PARTICLES, STRINGS, BRANES AND THE DIMENSION OF OUR UNIVERSE

Master Thesis

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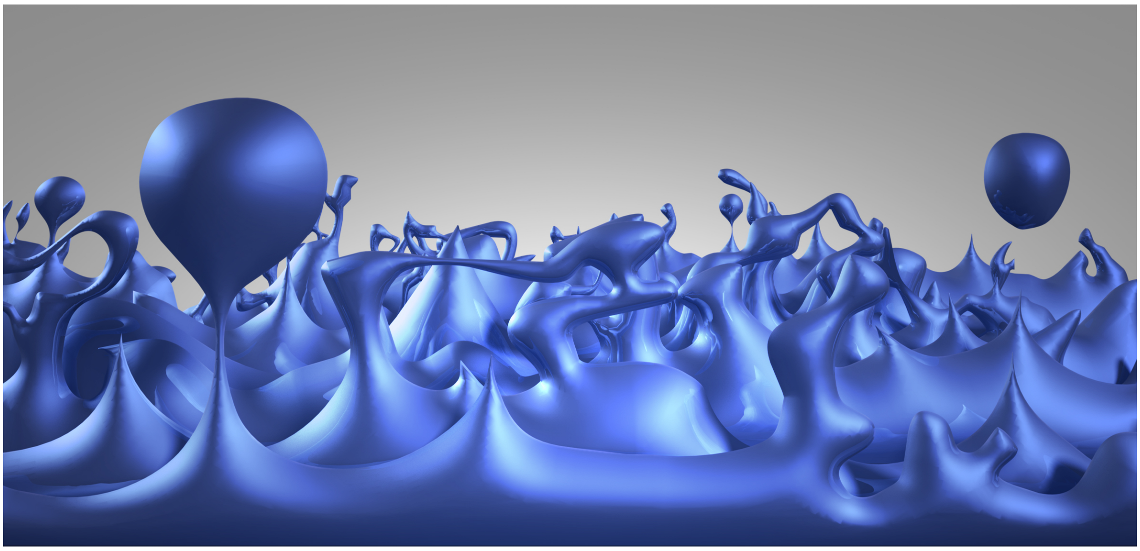

FROM STRINGS TO MINIMAL LENGTH AND THE SPECTRAL DIMENSION

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Illustration on title page: NASA/CXC/M.Weiss
http://chandra.harvard.edu/photo/2015/quantum/quantum_ill.jpg
The illustration shows an artists view of how spacetime could look like at the smallest scales.

Marco Knipfer: *Particles, Strings, Branes and the Dimension of our Universe,* From Strings To Minimal Length And The Spectral Dimension,
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There was a long history of speculation
that in quantum gravity,
unlike Einstein's classical theory,
it might be possible for the
topology of spacetime to change.

— Edward Witten [82]

Dedicated to the search of Quantum Gravity.

ABSTRACT

In thesis I investigate the possibility that at the smallest length scale (Planck scale) the very notion of "dimension" needs to be revisited. Due to *quantum effects* spacetime might become very turbulent at these scales and properties like those of *fractals* emerge, including a *scale dependent dimension*. It seems that this "spontaneous dimensional reduction" and the appearance of a minimal physical length are very general effects that most approaches to quantum gravity share. Main emphasis is given to the *spectral dimension* and its calculation for strings and p-branes.

ZUSAMMENFASSUNG

In dieser Arbeit untersuche ich die Möglichkeit, dass bei den kleinsten Abständen (Planck Skala) sogar der Begriff der „Dimension" überdacht werden muss. Durch *Quanteneffekte* kann die Raumzeit auf diesen Längen sehr stark fluktuieren und Eigenschaften von *Fraktalen*, wie eine *skalenabhängige Dimension*, annehmen. Diese „spontane dimensionale Reduktion" und das Aufkommen einer minimalen physikalischen Länge scheinen allgemeine Effekte zu sein, die die meisten Ansätze zu einer Theorie der Quantengravitation gemeinsam haben. Das Hauptaugenmerk liegt auf der sogenannten *spektralen Dimension* und ihrer Berechnung für Strings und p-Branen.

Ah, if we could only do the integral [. . .]. But we can't.

— Anthony Zee [91]

ACKNOWLEDGMENTS

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CONTENTS

LIST OF FIGURES

LIST OF TABLES

Part I

INTRODUCTION

In this part we are going to review the most successful physical theories that have been developed by mankind so far: Einstein's *General Theory of Relativity* (GR) describing Gravity, and *Quantum Field Theory* (QFT) describing all other known fundamental forces.

1

SETTING THE STAGE

General Relativity (GR) and *Quantum Field Theory* (QFT) are the most complete and best tested theories we currently have. GR describes the *large distances* such as a satellite orbiting the Earth, the Earth orbiting the Sun or even bigger: Cosmology. QFT on the other hand describes the *small distances* like an electron moving in a wire, "orbiting" a nucleus in an atom or particles hitting each other in colliders.

In their respective domains they (GR and QFT) are predicting experiments exceptionally well. They have been tested over and over again with ever growing experimental efforts and succeeded every time. So there is one theory for the small scales and one for the big scales. The natural thing to ask is then: Is there a physical situation for which *both* are important simultaneously and is there a more complete theory combining General Relativity and Quantum Theory?

It turns out the quest of finding such a combination is harder than their founders imagined. *Black Holes* (BHs) are relevant for addressing the questions, because at small scales (BH center) gravity becomes very intense. The *singularity* in a GR Black Hole means that GR predicts its own breakdown. Another example is the Big Bang singularity. At the earliest stages of our universe densities were so high that QFT and GR are both relevant. Those singularities are problems that cannot be solved without a theory of *Quantum Gravity* (QG).

There also exists the information loss problem of BHs. It turns out that when working with QFT in the presence of black holes, they emit radiation and have a *temperature* as well as an entropy. The problem is that no information can exit the BH. So where did the information that fell into the BH go, when it has completely evaporated?

The hope is that a theory of Quantum Gravity will make these and more things clearer and solve the problems. For the last half century many different theories have been developed with the aim of unifying Quantum Theory and Gravity. One such theory is *String Theory*. Despite several approaches to Quantum Gravity on the market, each of them has its own problems. We believe there may exist model independent features of Quantum Gravity. In this thesis we explore one of such possible model independent features: *Dimensional flow*, *i. e.* a spacetime dimension that is scale dependent.

It has been realized that quantum fluctuations of the spacetime itself can lead to the fractalization of spacetime and thus the possibility for the dimension to change with scale. The standard notion of a smooth manifold breaks down and a new *quantum manifold* is needed. There is a connection between quantum fluctuations, loss of resolution, minimal length and dimensional flow. The quantum fluctuations of spacetime are shown in an artistic view on the title page of this thesis.

We mention String Theory here, because the initial proposal of a variable dimension of spacetime was in a String paper by Atick and Witten [14] in 1988 where they showed that a gas of strings above the Hagedorn temperature only has two degrees of freedom and thus behaves effectively as if spacetime was two dimensional. The Nobel laureate 't Hooft [47] in 1993 argued that Black Holes at the Planck scale behave as if they were living on a two-dimensional lattice. This is connected to the entropy of a Black Hole and the idea behind holography.

Those indications of a dimensional reduction led to new notions of dimension and its investigation in basically all theories of Quantum Gravity. Interestingly a common feature in many theories is that at small scales the dimension decreases to the value two. Although, as we will show, not all theories have this behaviour.

Dimensional reduction is not only such a fascinating topic because it is a common feature of theories of Quantum Gravity. It also gives a hint at solving the problem of quantizing Gravity. If one naively tries to quantize Gravity in the framework of QFT, then one quickly realizes that GR in $3+1$ dimension is non-renormalizable and thus of limited use. Interestingly in two dimensions this problem goes away and Gravity is renormalizable. In the following this dimensional flow and how it behaves for different theories will be investigated.

2

GENERAL RELATIVITY

In order to understand the problems of GR a basic understanding of *differential geometry* and *curved spacetime* is needed. In this chapter a small review of Einsteins Theory of Relativity, also known as *General Relativity* (GR), will be given. It is based on the two books [30, 61]. This review has two purposes: The first purpose is to make the reader familiar with the notations and conventions used in this thesis. The second is to highlight some particular details in GR that are less often covered in introductory text books and will be needed in the following chapters about Quantum Gravity.

2.1 REVIEW OF THE BASIC EQUATIONS

Einstein developed his *Special Theory of Relativity* (SR) [36] in 1905, where he introduced the speed of light c as constant (same in every reference frame) and combined space and time into the notion of spacetime. SR basically replaced Newtonian mechanics. In the limit $c \to \infty$ Newtonian mechanics (not Gravity) is recovered from SR.

Ten years later, in November 1915, he presented his General Theory of Relativity [37] (published in 1916). Gravitation is implemented into SR by letting spacetime be curved. The effect of this curvature is what one usually thinks of as the gravitational force. For small curvatures Newtonian Gravity can be recovered.

Let us now dive into the mathematics: Observer A sees a source that instantaneously emits light out into every direction. This gives us a sphere of light. Now we have observer B moving with speed v relative to A and let the positions of A and B be the same for the instant that light is sent out. Since the speed of light is the same in every frame, he will also see a sphere of light. This gives us the equation

$$-(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = -(c\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 \,. \tag{1}$$

The left hand side describes the sphere of light that observer A sees and the right hand side the one that observer B sees. In the following we will set $c = 1$. Motivated by this we introduce the *spacetime* *interval* $(\Delta s)^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$, which describes the spacetime distance between two events. We write

$$(\Delta s)^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \,. \tag{2}$$

with $\{x^\mu\} = (t, x, y, z)$ and the metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Until now the distances Δx^μ were finite. The step to general relativity is to make them infinitesimal and let spacetime be curved. What curvature exactly means will be explained later. Then generally the metric will depend on position in spacetime, though a position dependent metric does not imply curved spacetime.[1] We end up with

$$(ds)^2 = g(x)_{\mu\nu} dx^\mu dx^\nu \,. \tag{3}$$

The metric $g(x)_{\mu\nu}$ describes how spacetime is curved and for the special case $g_{\mu\nu} = \eta_{\mu\nu}$, which we call flat or Minkowski spacetime, we recover Special Relativity.

Equation (3) describes the spacetime distance between two infinitesimally separated events. To make this clear we look at a particle of mass m. This particle moves on a curve $\mathcal{C}$. To get the length of the curve we add the spacetime distances from the start to the end of the curve. In our convention $(ds)^2 < 0$ for physical particles. Thus we define $(d\tau)^2 = -(ds)^2$, then we integrate $d\tau$:

$$\tau = \int_{\tau_i}^{\tau_f} d\tau = \int_{\tau_i}^{\tau_f} \sqrt{g_{\mu\nu} dx^\mu dx^\nu} \quad = \int_{\tau_i}^{\tau_f} \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}\, d\tau \,, \tag{4}$$

where we have parametrized the curve $x^\mu(\tau)$. A reparametrization of the curve is always possible, but the special choice where the square of the 4-velocity is one

$$\mathbf{u}^2 = \frac{dx(\tau)^\mu}{d\tau} \frac{dx(\tau)_\mu}{d\tau} = -1 \tag{5}$$

is the proper time τ and defines a preferred parameter along the curve $\mathcal{C}$.

The next step is to find out on what trajectories a particle moves in curved spacetime. We take a given metric $g_{\mu\nu}$ and calculate when the proper time (4) is extremal. It turns out that this is the case if the *Geodesic Equation*

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\beta} = 0 \Leftrightarrow u_\mu \nabla^\mu u^\alpha = 0 \tag{6}$$

is fulfilled. The Christoffel symbols $\Gamma^\mu_{\alpha\beta}$ are defined by

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right) \tag{7}$$

and we will soon explain the covariant derivative ∇_λ. The geodesic equation (7) describes how a test particle will move in a given spacetime $g_{\mu\nu}$. The second part says that a geodesic is the curve where the

Geodesic: As straight as possible in curved spacetime

1 *E. g.* in polar coordinates $g_{\mu\nu}$ depends on x, but we have flat space just in curvylinear coordinates.

derivative of its tangent along the curve vanishes, *i. e.* a geodesic is the generalization of a straight in a curved manifold.

