Sharing Decorations for Improvements in a Functional Core Language with Call-By-Need Operational Semantics

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Abstract. The calculus LRP is a polymorphically typed call-by-need lambda calculus extended by data constructors, case-expressions, seq-expressions and type abstraction and type application. This report is devoted to the extension LRPw of LRP by scoped sharing decorations. The extension cannot be properly encoded into LRP if improvements are defined w.r.t. the number of lbeta, case, and seq-reductions, which makes it necessary to reconsider the claims and proofs of properties. We show correctness of improvement properties of reduction and transformation rules and also of computation rules for decorations in the extended calculus LRPw. We conjecture that conservativity of the embedding of LRP in LRPw holds.

1 Introduction

In this technical report we consider improvements in the polymorphically typed, extended call-by-need functional language LRP and its extension by shared-worked decorations LRPw. Since it is known that the extension cannot be encoded in LRP (see Proposition 3.12), it is necessary to reconsider the claims and proofs of properties. The goal of the report is to show that known improvement laws for LRP also hold in the extended calculus LRPw, that a context lemma for improvement holds in LRPw and that several computation rules which simplify the reasoning with decorated expressions are invariant w.r.t. the improvement relation. The results of this report allow to use shared work decorations as a reasoning tool, e.g. for proving improvement laws on list-processing expressions and functions.

For reasoning on the correctness of program transformations, a notion of program semantics is required. We adopt the well-known and natural notion of contextual equivalence for our investigations: Contextual equivalence identifies two programs as equal if exchanging one program by the other program in any surrounding larger program (the so-called context) is not observable. Due to the quantification over all contexts it is sufficient to only observe the termination behavior of the programs, since e.g. different values like `True` and `False` can be distinguished by plugging them into a context C s.t. $C[\texttt{True}]$ terminates while $C[\texttt{False}]$ diverges. A program transformation is correct if it preserves the semantics, i.e. it preserves contextual equivalence. For reasoning whether program transformation are also optimizations, i.e. so-called *improvements*, we adopt the improvement theory originally invented by Moran and Sands [2], but slightly modified and adapted in [7] for the calculus LR. The calculus LR [11] is an untyped call-by-need lambda calculus extended by data-constructors, `case`-expressions, `seq`-expressions, and `letrec`-expressions. This calculus e.g. models the (untyped) core language of Haskell. In [11] the calculus LR was introduced and analyzed in the setting of a strictness analysis using abstract reduction where also several results on the reduction length w.r.t. program transformation

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were proved. The calculus LRP is the polymorphically typed variant of LR. Typing in LRP is by `let`-polymorphism [3, 4, 1, 12]. Polymorphism is made explicit in the syntax and there are also reduction rules for computing the specific types of functions. The type erasure of reduction sequences exactly leads to the untyped reduction sequences in LR, so that the untyped and typed calculus are compatible. The transfer of the results on improvement in LR to LRP is straightforward and can be found in [8].

In [2] a tick-algebra was introduced to prove correctness of improvement laws in a modular way. A tick $\checkmark^n$ can be attached to an expression to add a fixed amount of work to the expression (i.e. n execution steps). Several laws for computing with ticks are formulated and proved correct. In this paper we introduce the calculus LRPw which extends LRP in a similar way, where ticks are called decorations, but they are extended to a formalism that can express *work* which is *shared* between several subexpressions, which makes reasoning more comfortable and also more exact. In LRPw there are the two new (compared to LRP) constructs: Bindings of the form $a := n$ and decorations of the form $s^{[a]}$. Here $s^{[a]}$ means that the work expressed by the binding for a (i.e. n essential steps, if the binding is $a := n$) has to be done before the expression s can be further evaluated. If decoration a occurs at several subexpression, then the work is shared between the subexpressions (and thus at most performed once). The bindings $a := n$ occur in usual `letrec`-expressions and thus also define the scope of the sharing, and a notion of α-equivalence w.r.t. the labels a. This makes a formal treatment possible. As shorthand notation we will use the notation $s^{[a \mapsto n]}$ for shared work. However, this notation is imprecise and requires a definition of its semantics in the calculus LRPw (to fix the scoping of a).

As an example for the usefulness of shared-work decoration, consider the expression $\texttt{let from x = x : (from (x + 1)) in from (2 * 21)}$ which generates an infinite list of numbers $[42, 43, \ldots]$. For simplicity in this example we assume a work amount of 1 for arithmetic operations. The work for computing the product `(2*21)` is shared between all list elements, which can be expressed by our decorations: we can rewrite this list as $(\texttt{42}^{[a \mapsto 1]}\texttt{: let from x = x : (from (x + 1)) in from (43}^{[1,a \mapsto 1]}\texttt{)}^{[\mathtt{a}' \mapsto 1]}$ which exactly shows that there is shared work between the head and the tail of the list. Clearly, this can be iterated for further partial evaluation of the tail. Moreover, since we provide computation rules for the shared decorations, we can further compute with the decorations. Using the tick-notation of [2] such exact computations seem to be impossible.

We develop the improvement theory in the calculus LRPw and prove correctness and result w.r.t. improvement for the reduction rules and for several other program transformations. We develop computation rules for the shared-work decoration and proof their soundness.

Outline. Section 2 introduces the different calculi LRP and LRPw, and transfers the basic definitions, lemmas and correctness proofs of program transformations from LRP to LRPw. Section 3 defines the work decorations and proves a theorem that provides several computation rules for work decorations. Section 4 contains a proof that an improvement simulation on lists is correct for improvement and can be used as a tool. Some lengthy proofs can be found in the appendix.

2 The Polymorphically Typed Lazy Lambda Calculus LRPw

The extended call-by-need lambda calculus LRP (see e.g. [9, 6]), is a polymorphically typed variant of the calculus LR [11].

The calculus LRPw extends the calculus LRP by shared work decorations, where the decoration of the shared position is explicitely represented by two new constructs: There are new `letrec`-bindings $a_i := n_i$ meaning that a work load of n essential reduction steps is associated with label a where the shared position is the top of the `letrec`-expression, the construct $s^{[a]}$ means that before expression s can be evaluated the work associated with label a has to be evaluated.

Let $\mathcal{K}$ be a fixed set of type constructors, s.t. every $K \in \mathcal{K}$ has an arity $ar(K) \geq 0$ and an associated finite, non-empty set D_K of data constructors, s.t. every $c_{K,i} \in D_K$ has an arity $ar(c_{K,i}) \geq 0$. We assume that $\mathcal{K}$ includes type constructors for lists, pairs and Booleans together with the data constructors `Nil` and `Cons`, where we often use the Haskell notation of an infix colon; pairs as mixfix brackets, and the constants `True` and `False`.

Variables: We assume type variables $a, a_i \in \mathit{TVar}$ and term variables $x, x_i \in \mathit{Var}$

Labels: We assume label names a, b, a_i, b_i used for sharing work.

Types: Types Typ and polymorphic types PTyp are generated by the following grammar:

$$\begin{array}{ll} \tau \in \mathit{Typ} & ::= a \mid (\tau_1 \to \tau_2) \mid K\ \tau_1 \ldots \tau_{ar(K)} \\ \rho \in \mathit{PTyp} & ::= \tau \mid \boldsymbol{\lambda} a.\rho \end{array}$$

Expressions: Expression Expr_F, patterns $\mathit{pat}_{K,i}$, and polymorphic abstractions PExpr_F are generated by the following grammar:

$$\begin{array}{ll} s,t \in \mathit{Expr}_F & ::= u \mid x :: \rho \mid (s\ \tau) \mid (s\ t) \mid (\texttt{seq}\ s\ t) \mid (\texttt{letrec}\ \mathit{Bind}_1, \ldots, \mathit{Bind}_n\ \texttt{in}\ t) \\ & \quad \mid (s^{[a]}) \\ & \quad \mid (c_{K,i} :: \tau\ s_1\ \ldots\ s_{ar(c_{K,i})}) \mid (\texttt{case}_K\ s\ (\mathit{pat}_{K,1}\ \texttt{->}\ t_1) \ldots (\mathit{pat}_{K,|D_K|}\ \texttt{->}\ t_{|D_K|})) \\ \mathit{pat}_{K,i} & ::= (c_{K,i} :: \tau\ x_1 :: \tau_1 \ldots x_{ar(c_{K,i})} :: \tau_{ar(c_{K,i})}) \\ \mathit{Bind}_i & ::= x_i :: \rho_i = s_1 \mid a_i := n_i \quad \text{where } a_i \text{ is a label and } n_i \text{ is a nonnegative integer} \\ u \in \mathit{PExpr}_F & ::= \Lambda a_1 \ldots . \Lambda a_k . \lambda x :: \tau . s \end{array}$$

Typing rules:

$$\frac{u :: \rho}{\Lambda a.u :: \boldsymbol{\lambda} a.\rho} \qquad \frac{s :: \tau_1 \quad \mathit{pat}_i :: \tau_1 \quad t_i :: \tau_2}{(\texttt{case}_K\ s\ (\mathit{pat}_1\ \texttt{->}\ t_1) \ldots (\mathit{pat}_{|D_K|}\ \texttt{->}\ t_{|D_K|})) :: \tau_2} \qquad \frac{s :: \tau_2}{(\lambda x :: \tau_1 . s) :: \tau_1 \to \tau_2}$$

$$\frac{s :: \boldsymbol{\lambda} a.\rho}{(s\ \tau) :: \rho[\tau/a]} \qquad \frac{s :: \tau_1 \to \tau_2 \quad t :: \tau_1}{(s\ t) :: \tau_2}$$

$$\frac{s_1 :: \rho_1 \quad \ldots \quad s_n :: \rho_n \quad t :: \rho}{(\texttt{letrec}\ a_1 := n_1, \ldots, a_m := n_m,\ x_1 :: \rho_1 = s_1, \ldots, x_n :: \rho_n = s_n\ \texttt{in}\ t) :: \rho}$$

$$\frac{s :: \tau \quad t :: \tau'}{(\texttt{seq}\ s\ t) :: \tau'} \qquad \frac{\begin{array}{c} s_1 :: \tau_1, \ldots, s_{ar(c)} :: \tau_{ar(c)} \qquad \tau = \tau_1 \to \ldots \to \tau_{ar(c)} \to \tau_{ar(c)+1} \\ \mathit{type}(c) = \boldsymbol{\lambda} a_1, \ldots, a_m . \tau'' \qquad \exists \tau_1', \ldots, \tau_m' : \tau''[\tau_1'/a_1, \ldots, \tau_m'/a_m] = \tau \end{array}}{(c :: \tau\ s_1 \ldots s_{ar(c)}) :: \tau_{ar(c)+1}}$$

Labeling algorithm: Labeling of s starts with s^{top}. The rules from below are applied until no more labeling is possible or until a fail occurs, where $a \vee b$ means label a or label b.

$$\begin{array}{ll} (s\ t)^{\mathsf{sub} \vee \mathsf{top}} & \to (s^{\mathsf{sub}}\ t)^{\mathsf{vis}} \\ (\texttt{letrec}\ \mathit{Env}\ \texttt{in}\ s)^{\mathsf{top}} & \to (\texttt{letrec}\ \mathit{Env}\ \texttt{in}\ s^{\mathsf{sub}})^{\mathsf{vis}} \\ (\texttt{letrec}\ x = s, \mathit{Env}\ \texttt{in}\ C[x^{\mathsf{sub}}]) & \to (\texttt{letrec}\ x = s^{\mathsf{sub}}, \mathit{Env}\ \texttt{in}\ C[x^{\mathsf{vis}}]) \\ (\texttt{letrec}\ x = s, y = C[x^{\mathsf{sub}}], \mathit{Env}\ \texttt{in}\ t) & \to (\texttt{letrec}\ x = s^{\mathsf{sub}}, y = C[x^{\mathsf{vis}}], \mathit{Env}\ \texttt{in}\ t), \text{if } C \neq [\cdot] \\ (\texttt{letrec}\ x = s, y = x^{\mathsf{sub}}, \mathit{Env}\ \texttt{in}\ t) & \to (\texttt{letrec}\ x = s^{\mathsf{sub}}, y = x^{\mathsf{nontarg}}, \mathit{Env}\ \texttt{in}\ t) \\ (\texttt{seq}\ s\ t)^{\mathsf{sub} \vee \mathsf{top}} & \to (\texttt{seq}\ s^{\mathsf{sub}}\ t)^{\mathsf{vis}} \\ (\texttt{case}_K\ s\ \mathit{alts})^{\mathsf{sub} \vee \mathsf{top}} & \to (\texttt{case}_K\ s^{\mathsf{sub}}\ \mathit{alts})^{\mathsf{vis}} \\ \texttt{letrec}\ x = s^{\mathsf{vis} \vee \mathsf{nontarg}}, y = C[x^{\mathsf{sub}}] \ldots & \to \texttt{Fail} \\ \texttt{letrec}\ x = C[x^{\mathsf{sub}}], \mathit{Env}\ \texttt{in}\ t & \to \texttt{Fail} \end{array}$$

Fig. 1. Syntax of expressions and types, typing rules, and rules for labeling

The syntax of expressions and types of LRPw is defined in Fig. 1, where we assume that variables have a fixed type, written as $x :: \rho$. The calculus LRPw extends the lambda-calculus by recursive let-expressions, data constructors, `case`-expressions (for every type constructor K), `seq`-expressions and by type abstractions $\Lambda a.s$ and type applications $(s\ \tau)$ in order to express polymorphic functions and type instantiation, and by the shared work-decorations $a := n$ and $^{[a]}$. Polymorphically typed variables are only permitted for usual bindings of let-environments; at other places, the language is monomorphic where the concrete types can be computed through type reductions. For example, the identity can be written as $\Lambda a.\lambda x :: a.x$, and an application to the constant `True` is written $(\Lambda a.\lambda x :: a.x)$ `Bool True`. The reduction is $(\Lambda a.\lambda x :: a.x)$ `Bool True` $\rightarrow (\lambda x ::$ `Bool`$.x)$ `True` $\rightarrow$ (`letrec` $x =$ `True in` x). An expression s is *well-typed* with type τ (polymorphic type ρ, resp.), written as $s :: \tau$ (or $s :: \rho$, resp.), if s can be typed with the typing rules in Fig. 1 with type τ (ρ, resp.).

The calculus LR [11] is the untyped variant of LRPw (without shared work-decorations), where types and type-reduction are removed. In the following we often ignore the types and omit the types at variables and also sometimes omit the type reductions. We use some abbreviations: We write $\lambda x_1, \ldots, x_n.s$ instead of $\lambda x_1.\ldots.\lambda x_n.s$. A `letrec`-environment (or a part of it) is abbreviated by Env, and with $\{x_{g(i)} = s_{f(i)}\}_{i=j}^{m}$ we abbreviate the bindings $x_{g(j)} = s_{f(j)}, \ldots, x_{g(m)} = s_{g(m)}$. We write `if` s `then` t_1 `else` t_2 instead of $\texttt{case}_{\texttt{Bool}}\ s$ (`True ->` t_1) (`False ->` t_2). Alternatives of `case`-expressions are abbreviated by *alts*. Constructor applications $(c_{K,i}\ s_1\ \ldots\ s_{ar(c_{K,i})})$ are abbreviated using vector notation, omitting the index as $c\,\overrightarrow{s}$.

We use $FV(s)$ and $BV(s)$ to denote the free and bound variables of an expression s, and $FN(s)$ and $BN(s)$ to denote the free and bound label-names of an expression s. An expression s is closed iff $FV(s) = \emptyset$ and $FN(s) = \emptyset$. In an environment $Env = \{x_i = t_i\}_{i=1}^{n}$, we define $LV(Env) = \{x_1, \ldots, x_n\}$.

A *value* is an abstraction $\lambda x.s$, a type abstraction $\Lambda a_1.\ldots.\Lambda a_n.\lambda x.s$, or a constructor application $c\,\overrightarrow{s}$.

A *context* C is an expression with exactly one hole $[\cdot]$ at expression position. The reduction rules of the calculus are in Fig. 2. The operational semantics of LRPw is defined by the normal order reduction strategy which is a call-by-need strategy, i.e. a call-by-name strategy adapted to sharing. The labeling algorithm shown in Fig. 1 is used to detect the position to which a reduction rule is applied according to normal order, and the labelings in the expressions in Fig 2 indicate the exact place and positions of the expressions and subexpressions involved in the reduction step. It uses the labels: top,sub,vis,nontarg where top means reduction of the top term, sub means reduction of a subterm, vis marks already visited subexpressions, and nontarg marks already visited variables that are not target of a (cp)-reduction. Note that the labeling algorithm does not descend into sub-labeled `letrec`-expressions. The rules of the labeling algorithm are in Fig. 1. If the labeling algorithm terminates, then we say the termination is successful, and a potential normal order redex is found, which can only be the direct superterm of the sub-marked subexpression. It is possible that there is no normal order reduction: in this case either the evaluation is already finished, or it is a dynamically detected error (like a type-error), or the labeling fails.

Definition 2.1. *Let t be an expression. Then a* normal order reduction step $t \xrightarrow{\text{LRPw}} t'$ *is defined by first applying the labeling algorithm to t, and if the labeling algorithm terminates successfully, then one of the rules in Fig. 2 has to be applied resulting in t', if possible, where the labels* sub, vis *must match the labels in the expression t.*

A weak head normal form (WHNF) *is a value v, or an expression* `letrec` Env `in` v, *where v is a value, or an expression* `letrec` $x_1 = c\,\overrightarrow{t}, \{x_i = x_{i-1}\}_{i=2}^{m}, Env$ `in` x_m.

An expression s converges, *denoted as $s{\downarrow_{\text{LRPw}}}$, iff there is a normal-order reduction $s \xrightarrow{\text{LRPw},*} s'$, where s' is a WHNF. This may also be denoted as $s \downarrow_{\text{LRPw}} s'$. If not $s{\downarrow}$, we write $s{\uparrow}$. With $\bot$ we denote a diverging, closed expression.*

Note that there are diverging expressions of any type, for example `letrec` $x :: \rho = x$ `in` x.

The calculus LRP is the subcalculus of LRPw which does not have the syntactic constructs $a := n$

(lbeta) $C[((\lambda x.s)^{\mathsf{sub}}\ r)] \to C[\mathtt{letrec}\ x = r\ \mathtt{in}\ s]$

(Tbeta) $((\Lambda a.u)^{\mathsf{sub}}\ \tau) \to u[\tau/a]$

(cp-in) $\mathtt{letrec}\ x_1 = v^{\mathsf{sub}}, \{x_i = x_{i-1}\}_{i=2}^{m}, Env\ \mathtt{in}\ C[x_m^{\mathsf{vis}}] \to \mathtt{letrec}\ x_1 = v, \{x_i = x_{i-1}\}_{i=2}^{m}, Env\ \mathtt{in}\ C[v]$
where v is a polymorphic abstraction

(cp-e) $\mathtt{letrec}\ x_1 = v^{\mathsf{sub}}, \{x_i = x_{i-1}\}_{i=2}^{m}, Env, y = C[x_m^{\mathsf{vis}}]\ \mathtt{in}\ r$
$\to \mathtt{letrec}\ x_1 = v, \{x_i = x_{i-1}\}_{i=2}^{m}, Env, y = C[v]\ \mathtt{in}\ r$ where v is a polymorphic abstraction

(llet-in) $(\mathtt{letrec}\ Env_1\ \mathtt{in}\ (\mathtt{letrec}\ Env_2\ \mathtt{in}\ r)^{\mathsf{sub}}) \to (\mathtt{letrec}\ Env_1, Env_2\ \mathtt{in}\ r)$

(llet-e) $\mathtt{letrec}\ Env_1, x = (\mathtt{letrec}\ Env_2\ \mathtt{in}\ t)^{\mathsf{sub}}\ \mathtt{in}\ r \to \mathtt{letrec}\ Env_1, Env_2, x = t\ \mathtt{in}\ r$

(lapp) $C[((\mathtt{letrec}\ Env\ \mathtt{in}\ t)^{\mathsf{sub}}\ s)] \to C[(\mathtt{letrec}\ Env\ \mathtt{in}\ (t\ s))]$

(lcase) $C[(\mathtt{case}_K\ (\mathtt{letrec}\ Env\ \mathtt{in}\ t)^{\mathsf{sub}}\ alts)] \to C[(\mathtt{letrec}\ Env\ \mathtt{in}\ (\mathtt{case}_K\ t\ alts))]$

(seq-c) $C[(\mathtt{seq}\ v^{\mathsf{sub}}\ t)] \to C[t]$ if v is a value

(seq-in) $(\mathtt{letrec}\ x_1 = (c\ \overrightarrow{s})^{\mathsf{sub}}, \{x_i = x_{i-1}\}_{i=2}^{m}, Env\ \mathtt{in}\ C[(\mathtt{seq}\ x_m^{\mathsf{vis}}\ t)]) \to (\mathtt{letrec}\ x_1 = (c\ \overrightarrow{s}), \{x_i = x_{i-1}\}_{i=2}^{m}, Env\ \mathtt{in}\ C[t])$

(seq-e) $(\mathtt{letrec}\ x_1 = (c\ \overrightarrow{s})^{\mathsf{sub}}, \{x_i = x_{i-1}\}_{i=2}^{m}, Env, y = C[(\mathtt{seq}\ x_m^{\mathsf{vis}}\ t)]\ \mathtt{in}\ r)$
$\to (\mathtt{letrec}\ x_1 = (c\ \overrightarrow{s}), \{x_i = x_{i-1}\}_{i=2}^{m}, Env, y = C[t]\ \mathtt{in}\ r)$

(lseq) $C[(\mathtt{seq}\ (\mathtt{letrec}\ Env\ \mathtt{in}\ s)^{\mathsf{sub}}\ t)] \to C[(\mathtt{letrec}\ Env\ \mathtt{in}\ (\mathtt{seq}\ s\ t))]$

(case-c) $C[\mathtt{case}_K\ (c\ \overrightarrow{t})^{\mathsf{sub}} \dots ((c\ \overrightarrow{y}) \to t) \dots] \to C[\mathtt{letrec}\ \{y_i = t_i\}_{i=1}^{ar(c)}\ \mathtt{in}\ t]$ if $ar(c) \geq 1$

(case-c) $C[(\mathtt{case}_K\ c^{\mathsf{sub}} \dots (c \to t) \dots)] \to C[t]$ if $ar(c) = 0$

(case-in) $\mathtt{letrec}\ x_1 = (c\ \overrightarrow{t})^{\mathsf{sub}}, \{x_i = x_{i-1}\}_{i=2}^{m}, Env\ \mathtt{in}\ C[\mathtt{case}_K\ x_m^{\mathsf{vis}} \dots ((c\ \overrightarrow{z}) \to t) \dots]$
$\to \mathtt{letrec}\ x_1 = (c\ \overrightarrow{y}), \{y_i = t_i\}_{i=1}^{ar(c)}, \{x_i = x_{i-1}\}_{i=2}^{m}, Env$
$\mathtt{in}\ C[\mathtt{letrec}\ \{z_i = y_i\}_{i=1}^{ar(c)}\ \mathtt{in}\ t]$ if $ar(c) \geq 1$ and where y_i are fresh variables

(case-in) $\mathtt{letrec}\ x_1 = c^{\mathsf{sub}}, \{x_i = x_{i-1}\}_{i=2}^{m}, Env\ \mathtt{in}\ C[\mathtt{case}_K\ x_m^{\mathsf{vis}} \dots (c \to t) \dots]$
$\to \mathtt{letrec}\ x_1 = c, \{x_i = x_{i-1}\}_{i=2}^{m}, Env\ \mathtt{in}\ C[t]$ if $ar(c) = 0$

(case-e) $\mathtt{letrec}\ x_1{=}(c\ \overrightarrow{t})^{\mathsf{sub}}, \{x_i{=}x_{i-1}\}_{i=2}^{m}, u{=}C[\mathtt{case}_K\ x_m^{\mathsf{vis}} \dots ((c\ \overrightarrow{z}) \to r) \dots], Env\ \mathtt{in}\ s$
$\to \mathtt{letrec}\ x_1{=}(c\ \overrightarrow{y}), \{y_i{=}t_i\}_{i=1}^{n}, \{x_i{=}x_{i-1}\}_{i=2}^{m}, u{=}C[\mathtt{letrec}\ \{z_i{=}y_i\}_{i=1}^{n}\ \mathtt{in}\ r], Env$
$\mathtt{in}\ s$ where $n = ar(c) \geq 1$ and y_i are fresh variables

(case-e) $\mathtt{letrec}\ x_1 = c^{\mathsf{sub}}, \{x_i = x_{i-1}\}_{i=2}^{m}, u = C[\mathtt{case}_K\ x_m^{\mathsf{vis}}\ \dots\ (c \to r_1) \dots], Env\ \mathtt{in}\ r_2$
$\to \mathtt{letrec}\ x_1 = c, \{x_i = x_{i-1}\}_{i=2}^{m}, u = C[r_1], Env\ \mathtt{in}\ r_2$ if $ar(c) = 0$

(letwn-in) $\mathtt{letrec}\ Env, a := n,\ \mathtt{in}\ C[(s^{[a]})^{\mathsf{sub}}] \to \mathtt{letrec}\ Env, a := n-1\ \mathtt{in}\ C[s^{[a]}]$ if $n > 0$

(letwn-e) $\mathtt{letrec}\ a := n, x = C[(s^{[a]})^{\mathsf{sub}}], Env\ \mathtt{in}\ r \to \mathtt{letrec}\ a := n-1, x = C[s^{[a]}], Env\ \mathtt{in}\ r$ if $n > 0$

(letw0-in) $\mathtt{letrec}\ Env, a := 0,\ \mathtt{in}\ C[(s^{[a]})^{\mathsf{sub}}] \to \mathtt{letrec}\ Env, a := 0\ \mathtt{in}\ C[s]$

(letw0-e) $\mathtt{letrec}\ a := 0, x = C[(s^{[a]})^{\mathsf{sub}}], Env\ \mathtt{in}\ r \to \mathtt{letrec}\ a := 0, x = C[s], Env\ \mathtt{in}\ r$

Fig. 2. Reduction rules

and $s^{[a]}$, and the operational semantics of LRP does not have the reduction rules (letwn) and (letw0). WHNFs are defined as in LRPw. Convergence $\downarrow_{\mathrm{LRP}}$ is defined accordingly.

