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No. 589

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Market Fragility and the Paradox of the Recent Stock-Bond Dissonance*

Christos Koulovatianos[a,b,♣], Jian Li[a], and Fabienne Weber[a]

November 7, 2017

* We thank Rajnish Mehra, Julien Penasse, and Volker Wieland for thought-provoking discussions. A much shorter version of this paper under the same title is under review by the journal "Economics Letters". Weber thanks the Fond National de la Recherche (FNR), Luxembourg, and the Central Bank of Luxembourg (BCL) for financial support; yet, none of the views expressed herein necessary reflect views of the BCL.

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Abstract

After the Lehman-Brothers collapse, the stock index has exceeded its pre-Lehman-Brothers peak by 36% in real terms. Seemingly, markets have been demanding more stocks instead of bonds. Yet, instead of observing higher bond rates, paradoxically, bond rates have been persistently negative after the Lehman-Brothers collapse. To explain this paradox, we suggest that, in the post-Lehman-Brothers period, investors changed their perceptions on disasters, thinking that disasters occur once every 30 years on average, instead of disasters occurring once every 60 years. In our asset-pricing calibration exercise, this rise in perceived market fragility alone can explain the drop in both bond rates and price-dividend ratios observed after the Lehman-Brothers collapse, which indicates that markets mostly demanded bonds instead of stocks.

Keywords: asset pricing, disaster risk, price-dividend ratio, bond returns

JEL classification: G12, G01, E44, E43

1. Introduction

Since the first oil crisis of 1973, the US stock exchange has been marked by two major setback episodes of its *aggregate dividend index*: the dot-com bust and the Lehman-Brothers collapse (see Figure 1). These two disaster episodes mark two subperiods, as depicted by Figure 1: the pre- and post-Lehman-Brothers regimes.

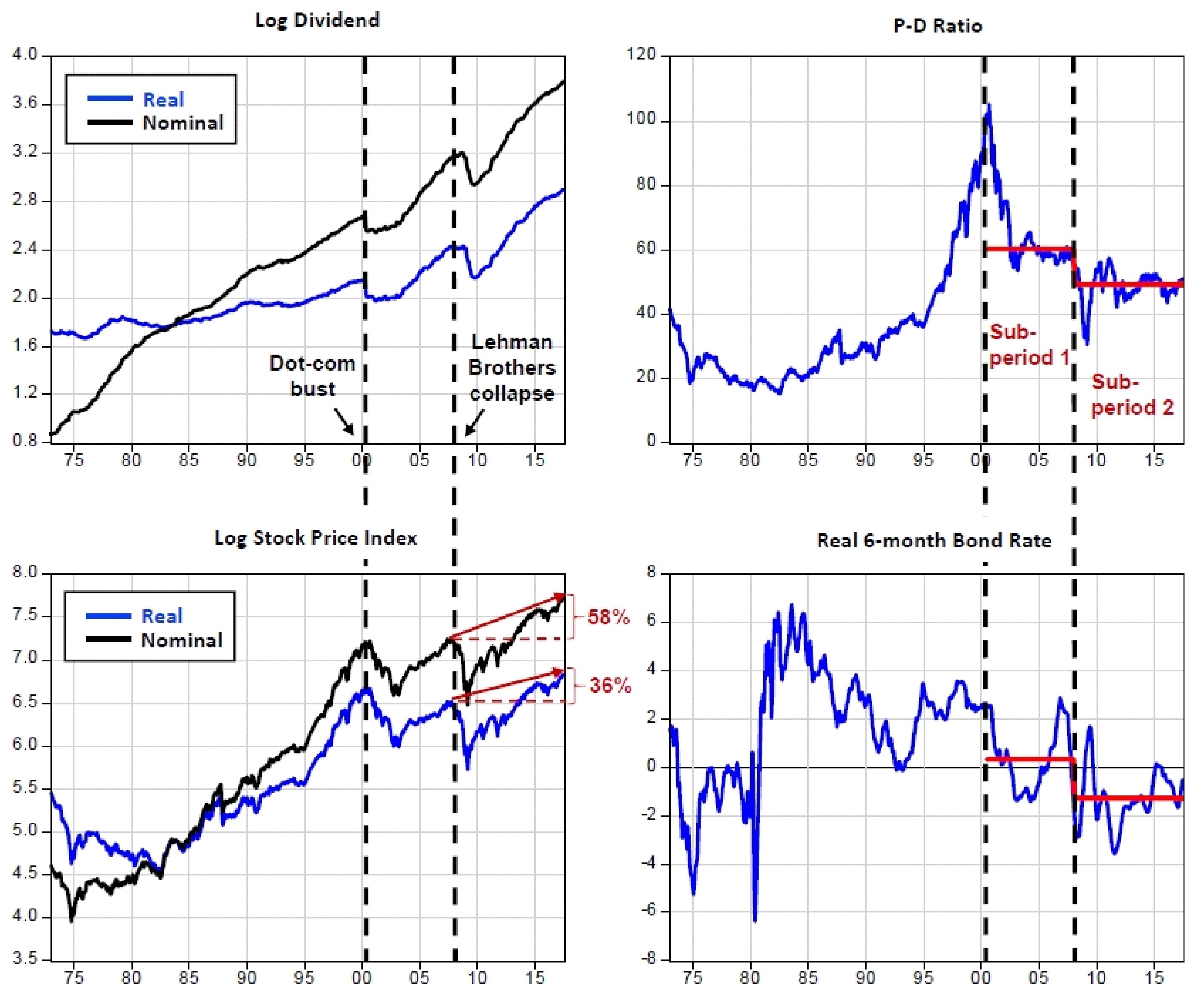

Figure 1 - US Data. Flat lines are statistics reported in Table 1. Sources: Datastream (TOTMKUS) and Board of Governors of the Federal Reserve System (US), 6-Month Treasury Bill: Secondary Market Rate (TB6MS).

In the post-Lehman-Brothers period (subperiod 2), by the end of July 2017, the stock market has grown above and beyond its pre-Lehman-Brothers peak by more than 36% in real terms (58% in nominal terms). On the other hand, we see that the real 6-month bond rate has decreased significantly from a mean bond rate of 0.33% in subperiod 1 to -1.29% in subperiod 2. This high increase in stock prices together with persistently negative bond rates after the Lehman-Brothers collapse, looks like a stock-bond dissonance according to standard asset pricing theory. We further observe that the price-dividend ratio has significantly fallen from subperiod 1 to subperiod 2 from 60.24 to 49.26 as depicted in Table 1. These two empirical observations are essential for explaining the current stock-bond dissonance. Later in the subsection 5.1 these are our four targets which we want to match.

subperiods	**1**	**2**
	2000/7–2008/1	**2008/2–2017/7**
mean real interest rate	0.33% (1.32%)	−1.29% (1.12%) [b]
median P-D ratio [a]	60.24 (3.05)	49.26 (2.31) [c]

Table 1 – Descriptive statistics. Bond rate, and P-D-ratio statistics that appear in Figure 1.

[a] Medians are reported when normality tests fail. Standard errors are reported in parentheses for means and median absolute deviations for medians.

[b] Difference-of-means t-test for difference from previous subperiod's statistic is 9.53 (p-value is 0).

[c] Wilcoxon signed-ranks test for difference from previous subperiod's median is 12.13 (p-value is 0).

According to standard asset-pricing theory, it is reasonable to think that investors have rebalanced their portfolios, demanding more stocks instead of bonds. If this was true, then the bond price would have fallen, leading to an increase in the bond rate. However, in the post-Lehman-Brothers period, the bond rate has significantly decreased and persistently stayed in a negative regime. *This stock-bond dissonance, (a) the persistent drop in bond rates, and (b) the persistent growth of stock prices, is a paradox.*

To explain this stock-bond dissonance we use the Lucas (1978) asset pricing model with rare disasters as in Barro (2006). We focus on two key observations in the data, which are the significant drop in the bond rate and the significant drop in the price-dividend ratio, which we try to match with the model. We show that a change in investors' expectations about the frequency of rare disasters can explain the observed dissonance. More precisely, after the Lehman-Brothers collapse, the investors have the perception of higher market fragility, i.e. higher disaster risk hitting the real economy, such as a sudden drop in dividends. In our calibration exercise we show that without changing the investors' preference parameters, nor market fundamentals, in both subperiods, the model can match the data only by allowing for an increase in investors' perceived frequency of a rare disaster.

