Infinite Infrared Regularization in Krein Spaces

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Indefiniteness in Quantum Field Theory

Indefiniteness in Quantum Field Theory

The massless scalar field in $1+1$ dimensions has the two point function

$$\mathfrak{W}_2(x,y) = -D^+(x-y), \quad -D^+(x) \stackrel{\text{def}}{=} \lim_{\varepsilon \searrow 0} \frac{1}{4\pi} \ln\left(x^2 + \mathrm{i}\varepsilon x^0\right),$$

wherein analytic regularization yields a distribution on $\mathscr{S}(\mathbb{R}^2)$.
Upon Fourier transformation it becomes, in lightcone coordinates $p_\pm \stackrel{\text{def}}{=} p_0 \pm p_1$,

$$\widetilde{D^+}(p) = \left[\left(\frac{1}{p_+}\right)_+ \delta(p_-) + \left(\frac{1}{p_-}\right)_+ \delta(p_+)\right] \Theta(p_0).$$

The tempered distribution

$$\left(\frac{1}{p}\right)_+ (\widetilde{f}) \stackrel{\text{def}}{=} \int_0^\infty \mathrm{d}p \, \frac{\widetilde{f}(p) - \widetilde{f}(0)}{p}$$

is **not positive definite** in general, when $\widetilde{f}(0) \neq 0$.

Is Indefiniteness Generic in QFT?

Or, is it an artifact of the chosen regularization? Two fundamental axioms of QFT:

Locality: Whenever $(x_j - x_{j+1})^2 < 0$, then

$$\mathfrak{W}(x_1, \ldots, x_j, x_{j+1}, \ldots, x_n) = \mathfrak{W}(x_1, \ldots, x_{j+1}, x_j, \ldots, x_n).$$

Positivity: For $\underline{f}, \underline{g} \in \underline{\mathscr{S}}$, the **sesquilinear form** $\left(\underline{f}\,, \underline{g}\right) \equiv \sum_{m,n} \mathfrak{W}_{m+n}(f_n^* \otimes g_m)$ shall be positive, *i.e.*, $\left(\underline{f}\,, \underline{f}\right) \geq 0$.

Here, $f_n \otimes g_m(x_1, \ldots, x_{m+n}) \stackrel{\text{def}}{=} f_n(x_1, \ldots, x_n) g_m(x_{n+1}, \ldots, x_{m+n})$, and

$$\underline{\mathscr{S}} \stackrel{\text{def}}{=} \left\{\underline{f} = (f_0, f_1, \ldots) | f_0 \in \mathbb{C},\ f_i \in \mathscr{S}(\mathbb{R}^{4i}),\ \#\{i, f_i \neq 0\} < \infty\right\}.$$

Locality and Positivity are in conflict, when trying to construct interacting theories:

Theorem. ([Strocchi 1993]) *A local formulation of a field theory with nontrivial gauge invariance, and satisfying all Wightman axioms, contains no charged states in the physical Hilbert space.*

The mechanism behind this phenomenon is the presence **virtual states**, *i.e.*, physical equivalence classes of the potential, in theories with nontrivial gauge invariance [Bogolubov *et al.* 1990, Chapter 10].

Physicists Strategy

1. Retain Locality throughout by fixing a local gauge.
2. Perform a **functional regularization** of emerging singularities by choosing the right test function space.
3. Reconstruct the state space as an indefinite inner product space (Wightman reconstruction from the n-point functions.)
4. Try to isolate the negative degrees of freedom.
5. Boldly disregard them as unphysical.

To carry out 3. and 4. one replaces positivity by the weaker

Hilbert space structure condition (HS). Assume that there exist Hilbert seminorms $p_n = (\,.\,,\,.\,)^{1/2}$ on $\mathscr{S}(\mathbb{R}^{4n})$ such that $\left|\mathfrak{W}_{n+m}(f_n^* \otimes g_m)\right| \leq p_n(f_n)p_m(g_m)$ for all $f_n \in \mathscr{S}(\mathbb{R}^{4n})$, $g_m \in \mathscr{S}(\mathbb{R}^{4m})$.