Lemma 2.2. *For every* LRPw*-expression s which is also an* LRP*-expression (i.e. s has no decorations and no $a := n$-construct): $s \downarrow_{\mathrm{LRPw}} \iff s \downarrow_{\mathrm{LRP}}$.*

Remark 2.3. The relation between the typed reduction in LRP and the untyped reduction in LR [11, 7] is that the removal of types and the reduction (Tbeta) results exactly in the untyped normal-order reduction. This also holds for WHNFs and the convergence notions. An immediate consequence is that the untyped contextual approximations and equivalences can be inherited to the typed LRP, since the typed contexts are also untyped ones.

We define some special context classes:

Definition 2.4. *A* reduction context *R is any context, such that its hole will be labeled with* sub *or* top *by the labeling algorithm in Fig. 1. A* weak reduction context*, R^-, is a reduction context, where the hole is not within a* letrec*-expression.* Surface contexts *S are contexts where the hole is not in an abstraction,* top contexts *T are surface contexts where the hole is not in an alternative of a case, and* weak top contexts *are top contexts where the hole does not occur in a* letrec*. A context C is* strict *iff $C[\bot] \sim_c \bot$.*

(gc1)	$\texttt{letrec}\ \{x_i = s_i\}_{i=1}^{n}, Env\ \texttt{in}\ t \to \texttt{letrec}\ Env\ \texttt{in}\ t$, if for all $i : x_i \notin FV(t, Env)$
(gc2)	$\texttt{letrec}\ x_1 = s_1, \ldots, x_n = s_n\ \texttt{in}\ t \to t$, if for all $i : x_i \notin FV(t)$
(gcW1)	$\texttt{letrec}\ Env, a_1 := n_1, \ldots, a_m := n_m\ \texttt{in}\ s \to \texttt{letrec}\ Env\ \texttt{in}\ s$, if labels $a_1, \ldots, a_m$ do not occur in Env or t
(gcW2)	$\texttt{letrec}\ a_1 := n_1, \ldots, a_m := n_m\ \texttt{in}\ s \to s$, if labels $a_1, \ldots, a_m$ do not occur in t
(cpx-in)	$\texttt{letrec}\ x = y, Env\ \texttt{in}\ C[x] \to \texttt{letrec}\ x = y, Env\ \texttt{in}\ C[y]$, if $y \in Var, x \neq y$
(cpx-e)	$\texttt{letrec}\ x{=}y, z{=}C[x], Env\ \texttt{in}\ t \to \texttt{letrec}\ x{=}y, z{=}C[y], Env\ \texttt{in}\ t$, if $y \in Var, x \neq y$
(cpax)	$\texttt{letrec}\ x = y, Env\ \texttt{in}\ s \to \texttt{letrec}\ x = y, Env[y/x]\ \texttt{in}\ s[y/x]$, if $y \in Var, x \neq y, y \in FV(s, Env)$
(cpcx-in)	$\texttt{letrec}\ x = c\,\overrightarrow{t}, Env\ \texttt{in}\ C[x] \to \texttt{letrec}\ x = c\,\overrightarrow{y}, \{y_i = t_i\}_{i=1}^{ar(c)}, Env\ \texttt{in}\ C[c\,\overrightarrow{y}]$
(cpcx-e)	$\texttt{letrec}\ x = c\,\overrightarrow{t}, z = C[x], Env\ \texttt{in}\ t \to \texttt{letrec}\ x = c\,\overrightarrow{y}, \{y_i = t_i\}_{i=1}^{ar(c)}, z = C[c\,\overrightarrow{y}], Env\ \texttt{in}\ t$
(abs)	$\texttt{letrec}\ x = c\,\overrightarrow{t}, Env\ \ \texttt{in}\ s \to \texttt{letrec}\ x = c\,\overrightarrow{x}, \{y_i = t_i\}_{i=1}^{ar(c)}, Env\ \texttt{in}\ s$
(abse)	$(c\,\overrightarrow{t}) \to \texttt{letrec}\ \{y_i = t_i\}_{i=1}^{ar(c)}\ \texttt{in}\ c\,\overrightarrow{x}$
(xch)	$\texttt{letrec}\ x = t, y = x, Env\ \texttt{in}\ r\ \to\ \texttt{letrec}\ y = t, x = y, Env\ \texttt{in}\ r$
(lwas)	$T[\texttt{letrec}\ Env\ \texttt{in}\ t] \to \texttt{letrec}\ Env\ \texttt{in}\ T[t]$ if T is a weak top context with hole depth 1
(letsh1)	$\texttt{letrec}\ Env, Env'\ \texttt{in}\ T[s] \to \texttt{letrec}\ Env'\ \texttt{in}\ T[(\texttt{letrec}\ Env\ \texttt{in}\ s)]$
(letsh2)	$\texttt{letrec}\ Env, Env', y = T[s]\ \texttt{in}\ r \to \texttt{letrec}\ Env', y = T[(\texttt{letrec}\ Env\ \texttt{in}\ s)]\ \texttt{in}\ r$
(letsh3)	$\texttt{letrec}\ Env\ \texttt{in}\ T[s] \to T[(\texttt{letrec}\ Env\ \texttt{in}\ s)]$ where in the (letsh)-rules, the variables $LV(Env)$ only occur in s, and T is weak top context that does not bind the variables in Env.
(ucp1)	$\texttt{letrec}\ Env, x = t\ \texttt{in}\ S[x] \to \texttt{letrec}\ Env\ \texttt{in}\ S[t]$
(ucp2)	$\texttt{letrec}\ Env, x = t, y = S[x]\ \texttt{in}\ r \to \texttt{letrec}\ Env, y = S[t]\ \texttt{in}\ r$
(ucp3)	$\texttt{letrec}\ x = t\ \texttt{in}\ S[x] \to S[t]$ where in the (ucp)-rules, $x \notin FV(S, Env, t, r)$ and S is a surface context

Fig. 3. Extra Transformation Rules

A program transformation P is binary relation on expressions. We write $s \xrightarrow{P} t$, if $(s,t) \in P$. For a set of contexts X and a transformation P, the transformation (X, P) is the closure of P w.r.t. the contexts in P, i.e. $s \xrightarrow{X,P} t$ iff there exists $C \in X$ with $C[s] \xrightarrow{P} C[t]$.

Definition 2.5. *We define several unions of the program transformations in Figs. 2 (ignoring the labels) and 3: (case) is the union of (case-c), (case-in), (case-e); (seq) is the union of (seq-c), (seq-in), (seq-e); (cp) is the union of (cp-in), (cp-e); (llet) is the union of (llet-in), (llet-e); (lll) is the union of (lapp),(lcase),(lseq),(llet-in),(llet-e); (letwn) is the union of (letwn-in), (letwn-e); (letw0) is the union of (letw0-in), (letw0-e); (letw) is the union of (letwn), (letw0); (gc) is the union of (gc1), (gc2); (cpx) is the union of (cpx-in), (cpx-e); (cpcx) is the union of (cpcx-in), (cpcx-e); (letsh) is the union of (letsh1), (letsh2), (letsh3), and (ucp) is the union of (ucp1), (ucp2), (ucp3).*

2.1 Improvement in LRP and LRPw

The main measure for estimating the time consumption of computation in this paper is a measure counting *essential* reduction steps in the normal-order reduction of expressions. We omit the type reductions in this measure, since these are always terminating and usually can be omitted after compilation. See [9] for more detailed explanations.

We define the essential reduction length for both calculi, where we allow some freedom in which reduction rules (as a subset of $\{lbeta, case, seq, letwn\}$) should be seen as essential. Clearly, we require that *letwn*-reductions are always counted (since they should represent work). We also require that (lbeta)-reductions are always counted, since there are expressions which have no (case)- or (seq)-reductions but an unbounded number of (lbeta)-reductions (see [8]).

Definition 2.6. *Let $\mathfrak{A} = \{lbeta, case, seq, letwn\}$, $A_{min} = \{letwn, lbeta\}$ and $A_{min} \subseteq A \subseteq \mathfrak{A}$. Let $L \in \{\mathrm{LRP}, \mathrm{LRPw}\}$ and let t be a closed L-expression with $t{\downarrow_L}t_0$. Then $\mathtt{rln}_A(t)$ is the number of a-*

reductions in the normal order reduction $t{\downarrow_L}t_0$ where $a \in A$. It is consistent to define the measure as ∞, if $t{\uparrow_L}$. For a reduction $t \xrightarrow{L,*} t'$, we define $\mathtt{rln}(t \xrightarrow{L,*} t')$ as the number of (lbeta)-, (case)-, (seq)-, and (in LRPw*) (letwn)-reductions in it.*

We define contextual equivalence and the improvement relation for both calculi LRPw and LRP:

Definition 2.7. *For $L \in \{\mathrm{LRP}, \mathrm{LRPw}\}$, let s, t be two L-expressions of the same type ρ and let $A_{min} \subseteq A \subseteq \mathfrak{A}$.*

- *s is* contextually smaller than *t, $s \leq_{c,L} t$, iff for all L-contexts $C[\cdot :: \rho]$: $C[s]{\downarrow_L} \implies C[t]{\downarrow_L}$.*
- *s and t are* contextually equivalent*, $s \sim_{c,L} t$, iff for all L-contexts $C[\cdot :: \rho]$: $C[s]{\downarrow_L} \iff C[t]{\downarrow_L}$.*
- *s A-*improves *t, $s \preceq_{A,L} t$, iff $s \sim_{c,L} t$ and for all L-contexts $C[\cdot :: \rho]$ s.t. $C[s], C[t]$ are closed: $\mathtt{rln}_A(C[s]) \leq \mathtt{rln}_A(C[t])$. If $s \preceq_{A,L} t$ and $t \preceq_{A,L} s$, we write $s \approx_{A,L} t$.*

A program transformation P is correct *(in L) if $P \subseteq \sim_{c,L}$ and it is an A-*improvement *iff $\xrightarrow{P} \subseteq (\preceq_{A,L})^{-1}$.*

The following context lemma for contextual equivalence holds in LRP and also in LRPw. The proof is standard, so we omit it.

Lemma 2.8 (Context Lemma for Equivalence). *Let $L \in \{\mathrm{LRP}, \mathrm{LRPw}\}$ and let s, t be L-expressions of the same type. Then $s \leq_c t$ iff for all $C \in \{R, S, T\}$: $C[s]{\downarrow_L} \implies C[t]{\downarrow_L}$.*

Let $\eta \in \{\leq, =, \geq\}$ be a relation on non-negative integers, X be a class of contexts X (we will instantiate X with: all contexts C; all reduction contexts R; all surface contexts S; or all top-contexts T), and let $A_{min} \subseteq A \subseteq \mathfrak{A}$. For expressions s, t of type ρ, let $s \bowtie_{A,\eta,X} t$ iff for all X-contexts $X[\cdot : \rho]$, s.t. $X[s], X[t]$ are closed: $\mathtt{rln}_A(X[s])\ \eta\ \mathtt{rln}_A(X[t])$. In particular, $\bowtie_{A,\leq,C} = \preceq_A$, $\bowtie_{A,\geq,C} = \succeq_A$, and $\bowtie_{A,=,C} = \approx_A$.

In the following we formulate statements for the calculus LRPw, if not stated otherwise.

The context lemma for improvement shows that it suffices to take reduction contexts into account for proving improvement. Its proof is similar to the ones for context lemmas for contextual equivalence in call-by-need lambda calculi (see [7, 11, 5]).

Lemma 2.9 (Context Lemma for Improvement). *Let s, t be expressions with $s \sim_c t$, $\eta \in \{\leq, =, \geq\}$, and let $X \in \{R, S, T\}$. Then $s \bowtie_{A,\eta,X} t$ iff $s \bowtie_{A,\eta,C} t$.*

Proof. The proof is nearly a complete copy of the proof of the context lemma for improvement in LRP (see [7]). However, for the sake of completeness we include it:

One direction is trivial. For the other direction we prove a more general claim using multicontexts where we assume A to be fixed as stated in the lemma:

> *For all $n \geq 0$ and for all $i = 1, \ldots, n$ let s_i, t_i be expressions with $s_i \sim_c t_i$ and $s_i \bowtie_{A,\eta,R} t_i$. Then for all multicontexts M with n holes such that $M[s_1, \ldots, s_n]$ and $M[t_1, \ldots, t_n]$ are closed: $\mathtt{rln}_A(M[s_1, \ldots, s_n])\ \eta\ \mathtt{rln}_A(M[t_1, \ldots, t_n])$.*

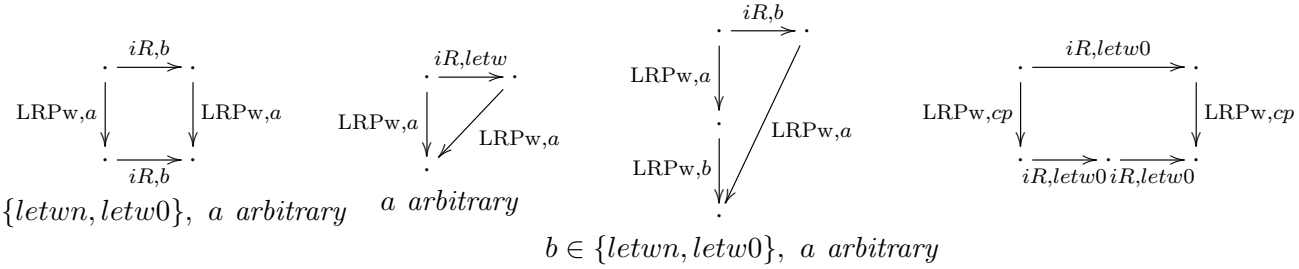
The proof is by induction on the pair (k, k') where k is the number of normal order reductions of $M[s_1, \ldots, s_n]$ to a WHNF, and k' is the number of holes of M. If M (without holes) is a WHNF, then the claim holds. If $M[s_1, \ldots, s_n]$ is a WHNF, and no hole is in a reduction context, then also $M[t_1, \ldots, t_n]$ is a WHNF and $\mathtt{rln}_A(M[s_1, \ldots, s_n]) = 0 = \mathtt{rln}_A(M[t_1, \ldots, t_n])$.

If in $M[s_1, \ldots, s_n]$ one s_i is in a reduction context, then one hole, say i of M is in a reduction context and the context $M[t_1, \ldots, t_{i-1}, \cdot, t_{i+1}, \ldots, t_n]$ is a reduction context. By the induction hypothesis, using the multi-context $M[\ldots, \cdot, s_i, \cdot, \ldots]$, we have $\mathtt{rln}_A(M[s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n])\ \eta\ \mathtt{rln}_A(M[t_1, \ldots, t_{i-1}, s_i, t_{i+1}, \ldots, t_n])$, and from the assumption we have $\mathtt{rln}_A(M[t_1, \ldots, t_{i-1}, s_i, t_{i+1}, \ldots, t_n])\ \eta\ \mathtt{rln}_A(M[t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n])$, and hence $\mathtt{rln}_A(M[s_1, \ldots, s_n])\ \eta\ \mathtt{rln}_A(M[t_1, \ldots, t_n])$.

If in $M[s_1,\ldots,s_n]$ there is no s_i in a reduction context, then $M[s_1,\ldots,s_n] \xrightarrow{\text{LRPw},a} M'[s'_1,\ldots,s'_{n'}]$, may copy or shift some of the s_i where $s'_j = \rho(s_i)$ for some permutation ρ on variables and on the sharing labels. However, the reduction type is the same for the first step of $M[s_1,\ldots,s_n]$ and $M[t_1,\ldots,t_n]$, i.e. $M[t_1,\ldots,t_n] \xrightarrow{\text{LRPw},a} M'[t'_1,\ldots,t'_{n'}]$ with $(s'_j,t'_j) = (\rho(s_i),\rho(t_i))$. We take for granted that the renaming can be carried through. The $\mathtt{rln}_A(.)$-count on both sides is $m = 0$ or $m = 1$, depending on whether or not $a \in A$ holds. Thus we can apply the induction hypothesis to $M'[s'_1,\ldots,s'_{n'}]$ and $M[t'_1,\ldots,t'_{n'}]$, and so we have $\mathtt{rln}_A(M[s_1,\ldots,s_n]) = m + \mathtt{rln}_A(M'[s'_1,\ldots,s'_{n'}])$ η $m + \mathtt{rln}_A(M'[t'_1,\ldots,t'_{n'}]) = \mathtt{rln}_A(M[t_1,\ldots,t_n])$.

We now use the context lemma and the context lemma for improvement to show several properties about the reduction rules and the additional transformation rules.

Lemma 2.10. *A complete set of forking and commuting diagrams for internal (letw)-transformations applied in reduction contexts can be read off the following diagrams:*



Proof. The first diagram describes the case where the transformation and the normal order reduction commute. It also includes cases where a (letw-in)-transformation is flipped into an (letw-e)-transformation, if the normal order reduction is (LRPw,llet). The second diagram describes the case where the a-labeled expression of the (letw)-transformation is removed by the normal order reduction, which may be the case if the expression is inside an unused alternative of `case` or inside the first argument of `seq`. The third diagram describes the case where the internal (letw)-transformation becomes a normal-order reduction. There are several cases where this may happen, e.g. for expressions of the form `letrec` *Env* `in letrec` $a := n$ `in` $C[s^{[a]}]$ where the normal order reduction is (LRPw,llet). The fourth diagram describes the case where an a-labeled expression is inside an abstraction which is copied by (LRPw,cp). If the transformation is a (letwn), then the transformations commute, but if the transformation is (letw0), then the transformation is duplicated, since it has to remove the a-label twice.

Lemma 2.11. *If $s \xrightarrow{iR,letw} t$ then s is a WHNF iff t is a WHNF.*

Lemma 2.12. *Let R be a reduction context and $s \xrightarrow{letw} t$. Then $R[s]\downarrow \iff R[t]\downarrow$.*

Proof. We split the proof in several parts:

- $R[s]\downarrow \implies R[t]\downarrow$: Assume that $R[s]\downarrow$ holds, and let $R[s] \xrightarrow{\text{LRPw},k} r$ where r is a WHNF. We show $R[t] \xrightarrow{\text{LRPw},k'} r'$ where r' is a WHNF, and $k' \le k$. We use induction on k. The base case $k = 0$ is covered by Lemma 2.11. For the induction step let $R[s] \xrightarrow{no} r_1 \xrightarrow{\text{LRPw},k-1} r$. If $R[s] \xrightarrow{\text{LRPw},letw} R[t]$, then $r_1 = R[t]$ and $R[t] \xrightarrow{\text{LRPw},k-1} r$ and thus the claim holds. If the reduction is internal, then apply a forking diagram to $r_1 \xleftarrow{no} R[s] \xrightarrow{\text{LRPw},letw} R[t]$.
  1. If the first diagram is applied, then $r_1 \xrightarrow{iR,letw} r'_1$, $R[t] \xrightarrow{no} r'_1$ and $r_1 \xrightarrow{\text{LRPw},k-1} r$. We apply the induction hypothesis to r_1 and r'_1 which shows $r'_1 \xrightarrow{\text{LRPw},k''} r'$ where r' is a WHNF and $k'' \le k-1$. Thus $R[t] \xrightarrow{\text{LRPw},k'} r'$ where r' is a WHNF and $k' \le k$
  2. If the second diagram is applied, then $R[t] \xrightarrow{no} r_1 \xrightarrow{\text{LRPw},k-1} r$ and thus the claim holds.

3. If the third diagram is applied, then $R[t] \xrightarrow{no} r_2 \xrightarrow{\text{LRPw},k-2} r$ (where $r_1 \xrightarrow{no} r_2$) and the claim holds.
4. In case of diagram (4) we apply the induction hypothesis twice for each $(iR, letw)$-transformation, which shows that $R[t] \xrightarrow{\text{LRPw},cp} r'_1 \xrightarrow{\text{LRPw},k''} r'$ where r' is a WHNF, $k'' \leq k-1$. Thus the claim holds.

– $R[t]\downarrow \implies R[s]\downarrow$. Let $\#cp(r)$ be the number of (LRPw,cp) reductions in the normal order reductions from r to a WHNF and $\#cp(r) = \infty$ if $r \uparrow$ Assume that $R[t] \xrightarrow{\text{LRPw},k} r$ where r is a WHNF. We show $R[s]\downarrow$ and $\#cp(R[s]) \leq \#cp(R[t])$ by induction on the measure $(\#cp(R[t]), k)$. For the base case (0,0) $R[t]$ is a WHNF and thus by Lemma 2.11 also $R[s]$ is a WHNF and the claim holds. For the induction step let $(l,k) > (0,0)$. Then $R[t] \xrightarrow{no} t' \xrightarrow{\text{LRPw},k-1} r$ where r is a WHNF. If $R[s] \xrightarrow{\text{LRPw},letw} R[t]$ then the claim holds: $R[s]\downarrow$ and $\#cp(R[s]) = \#cp(R[t])$. If the transformation is internal, then we apply a commuting diagram to $R[s] \xrightarrow{iR,letw} R[t] \xrightarrow{no} t_1$.
1. For the first diagram we have an expression s_1 s.t. $R[s] \xrightarrow{\text{LRPw},a} s_1$, $s_1 \xrightarrow{iR,letw} s_2$ and the measure for t_1 is $(\#cp(t_1), k-1)$ which is strictly smaller than (l,k) (since $\#cp(t_1) \leq l$). Thus we can apply the induction hypothesis and derive $s_1\downarrow$ and $\#cp(s_1) \leq \#cp(t_1)$. This shows $R[s]\downarrow$ and $\#cp(R[s]) \leq \#(R[t])$.
2. For the second diagram the claim obviously holds.
3. For the third diagram, the claim also holds.
4. For the last diagram, we apply the induction hypothesis twice, which is possible since $\#cp(\cdot)$ is strictly decreased.

Theorem 2.13. *The transformations (letw0) and (letwn) are correct.*

Proof. Correctness of the transformation (letw) follows from Lemma 2.12 and the context lemma.

Lemma 2.14. *Let $A_{min} \subseteq A \subseteq \mathfrak{A}$. If $s \xrightarrow{letw0} t$, then for all reduction contexts R, s.t. $R[s]$, $R[t]$ are closed: $\mathtt{rln}_A(R[s]) = \mathtt{rln}_A(R[t])$*

Proof. Since $(letw0)$ is correct we know that $\mathtt{rln}_A(R[s]) = \infty \iff \mathtt{rln}_A(R[t]) = \infty$. So suppose that $\mathtt{rln}_A(R[s]) = n$. We show $\mathtt{rln}_A(R[t]) = n$ by induction on a normal order reduction $R[s] \xrightarrow{\text{LRPw},k} s'$ where s' is a WHNF. The base case is covered by Lemma 2.11. For the induction step, let $R[s] \xrightarrow{no} s_1 \xrightarrow{\text{LRPw},k-1} s'$. If $R[s] \xrightarrow{\text{LRPw},letw0} R[t]$, then $\mathtt{rln}_A(R[s]) = \mathtt{rln}_A(R[t]) = \mathtt{rln}_A(s_1)$ and the claim holds. If the transformation is internal, then we apply a forking diagram to s_1. For the first diagram we have $s_1 \xrightarrow{iR,letw0} t_1$ and we apply the induction hypothesis to s_1 and thus have $\mathtt{rln}_A(s_1) = \mathtt{rln}_A(t_1)$. This also shows $\mathtt{rln}_A(R[s]) = \mathtt{rln}_A(R[t])$. For the second diagram the claim holds. For the third diagram the claim also holds, since the additional (LRPw,letw0)-reduction in the normal order reduction for $R[s]$ is not counted in the $\mathtt{rln}_A$-measure. For the fourth diagram we have $s_1 \xrightarrow{iR,letw0} s'_1 \xrightarrow{iR,letw0} t_1 \xleftarrow{\text{LRPw},cp} R[t]$. We apply the induction hypothesis twice: For s_1 we get $\mathtt{rln}_A(s_1) = \mathtt{rln}_A(s'_1)$ and for s'_1 we get $\mathtt{rln}_A(s'_1) = \mathtt{rln}_A(t_1)$ which finally shows $\mathtt{rln}_A(R[t]) = \mathtt{rln}_A(t_1) = \mathtt{rln}_A(s_1) = \mathtt{rln}_A(R[s])$.