Our model builds upon investors' increasing fear for more frequent market disruptions. As in Barro (2006) a rare disaster can be any low-probability event that triggers a sharp drop in per capita GDP or consumption. An economic disaster can be triggered by economic events that affect the business sector and specifically the aggregate dividend index (Great Depression in 1929, the 2008-2009 Global Financial Crisis), or by natural diasters, wartime destruction (World War I, World War II, nuclear conflicts). As in asset-pricing literature with rare disaster risks, e.g., Barro (2006, 2009), Gabaix (2012), Gourio (2012), and Wachter (2013), we assume that rare disasters are exogenous events. Although bonds are not a perfect hedge against disaster risks, investors substitute bonds for stocks in case of higher market fragility.

According to our model, an explanation to the paradox is based on investor perceptios about disaster risk: investors think that disasters occur once every 30 years on average compared to once every 60 years on average before the Lehman-Brothers collapse. Our argument is based on the fact that there is no decrease in the dividend growth rate nor an increase in its dividend volatility (outside crashes). So, without changing perceptions about disaster risk (market fragility), the drop in the price-dividend ratio or the drop in the bond rate.

Our sensitivity analysis supports our market-fragility explanation. Being aware that disaster risk is considered to be "dark matter", in our sensitivity analysis, we use a range of initial disaster probabilities from 1.7% to 2.5% in subperiod 1 and doubling the frequency for subperiod 2. Our model still performs relatively well which reconfirms our working hypothesis of an increase in market fragility.

We also calibrate our model allowing for the possibility of a partial default in government bonds, making them not completely risk-free. We show that the sovereign-default risk is less important quantitatively to explain the observed stock-bond dissonance. Indeed sovereign-default risk raises the government bond rate as markets require a default premium. Hence, it is market fragility alone that can explain the persistently negative government bond rates.

Our market-fragility explanation is in line with a number of studies focusing on rare disaster risks in asset pricing. First, an influential body of literature suggests that disaster risk is variable. More specifically we refer to Gabaix (2012), Gourio (2012), and Wachter (2013), who demonstrate that this variability can explain many asset-pricing puzzles. In addition, Marfe and Penasse (2017) find empirical evidence for disaster-risk variablity. Another body of literature assumes imperfect information about rare disaster risk and argues that parameter learning implies more pessimistic disaster-risk beliefs after a rare disaster

(Collin-Dufresne et al., 2016, Koulovatianos and Wieland, 2017, and Kozlowski et al., 2017). All these studies agree that after the Lehman-Brothers collapse, beliefs about rare disaster risk should be more pessimistic, backing up the working hypothesis examined in this paper. Yet, for the sake of simplicity, here we employ only rational expectations and an unexpected post-disaster structural break.

Due to challenges in observing disaster risk, John Campbell in his 2008 Princeton Finance lectures called disaster risk the "dark matter for economists". But ever since much progress has been made regarding disaster-risk estimation and its role in calibration. Chen, Joslin and Tran (2012) demonstrate that small changes in the distribution of heterogeneous beliefs can have substantial impact on the aggregate-market implications of disaster risk. Chen, Dou and Kogan (2017) also stress that disaster risk is difficult to infer, and offer a comprehensive robustness measure for estimating asset-pricing models with disaster risk.

2. More empirical details on the paradox

To describe the aggregate dividend process described in Figure 1, we assume that, as in Barro (2006, 2009), dividends, D_t, follow the process,

$$\ln(D_{t+1}) = \mu - \frac{\sigma^2}{2} + \ln(D_t) + \sigma\varepsilon_{t+1} + \nu_{t+1}\ln\left(1-\zeta_{t+1}\right) \ , \tag{1}$$

in which the random term $\varepsilon_{t+1} \sim N(0,1)$, is i.i.d. normal with mean 0 and variance 1. The random term, ν_{t+1},

$$\nu_{t+1} = \begin{cases} 1 & , \quad \text{with Prob. } \ \lambda \\ 0 & , \quad \text{with Prob. } \ 1-\lambda \end{cases} \ , \tag{2}$$

introduced low-probability, rare disasters to the dividend process, i.e., with probability $\lambda \in (0,1)$ dividends are hit by a negative rare disaster shock of size ζ_{t+1}. Variable $\zeta_{t+1} \in (0,1)$ is a random variable with given time-invariant distribution and compact support, $\mathcal{Z} \subset (0,1)$.

An interesting feature of the post-Lehman-Brothers stock prices is that the P-D ratio has fallen significantly (see Figure 1 and Table 1). For explaining the persistent drop in the P-D ratio, it would be reasonable to focus on changes in fundamentals. The three parameters involved in equations (1) and (2) are μ, σ, and λ. We first examine whether the transition from subperiod 1 to subperiod 2 has been marked by any changes in the trend, μ, and in the component of volatility that is not related to disasters, namely parameter σ. *Interestingly, neither μ, nor σ have changed across subperiods 1 and 2.*

2.1 What remained constant across subperiods 1 and 2: the dividend trend and the non-crash dividend volatility

In order to see that neither μ, nor σ changed across subperiods 1 and 2, we first need to obtain an estimate for σ, the non-disaster-shock dividend volatility, by excluding crash-episode periods. The criterion for determining non-crash periods is explained by Figure 2.

Figure 2 depicts the spread between the 3-month London Interbank Offered Rate (LIBOR) and the 3-month Overnight Indexed Swap (OIS). According to Thornton (2009, p. 1), the LIBOR-OIS spread is "*a measure of the health of banks because it reflects what banks believe is the risk of default associated with lending to other banks.*" This interpretation of the LIBOR-OIS spread, and the overall pattern revealed by Figure 2, motivate that a systematic rise of the LIBOR-OIS spread above 50 basis points indicates times of problems in the banking sector. The first green vertical line indicates the date at which the LIBOR-OIS spread suddenly increased beyond the 50-basis-points threshold. That date was August 9, 2007, when BNP Paribas, France's largest bank, announced that it would halt redemptions on three investment funds (see St. Louis Fed, 2007). Certainly, this date marked the start of a broader period of uneasiness regarding the solvency of the banking sector. Nevertheless, the 50-basis-points threshold of the LIBOR-OIS spread has been exceeded systematically only since February 2008. Therefore, February 2008, marked by the red vertical line in Figure 2, is the cutoff month separating subperiod 1 (the pre-subprime-crisis phase) and subperiod 2 (the post-subprime-crisis phase) in our sample.

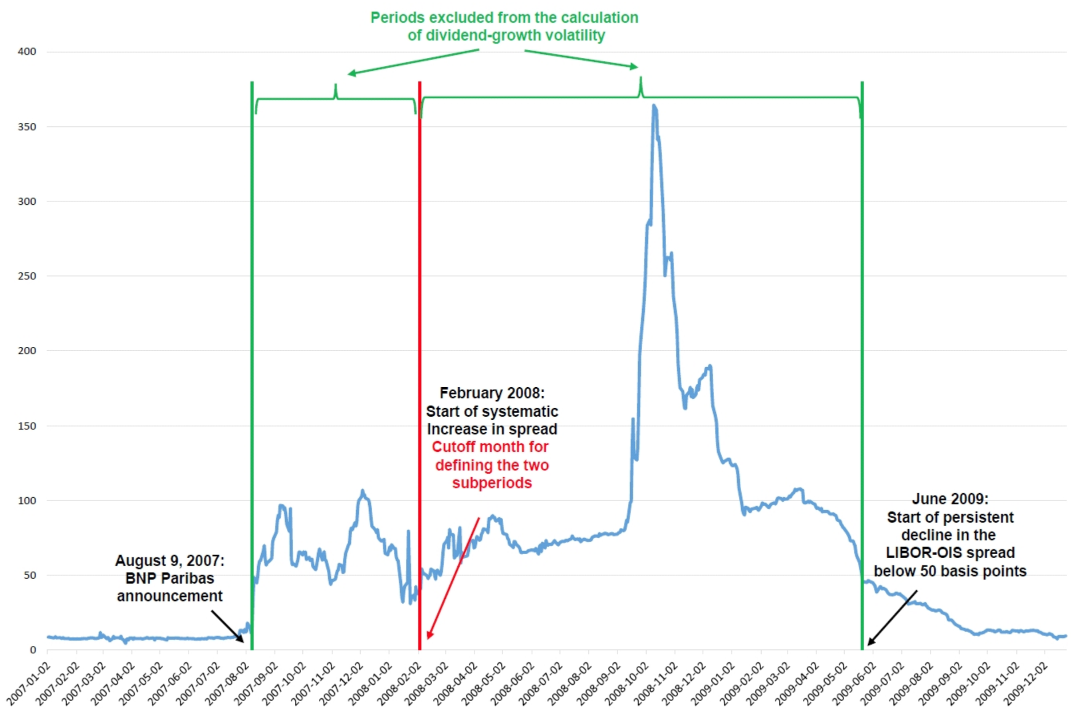

Figure 2 - Three-month LIBOR-OIS Spread (US daily data). Source for London Interbank Offered Rate (LIBOR): Federal Reserve Bank of St. Louis, Economic Research Division. Source for Overnight Indexed Swap (OIS): Datastream.

