Geometry of Prym–Teichmüller Curves and $\mathbb{C}$-linear Manifolds

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Introduction

Describing the solutions of polynomial equations is, arguably, one of the oldest and basic problems in mathematics. The study of spaces of such solutions, varieties, therefore combines techniques from almost all branches of modern mathematics. From a geometric perspective, it is particularly fruitful to consider solutions in the complex numbers, as this allows the use of not only algebraic methods, but also techniques from topology and complex analysis. Moreover, given a geometric object that is cut out by some polynomial equations, we may perturb the coefficients of these polynomials and thus obtain a natural notion of "nearby" objects and continuously varying families. It is therefore natural to study moduli spaces, the space of all varieties satisfying fixed geometric properties, which can again be endowed with an algebraic structure. Already the first non-trivial examples, one-dimensional varieties, i.e. curves or equivalently Riemann surfaces, give rise to interesting moduli spaces. Topologically, a curve is determined by its genus g and we denote the moduli space of complex, projective curves of genus g by $\mathcal{M}_g$.

But what is the best way to picture a complex algebraic curve, let alone a family of curves? As is often the case, this becomes easier by adding more structure. Consider a polygon in the complex plane $\mathbb{C}$ with sides that are pair-wise parallel and of equal length. By identifying these pairs of sides by translations, we obtain a topological surface. Moreover, the embedding of the polygon into the complex plane endows this surface with a complex structure and even with a flat metric outside of the finitely many vertices of the polygon. We thus obtain a *flat surface* (or *translation surface*). In fact, we have even more: as the sides were identified by translation, the differential $\mathrm{d}z$ on $\mathbb{C}$ descends to a holomorphic differential form that can be extended to the entire Riemann surface, obtaining zeros exactly at the vertices of the polygon. Conversely, any holomorphic differential form on a curve X yields such charts and a flat structure outside of the zeros by integration. A flat surface is therefore a tuple (X, ω) consisting of a complex smooth projective curve (i.e. compact Riemann surface) together with a non-zero holomorphic differential ω, see e.g. [Zor06] for details. We denote the moduli space of flat surfaces of genus g by $\Omega\mathcal{M}_g$ and note that it comes with a natural projection to $\mathcal{M}_g$.

Adding the flat structure has many fascinating implications. The flat structure depends in a holomorphic way on the (relative) periods of the differential ω (equivalently: on the sides of the corresponding polygon). Moreover, small perturbations yield a new differential with the same orders of zeros as ω. Denote by μ the partition of $2g-2$ consisting of the orders of the zeros of ω, i.e. $\mu = (a_1, \dots, a_n)$ for some n and there exist pairwise distinct $p_i \in X$ such that

$$\operatorname{div} \omega = \sum_{i=1}^{n} a_i p_i.$$

Then the subspaces $\Omega\mathcal{M}_g(\mu) \subseteq \Omega\mathcal{M}_g$ consisting only of differentials with associated partition μ yield a stratification of $\Omega\mathcal{M}_g$. Now, on each stratum the absolute and relative (with respect to the zeros) periods of the differential give a holomorphic coordinate system: locally $\Omega\mathcal{M}_g(\mu)$ is biholomorphic to $\mathbb{C}^{2g+n-1}$; again, see e.g. [Zor06] for details. Furthermore, by identifying $\mathbb{C}$ with $\mathbb{R}^2$, we obtain a natural action of the group $\mathrm{SL}_2(\mathbb{R})$ that shears the polygon and thus acts on the corresponding flat surface. Observe that this action respects the above stratification. Orbit closures of flat surfaces have been studied extensively over the last decades, incorporating many techniques from e.g. dynamical systems, Teichmüller theory and ergodic theory. Recently, there has been a major break-through: Eskin, Mirzakhani and Mohammadi have shown that any $\mathrm{SL}_2(\mathbb{R})$-orbit closure is in fact cut out by $\mathbb{R}$-linear equations in period coordinates [EMM15]. Note that the $\mathrm{SL}_2(\mathbb{R})$-action is transcendental (the action is on the periods of the differential ω). Remarkably, Filip has shown that any orbit closure is in fact an algebraic variety and is defined over $\overline{\mathbb{Q}}$ [Fil16]. However, a classification of all orbit closures is still a wide open problem, see [EFW17].

A particularly interesting case is the "minimal" case of closed orbits, i.e. 2-dimensional orbits. The projection of such an orbit closure to the moduli space of curves, $\mathcal{M}_g$, yields a special algebraic curve, a *Teichmüller curve*. Not many families of (primitive) Teichmüller curves are known, see [McM07], [McM06], [KS00], [BM10b] and [MMW17]; see e.g. [MMW17] for a more detailed summary of known results. A flat surface (X, ω) can only generate a Teichmüller curve if the stabiliser of the $\mathrm{SL}_2(\mathbb{R})$-action is sufficiently large. McMullen observed (originally in the genus 2 case) that this can sometimes be achieved by requiring a subvariety of $\mathrm{Jac}(X)$, the Jacobian of X, that contains the differential ω, to admit real multiplication with ω as an eigenform (see section 1.2 for details). In fact, Möller showed that this is a necessary condition [Möl06].

More precisely, McMullen constructed for every quadratic discriminant D a *Weierstraß curve* and thereby classified all Teichmüller curves in $\mathcal{M}_2$ by analysing when the Jacobian of the generating flat surface (X, ω) admits real multiplication by the associated quadratic order $\mathcal{O}_D$ that respects ω (see section 1.2 for details). However, for genus 3, requiring real multiplication on the entire Jacobian (i.e. being *algebraically primitive*) is too strong a restriction for obtaining infinite families, cf. [BHM16]. By relaxing this condition, McMullen constructed, again for every quadratic discriminant D, a *Prym–Teichmüller curve* $W_D(4)$ in genus 3 and $W_D(6)$ in genus 4 (see section 1.2 for precise definitions). Recently, Eskin, Filip and Wright have shown that all but finitely many Teichmüller curves in genus 3 are of this form [EFW17]. This was suggested by several recent strong finiteness results, see [BM12], [BHM16], [ANW14], [MW14], [NW14], [AN15] and [LNW15]. Moreover, Eskin, McMullen, Mukamel and Wright announced the existence of six exceptional orbit closures, two of which contain an infinite collection of Teichmüller curves in genus 4. One of them is treated in [MMW17].

Having constructed these Teichmüller curves, it is natural to ask for their topological type. More precisely, any Teichmüller curve $\mathcal{C}$ is a one-dimensional sub-orbifold of $\mathcal{M}_g$. Therefore, denoting by χ the (orbifold) Euler characteristic, by h_0 the number of connected components, by C the number of cusps and by e_d the number of points of order d, these invariants determine the genus g:

$$2h_0 - 2g = \chi + C + \sum_d e_d \left(1 - \frac{1}{d}\right),$$

i.e. they determine the topological type of $\mathcal{C}$.

For the Prym–Weierstraß curves in $\mathcal{M}_2$, the situation is as follows. In genus 2, cusps and connected components were determined by McMullen [McM05a], the Euler characteristic was computed by Bainbridge [Bai07], and the number and types of orbifold points were established by Mukamel [Muk14]. See also Hubert-Lelièvre [HL06] for other results related to the number of elliptic points on translation surfaces in the minimal stratum in genus 2.

The main goal of this thesis is to determine the topological type, i.e. the genus, of the Prym–Teichmüller curves in $\mathcal{M}_3$ and $\mathcal{M}_4$.

In genus 3 and 4, Möller [Möl14] calculated the Euler characteristic and Lanneau and Nguyen [LN14] classified the cusps of the Prym–Teichmüller curves. Moreover, the number of connected components in genus 3 was also determined in [LN14]. In the case of genus 4, Lanneau has communicated to the author that the Prym locus is always connected [LN16].

In chapter 1, we establish the following result in joint work with David Torres-Teigell.

Except for some extra symmetries occurring for small D, we describe the orbifold points of the Prym–Teichmüller curves in $\mathcal{M}_3$ in terms of integral solutions of ternary quadratic forms, which lie in some fundamental domain. More precisely, for any positive discriminant D, we define

$$\begin{aligned}
\mathcal{H}_2(D) &:= \{(a,b,c) \in \mathbb{Z}^3 : a^2 + b^2 + c^2 = D,\ \gcd(a,b,c,f_0) = 1\},\ \text{and}\\
\mathcal{H}_3(D) &:= \{(a,b,c) \in \mathbb{Z}^3 : 2a^2 - 3b^2 - c^2 = 2D,\ \gcd(a,b,c,f_0) = 1,\\
&\qquad -3\sqrt{D} < a < -\sqrt{D},\ c < b \le 0,\\
&\qquad (4a - 3b - 3c < 0) \vee (4a - 3b - 3c = 0\ \wedge\ c < 3b)\},
\end{aligned}$$

where f_0 denotes the conductor of D. The extra conditions in the definition of $\mathcal{H}_3(D)$ restrict the solutions to a certain fundamental domain. In particular, even though the quadratic form is indefinite, these conditions ensure that the set $\mathcal{H}_3(D)$ is finite for all D.

Theorem 0.1 (Theorem 1.1.1)**.** *For non-square discriminant $D > 12$, the Prym–Teichmüller curves $W_D(4)$ for genus three have orbifold points of order 2 or 3.*

More precisely, the number $e_3(D)$ of orbifold points of order 3 is $|\mathcal{H}_3(D)|$; the number $e_2(D)$ of orbifold points of order 2 is $|\mathcal{H}_2(D)|/24$ if D is even and there are no points of order 2 when D is odd.

The curve $W_8(4)$ has one point of order 3 and one point of order 4; the curve $W_{12}(4)$ has a single orbifold point of order 6.

Note also that $W_D(4)$ is empty for $D \equiv 5 \bmod 8$ by [Möl14]. The topological invariants of $W_D(4)$ for D up to 248 are given in Table 1.2 on page 44.

Moreover, we provide flat prototypes of the orbifold points in section 1.7.

The Weierstraß curves in $\mathcal{M}_2$ and the Prym–Weierstraß curves in $\mathcal{M}_3$ have two connected components if $D \equiv 1 \bmod 8$. In genus 2, McMullen [McM05b] provided a spin invariant to determine on which component a cusp lies. Using this, Bouw and Möller [BM10a] proved that the two components are Galois conjugate and therefore homeomorphic. In particular, they have the same number of cusps and orbifold points.

In chapter 2, we obtain analogous results for the Prym–Teichmüller curves in $\mathcal{M}_3$.

More precisely, we provide a spin invariant that determines, for each cusp prototype of [LN14], on which component it lives. Moreover, we explicitly describe the Galois conjugation of the cusps and show that the two components are again Galois-conjugate. In particular:

Theorem 0.2 (Theorem 2.1.1). *Let $D \equiv 1 \bmod 8$, which is not a square. Then the two components of $W_D(4)$ are homeomorphic (as orbifolds). In particular, they have the same number of cusps and elliptic points.*

This completes the topological classification of the Prym–Teichmüller curves in $\mathcal{M}_3$.

In chapter 3, again in joint work with David Torres-Teigell, we classify the orbifold points on the Prym–Teichmüller curves in $\mathcal{M}_4$.

In particular, we obtain the following result.

Theorem 0.3 (Theorem 3.1.1). *For discriminant $D > 12$, the Prym–Teichmüller curves $W_D(6)$ have orbifold points of order 2 and 3. More precisely:*

- *the number of orbifold points of order 2 is*

$$e_2(D) = \begin{cases} 0\,, & \text{if } D \text{ is odd,} \\ h(-D) + h(-D/4)\,, & \text{if } D \equiv 12 \bmod 16, \\ h(-D)\,, & \text{if } D \equiv 0, 4, 8 \bmod 16, \end{cases}$$

  where $h(-D)$ is the class number of the quadratic order $\mathcal{O}_{-D}$;

- *the number of orbifold points of order 3 is*

$$e_3(D) = \#\{a, i, j \in \mathbb{Z} : a^2 + 3j^2 + (2i - j)^2 = D,\ \gcd(a, i, j) = 1\}/12;$$

- *$W_5(6)$ has one point of order 3 and one point of order 5;*
- *$W_8(6)$ has one point of order 2 and one point of order 3;*
- *$W_{12}(6)$ has one point of order 2 and one point of order 6.*

Observe that, essentially due to technical aspects relating the polarisation of the abelian subvariety of the Jacobian that admits real multiplication (see section 3.2 for details), the result is much more concise (and more similar to the case in genus 2) than the classification in genus 3. The topological invariants of $W_D(6)$ for non-square discriminants $D \leq 181$ are listed in Table 3.3 on page 91.

Again, we provide flat prototypes of the orbifold points in section 3.6.

Furthermore, using the classification of cusps [LN14] and work of Mukamel [Muk14], we are able to provide exact asymptotics for the growth of the genus of $W_D(6)$ with respect to the discriminant D.

Theorem 0.4 (Theorem 3.1.4). *There exist constants $C_1, C_2 > 0$, independent of D, such that*

$$C_1 \cdot D^{3/2} < g(W_D(6)) < C_2 \cdot D^{3/2}.$$

Moreover, $g(W_D(6)) = 0$ if and only if $D \leq 20$.

The general technique for counting orbifold points on Teichmüller curves in genus 2, 3 and 4 is the following: an orbifold point is always a flat surface (X, ω) along with some holomorphic automorphism α, which admits ω as an eigendifferential. The topological action of α can be determined using flat geometry. Then, one can attempt to describe the locus inside the moduli space of curves admitting an automorphism of this form and count the intersection points of this locus and the Teichmüller curve.

In chapter 4, we attempt to understand these loci more conceptually.

More precisely, we introduce the notion of a $\mathbb{C}$-linear manifold, i.e. a submanifold $\mathcal{M} \subseteq \Omega\mathcal{M}_g(\mu)$ that is locally cut out by $\mathbb{C}$-linear equations in period coordinates. While there are many results on $\mathbb{R}$-linear manifolds (see e.g. [Wri15b] for a summary of known results), $\mathbb{C}$-linear manifolds that are not $\mathbb{R}$-linear have hardly been studied, except for some general results in [Möl08]. However, [Möl08, Definition 6.4] included an extra condition (requiring the existence of a certain compactification) that, while it made sense at the time, does not seem warranted from todays perspective.

First, we provide a large class of examples of $\mathbb{C}$-linear manifolds that are not $\mathbb{R}$-linear: spaces of eigenforms of cyclic covers of $\mathbb{P}^1$. These have been used, e.g., in the previous chapters to count orbifold points of Prym–Teichmüller curves.

Theorem 0.5 (Theorem 4.3.1). *Any family of eigenspaces of a family of cyclic covers of $\mathbb{P}^1$ is a $\mathbb{C}$-linear manifold.*

Note that some technical details and calculations regarding cyclic covers are included in Appendix A, as they are scattered throughout the literature and consistent notation is important to avoid confusion.

We then introduce the notion of a *covering construction* of a $\mathbb{C}$-linear manifold. While there are several notions of covering constructions for $\mathbb{R}$-linear manifolds (cf. [Api16], [MMW17]), these use the $\mathrm{SL}_2(\mathbb{R})$-action and the results of [EMM15] to show that the cover of an $\mathbb{R}$-linear manifold is again an $\mathbb{R}$-linear manifold. As there is no $\mathrm{SL}_2(\mathbb{R})$-action in the general $\mathbb{C}$-linear case, we provide a slightly technical alternative.

Let $\mathcal{M} \subseteq \Omega\mathcal{M}_g(\mu)$ be a linear manifold, $(X, \omega) \in \mathcal{M}$, ρ a monodromy representation of (X, ω) and $\mathcal{M}(\rho) \subseteq \Omega\mathcal{M}_h(\nu)$ the cover associated to ρ (see Definition 4.4.14 for details). Then we show:

Theorem 0.6 (Theorem 4.4.1)**.** *$\mathcal{M}(\rho)$ is a linear manifold.*

Our construction is split into several parts. We rigidify the manifolds by adding extra structure (essentially passing to Teichmüller space). We then show that, using this construction, covers of linear manifolds are again linear (Proposition 4.4.12) and quotients of (suitable) linear manifolds are again linear (Proposition 4.4.13). This allows us to define arbitrary covers of linear manifolds and yields a natural notion of a *primitive* linear manifold (Definition 4.4.15).

However, a classification of *primitive* $\mathbb{C}$-linear manifolds even in low genus is still open and a long-term project. Also, the question of algebraicity, i.e. if an analogous result to [Fil16] can be shown in the $\mathbb{C}$-linear case, is still wide open; see also the discussion in section 4.5.

Finally, note that chapter 1 has appeared as [TTZ16] (joint work with David Torres-Teigell), chapter 2 has been accepted for publication and will appear as [Zac16] and chapter 3 has appeared as [TTZ17] (again joint work with David Torres-Teigell). The chapters are essentially verbatim copies of the published versions.

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Special thanks also goes to David Torres-Teigell, my co-author and former office mate. Not only did he teach me much about Riemann surfaces and their automorphisms, but we also had a lot of fun during his Humboldt-Stipendium in Frankfurt. Thanks also go to all the current and former members of the workgroup. I am particularly grateful to my current and former office mates, Markus Rennig and Ralf Lehnert, who put up with me on a daily basis and kindly ignored my tendency to fill the office with piles of paper and dead plants. Matteo Costantini was an invaluable partner in uncountable discussions over the last couple of months. In this, he was reliably aided by Martin Lüdtke, who painstakingly scrutinised every unclean definition and logical fallacy I came up with. Quentin Gendron was also always available for discussions, in and outside of university and even after moving to Hannover; moreover, I am particularly indebted to him for introducing me to squash and many fascinating aspects of the Breton culture. An important part of working in Frankfurt were the frequent "Besprechungen"; I am very grateful to Kolja Hept and Thorsten Jörgens for keeping this tradition alive over the years. I am deeply indebted to André Kappes for many mathematical discussions, both in Karlsruhe and later in Frankfurt, and in particular for his encouragement to do a PhD in Frankfurt. Furthermore, I would also like to thank all my other mathematical and worldly friends that I have not yet named; this thesis would not have been possible without all the support and encouragement I received from them.

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1. Orbifold points on Prym–Teichmüller curves in genus three

Prym–Teichmüller curves $W_D(4)$ constitute the main examples of known primitive Teichmüller curves in the moduli space $\mathcal{M}_3$. In this chapter, we determine, for each non-square discriminant $D > 1$, the number and type of orbifold points in $W_D(4)$. These results, together with the formulas of Lanneau-Nguyen and Möller for the number of cusps and the Euler characteristic, complete the topological characterisation of Prym–Teichmüller curves in genus 3.

Crucial for the determination of the orbifold points is the analysis of families of genus 3 cyclic covers of degree 4 and 6, branched over four points of $\mathbb{P}^1$. As a side product of our study, we provide an explicit description of the Jacobians and the Prym–Torelli images of these two families, together with a description of the corresponding flat surfaces. The content of this chapter is joint work with David Torres-Teigell and has appeared as [TTZ16].

1.1. Introduction

A *Teichmüller curve* is an algebraic curve in the moduli space $\mathcal{M}_g$ of genus g curves that is totally geodesic for the Teichmüller metric. Teichmüller curves arise naturally from *flat surfaces*, i.e. elements (X, ω) of the bundle $\Omega\mathcal{M}_g$ over $\mathcal{M}_g$, consisting of a curve X with a holomorphic 1-form $\omega \in \Omega(X)$. The bundle $\Omega\mathcal{M}_g$ is endowed with an $\mathrm{SL}_2(\mathbb{R})$-action, defined by affine shearing of the flat structure induced by the differential. In the rare case that the closure of the projection to $\mathcal{M}_g$ of the $\mathrm{SL}_2(\mathbb{R})$-orbit of an element (X, ω) is an algebraic curve, i.e. that (X, ω) has many real symmetries, we obtain a Teichmüller curve.

Only few examples of families of (primitive) Teichmüller curves are known, see [McM07], [McM06], [KS00] and [BM10b]. In genus 2, McMullen was able to construct the *Weierstraß curves*, and thereby classify all Teichmüller curves in $\mathcal{M}_2$ by analysing when the Jacobian of the flat surface admits real multiplication that respects the 1-form. However, for genus 3, requiring real multiplication on the entire Jacobian (i.e. being algebraically primitive) is too strong a restriction for obtaining infinite families, cf. [BHM16]. By relaxing this condition McMullen constructed the *Prym–Teichmüller curves* $W_D(4)$ in genus 3 and $W_D(6)$ in genus 4 (see section 1.2 for definitions). Recently Eskin–Filip–Wright have announced that all but finitely many Teichmüller curves in genus 3 are of this form. This was suggested by several recent strong finiteness results, see [BM12], [BHM16], [ANW14], [MW14], [NW14], [AN15] and [LNW15].

While the situation for genus 2 is fairly well understood, things are less clear for higher genus. As curves in $\mathcal{M}_g$, Teichmüller curves carry a natural orbifold structure. As such, one is primarily interested in their homeomorphism type, i.e. the genus, the number of cusps, components, and the number and type of orbifold points. In genus two, this was solved for the Weierstraß curves by McMullen [McM05a], Bainbridge [Bai07] and Mukamel [Muk14]. See also Hubert–Lelièvre [HL06] for other results related to the number of elliptic points on translation surfaces in the minimal stratum in genus 2.

For the Prym–Teichmüller curves in genus 3 and 4, the Euler characteristics were calculated by Möller [Möl14] and the number of components and cusps were counted by Lanneau and Nguyen [LN14]. The primary aim of this paper is to describe the number and type of orbifold points occurring in genus 3, thus completing the topological characterisation of $W_D(4)$ for all (non-square) discriminants D via the formula

$$2h_0 - 2g = \chi + C + \sum_d e_d \left(1 - \frac{1}{d}\right) \tag{1.1}$$

where g denotes the genus of $W_D(4)$, h_0 the number of components, χ the Euler characteristic, C the number of cusps and e_d the number of orbifold points of order d, that is points whose stabiliser in the uniformising group has order d. As $W_D(4)$ is either connected or the connected components are homeomorphic by [Zac16], this characterises all Teichmüller curves inside the loci $W_D(4)$.

Except for some extra symmetries occurring for small D, we describe the orbifold points in terms of integral solutions of ternary quadratic forms, which lie in some fundamental domain. More precisely, for any positive discriminant D, we define

$$\begin{aligned}
\mathcal{H}_2(D) &:= \{(a,b,c) \in \mathbb{Z}^3 : a^2+b^2+c^2 = D\ ,\ \gcd(a,b,c,f_0)=1\,\},\ \text{and}\\
\mathcal{H}_3(D) &:= \{(a,b,c) \in \mathbb{Z}^3 : 2a^2-3b^2-c^2 = 2D\ ,\ \gcd(a,b,c,f_0)=1\ ,\\
&\qquad\qquad -3\sqrt{D} < a < -\sqrt{D}\ ,\ c < b \le 0\ ,\\
&\qquad\qquad (4a-3b-3c<0) \vee (4a-3b-3c=0\ \wedge\ c<3b)\,\},
\end{aligned}$$

where f_0 denotes the conductor of D. The extra conditions in the definition of $\mathcal{H}_3(D)$ restrict the solutions to a certain fundamental domain. In particular, even though the quadratic form is indefinite, these conditions ensure that the set $\mathcal{H}_3(D)$ is finite for all D.

Theorem 1.1.1. *For non-square discriminant $D > 12$, the Prym–Teichmüller curves $W_D(4)$ for genus three have orbifold points of order 2 or 3.*

More precisely, the number $e_3(D)$ of orbifold points of order 3 is $|\mathcal{H}_3(D)|$; the number $e_2(D)$ of orbifold points of order 2 is $|\mathcal{H}_2(D)|/24$ if D is even and there are no points of order 2 when D is odd.

The curve $W_8(4)$ has one point of order 3 and one point of order 4; the curve $W_{12}(4)$ has a single orbifold point of order 6.

Let us recall that $W_D(4)$ is empty for $D \equiv 5 \bmod 8$ (see [Möl14, Prop. 1.1]).

Theorem 1.1.1 combines the content of Theorem 1.5.1 and Theorem 1.5.6. The topological invariants of $W_D(4)$ for D up to 248 are given in Table 1.2 on page 44.

Our approach to solving this problem is purely algebraic and therefore the use of tools from the theory of flat surfaces will be sporadic.

Two families of curves will play a special role in determining orbifold points on Prym–Teichmüller curves, namely the *Clover family* and the *Windmill family*, which will be introduced in section 1.3. They parametrise certain genus 3 cyclic covers of $\mathbb{P}^1$ of degree 4 and 6, respectively. There are two special points in these families, namely the *Fermat curve* of degree 4, which is the only element of the Clover family with a cyclic group of automorphisms of order 8, and the exceptional *Wiman curve* of genus 3, which is the unique intersection of the two families and the unique curve in genus 3 that admits a cyclic group of automorphisms of order 12.

The fact that orbifold points in $W_D(4)$ correspond to points of intersection with these two families will follow from the study of the action of the Veech group $\mathrm{SL}(X, \omega)$ carried out in section 1.2. A consequence of this study is that orbifold points of order 4 and 6 correspond to the Fermat and Wiman curves, respectively, while points of order 2 and 3 correspond to generic intersections with the Clover family and the Windmill family, respectively.

In order to determine these points of intersection, we will need a very precise description of the two families or, more precisely, of their images under the Prym–Torelli map. To this end, we explicitly compute the period matrices of the two families in section 1.4. While the analysis of different types of orbifold points was rather uniform up to this point, the Clover family and the Windmill family behave quite differently under the Prym–Torelli map. In particular, any two images of the Clover family under the Prym–Torelli map are isomorphic (as polarised abelian varieties).

Theorem 1.1.2. *The Prym–Torelli image of the Clover family $\mathcal{X}$ is isogenous to the point $E_{\mathrm{i}} \times E_{\mathrm{i}}$ in the moduli space $\mathcal{A}_{2,(1,2)}$ of abelian surfaces with $(1,2)$-polarisation, where E_{i} denotes the elliptic curve corresponding to the square torus $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\mathrm{i})$. Orbifold points on $W_D(4)$ of order 2 and 4 correspond to intersections with this family.*

In contrast, the image of the Windmill family under the Prym–Torelli map lies in the Shimura curve of discriminant 6. We show this by giving a precise description of the endomorphism ring of a general member of this family (see Proposition 1.4.6).

Theorem 1.1.3. *The closure of the Prym–Torelli image of the Windmill family $\mathcal{Y}$ in $\mathcal{A}_{2,(1,2)}$ is the (compact) Shimura curve parametrising $(1,2)$-polarised abelian surfaces with endomorphism ring isomorphic to the maximal order in the indefinite rational quaternion algebra of discriminant 6. Orbifold points of $W_D(4)$ of order 3 and 6 correspond to intersections with this family.*

The relationship between the Clover family, the Windmill family, and a Prym–Teichmüller curve is illustrated in Figure 1.1.

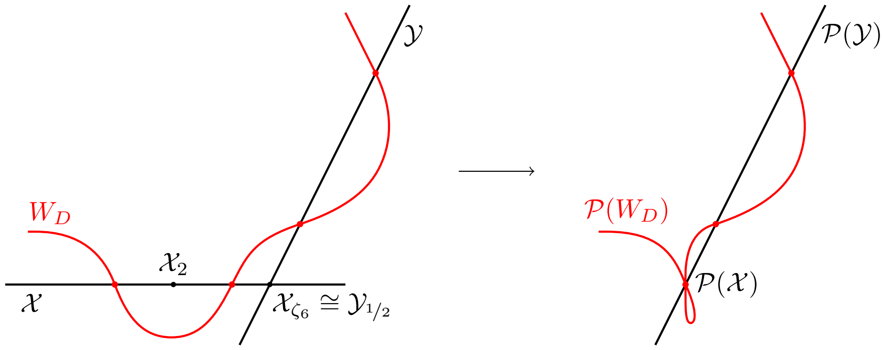

Figure 1.1.: The Clover family, the Windmill family, and the curve W_D inside $\mathcal{M}_3$ and their image under the Prym–Torelli map in $\mathcal{A}_{2,(1,2)}$.

In section 1.5, we finally determine the intersections of the Prym–Teichmüller curve $W_D(4)$ with the Clover family and the Windmill family by studying which points in their Prym–Torelli images admit real multiplication by the quadratic order $\mathcal{O}_D$ and by determining the corresponding eigenforms for this action. An immediate consequence is the following result.

Corollary 1.1.4. *The only Prym–Teichmüller curves in $\mathcal{M}_3$ with orbifold points of order 4 or 6 are $W_8(4)$ of genus zero with one cusp, one point of order 3 and one point of order 4, and $W_{12}(4)$ of genus zero with two cusps and one point of order 6.*

Note that our result extends that of Mukamel in [Muk14] to genus 3, although our approach and techniques differ in almost every detail. In the following we give a brief summary of the techniques used to classify orbifold points of Weierstraß curves in genus 2, to illustrate the similarities with and differences to our case.

The first difference is that, while in genus 2 all curves are hyperelliptic, this is never the case for genus 3 curves on Prym–Teichmüller curves by Lemma 1.2.7. Luckily, the Prym involution is a satisfactory substitute in all essential aspects. In particular, while Mukamel obtains restrictions on the types of orbifold points in genus 2 by observing the action on the Weierstraß points, we acquire an analogous result in genus 3 by relating symmetries of Prym forms to automorphisms of elliptic curves (Proposition 1.2.1).

At this point, however, the similarities between the genus 2 and 3 cases seem to end. Mukamel shows that the orbifold points on genus 2 Weierstraß curves correspond to curves admitting an embedding of the dihedral group D_8 into their automorphism group and whose Jacobians are therefore isogenous to products of elliptic curves that admit complex multiplication. He then identifies the space of genus 2 curves admitting a faithful D_8 action with the modular curve $\mathbb{H}/\Gamma_0(2)$. In this model, the curves admitting complex multiplication are well-known to correspond to the imaginary quadratic points in the fundamental domain. Thus counting orbifold points in genus 2 is equivalent to computing class numbers of imaginary quadratic fields, as in the case of Hilbert Modular Surfaces. Moreover, this period domain permits associating concrete flat surfaces to the orbifold points via his "pinwheel" construction.

By contrast, in genus 3, each orbifold point may lie on the Clover family *or* the Windmill family (Proposition 1.3.1). As mentioned above, these two cases behave quite differently. Moreover, in genus 3 we are no longer dealing with the entire Jacobian, but only with the Prym part, i.e. part of the Jacobian collapses and the remainder carries a non-principal $(1, 2)$ polarisation (see section 1.2). In particular, while in Mukamel's case the appearing abelian varieties could all be obtained by taking products of elliptic curves, in genus 3 one is forced to construct the Jacobians "from scratch" via Bolza's method (section 1.4). In addition, the whole Clover family collapses to a single point under the Prym–Torelli map, making it more difficult to keep track of the differentials. All this adds a degree of difficulty to pinpointing the actual intersection points of the Clover family and the Windmill family with a given $W_D(4)$. One consequence is that we obtain class numbers determining the number of orbifold points that are associated to slightly more involved quadratic forms (section 1.5).

Finally, we provide flat pictures of the orbifold points of order 4 and 6 in section 1.7.

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1.2. Orbifold points on Prym–Teichmüller curves

The aim of this section is to prove the following statement.

Proposition 1.2.1. *A flat surface (X, ω) parametrised by a point in $W_D(4)$ is an orbifold point of order n if and only if there exists $\sigma \in \mathrm{Aut}(X)$ of order $2n$ satisfying $\sigma^*\omega = \zeta_{2n}\omega$, where ζ_{2n} is some primitive order $2n$ root of unity.*

The different possibilities are listed in Table 1.1.

	ord(σ)	Branching data
(i)	4	$(0; 4, 4, 4, 4)$
(ii)	6	$(0; 2, 3, 3, 6)$
(iii)	8	$(0; 4, 8, 8)$
(iv)	12	$(0; 3, 4, 12)$

Table 1.1.: Possible orders of σ and their corresponding branching data.

Before proceeding with the proof, we briefly recall some notation and background information.

Orbifold Points If G is a finite group acting on a Riemann surface X of genus $g \geq 2$, we define the *branching data* (or signature of the action) as the signature of the orbifold quotient X/G, that is $\Sigma \coloneqq (\gamma; m_1, \ldots, m_r)$, where γ is the genus of the quotient X/G and the projection is branched over r points with multiplicities m_i.

Recall that an *orbifold point* of an orbifold $\mathbb{H}/\Gamma$ is the projection of a fixed point of the action of Γ, i.e. a point $s \in \mathbb{H}$ so that $\mathrm{Stab}_\Gamma(s) = \{A \in \Gamma : A \cdot s = s\}$ is strictly larger than the kernel of the action of Γ. Observe that this is equivalent to requiring the image of $\mathrm{Stab}_\Gamma(s)$ in $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{Aut}(\mathbb{H})$, which we denote by $\mathrm{PStab}_\Gamma(s)$, to be non-trivial. We call the cardinality of $\mathrm{PStab}_\Gamma(s)$ the *(orbifold) order* of s.

In the case of a Teichmüller curve, the close relationship between the uniformising group Γ and the affine structure of the fibers permits a characterisation of orbifold points in terms of flat geometry. To make this precise, we need some more notation.

Teichmüller curves Recall that a *flat surface* (X, ω) consists of a curve X together with a non-zero holomorphic differential form ω on X, which induces a flat structure by integration. Hence we may consider the moduli space of flat surfaces $\Omega\mathcal{M}_g$ as a bundle over the moduli space of genus g curves $\mathcal{M}_g$. Recall that there is a natural $\mathrm{SL}_2(\mathbb{R})$ action on $\Omega\mathcal{M}_g$ by shearing the flat structure, which respects – in particular – the zeros of the differentials. Every Teichmüller curve arises as the projection to $\mathcal{M}_g$ of the (closed) $\mathrm{SL}_2(\mathbb{R})$ orbit of some (X, ω). As $\mathrm{SO}(2)$ acts holomorphically on the fibers, we obtain the following commutative diagram

$$\begin{array}{ccc}
\mathrm{SL}_2(\mathbb{R}) & \xrightarrow{\quad F \quad} & \Omega\mathcal{M}_g \\
\downarrow & & \downarrow \\
\mathbb{H} \cong \mathrm{SO}(2)\backslash \mathrm{SL}_2(\mathbb{R}) & \xrightarrow{\quad f \quad} & \mathbb{P}\Omega\mathcal{M}_g \\
\downarrow & & \downarrow \pi \\
\mathcal{C} = \mathbb{H}/\Gamma & \longrightarrow & \mathcal{M}_g
\end{array}$$

where the map F is given by the action $A \mapsto A \cdot (X, \omega)$ and $\mathcal{C}$ is uniformised by

$$\Gamma = \mathrm{Stab}(f) \coloneqq \{A \in \mathrm{SL}_2(\mathbb{R}) : f(A \cdot t) = f(t)\ ,\ \forall t \in \mathbb{H}\} = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot \mathrm{SL}(X, \omega) \cdot \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right).$$

Here, $\mathrm{SL}(X, \omega)$ is the *affine group* of (X, ω), i.e. the derivatives of homeomorphisms of X that are affine with regard to the flat structure.

Given $t \in \mathbb{H}$, we will write $A_t \in \mathrm{SL}_2(\mathbb{R})$ for (a representative of) the corresponding element in $\mathrm{SO}(2)\backslash \mathrm{SL}_2(\mathbb{R})$ and (X_t, ω_t) for (a representative of) $f(t) = [A_t \cdot (X, \omega)] \in \mathbb{P}\Omega\mathcal{M}_g$.

For proofs and details, see e.g. [Möl11b], [Kuc12], [McM03].

In the following, we will be primarily interested in a special class of Teichmüller curves.

Prym–Teichmüller curves To ensure that the $\mathrm{SL}_2(\mathbb{R})$ orbit of a flat surface is not too large, the flat structure must possess sufficient real symmetries. McMullen observed that in many cases this can be achieved by requiring the Jacobian to admit real multiplication that "stretches" the differential. However, it turns out that for genus greater than 2, requiring the whole Jacobian to admit real multiplication is in general too strong a restriction to obtain infinite families (cf. e.g. [BHM16]).

More precisely, for positive $D \equiv 0, 1 \bmod 4$ non-square, we denote by $\mathcal{O}_D = \mathbb{Z}[T]/(T^2 + bT + c)$ with $D = b^2 - 4c$, the unique (real) quadratic order associated to D and say that a (polarised) abelian surface A has *real multiplication* by $\mathcal{O}_D$ if it admits an embedding $\mathcal{O}_D \hookrightarrow \mathrm{End}(A)$ that is self-adjoint with respect to the polarisation. We call the real multiplication by $\mathcal{O}_D$ *proper*, if the embedding cannot be extended to any quadratic order containing $\mathcal{O}_D$.

Now, consider a curve X with an involution ρ. The projection $\pi\colon X \to X/\rho$ induces a morphism $\mathrm{Jac}(\pi)\colon \mathrm{Jac}(X) \to \mathrm{Jac}(X/\rho)$ of the Jacobians and we call the kernel $\mathcal{P}(X, \rho)$ of $\mathrm{Jac}(\pi)$ the *Prym variety* associated to (X, ρ). In the following, we will always require the Prym variety to be 2-dimensional, hence the construction only works for X of genus 2, 3, 4 or 5. Denoting by $\Omega(X)^+$ and $\Omega(X)^-$ the $+1$ and -1-eigenspaces of $\Omega(X)$ with respect to ρ, and by $\mathrm{H}_1^+(X, \mathbb{Z})$ and $\mathrm{H}_1^-(X, \mathbb{Z})$ the corresponding intersections $\mathrm{H}_1(X, \mathbb{Z}) \cap (\Omega(X)^\pm)^\vee$, the Prym variety $\mathcal{P}(X, \rho)$ agrees with $(\Omega(X)^-)^\vee/\mathrm{H}_1^-(X, \mathbb{Z})$. Observe that, when X has genus 3, the Prym variety $\mathcal{P}(X, \rho)$ is no longer principally polarised but carries a $(1, 2)$-polarisation. See for instance [BL04, Chap. 12] or [Möl14] for details.

Starting with a flat surface (X, ω) where X admits an involution ρ satisfying $\rho^*\omega = -\omega$ and identifying $\mathrm{Jac}(X)$ with $\Omega(X)^\vee/\mathrm{H}_1(X, \mathbb{Z})$, the differential ω is mapped into the Prym part and hence, whenever $\mathcal{P}(X, \rho)$ has real multiplication by $\mathcal{O}_D$, we obtain an induced action of $\mathcal{O}_D$ on ω. We denote by $\mathcal{E}_D(2g - 2) \subset \Omega\mathcal{M}_g$ the space of (X, ω) such that

1. X admits an involution ρ such that $\mathcal{P}(X, \rho)$ is 2-dimensional,
2. the form ω has a single zero and satisfies $\rho^*\omega = -\omega$, and
3. $\mathcal{P}(X, \rho)$ admits proper real multiplication by $\mathcal{O}_D$ with ω as an eigenform,

and by $\mathbb{P}\mathcal{E}_D(2g - 2)$ the corresponding quotient by the $\mathrm{SO}(2)$ action. McMullen showed [McM03; McM06] that by defining $W_D(2g - 2)$ as the projection of the locus $\mathcal{E}_D(2g - 2)$ to $\mathcal{M}_g$, we obtain (possibly a union of) Teichmüller curves for every discriminant D in $\mathcal{M}_2$, $\mathcal{M}_3$ and $\mathcal{M}_4$. In the genus 2 case, the Prym involution is given by the hyperelliptic involution and the curve $W_D(2)$ is called the *Weierstraß curve*, while the curves $W_D(4)$ and $W_D(6)$ in $\mathcal{M}_3$ and $\mathcal{M}_4$, respectively, are known as *Prym–Teichmüller curves*. As we are primarily interested in the genus 3 case, we shall frequently refer to $W_D(4)$ simply by W_D.

We are now in a position to give a precise characterisation of orbifold points on Teichmüller curves in terms of flat geometry.

Proposition 1.2.2. *Let $\mathbb{H}/\Gamma$ be a Teichmüller curve generated by some $(X, \omega) = (X_\mathrm{i}, \omega_\mathrm{i})$. Then the following are equivalent.*

- *The point $t \in \mathbb{H}$ projects to an orbifold point in $\mathbb{H}/\Gamma$.*

- *There exists an elliptic matrix $C \in \mathrm{SL}(X,\omega)$, $C \neq \pm 1$ such that $A_t C A_t^{-1} \in \mathrm{SO}(2)$.*
- *The corresponding flat surface (X_t, ω_t) admits a (holomorphic) automorphism σ satisfying $[\sigma^* \omega_t] = [\omega_t]$ and $\sigma^* \omega_t \neq \pm \omega_t$.*

Proof. By the above correspondence, $t \in \mathbb{H}$ corresponds to some $(X_t, [\omega_t]) \in \mathbb{P}\Omega\mathcal{M}_g$ and equivalently to some $A_t \in \mathrm{SO}(2)\backslash \mathrm{SL}_2(\mathbb{R})$ with $[A_t \cdot (X,\omega)] = (X_t, [\omega_t])$.

Now, $C \in \mathrm{SL}(X,\omega)$ is in the stabiliser of A_t if and only if there exists $B \in \mathrm{SO}(2)$ such that

$$A_t C = B A_t, \text{ i.e. } A_t C A_t^{-1} \in \mathrm{SO}(2).$$

But then, by definition, $C \in \mathrm{SL}(X,\omega)$ is elliptic. Moreover, $C' := A_t C A_t^{-1}$ lies in $\mathrm{SL}(A_t \cdot (X,\omega)) = \mathrm{SL}(X_t, \omega_t)$, and as $C' \in \mathrm{SO}(2)$, the associated affine map is in fact a holomorphic automorphism σ of X_t. In particular, $\sigma^* \omega_t = \zeta \omega_t \in [\omega_t]$, where ζ is the corresponding root of unity.

Finally, observe that C acts trivially on $\mathrm{SO}(2)\backslash \mathrm{SL}_2(\mathbb{R})$ if and only if for every $A \in \mathrm{SL}_2(\mathbb{R})$ there exists $B \in \mathrm{SO}(2)$ so that

$$AC = BA, \text{ i.e. } ACA^{-1} \in \mathrm{SO}(2)\ \forall A \in \mathrm{SL}_2(\mathbb{R})$$

and this is the case if and only if $C = \pm 1$. □

Corollary 1.2.3. *There is a one-to-one correspondence between*

- *elements in* $\mathrm{Stab}_\Gamma(t)$,
- *elements in* $\mathrm{SL}\left(A_t \cdot (X,\omega)\right) \cap \mathrm{SO}(2)$, *and*
- *holomorphic automorphisms σ of X_t satisfying $\sigma^* \omega_t \in [\omega_t]$.*

Definition. *Given $(X, [\omega])$, we denote the group of automorphisms of $(X, [\omega])$, i.e. holomorphic automorphisms of X satisfying $\sigma^* \omega \in [\omega]$, by* $\mathrm{Aut}(X, [\omega])$.

In the case of Weierstraß and Prym–Teichmüller curves, we can say even more.

Corollary 1.2.4. *Let $W_D(2g-2)$ be as above, let $(X_t, [\omega_t]) \in \mathbb{P}\mathcal{E}_D(2g-2)$ correspond to an orbifold point and let σ be a non-trivial automorphism of $(X_t, [\omega_t])$. Let $\pi : X_t \to X_t/\sigma$ denote the projection. Then π has a totally ramified point.*

Proof. As $[\sigma^* \omega] = [\omega]$ and ω has a single zero, this must be a fixed point of σ, hence a totally ramified point. □

Note that the Prym–Teichmüller curves $W_D(4)$ and $W_D(6)$ lie entirely inside the branch locus of $\mathcal{M}_3$ and $\mathcal{M}_4$ respectively, as all their points admit involutions. In particular, the Prym involution ρ_t on each (X_t, ω_t) acts as -1, i.e. $\rho_t^* \omega_t = -\omega_t$, and therefore it does not give rise to orbifold points.

Corollary 1.2.5. *The Prym involution is the only non-trivial generic automorphism of $W_D(2g-2)$, i.e. the index* $[\mathrm{Stab}_\Gamma(s) : \mathrm{PStab}_\Gamma(s)]$ *is always* 2.

Moreover, Proposition 1.2.2 gives a strong restriction on the type of automorphisms inducing orbifold points.

Lemma 1.2.6. *The point in $W_D(2g-2)$ corresponding to a flat surface $(X,[\omega])$ is an orbifold point of order n if and only if $\operatorname{Aut}(X,[\omega])$ is generated by an automorphism σ of order $2n$. Moreover, σ^n is the Prym involution.*

Proof. Let $Q \in X$ be the (unique) zero of ω. By the above, the automorphisms of $(X,[\omega])$ lie in the Q-stabiliser of $\operatorname{Aut}(X)$. But these are (locally) rotations around Q, hence the stabiliser (and consequently $\operatorname{Aut}(X,[\omega])$) is cyclic. As the Prym involution ρ is an automorphism of $(X,[\omega])$, the group $\operatorname{Aut}(X,[\omega])$ has even order.

Conversely, any automorphism σ fixing Q satisfies $[\sigma^*\omega]=[\omega]$. The remaining claims follow from Corollary 1.2.5. □

To determine the number of branch points in the genus 3 case, we start with the following observation (cf. [Möl14, Lemma 2.1]).

Lemma 1.2.7. *The curve W_D is disjoint from the hyperelliptic locus in $\mathcal{M}_3$.*

Proof. Let $(X,[\omega])$ correspond to a point on W_D, denote by ρ the Prym involution on X and assume that X is hyperelliptic with involution σ. As X is of genus 3, $\sigma \neq \rho$. But σ commutes with ρ and therefore $\tau := \sigma \circ \rho$ is another involution.

Recall that σ acts by -1 on all of $\Omega(X)$ and its decomposition into ρ-eigenspaces $\Omega(X)^{\pm}$. The -1 eigenspace of τ is therefore $\Omega(X)^+$ and the $+1$ eigenspace is $\Omega(X)^-$. In particular, any Prym form on X is τ invariant, i.e. a pullback from X/τ.

However, by checking the dimensions of the eigenspaces, we see that X/τ is of genus 2, hence $X \to X/\tau$ is unramified by Riemann-Hurwitz and we cannot obtain a form with a fourfold zero on X by pullback, i.e. $(X,\omega) \notin \mathcal{E}_D(4)$, a contradiction. □

We now have all we need to prove Proposition 1.2.1.

Proof of Proposition 1.2.1. Starting with Proposition 1.2.2 and Lemma 1.2.6, observe that, since $\sigma \neq \rho$, the automorphism σ descends to a non-trivial automorphism $\overline{\sigma}$ of the elliptic curve X/ρ. Since it has at least one fixed point, $X/\sigma \cong \mathbb{P}^1$ and it is well-known that $\overline{\sigma}$ can only be of order 2, 3, 4 or 6.

For the number of ramification points, since X has genus 3, by Riemann-Hurwitz

$$4 = -4n + 2n \sum_{d \mid 2n} \left(1 - \frac{1}{d}\right) e_d\,,$$

where e_d is the number of points over which σ ramifies with order d. A case by case analysis using Lemma 1.2.7 shows that the only possibilities are those listed in Table 1.1. □

Remark 1.2.8. *Automorphism groups of genus 3 curves were classified by Komiya and Kuribayashi in [KK79] (P. Henn studied them even earlier in his PhD dissertation [Hen76]). One can also find a complete classification of these automorphism groups together with their branching data in [Bro91, Table 5], including all the information in our Table 1.1.*

1.3. Cyclic covers

Proposition 1.2.1 classified orbifold points of W_D in terms of automorphisms of the complex curve. The aim of this section is to express these conditions as intersections of W_D with certain families of cyclic covers of $\mathbb{P}^1$ in $\mathcal{M}_3$.

Let $\mathcal{X} \to \mathbb{P}^* := \mathbb{P}^1 - \{0, 1, \infty\}$ be the family of projective curves with affine model

$$\mathcal{X}_t \colon y^4 = x(x-1)(x-t)$$

and $\mathcal{Y} \to \mathbb{P}^*$ the family of projective curves with affine model

$$\mathcal{Y}_t \colon y^6 = x^2(x-1)^2(x-t)^3.$$

The family $\mathcal{X}$ has been intensely studied, notably in [Guà01] and [HS08]. In fact, it is even a rare example of a curve that is both a Shimura and a Teichmüller curve (cf. [Möl11a], see Remark 1.3.7 below). Because of the flat picture of its fibers (cf. section 1.7), we will refer to it as the Clover family.

The family $\mathcal{Y}$ is related to the Shimura curve of discriminant 6, which has been studied for instance in [Voi09] and [PS11]. We will refer to it as the Windmill family, again as a reference to the flat picture (cf. section 1.7).

Proposition 1.3.1. *If $(X, [\omega])$ corresponds to an orbifold point on W_D then X is isomorphic to some fiber of $\mathcal{X}$ or $\mathcal{Y}$.*

Moreover, this orbifold point is of order six if and only if X is isomorphic to $\mathcal{X}_{\zeta_6} \cong \mathcal{Y}_{1/2}$ the (unique) intersection point of $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{M}_3$; it is of order four if and only if X is isomorphic to $\mathcal{X}_{-1}$; it is of order two if it corresponds to a generic fiber of $\mathcal{X}$ and of order three if it corresponds to a generic fiber of $\mathcal{Y}$.

To state the converse, we need to pick a Prym eigenform on the appropriate fibers of $\mathcal{X}$ and $\mathcal{Y}$.

First, let us briefly review some well-known facts on the theory of cyclic coverings which will be applicable to both the Clover family $\mathcal{X}$ and the Windmill family $\mathcal{Y}$. For more background and details, see for example [Roh09].

Consider the family $\mathcal{Z} \to \mathbb{P}^*$ of projective curves with affine model

$$\mathcal{Z}_t \colon y^d = x^{a_1}(x-1)^{a_2}(x-t)^{a_3}\,.$$

Since we will be interested in coverings of $\mathbb{P}^1$ ramified over 4 points, we can choose the exponent a_4 at ∞ so that $\sum a_i \equiv 0 \mod d$, with $0 < a_i < d$. Moreover, we will suppose $\gcd(a_1, a_2, a_3, a_4, d) = 1$ so that the curve is connected. Note that any (connected) family of cyclic covers, ramified over four points, may be described in this way.

Let us define $g_i = \gcd(a_i, d)$, for $i = 1, \dots, 4$. For each fiber $\mathcal{Z}_t$, the map $\pi_t = \pi\colon (x, y) \mapsto x$ yields a cover $\mathcal{Z}_t \to \mathbb{P}^1$ of degree d ramified over 0, 1, t and ∞ with branching orders d/g_1, d/g_2, d/g_3 and d/g_4 respectively. Then, by Riemann-Hurwitz, the genus of $\mathcal{Z}_t$ is $d + 1 - (\sum_{i=1}^{4} g_i)/2$.

Note that the number of preimages of 0, 1, t and ∞ is g_1, g_2, g_3 and g_4 respectively. Denote for instance $\pi^{-1}(0) = \{P_j\}$, with $j = 0, \dots, g_1 - 1$. The following map

$$z \mapsto \left(z^{\frac{d}{g_1}}, \zeta_d^j z^{\frac{a_1}{g_1}} \sqrt[d]{(z^{\frac{d}{g_1}} - 1)^{a_2} (z^{\frac{d}{g_1}} - t)^{a_3}} \right), \qquad |z| < \varepsilon \tag{1.2}$$

gives a parametrisation of a neighbourhood of P_j. In a similar way, one can find local parametrisations around the preimages of the rest of the branching values.

The map π corresponds to the quotient $\mathcal{Z}_t / \langle \alpha^{\mathcal{Z}} \rangle$ by the action of the cyclic group of order d generated by the automorphism

$$\alpha^{\mathcal{Z}} := \alpha_t^{\mathcal{Z}} \colon (x, y) \mapsto (x, \zeta_d y), \tag{1.3}$$

where $\zeta_d = \exp(2\pi \mathrm{i}/d)$. When there is no ambiguity we will simply write α for $\alpha_t^{\mathcal{Z}}$. Since these automorphisms will be of great importance, we emphasise their expression in our particular cases.

Definition. *The automorphism*

$$\begin{array}{rccc} \alpha_t^{\mathcal{X}}\colon & \mathcal{X}_t & \to & \mathcal{X}_t \\ & (x, y) & \mapsto & (x, \zeta_4 y) \end{array}$$

generates a group of order 4 acting on $\mathcal{X}_t$*.*

The automorphism

$$\begin{array}{rccc} \alpha_t^{\mathcal{Y}}\colon & \mathcal{Y}_t & \to & \mathcal{Y}_t \\ & (x, y) & \mapsto & (x, \zeta_6 y) \end{array}$$

generates a group of order 6 acting on $\mathcal{Y}_t$*.*

By Lemma 1.2.6, the Prym involutions are given by

$$\begin{aligned} \rho^{\mathcal{X}} &:= \rho_t^{\mathcal{X}} := (\alpha^{\mathcal{X}})^2 \colon (x, y) \mapsto (x, -y), \text{ and} \\ \rho^{\mathcal{Y}} &:= \rho_t^{\mathcal{Y}} := (\alpha^{\mathcal{Y}})^3 \colon (x, y) \mapsto (x, -y). \end{aligned}$$

We will denote by $\mathcal{P}(\mathcal{X}_t)$ and $\mathcal{P}(\mathcal{Y}_t)$ the corresponding Prym varieties.

Note that different fibers of the families $\mathcal{X}$ and $\mathcal{Y}$ can be isomorphic.

In fact, in the case of the Clover family $\mathcal{X}$ any isomorphism $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ preserving the set $\{0, 1, \infty\}$ lifts to isomorphisms $\mathcal{X}_t \cong \mathcal{X}_{\phi(t)}$ for each t. As a consequence, our family is parametrised by $\mathbb{P}^*/\mathfrak{S}_3$, where we take the symmetric group $\mathfrak{S}_3$ to be generated by $z \mapsto 1 - z$ and $z \mapsto 1/z$. The corresponding modular maps yield curves in $\mathcal{M}_3$ and $\mathcal{A}_3$.