The context lemma for improvement and the previous lemma imply:

Corollary 2.15. *For $A_{min} \subseteq A \subseteq \mathfrak{A}$: $(letw0) \subseteq \approx_A$.*

Lemma 2.16. *Let $A_{min} \subseteq A \subseteq \mathfrak{A}$. If $s \xrightarrow{letwn} t$, then for all reduction contexts R s.t. $R[s]$ and $R[t]$ are closed: $\mathtt{rln}_A(R[s]) = \mathtt{rln}_A(R[t])$ or $\mathtt{rln}_A(R[s]) = 1 + \mathtt{rln}_A(R[t])$.*

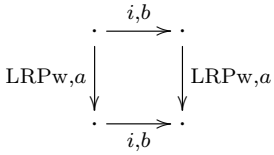
Proof. Since $(letwn)$ is correct we know that $\mathtt{rln}_A(R[s]) = \infty \iff \mathtt{rln}_A(R[t]) = \infty$. So suppose that $\mathtt{rln}_A(R[s]) = n$. We show $\mathtt{rln}_A(R[t]) = n$ or $\mathtt{rln}_A(R[t]) = n+1$ by induction on a normal order reduction $R[s] \xrightarrow{\text{LRPw},k} s'$ where s' is a WHNF. The base case is covered by Lemma 2.11. For the

induction step, let $R[s] \xrightarrow{no} s_1 \xrightarrow{\text{LRPw},k-1} s'$. If $R[s] \xrightarrow{\text{LRPw},letwn} R[t]$, then $\mathtt{rln}_A(R[s]) = 1 + \mathtt{rln}_A(R[t])$ and the claim holds. If the transformation is internal, then we apply a forking diagram to s_1. For the first diagram we have $s_1 \xrightarrow{iR,letwn} t_1$ and we apply the induction hypothesis to s_1 and thus have $\mathtt{rln}_A(s_1) = 1 + \mathtt{rln}_A(t_1)$ or $\mathtt{rln}_A(s_1) = \mathtt{rln}_A(t_1)$. This also shows $\mathtt{rln}_A(R[s]) = 1 + \mathtt{rln}_A(R[t])$ or $\mathtt{rln}_A(R[s]) = \mathtt{rln}_A(R[t])$. For the second diagram we have $\mathtt{rln}_A(R[s]) = \mathtt{rln}_A(R[t])$. For the third diagram we have $\mathtt{rln}_A(R[s]) = 1 + \mathtt{rln}_A(R[t])$. The fourth diagram is not applicable, since the given transformation is (letwn).

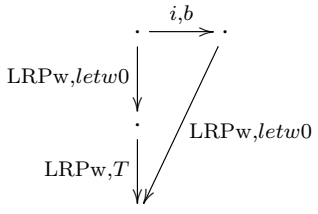
Corollary 2.17. *For $A_{min} \subseteq A \subseteq \mathfrak{A}$, (letwn) $\subseteq \succeq_A$.*

Proposition 2.18. *All reduction rules are correct.*

Proof. For the (letwn)-rules this is already proved. For the other rules, correctness was shown in the untyped calculus LR in [11], which can be directly transfered to LRP. However, LRPw has shared-work decorations and the (letwn)-rules as normal order reduction. To keep the proof compact, we only consider these new cases. The reasoning to show correctness of the reduction rules in LRPw is the same as for LR, since all additional diagrams between an internal transformation step (i, b) and a (LRPw, $letw$)-reduction are:



$a \in \{letwn, letw0\}, b \in \{lbeta, cp, case, seq, lll\}$



$b \in \{lbeta, cp, case, seq, lll\}$

The first case is the case where the (LRPw,letw) and the transformation commute, the second case is that the internal transformation becomes a normal order reduction after removing the a label. However, these cases are already covered by the diagram proofs in LR (see [11]) and thus can easily added.

We define a translation from expressions with work-decorations into decoration-free expressions, by removing the work-decorations and the corresponding bindings:

Definition 2.19. *Let t be an expression in* LRPw*, and* $\mathtt{rmw}(t)$ *be derived from t by removing the work-syntax, i.e.*

$$
\begin{array}{ll}
\mathtt{rmw}(\mathtt{letrec}\ x_1 = s_1, \ldots, x_n = s_n, a_1 := n_1, \ldots, a_m := n_m\ \mathtt{in}\ s) = & \\
\qquad \mathtt{letrec}\ x_1 = \mathtt{rmw}(s_1), \ldots, x_n = \mathtt{rmw}(s_n)\ \mathtt{in}\ \mathtt{rmw}(s) & \text{for } m \geq 0, n \geq 1 \\
\mathtt{rmw}(\mathtt{letrec}\ a_1 := n_1, \ldots, a_m := n_m\ \mathtt{in}\ s) & = \mathtt{rmw}(s), \\
\mathtt{rmw}(s^{[a]}) & = \mathtt{rmw}(s) \\
\mathtt{rmw}(f[s_1, \ldots, s_n]) & = f[\mathtt{rmw}(s_1), \ldots, \mathtt{rmw}(s_n)] \\
 & \quad \text{for all other language constructs } f.
\end{array}
$$

Proposition 2.20. *Let t be an expression in* LRPw*, then $t \downarrow_{\text{LRPw}} \iff \mathtt{rmw}(t) \downarrow_{\text{LRPw}}$.*

Proof. Observing that $t \xrightarrow{\text{LRPw}} t'$ implies $\mathtt{rmw}(t) = \mathtt{rmw}(t')$ or $\mathtt{rmw}(t) \xrightarrow{\text{LRPw}} \mathtt{rmw}(t')$, the proof is obvious.

An immediate consequence is the following theorem:

Theorem 2.21. *The embedding of* LRP *into* LRPw *w.r.t. $\sim_c$ is conservative.*

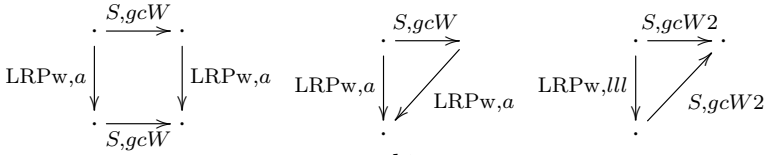
Considering conservativity of the embedding of LRP into LRPw w.r.t. the improvement relation $\approx_A$, we are able to show that for the case $seq \notin A$ this embedding is an isomorphism w.r.t. $\approx_A$ (this will be proved in Theorem 3.10). However, we do not know whether the embedding of LRP into LRPw is conservative w.r.t. the improvement relation $\preceq_A$ if $(seq) \in A$. We conjecture that the embedding of LRP into LRPw is conservative w.r.t. the improvement relation $\preceq_A$. However, we did not find a proof. A naive proof which tries to encode the work decorations by usual expressions fails, since there are work decorations which cannot be encoded (see Proposition 3.12) if $(seq) \in A$. However, conservativity is not really necessary. It would allow to lift results on improvements from LRP to LRPw more easily. Our goal to use the calculus LRPw as a proof technique to show results on improvements for LRP is possible:

Lemma 2.22. *Let $A_{min} \subseteq A \subseteq \mathfrak{A}$. Let s, t be* LRP*-expressions s.t. $s \preceq_{A,\mathrm{LRPw}} t$. Then also $s \preceq_{A,\mathrm{LRP}} t$ holds.*

Proof. This holds, since every LRP-context is also an LRPw-context and on decoration-free expressions the `rln`-length is the same in both calculi.

We prove correctness and (invariance w.r.t. $\approx$) for (gcW), the transformation which performs garbage collection of $a := n$-bindings which have no corresponding $^{[a]}$-label.

Lemma 2.23. *A complete set of forking and commuting diagrams for (S,gcW) can be read off the following diagrams:*



Proof. The first diagram covers the case where the transformation and the reduction commute. There are also cases where a (gcW2) becomes a (gcW1)-transformation, e.g. in `letrec` $x =$ (`letrec` $a := n$ `in` s) `in` $r \xrightarrow{S,gcW2}$ `letrec` $x = s$ `in` r where `letrec` $x =$ (`letrec` $a := n$ `in` s) `in` $r \xrightarrow{\mathrm{LRPw},llet}$ `letrec` $x = s, a := n$ `in` r. The second diagram covers the case where the (gcW)-redex is removed by the normal order reduction, e.g. if it is in an unused alternative of `case` or inside the first argument of `seq`. The last diagram covers the case where the `letrec`-expression of the redex of (LRPw,lll) is removed by (gcW2).

Lemma 2.24. *If $s \xrightarrow{S,gcW} t$ then*

- *If s is a WHNF, then t is a WHNF.*
- *If t is a WHNF, then $s \xrightarrow{\mathrm{LRPw},llet,0\vee1} s'$ where s' is a WHNF*

Proof. The first item can be easily verified. For the second item it may be the case that s is not a WHNF, but t is a WHNF, e.g. `letrec` $a := n$ `in` $r \xrightarrow{gcW2} r$ where r is a WHNF.

Proposition 2.25. *The transformation (gcW) is correct and for $A_{min} \subseteq A \subseteq \mathfrak{A}$: (gcW) $\subseteq \approx_A$.*

Proof. We first show correctness. Let $s \xrightarrow{S,gcW} t$

- $s\downarrow \implies t\downarrow$: This can be shown by induction on the length k in $s \xrightarrow{\mathrm{LRPw},k} s'$ where s' is a WHNF. For the base case Lemma 2.24 shows $t\downarrow$. For the induction step we apply a forking diagram. For the first diagram we have $s \xrightarrow{\mathrm{LRPw},a} s_1$, $s_1 \xrightarrow{S,gcW} t_1$, $t \xrightarrow{\mathrm{LRPw},a} t_1$. Applying the induction hypothesis to s_1 and t_1 shows $t_1\downarrow$ and thus $t\downarrow$. For the second diagram $t\downarrow$ obviously holds. For the third diagram we have $s \xrightarrow{\mathrm{LRPw},lll} s_1$, $s_1 \xrightarrow{gcW2} t$. We apply the induction hypothesis to s_1 and t which shows $t\downarrow$.

– $t\downarrow \Longrightarrow s\downarrow$: We use an induction in the length k in $t \xrightarrow{\text{LRPw},k} t'$ where t' is a WHNF. For the base case $k = 0$ Lemma 2.24 shows that $s\downarrow$. For the induction step we apply a forking diagram. For the first and the second diagram the cases are analogous to the previous part. For the third diagram we apply the diagram as long as possible which terminates, since there are no infinite sequences of (LRPw, , lll)-reductions. Then we get an expression s' with either $s \xrightarrow{\text{LRPw},lll,+} s'$ where s' is a WHNF and thus $s\downarrow$, or we apply the first or second diagram to t and s', and then the induction hypothesis (in case of diagram 1). In any case we derive $s\downarrow$.

The two items and the context lemma for $\sim_c$ show that (gcW) is correct. Now we consider improvement. Let $s \xrightarrow{S,gcW} t$. We show $\mathtt{rln}_A(s) = \mathtt{rln}_A(t)$. The context lemma for improvement then implies $(gcW) \subseteq \approx_A$. Since (gcW) is correct we already have $\mathtt{rln}_A(s) = \infty \iff \mathtt{rln}_A(t) = \infty$. Now let $s\downarrow s'$ (where $s \xrightarrow{\text{LRPw},k} s'$) and $\mathtt{rln}_A(s) = n$. We show $\mathtt{rln}_A(t) = n$ by induction on k. If $k = 0$ then Lemma 2.24 shows $\mathtt{rln}_A(s) = 0 = \mathtt{rln}_A(t)$. If $k > 0$ then we again apply the forking diagrams. The cases are completely analogous as for the correctness proof, where have to verify, that the first and the second diagram do either introduce nor remove normal order reductions, and the third diagram may only remove (LRPw, lll)-reductions which are not counted by the $\mathtt{rln}_A$-measure.

The following results from [11, 9] on the lengths of reductions also hold in the calculus LRPw, since the overlappings for (LRPw, $letw$) and the corresponding transformation are analogous to already covered cases.

Theorem 2.26. *Let t be a closed* LRPw*-expression with $t\downarrow t_0$ and $A_{min} \subseteq A \subseteq \mathfrak{A}$.*

1. *If $t \xrightarrow{C,\ a} t'$, and $a \in \mathfrak{A}$, then $\mathtt{rln}_A(t) \geq \mathtt{rln}_A(t')$.*
2. *Let t be a closed* LR*-expression with $t\downarrow t_0$ and $t \xrightarrow{C,cp} t'$, then $\mathtt{rln}_A(t) = \mathtt{rln}_A(t')$.*
3. *If $t \xrightarrow{S,a} t'$, and $a \in \mathfrak{A}$, then $\mathtt{rln}_A(t) \geq \mathtt{rln}_A(t')$ and $\mathtt{rln}_A(t') \geq \mathtt{rln}_A(t) - 1$ if $a \in A$, and $\mathtt{rln}_A(t') = \mathtt{rln}_A(t)$ if $a \notin A$.*
4. *If $t \xrightarrow{C,\ a} t'$, and $a \in \{lll, gc\}$, then $\mathtt{rln}_A(t) = \mathtt{rln}_A(t')$.*
5. *If $t \xrightarrow{C,a} t'$, and $a \in \{cpx, cpax, xch, cpcx, abs, lwas\}$, then $\mathtt{rln}_A(t) = \mathtt{rln}_A(t')$.*
6. *If $t \xrightarrow{C,\ ucp} t'$, then $\mathtt{rln}_A(t) = \mathtt{rln}_A(t')$.*

Corollary 2.27. *For $A_{min} \subseteq A \subseteq \mathfrak{A}$:*

1. *If $s \xrightarrow{S,a} s'$ where a is any rule from Figs. 2 and 3, then $s' \preceq_A s$.*
2. *If $s \xrightarrow{C,a} s'$ where a is (lll), (cp), (letw0) or any rule of Fig. 3. Then $s' \approx_A s$.*

Proof. The claims follow from Theorem 2.26 and the context lemma, and for the rule (letsh) the claim holds, since it is a composition of $(lwas)$ and $(llet)$ and their inverses. For (gcW) this follows from Proposition 2.25. For (letw0) it follows from Corollary 2.15, and for (letwn) it follows from Corollary 2.17.

3 Work Decorations

In this section we consider another notation for work decorations.

Definition 3.1. *For* LRPw *we use the following notation:*

work-decoration *: If $n \in \mathbb{N}$, then $s^{[n]}$ is an expression, where $[n]$ is called a* (unshared) work decoration*. The semantics of $s^{[n]}$ is* `letrec` $a := n$ `in` $s^{[a]}$ *where a is a fresh label.*

sharing decoration *: If a is label and $n \in \mathbb{N}$, then $C[s_1^{[a\mapsto n]}, \ldots, s_m^{[a\mapsto n]}]$ is an expression. The semantics of $C[s_1^{[a\mapsto n]}, \ldots, s_m^{[a\mapsto n]}]$ is* `letrec` $a := n$ `in` $C[s_1^{[a]}, \ldots, s_m^{[a]}]$

further notation: *For convenience, we also write several decorations in the form* $[n, a_1 \mapsto m_1, \ldots, a_k \mapsto m_k]$ *(where the* a_i *are distinct). We also write* $labels(X)$ *for the set of labels occuring in an expression or decoration* X*. The semantics of the expressions can be derived from the previous cases, where the nondeterminism in the translation is irrelevant, since (lll)-transformations allow to reorder and combine the corresponding environments without changing the* `rln`*-measure. We may also use the abstract notation* $[n, p]$ *for a sharing decoration with constant* n*, and further sharing decorations* p*.*

Note that LRPw contains expressions, which cannot be expressed by this notation. E.g., the expression $\lambda x.\texttt{letrec}\ a = n\ \texttt{in}\ C[s^{[a]}, t^{[a]}]$, since the semantic translation of $\lambda x.C[s^{[a]}, t^{[a]}]$ is $\texttt{letrec}\ a = n\ \texttt{in}\ \lambda x.C[s^{[a]}, t^{[a]}]$ which is a different expression.

We show that the (non-shared) work-decorations are redundant, and can be encoded by usual LRP-expressions.

Proposition 3.2. *The work decorations* $s^{[n]}$ *can be encoded as* $\texttt{letrec}\ x = (id^n)\ \texttt{in}\ (x\ s)$ *and thus are redundant.*

Proof. The proof is in Appendix A.

3.1 Computation Rules for Decorations

In this section we develop the computation rules with decorations.

First we define a combination of labels, since addition has to be modified. Here we assume that labels are sets consisting of exactly one nonnegative integer (a work-decoration) and several sharing decorations.

Definition 3.3. *The combination* $p_1 \oplus p_2$ *of two decorations* $p_1 = [n_1, p_1']$ *and* $p_2 = [n_2, p_2']$ *is defined as* $[n_1 + n_2, p_3]$*, where* $p_3 = p_1 \cup p_2$*.*
For two decorations $p_1 = [n_1, p_1']$ *and* $p_2 = [n_2, p_2']$ *we write* $p_1 \leq p_2$*, iff* $n_1 \leq n_2$ *and for all labels* a *that occur in* p_1, p_2*: if* $a \mapsto m_1$ *is in* p_1*, and* $a \mapsto m_2$ *is in* p_2*, then* $m_1 \leq m_2$*.*

For example, $[1, a_1 \mapsto 3, a_2 \mapsto 5] \oplus [2, a_1 \mapsto 3, a_3 \mapsto 7] = [3, a_1 \mapsto 3, a_2 \mapsto 5, a_3 \mapsto 7]$.

A corollary from the theorem on reduction lengths (Theorem 2.26) is:

Corollary 3.4. *Let* $A_{min} \subseteq A \subseteq \mathfrak{A}$ *and let* S *be a surface context. If* $s \xrightarrow{S} s'$ *by any reduction or transformation rule from Figs. 2 and 3, then* $s' \preceq_A s$ *and* $s \preceq_A s'^{[1]}$*.*

In Appendix B the following computation rules are proved:

Theorem 3.5. *Let* $A_{min} \subseteq A \subseteq \mathfrak{A}$*.*

1. *If* $s \xrightarrow{\text{LRPw},a} t$ *with* $a \in A$ *then* $s \approx_A t^{[1]}$*, and if* $a \notin A$*, then* $s \approx_A t$*.*
2. $R[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}] \approx_A \texttt{letrec}\ a := n\ \texttt{in}\ R[s]^{[a]}$ *and thus in particular* $R[s^{[n]}] \approx_A R[s]^{[n]}$*.*
3. $\texttt{rln}_A(\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}) = n + \texttt{rln}_A(s')$ *where* s' *is* s *where all* $^{[a]}$*-labels are removed. In particular this also shows* $\texttt{rln}_A(s^{[n]}) = n + \texttt{rln}_A(s)$
4. *For every reduction context* R*:* $\texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}]) = n + \texttt{rln}_A(R[s'])$ *where* s' *is* s *where all* $^{[a]}$*-labels are removed. In particular, this shows* $\texttt{rln}_A(R[s^{[n]}]) = n + \texttt{rln}_A(R[s])$*.*
5. $(s^{[n]})^{[m]} \approx_A s^{[n+m]}$
6. *For all surface contexts* S_1, S_2*:* $S_1[\texttt{letrec}\ a := n\ \texttt{in}\ S_2[s^{[a]}]] \preceq_A \texttt{letrec}\ a := n\ \texttt{in}\ S_1[S_2[s]]^{[a]}$ *and if* $S_1[S_2]$ *is strict, also* $S_1[\texttt{letrec}\ a := n\ \texttt{in}\ S_2[s^{[a]}]] \approx_A \texttt{letrec}\ a := n\ \texttt{in}\ S_1[S_2[s]]^{[a]}$*.*
   In particular, this shows for all surface contexts S *and expressions* s*:* $S[s^{[k]}] \preceq_A S[s]^{[k]}$*, and if* S *is strict, then* $S[s^{[k]}] \approx_A S[s]^{[k]}$*.*
7. $\texttt{letrec}\ a := n, b := m\ \texttt{in}\ (s^{[a]})^{[b]} \approx_A \texttt{letrec}\ a := n, b := m\ \texttt{in}\ (s^{[b]})^{[a]}$
8. $\texttt{letrec}\ a := n\ \texttt{in}\ (s^{[a]})^{[a]} \approx_A \texttt{letrec}\ a := n\ \texttt{in}\ (s^{[a]})$
9. $(t^{p_1})^{p_2} \approx_A t^{p_1 \oplus p_2}$*.*

10. *Let* $S[\cdot,\ldots,\cdot]$ *be a multi-context where all holes are in surface position. Then* $\texttt{letrec}\ a := n\ \texttt{in}\ S[s_1^{[a]},\ldots,s_n^{[a]}] \preceq_A \texttt{letrec}\ a := n\ \texttt{in}\ S[s_1,\ldots,s_n]^{[a]}$. *If some hole* $\cdot_i$ *with* $i \in \{1,\ldots,n\}$ *is in strict position in* $S[\ldots,.]$, *then* $\texttt{letrec}\ a := n\ \texttt{in}\ S[s_1^{[a]},\ldots,s_n^{[a]}] \approx_A \texttt{letrec}\ a := n\ \texttt{in}\ S[s_1,\ldots,s_n]^{[a]}$.
11. *Let* $S[\cdot,\ldots,\cdot]$ *be a multi-context where all holes are in surface position. Let* $S[s_1,\ldots,s_n]$ *be closed. Then* $S[s_1^{[p_1,a\mapsto m]},\ldots,s_n^{[p_n,a\mapsto m]}] \preceq_A S[s_1^{p_1},\ldots,s_n^{p_n}]^{[m]}$.
If some hole $\cdot_i$ *with* $i \in \{1,\ldots,n\}$ *is in strict position in* $S[\ldots,.]$, *then* $S[s_1^{[p_1,a\mapsto m]},\ldots,s_n^{[p_n,a\mapsto m]}] \approx_A S[s_1^{p_1},\ldots,s_n^{p_n}]^{[m]}$.
By iteratively applying the claim this shows for all surface contexts S *and expressions* s: $S[s^p] \preceq_A S[s]^p$, *and if* S *is strict, then* $S[s^p] \approx_A S[s]^p$.
12. *The following transformation is correct and invariant w.r.t.* $\approx_A$: *Replace* $(\texttt{letrec}\ x = s^{[n,p]}, Env\ \texttt{in}\ t)$ *by* $\texttt{letrec}\ x = s[x^{[a\mapsto n,p]}/x],\ Env[x^{[a\mapsto n,p]}/x]\ \texttt{in}\ t[x^{[a\mapsto n,p]}/x]$, *where* a *is a fresh label and all occurrences of* x *are in surface position.*
13. *If a label-name* a *occurs exactly once in a surface context, then it can be changed into an unshared work-decoration.*
14. *If* p, p' *are two decorations with* $p \le p'$, *and* $s \preceq_A t$, *then* $s^{[p]} \preceq_A t^{[p']}$.

An immediate consequence is:

Proposition 3.6. *The following variant of reduction is correct w.r.t. the* LRPw*-semantics:*

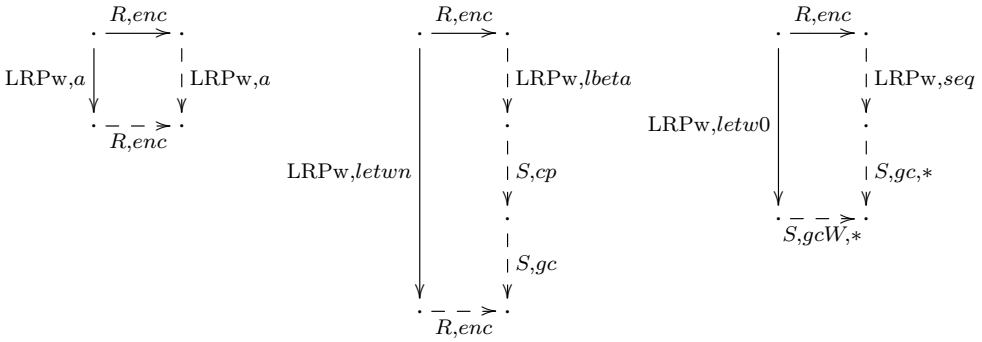
The reduction on LRP *expressions with shared work-decorations is as follows: If* $n > 0$ *and* $t = R[t_1^{[n]}]$, *then* $R[t_1^{[n]}] \xrightarrow{\text{LRPw}} R[t_1^{[n-1]}]$ *where this reduction contributes to the* `rln`*-measure. If* $n = 0$ *then* $R[t_1^{[0]}] \xrightarrow{\text{LRPw}} R[t_1]$ *where this reduction is not counted.*
If $n > 0$ *and* $t = R[t_1^{[a\mapsto n]}]$ *where* R *is a reduction context, where no decorations are on the path to the hole, then* $R[t^{[a\mapsto n]}] \xrightarrow{\text{LRPw}} R'[t^{[a\mapsto n-1]}]$, *where all* $[a \mapsto n]$*-decorations in* R *and* t *are changed into* $[a \mapsto n-1]$. *The reduction step also counts as one* `rln`*-reduction step, i.e.* $\texttt{rln}(R[t^{[a\mapsto n]}]) = 1 + \texttt{rln}(R[t^{[a\mapsto n-1]}])$. *If* $t = R[t_1^{[a\mapsto 0]}]$ *where* R *is a reduction context, where no decorations are on the path to the hole, then* $R[t^{[a\mapsto 0]}] \xrightarrow{\text{LRPw}} R'[t]$, *this reduction is not counted by the* `rln`*-measure.*

We are now able to show that the embedding of LRP into LRPw is an isomorphism w.r.t. $\approx_A$ provided that $seq \notin A$. Let $\texttt{id}^k$ be an abbreviation of $\underbrace{\texttt{id}\ \ldots\ \texttt{id}}_{k}$, where $\texttt{id} := \lambda x.x$. We consider the following program transformation (*enc*), which replaces an $a := n$-binding by (lbeta)-redexes:

$$(enc)\quad \texttt{letrec}\ a := n, Env\ \texttt{in}\ s \xrightarrow{enc} \texttt{letrec}\ x_a := (\texttt{id}\ \ldots\ \texttt{id})^{n+1}, Env[\texttt{seq}\ x_a\ t/t^{[a]}]\ \texttt{in}\ s[\texttt{seq}\ x_a\ t/t^{[a]}]$$

where $[\texttt{seq}\ x_a\ t/t^{[a]}]$ means that every subterm which is labeled with a is replaced by the corresponding `seq`-expression.