Table 2 presents statistics regarding the average dividend growth rate (means and medians), and also a measure of variability of the dividend growth rate, but focusing on the non-disaster-shock dividend volatility across subperiods 1 and 2.

subperiods	**1**	**2**
	2000/7–2008/1	**2008/2–2017/7**
mean dividend growth rate	5.59 [a] (12.02)	4.89 [b] (17.27)
median dividend growth rate	4.88 (6.23)	6.66 (5.67)
standard deviation of dividend growth rate (excluding crash period)	12.22	11.57

Table 2 – Descriptive statistics of dividends in real terms appearing in Figure 1. All numbers are percentages. Standard errors are reported in parentheses for means, and median absolute deviations are reported for medians.

[a] Normality test does not fail (Jarque-Bera test statistic is 3.19 with p-value 20.3%).

[b] Normality test fails (Jarque-Bera test statistic is 564.43 with p-value 0%).

Regarding measures of the average dividend growth rate, in Table 2 Jarque-Bera test statistics are reported, testing normality of distributions. While the dividend growth rate does not fail a normality test in subperiod 1, normality is rejected in subperiod 2. For this reason, a test of equality of means across subperiods 1 and 2 is not appropriate. Instead, in Table 3 we report a number of equality tests for the medians of the dividend growth rate across subperiods 1 and 2.

Test for Equality of Medians Between Series

Method	df	Value	Probability
Wilcoxon/Mann-Whitney		0.849519	0.3956
Wilcoxon/Mann-Whitney (tie-adj.)		0.849519	0.3956
Med. Chi-square	1	1.446670	0.2291
Adj. Med. Chi-square	1	1.128270	0.2881
Kruskal-Wallis	1	0.723697	0.3949
Kruskal-Wallis (tie-adj.)	1	0.723697	0.3949
van der Waerden	1	0.293416	0.5880

Category Statistics

Variable	Count	Median	> Overall Median	Mean Rank	Mean Score
GR_D_1	91	4.881276	41	99.05495	-0.041503
GR_D_2	114	6.663943	61	106.1491	0.033129
All	205	6.315151	102	103.0000	1.13E-16

Table 3 - Tests of equality of medians of dividend growth across subperiods 1 and 2. Variable "GR_D_1" is the dividend growth rate in subperiod 1 and Variable "GR_D_2" is the dividend growth rate in subperiod 2. This is a standardized Eviews output.

Since according to all median tests reported in Table 3 the null hypothesis of equality between the two medians cannot be rejected, *in our calibration of the model below, we use the total-sample median spanning the two subperiods, from July 2000 until July 2017 (205 months in total) which is equal to 6.32% (see also the last line in Table 3).*

Regarding the measure of non-disaster dividend volatility (volatility of dividend growth excluding crash periods), for subperiod 1 we include the months from 2000/7 until 2007/7 and for subperiod 2 we include the months from 2009/6 until 2017/7.

Test for Equality of Variances Between Series

Method	df	Value	Probability
F-test	(97, 84)	1.117252	0.6040
Siegel-Tukey		1.282917	0.1995
Bartlett	1	0.275745	0.5995
Levene	(1, 181)	0.971887	0.3255
Brown-Forsythe	(1, 181)	0.954303	0.3299

Category Statistics

Variable	Count	Std. Dev.	Mean Abs. Mean Diff.	Mean Abs. Median Diff.	Mean Tukey-Siegel Rank
GRD1_NO	85	12.22474	8.994378	8.981747	86.60000
GRD2_NO	98	11.56549	7.768607	7.765153	96.68367
All	183	11.90850	8.337954	8.330238	92.00000

Bartlett weighted standard deviation: 11.87599

Table 4 - Tests of equality of variances of dividend growth across subperiods 1 and 2, excluding crash periods. Variable "GRD1_NO" is the dividend growth rate in subperiod 1 for which crash periods are excluded, and variable "GRD2_NO" is the dividend growth rate in subperiod 2, for which crash periods are excluded as well. This is a standard Eviews output.

Based on Table 4, the standard deviations for these no-disaster subperiods are 12.22% and 11.57%, but all tests cannot reject the null hypothesis that these two standard deviations are equal. So, *throughout the rest of the paper, in the calibration below, we use the total-sample standard deviation, spanning the two subperiods, from July 2000 until July 2017 (183 months in total, after excluding the period from August 2007 until May 2009), which is equal to 11.91% (see also the penultimate line in Table 4).*

2.2 What changed across subperiods 1 and 2: perceptions about disaster risk

While neither μ, nor σ changed across subperiods 1 and 2, parameter λ, seems to have changed. Our working hypothesis is that there has been a pessimistic shift in rare-disaster beliefs about parameter λ after the Lehman-Brothers collapse, i.e. λ mhas increased. This working hypothesis is corroborated by an increase in the "SKEW" index, depicted by Figure 3.

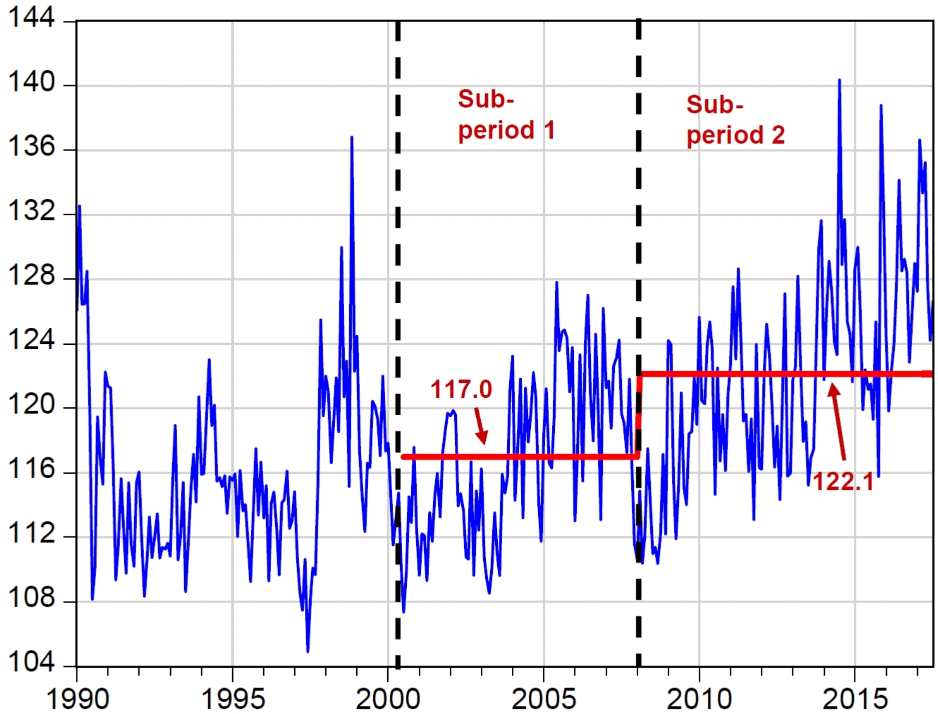

Figure 3 - The SKEW index, US monthly data. Source: Chicago Board Options Exchange.