As for the Windmill family $\mathcal{Y}$, for each $t \in \mathbb{P}^*$ the curves $\mathcal{Y}_t$ and $\mathcal{Y}_{1-t}$ are isomorphic via the map $(x, y) \mapsto (1 - x, \zeta_{12} y)$, which induces the automorphism $z \mapsto 1 - z$ on $\mathbb{P}^1$. Since any isomorphism between fibers $\mathcal{Y}_t$ and $\mathcal{Y}_{t'}$ must descend to an isomorphism of $\mathbb{P}^1$ interchanging branching values of the same order, it is clear that no other two fibers are isomorphic, and therefore the family is actually parametrised by $\mathbb{P}^*/\sim$, where $z \sim 1 - z$. In subsection 1.4.2 we will give a more explicit description of this family in terms of its Prym–Torelli image.

The discussion above proves the following.

Lemma 1.3.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be the families defined above.*

1. *The map $\mathbb{P}^* \to \mathcal{M}_3$, $t \mapsto \mathcal{X}_t$ is of degree 6. It ramifies over $\mathcal{X}_{-1}$ that has 3 preimages $\{\mathcal{X}_t : t = -1, 1/2, 2\}$ and $\mathcal{X}_{\zeta_6}$ that has 2 preimages $\{\mathcal{X}_t : t = \zeta_6^{\pm 1}\}$.*

   The only fibers with a cyclic group of automorphisms or order larger than 4 are $\mathcal{X}_{-1}$ that admits a cyclic group of order 8 and $\mathcal{X}_{\zeta_6}$ that admits a cyclic group of order 12.

2. *The map $\mathbb{P}^* \to \mathcal{M}_3$, $t \mapsto \mathcal{Y}_t$ is of degree 2. It ramifies only over $\mathcal{Y}_{1/2}$ that has a single preimage.*

   The only fiber with a cyclic group of automorphisms of order larger than 6 is $\mathcal{Y}_{1/2}$ that admits a cyclic group of order 12.

Proof of Proposition 1.3.1. If $(X, [\omega])$ corresponds to an orbifold point on W_D, then X must belong to one of the families in Table 1.1.

First of all, note that curves of type (iii) admit an automorphism of order 4 with branching data $(0; 4, 4, 4, 4)$, and therefore they also belong to family (i). Similarly, those of type (iv) admit automorphisms of order 4 and 6 with branching data $(0; 4, 4, 4, 4)$ and $(0; 2, 3, 3, 6)$ respectively, and therefore they belong both to families (i) and (ii). As a consequence we can suppose that X belongs either to (i) or (ii).

Let us suppose that X is of type (i). Looking at the branching data, one can see that X is necessarily isomorphic to one of the following two curves for some $t \in \mathbb{P}^*$

$$y^4 = x(x-1)(x-t)\,,$$
$$y^4 = x^3(x-1)^3(x-t)\,.$$

However curves of the second kind are always hyperelliptic, with hyperelliptic involution given by

$$\tau\colon (x, y) \mapsto \left(\frac{tx - t}{x - t}, t(t-1)\frac{y}{(x-t)^2}\right).$$

As points of W_D cannot correspond to hyperelliptic curves by Lemma 1.2.7, the curve X is necessarily isomorphic to some $\mathcal{X}_t$.

If X is of type (ii), the branching data tells us that X must be isomorphic to some fiber $\mathcal{Y}_t$.

The claim about the order of the orbifold points follows from Lemma 1.2.6 and Lemma 1.3.2. □

Remark 1.3.3. *Let us note here that the special fiber $\mathcal{X}_{-1}$ is isomorphic to the Fermat curve $x^4 + y^4 + z^4 = 0$ and that the unique intersection point of the Clover family and the Windmill family, that is $\mathcal{X}_{\zeta_6} \cong \mathcal{Y}_{1/2}$, is isomorphic to the exceptional Wiman curve of genus 3 with affine equation $y^3 = x^4 + 1$.*

1.3.1. Differential forms

By the considerations in section 1.2, we are only interested in differential forms with a single zero in a fixed point of the Prym involution.

Lemma 1.3.4. *Let $t \in \mathbb{P}^*$.*

1. *The space of holomorphic 1-forms on each fiber $\mathcal{X}_t$ of the Clover family is generated by the following eigenforms for the action of $\alpha^{\mathcal{X}}$:*

$$\omega_1^{\mathcal{X}} = \frac{\mathrm{d}x}{y^3}, \qquad \omega_2^{\mathcal{X}} = \frac{x\mathrm{d}x}{y^3}, \qquad \omega_3^{\mathcal{X}} = \frac{\mathrm{d}x}{y^2}.$$

   In particular, the spaces of odd and even forms for the action of $\rho^{\mathcal{X}}$ are given by $\Omega(\mathcal{X}_t)^- = \langle \omega_1^{\mathcal{X}}, \omega_2^{\mathcal{X}} \rangle$ and $\Omega(\mathcal{X}_t)^+ = \langle \omega_3^{\mathcal{X}} \rangle$, respectively.

2. *The space of holomorphic 1-forms on each fiber $\mathcal{Y}_t$ of the Windmill family is generated by the following eigenforms for the action of $\alpha^{\mathcal{Y}}$:*

$$\omega_1^{\mathcal{Y}} = \frac{\mathrm{d}x}{y}, \qquad \omega_2^{\mathcal{Y}} = \frac{y\mathrm{d}x}{x(x-1)(x-t)}, \qquad \omega_3^{\mathcal{Y}} = \frac{y^4\mathrm{d}x}{x^2(x-1)^2(x-t)^2}.$$

   In particular, the spaces of odd and even forms for the action of $\rho^{\mathcal{Y}}$ are given by $\Omega(\mathcal{Y}_t)^- = \langle \omega_1^{\mathcal{Y}}, \omega_2^{\mathcal{Y}} \rangle$ and $\Omega(\mathcal{Y}_t)^+ = \langle \omega_3^{\mathcal{Y}} \rangle$, respectively.

Proof. By writing their local expressions, one can check that all these forms are holomorphic. The action of ρ can be checked in the affine coordinates. □

By analysing the zeroes one obtains the following lemma.

Lemma 1.3.5. *Let $t \in \mathbb{P}^*$.*

1. *The forms in $\mathbb{P}\Omega(\mathcal{X}_t)^-$ having a 4-fold zero at a fixed point of $\rho^{\mathcal{X}}$ are*
   - *$\omega_1^{\mathcal{X}}$ which has a zero at the preimage of ∞,*
   - *$\omega_2^{\mathcal{X}}$ which has a zero at the preimage of 0,*
   - *$-\omega_1^{\mathcal{X}} + \omega_2^{\mathcal{X}}$ which has a zero at the preimage of 1, and*
   - *$-t\omega_1^{\mathcal{X}} + \omega_2^{\mathcal{X}}$ which has a zero at the preimage of t.*

   They all form an orbit under $\mathrm{Aut}(\mathcal{X}_t)$.

2. *For $t \neq 1/2$, the only form in $\mathbb{P}\Omega(\mathcal{Y}_t)^-$ which has a 4-fold zero at a fixed point of $\rho^{\mathcal{Y}}$ is $\omega_2^{\mathcal{Y}}$.*

Note that by Lemma 1.3.2 the case of $\mathcal{Y}_t$ for $t = 1/2$ is already included in the study of the $\mathcal{X}$ family (see also Remark 1.3.3).

Proof. 1. For any $\mathcal{X}_t$, the preimages of 0, 1, t and ∞ are the only fixed points of $\rho^{\mathcal{X}}$. Using local charts, it is easy to see that these are the only forms with 4-fold zeroes at those points.

The last statement follows from the fact that $\mathrm{Aut}(\mathcal{X}_t)$ permutes the preimages of 0, 1, t and ∞.

2. Observe that the differential $\mathrm{d}x$ does not vanish on $\mathcal{Y}_t$ away from the preimages of 0, 1, t and ∞. Under the parametrisations Equation 1.2, the local expression of $\mathrm{d}x$ around the preimages of 0 and 1 is $\mathrm{d}x = 3z^2\mathrm{d}z$ and around the preimages of t is $\mathrm{d}x = 2z\mathrm{d}z$. Looking at the local expressions, one can see that $\omega_1^{\mathcal{Y}}$ has simple zeroes at the (four) preimages of 0 and 1, and $\omega_2^{\mathcal{Y}}$ has a 4-fold zero at infinity.

Again using local charts, it is easy to see that a form $u\omega_1^{\mathcal{Y}} + v\omega_2^{\mathcal{Y}}$, $u, v \in \mathbb{C}$, can have at most 2-fold zeroes at the preimages of t.

On the other hand, if $u\omega_1^{\mathcal{Y}} + v\omega_2^{\mathcal{Y}}$ has a 4-fold zero at ∞, then the local expression above implies that $u = 0$. □

We can now state the converse of Proposition 1.3.1.

Proposition 1.3.6. *Let $t \in \mathbb{P}^*$ and let $\mathcal{O}_D$ be some real quadratic order.*

1. *If $\mathcal{P}(\mathcal{X}_t)$ admits proper real multiplication by $\mathcal{O}_D$ with $\omega_1^{\mathcal{X}}$ as an eigenform then $\omega_2^{\mathcal{X}}$, $-\omega_1^{\mathcal{X}} + \omega_2^{\mathcal{X}}$ and $-t\omega_1^{\mathcal{X}} + \omega_2^{\mathcal{X}}$ are also eigenforms and $(\mathcal{X}_t, \omega_1^{\mathcal{X}})$ corresponds to an orbifold point on W_D.*

   Moreover, if $\mathcal{X}_t \cong \mathcal{X}_{-1}$, then $(\mathcal{X}_t, \omega_1^{\mathcal{X}})$ is of order 4; if $\mathcal{X}_t \cong \mathcal{X}_{\zeta_6}$, then $(\mathcal{X}_t, \omega_1^{\mathcal{X}})$ is of order 6; otherwise, $(\mathcal{X}_t, \omega_1^{\mathcal{X}})$ is of order 2.

2. *If $\mathcal{P}(\mathcal{Y}_t)$ admits proper real multiplication by $\mathcal{O}_D$ with $\omega_2^{\mathcal{Y}}$ as an eigenform then $(\mathcal{Y}_t, \omega_2^{\mathcal{Y}})$ corresponds to an orbifold point on W_D.*

   Moreover, if $\mathcal{Y}_t = \mathcal{Y}_{1/2}$, then $(\mathcal{Y}_t, \omega_2^{\mathcal{Y}})$ is of order 6; otherwise, $(\mathcal{Y}_t, \omega_2^{\mathcal{Y}})$ is of order 3.

Proof. By the previous lemma, if one of the four forms on $\mathcal{X}_t$ is an eigenform for some choice of real multiplication $\mathcal{O}_D \hookrightarrow \mathrm{End}\,\mathcal{P}(\mathcal{X}_t)$, then the other three are also eigenforms for the choice of real multiplication conjugate by the corresponding automorphism. The statements about the points of higher order follow from Lemma 1.3.9 and Lemma 1.3.10.

The rest of the claims follows from Proposition 1.2.1 and Lemma 1.3.2. □

Remark 1.3.7. *Note that, while the Clover family $\mathcal{X}$ is the same curve inside $\mathcal{M}_3$ that is studied in [HS08] and [Möl11a], the flat structures we consider on the fibers are different and the families are actually disjoint in $\Omega\mathcal{M}_3$. More precisely, we are interested in Prym–Teichmüller curves, i.e. a differential in the -1 eigenspace for the Prym involution, while the Wollmilchsau Teichmüller curve is constructed as a cover of the elliptic curve $\mathcal{X}_t/\rho$, i.e. has the flat structure of the differential in the $+1$ eigenspace. In particular, for our choices of differential $(\mathcal{X}_t, \omega_t)$, the (projection of the) $\mathrm{SL}_2(\mathbb{R})$ orbit will never be the curve $\mathcal{X}$, but the Prym–Teichmüller curve $W_D(4)$ whenever the real-multiplication condition is satisfied.*

1.3.2. Homology To calculate the Jacobians of the fibers of the Clover family $\mathcal{X}$ and the Windmill family $\mathcal{Y}$, we also need a good understanding of their homology.

Consider again the general family $\mathcal{Z} \to \mathbb{P}^*$ introduced at the beginning of section 1.3. Set $\mathbb{P}^*_t := \mathbb{P}^* - \{t\}$ and $\mathcal{Z}^*_t := \pi^{-1}(\mathbb{P}^*_t)$, where $\pi\colon \mathcal{Z}_t \to \mathbb{P}^1$ is the projection onto the x coordinate. We thus obtain an unramified cover and the sequence

$$1 \to \pi_1(\mathcal{Z}^*_t) \to \pi_1(\mathbb{P}^*_t) \to C_d \to 1,$$

where C_d denotes the cyclic group of order d, is exact. Let σ_P denote a simple counter-clockwise loop around the point $P \in \mathbb{P}^1$. Then $\pi_1(\mathbb{P}^*_t)$ is generated by $\sigma_0, \sigma_1, \sigma_t$ and σ_∞ and their product is trivial. Observe that these four loops are mapped to elements of order $d/\gcd(a_1,d)$, $d/\gcd(a_2,d)$, $d/\gcd(a_3,d)$ and $d/\gcd(a_4,d)$ in C_d respectively. Moreover, cycles in $\pi_1(\mathbb{P}^*_t)$ whose image in C_d is trivial lift to cycles in $\mathrm{H}_1(\mathcal{Z}_t,\mathbb{Z})$.

For cycles $F, G \in \mathrm{H}_1(\mathcal{Z}_t,\mathbb{Z})$, we pick representatives intersecting at most transversely and define the intersection number $F \cdot G := \sum F_p \cdot G_p$, where the sum is taken over all $p \in F \cap G$ and for any such p, we define $F_p \cdot G_p := +1$ if G approaches F "from the right in the direction of travel" and $F_p \cdot G_p := -1$ otherwise, cf. Figure 1.2.

In the following, we identify $\mathrm{Gal}(\mathcal{Z}_t/\mathbb{P}^1) = C_d$ with the dth complex roots of unity and choose the generator α as $\exp(2\pi\mathrm{i}/d)$ (this generator corresponds to $\alpha^{\mathcal{Z}}$ in Equation 1.3, hence the notation). Since all the fibers are topologically equivalent, let us suppose for simplicity $t \in \mathbb{R}$, $t > 1$. Then, the simply-connected set $\mathbb{P}^1 - [0,\infty]$ contains no ramification points and therefore has d disjoint preimages $S_1, \ldots, S_d$, which we call *sheets* of $\mathcal{Z}_t$. These are permuted transitively by α and we choose the numbering so that $\alpha(S_{[n]}) = S_{[n+1]}$, where $[n] := n \bmod d$. The sheet changes are given by the monodromy: a path travelling around 0 in a counter-clockwise direction on sheet $[n]$ continues onto sheet $[n + a_0]$ after crossing the interval $(0,1)$ and similarly for the other branch points.

We are now in a position to explicitly describe the fiberwise homology of $\mathcal{X}$ and $\mathcal{Y}$.

Let $F^{\mathcal{X}}$ denote the lift of $\sigma_1^{-1}\sigma_0$ that starts on sheet number 1 of $\mathcal{X}_t$ and let $G^{\mathcal{X}}$ denote the lift of $\sigma_t^{-1}\sigma_1$ that also starts on sheet 1 (see Figure 1.2). Observe that $F^{\mathcal{X}} \cdot G^{\mathcal{X}} = +1$.

Similarly, denote by $F^{\mathcal{Y}}$ and $G^{\mathcal{Y}}$ the lifts of $\sigma_1^{-1}\sigma_0$ and $\sigma_\infty^{-3}\sigma_t$, that start on sheet 1 and 5 of $\mathcal{Y}_t$, respectively (see Figure 1.3). Observe that $F^{\mathcal{Y}} \cdot G^{\mathcal{Y}} = 0$.

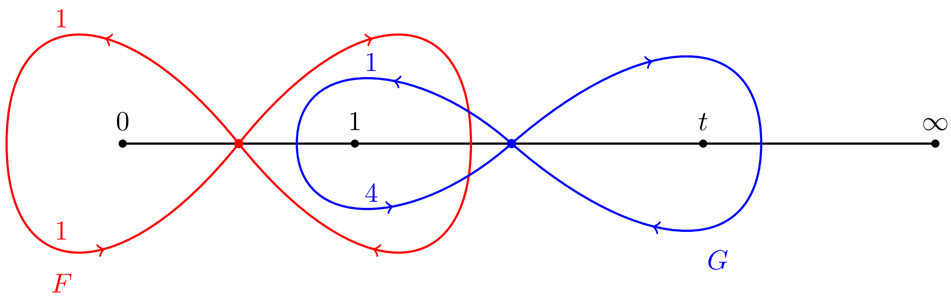


Figure 1.2.: The cycles $F^{\mathcal{X}}$ and $G^{\mathcal{X}}$ on $\mathcal{X}_t$. The upper-left parts of both cycles lie on sheet number 1. Observe that $F^{\mathcal{X}} \cdot G^{\mathcal{X}} = 1$.

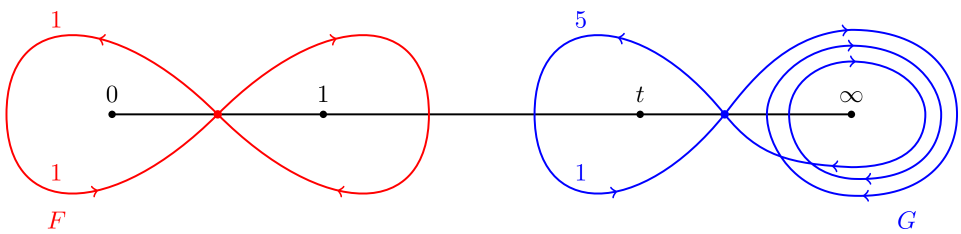


Figure 1.3.: The cycles $F^{\mathcal{Y}}$ and $G^{\mathcal{Y}}$ on $\mathcal{Y}_t$. The upper-left parts lie on sheets number 1 and 5, respectively.

For ease of notation, we will drop superscripts in the following lemma, as no confusion can arise.

Lemma 1.3.8. *Let $t \in \mathbb{P}^*$.*

1. *The cycles $F, \alpha F, \alpha^2 F, G, \alpha G, \alpha^2 G$ yield a basis of $\mathrm{H}_1(\mathcal{X}_t, \mathbb{Z})$. Moreover, the cycles*
$$F + \alpha F + G + \alpha G,\ -G + \alpha^2 G,\ \alpha F + \alpha^2 F - G + \alpha^2 G,\ F + 2\alpha F + \alpha^2 F$$
span a $(1,2)$-polarised, ρ-anti-invariant sublattice of $\mathrm{H}_1(\mathcal{X}_t, \mathbb{Z})$, which we denote by $\mathrm{H}_1^-(\mathcal{X}_t, \mathbb{Z})$. The complementary ρ-invariant sublattice, $\mathrm{H}_1^+(\mathcal{X}_t, \mathbb{Z})$, is spanned by $F + \alpha^2 F, G + \alpha^2 G$.

2. *The cycles $F, \alpha F, \alpha^3 F, \alpha^4 F, G, \alpha G$ yield a basis of $\mathrm{H}_1(\mathcal{Y}_t, \mathbb{Z})$. Moreover, the cycles*
$$F - \alpha^3 F,\ \alpha^4 F - \alpha F,\ G,\ \alpha G$$
span a $(1,2)$-polarised, ρ-anti-invariant sublattice of $\mathrm{H}_1(\mathcal{Y}_t, \mathbb{Z})$, which we denote by $\mathrm{H}_1^-(\mathcal{Y}_t, \mathbb{Z})$. The complementary ρ-invariant sublattice, $\mathrm{H}_1^+(\mathcal{Y}_t, \mathbb{Z})$, is spanned by $F + \alpha^3 F, \alpha^4 F + \alpha F$.

Proof. *1.* An elementary but somewhat tedious calculation yields the intersection matrix
$$\begin{pmatrix} 0 & 1 & 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 & 0 \end{pmatrix}$$

for the above cycles on $\mathcal{X}_t$. As it has rank 6 and determinant 1, these cycles span all of $\mathrm{H}_1(\mathcal{X}_t, \mathbb{Z})$. Furthermore, this immediately provides us with the relations

$$\alpha^3 F = -F - \alpha F - \alpha^2 F \quad \text{and} \quad \alpha^3 G = -G - \alpha G - \alpha^2 G,$$

which confirms the claimed anti-invariance. The change to the second set of cycles yields

$$\begin{pmatrix} 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 2 & & \\ -1 & 0 & 0 & 0 & & \\ 0 & -2 & 0 & 0 & & \\ & & & & 0 & 2 \\ & & & & -2 & 0 \end{pmatrix}$$

where the upper-left block is the anti-invariant and the lower-right block is the invariant part. Calculating determinants, we see that both blocks have determinant 4, proving the claim about the polarisation.

2. Proceeding as before, one finds the following intersection matrix for the cycles on $\mathcal{Y}_t$

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

proving that they generate $\mathrm{H}_1(\mathcal{Y}_t, \mathbb{Z})$, and the following one for the second set of cycles

$$\begin{pmatrix} 0 & 2 & 0 & 0 & & \\ -2 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 1 & & \\ 0 & 0 & -1 & 0 & & \\ & & & & 0 & 2 \\ & & & & -2 & 0 \end{pmatrix},$$

yielding the $(1, 2) \times (2)$-polarisation on the product. □

1.3.3. Special points We briefly summarise some of the subtleties occurring at those points admitting additional symmetries.

The curve $\mathcal{X}_2$ In the Clover family $\mathcal{X}$, the fibers over $1/2$, -1 and 2 form an orbit under the action of $\mathfrak{S}_3$. Over these points, $\alpha^{\mathcal{X}}$ extends to an automorphism $\beta^{\mathcal{X}}$ satisfying $(\beta^{\mathcal{X}})^2 = \alpha^{\mathcal{X}}$, i.e. a symmetry of order 8, making them all isomorphic to the well-known *Fermat curve*. More precisely, $\beta^{\mathcal{X}}$ may be obtained by lifting the automorphism that permutes two of the branch points and fixes the remaining pair on $\mathbb{P}^1$. Note that this may be achieved in two ways, e.g. for $t = 2$, we obtain

$$\beta_1^{\mathcal{X}} : (x, y) \mapsto \left(\frac{x}{x-1}, \zeta_8 \frac{y}{x-1}\right) \text{ and}$$
$$\beta_2^{\mathcal{X}} : (x, y) \mapsto (2 - x, \zeta_8 y).$$

Observe that $\beta_1^{\mathcal{X}}$ fixes 2 and 0 while interchanging 1 and ∞, while $\beta_2^{\mathcal{X}}$ fixes 1 and ∞ while interchanging 2 and 0. It is straight-forward to check the analogous statement of Lemma 1.3.5 in this case.

Lemma 1.3.9. *Let t be one of $1/2$, -1 or 2. Then the two forms from Lemma 1.3.5 with zeros at the fixed points of $\beta_1^{\mathcal{X}}$ are eigenforms for $\beta_1^{\mathcal{X}}$, while the other two forms are eigenforms for $\beta_2^{\mathcal{X}}$.*

The curve $\mathcal{Y}_{1/2}$ (or $\mathcal{X}_{\zeta_6}$) The only member of the Windmill family $\mathcal{Y}$ whose automorphism group contains a cyclic group of order larger than 6 is $\mathcal{Y}_{1/2}$, admitting an automorphism of order 12, $\beta^{\mathcal{Y}}(x, y) = (1 - x, \zeta_{12}^7 y)$, satisfying $(\beta^{\mathcal{Y}})^2 = \alpha^{\mathcal{Y}}$. In contrast to the case of $\mathcal{X}_2$, however, the automorphism $\beta^{\mathcal{Y}}$ generates the full automorphism group.

Recall that, by Proposition 1.3.1, the curve $\mathcal{Y}_{1/2}$ is isomorphic to the curve $\mathcal{X}_{\zeta_6}$ of the Clover family. However, here we will use the model of the curve as a member of the Windmill family.

Note first that $\beta^{\mathcal{Y}}$ descends to the automorphism $z \mapsto 1 - z$ of $\mathbb{P}^1$. Moreover $\beta^{\mathcal{Y}}$ fixes ∞ with rotation number ζ_{12} and therefore $\beta^{\mathcal{Y}}$ acts as $(1^+, 1^-, 2^+, \ldots, 6^+, 6^-)$ on the half-sheets, where we write k^+ (respectively k^-) for the upper half-plane (respectively lower half-plane) corresponding to the kth sheet.

By letting the initial points of $F^{\mathcal{Y}}$ and $G^{\mathcal{Y}}$ go to 1 and ∞, respectively, and shrinking the cycles around the preimages of 0, 1, t and ∞ one can use the (equivalent) choice of cycles pictured in Figure 1.4.

After the shrinking process, the cycles $F^{\mathcal{Y}}$ and $G^{\mathcal{Y}}$ in $\mathcal{Y}_{1/2}$ have the shape depicted in Figure 1.5.

Taking all this into account, one can easily calculate the analytic and rational representations of $\beta^{\mathcal{Y}}$.

Lemma 1.3.10. *The analytic and rational representations of $\beta^{\mathcal{Y}}$ with respect to the bases $\mathrm{H}_1(\mathcal{Y}_{1/2}, \mathbb{Z}) = \langle F^{\mathcal{Y}}, \alpha^{\mathcal{Y}} F^{\mathcal{Y}}, (\alpha^{\mathcal{Y}})^3 F^{\mathcal{Y}}, (\alpha^{\mathcal{Y}})^4 F^{\mathcal{Y}}, G^{\mathcal{Y}}, \alpha^{\mathcal{Y}} G^{\mathcal{Y}} \rangle_{\mathbb{Z}}$ and $\Omega(\mathcal{Y}_{1/2}) = \langle \omega_1^{\mathcal{Y}}, \omega_2^{\mathcal{Y}}, \omega_3^{\mathcal{Y}} \rangle$ are given, respectively, by*

$$A_{\beta^{\mathcal{Y}}} = \begin{pmatrix} \zeta_{12}^{-1} & 0 & 0 \\ 0 & \zeta_{12}^7 & 0 \\ 0 & 0 & \zeta_{12}^{-2} \end{pmatrix} \qquad R_{\beta^{\mathcal{Y}}} = \begin{pmatrix} 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & -1 & 1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 2 \end{pmatrix}.$$

In particular, $\omega_2^{\mathcal{Y}}$ is an eigenform for $\beta^{\mathcal{Y}}$.

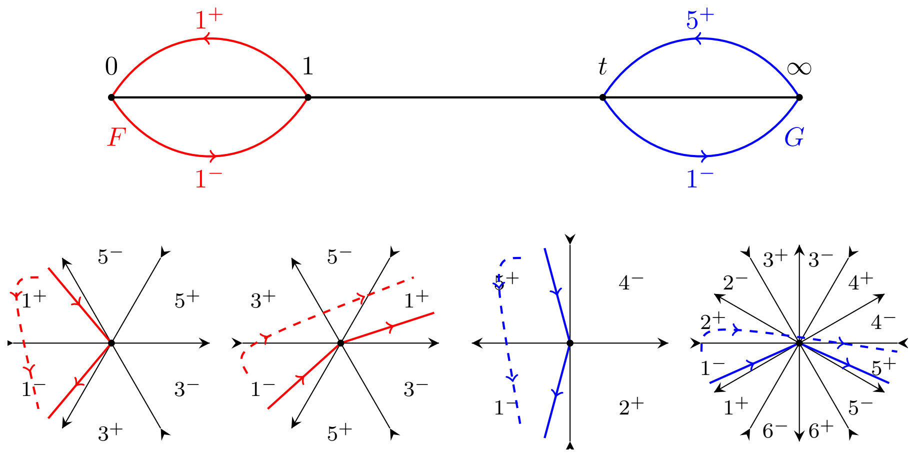


Figure 1.4.: The shrunk cycles $F^{\mathcal{Y}}$ and $G^{\mathcal{Y}}$, and the process of shrinking around the preimages of 0, 1, t and ∞, respectively.

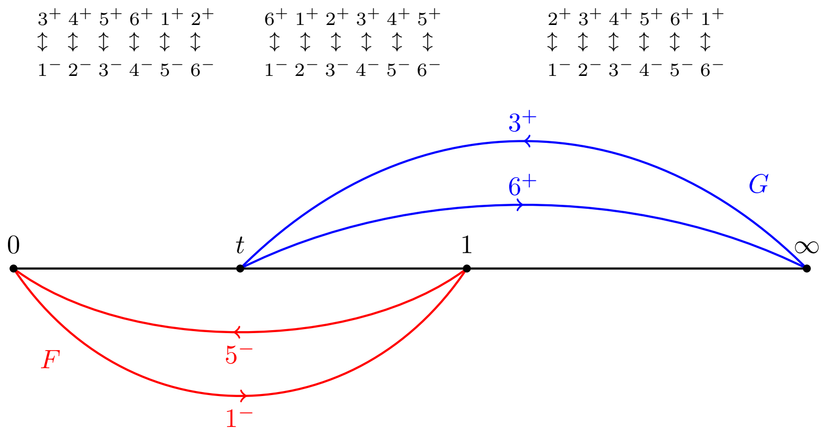


Figure 1.5.: The cycles $F^{\mathcal{Y}}$ and $G^{\mathcal{Y}}$ in $\mathcal{Y}_{1/2}$.

1.3.4. Stable reduction of degenerate fibers While the Windmill family is not compact in $\mathcal{M}_3$, it turns out that all fibers of its closure in $\overline{\mathcal{M}_3}$, the Deligne-Mumford compactification, admit compact Jacobians, i.e. that the Torelli image of $\overline{\mathcal{Y}}$ is contained in $\mathcal{A}_3$. Moreover, this analysis will be invaluable when constructing a fundamental domain for $\mathcal{Y}$ later.

The degenerate fibers of $\mathcal{Y}$ The degenerate fibers of the Windmill family $\mathcal{Y}$ correspond to $t = 0, 1, \infty$. To describe them, we resort to the theory of *admissible covers*. For a brief overview of the tools needed in this special case, see e.g. [BM10b, §4.1] and the references therein.

The stable reduction when $t \to 1$ (equivalently, when $t \to 0$) yields the two components

$$\begin{aligned} \overline{\mathcal{Y}}_1^1 &: y^6 = x^2(x-1)^5\,, \quad \text{of genus 2,} \\ \overline{\mathcal{Y}}_1^2 &: y^6 = x^2(x-1)^3\,, \quad \text{of genus 1.} \end{aligned}$$

The stable reduction when $t \to \infty$ yields the three components

$$\begin{aligned} \overline{\mathcal{Y}}_\infty^1 &: y^6 = x^2(x-1)^2\,, \quad \text{consisting of two components of genus 1,} \\ \overline{\mathcal{Y}}_\infty^2 &: y^6 = x^3(x-1)^5\,, \quad \text{of genus 1.} \end{aligned}$$

A simple calculation gives the following lemma.

Lemma 1.3.11. *The degeneration of the $(\alpha^{\mathcal{Y}})^*$-eigenforms of Lemma 1.3.4 for $t \to 1$ is given by*

$$\omega_1^1 = \frac{\mathrm{d}x}{y} \ \textit{on}\ \overline{\mathcal{Y}}_1^1\,, \qquad \omega_2^1 = \frac{y\mathrm{d}x}{x(x-1)} \ \textit{on}\ \overline{\mathcal{Y}}_1^2\,, \qquad \omega_3^1 = \frac{y^4\mathrm{d}x}{x^2(x-1)^4} \ \textit{on}\ \overline{\mathcal{Y}}_1^1\,,$$

and for $t \to \infty$ by

$$\omega_1^\infty = \frac{\mathrm{d}x}{y} \ \textit{on}\ \overline{\mathcal{Y}}_\infty^2\,, \qquad \omega_2^\infty = \frac{y\mathrm{d}x}{x(x-1)} \ \textit{on}\ \overline{\mathcal{Y}}_\infty^1\,, \qquad \omega_3^\infty = \frac{\mathrm{d}x}{y^2} \ \textit{on}\ \overline{\mathcal{Y}}_\infty^1\,,$$

where the differentials are identically zero on the components where they are not defined.

Via the shrinking process introduced above, one can compute the degeneration of the cycles in both cases (see Figure 1.7). In the following lemma, we sum up some results about the homology of the degenerate fibers that we will need in section 1.4.

Lemma 1.3.12. *Let F^∞, G^∞ and F^1, G^1 denote the cycles on $\overline{\mathcal{Y}}_\infty$ and $\overline{\mathcal{Y}}_1$ corresponding to the degeneration of $F^{\mathcal{Y}}$ and $G^{\mathcal{Y}}$.*

1. *F^∞ and G^∞ live in $\overline{\mathcal{Y}}_\infty^1$ and $\overline{\mathcal{Y}}_\infty^2$ respectively.*

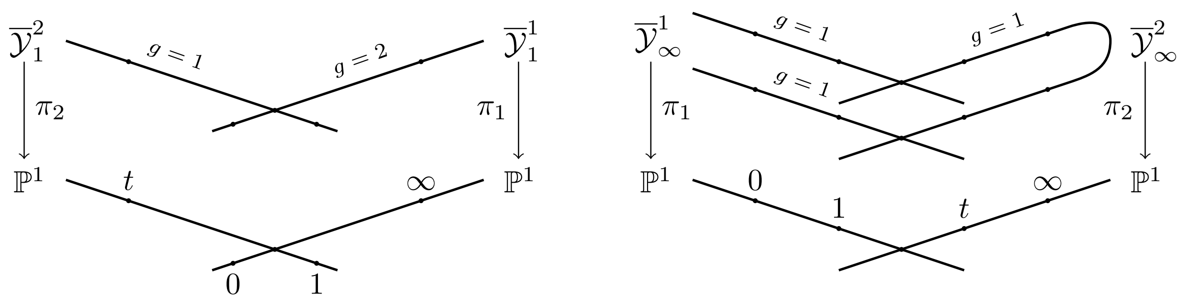


Figure 1.6.: The stable fibers $\overline{\mathcal{Y}}_1$ and $\overline{\mathcal{Y}}_\infty$.

2. *There is a decomposition of cycles $F^1 = F_1^1 + F_2^1$ and $G^1 = G_1^1 + G_2^1$, where F_k^1, G_k^1 are cycles in the component $\overline{\mathcal{Y}}_1^k$ going through the nodal point.*

   Moreover, one has the following intersection matrices for the sets of cycles $\{F_k^1, \alpha^{\mathcal{Y}} F_k^1, (\alpha^{\mathcal{Y}})^3 F_k^1, (\alpha^{\mathcal{Y}}$ for $k = 1, 2$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 \\ -1 & -1 & 1 & 1 & 0 & 2 \\ 0 & -1 & 0 & 1 & -2 & 0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 0 & -1 & 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \end{pmatrix},$$

   respectively.

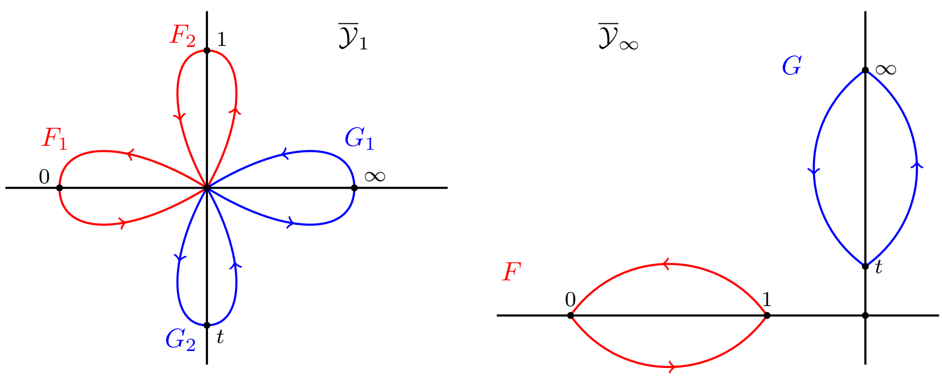


Figure 1.7.: The bases of homology in $\overline{\mathcal{Y}}_1$ and $\overline{\mathcal{Y}}_\infty$ as lifts of cycles in $\mathbb{P}^1$ by π_1 and π_2.

Proof. In the case of $\overline{\mathcal{Y}}_\infty$, it is obvious from Figure 1.3 that the degeneration of the cycles $F^{\mathcal{Y}}$ and $G^{\mathcal{Y}}$ lie in $\overline{\mathcal{Y}}_\infty^1$ and $\overline{\mathcal{Y}}_\infty^2$ respectively.

The case $\overline{\mathcal{Y}}_1$ is more delicate. It follows again from Figure 1.3 that the degeneration of both $F^{\mathcal{Y}}$ and $G^{\mathcal{Y}}$ are the union of cycles in $\overline{\mathcal{Y}}_1^1$ and $\overline{\mathcal{Y}}_1^2$ meeting at the nodal point. In fact, since the points in $\overline{\mathcal{Y}}_1$ corresponding to the preimages of 0 and 1 (respectively t and

∞) lie in different components, it is clear that F^1 (respectively G^1) will decompose as the sum $F_1^1 + F_2^1$ (respectively $G_1^1 + G_2^1$) of cycles in $\overline{\mathcal{Y}}_1^1$ and $\overline{\mathcal{Y}}_1^2$.

Consider first the component $\overline{\mathcal{Y}}_1^2$, isomorphic to $y^6 = x^2(x-1)^3$. Note that the preimages of 0 and 1 under π_2 correspond to the preimages of 1 and t in the general member of our family $\mathcal{Y}_t$. Let us denote by $Q \in \overline{\mathcal{Y}}_1^2$ the nodal point and suppose, for simplicity, that its image $q \in \mathbb{P}^1$ under π_2 lies in the interval $[1,0]$. Removing this interval and proceeding as before we get the picture in Figure 1.8, where the sheet changes follow from studying the behaviour of $F^{\mathcal{Y}}$ and $G^{\mathcal{Y}}$ around the preimages of 1 and t in the general member of our family (see Figure 1.4).

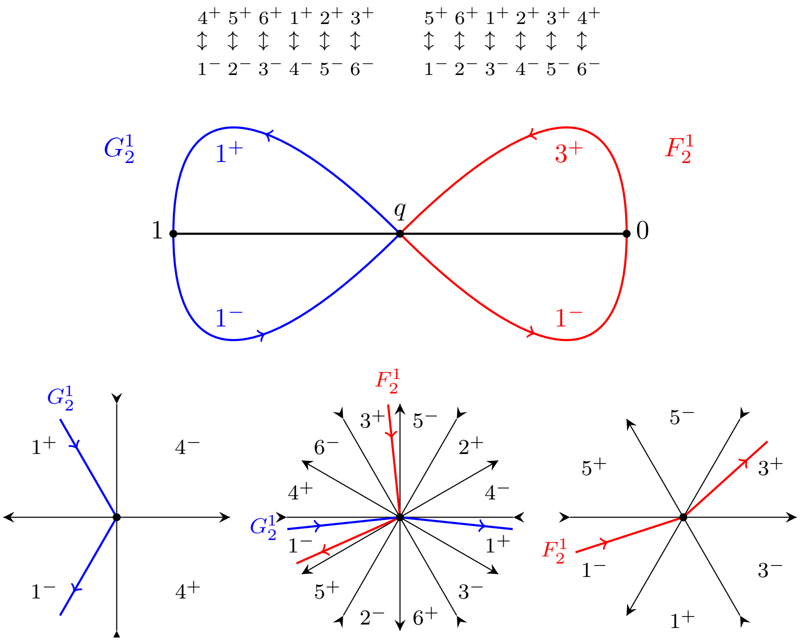


Figure 1.8.: Degenerate cycles on $\overline{\mathcal{Y}}_1^2$ and their behaviour around the preimages of t, q and 1, respectively. Note that $F_2^1 \cdot G_2^1 = -1$.

One can get a similar picture for the other component $\overline{\mathcal{Y}}_1^1$. Now a tedious but straightforward calculation yields the intersection matrices. □

1.4. The Prym–Torelli images

To understand the orbifold points of W_D, by Proposition 1.3.6, we must determine which $\mathcal{X}_t$ and $\mathcal{Y}_t$ admit real multiplication that satisfies the eigenform condition. Therefore, the aim of this section is to concretely calculate the period matrices of the families of Prym varieties $\mathcal{P}(\mathcal{X}_t)$ of the Clover family and $\mathcal{P}(\mathcal{Y}_t)$ of the Windmill family.

1.4.1. The Prym variety $\mathcal{P}(\mathcal{X}_t)$ In the case of the Clover family $\mathcal{X}$, all the fibers $\mathcal{X}_t$ are sent to the same Prym variety by the Prym–Torelli map.

Proposition 1.4.1. *For all $t \in \mathbb{P}^*$, the Prym variety $\mathcal{P}(\mathcal{X}_t)$ is isomorphic to $\mathbb{C}^2/\Lambda$, where $\Lambda = \mathcal{P}\Pi^{\mathcal{X}} \cdot \mathbb{Z}^4$ for*

$$\mathcal{P}\Pi^{\mathcal{X}} = \begin{pmatrix} -\frac{1+\mathrm{i}}{2} & 1 & 1 & 0 \\ 1 & -(1+\mathrm{i}) & 0 & 2 \end{pmatrix},$$

together with the polarisation induced by the intersection matrix

$$E^{\mathcal{X}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix}.$$

In particular, the image of the Clover family $\mathcal{X}$ under the Prym–Torelli map is a single point.

Calculating Jacobians of curves with automorphisms can be done by a method attributed to Bolza, see [BL04, Chap. 11.7] for details. The idea is to determine, for a given automorphism σ and fixed choices of basis, the analytic and rational representations A_σ and R_σ of the automorphisms and use this information to find relations in the period matrix Π, using the identity

$$\begin{pmatrix} A_\sigma & 0 \\ 0 & \overline{A_\sigma} \end{pmatrix} \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} = \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} R_\sigma.$$

All members of the Clover family $\mathcal{X}$ admit, apart from the automorphism $\alpha^{\mathcal{X}}$, the following two involutions

$$\gamma\colon (x,y) \mapsto \left(\frac{t}{x}, \frac{y\sqrt{t}}{x}\right) \quad \text{and} \quad \delta\colon (x,y) \mapsto \left(\frac{t(x-1)}{x-t}, \frac{-y\sqrt{t(t-1)}}{x-t}\right).$$

The automorphisms γ and δ are lifts by π of the automorphisms of $\mathbb{P}^1 = X/\langle\alpha\rangle$ given by $z \mapsto t/z$ and $z \mapsto (tz-t)/(z-t)$, respectively. In particular, these two involutions of $\mathbb{P}^1$ generate a Klein four-group acting on the fixed points of $\rho^{\mathcal{X}}$.

Note that, although γ and δ do not commute, one has $\gamma\delta\gamma^{-1}\delta^{-1} = \alpha^2$, and $\langle\alpha,\gamma,\delta\rangle$ has order 16. In fact, for a general member of the family this is the whole group of automorphisms of the curve (see [KK79, (6)(b)] or [Bro91]).

By Lemma 1.3.4, the action of α^* on $\Omega(\mathcal{X}_t)$ is given by

$$\begin{pmatrix} \mathrm{i} & 0 & 0 \\ 0 & \mathrm{i} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

in the eigenform basis. The automorphisms γ and δ induce analytic representations

$$\gamma^* = \begin{pmatrix} 0 & -\sqrt{t} & 0 \\ -\frac{1}{\sqrt{t}} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \delta^* = \begin{pmatrix} \frac{-t}{\sqrt{t(t-1)}} & \frac{-t}{\sqrt{t(t-1)}} & 0 \\ \frac{1}{\sqrt{t(t-1)}} & \frac{t}{\sqrt{t(t-1)}} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

To calculate the rational representation, let us suppose again $t \in \mathbb{R}$, $t > 1$. Keeping track of the action of γ and δ on the branching points of π and on the half-sheets of the cover, one can write down the action of these automorphisms in the homology

$$\begin{aligned} \gamma F^{\mathcal{X}} &= -\alpha^2 F^{\mathcal{X}} + G^{\mathcal{X}} + \alpha G^{\mathcal{X}}\,, & \gamma G^{\mathcal{X}} &= -G^{\mathcal{X}}\,, \\ \delta F^{\mathcal{X}} &= -F^{\mathcal{X}}\,, & \delta G^{\mathcal{X}} &= -\alpha F^{\mathcal{X}} - \alpha^2 F^{\mathcal{X}} - \alpha^2 G^{\mathcal{X}}\,. \end{aligned}$$

Remark 1.4.2. *Observe that γ and δ act as involutions and the quotient is $\mathcal{X}_t/\gamma \cong \mathcal{X}_t/\delta \cong E_{\mathrm{i}}$, where E_{i} is the unique elliptic curve with an order four automorphism. Indeed, $\mathcal{X}_t$ is not hyperelliptic and δ and γ have fixed points (e.g. preimages of $\sqrt{t}$ and $t - \sqrt{t(t-1)}$ on $\mathcal{X}_t$), therefore the quotient has genus 1. Moreover, α commutes with both δ and γ, hence descends to an order four automorphism of the quotient elliptic curve.*

Proof of Proposition 1.4.1. To calculate the Jacobian $\mathrm{Jac}(\mathcal{X})$ write $f_i \coloneqq f_i^{\mathcal{X}}(t) = \int_{F^{\mathcal{X}}} \omega_i^{\mathcal{X}}$ and $g_i \coloneqq g_i(t) = \int_{G^{\mathcal{X}}} \omega_i^{\mathcal{X}}$. From the action of α one can deduce that the Jacobian of $\mathcal{X}_t$ in the bases of Lemmas 1.3.4 and 1.3.8 is given by the period matrix

$$\Pi_t^{\mathcal{X}} = \begin{pmatrix} f_1 & \mathrm{i}f_1 & -f_1 & g_1 & \mathrm{i}g_1 & -g_1 \\ f_2 & \mathrm{i}f_2 & -f_2 & g_2 & \mathrm{i}g_2 & -g_2 \\ f_3 & -f_3 & f_3 & g_3 & -g_3 & g_3 \end{pmatrix}.$$

Using the actions of γ and δ both on $\Omega(\mathcal{X}_t)$ and $\mathrm{H}_1(\mathcal{X}_t, \mathbb{Z})$ one gets the relations

$$f_1 = -\sqrt{t} f_2 - g_1(1+\mathrm{i})\,, \quad g_2 = \frac{g_1}{\sqrt{t}}\,, \quad g_1 = \frac{-f_2\sqrt{t}(1-\sqrt{t}+\sqrt{t-1})}{(1+\mathrm{i})(\sqrt{t-1}-\sqrt{t})}\,.$$

By changing to the basis of $\mathrm{H}_1^-(\mathcal{X}_t, \mathbb{Z}) \oplus \mathrm{H}_1^+(\mathcal{X}_t, \mathbb{Z})$ given in Lemma 1.3.8 one gets

$$\begin{pmatrix} (1+\mathrm{i})(f_1+g_1) & -2g_1 & -2g_1 + (\mathrm{i}-1)f_1 & 2\mathrm{i}f_1 & 0 & 0 \\ (1+\mathrm{i})(f_2+g_2) & -2g_2 & -2g_2 + (\mathrm{i}-1)f_2 & 2\mathrm{i}f_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2f_3 & 2g_3 \end{pmatrix}$$

and sees that the Jacobian $\mathrm{Jac}(\mathcal{X}_t)$ is isogenous to the product $\mathcal{P}(\mathcal{X}_t) \times \mathrm{Jac}(\mathcal{X}_t/\rho^{\mathcal{X}})$, where $\mathcal{P}(\mathcal{X}_t)$ is $(1,2)$-polarised and $\mathrm{Jac}(\mathcal{X}_t/\rho^{\mathcal{X}})$ is (2)-polarised. Note that the polarisation on $\mathcal{P}(\mathcal{X}_t)$ is given by the principal 4×4 minor in the intersection matrix in the proof of Lemma 1.3.8, which agrees with $E^{\mathcal{X}}$.

Finally, we can change the basis of $\Omega(\mathcal{X}_t)^-$ by the matrix

$$Q_t = \frac{1}{\sqrt{t-1}f_2}\begin{pmatrix} -\frac{(1+\mathrm{i})(\sqrt{t}-\sqrt{t-1})}{4\sqrt{t}} & -\frac{1+i}{4} \\ \frac{\mathrm{i}}{2\sqrt{t}} & \frac{i(\sqrt{t}-\sqrt{t-1})}{2} \end{pmatrix}, \tag{1.4}$$

to get the period matrix

$$\begin{pmatrix} \mathcal{P}\Pi^{\mathcal{X}} & 0 \\ 0 & \mathcal{E}\Pi_t^{\mathcal{X}} \end{pmatrix} \text{ where } \mathcal{P}\Pi^{\mathcal{X}} := \begin{pmatrix} -\frac{1+\mathrm{i}}{2} & 1 & 1 & 0 \\ 1 & -(1+\mathrm{i}) & 0 & 2 \end{pmatrix} \text{ and } \mathcal{E}\Pi_t^{\mathcal{X}} := \begin{pmatrix} 2f_3 & 2g_3 \end{pmatrix}.$$

Note that $\mathcal{P}\Pi^{\mathcal{X}}$ no longer depends on t, proving the final statement. □

Remark 1.4.3. *These results are equivalent to those of Guàrdia in [Guà01]. However, we cannot simply apply his results for two reasons. First, we are not restricted to real branching values and in particular the curve $\mathcal{X}_{\zeta_6}$ plays a special role. More importantly, in order to study the points of intersection with the Prym–Teichmüller curves W_D, we need to keep track of the differential forms with a 4-fold zero in each fiber of the family. As a consequence, we need an explicit expression of the elements of $\Omega(\mathcal{X}_t)^-(4)$, i.e. the ρ-anti-invariant differential forms with a 4-fold zero, in the basis in which the period matrix $\mathcal{P}\Pi^{\mathcal{X}}$ above is written.*

The endomorphism ring $\operatorname{End}\mathcal{P}(\mathcal{X}_t)$ To see when $\mathcal{P}(\mathcal{X}_t)$ has real multiplication by a given order, we need a good understanding of the endomorphism ring. First, however, we describe the endomorphism algebra.

Proposition 1.4.4. *The endomorphism algebra $\operatorname{End}_{\mathbb{Q}}\mathcal{P}(\mathcal{X}_t)$ is the algebra isomorphic to $M_2(\mathbb{Q}[\mathrm{i}])$ generated by the identity and the automorphisms α, γ, δ and $\gamma\delta$.*

Proof. Note that the automorphisms α, γ and δ of $\mathcal{X}_t$ preserve the spaces $\Omega(\mathcal{X}_t)^-$ and $\mathrm{H}_1^-(\mathcal{X}_t,\mathbb{Z})$, so they induce automorphisms of the Prym variety. One can construct their analytic and rational representations in the bases of Lemmas 1.3.4 and 1.3.8 to obtain

$$A_\alpha = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \qquad R_\alpha = \begin{pmatrix} 1 & -2 & -2 & 0 \\ -1 & 1 & 0 & -2 \\ 2 & -2 & -1 & 2 \\ -1 & 2 & 1 & -1 \end{pmatrix};$$

$$A_\gamma = \begin{pmatrix} 0 & \frac{1-\mathrm{i}}{2} \\ 1+\mathrm{i} & 0 \end{pmatrix}, \quad R_\gamma = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

$$A_\delta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad R_\delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}.$$

Since A_α lies in the center of $M_2(\mathbb{C})$ and the involutions γ and δ anti-commute, the endomorphism algebra $\operatorname{End}_{\mathbb{Q}} \mathcal{P}(\mathcal{X}_t)$ must contain the (definite) quaternion algebra $F = \langle A_\alpha, A_\gamma, A_\delta \rangle_{\mathbb{Q}} \cong M_2(\mathbb{Q}[\mathrm{i}])$. Since $|F : \mathbb{Q}| = 16$, this already has to be the entire algebra $\operatorname{End}_{\mathbb{Q}} \mathcal{P}(\mathcal{X}_t)$ (see [BL04, Prop. 13.4.1]). □

Proof of Theorem 1.1.2. Proposition 1.4.4 implies that $\mathcal{P}(\mathcal{X}_t) \cong E_{\mathrm{i}} \times E_{\mathrm{i}}$, and the claim about the orbifold points follows from Proposition 1.3.1. □

Recall that, for any polarised abelian variety, the Rosati involution $\cdot'$ on the endomorphism ring is induced by the polarisation. Therefore, given an element $\varphi \in \operatorname{End}_{\mathbb{Q}} \mathcal{P}(\mathcal{X}_t)$ with rational representation R_φ, its image φ' under the Rosati involution has rational representation $E^{-1} R_\varphi^T E$, where $E = E^{\mathcal{X}}$ is the polarisation matrix from above. It is then easy to check that $\alpha' = -\alpha$, $\gamma' = \gamma$, $\delta' = \delta$ and $(\gamma\delta)' = -\gamma\delta$. Under the embedding $F \hookrightarrow M_2(\mathbb{C})$ given by the analytic representation, the Rosati involution is the restriction of the involution

$$\begin{array}{ccc} M_2(\mathbb{C}) & \to & M_2(\mathbb{C}) \\ B & \mapsto & A^{-1} B^H A \end{array}, \quad \text{for } A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.5}$$

where B^H denotes the hermitian transpose.

This gives us a simple criterion to check whether a specific rational endomorphism actually lies in $\operatorname{End} \mathcal{P}(\mathcal{X}_t)$.

1.4.2. The Prym variety $\mathcal{P}(\mathcal{Y}_t)$

In the case of the Windmill family $\mathcal{Y}$, we have the following characterisation.