Towards Krein Spaces

Let $\mathcal{V}$ be a vector space equipped with an indefinite inner product $\langle . \, , . \rangle$.
A locally convex topology τ on $\mathcal{V}$ is called **majorant**,
if the inner product $\langle . \, , .. \rangle$ is jointly τ–continuous.
Let $\mathcal{K} \stackrel{\text{def}}{=} \overline{\mathcal{V}}^{\tau}$ be the τ–completion of $\mathcal{V}$

Recall [Bognár 1974, Azizov/Iokhvidov 1989, Strocchi 1993]

Lemma (A). *i) To every majorant exists a weaker one defined by a single seminorm.*
ii) A seminorm p defines a majorant if $|\langle x \, , x \rangle| \leq \alpha p(x)^2, \quad \alpha > 0, \; \forall x \in \mathcal{V}$.

Proposition (B). *If a majorant is normed, i.e., defined by $\|.\| \stackrel{\text{def}}{=} (. \, , .)^{1/2}$*
*then exists the **metric (Gram) operator**:*

$$\langle x \, , y \rangle = (x \, , Jy), \quad \forall x, y \in \mathcal{K}.$$

Definition (C). *$\mathcal{K}$ is a **Krein space** iff J is completely invertible.*
($\Longleftrightarrow \mathcal{K} = \mathcal{K}^{+} \; (\dot{+}) \; \mathcal{K}^{-}$, $\mathcal{K}^{\pm}$ complete w.r.t. the weak topology)

Lemma (D). *A majorant Hilbert topology is **maximal**, i.e., no weaker majorant exists,*
iff J has bounded inverse, i.e., $\mathcal{K}$ is a Krein space.

A–D render HS effective: They allow the construction of the largest Krein space closure to a given indefinite inner product, and the separation of the negative part, to arrive at a Hilbert space containing all physical information.

The Gelfand-Shilov Scheme of Test Function Spaces

$$\mathscr{S}_\alpha(\mathbb{R}^n) \stackrel{\text{def}}{=} \left\{ f \in C^\infty(\mathbb{R}^n) \,\middle|\, \forall k, q \in \mathbb{N}_0 \colon \sup_{x, |\sigma| \le q} |x^k D_x^\sigma f(x)| \le C_q A^k k^{k\alpha} \right\},$$

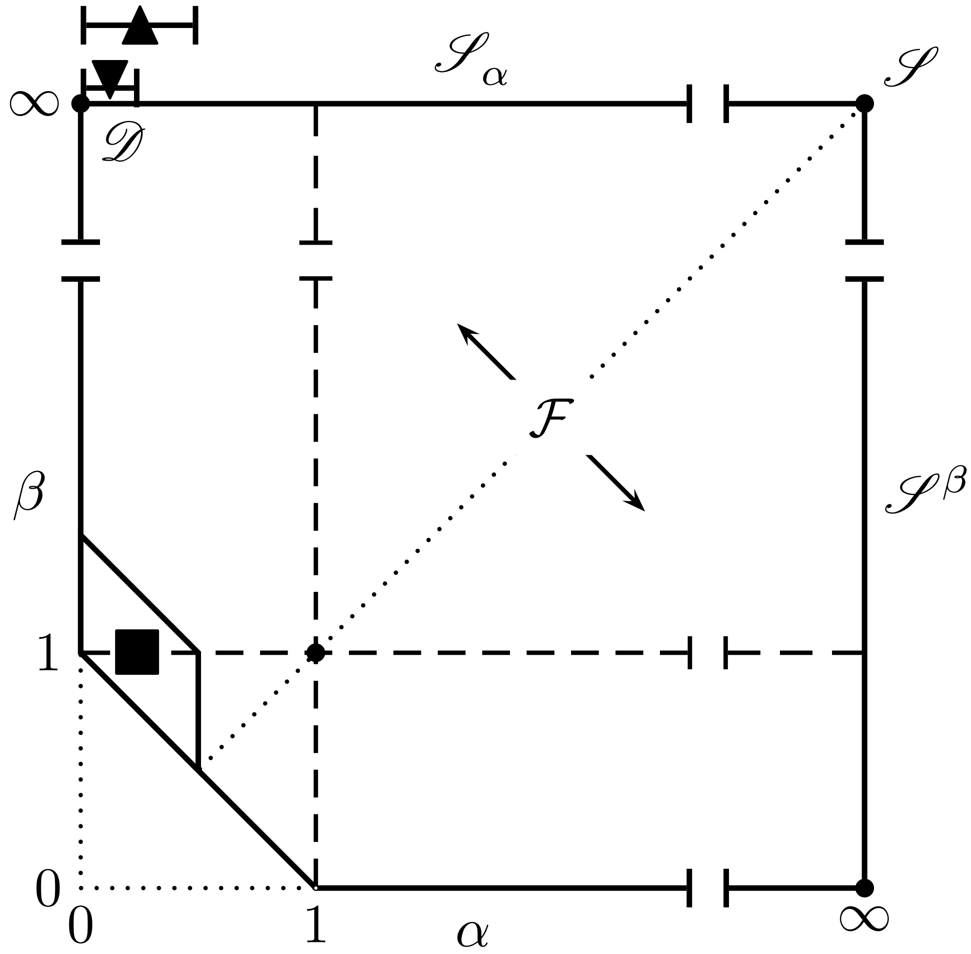
$$\mathscr{S}^\beta(\mathbb{R}^n) \stackrel{\text{def}}{=} \left\{ f \in C^\infty(\mathbb{R}^n) \,\middle|\, \forall k, q \in \mathbb{N}_0 \colon \sup_{x, |\sigma| \le q} |x^k D_x^\sigma f(x)| \le C_k B^q q^{q\beta} \right\},$$