The SKEW index partially reveals the investors' beliefs on market fragility. despite that it is not a perfect proxy for the rare disaster risk hitting the dividend index. Figure 3 plots the Chicago Board Options Exchange (CBOE) Skew index, commonly known as "SKEW". According to Chicago Board Options Exchange CBOE (2010), the SKEW is an indicator based on options, measuring the perceived tail risk of the distribution of Standard and Poor's (S&P) 500 log returns at a 30-day horizon. The SKEW measures tail risk, and specifically the risk related to an increase in the probability of extreme negative outlier returns, two or more standard deviations below the mean. Details on the formal definition of SKEW is provided by Chicago Board Options Exchange CBOE (2010, p. 5).

The main point made by Figure 3 is that the mean level of the SKEW index has increased in subperiod 2. Interestingly, the SKEW index is well approximated by a normal distribution in both subperiods (the Jarque-Bera statistic is 3.71, implying a p-value of 0.16 in subperiod 1, while for subperiod 2 the Jarque-Bera statistic is 2.51, implying a p-value of 0.29).

subperiods	**1**	**2**
	2000/7–2008/1	**2008/2–2017/7**
mean SKEW	117.0 (4.99)	122.1 (6.28)[a]

Table 5 – Descriptive statistics of the SKEW index appearing in Figure 3.

Standard errors in parentheses.

[a] Difference-of-means t-test for difference from previous subperiod's statistic is -6.37 (p-value is 0).

Table 5 presents a formal statistical test revealing that the mean SKEW has increased significantly in subperiod 2. This evidence supports our working hypothesis that, after the Lehman-Brothers collapse, beliefs about rare disasters have become more pessimistic. The risk interpretation of the changes reported by Figure 3 and Table 5 is given by Chicago Board Options Exchange CBOE (2010, p. 8). Specifically, the estimated risk-adjusted probability that the S&P 500 may experience a sudden drop of two standard deviations in the next 30 days has increased from 6.89% in subperiod 1 to 8.27% in subperiod 2 on average. Similarly, the estimated risk-adjusted probability of a sudden drop of three standard deviations has increased from 1.16% in subperiod 1 to 1.46% in subperiod 2 on average. Although these estimates are not a perfect proxy of the risk of a rare disaster hitting the dividend index, they reveal that beliefs about tail risks and market fragility have been elevated after the Lehman-Brothers collapse. In the rest of the paper *we use an asset-pricing model in order to investigate whether a change in λ alone across subpriods 1 and 2 is capable of replicating key asset-pricing features summarized by Table 1, and we obtain a model-based sense of the increase in parameter λ.*

3. Model

In this section we present our model with disaster risk. We follow the classic Lucas-tree setup (Lucas, 1978). There is a risky asset, the stock composite index (the market portfolio), and a one-year zero-coupon bond. Our stylized asset-pricing model that uses i.i.d. disaster shocks hitting the dividend process, and summarized by equations (1) and (2), implies a flat term structure on bond rates in equilibrium, so there is no need to introduce bonds with different maturity. The one-year zero-coupon bond is not entirely risk-free. In the case of a rare disaster hitting the dividend process, the probability of a partial default on government bonds exists. So we do not only have market fragility in our model but also a sovereign-default risk.

The budget constraint of an investor is,

$$\underbrace{S_{t-1}D_t}_{\substack{\shortparallel \\ \text{Income}}} = \underbrace{P_t\left(S_t - S_{t-1}\right)}_{\substack{\shortparallel \\ \text{Investment in Stocks}}} + \underbrace{Q_t B_t - (1-\delta)^{\nu_t \nu_t^B} B_{t-1}}_{\substack{\shortparallel \\ \text{Investment in Bonds}}} + \underbrace{C_t}_{\substack{\shortparallel \\ \text{Consumption}}} , \tag{3}$$

in which S_{t-1} and B_{t-1} is the number of stocks and bonds held by the investor in the beginning of period t, while P_t and Q_t are the stock and bond prices in period t. The term

$(1-\delta)^{\nu_t \nu_t^B}$ multiplying B_{t-1} in (3) states that if there is no dividend disaster ($\upsilon_t = 0$), then the zero-coupon bond pays 1 unit of the consumable good at the maturity date; in periods that a dividend disaster occurs ($\upsilon_t = 1$), then a probabilistic sovereign-default process is triggered, governed by ν_t^B,

$$\nu_t^B = \begin{cases} 1 \quad , \quad \text{with Prob. } \pi \\ 0 \quad , \quad \text{with Prob. } 1-\pi \end{cases} , \tag{4}$$

with $\pi \in [0,1)$. If both a dividend-disaster and a default occur ($\nu_t = \nu_t^B = 1$), then the zero-coupon bond pays $1-\delta$ units of the consumable good, i.e., it defaults by the fraction $\delta \in [0,1]$.[1] Variables ε_{t+1}, ν_{t+1}, ν_{t+1}^B, and ζ_{t+1} are independent among each other and also independent and identically distributed (i.i.d.) over time.

Preferences are recursive, of the form of Epstein-Zin-Weil (EZW), with utility in period t, denoted by J_t, given by the recursion,

$$J_t = \left\{ (1-\beta) C_t^{1-\frac{1}{\eta}} + \beta \left[E_t \left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{1-\frac{1}{\eta}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\eta}}} , \tag{5}$$

in which $\eta > 0$ is the intertemporal elasticity of substitution (IES), $\gamma > 0$ is the coefficient of relative risk aversion, and $\beta \in (0,1)$ is the utility discount factor that is inversely related to the rate of time preference, $\rho = (1-\beta)/\beta$.

3.1 Asset prices

Equation (1) implies that dividend growth is random following i.i.d. shocks over time. In Appendices A through C we prove that these i.i.d. shocks imply a constant P-D ratio over time, denoted by x,

$$\frac{P_t}{D_t} = x = \frac{\omega}{1-\omega} \quad \text{with} \quad \omega = \beta e^{\left(1-\frac{1}{\eta}\right)\left(\mu - \gamma \frac{\sigma^2}{2}\right)} (1-\lambda\xi)^{\frac{1-\frac{1}{\eta}}{1-\gamma}} \;, \; t = 0,1,..., \tag{6}$$

[1] The concept of sovereign default follows Barro (2006, p. 836) who observes that in periods of rare market disasters the probability of a sovereign default increases. We thank an anonymous referee for raising this point.

in which $\xi = 1 - E_\zeta\left[(1-\zeta)^{1-\gamma}\right]$, with $E_\zeta(\cdot)$ denoting expectation with respect to variable ζ only. The expected bond rate, denoted by r^B, is

$$E\left(r^B\right) = \frac{1}{\beta} e^{\frac{1}{\eta}\mu - \gamma\left(1+\frac{1}{\eta}\right)\frac{\sigma^2}{2}} \frac{(1-\lambda\xi)^{\frac{\frac{1}{\eta}-\gamma}{1-\gamma}}(1-\lambda\pi\delta)}{1-\lambda\left\{1-E_\zeta\left[(1-\zeta)^{-\gamma}\right](1-\pi\delta)\right\}} - 1\ ,\quad t = 0, 1, ... \tag{7}$$

3.2 Empirical implications and tests of the model

The flat P-D ratio implied by equation (6) is not a bad approximation of the P-D ratio dynamics in both subperiods 1 and 2. As Figure 1 indicates, after the P-D ratio overreactions to the disaster episodes calmed down, P-D ratios remained almost constant throughout subperiods 1 and 2, but at different levels.

subperiods	**1**	**2**
	2000/7–2008/1	**2008/2–2017/7**
estimator α_1 in equation (9)	1.05 (0.37) [a]	0.88 (0.07) [a]
ADF statistic for unit root of $\ln(P_t)$	−2.09 [b]	−0.52 [b]
ADF statistic for unit root of $\ln(D_t)$	1.69 [b]	1.39 [b]

Table 6 – Cointegration coefficients (standard errors in parentheses) and ADF unit-root tests.

[a] Max eigenvalue test indicates one cointegrating equation at the 5% level.

[b] ADF test rejects a unit root (1% critical value is -3.50, 5% critical value is -2.89).