Proposition 1.4.5. *For all $t \in \mathbb{P}^*$, the Prym variety $\mathcal{P}(\mathcal{Y}_t) = \mathbb{C}^2/\Lambda_t$, where $\Lambda_t = \mathcal{P}\Pi_t^{\mathcal{Y}} \cdot \mathbb{Z}^4$ for*

$$\mathcal{P}\Pi_t^{\mathcal{Y}} = \begin{pmatrix} 2f & 2\zeta_6^2 f & 1 & \zeta_6^{-1} \\ 2 & 2\zeta_6^{-2} & 2f & 2\zeta_6 f \end{pmatrix},$$

where $f := f(t) = \int_{F^{\mathcal{Y}}} \omega_1^{\mathcal{Y}}$ is the period map, together with the polarisation induced by the intersection matrix

$$E^{\mathcal{Y}} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

As above, we use Bolza's method for calculating the period matrix. Fortunately, in this case it suffices to regard $\alpha := \alpha^{\mathcal{Y}}$.

By Lemma 1.3.4, the action of α^* on $\Omega(\mathcal{Y}_t)$ is given by

$$\begin{pmatrix} \zeta_6^{-1} & 0 & 0 \\ 0 & \zeta_6 & 0 \\ 0 & 0 & \zeta_6^4 \end{pmatrix}$$

in the eigenform basis.

Proof of Proposition 1.4.5. Again, we write $f_i := f_i^{\mathcal{Y}}(t) = \int_{F^{\mathcal{Y}}} \omega_i^{\mathcal{Y}}$ and $g_i := g_i^{\mathcal{Y}}(t) = \int_{G^{\mathcal{Y}}} \omega_i^{\mathcal{Y}}$. Since $\alpha^3(G^{\mathcal{Y}}) = \rho^{\mathcal{Y}}(G^{\mathcal{Y}}) = -G^{\mathcal{Y}}$ and $(\rho^{\mathcal{Y}})^*\omega_3^{\mathcal{Y}} = -\omega_3^{\mathcal{Y}}$, one has $g_3 = 0$. Using the action of α on $\Omega(\mathcal{Y}_t)$, one gets that, in these bases, the period matrix of $\mathcal{Y}_t$ reads

$$\Pi_t^{\mathcal{Y}} = \begin{pmatrix} f_1 & \zeta_6^{-1} f_1 & -f_1 & \zeta_6^2 f_1 & g_1 & \zeta_6^{-1} g_1 \\ f_2 & \zeta_6 f_2 & -f_2 & \zeta_6^{-2} f_2 & g_2 & \zeta_6 g_2 \\ f_3 & \zeta_6^{-2} f_3 & f_3 & \zeta_6^{-2} f_3 & 0 & 0 \end{pmatrix}. \tag{1.6}$$

Moreover, by normalising $g_1 = f_2 = f_3 = 1$ and using Riemann's relations, one sees that

$$\Pi_t^{\mathcal{Y}} E^{-1} \left(\Pi_t^{\mathcal{Y}}\right)^T = 0 \Rightarrow g_2 = 2f_1, \text{ and}$$
$$\mathrm{i}\Pi_t^{\mathcal{Y}} E^{-1} \left(\overline{\Pi_t^{\mathcal{Y}}}\right)^T > 0 \Rightarrow 2|f_1|^2 - 1 < 0.$$

Writing $f := f_1$, we finally get

$$\Pi_t^{\mathcal{Y}} = \begin{pmatrix} f & \zeta_6^{-1} f & -f & \zeta_6^2 f & 1 & \zeta_6^{-1} \\ 1 & \zeta_6 & -1 & \zeta_6^{-2} & 2f & 2\zeta_6 f \\ 1 & \zeta_6^{-2} & 1 & \zeta_6^{-2} & 0 & 0 \end{pmatrix}. \tag{1.7}$$

As above, the Jacobian $\mathrm{Jac}(\mathcal{Y}_t)$ is isogenous to the variety $\mathcal{P}(\mathcal{Y}_t) \times \mathrm{Jac}(\mathcal{Y}_t/\rho^{\mathcal{Y}})$, whose period matrix is obtained by changing to the basis of $\mathrm{H}_1^-(\mathcal{Y}_t, \mathbb{Z}) \oplus \mathrm{H}_1^+(\mathcal{Y}_t, \mathbb{Z})$ of Lemma 1.3.8, yielding

$$\begin{pmatrix} \mathcal{P}\Pi_t^{\mathcal{Y}} & 0 \\ 0 & \mathcal{E}\Pi_t^{\mathcal{Y}} \end{pmatrix}, \text{ where } \mathcal{P}\Pi_t^{\mathcal{Y}} := \begin{pmatrix} 2f & 2\zeta_6^2 f & 1 & \zeta_6^{-1} \\ 2 & 2\zeta_6^{-2} & 2f & 2\zeta_6 f \end{pmatrix} \text{ and } \mathcal{E}\Pi_t^{\mathcal{Y}} := \begin{pmatrix} 2 & 2\zeta_6^{-2} \end{pmatrix}.$$

The polarisation on $\mathcal{P}(\mathcal{Y}_t)$ is again given by the principal 4×4 minor in the intersection matrix in the proof of Lemma 1.3.8, which agrees with $E^{\mathcal{Y}}$. □

The endomorphism ring $\mathrm{End}\,\mathcal{P}(\mathcal{Y}_t)$ In this section we study the endomorphism ring $\mathrm{End}\,\mathcal{P}(\mathcal{Y}_t)$ and the endomorphism algebra $\mathrm{End}_{\mathbb{Q}}\,\mathcal{P}(\mathcal{Y}_t)$ in order to get a description of the Windmill family $\mathcal{Y}$ as a Shimura curve. More precisely, let $\mathfrak{M}$ denote the maximal order

$$\mathfrak{M} = \mathbb{Z}\left[\frac{\mathbf{1}+\mathbf{j}}{2}, \frac{\mathbf{1}-\mathbf{j}}{2}, \frac{\mathbf{i}+\mathbf{ij}}{2}, \frac{\mathbf{i}-\mathbf{ij}}{2}\right] \tag{1.8}$$

in the quaternion algebra

$$F := \left\{x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{ij} \;:\; x_k \in \mathbb{Q}\,,\, \mathbf{i}^2 = 2\,,\, \mathbf{j}^2 = -3\right\} \cong \left(\frac{2, -3}{\mathbb{Q}}\right).$$

We will prove the following.

Proposition 1.4.6. *The Prym–Torelli map gives an isomorphism between the compactification $\overline{\mathcal{Y}}$ of the Windmill family $\mathcal{Y}$ and the (compact) Shimura curve whose points correspond to abelian surfaces with a $(1,2)$ polarisation, endomorphism ring* $\mathrm{End}\,A \cong \mathfrak{M}$ *and Rosati involution given by Equation 1.9. This curve is isomorphic to* $\mathbb{H}/\Delta(2,6,6)$.

Recall that a (compact hyperbolic) triangle group is a Fuchsian group constructed in the following way. Let l, m and n be positive integers such that $1/l + 1/m + 1/n < 1$ and consider a hyperbolic triangle T in the hyperbolic plane with vertices v_l, v_m and v_n with angles π/l, π/m and π/n respectively. The subgroup $\Delta(l, m, n)$ of $\mathrm{PSL}_2(\mathbb{R})$ generated by the positive rotations through angles $2\pi/l$, $2\pi/m$ and $2\pi/n$ around v_l, v_m and v_n respectively is called a *triangle group of signature* (l, m, n). The triangle T is unique up to conjugation in $\mathrm{PSL}_2(\mathbb{R})$ and, therefore, so is the associated triangle group described above ([Bea83, §7.12]). Note that the quadrilateral consisting of the union of the triangle T and any of its reflections serves as a fundamental domain for $\Delta(l, m, n)$ (see Figure 1.9 for a fundamental domain of $\Delta(2, 6, 6)$ inside the hyperbolic disc $\mathbb{D}$).

Let us now calculate $\operatorname{End}\mathcal{P}(\mathcal{Y}_t)$. Since the automorphism α of $\mathcal{Y}_t$ induces an automorphism of $\mathcal{P}(\mathcal{Y}_t)$ and $\mathbf{j} := 2\alpha - 1$ satisfies $\mathbf{j}^2 = -3$, there is always an embedding $\mathbb{Q}(\sqrt{-3}) \hookrightarrow \operatorname{End}_{\mathbb{Q}}\mathcal{P}(\mathcal{Y}_t)$. However, the full endomorphism algebra of an abelian surface is never an imaginary quadratic field (see [BL04, Ex. 9.10(4)], for example) and one can check that the analytic and rational representations $A_{\mathbf{i}}$ and $R_{\mathbf{i}}$ defined below yield an element of $\operatorname{End}_{\mathbb{Q}}\mathcal{P}(\mathcal{Y}_t)$. It is then easy to see that the endomorphism algebra of the general member of our family agrees with the (indefinite) quaternion algebra F.

Abelian varieties with given endomorphism structure have been intensely studied, notably by Shimura [Shi63]. Shimura explicitly constructs moduli spaces for such families in much greater generality than we require here. However, his results specialise to our situation. To emulate his construction, we begin by observing that since $F \otimes \mathbb{R} \cong M_2(\mathbb{R})$, we can see F as a subalgebra of $M_2(\mathbb{R})$. The following matrices show the relation between the embedding $F \hookrightarrow M_2(\mathbb{R})$, the analytic representation $A\colon F \hookrightarrow M_2(\mathbb{C})$ and the rational representation $R\colon F \hookrightarrow M_4(\mathbb{Q})$

$$\mathbf{i} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix}, \quad A_{\mathbf{i}} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \qquad R_{\mathbf{i}} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & -2 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix};$$

$$\mathbf{j} = \begin{pmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}, \quad A_{\mathbf{j}} = \begin{pmatrix} -i\sqrt{3} & 0 \\ 0 & i\sqrt{3} \end{pmatrix}, \quad R_{\mathbf{j}} = \begin{pmatrix} -1 & 2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 2 & 1 \end{pmatrix};$$

$$\mathbf{ij} = \begin{pmatrix} 0 & \sqrt{6} \\ \sqrt{6} & 0 \end{pmatrix}, \quad A_{\mathbf{ij}} = \begin{pmatrix} 0 & i\sqrt{3} \\ -2i\sqrt{3} & 0 \end{pmatrix}, \quad R_{\mathbf{ij}} = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 \\ 2 & 2 & 0 & 0 \\ -4 & 2 & 0 & 0 \end{pmatrix}.$$

By checking which elements of F have integral rational representation, one can see that the endomorphism ring $\operatorname{End}_{\mathbb{Q}}\mathcal{P}(\mathcal{Y}_t)$ of the general member of our family agrees with the maximal order $\mathfrak{M}$ defined above.

Proceeding as in the case of the Clover family and writing $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{ij}$ for an element of F, we note that, by the Skolem-Noether theorem, the quaternion conjugation and the Rosati involution are conjugate. It is not difficult to check that, here, the Rosati involution is given by

$$x' := \mathbf{j}^{-1}\overline{x}\mathbf{j} = x_0 + x_1\mathbf{i} - x_2\mathbf{j} + x_3\mathbf{ij}\,, \tag{1.9}$$

where $\overline{x} = x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{ij}$ is the usual conjugation in F. Note that the Rosati involution in $F \hookrightarrow M_2(\mathbb{R})$ agrees with transposition and that, under the embedding $F \hookrightarrow M_2(\mathbb{C})$ given by the analytic representation, it is again the restriction of the involution

$$\begin{array}{lcl} M_2(\mathbb{C}) & \to & M_2(\mathbb{C}) \\ B & \mapsto & A^{-1}B^H A \end{array}\;, \quad \text{for } A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.10}$$

Proof of Proposition 1.4.6. Let us give the construction of the Windmill family as a Shimura curve. Following [Shi63], one can define the isomorphism

$$\begin{array}{rccl} \Phi: & \mathfrak{M} & \longrightarrow & \Lambda_t \\ & a & \longmapsto & A_a \cdot y\,, \qquad y = \binom{2f}{2} \end{array}$$

where A_a denotes the analytic representation of a, and check that the polarisation satisfies $E(\Phi(a), y) = \operatorname{tr}(a \cdot T)$ for $T = \frac{1}{3}\mathbf{j} \in F$. The family of abelian varieties A with a $(1,2)$ polarisation together with an embedding $\mathfrak{M} \hookrightarrow \operatorname{End} A$ and Rosati involution induced by Equation 1.9 is then given by the Shimura curve $\mathbb{H}/\Gamma(T,\mathfrak{M})$, where $\Gamma(T,\mathfrak{M})$ agrees with the group of elements of norm 1 of $\mathfrak{M}$. By [Tak77] this is a quadrilateral group of signature $\langle 0; 2, 2, 3, 3\rangle$.

However, for each such variety A, there exist two different embeddings $F \hookrightarrow \operatorname{End}_{\mathbb{Q}} A$ which differ by quaternion conjugation on F. As a consequence, the map $\mathbb{H}/\widetilde{\Gamma}(T,\mathfrak{M}) \to \mathcal{A}_{2,(1,2)}$ has degree 2, and the Shimura curve constructed above is a double cover of its image, which is uniformised by the triangle group $\Delta(2,6,6)$ extending $\Gamma(T,\mathfrak{M})$ (see [Tak77]).

Now, the Prym–Torelli image of $\overline{\mathcal{Y}}$ lies entirely in this family and the proposition follows. □

Proof of Theorem 1.1.2. It is just a reformulation of Proposition 1.4.6 and Proposition 1.3.1. □

Remark 1.4.7. *Cyclic coverings of this type are well-known and have been intensely studied. For example, it immediately follows from the results of Deligne and Mostow [DM86, §14.3] that the compactified Windmill family $\overline{\mathcal{Y}}$ is parametrised by $\mathbb{H}/\Delta(2,6,6)$. More precisely, the monodromy data of the Windmill family yields (using their notation) $\mu_1 = \mu_2 = 1/3$, $\mu_3 = 1/2$, and $\mu_4 = 5/6$, hence we obtain a map from $\mathbb{P}^1$ into $\mathbb{H}/\Delta(3,6,6)$. Taking the quotient by the additional symmetry in the branching data here present, it descends to a map from the basis of $\mathcal{Y}$ into $\mathbb{H}/\Delta(2,6,6)$, as above.*

In our case, the lift of the period map $f = f(t)$ from $\mathbb{P}^1$ to the disc of radius $1/\sqrt{2}$ gives us a particular model of the Shimura curve introduced above as the quotient of this disc with the hyperbolic metric by the action of a specific triangle group $\Delta(2,6,6)$. In order to find a fundamental domain for this group, we will study the value of the period map at the special points of the compactification $\overline{\mathcal{Y}}$ of the Windmill family $\mathcal{Y}$, namely the curves $\mathcal{Y}_{1/2}$, $\overline{\mathcal{Y}}_1$ and $\overline{\mathcal{Y}}_\infty$. In particular, we will prove the following.

Proposition 1.4.8. *The* $\Delta(2,6,6)$ *group uniformising* $\overline{\mathcal{Y}}$ *is generated by the hyperbolic triangle with vertices* $f(1/2) = \frac{3-\sqrt{3}+\mathrm{i}(\sqrt{3}-1)}{4}$ *of angle* $\pi/2$, $f(1) = \frac{1}{2}\zeta_6$ *of angle* $\pi/6$ *and* $f(\infty) = 0$ *of angle* $\pi/6$ *inside the disc of radius* $1/\sqrt{2}$ *(see Figure 1.9). These vertices correspond to the curves* $\mathcal{Y}_{1/2}$, $\overline{\mathcal{Y}}_1$ *and* $\overline{\mathcal{Y}}_\infty$ *respectively.*

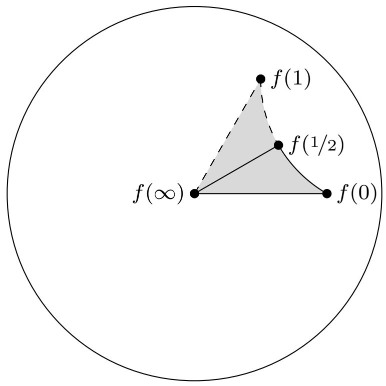


Figure 1.9.: Fundamental domain of $\Delta(2,6,6)$ on the disc of radius $1/\sqrt{2}$ with vertices 0, $1/2$ and $\frac{1}{4}(3-\sqrt{3}) + \frac{\mathrm{i}}{4}(\sqrt{3}-1)$ corresponding to special fibers of $\overline{\mathcal{Y}}$.

Proof. It follows from Lemma 1.3.2(2) that the curve $\mathcal{Y}_{1/2}$ corresponds to the point of order 2 in the triangle group and, therefore, $\overline{\mathcal{Y}}_1$ and $\overline{\mathcal{Y}}_\infty$ correspond to the two points of order 6. Consider Equation 1.7, giving the period matrix $\Pi_t^{\mathcal{Y}}$ of the general member of the Windmill family.

In the case of $\overline{\mathcal{Y}}_\infty$, it follows from Lemma 1.3.11 and Lemma 1.3.12 that $\int_{F^{\mathcal{Y}}} \omega_1^\infty = \int_{G^{\mathcal{Y}}} \omega_2^\infty = \int_{G^{\mathcal{Y}}} \omega_3^\infty = 0$ and one has the following period matrix

$$\Pi_\infty^{\mathcal{Y}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & \zeta_6^{-1} \\ 1 & \zeta_6 & -1 & \zeta_6^{-2} & 0 & 0 \\ 1 & \zeta_6^{-2} & 1 & \zeta_6^{-2} & 0 & 0 \end{pmatrix}.$$

In particular $f(\infty) = 0$.

Similarly, using Lemma 1.3.10 and the fact that $A_{\beta^{\mathcal{Y}}} \Pi_{1/2}^{\mathcal{Y}} = \Pi_{1/2}^{\mathcal{Y}} R_{\beta^{\mathcal{Y}}}$ one gets

$$\Pi_{1/2}^{\mathcal{Y}} = \begin{pmatrix} \vartheta & \zeta_6^{-1}\vartheta & -\vartheta & \zeta_6^{2}\vartheta & 1 & \zeta_6^{-1} \\ 1 & \zeta_6 & -1 & \zeta_6^{-2} & 2\vartheta & 2\zeta_6\vartheta \\ 1 & \zeta_6^{-2} & 1 & \zeta_6^{-2} & 0 & 0 \end{pmatrix},$$

where

$$\vartheta = \frac{3-\sqrt{3}+\mathrm{i}(1-\sqrt{3})}{4}.$$

Finally, in the case $\overline{\mathcal{Y}}_1$, it follows again from Lemma 1.3.12 that $G_2^1 = \alpha F_2^1 - F_2^1$. Comparing this with the entries of the period matrix $\Pi_t^{\mathcal{Y}}$ in Equation 1.6, one finds that

$2f(1) = g_2(1) = \zeta_6 - 1$. Therefore $f(1) = \frac{1}{2}\zeta_6^2$ and

$$\Pi_1^{\mathcal{Y}} = \begin{pmatrix} \frac{1}{2}\zeta_6^2 & \frac{1}{2}\zeta_6 & -\frac{1}{2}\zeta_6^2 & \frac{1}{2}\zeta_6^{-2} & 1 & \zeta_6^{-1} \\ 1 & \zeta_6 & -1 & \zeta_6^{-2} & \zeta_6^2 & -1 \\ 1 & \zeta_6^{-2} & 1 & \zeta_6^{-2} & 0 & 0 \end{pmatrix}.$$

Now, since $f(\infty) = 0$ is a point of order 6 of $\Delta(2,6,6)$, the point $\frac{1}{2}\zeta_6^2$ corresponding to $\overline{\mathcal{Y}}_1$ (respectively the point ϑ corresponding to $\mathcal{Y}_{1/2}$) is equivalent to $f(1) = \frac{1}{2}\zeta_6$ (respectively to $f(1/2) = \overline{\vartheta}$), by reflecting along the sides of the triangle. We may therefore choose our fundamental domain as claimed. □

1.5. Orbifold points in W_D

In this section we will finally determine the orbifold points on W_D. By Proposition 1.3.6, these correspond precisely to the fibers of the Clover family $\mathcal{X}$ and of the Windmill family $\mathcal{Y}$ whose Prym variety admits proper real multiplication by $\mathcal{O}_D$, together with an eigenform for real multiplication having a 4-fold zero at a fixed point of the Prym involution. Remember that $\mathcal{O}_D$ is defined as $\mathbb{Z}[T]/(T^2 + bT + c)$, where $D = b^2 - 4c$. In particular, $\mathcal{O}_D$ is generated as a $\mathbb{Z}$-module by

$$T := \begin{cases} \dfrac{\sqrt{D}}{2}\,, & \text{if } D \equiv 0 \bmod 4\,; \\ \dfrac{1+\sqrt{D}}{2}\,, & \text{if } D \equiv 1 \bmod 4\,. \end{cases}$$

We will write $T_D := T$ whenever we want to stress the dependence on D.

Let D be a discriminant with conductor f_0 and let us recall the sets

$$\begin{aligned} \mathcal{H}_2(D) &:= \{(a,b,c) \in \mathbb{Z}^3 \ : a^2 + b^2 + c^2 = D\ ,\ \gcd(a,b,c,f_0) = 1\},\ \text{and} \\ \mathcal{H}_3(D) &:= \{(a,b,c) \in \mathbb{Z}^3 \ : 2a^2 - 3b^2 - c^2 = 2D\ ,\ \gcd(a,b,c,f_0) = 1\ , \\ &\qquad -3\sqrt{D} < a < -\sqrt{D}\ ,\ c < b \le 0\ , \\ &\qquad (4a - 3b - 3c < 0) \vee (4a - 3b - 3c = 0\ \wedge\ c < 3b)\,\} \end{aligned}$$

defined in section 1.1.

The number of orbifold points on W_D of orders 2, 3, 4 and 6 are given by the following formulas.

$$e_2(D) := \begin{cases} 0\,, & D \equiv 1 \bmod 4 \quad \text{or} \quad D = 8, 12\,; \\ |\mathcal{H}_2(D)|/24\,, & \text{otherwise}\,; \end{cases}$$

$$e_3(D) := \begin{cases} 0\,, & D = 12\,; \\ |\mathcal{H}_3(D)|\,, & \text{otherwise}\,; \end{cases}$$

$$e_4(D) := \begin{cases} 1\,, & D = 8\,; \\ 0\,, & \text{otherwise}\,; \end{cases}$$

$$e_6(D) := \begin{cases} 1\,, & D = 12\,; \\ 0\,, & \text{otherwise}\,. \end{cases}$$

1.5.1. Points of order 2 and 4

Theorem 1.5.1. *The curve W_8 has one orbifold point of order 4. Moreover, no other W_D has orbifold points of order 4.*

Let D be a discriminant with conductor f_0. The number of orbifold points of order 2 in W_D is the generalised class number $e_2(D)$ defined above.

Let us recall that the Prym image of any fiber of the Clover family $\mathcal{X}$ is given by $\mathcal{P}(\mathcal{X}_t) = \mathbb{C}^2/\Lambda$, where $\Lambda = \mathcal{P}\Pi^{\mathcal{X}} \cdot \mathbb{Z}^4$ for

$$\mathcal{P}\Pi^{\mathcal{X}} = \begin{pmatrix} -\frac{1+i}{2} & 1 & 1 & 0 \\ 1 & -(1+i) & 0 & 2 \end{pmatrix}$$

and that we have $\operatorname{End}_{\mathbb{Q}} \mathcal{P}(\mathcal{X}_t) \cong M_2(\mathbb{Q}[\mathrm{i}])$.

We will first study the possible embeddings of $\mathcal{O}_D$ in $\operatorname{End} \mathcal{P}(\mathcal{X}_t)$ as self-adjoint endomorphisms.

Lemma 1.5.2. *Let A be an element of $\operatorname{End} \mathcal{P}(\mathcal{X}_t)$. The following are equivalent:*

(i) A is a self-adjoint endomorphism such that $A^2 = D \cdot \mathrm{Id}$;

(ii) $A := A_{\sqrt{D}}(a,b,c) = a \cdot A_\gamma + b \cdot A_\delta + c\mathrm{i} \cdot A_{\gamma\delta}$ for some $a, b, c \in \mathbb{Z}$ such that $a^2 + b^2 + c^2 = D$.

Proof. By Proposition 1.4.4, any element of $\operatorname{End}_{\mathbb{Q}} \mathcal{P}(\mathcal{X}_t)$ can be written as $A = a \cdot A_\gamma + b \cdot A_\delta + c \cdot A_{\gamma\delta} + d \cdot \mathrm{Id}$, with $a, b, c, d \in \mathbb{Q}[\mathrm{i}]$. By Equation 1.5 it is clear that A is self-adjoint if and only if $a, b, d \in \mathbb{Q}$ and $c \in \mathbb{Q} \cdot \mathrm{i}$. On the other hand, only scalars or pure quaternions satisfy $A^2 \in \mathbb{Q}$, hence $d = 0$. A simple calculation shows that this implies $D = A^2 = a^2 + b^2 + c^2$.

Now, one can check that the rational representation of such an element is given by

$$R_{\sqrt{D}}(a,b,c) = \begin{pmatrix} a+b+c & -2c & 0 & 2a+2c \\ a & -a-b-c & -a-c & 0 \\ 0 & 2b & a+b+c & 2a \\ -b & 0 & -c & -a-b-c \end{pmatrix},$$

therefore A induces an endomorphism if and only if $a, b, c \in \mathbb{Z}$. □

The analytic representation

$$A_{\sqrt{D}}(a,b,c) = \begin{pmatrix} b & a\cdot\frac{1-\mathrm{i}}{2} - c\cdot\frac{1+\mathrm{i}}{2} \\ a(1+\mathrm{i}) - c(1-\mathrm{i}) & -b \end{pmatrix}$$

has eigenvectors

$$\omega(a,b,c)^+ = \begin{pmatrix} \frac{-1+\mathrm{i}}{2}\cdot\frac{a-c\,\mathrm{i}}{b+\sqrt{D}} \\ 1 \end{pmatrix} \text{ and } \omega(a,b,c)^- = \begin{pmatrix} \frac{-1+\mathrm{i}}{2}\cdot\frac{a-c\,\mathrm{i}}{b-\sqrt{D}} \\ 1 \end{pmatrix}. \tag{1.11}$$

The eigenvectors (almost) determine the triple (a,b,c) and the discriminant D.

Lemma 1.5.3. *$A_{\sqrt{D}}(a,b,c)$ and $A_{\sqrt{D'}}(a',b',c')$ have the same eigenvectors if and only if*

(i) *$D = m^2E$ and $D' = m'^2E$ for some discriminant E, with $\gcd(m,m') = 1$, and*

(ii) *Both (a,b,c) and (a',b',c') are integral multiples of a triple $(a_0,b_0,c_0) \in \mathbb{Z}^3$ with $a_0^2 + b_0^2 + c_0^2 = D_0$.*

In particular, $A_{\sqrt{D}}(a,b,c)$ and $A_{\sqrt{D}}(a',b',c')$ have the same eigenvectors if and only if $(a',b',c') = \pm(a,b,c)$. More precisely: $\omega(a,b,c)^+ = \omega(-a,-b,-c)^-$ and $\omega(a,b,c)^- = \omega(-a,-b,-c)^+$.

Proof. Suppose $A_{\sqrt{D}}(a,b,c)$ and $A_{\sqrt{D'}}(a',b',c')$ have the same eigenvectors, so that

$$\frac{a-c\,\mathrm{i}}{b+\sqrt{D}} = \frac{a'-c'\,\mathrm{i}}{b'\pm\sqrt{D'}}\,.$$

This immediately implies that there has to be some discriminant E such that $D = m^2E$ and $D' = m'^2E$, where we choose $\gcd(m,m') = 1$.

The equality above is equivalent to

$$\begin{aligned} ab' \pm am'\sqrt{E} &= a'b + a'm\sqrt{E} \\ cb' \pm cm'\sqrt{E} &= c'b + c'm\sqrt{E}\,. \end{aligned}$$

Since E is not a square, this means $am' = \pm a'm$, $ab' = a'b$, $cm' = \pm c'm$ and $cb' = c'b$. Since m and m' are coprime we have

$$\begin{aligned} a &= ma_0\,, & b &= mb_0\,, & c &= mc_0\,, \quad \text{and} \\ a' &= \pm m'a_0\,, & b' &= \pm m'b_0\,, & c' &= \pm m'c_0\,. \end{aligned}$$

for some triple $(a_0,b_0,c_0) \in \mathbb{Z}^3$. Dividing both sides of $a^2+b^2+c^2 = D$ by m^2, we obtain $a_0^2 + b_0^2 + c_0^2 = E$.

The converse is immediate. □

Lemma 1.5.4. *Suppose $\mathcal{P}(\mathcal{X}_t)$ admits real multiplication by $\mathcal{O}_D$. Then $D \equiv 0 \bmod 4$.*

Moreover, there is a bijection between the choices of real multiplication $\mathcal{O}_D \hookrightarrow \operatorname{End}\mathcal{P}(\mathcal{X}_t)$ and the choices of triples (a, b, c) as in Lemma 1.5.2.

Proof. Let $\mathcal{O}_D \hookrightarrow \operatorname{End}\mathcal{P}(\mathcal{X}_t)$ be a choice of real multiplication. The rational representation R_T of the element $T \in \mathcal{O}_D$ will be given by $R_{\sqrt{D}}(a, b, c)/2$ or $(\mathrm{Id} + R_{\sqrt{D}}(a, b, c))/2$ for some (a, b, c) satisfying the conditions of Lemma 1.5.2, depending on whether $D \equiv 0$ or $1 \bmod 4$ respectively. Therefore

$$R_T(a,b,c) = \begin{cases} \begin{pmatrix} \frac{a+b+c}{2} & -c & 0 & a+c \\ \frac{a}{2} & -\frac{a+b+c}{2} & -\frac{a+c}{2} & 0 \\ 0 & b & \frac{a+b+c}{2} & a \\ -\frac{b}{2} & 0 & -\frac{c}{2} & -\frac{a+b+c}{2} \end{pmatrix}, & \text{if } D \equiv 0 \bmod 4, \\ \begin{pmatrix} \frac{1+a+b+c}{2} & -c & 0 & a+c \\ \frac{a}{2} & \frac{1-a-b-c}{2} & -\frac{a+c}{2} & 0 \\ 0 & b & \frac{1+a+b+c}{2} & a \\ -\frac{b}{2} & 0 & -\frac{c}{2} & \frac{1-a-b-c}{2} \end{pmatrix}, & \text{if } D \equiv 1 \bmod 4. \end{cases}$$

A simple parity check shows that $R_T(a, b, c)$ is always integral for $D \equiv 0 \bmod 4$ and never integral for $D \equiv 1 \bmod 4$.

Conversely, every choice of (a, b, c) gives a different embedding $\mathcal{O}_D \hookrightarrow \operatorname{End}\mathcal{P}(\mathcal{X}_t)$ by Lemma 1.5.3. □

Lemma 1.5.5. *Let $D \equiv 0 \bmod 4$ be a discriminant with conductor f_0. A form ω is an eigenform for real multiplication by $\mathcal{O}_D$ if and only if it is the eigenform of some $A_{\sqrt{D}}(a, b, c)$ with $\gcd(a, b, c, f_0) = 1$.*

Proof. By the previous lemma, any choice of real multiplication corresponds to a triple $(a, b, c) \in \mathbb{Z}^3$ as in Lemma 1.5.2.

By Lemma 1.5.3, such an embedding $\mathcal{O}_D \hookrightarrow \operatorname{End}\mathcal{P}(\mathcal{X}_t)$, $T \mapsto A_{\sqrt{D}}(a, b, c)/2$ is proper if and only if $\gcd(a, b, c, f_0) = 1$. □

Proof of Theorem 1.5.1. By Lemma 1.5.5, the set $\mathcal{H}_2(D)$ counts choices of proper real multiplication $\mathcal{O}_D \hookrightarrow \operatorname{End}\mathcal{P}(\mathcal{X}_t)$. Since every tuple $(a, b, c) \in \mathcal{H}_2(D)$ gives two eigenforms and, by Lemma 1.5.3, (a, b, c) and $(-a, -b, -c)$ give the same eigenforms, there are exactly $|\mathcal{H}_2(D)|$ eigenforms for real multiplication in $\mathcal{P}(\mathcal{X}_t)$ for each $D \equiv 0 \bmod 4$, up to scaling. By [Möl14, Prop. 4.6], each of them corresponds precisely to one element in some $\mathbb{P}\Omega(\mathcal{X}_t)^-(4)$. Recall also that, for each $t \in \mathbb{P}^*$, the isomorphism induced by the matrix Q_t, defined in Equation 1.4, allows us to see the four differentials of $\mathcal{X}_t$ given by Lemma 1.3.5 in the basis of differentials associated to $\mathcal{P}\Pi^{\mathcal{X}}$.

In the case $D = 8$, one has

$$|\mathcal{H}_2(8)| = |\{(\pm 2, \pm 2, 0), (\pm 2, 0, \pm 2), (0, \pm 2, \pm 2)\}| = 12.$$

Using Q_t, it is easy to see that the eigenforms associated to the elements of $\mathcal{H}_2(8)$ correspond to the elements of $\mathbb{P}\Omega(\mathcal{X}_2)^-(4)$. More precisely, these eigenforms coincide, up to scaling, with the images $Q_t(\omega_1^{\mathcal{X}})$, $Q_t(\omega_2^{\mathcal{X}})$, $Q_t(-\omega_1^{\mathcal{X}}+\omega_2^{\mathcal{X}})$ and $Q_t(-t\omega_1^{\mathcal{X}}+\omega_2^{\mathcal{X}})$, for $t=-1, 1/2, 2$ (recall that $\mathcal{X}_2 \cong \mathcal{X}_{-1} \cong \mathcal{X}_{1/2}$). For example, by Equation 1.11 the matrix $A_{\sqrt{8}}(2,2,0)$ has as an eigenvector $\left(\frac{1-i}{-2-\sqrt{8}}, 1\right)$, which is a multiple of $Q_2(\omega_1^{\mathcal{X}})$. As a consequence of Lemma 1.3.9, the curve W_8 has one orbifold point of order 4 and no orbifold points of order 2. In particular, no other W_D can contain a point of order 4.

Arguing the same way for $D=12$ and using Lemma 1.3.10, one finds the (unique) orbifold point of order 6 on W_{12} in accordance with Theorem 1.5.6.

Now, let $D \neq 8, 12$. By Proposition 1.3.6, we know that $\mathcal{X}_t \ncong \mathcal{X}_2$. As, by Lemma 1.3.5, for each $t \in \mathbb{P}^*$ the set $\mathbb{P}\Omega(\mathcal{X}_t)^-(4)$ has four elements and the map $t \mapsto \mathcal{X}_t$ is generically $6:1$ (cf. Lemma 1.3.2), we have to divide $|\mathcal{H}_2(D)|$ by $4 \cdot 6 = 24$ to get the correct number of orbifold points. □

1.5.2. Points of order 3 and 6

Theorem 1.5.6. *The curve W_{12} has one orbifold point of order 6. Moreover, no curve W_D has orbifold points of order 6.*

Let $D \neq 12$ be a discriminant with conductor f_0. The number of orbifold points of order 3 in W_D is the generalised class number $e_3(D)$ defined above.

In the case of the Windmill family $\mathcal{Y}$ we are, by Lemma 1.3.5, only interested in the case where $\omega_2^{\mathcal{Y}}$ is an eigenform for real multiplication. Using the bases constructed in Lemmas 1.3.4 and 1.3.8, we get the following.

Lemma 1.5.7. *The curve $\mathcal{Y}_t$ is an orbifold point of W_D if and only if the matrix*

$$A_T := \begin{pmatrix} T_D & 0 \\ 0 & -T_D \end{pmatrix}$$

is the analytic representation of an endomorphism of $\mathcal{P}(\mathcal{Y}_t)$ and $A_{T'}$ is not for all discriminants D' dividing D, where $T' = T_{D'}$.

The orbifold order of $\mathcal{Y}_t$ is 6 if $\mathcal{Y}_t \cong \mathcal{Y}_{1/2}$ and 3 otherwise.

Proof. The form ω_2 is an eigenform for real multiplication by $\mathcal{O}_D$ on $\mathcal{P}(\mathcal{Y}_t)$ if and only if there is a matrix $\left(\begin{smallmatrix} T & 0 \\ \gamma & -T \end{smallmatrix}\right)$ for some $\gamma \in \mathbb{C}$ representing a self-adjoint endomorphism of $\mathcal{P}(\mathcal{Y}_t)$ and, moreover, the corresponding action of $\mathcal{O}_D$ is proper. By the explicit description of the Rosati involution in this basis Equation 1.10, the self-adjoint condition implies $\gamma = 0$. Moreover, the action of $\mathcal{O}_D$ is proper if and only if $A_{T'}$ does not induce an endomorphism for every discriminant $D'|D$, where $T' = T_{D'}$.

The claim about the orbifold order follows from Proposition 1.3.1 and Proposition 1.3.6. □

Using the period matrix $\mathcal{P}(\mathcal{Y}_t)$ we can compute the rational representation R_T for such an A_T in terms of f and find conditions for R_T to be integral. Remember that the parameter $f = f(t)$ lives in the disc of radius $1/\sqrt{2}$.

Proposition 1.5.8. *Let $f \in \mathbb{C}$ such that $|f|^2 < 1/2$ and let $\mathcal{P}(\mathcal{Y}_t)$ be as above. The matrix A_T induces a self-adjoint endomorphism of the corresponding Prym variety if and only if there exist integers $a, b, c \in \mathbb{Z}$ such that*

(i) $2a^2 - 3b^2 - c^2 = 2D$, *and*

(ii) $f = f(a, b, c, D) := \dfrac{\sqrt{3}b\mathrm{i} + c}{2(a - \sqrt{D})}$.

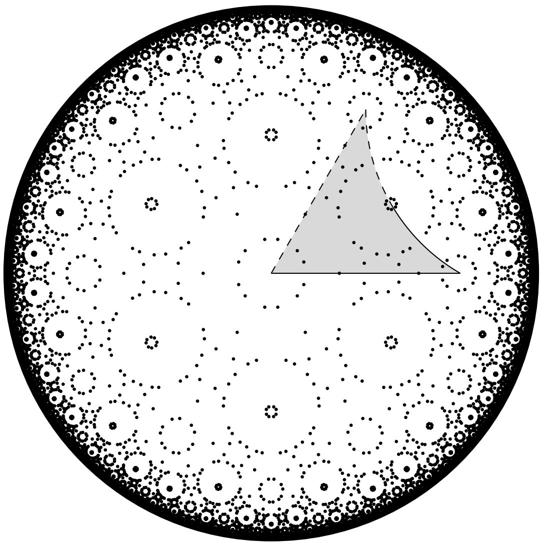


Figure 1.10.: Points in the disc of radius $1/\sqrt{2}$ satisfying the conditions of Proposition 1.5.8 for $D = 3257$ together with the fundamental domain of $\Delta(2, 6, 6)$.

Proof. Given an element of $\operatorname{End}_{\mathbb{Q}} \mathcal{P}(\mathcal{Y}_t)$ with analytic representation A, its rational representation R is given by

$$R = \begin{pmatrix} \mathcal{P}(\mathcal{Y}_t) \\ \overline{\mathcal{P}(\mathcal{Y}_t)} \end{pmatrix}^{-1} \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix} \begin{pmatrix} \mathcal{P}(\mathcal{Y}_t) \\ \overline{\mathcal{P}(\mathcal{Y}_t)} \end{pmatrix} .$$

Suppose that A_T induces a self-adjoint endomorphism. In particular, the matrix $A_{\sqrt{D}} = \left(\begin{smallmatrix} \sqrt{D} & 0 \\ 0 & -\sqrt{D} \end{smallmatrix}\right)$ also induces an endomorphism and a tedious but straightforward calculation shows that the corresponding rational representation is

$$R_{\sqrt{D}} = \begin{pmatrix} B_1 & 0 & B_3 & B_2 \\ 0 & B_1 & B_2 & B_4 \\ 2B_4 & -2B_2 & -B_1 & 0 \\ -2B_2 & 2B_3 & 0 & -B_1 \end{pmatrix} ,$$

where

$$\begin{aligned} B_1 &= \frac{\sqrt{D}(2|f|^2+1)}{2|f|^2-1}, \\ B_2 &= -\frac{2\sqrt{3}\sqrt{D}(|f|^2-f^2)\mathrm{i}}{3f(2|f|^2-1)}, \\ B_3 &= \frac{\sqrt{3}\sqrt{D}(|f|^2-f^2)\mathrm{i}}{3f(2|f|^2-1)} + \frac{\sqrt{D}(|f|^2+f^2)}{f(2|f|^2-1)} \quad \text{and} \\ B_4 &= \frac{\sqrt{3}\sqrt{D}(|f|^2-f^2)\mathrm{i}}{3f(2|f|^2-1)} - \frac{\sqrt{D}(|f|^2+f^2)}{f(2|f|^2-1)}. \end{aligned}$$

We define $a \coloneqq B_1 \in \mathbb{Z}$ and from the expression above we get that

$$|f|^2 = \frac{1}{2} \cdot \frac{a+\sqrt{D}}{a-\sqrt{D}}. \tag{1.12}$$

Moreover, since $|f|^2 - f^2 = -2\mathrm{i} \cdot f \operatorname{Im} f$, $|f|^2 + f^2 = 2f \operatorname{Re} f$ and $2|f|^2 - 1 = \frac{2\sqrt{D}}{a-\sqrt{D}}$, the expressions above imply

$$b \coloneqq B_2 = \frac{2(a-\sqrt{D})\operatorname{Im}(f)}{\sqrt{3}} \quad \text{and} \quad c \coloneqq 2B_3 - B_2 = -2B_4 + B_2 = 2(a-\sqrt{D})\operatorname{Re}(f),$$

so that

$$f = \frac{c+\sqrt{3}b\mathrm{i}}{2(a-\sqrt{D})},$$

and Equation 1.12 implies that $2a^2 - 3b^2 - c^2 = 2D$, as claimed.

Conversely, suppose that $a, b, c \in \mathbb{Z}$ satisfy the conditions of the proposition and define $f = f(a, b, c, D)$ as above. The rational representation of A_T (at the point corresponding to f) is given by $R_T = R_{\sqrt{D}}/2$ or $(\mathrm{Id} + R_{\sqrt{D}})/2$, depending on whether $D \equiv 0$ or $1 \bmod 4$, respectively, and therefore

$$R_T = \begin{cases} \begin{pmatrix} \frac{a}{2} & 0 & \frac{b+c}{2} & b \\ 0 & \frac{a}{2} & b & \frac{b-c}{2} \\ b-c & -2b & -\frac{a}{2} & 0 \\ -2b & b+c & 0 & -\frac{a}{2} \end{pmatrix}, & \text{if } D \equiv 0 \bmod 4, \\ \\ \begin{pmatrix} \frac{1+a}{2} & 0 & \frac{b+c}{2} & b \\ 0 & \frac{1+a}{2} & b & \frac{b-c}{2} \\ b-c & -2b & \frac{1-a}{2} & 0 \\ -2b & b+c & 0 & \frac{1-a}{2} \end{pmatrix}, & \text{if } D \equiv 1 \bmod 4. \end{cases}$$

Considering the equality $2a^2 - 3b^2 - c^2 \equiv 2D \bmod 8$, one sees that

- a, b and c are even if $D \equiv 0 \bmod 4$, and
- a is odd and b and c are even if $D \equiv 1 \bmod 4$

and therefore $R_T \in M_4(\mathbb{Z})$ in both cases. □

To compute the number of orbifold points on W_D, we now count, for each discriminant D, how many points $f(a,b,c,D)$ in the fundamental domain of $\Delta(2,6,6)$ satisfy the previous conditions. Recall from subsection 1.4.2 that we consider the fundamental domain for the triangle group $\Delta(2,6,6)$ depicted in Figure 1.9.

Lemma 1.5.9. *Let $\widetilde{\mathcal{H}}_3(D)$ be the set of triples of integers (a,b,c) such that*

(i) $2a^2 - 3b^2 - c^2 = 2D$;

(ii) $-3\sqrt{D} < a < -\sqrt{D}$;

(iii) $c < b \leq 0$;

(iv) Either $4a - 3b - 3c < 0$, *or* $4a - 3b - 3c = 0$ *and* $c < 3b$.

The set $\widetilde{\mathcal{H}}_3(D)$ agrees with the triples (a,b,c) in Proposition 1.5.8 that yield a point $f(a,b,c,D)$ in the fundamental domain of $\Delta(2,6,6)$.

Remark 1.5.10. *Note that $\widetilde{\mathcal{H}}_3(D)$ agrees with the set $\mathcal{H}_3(D)$ defined above except for the condition on the* gcd. *This condition will ensure that the embedding of $\mathcal{O}_D$ into* $\operatorname{End}\mathcal{P}(\mathcal{Y}_t)$ *is proper.*

Proof. Recall that we are using the fundamental domain depicted in Figure 1.9, whose vertices have been calculated in Proposition 1.4.8. Condition *(ii)* ensures that $0 \leq |f|^2 \leq 1/4$ and condition *(iii)* that $0 \leq \arg f < \pi/3$. Now, the geodesic joining $f(0)$ and $f(1)$ is an arc of circumference $|z - (3+\sqrt{3}\mathrm{i})/4|^2 = 1/4$. Therefore, f lives on the (open) half-disc containing the origin, determined by this geodesic, if and only if

$$\left| f - \frac{3+\sqrt{3}\mathrm{i}}{4} \right|^2 = \left(\frac{c}{2(a-\sqrt{D})} - \frac{3}{4} \right)^2 + \left(\frac{\sqrt{3}b}{2(a-\sqrt{D})} - \frac{\sqrt{3}}{4} \right)^2 \geq \frac{1}{4}\,.$$

Expanding this expression and using the previous conditions, one gets the first part of condition *(iv)*. Since the sides joining $f(1)$ and $f(1/2)$, and $f(1/2)$ and $f(0)$ are identified by an element of order 2 in $\Delta(2,6,6)$, we need to count only the points f that lie on one of them, say the arc of the geodesic joining $f(1)$ and $f(1/2)$. Proceeding as before, we obtain the second part of condition *(iv)*. □

Proof of Theorem 1.5.6. First note that if $D = g^2 D'$, then

$$f(a,b,c,D) = f(a',b',c',D') \quad \text{if and only if } a = ga',\ b = gb' \text{ and } c = gc'\,. \tag{1.13}$$

Since 12 is a fundamental discriminant, Lemma 1.5.7 and Lemma 1.5.9 imply that W_{12} has one orbifold point of order 6. Moreover, this is the only curve with an orbifold point of order 6 because, by Equation 1.13 above, the point $f(a,b,c,D)$ can only correspond to $t = 1/2$ if one has $D = f_0^2 D_0$ for $D_0 = 12$.

Now let $D \neq 12$. By Lemma 1.5.7 and Lemma 1.5.9, we only need to prove that $\mathcal{H}_3(D)$ is the set of triples in $\widetilde{\mathcal{H}}$ which are not contained in any $\widetilde{\mathcal{H}}_3(D')$, for discriminants $D'|D$. This is true since, by Equation 1.13, $(a,b,c) \in \widetilde{\mathcal{H}}_3(D)$ is not contained in any $\widetilde{\mathcal{H}}_3(D')$ if and only if $\gcd(a,b,c,f_0) = 1$. □

1.6. Examples

Example 1 (W_{12} and W_{20}). The curve W_{12} has genus zero, two cusps and one orbifold point of order 6, and the curve W_{20} has genus zero, four cusps and one elliptic point of order 2, cf. [Möl14, Ex. 4.4]. Our results agree with this. These are the curves $V(S_1)$ and $V(S_2)$ in [McM06].

Example 2 (W_8). By Theorem 1.5.1 and Theorem 1.5.6, we find that W_8 has one orbifold point of order 3 and one orbifold point of order 4. By [LN14, Thm. C.1] the number of cusps is $C(W_8) = 1$, the curve is connected, and by [Möl14, Thm. 0.2] the Euler characteristic is $\chi(W_8) = -5/12$. We can then use Equation 1.1 to compute its genus as $g(W_8) = 0$.

Example 3 (W_{2828}). Theorem 1.5.1 and Theorem 1.5.6 also tell us that W_{2828} has six orbifold points of order 2. They correspond to the $|\mathcal{H}_2(2828)| = 144$ eigenforms for real multiplication by $\mathcal{O}_{2828}$ in $\mathcal{P}(\mathcal{X}_t)$, as in Equation 1.11, divided by 24. In Figure 1.11, we depict the first coordinate of these eigenforms in the complex plane.

As for the orbifold points of order 3, there are twenty of them. They correspond to the twenty points on the Shimura curve isomorphic to $\mathbb{D}/\Delta(2,6,6)$ admitting proper real multiplication by $\mathcal{O}_{2828}$. In Figure 1.11, we depict the preimage of these 20 points in $\mathbb{D}$, that is the points $f(a,b,c,2828)$ as in Proposition 1.5.8.

The number of cusps is $C(W_{2828}) = 68$, the curve is connected, and the Euler characteristic is $\chi(W_{2828}) = -8245/3$. Therefore, by Equation 1.1, the genus is $g(W_{2828}) = 1333$.

1.7. Flat geometry of orbifold points

In this section we will briefly describe the translation surfaces corresponding to the Windmill family and to the Clover family.

Recall that, by Lemma 1.3.5, the general member $\mathcal{Y}_t$ of the Windmill family has only one differential with a single zero, namely $\omega_2^{\mathcal{Y}}$. Flat surfaces $(\mathcal{Y}_t, \omega_2^{\mathcal{Y}})$ arise from the following *double windmill* construction, which also explains the name Windmill family: for each period $\tau \in \mathbb{C}$ consider the "blade" depicted on the left side of Figure 1.12, where $\overrightarrow{AF} = \tau$,

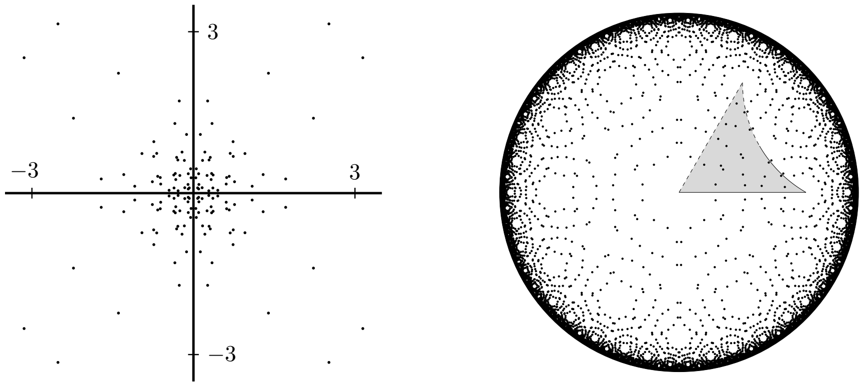


Figure 1.11.: Orbifold points of order 2 and 3 in W_{2828}.

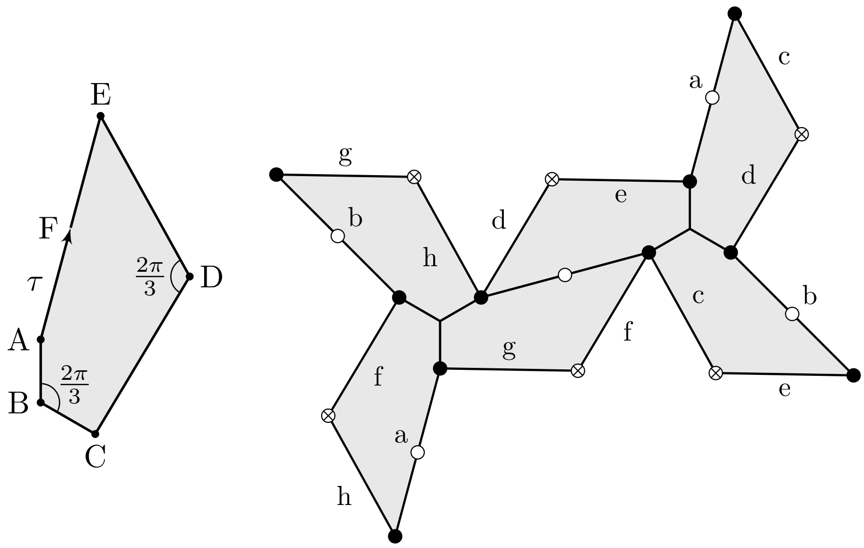


Figure 1.12.: Double windmill for the period $\tau \in \mathbb{C}$.

$|AF| = |EF|$, $|AB| = |BC|$ and $|CD| = |DE|$. We normalise the differential by fixing the edge $\overrightarrow{AB}$ to be i. Now take 6 copies of the blade and glue them together with side pairing as in the right side of the picture. One can check that this yields a genus 3 curve and that the corresponding differential has a unique zero, namely the black point in the picture. Moreover, there is an obvious order 6 automorphism α of the curve, induced by the composition of a rotation of order three on each of the two windmills and a rotation of order two of the whole picture around the white point on the common side of the two windmills. This automorphism fixes the black point and exchanges cyclically the three white points, the two centers of the windmills and the two crossed points, respectively.

It is again easy to check that α^3 corresponds to the Prym involution. Therefore, the corresponding curve belongs to the Windmill family. The black point corresponds to the preimage of ∞ under the cyclic cover $\mathcal{Y}_t \to \mathbb{P}^1$, the three white points correspond to the preimages of t, the two crossed points to the preimages of 1 and the centers of the windmills to the two preimages of 0.

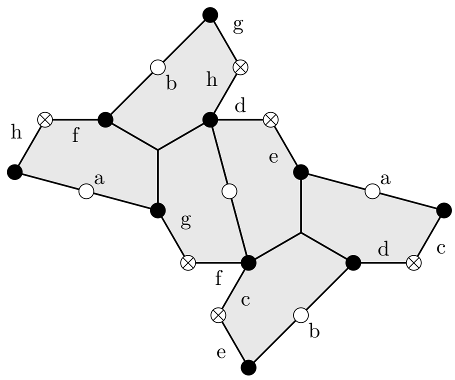


Figure 1.13.: Double windmill corresponding to the special point $(\mathcal{Y}_{1/2}, \omega_2^{\mathcal{Y}})$.