It is of importance to understand that in in the formulation of General Relativity Gravity is not a force. Particles always follow the path that is as straight as possible, but since spacetime might be curved, this path might be not a straight line, though no force is acting on the particle.

The Christoffel symbols also appear when one defines the *covariant derivative*, which is *e. g.* for a vector given by

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \,. \tag{8}$$

The covariant derivative is introduced here, because the partial derivative of a tensor does not transform as a tensor. The Christoffel symbols are also no tensors. However the combination of the two terms in equation (8) transforms like a tensor, *i. e.* the non-tensorial parts cancel each other out in the transformation. It is important to note that Γ is a special choice by making the assumptions [30]:

1. $\nabla(T+S) = \nabla T + \nabla S$,
2. $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$,
3. $\nabla_\mu (T^\lambda_{\lambda\rho}) = (\nabla T)_\mu{}^\lambda{}_{\lambda\rho}$,
4. $\nabla_\mu \phi = \partial_\mu \phi$ for scalar ϕ,
5. $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$: no torsion,
6. $\nabla_\rho g_{\mu\nu} = 0$: metric compatibility.

Assumption (5) is a big one, namely that the torsion tensor $T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}$ vanishes. There are theories which incorporate torsion and it is even possible to formulate General Relativity *only* with torsion and without curvature, called *Teleparallel Gravity* [32]. From these assumptions and especially from the equation of metric compatibility one can derive equation (7) for the Christoffel symbols Γ, which are also often called *Levi-Civita connection* (if the torsion vanishes). In the following we will restrict ourself to spacetimes without torsion.

Now we know how particles move when they are subject to a specific spacetime metric. But how do we determine the metric? It turns out one can again use the action principle, but we need to do some work in before. We need to get a measure for curvature, so what are the effects of curvature?

- Loss of parallelism: Two (free) particles with initially parallel momenta will not move on parallel lines. Parallel transporting a vector in a closed loop will not yield the same vector.
- There no longer exist global inertial frames, only locally.

We are going to parallel transport a vector along a closed loop and measure the difference to the initial vector. We infinitesimally transport a vector along A^μ, then along B^ν. Afterwards we transport it back along $-A^\mu$ and $-B^\nu$. The difference in the vector should have the generic form

$$\delta V^\rho = R^\rho{}_{\mu\nu} V^\sigma A^\mu B^\nu \, . \tag{9}$$

The coefficient $R^\rho{}_{\mu\nu}$ in equation (9) is the *Riemann tensor*, which is of big importance in differential geometry. From the above considerations we end up with

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma \tag{10}$$

and evaluating the covariant derivatives leaves us with

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - (\mu \leftrightarrow \nu) \, . \tag{11}$$

The *Riemann Tensor* $R^\rho_{\sigma\mu\nu}$ tells us all about the curvature. From it one can construct the *Ricci Tensor* $R_{\sigma\nu} = R^\rho_{\sigma\rho\nu}$ and the *Ricci Scalar* $R = R^\mu_\mu$.

The Ricci Scalar leads us to the *Einstein-Hilbert action*

$$S_{EH} = \frac{1}{\kappa} \int d^4x \, \sqrt{-g} \, R \, , \tag{12}$$

where g is the determinant of the metric and κ is a constant in order to make the action dimensionless (which will become important when talking about the renormalizability of Gravity). The action equation (12) is not the only possible action, but it is the simplest one. Variation of the action $S = \frac{1}{16\pi G} S_{EH} + S_M$ yields the *Einstein Field Equations* (EFEs)

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \, , \tag{13}$$

with the energy momentum tensor $T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}$. Adding a constant to R gives rise to the *cosmological constant* Λ.

The left hand side of the EFEs describes how spacetime curves due to the presence of matter/energy and the right hand side describes the matter distributions. With the Geodesic Equation (6) and the EFEs (13) we know how matter moves and how spacetime reacts to matter.

2.2 BLACK HOLES AND THE BREAKDOWN OF GR

Shortly after Einstein published his work on General Relativity the German physicist Karl Schwarzschild found a solution that is now called the *Schwarzschild Solution* [77]. It is mysterious how Schwarzschild was able to find this solution, since at that time he was at war doing ballistics calculations. Back then Einstein himself did not believe that an analytic solution existed. It is unclear how Schwarzschild could have had the time and clarity to find a solution to them while at war.

Birkhoff's Theorem tells us that:

> Let the geometry of a given region of spacetime (1) be *spherically symmetric*, and (2) be a solution to the Einstein field equations in *vacuum*. Then that geometry is necessarily a *piece* of the Schwarzschild geometry [61],

where the term *vacuum* has led to much confusion, and will be explained below. The Schwarzschild geometry is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \qquad (14)$$

with the angular part $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$. Outside a spherically symmetric mass distribution the metric is always Schwarzschild, this is what was meant in the quote above.

Schwarzschild is no vacuum solution!

There has been a lot of confusion about the source of the curvature of this solution (14). Vacuum means $T_{\mu\nu} = 0$, but what curves spacetime? One reason for this confusion might be the derivation of the metric. One starts with vacuum ($T_{\mu\nu} = 0$) and calculates a generic spherically symmetric metric. The integration constant M can be identified with the mass by requiring that the weak field limit gives Newton's potential. The problem is that one only has the manifold without one point, namely the singularity. A careful calculation with distributions shows that actually the Schwarzschild geometry is curved by a point source, namely [17, 66]

$$T^0_0(\vec{x}) = -M\delta^{(3)}(\vec{x})\,. \qquad (15)$$

Often people forget about the distributional nature of the mass density and think that the Schwarzschild metric is a solution in vacuum everywhere [31].

Singularities in Schwarzschild

The Schwarzschild Metric (14) has two singularities:

- $r = 2M = r_S$: This is the *Schwarzschild radius*. In Schwarzschild coordinates as in equation (14) there is a singularity, but it can be removed by adopting Kruskal–Szekeres coordinates. It is only a coordinate singularity. Nevertheless something interesting happens here: Once one gets closer than r_S, there is no way back. What falls in, stays in. Not even light can escape, thus the name *black hole*.

- $r = 0$: Here we have a true singularity. A simple check reveals $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6} \to \infty$. This is connected to the fact that there is a delta source at the origin. It is reasonable to believe that Quantum Gravity would do something with this singularity. One could say that here General Relativity announces its own demise. Quantum Gravity considerations, *e. g.* the Generalized Uncertainty Principle (GUP, section 5.3) or Noncommutative Geometry (section 5.4), handle this central singularity by smearing out the delta source, although in the GUP case the singularity is only smoothened and not completely removed.

This singularity might be one of the most urgent problems why a theory of Quantum Gravity is needed.

The important point is that in Black Holes General Relativity and Quantum Mechanics both have to play a role and thus Black Holes are a good candidate for testing Quantum Gravity.

3

QUANTUM FIELD THEORY

In this chapter a small introduction/review of quantum field theory will be given. The quantization will only be performed in the *Feynman Path Integral* formalism and not the canonical or the Schwinger way. A small teaser of *renormalization* will be given in view of the non-renormalizability of General Relativity. Of special interest is the notion of *propagator*, since the spectral dimension is basically a property of the propagator.

3.1 QUANTIZATION OF THE SCALAR FIELD IN THE PATH INTEGRAL FORMALISM

Many books on Quantum Field Theory [20, 72, 76] are using the metric $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$, however here we will stick with the one we used in General Relativity, namely $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ and most of this chapter will be based on [83].

We start with a complex scalar field ϕ. The field $\phi(x) = \phi(t, \vec{x})$ has one complex value at every point x in spacetime. The Lagrangian of the free field theory of a scalar field is

$$\mathcal{L}_0(\phi, \partial_\mu \phi) = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \ , \tag{16}$$

which determines the action $S = \int d^4x \ \mathcal{L}$. Note that S is scale invariant only if $m = 0$. The equation of motion from the Lagrangian (16) is the *Klein-Gordon equation*

$$(\Box + m^2)\phi(x) = 0 \ , \quad \Box = -\eta^{\mu\nu}\partial_m u \partial_\nu \ . \tag{17}$$

We have the *path integral*

$$Z_0[J] \equiv \langle 0|0 \rangle_J = \int \mathcal{D}\phi e^{i \int d^4x [\mathcal{L}_0 + J\phi]} \tag{18}$$

$$= e^{\frac{i}{2} \int d^4x d^4x' J(x) \Delta(x-x') J(x')} \ , \tag{19}$$

where $\mathcal{D}\phi$ means that one integrates over all possible field configurations and $\Delta(x - x')$ is the *Feynman propagator*

$$\Delta(x - x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-x')}}{k^2 + m^2 - i\epsilon} \ , \tag{20}$$

which is a Green's function of the Klein-Gordon equation (16), *i.e.* it fulfills

$$(\Box + m^2)\Delta(x - x') = \delta^{(4)}(x - x') \,. \tag{21}$$

In equation (20) the $i\epsilon$ moves the poles of the integrand such that the Feynman propagator is obtained. Using another pole prescription gives another propagator with different properties.

The Feynman propagator is the two-point function

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \frac{1}{i}\Delta(x_2 - x_1) \,, \tag{22}$$

where T means time ordering. Equation (22) is consistent with the $i\epsilon$ pole prescription. General n-point functions can be generated from the *generating functional* $Z_0[J]$ by

Z[j] generates n-point functions

$$\begin{aligned}\langle 0|T\phi(x_1)\cdots\phi(x_{2n})|0\rangle &= \frac{1}{i}\frac{\delta}{\delta J(x_1)}\cdots\frac{1}{i}\frac{\delta}{\delta J(x_{2n})}Z_0[J]\Big|_{J=0} \\ &= \frac{1}{i^n}\sum_{\text{pairings}}\Delta(x_{i_1} - x_{i_2})\cdots\Delta(x_{i_{2n-1}} - x_{i_{2n}}) \,.\end{aligned} \tag{23}$$

The last equality in equation (23) is also called *Wick's theorem*.

For interacting field theories one adds an interaction term $\mathcal{L}_I$ to the Lagrangian

$$\mathcal{L}(\phi, \partial_\mu\phi) = \mathcal{L}_0 + \mathcal{L}_I \,, \tag{24}$$

for example $\mathcal{L}_I = -\frac{\lambda}{4!}\phi^4(x)$ in the so called ϕ^4-theory. Usually one cannot evaluate the path integral directly, thus a trick is used:

$$Z[J] = \int \mathcal{D}\phi e^{i\int d^4x(\mathcal{L}_0 + \mathcal{L}_I + J\phi)} \tag{25}$$

$$= e^{i\int d^4x\,\mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)}\int \mathcal{D}\phi e^{i\int d^4x(\mathcal{L}_0 + J\phi)} \tag{26}$$

$$= e^{i\int d^4x\,\mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)} Z_0[J] \tag{27}$$

$$= e^{i\int d^4x\,\mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)} e^{\frac{i}{2}\int d^4x d^4x' J(x)\Delta(x-x')J(x')} \,. \tag{28}$$

The trick is to write ϕ in the interaction term as derivative with respect to the source J. Then the exponential with the derivatives can be applied order by order (Taylor expansion) to $Z_0[J]$. Feynman diagrams are nothing else than a nice way to do a double Taylor expansion of equation (28) with pictures. The order of expansion of the first exponential gives the number of vertices and the second exponential gives the number of propagators. This is the idea behind perturbation theory in Quantum Field Theory, which only converges for small coupling constant (λ in this case).