For empirical evidence on the validity of (6), notice that another way of writing (6) is,

$$\ln(P_t) = \ln(x) + \ln(D_t)\ . \tag{8}$$

In Table 6 we report estimates of α_1 in the cointegrating equation,

$$\ln(P_t) = \alpha_0 + \alpha_1 \ln(D_t) + u_t \ , \tag{9}$$

whenever cointegration is applicable, based on Augmented Dickey-Fuller (ADF) unit-root tests.

Table 6 provides evidence that, in subperiods 1 and 2, $\ln(P_t)$ and $\ln(D_t)$ are both integrated of order 1, and that the estimates of α_1 do not differ much from 1.[2] In subperiod 1, coefficient α_1 is not significantly different from 1. Although in subperiod 2 coefficient α_1 differs from 1, the value of 0.88 supports that equation (8) is not a bad big-picture approximation of financial markets in the US after the Lehman-Brothers collapse. This evidence validates using the Barro (2006, 2009) model for asset-pricing purposes during subperiods 1 and 2, despite that these subperiods have relatively short length of about 8 years each.

4. Calibration

4.1 Benchmark calibration and key targets

We summarize our calibrating parameter values in Table 7, focusing on the benchmark case of no sovereign fragility ($\pi = \delta = 0$).

[2] ADF tests showing that $\ln(P_t)$ and $\ln(D_t)$ are not integrated of order 2 or above can be provided by the authors upon request.

subperiods	**1**	**2**
	2000/7–2008/1	**2008/2–2017/7**
$\boldsymbol{\eta}$	1.85	same
$\boldsymbol{\rho}$	2.91%	same
$\boldsymbol{\gamma}$	3.92	same
$\boldsymbol{\mu}$	6.32% (data)	same
$\boldsymbol{\sigma}$	11.91% (data)	same
$\boldsymbol{\alpha}$	7.08 (data)	same
$\boldsymbol{\lambda}$	1.7% (benchmark)	**3.5%**

Table 7 – Calibrating parameter values. $E_\zeta(\zeta) = 23.28\%$.

Parameter α is a newly introduced parameter. It is based on the finding by Barro and Jin (2011) that, after transforming disaster sizes using the formula $z = 1/(1-\zeta)$, empirically, variable z is Pareto distributed with density $f(z) = \alpha z_0^\alpha / z^{\alpha+1}$, in which z_0 is the minimum value of z. Our chosen value for ζ_0 (obeying $z_0 = 1/(1-\zeta_0)$, the minimum cutoff disaster size) is 12.5%. We independently estimate α, and choose calibrating values of α from the 95% confidence interval of its estimated value.[3] For estimating α, we use the database by Barro and Jin (2011) that refers to GDP disasters, which is downloadable from,

https://scholar.harvard.edu/barro/publications/size-distribution-macroeconomic-disasters-data

Barro and Jin (2011) present a sensitivity analysis of all their results considering that the lower bound for the disasters, ζ_0, ranges from 9.5% to 14.5%, while Barro (2006, 2009)

[3] We use this estimated Pareto distribution in order to compute all expressions involving the expectation $E_\zeta(\cdot)$. Barro and Jin (2011) demostrate that the goodness of fit to disaster-size data increases if one uses two Pareto distributions, each being effective for a different interval of the support of z. Yet, a single Pareto distribution also gives a good approximation, so we use this for simplicity.

also work with $\zeta_0 = 15\%$. We pick a value somewhere in the middle of this range, setting $\zeta_0 = 12.5\%$, which leaves 110 disasters out of 157 in the Barro and Jin (2011) sample. Using the maximum-likelihood estimation of the "shape" parameter α in the Pareto distribution, we obtain an estimate for α equal to 5.986 (standard error 0.57), and a 95% confidence interval implying that $\hat{\alpha} \in [4.87, 7.11]$.[4]

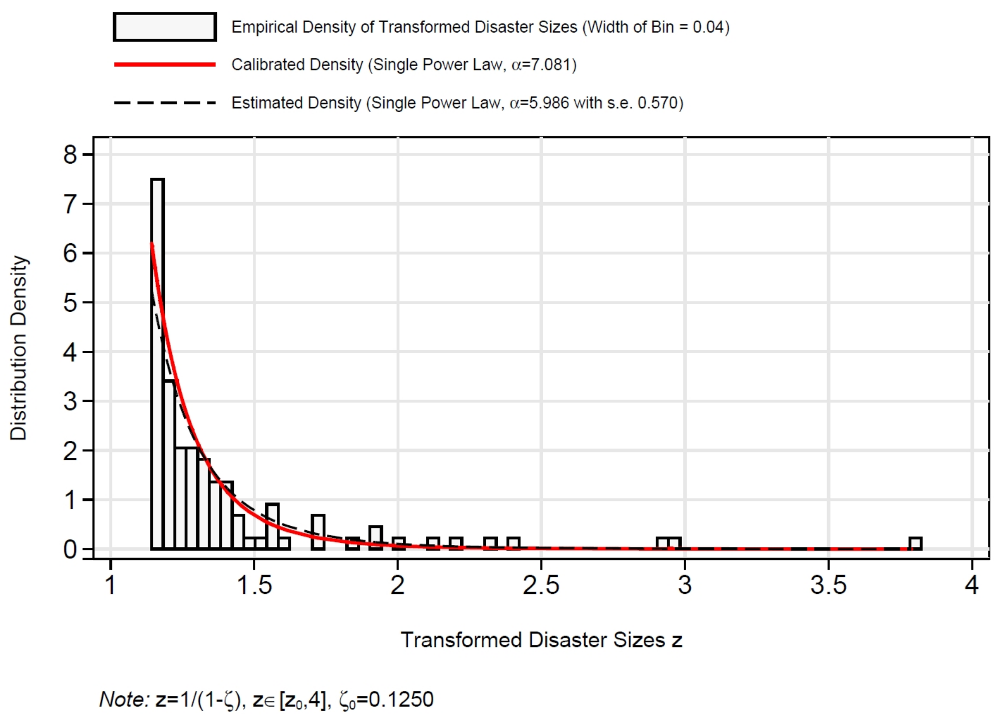

Figure 4 - Goodness of fit of transformed disaster-size data above the 12.5% threshold.

Source: Barro and Jin (2011).

[4] Specifically, we use the package 'PARETOFIT', a module to fit a Type 1 Pareto distribution by Stephen P. Jenkins, which is implementable using Stata and downloadable from,

http://fmwww.bc.edu/RePEc/bocode/p

In our calibration exercise we compute all expectations involving ζ, using $E_{\zeta}(\cdot)$ based on a Pareto distribution for the transformed variable $z = 1/(1-\zeta)$ with a calibrating parameter α^*, taken from this 95% confidence interval, i.e., $\alpha^* \in [4.87, 7.11]$. As Figure 4 reveals, our calibrating value, $\alpha^* = 7.081$ fits the disaster data very well, doing approximately the same good job in fitting the disaster distribution as the point estimate $\hat{\alpha} = 5.986$.

We have four targets: r_1^B, r_2^B, $(P/D)_1$, and $(P/D)_2$, denoting the bond rate and the P-D ratio in the two subperiods. Parameter values μ, σ, and α are directly inferred from data. We use parameter $\lambda_1 = 1.7\%$ as a benchmark value. Then our calibration exercise is to match the four targets using four parameter values, the three preference parameters, η, ρ, and γ, which are constant across the two subperiods, and also λ_2, which is the fourth parameter value.

	$\mathbf{r}^B$	$\mathbf{r}^B$	**P-D ratio**	**P-D ratio**
subperiods	**Model**	**Data**	**Model**	**Data**
2000/7–2008/1	0.33%	0.33%	60.61	60.24
2008/2–2017/7	−1.29%	−1.29%	48.34	49.26

Table 8 – Model vs. data. Case with no sovereign fragility: $\pi = \delta = 0$.