$$\mathscr{S}_\alpha^\beta(\mathbb{R}^n) \stackrel{\text{def}}{=} \left\{ f \in C^\infty(\mathbb{R}^n) \,\middle|\, \forall k, q \in \mathbb{N}_0 \colon \sup_{x, |\sigma| \le q} |x^k D_x^\sigma f(x)| \le C A^k B^q k^{k\alpha} q^{q\beta} \right\}.$$

- Upper index β controls **regularity**, lower index α **growth**
- Spaces with $\beta = 1$ contain only **real analytic** functions, for $\beta < 1$ only **entire** fn's
- **Fourier transformation** exchanges indices $\mathcal{F} \colon \mathscr{S}_\alpha^\beta \xrightarrow{\sim} \mathscr{S}_\beta^\alpha$
- Limits: $\mathscr{S}_\alpha \cong \varinjlim_{\beta \to \infty} \mathscr{S}_\alpha^\beta$ and $\mathscr{S}^\beta \cong \varinjlim_{\alpha \to \infty} \mathscr{S}_\alpha^\beta$
- Old acquaintance: **Schwartz space** $\mathscr{S} \cong \varinjlim_{\alpha, \beta \to \infty} \mathscr{S}_\alpha^\beta$;
  Compactly supported test functions $\mathscr{D} \cong \mathscr{S}_0(\mathbb{R}^n)$;
  Test functions for **Fourier hyperfunctions** $\mathscr{P}_* \cong \mathscr{S}_1^1$.

Strength of Singularity: Examples

Suitable test function spaces for certain QFT models [Moschella/Strocchi 1992].



▲ The two-point function of the $1+1$-*dimensional dipole field*, solution of $\Box^2\varphi = 0$, is in ${\mathscr{S}_\alpha}'$, for $\alpha < 1/2$.

▼ The *vertex operator* $:\exp(\mathrm{i}g\varphi):$ of this field is in ${\mathscr{S}_\alpha}'$, for $\alpha < 1/4$.

■ In the temporal variables, $:\exp(\mathrm{i}g\varphi):$ for the dipole field in $3+1$ dimensions is in ${\mathscr{S}_\alpha^\beta}'$, $\alpha + \beta < 3/2$, $\alpha < 1/2$.