Feynman diagrams are a double Taylor expansion of Z[J]

In order to describe other particles like fermions, photons and gluons other kinds of fields need to be introduced, but the idea stays the same. Here we will not go into detail how to calculate scattering amplitudes, but address one more important point: Renormalization.

3.2 RENORMALIZATION

In calculations one quickly encounters terms involving $\Delta(0)$, *i.e.* a propagator that ends where it started, a loop. A look at the definition of the propagator shows that $\Delta(0)$ diverges in the ultraviolet (large momenta). The first step to handling those infinities is to separate the finite part from the infinite for example in $\Delta(0)$. This is called *regularization*.

REGULARIZATION: In [83] an ultraviolet "cutoff" is implemented by changing the propagator to

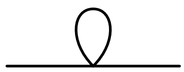
$$\Delta(x-y) \to \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{k^2+m^2-i\epsilon} \left(\frac{\Lambda^2}{k^2+\Lambda^2-i\epsilon}\right)^2 . \tag{29}$$

This extra term goes away in the limit $\Lambda \to \infty$ and for finite Λ it acts as a momentum suppression in the integral. For $\Lambda \gg m$ the loop integral (29) with $x = y$ goes like $\int_0^\infty dk\, k^{-3}\Lambda^4 \sim \Lambda^2$ and

$$\Delta(0) = \frac{i}{16\pi^2}\Lambda^2 . \tag{30}$$

Of course this still diverges if $\Lambda \to \infty$, but we now found a way to extract the infinity and later we will show how to handle it. Interestingly the factor in equation (29) is the square of the factor that naturally arises in the context of the Generalized Uncertainty Principle (GUP), see section 6.4.4.

RENORMALIZATION: Another way of extracting the infinities is *dimensional regularization*. Here we will give a quick sketch of dimensional regularization in ϕ^4-theory and what renormalization is about. In d-dimensional spacetime it turns out that the singularity of the propagator is a simple pole in $d-4$. Introducing an arbitrary mass parameter μ we get [76] for the $\mathcal{O}(g)$-correction to the free propagator



$$= \frac{1}{2} g\mu^{4-d} \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2-m^2} , \tag{31}$$

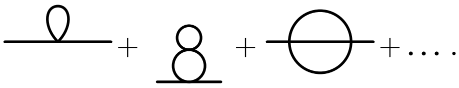
which, introducing $d = 4-\epsilon$, can be shown to be [76]

$$= \frac{igm^2}{16\pi^2\epsilon} + \text{finite} . \tag{32}$$

The (connected) 2-point function $G_c^{(2)}(p)$ is the free 2-point function $G_0(p)$ plus loop corrections. Summing all loop corrections one obtains

$$G_c^{(2)} = \frac{i}{p^2-m^2-\Sigma(p)} , \tag{33}$$

with the *self energy* $\Sigma(p)$ given by the sum of all 1PI (one particle irreducible) graphs

$$\frac{1}{i}\Sigma(p) =$$  (34)

Looking at equation (33) we recognize that the self energy changes the mass to

The difference between the mass m and the renormalized mass m_1

$$m^2 \to m^2 + \Sigma(p)\,. \tag{35}$$

To $\mathcal{O}(g)$ the self energy is given by equation (32). Now we interpret the original mass m to be infinite and of no direct physical significance

$$m^2 = m_1^2 + \frac{m^2 g}{16\pi^2\epsilon} \approx m_1^2\left(1 + \frac{g}{16\pi^2\epsilon}\right)\,, \tag{36}$$

where the second step is correct in order $\mathcal{O}(g)$. The *renormalized mass* m_1 is finite. The same game can be played for the coupling constant and for second order one needs something called "wave function renormalization", but the idea stays the same. Another way of achieving the same thing is by not changing m to m_1, but by introducing counter terms in the Lagrangian with the same effect. If this can be done to all orders then the theory is renormalizable.

RENORMALIZABILITY: Following [83] we are going to use dimensional analysis to give an argument for the non-renormalizability of *General Relativity*. Remember that $\hbar = c = 1$. Counting in dimensions of mass we have $[m] = +1, [x^\mu] = -1, [\partial^\mu] = +1, [d^d x] = -d$. Since $Z[J] = \int \mathcal{D}\phi e^S = \int \mathcal{D}\phi \exp\left[i\int d^d x(\mathcal{L} + J\phi)\right]$ the action S must have dimension zero $[S] = 0$, because it appears in the exponent. Then $S = \int d^d x \mathcal{L}$ leads us to $[\mathcal{L}] = d, [\phi] = \frac{1}{2}(d-2)$. Interaction terms like $g_n\phi^n$ require

$$[g_n] = d - \frac{1}{2}n(d-2)\,. \tag{37}$$

The dependence of a scattering amplitude on the coupling g must be given by a dimensionless parameter $g \cdot m^{-[g]}$ or for high energies ($s \gg m^2$) $g \cdot s^{-[g]/2}$, with the Mandelstam variable s. Here we see, that if $[g] < 0$ the term depends on s to a positive power and diverges. Such theories are non-renormalizable. From equation (37) we also see that in $d = 4$ dimensions theories with $n > 4$ fail. Thus for example ϕ^5 theory is non-renormalizable.

Gravity has the Einstein Hilbert action (12)

$$S = \frac{1}{\kappa}\int d^4x\sqrt{-g}R\,. \tag{38}$$

With $[R] = 2$ we have that also $[\kappa] = 2$ in order to render S dimensionless. Thus the coupling constant $\frac{1}{\kappa}$ has dimension -2 and the theory is power counting non-renormalizable. Note that if spacetime had only $(1+1)$ dimensions, then the coupling constant would be dimensionless and the theory power counting renormalizable. This is the reason why it is so exciting that in many approaches one finds that at the Planck scale (high energies) spacetime seems to be effectively two dimensional. Then Gravity would be renormalizable!

General Relativity is non-renormalizable

Thus there are at least three possibilities of solving the problem of non-renormalizability for Gravity:

1. Quantum Gravity is perturbatively nonrenormalizable, but maybe just perturbation theory breaks down. In perturbation theory one would need an infinite amount of parameters. Weinberg realized that if the theory at high energies only needs a finite amount of finite parameters then everything is fine [87]. This is called *Asymptotic Safety*.

2. A whole new theory of Quantum Gravity is needed and the naive way of quantizing Gravity is doomed.

3. At the Planck scale spacetime becomes two dimensional and Gravity is renormalizable.

This thesis is about the third possibility and the dimensional flow, *i. e.*, how the very dimension can depend on scale.

Part II

THE SHOWCASE

This part is all about what happens when General Relativity and Quantum Theory (QT) meet. An introduction to Quantum Field Theory in curved spacetime and to String Theory will be given. Then the focus will be on the very notion of *dimension*. Not only does String Theory — a candidate theory for combining GR and QT — predict 10 or 11 dimensions, but also there exist good arguments that at the Planck scale spacetime itself will be fluctuating so rapidly that it cannot be described by the methods of GR anymore. We will show arguments that the dimensionality of spacetime in this very rapidly fluctuating regime will change and that the spacetime will become a fractal.

4

COMBINING GR AND QFT

This chapter is about the combination of General Relativity and Quantum Field Theory. String Theory, Loop Quantum Gravity and other proposals basically all try to achieve this goal. In the following sections it will be shown what the features and problems of the combination of GR and QFT are and why maybe a fundamentally different theory (like String Theory, LQG, ...) seems to be needed.

4.1 QFT IN CURVED SPACETIME

Quantum Field Theory in curved spacetime means that fields are quantized on a curved background. The background itself is not quantized, only the fields on the background are quantized and no backreaction to the curvature is considered. For free matter fields interacting with gravity (gravitons) truncating at the one loop level is equivalent to one loop quantum gravity [20], *i. e.*, first order quantum corrections to GR. This is because the one-loop calculation contains all the terms of the "complete theory" up to order $\hbar$.

SCALAR FIELD IN CURVED SPACETIME: QFT in curved spacetime has some striking differences to QFT in Minkowski space. In the following we will develop some of the new features by means of a free scalar field [20].

The Lagrangian density is generalized to

$$\mathcal{L}(x) = \frac{1}{2}\sqrt{-g(x)}\, g(x)\mu\nu(\nabla_\mu\phi(x))(\nabla_\nu\phi(x)) - \frac{1}{2}[m^2 + \xi R(x)]\phi^2(x)\,, \tag{39}$$

with the metric $g(x)^{\mu\nu}$, its determinant $g(x)$, the covariant derivative ∇_μ (which is in the case of the scalar field the same as the partial derivative) and the Ricci scalar $R(x)$. The coupling ξ of the field to the Ricci scalar is a priori unknown. Special choices are $\xi = 0$ (minimal coupling) and $\xi = {}^{(n-2)}/_{4(n-1)}$ (conformal coupling).

The vanishing of the variation of the action $S = \int d^n x \mathcal{L}(x)$ gives the Klein Gordon Equation in curved spacetime ($\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$)

$$\left[\Box_x + m^2 + \xi R(x)\right]\phi(x) = 0\,. \tag{40}$$

The problem or new feature arises when the field ϕ is expanded in modes. Say we expand the field in a complete orthonormal set of mode solutions $u_i(x)$

$$\phi(x) = \sum_i \left[a_i u_i(x) + a_i^\dagger u_i^\star \right], \tag{41}$$

with the usual properties. The standard way now is to identify positive and negative frequency modes. Then the operators a_i and $a_i^\star$ are identified as lowering and raising (annihilation and creation) operators respectively.

In GR one has to identify the positive frequency modes with respect to a Killing vector (∂_t in Minkowski space). We can state that if some timelike Killing vector field ξ exists, then positive frequency modes u_j can be identified by

$$\mathcal{L}_\xi u_j = -i\omega u_j, \quad \omega > 0 \tag{42}$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to the vector ξ.

Generally such a Killing vector will not exist, but even if we have one, new phenomena appear: Because of the *principle of general covariance* physically it does not matter what coordinate system is used. Thus let us use another complete orthonormal set of modes $\bar{u}_j(x)$ and expand the field in them. There exists the possibility of relating the two sets of modes via a Bogolubov transformation (see [20])

$$a_i = \sum_j \left(\alpha_{ji} \bar{a}_j + \beta_{ji}^\star \bar{a}_j^\dagger \right) , \tag{43}$$

$$\bar{a}_j = \sum_i \left(\alpha_{ji}^\star a_i - \beta_{ji}^\star a_i^\dagger \right) . \tag{44}$$

Since in general $\beta_{ji} \neq 0$, the two observers will not even agree on the vacuum

$$a_i \left|\bar{0}\right\rangle = \sum_j \beta_{ji}^\star \left|\bar{1}_j\right\rangle \neq 0 , \tag{45}$$

which is a result of the mixing of creation and annihilation operators in the transformation. The expectation value of the number operator $N_i = a_i^\dagger a_i$ for the $\bar{u}$-vacuum $|\bar{0}\rangle$ is

$$\langle\bar{0}|N_i|\bar{0}\rangle = \sum_j |\beta_{ji}|^2 . \tag{46}$$

Two different observers will not agree on the particle number and not even on the vacuum. Where one observer sees vacuum, the other sees a boiling see of particles.

The notion of particle number is ill defined

This means that the very notion of particles is not a good concept in curved spacetime. One might ask why we still count particle numbers in flat space. The answer is that Minkowski space has as a symmetry group the Poincaré group and inertial observers agree on the vacuum state (and particle number). As long as $\bar{u}_j$ are only a linear combination of u_j and not $u_j^\star$ then $\beta_{ij} = 0$ and both will agree on the vacuum.