Lemma 3.7. *A complete set of forking and commuting diagrams for* (R, enc) *can be read off the following diagrams*



Lemma 3.8. *If $s \xrightarrow{R,enc} t$ then s is a WHNF iff t is a WHNF.*

Proposition 3.9. *Let $A_{min} \subseteq A \subset \mathfrak{A}$, s.t. $seq \notin A$. Then $(enc) \subseteq \approx_A$*

Proof. We first show correctness and use the context lemma. Let $s \xrightarrow{R,enc} t$. Then we have to show two parts:

- $s\downarrow \implies t\downarrow$: We use induction on the length k of the reduction $s \xrightarrow{\mathrm{LRPw},k} s_k$, where s_k is a WHNF. If $k = 0$ then Lemma 3.8 shows that t is a WHNF thus $t\downarrow$. For the induction step we apply a forking diagram to $t \xleftarrow{R,enc} s \xrightarrow{\mathrm{LRPw}} s_1$. For the first diagram, we have $t \xrightarrow{\mathrm{LRPw}} t_1$ s.t. $s_1 \xrightarrow{\mathrm{LRPw}} t_1$. Since $s_1 \xrightarrow{\mathrm{LRPw},k-1} s_k$, the induction hypothesis shows $t_1\downarrow$ and thus also $t\downarrow$. For the second diagram, we have $t \xrightarrow{\mathrm{LRPw},lbeta} t' \xrightarrow{C,cp}\xrightarrow{C,gc} t_1$ s.t. $s_1 \xrightarrow{enc} t_1$. The induction hypothesis shows $t_1\downarrow$ and correctness of (cp) and (gc) shows $t'\downarrow$ and thus $t\downarrow$. For the last diagram, we have $t \xrightarrow{\mathrm{LRPw},seq} t' \xrightarrow{C,gc,*}\xleftarrow{C,gcW,*} s_1$. Correctness of (gc) and (gcW) shows $t'\downarrow$ and thus $t\downarrow$.
- $t\downarrow \implies s\downarrow$: Let $t \xrightarrow{\mathrm{LRPw},k} t_k$ where t_k is a WHNF. We use induction on $(\mathtt{rln}_{\mathfrak{A}}(t), k)$. For the case $(0,0)$, Lemma 3.8 shows that s is a WHNF thus $s\downarrow$. For the induction step we apply a commuting diagram to $s \xrightarrow{R,enc} t \xrightarrow{\mathrm{LRPw}} t_1$. Note that if $\mathtt{rln}_{\mathfrak{A}}(t) = 0$ (but $k > 0$) then only the first diagram is applicable.

  For the first diagram, we have $s \xrightarrow{\mathrm{LRPw}} s_1$ s.t. $s_1 \xrightarrow{\mathrm{LRPw}} t_1$. Since $t_1 \xrightarrow{\mathrm{LRPw},k-1} t_k$, and $\mathtt{rln}_{\mathfrak{A}}(t_1) \leq \mathtt{rln}_{\mathfrak{A}}(t)$, the induction hypothesis is applicable and shows $s_1\downarrow$ and thus also $s\downarrow$.

  For the second diagram, we have $s \xrightarrow{\mathrm{LRPw},letwn} s' \xrightarrow{R,enc} s'' \xleftarrow{S,gc}\xleftarrow{S,cp} t_1$. We have $\mathtt{rln}_{\mathfrak{A}}(t_1) < \mathtt{rln}_{\mathfrak{A}}(t)$ and by Theorem 2.26 we have $\mathtt{rln}_{\mathfrak{A}}(s'') < \mathtt{rln}_{\mathfrak{A}}(t)$. Thus we can apply the induction hypothesis to s'' which shows $s'\downarrow$ and thus $s\downarrow$.

  For the third diagram, we have $s \xrightarrow{\mathrm{LRPw},letw0} s' \xrightarrow{S,gcW,*}\xleftarrow{S,gc,*} t_1$. Correctness of the rules (letw0), (gcW) and (gc) shows $s\downarrow$.

For proving $(enc) \subseteq \approx_A$ we use the context lemma for improvement. Thus it is sufficient to show that for $s \xrightarrow{R,enc} t$, the equarion $\mathtt{rln}_A(s) = \mathtt{rln}_A(t)$ holds. Clearly $\mathtt{rln}_A(s) = \infty \iff \mathtt{rln}_A(t) = \infty$. So let $s \xrightarrow{\mathrm{LRPw},k} s_k$ where s_k is a WHNF. By induction on k, we show $\mathtt{rln}_A(s) = \mathtt{rln}_A(t)$. If $k = 0$, then s is a WHNF, and Lemma 3.8 implies that t is a WHNF, and thus $\mathtt{rln}_A(s) = 0 = \mathtt{rln}_A(t)$. If $k > 0$ then we apply a forking diagram to $s_1 \xleftarrow{\mathrm{LRPw}} s \xrightarrow{R,enc} t$. For the first diagram, we have $t \xrightarrow{\mathrm{LRPw}} t_1$ s.t. $s_1 \xrightarrow{\mathrm{LRPw}} t_1$. Since $s_1 \xrightarrow{\mathrm{LRPw},k-1} s_k$, the induction hypothesis shows $\mathtt{rln}_A(s_1) = \mathtt{rln}_A(t_1)$ and thus also $\mathtt{rln}_A(s) = \mathtt{rln}_A(t)$.

For the second diagram, we have $t \xrightarrow{\mathrm{LRPw},lbeta} t' \xrightarrow{C,cp}\xrightarrow{C,gc} t_1$ s.t. $s_1 \xrightarrow{enc} t_1$. Clearly $\mathtt{rln}_A(s) = 1 + \mathtt{rln}_A(s_1)$ and $\mathtt{rln}_A(t) = 1 + \mathtt{rln}_A(t')$ Now the induction hypothesis shows $\mathtt{rln}_A(s_1) = \mathtt{rln}_A(t_1)$ and Theorem 2.26 shows $\mathtt{rln}_A(t_1) = \mathtt{rln}_A(t')$ which shows the claim.

For the last diagram, we have $t \xrightarrow{\mathrm{LRPw},seq} t' \xrightarrow{C,gc,*}\xleftarrow{C,gcW,*} s_1$. Then $\mathtt{rln}_A(s) = \mathtt{rln}_A(s_1)$ and (since $seq \notin A$) $\mathtt{rln}_A(t) = \mathtt{rln}_A(t')$. Finally, Theorem 2.26 and Proposition 2.25 show $\mathtt{rln}_A(t') = \mathtt{rln}_A(s_1)$.

Theorem 3.10. *Let $A_{min} \subseteq A \subset \mathfrak{A}$, s.t. $seq \notin A$. Then every decorated expression s can be represented as an* LRP*-expression s' with $s \approx_A s'$. This means the embedding of* LRP *into* LRPw *is an isomorphism w.r.t $\approx_A$.*

Proof. It suffices to show that $s \approx_{\mathrm{LRP},A} t$ implies $s \approx_{\mathrm{LRPw},A} t$. Let $s \approx_{\mathrm{LRP},A} t$ and let C be an LRPw-context. Then $\mathtt{rln}_A(C[s]) = \mathtt{rln}_A(C[t])$ in LRPw and thus $s \approx_{\mathrm{LRPw},A} t$: We apply (enc)-transformations to $C[s]$ and $C[t]$ (without changing s, t, since they do neither contain $a := n$ labels nor $\cdot^{[a]}$ labels.) until we get expressions $C'[s]$, $C'[t]$ s.t. both are free of bindings $a := n$ and labels $\cdot^{[a]}$. We have $C'[s] \approx_A C[s]$ and $C'[t] \approx_A C[t]$ by Proposition 3.9 and thus $\mathtt{rln}_A(C'[s]) = \mathtt{rln}_A(C[s])$ and $\mathtt{rln}_A(C'[t]) = \mathtt{rln}_A(C[t])$. Since C' is an LRP-context, the precondition $s \approx_{\mathrm{LRP},A} t$ shows $\mathtt{rln}_A(C[s]) = \mathtt{rln}_A(C[t])$ which shows the claim.

(cp-in) $\texttt{letrec}\ x_1 = (\lambda y.t)^{[n,p_1]}, \{x_i = x_{i-1}\}_{i=2}^{m}, Env\ \texttt{in}\ C[x_m^{p_2}]$
$\rightarrow \texttt{letrec}\ x_1 = \lambda y.t^{[a\mapsto n,p_1]}, \{x_i = x_{i-1}\}_{i=2}^{m}, Env\ \texttt{in}\ C[(\lambda y.t)^{([a\mapsto n,p_1]\oplus p_2)}]$
where p_2 is nontrivial only if $x_m^{p_2}$ is not the right hand side of a binding
(llet-e) $(\texttt{letrec}\ Env_1, x = (\texttt{letrec}\ Env_2\ \texttt{in}\ t)^p\ \texttt{in}\ r) \rightarrow (\texttt{letrec}\ Env_1, Env_2, x = t^p\ \texttt{in}\ r)$

The standard cases are usually dealt with shifting the decoration up, since the decoration is in a strict position, and/or using further rules (locally) from Theorem 3.5.

Fig. 4. The non-standard cases of decoration modification of reduction rules of LRPw (variants omitted)

(cpcx-in) $\texttt{letrec}\ x = c\overrightarrow{t}^{[n,p_1]}, Env\ \texttt{in}\ C[x^{p_2}] \rightarrow \texttt{letrec}\ x = c\overrightarrow{y}^{[a\mapsto n,p_1]}, \{y_i = t_i\}_{i=1}^{ar(c)}, Env\ \texttt{in}\ C[c\overrightarrow{y}^{[a\mapsto n,p_1]\oplus p_2}]$
(xch) $\texttt{letrec}\ x = t^p, y = x, Env\ \texttt{in}\ r \ \rightarrow\ \texttt{letrec}\ y = t^p, x = y, Env\ \texttt{in}\ r$ where $y = x^q$ is not permitted.
(lwas) $T[\texttt{letrec}\ Env\ \texttt{in}\ t^p] \rightarrow \texttt{letrec}\ Env\ \texttt{in}\ T[t^p]$
if T is a weak top context with hole depth 1
(ucp1) $\texttt{letrec}\ Env, x = t^p\ \texttt{in}\ S[x] \rightarrow \texttt{letrec}\ Env\ \texttt{in}\ S[t^p]$

The standard cases are usually dealt with shifting the decoration up, since the decoration is in a strict position, and/or using further rules (locally) from Theorem 3.5.

Fig. 5. Non-standard cases of decoration modification in the Extra Transformation Rules (variants omitted)

Remark 3.11. For a surface context C, the sharing decorated expression $s := C[s_1^{[n_1,a\mapsto h]}, \ldots, s_m^{[n_m,a\mapsto h]}]$ can be given a semantics in the case $n_i > 0$ for all i.: Define

$$sem(s) := \texttt{letrec}\ x_0 = \texttt{id}^{[h]}\ \texttt{in}\ C[(x_0\ s_1^{[n_1-1])}, \ldots, (x_0\ s_m^{[n_m-1]})]).$$

In the case $seq \in A$ and $n_i = 0$ for some i , the equivalence classes of expressions w.r.t. $\approx_A$ are properly extended (see Proposition 3.12)

It can easily be verified, that $sem(s) \xrightarrow{T,*} s'$ with $s' \approx_A C[s_1, \ldots, s_m]$, where the reduction requires $h + \sum_{i=1}^{m} n_i$ rln-reduction steps: $h + n_i$ steps for each $(x_0\ s_i^{[n_i-1]})$ where h is the number of shared rln-reduction steps.

In the exceptional case $n_1 = 0$ there are some cases, which can be given a semantics: for example $(Z^{[a\mapsto 1]}, Z^{[a\mapsto 1]}) \approx_A \texttt{letrec}\ x = \texttt{id}\ Z\ \texttt{in}\ (x, x)$. Generalizing, if the sharing decorated subexpressions are syntactically equal, then the construction may be applied in certain cases.

Proposition 3.12. *Let $A = \mathfrak{A}$. Then the decorated expression $(Z^{[a\mapsto 1]}, \texttt{Nil}^{[a\mapsto 1]})$ is not equivalent w.r.t. $\approx_A$ to any* LRP*-expression.*

Proof. Assume there is such an expression s. Then $s \sim_c (Z, \texttt{Nil})$ and $\texttt{rln}_A(s) = 0$, so we can assume that s is a WHNF. Using the correctness w.r.t. $\approx_A$ of program transformations and that $Z \not\sim_c \texttt{Nil}$, we can assume that s is of the form $\texttt{letrec}\ x = s_1, y = s_2, Env\ \texttt{in}\ (x, y)$. We see that s_1 as well as s_2 alone have $\texttt{rln}_A$-count 1 in the environment. Using that (lll), (cpx) and (gc) are correct program transformations w.r.t $\approx_A$, we can assume that s_1, s_2 are applications, `seq`- or a case-expressions. But then every of them requires at least one $\texttt{rln}_A$-reduction that is independent of the other to become a WHNF. Hence the context $C := \texttt{let}\ z = [\cdot]\ \texttt{in}\ \texttt{seq}\ (\texttt{fst}\ z)\ (\texttt{snd}\ z)$ applied to $(Z^{[a\mapsto 1]}, \texttt{Nil}^{[a\mapsto 1]})$ requires $6 = 5 + 1$ steps: 2 for `fst`, 2 for `snd`, 1 for `seq`, and 1 for the shared evaluation of $Z^{[a\mapsto 1]}$, whereas s requires at least $7 = 5 + 2$: the 2 reductions are the minimum to reach a WHNF for the first as well for the second component.

We show how the decorations are implicitly modified under reductions and transformations, where the reduction are invariant under $\approx$. See figure 4 for the reduction rules of LRP w.r.t. decorations.

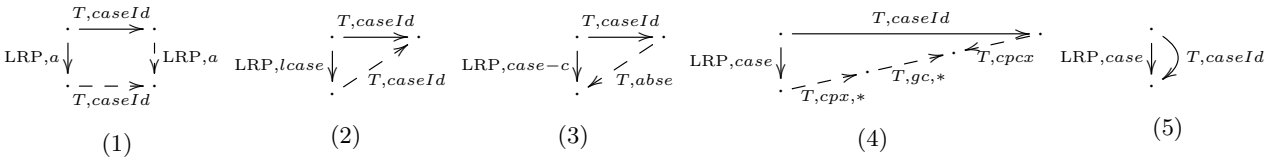


Fig. 6. Diagrams for (caseId)

3.2 More Transformations and Improvements

Let (caseId) be defined as:

$$(\mathtt{case}_K\ s\ (pat_1 \to pat_1) \dots (pat_{|D_K|} \to pat_{|D_K|})) \to s$$

The rule (caseId) is the heart (of the correctness proof) of other type-dependent transformations, like rules involving map, filter, fold, asf., and it is only correct under typing, i.e. in LRP and LRPw, but not in LR, which can be seen by trying the case $s = \lambda x.t$.

We show that (caseId) is an improvement in LRPw.

Lemma 3.13. *Let $s \xrightarrow{T,caseId} t$. If s is a WHNF, then t is a WHNF. If t is a WHNF, then $s \xrightarrow{\text{LRPw},lll,*} \xrightarrow{\text{LRPw},case,0\vee1} \xrightarrow{\text{LRPw},lll,*} s'$ where s' is a WHNF.*

Lemma 3.14. *Let $A_{min} \subseteq A \subseteq \mathfrak{A}$. If $s{\downarrow} \wedge s \xrightarrow{T,caseId} t$, then $t{\downarrow}$ and $\mathtt{rln}_A(s) \geq \mathtt{rln}_A(t)$.*

Proof. Let $s \xrightarrow{T,caseId} t$ and $s \xrightarrow{\text{LRPw},k} s'$ where s' is a WHNF. We use induction on k. For $k = 0$ Lemma 3.13 shows the claim. For the induction step, let $s \xrightarrow{\text{LRPw}} s_1$. The diagrams in Fig. 6 describe all cases how the fork $s_1 \xleftarrow{\text{LRPw}} s \xrightarrow{T,caseId}$ can be closed. For diagram (1) we apply the induction hypothesis to $s_1 \xrightarrow{T,caseId} t_1$ which shows $t_1{\downarrow}$, $\mathtt{rln}_A(s_1) \geq \mathtt{rln}_A(t_1)$ and thus also $t{\downarrow}$ and $\mathtt{rln}_A(s) \geq \mathtt{rln}_A(t)$. For diagram (2) the induction hypothesis shows the claim. For diagram (3) we have $t{\downarrow}$, since (abse) is correct. Moreover, $t \xrightarrow{T,abse} s'$ is equivalent to $s' \xrightarrow{T,ucp\vee gc,*} t$ and Theorem 2.26 shows $\mathtt{rln}_A(s') = \mathtt{rln}_A(t)$. Thus also $\mathtt{rln}(s) \geq \mathtt{rln}(t)$. For diagram (4) we have $t{\downarrow}$, since (cpcx), (gc), and (cpx) are correct. Theorem 2.26 shows that $\mathtt{rln}_A(s) \geq \mathtt{rln}_A(s') = \mathtt{rln}_A(t)$, since (cpcx), (cpx) and (gc) do not change the measure $\mathtt{rln}(\cdot)$. For diagram(5) the claim obviously holds.

Theorem 3.15. *(caseId) is an improvement, i.e. for $A_{min} \subseteq A \subseteq \mathfrak{A}$: $(caseId) \subseteq \succeq_A$.*

Proof. Lemma 3.13 and the diagrams in Fig. 6 can be used to show (by induction on the sequence for t) that if $s \xrightarrow{T,caseId} t$ and $t{\downarrow}$, then $s{\downarrow}$, since the used existentially quantified transformations are correct and diagram 2 can only be applied finitely often. Then the context lemma for $\sim_c$ (which states that convergence preservation and reflection in reduction contexts suffices to show $\sim_c$, see e.g. [5]) and Lemma 3.14 show that (caseId) is correct. Finally, the context lemma for improvement (Lemma 2.9) and Lemma 3.14 show that (caseId) is an improvement.

4 A Head-Centered Improvement Simulation for Lists

For $A_{min} \subseteq A \subseteq \mathfrak{A}$, we define an improvement simulation $\sqsubseteq_{A,h,\tau}$ on lists of the same type, $\mathtt{List}\ \tau$, for proving $\preceq$-relations between functions on lists.

Definition 4.1. *Let $A_{min} \subseteq A \subseteq \mathfrak{A}$. Let τ be a type, and $\mathfrak{L}_\tau := \{(s,t) \mid s,t :: \mathtt{List}(\tau), FV(s) = FV(t) = \emptyset$, decorations are only in surface contexts in s,t, and $s \sim_c t\}$. We define the following operator $F_{A,h} :: \mathfrak{L}_\tau \to \mathfrak{L}_\tau$: Let $\eta \subseteq \mathfrak{L}_\tau$, and $s\ \eta\ t$.*

1. *If $s \sim_c \bot \sim_c t$, then $s\ F_{A,h}(\eta)\ t$.*
2. *If $s \approx_A \mathtt{Nil}^{[k]}$, $t \approx_A \mathtt{Nil}^{[k']}$ and $k \leq k'$, then $s\ F_{A,h}(\eta)\ t$.*

3. *If* $s \preceq_A (s_1^{[p_1]} : s_2^{[k_2]})^{[k_3]}$, *and* $(t_1^{[p_1']} : t_2^{[k_2']})^{[k_3']} \preceq_A t$, *for some expressions* s_1, s_2, t_1, t_2, *with* $FV(s_1) = FV(s_2) = FV(t_1) = FV(t_2) = \emptyset$ *and decorations where* s_2, t_2 *may contain further sharing decorations, but only in surface context positions; where we also assume that there may be common labels in the expressions. and the following conditions hold:*
   (a) $p_1 \leq p_1'$, $k_2 \leq k_2'$, *and* $k_3 \leq k_3'$.
   (b) $s_1 \preceq_A t_1$ *and* s_1, t_1 *are decoration-free.*
   (c) *The set* D_0 *of labels in* s *is also the set of labels in* t, *and the set of labels* D_i *in* s_i, $i = 1, 2$, *are also the set of labels in* t_i, $i = 1, 2$.
   (d) *For labels in* D_0, *we assume that these are free in the expressions, and that the relations* $s \preceq_A (s_1^{[p_1]} : s_2^{[k_2]})^{[k_3]}$ *and* $(t_1^{[p_1']} : t_2^{[k_2']})^{[k_3']} \preceq_A t$, *hold under this freeness-assumption.*
   (e) $s_2 \ \eta \ t_2$.
   Then $s \ F_h(\eta) \ t$.

Let $\sqsubseteq_{A,h,\tau}$ *be the greatest fixpoint of* $F_{A,h}$. □

To ease reading we leave out the index τ in the following and simply write $\sqsubseteq_{A,h}$ instead of $\sqsubseteq_{A,h,\tau}$ unless the type τ becomes relevant.

Clearly, the operator $F_{A,h}$ is monotone, and thus $\sqsubseteq_{A,h}$ is well-defined, i.e. the fixpoint exists.

Moreover, due to determinism of normal-order reduction, $F_{A,h}$ is lower-continuous, and thus Kleene's fixpoint theorem can be applied, which implies the following inductive characterization of $\sqsubseteq_{A,h}$: Let $\sqsubseteq_{A,h,0} = \mathfrak{L}_\tau$, and $\sqsubseteq_{A,h,i} = F(\sqsubseteq_{A,h,i-1})$ for $i > 0$. Then $\sqsubseteq_{A,h} = \bigcap_{i=0}^{\infty} \sqsubseteq_{A,h,i}$. Thus for $(s,t) \in \mathfrak{L}_\tau$ we can show $s \sqsubseteq_{A,h} t$ by proving $s \sqsubseteq_{A,h,i} t$ for all i.

Theorem 4.2. *If* $s \sqsubseteq_{\mathfrak{A},h} t$, *then also* $s \preceq_{\mathfrak{A}} t$.

Proof. We show a generalized claim. Using this claim with a single-hole surface-context T and $n = 1$ shows that $s \preceq_{\mathfrak{A},T} t$, and thus using the context lemma for improvement, also the claim of the theorem follows. The claim is:

> Let $C[\cdot, \ldots, \cdot]$ be a multicontext, where the holes are in surface-contexts, for $i = 1, \ldots, n$ let s_i, t_i be closed and of the same type such that for each pair s_i, t_i the relation $s_i \sqsubseteq_{\mathfrak{A},h} t_i$ (and thus $s_i, t_i :: \mathtt{List}(\tau)$) holds. Then $\mathtt{rln}_{\mathfrak{A}}(C[s_1, \ldots, s_n]) \leq \mathtt{rln}_{\mathfrak{A}}(C[t_1, \ldots, t_n])$ holds.

For the proof we assume that for the first input pair (s,t), the infinite sequence of the expansion (including the decorations asf.) according to Definition 4.1 is fixed, and so we make the same choices even if copies of s, t appear in the expressions. Hence we can use the Kleene-criterion for computing the fixed point.

First observe that $\mathtt{rln}_{\mathfrak{A}}(C[t_1, \ldots, t_n]) = \infty$ if, and only if $\mathtt{rln}_{\mathfrak{A}}(C[s_1, \ldots, s_n]) = \infty$, which follows from finiteness of decorations and from $s_i \sim_c t_i$.

In other cases we show the claim by induction on the lexicographically ordered measure (μ_1, μ_2, μ_3) where $\mu_1 = \mathtt{rln}_{\mathfrak{A}}(C[t_1, \ldots, t_n])$, μ_2 is the number of holes in C and $\mu_3 = \mathtt{rlnall}(C[t_1, \ldots, t_n])$.