The key element of our calibration exercise is that, in subperiod 2, after the Lehman-Brothers collapse, the disaster-risk parameter, λ, has more than doubled, reflecting that disasters occur in slightly less than 30 years ($1/3.5\% \simeq 29$) on average.[5] *In Table 8 we can see that this simple modification in perceived market fragility is capable of replicating the*

[5] Barro's (2009) benchmark suggests disasters occurring once every 60 years ($1/1.7\% = 59$). This change is consistent with models of rational learning about disaster risk implying that perceived disaster risk increases after a disaster episode and then remains high for a long period afterwards (see Koulovatianos and Wieland, 2017).

persistent changes in the bond rate and the P-D ratio that occurred after the Lehman-Brothers collapse.

4.2 Sensitivity analysis: varying the disaster probability in the first subperiod

In this subsection we provide a sensitivity analysis using a different benchmark for λ_1: instead of fixing λ_1 to 1.7%. We vary its values in a range from 1.5% to 2.5%, doubling λ_2 in each calibration exercise. The key message here is that none of the matching preference parameters η, ρ, and γ, change drastically.

		Benchmark	**Sensitivity analysis changing λ**										
			Parameters										
λ_1		0.017	0.015	0.016	0.017	0.018	0.019	0.020	0.021	0.022	0.023	0.024	0.025
λ_2		0.035	0.030	0.032	0.034	0.036	0.038	0.040	0.042	0.044	0.046	0.048	0.050
ρ		0.029	0.026	0.028	0.027	0.028	0.028	0.028	0.029	0.030	0.032	0.030	0.033
γ		3.92	4.06	3.98	3.93	3.88	3.82	3.77	3.73	3.70	3.68	3.58	3.60
η		1.85	1.44	1.68	1.59	1.69	1.77	1.84	1.95	2.08	2.29	2.16	2.55
α		7.08	7.08	7.08	7.08	7.08	7.08	7.08	7.08	7.08	7.08	7.08	7.08
	Data		**Model**										
r_1	**0.33**	0.33	0.33	0.33	0.33	0.33	0.33	0.33	0.33	0.33	0.33	0.33	0.33
r_2	**-1.29**	-1.29	-1.29	-1.23	-1.29	-1.29	-1.29	-1.29	-1.29	-1.30	-1.29	-1.29	-1.29
$PDratio_1$	**60.24**	60.61	55.99	59.01	59.50	60.22	61.56	62.69	63.03	62.23	60.39	66.93	60.35
$PDratio_2$	**49.26**	48.34	49.26	49.32	49.99	49.32	49.26	49.08	48.13	46.54	44.11	47.93	42.37
$E(\zeta)$		0.233	0.233	0.233	0.233	0.233	0.233	0.233	0.233	0.233	0.233	0.233	0.233

Table 9 - Sensitivity analysis examining the impact of changing λ_1.

We perform a sensitivity analysis focusing on changing the disaster probability parameter λ. Compared to the benchmark value of λ_1 at 1.7%, we expand the parameter space ranging from $\lambda_1 = 1.5\%$ to $\lambda_1 = 2.5\%$. Keeping all parameters inferred from data constant, namely

μ, σ, and α, in each calibration exercise we gradually increase λ_1 by 0.1% percentage points, doubling λ_2 at the same time.

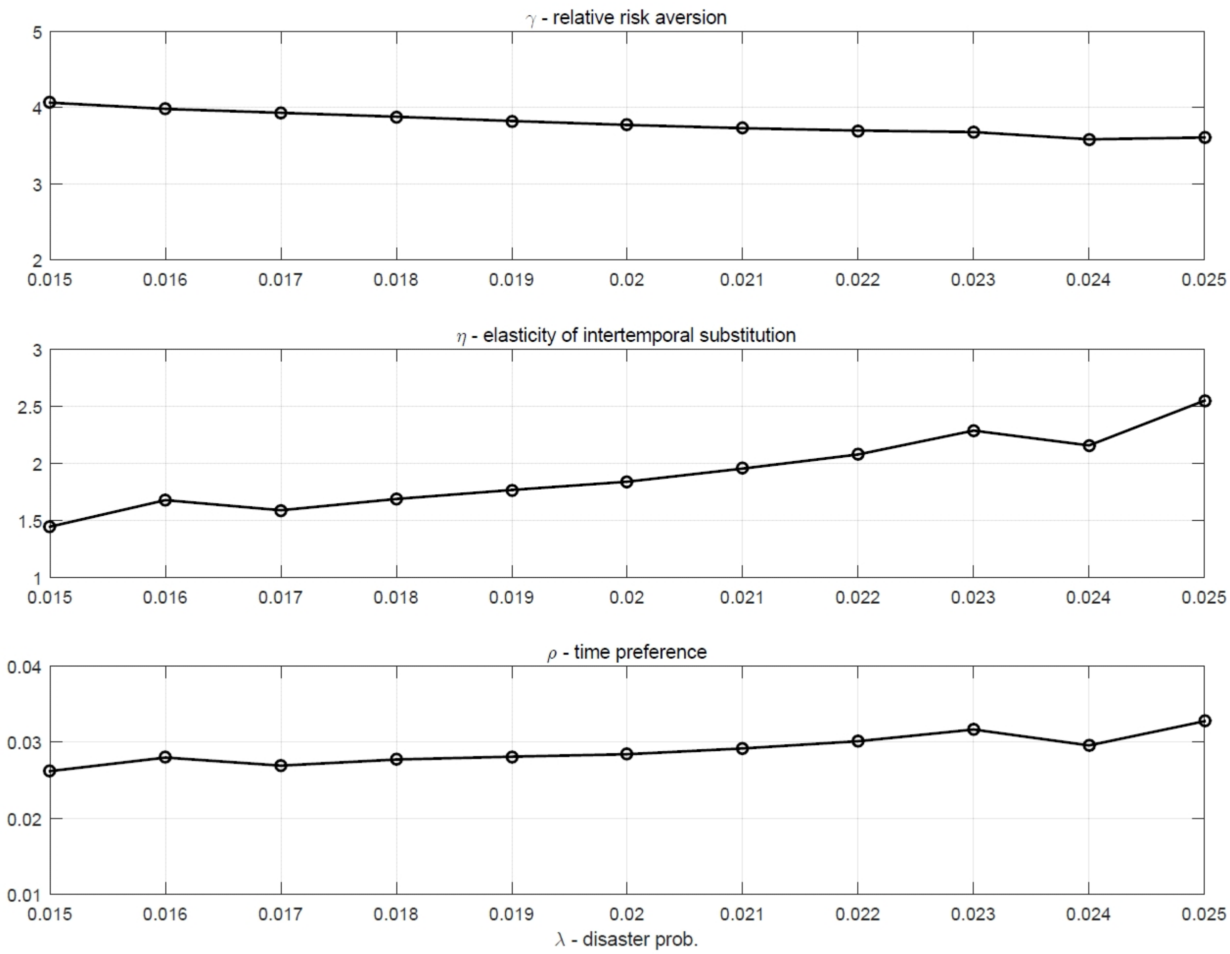

Figure 5 - Sensitivity analysis examining the impact of changing λ_1 (and setting $\lambda_2 = 2\lambda_1$) on preference parameters, γ, η, and ρ that provide the best fit to the four data targets, keeping μ, σ, and α constant.

Fitting the four targets of Table 8 through a minimum-distance approach, Figure 5 and Table 9 report how the three preference parameters, η, ρ, and γ, change as we vary the anchor value of λ_1 each time. In Table 9, we provide the re-calibrated parameters and the corresponding matched values. Under this sensitivity analysis, our model still performs relatively well. The model simulated values do not vary much compared to the benchmark results, and the results are quite close to the actual data targets. Regarding the preference parameters η, ρ, and γ, Figure 5 plots the corresponding variations, and none of the matching preference parameters change drastically. Importantly, according to Chen, Dou, and Kogan (2017, Figure 1, p. 23), we are confident that our calibration parameters are in the "acceptable calibration" area for models like ours.