Example 4. The special point $\mathcal{Y}_{1/2}$ has an extra automorphism β of order 12. The corresponding flat surface is depicted below in Figure 1.13. The automorphism β corresponds to first rotating each of the blades by $\pi/2$ around each of the white points and reglueing, and then composing with α.

Example 5. Each component of the Prym–Teichmüller curve W_{17} has one orbifold point of order 3 (cf. [Zac16]). Using the lengths described in [LN14] and Mukamel's implemented algorithm from [Muk12], one finds that this orbifold point corresponds to the S-shaped table depicted in Figure 1.14, where

$$\begin{aligned} d &= \left(\frac{11\sqrt{17} - 35}{52}, -\sqrt{3} \cdot \frac{(17\sqrt{17} - 73)}{52} \right) , \\ a &= \left(\frac{\sqrt{17} - 1}{2}, 0 \right) , \\ b = e &= \left(\frac{-\sqrt{17} + 5}{2}, 0 \right) , \\ f = c &= \left(\frac{-3\sqrt{17} - 33}{52}, \sqrt{3} \cdot \frac{(7\sqrt{17} - 27)}{52} \right) . \end{aligned}$$

Since the automorphism α of order 6 fixes the zero of the differential and interchanges cyclically the preimages of 0, 1 and t respectively, one can easily detect these points. The three preimages of t are, together with the preimage of ∞, the fixed points of the Prym involution, which is just a rotation of the whole picture through an angle of π. Therefore, they correspond to the center of the S-table and to the midpoints of edges a and d.

As for the preimages of 0 and 1, they can be found as the fixed points of α^2. Since the angle around the zero of the differential is 10π, the automorphism α corresponds to a rotation of angle $10\pi/6$ around that point.

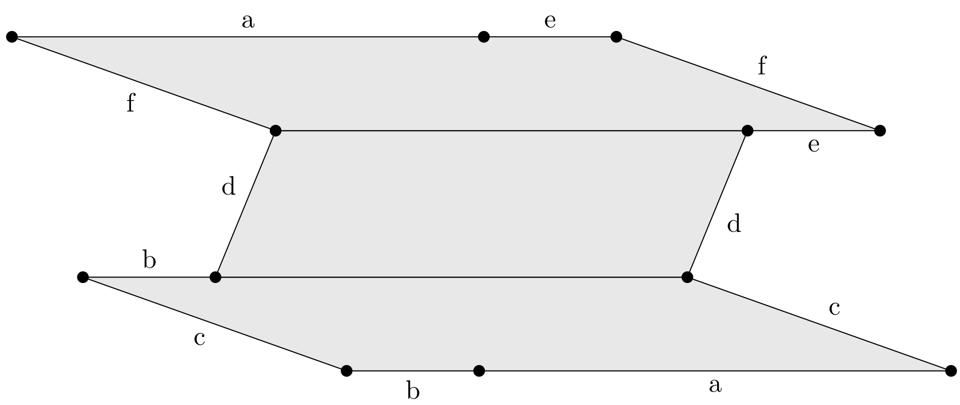


Figure 1.14.: S-shaped table for the orbifold point of order 3 on W_{17} (Y axis scaled by a factor of 5).

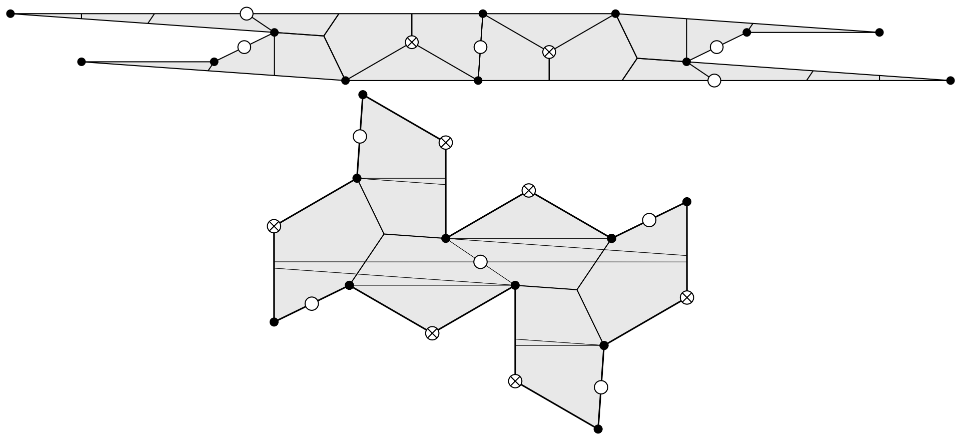


Figure 1.15.: Double windmill cut and pasted from the S-shaped table corresponding to the orbifold point of order three on W_{17}.

Cutting appropriately the S-shaped table into pieces and reglueing them yields the double windmill in Figure 1.15. Note that in this case the differential is not normalised in the same way as in our construction.

One can similarly construct the flat surfaces associated to the Clover family via the following *four-leaf clover* construction, which is again responsible for the name. Let us consider the differential $\omega_1^{\mathcal{X}}$ in $\Omega(\mathcal{X}_t)$, which by Lemma 1.3.5 has a zero at the preimage of ∞. Flat surfaces $(\mathcal{X}_t, \omega_1^{\mathcal{X}})$ can be constructed in the following way: for each period $\tau \in \mathbb{C}$ we consider the "blade" on the left side of Figure 1.16, where $\overrightarrow{AF} = \tau$, $|AF| = |EF|$, $|AB| = |BC|$ and $|CD| = |DE|$. We again normalise the differential by fixing the edge $\overrightarrow{AB}$ to be i. Now we glue 4 copies of the blade with side pairings as in the right side of the picture. Again, this yields a genus 3 curve together with an abelian differential with a single zero, namely the black point in the picture. The order 4 automorphism α induced by a rotation of order four around the center of the windmill fixes four points: the center, the black point, the white point and the crossed point. The square α^2 corresponds to the

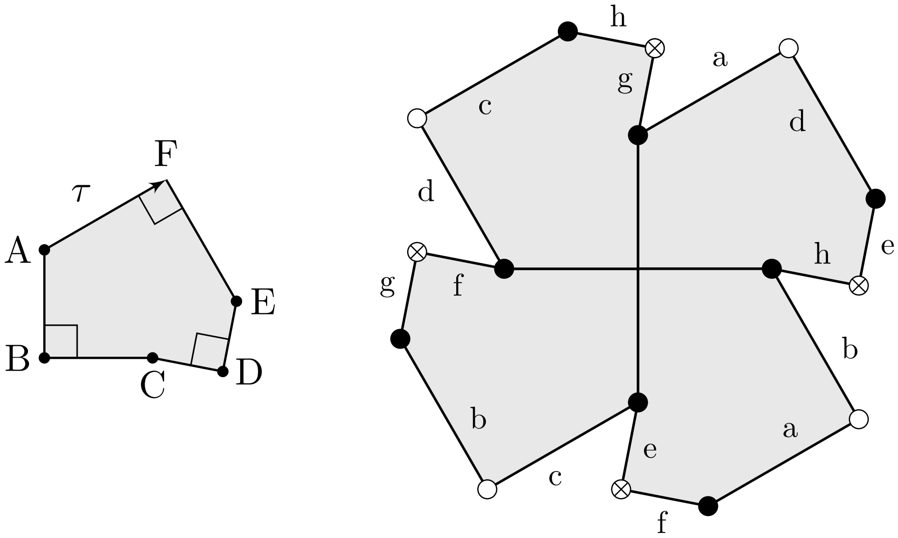


Figure 1.16.: Flat surface corresponding to $(\mathcal{X}_t, \omega_1^{\mathcal{X}})$ for the period $\tau \in \mathbb{C}$.

Prym involution, and therefore the corresponding curve belongs to the Clover family. In our construction, the black point corresponds to the preimage of ∞ under the cyclic cover $\mathcal{X}_t \to \mathbb{P}^1$, the white point to the preimage of t, the crossed point to the preimage of 1 and the center of the windmill to the preimage of 0.

D	χ	C	g	e_2	e_3	D	χ	C	g	e_2	e_3
17	−5/3	3	0	0	1	137	−40	22	9	0	3
20	−5/2	4	0	1	0	140	−95/3	12	9	2	4
24	−5/2	4	0	1	0	145	−160/3	32	11	0	2
28	−10/3	4	0	0	2	148	−125/2	36	14	1	0
32	−5	7	0	0	0	152	−205/6	12	10	3	4
33	−5	7	0	0	0	153	−50	30	11	0	0
40	−35/6	6	0	1	2	156	−130/3	16	14	0	2
41	−20/3	8	0	0	1	160	−70	42	15	0	0
44	−35/6	6	0	1	2	161	−160/3	22	16	0	2
48	−10	10	1	0	0	164	−60	32	14	4	0
52	−25/2	12	1	1	0	168	−45	16	15	2	0
56	−25/3	6	1	2	2	172	−105/2	22	14	1	6
57	−35/3	11	1	0	1	176	−70	30	21	0	0
60	−10	8	2	0	0	177	−65	31	18	0	0
65	−40/3	12	1	0	2	180	−75	32	22	2	0
68	−15	14	1	2	0	184	−185/3	22	19	2	4
72	−25/2	10	2	1	0	185	−190/3	26	19	0	2
73	−55/3	17	1	0	2	188	−140/3	12	17	0	4
76	−95/6	14	1	1	2	192	−80	36	23	0	0
80	−20	16	3	0	0	193	−245/3	39	21	0	4
84	−25	16	5	2	0	200	−325/6	18	17	3	4
88	−115/6	16	1	1	4	201	−245/3	37	23	0	1
89	−65/3	15	4	0	1	204	−65	28	19	2	0
92	−50/3	8	4	0	4	208	−100	48	27	0	0
96	−30	20	6	0	0	209	−235/3	35	22	0	2
97	−85/3	21	4	0	2	212	−175/2	28	30	3	0
104	−125/6	10	5	3	2	216	−135/2	32	18	3	0
105	−30	18	7	0	0	217	−290/3	42	27	0	4
108	−45/2	14	5	1	0	220	−230/3	32	22	0	4
112	−40	24	9	0	0	224	−100	34	34	0	0
113	−30	18	6	0	3	228	−105	46	30	2	0
116	−75/2	20	9	3	0	232	−165/2	30	25	1	6
120	−85/3	12	8	2	2	233	−265/3	29	29	0	5
124	−100/3	16	9	0	2	236	−425/6	26	22	3	2
128	−40	22	10	0	0	240	−120	40	41	0	0
129	−125/3	25	9	0	1	241	−355/3	49	35	0	2
132	−45	30	8	2	0	244	−275/2	52	43	3	0
136	−115/3	20	9	2	2	248	−70	14	26	4	6

Table 1.2.: Topological invariants of the Prym–Teichmüller curves W_D for D up to 248. For $D \equiv 1 \bmod 8$, we give the homeomorphism type of one of the two homeomorphic components, cf. chapter 2 or [Zac16].

2. The Galois action and a spin invariant for Prym–Teichmüller curves in genus 3

Given a Prym–Teichmüller curve in $\mathcal{M}_3$, the aim of this chapter is to provide an invariant that sorts the cusp prototypes of Lanneau and Nguyen by component. This can be seen as an analogue of McMullen's genus 2 spin invariant, although the source of this invariant is different. Moreover, we describe the Galois action on the cusps of these Teichmüller curves, extending the results of Bouw and Möller in genus 2. We use this to show that the components of the genus 3 Prym–Teichmüller curves are homeomorphic. The content of this chapter appears in published form as [Zac16].

2.1. Introduction

A *Teichmüller curve* is a curve inside the moduli space $\mathcal{M}_g$ of smooth projective genus g curves that is totally geodesic for the Teichmüller metric. Every Teichmüller curve arises as the projection of the $\mathrm{GL}_2^+(\mathbb{R})$ orbit of a flat surface (see section 2.2 and the references therein for background and definitions). Only a few infinite families of primitive Teichmüller curves are known. McMullen constructed several primitive families in low genera, among them, for every non-square discriminant D, the *Prym–Teichmüller* or *Prym–Weierstraß* curves W_D in genus 3 [McM06].

This family is fairly well understood. In particular, Möller calculated the Euler characteristic [Möl14], Lanneau and Nguyen enumerated the cusps and connected components [LN14], and the number and type of orbifold points are determined in [TTZ16]. The aim of this note is to complete the classification of the topological components by showing that the connected components of W_D are always homeomorphic.

To be more precise, in [LN14], Lanneau and Nguyen show that W_D has at most two components for any D and has two components if and only if $D \equiv 1 \bmod 8$.

Theorem 2.1.1. *Let $D \equiv 1 \bmod 8$, which is not a square. Then the two components of W_D are homeomorphic (as orbifolds). In particular, they have the same number of cusps and elliptic points.*

A similar result was obtained by Bouw and Möller [BM10a] for Teichmüller curves in genus 2. Note that a Teichmüller curve is always defined over a number field but is never compact. Both approaches rely on determining the stable curves associated to the cusps of the Teichmüller curve and describing explicitly the Galois action on these cusps. At this point, it is crucial that we are able to determine of a pair of cusps if they lie on the same component or not. In genus 2, Bouw and Möller could use McMullen's spin invariant [McM05b] to achieve this.

However, while Lanneau and Nguyen list prototypes corresponding to the cusps of Prym–Teichmüller curves [LN14], they do not provide an effective analogue of the spin invariant. Here we give such an invariant, which is, moreover, easy to compute.

Theorem 2.1.2. *Let $D \equiv 1 \bmod 8$, which is not a square. Given a cusp prototype $[w, h, t, e, \varepsilon]$ (see section 2.2), the associated cusp of W_D lies on the component W_D^i if and only if*

$$2i \equiv e + \varepsilon \bmod 4,$$

for $i = 1, 2$.

In section 2.3, we prove Theorem 2.1.2 essentially using topological arguments. More precisely, we analyse the intersection pairing on a certain intrinsic subspace of homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients. This is similar to the approach of [McM05b] where the Arf invariant of a quadratic form that was associated to the flat structure was analysed on such a subspace, but the nature of these subspaces is different (cf. [LN14, Remark 2.9]). Note also that in genus 3 the two components lie on disjoint Hilbert Modular Surfaces (cf. [Möl14, Proposition 4.6]) and that the $(1, 2)$-polarisation of the Prym variety plays a special role in this case, essentially yielding a much more compact formula (cf. [McM05b, Theorem 5.3]).

In section 2.4, we proceed to give an explicit description of the Galois action on Lanneau and Nguyen's cusp prototypes (Proposition 2.4.6) and combine this with Theorem 2.1.2 to show that Galois-conjugate cusps always lie on different components of W_D, thus proving Theorem 2.1.1.

Acknowledgements I am very grateful to my advisor, Martin Möller, for many helpful discussions and comments and to Matteo Costantini and Robert Kucharczyk for valuable conversations. I also thank the anonymous referees for useful suggestions, in particular regarding Lemma 2.2.1. I thank [Par] for computational assistance.

2.2. Cusp prototypes

A *flat surface* is a pair (X, ω) where X is a compact Riemann surface of genus g and $\omega \in \mathrm{H}^0(X, \omega_X)$ is a holomorphic 1-form on X. Note that X obtains a flat structure away from the zeros of ω via integrating ω and affine shearing of this flat structure gives an action of $\mathrm{GL}_2^+(\mathbb{R})$. A *Teichmüller curve* is a $\mathrm{GL}_2^+(\mathbb{R})$ orbit of a flat surface that projects to an algebraic curve inside the moduli space $\mathcal{M}_g$. See e.g. [Möl11b] for background on Teichmüller curves and flat surfaces. Not many families of primitive Teichmüller curves are known; McMullen constructed families in low genera by requiring a factor of the Jacobian of X to admit real multiplication, the (Prym–)Weierstraß curves. We briefly review the construction in genus 3, the case with which we are concerned.

Prym varieties and real multiplication Let $D \equiv 0, 1 \bmod 4$ be a (positive) non-square discriminant and denote by $\mathcal{O}_D$ the corresponding order in the real quadratic number field $\mathbb{Q}(\sqrt{D})$. Let X be a genus 3 curve and ρ an involution with X/ρ of genus 1. Then we define the *Prym Variety* $\mathrm{Prym}(X, \rho)$ as the connected component of the identity of $\ker(\mathrm{Jac}(X) \to \mathrm{Jac}(X/\rho))$ and we say that (X, ρ) *admits real multiplication by* $\mathcal{O}_D$ if there exists an injective ring homomorphism $\iota\colon \mathcal{O}_D \to \mathrm{End}\,\mathrm{Prym}(X, \rho)$, such that

- every endomorphism $\iota(s)$ is self-adjoint with respect to the intersection pairing on H_1, and
- ι cannot be extended to any $\mathcal{O}_{D'} \supset \mathcal{O}_D$.

In other words, the ρ-anti-invariant part $\mathrm{H}_1(X, \mathbb{Z})^-$ of the homology admits a symplectic $\mathcal{O}_D$-module structure and $\mathcal{O}_D$ is maximal in this respect.

Prym–Weierstraß curves Denote by W_D the space of genus 3 flat surfaces (X, ω, ρ, ι) with an involution ρ that admit real multiplication ι as above and where additionally ω has a single (4-fold) zero, is ρ-anti-invariant, and is an eigenform for the induced action of $\mathcal{O}_D$ on $\mathrm{H}^0(X, \omega_X)$. McMullen [McM06] showed that W_D is a union of Teichmüller curves, the *genus* 3 *Prym–Weierstraß* or *Prym–Teichmüller curves of discriminant* D. Prym–Weierstraß curves have been studied intensely, see e.g. [McM06], [Möl14], [LN14] and [TTZ16]. Note, in particular, that W_D is empty for $D \equiv 5 \bmod 8$.

Again, we note that we explicitly exclude the case that $D = d^2$ is a square, see [LN14, Appendix B] for some results in this case.

Cusps Recall that a Teichmüller curve $\mathcal{C}$ is never compact. We describe the cusps first in the terminology of flat surfaces. Let (X, ω) be a flat surface generating $\mathcal{C}$ and consider a direction $v \in \mathbb{P}^1(\mathbb{R})$. Recall that a geodesic segment is said to be a *saddle connection* if its endpoints are (not necessarily distinct) zeros of ω and its interior contains no zeros of ω. The direction v is said to be *periodic* if all geodesics in direction v are either closed or saddle connections. We say that a *cylinder* is a maximal union of homotopic geodesics on (X, ω) and any closed geodesic inside a cylinder is a *core curve*. The length of a core curve is the *width* of the cylinder. A cylinder is called *simple* if each boundary consists of a saddle connection. The cusps of $\mathcal{C}$ are in one-to-one correspondence with the parabolic *cylinder decompositions* on (X, ω), see e.g. [McM05b, §4], [Vee89, §2] or [Möl11b, §5.4].

Prototypes To describe the cusps of W_D, Lanneau and Nguyen introduce prototypes that encode the cylinder decompositions [LN14, §3,4 and C]. We briefly summarise the results we need.

The following result is a slight refinement of [LN14, Proposition 3.2].

Lemma 2.2.1. *Given D non-square and a point (X, ω) on W_D, any periodic direction decomposes (X, ω) into three cylinders.*

Proof. By [LN14, Proposition 3.2], any periodic direction decomposes (X, ω) into either three cylinders, or two cylinders that are permuted by the Prym involution or one cylinder (that is fixed by the Prym involution). Obviously, in the last two cases, the ratio of cylinder circumferences is 1. However, [Wri15a, Theorem 1.9] asserts that adjoining the ratio of cylinder circumferences to $\mathbb{Q}$ gives the trace field of (X, ω), which is $\mathbb{Q}(\sqrt{D})$ (cf. [McM06, Corollary 3.6]), a contradiction. □

Remark 2.2.2. *Lemma 2.2.1 can be seen as a converse to [LN14, Corollary 3.4].*

Following [LN14], after rescaling, applying Dehn-twists, and normalising so that the horizontal direction is periodic, this decomposition may be encoded in a combinatorial prototype

$$P_D = [w, h, t, e, \varepsilon] \in \mathbb{Z}^5$$

subject to the following conditions:

$$\begin{cases} D = e^2 + 8wh, \ \varepsilon = \pm 1, \ w, h > 0, \\ w > \frac{\lambda}{2}, \ 0 \le t < \gcd(w, h), \ \gcd(w, h, t, e) = 1, \end{cases}$$

where we set

$$\lambda := \lambda_P := \frac{e + \sqrt{D}}{2}. \tag{2.1}$$

Moreover, if $\varepsilon = 1$, the stronger condition $w > \lambda$ is required.

Conversely, given a combinatorial prototype, we obtain a three-cylinder decomposition into one of the following three geometric types (see Figure 2.1):

- $A+$: If $\varepsilon = 1$ and $\lambda < w$, we obtain a cylinder decomposition with a single (short) simple cylinder of width and height λ and two cylinders of width w, height h and twist t.
- $A-$: If $\varepsilon = -1$ and $\lambda < w$, we obtain a cylinder decomposition with two (short) simple cylinders of width and height $\lambda/2$ and a third cylinder of width w, height h and twist t.
- B: If $\varepsilon = -1$ and $\lambda/2 < w < \lambda$, we obtain a cylinder decomposition with no simple cylinders but again two short cylinders of width and height $\lambda/2$ and a third cylinder of width w, height h and twist t.

Each geometric prototype corresponds to exactly one cusp of W_D.

2.3. Components and spin

In analogy to the situtation in genus 2, Lanneau and Nguyen showed that, for any discriminant D, the locus W_D has at most two components [LN14, Theorem 2.8, 2.10]. More precisely, W_D has two components if and only if $D \equiv 1 \bmod 8$. In the following, we denote these components by W_D^1 and W_D^2.

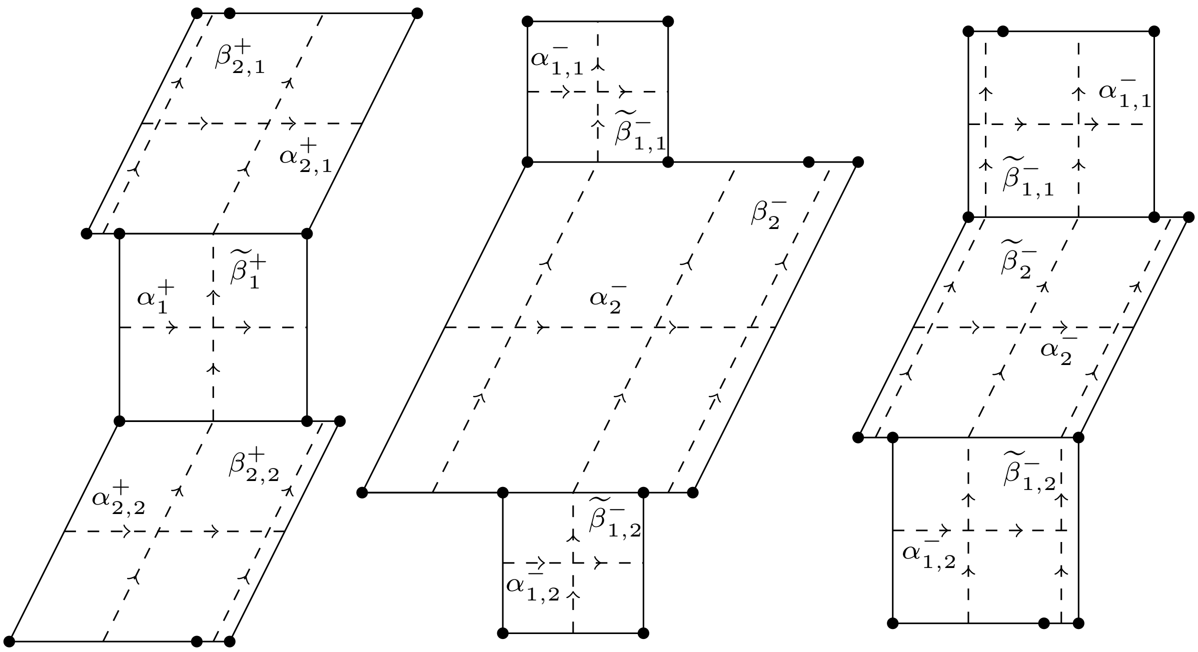


Figure 2.1.: Prototypes of geometric type $A+$, $A-$ and B. Observe that all α_i are drawn in a horizontal direction, the β_i are drawn vertical. We set $\alpha_i = \alpha_{i,1} + \alpha_{i,2}$ and $\beta_i = \beta_{i,1} + \beta_{i,2}$ when appropriate, and furthermore, for the $A+$ prototype, $\beta_1^+ = \widetilde{\beta}_1^+ - \beta_2^+$, for the $A-$ prototype, $\beta_{1,i}^- = \widetilde{\beta}_{1,i}^- - \beta_2^-$, and, for the B prototype, $\beta_2^- = \widetilde{\beta}_{1,1}^- + \widetilde{\beta}_{1,2}^- - \widetilde{\beta}_2^-$ and $\beta_{1,i}^- = \widetilde{\beta}_{1,i}^- - \beta_2^-$. Thus the α_i and β_i give symplectic bases whose periods describe the cylinder heights and widths.

The aim of this section is to provide an analogue of McMullen's spin invariant in genus 2 [McM05b], i.e. an invariant that determines if a cusp prototype is associated to a cusp on W_D^1 or W_D^2.

To each geometric prototype $P_D = [w, h, t, e, \varepsilon]$, Lanneau and Nguyen associate a basis $\mathfrak{b} = \mathfrak{b}(P_D)$ of $\mathrm{H}_1(X, \mathbb{Z})^-$ "spanning cylinders", cf. [LN14, §4]. We will see that, in fact, the behaviour of the basis will depend only on ε, i.e. geometric type $A-$ and B will not be distinguished. Hence, we denote the bases by

$$\mathfrak{b}^\varepsilon = (\alpha_1^\varepsilon, \alpha_2^\varepsilon, \beta_1^\varepsilon, \beta_2^\varepsilon),$$

where α_i and β_i are as in Figure 2.1. In particular, the periods (with respect to ω) are

$$\int_{\alpha_1^+} \omega = \lambda, \quad \int_{\alpha_2^+} \omega = 2w, \quad \int_{\beta_1^+} \omega = \mathrm{i}\lambda, \quad \int_{\beta_2^+} \omega = 2t + 2\mathrm{i}h \tag{2.2}$$

if P_D is of geometric type $A+$ (i.e. $\varepsilon = 1$) and

$$\int_{\alpha_1^-} \omega = \lambda, \quad \int_{\alpha_2^-} \omega = w, \quad \int_{\beta_1^-} \omega = \mathrm{i}\lambda, \quad \int_{\beta_2^-} \omega = t + \mathrm{i}h \tag{2.3}$$

if P_D is of geometric type $A-$ or B (i.e. $\varepsilon = -1$).

Moreover, the intersection form on $\mathrm{H}_1(X, \mathbb{Z})^-$ is of type $(1, 2)$. Clearly, it is described by the matrices

$$\langle \cdot, \cdot \rangle_{\mathfrak{b}^+} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \langle \cdot, \cdot \rangle_{\mathfrak{b}^-} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{2.4}$$

In particular, $\langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle = 0$ for any i, j and $\langle \alpha_i, \beta_j \rangle$ is nonzero iff $i = j$.

Recall that, for $D \equiv 1 \bmod 4$, the quadratic order is $\mathcal{O}_D = \mathbb{Z} \oplus T\mathbb{Z}$, where

$$T = \frac{1 + \sqrt{D}}{2}.$$

As $(X, \omega) \in W_D$ admits real multiplication ι, $\mathrm{H}_1(X, \mathbb{Z})^-$ is an $\mathcal{O}_D$-module. In particular, for odd D, we may view T as an endomorphism $\iota(T)$ on $\mathrm{H}_1(X, \mathbb{Z})^-$. We now describe this endomorphism on the cusp prototypes. Note that this calculation essentially appears already in [LN14, §4], but due to differences in notation and for the convenience of the reader, we briefly restate the result.

Lemma 2.3.1. *Let D be an odd discriminant. Given a prototype $P_D = [w, h, t, e, \varepsilon]$ associated to a flat surface (X, ω) the endomorphism $\iota(T)$ acts on $\mathrm{H}_1(X, \mathbb{Z})^-$ in the basis $\mathfrak{b}(P_D) = \mathfrak{b}^\varepsilon$ by $\iota(T)_{P_D} = \iota(T)^\varepsilon$, where*

$$\iota(T)^+ = \begin{pmatrix} \frac{e+1}{2} & 2w & 0 & 2t \\ h & -\frac{e-1}{2} & -t & 0 \\ 0 & 0 & \frac{e+1}{2} & 2h \\ 0 & 0 & w & -\frac{e-1}{2} \end{pmatrix} \text{ and } \iota(T)^- = \begin{pmatrix} \frac{e+1}{2} & w & 0 & t \\ 2h & -\frac{e-1}{2} & -2t & 0 \\ 0 & 0 & \frac{e+1}{2} & h \\ 0 & 0 & 2w & -\frac{e-1}{2} \end{pmatrix}.$$

Note that e is odd iff D is odd.

Proof. Note first that $T = \lambda - \frac{e-1}{2}$ (cf. (2.1)) and that any $\gamma \in \mathrm{H}_1(X, \mathbb{Z})^-$ satisfies

$$\int_{\iota(T)\cdot\gamma} \omega = \int_\gamma \iota(T)^\vee \omega = T \cdot \int_\gamma \omega,$$

as ω is an eigenform. Now, using the periods of $\mathfrak{b}^\pm$ in (2.2) and (2.3), as well as the identities $\lambda^2 = e\lambda + 2wh$ and $T \cdot \lambda = 2wh + \lambda\frac{e+1}{2}$, the representations $\iota(T)^\pm$ are obtained by a straight-forward calculation. □

We are now in a position to describe the restriction of the intersection pairing $\langle \cdot, \cdot \rangle$ to the image of the endomorphism $\iota(T)$ in $\mathrm{H}_1(X, \mathbb{Z}/2\mathbb{Z})^-$ (for D odd). To ease notation, we will no longer distinguish T and $\iota(T)$, as no confusion can arise.

Proposition 2.3.2. *Let D be an odd discriminant and T the endomorphism from above. Let (X, ω) be the geometric prototype associated to the cusp prototype $P_D = [w, h, t, e, \varepsilon]$. Then*

$$\langle \cdot, \cdot \rangle|_{\operatorname{Im} T} \equiv 0 \bmod 2 \iff e + \varepsilon \equiv 0 \bmod 4,$$

where $\langle \cdot, \cdot \rangle|_{\operatorname{Im} T}$ is the restriction of the intersection pairing on $\mathrm{H}_1(X, \mathbb{Z})^-$ to the image of T.

Proof. We begin by observing that, as T is self-adjoint by the condition on real multiplication, we have $\langle T\gamma, T\delta \rangle = \langle T^2\gamma, \delta \rangle$ for any $\gamma, \delta \in \mathrm{H}_1(X, \mathbb{Z})^-$. Moreover, by (2.4), any two elements $b_1, b_2 \in \mathfrak{b}^\varepsilon$ satisfy

$$\langle b_1, b_2 \rangle \not\equiv 0 \bmod 2 \iff \{b_1, b_2\} = \begin{cases} \{\alpha_1^+, \beta_1^+\}, & \text{if } \varepsilon = 1, \\ \{\alpha_2^-, \beta_2^-\}, & \text{if } \varepsilon = -1. \end{cases}$$

Therefore, by checking mod 2 the $1,1$ entry of $(T^+)^2$ and the $2,2$ entry of $(T^-)^2$, we find (using $D = e^2 + 8wh$) that

$$\langle \cdot, \cdot \rangle_\pm|_{\operatorname{Im} T^\pm} \equiv 0 \bmod 2 \iff e \pm 1 = e + \varepsilon \equiv 0 \bmod 4,$$

as claimed. □

Remark 2.3.3. *Note that Lanneau and Nguyen use a similar idea (restriction of the intersection pairing to the image of an operator* mod 2*) to show that there are in fact two distinct components of W_D for $D \equiv 1 \bmod 8$ [LN14, Theorem 6.1]. However, they use a different operator $T = T(P)$ for every prototype and this does not seem a feasible invariant.*

Proof of Theorem 2.1.2. Let D be an odd discriminant. We denote by $\mathcal{X} \to W_D$ the universal family over the Teichmüller curve W_D, see [Möl06, §1.4]. By definition of W_D, each fibre $\mathcal{X}_t$ has an involution ρ_t and the real multiplication gives endomorphisms T_t of $\mathrm{H}_1(\mathcal{X}_t, \mathbb{Z})^-$, allowing us to consider the restriction of the intersection form $\langle \cdot, \cdot \rangle_t$ to the image of T_t and take $\mathbb{Z}/2\mathbb{Z}$ coefficients. In particular, the map

$$t \mapsto \langle \cdot, \cdot \rangle|_{\operatorname{Im} T_t} \bmod 2$$

is continuous and as the range (the space of bilinear operators on an $\mathbb{F}_2$ vector space) is discrete, it is locally constant. Now, Proposition 2.3.2 asserts that two cusp prototypes P_D, P'_D are associated to cusps on the same component if and only if $e + \varepsilon \equiv e' + \varepsilon' \bmod 4$ and, as any such e must be odd, this yields the claim. □

2.4. The Galois action on the components

The aim of this section is to prove Theorem 2.1.1. The idea is to show that, for $D \equiv 1 \bmod 8$, the two components of W_D are in fact Galois-conjugate in analogy to the situation in genus 2 (cf. [BM10a, Theorem 3.3]).

To achieve this, we first describe algebraic models of the stable curves associated to the cusps of W_D and then describe the Galois-action on these curves explicitly.

Stable curves While a Teichmüller curve $\mathcal{C}$ is never compact, it admits a smooth completion $\overline{\mathcal{C}}$. Moreover, after passing to a finite cover, we may pull back the universal family over $\mathcal{M}_g$ to $\mathcal{C}$, thus obtaining a family of curves, which we – by abuse of notation – also denote by $\mathcal{X} \to \mathcal{C}$ and which extends to a family of stable curves $\overline{\mathcal{X}} \to \overline{\mathcal{C}}$, cf. [Möl06, §1.4].

Much of the geometry of the stable fibres is given by the flat structure. By the above, given a flat surface (X, ω) on $\mathcal{X}$ together with a periodic direction v, we may associate a cusp $(X_\infty, \omega_\infty)$ to (X, ω, v), where X_∞ is a stable curve and ω_∞ is a stable differential on X_∞, see e.g. [Möl11b, §2.5 and §5.4]. In particular, X_∞ is obtained from X topologically by contracting the core curves of cylinders and ω_∞ has poles with residue equal to the cylinder widths at the nodes of X_∞.

Lemma 2.4.1. *Let $c \in \overline{W_D} \setminus W_D$ be a point such that the fibre $X_\infty = \overline{\mathcal{X}}_c$ is singular. Then X_∞ is a trinodal curve, i.e. X_∞ is a projective line with three pairs of points identified.*

Proof. This follows immediately from [Möl11b, Corollary 5.11]: let $(X_\infty, \omega_\infty)$ be the stable flat surface associated to c. Then, as every component of X_∞ must contain a zero of ω_∞, the stable curve X_∞ is irreducible. Moreover, $(X_\infty, \omega_\infty)$ is obtained by contracting the core curves of a cylinder decomposition on some $(X, \omega) \in W_D$. But by Lemma 2.2.1, any such (X, ω) decomposes into three cylinders, hence X_∞ is obtained topologically by contracting three (homologically independent) curves on a genus 3 Riemann surface and therefore has geometric genus 0 and three nodes. □

Using the prototypes of [LN14] from section 2.2, we can describe the singular fibres of $\overline{W_D}$ more explicitly, in the spirit of [BM10a, Proposition 3.2].

Proposition 2.4.2. *The stable curve above the cusp associated to the combinatorial prototype $[w, h, t, e, \varepsilon]$ may be normalised by a projective line with six marked points: $\pm 1, \pm x_1$, and $\pm x_3$, where*

$$x_1 = -s - \sqrt{\frac{1-s^2}{3}} \quad \text{and} \quad x_3 = -s + \sqrt{\frac{1-s^2}{3}} \quad \text{for} \quad s = \begin{cases} \dfrac{e+\sqrt{D}}{4w}, & \text{if } \varepsilon = 1, \\ \dfrac{2w}{e+\sqrt{D}}, & \text{if } \varepsilon = -1, \end{cases}$$

and the pairs of points $(+1, -1)$, $(x_1, -x_3)$, *and* $(x_3, -x_1)$ *are identified in the stable model.*

In particular, the absolute value of s *uniquely determines the stable fibres.*

Proof. By Lemma 2.4.1, the normalisation of the stable curve X_∞ associated to a cusp of W_D is a projective line with three pairs of marked points which we denote by $x_1, y_1, x_2, y_2, x_3, y_3$.

Now, the stable differential ω_∞ has poles at the nodes of X_∞ and the residues at each node must add up to zero, i.e. we have the crossratio equation

$$\omega_\infty = \left(\sum_{i=1}^{3} \frac{r_i}{z - x_i} - \frac{r_i}{z - y_i}\right) \mathrm{d}z = \frac{C\mathrm{d}z}{\prod_{i=1}^{3}(z - x_i)(z - y_i)}, \tag{2.5}$$

for the residues r_i, some constant C, and after choosing coordinates so that the unique zero of ω_∞ is at ∞.

Moreover, the Prym involution ρ acts on X_∞, hence also on the normalisation, where we choose coordinates so that it acts as $z \mapsto -z$ (fixing the zero at ∞) and $x_2 = 1$. Recall that the stable fibre was obtained topologically by contracting the core curves of the three cylinders and that two cylinders are exchanged by the involution, one is fixed. We therefore find

$$y_1 = -x_3, \quad y_2 = -x_2 = -1, \quad y_3 = -x_1, \quad \text{and} \quad r_1 = r_3.$$

Comparing coefficients in (2.5), we obtain

$$x_1 = -x_3 - 2s \quad \text{and} \quad x_3 = -s \pm \sqrt{\frac{1 - s^2}{3}} \quad \text{for} \quad s = \frac{r_2}{2r_1}.$$

Observe that the choice of sign in x_3 interchanges the values of x_1 and x_3 and that $-s$ gives the same set of points.

Now, consider the cusp associated to the prototype $[w, h, t, e, \varepsilon]$. If $\varepsilon = 1$, we have $r_1 = r_3 = w$ and $r_2 = \lambda$, while $\varepsilon = -1$ implies $r_1 = r_3 = \lambda/2$ and $r_2 = w$ (cf. Figure 2.1). This determines s.

Conversely, $|s|$ determines the points x_i. Identifying the points ± 1, x_1 and $-x_3$, and x_3 and $-x_1$, we obtain a stable curve with three nodes and an involution. □

Remark 2.4.3. *Note that replacing* s *with* $-s$ *in Proposition 2.4.2 gives the same six points on* $\mathbb{P}^1$*, i.e. the same stable curve. This ambiguity corresponds to the action of the Prym involution on the stable curve.*

Remark 2.4.4. *Observe that the stable curve does not "see" the twist parameter* t*, as it only depends on the cylinder widths. In particular, cusp prototypes that differ only in their twist parameter cannot be distinguished by the associated stable curves. This motivates the following definition.*

Definition. *Given a prototype* $P = [w, h, t, e, \varepsilon]$*, we define the associated* algebraic cusp prototype *as* $[w, h, e, \varepsilon]$.

The Galois action As Teichmüller curves are rigid, they are defined (as algebraic varieties) over a number field [McM09; MV11] and one can show that the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the set of all Teichmüller curves (cf. [Möl06, §5], [MV11, §6]), hence also on the set of cusps of Teichmüller curves.

Moreover, given $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, let $\mathcal{X} \to \mathcal{C}$ denote the universal family over a Teichmüller curve. Then the associated family of Jacobians splits with one factor admitting real multiplication, see e.g. [Möl11b, Corollary 5.7]. Now, σ acts on the universal family $\mathcal{X}$, as well as on the associated family of Jacobians, preserving the splitting and the real multiplication. In the case that $\mathcal{C}$ is a Prym–Weierstraß curve in $\mathcal{M}_3$, this implies that σ acts on the family of Prym-varieties over $\mathcal{C}$. In particular, a (fibrewise) ρ-anti-invariant eigenform for real multiplication by some $\mathcal{O}_D$ with a single zero is mapped again to a (fibrewise) ρ-anti-invariant eigenform for real multiplication by $\mathcal{O}_D$, as the splitting of the family, the real multiplication and the multiplicities of the zeros are all preserved by σ.

Note that Galois conjugation on curves in the moduli stack of curves preserves the number of cusps and number and type of orbifold points. In fact, the number of isomorphic fibers of the universal family over an orbifold chart near an orbifold point detects the orbifold order and is preserved by Galois conjugation.

Remark 2.4.5. *While a Teichmüller curve $\mathcal{C}$ and its Galois conjugate $\mathcal{C}^\sigma$ are homeomorphic* as orbifolds*, they are in general, however, not isomorphic as complex curves. Indeed, by the calculations of the explicit equation of the Teichmüller curve W_{17}^{ε} in $\mathcal{M}_2$ in [BM10a, §7] (the equations are also given in [MZ16, §6] with a different normalisation) it is not difficult to check that the two components are not isomorphic: using the notation of [MZ16, (35)], one can calculate the modular j function and clearly $j(\kappa_0) \neq j(\kappa_0^\sigma)$.*

Using the algebraic description of the stable curves, we may describe the Galois action on the cusps of W_D. As this is again independent of the twist parameter t, the action is given only on algebraic cusp prototypes.

Proposition 2.4.6. *Let $P = [w, h, e, \varepsilon]$ be an algebraic cusp prototype, let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be a Galois automorphism that maps $\sqrt{D}$ to $-\sqrt{D}$, and denote by P^σ the prototype corresponding to the σ-conjugate cusp. Then, if $\varepsilon = 1$,*

$$P^\sigma = \begin{cases} [h, w, e, -\varepsilon], & \text{if } h > \lambda/2, \\ [w, h, -e, \varepsilon], & \text{if } h < \lambda/2, \end{cases}$$

and if $\varepsilon = -1$,

$$P^\sigma = \begin{cases} [h, w, e, -\varepsilon], & \text{if } h > \lambda, \\ [w, h, -e, \varepsilon], & \text{if } h < \lambda, \end{cases}$$

where $2\lambda = e + \sqrt{D}$, as above.

Proof. Let $P = [w, h, e, \varepsilon]$ be an algebraic cusp prototype. By Proposition 2.4.2, the conjugate cusp will depend only on the action of σ on s. Recall that for $\varepsilon = 1$, we have

$$s = s(P) = s^+ = \frac{e + \sqrt{D}}{4w}, \quad \text{i.e.} \quad (s^+)^\sigma = -\frac{-e + \sqrt{D}}{4w},$$

while for $\varepsilon = -1$

$$s = s(P) = s^- = \frac{2w}{e+\sqrt{D}} = \frac{-e+\sqrt{D}}{4h}, \quad \text{i.e.} \quad (s^-)^\sigma = -\frac{e+\sqrt{D}}{4h},$$

as $D = e^2 + 8wh$.

Now, consider a prototype $P' = [w', h', e', \varepsilon']$ such that $|s(P)^\sigma| = |s(P')|$ (recall that by Remark 2.4.3, s is determined only up to sign, due to the action of the Prym involution). Comparing coefficients in $\mathbb{Q}(\sqrt{D})$ and as $w, h > 0$, it is clear that either $\varepsilon' = -\varepsilon$ and $e' = e$ or $e' = -e$ and $\varepsilon' = \varepsilon$. In the first case, $w' = h$ and $h' = w$, while in the second case $w' = w$ and $h' = h$.

Moreover, observe (using again that $D = e^2 + 8wh$) that

$$h < \frac{\lambda}{2} \iff e + \sqrt{D} > 4h = \frac{D-e^2}{2w} \iff \frac{\sqrt{D}-e}{2} < w,$$

and that any valid prototype $[w', h', e', 1]$ must satisfy $w' > \lambda'$. Hence, comparing h to λ (respectively $\lambda/2$), determines which of the above described choices for P' gives a valid prototype and thus yields the claim. □

We now combine Theorem 2.1.2 with Proposition 2.4.6 to show that, when D is odd any two conjugate cusps are on different components.

Proposition 2.4.7. *Let $D \equiv 1 \bmod 8$ and $P_D = [w, h, e, \varepsilon]$ be an algebraic cusp prototype. Then P_D and P_D^σ are on different components of W_D.*

In particular, the cusps associated to $\left[\frac{D-1}{8}, 1, -1, -1\right]$ and $\left[\frac{D-1}{8}, 1, 1, -1\right]$ lie on W_D^1 and W_D^2, respectively, and are conjugate.

Proof. Let $P_D = [w, h, e, \varepsilon]$ be an algebraic cusp prototype and denote by

$$c(P) = e + \varepsilon \bmod 4,$$

the component (see Theorem 2.1.2) of W_D that the associated cusp(s) of P lie on. Then, by Proposition 2.4.6, we have

$$c(P^\sigma) \equiv -e + \varepsilon \equiv e - \varepsilon \bmod 4,$$

as both e and ε are $\pm 1 \bmod 4$. In particular, $c(P) \not\equiv c(P^\sigma) \bmod 4$, hence the cusps lie on alternate components. □

Proof of Theorem 2.1.1: Let $D \equiv 1 \bmod 8$, non-square, and W_D^i be a Teichmüller curve. Now, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on W_D^i and as this action extends to an action on the families of curves and their Jacobians, respects the (Prym) splitting, and maps eigenforms for real multiplication to eigenforms (for the same D), it preserves the locus W_D. Hence, any given element of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts either trivially or interchanges the two components. But by Proposition 2.4.7, there exists an automorphism that *does not* fix W_D^i and therefore the components are Galois-conjugate. In particular, they are homeomorphic. □

3. Orbifold points on Prym–Teichmüller curves in genus four

For each discriminant $D > 1$, McMullen constructed the Prym–Teichmüller curves $W_D(4)$ and $W_D(6)$ in $\mathcal{M}_3$ and $\mathcal{M}_4$, which constitute one of the few known infinite families of geometrically primitive Teichmüller curves. In this chapter, we determine for each D the number and type of orbifold points on $W_D(6)$. These results, together with the results of the chapter 1 (cf. [TTZ16]) and chapter 2 (cf. [Zac16]) regarding the genus 3 case and with results of Lanneau–Nguyen (cf. [LN14]) and Möller (cf. [Möl14]), complete the topological characterisation of all Prym–Teichmüller curves and determine their genus.

The study of orbifold points relies on the analysis of intersections of $W_D(6)$ with certain families of genus 4 curves with extra automorphisms. As a side product of this study, we give an explicit construction of such families and describe their Prym–Torelli images, which turn out to be isomorphic to certain products of elliptic curves. We also give a geometric description of the flat surfaces associated to these families and describe the asymptotics of the genus of $W_D(6)$ for large D. The content of this chapter is joint work with David Torres-Teigell and has appeared as [TTZ17].

3.1. Introduction

A *flat surface* is a pair (X, ω) where X is a compact Riemann surface of genus g and ω is a holomorphic differential on X. By integration, the differential endows X with a flat structure away from the zeros of ω. Consider now $\Omega\mathcal{M}_g$, the moduli space of flat surfaces which is a natural bundle over the moduli space $\mathcal{M}_g$ of smooth projective curves of genus g. There is a natural $\mathrm{SL}_2(\mathbb{R})$ action on $\Omega\mathcal{M}_g$ by affine shearing of the flat structure and we consider the projections of orbit closures to $\mathcal{M}_g$. In the rare case that the $\mathrm{SL}_2(\mathbb{R})$ orbit of (X, ω) projects to an (algebraic) curve in $\mathcal{M}_g$ we call this the *Teichmüller curve generated by* (X, ω) in $\mathcal{M}_g$.

Not many families of (primitive) Teichmüller curves are known, see e.g. [MMW17] for a brief overview. Among them, McMullen constructed the *Weierstraß curves* in genus 2 [McM03] and generalised this construction to the *Prym–Teichmüller curves* in genus 3 and 4 [McM06]. Recently, Eskin, McMullen, Mukamel and Wright announced the existence of six exceptional orbit closures, two of which contain an infinite collection of Teichmüller curves. One of them is treated in [MMW17].

Any Teichmüller curve $\mathcal{C}$ is a sub-orbifold of $\mathcal{M}_g$. Therefore, denoting by χ the (orbifold) Euler Characteristic, by h_0 the number of connected components, by C the number of cusps and by e_d the number of points of order d, these invariants determine the genus g:

$$2h_0 - 2g = \chi + C + \sum_d e_d \left(1 - \frac{1}{d}\right),$$

i.e. they determine the topological type of $\mathcal{C}$.

For the Prym–Weierstraß curves, the situation is as follows. In genus 2, cusps and connected components were determined by McMullen [McM05a], the Euler characteristic was computed by Bainbridge [Bai07], and the number and types of orbifold points were established by Mukamel [Muk14]. In genus 3 and 4, Möller [Möl14] calculated the Euler characteristic and Lanneau and Nguyen [LN14] classified the cusps. The number of connected components in genus 3 were also determined in [LN14] (see also [Zac16]) and the number and type of their orbifold points in genus 3 were established in [TTZ16]. In the case of genus 4, Lanneau has recently communicated to the authors that the Prym locus is always connected [LN16]. The present paper classifies the orbifold points of these curves.

Theorem 3.1.1. *For discriminant $D > 12$, the Prym–Teichmüller curves $W_D(6)$ have orbifold points of order 2 and 3. More precisely:*

- *the number of orbifold points of order 2 is*

$$e_2(D) = \begin{cases} 0\,, & \text{if } D \text{ is odd,} \\ h(-D) + h(-D/4)\,, & \text{if } D \equiv 12 \bmod 16, \\ h(-D)\,, & \text{if } D \equiv 0, 4, 8 \bmod 16, \end{cases}$$

  where $h(-D)$ is the class number of $\mathcal{O}_{-D}$;

- *the number of orbifold points of order 3 is*

$$e_3(D) = \#\{a, i, j \in \mathbb{Z} : a^2 + 3j^2 + (2i - j)^2 = D,\ \gcd(a, i, j) = 1\}/12;$$

- *$W_5(6)$ has one point of order 3 and one point of order 5;*
- *$W_8(6)$ has one point of order 2 and one point of order 3;*
- *$W_{12}(6)$ has one point of order 2 and one point of order 6.*

Theorem 3.1.1 combines the results of Theorem 3.3.1, Theorem 3.4.1, and Theorem 3.5.1 and thus completes the topological classification of the Prym–Weierstraß curves. The topological invariants of $W_D(6)$ for nonsquare discriminants $D \leq 181$ are listed in Table 3.3 on page 91.

Recall that the orbifold locus of $W_D(6)$ consists of flat surfaces (X, ω) where ω is not only an eigenform for the real multiplication but *also* for some (holomorphic) automorphism α of X. To describe this locus, it is therefore natural to consider instead families $\mathcal{F}$ of curves with a suitable automorphism α and consider the α-eigenspace decomposition of $\Omega\mathcal{F}$. We isolate suitable eigendifferentials ω with a single zero, and check whether the Prym part of $(X, \omega) \in \Omega\mathcal{F}$ admits real multiplication respecting ω, i.e. find the intersections of $\mathcal{F}$ with $W_D(6)$ for some D.

To be more precise, it is essentially topological considerations that not only show the possible orders d of orbifold points that can occur on a curve $W_D(6)$, but in fact determine the possibilities for the group $\operatorname{Aut} X$, in the case that (X, ω) is an orbifold point (see section 3.2). It turns out that there are essentially two relevant families: curves admitting a D_8 action – giving points of order 2 – and curves admitting a $C_6 \times C_2$ action – giving points of order 3. Because of the flat picture of the single-zero differentials on these families, we will call them the *Turtle family* (Figure 3.1) and the *Hurricane family* (Figure 3.2), see section 3.6 for details. Additionally, these families intersect, giving the (unique) point of order 6 on $W_{12}(6)$. Also, there is a unique point with a C_{10} action, giving the point of order 5 on $W_5(6)$. Any orbifold point on $W_D(6)$ must necessarily lie on one of these families (Proposition 3.2.1).

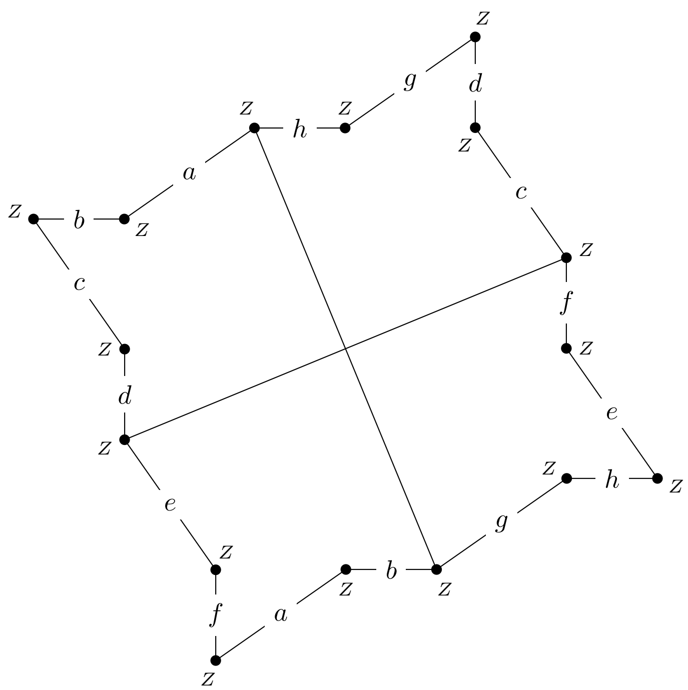


Figure 3.1.: A C_4-eigendifferential of genus 4 with a single zero (the Turtle): the canonical 4-cover of the 4-differential on an elliptic curve pictured in Figure 3.3 (section 3.6).

The difficulty when studying these families comes from obtaining the eigenforms in a basis where we can calculate the endomorphism ring in order to study real multiplication or, equivalently, understanding the analytic representation of suitable real multiplication in the eigenbasis of the automorphism on the Prym variety.