The base case is that there is no reduction of $C[t_1, \ldots, t_n]$ and there is no hole in a reduction context. Then the context itself is either a WHNF, and $\mathtt{rln}_A(C[t_1, \ldots, t_n]) = 0 = \mathtt{rln}_A(C[s_1, \ldots, s_n])$, or the context is stuck, in which case both expressions are divergent, and $\mathtt{rln}_A(C[t_1, \ldots, t_n]) = \infty = \mathtt{rln}_A(C[s_1, \ldots, s_n])$.

The other cases are that a hole of C is in a reduction context, or a reduction is possible for $C[t_1, \ldots, t_n])$, and we will show that we can apply the **induction hypothesis**.

If no hole of C is in a reduction context, then $C[t_1, \ldots, t_n] \xrightarrow{no} C'[t_1, \ldots, t_n]$ as well as $C[s_1, \ldots, s_n] \xrightarrow{no} C'[s_1, \ldots, s_n]$, where C' has n or less than n holes, since all holes are in surface contexts. We can apply the induction hypothesis after the reduction, since μ_1 remains equal or is decreased by 1, μ_2 remains equal or is decreased, and μ_3 is strictly decreased.

Note that this reduction may change the value of sharing decorations. Since we have assumed in condition (3d) that the relations hold also under the change of the value, the induction hypothesis is applicable.

Now we consider the case that some t_j is in a reduction context in $C[t_1,\ldots,t_n]$. Then we can assume w.l.o.g. that the hole j is in a reduction context in C, independent of the expressions in the holes. Hence s_j as well as t_j are in a reduction context in $C[s_1,\ldots,s_n]$ and $C[t_1,\ldots,t_n]$, respectively.

We check the cases from the definition of $\sqsubseteq_{\mathfrak{A},h}$.

1. If $s_j \sim_c \bot$, then $t_j \sim_c \bot$, and $C[s_1,\ldots,s_j,\ldots s_n] \sim_c \bot \sim_c C[t_1,\ldots t_j,\ldots,t_n]$.
2. If $s_j \approx_{\mathfrak{A}} \mathtt{Nil}^{[k]}$, then $t_j \approx_{\mathfrak{A}} \mathtt{Nil}^{[k']}$ with $k \le k'$. Since $\mathtt{Nil}^{[k]} \preceq_{\mathfrak{A}} \mathtt{Nil}^{[k']}$, it is sufficient to show $\mathtt{rln}_{\mathfrak{A}}(C[s_1,\ldots,\mathtt{Nil}^{[k']},\ldots s_n]) \le \mathtt{rln}_{\mathfrak{A}}(C[t_1,\ldots,\mathtt{Nil}^{[k']},\ldots,\ldots t_n])$, which follows from the induction hypothesis, since $C[\cdot_1,\ldots,\cdot_{j-1},\mathtt{Nil}^{[k']},\cdot_{j+1},\ldots,\cdot_n]$ has $n-1$-holes (which strictly decreases μ_2), and μ_1 is unchanged.
3. If $s_j \preceq_{\mathfrak{A}} (s_{j,1}^{[p_{j,1}]} : s_{j,2}^{[k_{j,2}]})^{[k_{j,3}]}$ then due to the preconditions there is a representation $(t_{j,1}^{[p'_{j,1}]} : t_{j,2}^{[k'_{j,2}]})^{[k'_{j,3}]} \preceq_{\mathfrak{A}} t_j$ with $p_{j,1} \le p'_{j,1}$, $k_{j,2} \le k'_{j,2}$, $k_{j,3} \le k'_{j,3}$, $s_{j,1} \preceq_{\mathfrak{A}} t_{j,1}$ and $s_{j,2} \sqsubseteq_{\mathfrak{A},h} t_{j,2}$.

   It suffices to show that $\mathtt{rln}_{\mathfrak{A}}(C[s_1,\ldots,(s_{j,1}^{[p_{j,1}]} : s_{j,2}^{[k_{j,2}]})^{[k_{j,3}]},\ldots,s_n]) \le \mathtt{rln}_{\mathfrak{A}}(C[t_1,\ldots,(t_{j,1}^{[p'_{j,1}]} : t_{j,2}^{[k'_{j,2}]})^{[k'_{j,3}]},\ldots,t_n])$ to prove the claim. The assumption $s_{j,1} \preceq_{\mathfrak{A}} t_{j,1}$ implies that it is sufficient to show $\mathtt{rln}_{\mathfrak{A}}(C[s_1,\ldots,(t_{j,1}^{[p_{j,1}]} : s_{j,2}^{[k_{j,2}]})^{[k_{j,3}]},\ldots,s_n]) \le \mathtt{rln}_{\mathfrak{A}}(C[t_1,\ldots,(t_{j,1}^{[p'_{j,1}]} : t_{j,2}^{[k'_{j,2}]})^{[k'_{j,3}]},\ldots,t_n])$. Since $p_{j,1} \le p_{j,1'}$, it is sufficient to show $\mathtt{rln}_{\mathfrak{A}}(C[s_1,\ldots,(t_{j,1}^{[p'_{j,1}]} : s_{j,2}^{[k_{j,2}]})^{[k_{j,3}]},\ldots,s_n]) \le \mathtt{rln}_{\mathfrak{A}}(C[t_1,\ldots,(t_{j,1}^{[p'_{j,1}]} : t_{j,2}^{[k'_{j,2}]})^{[k'_{j,3}]},\ldots,t_n])$. Note that this change may have an effect on the inner decorations of $s_{j,2}$.

   Similarly, since $k_{j,3} \le k'_{j,3}$, and the hole j is in a reduction context, it is sufficient to show $\mathtt{rln}_{\mathfrak{A}}(C[s_1,\ldots,(t_{j,1}^{[p'_{j,1}]} : s_{j,2}^{[k_{j,2}]}),\ldots,s_n]) \le \mathtt{rln}_{\mathfrak{A}}(C[t_1,\ldots,(t_{j,1}^{[p'_{j,1}]} : t_{j,2}^{[k'_{j,2}]}),\ldots,t_n])$. At this place in the proof we we switch to the language LRPw: Let D_f, the set fresh labels, be $D_f = D_1 \setminus D_0$. The expressions in hole j are $(\mathtt{letrec}\ a_1 = h_1, a_2 = h_2 \ldots\ \mathtt{in}\ t_{j,1} : s_{j,2}^{[k_{j,2}]}$, and $(\mathtt{letrec}\ a_1 := h_1, a_2 := h_2 \ldots\ \mathtt{in}\ t_{j,1} : t_{j,2}^{[k'_{j,2}]}$ where the a_i are the (fresh) labels in D_f.

   We integrate the subcontext $(\mathtt{letrec}\ a_1 := h_1, a_2 := h_2 \ldots\ \mathtt{in}\ (t_{j,1} : [.]))$ into the multi-context C, resulting in the context $C' = C[\ldots,\underbrace{(\mathtt{letrec}\ a_1 := h_1, a_2 := h_2 \ldots\ \mathtt{in}\ (t_{j,1} : [.]))}_{j},\ldots,.]$ and obtain that the number of holes of C' is again n and μ_1 is unchanged. The next normal-order reductions are shifting this let-environment to the top. However, the induction hypothesis could not be applied, since the number of `rlnall`-reductions may have been increased. Now consider the next normal order reduction for $C''[t_1,\ldots,t_{j,2}^{[k'_{j,2}]},\ldots,t_n] = C'''[t_1,\ldots,(t_{j,1}^{[p'_{j,1}]} : t_{j,2}^{[k'_{j,2}]}),\ldots,t_n]$. If there is no such reduction, then $C''[t_1,\ldots,t_{j,2}^{[k'_{j,2}]},\ldots,t_n]$ is a WHNF. Then $C''[s_1,\ldots,s_{j,2}^{[k_{j,2}]},\ldots,s_n]$ is also a WHNF and thus $\mathtt{rln}_{\mathfrak{A}}(C'''[t_1,\ldots,(t_{j,1}^{[p'_{j,1}]} : t_{j,2}^{[k'_{j,2}]}),\ldots,t_n]) = 0 = \mathtt{rln}_{\mathfrak{A}}(C'''[s_1,\ldots,(t_{j,1}^{[p'_{j,1}]} : s_{j,2}^{[k_{j,2}]}),\ldots,s_n])$ which shows the claim and $\mu_1 = 0$. If a normal order reduction for $C'''[t_1,\ldots,(t_{j,1}^{[p'_{j,1}]} : t_{j,2}^{[k'_{j,2}]}),\ldots,t_n]$ exists, then – due to typing – it must be a (seq)- or (case)-reduction.

   The reduction strictly decreases the measure μ_1, but we have to look for the form of the results: If the (seq)-reduction removes $(t_{j,1}^{[p'_{j,1}]} : s_{j,2}^{[k_{j,2}]})$ and $(t_{j,1}^{[p'_{j,1}]} : t_{j_2}^{[k'_{j,2}]})$, then we can apply the induction hypothesis.

   If the (seq)-reduction does not remove $(t_{j,1}^{[p'_{j,1}]} : s_{j,2}^{[k_{j,2}]})$ and $(t_{j,1}^{[p'_{j,1}]} : t_{j_2}^{[k'_{j,2}]})$, since the `seq`-expression is of a form `seq` $x\ r$, then this expression is part of the context and replaced by r, and so we can apply the induction hypothesis.

   If the reduction is a (case)-reduction, then the expressions $t_{j,1}, s_{j,2}$ and also $t_{j,1}, t_{j,2}$ are moved into a `letrec`-environment and remain in surface-context position. Since μ_1 is strictly decreased, the preconditions hold for the result, we can apply the induction hypothesis, which shows the claim. □

For the case $A \neq \mathfrak{A}$, we make a separate definition and a separate proof though the definitions of proofs have a lot in comman. The advantage is that the (slightly different) proofs can be separately checked.

Definition 4.3. *Let $A_{min} \subseteq A \subset \mathfrak{A}$. Let τ be a type, and $\mathfrak{L}_\tau := \{(s,t) \mid s,t :: \mathtt{List}(\tau), FV(s) = FV(t) = \emptyset$, decorations are only in surface contexts in s,t, and $s \sim_c t\}$. We define the following operator $F_{A,lbc} : \mathfrak{L}_\tau \to \mathfrak{L}_\tau$: Let $\eta \subseteq \mathfrak{L}_\tau$, and $s\ \eta\ t$.*

1. *If $s \sim_c \bot \sim_c t$, then $s\ F_{A,h}(\eta)\ t$.*
2. *If $s \approx_A \mathtt{Nil}^{[k]}$, $t \approx_A \mathtt{Nil}^{[k']}$ and $k \leq k'$, then $s\ F_{A,h}(\eta)\ t$.*
3. *If $s \preceq_A (s_1^{[p_1]} : s_2^{[k_2]})^{[k_3]}$, and $(t_1^{[p_1']} : t_2^{[k_2']})^{[k_3']} \preceq_A t$, for some expressions s_1, s_2, t_1, t_2 with $FV(s_1) = FV(s_2) = FV(t_1) = FV(t_2) = \emptyset$, and decorations where s_2, t_2 may contain further sharing decorations, but only in surface context positions, and where we also assume that there may be common labels in the expressions, and the following conditions hold:*
   (a) $p_1 \leq p_1'$, $k_2 \leq k_2'$, and $k_3 \leq k_3'$.
   (b) If $k_3' = 0$, then also $(t_1^{[p_1']} : t_2^{[k_2']}) \preceq_{\mathfrak{A}} t$.
   (c) $s_1 \preceq_A t_1$ and s_1, t_1 are decoration-free.
   (d) The set D_0 of labels in s is also the set of labels in t, and the set of labels D_i in s_i, $i = 1,2$, are also the set of labels in t_i, $i = 1,2$. The set of fresh labels D_f is $D_1 \setminus D_0$.
   (e) For labels in D_0, we assume that these are free in the expressions, and the relations $s \preceq_A (s_1^{[p_1]} : s_2^{[k_2]})^{[k_3]}$, $(t_1^{[p_1']} : t_2^{[k_2']})^{[k_3']} \preceq_A t$ and $(t_1^{[p_1']} : t_2^{[k_2']}) \preceq_{\mathfrak{A}} t$ hold under this assumption.
   (f) $s_2\ \eta\ t_2$.

   Then $s\ F_{lbc}(\eta)\ t$.

Let $\sqsubseteq_{A,lbc,\tau}$ be the greatest fixpoint of $F_{A,lbc}$. □

For A with $A_{min} \subseteq A \subseteq \mathfrak{A}$, in particular for the case $A \neq \mathfrak{A}$, the simulation $\sqsubseteq_{A,lbc}$ is correct for $\preceq_A$, where the proof of Theorem 4.2 is modified at several places.

Theorem 4.4. *Let $A_{min} \subseteq A \subseteq \mathfrak{A}$. If $s \sqsubseteq_{A,lbc} t$, then also $s \preceq_A t$.*

Proof. We show a generalized claim. Using this claim with a single-hole surface-context T and $n = 1$ shows that $s \preceq_{A,T} t$, and thus using the context lemma for improvement, also the claim of the theorem follows. The claim is:

> Let $C[\cdot,\ldots,\cdot]$ be a multicontext, where the holes are in surface-contexts, for $i = 1,\ldots,n$ let s_i, t_i be closed and of the same type such that for each pair s_i, t_i : $s_i \sqsubseteq_{A,lbc} t_i$ (and thus $s_i, t_i :: \mathtt{List}(\tau)$).
> Then $\mathtt{rln}_A(C[s_1,\ldots,s_n]) \leq \mathtt{rln}_A(C[t_1,\ldots,t_n])$ holds.

For the proof we assume that for the first input pair (s,t), the infinite sequence of the expansion (including the decorations asf.) according to Definition 4.3 is fixed, and so we make the same choices even if copies of s,t appear in the expressions. Hence we can use the Kleene-criterion for computing the fixed point.
First observe that $\mathtt{rln}_A(C[t_1,\ldots,t_n]) = \infty$ if, and only if $\mathtt{rln}_A(C[s_1,\ldots,s_n]) = \infty$, which follows from finiteness of decorations and from $s_i \sim_c t_i$.

In other cases we show the claim by induction on the lexicographically ordered measure $(\mu_1, \mu_2, \mu_3, \mu_4)$ where $\mu_1 = \mathtt{rln}_A(C[t_1,\ldots,t_n])$, μ_2 is the number of holes in C, $\mu_3 = \mathtt{rln}_{\mathfrak{A}}(C[t_1,\ldots,t_n])$ and $\mu_4 = \mathtt{rlnall}(C[t_1,\ldots,t_n])$.

The base case is that there is no reduction of $C[t_1,\ldots,t_n]$ and there is no hole in a reduction context. Then the context itself is either a WHNF, and $\mathtt{rln}_A(C[t_1,\ldots,t_n]) = 0 = \mathtt{rln}_A(C[s_1,\ldots,s_n])$, or the context is stuck, in which case both expressions are divergent, and $\mathtt{rln}_A(C[t_1,\ldots,t_n]) = \infty = \mathtt{rln}_A(C[s_1,\ldots,s_n])$.

The other cases are that a hole of C is in a reduction context, or a reduction is possible for $C[t_1,\ldots,t_n]$, and we will show that we can apply the induction hypothesis.

If no hole of C is in a reduction context, then $C[t_1,\ldots,t_n] \xrightarrow{no} C'[t_1,\ldots,t_n]$ as well as $C[s_1,\ldots,s_n] \xrightarrow{no} C'[s_1,\ldots,s_n]$, where C' has n or less than n holes, since all holes are in surface contexts. We can apply the induction hypothesis after the reduction, since μ_1 remains unchanged or is decreased by 1, μ_2 remains equal or is decreased, and (μ_3,μ_4) is strictly decreased. Note that this reduction may change the value of sharing decorations. Since we have assumed that the relations hold in condition (3e) also under the change of the value, the induction hypothesis is applicable.

Now we consider the case that some t_j is in a reduction context in $C[t_1,\ldots,t_n]$. Then we can assume w.l.o.g. that the hole j is in a reduction context in C, independent of the expressions in the holes. Hence s_j as well as t_j are in a reduction context in $C[s_1,\ldots,s_n]$ and $C[t_1,\ldots,t_n]$, respectively.

We check the cases from the definition of $\sqsubseteq_{A,lbc}$.

1. If $s_j \sim_c \bot$, then $t_j \sim_c \bot$, and $C[s_1,\ldots,s_j,\ldots s_n] \sim_c \bot \sim_c C[t_1,\ldots t_j,\ldots,t_n]$.
2. If $s_j \approx_A \texttt{Nil}^{[k]}$, then $t_j \approx_A \texttt{Nil}^{[k]}$. Since $\texttt{Nil}^{[k]} \preceq_A \texttt{Nil}^{[k']}$, it is sufficient to show $\texttt{rln}_A(C[s_1,\ldots,\texttt{Nil},\ldots s_n]) \leq \texttt{rln}_A(C[t_1,\ldots,\texttt{Nil},\ldots t_n])$, which follows from the induction hypothesis, since $C[\cdot_1,\ldots,\cdot_{j-1},\texttt{Nil},\cdot_{j+1},\ldots,\cdot_n]$ has $n-1$-holes (which strictly decreases μ_2), and μ_1 is unchanged.
3. If $s_j \preceq_A (s_{j,1}^{[p_{j,1}]} : s_{j,2}^{[k_{j,2}]})^{[k_{j,3}]}$ then due to the preconditions there is a representation $(t_{j,1}^{[p'_{j,1}]} : t_{j,2}^{[k'_{j,2}]})^{[k'_{j,3}]} \preceq_A t_j$ such that $k \leq k'$, $p_{j,1} \leq p'_{j,1}$, $k_{j,2} \leq k'_{j,2}$, $k_{j,3} \leq k'_{j,3}$, and $s_{j,1} \preceq_A t_{j,1}$ and $s_{j,2} \sqsubseteq_{A,lbc} t_{j,2}$.

   If $k'_{j,3} > 0$, then it is sufficient to show the claim for the two expressions $C[s_1,\ldots,(s_{j,1}^{[p_{j,1}]} : s_{j,2}^{[k_{j,2}]})^{[k_{j,3}]},\ldots,s_n])$ and $C[t_1,\ldots,(t_{j,1}^{[p'_{j,1}]} : t_{j,2}^{[k'_{j,2}]})^{[k'_{j,3}]},\ldots,t_n]$, which follows from the induction hypothesis, since $k'_{j,3} > 0$.

   If $k'_{j,3} = 0$, then also $k_{j,3} = 0$ and we have to show the claim for the two expressions $C[s_1,\ldots,(s_{j,1}^{[p_{j,1}]} : s_{j,2}^{[k_{j,2}]}),\ldots,s_n])$ and $C[t_1,\ldots,(t_{j,1}^{[p'_{j,1}]} : t_{j,2}^{[k'_{j,2}]}),\ldots,t_n]$. Note that $(t_{j,1}^{[p'_{j,1}]} : t_{j,2}^{[k'_{j,2}]}) \preceq_{\mathfrak{A}} t_j$, hence μ_3 is not increased.

   The assumption $s_{j,1} \preceq_A t_{j,1}$ implies that it is sufficient to show the claim for $C[s_1,\ldots,(t_{j,1}^{[p_{j,1}]} : s_{j,2}^{[k_{j,2}]}),\ldots,s_n])$ and $C[t_1,\ldots,(t_{j,1}^{[p'_{j,1}]} : t_{j,2}^{[k'_{j,2}]}),\ldots,t_n]$. Since $p_{j,1} \leq p'_{j,1}$ it is sufficient to show the claim for $C[s_1,\ldots,(t_{j,1}^{[p'_{j,1}]} : s_{j,2}^{[k_{j,2}]}),\ldots,s_n])$ and $C[t_1,\ldots,(t_{j,1}^{[p'_{j,1}]} : t_{j,2}^{[k'_{j,2}]}),\ldots,t_n]$. Note that this change may have an effect on the inner decorations of $s_{j,2}$.

   At this place in the proof we switch to the language LRPw: Let D_f, the set fresh labels, be $D_f = D_1 \setminus D_0$. The expressions in hole j are $(\texttt{letrec}\ a_1 := h_1, a_2 := h_2 \ldots\ \texttt{in}\ t_{j,1} : s_{j,2}^{[k_{j,2}]}$, and $(\texttt{letrec}\ a_1 := h_1, a_2 := h_2 \ldots\ \texttt{in}\ t_{j,1} : t_{j,2}^{[k'_{j,2}]}$ where the a_i are the (fresh) labels in D_f.

   We integrate the subcontext $(\texttt{letrec}\ a_1 := h_1, a_2 := h_2 \ldots\ \texttt{in}\ (t_{j,1} : [.]))$ into the multi-context C, resulting in the context $C' = C[\ldots,\underbrace{(\texttt{letrec}\ a_1 := h_1, a_2 := h_2 \ldots\ \texttt{in}\ (t_{j,1} : [.]))}_{j},\ldots,.]$ and obtain that the number of holes of C' is again n and μ_1 is unchanged. The next normal-order reductions are shifting this let-environment to the top. However, the induction hypothesis could not be applied, since the number of `rlnall`-reductions may have been increased. Note that we have modified the right-hand side, but from $(t_{j,1}^{[p'_{j,1}]} : t_{j,2}^{[k'_{j,2}]}) \preceq_{\mathfrak{A}} t_j$, we see that $\texttt{rln}_{\mathfrak{A}}(C[t_1,\ldots,(t_{j,1}^{[p'_{j,1}]} : t_{j,2}^{[k'_{j,2}]}),\ldots,t_n]) \leq \texttt{rln}_{\mathfrak{A}}(C[t_1,\ldots,t_n])$.

   Now consider the next normal order reduction for $C[t_1,\ldots,t_{j,2}^{[k'_{j,2}]},\ldots,t_n] = C[t_1,\ldots,(t_{j,1}^{[p'_{j,1}]} : t_{j,2}^{[k'_{j,2}]}),\ldots,t_n]$. If there is no such reduction, then $C[t_1,\ldots,(t_{j,1}^{[p'_{j,1}]} : t_{j,2}^{[k'_{j,2}]}),\ldots,t_n]$ is a WHNF. Then $C[s_1,\ldots,(t_{j,1}^{[p'_{j,1}]} : s_{j,2}^{[k_{j,2}]}),\ldots,s_n]$ is also a WHNF and the measure is $\mu_1 = 0$. Then $C[s_1,\ldots,(t_{j,1}^{[p'_{j,1}]} : s_{j,2}^{[k_{j,2}]}),\ldots,s_n]$ and $C[t_1,\ldots,(t_{j,1}^{[p'_{j,1}]} : t_{j,2}^{[k'_{j,2}]}),\ldots,t_n]$ do not have a normal-order reduction, and the claim is shown.

If a normal order reduction for $C[t_1, \ldots, (t_{j,1}^{[p'_{j,1}]} : t_{j,2}^{[k'_{j,2}]}), \ldots, t_n]$ exists, then – due to typing – the reduction rule b must be a (seq)- or (case)-reduction. Either μ_1 is strictly decreased or μ_1, μ_2 are the same and (μ_3, μ_4) is strictly decreased, hence the global order is strictly decreased.
We have to look at the results whether we can apply the induction hypothesis:
If the (seq)-reduction removes $(t_{j,1}^{[p'_{j,1}]} : s_{j,2}^{[k_{j,2}]})$ and $(t_{j,1}^{[p'_{j,1}]} : t_{j_2}^{[k'_{j,2}]})$, then we the expression has a form such that the induction hypothesis can be applied.
If the (seq)-reduction does not remove $(t_{j,1}^{[p'_{j,1}]} : s_{j,2}^{[k_{j,2}]})$ and $(t_{j,1}^{[p'_{j,1}]} : t_{j_2}^{[k'_{j,2}]})$, since the `seq`-expression is of a form `seq` x r, then this expression is part of the context and replaced by r, and so we can apply the induction hypothesis.
If the reduction is a (case)-reduction, then the expressions $t_{j,1}^{[p'_{j,1}]}, s_{j,2}^{[k_{j,2}]}$ and also $t_{j,1}^{[p'_{j,1}]}, t_{j,2}^{[k'_{j,2}]}$ are moved into a `letrec`-environment and remain in surface-context position (at the same places in the $C[s..]$ and the $C[t..]$-expression). Thus we can apply the induction hypothesis.
Since the order is strictly decreased and the form of the expressions permits the application of the induction hypothesis, the claim is shown. □

5 A List Induction Scheme

The goal of this section is to show that there is a specialization of the list induction scheme (CC-list induction scheme) for improvement that can also be used for $\sim_c$, for example to show that $a \mathbin{++} (b \mathbin{++} c) \sim_c (a \mathbin{++} b) \mathbin{++} c$ for any a, b, c.

In this section A is irrelevant, so we mean by `rln` the measure $\texttt{rln}_{\mathfrak{A}}$.

5.1 CC-List Induction Scheme for Equivalence

Definition 5.1. *The* constructor depth *of a context C is the number of constructors on the path to the hole.*

Lemma 5.2. *If C has constructor depth n, $(case) \in A$, and $\texttt{rln}(C[s]) < n$ for some s, then C is not strict.*

Proof. This holds, since looking one constructor depth deeper requires at least one normal-order case-reduction.