Disagreements on the vacuum arise also in flat space. It is sufficient to look at a particle detector on a trajectory with constant acceleration[1] a^μ. If we take as vacuum the Minkowski vacuum then this detector will see a bath of thermal radiation (this is called "Unruh" effect) with temperature [20]

$$T = \frac{a}{2\pi k_B} . \tag{47}$$

From the Unruh effect one quickly gets to the "Hawking temperature" [45] of a Schwarzschild Black Hole. Assume the vacuum is what a free falling observer would see as vacuum. Close to the horizon of the Black Hole a static observer would need to be accelerated with [30]

$$a = \frac{GM}{r\sqrt{r - 2GM}} \tag{48}$$

in order to be static. An observer at infinity will see thermal radiation emitted from the Black Hole that is proportional to a^{-1} and (including the redshift) we arrive at

Black Holes have a temperature

$$T = \frac{\kappa}{2\pi} , \tag{49}$$

with the surface gravity $\kappa = 1/4GM$. Denoting the mass of our sun with $M_\odot$ and plugging in the numbers one gets as a quick rule of thumb [55]

$$T_{BH} = \frac{\hbar c^3}{8\pi k_B GM} \approx 6.17 \cdot 10^{-8} \left(\frac{M_\odot}{M_{BH}} \right) K , \tag{50}$$

which leads to a temperature of $\sim 10^{-9}$K for a Black Hole of 10 solar masses, much less than the Cosmic Microwave Background of about 3K.

GRAVITY IS NOT QUANTIZED: We stress again that in this section Gravity is not quantized. We were solely looking at quantum fields propagating on a curved background. Also this is only valid as long as the backreaction of the fields on the curved background is neglected. Matter does not only respond to gravity, but it also curves spacetime, *i.e.*, it also creates gravity. QFT in curved spacetime neglects the latter one. There are very good reasons to believe that Gravity should be quantized at the fundamental level. General Relativity is in remarkable good agreement with observations, but at least at the Planck scale it breaks down. Gravity is troubled by problems like

1 this is commonly dubbed "Rindler space".

singularities and the information loss problem. Quantum theory and thus a quantization of gravity might be the solution. Other (and more detailed) arguments for why the gravitational field must be quantized are given in [3, 55].

4.2 QUANTIZATION OF GRAVITY: MAIN APPROACHES

Quantum Field Theory in a curved background makes interesting predictions as shown in the previous section. A full Theory of Quantum Gravity however is a much bigger task, also because of the paucity of experimental data. There are many approaches that people are currently investigating and this section is to give an overview.

Claus Kiefer organizes the main approaches as follows [54]:

- *Quantum General Relativity:* One starts from classical GR and applies quantization rules as for example with the electromagnetic field.
  - *Covariant approaches* make use of 4d covariance and include perturbation theory, effective field theories, renormalization-group approaches and path integral methods.
  - *Canonical approaches* apply the Hamiltonian formalism and include Quantum Geometrodynamics and Loop Quantum Gravity.
- *String Theory:* The basic idea is that all particles (and thus also forces) are some excitation of a fundamental object that is a string. Chapter 5 gives a short introduction to String Theory.
- Other approaches like quantization of topology and causal sets.

Here a few examples of those theories will be described in order to give a feeling for the problems of Quantum Gravity and why we still do not posses a full theory.

PERTURBATIVE QUANTUM GRAVITY AND ASYMPTOTIC SAVETY: One starts by splitting the metric into a background part $\bar{g}_{\mu\nu}$ and a small perturbation $h_{\mu\nu}$:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \,. \tag{51}$$

This framework is an effective field theory for Quantum Gravity and for example gravitational waves can be calculated from the classical version. For the quantum theory the field $h_{\mu\nu}$ is quantized as in any other Quantum Field Theory. As mentioned in section 3.2 this theory is non-renormalizable. To every loop order there are more divergences that need to be renormalized and appropriate parameters need to be introduced and fixed by measurements. In the end an infinite amount of parameters would be needed which makes the theory useless [55].

Even if this ansatz gives a non-renormalizable theory, it might give useful results for low energies. A high energy cutoff is used and beyond this one does not expect the theory to hold any more. This framework is called *effective field theory* and it is possible to calculate first order corrections to the Newton potential [21].

Also there is the possibility that Gravity in the end is non-perturbatively renormalizable, *i.e.* a nontrivial UV fixpoint in the renormalization group flow could cure divergencies. Thus the hope is that only perturbative description breaks down for Gravity, but not the Quantum Theory itself.

The other way of quantization is by means of path integrals. Also in this approach gravity makes problems [55]. The path integral

$$Z[g] = \int \mathcal{D}g_{\mu\nu}(x) e^{iS[g_{\mu\nu}(x)]/\hbar}, \tag{52}$$

even in the euclidean formalism $t \to -i\tau$ does not converge, because it is unbounded from below. One attempt of doing the Lorentzian calculation is called *dynamical triangulation*. Here spacetime is divided into small pieces called simplices they are numerically summed via Monte Carlo calculations.

LOOP QUANTUM GRAVITY: A similar framework is Loop Quantum Gravity (LQG) where also Monte Carlo simulations are used heavily. The idea behind LQG is that space itself is granular at very small scales because it is quantized. Space and time are just semiclassical approximations and quantum states are no quantum states on spacetime, but quantum states of spacetime [75]. In this framework space is a network of loops called spin network and its time evolution has the name spin foam. LQG still lacks the limit to GR (semiclassical limit), but it describes a quantized spacetime.

5

STRING THEORY

Quantum mechanics brought
an unexpected fuzziness
into physics because of quantum uncertainty,
the Heisenberg uncertainty principle.
String theory does so again
because a point particle
is replaced by a string,
which is more spread out.

— Edward Witten [35]

In *String Theory* the basic idea is that there are no point particles, but only one kind of "matter", namely strings. Those relativistic, quantum mechanical strings can vibrate and the different vibration modes correspond to different elementary particles, see figure 1. A string endpoint can also meet its other endpoint which gives rise to the distinction of "open strings" and "closed strings". Closed strings always exist in the theory one massless mode of the closed string is identified with the graviton.

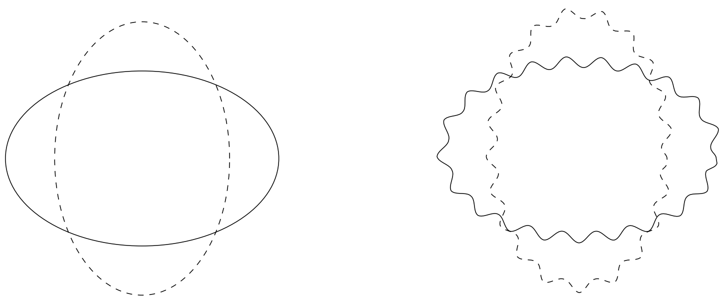


Figure 1: Different excitations of strings give different particles.

In this chapter a short introduction to String Theory will be given. The goal is to show the following points that will be important for the rest of the thesis:

- The necessity of extra dimensions in String Theory,
- Why String Theory introduces a minimal length,

- How the Generalized Uncertainty Principle emerges from the scattering of strings,
- What connects noncommutative geometry to String Theory.

The biggest part of this chapter is based on the String Theory textbook [18].

5.1 BRIEF INTRODUCTION TO STRING THEORY

5.1.1 *The "Why"*

In the introduction to this thesis it has already been shown that there are problems unifying Quantum Mechanics and Gravitation. Why is String Theory such a big research area and why are people so interested in it? There are some convincing general features of String Theory:

THE NATURAL INCORPORATION OF GRAVITY: First when String Theory was introduced to describe the strong nuclear force, people could not get rid of a spin 2 particle that they did not want in the theory. It turns out that this particle is the graviton and String Theory naturally incorporates Gravity. Quantum Field Theory has problems with Gravity, but String Theory even requires it.

A NATURAL UV CUTOFF: Because of the spatial extension of strings, as opposed to point particles, the interactions do no longer happen at a single point, but are also extended. There are no short-distance singularities. This is connected to renormalization, minimal length and the GUP.

5.1.2 *The "How"*

As a starting point an action for the string is needed. Recall that the action for a relativistic particle is basically its world line length. Thus it is very natural to have the *world sheet area* playing the role of the action in String Theory. Let the string be parametrized by τ, σ that are also written as σ_0, σ_1 and $X^\mu(\tau,\sigma)$ give the position in D-dimensional spacetime. This is shown in figure 2. The "Nambu-Goto action" is in a flat background metric

string action = world sheet area

$$S_{NG} = -T\int d\sigma d\tau \sqrt{\left(\dot{X}\cdot X'\right)^2 - \dot{X}^2 X'^2}\,, \tag{53}$$

where $X \cdot Y \equiv X^\mu Y_\mu$ is the scalar product. The dot and prime denote a derivative with respect to τ and σ respectively. This action contains a square root and thus is not well suited for quantization. One can

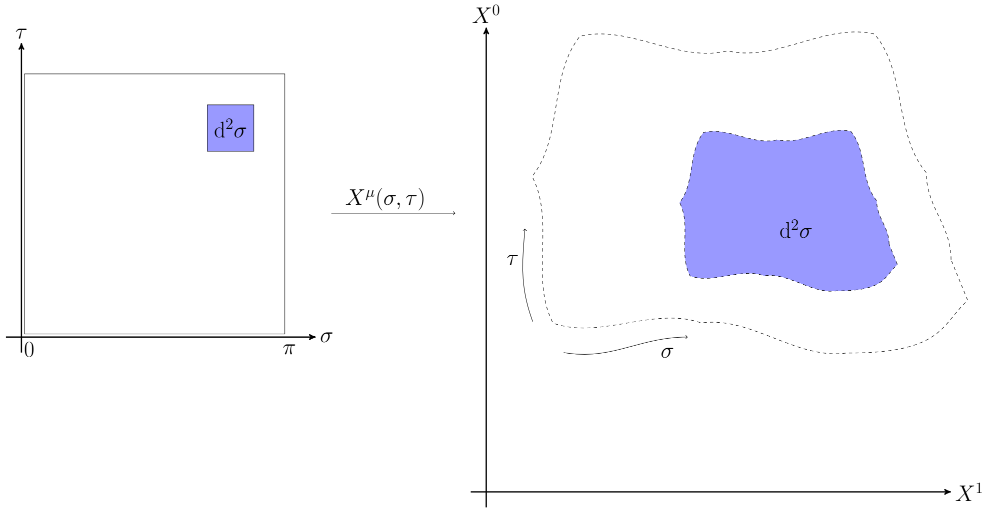


Figure 2: The worldsheet is parametrized by τ and σ.

find a classical equivalent action called the "Polyakov" action. The modern name for it is the "string sigma-model action"

$$S_\sigma = -\frac{T}{2}\int d^2\sigma\sqrt{-h}\,h^{\alpha\beta}\partial_\alpha X\cdot\partial_\beta X\,. \tag{54}$$

The indices α and β take values τ and σ (or equivalently 0 and 1). The metric $h^{\alpha\beta}$ is the world sheet metric and h denotes its determinant. One recovers the Nambu-Goto action by plugging the equations of motion for $h^{\alpha\beta}$ into S_σ. Thus the two actions are classically equivalent.

Since there is no kinetic term for $h_{\alpha\beta}$, the world-sheet energy-momentum tensor $T_{\alpha\beta}$ vanishes

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S_\sigma}{\delta h^{\alpha\beta}} \tag{55}$$

$$= \partial_\alpha X\cdot\partial_\beta X - \frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X\cdot\partial_\delta X = 0\,. \tag{56}$$

It is possible to use the symmetries[1] of S_σ in order to gauge $h_{\alpha\beta} = \eta_{\alpha\beta} = \mathrm{diag}(-1, 1)$.

The equations of motion are

$$\partial_\alpha\partial^\alpha X^\mu = \left(-\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial\sigma^2}\right)X^\mu = 0 \tag{57}$$

Because a gauge has been chosen, $T_{\mu\nu} = 0$ has to be imposed as constraint equations

$$T_{01} = T_{10} = \dot{X}\cdot X' = 0\,,$$
$$T_{00} = T_{11} = \frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0\,. \tag{58}$$

1 Poincaré, Reparametrization and Weyl, here the last two are needed. By fixing $h_{\alpha\beta}$ a residual symmetry, namely reparametrizations that are also Weyl transformations, remains.