4.3 Sensitivity analysis: sovereign-default risk

We also introduce sovereign-default risk, setting $\delta = E_{\zeta}(\zeta) = 23.28\%$, and $\pi = 40\%$, as in Barro (2006). As it is obvious from formulas (6) and (7), sovereign-default risk leaves P-D ratios unaltered, but raises r^B, as markets require a default premium. Using our calibrating values from Table 3, the resulting interest rates are $r_1^B = 0.75\%$ and $r_2^B = -0.50\%$. In light of our sensitivity analysis, it seems that even if sovereign-default risk is present, it is, instead market fragility that is most likely to explain recent asset-price trends. Especially if we think that higher sovereign-default risk emerged after the Lehman-Brothers collapse, this risk element would push bond rate upward instead of downward.Conclusion

5. Conclusion: market fragility can resolve the paradox

The first part of the stock-bond-dissonance paradox refers to why bond rates have been so persistently low. Since the Lehman-Brothers collapse, the rise in stock prices creates the plausible impression that markets have increased their demand for stocks, lowering the demand for bonds. However, if fewer bonds had been demanded in the post-Lehman-Brothers era, then bond rates should have increased. Our approach to this part of the paradox has been to focus on explaining the simultaneous drop in the P-D ratio through increased market fragility, captured by the size of parameter λ in our model. Our theory says that there is no paradox: the drop in the P-D ratio implies that this substitution between stocks and bonds did not necessarily happen; *instead markets must have increased the demand for bonds, while decreasing the demand for stocks in a subtle manner.*

The other part of the paradox, referring to why stock prices have grown so much and so fast, can be explained by the fact that no disasters have occurred after 2008. To an extent, the dividend trend, captured by parameter μ in the model, reflects the incremental productivity growth of firms in the stock exchange. Disasters affect the perceived effective growth of dividends, so different perceptions of disaster risk before and after the financial crisis affect only the perceived but not the actual growth of dividends and prices. Therefore, *the fast rise of stock prices can be explained by the coincidence, that no disasters have occurred after 2008.*

For explaining the persistently negative bond returns, we do not rule out that the Fed policy contributed to the high demand for bonds. Yet, according to our approach, it is market fragility (perhaps bank fragility that followed the 2008 financial crisis, consistent with a rise in λ in our model), that led the Fed to its aggressive quantitative easing policy.

Our suggested market fragility explanation for resolving the paradox, points at a first message: it is crucial to avoid misinterpreting seemingly good market trends as market robustness at times of underlying market fragility. Market fragility always implies weaker investment in the real economy. This weakness alters the effects of planned fiscal and monetary policies. Our arguments in this study may serve as a starting point for new research on better identifying underlying market fragility and its sources.

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Appendix

for

Market Fragility and the Paradox of the Recent Stock-Bond Dissonance*

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November 7, 2017

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1. Appendix A – Proof of equations (6) and (7) in the paper

Using the transformation,

$$W_t = S_{t-1}\left(D_t + P_t\right) + (1-\delta)^{\nu_t \nu_t^B} B_{t-1}\ , \tag{A.1}$$

the budget constraint (3) becomes,

$$W_{t+1} = R_{t+1}^P \left(W_t - C_t\right)\ , \tag{A.2}$$

in which R_t^P is the gross portfolio return defined as,

$$R_t^P = \phi_{t-1}^S R_t^S + \phi_{t-1}^B R_t^B\ , \tag{A.3}$$

with

$$R_t^S = \frac{D_t + P_t}{P_{t-1}} \quad \text{and} \quad R_t^B = \frac{(1-\delta)^{\nu_t \nu_t^B}}{Q_{t-1}} \tag{A.4}$$

being the gross returns of stocks and bonds, and with $\phi_t^S = P_t S_t / \left(P_t S_t + Q_t B_t\right)$ and $\phi_t^B = Q_t B_t / \left(P_t S_t + Q_t B_t\right)$ being the portfolio weights.

Using (A.2) the Bellman equation is,

$$V_t\left(W_t\right) = \max_{c_t \geq 0, \phi_t^S, \phi_t^B} \left\{ (1-\beta)\, C_t^{1-\frac{1}{\eta}} + \beta \left\{ E_t \left[V_{t+1} \left(R_{t+1}^P \left(W_t - C_t\right)\right)^{1-\gamma} \right] \right\}^{\frac{1-\frac{1}{\eta}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\eta}}}, \tag{A.5}$$

subject to (A.3) and subject to the stochastic structure given by (1) and (2). Under a general stochastic structure, the value function, $V_t(\cdot)$, is of the form,[6]

$$V\left(W_t\right) = \psi_t W_t\ ,\ t = 0, 1, \ldots\ . \tag{A.6}$$

[6] Equation (A.6) corresponds to Epstein and Zin (1991, p. 267, eq. 9).

A useful implication of (A.6) is,[7]

$$\frac{C_t}{W_t} = (1-\beta)^{\eta}\,\psi_t^{1-\eta}\ ,\ t = 0, 1, \ldots\ . \tag{A.7}$$

In addition, (A.6) implies the key asset pricing equation of this model, which is,[8]

$$E_t\left[\beta^{\frac{1-\gamma}{1-\frac{1}{\eta}}}\left(\frac{C_{t+1}}{C_t}\right)^{\frac{1-\gamma}{1-\eta}}\left(R_{t+1}^P\right)^{\frac{\frac{1}{\eta}-\gamma}{1-\frac{1}{\eta}}} R_{t+1}^i\right] = 1\ ,\quad i \in \{S, B\}\ . \tag{A.8}$$

In the standard textbook Lucas (1978) asset-pricing model the key simplifying assumption is that all investors are identical, all having the same amount of S_{-1} stocks in period 0, and all having $B_{t-1} = 0$ for all $t \in \{0, 1, \ldots\}$, i.e., bonds in zero net supply in all periods. Identical investors do not trade stocks in equilibrium. Combining these simplifying assumptions of no trade in equilibrium with the budget constraint, and also with equations (3) and (A.1), we obtain,

$$S_t = S_{-1}\ ,\ \ C_t = S_{-1}D_t\ ,\ \ W_t = S_{-1}\left(D_t + P_t\right)\ ,\ \ \phi_t^S = 1\,,\ \text{and}\ \phi_t^B = 0\ ,\ \ t = 0, 1, \ldots\ . \tag{A.9}$$

Combining (A.3) with (A.9) gives,

$$R_t^P = R_t^S\ , \tag{A.10}$$

while equation (A.4) implies,

$$R_{t+1}^S = \frac{1 + x_{t+1}}{x_t}\frac{D_{t+1}}{D_t}\ ,\quad \text{with}\quad x_t \equiv \frac{P_t}{D_t}\ . \tag{A.11}$$

In addition, equation (A.9) implies that $C_{t+1}/C_t = D_{t+1}/D_t$, so substituting this result into (A.8) for $i = S$, together with (A.10), (A.11), and (1), equation (A.8) becomes,

$$E_t\left\{\left(\beta\frac{1 + x_{t+1}}{x_t}\right)^{\frac{1-\gamma}{1-\frac{1}{\eta}}}\left[e^{\mu - \frac{\sigma^2}{2} + \sigma\varepsilon_{t+1}}\left(1 - \zeta_{t+1}\right)^{\nu_{t+1}}\right]^{1-\gamma}\right\} = 1\ . \tag{A.12}$$

[7] Equation (A.7) should correspond to Epstein and Zin (1991, p. 268, eq. 12), but equations (A.7) and Epstein and Zin (1991, p. 268, eq. 12) are different. See Appendix B for a proof ot equation (A.7).
[8] See Epstein and Zin (1991, p. 268, eq. 16).

In Appendix C we prove that, as a consequence of our assumption that variables ε_{t+1}, ν_{t+1}, and ζ_{t+1} are i.i.d. over time, the P-D ratio is also constant over time, i.e.,

$$x_t = x \ , \ \ t = 0, 1, ... \ . \tag{A.13}$$

Substituting (A.13) into (A.12), proves the formula given by (6).