The following induction scheme is slightly different and covers more cases than the scheme that requires $C_1[xs] \sim C_2[xs] \implies C_1[x : xs] \sim_c C_2[x : xs]$ instead of (3) of Definition 5.3.

Definition 5.3 (CC-List Induction Scheme for Equivalence). *Let C_1, C_2 be surface contexts such that $C_1[x], C_2[x]$ are of the same type for a fresh variable x of type* `List`(τ).
Let the following hold:

1. $C_1[\bot] \sim_c C_2[\bot]$.
2. $C_1[\texttt{Nil}] \sim_c C_2[\texttt{Nil}]$.
3. *For fresh variables xh and xs and some expression t the following holds:* $(C_1[xh : xs]) \sim_c (t : C_1[xs])$ *and* $(C_2[xh : xs]) \sim_c (t : C_2[xs])$.

Then C_1, C_2 satisfy the CC-list induction scheme for equivalence.

We show that the CC-list induction scheme is sufficient for contextual equivalence:

Theorem 5.4. *If the contexts C_1, C_2 satisfy the CC-list induction scheme for equivalence, then for a fresh variable x and for all expressions s:*
`letrec` $x = s$ `in` $C_1[x] \sim_c$ `letrec` $x = s$ `in` $C_2[x]$.

Proof. We use the context lemma: Let s be an arbitrary expression and S be a surface context such that $S[\texttt{letrec } x = s \texttt{ in } C_1[x]]$, $S[\texttt{letrec } x = s \texttt{ in } C_2[x]]$ are closed.
The goal is to show that $S[\texttt{letrec } x = s \texttt{ in } C_1[x]]\downarrow \iff S[\texttt{letrec } x = s \texttt{ in } C_2[x]]\downarrow$.
The assumptions show that if the local evaluation of x in $S[\texttt{letrec } x = s \texttt{ in } x]$ diverges, i.e., results in $\bot$, then $S[\texttt{letrec } x = s \texttt{ in } C_1[x])]\downarrow \iff S[\texttt{letrec } x = s \texttt{ in } C_2[x])]\downarrow$.
The assumptions also show that if the local evaluation of x in $S[\texttt{letrec } x = s \texttt{ in } x]$ results in $\texttt{Nil}$, then $S[\texttt{letrec } x = s \texttt{ in } C_1[x])]\downarrow \iff S[\texttt{letrec } x = s \texttt{ in } C_2[x])]\downarrow$.

Now assume the local evaluation of x in $S[\texttt{letrec } x = s \texttt{ in } C_1[x])]$ results in $S'[\texttt{letrec } x = s_h : s_t \texttt{ in } C_1[x])]$, then this is $\sim_c S'[\texttt{letrec } xh = s_h, xt = s_t \texttt{ in } C_1[xh : xt])] \sim_c S'[\texttt{letrec } xh = s_h, xt = s_t \texttt{ in } t : C_1[xt])]$, where the constructor-depth of S and S' are the same. (The relation $\texttt{rln}(S[\texttt{letrec } x = s \texttt{ in } C_1[x]]) \geq \texttt{rln}(S'[\texttt{letrec } xh = s_h, xt = s_t \texttt{ in } t : C_1[xt])])$ holds.)

Similarly for the C_2-part: $S[\texttt{letrec } x = s \texttt{ in } C_2[x])] \sim_c S'[\texttt{letrec } xh = s_h, xt = s_t \texttt{ in } t : C_2[xt])]$. Now, standard methods using induction show, basically on the constructor depth of the hole in $S[\texttt{letrec } x = s \texttt{ in } [\cdot]]$ and $S'[\texttt{letrec } xh = s_h, xt = s_t \texttt{ in } t : [\cdot])]$, that $S[\texttt{letrec } x = s \texttt{ in } C_1[x]]\downarrow \iff S[\texttt{letrec } x = s \texttt{ in } C_2[x]]\downarrow$. Hence by the context lemma for equivalence, we obtain $\texttt{letrec } x = s \texttt{ in } C_1[x] \sim_c \texttt{letrec } x = s \texttt{ in } C_2[x]$. □

The append-function `++` can be defined in LRP as a recursive `letrec`-binding:

$$\begin{aligned} Env_{\texttt{++}} := (\texttt{++}) \;= \lambda xs, ys.(&\texttt{case}_{List}\ xs\ (\texttt{Nil} \to ys) \\ &((z : zs) \to z : ((\texttt{++})\ zs\ ys))) \end{aligned}$$

Using the CC-induction scheme we are able to show that left-associative and right-associative bracketing of append are contextually equialent:

Proposition 5.5. *Then right-associative bracketing for* `++` *is contextually equivalent to left-associative bracketing.*

Proof. With $C_1 = \texttt{letrec } Env_{\texttt{++}} \texttt{ in } [\cdot] \texttt{++} (b \texttt{++} c)$ and $C_2 = \texttt{letrec } Env_{\texttt{++}} \texttt{ in } ([\cdot] \texttt{++} b) \texttt{++} c$, the preconditions of the induction scheme 1 (see Definition 5.3) hold and we can apply Theorem 5.4:

- $C_1[x] \sim_c C_2[x]$ can be shown by standard inductive reasoning on contextual equivalence.
- $C_1[\bot] \sim_c \bot \sim_c C_2[\bot]$ and thus $C_1[\bot] \sim_c C_2[\bot]$.
- $C_1[\texttt{Nil}] \sim_c (b \texttt{++} c)$ and $C_2[\texttt{Nil}] \sim_c b \texttt{++} c)$
- $C_1[xh{:}xs] \sim_c (xh{:}C_1[xs])$ and $C_2[xh{:}xs] \sim_c (xh{:}C_2[xs])$. □

Example 5.6. Let `filter`, `id`, `not`, and `not3` be defined as follows:

$$\begin{aligned} \texttt{id} &= \lambda x.x \\ \texttt{filter} &= \lambda p, xs.\texttt{case}_{\texttt{List}}\ xs \\ &\quad (\texttt{Nil -> Nil}) \\ &\quad ((y : ys) \texttt{ -> } (\texttt{case}_{\texttt{Bool}}\ y \\ &\qquad\qquad (\texttt{True -> } y : (\texttt{filter}\ p\ ys)) \\ &\qquad\qquad (\texttt{False -> filter}\ p\ ys))) \\ \texttt{not} &= \lambda x.\texttt{case}_{\texttt{Bool}}\ x\ (\texttt{True -> False})\ (\texttt{False -> True}) \\ \texttt{not3} &= (\lambda x.\texttt{not}\ (\texttt{not}\ (\texttt{not}\ x))) \end{aligned}$$

Let *Env* be a `letrec`-environment where the required definitions of `id`, `map`, `filter` are included, and let $\mathbb{L} := \texttt{letrec } Env\ [\cdot]$.

CC-Induction scheme (together with Theorem 5.4) can easily be applied to show that the following transformations are correct:

- $\mathbb{L}[\texttt{map id } xs] \;\to\; \mathbb{L}[xs]$.
- $\mathbb{L}[\texttt{filter } (\lambda x.\texttt{True})\ xs] \;\to\; \mathbb{L}[xs]$.
- $\mathbb{L}[\texttt{map not3 } xs] \to \mathbb{L}[\texttt{map not } xs]$

6 Improvement for Folds

We analyse various fold-applications and exhibit improvement transformations between fold expressions.

Lemma 6.1. *Let $(case) \in A$. Let s be closed expression of type τ, where τ does not contain $\to$-types, and let $n > 0$. Then there is a closed expression v_n of depth at most n subject to the following condition:*

1. *The nodes at depth $k \leq n$ are constructor-nodes or $\bot$, whereas deeper nodes may be arbitrary;*
2. *Every node at a depth $k \leq n$ (at position q) may carry a sharing decoration W_q.*
3. *For all (type-correct) contexts C with $\mathtt{rln}(C[s]) \leq n$ the equation $\mathtt{rln}(C[s]) = \mathtt{rln}(C[v_n])$ holds.*

Proof. We apply Theorem 3.5 for sharing decorations.
If $s\Uparrow$, then the representation is $\bot$.

If $s\downarrow$, then $s \approx (\mathtt{letrec}\ Env\ \mathtt{in}\ c\ s_1 \dots s_m)^{[k_0]}$ for some k_0, where c is a constructor matching the type of s. This is the same as $s \approx (\mathtt{letrec}\ Env\ \mathtt{in}\ (c\ s_1 \dots s_m)^{[k_0]})$.
We proceed by locally evaluating one s_i after the other. If the evaluation of s_i diverges, then we replace it by $\bot$. Otherwise we replace it by the result and keep the sharing decorations such that the overall expression is invariant w.r.t. $\approx$. Iterating this evaluation also for deeper positions, i.e. until all nodes of constructor-depth $\leq n$ are developed, we obtain an expression v', almost as claimed with $v' \approx s$, but only if the letrec-environments would be removed. If the constructor-depth is at least n, then we copy all the letrec-environments down to the nodes at constructor-depth n, (and after that we remove them at the source position) and thus obtain v_n. We could even replace all nodes of constructor-depth n by $\bot$. This copy-operations and the final replacements do not change the $\mathtt{rln}$-number for a request $C[.]$ with $\mathtt{rln}(C[s]) \leq n$, since normal-order reductions with not more than n case-reductions cannot test the contents of the nodes at constructor depth $\geq n$.

Corollary 6.2. *Let s be an epression of type $\mathtt{List}(\tau)$, where τ does not contain $\to$-types, let $(case) \in A$. and let $n > 0$.*

Then there is a closed expression $v_n = (s_1^{W_1} : (s_2^{W_2} : (\dots)^{W_{2,tl}})^{W_{1,tl}})^{W_0}$, with sharing work decoration $W_i, W_{i,tl}$, such that $s \sim_c v_n$, in v_n either the last s_i is $\bot$, or v_n corresponds to a list of length at least n, i.e., the n-th tail is converging, and for all C with $\mathtt{rln}(C[s]) \leq n$, also $\mathtt{rln}(C[s]) = \mathtt{rln}(C[v_n])$.

Note that $s \preceq v_n$ holds by using the knowledge about the pcgE-transformation [8]. However, there is a small gap to show that (pcgE) is also an improvement in LRPw. We outline how to bridge this gap: The proof for showing that (pcgE) is an improvement in LRPw uses the same the method and diagrams as in [8] for LRP. However, a required lemma is that copying arbitrary expressions is a correct program transformation in LRPw. This lemma can be established by using the methods in [10] applied to the *untyped* variant of LRPw, i.e. the calculus LR extended by $a := n$ and $^{[a]}$ constructs. Showing correctness of copy in this calculus is straightforward by using infinite trees as in [10] for the calculus LR: The new constructs are simply kept in the infinite trees. Finally, the correctness in the untyped setting has to be lifted into the typed setting which again is straightforward.

7 Computing Decorations

In this section we present an algorithm to compute work decorations by partially evaluating expressions.

Algorithm 7.1 *Let $A_{min} \subseteq A \subseteq \mathfrak{A}$. The following procedure computes decorations for closed expressions s: If $s \Uparrow$, then s can be written as $\bot$. If $s \downarrow s_0$, and $n = \mathtt{rln}_A(s)$, then s can be written as $s_0^{[n]}$. For subexpressions, we can make use of so-called* local evaluation*, where a subexpression is labeled with* top*, and then the label-shifting and reduction follows, where the reductions are only permitted if they*

fulfill the label-conditions for normal-order reductions, and where for a reduction sequence, the label `top` *remains at the (starting local) subexpression until the reduction ends.*
An interesting special case is a closed expression of the form

$$\texttt{letrec}\ \mathit{Env}\ \texttt{in}\ (c\ s_1 \dots s_n),$$

where the sharing decorations can be defined (and perhaps also computed) in some cases. Therefore, let us assume that the local evaluation of all s_i terminates, and that for every i: after the evaluation of s_i to s_i', the environment is no longer needed for s_i'.

1. *First determine the numbers n_J for $\emptyset \neq J \subseteq \{1,\dots,n\}$, which count the necessary number of $\texttt{rln}_A$-reductions in the common local evaluation of the subexpressions $s_j, j \in J$ in s.*
2. *Then determine the numbers b_J, for $j \subseteq \{1,\dots,n\}$, which can be interpreted as the number of $\texttt{rln}_A$-reductions required for exactly the set $s_i, i \in J$, but not for subsets. This can be done using the* inclusion-exclusion principle *of combinatorics:*
   For example $b_{\{1,2,3\}}$ is
   $n_{\{1,2,3\}} - \sum_{K\subseteq\{1,2,3\},|K|=2} n_K + \sum_{K\subseteq\{1,2,3\},|K|=1} n_K$.
   and in general $b_{\{1,\dots,n\}}$ is
   $$\begin{array}{ll} & \sum_{K\subseteq\{1,\dots,n\},|K|=1} n_K \\ - & \sum_{K\subseteq\{1,\dots,n\},|K|=2} n_K \\ \dots & \\ - & (-1)^{n-1} \sum_{K\subseteq\{1,\dots,n\},|K|=n-1} n_K \end{array}$$
   Using this approach, all $b_J \in \mathbb{N}$ can be determined from the numbers n_J by using the corresponding formulas, and using only addition and subtraction.
3. *The computed expression is then of the form*
   $$(c\ {t_1}^{p_1} \dots {t_n}^{p_n})$$
   with the following conditions and computations:
   (a) For $i = 1,\dots,n$, t_i is the result of the local evaluation of s_i.
   (b) The sharing decorations p_i consist of all $a_J \mapsto b_J$ with $i \in J \subseteq \{1,\dots,n\}$
   (c) For all $\emptyset \neq J \subseteq \{1,\dots,n\}$ the sharing decorations are $a_J \mapsto b_J$.
   Note that some sharing decorations are in fact (unshared) decorations; in particular $a_i \mapsto b_i$.

For $n = 2$, the expression is `letrec` *Env* `in` *$(c\ s_1\ s_2)$, Then n_1 is the rln_A-reduction count for s_1, n_2 for s_2, and $n_{1,2}$ for evaluating both s_1, s_2. The numbers are: $b_{1,2} = (n_1 + n_2) - n_{1,2}$, and $b_1 = n_1 - b_{1,2}$, $b_2 = n_2 - b_{1,2}$. The expression is then represented as $(t_1^{[b_1, a_1 \mapsto b_{1,2}]}, t_2^{[b_2, a_1 \mapsto b_{1,2}]})$.*

However, note that this method is insufficient for the general case where t_i may have a common environment after evaluation.

8 Conclusion

We have provided the necessary proofs of all the computation rules for unshared and shared decorations. There is also a proof of the simulation proof method for improvement.

References

1. Simon Marlow, editor. *Haskell 2010 – Language Report.* 2010. `www.haskell.org`.
2. Andrew Moran and David Sands. Improvement in a lazy context: An operational theory for call-by-need. In *Proc. POPL 1999*, pages 43–56. ACM Press, 1999.
3. Benjamin C. Pierce. *Types and Programming Languages.* The MIT Press, 2002.
4. Andrew M. Pitts. Parametric polymorphism and operational equivalence. *Math. Structures Comput. Sci.*, 10:321–359, 2000.

5. Manfred Schmidt-Schauß and David Sabel. On generic context lemmas for higher-order calculi with sharing. *Theoret. Comput. Sci.*, 411(11-13):1521 – 1541, 2010.
6. Manfred Schmidt-Schauß and David Sabel. Contextual equivalences in call-by-need and call-by-name polymorphically typed calculi (preliminary report). In *Proc. WPTE 2014*, volume 40 of *OASICS*, pages 63–74. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2014.
7. Manfred Schmidt-Schauß and David Sabel. Improvements in a functional core language with call-by-need operational semantics. In Elvira Albert, editor, *Proc. PPDP '15*, pages 220–231, New York, NY, USA, 2015. ACM.
8. Manfred Schmidt-Schauß and David Sabel. Improvements in a functional core language with call-by-need operational semantics. Frank report 55, Institut für Informatik, Goethe-Universität Frankfurt am Main, November 2015. `http://www.ki.informatik.uni-frankfurt.de/papers/frank/`.
9. Manfred Schmidt-Schauß and David Sabel. Improvements in a functional core language with call-by-need operational semantics. Frank report 55, Institut für Informatik, Goethe-Universität Frankfurt am Main, March 2015. `http://www.ki.informatik.uni-frankfurt.de/papers/frank/`.
10. Manfred Schmidt-Schauß, David Sabel, and Elena Machkasova. Simulation in the call-by-need lambda-calculus with letrec, case, constructors, and seq. *Log. Methods Comput. Sci.*, 11(1), 2015.
11. Manfred Schmidt-Schauß, Marko Schütz, and David Sabel. Safety of Nöcker's strictness analysis. *J. Funct. Programming*, 18(04):503–551, 2008.
12. Dimitrios Vytiniotis and Simon Peyton Jones. Evidence Normalization in System FC (Invited Talk). In Femke van Raamsdonk, editor, *Proc. RTA 2013*, volume 21 of *LIPIcs*, pages 20–38, Dagstuhl, Germany, 2013. Schloss Dagstuhl–Leibniz-Zentrum für Informatik.

A Redundancy of rln-Decorations

We prove Proposition 3.2:

Proposition A.1. *The sharing rln-decorations $s^{[n]}$ can be encoded as* $\texttt{letrec}\ x = (id^n)\ \texttt{in}\ (x\ s)$ *and thus are redundant.*

Proof. Let $A_{min} \subseteq A \subseteq \mathfrak{A}$. We show that $s^{[n]} = \texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]} \approx_A \texttt{letrec}\ x = (id^n)\ \texttt{in}\ (x\ s)$: Let R be a reduction context. It suffices to show $R[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}] \sim_c R[\texttt{letrec}\ x = (id^n)\ \texttt{in}\ (x\ s)]$ and $\texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}]) = \texttt{rln}_A(R[\texttt{letrec}\ x = (id^n)\ \texttt{in}\ (x\ s)])$, since then the context lemma for $\sim_c$ and the context lemma for improvement show the claim.

- If R is a weak reduction context, then

  $$R[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}] \xrightarrow{\text{LRPw},lll,*} \texttt{letrec}\ a := n\ \texttt{in}\ R[s^{[a]}]$$

  and thus $\texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}]) = \texttt{rln}_A(\texttt{letrec}\ a := n\ \texttt{in}\ R[s^{[a]}])$. Now

  $$\texttt{letrec}\ a := n\ \texttt{in}\ R[s^{[a]}] \xrightarrow{\text{LRPw},letwn,n} \texttt{letrec}\ a := 0\ \texttt{in}\ R[s^{[a]}] \xrightarrow{\text{LRPw},letw0} \texttt{letrec}\ a := 0\ \texttt{in}\ R[s]$$

  and thus $\texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}]) = n + \texttt{rln}_A(R[\texttt{letrec}\ a := 0\ \texttt{in}\ s])$. Now Proposition 2.25 shows $\texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}]) = n + \texttt{rln}_A(R[s])$.
  For $R[\texttt{letrec}\ x = (id^n)\ \texttt{in}\ (x\ s)]$ one can verify that

  $$\begin{array}{l} R[\texttt{letrec}\ x = (id^n)\ \texttt{in}\ (x\ s)] \\ \quad \xrightarrow{\text{LRPw},lll,*} (\xrightarrow{\text{LRPw},lbeta} \xrightarrow{\text{LRPw},llet})^{n-1} \xrightarrow{\text{LRPw},cp} \xrightarrow{\text{LRPw},lbeta} \xrightarrow{\text{LRPw},lll,*} \\ \texttt{letrec}\ x = x_1, x_1 = x_2, x_{n-1} = id, x_n = s\ \texttt{in}\ R[x_n] \end{array}$$

  and thus $\texttt{rln}_A(R[\texttt{letrec}\ x = (id^n)\ \texttt{in}\ (x\ s)]) = n + \texttt{rln}_A(\texttt{letrec}\ x = x_1, x_1 = x_2, x_{n-1} = id, x_n = s\ \texttt{in}\ R[x_n])$. Finally, since

  $$\begin{array}{ll} & \texttt{letrec}\ x = x_1, x_1 = x_2, x_{n-1} = id, x_n = s\ \texttt{in}\ R[x_n] \\ \xrightarrow{ucp} & \texttt{letrec}\ x = x_1, x_1 = x_2, x_{n-1} = id, x_n = s\ \texttt{in}\ R[s] \\ \xrightarrow{gc2} & R[s] \end{array}$$

  and Theorem 2.26 shows that (ucp) and (gc2) do not change the $\texttt{rln}_A$-measure, we have $\texttt{rln}_A(R[\texttt{letrec}\ x = (id^n)\ \texttt{in}\ (x\ s)]) = n + \texttt{rln}_A(R[s])$. Concluding, this shows $R[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}] \sim_c R[\texttt{letrec}\ x = (id^n)\ \texttt{in}\ (x\ s)]$ since the left expression can be transformed into the right expression by correct program transformations and it also shows $\texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}]) = \texttt{rln}_A(R[\texttt{letrec}\ x = (id^n)\ \texttt{in}\ (x\ s)])$.
- If R is not a weak reduction context, then there are two cases:
  1. $R[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}] \xrightarrow{\text{LRPw},lll,*} \texttt{letrec}\ Env, a := n\ \texttt{in}\ R_1^-[s^{[a]}]$ and
     $R[\texttt{letrec}\ x = (id^n)\ \texttt{in}\ (x\ s)] \xrightarrow{\text{LRPw},lll,*} \texttt{letrec}\ Env, x = (id^n)\ \texttt{in}\ R_1^-[(x\ s)]$ where R_1^- is a weak reduction context.
     For the left expression:

     $$\begin{array}{ll} & \texttt{letrec}\ Env, a := n\ \texttt{in}\ R_1^-[s^{[a]}] \\ \xrightarrow{\text{LRPw},letwn,n} & \texttt{letrec}\ Env, a := 0\ \texttt{in}\ R_1^-[s^{[a]}] \\ \xrightarrow{\text{LRPw},letw0} & \texttt{letrec}\ Env, a := 0\ \texttt{in}\ R_1^-[s] \\ \xrightarrow{gcW} & \texttt{letrec}\ Env\ \texttt{in}\ R_1^-[s] \end{array}$$

     and thus $R[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}]) \sim_c \texttt{letrec}\ Env\ \texttt{in}\ R_1^-[s]$ and $\texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}]) = n + \texttt{rln}_A(\texttt{letrec}\ Env\ \texttt{in}\ R_1^-[s])$.

For the right expression

$$\begin{array}{ll}
 & \texttt{letrec}\ Env, x = (id^n)\ \texttt{in}\ R_1^-[(x\ s)] \\
(\xrightarrow{\text{LRPw},lbeta}\xrightarrow{\text{LRPw},llet})^{n-1} & \texttt{letrec}\ Env, x = x_1, x_1 = x_2, \ldots, x_{n-1} = id\ \texttt{in}\ R_1^-[(x\ s)] \\
\xrightarrow{\text{LRPw},cp} & \texttt{letrec}\ Env, x = x_1, x_1 = x_2, \ldots, x_{n-1} = id\ \texttt{in}\ R_1^-[(id\ s)] \\
\xrightarrow{\text{LRPw},lbeta} & \texttt{letrec}\ Env, x = x_1, x_1 = x_2, \ldots, x_{n-1} = id\ \texttt{in}\ R_1^-[\texttt{letrec}\ x_n = s\ \texttt{in}\ x_n] \\
\xrightarrow{\text{LRPw},lll,*} & \texttt{letrec}\ Env, x = x_1, x_1 = x_2, \ldots, x_{n-1} = id, x_n = s\ \texttt{in}\ R_1^-[x_n] \\
\xrightarrow{ucp} & \texttt{letrec}\ Env, x = x_1, x_1 = x_2, \ldots, x_{n-1} = id\ \texttt{in}\ R_1^-[s] \\
\xrightarrow{gc} & \texttt{letrec}\ Env\ \texttt{in}\ R_1^-[s]
\end{array}$$

and thus $R[\texttt{letrec}\ x = (id^n)\ \texttt{in}\ (x\ s)] \sim_c \texttt{letrec}\ Env\ \texttt{in}\ R_1^-[s]$ and $\texttt{rln}_A(R[\texttt{letrec}\ x = (id^n)\ \texttt{in}\ (x\ s)]) = n + \texttt{rln}_A(\texttt{letrec}\ Env\ \texttt{in}\ R_1^-[s])$
Together this shows $R[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}] \sim_c R[\texttt{letrec}\ x = (id^n)\ \texttt{in}(x\ s)]$ and $\texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}]) = \texttt{rln}_A(R[\texttt{letrec}\ x = (id^n)\ \texttt{in}\ (x\ s)])$.