When deriving the equations of motion from the action S_σ by variation and $\delta S = 0$ boundary terms appear. Thus suitable boundary conditions have to be chosen. The possibilities are

- *Closed string:* $X^\mu(\sigma,\tau) = X^\mu(\sigma+\pi,\tau)$
- *Open String, Neumann boundary conditions:* $X'_\mu\Big|_{\sigma=0,\pi} = 0$, no momentum is flowing through the endpoints of the string.
- *Open String, Dirichlet boundary conditions:* $\delta X^\mu\Big|_{\sigma=0,\pi} = 0$, meaning that the endpoints of the string are fixed at D-branes.

For an open string one can choose Neumann or Dirichlet boundary conditions for every μ separately. In *light-cone coordinates*

$$\sigma^\pm = \tau \pm \sigma \,, \tag{59}$$

the equations of motion (57) and the constraints (58) are

$$0 = \partial_+\partial_- X^\mu \,, \tag{60}$$
$$0 = T_{++} = \partial_+ X^\mu \partial_+ X_\mu \,, \tag{61}$$
$$0 = T_{--} = \partial_- X^\mu \partial_- X_\mu \,, \tag{62}$$

and $T_{+-} = T_{-+} = 0$ is fulfilled identically.

Equation (57) and equivalently (60) is a wave equation solved by $X^\mu(\sigma,\tau) = X^\mu_R(\tau-\sigma) + X^\mu_L(\tau+\sigma)$. For simplicity we will focus on the *closed string*. Expanding X^μ in modes

$$X^\mu_R = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau-\sigma) + \frac{i}{2}l_s \sum_{n\neq 0} \frac{\alpha^\mu_n}{n} e^{-2in(\tau-\sigma)} \,, \tag{63}$$
$$X^\mu_L = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau+\sigma) + \frac{i}{2}l_s \sum_{n\neq 0} \frac{\tilde\alpha^\mu_n}{n} e^{-2in(\tau+\sigma)} \,, \tag{64}$$

with the center-of-mass position x^μ and the total momentum of the string p^μ. Since X^μ_R and X^μ_L have to be real, $\alpha^\mu_{-n} = (\alpha^\mu_n)^*$ and $\tilde\alpha^\mu_{-n} = (\tilde\alpha^\mu_n)^*$. For historical reasons there are three equivalent constants that are often used: The *string length* l_s, the *string tension* T and the *Regge slope* α'. Note that the prime in α' is just a label and no σ-derivative[2]. The three constants are related by the equations

$$T = \frac{1}{2\pi\alpha'} \quad \text{and} \quad \frac{l_s^2}{2} = \alpha' \,. \tag{65}$$

It turns out that $a^\mu_m = \frac{1}{\sqrt{m}}\alpha^\mu_m$ and $a^{\mu\dagger}_m = \frac{1}{\sqrt{m}}\alpha^\mu_{-m}$ fulfill the same algebra as the raising/lowering operators for a harmonic oscillator

$$[a^\mu_M, a^{\nu\dagger}_n] = [\tilde a^\mu_M, \tilde a^{\nu\dagger}_n] = \eta^{\mu\nu}\delta_{m,n}, \quad m,n > 0 \,. \tag{66}$$

The important difference to the harmonic oscillator is that $[a^0_m, a^{0\dagger}_m] = -1$, leading to negative norm states like

Negative norm states

$$\left\langle 0 \middle| a^0_m a^{0\dagger}_m \middle| 0 \right\rangle = -1\,. \tag{67}$$

States with a negative norm would violate causality and unitarity and are thus a bad sign. Nevertheless it is possible to handle them. In order to get rid of those negative norm states one first needs to introduce the *Virasoro generators*

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n}\cdot\alpha_n \quad \text{and} \quad \tilde{L}_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n\,. \tag{68}$$

One can show that $T_{\mu\nu} = 0$ needs $L_m = 0$ (classically).

For the quantization here only a brief sketch will be given. The details can be found in [18]. From the classical theory we do not know in what order the α^μ_m should be. Because of this we *normal-order* the Virasoro generators

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\alpha_{m-n}\cdot\alpha_n: \quad \text{and} \quad \tilde{L}_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n:\,. \tag{69}$$

Only L_0 is affected by normal-ordering, it gets an additive constant a which represents our ignorance of the order of operators. Also quantum mechanically the constraint $L_m = 0$ becomes

$$L_m\left|\phi\right\rangle = 0 \quad \text{for} \quad m > 0 \quad \text{and}\left\langle\phi\right| L_m = 0 \quad \text{for} \quad m < 0\,, \tag{70}$$

since $L_{-m} = L^\dagger_m$. The *mass-shell* condition is

$$(L_0 - a)\left|\phi\right\rangle = (\tilde{L}_0 - a)\left|\phi\right\rangle = 0\,, \tag{71}$$

leading to the *level-matching condition*

$$(L_0 - \tilde{L}_0)\left|\phi\right\rangle = 0\,, \tag{72}$$

relating left- and right-moving modes. Equations (70) and (71) characterize *physical* states in the quantum theory.

Without proof (see e.g. [41] or [18]) we state that all negative norm states decouple from the physical states and the physical states all have positive norm for the special case $a = 1$ and $D = 26$. This special case in Bosonic String Theory goes by the name "critical". It is the reason why String Theory predicts 26 dimensions. If Supersymmetry is included (Super String Theory) the same reasoning gives $D = 10$.

The spectrum of Bosonic String Theory contains

2 Historically α' was the slope of the squared mass as a function of spin, so in this sense the prime was a derivative.

- a Tachyon with $M^2 < 0$ which is removed when Supersymmetry is included,
- a massless spin 2 particle which can be identified with the graviton[3],
- a photon (open string),
- Dilaton, Kalb-Ramond field, ...

We have seen that Bosonic String Theory predicts 26 dimensions when quantized in the form of *covariant quantization*. Another method is *light-cone quantization* which also gives $a = 1$ and $D = 26$. Here we summarize the two quantization schemes [60, 86].

Different quantization schemes give same result

- *Covariant quantization* is manifest Lorentz invariant. All X^μ are treated as operators with commutator relations and the constraints $T_{\alpha\beta} = 0$ are imposed on the states. This method is analogous to the *Gupta-Bleuler* quantization in Quantum Electro Dynamics and introduces negative norm states which can be removed by the correct choice of a and D.
- *Light-cone quantization* does not have negative norm states, but also is not manifest Lorentz invariant because of the appearance of a *Weyl anomaly*. A gauge is chosen before quantization and the constraints are solved in the classical theory. Thus only the physical degrees of freedom are quantized and no negative norm states appear. In the end Lorentz invariance is recovered by setting $a = 1$ and $D = 26$ which cancels the Weyl anomaly.

5.2 EXTRA SPATIAL DIMENSIONS

As motivated in the above section, String Theory needs extra dimensions (XDs) for being mathematically consistent. The need for XDs in String Theory basically arises because of the relative minus sign of the time coordinate. Thus additional spatial dimensions are considered, because an additional time dimension would only make the situation worse. Timelike extra dimensions have been considered in the literature and even proven to give a well-posed initial value problem [34], but here we will only consider spacelike extra dimensions.

The first theory with XDs was by Nordström in 1914 [68][4] in an attempt of unifying gravity with Special Relativity before Einstein's General Relativity. Kaluza and Klein [52, 56] introduced an extra spatial dimension in order to unify gravity and electromagnetism. This theory has some problems and is no longer considered in its original form, but the idea has survived.

3 Actually massless spin 2 particles are equivalent to General Relativity [86].
4 translated version from 2007 given here.

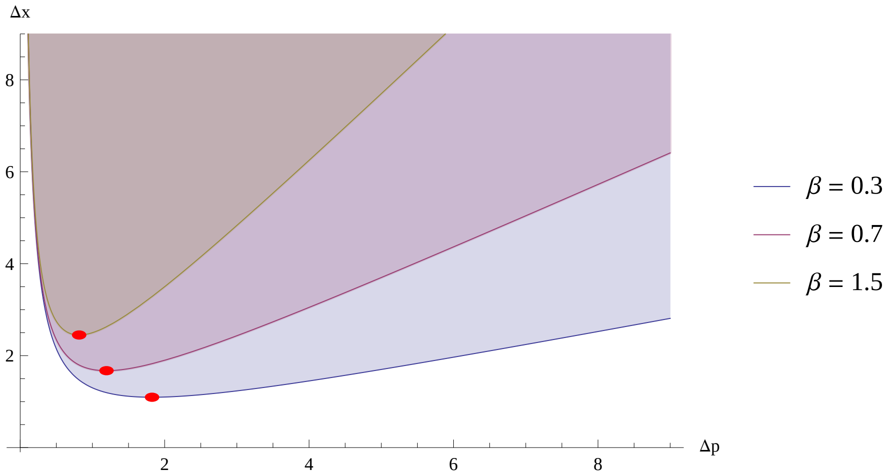


Figure 3: Plot of the Generalized Uncertainty Principle according to equation (73) for different values of β. One can see that there exists a minimal position uncertainty depending on β.

Of course the question arises: If there are more than 3+1 dimensions, why do we not see them? The standard answer is that they are curled up, *i.e.* compact, and their radius is so small that they have not been observed yet. The way of compactification determines many properties of the theory. In contrary to these *universal extra dimensions* are the ADD-extra dimensions [13] that allow particles to only live on a 4D submanifold (brane) and gravity can reach into the whole "bulk". The ADD model is a candidate for solving the weak hierarchy problem[5]. Another model has been proposed by Randall and Sundrum [73] and it is often called *warped extra dimensions*.

5.3 GENERALIZED UNCERTAINTY PRINCIPLE AND MINIMAL LENGTH

String Theory introduces a generalization [6, 57] of the Heisenberg Uncertainty Principle (HUP). This Generalized Uncertainty Principle (GUP) has the form

$$\Delta x \Delta p \geqslant \frac{\hbar}{2}\left(1+\beta(\Delta p)^2\right) \tag{73}$$

and there are many indications that such a modification is necessary in the Quantum Gravity regime.

MINIMAL LENGTH: The basic idea is that equation (73) has a minimum (see figure 3) for Δx, namely $\Delta x_{min} = \sqrt{\beta}$. Thus the minimum of the position uncertainty is nonzero and the theory contains a minimal length. When strings are scattered they do not interact at a single point, since the string has an extension. Thus one can argue that the

5 The weak hierarchy problem is the strange fact that gravity is about 24 orders of magnitude weaker than the electroweak force.

GUP is a way to introduce a minimal length (motivated by Quantum Gravity) into Quantum Mechanics. The idea of a minimal length is not so far-fetched, even Heisenberg [46], Pauli [71], Snyder [80] and Yang [90] have thought about it. A review of minimal length scale scenarios and how in different theories a minimal length emerges is given in [50].

Minimal length in GUP

GEDANKEN EXPERIMENT: The HUP appears in the gedanken experiment of the Heisenberg Microscope. Including the gravitational force a modification in the form of the GUP arises [2]. The modification is due to the energy of the photon. When trying to probe smaller distances, photons of higher energies are used, but eventually their energy becomes so large that they disturb spacetime strong enough to influence the measurement.

Colliding particles with ever higher energies does not increase resolution infinitely

More easily imagine the scattering of two particles. Bigger energy relates to higher spatial resolution[6], *i. e.* smaller Δx. If the energy becomes large enough, eventually a Black Hole will form. Thus no distances smaller than this Black Hole can be probed. The radius of a Schwarzschild Black Hole is $r_S = 2M$ in natural units and since the mass is related to the momentum of the photon we arrive at a gravitational uncertainty $\Delta x_G \sim \Delta p$. The GUP is obtained by adding the usual HUP $\Delta x \sim \frac{1}{\Delta P}$ and the gravitational uncertainty linearly.