We begin by analysing $\mathcal{M}_4(D_8)$, the 2-dimensional locus of genus 4 curves with a specific D_8 action, see section 3.3. The Turtle family is a 1-dimensional sub-locus of this moduli space.

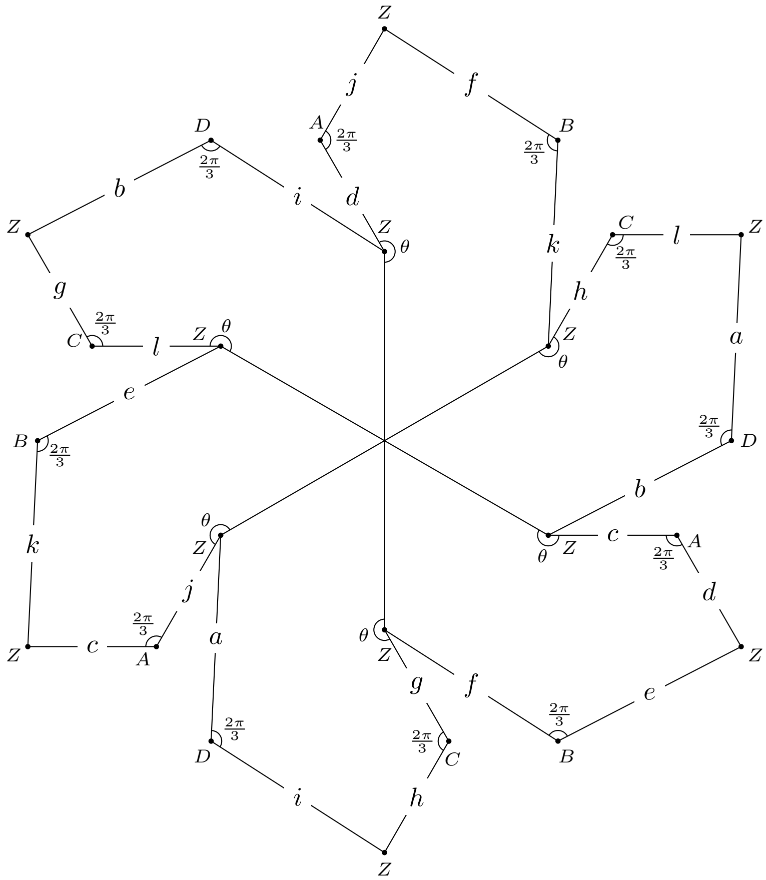


Figure 3.2.: A C_6-eigendifferential of genus 4 with a single zero (the Hurricane): the canonical 6-cover of the 6-differential on $\mathbb{P}^1$ pictured in Figure 3.3 (section 3.6).

As a by-product, we give an explicit description of $\mathcal{M}_4(D_8)$. For an elliptic curve E, let ϕ denote the elliptic involution.

Theorem 3.1.2. *The family* $\mathcal{M}_4(D_8)$ *is in bijection with the family*

$$\mathcal{E} = \{(E, [P]) : E \in \mathcal{M}_{1,1},\ [P] \in (E \setminus E[2])/\phi\}$$

of elliptic curves with a distinguished base point, together with an elliptic pair.

In particular, this family is 2*-dimensional; however, the sub-locus* $\mathcal{X}$ *of curves admitting a* C_4*-eigenform with a single zero is* 1*-dimensional and in bijection with* $\mathcal{M}_{1,1}\setminus\{E_2\}$.

This bijection is induced by the construction of this family as a fibre product of two (isomorphic) families of elliptic curves over a base projective line.

To determine which points admit real multiplication with a common eigenform, we fix an eigenbasis of ΩX and consider the *Prym–Torelli map* PT, which associates the corresponding Prym variety to a Prym pair (X, ρ). We show that, in the D_8 case, the Prym variety of such a pair is always isomorphic to the product $E \times E$, where E is an elliptic curve arising as a quotient of X, and then the Prym variety admits suitable real multiplication if and only if the elliptic curve has complex multiplication, accounting for the class numbers.

The Hurricane family behaves quite differently, see section 3.4. We denote by E_ζ the unique elliptic curve with an automorphism ψ of order 6.

Theorem 3.1.3. *The Hurricane family agrees with the family*

$$\mathcal{Y}_t : y^6 = x(x-1)^2(x-t)^2$$

of cyclic covers of $\mathbb{P}^1$.

However, the Prym–Torelli image of $\mathcal{Y}$ *is the single point* $E_\zeta \times E_\zeta$.

The Hurricane family has the advantage that it is 1-dimensional and can be understood in terms of cyclic covers of $\mathbb{P}^1$. However, due to the large automorphism group, the whole family collapses to a single point under the Prym–Torelli map, which of course admits real multiplication in many different ways. Now, each fibre $\mathcal{Y}_t$ gives a different C_6-eigenbasis in $\Omega E_\zeta \times \Omega E_\zeta$ and checking when *this* basis is also an eigenbasis for some real multiplication gives the intersections of $\mathcal{Y}$ and some $W_D(6)$.

The Hurricane family can also be constructed as a family of fibre products over certain quotient curves. More precisely, all fibres $\mathcal{Y}_t$ of $\mathcal{Y}$ can be seen as a fibre product of two copies of E_ζ over a projective line quotient $\mathbb{P}^1$. However, in contrast to the D_8 case, the base of the Hurricane family will not be isomorphic to a modular curve, but it will be a dense subset inside the curve E_ζ.

More precisely, denote by E_ζ^* the curve E_ζ with the 2-torsion points and the ψ-orbit of order 3 removed and let ϕ be again the elliptic involution of E_ζ. There is a generically 6-to-1 map between the set of elliptic pairs of points E_ζ^*/ϕ and the fibres of $\mathcal{Y}$ (cf. Proposition 3.4.10).

Moreover, for each isomorphism class $[Y] \in \mathcal{Y}$, there exist generically 12 elements (up to scale) in $\Omega E_\zeta \times \Omega E_\zeta$ defining a C_6-eigendifferential with a single zero on the curves in $[Y]$ (cf. Proposition 3.4.15). This fact explains the factor of 12 in the formula for the number of orbifold points of order 3.

Using the work of Möller [Möl14] and Lanneau–Nguyen [LN14], Theorem 3.1.1 lets us calculate the genus of the Prym–Weierstraß curves $W_D(6)$. In section 3.7, we describe the asymptotic growth rate of the genus, $g(W_D(6))$ with respect to the discriminant D.

Theorem 3.1.4. *There exist constants* $C_1, C_2 > 0$*, independent of* D*, such that*

$$C_1 \cdot D^{3/2} < g(W_D(6)) < C_2 \cdot D^{3/2}.$$

Moreover, $g(W_D) = 0$ *if and only if* $D \leq 20$.

Table 3.1.: Topological invariants of the six Teichmüller curves $W_D(6)$ that have genus 0. The number of cusps is described in [LN14], the Euler characteristic in [Möl14]. For larger D, see Table 3.3 on page 91.

D	g	e_2	e_3	e_5	e_6	C	χ
5	0	0	1	1	0	1	$-7/15$
8	0	1	1	0	0	2	$-7/6$
12	0	1	0	0	1	3	$-7/3$
13	0	0	2	0	0	3	$-7/3$
17	0	0	1	0	0	6	$-14/3$
20	0	2	1	0	0	5	$-14/3$

The topological invariants of the geometrically primitive genus 0 Prym–Teichmüller curves are summarized in Table 3.1.

Theorem 3.1.1 can be seen as the next and final step after [Muk14] and [TTZ16] in the study of orbifold points on Prym–Weierstraß curves, thus bringing closure to the topological characterization of such curves. While the general method is similar in the genus 2, 3 and 4 cases (namely, studying the intersection of the Teichmüller curves with certain families), the specific phenomena occurring are different.

In genus 2, the situation was simpler essentially due to the fact that the Prym variety was the entire Jacobian [Muk14]. While the relevant family also had a generic D_8 automorphism group, this was a 1-dimensional object, while in $\mathcal{M}_4$ this locus is a surface where the C_4-eigendifferentials are contained in an embedded Modular curve.

In genus 3, the defining phenomenon was the fact that the Prym variety was a $(2,1)$-polarised abelian sub-variety of the Jacobian [TTZ16]. Also, for the first time, two 1-dimensional families occured and in the case of C_4 curves, the Prym–Torelli image also collapsed to a point. However, both these families could be described as cyclic covers of $\mathbb{P}^1$ in which case the eigenspace decomposition of ΩX is well understood. The main technical difficulty in those cases was the explicit calculation of period matrices using Bolza's method. Moreover, the formulas obtained were of a slightly different flavour, as the C_6-family turned out to be isomorphic to the *compact* Shimura curve $\mathbb{H}/\Delta(2,6,6)$, giving a more general class number than in the other cases.

In contrast, in genus 4, for the first time a 2-dimensional locus plays a central role: indeed, the space $\mathcal{M}_4(D_8)$ can be seen as a cyclic cover over an elliptic curve, which makes the computation of the eigendifferentials with a single zero more difficult, cf. Theorem 3.3.8. On the other hand, while in almost all cases the Prym variety is isogenous to a product of elliptic curves (this is the reason for the abundance of modular curves and class numbers in the formulas), it turns out that in genus 4 the Prym varieties are actually *isomorphic* to this product. This results in a closer relationship of the endomorphism rings in this case and is the reason we obtain an exact class number of a negative discriminant order in Theorem 3.1.1.

Table 3.2.: Families parametrising possible orbifold points on Prym–Weierstraß curves. Here, $\mathcal{X}_t$ is a general fibre of $\mathcal{X}$ and E_t is an elliptic curve that appears as a quotient of $\mathcal{X}_t$. For genus 2 see [Muk14], for genus 3 see [TTZ16].

$g(\mathcal{X}_t)$	d	$\dim \mathcal{X}$	$\dim \mathrm{PT}(\mathcal{X})$	$\mathrm{Aut}(\mathcal{X}_t)$	$\mathrm{End}(\mathcal{P}(\mathcal{X}_t, \sigma^d))$
2	2	1	1	D_8	order in $M_2(\mathrm{End}(E_t))$
3	2	1	0	$C_2 \ltimes (C_2 \times C_4)$	order in $M_2(\mathbb{Q}[i])$
3	3	1	1	C_6	order in $\left(\frac{2,-3}{\mathbb{Q}}\right)$
4	2	2 (1)	1	D_8	$M_2(\mathrm{End}(E_t))$
4	3	1	0	$C_6 \times C_2$	$M_2(\mathbb{Z}[\zeta_6])$

In particular, the technical approach in this paper is completely different than the one in [TTZ16], since the computational aspects of Bolza's method have been replaced by a more conceptual description of the families.

The occuring positive dimensional families are summarised and compared to the families occuring in genus 2 and 3 in Table 3.2.

Finally, in section 3.6, we provide the flat pictures associated to the eigendifferentials in the Turtle family and the Hurricane family.

Acknowledgements We are very grateful to Martin Möller for many useful discussions and his constant encouragement to also complete the genus 4 case. We also thank [Par] for computational assistance.

3.2. The orbifold locus of W_D

The aim of this section is to describe the orbifold locus of $W_D(6)$ as the intersection with families of curves with a prescribed automorphism group in $\mathcal{M}_4$. In particular, Proposition 3.2.1 determines the possible orders of orbifold points that may occur.

As usual, we write $(g; n_1, \ldots, n_r)$ for orbifolds of genus g with r points of order $n_1, \ldots, n_r$. Recall that, given an automorphism α of order N on X, points of order n_i on X/α correspond to orbits of length N/n_i on X. Moreover, we denote by ζ_d a primitive dth root of unity.

Proposition 3.2.1. *Let $(X, \omega) \in \Omega W_D$ be a flat surface that parametrises an orbifold point of W_D of order d. Then there exists a holomorphic automorphism $\alpha \in \mathrm{Aut}\, X$ of order $2d$ that satisfies $\alpha^*\omega = \zeta_{2d}\omega$ and one of the following conditions:*

1. *the order of α is 4 and X/α has signature $(1; 4, 4)$;*
2. *the order of α is 6 and X/α has signature $(0; 3, 3, 6, 6)$;*
3. *the order of α is 10 and X/α has signature $(0; 5, 10, 10)$;*

4. the order of α is 12 *and* X/α *has signature* $(0; 3, 12, 12)$.

Remark 3.2.2. *Observe that family (1) is* 2*-dimensional, family (2) is* 1*-dimensional and families (3) and (4) consist of a finite number of points in* $\mathcal{M}_4$.

Before we proceed to the proof, we first recall some background and notation.

Flat surfaces and Teichmüller curves A *flat surface* is a pair (X, ω) where X is a Riemann surface (or equivalently a smooth irreducible complex curve) of genus g and $\omega \in \Omega X$ is a holomorphic differential form on X. Note that X may be endowed with a flat structure away from the zeros of ω via integration. We will denote the moduli space of flat surfaces by $\Omega\mathcal{M}_g$ and note that it can be viewed as a bundle $\Omega\mathcal{M}_g \to \mathcal{M}_g$ over the moduli space of smooth, irreducible, complex curves of genus g. The space $\Omega\mathcal{M}_g$ is naturally stratified by the distribution of the zeros of the differentials; given a partition $\mu = (\mu_1, \ldots, \mu_r)$ of $2g-2$, denote by $\Omega\mathcal{M}_g(\mu)$ the corresponding stratum and given a family $\mathcal{F}$ of curves in $\mathcal{M}_g$ we set $\Omega\mathcal{F}(\mu) \coloneqq \Omega\mathcal{M}_g(\mu) \cap \Omega\mathcal{F}$. We will use exponential notation for repeated indices, so that, for instance $(1, \ldots, 1) = (1^{2g-2})$.

Recall that $\Omega\mathcal{M}_g$ admits a natural $\mathrm{GL}_2(\mathbb{R})$ action by affine shearing of the flat structures. A *Teichmüller curve* is the (projection of a) $\mathrm{GL}_2(\mathbb{R})^+$ orbit that projects to an algebraic curve in $\mathcal{M}_g$. See for instance [Möl11b] for background on Teichmüller curves and flat surfaces.

Prym-Teichmüller curves in genus 4 McMullen [McM06] constructed families of primitive Teichmüller curves in genus 2, 3 and 4, the *Prym–Teichmüller* (or *Prym–Weierstraß*) curves $W_D(2g-2)$. We briefly recall the construction in the genus 4 case. For brevity, we denote the curve $W_D(6)$ by W_D in the following.

Let X be of genus 4 admitting a holomorphic involution ρ. We say that ρ is a *Prym involution* if X/ρ has genus 2. In particular, this gives a decomposition $\Omega X = \Omega X^+ \oplus \Omega X^-$ into 2-dimensional ρ-eigenspaces with eigenvalues 1 and -1 respectively. It also determines sublattices $H_1(X,\mathbb{Z})^+, H_1(X,\mathbb{Z})^- \subset H_1(X,\mathbb{Z})$ consisting of ρ-invariant and ρ-anti-invariant cycles that satisfy $H_1(X,\mathbb{Z})^\pm = H_1(X,\mathbb{Z}) \cap (\Omega X^\pm)^*$. All this implies that the *Prym variety*

$$\mathcal{P}(X,\rho) \coloneqq \frac{(\Omega X^-)^*}{H_1(X,\mathbb{Z})^-} = \ker(\mathrm{Jac}(X) \to \mathrm{Jac}(X/\rho))^0$$

is a 2-dimensional, $(2,2)$-polarised abelian sub-variety of the Jacobian $\mathrm{Jac}(X)$ (see [Möl14] or [BL04, Ch. 12] for details).

For any discriminant $D \equiv 0, 1 \bmod 4$, write $D = b^2 - 4ac$ for some $a, b, c \in \mathbb{Z}$. The (unique) quadratic order of discriminant D is defined as $\mathcal{O}_D = \mathbb{Z}[T]/(aT^2 + bT + c)$, which agrees with

$$\mathcal{O}_D = \mathbb{Z} \oplus T_D\mathbb{Z}, \text{ where } T_D = \begin{cases} \frac{\sqrt{D}}{2}, & \text{if } D \equiv 0 \bmod 4 \text{ and} \\ \frac{\sqrt{D}+1}{2}, & \text{if } D \equiv 1 \bmod 4, \end{cases}$$

provided D is not a square. If $D = f^2$, the order $\mathcal{O}_D = \mathbb{Z}[T]/(T^2 - fT)$ is isomorphic to the subring $\{(a,b) \in \mathbb{Z} \oplus \mathbb{Z} : a \equiv b \bmod f\}$.

Now let $D > 0$ be a positive discriminant. We say that a polarised abelian surface A has *real multiplication* by $\mathcal{O}_D$ if it admits an embedding $\mathcal{O}_D \hookrightarrow \operatorname{End} A$ that is self-adjoint with respect to the polarisation. We call the real multiplication by $\mathcal{O}_D$ *proper*, if the embedding cannot be extended to any quadratic order containing $\mathcal{O}_D$.

Denote by ΩW_D the locus of $(X, \omega) \in \Omega\mathcal{M}_4(6)$ such that

1. X admits a Prym involution ρ, so that $\mathcal{P}(X, \rho)$ is 2-dimensional,
2. the form ω has a single zero and satisfies $\rho^*\omega = -\omega$, and
3. $\mathcal{P}(X, \rho)$ admits proper real multiplication by $\mathcal{O}_D$ with ω as an eigenform.

McMullen showed that the projection W_D of ΩW_D to $\mathcal{M}_4$ gives (a union of) Teichmüller curves for every discriminant D [McM06]. In fact, Lanneau has communicated to the authors that W_D is connected for all D [LN16].

Orbifold points on Prym–Teichmüller curves An orbifold point of order d on W_D corresponds to a flat surface $(X, \omega) \in \Omega W_D$ such that

- there exists a holomorphic automorphism $\alpha \in \operatorname{Aut} X$, such that $\alpha^*\omega = \lambda\omega$ for some $\lambda \in \mathbb{C}^* \setminus \{\pm 1\}$;
- the element $\rho = \alpha^d$ is a Prym involution satisfying $\rho^*\omega = -\omega$;
- ω is an eigenform for real multiplication on the Prym variety $\mathcal{P}(X, \rho)$.

Note that this implies that α is of order $2d$ and must have a fixed point (at the single zero of ω). Details and background can be found in [TTZ16].

Definition. *We will say that* (ω_1, ω_2) *is an* α-eigenbasis *of* ΩX^- *if the* ω_i *are both eigenforms for the action of* α^*.

To study these points, we study the locus of curves in $\mathcal{M}_4$ with an appropriate automorphism α and an eigenform with a single zero.

Proof of Proposition 3.2.1. Let (X, ω) correspond to an orbifold point in W_D of order d. The Prym involution ρ in genus 4 gives a genus 2 quotient with two fixed points, i.e. $X/\rho \cong (2; 2^2)$. By the argument above, the curve X must possess an automorphism α of order $2d$ that admits ω as an eigenform with eigenvalue ζ_{2d}, has at least one fixed point and satisfies $\alpha^d = \rho$.

The automorphism α descends to an automorphism of X/ρ of order d and, looking at possible orders of automorphisms on curves of genus 2, one sees that $d = 2, 3, 4, 5, 6, 8$ or 10 (see for instance [Sch69; Bro91]). Now, points of odd order k on X/α (equivalently, α-orbits of length $2d/k$ on X) give unramified points on X/ρ, since they are not fixed by $\rho = \alpha^d$ (more precisely, their preimages on X are not fixed). Points of even order $2k$ on X/α (equivalently, α-orbits of length d/k on X) give d/k ramified points on X/ρ.

Since there are only two ramification points on X/ρ and at least one of them necessarily comes from a fixed point of α, the automorphism α has two fixed points and no more ramification points of even order. A case-by-case analysis using Riemann-Hurwitz yields the four options in the statement. □

Products of elliptic curves In the analysis of orbifold points on Prym–Teichmüller curves in genus 4, Klein-four actions and products of elliptic curves will be omnipresent. The following result will be a crucial technical tool.

Proposition 3.2.3. *Let (X, ρ) be a genus 4 curve in the Prym locus admitting a Klein four-group of automorphisms $V_4 = \langle \rho, \beta \rangle$ such that both X/β and $X/\rho\beta$ have genus 1. Then $\mathcal{P}(X, \rho) \cong X/\beta \times X/\rho\beta$ as $(2, 2)$-polarised abelian varieties.*

Remark 3.2.4. *Note that this is in stark contrast to the situation in genus 2 and 3. Although in those cases V_4 actions were also ubiquitous, the Prym variety was always only* isogenous *to a product of elliptic curves (cf. [Muk14, Proposition 2.13] and [TTZ16, Theorem 1.2]) as the Prym variety is $(1, 1)$, respectively $(2, 1)$, polarised in those situations. In the genus 4 case, the result above yields an even stronger relationship between the geometry of the quotient elliptic curves and the Prym variety.*

Let us first recall some general facts about elliptic curves. An elliptic curve $E := (E, O) \in \mathcal{M}_{1,1}$ is a smooth genus 1 curve together with a chosen base point $O \in E$. It always admits the structure of a group variety with neutral element O. The set of 2-torsion points with respect to this group law consists of four elements and is usually denoted by $E[2]$.

Every elliptic curve is isomorphic to $E_\lambda := \{v^2 = u(u-1)(u-\lambda)\}$ for some $\lambda \in \mathbb{C}\backslash\{0, 1\}$, where we choose the base point O to be the point at infinity. Permuting $\{0, 1, \infty\}$ gives an isomorphism between the elliptic curves corresponding to

$$\lambda\,,\ 1-\lambda\,,\ \frac{1}{\lambda}\,,\ \frac{1}{1-\lambda}\,,\ \frac{\lambda-1}{\lambda}\,,\ \frac{\lambda}{\lambda-1}\,.$$

By the uniformisation theorem, every elliptic curve can also be represented as the quotient of $\mathbb{C}$ by a lattice $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$ for some τ in the upper half-plane $\mathbb{H}$. Points in the same $\mathrm{SL}_2\,\mathbb{Z}$-orbit yield isomorphic elliptic curves, and therefore one can realise the moduli space of elliptic curves $\mathcal{M}_{1,1}$ as the quotient $\mathbb{H}/\,\mathrm{SL}_2\,\mathbb{Z}$. The relationship between τ and λ is given by the modular λ-function.

Each elliptic curve carries a natural *elliptic involution* ϕ, the set of fixed points of which agrees with $E[2] = \mathrm{Fix}(\phi)$. In the model E_λ, the elliptic involution is given by $(u, v) \mapsto (u, -v)$ and one has $E_\lambda[2] = \{(0, 0), (1, 0), (\lambda, 0), \infty\}$. The quotient by the elliptic involution is isomorphic to $\mathbb{P}^1$.

The general element of $\mathcal{M}_{1,1}$ has no further automorphisms fixing the base point. The only exceptions, which correspond to the orbifold points of $\mathbb{H}/\,\mathrm{SL}_2\,\mathbb{Z}$, are E_2 (corresponding to $\tau = \mathrm{i}$ in the upper half-plane) with a cyclic automorphism group of order 4, and E_{ζ_6} (corresponding to $\tau = \zeta_6$ in the upper half-plane), where $\zeta_6 = e^{2\pi\mathrm{i}/6}$, with a cyclic automorphism group of order 6.

Proof of Proposition 3.2.3. Consider the quotients $p_1\colon X \to X/\beta$ and $p_2\colon X \to X/\rho\beta$. Since X/V_4 has genus 0, the images of the pullbacks $p_1^*\colon \Omega(X/\beta) \to \Omega X$ and $p_2^*\colon \Omega(X/\rho\beta) \to \Omega X$ must both lie in ΩX^-, the -1-eigenspace of ρ and in fact generate ΩX^-. Therefore, denoting also by p_i^* the induced map between Jacobians and identifying the elliptic curves with their Jacobians, one has

$$\begin{array}{ccccc} \Omega(X/\beta) \times \Omega(X/\rho\beta) & \xrightarrow{\cong} & \Omega X^- & \hookrightarrow & \Omega X \\ \downarrow & & \downarrow & & \downarrow \\ X/\beta \times X/\rho\beta & \xrightarrow{(p_1^*, p_2^*)} & \mathcal{P}(X,\rho) & \hookrightarrow & \operatorname{Jac} X \end{array}$$

Since the polarisations induced from $\operatorname{Jac} X$ on $X/\beta \times X/\rho\beta$ and on $\mathcal{P}(X,\rho)$ are both of type $(2,2)$, the map (p_1^*, p_2^*) is necessarily an isomorphism of polarised abelian varieties. □

In particular, the proof shows that, in the above situation, we have a natural decomposition

$$\Omega X^- = \Omega X^+_\beta \oplus \Omega X^+_{\rho\beta}$$

into a β and $\rho\beta$ invariant subspace consisting of the differential forms that arise as pullbacks from the two quotient elliptic curves.

Definition. *Let X be a genus 4 curve with a V_4 action. We say that (η_1, η_2) is a* product basis *of ΩX^- if $\eta_1 \in \Omega X^+_\beta$ and $\eta_2 \in \Omega X^+_{\rho\beta}$.*

Note that any product basis is a β-eigenbasis. More precisely, we have

$$\beta^*\eta_1 = \eta_1 \quad \text{and} \quad \beta^*\eta_2 = -\eta_2, \tag{3.1}$$

as η_2 is ρ-anti-invariant.

3.3. Points of order 2 and 6

The aim of this section is to prove the following formula describing the number of points of order 2 on each Teichmüller curve W_D and the unique point of order 6 on W_{12} (cf. Theorem 3.1.1). Let $h(-C)$ denote the class number of the imaginary quadratic order $\mathcal{O}_{-C}$.

Theorem 3.3.1. *Let $D \neq 12$ be a positive discriminant.*

- *If $D \equiv 1 \bmod 4$ then W_D has no orbifold points of order 2.*
- *If $D \equiv 12 \bmod 16$ then W_D has $h(-D) + h(-\frac{D}{4})$ orbifold points of order 2.*
- *Otherwise, W_D has $h(-D)$ orbifold points of order 2.*

Moreover, W_{12} has one point of order 2 and one point of order 6.

To prove this theorem, we begin by a careful analysis of genus 4 curves admitting an automorphism of order 4 with two fixed points.

Curves admitting an automorphism of order 4 By Proposition 3.2.1, for (X, ω) to parametrise a point of order 2 on W_D, the curve X must necessarily lie in the locus of curves with an automorphism of order 4 with two fixed points. In fact, all such curves admit a faithful D_8 action.

Lemma 3.3.2. *Let X be a curve of genus 4 and $\alpha \in \operatorname{Aut} X$ an automorphism of order 4 with two fixed points.*

Then X/α^2 is of genus 2 and there exists an involution $\beta \in \operatorname{Aut} X$ such that $\alpha\beta = \beta\alpha^{-1}$, i.e. $\langle \alpha, \beta \rangle \leq \operatorname{Aut} X$ is a D_8.

Proof. Since the quotient X/α by such an automorphism yields a curve of genus 1 with two orbifold points of order 4, this is just case $N2$ in [BC99]. The proof of Bujalance and Conder relies on a previous result by Singerman [Sin72, Thm. 1] stating that every Fuchsian group with signature $(1; t, t)$ is included in a Fuchsian group with signature $(0; 2^4, t)$. This group corresponds to the quotient $X/\langle \alpha, \beta \rangle$. □

In terms of the corresponding curves, the situation is the following. The automorphism α descends to an involution $\overline{\alpha}$ of the genus 2 curve X/α^2 different from the hyperelliptic involution $\overline{\beta}$. The hyperelliptic involution lifts to an involution β on X which, together with α, generates the dihedral group. We will denote by $p_1 : X \to X/\beta$ and $p_2 : X \to X/\alpha^2\beta$ the corresponding projections.

Definition. *We will denote by $\mathcal{M}_4(D_8)$ the family of genus 4 curves admitting an automorphism of order 4 with two fixed points.*

Remark 3.3.3. *Note that by Lemma 3.3.2, this family agrees with the moduli space of Riemann surfaces of genus 4 with D_8-symmetry, where we fix the topological action as in the lemma. Moreover, moduli spaces of curves with automorphisms have been studied intensively, see e.g. [GDH92] for background and notations.*

It turns out that such a curve is essentially determined by its genus 1 quotients.

Proposition 3.3.4. *The family $\mathcal{M}_4(D_8)$ is in bijection with the family*

$$\mathcal{E} = \{(E, [P]) \,:\, E \in \mathcal{M}_{1,1},\ [P] \in (E \backslash E[2])/\{\pm 1\}\}$$

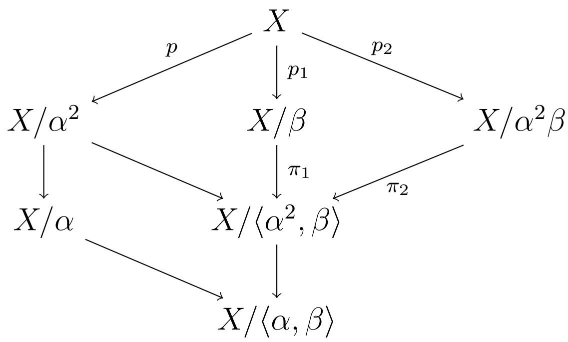
of elliptic curves with a distinguished base point, together with an elliptic pair.

The bijection is given by $X \mapsto (X/\beta,\ [p_1(\operatorname{Fix}(\alpha\beta))])$, where the origin of the elliptic curve is chosen to be the point $p_1(\operatorname{Fix}(\alpha))$.

Remark 3.3.5. *Note that this is a 2-dimensional locus inside $\mathcal{M}_4$. However, we will show in Theorem 3.3.8 that the sub-locus $\mathcal{X}$ where X admits an α-eigenform in ΩX^- with a single zero is in fact 1-dimensional.*

This classification is obtained by a careful analysis of the ramification data of $D_8 \cong \langle \alpha, \beta \rangle \leq \operatorname{Aut} X$.

Consider the following diagram of ramified covers:



Observe that all maps in the diagram are of degree 2.

The involutions β and $\alpha^2\beta$ each have 6 fixed points on X. Together they form three orbits of length 4 under $\langle \alpha, \beta \rangle$. Similarly, $\alpha\beta$ and $\alpha^{-1}\beta$ have 2 fixed points each, forming a whole orbit under $\langle \alpha, \beta \rangle$ of length 4. Now, the four points of order 2 in $X/\langle \alpha, \beta \rangle$ correspond to the (three) orbits of the fixed points of β and $\alpha^2\beta$ plus the orbit of the fixed points of $\alpha\beta$ and $\alpha^{-1}\beta$.

Looking at the ramification data of α and β, one sees that the quotients X/β and $X/\alpha^2\beta$ by β and $\alpha^2\beta = \alpha\beta\alpha^{-1}$ respectively correspond to curves of genus 1. Choosing the image of $\operatorname{Fix}(\alpha)$ as an origin on each quotient, they are in fact isomorphic as elliptic curves, since β and $\alpha^2\beta$ are conjugate.

Also, the above-described action of $\alpha^2\beta$ and β may be described purely in terms of the quotient maps: the six branch points of p_1 are mapped via p_2 to the three 2-torsion points on $X/\alpha^2\beta$, while p_1 maps the six branch points of p_2 to the three 2-torsion points on X/β.

Proof of Proposition 3.3.4. Denote by ϕ the elliptic involution on E and let $\varphi\colon E \to \mathbb{P}^1$ be the corresponding quotient map, which we normalise such that $\varphi(O) = \infty$ and $\varphi(P) = 0$. We define $X = X_{(E,[P])}$ as the fibre product of the diagram

$$E \xrightarrow{\varphi} \mathbb{P}^1 \xleftarrow{-\varphi} E.$$

Note that, although there is a degree of freedom in choosing φ, this does not affect the construction.

It is obvious from the ramification data that $X_{(E,[P])}$ has genus 4, and the automorphisms $(Q_1, Q_2) \mapsto (Q_2, \phi(Q_1))$ and $(Q_1, Q_2) \mapsto (Q_1, \phi(Q_2))$ of $E \times E$ restrict to automorphisms α and β of $X_{(E,[P])}$ generating a D_8. It is straightforward to check that the map $(E, [P]) \mapsto X_{(E,[P])}$ thus defined is inverse to $X \mapsto (X/\beta,\ [p_1(\operatorname{Fix}(\alpha\beta))])$. $\square$

In particular, these curves satisfy the assumptions of Proposition 3.2.3, and therefore their Prym varieties are isomorphic to a product of elliptic curves.

Corollary 3.3.6. *Let $X \in \mathcal{M}_4(D_8)$. Then $\mathcal{P}(X, \alpha^2) \cong E \times E$ as polarised abelian varieties, where $E \cong X/\beta \cong X/\alpha^2\beta$.*

As the quotient elliptic curves are isomorphic, we pick some differential form η_E on E and denote by

$$\eta_i = p_i^* \eta_E, \quad \text{for } i = 1, 2, \tag{3.2}$$

the corresponding product basis. Using the explicit description of X as a fibre product and the expression of α used in the proof of Proposition 3.3.4, one can easily describe the action of α^* on these differentials to see that

$$\alpha^*\eta_1 = -\eta_2 \quad \text{and} \quad \alpha^*\eta_2 = \eta_1 \, .$$

In particular, α interchanges the spaces ΩX_β^+ and $\Omega X_{\rho\beta}^+$.

The eigenspace decomposition For a D_8 curve to parametrise an orbifold point, it must necessarily admit an α-eigenform with a single (6-fold) zero. To determine the possible eigenforms, we must analyse the decomposition of ΩX into α-eigenspaces. We denote, as usual, by ΩX^- and ΩX^+ the -1- and $+1$-eigenspaces of ΩX with respect to the (Prym) involution α^2.

Proposition 3.3.7. *Let $X \in \mathcal{M}_4(D_8)$. There is a natural splitting*

$$\Omega X^- = \Omega X_\alpha^{\mathrm{i}} \oplus \Omega X_\alpha^{-\mathrm{i}}$$

into $\pm\mathrm{i}$-eigenspaces of α. The spaces $\Omega X_\alpha^{\pm\mathrm{i}}$ are interchanged by β.

Proof. The quotient X/α has genus 1, so it is obvious that ΩX^+ decomposes into α-eigenspaces of dimension 1 with eigenvalue $+1$ and -1.

On the other hand, since $\alpha\beta = \beta\alpha^{-1}$, if $\alpha^*\omega = \lambda\omega$ for some $\lambda \in \mathbb{C}$, clearly $\alpha^*(\beta^*\omega) = \lambda^{-1}\beta^*\omega$. In particular, the eigenvalues of α^* on ΩX^- can only be $\pm\mathrm{i}$, therefore the space necessarily decomposes as the sum of the α^* i-eigenspace and the $-$i-eigenspace. □

Note that any α-eigenbasis (ω_1, ω_2) of ΩX^- will satisfy $\omega_1 \in \Omega X_\alpha^{\mathrm{i}}$ and $\omega_2 \in \Omega X_\alpha^{-\mathrm{i}}$, up to renumbering. Moreover, any product basis (η_1, η_2) as in Equation 3.2 gives rise to an α-eigenbasis

$$\omega_1 = \eta_1 + \mathrm{i}\eta_2, \quad \omega_2 = \eta_1 - \mathrm{i}\eta_2 \, . \tag{3.3}$$

Now, while the family $\mathcal{M}_4(D_8)$ of curves admitting a D_8 action is 2-dimensional, it turns out that requiring an α-eigenform with a single zero reduces the dimension of the locus we are interested in by one. Let us define

$$\mathcal{X} = \left\{ X \in \mathcal{M}_4(D_8) : \exists\, \omega \in \Omega X^- \ \alpha\text{-eigenform with a single zero} \right\} .$$

Because of the flat picture of the elements (X, ω) in $\Omega\mathcal{X}^-(6)$, we will call $\mathcal{X}$ the *Turtle family* (see section 3.6).

Theorem 3.3.8. *The map*

$$\begin{array}{rcl}\mathcal{X} & \longrightarrow & \mathcal{M}_{1,1}\\ X & \longmapsto & X/\beta\,,\end{array}$$

where the origin of X/β is chosen to be $p_1(\mathrm{Fix}(\alpha))$, induces a bijection between $\mathcal{X}$ and $\mathcal{M}_{1,1}\backslash\{E_2\}$.

The only curve in $\mathcal{X}$ where α is extended by an automorphism of order 12 is the one corresponding to $X/\beta \cong E_{\zeta_6}$. It agrees with family (4) in Proposition 3.2.1.

Proof. By Equation 3.3 α-eigenforms in ΩX^+ are given, up to scale, by $\omega_1 = p_1^*\eta_E + \mathrm{i}p_2^*\eta_E$ and $\omega_2 = p_1^*\eta_E - \mathrm{i}p_2^*\eta_E$. We will proceed in several steps.

Step 1: The α-eigenforms in ΩX^- can have a zero at most at one of the (two) fixed points of α.

Otherwise, every differential in ΩX^- would vanish at both fixed points of α. In particular, so would $p_1^*\eta_E$ and $p_2^*\eta_E$. But the maps p_i are unramified at $\mathrm{Fix}(\alpha)$ and we know that η_E has no zeroes in E. Note that, since zeroes of α-eigenforms outside $\mathrm{Fix}(\alpha)$ must be permuted by α, this immediately implies that the differentials ω_1 and ω_2 lie either in $\Omega\mathcal{X}(1^4, 2)$ or in $\Omega\mathcal{X}(6)$. Hence it remains to show that $\omega_1, \omega_2 \in \Omega\mathcal{X}(6)$.

Step 2: Note that $p_i^*\eta_E$ vanishes only at the six branch points of p_i.

In particular neither ω_1 nor ω_2 vanish at $\mathrm{Fix}(\beta) \cup \mathrm{Fix}(\alpha^2\beta)$, as the two sets of fixed points are disjoint.

Step 3: Choose λ so that $E \cong E_\lambda = \{y^2 = x(x-1)(x-\lambda)\}$ with the point at infinity as a distinguished point, and let $[P] = (A, \pm B)$ in these coordinates. Then we claim that $\omega_1, \omega_2 \in \Omega\mathcal{X}(6)$ if and only if $3A - 1 - \lambda = 0$.

In fact note that, in this case, the map $\varphi\colon E_\lambda \to \mathbb{P}^1$ in the proof of Proposition 3.3.4 can be chosen to be $(x, y) \mapsto x - A$, and the points in $X_{(E_\lambda,[P])}$ outside of the branch loci of the maps p_i can be seen as pairs of points

$$\begin{aligned}Q = \Big(&(x, \varepsilon_1\sqrt{x(x-1)(x-\lambda)}), (-x+2A, \varepsilon_2\mathrm{i}\sqrt{(x-2A)(x-2A+1)(x-2A+\lambda)})\Big)\\ &\in E_\lambda \times E_\lambda\,,\end{aligned}$$

where $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$. Normalising $\eta_E = \mathrm{d}x/y$ and evaluating a local expression around Q yields

$$\begin{aligned}\omega_1(Q) &= \frac{\varepsilon_1}{\sqrt{x(x-1)(x-\lambda)}} + \frac{\varepsilon_2}{\sqrt{(x-2A)(x-2A+1)(x-2A+\lambda)}} =\\ &= \frac{\varepsilon_1\sqrt{(x-2A)(x-2A+1)(x-2A+\lambda)} \pm \varepsilon_2\sqrt{x(x-1)(x-\lambda)}}{\varepsilon_1\varepsilon_2\sqrt{x(x-1)(x-\lambda)(x-2A)(x-2A+1)(x-2A+\lambda)}}\,,\end{aligned}$$

and similarly for ω_2.

Now, comparing the addends in the numerator and taking squares one sees that the differential ω_1 will vanish at Q, for (exactly) two choices $(\varepsilon_1, \varepsilon_2)$ and $(-\varepsilon_1, -\varepsilon_2)$, whenever

$$2(3A - \lambda - 1)x^2 - 4A(3A - \lambda - 1)x + 2A(2A - \lambda)(2A - 1) = 0\,.$$

In particular, if (and only if) $3A - \lambda - 1 = 0$ the differentials do not vanish in the affine part of E_λ, hence the zeroes of ω_1 and ω_2 must be at infinity, i.e. in $\mathrm{Fix}(\alpha)$, and Step 1 implies that there is only a single zero on $X_{(E_\lambda,[P])}$.

Step 4: The point $P = (A, B) \in E_\lambda$ can be uniquely chosen as a non-2-torsion point subject to the condition $A = (\lambda + 1)/3$ from above if and only if $\lambda \neq -1, 1/2, 2$.

Note that these three values of λ give rise to the same elliptic curve, namely the square torus E_2. In particular, for all $P \in E_2 \setminus E_2[2]$ the curve $X_{(E_2,[P])}$ has no α-eigenform with a single zero.

Therefore, for any $(E, O) \in \mathcal{M}_{1,1} \setminus \{E_2\}$, there is a unique choice of $[P]$, such that the fibre product $X_{(E,[P])}$ admits a D_8 action together with an α-eigenform that has a single 6-fold zero.

It remains to check when α can be extended, i.e. when there exists an $\alpha' \in \mathrm{Aut}\, X_{(E,[P])}$ that satisfies $\alpha \in \langle \alpha' \rangle$.

However, the proof of Proposition 3.2.1 shows that this can happen only if α' is of order 12. In this case, $(\alpha')^6 = \alpha^2$ commutes with β, hence descends to an automorphism of order 6 on the elliptic curve X/β which must therefore be isomorphic to E_{ζ_6}.

On the other hand, denote by $\psi \in \mathrm{Aut}\, E_{\zeta_6}$ the automorphism of order 6 on E_{ζ_6}. It is easy to see that the automorphism $(Q_1, Q_2) \mapsto (\psi(Q_2), \psi^4(Q_1))$ on $E_{\zeta_6} \times E_{\zeta_6}$ restricts to an automorphism of order 12, extending α, on the curve of $\mathcal{X}$ corresponding to this elliptic curve. In fact, the corresponding fibre product is the curve $y^6 = x(x+1)^2(x-1)^2$, see also section 3.4. □

Moreover, we have the following corollary.

Corollary 3.3.9. *Let X be a genus 4 curve admitting an automorphism α of order 4 with two fixed points. If additionally $(X, \omega) \in \Omega W_D$ then $X \in \mathcal{X}$ and $\omega = \omega_1$ or ω_2. In particular, ω is an α-eigenform and (X, ω) is a point of order 2 on W_D.*

To check which $(X, \omega_i) \in \Omega\mathcal{X}$ are on W_D, we need to check when $\mathcal{P}(X, \alpha^2)$ admits real multiplication with ω_i as an eigenform. Note that β^* interchanges ω_1 and ω_2, and therefore it is enough to focus on one of the two eigenforms.

First, we need the following explicit description of the endomorphism ring of the Prym variety. Recall that the endomorphism ring $\mathrm{End}(E)$ of an elliptic curve is either $\mathbb{Z}$ or an order in an imaginary quadratic field.

Lemma 3.3.10. *Let (η_1, η_2) be the product basis of ΩX^- as in Equation 3.2. Then*

$$\mathrm{End}(\mathcal{P}(X, \alpha^2)) = M_2(\mathrm{End}(E))\,,$$

where $E \cong X/\beta$. Self-adjoint endomorphisms correspond to matrices satisfying $M^T = M^\sigma$, where M^σ denotes conjugation by the non-trivial Galois automorphism of End(E) *on each entry.*

Moreover, ω_1 corresponds to the η-representation $(1,\ \mathrm{i})$ *and ω_2 to the representation* $(1,\ -\mathrm{i})$.

Proof. The first part of the lemma follows immediately from Corollary 3.3.6. The claim about the eigenforms follows from Equation 3.3. □

We now have all the ingredients assembled to prove the formula for the points of order 2.

Proof of Theorem 3.3.1. Recall that X classifies an orbifold point of order 2 on W_D if and only if $\mathcal{P}(X, \alpha^2)$ admits proper self-adjoint real multiplication with the α-eigenforms ω_1 or ω_2 as an eigenform. Since these are interchanged by β, it is enough to focus our attention on ω_1.

Again, we set $E = X/\beta$. Note that, by Theorem 3.3.8, E must not be isomorphic to E_{ζ_6}.

Assume that D is not a square. Now, in the η-basis of $E \times E$, the form ω_1 has the representation $(1,\ \mathrm{i})$ (cf. Lemma 3.3.10). In other words, $(X, \omega_1) \in \Omega\mathcal{X}$ is an orbifold point on W_D if and only if there exists $T_D \in M_2(\mathrm{End}(E))$, where

$$T_D = \begin{cases} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{-D}}{2} \\ \frac{\sqrt{-D}}{2} & \frac{1}{2} \end{pmatrix}, & \text{if } D \equiv 1 \bmod 4, \text{ and} \\ \begin{pmatrix} 0 & -\frac{\sqrt{-D}}{2} \\ \frac{\sqrt{-D}}{2} & 0 \end{pmatrix}, & \text{if } D \equiv 0 \bmod 4, \end{cases}$$

while there is no $T_{D'} \in M_2(\mathrm{End}(E))$ for $D = f^2 D'$.

As End(E) is integral over $\mathbb{Z}$ and $1/2$ is not, the case $D \equiv 1 \bmod 4$ can never occur. The other case occurs whenever $\sqrt{-D}/2 \in \mathrm{End}(E)$, and this happens if and only if E has complex multiplication by the order $\mathcal{O}_{-D} \subset \mathbb{Q}(\sqrt{-D})$.

To determine precisely which orders $\mathcal{O}_{-C}$ contain such a maximal T_D, note that, by definition, $\sqrt{-D}/2 \in \mathcal{O}_{-C}$ if and only if $D = b^2 C$ for some integer b. Moreover, C must be congruent with 0 or $3 \bmod 4$ so that $-C$ is a discriminant.

For $b > 2$ the action is never proper, and therefore we can assume $b = 1$ or 2.

The case $b = 1$ implies that elliptic curves E not isomorphic to E_{ζ_6} admitting complex multiplication by $\mathcal{O}_{-D}$ always determine an orbifold point of order 2 on W_D.

As for $b = 2$, there are several options. If $D/4 \equiv 1 \bmod 4$, then $-D/4 \equiv 3 \bmod 4$ is not a discriminant. If, however, $C = D/4 \equiv 3 \bmod 4$, then $-C$ is a discriminant and complex multiplication by $\mathcal{O}_{-C} = \mathcal{O}_{-D/4}$ also gives proper real multiplication by $\mathcal{O}_D$ on the Prym part. Finally, if $C = D/4 \equiv 0 \bmod 4$, then $-C$ is a discriminant but the Prym then admits real multiplication by $\mathcal{O}_C$, hence the real multiplication by $\mathcal{O}_D$ is not proper in these cases.

Moreover, observe that $E \cong E_{\zeta_6}$ if and only if $C = 3$, i.e. $D = 12$. On the other hand, if $D = 12$, there exists precisely one elliptic curve with proper complex multiplication by $\mathcal{O}_{-12}$ and hence $W_{12}(6)$ admits one point of order 2 and one point of order 6.

Finally, as it is well-known that there are $h(-C)$ elliptic curves admitting complex multiplication by $\mathcal{O}_{-C}$, this proves the result.

For the square discriminant case $D = f^2$, one can follow the same reasoning as above and use the fact that $\mathcal{O}_D = \mathbb{Z}[T]/(T^2 - fT)$ to deduce that the generator $T \in M_2(\mathrm{End}(E))$ must agree with

$$T = \begin{pmatrix} \frac{f}{2} & -\mathrm{i}\frac{f}{2} \\ \mathrm{i}\frac{f}{2} & \frac{f}{2} \end{pmatrix}$$

and an analysis similar to the one above proves the theorem. □

3.4. Points of order 3

In this section we prove the formula for the orbifold points of order 3 on W_D.

Recall the numbers

$$e_3(D) = \#\{a, i, j \in \mathbb{Z} : a^2 + 3j^2 + (2i - j)^2 = D,\ \gcd(a, i, j) = 1\}/12.$$

We have the following description of the orbifold points of order 3.

Theorem 3.4.1. *Let $D \neq 12$ be a positive discriminant. Then W_D has $e_3(D)$ orbifold points of order* 3.

To describe the points of order 3 on W_D, we again describe the intersection with the locus of curves with a fixed type of automorphism.

Curves admitting an automorphism of order 6 By Proposition 3.2.1, for an (X, ω) to parametrise a point of order 3 on W_D the curve X must necessarily admit an automorphism α of order six with two fixed points and two orbits of length 2 admitting ω as an eigenform. Note that in particular X/α has genus 0. Cyclic covers of the projective line have been thoroughly studied by several authors (see for example [Roh09; Bou05]), see also [TTZ16] for a brief summary of the facts required here).

Now, there are two families of cyclic covers of $\mathbb{P}^1$ of degree 6 with the given branching data, namely:

$$\mathcal{Y}_t\ :\ y^6 = x(x-1)^2(x-t)^2\,, \quad t \in \mathbb{P}^1\backslash\{0, 1, \infty\}$$

and

$$\mathcal{Z}_t \,:\, y^6 = x(x-1)^2(x-t)^4\,, \quad t \in \mathbb{P}^1 \backslash \{0,1,\infty\}\,.$$

Denote by $\alpha = \alpha_t$ the automorphisms $(x,y) \mapsto (x, \zeta_6 y)$ of order 6 on $\mathcal{Y}_t$ and on $\mathcal{Z}_t$. Note that both $\mathcal{Y}_t/\alpha^3$ and $\mathcal{Z}_t/\alpha^3$ have genus 2, so $\rho = \alpha^3$ is actually a Prym involution.

The following proposition shows immediately that no member of the $\mathcal{Z}$ family can belong to a Teichmüller curve W_D.

Lemma 3.4.2. *The space $\Omega\mathcal{Z}^-$ is disjoint from the minimal stratum $\Omega\mathcal{M}_4(6)$.*

Proof. It is easy to check (see for example [Bou05]) that for each t the space $\Omega\mathcal{Z}_t^-$ is generated by the differentials

$$\xi_1 = \frac{y\,\mathrm{d}x}{x(x-1)(x-t)} \quad \text{and} \quad \xi_2 = \frac{\mathrm{d}x}{y}\,.$$

They both lie in the stratum $\Omega\mathcal{M}_4(1^2,4)$. In fact, a local calculation shows that

$$\begin{aligned} \operatorname{div}\xi_1 &= 4P_1 + R_1' + R_2' \quad \text{and} \\ \operatorname{div}\xi_2 &= 4P_2 + R_1'' + R_2'', \end{aligned}$$

where the P_i are the two fixed points of α and $\{R_i'\}$ and $\{R_i''\}$ are the α-orbits of length 2.

Now, any element of $\Omega\mathcal{Z}_t^-$ different from the generators can be written as a linear combination $\xi = a\,\xi_1 + b\,\xi_2$. But, since the ξ_i vanish at different points, such a differential can never have a zero at any point of $\operatorname{Fix}(\alpha)$, nor at any point in the two α-orbits of length 2. As a consequence $\xi \in \Omega\mathcal{Z}_t^-(1^6)$ and the result follows. □

The following lemma detects which fibres of the $\mathcal{Y}$ family are isomorphic, together with the special fibre having a larger automorphism group.

Lemma 3.4.3. *The isomorphism $z \mapsto 1/z$ of $\mathbb{P}^1$ lifts to an isomorphism $\mathcal{Y}_t \cong \mathcal{Y}_{1/t}$ for each $t \in \mathbb{P}^1\backslash\{0,1,\infty\}$.*

In particular, at the fixed point, the automorphism α_{-1} of the curve $\mathcal{Y}_{-1}$ extends to an automorphism $\gamma\colon (x,y) \mapsto (1/x, y/x)$ of order 12.

Proof. As the curve is given in coordinates explicitly as a cyclic cover of $\mathbb{P}^1$, this is a straight-forward calculation. □

The intersections of $\mathcal{Y}$ and W_D will give the orbifold points of order 3 on W_D. To make this statement more precise, we begin by the following observation.

Proposition 3.4.4. *For each t the space $\Omega\mathcal{Y}_t^-$ is generated by the α-eigenforms*

$$\omega_1 = \frac{y\,\mathrm{d}x}{x(x-1)(x-t)} \quad \textit{and} \quad \omega_2 = \frac{-y\,\mathrm{d}x}{\sqrt{t}(x-1)(x-t)}\,.$$

Up to scale, the only differentials in $\Omega\mathcal{Y}_t^-(6)$ are ω_1 and ω_2.

Proof. The local expressions show that these differentials are holomorphic for all t. They obviously span the α-eigenspace of eigenvalue ζ_6 and therefore generate $\Omega\mathcal{Y}_t^-$ (cf. [Bou05]).

It is easy to see that ω_1 (resp. ω_2) has a single zero at the single point at infinity (has a single zero at $(0,0)$). Now, for every $a \neq 0$ the zeroes of the differential

$$\omega_a := \omega_1 + a\omega_2 = \frac{y\mathrm{d}x}{(x-1)(x-t)}\left(\frac{1}{x} + a\right)$$

are located at the points with x-coordinate $^{-1}/_{a}$. They are either six simple zeroes if $a \neq -1, ^{-1}/_{t}$, or three zeroes of order 2 otherwise. □

Remark 3.4.5. *Note that, in contrast to the family $\mathcal{X}$ of curves with a D_8 action, the α-eigenspace inside $\Omega\mathcal{Y}_t^-$ is in fact 2-dimensional. However, we will only be interested in the two 1-dimensional subspaces of eigenforms with a single zero.*

Because of the flat picture of the differentials $(\mathcal{Y}_t, \omega_i)$, we will call $\mathcal{Y}$ the *Hurricane family* (see section 3.6). Note that (ω_1, ω_2) yields an α-eigenbasis of $\Omega\mathcal{Y}_t^-$. The following is a consequence of Lemma 3.4.2 and Proposition 3.4.4.