A Model with Infinite Infrared Singularity

$$\langle f\,,g\rangle \stackrel{\text{def}}{=} (f\,,g)_{L^2} - \sum_{k=0}^{\infty} c_k^2 \overline{f}^{(k)}(0) g^{(k)}(0), \quad c_k \in \mathbb{R}\setminus\{0\},$$

is an **indefinite product**, characterized by the **infra-exponential symbol of order** $1/(2\delta)$

$$J(\xi) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} c_k^2 \xi^k, \quad |J(\xi)| \le C_\varepsilon \mathrm{e}^{\varepsilon|\xi|^{1/(2\delta)}},\ \forall\varepsilon > 0.$$

It is well defined on $\mathscr{S}_\alpha^\beta(\mathbb{R})$ for any α and $\boxed{\beta < \delta}$ ($\beta \le \delta$ suffices), since then

$$|c_k^2| \le C\frac{\theta^k}{D^k \mathrm{e}^{2k\delta} k^{2k\delta}}, \quad \forall D > 0\ \exists\theta \in (0,1), C > 0,$$
$$|f^{(k)}(0)| \le C_f (B+\rho)^k k^{k\beta},\ \text{some } B, C_f > 0, \quad \forall\rho > 0.$$

Task: Infinite Infrared Regularization

- Construct the maximal Krein space completion of $\left\{\mathscr{S}_\alpha^{\beta<\delta}, \langle .\,,.\rangle\right\}$;
- Identify its largest, positive definite, subspace,
- which is invariant under the Heisenberg algebra $\left\langle \hat{q} \stackrel{\text{def}}{=} x\cdot, \quad \hat{p} \stackrel{\text{def}}{=} -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\right\rangle$.

Heuristics:

- The singularity outside the origin matters not for the regularization.
- The Heisenberg algebra is ubiquitous as an observable algebra.

Neutral Decomposition

Set $\boxed{\gamma_k \stackrel{\text{def}}{=} k^{k\delta}}$ and construct neutral functions $\mathcal{N} \stackrel{\text{def}}{=} \operatorname{span}\{\chi_k\}_{k\in\mathbb{N}_0} \subset \mathscr{S}_\alpha^\beta$ satisfying

$$\text{i) } \|\chi_k\|_{L^2}^2 = c_k^2\gamma_k^2 \qquad \text{ii) } \chi_k^{(i)}(0) = \delta_{ik}\cdot\gamma_k$$
$$\text{iii) } (\chi_k\,,\chi_l)_{L^2} = 0,\ \forall k\neq l \quad \Longrightarrow \quad \text{iv) } \langle\chi_k\,,\chi_l\rangle = 0,\ \forall k,l$$

The decomposition $f = f^{N+} + \sum_{i=0}^{N} f^i\chi_i$, with $f^i = \dfrac{f^{(i)}(0)}{\gamma_i}$,

defines a majorant τ on $\left\{\mathscr{S}_\alpha^{\beta<\delta}, \langle.\,,.\rangle\right\}$ through the seminorm

$$p(f)^2 \stackrel{\text{def}}{=} \lim_{N\to\infty}\left[\langle f^{N+}\,, f^{N+}\rangle + \sum_{i=0}^{N}\left\{|\langle f\,,\chi_i\rangle|^2 + |f^i|^2\right\}\right].$$

In $\boxed{\mathcal{K} \stackrel{\text{def}}{=} \overline{\mathscr{S}_\alpha^\beta}^\tau}$, this gives rise to a Hilbert scalar product $(.\,,.)$ with $\|.\| \stackrel{\text{def}}{=} p(.) = (.\,,.)^{1/2}$

$$(f\,,g) \stackrel{\text{def}}{=} \langle f^+\,,g^+\rangle + \sum_{i=0}^{\infty}\left\{\langle f\,,\chi_i\rangle\langle\chi_i\,,g\rangle + \overline{f^i}g^i\right\}, \quad \forall f,g\in\mathcal{K},$$

where $f^+ \stackrel{\text{def}}{=} \lim_{N\to\infty} f^{N+}$ (convergent in $\mathcal{K}$).

Finiteness of $\|.\|$

$$\Big\|\sum_{i=0}^{\infty} f^i \chi_i\Big\|^2_{L^2} \quad < \infty \ \text{ and } f \in L^2 \implies f^{N+} \overset{N\to\infty}{\longrightarrow} f^+ \in L^2$$

$$\implies \lim_{N\to\infty} \left\langle f^{N+} , f^{N+} \right\rangle \ < \infty.$$

$$|\langle f \, , \chi_i \rangle|^2 + |f^i|^2 \ \ \le |(f \, , \chi_i)_{L^2}|^2 + c_i^4 \gamma_i^2 |f^{(i)}(0)|^2 + \frac{|f^{(i)}(0)|^2}{\gamma_i^2}.$$