For proving equation (7) we substitute (A.4), (A.10), (A.11), and (A.13) into (A.8), for $i = B$, to obtain,

$$\beta^{\frac{1-\gamma}{1-\frac{1}{\eta}}} \left(\frac{1+x}{x}\right)^{\frac{\frac{1}{\eta}-\gamma}{1-\frac{1}{\eta}}} E_t \left\{ \left[e^{\mu - \frac{\sigma^2}{2} + \sigma\varepsilon_{t+1}} \left(1-\zeta_{t+1}\right)^{\nu_{t+1}} \right]^{-\gamma} (1-\delta)^{\nu_{t+1}\nu_{t+1}^B} \frac{1}{Q_t} \right\} = 1 \ . \tag{A.14}$$

Equation (A.14) implies a constant value for Q_t over time, $Q_t = Q$ for all $t \in \{0, 1, ...\}$. The bond price, Q, is set a-priori to the realization of disasters and defaults. This is the reason why Q is constant over time. Yet, because of the sovereign default risk, the ex-post bond return is variable over time. Specifically, with probability $\lambda\pi$, the ex-post (post-default) maturity price of the bond is $1 - \delta$, making the ex-post bond return equal to $\underline{r}^B = (1-\delta)/Q - 1$. So,

$$r_t^B = \begin{cases} r^B = \frac{1}{Q} - 1 & , \quad \text{with Prob. } \ 1 - \lambda\pi \\ \underline{r}^B = \frac{1-\delta}{Q} - 1 & , \quad \text{with Prob. } \ \lambda\pi \end{cases} . \tag{A.15}$$

In equation (7) we refer to the expected return implied by equation (A.15), which is given by $E\left(r^B\right) = (1 - \lambda\pi\delta)/Q - 1$. □

2. Appendix B – Proof of equation (A.7)

Take equation (A.6) as an initial guess for the functional form of the value function, considering that ψ_t is an unknown stochastic process. Substituting (A.6) into equation (A.5) we obtain,

$$\psi_t W_t = \max_{c_t \geq 0, \phi_t^S, \phi_t^B} \left[(1-\beta) C_t^{1-\frac{1}{\eta}} + \beta\omega_t \cdot (W_t - C_t)^{1-\frac{1}{\eta}} \right]^{\frac{1}{1-\frac{1}{\eta}}} , \tag{A.16}$$

in which, $\omega_t \equiv \left\{ E_t \left[\psi_{t+1}^{1-\gamma} \left(R_{t+1}^{P} \right)^{1-\gamma} \right] \right\}^{\frac{1-\frac{1}{\eta}}{1-\gamma}}$. Taking first-order conditions on equation (A.5) with respect to C_t gives,

$$C_t = \left(\frac{\beta}{1-\beta} \omega_t \right)^{-\eta} (W_t - C_t) \ . \tag{A.17}$$

Equation (A.17) implies,

$$C_t^{1-\frac{1}{\eta}} = \left(\frac{\beta}{1-\beta} \omega_t \right)^{1-\eta} (W_t - C_t)^{1-\frac{1}{\eta}} \ . \tag{A.18}$$

Substituting (A.18) into (A.16), imposes optimality conditions on (A.16). So, the max operator in (A.16) can be eliminated after substituting (A.18) into (A.16), which gives,

$$\psi_t^{1-\frac{1}{\eta}} W_t^{1-\frac{1}{\eta}} = \beta \omega_t \left[\left(\frac{\beta}{1-\beta} \omega_t \right)^{-\eta} + 1 \right] (W_t - C_t)^{1-\frac{1}{\eta}} \ . \tag{A.19}$$

Using (A.18) and substituting it into (A.19) results in,

$$\psi_t^{1-\frac{1}{\eta}} \left(\frac{C_t}{W_t} \right)^{\frac{1}{\eta}-1} = (1-\beta) \left[1 + \left(\frac{\beta}{1-\beta} \omega_t \right)^{\eta} \right] \ . \tag{A.20}$$

Equation (A.17) implies,

$$\frac{C_t}{W_t} = \left[1 + \left(\frac{\beta}{1-\beta} \omega_t \right)^{\eta} \right]^{-1} \ . \tag{A.21}$$

Substituting (A.21) into (A.20) gives,

$$\psi_t^{1-\frac{1}{\eta}} = (1-\beta) \left[1 + \left(\frac{\beta}{1-\beta} \omega_t \right)^{\eta} \right]^{\frac{1}{\eta}} \ . \tag{A.22}$$

Equation (A.22) reconfirms that the guess given by equation (A.6) is valid. Combining (A.21) with (A.22) leads to equation (A.7). □

3. Appendix C – Proof that the price-dividend ratio is constant

Since variables ε_{t+1}, ν_{t+1}, and ζ_{t+1} are i.i.d. over time, through integral-variable transformation, equation (A.12) implies that,

$$E_t\left(\frac{1+x_{t+1}}{x_t}\right) = E_{t+1}\left(\frac{1+x_{t+2}}{x_{t+1}}\right) = \kappa \ , \ \ t = 0, 1, ... \ . \tag{A.23}$$

To see that $x_{t+1} = x_t = x$ for $t = 0, 1, ...$, fix some $x_t = \bar{x}_t > 0$, assuming that $\bar{x}_t$ is a solution to the asset-pricing model. Consider conditional expectations for (A.23), namely,

$$E_t\left(\frac{1+x_{t+1}}{x_t} \ \mid x_t = \bar{x}_t\right) = \kappa \ . \tag{A.24}$$

Equation (A.24) implies a unique solution for $E_t\left(x_{t+1}\right)$. Let that unique solution be $\bar{x}_{t+1} = E_t\left(x_{t+1}\right)$. Using $\bar{x}_{t+1}$, consider equation (A.24) one period ahead to obtain $\bar{x}_{t+2} \equiv E_{t+1}\left(x_{t+2}\right)$. Notice that, due to additive separability, and since the choice of $t \in \{0, 1, ...\}$ was arbitrary, equation (A.24) implies,

$$\frac{1+\bar{x}_{t+1}}{\bar{x}_t} = \frac{1+\bar{x}_{t+2}}{\bar{x}_{t+1}} \ , \ \ t = 0, 1, ... \ . \tag{A.25}$$

Using $g_{t+1} \equiv \bar{x}_{t+1}/\bar{x}_t$ equation (A.25) implies,

$$g_{t+2} - g_{t+1} = \frac{1}{\bar{x}_t}\left(1 - \frac{1}{g_{t+1}}\right) \ , \ \ t = 0, 1, ... \ . \tag{A.26}$$

If $g_{t+1} \neq 1$, since $\bar{x}_t > 0$, we can easily verify that equation (A.26) implies unstable dynamics for x_t. If $g_{t+1} > 1$ for some t, then (A.26) implies $g_{t+s} > 1$ for all $s \in \{0, 1, ...\}$, and $x_t \to \infty$. If $g_{t+1} < 1$ for some t, then eventually $g_{\hat{t}} < 0$ for some $\hat{t} > t$, leading to $x_{\hat{t}} < 0$. For $\eta \neq 1$ (which is of interest for matching the data), both of these possibilities lead to a non-well-defined value function. To see this, use equations (A.7), (A.11), (A.9), and (A.6) to obtain,

$$V_t\left(W_t\right) = \psi_t W_t = (1-\beta)^{\frac{\eta}{\eta-1}}\left(1+x_t\right)^{\frac{\eta}{\eta-1}} S_{-1} D_t \ . \tag{A.27}$$

For $x_t \to \infty$, either $V_t(W_t) \to \infty$ (if $\eta > 1$), or $V_t(W_t) \to 0$, (if $\eta < 1$), even if $0 < D_t < \infty$. None of these possibilities implies a well-defined value function or a maximum value, given that the EZW utility function represents a cardinal certainty-equivalent time aggregator measured in consumption units (in addition, the solution $g_t = 1$ for all $t \in \{0, 1, ...\}$, gives $V_t(W_t) > 0$ bounded away from 0 for all $t \in \{0, 1, ...\}$). Since from equation (A.9) $C_t/W_t = 1/(1 + x_t)$, having $x_{\hat{t}} < 0$ for some $\hat{t} > t$, implies $C_{\hat{t}} > W_{\hat{t}}$, and equation (A.2) then gives $W_{\hat{t}+1} < 0$ if $R^P_{\hat{t}+1} > 0$, i.e. $V_{\hat{t}}(W_{\hat{t}}) < 0$, which is unacceptable, given the consumption-unit cardinality of EZW preferences. But even if $R^P_{\hat{t}+1} < 0$, having negative stock prices in an exchange economy of identical agents is impossible in equilibrium. So, the only acceptable equilibrium solution for (A.26) is $g_t = 1$ for all $t \in \{0, 1, ...\}$, which leads to equation (A.13), proving the result. $\square$

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