Corollary 3.4.6. *Let X be a genus 4 curve admitting an automorphism α of order 6 with two fixed points and two orbits of length 2. If $(X,\omega) \in \Omega X^-(6)$ then $X \in \mathcal{Y}$ and $\omega = \omega_1$ or ω_2. In particular, ω is an α-eigenform.*

Corollary 3.4.7. *A flat surface (X,ω) parametrising a point on W_D corresponds to an orbifold point of order 3 if and only if there is some $t \in \mathbb{P}^1\backslash\{0,1,-1,\infty\}$ such that $X \cong \mathcal{Y}_t$ and $[\omega] = [\omega_1]$ or $[\omega] = [\omega_2]$.*

It corresponds to an orbifold point of order 6 if and only if $X \cong \mathcal{Y}_{-1}$ and $[\omega] = [\omega_1]$ or $[\omega] = [\omega_2]$

Proof. This is a consequence of Proposition 3.2.1, Lemma 3.4.2 and Proposition 3.4.4. □

We must therefore analyse when the Prym part of $\mathcal{Y}_t$ admits real multiplication. Recall that the elliptic curve E_ζ, where $\zeta := \exp(2\pi\mathrm{i}/6)$, is the only elliptic curve admitting an automorphism of order 6 fixing the base point. It corresponds to the hexagonal lattice, i.e.

$$E_\zeta \cong \mathbb{C}/\Lambda_\zeta, \quad \text{with} \quad \Lambda_\zeta = \mathbb{Z} \oplus \zeta\mathbb{Z}.$$

Next, we collect some useful observations.

Lemma 3.4.8. *Any curve $\mathcal{Y}_t$ admits an involution β commuting with α, i.e. such that $\langle\alpha,\beta\rangle \cong C_6 \times C_2$. Moreover, one has $\mathcal{P}(\mathcal{Y}_t, \rho) \cong E_\zeta \times E_\zeta$.*

The general member $\mathcal{Z}_t$ of the $\mathcal{Z}$ family has an automorphism group equal to C_6.

Proof. By Theorems 1 and 2 in [Sin72] there is only one Fuchsian group containing a generic Fuchsian group of signature $(0;3,3,6,6)$. The signature of such supergroup is $(0;2,2,3,6)$, and the inclusion is of index 2 and therefore normal. As a consequence, the automorphism group of any general fibre in the $\mathcal{Y}$ family or the $\mathcal{Z}$ family is at most an extension of index two of C_6.

In the case of $\mathcal{Y}_t$, the inclusion induces an extra automorphism $\beta := \beta_t$, given by $(x,y) \mapsto (x/t, \sqrt{t}\, y/x)$.

In particular α^3 and β generate a Klein four-group such that the quotients $\mathcal{Y}_t/\beta$ and $\mathcal{Y}_t/\alpha^3\beta$ have genus 1. Therefore they satisfy the conditions of Proposition 3.2.3 and $\mathcal{P}(\mathcal{Y}_t, \rho) \cong \mathcal{Y}_t/\beta \times \mathcal{Y}_t/\alpha^3\beta$. Since α induces an automorphism ψ of order 6 on both $\mathcal{Y}_t/\beta$ and $\mathcal{Y}_t/\alpha^3\beta$, they are necessarily isomorphic to the elliptic curve E_ζ.

As for the $\mathcal{Z}$ family, any such automorphism would induce an automorphism of $\mathcal{Z}_t/\alpha \cong \mathbb{P}^1$ permuting orbifold points of the same order. Since the exponents at 0 and ∞ and at 1 and t are different, there cannot be such an automorphism. $\square$

We will write again $p_1 \colon \mathcal{Y}_t \to \mathcal{Y}_t/\beta$ and $p_2 \colon \mathcal{Y}_t \to \mathcal{Y}_t/\alpha^3\beta$ for the corresponding projections. The following lemma gives an explicit formula for these two maps that will be needed later to compute the explicit pullbacks of the differentials on E_ζ.

Lemma 3.4.9. *Consider the Weierstraß equation $\{v^2 = u^3 - 1\}$ defining E_ζ. In this model, the maps p_1 and p_2 are given by*

$$\begin{aligned} p_1 \colon \mathcal{Y}_t &\to E_\zeta \\ (x,y) &\mapsto \left(\frac{-1}{(1+\sqrt{t})^{2/3}} \frac{(x-1)(x-t)}{y^2}, \frac{\mathrm{i}}{(1+\sqrt{t})} \frac{(x-1)(x-t)(x+\sqrt{t})}{y^3} \right), \end{aligned}$$

$$\begin{aligned} p_2 \colon \mathcal{Y}_t &\to E_\zeta \\ (x,y) &\mapsto \left(\frac{-1}{(1-\sqrt{t})^{2/3}} \frac{(x-1)(x-t)}{y^2}, \frac{\mathrm{i}}{(1-\sqrt{t})} \frac{(x-1)(x-t)(x-\sqrt{t})}{y^3} \right). \end{aligned}$$

These maps are only unique up to composition with (a power of) α.

Proof. The map $\mathcal{Y}_t \to \mathcal{Y}_t/\beta$ induces an isomorphism between the function field $\mathbb{C}(\mathcal{Y}_t/\beta)$ and the subfield $\mathbb{C}(\mathcal{Y}_t)^{\langle\beta\rangle} \subset \mathbb{C}(\mathcal{Y}_t)$ fixed by β^*. This subfield is generated by the rational functions

$$\widetilde{u} := x + \beta(x) + 2\sqrt{t} = \frac{(x+\sqrt{t})^2}{x}, \quad \widetilde{v} := y + \beta(y) = y\frac{x+\sqrt{t}}{x}.$$

Using the equation of $\mathcal{Y}_t$ it is easy to check that the generating functions $\widetilde{u}$ and $\widetilde{v}$ satisfy the relation $\widetilde{v}^6 = \widetilde{u}^3(\widetilde{u}-c)^2$, where $c = (1+\sqrt{t})^2$. One can then check the ramification points of the degree 6 function $(\widetilde{u}, \widetilde{v}) \mapsto \widetilde{u}$ and easily deduce the isomorphism

$$\begin{aligned} \widetilde{E} \,:\, \widetilde{v}^6 = \widetilde{u}^3(\widetilde{u}-c)^2 &\to E_\zeta \,:\, v^2 = u^3 - 1 \\ (\widetilde{u}, \widetilde{v}) &\mapsto (u,v) = \left(\frac{-\widetilde{u}(\widetilde{u}-c)}{c^{1/3}\,\widetilde{v}^2}, \frac{\mathrm{i}\,\widetilde{v}^3}{c^{1/2}\,\widetilde{u}(\widetilde{u}-c)} \right) \end{aligned}$$

Finally, replacing $\widetilde{u}$ and $\widetilde{v}$ by their values in terms of the coordinates x and y, one gets the formula for p_1.

The same argument replacing β by $\alpha^3\beta$ yields the result for p_2. □

Fibre products Similarly to the case of the D_8 family, one can also construct the Hurricane family $\mathcal{Y}$ of genus 4 curves with a $C_6 \times C_2$ action as a certain family of fibre products over two isomorphic elliptic curves. In order to do so, let ψ denote the automorphism of order 6 on E_ζ and consider the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{\;p_1\;} & X/\beta \cong E_\zeta \\ \big\downarrow & & \big\downarrow \varphi \\ X/\alpha^3 & \longrightarrow & X/\langle \alpha^3, \beta \rangle \cong \mathbb{P}^1 \\ \big\downarrow & & \big\downarrow \\ \mathbb{P}^1 \cong X/\alpha & \longrightarrow & X/\langle \alpha, \beta \rangle \cong \mathbb{P}^1 \end{array}$$

Clearly, $\overline{\beta}$ is the hyperelliptic involution on X/α^3. On $X/\alpha \cong \mathbb{P}^1$ the involution $\overline{\beta}$ has two fixed points and the preimages of these points give the six Weierstraß points on X/α^3. Moreover, $X \to X/\alpha^3$ is ramified only over the two fixed points of α, while the map $X/\alpha^3 \to X/\alpha$ also branches at R' and R'', the preimages (on X) being $\{R_1', R_2'\}$ and $\{R_1'', R_2''\}$, respectively.

Now, α and β have no common fixed points, hence the image of the (two) fixed points of α on X gives the (unique) fixed point O of $\psi = \overline{\alpha}$ on E_ζ. Additionally, $\overline{\beta}$ interchanges R' and R'', hence we may name the fibres such that the images R_1 of $\{R_1', R_1''\}$ and R_2 of $\{R_2', R_2''\}$ form the unique ψ-orbit of order 3 on E_ζ.

On the other hand, the six Weierstraß points of X/α^3 have 12 preimages on X with β acting on each fibre. Three fibres form the six fixed points of β on X, i.e. the branch points of p_1, while the other three give the fixed points of $\alpha^3\beta$, which are equivalently the fixed points of the elliptic involution $\phi = \overline{\alpha}^3$ on X/β, i.e. the three 2-torsion points. The situation is exactly reversed for the projection $p_2\colon X \to X/\alpha^3\beta \cong E_\zeta$.

Finally, note that in Lemma 3.4.9 the coordinates on E_ζ were chosen such that the projection $\varphi\colon E_\zeta \to \mathbb{P}^1 \cong E_\zeta/\phi$ to the quotient by the elliptic involution maps O to ∞ and both R_1 and R_2 to 0. Observe that ψ then descends to an automorphism of order 3 on the quotient that fixes 0 and ∞. In particular, we can assume $\varphi(\psi(S)) = \zeta_6^2\varphi(S)$, for each $S \in E_\zeta$.

Now, for each $P \in E_\zeta^* := E_\zeta \setminus (E_\zeta[2] \cup \{R_1, R_2\})$ consider the map $\varphi_P\colon E_\zeta \to \mathbb{P}^1$, $Q \mapsto \varphi(P) \cdot \varphi(Q)$. We define $Y = Y_P$ as the fibre product of the diagram

$$E_\zeta \xrightarrow{\varphi} \mathbb{P}^1 \xleftarrow{\varphi_P} E_\zeta.$$

This fibre product admits a group of automorphisms isomorphic to $C_6 \times C_2$ given by the restriction of the following automorphisms of $E_\zeta \times E_\zeta$:

$$\alpha(Q_1, Q_2) = (\psi(Q_1), \psi(Q_2)), \quad \beta(Q_1, Q_2) = (Q_1, \psi^3(Q_2)).$$

By Lemma 3.4.8, every Y_P is therefore a fibre of the $\mathcal{Y}$ family.

Proposition 3.4.10. *The map $P \mapsto Y_P$ gives a 6-to-1 map between the set of elliptic pairs of points E_ζ^*/ϕ and the fibres of $\mathcal{Y}$.*

It descends to a 2-to-1 map between the set E_ζ^*/ψ of regular orbits of ψ and the fibres of $\mathcal{Y}$. Moreover, the only ramification value of this map corresponds to the curve $\mathcal{Y}_{-1}$ admitting an automorphism of order 12.

Proof. Let $P \in E_\zeta^*$. Note that the construction does not depend on the choice of $\{P, \phi(P)\}$. In fact, even for any choice of a different point in the orbit $\{\psi^j(P)\}_{j=0}^5$ the automorphism of $E_\zeta \times E_\zeta$ given by $(Q_1, Q_2) \mapsto (Q_1, \psi^{-j}(Q_2))$ induces an isomorphism between Y_P and $Y_{\psi^j(P)}$.

Now, for the point $P' \in E_\zeta^*$ such that $\phi(P') = 1/\phi(P)$, the automorphism $(Q_1, Q_2) \mapsto (Q_2, Q_1)$ induces an isomorphism between Y_P and $Y_{P'}$.

On the other hand, for any $Y \in \mathcal{Y}$ take $x \in \mathrm{Fix}(\beta)$ and write $P = [x] \in Y/\beta \cong E_\zeta$ for its image in the quotient. It is straightforward to check that $Y \cong Y_P$. Any other choice of $x \in \mathrm{Fix}(\beta)$ or $x \in \mathrm{Fix}(\alpha^3\beta)$ determines different points in $\{\psi^j(P), \psi^j(P')\}_{j=0}^5$, defining the same fibre product. □

Remark 3.4.11. *Note that the action of ψ on the point P corresponds to the action of α on the maps p_i mentioned in Lemma 3.4.9. The remaining factor of 2 comes from the (generic) identification of $\mathcal{Y}_t$ with $\mathcal{Y}_{1/t}$.*

Eigenforms with a single zero By Lemma 3.4.8, all Prym varieties in the $\mathcal{Y}$ family are isomorphic. To understand $\mathrm{End}(\mathcal{P}(\mathcal{Y}_t, \rho))$, where $\rho = \alpha^3$, denote by (η_1, η_2) again the product basis of $\Omega\mathcal{Y}_t^-$ given by

$$\eta_i = p_i^* \eta_E, \quad \text{for } i = 1, 2. \tag{3.4}$$

It is well known that $\mathcal{O}_\zeta := \mathrm{End}(E_\zeta) = \mathbb{Z} \oplus \mathbb{Z}\zeta_6^2$ are the *Eisenstein integers.*

Lemma 3.4.12. *Let (η_1, η_2) be the product basis of $\Omega\mathcal{Y}_t^-$ from Equation 3.4. Then*

$$\mathrm{End}(\mathcal{P}(\mathcal{Y}_t, \alpha^3)) = M_2(\mathrm{End}(E_\zeta)) = M_2(\mathcal{O}_\zeta).$$

Self-adjoint endomorphisms correspond to matrices $M^T = M^\sigma$, where M^σ denotes conjugation by the non-trivial Galois automorphism of $\mathcal{O}_\zeta$ on each entry.

Proof. This is an immediate consequence of Lemma 3.4.8. □

While the product basis gives an easy understanding of the endomorphism ring, and while in fact any differential in $\Omega\mathcal{Y}_t^-$ is an α-eigendifferential, we are interested in α-eigendifferentials *with a single zero* that are also eigenforms for real multiplication of the Prym variety. By Proposition 3.4.4, these are precisely the differentials ω_1 and ω_2 on $\mathcal{Y}_t$.

To check whether ω_1 or ω_2 are eigenforms for real multiplication, we must therefore keep track of these differentials in the product basis. For this, we set

$$\mu := \mu_t := \left(\frac{1-\sqrt{t}}{1+\sqrt{t}}\right)^{1/3}.$$

The relationship between the α-eigenbasis and the product basis can be summarised as follows:

Lemma 3.4.13. *Denote by (η_1, η_2) the product basis. Then*

$$[\omega_1] = [-\mu_t\,\eta_1 + \eta_2]\,, \quad [\omega_2] = [\mu_t\,\eta_1 + \eta_2]$$

gives an α-eigenbasis with ω_i having each a single zero on $\mathcal{Y}_t$.

In particular, for each isomorphism class of curves $[Y] \in \mathcal{Y}$, with $[Y] \neq [\mathcal{Y}_{-1}]$, there exist 12 elements $\mu_t \in \mathbb{C}^*$ such that $\mu_t\,\eta_1 + \eta_2$ are precisely the α-eigendifferentials with a single zero on $[Y]$.

On the curves $[\mathcal{Y}_{-1}]$ there are six different values of μ giving eigendifferentials with a single zero.

Proof. The differential $\omega_1 + \omega_2$ (resp. $\omega_1 - \omega_2$) is β-invariant (resp. $\alpha^3\beta$-invariant). Therefore there exist k_1, k_2 such that $\omega_1 + \omega_2 = k_1\,\eta_1$ and $\omega_1 - \omega_2 = k_2\,\eta_2$, where η_E is a fixed differential on E_ζ.

In particular

$$\frac{-x}{\sqrt{t}} = \frac{\omega_2}{\omega_1} = \frac{k_1\,\eta_1 - k_2\,\eta_2}{k_1\,\eta_1 + k_2\,\eta_2}.$$

One can solve for k_1/k_2 to get

$$\frac{k_1}{k_2} = -\frac{(x-\sqrt{t})\eta_2}{(x+\sqrt{t})\eta_1},$$

and then, using Lemma 3.4.9 and choosing $\eta_E = \mathrm{d}u/v$ in that model,

$$\frac{k_1}{k_2} = -\left(\frac{1-\sqrt{t}}{1+\sqrt{t}}\right)^{1/3} = -\mu\,.$$

Now, solving t in terms of μ gives

$$t = \left(\frac{\mu^3-1}{\mu^3+1}\right)^2,$$

and Lemma 3.4.3 implies the rest of the claims. □

Note that every value of μ gives two eigendifferentials with single zeros on (generically) two different fibres of $\mathcal{Y}$, which are identified by six different isomorphisms.

Lemma 3.4.14. *For $t \neq -1$ we have that in $\mathbb{P}\Omega\mathcal{Y}(6)$*

$$(\mathcal{Y}_t, \omega_1) \cong (\mathcal{Y}_t, \omega_2) \cong (\mathcal{Y}_{1/t}, \omega_1) \cong (\mathcal{Y}_{1/t}, \omega_2)$$

as flat surfaces and $(\mathcal{Y}_t, \omega_1) \ncong (X, \omega)$ for all other $(X, \omega) \in \mathbb{P}\Omega\mathcal{Y}(6)$.

Proof. This is clear by Proposition 3.4.4, Lemma 3.4.3 and the fact that β interchanges the classes of ω_1 and ω_2. □

In particular, we do not have to distinguish between the classes of ω_1 and ω_2. This relationship becomes more explicit when expressed in the fibre product construction.

Proposition 3.4.15. *Let $(\mu\,\eta_E, \eta_E) \in \Omega E_\zeta \times \Omega E_\zeta$, $\mu \neq 0$ and let $P = (A, B) \in E_\zeta^*$. The corresponding α-eigendifferential $\mu\,\eta_1 + \eta_2$ on Y_P has a single zero at a fixed point of α if and only if $A = \mu^2$.*

In particular, this induces a 12-to-1 map

$$\begin{array}{lcl} \mathbb{C}^* & \to & \mathbb{P}\,\Omega\mathcal{Y}(6) \\ \mu & \mapsto & [(Y_P, [\mu\,\eta_1 + \eta_2])]\,, \end{array}$$

where $[P] = (\mu^2, \pm\sqrt{\mu^6 - 1})$ is an elliptic pair on E_ζ.

Proof. For each $P \in E_\zeta^*$ we will consider the differentials $\eta_1 = p_1^*\eta_E$ and $\eta_2 = p_2^*\eta_E$ on Y_P. The proof of this theorem will proceed in a similar way to the proof of Theorem 3.3.8 up until Step 3.

Step 1: The α-eigenforms in ΩY_P^- can have zeroes at most at one of the fixed points of α.

Otherwise, every differential in ΩY_P^- would vanish at both fixed points of α. In particular, so would η_1 and η_2, but the maps p_i are unramified at $\mathrm{Fix}(\alpha)$ and we know that η_E has no zeroes in E.

Again, zeroes of α-eigenforms must be permuted by α, the orbits of which have length 1, 2 or 6. This immediately implies that α-eigenforms lie either in $\Omega Y_P(1^6)$ if the zeroes are located at regular points, in $\Omega Y_P(6)$ if it only has zeroes at a fixed point of α, or in $\Omega Y_P(3^2)$ if it has zeroes at the two points of the orbit of length 2 (see the proof of Proposition 3.4.4). Again, we just need to prove that $\omega_1, \omega_2 \in \Omega Y_P(6)$.

Step 2: Again, $p_i^*\eta_E$ vanishes only at the six branch points of p_i. In particular both η_1 and η_2 lie in $\Omega Y_P(1^6)$.

Step 3: Let $E_\zeta \cong \{v^2 = (u^3 - 1)\}$ with the point at infinity as a distinguished point, and let $(\mu\,\eta_E, \eta_E) \in \Omega E_\zeta \times \Omega E_\zeta$. We claim that, given a point $P = (A, B)$ in these coordinates, the differential $\mu\,\eta_1 + \eta_2$ on Y_P has a single zero if and only if $A = \mu^2$.

Note that we can normalise $\varphi : E_\zeta \to \mathbb{P}^1$ to be $(u, v) \mapsto u$. By construction of Y_P as the fibre product of the maps $\varphi, \varphi_P : E_\zeta \to \mathbb{P}^1$, points in Y_P outside of the branch loci of the maps p_i can then be seen as pairs

$$Q = \left(\left(u, \varepsilon_1 \sqrt{u^3 - 1}\right), \left(\frac{u}{A}, \varepsilon_2 \sqrt{\frac{u^3 - A^3}{A^3}}\right) \right) \in E_\zeta \times E_\zeta \,,$$

where $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$. Normalising $\eta_E = \mathrm{d}x/y$, and evaluating locally around Q yields

$$\mu\, \eta_1 + \eta_2(Q) = \frac{\varepsilon_1 \mu}{\sqrt{u^3 - 1}} + \frac{\varepsilon_2 \sqrt{A^3}}{A\sqrt{u^3 - A^3}} = \frac{\varepsilon_1 \mu A \sqrt{u^3 - A^3} + \varepsilon_2 \sqrt{A^3(u^3 - 1)}}{A\sqrt{(u^3 - 1)(u^3 - A^3)}} \,.$$

Comparing again the addends in the numerator and taking squares, one sees that this differential vanishes at Q (for two choices $(\varepsilon_1, \varepsilon_2)$ and $(-\varepsilon_1, -\varepsilon_2)$) whenever

$$u^3 = A \cdot \frac{\mu^2 A^2 - 1}{\mu^2 - A} \,.$$

In particular, whenever the right-hand side is different from 0, 1 and ∞ one has that the differential $\mu\, \eta_1 + \eta_2$ necessarily has 6 simple zeroes. The case $u^3 = 1$ corresponds to $\mu = 0$, which has been treated in Step 2. The case $u = 0$ corresponds to $A = \pm 1/\mu$ and yields the differentials with zeroes at the two points of the α-orbit of length 2.

Finally, if $A = \mu^2$ the zeroes of the differential must be in $\mathrm{Fix}(\alpha)$, and Step 1 then implies that there is a single zero.

As a consequence, to each $\mu \in \mathbb{C}^*$ we can associate the elliptic pair of points

$$[P] = (\mu^2, \pm\sqrt{\mu^6 - 1}) \in E_\zeta^*,$$

defining the curve Y_P together with the differential with a single zero $\mu\, \eta_1 + \eta_2$. By Proposition 3.4.10 and the fact that $\pm\mu$ give the same elliptic pair and, by Lemma 3.4.14, the same class of flat surfaces, the association is 12-to-1. □

We are now finally in a position to prove the formula for $e_3(D)$.

Proof of Theorem 3.4.1. First, let D be a nonsquare discriminant and recall the order $\mathcal{O}_D = \mathbb{Z} \oplus T_D\mathbb{Z}$ associated to D, where

$$T_D = \begin{cases} \frac{\sqrt{D}}{2}, & D \equiv 0 \bmod 4, \\ \frac{\sqrt{D}+1}{2}, & D \equiv 1 \bmod 4. \end{cases}$$

Then, for $i = 1, 2$, $(\mathcal{Y}_t, \omega_i)$ lies on W_D if and only if $\mathcal{P}(\mathcal{Y}_t, \rho)$ admits real multiplication with ω_i as an eigenform. By Lemma 3.4.12 and Lemma 3.4.13 this is equivalent to the existence of some self-adjoint matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_\zeta) \quad \text{such that} \quad A \cdot \begin{pmatrix} \pm\mu \\ 1 \end{pmatrix} = T \cdot \begin{pmatrix} \pm\mu \\ 1 \end{pmatrix} .$$

By Lemma 3.4.14, it suffices to consider $+\mu$. Moreover, by self-adjointness, we have $c = b^\sigma$, the Galois conjugate in $\mathcal{O}_\zeta$, and $a, d \in \mathbb{Z}$. The eigenform condition then yields

$$(a - T_D)\mu + b = 0 \quad \text{and} \quad b^\sigma \mu + d - T_D = 0.$$

The first equation gives

$$\mu = \frac{b}{T_D - a}$$

and substituting this into the second equation yields

$$bb^\sigma - ad = T_D^2 - (a+d)T_D.$$

First, we consider the case $D \equiv 0 \bmod 4$. Then this gives

$$bb^\sigma - ad = \frac{D}{4} - (a+d)\frac{\sqrt{D}}{2}.$$

As the right side of the equation must be an integer, we find $a = -d$ and hence

$$D = 4bb^\sigma + (2a)^2, \quad \text{for} \quad D \equiv 0 \bmod 4.$$

Similarly, for $D \equiv 1 \bmod 4$, we obtain $d = a - 1$ and thus

$$D = 4bb^\sigma + (2a-1)^2, \quad \text{for} \quad D \equiv 1 \bmod 4.$$

It is well-known that the norm squared of an element in $\mathcal{O}_\zeta$ is given by

$$bb^\sigma = i^2 - ij + j^2 = \frac{3j^2 + (2i-j)^2}{4}, \quad \text{for} \quad b = i + \zeta_6^2 j.$$

Hence, $\mathcal{P}(\mathcal{Y}_t, \rho)$ admits a real multiplication by $\mathcal{O}_D$ with ω_i as an eigenform for every $a, i, j \in \mathbb{Z}$ such that

$$a^2 + 3j^2 + (2i-j)^2 = D.$$

Clearly, this real multiplication is proper if and only if $\gcd(a, i, j) = 1$.

By Lemma 3.4.13 or equivalently Proposition 3.4.15, this gives 12 times the cardinality of points of order 3.

A similar analysis in the square discriminant case $D = f^2$ yields, with the same notation as above, $d = f - a$ and $bb^\sigma - a(f-a) = 0$. Multiplying by 4 and adding f^2 to both sides of the equation one gets

$$D = 4bb^\sigma + (2a - f)^2,$$

and the same argument as above proves the result. □

3.5. Points of order 5

In this section we will find the orbifold points of order 5 on the Teichmüller curves W_D.

Theorem 3.5.1. *The Teichmüller curve W_5 has one orbifold point of order 5. For any other discriminant, W_D has no orbifold points of order 5.*

Curves admitting an automorphism of order 10 By Proposition 3.2.1, flat surfaces (X, ω) parametrising a point of order 5 on W_D will correspond to cyclic covers of degree 10 of $\mathbb{P}^1$ ramified over three points with ramification order 5, 10 and 10. There are two such curves:

$$\mathcal{V} \,:\; y^{10} = x(x-1)^2\,, \quad \text{and} \quad \mathcal{U} \,:\; y^{10} = x(x-1)^8\,.$$

Calculations similar to the ones in the proof of Lemma 3.4.2 and Proposition 3.4.4 give us the differentials with a single zero on these curves.

Proposition 3.5.2. *The space $\Omega\mathcal{V}^-$ is generated by the α-eigenforms*

$$\omega_1 = \frac{y\mathrm{d}x}{x(x-1)}\,,\; \omega_2 = \frac{y^3\mathrm{d}x}{x(x-1)}\,.$$

Up to scale, the only differential in $\Omega\mathcal{V}^-(6)$ is ω_1.

The space $\Omega\mathcal{U}^-$ is disjoint from the minimal stratum $\Omega\mathcal{M}_4(6)$.

In particular one has the following corollary.

Corollary 3.5.3. *Let X be a genus 4 curve admitting an automorphism α of order 10 with two fixed points and an orbit of length 2. If $(X, \omega) \in \Omega X^-(6)$ then $X = \mathcal{V}$ and, up to scale, $\omega = \omega_1$. In particular, ω is an α-eigenform.*

The action of α on $\mathcal{P}(\mathcal{V}, \alpha^5)$ induces an embedding $\mathbb{Q}(\zeta_{10}) \hookrightarrow \mathrm{End}_{\mathbb{Q}}(\mathcal{P}(\mathcal{V}, \alpha^5))$ and, in particular, determines an element $T_5 = \alpha + \alpha^{-1} = (\sqrt{5}+1)/2$ for which ω_1 is an eigenform.

Proof of Theorem 3.5.1. By the argument above and the maximality of $\mathcal{O}_5$, the Prym variety $\mathcal{P}(\mathcal{V}, \alpha^5)$ admits proper real multiplication by $\mathcal{O}_5$ with ω_1 as an eigenform.

Now Teichmüller curves ΩW_D and ΩW_E are disjoint for different discriminants D and E. Therefore, as ω_1 is, up to scale, the only differential with a single zero on $\mathcal{V}$, there can be no other W_D with a point of order 5. □

3.6. Flat geometry of orbifold points

In this section we will describe, up to scale, the translation surfaces corresponding to the Turtle family $\mathcal{X}$, the Hurricane family $\mathcal{Y}$ and the curve $\mathcal{V}$. We use the notion of k-differentials and $(1/k)$-translation structures, cf. [BCGGM16, §2.1, 2.3].

Note that, whereas in the first two cases we have a 1-dimensional family of flat surfaces, in the case of $\mathcal{V}$ the construction is unique (cf. Corollary 3.5.3). The case of a flat surface with a symmetry of order twelve, also unique, is given by $\mathcal{X} \cap \mathcal{Y}$, the intersection of the Turtle family and the Hurricane family (cf. Theorem 3.3.8).

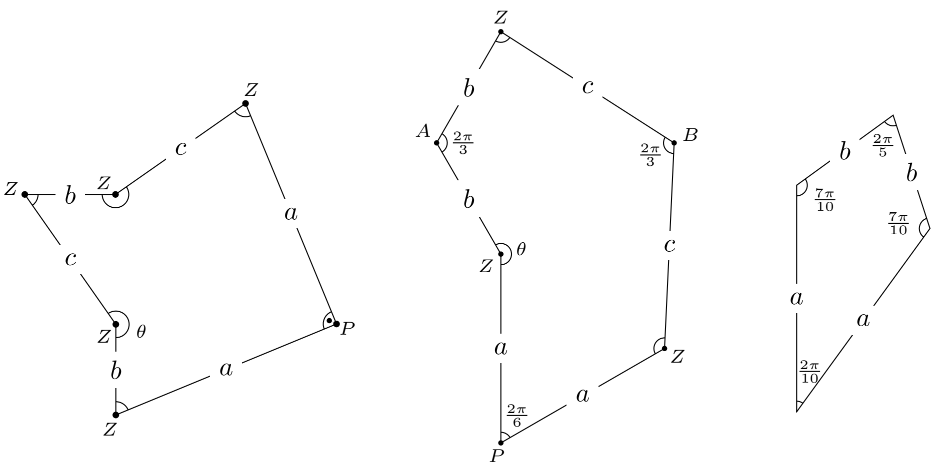


Figure 3.3.: LEFT: A 4-differential of genus 1 with a zero and pole, each of order 3. The length of c and the angle θ give a complex parameter. CENTER: A 6-differential of genus 0 with a single zero of order 1, a pole of order 5 and two poles of order 4 each. The length of b and the angle θ give a complex parameter. RIGHT: The unique (up to scale) 10-differential on $\mathbb{P}^1$ with poles of order 3, 8 and 9.

Points of order 2 We briefly describe the construction of flat surfaces $(\mathcal{X}_\kappa, \eta_\kappa) \in \Omega\mathcal{X}(6)$ (that is curves X_κ with a four-fold symmetry α together with a differential η_κ with a six-fold zero) in terms of a parameter $\kappa = \kappa(c, \theta)$.

By Proposition 3.2.1, the quotient X_κ/α is of genus 1 with two fixed points. Therefore, a 4-differential ξ of genus 1 with a zero and a pole, each of order 3, at the two fixed points will have (X_κ, η_κ) as a canonical cover, i.e. $\eta_\kappa^4 = \pi^*\xi$, cf. [BCGGM16]. The polygon corresponding to ξ is given in Figure 3.3 with an angle of $2\pi/4$ at the pole and $7 \cdot 2\pi/4$ at the zero. Note that the three pairs of sides are identified by translation and rotation by angle $\pi/2$ and that the side c can be chosen as a complex parameter (i.e. the length of c and the angle θ). The "unfolded" canonical cover, resembling a turtle, is pictured in Figure 3.1 (section 3.1).

Points of order 3 Similarly, we can construct flat surfaces $(Y_\tau, \eta_\tau) \in \Omega\mathcal{Y}(6)$ admitting a six-fold symmetry α and a six-fold zero in terms of a parameter $\tau = \tau(b, \theta)$.

By Proposition 3.2.1, the quotient X/α is of genus 0 with two fully ramified points and two points that are fully ramified over an intermediate cover of degree 3. For the flat picture, this implies that we have a zero with angle $7 \cdot 2\pi/6$, a pole with angle $2\pi/6$ and two poles with angles $2\pi/3$, see Figure 3.3 where the sides are identified by translation and rotation of multiplies of $2\pi/6$ to give a surface of genus 0. Equivalently, this is a 6-differential on $\mathbb{P}^1$ with a single zero of order 1, a pole of order 5 and two poles of order 4, admitting a canonical cover with only a single zero, cf. [BCGGM16, §2]. The "unfolded" canonical cover, resembling a hurricane, is pictured in Figure 3.2 (section 3.1).

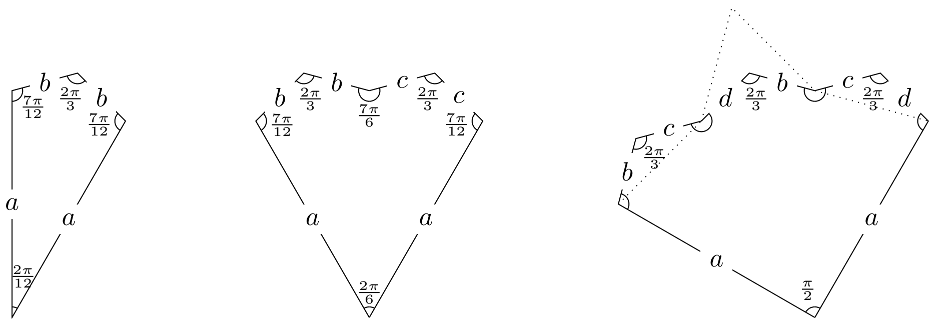


Figure 3.4.: LEFT: A 12-differential of genus 0 with a pole of order 5 that pulls back to the zero, and poles of order 11 and 8. Note that this is unique up to scaling. CENTER: By taking the canonical double cover, we obtain a 6-differential on $\mathbb{P}^1$ as in Figure 3.3. RIGHT: By taking a triple cover, we obtain a 4-differential on the elliptic curve E_ζ with an automorphism of order 6. If we cut and re-glue as indicated, we obtain a polygon as in Figure 3.3.

Point of order 6 By Theorem 3.3.1, there is a unique Prym differential (X, ω) with a symmetry of order 12 situated on $W_{12}(6)$.

By Proposition 3.2.1, we may picture this as a degree 12 cyclic cover of $\mathbb{P}^1$ with two points of order 12 and one point of order 3. Hence, by [BCGGM16], (X, ω) is the canonical cover of a 12-differential on $\mathbb{P}^1$ with a pole of order 5 that pulls back to the zero, and poles of order 11 and 8 at the totally ramified point and the point of order 3, respectively. Equivalently, we may glue a quadrilateral with two angles of $7\pi/12$ each and angles of $2\pi/12$ and $2\pi/3$ to give a surface of genus 0, see Figure 3.4. By "unfolding" once, i.e. taking the canonical 2-cover, we obtain the 6-differential on $\mathbb{P}^1$ that exhibits (X, ω) as a fibre in the Hurricane family (see Figure 3.4). Taking the canonical degree 3 cover, we can cut and re-glue as indicated in Figure 3.4 to obtain the 4-differential that is a C_{12}-eigenform on the elliptic curve with an automorphism of order 6 in the shape of the Turtle family (see Figure 3.3).

Point of order 5 Finally, by Theorem 3.5.1 there is a unique point of order 5, i.e. an (X, ω) with a symmetry of order 10 and a six-fold-zero differential.

More precisely, X is a degree 10 cyclic cover of $\mathbb{P}^1$ ramified over three points, two of order 10 and one of order 5. Hence, (X, ω) is the canonical cover of a 10-differential ξ on $\mathbb{P}^1$ with a pole of order 3 (that pulls back to the single zero on X) and poles of order 9 and 8 at the second fixed point and the point of order 5 respectively, cf. [BCGGM16, Prop. 2.4]. Equivalently, the flat picture has angles of size $2\pi/10$, $2\pi/5$ and two angles of size $7 \cdot 2\pi/10$ each, see Figure 3.3 where the sides are identified by translation and rotation of multiplies of $2\pi/10$ to give a surface of genus 0. Note that this differential is unique up to scaling. The "unfolded" canonical cover is shown in Figure 3.5.

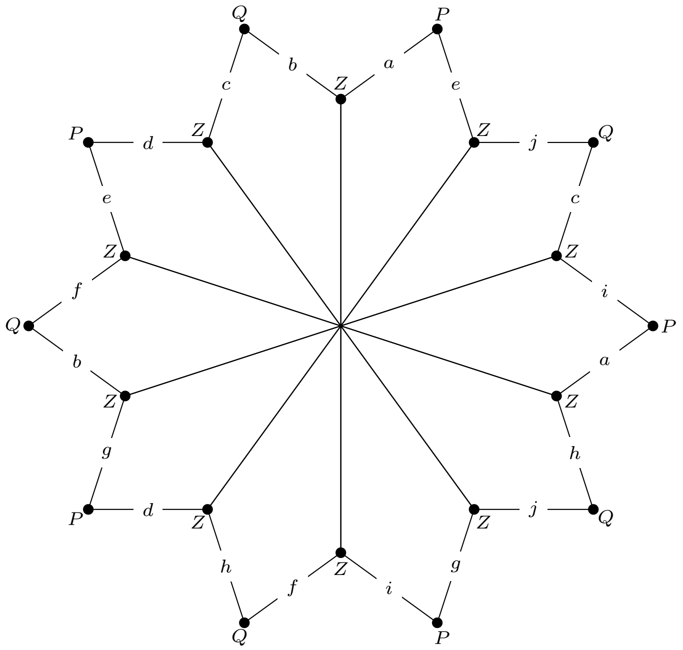


Figure 3.5.: The canonical 10-cover of the 10-differential in Figure 3.3: a C_{10}-eigendifferential of genus 4 with a single zero Z.

3.7. Genus asymptotics

The aim of this section is to describe the asymptotic behaviour of the genus $g(W_D)$ of W_D with respect to D.

As additional boundary components make the calculation of the Euler characteristic for $D = d^2$ more tedious (cf. [Bai07, §13]), we will assume throughout this section that W_D is primitive, i.e. that D is not a square.

Theorem 3.7.1. *There exist constants $C_1, C_2 > 0$, independent of D, such that*

$$C_1 \cdot D^{3/2} < g(W_D) < C_2 \cdot D^{3/2}.$$

More precisely, we give the following explicit upper bound on the genus.

Proposition 3.7.2. *The genus of W_D satisfies $g(W_D) < 1 + D^{3/2} \cdot \frac{35}{48\pi^2}$.*

We also give an explicit lower bound.

Proposition 3.7.3. *The genus of W_D satisfies*

$$g(W_D) \geq \frac{3}{200} D^{3/2} - \frac{D}{6} - D^{3/4} - 150.$$

Corollary 3.7.4. *The only curves W_D with $g(W_D) = 0$ are the loci for $D \leq 20$.*

Proof. By Proposition 3.7.3, $g(W_D) > 0$ whenever $D > 1050$. The smaller values of D were checked by computer. □

Recall that the genus of W_D, $g(W_D)$, is given by $(D > 12)$

$$g(W_D) = h_0(W_D) - \frac{\chi(W_D)}{2} - \frac{C(W_D)}{2} - \frac{e_2(W_D)}{4} - \frac{e_3(W_D)}{3},$$

where $\chi(W_D)$ is the (orbifold) Euler characteristic of W_D, $C(W_D)$ denotes the number of cusps and $e_d(W_D)$ the number of points of order d on W_D. Moreover, by [LN16], $h_0(W_D) = 1$ and by [Möl14, Theorem 4.1],

$$\chi(W_D) = -7\chi(X_D),$$

where X_D is the Hilbert modular surface of discriminant D. Moreover, $\chi(X_D)$ was calculated, for fundamental D, by Siegel in terms of the Dedekind zeta function ζ_D of $\mathbb{Q}(\sqrt{D})$. For non-fundamental D we write $D = f^2 D_0$, where f is the conductor of D and $\left(\frac{D_0}{p}\right)$ for the Legendre symbol, if p is a prime. Furthermore, we set

$$F(D) = \prod_{p|f} \left(1 - \left(\frac{D_0}{p}\right) p^{-2}\right),$$

where the product runs over all prime divisors p of f, and thus have

$$\chi(X_D) = \chi(X_{f^2 D_0}) = 2f^3 \zeta_{D_0}(-1) F(D) = D^{3/2} \zeta_{D_0}(2) \frac{F(D)}{2\pi^4},$$

using the functional equation of ζ_{D_0}, cf. [Bai07, §2.3]. Finally, using Euler products, we obtain the classical bounds

$$\zeta(2)^2 = \frac{\pi^4}{36} > \zeta_{D_0}(2) > \zeta(4) = \frac{\pi^4}{90}$$

and

$$\frac{\zeta(2)}{\zeta(4)} = \frac{15}{\pi^2} > F(D) > \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

We can now give an upper bound on $g(W_D)$.

Proof of Proposition 3.7.2. As $C(W_D), e_2(W_D), e_3(W_D) > 0$, these terms may be neglected yielding

$$g(W_D) \leq 1 + \frac{7}{2}\chi(X_D) < 1 + D^{3/2} \cdot \frac{35}{48\pi^2},$$

using the bounds given above. □

Obtaining a lower bound is slightly more involved, as it involves bounding the number of cusps and orbifold points from above. In general, the cusps are hardest to control, but by [LN14] and [McM05a], we have

$$C(W_D(6)) = C(W_D(2)),$$

i.e. the Teichmüller curves of discriminant D in $\mathcal{M}_2$ and $\mathcal{M}_4$ have the same number of cusps. Moreover, denote by P_D the *product locus* in $\mathcal{A}_2$, i.e. abelian surfaces that are polarized products of elliptic curves. This is a union of modular curves and, again by [McM05a],

$$C(W_D) = C(P_D).$$

To bound the cusps we may therefore proceed in complete analogy to [Muk14, §6].

Lemma 3.7.5. *The cusps are bounded from above by*

$$\frac{C(W_D)}{2} \leq D^{3/4} + 150 + \frac{5}{4}\chi(X_D).$$

Proof. By [Bai07, Theorem 2.22], $\chi(P_D) = -\frac{5}{2}\chi(X_D)$. Moreover, by [Muk14, Proposition 6.5], the number of connected components of P_D can be bounded by

$$h_0(P_D) \leq D^{3/4} + 150.$$

Therefore, we may write

$$\begin{aligned} -\frac{C(W_D)}{2} &= -\frac{C(P_D)}{2} = g(P_D) - h_0(P_D) + \frac{\chi(P_D)}{2} + \sum_d \left(1 - \frac{1}{d}\right) e_d(P_D) \\ &\geq -h_0(P_D) + \frac{\chi(P_D)}{2} \geq -D^{3/4} - 150 - \frac{5}{4}\chi(X_D), \end{aligned}$$

which yields the claim. □

Next, we must bound the number of orbifold points.

Lemma 3.7.6. *The number of points of order* 2 *satisfies* $e_2(D) < \frac{D}{2}$.

Proof. By Theorem 3.3.1, we have $e_2(D) \leq h(-D) + h(-\frac{D}{4})$. Now, it is well-known that class numbers of imaginary quadratic fields may be computed by counting reduced quadratic forms (cf. e.g. [Coh93, §5.3]), giving $h(-D) < \frac{D}{3}$ and thus proving the claim. □

Lemma 3.7.7. *The number of points of order* 3 *satisfies* $e_3(D) < \frac{D}{6}$.

Proof. By Theorem 3.4.1,

$$e_3(D) \leq \#\{a, i, j \in \mathbb{Z} : a^2 + 3j^2 + (2i - j)^2 = D\}/12.$$

The integers a and j essentially determine i. Moreover, a must have the same parity as D giving at most $\sqrt{D}/2$ choices (up to sign) and j ranges (again up to sign) over at most $\sqrt{D}$ possibilities. Accounting for sign choices and dividing by 12 yields the claim. □

Remark 3.7.8. *The bound from Lemma 3.7.7 can be improved. Indeed, by the theory of modular forms of half-integral weight, integral solutions of positive definite quadratic forms can always be realized as coefficients of a suitable modular form, see [Shi73] or e.g. [Leh92] for the concrete case at hand. In particular, the integral solutions of* $a^2 + 3j^2 + (2i - j)^2$ *are coefficients of a modular form of weight* $3/2$*, level* 12 *and Kronecker character* 12*. Hence,*

$$e_3(D) < C \cdot D^{3/4}$$

for some constant C *that is independent of* D *(cf. e.g. [MZ16, Theorem 2.1] for growth rates of coefficients of modular forms).*

This permits us to also give a lower bound for $g(W_D)$, proving Theorem 3.7.1.

Proof of Proposition 3.7.3. By the above bounds, we have

$$\begin{aligned} g(W_D) &= h_0(W_D) - \frac{\chi(W_D)}{2} - \frac{C(W_D)}{2} - \frac{e_2(W_D)}{4} - \frac{e_3(W_D)}{3} \\ &> \frac{9}{4}\chi(X_D) - \frac{D}{6} - D^{3/4} - 150, \end{aligned}$$

which yields the claim by the above bounds on $\chi(X_D)$. □

D	g	χ	C	e_2	e_3	e_5	e_6
5	0	$-7/15$	1	0	1	1	
8	0	$-7/6$	2	1	1		
12	0	$-7/3$	3	1	0		1
13	0	$-7/3$	3	0	2		
17	0	$-14/3$	6	0	1		
20	0	$-14/3$	5	2	1		
21	1	$-14/3$	4	0	1		
24	1	-7	6	2	0		
28	1	$-28/3$	7	2	2		
29	1	-7	5	0	3		
32	1	$-28/3$	7	2	2		
33	2	-14	12	0	0		
37	1	$-35/3$	9	0	4		
40	2	$-49/3$	12	2	2		
41	3	$-56/3$	14	0	1		
44	3	$-49/3$	9	4	2		
45	4	-14	8	0	0		
48	4	$-56/3$	11	2	1		
52	4	$-70/3$	15	2	2		
53	4	$-49/3$	7	0	5		
56	6	$-70/3$	10	4	2		
57	7	$-98/3$	20	0	1		
60	8	-28	12	4	0		
61	6	$-77/3$	13	0	4		
65	8	$-112/3$	22	0	2		
68	6	-28	14	4	3		
69	10	-28	10	0	0		
72	10	-35	16	2	0		
73	10	$-154/3$	32	0	2		
76	11	$-133/3$	21	4	2		
77	9	-28	8	0	6		
80	10	$-112/3$	16	4	2		
84	14	$-140/3$	18	4	1		
85	12	-42	16	0	6		
88	15	$-161/3$	22	2	4		
89	17	$-182/3$	28	0	1		
92	15	$-140/3$	13	6	4		
93	15	-42	12	0	3		
96	18	-56	20	4	0		
97	21	$-238/3$	38	0	2		
101	14	$-133/3$	15	0	5		
104	18	$-175/3$	20	6	2		
105	27	-84	32	0	0		
108	21	-63	21	4	0		
109	18	-63	25	0	6		
112	22	$-224/3$	29	2	4		
113	26	-84	32	0	3		
116	21	-70	25	6	3		
117	21	-56	16	0	0		
120	29	$-238/3$	20	4	2		
124	31	$-280/3$	29	6	2		
125	21	$-175/3$	15	0	5		
128	25	$-224/3$	22	4	4		
129	37	$-350/3$	44	0	1		
132	29	-84	26	4	0		
133	27	$-238/3$	22	0	8		
136	35	$-322/3$	36	4	2		
137	37	-112	38	0	3		
140	33	$-266/3$	18	8	4		
141	34	-84	18	0	0		
145	46	$-448/3$	58	0	2		
148	39	$-350/3$	37	2	4		
149	30	$-245/3$	19	0	7		
152	37	$-287/3$	18	6	4		
153	45	-140	52	0	0		
156	46	$-364/3$	26	8	2		
157	36	$-301/3$	25	0	8		
160	44	$-392/3$	40	4	4		
161	55	$-448/3$	40	0	2		
164	37	-112	34	8	3		
165	42	$-308/3$	18	0	4		
168	51	-126	24	4	0		
172	53	-147	37	4	6		
173	37	-91	13	0	9		
176	49	$-392/3$	29	6	4		
177	66	-182	52	0	0		
180	52	-140	36	4	0		
181	49	-133	33	0	6		

Table 3.3.: Topological invariants of the Teichmüller curves $W_D(6)$ for nonsquare discriminant. The number of cusps is described in [LN14], the Euler characteristic in [Möl14].

4. $\mathbb{C}$-Linear manifolds

The general technique for counting orbifold points on Teichmüller curves in genus 2, 3 and 4 is the following: an orbifold point is always a flat surface (X, ω) together with some holomorphic automorphism α which admits ω as an eigendifferential. The topological action of α can be determined using flat geometry. Then, one can attempt to describe the locus inside the moduli space of flat surfaces admitting an automorphism and an eigendifferential of this form and count the intersection points of this locus and the Teichmüller curve.

The aim of this chapter is to study these loci more conceptually.

In particular, such an eigenform condition can be thought of as a $\mathbb{C}$-linear equation in period coordinates. We will show that this is in fact enough to determine these loci, i.e. that they are cut out by $\mathbb{C}$-linear equations in period coordinates.

More precisely, we introduce the notion of a $\mathbb{C}$-linear manifold, i.e. a closed submanifold $\mathcal{M} \subseteq \Omega\mathcal{M}_g(\mu)$ that is locally cut out by $\mathbb{C}$-linear equations in period coordinates. Note that $\mathbb{R}$-linear manifolds, *affine invariant manifolds*, have been studied intensely over the last decades, see e.g. [Wri15b] for a summary of known results. In particular, they are closely related to the $\mathrm{GL}_2(\mathbb{R})$-action on the moduli space of flat surfaces: indeed, any affine invariant manifold is $\mathrm{GL}_2(\mathbb{R})$-invariant and a deep result of Eskin–Mirzakhani–Mohammadi asserts that the converse is also true, i.e. that any $\mathrm{GL}_2(\mathbb{R})$-orbit closure is in fact an affine invariant manifold [EMM15]. Moreover, Filip has shown that any affine invariant manifold is in fact algebraic [Fil16].

By contrast, $\mathbb{C}$-linear manifolds that are not $\mathbb{R}$-linear have hardly been studied, except for some general results in [Möl08]. However, [Möl08, Definition 6.4] included an extra condition (requiring the existence of a certain compactification) that, while it made sense at the time, does not seem warranted from today's perspective. This limits the applicability of the results of [Möl08] to our situation.

First, we provide a large class of examples of $\mathbb{C}$-linear manifolds that are not $\mathbb{R}$-linear: spaces of eigenforms of cyclic covers of $\mathbb{P}^1$. These have been used, e.g., in the previous chapters to count orbifold points of Prym–Teichmüller curves.

Theorem 4.0.1 (Theorem 4.3.1)**.** *Any family of eigenspaces of a family of cyclic covers of $\mathbb{P}^1$ is a $\mathbb{C}$-linear manifold.*

In order to have a chance at achieving any meaningful classification, we need some notion of *primitive* $\mathbb{C}$-linear manifold, i.e. a manifold that is not a cover of another. The typical example of an imprimitive $\mathbb{C}$-linear manifold arises from an eigenspace of a cyclic cover where the (exponent of the) eigenvalue is not coprime to the degree of the cover, i.e. arises via pullback from an intermediate cover (see section A.4 for details).

We therefore proceed to define a *covering construction* of a ℂ-linear manifold. Of course, any such covering construction should be induced by ramified covering maps on the "points": if $\mathcal{N}$ is a cover of $\mathcal{M}$ then the flat surfaces in $\mathcal{N}$ should be *translation covers*, i.e. of the form $(Y, f^*\omega)$, where (X, ω) is some flat surface in $\mathcal{M}$ and $f: Y \to X$ is a ramified cover. Any such cover is uniquely determined by a monodromy representation ρ of the Riemann surface X (with some marked points, see section 4.4 for details and notation).

Let $\mathcal{M} \subseteq \Omega\mathcal{M}_g(\mu)$ be a linear manifold, $(X, \omega) \in \mathcal{M}$, ρ a monodromy representation of (X, ω) and $\mathcal{M}(\rho) \subseteq \Omega\mathcal{M}_h(\nu)$ the cover associated to ρ (see Definition 4.4.14 for details). Then we show:

Theorem 4.0.2 (Theorem 4.4.1)**.** *$\mathcal{M}(\rho)$ is a linear manifold.*

While there are several notions of covering constructions for ℝ-linear manifolds (cf. [Api16], [MMW17]), these typically use the $\mathrm{SL}_2(\mathbb{R})$-action and the results of [EMM15] to show that the cover of an ℝ-linear manifold is again an ℝ-linear manifold. As there is no $\mathrm{SL}_2(\mathbb{R})$-action in the general ℂ-linear case, we provide a slightly technical alternative. We use the fact that linearity of a manifold can be checked in terms of local systems (Lemma 4.1.3).

This implies that we need to sufficiently rigidify the manifolds involved (by essentially passing to Teichmüller space) to obtain an actual ramified covering (Lemma 4.4.6). Once such a map is constructed, we may transport the local systems freely between the manifolds involved.

We thus split the covering construction into two parts: constructing covers of a manifold where the translation covers have prescribed monodromy groups (Proposition 4.4.12) and constructing quotients of a manifold where all points admit a suitable G-action (Proposition 4.4.13). In particular, these two constructions are inverse to each other.

The main result is that covers and (suitable) quotients of linear manifolds are again linear (Proposition 4.4.12 and Proposition 4.4.13). This allows us to define arbitrary covers of linear manifolds that are again linear and we thus obtain a natural notion of *primitive* ℂ-linear manifold, see Definition 4.4.15 for a precise definition.

However, a classification of *primitive* ℂ-linear manifolds is an open project. Also, the question of algebraicity, i.e. if an analogous result to [Fil16] can be shown in the ℂ-linear case, is still wide open. We discuss questions and open problems in section 4.5.