$$|(f \, , \chi_i)_{L^2}|^2 \le \|f\|^2_{L^2} c_i^2 \gamma_i^2 \ \le \|f\|^2_{L^2} \frac{C\theta^i}{D^i \mathrm{e}^{2i\delta}}.$$

$$c_i^4 \gamma_i^2 |f^{(i)}(0)|^2 \le \frac{C^2 C_f^2}{i^{2i(\delta-\beta)}} \left(\frac{\theta(B+\rho)}{\mathrm{e}^{2\delta} D} \right)^{2i}.$$

$$\frac{|f^{(i)}(0)|^2}{\gamma_i^2} \le \frac{C_f^2}{i^{2i(\delta-\beta)}} (B+\rho)^{2i}.$$

$$\implies \sum_{i=0}^{\infty} \left\{ |\langle f \, , \chi_i \rangle|^2 + |f^i|^2 \right\} < \infty \ \implies p(f)^2 = \|f\|^2 < \infty.$$

Separation and Decomposition

Set $\mathcal{P} \stackrel{\text{def}}{=} \left\{ f \in \mathscr{S}_\alpha^\beta \;\middle|\; f^{(k)}(0) = 0,\ \forall k \in \mathbb{N}_0 \right\}$. On $\mathcal{P}$, $\langle . , . \rangle = (. , .)_{L^2}$ is positive definite.

Lemma. *The mapping* $P \colon \mathscr{S}_\alpha^\beta \longrightarrow \mathcal{K}; \quad f \longmapsto f^+$ *is continuous in the topology* τ, *has a continuous extension to* $\mathcal{K}$, *and is an* $(. , .)$*–orthogonal projection onto* $\overline{\mathcal{P}}^\tau$, *which induces the decomposition* $\mathcal{K} = \overline{\mathcal{P}}^\tau \oplus \overline{\mathcal{N}}^\tau$.

But: $\overline{\mathcal{P}}^\tau$ is too big (it contains neutral elements). Further decomposition is necessary.

Lemma. *The functionals* $F_i(f) \stackrel{\text{def}}{=} \langle \chi_i , f \rangle$, $f \in \mathscr{S}_\alpha^\beta$, *are* $\| . \|$*–bounded, and thus possess unique continuations to* $\mathcal{K}$. *The vector representatives* $v_i \in \mathcal{K}$, $F_i(.) = (v_i , .)$ *are contained in* $\overline{\mathcal{P}}^\tau$. *They satisfy*

i) $\langle \chi_i , v_i \rangle = (v_i , v_i) = 1$ *ii)* $(v_i , v_j) = \langle \chi_i , v_j \rangle = 0$, $i \neq j$

iii) $\langle v_i , v_i \rangle = 0$ *iv)* $\langle v_i , f \rangle = f^i$, *for all* $f \in \mathscr{S}_\alpha^\beta$.

The v_i span a closed Hilbert subspace $\boxed{\overline{\mathcal{N}}^{\tau \langle * \rangle}}$ of $\mathcal{K}$, isomorphic to the $\langle . , . \rangle$–dual of $\overline{\mathcal{N}}^\tau$.

Lemma. *Set* $\boxed{\mathcal{H} \stackrel{\text{def}}{=} \overline{\mathcal{P}}^{\tau +}}$ *where the topology* τ_+ *is induced by the quadratic Hilbert norm* $p_+(.)^2 \stackrel{\text{def}}{=} \langle . , . \rangle$ *on* $\mathcal{P}$. *Then holds the* $(. , .)$*-orthogonal decomposition*

$$\mathcal{K} = \mathcal{H} \oplus \overline{\mathcal{N}}^{\tau \langle * \rangle} \oplus \overline{\mathcal{N}}^\tau .$$

Gram Operator and Physical Subspace

τ majorant $\overset{\text{Prop. B}}{\Longrightarrow}$ Gram Operator $\langle .\, , .\rangle = (.\, , J.)$ exists.