STRING THEORY: The first argument for the GUP from String Theory is as above mentioned that strings do not interact at a single point. Another one was developed by Susskind in [84, 85] and summarized in [50]. Looking at the string in light-cone gauge, its transverse extension of the ground state is $(\Delta X_\perp)^2 \approx l_s^2 \log(l_s E)$. In the longitudinal direction one gets

$$(\Delta X_-)^2 \approx \left(\frac{l_s}{P_+}\right)^2 E^2 \,. \tag{74}$$

Again adding the HUP and the above uncertainty[7], considering $E \sim p$ for high energies, we arrive at the GUP. Also from studies that looked at the scattering of strings [5, 42] the same conclusions can be drawn.

OTHER THEORIES OF QUANTUM GRAVITY: A minimal length (and thus also the GUP) arise in other theories of Quantum Gravity, too [50]. For example Loop Quantum Gravity [75] has a minimal area and volume which is a strong indicator for a minimal length.

6 in the sense of the usual HUP.
7 adding the ΔX_--uncertainty, since it grows faster with energy/momentum.

5.4 NONCOMMUTATIVE GEOMETRY

When a physicist talks about *Noncommutative Geometry* (NCG), the implementation of some kind of "fuzziness" of spacetime in the form of noncommuting coordinates

$$[x^i, x^j] = i\theta^{ij} \; , \quad \theta^{ij} = -\theta^{ji} \; , \tag{75}$$

is meant. Although even the founders of Quantum Theory considered noncommuting coordinates in order to cure infinities[8], NCG started being researched again when it was realized that in certain circumstances in string theory the target spacetime coordinates become noncommuting [78, 89]. In NCG a minimal length arises since the matrix θ^{ij} discretizes spacetime analogue to how phase space is discretized by the Planck constant $\hbar$. A large review of NCG and its applications with special attention to Black Holes is [64]. For the purposes of this thesis it is enough to note that the effects of NCG can be implemented by a modification of the momentum measure of the form [64, 79] $\exp\left(-\frac{\theta}{2}\vec{p}^2\right)$ with θ being the average magnitude of the elements of θ^{ij}.

5.5 STRING AND P-BRANE PROPAGATOR

In section 6.4 we will be talking about the spectral dimension which is deeply linked to the notion of a propagator and especially the "heat kernel". In this section an introduction to the closed bosonic string propagator in a loop space representation [10] will be given as well as the generalization to p-branes [12].

STRING PROPAGATOR: Here another way of looking at the dynamics of a string is taken. Other than the usual approach of quantizing the string from the Polyakov action (54), in [10] the space of all possible loop configurations (loop space) is looked at. *I. e.*, we are looking at a non self-intersecting spatial loop C_0 in loop space evolving to a final, also non self-intersecting loop C. Basically the Quantum Mechanics of strings in this loop space is developed in the Hamiltonian formalism. The propagator is derived by a path integral approach and is in agreement to a derivation using a loop space functional wave equation approach (both in [10]).

The resulting closed string propagator in loopspace is given by a integral over an "area-lapse" A of the kernel K

$$G(C, C_0) = \frac{1}{2i\hbar m^2}\int_0^\infty dA \; e^{\frac{-im^2 A}{2\hbar}} K(x(s), x_0(x); A) \; . \tag{76}$$

8 The infinities were handled by renormalization and noncommutativity was no longer needed.

The Kernel K can be computed to be

$$K(x(s), x_0(s); A) = \left(\frac{m^2}{2i\pi\hbar A}\right)^{\frac{3}{2}} e^{\frac{im^2}{4\hbar A}\Sigma^{\mu\nu}(C-C_0)\Sigma_{\mu\nu}(C-C_0)} , \quad (77)$$

where $\Sigma^{\mu\nu}$ is defined as the change of the *oriented surface element* (or loop coordinates) $\sigma^{\mu\nu}$

$$\sigma^{\mu\nu}(C) \equiv \oint_C x^\mu dx^\nu , \quad (78)$$

$$\Sigma^{\mu\nu}(C - C_0) = \sigma^{\mu\nu}(C) - \sigma^{\mu\nu}(C_0) , \quad (79)$$

and $m^2 = 1/2\pi\alpha'$. Actually $K(x(s), x_0(s); A)$ does not depend on s, but is a functional of the functions $x(s)$ and $x_0(s)$. The proper area A of the world sheet takes the role of an evolution parameter. The kernel fulfills the kernel wave equation

$$i\hbar\frac{\partial}{\partial A}K(x(s), x_0(s); A) = -\frac{\hbar^2}{2m^2}\left(\int_0^1 ds\sqrt{x'^2}\right)^{-1}\int_0^1 \frac{ds}{\sqrt{x'^2}}\frac{\delta^2}{\delta x^\mu(s)\delta x_\mu(s)}K\left(x(s), x_0(s); A\right) , \quad (80)$$

and has the initial value

$$K(x(s), x_0(s); A = 0) = \delta(x - x_0) , \quad (81)$$

where the right hand side is a functional delta, *i. e.* it is is only nonzero when the two functions are the same. With equations (76, 80, 81) we see that the propagator fulfills

$$\left[-\hbar^2\left(\int_0^1 ds\sqrt{x'^2}\right)^{-1}\int_0^1 \frac{ds}{\sqrt{x'^2}}\frac{\delta^2}{\delta x^\mu(s)\delta x_\mu(s)} + m^4\right]G(C, C_0) = -\delta(C - C_0) . \quad (82)$$

The $\delta(C - C_0)$ is the same as in equation (81), only $x(s)$ has been given the name C. Equation (82) is a Klein Gordon type equation integrated over the whole string. It is noted [10] that the propagator (76) with the kernel (77) are exact solutions and there has been no approximation in the derivation.

P-BRANE PROPAGATOR: Here we follow [12, 15] and also adopt $\hbar = 1$ as opposed to the previous part on the string propagator. In order to generalize the string propagator (76) to p-branes one makes two approximations [12, 15]:

- The *minisuperspace* approximation from Quantum Cosmology allows the separation of the center of mass dynamics from the deformations of the p-brane as we will see in the resulting propagator (83).

- The *quenching* is an approximation from lattice QCD where the weak quark flavours are ignored. Basically one looks at the system restricted to a large box and quantizes only long wavelengths, leaving short wavelengths "frozen".

The resulting propagator for a p-brane in the quenched minisuperspace approximation

Quenched minisuperspace p-brane propagator

$$G_p(x-x_0,\sigma) = \frac{i}{2M_0}\int_0^\infty ds\, e^{is\frac{M_0}{2}(p+1)} K_{cm}(x-x_0;s)K_p(\sigma;s) \quad (83)$$

is not so different from the one corresponding to a string (76). The "effective point particle mass" M_0 is defined as

$$M_0 \equiv m_{p+1}V_p\ , \quad (84)$$

with the *brane tension* m_{p+1} and the *proper volume* V_p of the p-dimensional boundary of the brane world manifold. The p-brane propagator (83) contains two kernels: The *center of mass kernel*

$$K_{cm}(x-x_0;s) = \left(\frac{\pi M_0}{is}\right)^{\frac{D}{2}} e^{i\frac{M_0(x-x_0)^2}{2s}}\ , \quad (85)$$

describing the center of mass of the brane like a point particle with mass M_0, and the *volume propagator*

$$K_p(\sigma;s) = \left(\frac{M_0}{i\pi V_p^2 s}\right)^{\frac{1}{2}\binom{D}{p+1}} e^{\frac{i}{(p+1)!}\frac{M_0}{2sV_p^2}\sigma^{\mu_1\cdots\mu_{p+1}}\sigma_{\mu_1\cdots\mu_{p+1}}}. \quad (86)$$

Here $\binom{D}{p+1}$ denotes the binomial coefficient. The volume propagator in the form (86) describes the transition between vacuum (zero volume state) and a state with one brane of proper volume V_p. In the quenching approximation the shape of the p-branes is ignored and only the volume distinguishes two branes.

The *volume multivector* $\sigma^{\mu_1\cdots\mu_{p+1}}$ is the generalization of the loop coordinates $\sigma^{\mu\nu}$ defined in equation (79). Given the coordinates of the world manifold boundary $y^\mu(u^1,\cdots,u^p)$ the volume multivector is

$$\sigma^{\mu_1\cdots\mu_{p+1}} \equiv \int y^{\mu_1}dy^{\mu_1}\wedge\cdots\wedge dy^{\mu_{p+1}},\quad p\geqslant 1\,. \quad (87)$$

It represents the target spacetime volume that is enclosed by the p-brane. In the following we will be using the shorthand notation $\sigma^{\mu_1\cdots\mu_{p+1}}\sigma_{\mu_1\ldots\mu_{p+1}} \equiv \sigma\cdot\sigma$.

Note the limit $K_{p=0}(\sigma;s) = \delta(0)$ with a proper definition of the delta function for $\sigma^{\mu_1\cdots\mu_{p+1}}$ and thus the propagator reduces to the point particle propagator in the case $p = 0$[9] except for a diverging factor $\delta(0)$.

9 One takes $\sigma = 0$, because a point has no volume. This is in contrast to the definition of σ (87) where $p\geqslant 1$ is needed.

Setting $p = 1$ in equation (83) yields

$$G_{p=1} = \frac{i}{2M_0} \int_0^\infty ds\, e^{isM_0} K_{cm}(x - x_0; s) \left(\frac{M_0}{i\pi^2 V_1^2 s} \right)^{\frac{1}{2}\binom{D}{2}} e^{\frac{i}{2} \frac{M_0}{sV_1^2} \sigma \cdot \sigma}, \tag{88}$$

where V_1 is the string proper length. We see that this is not exactly the propagator as calculated before for the string equation 76, but the kernel now includes K_{cm} and also $m^2/2$ is switched with M_0.

In [15] the authors play the game even further. If at the Planck scale the very notion of *dimension* breaks and becomes scale dependent, then also the dimension of a p-brane (*i.e.* the p) should become variable. In [15] a new object dubbed "Planckion" with a *polydimensional propagator* is introduced. It is a quantum superposition of p-branes with different dimensions. Since this Planckion ansatz has not gained any traction in the research here we will also not follow it any further.

6

FRACTAL DIMENSIONS

The fractional dimension of an object
is not unlike the artificial description
of the average family in Canada,
which for population description purposes
can be regarded as having 2.2 children.

—Brian H. Kaye [53]

6.1 MOTIVATION

From experiments we know that General Relativity describes our universe very good at large distances. It is only at short distances when General Relativity meets Quantum Mechanics that we run into troubles. General Relativity describes Gravity as the curvature of spacetime. Thus quantizing Gravity means that spacetime itself will get quantum properties. We can expect the smooth geometry to change to some sort of *quantum geometry*. This quantum geometry should also have some connection to a *minimal length scale*.

In the past years many papers have been written about how the dimension of spacetime could change with the probed scale. This change of dimension is often titled as "dimensional flow" or "Spontaneous Dimensional Reduction" [28]. It is an interesting observation that in many theories of Quantum Gravity the *spectral dimension*[1] for short distances goes to 2 [29]. This is not only a remarkable that those very different approaches lead to the same spectral dimension in the UV, but also because in two dimensions Gravity is renormalizable.

For the dimension to flow from 4 or more to 2 it obviously needs to take on fractal or non-integer values. This leads to the question on how to perform differentiation, integration and differential geometry on spaces with fractal dimension. Interestingly this can be done and a theory of this has been developed that is called "multi-fractional spacetime [22].

1 The spectral dimension will be explained later, for now take it as the dimension of spacetime.

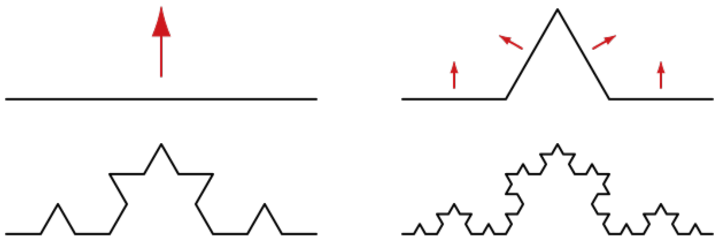


Figure 4: The Koch curve has fractal (Hausdorff) dimension $D_H = \ln(4)/\ln(3)$. Picture from [88].