4.1. Flat surfaces and period coordinates

Let X be a compact Riemann surface or equivalently a smooth projective complex curve and let $\omega \in \Omega(X) = \mathrm{H}^0(X, \Omega^1_X) = \mathrm{H}^{1,0}(X)$ be a non-zero differential form. By integration, ω endows X with a flat structure outside the zeros of ω. We therefore call the pair (X, ω) a *flat surface*. We denote the moduli space of flat surfaces by $\Omega\mathcal{M}_g$. Note that this space comes with a natural projection $\Omega\mathcal{M}_g \to \mathcal{M}_g$.

Moreover, due to the presence of automorphisms, these spaces are not algebraic varieties, but we may remedy this by, for example, adding a level-3-structure. We denote the corresponding spaces by $\mathcal{M}_g^{[3]}$ and $\Omega\mathcal{M}_g^{[3]}$, respectively. However, we will abuse notation and tacitly drop the exponent, unless the distinction explicitly affects our arguments. In particular, this guarantees the existence of universal families over all our objects.

4.1.1. Period coordinates Given a flat surface (X, ω), consider

$$\operatorname{div}\omega = \sum_{i=1}^{n} a_i p_i,$$

where p_i are the (distinct) zeros of ω and $\sum a_i = 2g-2$. Associating to (X, ω) the partition $(a_1, \dots, a_n)$ of $2g-2$ gives a natural stratification of $\Omega\mathcal{M}_g$; given a partition μ of $2g-2$, we denote the corresponding stratum by $\Omega\mathcal{M}_g(\mu)$.

The reason for considering this stratification is that the strata may be endowed with holomorphic *period coordinates*: for $(X_0, \omega_0) \in \Omega\mathcal{M}_g(a_1, \dots, a_n)$ as above, we set $N := 2g+n-1$ and pick a basis $\gamma_1, \dots, \gamma_N$ of the relative homology group $\mathrm{H}_1(X_0, Z(\omega_0); \mathbb{Z})$ with respect to $Z(\omega_0)$, the zeros of ω_0. The Gauß-Manin connection lets us identify this group with $\mathrm{H}_1(X, Z(\omega); \mathbb{Z})$ for all (X, ω) in a neighbourhood U_0 of (X_0, ω_0) by parallel transport. We thus obtain a map

$$\Omega\mathcal{M}_g(a_1, \dots, a_n) \supseteq U_0 \ni (X, \omega) \longmapsto \left(\int_{\gamma_i} \omega\right)_{i=1}^{N} \in \mathbb{C}^N,$$

which turns out to be locally biholomorphic (away from orbifold points), thus providing analytic charts with transition functions in $\mathrm{GL}(N, \mathbb{Z})$ (the change of basis in relative homology). Equivalently, we may view period coordinates as associating the relative cohomology class

$$U_0 \ni (X, \omega) \mapsto [\gamma \mapsto \int_\gamma \omega] \in \mathrm{H}^1(X, Z(\omega); \mathbb{C})$$

to (X, ω); again we use the Gauß-Manin connection to identify these cohomology groups on U_0 to yield complex coordinates on the stratum.

Let μ be a partition of $2g-2$ and $K \subseteq \mathbb{C}$ a subfield of $\mathbb{C}$.

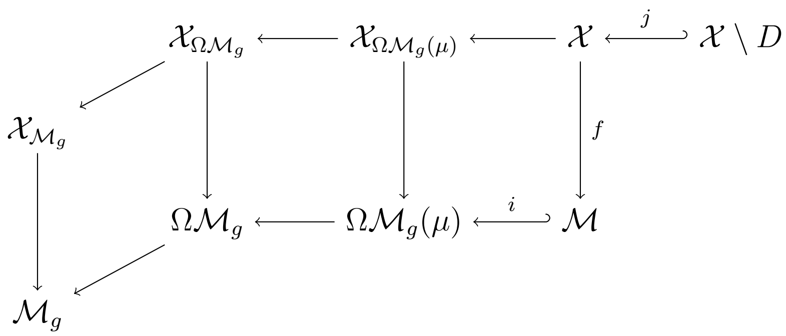
Definition 4.1.1. *A* K-linear manifold *is the image of a closed* algebraic *immersed manifold* $\mathcal{M} \to \Omega\mathcal{M}_g(\mu)$ *that is locally cut out by linear equations in period coordinates with coefficients in* $K \subseteq \mathbb{C}$.

Note that, in general, a K-linear manifold will not be embedded in the stratum (the image could have self-intersections). However, this will not be a problem for us, as we can work with the fibre-product. For notational convenience and as no confusion can arise, we treat $\mathcal{M}$ as an embedded manifold.

Remark 4.1.2. $\mathbb{R}$-*linear manifolds are also known as* affine invariant submanifolds *and have been studied extensively over the last decades. Affine invariant manifolds are closely related to dynamical systems: the group* $\mathrm{GL}_2(\mathbb{R})$ *acts naturally on the strata* $\Omega\mathcal{M}_g(\mu)$ *and any affine invariant manifold is invariant under this action. In fact, the converse is also true: a deep result of Eskin-Mirzakhani-Mohammadi [EMM15] asserts that any* $\mathrm{GL}_2(\mathbb{R})$-*orbit closure is an affine invariant manifold. Moreover, Filip [Fil16] showed that in the case* $K = \mathbb{R}$ *the algebraicity condition is always satisfied, i.e. any closed submanifold that is locally cut out by* $\mathbb{R}$-*linear equations in period coordinates is algebraic.*

In particular, Teichmüller curves are examples of affine invariant manifolds.

4.1.2. Bundles on manifolds Let $i\colon \mathcal{M} \hookrightarrow \Omega\mathcal{M}_g(\mu)$ be an algebraic submanifold. Then $\mathcal{M}$ comes equipped with a universal family $f\colon \mathcal{X} \to \mathcal{M}$, i.e. the fibre over $(X,\omega) \in \mathcal{M}$ is X, that is obtained by pullback of the universal family over $\mathcal{M}_g$:



Recall that we have tacitly endowed everything with a level-3 structure such that all these universal families do in fact exist. Moreover, note that the universal family over $\Omega\mathcal{M}_g(\mu)$ (and its pullback to any submanifold) comes equipped with a universal divisor D that restricts to the divisor of zeros of ω on every fibre: $D|_{(X,\omega)} = D_\omega \coloneqq \operatorname{div}\omega$. Again, we denote the reduced divisor associated to D_ω by $Z(\omega)$. We denote by $j\colon \mathcal{X}\setminus D \hookrightarrow \mathcal{X}$ the natural inclusion. There are several natural bundles on $\mathcal{M}$, which we briefly describe.

- The local system of (absolute) cohomology $\mathbb{V} \coloneqq \mathrm{R}^1 f_*\underline{\mathbb{C}}$, the fibre over $(X,\omega) \in \mathcal{M}$ being $\mathrm{H}^1(X,\mathbb{C})$, and the associated vector bundle $\mathbb{V}\otimes\mathcal{O}_\mathcal{M}$ of rank $2g$.
- The local system of relative cohomology $\mathbb{V}^{\mathrm{rel}} \coloneqq \mathrm{R}^1 f_* j_!\underline{\mathbb{C}}$, the fibre over $(X,\omega) \in \mathcal{M}$ being $\mathrm{H}^1(X, Z(\omega);\mathbb{C})$, and the associated vector bundle $\mathbb{V}^{\mathrm{rel}}\otimes\mathcal{O}_\mathcal{M}$ of rank N.
- The Hodge bundle $f_*\omega_{\mathcal{X}/\mathcal{M}}$ on $\mathcal{M}$ of rank g, the fibre over $(X,\omega) \in \mathcal{M}$ being $\mathrm{H}^{1,0}(X)$.
- The tangent bundle $\mathrm{T}\,\mathcal{M}$ of rank $\dim\mathcal{M}$, the fibre over $(X,\omega)\in\mathcal{M}$ being $\mathrm{T}_{(X,\omega)}\,\mathcal{M}$, the tangent space of $\mathcal{M}$ at (X,ω).

Note that, as all the involved families are flat, R^1 commutes with base change of vector bundles. Clearly, $f_*\omega_{\mathcal{X}/\mathcal{M}} \subseteq \mathbb{V}\otimes\mathcal{O}_\mathcal{M}$ is a subbundle. Moreover, by dualizing the exact sequence of relative homology we see that

$$0 \to \mathrm{H}^0(X,\mathbb{C}) = \mathbb{C} \to \mathrm{H}^0(Z(\omega),\mathbb{C}) \to \mathrm{H}^1(X,Z(\omega);\mathbb{C}) \to \mathrm{H}^1(X,\mathbb{C}) \to \mathrm{H}^1(Z(\omega),\mathbb{C}) = 0$$

(note that $\dim Z(\omega) = 0$) is exact. Equivalently, observe that we obtain the relative cohomology groups $\mathrm{H}^i(X, Z(\omega); \mathbb{C})$ as the sheaf cohomology of $j_!\underline{\mathbb{C}}$ on X and the above sequence is therefore an immediate consequence of the exactness of the sequence

$$0 \to j_!\underline{\mathbb{C}} \to \underline{\mathbb{C}} \to i_*\underline{\mathbb{C}} \to 0$$

of sheaves on X. In the relative situation, this induces a surjective map $\mathbb{V}^{\mathrm{rel}} \to \mathbb{V}$, the kernel being the relative cohomology cycles.

4.1.3. A criterion for linearity Consider first the case that $\mathcal{M} = \Omega\mathcal{M}_g(\mu)$ consists of the entire stratum. The linear structure of period coordinates identifies the tangent bundle $\mathrm{T}\,\mathcal{M}$ with $\mathbb{V}^{\mathrm{rel}} \otimes \mathcal{O}_{\mathcal{M}}$, the bundle of relative cohomology, in this case. If $\mathcal{M} \hookrightarrow \Omega\mathcal{M}_g(\mu)$ is a submanifold of a stratum, period coordinates identify a neighbourhood U_0 of a point $(X_0, \omega_0) \in \mathcal{M}$ with a subset of relative cohomology $\mathrm{H}^1(X_0, Z(\omega_0); \mathbb{C})$. In the case that $\mathcal{M}$ is locally linear, we may also identify U_0 with the tangent space $\mathrm{T}_{(X_0,\omega_0)}\,\mathcal{M}$ to $\mathcal{M}$ at (X_0, ω_0). This allows us to restate Möller's linearity criterion:

Lemma 4.1.3 ([Möl08, Thm 3.1])**.** *An algebraic submanifold $\mathcal{M} \subseteq \Omega\mathcal{M}_g(\mu)$ is linear if and only if there exists a sub-local system $\mathbb{L} \subseteq \mathbb{V}^{\mathrm{rel}}$ such that $i_*(\mathrm{T}\,\mathcal{M})$ and $\mathbb{L} \otimes \mathcal{O}_{\mathcal{M}}$ are locally isomorphic.*

Proof. The above discussion yields a local identification of $\mathbb{L}$ and $\mathrm{T}\,\mathcal{M}$. Conversely, if the tangent bundle is locally constant, $\mathcal{M}$ must be locally linear. Note that, as our description is local, the twist by the line bundle $\mathcal{O}_{\mathbb{P}}(1)$ appearing in [Möl08] can be ignored here. □

4.2. Automorphisms and cyclic covers

The first non-trivial examples of $\mathbb{C}$-linear manifolds will be cyclic covers of the projective line. Let α be a holomorphic automorphism of X of order d and consider the map $\pi\colon X \to Y \coloneqq X/\alpha$. It is a branched cover of order d and the branching behaviour is determined by the lengths of α-orbits on X.

4.2.1. Families of cyclic coverings By varying the branch points on Y we obtain a family $\mathcal{X}$ of covers of Y, of dimension

$$\dim \mathcal{X} = \#\{\text{branch points of } \pi\} - \dim \operatorname{Aut} Y.$$

In the following, we shall restrict to the case where the genus of Y is 0, i.e. $Y \cong \mathbb{P}^1$. In this case it is well-known that $\dim \operatorname{Aut} Y = 3$ and the advantage is that we may easily write down coordinates.

Now, for each fibre X of $\mathcal{X}$, we fix the following notation: Let $x_1, \ldots, x_n \in \mathbb{P}^1$ be the branch points of π on $\mathbb{P}^1$. For convenience, we choose coordinates on $\mathbb{P}^1$ so that

$0 \neq x_i \neq \infty$ for all i. Furthermore, we denote by $a_1, \ldots, a_n$ the *monodromy exponent*, i.e. the exponents that make X equal to the normalisation of

$$\pi\colon y^d = \prod_{i=1}^{n} (x - x_i)^{a_i} \to \mathbb{A}^1.$$

In particular, we may choose $0 < a_i < d$ and we have $\gcd(d, a_1, \ldots, a_n) = 1$ and $\sum a_i \in d\mathbb{Z}$, as X is connected. Note that every x_i has $\gcd(d, a_i)$ preimages on X. Moreover, the data $[d; a_1, \ldots, a_n]$ together with the points x_i determine the complex structure of X uniquely. See Appendix A for background and details on cyclic covers.

4.2.2. Decomposition of the VHS The automorphism α acts on the cohomology of each fibre $\mathrm{H}^1(X, \mathbb{C})$ and, as it is of finite order, this action is diagonalisable. We use the notation of [KM16, §6.5]. We identify $G \coloneqq \mathrm{Gal}(X/\mathbb{P}^1) \cong \mathbb{Z}/d\mathbb{Z}$ and pick a primitive d-th root of unity $\zeta \coloneqq \zeta_d$ so that the α generates G and its action is given by

$$\alpha\colon x \mapsto x, \quad y \mapsto \zeta y$$

and we set $\chi\colon G \to \mathbb{C}^\times$ to be the character of G given by $\chi(\alpha) = \zeta^{-1}$. We thus have the eigenspace decomposition

$$\mathrm{H}^1(X, \mathbb{C}) = \bigoplus_{k=1}^{d} \mathrm{H}^1(X, \mathbb{C})_{\chi^k}.$$

As the action of α respects the symplectic pairing on H^1, this decomposition is orthogonal. Note that the action of α also respects the Hodge decomposition $\mathrm{H}^1(X, \mathbb{C}) = \mathrm{H}^{1,0} \oplus \mathrm{H}^{0,1}$ and we therefore also obtain a decomposition of $\mathrm{H}^{1,0}$ and $\mathrm{H}^{0,1}$.

This decomposition carries over to the relative setting: By varying $x_1, \ldots, x_n$ in $B \coloneqq \mathcal{M}_{0,n}$, we obtain a family $f\colon \mathcal{X} \to B$. Moreover, this family is locally topologically trivial and therefore $R^1 f_* \mathbb{C}$ is a local system on B (with fibre $\mathbb{H}^1(X_t, \mathbb{C})$ for $t \in B$) and gives rise to a VHS on B. We write

$$R^1 f_* \mathbb{C} = \bigoplus_{k=1}^{d} \mathbb{L}_k$$

for the splitting of the VHS. Again the splitting is orthogonal with respect to the symplectic pairing. By abuse of notation, we use $\mathbb{L}_k$ to also denote the fibre of $\mathbb{L}_k$ over a point of B if no confusion can arise; we furthermore write $\mathcal{L}_k^{0,1} \subseteq \mathbb{L}_k \otimes \mathcal{O}_B$ for the subbundle of holomorphic forms.

4.2.3. Eigendifferentials We briefly review some well-known facts about cyclic covers, see Appendix A for more background and details. By representation theory, more precisely the Chevalley-Weil formula, we know that

$$\operatorname{rk} \mathbb{L}_k = \#\left\{ i : a_i \neq 0 \bmod \frac{d}{\gcd(k,d)} \right\} - 2 \quad \text{and} \quad \operatorname{rk} \mathbb{L}_k^{1,0} = \sum_{i=1}^{d} \left\langle \frac{k a_i}{d} \right\rangle - 1,$$

where $\langle x \rangle := x - \lfloor x \rfloor$ denotes the fractional part of x. Note that, in particular, $\operatorname{rk} \mathbb{L}_k = n-2$ if $\gcd(k,d) = 1$. Moreover, in this case, a basis of $\mathbb{L}_k^{1,0}$ is given by

$$\omega_k^l = x^l f_k \frac{\mathrm{d}x}{y^k}, \quad 0 \le l \le \dim \mathbb{L}_k^{1,0} - 1, \quad \text{and} \quad f_k = \prod_{i=1}^{n} (x - x_i)^{\lfloor \frac{ka_i}{d} \rfloor}. \tag{4.1}$$

For general k, set $e = \frac{d}{\gcd(d,k)}$ and consider the intermediate cover $\pi' \colon X \to X/\alpha^e$, where $\pi_e \colon X/\alpha^e \to X/\alpha \cong \mathbb{P}^1$ is a cyclic cover of degree e and π' is of degree $\frac{d}{e}$. Note that $\mathbb{L}_k$ (on X) is the pullback of the corresponding eigenspace on X/α^e, where the above results apply. More precisely: given an l-eigenform on X/α^e, i.e. some form ω_l such that $\alpha^* \omega_l = \zeta_e^l \omega_l$ for some primitive e-th root of unity ζ_e, clearly

$$\alpha^*(\pi')^* \omega_l = (\pi')^* \alpha^* \omega_l = \zeta_e^l (\pi')^* \omega_l = \zeta_d^{\frac{d}{e} l} (\pi')^* \omega_l.$$

Therefore, the l-eigenforms on X/α^e pull back to $\frac{d}{e}l$-eigenforms on X. Choosing $l = \frac{k}{\gcd(d,k)}$, we clearly have $\gcd(e,l) = 1$ and $\frac{d}{e} l = k$. Moreover, by the above dimension formula, we see that the pullback is in fact all of $\mathbb{L}_k^{1,0}$:

$$\mathbb{L}_k^{1,0} = (\pi')^* \mathbb{L}^{1,0}_{\frac{k}{\gcd(k,d)}}.$$

4.2.4. Strata of eigendifferentials By explicitly calculating the normalisation of the algebraic model $y^d = \prod (x - x_i)^{a_i}$, one can calculate the divisor of each eigenform to check in which stratum it lies, see Appendix A for details. We give a brief review of the facts.

For $0 \le k < d$, we set $r(k) := \operatorname{rk} \mathbb{L}_k^{1,0} - 1 = \sum_{i=1}^{N} \left\langle \frac{ka_i}{d} \right\rangle - 2$ and, for every $1 \le i \le N$, we define the integers:

$$z(i,k) := \frac{d}{\gcd(d, a_i)} \left(1 - \left\langle \frac{ka_i}{d} \right\rangle \right) - 1.$$

Now, a generic eigendifferential in $\omega \in \mathbb{L}_k^{1,0}$ has a simple zero at every point over the fibres above $r(k)$ unramified "floating" points (i.e. points in $p_1, \ldots, p_{r(k)} \in \mathbb{P}^1 \setminus \{x_1, \ldots, x_N, \infty\}$) and possibly zeros over ramification points x_i as well as over the point $\infty \in \mathbb{P}^1$. More precisely: ω has the following configuration of zeros:

- a simple zero at each of the d preimages of the points $p_1, \ldots, p_{r(k)}$;
- for every $1 \le i \le N$: a zero of order $z(i,k)$ (if $z(i,k) \neq 0$) over each of the $\gcd(a_i, d)$ points over x_i.

Note that ω does not have any other zeros, in particular there is no zero above the point $\infty \in \mathbb{P}^1$.

Remark 4.2.1. *If $\gcd(k,d) > 1$, the formula for $z(i,k)$ is also valid. Observe, however, that in this case $z(i,k)$ is always strictly positive if $a_i \equiv 0 \bmod \frac{d}{\gcd(d,k)}$, i.e. the intermediate cover is not ramified over x_i.*

Definition 4.2.2. *With the above notation, we set $\mu(k)$ to be the stratum of a generic eigendifferential in $\mathbb{L}_k$.*

4.3. ℂ-linearity of cyclic covers

The aim of this section is to prove the following result, see also [Möl08, Prop 6.1]:

Theorem 4.3.1. *Any family of eigenspaces of a family of cyclic covers of* $\mathbb{P}^1$ *is a* ℂ*-linear manifold.*

More precisely: let $[d; a_1, \dots, a_n]$ *be a cyclic cover of* $\mathbb{P}^1$ *and* $\mathbb{L}_k$ *the* k*th isotypical component of* $\mathrm{H}^1(X, \mathbb{C})$ *(as defined above). Then*

$$\mathcal{M}(k; [d; a_1, \dots, a_n]) \coloneqq \{(X, \omega) \in \Omega\mathcal{M}(\mu(k)) : \omega \in \mathbb{L}_k\} \subseteq \Omega\mathcal{M}(\mu(k))$$

is a ℂ*-linear manifold.*

For the rest of this section, we fix a cyclic cover $[d; a_1, \dots, a_n]$ of $\mathbb{P}^1$ with branch points $x_1, \dots, x_n \in \mathbb{P}^1$ such that

$$\forall i : 0 \neq x_i \neq \infty, \quad 0 < a_i < d, \quad \sum_{i=1}^{n} a_i \in d\mathbb{Z}.$$

Moreover, for $x \in \mathbb{P}^1$, we denote by $\{p_1, \dots, p_m\} = \pi^{-1}(x)$ where we choose the numbering such that $\alpha(p_i) = p_{i+1}$ where the indices are to be understood modulo n. Moreover, we set $a(x) \coloneqq a_i$ if $x = x_i$ and $a(x) = d$ otherwise. We begin with some observations that will help us calculate the relative periods.

Lemma 4.3.2. *Let* $\omega \in \mathbb{L}_k^{1,0}$*. Then for any* $1 \leq i \leq m$

1. *there exists* $\lambda_i \in \mathbb{C}$*, independent of* ω*, such that* $\int_{p_1}^{p_i} \omega = \lambda_i \int_{p_1}^{p_2} \omega$*;*
2. $\int_{p_1}^{p_i} \omega = 0$ *if* $a(x) \neq 0 \bmod \frac{d}{\gcd(k,d)}$.

Proof. 1. This follows immediately from the fact that ω is an eigenform. Pick $\zeta \in \mathbb{C}$ such that $\alpha^*\omega = \zeta\omega$. As $\alpha^{i-1}(p_1) = p_i$ for any i, clearly

$$\int_{p_i}^{p_{i+1}} \omega = \int_{\alpha^{i-1}(p_1)}^{\alpha^{i-1}(p_2)} \omega = \int_{p_1}^{p_2} (\alpha^{i-1})^*\omega = \zeta^{i-1} \int_{p_1}^{p_2} \omega.$$

Using this, we see that

$$\int_{p_1}^{p_i} \omega = \sum_{l=1}^{i-1} \int_{p_l}^{p_{l+1}} \omega = \left(\sum_{l=0}^{i-2} \zeta^l\right) \int_{p_1}^{p_2} \omega.$$

2. Assume first that k and d are coprime. As α acts transitively on the fibre, $\alpha^m(p_i) = p_i$ for all i. Thus, also

$$\int_{p_1}^{p_i} \omega = \int_{\alpha^m(p_1)}^{\alpha^m(p_i)} \omega = \zeta^m \int_{p_1}^{p_i} \omega.$$

Note that $m < n$, as $\gcd(a(x), d) < d$, i.e. π is ramified over x. But as $\gcd(k, d) = 1$, ζ is an n-th primitive root of unity, in particular, $\zeta^m \neq 1$, as $m < n$, hence the claim follows.

In the general case, consider again the map $\pi'\colon X \to X/\alpha^e$ where $e = \frac{d}{\gcd(k,d)}$. Then there exists some $k' = \frac{k}{\gcd(k,d)}$-eigendifferential ω' such that $\omega = (\pi')^*\omega'$. But as k' and e are coprime, we are again in the above situation and thus

$$\int_{p_1}^{p_i} \omega = \int_{p_1}^{p_i} (\pi')^*\omega' = \int_{\pi'(p_1)}^{\pi'(p_i)} \omega' = 0,$$

as $\pi'(p_i)$ and $\pi'(p_1)$ lie on the same fibre of $\pi_e\colon X/\alpha^e \to X/\alpha$ (note that π_e is indeed branched at x, as $\gcd(a(x), e) < e$). □

Lemma 4.3.3. *Let $\omega \in \mathbb{L}_k^{1,0}$, $x' \in \mathbb{P}^1$ and $\{p'_1, \dots, p'_{m'}\} = \pi^{-1}(x')$. Then, for any $1 \leq i \leq m$ there exist $\lambda_i, \lambda'_i \in \mathbb{C}$, independent of ω, such that*

1. $\displaystyle\int_{p_1}^{p'_i} \omega = \lambda_i \int_{p_1}^{p_2} \omega + \lambda'_i \int_{p_1}^{p'_1} \omega;$
2. $\displaystyle\int_{p_1}^{p'_i} \omega = \lambda_i \int_{p_1}^{p_2} \omega,$ *if* $a(x') \neq 0 \bmod \frac{d}{\gcd(k,d)}$.

Proof. 1. Again, pick $\zeta \in \mathbb{C}$ such that $\alpha^*\omega = \zeta\omega$. As $\alpha^{i-1}(p_1) = p_i$ for any i and using Lemma 4.3.2, we see that

$$\begin{aligned}\int_{p_1}^{p'_i} \omega &= \int_{p_1}^{p_i} \omega + \int_{p_i}^{p'_i} \omega = \lambda_i \int_{p_1}^{p_2} \omega + \int_{\alpha^{i-1}(p_1)}^{\alpha^{i-1}(p'_1)} \omega \\ &= \lambda_i \int_{p_1}^{p_2} \omega + \int_{p_1}^{p'_1} (\alpha^{i-1})^*\omega = \lambda_i \int_{p_1}^{p_2} \omega + \zeta^{i-1} \int_{p_1}^{p'_1} \omega.\end{aligned}$$

2. This is an immediate consequence of the first part and Lemma 4.3.2. □

Proof of Theorem 4.3.1. Denote the length of $\mu(k)$ by N. Then $\dim \Omega\mathcal{M}_g(\mu(k)) = 2g + N - 1$. Moreover, $\dim \mathcal{M}(k; [d; a_1, \dots, a_n]) = n - 3 + \operatorname{rk} \mathbb{L}_k^{1,0}$, as the base space is $\mathcal{M}_{0,n}$ and $\dim \mathcal{M}_{0,n} = n - 3$ (for $n \geq 3$) while the fibres are the isotypical components $\mathbb{L}_k^{1,0}$. Hence we must show that $\mathcal{M}(k; [d; a_1, \dots, a_n])$ is (locally) cut out by $2g + N - 1 - (\operatorname{rk} \mathbb{L}_k^{1,0} + n - 3)$ linear equations in period coordinates.

For $(X, \omega) \in \mathcal{M}(k; [d; a_1, \dots, a_n])$, consider first the $2g$ absolute periods. Recall that $\mathbb{L}_k$ is a sub-local system of $R^1 f_*\mathbb{C}$ and ω is in a fibre of $\mathcal{L}_k^{1,0} \subset \mathbb{L}_k \otimes \mathcal{O}_B$. Moreover, as the decomposition of the VHS is orthogonal with respect to the symplectic pairing, this implies that the periods of ω with respect to any absolute homology cycle in the dual of the complement of $\mathbb{L}_k$ is zero; we therefore obtain $2g - \operatorname{rk} \mathbb{L}_k$ (linear) conditions on the absolute periods.

Next we consider the relative periods. Fix some zero $p \in X$ of ω and set $e := \frac{d}{\gcd(k,d)}$. Then the relative period coordinates are given by integrating ω over the $N - 1$ relative cycles

$[q-p]$, connecting p to another zero q of ω. By the considerations in subsection 4.2.4, ω has simple zeros at each of the d preimages of some points $p_1, \dots, p_{r(k)} \in \mathbb{P}^1 \setminus \{x_1, \dots, x_n\}$, where $r(k) = \operatorname{rk} \mathbb{L}_k^{1,0} - 1$, and, consequently, $N - dr(k)$ zeros at some of the preimages of the ramification points x_i. Note, however, that by Remark 4.2.1 there will always be a zero above x_i if $a_i \equiv 0 \bmod e$.

Combining Lemma 4.3.3 and Lemma 4.3.2, we obtain linear relations between all relative periods except for relative periods connecting p to

- one point in the fibre over each of $p_1, \dots, p_{r(k)}$ and
- one point in the fibre over each x_i with $a_i \equiv 0 \bmod e$.

In other words, we obtain $N - 1 - (\operatorname{rk} \mathbb{L}_k^{1,0} - 1 + \#\{i : a_i \equiv 0 \bmod e\})$ linear relations among the $N-1$ relative periods. Using the above relations in the absolute periods and that

$$\operatorname{rk} \mathbb{L}_k = \#\{i : a_i \not\equiv 0 \bmod e\} - 2,$$

we obtain $2g + N - 1 - (\operatorname{rk} \mathbb{L}_k^{1,0} + n - 3)$ linear relations in period coordinates as required. □

4.4. Covering constructions

To obtain any meaningful classification of $\mathbb{C}$-linear manifolds, we must first develop a notion of *covering construction*, i.e. when a linear manifold is a cover or the pull-back of some linear manifold in a lower genus stratum. The prototype is the case of the pullback of an eigenspace of a cyclic cover of degree d to some cyclic cover of degree de, obtaining an eigenspace for an eigenvalue of non-coprime order to the degree. Of course, the subtlety lies in detecting which $\mathbb{C}$-linear manifolds arise via such covering constructions.

Similar to the approach of Apisa ([Api16]), we consider fibre-wise ramified covers and therefore mark the ramification points. We thus consider strata of flat surfaces with marked points. We denote the zeros of ω by $Z(\omega)$. However, while Apisa uses the $\mathrm{SL}_2(\mathbb{R})$ action and the classification results of Eskin-Mirzakhani-Mohammadi [EMM15] to show that his covering constructions are again $\mathbb{R}$-linear manifolds, we will instead show this directly using Lemma 4.1.3, thereby refining a result of Möller [Möl08, Prop 3.4].

Let $\mathcal{M} \subseteq \Omega\mathcal{M}_g(\mu)$ be a linear manifold, $(X, \omega) \in \mathcal{M}$, ρ a monodromy representation of (X, ω) and $\mathcal{M}(\rho) \subseteq \Omega\mathcal{M}_h(\nu)$ the cover associated to ρ (see Definition 4.4.14 for details).

Theorem 4.4.1. *$\mathcal{M}(\rho)$ is a linear manifold.*

This will allow us to define the notion of a *primitive* linear manifold (Definition 4.4.15).

We begin by introducing strata of differentials with marked points to control ramified covers of flat surfaces inside linear manifolds.

Definition 4.4.2. *Let μ be a partition of $2g-2$ and $n \in \mathbb{N}$. Then we define the* stratum of abelian differentials with marked points

$$\Omega\mathcal{M}_g(\mu, 0^m) \coloneqq \{(X, \omega; q_1, \ldots, q_m) : (X, \omega) \in \Omega\mathcal{M}_g(\mu) \text{ and } q_i \in X \setminus Z(\omega)\}.$$

We denote the natural forgetful map by $f_m \colon \Omega\mathcal{M}_g(\mu, 0^m) \to \Omega\mathcal{M}_g(\mu)$. Moreover, for $(X, \omega; q_1, \ldots, q_m) \in \Omega\mathcal{M}_g(\mu, 0^m)$ we set $\Sigma \coloneqq Z(\omega) \cup \{q_1, \ldots, q_m\}$ and write $(X, \Sigma) \coloneqq (X, \omega; q_1, \ldots, q_m)$ if no confusion can arise.

As above, we may define period coordinates for strata with marked points: we integrate ω against the relative homology cycles $\mathrm{H}_1(X, \Sigma, \mathbb{C})$ to obtain local coordinates. Indeed, if we pick two points $q_1, q_2 \in X \setminus Z(\omega)$ and some $p \in Z(\omega)$, we have that

$$\int_p^{q_1} \omega = \int_p^{q_2} \omega \iff \int_{q_1}^{q_2} \omega = 0 \iff q_1 = q_2,$$

as we may calculate the integral in a chart that does not contain any zeros of ω. In particular, this implies that we can reconstruct the marked points on X knowing only the relative periods with respect to ω.

Lemma 4.4.3. *Let $\mathcal{M} \subseteq \Omega\mathcal{M}_g(\mu)$ be a linear manifold and $m \in \mathbb{N}$. Then $f_m^{-1}(\mathcal{M}) \subseteq \Omega\mathcal{M}_g(\mu, 0^m)$ is also a linear manifold. Moreover,* $\dim f_m^{-1}(\mathcal{M}) = \dim \mathcal{M} + m$.

Proof. Pick $(X, \omega) \in \mathcal{M}$ and a period coordinate chart $U \subset \Omega\mathcal{M}_g(\mu)$ such that $\mathcal{M}_U \coloneqq U \cap \mathcal{M}$ is cut out by linear equations. By definition, the preimage $f_m^{-1}(U)$ is also a coordinate chart for period coordinates (of $\Omega\mathcal{M}_g(\mu, 0^m)$) and the absolute periods as well as the relative periods involving only the zeros of ω satisfy the same (linear) relations in $f_m^{-1}(\mathcal{M}_U)$ as in $\mathcal{M}_U$. Moreover, there are no constraints on the choice of the points q_i; hence the dimension increases by m and all equations are linear. □

Covers and quotients of flat surfaces Strata with marked points allow us to define covering constructions. Our constructions are similar to the one in [Api15, §3] but differ in several important details. First, we review the notion of a *translation cover* of a flat surface; while this is treated in many places in the literature, we include the details as this will be vital for recognising which flat surfaces arise as translation covers. We begin by describing the case of Galois covers, as this will allow us to move freely in both directions: pulling back differentials "downstairs" (denoted (X, ω)) and pushing forward (invariant) differentials "upstairs" (denoted (Y, η)) along branched covers. As any cover is a composition of Galois covers, we can then combine these techniques to prove Theorem 4.4.1.

Given a differential with marked points $(X, \Sigma) \coloneqq (X, \omega; q_1, \ldots, q_m)$, pick a point $x \in X \setminus \Sigma$, $d \in \mathbb{N}$ and a monodromy representation $\rho \colon \pi_1(X \setminus \Sigma, x) \to S_d$ such that $G \coloneqq \rho(\pi_1(X \setminus \Sigma, x))$ has d elements. It is well-known that ρ determines a unique branched Galois cover $f_\rho \colon Y \to X$ of degree d that is unramified outside of Σ: indeed, set $Y' = \mathbb{H}/\ker\rho$. Then the inclusion $\ker\rho \subseteq \pi_1(X \setminus \Sigma, x)$ induces a unique holomorphic étale cover $\tilde{f}_\rho \colon Y' \to X \setminus \Sigma$ such that $\ker\rho = (\tilde{f}_\rho)_* \pi_1(Y', y)$ for any choice of $y \in \tilde{f}_\rho^{-1}(x)$. By the Riemann extension theorem, there exists a unique Riemann surface Y together with a holomorphic branched covering map f_ρ such that the following diagram commutes:

$$\begin{array}{ccc} Y' & \hookrightarrow & Y \\ {\scriptstyle \tilde{f}_\rho}\downarrow & & \downarrow{\scriptstyle f_\rho} \\ X \setminus \Sigma & \hookrightarrow & X \end{array}$$

We set $\Sigma' := Y \setminus Y'$ and observe that $\Sigma' := f_\rho^{-1}(\Sigma) \subset Y$. Now we define the holomorphic differential $\eta := f_\rho^* \omega$ on Y; note that $Z(\eta) \subseteq \Sigma'$. Moreover, the stratum of η is determined only by the choice of ρ and the stratum of ω. In summary: the data of a differential with marked points (X, Σ) of genus g together with a point $x \in X \setminus \Sigma$ and a normal subgroup $\ker \rho \leq \pi_1(X \setminus \Sigma, x)$ determines a unique flat surface (Y, η) of genus h together with d choices of a point y in the fibre of x.

Note moreover that, by construction, G acts freely and holomorphically on Y' and this action extends to a holomorphic action on Y and the holomorphic quotient map is ramified at most at the zeros of the G-invariant differential η. In fact, f_ρ (and $\tilde{f}_\rho$) is the quotient by G. This lets us reverse this construction.

Consider a flat surface (Y, η) of genus h together with a subgroup $G \subsetneq \mathrm{Aut}(Y)$ of holomorphic automorphisms that fix η and act freely on $Y \setminus Z(\eta)$. Then η descends to a holomorphic differential ω on the quotient $X := Y/G$ of genus g. Again, the stratum of (X, ω) is determined by the stratum of η and the action of G. Moreover, we additionally mark any ramification points $q_1, \ldots, q_m$ of $f_G \colon Y \to Y/G = X$ on X that are not zeros of ω. Clearly, by choosing any point $y \in Y \setminus Z(\eta)$ and the subgroup $(f_G)_* \pi_1(Y \setminus Z(\eta), y) \subseteq \pi_1(X \setminus \Sigma, x)$, the two constructions are inverse to each other.

Therefore, the data $(X, \Sigma, x; \rho)$ determines a stratum $\nu = \nu(\rho)$ and a unique flat surface $(X, \Sigma, x)^\rho := (Y, f_\rho^* \omega) \in \Omega\mathcal{M}_{h(\rho)}(\nu(\rho))$ of genus $h = h(\rho)$ with marked points Σ' given by the zeros of $\eta = f_\rho^* \omega$ together with a holomorphic action of $G = \pi_1(X \setminus \Sigma, x)/\ker \rho$. Note, however, that as a pointed flat surface there are d choices of (Y, η, y) corresponding to the d preimages of x permuted by the Deck group of f_ρ.

Conversely, the data $(Y, \eta, y; G)$ where G is a subgroup of $\mathrm{Aut}(Y)$ acting as above, determines a unique stratum $\mu = \mu(G)$ and a unique flat surface $(Y, \eta)/G := (X, \Sigma) \in \Omega\mathcal{M}_{g(G)}(\mu(G), 0^{m(G)})$ of genus $g = g(G)$ with $m = m(G)$ marked (ramification) points. Moreover, we recover the normal subgroup $(f_G)_* \pi_1(Y \setminus Z(\eta), y) \subseteq \pi_1(X \setminus \Sigma, x)$.

Covers of strata As the above construction is essentially topological, it is not well-defined on a stratum. To remedy this, we must include a Teichmüller marking. Fix a reference surface $(S, \Sigma) \in \Omega\mathcal{M}_g(\mu, 0^m)$ together with a point $s \in S \setminus \Sigma$ and consider $\Omega\mathcal{T}_{g,1}(\Sigma)$, the space of pointed flat surfaces in $\Omega\mathcal{M}_g(\mu, 0^m)$ endowed with a Teichmüller marking, i.e. an isotopy class of a homeomorphism $\phi \colon (S, \Sigma, s) \to (X, \Sigma, x)$ that maps s to x and restricts to a homeomorphism $S \setminus \Sigma \to X \setminus \Sigma$. In particular, ϕ induces an isomorphism $\phi_*\colon \pi_1(S \setminus \Sigma, s) \cong \pi_1(X \setminus \Sigma, x)$. Denote by $\mathrm{Mod}(\Sigma) = \mathrm{Mod}_{g,n+m}$ the mapping class group of S that fixes Σ (forgetting the extra marked point). Then the quotient of $\Omega\mathcal{T}_{g,1}(\Sigma)$ by $\mathrm{Mod}(\Sigma)$ coincides with the stratum $\Omega\mathcal{M}_g(\Sigma)$. Observe that $\Omega\mathcal{T}_{g,1}(\Sigma)$ is naturally endowed with period coordinates.

We now fix, once and for all, a monodromy representation $\rho\colon \pi_1(S \setminus \Sigma, s) \to S_d$ with image a group G as above. By covering space theory, this induces an étale cover $f_G\colon S' \setminus \Sigma' \to S \setminus \Sigma$ and we choose, once and for all, a point $s_i' \in f_G^{-1}(s)$, $(i = 1, \ldots, d)$. Denote by h the genus of S'. Then we can associate to any $(X, \Sigma, \phi, x) \in \Omega\mathcal{T}_{g,1}(\Sigma)$ a monodromy representation $\rho \circ \phi_*^{-1}\colon \pi_1(X \setminus \Sigma, x) \to S_d$ and thus a flat surface (Y, η) with a group $G(Y) \subseteq \mathrm{Aut}(Y, \eta)$ such that $(Y, \eta)/G(Y) = (X, \omega)$, as described above. Moreover, we may lift ϕ along the unramified coverings $S' \setminus \Sigma' \to S \setminus \Sigma$ and $Y \setminus Z(\eta) \to X \setminus \Sigma$ to obtain a homeomorphism $\psi\colon S' \setminus \Sigma' \to Y \setminus Z(\eta)$. Replacing ϕ by an isotopic homeomorphism yields a homeomorphism isotopic to ψ. By setting $y := \psi(s_i')$, we obtain a unique point

$$\sigma_{G,i}((X, \Sigma, \phi, x)) := (Y, \eta, \psi, y) \in \Omega\mathcal{T}_{h,1}(\nu)$$

and thus obtain d maps $\sigma_{G,i}\colon \Omega\mathcal{T}_{g,1}(\Sigma) \to \Omega\mathcal{T}_{h,1}(\nu)$.

Quotients of strata We can (essentially) reverse this construction. However, we must first describe the image of the maps $\sigma_{G,i}$ inside $\Omega\mathcal{T}_{h,1}(\nu)$.

To describe the subset of a stratum $\Omega\mathcal{M}_g(\nu)$ with a fixed group action, we pass again to the Teichmüller space. Again, we fix a reference surface $(S', \eta_0) \in \Omega\mathcal{M}_h(\nu)$ together with a point $s' \in S' \setminus Z(\eta_0)$ and a group $G \subseteq \mathrm{Aut}(S', \eta_0)$ as above. Hence we obtain a quotient surface $f_G\colon S' \to S = S'/G$ onto which the G-invariant differential η_0 descends. We denote the differential on S by ω_0 and by Σ the set of zeros of ω_0 and ramification points of f_G; the image of the (unramified) point s' is denoted by s.

Then we define $\Omega\mathcal{T}_{h,1}(G)$ as the space of flat surfaces $(Y, \eta, \psi, y, G(Y))$ with a Teichmüller marking $\psi\colon (S', Z(\eta_0), s') \to (Y, Z(\eta), y)$ (i.e. sends zeros of η_0 to zeros of η of the same order and maps s' so y) and a group $G(Y) \subseteq \mathrm{Aut}(Y, \eta)$ such that the induced group action $\psi^* G(Y)$ of $G(Y)$ on (S', η, s) is topologically equivalent to the action of G on S, i.e. there exists a homeomorphism ϕ between the quotients that lifts to a homeomorphism $\tilde{\phi}$ on S' that is isotopic to the identity:

$$\begin{array}{ccc} S' & \xrightarrow{\quad\tilde{\phi}\quad} & S' \\ \big\downarrow & & \big\downarrow \\ S'/G & \xrightarrow{\quad\phi\quad} & S'/\psi^* G(Y) \end{array}$$

Remark 4.4.4. *Note that this is in fact a necessary condition: indeed, the Teichmüller marking ψ is only defined up to isotopy. Consider isotopic homeomorphisms $\psi_0, \psi_1\colon (S', Z(\eta_0), s') \to (Y, Z(\eta), y)$ and an automorphism $\alpha\colon Y \setminus Z(\eta) \to Y \setminus Z(\eta)$. Then the homeomorphism $\psi_1^{-1}\psi_0\colon S' \to S'$ descends to a homeomorphism $S'/\psi_0^{-1}\alpha\psi_0 \to S'/\psi_1^{-1}\alpha\psi_1$ and is isotopic to the identity on S.*

In particular, this implies that $G(Y)$ acts freely on y and this lets us associate to every $(Y, \eta, \psi, y, G(Y))$ the quotient (X, Σ, ϕ, x), where $X = Y/G(Y)$, ω is the induced differential of genus g and Σ denotes the zeros of ω and ramification points of $G(Y)$, as above. Moreover, x is the image of y and ϕ is the induced Teichmüller marking $\phi\colon (S, \Sigma, s) \to (X, \Sigma, x)$.

Thus we obtain a natural map $\pi_G\colon \Omega\mathcal{T}_{h,1}(G) \to \Omega\mathcal{T}_{g,1}(\Sigma)$.

Moreover, this space is in fact a subset of the stratum of Teichmüller space.

Lemma 4.4.5. *The forgetful map $\Omega\mathcal{T}_{h,1}(G) \to \Omega\mathcal{T}_{h,1}(\nu)$ is an injection.*

Proof. This follows from the fact that different holomorphic automorphisms are not isotopic. Indeed, consider a surface Y' with groups of holomorphic automorphisms G and H that act freely and such that there exists a homeomorphism $\varphi\colon Y'/G \to Y'/H$ that lifts to a homeomorphism $\tilde{\varphi}\colon Y' \to Y'$ that is isotopic to the identity. Then $\tilde{\varphi}$ induces an isomorphism of the deck groups by conjugation. But as $\tilde{\varphi}$ is isotopic to the identity, this implies $G = H$, as isotopic holomorphic automorphisms are already equal. □

This allows us to talk about covers and quotients of (subsets of) strata.

Lemma 4.4.6. *With the above notation, given a flat surface $(S', \eta', s') \in \Omega\mathcal{M}_{h,1}(\nu)$ and a group $G \subseteq \mathrm{Aut}(S', \eta')$ as above, we obtain an unramified cover*

$$\pi_G\colon \Omega\mathcal{T}_{h,1}(G) \to \Omega\mathcal{T}_{g,1}(\Sigma)$$

of degree d, where $\Omega\mathcal{T}_{g,1}(\Sigma)$ has the reference surface $(S'/G, \Sigma, s)$ with the Teichmüller markings induced by G.

Conversely, given a flat surface $(S, \omega, s) \in \Omega\mathcal{M}_{g,1}(\mu, 0^m)$ together with a monodromy representation as above, we obtain d sections $\sigma_{G,i}\colon \Omega\mathcal{T}_{g,1}(\Sigma) \to \Omega\mathcal{T}_{h,1}(G)$ of π_G that correspond to the d choices of the point s' on the ramified cover $S' \to S$.

Proof. This follows from the above discussion. As the marked point is always chosen outside of the ramification locus of G, the map π_G is unramified of degree d. Moreover, covering space theory asserts that the constructions are unique (up to the choice of a point in the fibre) and that π_G is surjective. □

Representation theoretic viewpoint Recall that the Teichmüller space can be described by Fuchsian representations of the fundamental group of a surface. More precisely, denote by S a fixed reference surface of genus g together with n marked points $p_1, \ldots, p_n \in S$. Then we say that a representation $\mathcal{R}_{g,n}\colon \pi_1(S \setminus \{p_1, \ldots, p_n\}) \to \mathrm{PSL}_2(\mathbb{R})$ is *fuchsian*, if it is faithful with discrete image and the quotient of $\mathbb{H}$ by the image of $\mathcal{R}_{g,n}$ is of finite hyperbolic area and homeomorphic to $S \setminus \{p_1, \ldots, p_n\}$. However, this construction is up to the choice of the preimage on $\mathbb{H}$, the universal cover of S, of a base-point s, i.e. up to conjugation by $\mathrm{PSL}_2(\mathbb{R})$. In other words, $\mathcal{T}_{g,n}$ is the space of fuchsian representations up to conjugation by $\mathrm{PSL}_2(\mathbb{R})$.

This descriptions seems useful for describing our covering construcions, as it would allow us to fix, once and for all, a subgroup of $\pi_1(S \setminus \{p_1, \dots, p_n\})$ and restrict the representations to construct coverings. However, for Lemma 4.4.6, the choice of a basepoint on the curve was essential. The choice of lift to the fibre in the universal cover must still be remedied by conjugation, but this makes it more subtle. Moreover, the marked points add subtle geometric aspects to the otherwise completely algebraic description of the representations. Also, describing the locus with a suitable group action is not very practical in this language.

Therefore, describing the constructions using the language of fuchsian representations does not seem feasible.

Covers and quotients of linear manifolds We can now extend these constructions to any submanifold $\mathcal{M} \subseteq \Omega\mathcal{M}_g(\mu, 0^m)$ and submanifolds $\mathcal{N} \subseteq \Omega\mathcal{M}_h(\nu)$ admitting an appropriate group action.

Definition 4.4.7. *Fix $(S, \Sigma) \in \mathcal{M}$, $s \in X \setminus \Sigma$ and $\rho\colon \pi_1(S \setminus \Sigma, s) \to G \subseteq S_d$ as above.*

Then we denote by $\mathcal{T}_{g,1}\mathcal{M} \subseteq \Omega\mathcal{T}_{g,1}(\Sigma)$ the lift of $\mathcal{M}$ to the corresponding stratum of Teichmüller space. Furthermore, we denote by $\mathcal{T}_{h,1}\mathcal{M}(G) \subseteq \Omega\mathcal{T}_{h,1}(G)$ the preimage of $\mathcal{T}_{g,1}\mathcal{M}$ under π_G.

Clearly, the quotient manifold can only be constructed if $\mathcal{N} \subseteq \Omega\mathcal{M}_h(\nu)$ does in fact admit a lift to some $\Omega\mathcal{T}_{h,1}(G)$ for some group G.

Definition 4.4.8. *Fix $(S', \eta_0) \in \mathcal{N}$ and $s' \in S' \setminus \Sigma'$ as above. If there exists a $G \subseteq \mathrm{Aut}(S', \eta_0)$ such that the lift $\mathcal{T}_{h,1}\mathcal{N} \subseteq \Omega\mathcal{T}_{h,1}(\nu)$ is contained in $\Omega\mathcal{T}_{h,1}(G)$ (via the action induced by the Teichmüller marking with reference point $(S', Z(\eta_0), s')$), we denote this lift by $\mathcal{T}_{h,1}\mathcal{N}_G$. In this situation, we say that $\mathcal{N}$ admits a* compatible G-action *and denote by $\mathcal{T}_{g,1}\mathcal{N}/G \subseteq \Omega\mathcal{T}_{g,1}(\Sigma)$ the image of $\mathcal{T}_{h,1}\mathcal{N}_G$ under π_G.*

Remark 4.4.9. *Choose the section $\sigma_{i,G}$ of π_G corresponding to $s' \in S'$. Then $\mathcal{T}_{g,1}\mathcal{N}/G$ is the preimage of $\mathcal{T}_{h,1}\mathcal{N}_G$ under $\sigma_{i,G}$.*

The following observation ensures that these constructions descend under the action of the appropriate mapping class group. Denote, as above, by $\mathrm{Mod}(G) = \mathrm{Mod}_{g,n+m}$ the mapping class group of S' that fixes $Z(\eta_0)$ (forgetting the extra marked point).

Lemma 4.4.10. *π_G is equivariant with respect to the action of the appropriate mapping class groups.*

Proof. This is clear from the above constructions: the action of the mapping class group changes the marking and is thus compatible with the described constructions. □

In particular, this implies that $\mathcal{T}_{h,1}\mathcal{M}(G)$ and $\mathcal{T}_{g,1}\mathcal{N}/G$ are invariant under the mapping class groups $\mathrm{Mod}(G)$ and $\mathrm{Mod}(\Sigma)$. We denote the quotients by

$$\mathcal{M}(G) := (\mathcal{T}_{h,1}\mathcal{M}(G))/\operatorname{Mod}(G) \subseteq \Omega\mathcal{M}_h(\nu) \qquad \text{and}$$
$$\mathcal{N}/G := (\mathcal{T}_{g,1}\mathcal{N}/G)/\operatorname{Mod}(\Sigma) \subseteq \Omega\mathcal{M}_g(\mu, 0^m).$$

Remark 4.4.11. *Note that, given a linear manifold $\mathcal{M} \subseteq \mathcal{M}_g(\mu, 0^m)$ together with a monodromy representation ρ as above, the manifold $\mathcal{T}_{h,1}\mathcal{M}(G)$ (and thus $\mathcal{M}(G)$, etc.) is not at all uniquely determined. The following results will, however, be independent of the choices made in the construction.*

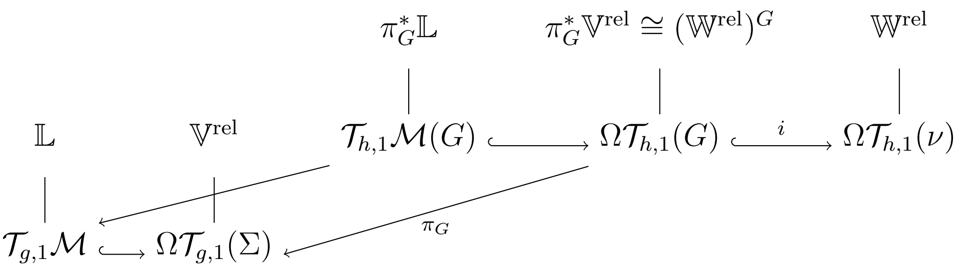
Linearity of covers Of course, this construction is only useful if the cover of a linear manifold is again a linear manifold. Apisa's covering construction uses the $\mathrm{SL}_2(\mathbb{R})$ action and the classification results of [EMM15] in an essential way [Api15, Lemma 3.3]. In our situation, this is obviously not possible.

Proposition 4.4.12. *Let $\mathcal{M} \subseteq \Omega\mathcal{M}_g(\mu, 0^m)$ be a linear manifold.*

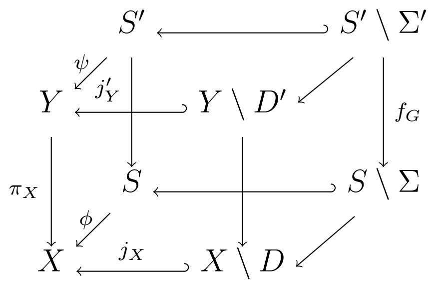
Then $\mathcal{T}_{h,1}\mathcal{M}(G) \subseteq \Omega\mathcal{T}_{h,1}(\nu)$ and $\mathcal{M}(G) \subseteq \Omega\mathcal{M}_h(\nu)$ are linear manifolds.