$$\langle f\, , \chi_i\rangle = (f\, , J\chi_i) = (f\, , v_i),\ \forall f \in \mathcal{K} \quad \Longrightarrow J\chi_i = v_i \qquad \text{(by definition of } v_i\text{)}$$

$$\langle f\, , v_i\rangle = (f\, , Jv_i) = \overline{f^i} = (f\, , \chi_i),\ \forall f \in \mathcal{K} \quad \Longrightarrow Jv_i = \chi_i \qquad \text{(by iv) above)}$$

(and $J|_{\mathcal{H}} = \mathbb{1}$) $\Longrightarrow J = J^{-1}$ is bounded $\overset{\text{Def. C \& Lemma D}}{\Longrightarrow}$ $\mathcal{K}$ is a Krein space

What is $\mathcal{H}$?

$$i^k \widehat{f}^{(k)}(0) = \left(i^k \frac{\mathrm{d}^k}{\mathrm{d}\xi^k} \int_{\mathbb{R}} \mathrm{e}^{-ix\xi} f(x)\mathrm{d}x \right)\Bigg|_{\xi=0} = i^k \int_{\mathbb{R}} (-ix)^k f(x)\mathrm{d}x = \mu^k(f)$$

$\mathcal{F}\colon \mathscr{S}_\beta^\alpha \longrightarrow \mathscr{S}_\alpha^\beta$ is an L^2-isometry $\Longrightarrow$

Theorem. *Let $0 \le \alpha \le \infty$, $1 < \beta < \infty$, and J be infra-exponential of order $1/(2\delta)$ with strictly positive Taylor coefficients. Then, $(\mathscr{S}_\alpha^{\beta<\delta}, \langle .\, , .\rangle)$ has a maximal completion to a Krein space $\mathcal{K}$. The maximal positive definite subspace of $\mathcal{K}$, invariant under the action of the Heisenberg algebra in the Schrödinger representation by selfadjoint operators on it, is $\mathcal{H} = \mathcal{F}L_0^2(\mathbb{R})$, where $L_0^2(\mathbb{R}) \overset{\text{def}}{=} \left\{ f \in L^2(\mathbb{R}) \;\middle|\; \mu^k(f) = 0,\ \forall k \in \mathbb{N}_0 \right\}$, where the kth moment $\mu^k(f)$ of a function $f \in L^2(\mathbb{R})$ is given by $\mu^k(f) \overset{\text{def}}{=} \int_{\mathbb{R}} x^k f(x)\mathrm{d}x$.*

Sufficient Conditions for Regularization

Theorem. *Let $\mathcal{V}$ be a space with non-degenerate, indefinite inner product $\langle . \, , . \rangle$.*

0) *There exists an orthogonal system $\{\widetilde{\chi}_i\}_{i \in \mathbb{N}_0}$ of mutually linearly independent, neutral vectors in $\mathcal{V}$.*

1) *For all $v \in \mathcal{V}$, the unique decomposition $v = \widetilde{v}^{N+} + \sum_{i=0}^{N} \widetilde{v}^i \widetilde{\chi}_i, \quad \widetilde{v}^i \in \mathbb{C}, \; N \in \mathbb{N}_0,$*

   becomes asymptotically positive in the sense that $0 \leq \lim_{N \to \infty} \langle \widetilde{v}^{N+} , \widetilde{v}^{N+} \rangle$.

2) *There exists a sequence of complex numbers $\{\gamma_i\}_{i \in \mathbb{N}_0}$ such that both sequences $\{\gamma_i \langle \widetilde{\chi}_i \, , v \rangle\}$ and $\{\widetilde{v}^i / \gamma_i\}$ are in $l^2(\mathbb{N}_0)$.*

Then, $\mathcal{K} = \overline{\mathcal{V}}^{\tau}$ as constructed above, is a Krein space with rank of negativity $\#\{\widetilde{\chi}_i \neq 0\}$.

In particular, setting $\chi_i \stackrel{\text{def}}{=} \gamma_i \widetilde{\chi}_i$, and $v^i = \widetilde{v}^i / \gamma_i$, one has

$$p(v)^2 \;= \lim_{N \to \infty} \left[\langle v^{N+} , f^{N+} \rangle + \sum_{i=0}^{N} \left\{ |\gamma_i \langle v \, , \widetilde{\chi}_i \rangle|^2 + |\widetilde{v}^i / \gamma_i|^2 \right\} \right] \stackrel{2)}{<} \infty .$$

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