This chapter will give an introduction to the various definitions of dimension that have been used to describe the dimensional flow. For this purpose we will also bring up the notion of a fractal.

6.2 WHAT IS A DIMENSION

When talking about the dimensional flow and quantum gravity one should first think about what the notion of a dimension really means. New definitions of dimensions first became popular when the theory of fractals was developed. Here we will follow Mandelbrot [59].

In many cases the other definitions of dimension will be the same as the *topological dimension* of the space, but often they are not. An example is be the *Koch curve*[2] with topological dimension $D_T = 1$ (it is a curve) and Hausdorff dimension $D_H = \log(4)/\log(3) \approx 1.26 > 1$ which will be derived in the following section.

One further thing to consider is how the different notions of dimension came up. The Hausdorff dimension has been developed by a mathematician, whereas the spectral dimension and thermal dimension have been introduced by physicists with some physical system in mind. Not every definition is applicable to every situation and they also need not coincide. The spectral dimension for example depends on the kind of particle used to "probe" spacetime.

6.3 HAUSDORFF DIMENSION AND FRACTALS

The probably most famous definition of dimension is the so called "Hausdorff (-Besicovitch) dimension" that was introduced by the German mathematician Felix Hausdorff (1868-1942) [44] and refined by Besicovitch [19]. We will introduce the Hausdorff dimension by the example of the *Koch curve* in figure 4. The Koch curve is constructed step by step. In each step each "segment" is split equally into three segments and the middle one becomes a "spike" consisting of 2 equal line segments of the same length as the outer two. Thus in each step

2 See next section.

the length of the curve gets multiplied by 4/3. The Koch curve is the limit for infinitely many steps and it is a fractal. The curve is everywhere continuous, but nowhere differentiable. Obviously the length of the Koch curve is infinite, which turns out to be a general feature of fractals.

When looking at the curve with only a finite resolution ϵ, the length is finite, since all "spikes" that are smaller than the resolution are not seen. If we note the length for a given resolution ϵ as $\mathcal{L}(\epsilon)$, we have that

$$\lim_{\epsilon \to 0} \mathcal{L}(\epsilon) = \infty \,. \tag{89}$$

It is possible to identify fractals with being *scale divergent*[3] [70]. In order to motivate the Hausdorff dimension let us consider a straight line and a square. If we divide the line in N equal lines, their length will be $r(N) = (1/N)^{1/D}, D = 1$. If we divide a square into 4 equal squares then the length of one side of one square will be $r(4) = (1/4)^{1/D}, D = 2$ and generally $r(N) = (1/N)^{1/D}$ or $D = -\frac{\ln(N)}{\ln(r)}$. Thus for "ordinary" objects their dimension D says something about their scaling behaviour. We call r the scale factor and N the number of segments.

Back to the Koch curve: In each step the length of every segment becomes 1/3 of the length of each segment before, but we have 4 times the number of segments. Thus $r = 1/3, N = 4$ and the *Hausdorff dimension*[4]

$$D_H = -\frac{\ln(4)}{\ln(1/3)} = \frac{\ln(4)}{\ln(3)} \approx 1.26 > 1 \,. \tag{90}$$

PATH OF A QM PARTICLE: Interestingly Hausdorff dimensions greater than the topological dimension are nothing as exotic as one might think. In deed, even in 1980 Abbott and Wise [1] have shown that the *path of a quantum mechanical particle* has Hausdorff dimension two and thus is a fractal. Let us retrace the calculation in [1]. We start with a particle at rest ($\vec{v} = 0$) at the origin with wave function

$$\psi_{\Delta x}(\vec{x}) = \frac{(\Delta x)^{\frac{3}{2}}}{\hbar^3} \int \frac{d^3 p}{(2\pi)^{\frac{3}{2}}} \tilde{\psi}\left(\frac{|\vec{p}|\Delta x}{\hbar}\right) e^{i\frac{\vec{p}\cdot\vec{x}}{\hbar}} \,, \tag{91}$$

where $\tilde{\psi}$ needs to have a form that localizes the particle around the origin with width Δx. We will just assume a Gaussian $\tilde{\psi}(|\vec{k}|) =$

3 This definition by Nottale is more restrictive than Mandelbrot's definition. We will however not discuss about definitions of fractals. For our purposes it is enough to define fractals as having a fractal dimension other than the topological one.

4 Hausdorff developed a mathematically more rigorous procedure with the Hausdorff measure and by covering the curve with spheres. We won't go into detail here. Our definition gives in the limit $\epsilon \to 0$ the same result in most cases.

$\left(\frac{2}{\pi}\right)^{\frac{3}{4}} e^{-k^2}$, where we switched to $\vec{k} = \frac{\Delta x}{\hbar}\vec{p}$. Next we want the average distance $\langle \Delta \ell \rangle$ that the particle moves per time Δt, thus we time evolve $\tilde{\psi}$ with $e^{iH\frac{\Delta t}{\hbar}} = e^{-i\vec{k}^2 \frac{\hbar \Delta t}{2m(\Delta x)^2}}$,

$$\psi_{\Delta x}(\vec{x}, \Delta t) = \frac{(\Delta x)^{\frac{3}{2}}}{\hbar^3} \int \frac{d^3 p}{(2\pi)^{\frac{3}{2}}} \tilde{\psi}\left(|\vec{k}|\right) e^{i\frac{\vec{k}\cdot\vec{x}}{\Delta x}} e^{-i\vec{k}^2 \frac{\hbar \Delta t}{2m(\Delta x)^2}} . \tag{92}$$

Then

$$\langle \Delta \ell \rangle = \int d^3\vec{x}\, |\vec{x}|\, |\psi_{\Delta x}(\vec{x}, \Delta t)|^2 . \tag{93}$$

Performing the integrals we get

$$\langle \Delta \ell \rangle \sim \frac{\hbar \Delta t}{m \Delta x} \sqrt{1 + \left(\frac{2m(\Delta x)^2}{\hbar \Delta t}\right)^2} \quad \xrightarrow{\text{small}\,\Delta x} \quad \hbar \frac{\Delta t}{m \Delta x} . \tag{94}$$

Now that we know the average distance traveled per time step Δt, we can measure the particle N-times with Δt between each measurement and get for the length of the path

$$\langle \ell \rangle = N \langle \Delta \ell \rangle \sim N \Delta t \frac{\hbar}{m \Delta x} = T \frac{\hbar}{m \Delta x} . \tag{95}$$

The length diverges for an infinite resolution $\Delta x \to 0$ and thus the path is a fractal. Also the interpretation of the divergence is natural: Since Δx is the width of the wavepackage it is linked to the momentum uncertainty of the particle by $\Delta x \Delta p \approx 1$. For vanishing Δx the momentum uncertainty diverges and the average distance the particle moves in a little time step Δt diverges.

The Hausdorff length is

$$\langle L \rangle = \langle \ell \rangle (\Delta x)^{D_H - 1} \sim (\Delta x)^{D_H - 2} , \tag{96}$$

and in order for it to be independent of Δx the Hausdorff dimension the path of a quantum mechanical particle has to be $D_H = 2$.

Path of a qm. particle has Hausdorff dimension 2

From here on we will differ from the original paper [1] and give another derivation of $D_H = 2$ closer to the definition of the Hausdorff dimension for the Koch curve, see equation (90). Using the uncertainty relation

$$\Delta t \Delta E = \hbar \,, \quad \Delta x \Delta p = \hbar \,, \tag{97}$$

$$\Longrightarrow \Delta t = \frac{2m(\Delta x)^2}{\hbar} . \tag{98}$$

The reasoning is like follows: We perform $N = \frac{T}{\Delta t}$ measurements with resolution Δx and measurement time T. For the next run we use a better resolution Δx and leave the time T the same, thus resulting in $N' = \frac{T}{\Delta t'}$ measurements, with Δt and Δx linked by equation (98). The Hausdorff dimension can be obtained by analogy to the Koch curve[5].

5 In the QM case the constant factors are left out, since only the change of N and $\langle \Delta \ell \rangle$ from one resolution to another is important and thus constant factors cancel out.

Koch-curve, s is the number of "resolution steps":	QM particle, no "resolution steps, since the resolution can be varied continuously:
$N = 4^s$ $\Delta\ell = \left(\frac{1}{3}\right)^s$ $\ell = N\Delta\ell = \left(\frac{4}{3}\right)^s = (4)^s\left(\frac{1}{3}\right)^s$ $D_H = \frac{\ln 4}{\ln 3}$	$N = \frac{T}{\Delta t} \sim \frac{1}{(\Delta x)^2}$ $\langle\Delta\ell\rangle \sim \hbar\frac{\Delta t}{m\Delta x} \sim \Delta x$ $\langle\ell\rangle = N\langle\Delta\ell\rangle \sim \frac{1}{(\Delta x)^2}\Delta x$ $D_H = \frac{\ln\frac{1}{(\Delta x)^2}}{\ln\frac{1}{(\Delta x)}} = 2$
N: Number of line segments $\Delta\ell$: Length of each segment	N: Number of measurements $\langle\Delta\ell\rangle$: Length of each segment

Abbott [1] also calculated the Hausdorff dimension for nonzero average momentum with results

$$D_H = \begin{cases} 1 & \text{for } \Delta x \gg \frac{\hbar}{|\vec{p}_{av}|}, \text{ classical,} \\ 2 & \text{for } \Delta x \ll \frac{\hbar}{|\vec{p}_{av}|}, \text{ quantum mechanical.} \end{cases} \quad . \tag{99}$$

Between those limits there has to be a transition between QM and classical.

HAUSDORFF DIMENSION IN A QUANTUM MANIFOLD: The Hausdorff dimension of the path has also been studied for a quantum particle with noncommutative geometry. This means that again the momentum measure has been modified with $\exp(-\frac{\ell^2}{\hbar^2}p^2)$. Following the same steps as before the length turns out to be

$$\langle l\rangle \sim \frac{\hbar T}{m\Delta x}\left(1+\frac{\ell^2}{(\Delta x)^2}\right)^{-\frac{d+1}{2}}\sqrt{1+\left(1+\frac{\ell^2}{(\Delta x)^2}\right)^2\frac{4m^2(\Delta x)^4}{\hbar^2(\Delta t)^2}}\ , \tag{100}$$

where d is the number of space dimensions. The general form of the Hausdorff dimension can be obtained by choosing D_H such that the Hausdorff length

$$\langle L_H\rangle = \langle l\rangle\,(\Delta x)^{D_H-1} \tag{101}$$

is independent of Δx. Then

$$d_H = 2 - \frac{d+1}{1+\frac{(\Delta x)^2}{\ell^2}} \tag{102}$$

flows from $d_H = 2$ for $\ell \ll \Delta x \ll \sqrt{\hbar\Delta t/m}$ to $d_H = 1-d$ for $\Delta x \ll \ell$. Thus in the UV limit the Hausdorff dimension is $d_H \leqslant 0$

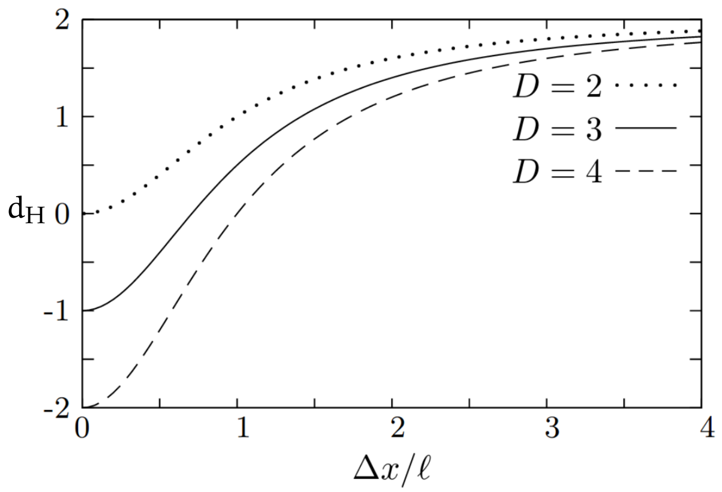


Figure 5: Plot of the Hausdorff dimension for NCG as in equation (102). D is the number of space dimensions. Plot from [65].

which corresponds to an empty set and can be interpreted as a complete "dissolution" of the path for scales smaller than the minimal length [65]. A plot of the Hausdorff dimension is figure 5. Also it has been shown [11] that the Hausdorff dimension of the world sheet of a string or more generally a p-brane changes at length scales close to α' to $d_H = p + 2$.