2.

$$\begin{array}{ll}
 & R[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}] \\
\xrightarrow{\text{LRPw},lll,*} & \texttt{letrec}\ Env, a := n, y_1 = R_1^-[s^{[a]}], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m,\ \texttt{in}\ R_0^-[y_m]
\end{array}$$

and

$$\begin{array}{ll}
 & R[\texttt{letrec}\ x = (id^n)\ \texttt{in}\ (x\ s)] \\
\xrightarrow{\text{LRPw},lll,*} & \texttt{letrec}\ Env, x = (id^n), y_1 = R_1^-[(x\ s)], \{y_i = R_i^{y_{i-1}}\}_{i=2}^m\ \texttt{in}\ R_0^-[y_m]
\end{array}$$

where $R_0^-, \ldots, R_m$ are weak reduction contexts.
For the left expression:

$$\begin{array}{ll}
 & \texttt{letrec}\ Env, a := n, y_1 = R_1^-[s^{[a]}], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m\ \texttt{in}\ R_0^-[y_m] \\
\xrightarrow{\text{LRPw},letwn,n} & \texttt{letrec}\ Env, a := 0, y_1 = R_1^-[s^{[a]}], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m\ \texttt{in}\ R_0^-[y_m] \\
\xrightarrow{\text{LRPw},letw0} & \texttt{letrec}\ Env, a := 0, y_1 = R_1^-[s], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m\ \texttt{in}\ R_0^-[y_m] \\
\xrightarrow{gcW} & \texttt{letrec}\ Env, y_1 = R_1^-[s], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m\ \texttt{in}\ R_0^-[y_m]
\end{array}$$

and thus

$$R[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}] \sim_c \texttt{letrec}\ Env, y_1 = R_1^-[s], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m\ \texttt{in}\ R_0^-[y_m]$$

and

$$\texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}]) = n + \texttt{rln}_A(\texttt{letrec}\ Env, y_1 = R_1^-[s], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m\ \texttt{in}\ R_0^-[y_m])$$

For the right expression:

$$\begin{array}{ll}
 & \texttt{letrec}\ Env, x = (id^n), y_1 = R_1^-[(x\ s)], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m\ \texttt{in}\ R_0^-[y_m] \\
(\xrightarrow{\text{LRPw},lbeta}\xrightarrow{\text{LRPw},llet})^{n-1} & \texttt{letrec}\ Env, x = x_1, \{x_i = x_{i+1}\}_{i=1}^{n-2}, x_{n-1} = id, \\
 & \qquad y_1 = R_1^-[(x\ s)], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m \\
 & \texttt{in}\ R_0^-[y_m] \\
\xrightarrow{\text{LRPw},cp}\xrightarrow{\text{LRPw},lbeta}\xrightarrow{lll,*} & \texttt{letrec}\ Env, x = x_1, \{x_i = x_{i+1}\}_{i=1}^{n-2}, x_{n-1} = id, x_n = s, \\
 & \qquad y_1 = R_1^-[x_n], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m \\
 & \texttt{in}\ R_0^-[y_m] \\
\xrightarrow{ucp}\xrightarrow{gc} & \texttt{letrec}\ Env, y_1 = R_1^-[s], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m\ \texttt{in}\ R_0^-[y_m]
\end{array}$$

and thus

$$R[\texttt{letrec}\ x = (id^n)\ \texttt{in}\ (x\ s)] \sim_c \texttt{letrec}\ Env, y_1 = R_1^-[s], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m\ \texttt{in}\ R_0^-[y_m]$$

and

$$\mathtt{rln}_A(R[\mathtt{letrec}\, x = (id^n)\ \mathtt{in}\ (x\ s)]) = n + \mathtt{rln}_A(\mathtt{letrec}\, Env, y_1 = R_1^-[s], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m\ \mathtt{in}\ R_0^-[y_m])$$

Together this shows

$$R[\mathtt{letrec}\ a := n\ \mathtt{in}\ s^{[a]}] \sim_c R[\mathtt{letrec}\ x = (id^n)\ \mathtt{in}\ (x\ s)]$$

and

$$\mathtt{rln}_A(R[\mathtt{letrec}\ a := n\ \mathtt{in}\ s^{[a]}]) = \mathtt{rln}_A(R[\mathtt{letrec}\ x = (id^n)\ \mathtt{in}\ (x\ s)])$$

B Proofs of Computation Rules

The following theorem summarizes the results proved in this section.

Theorem B.1. *Let $A_{min} \subseteq A \subseteq \mathfrak{A}$.*

1. *If $s \xrightarrow{\text{LRPw},a} t$ with $a \in A$, then $s \approx_A t^{[1]}$, and if $a \notin A$, then $s \approx_A t$.*
2. *$R[\mathtt{letrec}\ a := n\ \mathtt{in}\ s^{[a]}] \approx_A \mathtt{letrec}\ a := n\ \mathtt{in}\ R[s]^{[a]}$ and thus in particular $R[s^{[n]}] \approx_A R[s]^{[n]}$.*
3. *$\mathtt{rln}_A(\mathtt{letrec}\ a := n\ \mathtt{in}\ s^{[a]}) = n + \mathtt{rln}_A(s')$ where s' is s where all $^{[a]}$-labels are removed. In particular this also shows $\mathtt{rln}_A(s^{[n]}) = n + \mathtt{rln}_A(s)$*
4. *For every reduction context R: $\mathtt{rln}_A(R[\mathtt{letrec}\ a := n\ \mathtt{in}\ s^{[a]}]) = n + \mathtt{rln}_A(R[s'])$ where s' is s where all $^{[a]}$-labels are removed. In particular, this shows $\mathtt{rln}_A(R[s^{[n]}]) = n + \mathtt{rln}_A(R[s])$.*
5. *$(s^{[n]})^{[m]} \approx_A s^{[n+m]}$*
6. *For all surface contexts S_1, S_2: $S_1[\mathtt{letrec}\ a := n\ \mathtt{in}\ S_2[s^{[a]}]] \preceq_A \mathtt{letrec}\ a := n\ \mathtt{in} S_1[S_2[s]]^{[a]}$ and if $S_1[S_2]$ is strict, also $S_1[\mathtt{letrec}\ a := n\ \mathtt{in}\ S_2[s^{[a]}]] \approx_A \mathtt{letrec}\ a := n\ \mathtt{in} S_1[S_2[s]]^{[a]}$.*
7. *$\mathtt{letrec}\ a := n, b := m\ \mathtt{in}\ (s^{[a]})^{[b]} \approx_A \mathtt{letrec}\ a := n, b := m\ \mathtt{in}\ (s^{[b]})^{[a]}$*
8. *$\mathtt{letrec}\ a := n\ \mathtt{in}\ (s^{[a]})^{[a]} \approx_A \mathtt{letrec}\ a := n\ \mathtt{in}\ (s^{[a]})$*
9. *$(t^{p_1})^{p_2} \approx_A t^{p_1 \oplus p_2}$.*
10. *Let $S[\cdot, \ldots, \cdot]$ be a multi-context where all holes are in surface position. Then $\mathtt{letrec}\ a := n\ \mathtt{in}\ S[s_1^{[a]}, \ldots, s_n^{[a]}] \preceq_A \mathtt{letrec}\ a := n\ \mathtt{in}\ S[s_1, \ldots, s_n]^{[a]}$. If some hole $\cdot_i$ with $i \in \{1, \ldots, n\}$ is in strict position in $S[\ldots,.]$, then $\mathtt{letrec}\ a := n\ \mathtt{in}\ S[s_1^{[a]}, \ldots, s_n^{[a]}] \approx_A \mathtt{letrec}\ a := n\ \mathtt{in}\ S[s_1, \ldots, s_n]^{[a]}$.*
11. *Let $S[\cdot, \ldots, \cdot]$ be a multi-context where all holes are in surface position. Let $S[s_1, \ldots, s_n]$ be closed. Then $S[s_1^{[p_1, a \mapsto m]}, \ldots, s_n^{[p_n, a \mapsto m]}] \preceq_A S[s_1^{p_1}, \ldots, s_n^{p_n}]^{[m]}$.*
    If some hole $\cdot_i$ with $i \in \{1, \ldots, n\}$ is in strict position in $S[\ldots,.]$, then $S[s_1^{[p_1, a \mapsto m]}, \ldots, s_n^{[p_n, a \mapsto m]}] \approx_A S[s_1^{p_1}, \ldots, s_n^{p_n}]^{[m]}$.
12. *The following transformation is correct w.r.t. $\approx_A$: Replace $(\mathtt{letrec}\ x = s^{[n,p]}, Env\ \mathtt{in}\ t)$ by $\mathtt{letrec}\ x = s[x^{[a \mapsto n,p]}/x]$, $Env[x^{[a \mapsto n,p]}/x]\ \mathtt{in}\ t[x^{[a \mapsto n,p]}/x]$, where a is a fresh label and all occurrences of x are in surface position.*
13. *If a label-name a occurs exactly once in a surface context, then it can be changed into an unshared decoration.*
14. *If p, p' are two decorations with $p \leq p'$, and $s \preceq_A t$, then $s^{[p]} \preceq_A t^{[p']}$.*

Proof. (1) is proved in Theorem B.8.
(2) is proved in Proposition B.3.
(3) is proved in Lemma B.4.
(4) is proved in Corollary B.5.
(5) is proved in Proposition B.6.
(6) is proved in Corollary B.12.
(7) and (8) are proved in Proposition B.13, and (9) holds by iteratively applying items (4), (7) and (8) and by applying (lll) and (gcW)-transformations which are invariant w.r.t. $\approx_A$.
(10) is proved in Proposition B.14.

(11) is proved in Corollary B.15.
(12) is proved in Proposition B.16.
(13) follows from the semantics of the labels.
(14) follows from (1) and the context lemma for improvements.

Proposition B.2. *Let $A_{min} \subseteq A \subseteq \mathfrak{A}$. If $s \xrightarrow{\text{LRPw},letwn} t$, then $s \approx_A t^{[1]}$*

Proof. We use the context lemma for improvement and thus have to show for all reduction contexts R:
$\mathtt{rln}_A(R[s]) = \mathtt{rln}_A(R[\mathtt{letrec}\ a := 1\ \mathtt{in}\ t^{[a]}])$
There are three general cases for the reduction context R and two cases for s and t:

1. $s = \mathtt{letrec}\ b := n, Env\ \mathtt{in}\ R_0^-[r^{[b]}]$,
   $t = \mathtt{letrec}\ b := n-1, Env\ \mathtt{in}\ R_0^-[r^{[b]}]$
   (a) R is a weak reduction context. Then $\mathtt{rln}_A(R[s]) = \mathtt{rln}_A(R[\mathtt{letrec}\ a := 1\ \mathtt{in}\ t^{[a]}])$, since:

$$
\begin{array}{ll}
R[s] & \xrightarrow{\text{LRPw},lll,*} \ \mathtt{letrec}\ b := n, Env\ \mathtt{in}\ R[R_0^-[r^{[b]}]] \\
 & \xrightarrow{\text{LRPw},letwn} \mathtt{letrec}\ b := n-1, Env\ \mathtt{in}\ R[R_0^-[r^{[b]}]] \\
R[\mathtt{letrec}\ a := 1\ \mathtt{in}\ t^{[a]}] & \xrightarrow{\text{LRPw},lll,*} \ \mathtt{letrec}\ a := 1\ \mathtt{in}\ R[(\mathtt{letrec}\ b := n-1, Env\ \mathtt{in}\ R_0^-[r^{[b]}])^{[a]}] \\
 & \xrightarrow{\text{LRPw},lll,*} \ \mathtt{letrec}\ a := 1, b := n-1, Env\ \mathtt{in}\ R[R_0^-[r^{[b]}])^{[a]}] \\
 & \xrightarrow{\text{LRPw},letwn} \mathtt{letrec}\ a := 0, b := n-1, Env\ \mathtt{in}\ R[R_0^-[r^{[b]}])^{[a]}] \\
 & \xrightarrow{\text{LRPw},letw0} \mathtt{letrec}\ a := 0, b := n-1, Env\ \mathtt{in}\ R[R_0^-[r^{[b]}])] \\
 & \xrightarrow{gcW} \ \mathtt{letrec}\ b := n-1, Env\ \mathtt{in}\ R[R_0^-[r^{[b]}])]
\end{array}
$$

   (b) $R = \mathtt{letrec}\ Env'\ \mathtt{in}\ R'[\cdot]$, where R' is a weak reduction context. Then $\mathtt{rln}_A(R[s]) = \mathtt{rln}_A(R[\mathtt{letrec}\ a := 1\ \mathtt{in}\ t^{[a]}])$, since:

$$
\begin{array}{l}
\mathtt{letrec}\ Env'\ \mathtt{in}\ R'[s] \xrightarrow{\text{LRPw},lll,*} \ \mathtt{letrec}\ b := n, Env, Env'\ \mathtt{in}\ R'[R_0^-[r^{[b]}]] \\
\qquad\qquad \xrightarrow{\text{LRPw},letwn} \mathtt{letrec}\ b := n-1, Env, Env'\ \mathtt{in}\ R'[R_0^-[r^{[b]}]] \\
\mathtt{letrec}\ Env'\ \mathtt{in}\ R'[\mathtt{letrec}\ a := 1\ \mathtt{in}\ t^{[a]}] \\
\xrightarrow{\text{LRPw},lll,*} \ \mathtt{letrec}\ Env', a := 1\ \mathtt{in}\ R'[(\mathtt{letrec}\ b := n-1, Env\ \mathtt{in}\ R_0^-[r^{[b]}])^{[a]}] \\
\xrightarrow{\text{LRPw},lll,*} \ \mathtt{letrec}\ Env', a := 1, b := n-1, Env\ \mathtt{in}\ R'[R_0^-[r^{[b]}])^{[a]}] \\
\xrightarrow{\text{LRPw},letwn} \mathtt{letrec}\ Env', a := 0, b := n-1, Env\ \mathtt{in}\ R'[R_0^-[r^{[b]}])^{[a]}] \\
\xrightarrow{\text{LRPw},letw0} \mathtt{letrec}\ Env', a := 0, b := n-1, Env\ \mathtt{in}\ R'[R_0^-[r^{[b]}])] \\
\xrightarrow{gcW} \ \mathtt{letrec}\ Env', b := n-1, Env\ \mathtt{in}\ R'[R_0^-[r^{[b]}])]
\end{array}
$$

   (c) $R = \mathtt{letrec}\ Env', x = R'[\cdot]\ \mathtt{in}\ u$, where R' is a weak reduction context. Then $\mathtt{rln}_A(R[s]) = \mathtt{rln}_A(R[\mathtt{letrec}\ a := 1\ \mathtt{in}\ t^{[a]}])$, since:

$$
\begin{array}{l}
\mathtt{letrec}\ Env', x = R'[s]\ \mathtt{in}\ u \xrightarrow{\text{LRPw},lll,*} \ \mathtt{letrec}\ b := n, Env, Env', x = R'[R_0^-[r^{[b]}]]\ \mathtt{in}\ u \\
\qquad\qquad \xrightarrow{\text{LRPw},letwn} \mathtt{letrec}\ b := n-1, Env, Env', x = R'[R_0^-[r^{[b]}]]\ \mathtt{in}\ u \\
\mathtt{letrec}\ Env', x = R'[\mathtt{letrec}\ a := 1\ \mathtt{in}\ t^{[a]}]\ \mathtt{in}\ u \\
\xrightarrow{\text{LRPw},lll,*} \ \mathtt{letrec}\ Env', a := 1, x = R'[(\mathtt{letrec}\ b := n-1, Env\ \mathtt{in}\ R_0^-[r^{[b]}])^{[a]}]\ \mathtt{in}\ u \\
\xrightarrow{\text{LRPw},lll,*} \ \mathtt{letrec}\ Env', a := 1, b := n-1, Env, x = R'[R_0^-[r^{[b]}]^{[a]}]\ \mathtt{in}\ u \\
\xrightarrow{\text{LRPw},letwn} \mathtt{letrec}\ Env', a := 0, b := n-1, Env, x = R'[R_0^-[r^{[b]}]^{[a]}]\ \mathtt{in}\ u \\
\xrightarrow{\text{LRPw},letw0} \mathtt{letrec}\ Env', a := 0, b := n-1, Env, x = R'[R_0^-[r^{[b]}]]\ \mathtt{in}\ u \\
\xrightarrow{\text{LRPw},gcW} \ \mathtt{letrec}\ Env', b := n-1, Env, x = R'[R_0^-[r^{[b]}]]\ \mathtt{in}\ u
\end{array}
$$

2. $s = \texttt{letrec}\ b := n, Env, y_1 = R_0^-[r^{[b]}], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m\ \texttt{in}\ R_{m+1}^-[y_m]$
   $t = \texttt{letrec}\ b := n-1, Env, y_1 = R_0^-[r^{[b]}], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m\ \texttt{in}\ R_{m+1}^-[y_m]$
   (a) R is a weak reduction context. Then $\texttt{rln}_A(R[s]) = \texttt{rln}_A(R[\texttt{letrec}\ a := 1\ \texttt{in}\ t^{[a]}])$, since:

$$
\begin{array}{ll}
R[s] & \xrightarrow{\text{LRPw},lll,*} \ \texttt{letrec}\ b := n, Env, y_1 = R_0^-[r^{[b]}], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m\ \texttt{in}\ R[R_{m+1}^-[y_m]] \\
& \xrightarrow{\text{LRPw},letwn} \texttt{letrec}\ b := n-1, Env, y_1 = R_0^-[r^{[b]}], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m\ \texttt{in}\ R[R_{m+1}^-[y_m]] \\
R[\texttt{letrec}\ a := 1\ \texttt{in}\ t^{[a]}] & \\
\xrightarrow{\text{LRPw},lll,*} & \texttt{letrec}\ a := 1\ \texttt{in} \\
& \quad R[(\texttt{letrec}\ b := n-1, Env, y_1 = R_0^-[r^{[b]}], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m\ \texttt{in}\ R_{m+1}^-[y_m])^{[a]}] \\
\xrightarrow{\text{LRPw},letwn} & \texttt{letrec}\ a := 0\ \texttt{in} \\
& \quad R[(\texttt{letrec}\ b := n-1, Env, y_1 = R_0^-[r^{[b]}], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m\ \texttt{in}\ R_{m+1}^-[y_m])^{[a]}] \\
\xrightarrow{\text{LRPw},letw0} & \texttt{letrec}\ a := 0\ \texttt{in} \\
& \quad R[(\texttt{letrec}\ b := n-1, Env, y_1 = R_0^-[r^{[b]}], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m\ \texttt{in}\ R_{m+1}^-[y_m])] \\
\xrightarrow{gcW} & \quad R[(\texttt{letrec}\ b := n-1, Env, y_1 = R_0^-[r^{[b]}], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m\ \texttt{in}\ R_{m+1}^-[y_m])] \\
\xrightarrow{\text{LRPw},lll,*} & \texttt{letrec}\ b := n-1, Env, y_1 = R_0^-[r^{[b]}], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m\ \texttt{in}\ R[R_{m+1}^-[y_m])]
\end{array}
$$

   (b) $R = \texttt{letrec}\ Env'\ \texttt{in}\ R'[\cdot]$, where R' is a weak reduction context. Then $\texttt{rln}_A(R[s]) = \texttt{rln}_A(R[\texttt{letrec}\ a := 1\ \texttt{in}\ t^{[a]}])$, since:

$$
\begin{array}{ll}
\texttt{letrec}\ Env'\ \texttt{in}\ R'[s] & \\
\xrightarrow{\text{LRPw},lll,*} & \texttt{letrec}\ Env', b := n, Env, y_1 = R_0^-[r^{[b]}], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m\ \texttt{in}\ R'[R_{m+1}^-[y_m]] \\
\xrightarrow{\text{LRPw},letwn} & \texttt{letrec}\ Env', b := n-1, Env, y_1 = R_0^-[r^{[b]}], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m\ \texttt{in}\ R'[R_{m+1}^-[y_m]] \\
\texttt{letrec}\ Env'\ \texttt{in}\ R'[\texttt{letrec}\ a := 1\ \texttt{in}\ t^{[a]}] & \\
\xrightarrow{\text{LRPw},lll,*} & \texttt{letrec}\ Env', a := 1\ \texttt{in}\ R'[t^{[a]}] \\
\xrightarrow{\text{LRPw},letwn} & \texttt{letrec}\ Env', a := 0\ \texttt{in}\ R'[t^{[a]}] \\
\xrightarrow{\text{LRPw},letw0} & \texttt{letrec}\ Env', a := 0\ \texttt{in}\ R'[t] \\
\xrightarrow{gcW} & \texttt{letrec}\ Env'\ \texttt{in}\ R'[t] \\
\xrightarrow{\text{LRPw},lll,*} & \texttt{letrec}\ Env', b := n-1, Env, y_1 = R_0^-[r^{[b]}], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m\ \texttt{in}\ R'[R_{m+1}^-[y_m]]
\end{array}
$$

   (c) $R = \texttt{letrec}\ Env', x = R'[\cdot]\ \texttt{in}\ u$, where R' is a weak reduction context. Then $\texttt{rln}_A(R[s]) = \texttt{rln}_A(R[\texttt{letrec}\ a := 1\ \texttt{in}\ t^{[a]}])$, since:

$$
\begin{array}{ll}
\texttt{letrec}\ Env', x = R'[s]\ \texttt{in}\ u & \\
\xrightarrow{\text{LRPw},lll,*} & \texttt{letrec}\ Env', b := n, Env, y_1 = R_0^-[r^{[b]}], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m, x = R'[R_{m+1}^-[y_m]]\ \texttt{in}\ u \\
\xrightarrow{\text{LRPw},letwn} & \texttt{letrec}\ Env', b := n-1, Env, y_1 = R_0^-[r^{[b]}], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m, x = R'[R_{m+1}^-[y_m]]\ \texttt{in}\ u \\
\texttt{letrec}\ Env', x = R'[\texttt{letrec}\ a := 1\ \texttt{in}\ t^{[a]}]\ \texttt{in}\ u & \\
\xrightarrow{\text{LRPw},lll,*} & \texttt{letrec}\ Env', a := 1, x = R'[t^{[a]}]\ \texttt{in}\ u \\
\xrightarrow{\text{LRPw},letwn} & \texttt{letrec}\ Env', a := 0, x = R'[t^{[a]}]\ \texttt{in}\ u \\
\xrightarrow{\text{LRPw},letw0} & \texttt{letrec}\ Env', a := 0, x = R'[t]\ \texttt{in}\ u \\
\xrightarrow{gcW} & \texttt{letrec}\ Env', x = R'[t]\ \texttt{in}\ u \\
\xrightarrow{\text{LRPw},lll,*} & \texttt{letrec}\ Env', b := n-1, Env, y_1 = R_0^-[r^{[b]}], \{y_i = R_i^-[y_{i-1}]\}_{i=2}^m, x = R'[R_{m+1}^-[y_m]]\ \texttt{in}\ u
\end{array}
$$

Proposition B.3. *Let* $A_{min} \subseteq A \subseteq \mathfrak{A}$. $R[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}] \approx_A \texttt{letrec}\ a := n\ \texttt{in}\ R[s]^{[a]}$ *and thus in particular* $R[s^{[n]}] \approx_A R[s]^{[n]}$

Proof. We show $R[\texttt{letrec } a := n \texttt{ in } s^{[a]}] \approx_A \texttt{letrec } a := n \texttt{ in } R[s]^{[a]}$ by induction on n. If $n = 0$ then the claim holds, since $(letw0) \subseteq \approx_A$.

For the induction step assume that the claim holds for all k, with $k \leq n-1$.

We make a case distinction on the reduction context R.

