieber and J. Shoven. Cambridge: MIT Press, pp. 219–245.

Shiller, R. (2015). *U.S. Stock Price Data, Annual, with consumption, both short and long rates, and present value calculations*. [accessed at http://www.econ.yale.edu/~shiller/data.htm on May 26, 2015].

Siegel, J. J. (1998). *Stocks for the Long-Run: The Definitive Guide to Financial Market Returns and Long-Term Investment Strategies*. New York, London and Toronto: McGraw Hill.

Soares, R. R. (2005). "Mortality Reductions, Educational Attainment, and Fertility Choice". *American Economic Review* 95(3), pp. 580–601.

Statistisches Bundesamt (2014). *Bildungsstand der Bevölkerung, 2014*. Wiesbaden: Statistisches Bundesamt.

Steiner, V. and K. Wrohlich (2011). "Financial Student Aid and Enrollment in Higher Education: New Evidence from Germany". *The Scandinavian Journal of Economics* 114(1), pp. 124–147.

Storesletten, K, C Telmer, and A Yaron (2004). "Cyclical Dynamics in Idiosyncratic Labor Market Risk". *Journal of Political Economy* 112, pp. 695–717.

——— (2007). "Asset Pricing with Idiosyncratic Risk and Overlapping Generations". *Review of Economic Dynamics* 10, pp. 519–548.

Storesletten, K. C. (2000). "Sustaining Fiscal Policy Through Immigration". *Journal of Political Economy* 108, pp. 300–323.

United Nations, Department of Economic and Social Affairs, Population Division (2007). *World Population Prospects: The 2006 Revision*. New York: United Nations.

——— (2013). *World Population Prospects: The 2012 Revision*. New York: United Nations.

Willis, R and S Rosen (1979). "Education and Self-selection". *Journal of Political Economy* 87, pp. 7–36.

Winter, C. (2014). "Accouting for the Changing Role of Family Income in Determining College Entry". *The Scandinavian Journal of Economics* 116(4), pp. 909–963.

Curriculum Vitae

Persönliche Daten

Name	Christian Geppert

Wissenschaftliche Positionen

seit 08/2014	Wissenschaftlicher Mitarbeiter Johann Wolfgang Goethe-Universität, Frankfurt am Main Research Center SAFE, Applied Econometrics and International Economic Policy
09/2014 – 12/2014	Gastwissenschaftler Organisation for Economic Co-operation and Development (OECD), Paris Directorate for Employment, Labour and Social Affairs, Social Policy Division
04/2010 – 07/2014	Wissenschaftlicher Mitarbeiter Universität zu Köln Center for Macroeconomic Research

Bildungsweg

07/2015	Promotion zum Doktor rerum politicarum (Dr. rer. pol.) Johann Wolfgang Goethe-Universität, Frankfurt am Main Dissertation: Macroeconomic Effects of Demographic Change: The Role of Human Capital
04/2010 – 07/2015	Doktorand im Fach Volkswirtschaftslehre Johann Wolfgang Goethe-Universität Frankfurt am Main und Universität zu Köln Promotionsbetreuer: Prof. Dr. Alexander Ludwig
08/2013 – 12/2013	Gastwissenschaftler im Rahmen der Promotion Arizona State University, Tempe Betreuender Professor: Prof. Dr. Alexander Bick
04/2004 – 03/2010	Diplom im Studiengang Volkswirtschaftslehre Universität zu Köln Spezialisierung: Makroökonomik, Finanzierungslehre
07/2008 – 11/2008	Austauschstudent im Masterstudiengang The University of Sydney Spezialisierung: Finanzierungslehre
08/1994 – 06/2003	Allgemeine Hochschulreife Albert-Einstein-Schule, Schwalbach a. Ts. Spezialisierung: Mathematik, Wirtschaftswissenschaften