Proof. By Lemma 4.1.3, the tangent bundle of $\mathcal{M} \subseteq \Omega\mathcal{M}_g(\mu, 0^m)$ is a sub-local system $\mathbb{L} \subseteq \mathbb{V}^{\mathrm{rel}}$ of the local system of relative cohomology on $\mathcal{M}$. Moreover, local systems on $\mathcal{M}$ lift to local systems on $\mathcal{T}_{g,1}\mathcal{M}$. By abuse of notation, we also denote the lifts by $\mathbb{L}$ and $\mathbb{V}^{\mathrm{rel}}$. Note that $\mathbb{V}_g^{\mathrm{rel}}$ gives the period coordinates on $\Omega\mathcal{T}_{g,1}(\Sigma)$ and $\mathbb{L}$ is the tangent bundle of $\mathcal{T}_{g,1}\mathcal{M}$ inside $\mathbb{V}^{\mathrm{rel}}$.

Moreover, as π_G is unramified, $\pi_G^*\mathbb{L}$ is the tangent bundle on $\mathcal{T}_{h,1}\mathcal{M}(G)$. Also, pulling back the relative cohomology $\mathbb{W}^{\mathrm{rel}}$ to $\Omega\mathcal{T}_{h,1}(G)$ equips it with a G-action and we denote by $(\mathbb{W}^{\mathrm{rel}})^G$ the G-invariant sub-bundle. The situation is summarised in the following diagram:



Note that we may identify $(\mathbb{W}^{\mathrm{rel}})^G$ with $\pi_G^*\mathbb{V}^{\mathrm{rel}}$. Indeed, denote again by $f_g\colon \mathcal{X} \to \Omega\mathcal{T}_{g,1}(\Sigma)$ and $f_h\colon \mathcal{Y} \to \Omega\mathcal{T}_{h,1}(G)$ the corresponding universal families and by $j\colon \mathcal{X} \setminus D \hookrightarrow \mathcal{X}$ and $j'\colon \mathcal{Y} \setminus D' \hookrightarrow \mathcal{Y}$ the universal divisors (i.e. fibre-wise the zeros of the differential and the marked points). Moreover, let $\pi\colon \mathcal{Y} \to \mathcal{X}$ be the map induced by π_G. Then $\mathbb{V}^{\mathrm{rel}} = \mathrm{R}^1(f_g)_* j_! \mathbb{C}$ and $i^*\mathbb{W}^{\mathrm{rel}} = \mathrm{R}^1(f_h)_* j'_! \mathbb{C}$. But the maps j and j' are determined only by the Teichmüller marking. Indeed, fibre-wise we are in the following situation: let Y be a fibre of $\mathcal{Y}$ and X the corresponding fibre of $\mathcal{X}$. Denote by S and S' the reference surfaces inside the strata (with marked points Σ and Σ') and by ϕ and ψ the Teichmüller markings from the above construction of π_G. Then j and j' are induced by the following commuting cube:



In particular, they only depend on the fixed cover f_G and the Teichmüller marking and therefore we obtain an inclusion $\pi^* j_! \mathbb{C} \hookrightarrow j'_! \mathbb{C}$ in families. Moreover, by the same argument, the pulled-back relative cohomology classes are also G-invariant in families and, checking fibre-wise, this inclusion yields an isomorphism $\pi_G^* \mathbb{V}^{\mathrm{rel}} \cong (\mathbb{W}^{\mathrm{rel}})^G$ as claimed.

Now, $\mathcal{T}_{h,1}\mathcal{M}(G)$ is linear if $(\mathbb{W}^{\mathrm{rel}})^G$ is in fact a sub-local system of $\mathbb{W}^{\mathrm{rel}}$ (note that $\pi_G^* \mathbb{L}$ is a sub-local system of $\pi_G^* \mathbb{V}^{\mathrm{rel}} \cong (\mathbb{W}^{\mathrm{rel}})^G$ by construction). But again, this is the case, as the group action at each point is determined only by the Teichmüller marking. Thus the G-splitting of the relative cohomology group $\mathrm{H}^1(S', \Sigma'; \mathbb{C})$ is transported to every fibre by the Teichmüller marking and is thus compatible with the linear structure of $i^* \mathbb{W}^{\mathrm{rel}}$, which is also obtained through the Teichmüller marking.

Finally, by the definition of period coordinates, the quotient by the mapping class group is also a linear manifold. □

Linearity of quotients To construct general covers of linear manifolds, we also need our quotient construction to be well-behaved.

Proposition 4.4.13. *Let $\mathcal{N} \subseteq \Omega\mathcal{M}_h(\nu)$ be an algebraic linear manifold that admits a compatible G-action (cf. Definition 4.4.8) for some G.*

Then $\mathcal{T}_{g,1}\mathcal{N}/G \subseteq \Omega\mathcal{T}_{g,1}(\Sigma)$ and $\mathcal{N}/G \subseteq \Omega\mathcal{M}_g(\mu, 0^m)$ are linear manifolds.

Proof. The proof is similar to the proof of Proposition 4.4.12, except that we now pull back the local systems along the map $\sigma_{G,i}$. □

General covers We now discuss the case of general (i.e. not necessarily Galois) covers. Let $f\colon Y \to X$ be a (finite degree) unramified cover of (possibly punctured) Riemann surfaces and fix $x \in X$ and $y \in f^{-1}(x)$. Then (Y, y) is determined by the subgroup $f_*\pi_1(Y, y) \subseteq \pi_1(X, x)$; note that conjugating $f_*\pi_1(Y, y)$ in $\pi_1(X, x)$ gives the same cover f but changes the base point in the fibre. Equivalently, the cover $f\colon (Y, y) \to (X, x)$ is determined by the corresponding monodromy representation $\rho\colon \pi(X, x) \to S_d$. Moreover, we can construct the Galois closure of f: denote by N the normal core of $f_*\pi_1(Y, y)$, i.e. the intersection of all its conjugates in $\pi_1(X, x)$. Then $N \leq \pi_1(X, x)$ is a normal subgroup of finite index and thus induces a Galois covering $g\colon Y' \to X$ with Galois group $G = \pi_1(X,x)/N$:

$$\begin{array}{ccc} Y' & & \\ & \searrow^{h} & \\ g\big\downarrow & & Y \\ & \swarrow_{f} & \\ X & & \end{array}$$

Note that Y' is unique up to the choice of a point $y' \in f'^{-1}(x)$. Moreover, $h\colon Y' \to Y$ is also Galois with Galois group $H := f_*\pi_1(Y,y)/N$.

Starting with a monodromy representation $\rho\colon \pi_1(X,x) \to S_d$, we denote the corresponding groups by $G(\rho)$ and $H(\rho)$.

Definition 4.4.14. *Let $\mathcal{M} \subseteq \Omega\mathcal{M}_g(\mu, 0^m)$ be a linear manifold and $(X,\Sigma) \in \mathcal{M}$. Then we set $\mathcal{M}(\rho) = \mathcal{M}(G(\rho))/H(\rho)$.*

Proof of Theorem 4.4.1. The statement now follows immediately from Proposition 4.4.12 and Proposition 4.4.13. □

This finally gives us a well-defined notion of primitive linear manifold.

Definition 4.4.15. *Let $\mathcal{M} \subseteq \Omega\mathcal{M}_g(\mu, 0^m)$ and $\mathcal{N} \subseteq \Omega\mathcal{M}_h(\nu)$ be linear manifolds. Then we say that $\mathcal{N}$ is a* cover *of $\mathcal{M}$ if there exists a monodromy representation ρ such that* $\dim \mathcal{N} = \dim \mathcal{M}$ *and $\mathcal{N} \subseteq \mathcal{M}(\rho)$. Moreover, we say that a linear manifold $\mathcal{N}$ is* imprimitive *if there exists a linear manifold $\mathcal{M}$ such that $\mathcal{N}$ is a (non-trivial) cover of $\mathcal{M}$. A linear manifold $\mathcal{N}$ is* primitive *if it is not imprimitive.*

4.5. Questions and open problems

We end this section by formulating some questions and open problems in the theory of ℂ-linear manifolds, as directions for possible future research.

ℝ-linear manifolds have been extensively studied and there are many known examples. Building on Theorem 4.3.1, it is natural to ask for other new explicit examples of ℂ-linear manifolds.

Question 4.5.1. *What are other explicit examples of ℂ-linear manifolds that are not ℝ-linear?*

Motivated by the example of the Prym loci and the examples that appear in the classification of orbifold points on Teichmüller curves, a natural candidate would be a bundle of eigenforms for an endomorphism with a complex eigenvalue over a Shimura curve. In most known examples (e.g. [McM06], [Muk14], [MMW17], etc.) any such endomorphism is induced by an automorphism of the underlying Riemann surface. The difficulty in the case of a more general endomorphism is that the action on the relative periods of the eigendifferential is a priori not known, see also [Wri13] for related ideas.

Moreover, much progress has been achieved recently in the classification of affine invariant manifolds and in low genus the picture is now fairly complete [EFW17], [MMW17], [BM12], [BHM16], [ANW14], [MW14], [NW14], [AN15] and [LNW15]. It would be interesting to classify, at least in low genus, primitive $\mathbb{C}$-linear manifolds that are not $\mathbb{R}$-linear.

Question 4.5.2. *What primitive $\mathbb{C}$-linear manifolds that are not $\mathbb{R}$-linear arise in low genus?*

Such a classification was already attempted for genus 2 curves in [Möl08, Theorem 4.1]. However, due to the extra conditions imposed on the boundary in [Möl08, Definition 6.4], that classification is not complete.

A major breakthrough in the theory of affine invariant manifolds was achieved recently by Filip [Fil16], who was able to show that all affine invariant manifolds are in fact algebraic. A similar question can be asked for $\mathbb{C}$-linear manifolds.

Question 4.5.3. *Let $\iota\colon \mathcal{M} \to \Omega\mathcal{M}_g(\mu)$ be an immersion of a closed manifold $\mathcal{M}$ whose image is cut out locally by $\mathbb{C}$-linear equations in period coordinates. Is the image of ι necessarily algebraic?*

Filip used techniques of Mixed Hodge Structures in [Fil16]. It would be interesting to apply such techniques to $\mathbb{C}$-linear manifolds. Moreover, this might yield a more conceptual understanding of Theorem 4.4.1.

A. Cyclic covers of $\mathbb{P}^1$

In this appendix we collect some facts about cyclic covers of $\mathbb{P}^1$. In particular, we give an explicit description of the algebraic model, the canonical divisor and intermediate covers. All of this is fairly well-known but scattered throughout the literature.

A.1. The topological model

Let X be a Riemann surface with an automorphism α of order d such that $X/\alpha \cong \mathbb{P}^1$. Then $\pi\colon X \to X/\alpha \cong \mathbb{P}^1$ displays X as a (ramified) cyclic cover of $\mathbb{P}^1$ of order d. The Riemann-Hurwitz formula relates the ramification data of π to the genus g of X:

$$2g-2 = -2d + \sum_{p\in X}(e_p - 1),$$

where e_p is the ramification index of π at p, i.e. the number of preimages of $\pi(p')$ for some p' in a small neighbourhood of p. Note that $e_p = 1$ for all but finitely many points in X. In particular, for all $p \in X$,

$$\#\pi^{-1}(\pi(p)) = \frac{d}{e_p},$$

and $e_p = e_{p'}$ if $\pi(p) = \pi(p')$ (α acts transitively on each fibre).

We denote the points $p \in X$ with $e_p > 1$ as *ramification points* and their images on $\mathbb{P}^1$ as *branch points*. We define the *branch locus* as

$$S \coloneqq \{x \in \mathbb{P}^1 \mid \exists\, p \in X : e_p > 1 \text{ and } \pi(p) = x\}.$$

Moreover, π is Galois cover (with Galois group $\langle\alpha\rangle \cong \mathbb{Z}/d\mathbb{Z}$) and we obtain an unramified (topological) cover by considering

$$\pi'\colon X \setminus \pi^{-1}(S) \to \mathbb{P}^1 \setminus S.$$

As such, it corresponds to a unique normal subgroup of the fundamental group $\pi_1(\mathbb{P}^1 \setminus S)$. We denote the branch points of π by $x_1, \dots, x_n \in \mathbb{P}^1$ and choose generators $\gamma_1, \dots, \gamma_n$ of $\pi_1(\mathbb{P}^1 \setminus S)$, where γ_i is a simple counter-clockwise loops around x_i.

Fix an integer d and choose points $x_1, \dots, x_n \in \mathbb{C} \setminus \{0\}$ and integers $a_1, \dots, a_n$ satisfying $0 < a_i < d$ and $\sum a_i \in d\mathbb{Z}$. We denote a tuple

$$[d; a_1, \dots, a_n], \quad d \in \mathbb{Z}, \quad 0 < a_i < d, \quad \sum_{i=1}^{n} a_i \in d\mathbb{Z},$$

together with a tuple $(x_1, \dots, x_n) \in (\mathbb{C} \setminus \{0\})^n$ as a *(combinatorial) ramification datum.*

A.2. The algebraic model

We now give an explicit algebraic model for the complex curve associated to the ramified covering (Riemann surface) of section A.1. To this end, we construct cyclic covers over $\mathbb{A}^1$ and glue two of these together to obtain a cyclic cover over $\mathbb{P}^1$.

Fix a combinatorial ramification datum $[d; a_1, \ldots, a_n]$ and distinct points $x_1, \ldots, x_n \in \mathbb{C} \setminus \{0\}$ as above and set

$$f(x) = \prod_{i=1}^{n} (x - x_i)^{a_i} \quad \text{and} \quad \pi\colon C' = \operatorname{Spec} \mathbb{C}[x,y]/(y^d - f(x)) \to \operatorname{Spec} \mathbb{C}[x] = \mathbb{A}^1,$$

where π is induced by the inclusion map of rings. Hence, on the set of $\mathbb{C}$-valued points, π is the projection $(x_0, y_0) \mapsto x_0$. By abuse of notation, we will often identify the geometric point x_0 with the maximal ideal $(x - x_0) \leq \mathbb{C}[x]$.

Clearly, π is of degree d and unramified outside x_i. However, if $a_i > 1$, the model C' is singular at $\pi^{-1}x_i$. We therefore pass to the normalisation $C^{\mathrm{norm}} \to C'$ and consider the fibre. To ease notation we set $a = a_i$ and pass to local coordinates so that it suffices to consider the situation

$$C^{\mathrm{norm}} \to \operatorname{Spec} \mathbb{C}[x,y]/(y^d - x^a) \to \operatorname{Spec} \mathbb{C}[x].$$

The fibre over 0 consists of $e = \gcd(d, a)$ irreducible components. Indeed, if we set $d' = d/\gcd(d,a)$ and $a' = a/\gcd(d,a)$, clearly

$$\operatorname{Spec} \mathbb{C}[x,y]/(y^d - x^a) = \operatorname{Spec} \mathbb{C}[x,y]/\prod_{i=1}^{e} (y^{d'} - \zeta_e^i x^{a'}),$$

where ζ_e is some fixed eth root of unity. Hence, the fibre consists of a product of e irreducible components that are cusps with equation $y^{d'} = x^{a'}$ with d' and a' coprime. The normalisation map of such a cusp is well-known to be bijective. Indeed, consider the map of rings

$$\varphi\colon \mathbb{C}[x,y]/(y^{d'} - x^{a'}) \to \mathbb{C}[t^{a'}, t^{d'}], \quad x \mapsto t^{d'}, \quad y \mapsto t^{a'}.$$

This map is clearly well-defined and surjective. It is in fact an isomorphism as we can find integers $l, m \in \mathbb{Z}$ such that the map to the function field

$$\mathbb{C}[t] \ni t \mapsto x^l y^m \in \operatorname{Quot}\big(\mathbb{C}[x,y]/(y^{d'} - x^{a'})\big)$$

restricts to a map $\varphi^{-1}\colon \mathbb{C}[t^{a'}, t^{d'}] \to \mathbb{C}[x,y]/(y^{d'} - x^{a'})$ that is clearly the inverse to φ. The normalisation of $\mathbb{C}[t^{a'}, t^{d'}]$ (in $\mathbb{C}(t)$), on the other hand, is clearly the embedding into $\mathbb{C}[t]$ and the geometric map from $\mathbb{A}^1$ to the cusp is bijective, the fibre over the cusp (at the ideal $(t^{a'})$) being

$$\operatorname{Spec}\big(\mathbb{C}[t] \otimes_{\mathbb{C}[t^{a'}, t^{d'}]} \mathbb{C}[t^{a'}, t^{d'}]/t^{a'}\big) = \operatorname{Spec} \mathbb{C}[t]/t^{a'},$$

i.e. a single non-reduced point.

To summarise: the fibre of π over x_i consists of $\gcd(d, a_i)$ points and the local parameter in the normalisation is described by the ring homomorphism

$$\mathbb{C}[x]_{(x-x_i)} \to \left(\mathbb{C}[x,y]/(y^d - f(x))\right)_{(y,x-x_i)} \to \mathbb{C}[t^{\frac{a_i}{\gcd(d,a_i)}}, t^{\frac{d}{\gcd(d,a_i)}}] \hookrightarrow \mathbb{C}[t], \tag{A.1}$$

where $x - x_i \mapsto t^{d/\gcd(d,a_i)}$ and $y \mapsto t^{a_i/\gcd(d,a_i)}$ (and everything else is invertible).

We now glue two such covers together to obtain a ramified cover of $\mathbb{P}^1$. Denote by $\operatorname{Spec}\mathbb{C}[u] = \mathbb{A}^1$ a second copy of $\mathbb{A}^1$ that we glue to $\operatorname{Spec}\mathbb{C}[x]$ in the usual way to obtain a $\mathbb{P}^1$:

$$\operatorname{Spec}\mathbb{C}[x] \supset \operatorname{Spec}\mathbb{C}[x]_x \cong \operatorname{Spec}\mathbb{C}[u = 1/x]_u \subset \operatorname{Spec}\mathbb{C}[u].$$

Then we can set

$$g(u) = \prod_{i=1}^{n} \left(u - \frac{1}{x_i}\right)^{a_i} = u^{\sum a_i} f\left(\frac{1}{u}\right) \prod \left(\frac{-1}{x_i}\right)^{a_i}$$

(note that the product is simply a unit) and glue the cover

$$C'' = \operatorname{Spec}\mathbb{C}[u,z]/(z^d - g(u)) \to \operatorname{Spec}\mathbb{C}[u] = \mathbb{A}^1$$

to C' along the open preimages of $\mathbb{C}[x]_x$ and $\mathbb{C}[u = 1/x]_u$ and thus obtain a curve $\pi\colon C \to \mathbb{P}^1$. More precisely, we can set $N = \sum_{i=1}^n a_i$ and check in the function field $K(C)$:

$$z^d = g(u) = \varepsilon u^N f\left(\frac{1}{u}\right) = \varepsilon \frac{f(x)}{x^N} = \varepsilon \frac{y^d}{x^N},$$

where ε is some unit, which provides the desired isomorphism and incidentally also shows that C is unramified over $\infty \in \mathbb{P}^1$ if and only if $d|N$. In this case, we set $N' = N/d$ and observe that

$$u = \frac{1}{x}, \quad z = \varepsilon' \frac{y}{x^{N'}}, \tag{A.2}$$

where ε' is some unit, as both sides were dth powers in $K(C)$. We have thus shown:

Proposition A.2.1. *Given the combinatorial ramification data $[d; a_1, \ldots, a_n]$, the normalisation of the projective curve with affine model given by*

$$y^d = \prod_{i=1}^{n} (x - x_i)^{a_i} = f(x)$$

is the Riemann surface X with ramification data $[d; a_1, \ldots, a_n]$ (over $\mathbb{P}^1$) and the ramified maps of Riemann surfaces is the morphism π induced by the embedding $\mathbb{C}[x] \to \mathbb{C}[x,y]/(y^d - f(x))$.

A.3. The canonical divisor

Given a ramification datum, we can now, using the (normalisation of) the algebraic model computed in section A.2, compute the canonical divisor on the cyclic cover $\pi\colon X \to \mathbb{P}^1$.

We begin, again, by a topological observation: using the Riemann-Hurwitz formula, we can, by the above observations, calculate the degree of the canonical divisor as

$$\begin{aligned} 2g-2 &= -2d + \sum_{p \in X} (e_p - 1) = -2d + \sum_{i=1}^{n} \gcd(a_i, d) \left(\frac{d}{\gcd(a_i, d)} - 1 \right) \\ &= (n-2)d - \sum_{i=1}^{n} \gcd(a_i, d). \end{aligned} \tag{A.3}$$

On the other hand, consider the (meromorphic) global section of the canonical divisor $\mathrm{d}x$ on $\mathbb{P}^1$. Clearly, it has no poles or zeros on $\mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[x]$ but admits a double zero at infinity, as

$$\mathrm{d}x = \mathrm{d}\left(\frac{1}{u}\right) = -\frac{\mathrm{d}u}{u^2}$$

on the coordinate patch $\mathbb{C}[u = 1/x]$. Similarly, we consider the global section of the canonical divisor $\pi^* \mathrm{d}x = \mathrm{d}x$ on X: the defining equation yields the relation

$$dy^{d-1}\mathrm{d}y = f'(x)\mathrm{d}x = f(x) \sum_{i=1}^{n} \frac{a_i}{x - x_i} \mathrm{d}x$$

and this already shows $\operatorname{ord}_p \mathrm{d}x = 0$ if $y \neq 0$, i.e. if p is not a ramification point. At the ramification points, we must compute the order of $\mathrm{d}x$ on the normalisation. More precisely, over the ith branch point x_i, the normalisation has $\gcd(d, a_i)$ components and, for each component the normalisation map is given by Equation A.1 as $x \mapsto t^{d/\gcd(d,a_i)}$ yielding

$$\mathrm{d}x = \mathrm{d}(t^{d/\gcd(d,a_i)}) = \varepsilon t^{\frac{d}{\gcd(d,a_i)} - 1} \mathrm{d}t,$$

which results in a zero of order $\frac{d}{\gcd(d,a_i)} - 1$ at each of the $\gcd(d, a_i)$ ramification points in the fibre over x_i.

Finally, π is unramified over $\infty \in \mathbb{P}^1$ and at each of the d preimages we can calculate the order of $\mathrm{d}x$ with the coordinate change Equation A.2:

$$\mathrm{d}x = \mathrm{d}\left(\frac{1}{u}\right) = -\frac{\mathrm{d}u}{u^2}.$$

In other words, $\mathrm{d}x$ has a double pole at each of the d preimages over ∞. In particular, $\mathrm{d}x$ is not a holomorphic section of the canonical bundle. We can, however, compare our calculation to the Riemann-Hurwitz formula to do a reality check: combining the above, we have

$$\operatorname{div} \mathrm{d}x = \sum_{i=1}^{n} \gcd(d, a_i) \left(\frac{d}{\gcd(d, a_i)} - 1 \right) - 2d = 2g - 2$$

in accordance with Equation A.3.

To construct holomorphic global sections, we consider instead, for $k \in \mathbb{N}$, the differentials $\frac{\mathrm{d}x}{y^k}$. By the same argument as above, $\operatorname{ord}_p \frac{\mathrm{d}x}{y^k} = 0$ if p is no ramification point. Over the branch point x_i, we pass again to the normalisation of the $\gcd(d, a_i)$ cusps and by Equation A.1, we can use $x \mapsto t^{d/\gcd(d,a_i)}$ and $y \mapsto t^{a_i/\gcd(d,a_i)}$ to see that locally

$$\frac{\mathrm{d}x}{y^k} = \frac{\mathrm{d}t^{d/\gcd(d,a_i)}}{t^{ka_i/\gcd(d,a_i)}} = \varepsilon t^{\frac{d}{\gcd(d,a_i)}-1-k\frac{a_i}{\gcd(d,a_i)}}\mathrm{d}t,$$

and therefore that $\operatorname{ord}_p \frac{\mathrm{d}x}{y^k} = \frac{d}{\gcd(d,a_i)} - 1 - k\frac{a_i}{\gcd(d,a_i)}$ for each of the $\gcd(d, a_i)$ preimages of x_i. In particular, we observe that

$$\frac{d - \gcd(d, a_i)}{a_i} \geq k \tag{A.4}$$

for $\frac{\mathrm{d}x}{y^k}$ to be holomorphic at (each) preimage p of x_i.

Finally, we consider again the situation over the (non-branch point) $\infty \in \mathbb{P}^1$. At each of the d preimages, we use again the coordinate change Equation A.2 to see that

$$\frac{\mathrm{d}x}{y^k} = \varepsilon \frac{1}{z^k x^{kN'}}\mathrm{d}\left(\frac{1}{u}\right) = \varepsilon' u^{kN'-2}\frac{\mathrm{d}u}{z^k},$$

where z^k is a unit and $N' = \left(\sum a_i\right)/d$. In particular, we see that $\frac{\mathrm{d}x}{y^k}$ vanishes with order $kN' - 2$ at each point over ∞. Again, we observe that $\frac{\mathrm{d}x}{y^k}$ is holomorphic over each preimage of ∞ if and only if

$$k\sum_{i=1}^{n} a_i \geq 2d \tag{A.5}$$

is satisfied. Again, we calculate the degree as a reality check and see:

$$\begin{aligned}\deg \frac{\mathrm{d}x}{y^k} &= \sum_{i=1}^{n} \gcd(d, a_i)\left(\frac{d}{\gcd(d, a_i)} - 1 - k\frac{a_i}{\gcd(d, a_i)}\right) + d\left(\frac{k}{d}\sum_{i=1}^{n} a_i - 2\right) \\ &= 2g - 2,\end{aligned}$$

by Equation A.3, as above.

Note that if we consider any $x' \neq x_i$ for all i, we observe that

$$(x - x')^e \frac{\mathrm{d}x}{y^k}$$

has a zero of order e at x', behaves like $\frac{\mathrm{d}x}{y^k}$ outside of x' and ∞ and transforms as

$$(x - x')^e = \left(\frac{1}{u} - x'\right)^e = \frac{1}{u^e}(1 - ux')^e$$

at ∞. Hence it adds a pole of order e at (each point over) ∞, as $1 - ux'$ is invertible at $u = 0$. Similarly, if we set $x' = x_i$ for some i, we add a zero of order $\frac{d}{\gcd(d,a_i)}e$ to each of the $\gcd(d, a_i)$ preimages of x_i and again a pole of order e to each of the d preimages of ∞.

A.4. Intermediate covers

Consider a divisor $e \mid d$. We set $d' = \frac{d}{e}$. Then the projection $\pi\colon X \to X/\alpha$ factors through $X \to X/\alpha^e \to X/\alpha$. In particular, we may again give an algebraic model for X/α^e by

$$y^e = \prod_{i=1}^{n} (x - x_i)_i^a = f(x),$$

where we may now consider the a_i modulo e. We set $[a_i]_e \coloneqq a_i \bmod e$. Indeed, the map of rings

$$\mathbb{C}[x, y] \to \mathbb{C}[x, y]/(y^d - f(x)), \quad x \mapsto x, \quad y \mapsto y^{d'}$$

clearly factors through $\mathbb{C}[x, y]/(y^e - f(x))$. Note that $\pi_{d'}\colon X \to X/\alpha^e$ is (a ramified cover) of degree d', while $\pi_e\colon X/\alpha^e \to X/\alpha$ is of degree e. By abuse of notation, we do not distinguish between the algebraic models and their normalisations and the morphisms between the algebraic models and their normalisations[1] unless this is required for explicit computations.

As we want to pull back differential forms from X/α^e to X, we briefly discuss the ramification behaviour of these maps.

The discussion in section A.2 explains the ramification behaviour of π_e: each point $x_i \in \mathbb{P}^1 = X/\alpha$ has $\gcd(a_i, e)$ preimages (in the normalisation). In particular if the number of preimages of x_i on X is a multiple of e, then x_i is unramified with respect to π_e, otherwise it is ramified with index $e/\gcd(a_i, e)$.

We now turn our attention to the morphism $\pi' \coloneqq \pi_{d'}$. Clearly it is unramified outside the preimages of x_i on X/α^e. Consider a point x_i' in the $\gcd(a_i, e)$ preimages of x_i on X/α^e. Then each x_i' has $\gcd(a_i, d)/\gcd(a_i, e)$ preimages on X, hence each of the $\gcd(a_i, d)/\gcd(a_i, e)$ preimages of x_i' on X has a ramification index of

$$e_q = \frac{d}{e}\frac{\gcd(e, a_i)}{\gcd(d, a_i)} \iff \pi'(q) \in \pi_e^{-1}(x_i).$$

Equivalently, we may consider the normalisation of π' in this situation. Consider a preimage x_i' of $x_i \in X/\alpha$ on the normalisation of X/α^e and a preimage q of x_i' on the normalisation of X. The same arguments as in section A.2 imply that the induced map

[1] Note that, by the universal property of the normalisation, for any morphism $\pi\colon X \to Y$, there exists a unique morphism $\pi^{\mathrm{norm}}\colon X^{\mathrm{norm}} \to Y^{\mathrm{norm}}$ making the diagram

$$\begin{array}{ccc} X^{\mathrm{norm}} & \xrightarrow{\exists! \pi^{\mathrm{norm}}} & Y^{\mathrm{norm}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\pi} & Y \end{array}$$

commute, as the composition $X^{\mathrm{norm}} \to X \xrightarrow{\pi} Y$ is a morphism of a normal variety into Y and thus factors through the normalisation.

between the normalisations of the algebraic models of X/α^e and X/α is locally induced by

$$\mathbb{C}[t] \supseteq \mathbb{C}[t^{\frac{[a_i]_e}{\gcd(a_i,e)}}, t^{\frac{e}{\gcd(a_i,e)}}] \to \mathbb{C}[t^{\frac{[a_i]_d}{\gcd(a_i,d)}}, t^{\frac{d}{\gcd(a_i,d)}}] \subseteq \mathbb{C}[t], \quad t \mapsto t^{\frac{d}{e}\frac{\gcd(a_i,e)}{\gcd(a_i,d)}}, \tag{A.6}$$

i.e. this map of rings describes the π' locally on the fibres. Note that the the local exponent a_i for a non-ramified point is 0 so that the map is simply the identity $t \mapsto t$ in this case.

This allows us to pull back differentials. Consider the l-eigendifferential

$$\omega_l^r = \prod_{i=1}^{r}(x-p_i)\prod_{i=1}^{N}(x-x_i)^{\left\lfloor \frac{l[a_i]_e}{e}\right\rfloor}\frac{\mathrm{d}x}{y^k}$$

on X/α^e, where the points p_i are pairwise distinct and do not coincide with the x_j on $\mathbb{P}^1$. The discussion in section A.3 yields the local form of ω_l^r around every point on X/α^e: around a point away from all the preimages of the p_i and x_i, it is simply of the form $\mathrm{d}t$; at each of the e preimages of the points p_i it has a simple zero, i.e. is of the form $t\mathrm{d}t$ and around each of the $\gcd(a_i, e)$ preimages of the point x_i, it is of the form

$$t^{\frac{e}{\gcd(e,a_i)}(1+\left\lfloor \frac{l[a_i]_e}{e}\right\rfloor)-l\frac{[a_i]_e}{\gcd(e,a_i)}-1}\mathrm{d}t;$$

finally, around each of the e preimages of the point $\infty \in \mathbb{P}^1$ it is of the form

$$t^{\sum_{i=1}^{N}(\frac{l}{e}[a_i]_e-\left\lfloor \frac{l[a_i]_e}{e}\right\rfloor)-r-2}\mathrm{d}t.$$

We can therefore describe the divisor of the pullback $(\pi')^*\omega_l^r$ by applying Equation A.6 to each of the local situations. Note that

$$\mathrm{d}\big(t^{\frac{d}{e}\frac{\gcd(a_i,e)}{\gcd(a_i,d)}}\big) = t^{\frac{d}{e}\frac{\gcd(a_i,e)}{\gcd(a_i,d)}-1}\mathrm{d}t.$$

We thus obtain that $(\pi')^*\omega_l^r$ is again of the form $\mathrm{d}t$ around every preimage of every point outside of $\{p_i, x_j\}$ and has a simple zero at each of the d preimages of each point p_i; at each of the $\gcd(d, a_i)$ preimages of each point x_i it is of order

$$\frac{d}{\gcd(d,a_i)}\left(1+\left\lfloor \frac{l[a_i]_e}{e}\right\rfloor\right)-\frac{d}{e}\frac{l[a_i]_e}{\gcd(d,a_i)}-1 = \frac{d}{\gcd(d,a_i)}\left(1-\left\langle \frac{l[a_i]_e}{e}\right\rangle\right)-1;$$

finally, at each of the d preimages of ∞ it is of order

$$\sum_{i=1}^{N}\left(\frac{l}{e}[a_i]_e-\left\lfloor \frac{l[a_i]_e}{e}\right\rfloor\right)-r-2=0.$$

It is straight-forward to check that these orders do indeed add up to $2g(X)-2$.

Zusammenfassung

Die zentralen Objekte der Dissertation sind Familien von flachen Flächen. Dabei besteht eine *flache Fläche* (X, ω) aus einer kompakten riemannschen Fläche (äquivalent einer glatten projektiven komplexen algebraischen Kurve) und einer holomorphen Differentialform $\omega \in \mathrm{H}^{1,0}(X) \setminus \{0\}$. Indem wir ω integrieren, können wir X außerhalb der Nullstellen von ω mit einer flachen Metrik versehen, beziehungsweise mit einem komplexen Atlas, dessen Kartenwechselabbildungen lokal durch Translationen gegeben sind. Weiterhin bezeichnen wir den *Modulraum der projektiven komplexen algebraischen Kurven von Geschlecht* g mit $\mathcal{M}_g$ und den *Modulraum der flachen Flächen von Geschlecht* g mit $\Omega\mathcal{M}_g$. Es gibt eine natürliche Projektionsabbildung $\pi\colon \Omega\mathcal{M}_g \to \mathcal{M}_g$.

Die Gruppe $\mathrm{SL}_2(\mathbb{R})$ operiert auf dem Modulraum der flachen Flächen durch Scheren der flachen Struktur. In dem (seltenen) Fall, dass $\mathcal{C} = \pi(\mathrm{SL}_2(\mathbb{R}) \cdot (X, \omega))$ in $\mathcal{M}_g$ eindimensional (also eine Kurve) ist, nennen wir $\mathcal{C}$ eine *Teichmüllerkurve*.

Die Menge der bekannten Teichmüllerkurven ist recht überschaubar: Es gibt eine Reihe von „klassischen Beispielen“, die auf Veech und Ward zurückgehen und später von Bouw und Möller verallgemeinert wurden (sie liefern Beispiele für Teichmüllerkurven mit beliebig großem Fasergeschlecht). Weiterhin gibt es die *Weierstraßkurven*, eine unendliche Familie von Teichmüllerkurven im Modulraum $\mathcal{M}_2$, die von McMullen und Kalta entdeckt wurden. McMullen ist eine Beschreibung aller Teichmüllerkurven in $\mathcal{M}_2$ gelungen. Weiterhin hat er unendliche Familien in $\mathcal{M}_3$ und $\mathcal{M}_4$ konstruiert, die *Prym-Teichmüllerkurven*. Kürzlich haben Mukamel, McMullen und Wright neue Familien in Geschlecht 4 entdeckt. Andererseits gibt es eine Reihe starker Endlichkeitsaussagen, die vermuten lassen, dass es sich bei diesen Familien um ein spezielles Phänomen in niedrigem Geschlecht handelt.

Teichmüllerkurven sind als Kurven in $\mathcal{M}_g$ auf natürliche Weise Orbifolds. Topologisch lassen sie sich daher durch ihr Geschlecht g klassifizieren, welches sich aus, h_0, der Anzahl der Zusammenhangskomponenten, C, der Anzahl der Spitzen, χ, der Orbifold-Eulercharakteristik, und aus e_d, der Anzahl der Orbifoldpunkte von Ordnung d, mit Hilfe der Formel

$$2h_0 - 2g = \chi + C + \sum_d e_d \left(1 - \frac{1}{d}\right)$$

ermitteln lässt.

Die Weierstraßkurven in $\mathcal{M}_2$ sind ausführlich studiert worden. McMullen trieb die topologische Klassifikation voran, indem er die Anzahl der Spitzen und Zusammenhangskomponenten klassifizierte. Seine Doktoranden Bainbridge und Mukamel bestimmten die (Orbifold-)Eulercharakteristik und die Typen und Anzahl der Orbifoldpunkte der einzelnen Kurven, so dass das Geschlecht dieser Teichmüllerkurven bestimmt werden konnte.

Die Prym-Teichmüllerkurven $W_D(4)$ in $\mathcal{M}_3$ und $W_D(6)$ in $\mathcal{M}_4$ sind wie die Weierstraßkurven in $\mathcal{M}_2$ durch quadratische Diskriminanten indiziert. In diesen Fällen wurde die Eulercharakteristik von Möller berechnet, die Berechnung der Anzahl der Spitzen sowie der Zusammenhangskomponenten erfolgte durch Lanneau und Nguyen.

Das Hauptziel dieser Dissertation ist die Bestimmung des topologischen Typs aller Prym-Teichmüllerkurven in $\mathcal{M}_3$ und $\mathcal{M}_4$.

Das erste Resultat der Dissertation ist in gemeinsamer Arbeit mit David Torres-Teigell entstanden.

Mit Ausnahme einiger Spezialfälle für kleine Diskriminanten, die wir separat behandeln, beschreiben wir die Orbifoldpunkte auf den Prym-Teichmüllerkurven in $\mathcal{M}_3$ durch ganzzahlige Lösungen quadratischer Formen. Genauer definieren wir für jede positive Diskriminante D

$$\begin{aligned}\mathcal{H}_2(D) &:= \{(a,b,c) \in \mathbb{Z}^3 : a^2+b^2+c^2 = D,\ \gcd(a,b,c,f_0) = 1\},\ \text{und}\\ \mathcal{H}_3(D) &:= \{(a,b,c) \in \mathbb{Z}^3 : 2a^2-3b^2-c^2 = 2D,\ \gcd(a,b,c,f_0) = 1,\\ &\qquad\qquad -3\sqrt{D} < a < -\sqrt{D},\ c < b \le 0,\\ &\qquad\qquad (4a-3b-3c<0) \vee (4a-3b-3c = 0\ \wedge\ c < 3b)\},\end{aligned}$$

wobei f_0 den Führer von D bezeichnet. Die zusätzlichen Bedingungen in der Definition von $\mathcal{H}_3(D)$ schränken die Lösungen auf die innerhalb eines gewissen Fundamentalbereiches liegenden ein und sorgen dafür, dass es nur endlich viele solcher ganzzahligen Lösungen gibt.

Theorem 1 (Theorem 1.1.1)**.** *Für nicht-quadratische Diskriminanten $D > 12$ hat die entsprechende Prym-Teichmüllerkurve in $\mathcal{M}_3$ nur Orbifoldpunkte von Ordnung 2 und 3.*

Genauer: Die Zahl $e_3(D)$ der Orbifoldpunkte von Ordnung 3 ist $|\mathcal{H}_3(D)|$; die Zahl $e_2(D)$ der Orbifoldpunkte von Ordnung 2 ist $|\mathcal{H}_2(D)|/24$, falls D gerade ist und es gibt keine Punkte der Ordnung 2 falls D ungerade ist.

Weiterhin hat die Kurve $W_8(4)$ genau einen Punkt der Ordnung 3 und einen Punkt der Ordnung 4; die Kurve $W_{12}(4)$ hat einen einzigen Orbifoldpunkt der Ordnung 6.

Weiterhin geben wir eine Tabelle mit den topologischen Invarianten der Teichmüllerkurven (Tabelle 1.2 auf Seite 44), sowie Prototypen an, die die flache Struktur dieser Orbifoldpunkte explizit geometrisch beschreiben (siehe Abschnitt 1.7).

Für den Beweis des Theorems betrachten wir zwei Familien zyklischer Überlagerungen von $\mathbb{P}^1$ in $\mathcal{M}_3$, die *Kleeblattfamilie* $\mathcal{X}$ und die *Windmühlenfamilie* $\mathcal{Y}$ (die Namen erklären sich durch die flachen Prototypen in Abschnitt 1.7). Weiterhin betrachten wir ihr Bild unter der Prym-Torelli-Abbildung, das heißt wir ordnen jeder Kurve eine $(1,2)$-polarisierte abelsche zweidimensionale *Prym-Varietät* zu, die durch die Prym-Involution aus ihrer Jacobischen hervorgeht (siehe Abschnitt 1.2). Mit Hilfe klassischer Methoden, die auf Bolza zurückgehen, können wir sowohl die Periodenmatrizen als auch die Prym-Varietäten im Fall dieser zyklischen Überlagerungen explizit beschreiben und diese auf reelle Multiplikation untersuchen. Die Schnittpunkte mit den Prym-Teichmüllerkurven sind genau die Punkte, die ein geeignetes Differential und eine damit verträgliche reelle Multiplikation zulassen.

Wir erhalten zudem strukturelle Resultate bezüglich der Familien $\mathcal{X}$ und $\mathcal{Y}$ und sehen, dass sich diese sehr unterschiedlich verhalten.

Theorem 2 (Theorem 1.1.2)**.** *Das Bild der Prym-Torelli-Abbildung der Familie $\mathcal{X}$ ist isogen zu dem Punkt $E_{\mathrm{i}} \times E_{\mathrm{i}}$ im Modulraum $\mathcal{A}_{2,(1,2)}$ der abelschen Flächen mit $(1,2)$-Polarisierung. Dabei bezeichnet E_{i} die zum quadratischen Torus $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\mathrm{i})$ gehörige elliptische Kurve. Orbifoldpunkte der Ordnung 2 und 4 auf $W_D(4)$ entsprechen Schnittpunkten mit dieser Familie.*

Insbesondere unterscheidet sich diese Situation stark von der in Geschlecht 2 von Mukamel untersuchten. Daher sind durchgehend andere Methoden notwendig.

Im Gegensatz dazu liegt das Prym-Torelli-Bild der $\mathcal{Y}$-Familie dicht in einer Shimurakurve von Diskriminante 6.

Theorem 3 (Theorem 1.1.3)**.** *Der Abschluss des Prym-Torelli-Bildes der $\mathcal{Y}$-Familie in $\mathcal{A}_{2,(1,2)}$ ist die (kompakte) Shimurakurve, die $(1,2)$-polarisierte abelsche Flächen parametrisiert, deren Endomorphismenring isomorph zu der Maximalordnung in der indefiniten Quaternionenalgebra von Diskriminante 6 ist. Orbifoldpunkte der Ordnung 3 und 6 auf $W_D(4)$ entsprechen den Schnittpunkten mit dieser Familie.*

Mit Hilfe dieser expliziten Darstellungen gelingt es uns die Schnitte mit den Teichmüllerkurven $W_D(4)$ explizit zu zählen, um die oben genannten Formeln zu erhalten.

Für $D \equiv 1 \bmod 8$ zeigen Lanneau und Nguyen, dass die zugehörige Prym-Teichmüllerkurve in zwei Zusammenhangskomponenten zerfällt. Das gleiche Verhalten wurde auch schon von McMullen für die Weierstraßkurven in $\mathcal{M}_2$ beobachtet. McMullen gab für die Weierstraßkurven sowohl eine Klassifikation der Spitzen als auch eine *Spin-Invariante* an, mit der man entscheiden kann, auf welcher Komponente eine Spitze einer Teichmüllerkurve liegt. Mit Hilfe dieser Invarianten und der Klassifikation der Spitzen konnten Bouw und Möller zeigen, dass die Komponenten der Weierstraßkurven Galois-konjugiert und daher insbesondere homöomorph sind.

Zwar wurden die Spitzen der Prym-Teichmüllerkurven von Lanneau und Nguyen klassifiziert, es war aber nicht bekannt, wie sich die Spitzen auf die einzelnen Komponenten verteilen.

In einem weiteren Kapitel der Arbeit wird eine Spin-Invariante für Prym-Teichmüllerkurven beschrieben, mit deren Hilfe man bestimmen kann, welche Spitze auf welcher Komponente liegt. Mit Hilfe dieser kann das zweite Resultat der Dissertation gezeigt werden:

Theorem 4 (Theorem 2.1.1)**.** *Sei $D \equiv 1 \bmod 8$ kein Quadrat. Dann sind die zwei Zusammenhangskomponenten der zugehörigen Prym-Teichmüllerkurve in $\mathcal{M}_3$ homöomorph.*

Insbesondere hat also jede der Komponenten gleich viele Spitzen und Orbifoldpunkte und das gleiche Geschlecht.

Die Idee des Beweises ist es, in Analogie zu der Arbeit von Bouw und Möller die Galois-Operation auf den Spitzen explizit anzugeben. Mit Hilfe der Spin-Invariante (Theorem 2.1.2) zeigen wir, dass eine Spitze und ihre Galois-konjugierte Spitze nicht auf derselben Komponente liegen. Daher sind die Zusammenhangskomponenten Galois-konjugiert und damit insbesondere auch homöomorph.

Damit ist die topologische Klassifikation der Prym-Teichmüllerkurven in $\mathcal{M}_3$ abgeschlossen.

Das nächste Ergebnis der Dissertation ist wieder in gemeinsamer Arbeit mit David Torres-Teigell entstanden und liefert eine analoge Klassifikation der Orbifoldpunkte für die Prym-Teichmüllerkurven in $\mathcal{M}_4$.

Theorem 5 (Theorem 3.1.1)**.** *Für alle Diskriminanten $D > 12$ hat die entsprechende Prym-Teichmüllerkurve in $\mathcal{M}_4$ nur Orbifoldpunkte von Ordnung 2 und 3. Genauer gilt:*

- *Für die Anzahl $e_2(D)$ der Punkte von Ordnung 2 gilt*

$$e_2(D) = \begin{cases} 0\,, & \textit{falls } D \textit{ ungerade ist,} \\ h(-D) + h(-D/4)\,, & \textit{falls } D \equiv 12 \bmod 16 \textit{ ist und} \\ h(-D)\,, & \textit{falls } D \equiv 0,4,8 \bmod 16 \textit{ ist.} \end{cases}$$

  Dabei bezeichnet $h(-D)$ die Klassenzahl der quadratischen Ordnung $\mathcal{O}_{-D}$.

- *Für die Anzahl $e_3(D)$ der Punkte von Ordnung 3 gilt*

$$e_3(D) = \#\{a,i,j \in \mathbb{Z} : a^2 + 3j^2 + (2i-j)^2 = D,\ \gcd(a,i,j) = 1\}/12.$$

- *$W_5(6)$ hat einen Punkt der Ordnung 3 und einen Punkt der Ordnung 5.*
- *$W_8(6)$ hat einen Punkt der Ordnung 2 und einen Punkt der Ordnung 3.*
- *$W_{12}(6)$ hat einen Punkt der Ordnung 2 und einen Punkt der Ordnung 6.*

Bemerkenswert ist, dass es im Geschlecht-4-Fall, ähnlich wie im Geschlecht-2-Fall, einen direkten Zusammenhang zu elliptischen Kurven mit komplexer Multiplikation gibt und daher „konventionelle“ Klassenzahlen auftauchen, siehe Abschnitt 3.2.

Um genauer zu sein: Die Grundidee des Vorgehens ist ähnlich wie im Geschlecht-3-Fall. Allerdings stößt man in diesem Fall auf das Problem, dass der Lokus in $\mathcal{M}_4$ der Kurven mit der „richtigen“ Automorphismengruppe, der Diedergruppe D_8, mit dem man die Teichmüllerkurven schneiden möchte, zweidimensional ist (es handelt sich nicht mehr um eine zyklische Überlagerung des $\mathbb{P}^1$, sondern einer elliptischen Kurve). Sie lässt sich allerdings explizit durch elliptische Kurven beschreiben und eine Bedingung an das Differential, die von der Teichmüllerkurvendefinition herkommt, verringert die Dimension wieder um 1.

Theorem 6 (Theorem 3.1.2)**.** *Die Familie $\mathcal{M}_4(D_8)$ steht in Bijektion zu der Familie*

$$\mathcal{E} = \{(E, [P]) : E \in \mathcal{M}_{1,1},\ [P] \in (E \setminus E[2])/\phi\},$$

die aus elliptischen Kurven zusammen mit einem ausgezeichneten Basispunkt und einem elliptischen Punktepaar besteht. Hier bezeichnet ϕ die elliptische Involution.

Insbesondere ist diese Familie zweidimensional; allerdings ist die Teilmenge der Kurven, die eine C_4-Eigenform mit einer einzigen Nullstelle zulassen, nur eindimensional und steht in Bijektion zu $\mathcal{M}_{1,1} \setminus \{E_2\}$.

Die Bijektion geht in diesem Fall direkt aus einer expliziten Beschreibung von Kurven mit einer D_8-Wirkung als geeignetes Faserprodukt geeigneter elliptischer Kurven hervor, siehe Abschnitt 3.3. Wir beschreiben dann das Prym-Torelli-Bild dieser Familie und zeigen, dass die reelle Multiplikationsbedingung der Prym-Teichmüllerkurve gerade komplexer Multiplikation der zugehörigen elliptischen Kurve entspricht. Da die Klassenzahl die Anzahl elliptischer Kurven mit komplexer Multiplikation ist, lässt sich die Anzahl der Punkte der Ordnung 2 durch diese ausdrücken.

Für die Punkte der Ordnung 3 muss man wieder eine zyklische Überlagerung des $\mathbb{P}^1$ betrachten. Allerdings besteht das Prym-Torelli-Bild auf Grund der vielen Automorphismen nur aus einem einzigen Punkt (Theorem 3.1.3). Durch sorgfältige Analyse der Eigenbasis und der auftretenden reellen Multiplikation erhalten wir die obige Formel für die Punkte der Ordnung 3. Wir geben außerdem eine explizite Konstruktion solcher Kurven als Faserprodukt der zum Hexagon gehörenden elliptischen Kurve mit sich selbst an, siehe Abschnitt 3.4.

Des Weiteren geben wir wieder Prototypen für die flache Struktur mit Hilfe sogenannter k-Differentiale an, siehe Abschnitt 3.6. Außerdem können wir in diesem Fall genaue Aussagen zum Wachstum des Geschlechts machen.

Theorem 7 (Theorem 3.1.4)**.** *Es gibt Konstanten $C_1, C_2 > 0$, die unabhängig von D sind, so dass*

$$C_1 \cdot D^{3/2} < g(W_D(6)) < C_2 \cdot D^{3/2}.$$

Wir können die Konstanten recht explizit angeben und sehen daher insbesondere, dass $W_D(6)$ genau dann Geschlecht 0 hat, wenn $D \leq 20$.

Da bei der Klassifikation der Orbifoldpunkte die Analyse von Familien von flachen Flächen mit Automorphismen durchgehend eine wichtige Rolle spielte, werden diese im letzten Abschnitt der Dissertation etwas systematischer untersucht.

Die Ordnungen der Nullstellen eines holomorphen Differentials ω auf einer Kurve von Geschlecht g definieren eine Partition μ von $2g-2$. Andererseits können wir zu jeder Partition μ von $2g-2$ die Teilmenge $\Omega\mathcal{M}_g(\mu) \subseteq \Omega\mathcal{M}_g$ definieren, die aus den (X, ω) bestehen, deren Nullstellenordnungen der Partition μ entsprechen. Die $\Omega\mathcal{M}_g(\mu)$ bilden eine natürliche Stratifizierung des Modulraums $\Omega\mathcal{M}_g$. Die einzelnen Strata können mit *Periodenkoordinaten* versehen werden, so dass sie lokal biholomorph zu $\mathbb{C}^N$ für $N = 2g + |\mu| - 1$ sind. Ein bedeutendes Ergebnis von Eskin, Mirzakhani und Mohammadi besagt, dass jeder $\mathrm{SL}_2(\mathbb{R})$-Bahnabschluss durch $\mathbb{R}$-lineare Gleichungen beschrieben wird; ein bedeutendes Ergebnis von Filip besagt, dass diese immer algebraisch sind.

Wir führen den Begriff einer $\mathbb{C}$-*linearen Mannigfaltigkeit* ein, d.h. einer Mannigfaltigkeit $\mathcal{M} \subseteq \Omega\mathcal{M}_g(\mu)$, die lokal durch $\mathbb{C}$-lineare Gleichungen in Periodenkoordinaten ausgeschnitten wird. Zunächst geben wir eine Klasse von Beispielen an.

Theorem 8 (Theorem 4.3.1)**.** *Der Raum der Eigendifferentiale einer zyklischen Überlagerung von* $\mathbb{P}^1$ *ist eine* $\mathbb{C}$*-lineare Mannigfaltigkeit.*

Als letztes wird eine *Überlagerung* $\mathbb{C}$*-linearer Mannigfaltigkeiten* definiert. Da man sich in diesem Fall nicht der $\mathrm{SL}_2(\mathbb{R})$-Aktion und der damit zusammenhängenden Klassifikationssätze bedienen kann, ist diese recht technisch.

Theorem 9 (Theorem 4.4.1)**.** *Eine Überlagerung einer* $\mathbb{C}$*-linearen Mannigfaltigkeit ist eine* $\mathbb{C}$*-lineare Mannigfaltigkeit.*

Trotzdem ist eine Klassifikation *primitiver* (d.h. nicht aus einer Überlagerung hervorgehender) $\mathbb{C}$-linearer Mannigfaltigkeiten noch nicht absehbar. Im abschließenden Abschnitt 4.5 werden einige weiterführende offene Fragen formuliert.

Das Kapitel 1 ist als gemeinsame Arbeit mit David Torres-Teigell als [TTZ16] erschienen, Kapitel 2 wurde zur Veröffentlichung angenommen und wird als [Zac16] erscheinen und Kapitel 3 ist, wieder als gemeinsame Arbeit mit David Torres-Teigell, als [TTZ17] erschienen. Diese Kapitel unterscheiden sich nur marginal von den veröffentlichten Versionen.

Lebenslauf

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Publications

- Orbifold Points on Prym-Teichmüller Curves in Genus Four (with David Torres-Teigell) (arXiv:1609.00144 [math.AG])

  J. Inst. Math. Jussieu (2017). DOI: 10.1017/S1474748017000196

- The Galois Action and a Spin Invariant for Prym-Teichmüller Curves in Genus 3 (arXiv:1511.06257 [math.AG])

  accepted in: *Bulletin de la SMF*

- Orbifold Points on Prym-Teichmüller Curves in Genus Three (with David Torres-Teigell) (arXiv:1502.05381 [math.AG])

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