1. R is a weak reduction context. Then

$$\begin{array}{ll}
& R[\texttt{letrec } a := n \texttt{ in } s^{[a]}] \\
\approx_A & \texttt{letrec } a := n \texttt{ in } R[s^{[a]}] & \text{(since } (lll) \subseteq \approx_A) \\
\approx_A & \texttt{letrec } b := 1 \texttt{ in } (\texttt{letrec } a := n-1 \texttt{ in } R[s^{[a]}])^{[b]} & \text{(by Proposition B.2)} \\
\approx_A & \texttt{letrec } b := 1 \texttt{ in } R[\texttt{letrec } a := n-1 \texttt{ in } s^{[a]}]^{[b]} & \text{(since } (lll) \subseteq \approx_A) \\
\approx_A & \texttt{letrec } b := 1 \texttt{ in } (\texttt{letrec } a := n-1 \texttt{ in } (R[s]^{[a]}))^{[b]} & \text{(induction hypothesis)} \\
\approx_A & \texttt{letrec } a := n \texttt{ in } (R[s]^{[a]}) & \text{(by Proposition B.2)}
\end{array}$$

2. $R = \texttt{letrec } Env \texttt{ in } R'[\cdot]$ where R' is a weak reduction context.

$$\begin{array}{ll}
& R[\texttt{letrec } a := n \texttt{ in } s^{[a]}] \\
= & \texttt{letrec } Env \texttt{ in } R'[\texttt{letrec } a := n \texttt{ in } s^{[a]}] \\
\approx_A & \texttt{letrec } a := n, Env,\ R'[s^{[a]}] & \text{(since } (lll) \subseteq \approx_A) \\
\approx_A & \texttt{letrec } b := 1 \texttt{ in } (\texttt{letrec } a := n-1, Env \texttt{ in } R'[s^{[a]}])^{[b]} & \text{(by Proposition B.2)} \\
\approx_A & \texttt{letrec } b := 1 \texttt{ in } R[\texttt{letrec } a := n-1 \texttt{ in } s^{[a]}]^{[b]} & \text{(since } (lll) \subseteq \approx_A) \\
\approx_A & \texttt{letrec } b := 1 \texttt{ in } (\texttt{letrec } a := n-1 \texttt{ in } R[s]^{[a]})^{[b]} & \text{(induction hypothesis)} \\
= & \texttt{letrec } b := 1 \texttt{ in } (\texttt{letrec } a := n-1 \texttt{ in } (\texttt{letrec } Env \texttt{ in } R'[s])^{[a]})^{[b]} \\
\approx_A & \texttt{letrec } a := n \texttt{ in } (\texttt{letrec } Env \texttt{ in } R'[s])^{[a]}) & \text{(by Proposition B.2)} \\
= & \texttt{letrec } a := n \texttt{ in } (R[s])^{[a]}
\end{array}$$

3. $R = \texttt{letrec } Env, x = R'[\cdot] \texttt{ in } u$ where R' is a weak reduction context.

$$\begin{array}{ll}
& R[\texttt{letrec } a := n \texttt{ in } s^{[a]}] \\
= & \texttt{letrec } Env, x = R'[\texttt{letrec } a := n \texttt{ in } s^{[a]}] \texttt{ in } u \\
\approx_A & \texttt{letrec } Env, a := n, x = R'[s^{[a]}] \texttt{ in } u & \text{(since } (lll) \subseteq \approx_A) \\
\approx_A & \texttt{letrec } b := 1 \texttt{ in } (\texttt{letrec } Env, a := n-1, x = R'[s^{[a]}] \texttt{ in } u)^{[b]} & \text{(by Proposition B.2)} \\
\approx_A & \texttt{letrec } b := 1 \texttt{ in } (\texttt{letrec } Env, x = (R'[\texttt{letrec } a := n-1 \texttt{ in } s^{[a]}]) \texttt{ in } u)^{[b]} & \text{(since } (lll) \subseteq \approx_A) \\
= & \texttt{letrec } b := 1 \texttt{ in } (R[\texttt{letrec } a := n-1 \texttt{ in } s^{[a]}])^{[b]} \\
\approx_A & \texttt{letrec } b := 1 \texttt{ in } (\texttt{letrec } a := n-1 \texttt{ in } R[s]^{[a]}])^{[b]} & \text{(induction hypothesis)} \\
= & \texttt{letrec } b := 1 \texttt{ in } (\texttt{letrec } a := n-1 \texttt{ in } (\texttt{letrec } Env, x = R'[s] \texttt{ in } u)^{[a]})^{[b]} \\
\approx_A & \texttt{letrec } a := n \texttt{ in } (\texttt{letrec } Env, x = R'[s] \texttt{ in } u)^{[a]} & \text{(by Proposition B.2)} \\
= & \texttt{letrec } a := n \texttt{ in } R[s]^{[a]}
\end{array}$$

Lemma B.4. *Let $A_{min} \subseteq A \subseteq \mathfrak{A}$. Then $\texttt{rln}_A(\texttt{letrec } a := n \texttt{ in } s^{[a]}) = n + \texttt{rln}_A(s')$ where s' is s where all $^{[a]}$-labels are removed. In particular this also shows $\texttt{rln}_A(s^{[n]}) = n + \texttt{rln}_A(s)$*

Proof. The reduction $\texttt{letrec } a := n \texttt{ in } s^{[a]} \xrightarrow{\text{LRPw},letwn,n} \xrightarrow{C,letw0,*} \texttt{letrec } a := 0 \texttt{ in } s'$ shows $\texttt{rln}_A(s^{[n]}) = n + \texttt{rln}_A(\texttt{letrec } a := 0 \texttt{ in } s)$. Finally, $(gcW) \subseteq \approx$ shows the claim.

Corollary B.5. *Let $A_{min} \subseteq A \subseteq \mathfrak{A}$. For every reduction context R: $\texttt{rln}_A(R[\texttt{letrec } a := n \texttt{ in } s^{[a]}]) = n + \texttt{rln}_A(R[s'])$ where s' is s where all $^{[a]}$-labels are removed. In particular, this shows $\texttt{rln}_A(R[s^{[n]}]) = n + \texttt{rln}_A(R[s])$.*

Proof. By Proposition B.3 we have $\texttt{rln}_A(R[\texttt{letrec } a := n \texttt{ in } s^{[a]}]) = \texttt{rln}_A(\texttt{letrec } a := n \texttt{ in } R[s]^{[a]})$ and by Lemma B.4 we have $\texttt{rln}_A(\texttt{letrec } a := n \texttt{ in } R[s]^{[a]}) = n + \texttt{rln}_A(R[s'])$.

Proposition B.6. *Let* $A_{min} \subseteq A \subseteq \mathfrak{A}$. *Then* $(s^{[n]})^{[m]} \approx_A s^{[n+m]}$

Proof. Clearly $(s^{[n]})^{[m]} \sim_c s^{[n+m]}$ Let R be a reduction context, then $\mathtt{rln}_A(R[(s^{[n]})^{[m]}]) = m + \mathtt{rln}_A([R[s^{[n]}]) = m + n + \mathtt{rln}_A(R[s]) = \mathtt{rln}_A(R[s^{[n+m]}])$ by Corollary B.5. Now the context lemma for improvement shows the claim.

Lemma B.7. *Let* $A_{min} \subseteq A \subseteq \mathfrak{A}$. *If* $s \xrightarrow{\text{LRPw},a} t$ *and* $a \in \{lbeta, case - c, seq - c\}$, *then* $s \approx_A t^{[1]}$ *if* $a \in A$.

Proof. We use the context lemma for improvement and thus have to show for all reduction contexts R:

$\mathtt{rln}_A(R[s]) = \mathtt{rln}_A(R[t^{[1]}])$ (if $a \in A$)
By Corollary B.5 we have $\mathtt{rln}_A(R[t^{[1]}]) = 1 + \mathtt{rln}_A(R[t])$ and thus it suffices to show
$\mathtt{rln}_A(R[s]) = 1 + \mathtt{rln}_A(R[t])$
Let $s_0 \xrightarrow{a} t_0$ for $a \in \{lbeta, case - c, seq\}$ and assume $a \in A$. We very all cases for s and t:

1. $s = R_0^-[s_0]$, Then $R[s] \xrightarrow{\text{LRPw},a} R[t]$.
   $t = R_0^-[t_0]$
2. $s = \mathtt{letrec}\ Env\ \mathtt{in}\ R_0^-[s_0]$, We go through the cases for R:
   $t = \mathtt{letrec}\ Env\ \mathtt{in}\ R_0^-[t_0]$
   (a) R is a weak reduction context. Then

$$\begin{array}{ll} R[s] & \xrightarrow{\text{LRPw},lll,*} \mathtt{letrec}\ Env\ \mathtt{in}\ R[R_0^-[s_0]] \\ & \xrightarrow{\text{LRPw},a} \quad \mathtt{letrec}\ Env\ \mathtt{in}\ R[R_0^-[t_0]] \\ & \xleftarrow{\text{LRPw},lll,*} R[\mathtt{letrec}\ Env\ \mathtt{in}\ R_0^-[t_0]] = R[t] \end{array}$$

   (b) $R = \mathtt{letrec}\ Env'\ \mathtt{in}\ R'$, where R' is a weak reduction context. Then

$$\begin{array}{ll} R[s] & \xrightarrow{\text{LRPw},lll,*} \mathtt{letrec}\ Env, Env'\ \mathtt{in}\ R'[R_0^-[s_0]] \\ & \xrightarrow{\text{LRPw},a} \quad \mathtt{letrec}\ Env, Env'\ \mathtt{in}\ R'[R_0^-[t_0]] \\ & \xleftarrow{C,lll,*} \quad \mathtt{letrec}\ Env'\ \mathtt{in}\ R'[\mathtt{letrec}\ Env\ \mathtt{in}\ R_0^-[t_0]] \\ & = \quad R[t] \end{array}$$

   (c) $R = \mathtt{letrec}\ Env', u = R'\ \mathtt{in}\ r$, where R' is a weak reduction context. Then $\mathtt{rln}(R[s]) = \mathtt{rln}(R[\mathtt{letrec}\ a := 1\ \mathtt{in}\ t^{[a]}])$, since:

$$\begin{array}{ll} R[s] & \xrightarrow{\text{LRPw},lll,*} \mathtt{letrec}\ Env, Env', u = R'[R_0^-[s_0]]\ \mathtt{in}\ r \\ & \xrightarrow{\text{LRPw},a} \quad \mathtt{letrec}\ Env, Env', u = R'[R_0^-[t_0]]\ \mathtt{in}\ r \\ & \xleftarrow{C,lll,*} \quad \mathtt{letrec}\ Env', u = R'[\mathtt{letrec}\ Env\ \mathtt{in} R_0^-[t_0]]\ \mathtt{in}\ r \\ & = \quad R[t] \end{array}$$

3. $s = \mathtt{letrec}\ Env, y = R_0^-[s_0]\ \mathtt{in}\ u_0$ We go through the cases for R:
   $t = \mathtt{letrec}\ Env, y = R_0^-[t_0]\ \mathtt{in}\ u_0$
   (a) R is a weak reduction context. Then

$$\begin{array}{ll} R[s] & \xrightarrow{\text{LRPw},lll,*} \mathtt{letrec}\ Env, y = R_0^-[s_0]\ \mathtt{in}\ R[u_0] \\ & \xrightarrow{\text{LRPw},a} \quad \mathtt{letrec}\ Env, y = R_0^-[t_0]\ \mathtt{in}\ R[u_0] \\ & \xleftarrow{C,lll,*} \quad R[t] \end{array}$$

(b) $R = \texttt{letrec}\ Env'\ \texttt{in}\ R'$, where R' is a weak reduction context.

$$\begin{array}{ll} R[s] & \xrightarrow{\text{LRPw},lll,*} \texttt{letrec}\ Env', Env, y = R_0^-[s_0]\ \texttt{in}\ R'[u_0] \\ & \xrightarrow{\text{LRPw},a} \quad \texttt{letrec}\ Env', Env, y = R_0^-[t_0]\ \texttt{in}\ R'[u_0] \\ & \xleftarrow{C,lll,*} \quad R[t] \end{array}$$

(c) $R = \texttt{letrec}\ Env', u = R'\ \texttt{in}\ r$, where R' is a weak reduction context. Then

$$\begin{array}{ll} R[s] & \xrightarrow{\text{LRPw},lll,*} \texttt{letrec}\ Env', Env, y = R_0^-[s_0], u = R'[u_0]\ \texttt{in}\ r \\ & \xrightarrow{\text{LRPw},a} \quad \texttt{letrec}\ Env', Env, y = R_0^-[t_0], u = R'[u_0]\ \texttt{in}\ r \\ & \xleftarrow{C,lll,*} \quad R[t] \end{array}$$

Theorem B.8. *Let $A_{min} \subseteq A \subseteq \mathfrak{A}$. If $s \xrightarrow{\text{LRPw},a} t$ with $a \in \{lbeta, case, seq, letwn\}$, then $s \approx t^{[1]}$, provided $a \in A$.*

Proof. For (letwn) this was proved in Proposition B.2, for (case-c), (seq-c), and (lbeta) this was proved in Lemma B.7. For the remaining (case) and (seq)-reductions, it suffices to observe that these transformation can be expressed by using one (LRPw,case-c)-reduction (or (LRPw,seq-c)-reduction respectively) and (cpcx), (gc), and (lll) transformations. Since for all these transformation we have $u \xrightarrow{C,cpcx \vee gc \vee lll} u'$ implies $u \approx_A u'$ (Theorem 2.26) the claim follows.

Proposition B.9. *Let $A_{min} \subseteq A \subseteq \mathfrak{A}$. For any strict surface context S: $S[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}] \approx_A \texttt{letrec}\ a := n\ \texttt{in}\ S[s]^{[a]}$ and thus in particular $S[s^{[n]}] \approx_A S[s]^{[n]}$*

Proof. If $S[r] \sim_c \bot$ for all r, then $S[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}] \sim_c \bot \sim_c \texttt{letrec}\ a := n\ \texttt{in}\ S[s]^{[a]}$ and since for any reduction context R: $R[\bot]\!\uparrow$, $R[S[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}]]\!\uparrow$ and $R[\texttt{letrec}\ a := n\ \texttt{in}\ S[s]^{[a]}]\!\uparrow$ and thus $\texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}]) = \infty = \texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ S[s]^{[a]}])$ and the context lemma for improvement shows $S[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}] \approx_A \texttt{letrec}\ a := n\ \texttt{in}\ S[s]^{[a]}$.

Otherwise, for every r and any reduction context R: $R[S[r]] \xrightarrow{\text{LRPw},k} R'[r]$ where R' is a reduction context. and $\texttt{rln}_A(R[S[r]] \xrightarrow{\text{LRPw},k} R'[r]) = m$ for some $m \leq k$.

For $R[S[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}]]$, we have $\texttt{rln}_A(R[S[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}]]) = m + \texttt{rln}_A(R'[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}])$ and by Proposition B.3 we have $\texttt{rln}_A(R'[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}]) = \texttt{rln}_A(\texttt{letrec}\ a := n\ \texttt{in}\ R'[s]^{[a]})$. By Lemma B.4 we have $\texttt{rln}_A(\texttt{letrec}\ a := n\ \texttt{in}\ R'[s]^{[a]}) = n + \texttt{rln}(R'[s'])$ where s' is s where all $^{[a]}$-labels are removed. Thus $\texttt{rln}_A(R[S[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}]]) = m + n + \texttt{rln}_A(R'[s'])$

For $R[\texttt{letrec}\ a := n\ \texttt{in}\ S[s]^{[a]}]$, we have by Corollary B.5 $\texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ S[s]^{[a]}]) = n + \texttt{rln}_A(R[S[s']])$ where s' is s where all $^{[a]}$-labels are removed. Since $\texttt{rln}_A(R[S[s']]) = m + \texttt{rln}_A(R'[S[s']])$, we have $\texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ S[s]^{[a]}]) = n + m + \texttt{rln}_A(R'[S[s']])$.

Concluding we have shown $\texttt{rln}_A(R[S[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}]]) = \texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ S[s]^{[a]}])$ and clearly also $R[S[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}]] \sim_c R[\texttt{letrec}\ a := n\ \texttt{in}\ S[s]^{[a]}]$ holds. Thus the context lemma for improvement shows the claim.

Proposition B.10. *Let $A_{min} \subseteq A \subseteq \mathfrak{A}$. Let S be a surface context. Then $S[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}] \preceq_A \texttt{letrec}\ a := n\ \texttt{in}\ S[s]^{[a]}$ and in particular $S[s^{[n]}] \preceq_A S[s]^{[n]}$.*

Proof. Let R be a reduction context. If $R[S]$ is strict, then Proposition B.9 shows $R[S[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}]] \preceq_A R[\texttt{letrec}\ a := n\ \texttt{in}\ S[s]^{[a]}]$ and thus $\texttt{rln}_A(R[S[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}]]) \leq \texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ S[s]^{[a]}])$ for all reduction contexts.

If $R[S]$ is non-strict, then $\texttt{rln}_A(R[S[r]]) = m_R$ for any R and where m_R depends only depends the context $R[S]$. Then $\texttt{rln}_A(R[S[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}]]) = m_R$. From Corollary B.5 we have $\texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ S[s]^{[a]}]) = n + \texttt{rln}_A(R[S[s']])$ where s' is s where all $^{[a]}$-labels are removed. Thus $\texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ S[s]^{[a]}]) = n + m_R$. Since $S[\texttt{letrec}\ a := n\ \texttt{in}\ s^{[a]}] \sim_c \texttt{letrec}\ a := n\ \texttt{in}\ S[s]^{[a]}$ (by correctness of $(letw)$ and (gcW)), the context lemma for improvement shows the claim.

Proposition B.11. *Let $A_{min} \subseteq A \subseteq \mathfrak{A}$. Let S be a surface context. Then* $\texttt{letrec}\ a := n\ \texttt{in}\ S[s^{[a]}] \preceq_A \texttt{letrec}\ a := n\ \texttt{in}\ S[s]^{[a]}$*, and if S is strict, then* $\texttt{letrec}\ a := n\ \texttt{in}\ S[s^{[a]}] \approx_A \texttt{letrec}\ a := n\ \texttt{in}\ S[s]^{[a]}$.

Proof. First assume that S is strict. Let R be a reduction context. Then $S' := R[\texttt{letrec}\ a := n\ \texttt{in}\ S[\cdot]$ is also strict. If $S'[r] \sim_c \bot$ for all r, then $\texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ S[s^{[a]}])) = \infty$ and $\texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ S[s])^{[a]}) = n + \texttt{rln}_A(R[S[s]]) = n + \infty = \infty$

Now assume that S is not strict. Let R be a reduction context. By (lll)-transformations we have $R[\texttt{letrec}\ a := n\ \texttt{in}\ S[s^{[a]}]] \approx_A \texttt{letrec}\ a := n\ \texttt{in}\ R[S[s^{[a]}]$.

If $R[S[\cdot]]$ is strict, then we have $\texttt{letrec}\ a := n\ \texttt{in}\ R[S[s^{[a]}]] \approx_A \texttt{letrec}\ a := n\ \texttt{in}\ R[S[s]]^{[a]}$ (since $R[S[\cdot]]$ is a strict surface context) and $\texttt{letrec}\ a := n\ \texttt{in}\ R[S[s]]^{[a]} \approx_A \texttt{letrec}\ a := n\ \texttt{in}\ R[S[s]^{[a]}]$ (since R is a strict surface context).

By (lll)-transformations we have $\texttt{letrec}\ a := n\ \texttt{in}\ R[S[s]^{[a]}] \approx_A R[\texttt{letrec}\ a := n\ \texttt{in}\ S[s]^{[a]}]$. thus $\texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ S[s^{[a]}]]) = \texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ S[s]^{[a]}])$.

If $R[S[\cdot]]$ is non-strict, then $\texttt{rln}_A(R[S[r]]) = m_R$ for any r and where m_R only depends on the context $R[S]$. Then $\texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ S[s^{[a]}]]) = \texttt{rln}_A(\texttt{letrec}\ a := n\ \texttt{in}\ R[S[s^{[a]}]]]) = m_R$, since $\texttt{rln}_A$-length of the normal order reduction for $R[S[r]]$ is the same for $\texttt{letrec}\ a := n\ \texttt{in}\ R[S[r]]$, since only (lll)-reduction may be added. We also have $\texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ S[s]^{[a]}]) = n + \texttt{rln}_A(R[S[s]]) = n + m_R$ by Corollary B.5.

Thus in any case $\texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ S[s^{[a]}]]) \leq \texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ S[s]^{[a]}])$ and the expressions are contextually equivalent and thus the context lemma for improvement shows the claim.

Corollary B.12. *Let $A_{min} \subseteq A \subseteq \mathfrak{A}$. For all surface contexts S_1, S_2: $S_1[\texttt{letrec}\ a := n\ \texttt{in}\ S_2[s^{[a]}]] \preceq_A \texttt{letrec}\ a := n\ \texttt{in} S_1[S_2[s]]^{[a]}$ and if $S_1[S_2]$ is strict, also $S_1[\texttt{letrec}\ a := n\ \texttt{in}\ S_2[s^{[a]}]] \approx_A \texttt{letrec}\ a := n\ \texttt{in} S_1[S_2[s]]^{[a]}$.*

Proof. This follows from Propositions B.10 and B.11.

Proposition B.13. *Let $A_{min} \subseteq A \subseteq \mathfrak{A}$.*

1. $\texttt{letrec}\ a := n, b := m\ \texttt{in}\ (s^{[a]})^{[b]} \approx_A \texttt{letrec}\ a := n, b := m\ \texttt{in}\ (s^{[b]})^{[a]}$
2. $\texttt{letrec}\ a := n\ \texttt{in}\ (s^{[a]})^{[a]} \approx_A \texttt{letrec}\ a := n\ \texttt{in}\ (s^{[a]})$
3. $(t^{p_1})^{p_2} \approx_A t^{p_1 \oplus p_2}$.

Proof. 1. Let R be a reduction context. Then $R[\texttt{letrec}\ a := n, b := m\ \texttt{in}\ (s^{[a]})^{[b]}] \approx_A R[\texttt{letrec}\ b := m\ \texttt{in}\ (\texttt{letrec}\ a := n\ \texttt{in}\ (s^{[a]})^{[b]}])$, since (*llet*) $\subseteq \approx_A$. Applying Corollary B.5 two times shows $\texttt{rln}_A(R[\texttt{letrec}\ b := m\ \texttt{in}\ (\texttt{letrec}\ a := n\ \texttt{in}\ (s^{[a]})^{[b]}])) = m + \texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ (s'^{[a]})]) = m + n\texttt{rln}_A(R[s'])$ where s'' is s where all label $^{[a]}$ and $^{[b]}$ are removed. Completely analogously it can be shown that $\texttt{rln}_A(R[\texttt{letrec}\ a := n, b := m\ \texttt{in}\ (s^{[b]})^{[a]}]) = n + m + \texttt{rln}_A(R[s'])$. Clearly, $\texttt{letrec}\ a := n, b := m\ \texttt{in}\ (s^{[a]})^{[b]} \sim_c \texttt{letrec}\ a := n, b := m\ \texttt{in}\ (s^{[b]})^{[a]}$ and thus the context lemma for improvement shows the claim.

2. Corollary B.5 shows that for all reduction contexts R the equation $\texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ (s^{[a]})^{[a]}]) = n + \texttt{rln}_A(R[s']) = \texttt{rln}_A(R[\texttt{letrec}\ a := n\ \texttt{in}\ (s^{[a]})])$ holds, where s' is s where all $^{[a]}$-labels are removed. The expressions are also contextually equivalent and thus the context lemma for improvement shows the claim.
3. This follows from the previous parts and from Proposition B.6.

Proposition B.14. *Let $A_{min} \subseteq A \subseteq \mathfrak{A}$. Let $S[\cdot, \ldots, \cdot]$ be a multi-context where all holes are in surface position. Then* $\texttt{letrec}\ a := n\ \texttt{in}\ S[s_1^{[a]}, \ldots, s_n^{[a]}] \preceq_A \texttt{letrec}\ a := n\ \texttt{in}\ S[s_1, \ldots, s_n]^{[a]}$*. If some hole $\cdot_i$ with $i \in \{1, \ldots, n\}$ is in strict position in $S[\ldots,.]$, then* $\texttt{letrec}\ a := n\ \texttt{in}\ S[s_1^{[a]}, \ldots, s_n^{[a]}] \approx_A \texttt{letrec}\ a := n\ \texttt{in}\ S[s_1, \ldots, s_n]^{[a]}$.

Proof. This follows by repeated application of Corollary B.12 and Proposition B.13.

Corollary B.15. *Let $A_{min} \subseteq A \subseteq \mathfrak{A}$. Let $S[\cdot,\ldots,\cdot]$ be a multi-context where all holes are in surface position. Let $S[s_1,\ldots,s_n]$ be closed. Then $S[s_1^{[p_1,a\mapsto m]},\ldots,s_n^{[p_n,a\mapsto m]}] \preceq_A S[s_1^{p_1},\ldots,s_n^{p_n}]^{[m]}$.*
If some hole $\cdot_i$ with $i \in \{1,\ldots,n\}$ is in strict position in $S[\ldots,.]$, then $S[s_1^{[p_1,a\mapsto m]},\ldots,s_n^{[p_n,a\mapsto m]}] \approx_A S[s_1^{p_1},\ldots,s_n^{p_n}]^{[m]}$.

Proposition B.16. *Let $A_{min} \subseteq A \subseteq \mathfrak{A}$. The following transformation is correct w.r.t. $\approx_A$: Replace $(\texttt{letrec}\ x = s^{[n,p]}, Env\ \texttt{in}\ t)$ by $\texttt{letrec}\ x = s[x^{[a\mapsto n,p]}/x],\ Env[x^{[a\mapsto n,p]}/x]\ \texttt{in}\ t[x^{[a\mapsto n,p]}/x]$, where a is a fresh label and all occurrences of x are in surface position.*

Proof. Let R be a reduction context. If all occurences of x in $R[(\texttt{letrec}\ x = s^{[n,p]}, Env\ \texttt{in}\ t)]$ are in non-strict positions, then $\texttt{rln}_A(R[(\texttt{letrec}\ x = s^{[n,p]}, Env\ \texttt{in}\ t)]) = \texttt{rln}_A(R[(\texttt{letrec}\ x = s, Env\ \texttt{in}\ t)]) = \texttt{rln}(R[\texttt{letrec}\ x = s[x^{[a\mapsto n,p]}/x], Env[x^{[a\mapsto n,p]}/x]\ \texttt{in}\ t[x^{[a\mapsto n,p]}/x]])$. If there is a strict position of x in $R[(\texttt{letrec}\ x = s^{[n,p]}, Env\ \texttt{in}\ t)]$, then $\texttt{rln}_A(R[(\texttt{letrec}\ x = s^{[n,p]}, Env\ \texttt{in}\ t)]) = \texttt{rln}_A(\texttt{letrec}\ x = s[x^{[a\mapsto n,p]}/x], Env[x^{[a\mapsto n,p]}/x]\ \texttt{in}\ t[x^{[a\mapsto n,p]}/x])$, since the work corresponding to labels in p are evaluated once and also the work n is only evaluated once. The context lemma for improvement thus shows the claim.