6.4 SPECTRAL DIMENSION

The spectral dimension is a more physically motivated definition of dimension, namely by diffusion. The diffusion is not in Lorentzian spacetime, but in Euclidean spacetime with diffusion time variable s.

The first part of this chapter will be an introduction to the spectral dimension following [25] and in the second part an overview of the spectral dimension of different approaches to Quantum Gravity will be given. The third part will be calculations that have not been done in the literature about the spectral dimension measured by p-branes followed by other proposed notions of dimension.

6.4.1 *Introduction to the Spectral Dimension*

Starting from a Lagrangian $\mathcal{L}$ the propagator G is the inverse of the operator in the kinetic term[6] $\mathcal{L} \supset -\phi\mathcal{K}(-\Box)\phi$. The poles of this operator, *i. e.* $\mathcal{K}(k^2) = 0$, give the *dispersion relation*. We wrote a general expression with $\mathcal{K}$ here, since in the theories we are going to investigate the usual operator $\Box$ will be deformed. The deformed operator

6 The symbol $\supset$ in this context means that we do not write all summands of $\mathcal{L}$, but only the one of current interest. This notation is rather sloppy but handy, since $A \supset B$ usually means B is a subset of A.

will generally contain higher order derivatives and be a nonlocal operator $\mathcal{K}$ that is called *form factor* [25].

In the following only flat spacetime is used, because we are interested in "quantum features" of spacetime. Considering curved spacetime there would be contributions to the spectral dimension due to curvature. The spectral dimension of a curved spacetime will be discussed in section 6.4.3.

The *Schwinger representation* of the (Fourier transformed) propagator in momentum space for a massless scalar field in D spacetime dimensions is

$$\tilde{G}(k^2) = -\frac{1}{\mathcal{K}(k^2)} = -\int_0^\infty \mathrm{d}s\, e^{-s\mathcal{K}(k^2)}, \tag{103}$$

where s has dimensions of length^2. Fourier transforming to position space gives

$$\tilde{G}(x,x') = -\int_0^\infty \mathrm{d}s \int \frac{\mathrm{d}^D k}{(2\pi)^D} e^{ik\cdot(x-x')} e^{-s\mathcal{K}(k^2)} = -\int_0^\infty \mathrm{d}s\, K(x,x';s), \tag{104}$$

where in the last step we introduced the *heat kernel* $K(x,x';s)$ as

$$K(x,x';s) = \int \frac{\mathrm{d}^D k}{(2\pi)^D} e^{ik\cdot(x-x')} e^{-s\mathcal{K}(k^2)}. \tag{105}$$

The heat kernel $K(x,x';s)$ got its name because it is a solution to the *diffusion equation*[7]

$$[\partial_s + \mathcal{K}(-\Box_x)]\, K(x,x';s) = 0, \quad K(x,x';0) = \delta^{(D)}(x-x'), \tag{106}$$

as can be checked by plugging equation (105) into (106). The initial condition $K(x,x';0) = \delta^{(D)}(x-x')$ comes from the fact that G is a Green function

$$\begin{aligned} 0 &= \int \mathrm{d}s\ [\partial_s + \mathcal{K}(-\Box_x)]\, K(x,x';s) \\ &= K(x,x';\infty) - K(x,x';0) + \underbrace{\mathcal{K}(-\Box_x)\tilde{G}(x,x')}_{\delta^{(D)}(x-x')}. \end{aligned} \tag{107}$$

Assuming that $K(x,x';\infty) = 0$ we see the consistency with equation (106). The heat kernel $K(x,x';s)$ represents the probability density of diffusing from x to x'. Consequently the return probability $P(s)$ is the trace of the heat kernel over spacetime normalized over the whole volume

$$P(s) = \frac{1}{\int \mathrm{d}^D x} \int \mathrm{d}^D x\, K(x,x;s) = K(x,x;s) = \int \frac{\mathrm{d}^D k}{(2\pi)^D} e^{-s\mathcal{K}(k^2)}. \tag{108}$$

7 The spreading of heat is a form of diffusion.

In curved spacetime there would be factors of $\sqrt{-g}$ and $P(s)$ has a different form, see section 6.4.3. From the return probability $P(s)$ the spectral dimension of spacetime measured by a diffusion with form factor $\mathcal{K}$ is defined as

$$d_S(s) = -2\frac{\partial \ln P(s)}{\partial \ln s}, \tag{109}$$

in order to reproduce $d_S = D$ for a standard massless scalar particle, as will be shown now:

Spectral dimension is defined through diffusion in Euclidean spacetime

For a massless scalar particle the dispersion relation is $\mathcal{K}(k^2) = k^2 = 0$. This gives the diffusion equation

$$[\partial_s - \Box_x] K(x,x';s) = 0, \quad K(x,x';0) = \delta^{(D)}(x-x'), \tag{110}$$

with solution

$$K(x,x';s) = \frac{e^{-\frac{(x-x')^2}{4s}}}{(4\pi s)^{\frac{D}{2}}}. \tag{111}$$

The return probability $P(s) = K(x,x;s)$ plugged into equation (109) yields the spectral dimension $d_S = D$ as required. Basically the definition of the spectral dimension (109) extracts the scaling with s from $P(s)$.

MODIFICATIONS OF THE DIFFUSION EQUATION: In many approaches to Quantum Gravity the diffusion equation $[\partial_s - \Box_x] K(x,x';s) = 0$ gets modified in various ways:
Modification of

- the Laplacian $\Box$, *i.e.* a nonstandard form factor $\mathcal{K}$,
- the diffusion operator ∂_s,
- the initial condition (source) $K(x,x';0) = \delta^{(D)}(x-x')$ of the diffusion.

MASSIVE PARTICLES: It turns out that probing spacetime and getting a meaningful definition of a spectral dimension is only possible with massless test particles [67]. Including mass would multiply the heat kernel $K(x,x';s)$ by e^{-ms^2} leading to a term of $2sm^2 \subset d_S$ in the spectral dimension. This diverges for large s, where the limit should be 4 (or D). Fractals posses the property of being self-similar. They are also scale-invariant, since there is always a smaller piece of the fractal which is similar to the whole fractal. Mass breaks scale invariance and thus massive particles fail to probe the spectral dimension of spacetime. In [67] massive scale invariant particles named “un-particles” are used to calculate the “un-spectral dimension”. We will return to this point later.

6.4.2 *Review of Examples in the Literature*

Mathematically the spectral dimension d_S is defined only in the limit $s \to 0$, but it has turned out to be a good idea to not take the limit and have a *running spectral dimension* $d_S(s)$. In this chapter we will review the spectral dimension for different theories in the literature, including Causal Dynamical Triangulations, Asymptotically Safe Gravity and Loop Quantum Gravity.

New calculations will be then presented for the Generalized Uncertainty Principle and for p-branes. In the case of the standard massless scalar particle the s-dependence drops out in the spectral dimension, so here one still has $d_S(s) = d_S = D$. An interesting observation is, that the majority of theories predict a spectral dimension of $d_S = 2$ in the UV regime, even though the theories are following vastly different approaches.

CAUSAL DYNAMICAL TRIANGULATIONS (CDT): One could argue that the first time people recognized the spectral dimension to be a useful tool was when the paper [8] appeared in 2005. In this paper the spectral dimension $d_S(s)$ was "measured" by Monte Carlo simulations in the framework of *Causal Dynamical Triangulations*. Here only the resulting spectral dimension will be given, an introduction to CDT is e.g. [7].

The numerical results [8] for the spectral dimension fit best with

$$d_S(s) = 4.02 - \frac{119}{54 + s}\,, \tag{112}$$

flowing from $d_S(s \to \infty) = 4.02 \pm 0.1$ to $d_S(s = 0) = 1.80 \pm 0.25$. This result has basically ignited the research in the spectral dimension. People started calculating d_S for their favourite theory of Quantum Gravity and surprisingly the results are very often consistent with each other.

ASYMPTOTICALLY SAFE GRAVITY: Shortly after the CDT paper, the spectral dimension was calculated in the theory of Asymptotically Save Gravity [58]. Asymptotic Safety or "nonperturbative renormalizability" tackles the problem stated in section 3.2, namely that by counting dimensions (coupling constant) one concludes that Gravity is non-renormalizable. This is however based on perturbative methods. The basic idea behind Asymptotic Safety is that there might exist a nontrivial fixed point in the renormalization group flow which leads to finite quantities.

The spectral dimension in this framework has been calculated in [58]. Here no exact form for $d_S(s)$ is given, only the limits

$$d_S(s \to \infty) = 4\,, \tag{113}$$

$$d_S(s = 0) = 2\,, \tag{114}$$

which fits with what other theories predict and also what one would want for gravity to be (power counting) renormalizable in the UV.

LOOP QUANTUM GRAVITY (LQG): The main idea behind LQG is that space itself is granular at very small scales. Thus looking at the spectral dimension is a good idea, since the whole purpose of the spectral dimension is to see effects of a quantized spacetime on the dimension.

In LQG the spectral dimension flows in the context of spin foam models from 4 in the IR to 2 in the UV [62]. There have been other studies [26, 74] in LQG with similar results.

It is reassuring that in LQG which quantizes spacetime directly, one gets about the same results as in other theories, where only effects of spacetime quantization are built into QFT without actually quantizing spacetime.

QUANTUM GRAVITY AT A LIFSHITZ POINT: Hořava–Lifshitz gravity [48] is an attempt at a UV-completion of gravity by treating space and time not equivalent in the UV (this is dubbed "anisotropic") with Lorentz-Invariance being recovered in the IR. The spectral dimension

$$d_S = 1 + \frac{D-1}{z} \tag{115}$$

is obtained [49]. In the theory $z = 3$ in the UV and $z = 1$ in the IR, thus Hořava–Lifshitz gravity obtains the same limits $d_S(s \to \infty) = 4, d_S(s = 0) = 2$ as the other theories, even though it has a very different approach.

MINIMAL LENGTH SCENARIOS: In order to describe the irregular path of the diffusion on a quantum spacetime, an minimal length is implemented [63]. This minimal length ℓ is averaging the quantum fluctuations of spacetime. In this approach the condition $K(x, x'; 0) = \delta^{(D)}(x - x')$ is changed to $K(x, x'; 0) = \rho_\ell(x, x')$, a Gaussian with width ℓ (the minimal length). The motivation behind this smoothing out of the initial condition of the diffusion is: The introduction of a minimal length ℓ forbids a localization to a delta distribution and thus the closest possible localization is given by a Gaussian. This is consistent with the theory of *noncommutative geometry*, where one can calculate that the delta distribution is smeared out to a Gaussian.

The resulting spectral dimension is

$$d_S = \frac{s}{s + \ell^2} D\,, \tag{116}$$

running (for $D = 4$) from $d_S(s \to \infty) = 4$ to $d_S(s = \ell^2) = 2$. Smaller scales than $s = \ell$ give even smaller d_S and finally $d_S(s = 0) = 0$. It is argued that in the regime $s < \ell^2$ the spacetime completely dissolves and the very notion of spacetime becomes ill defined.

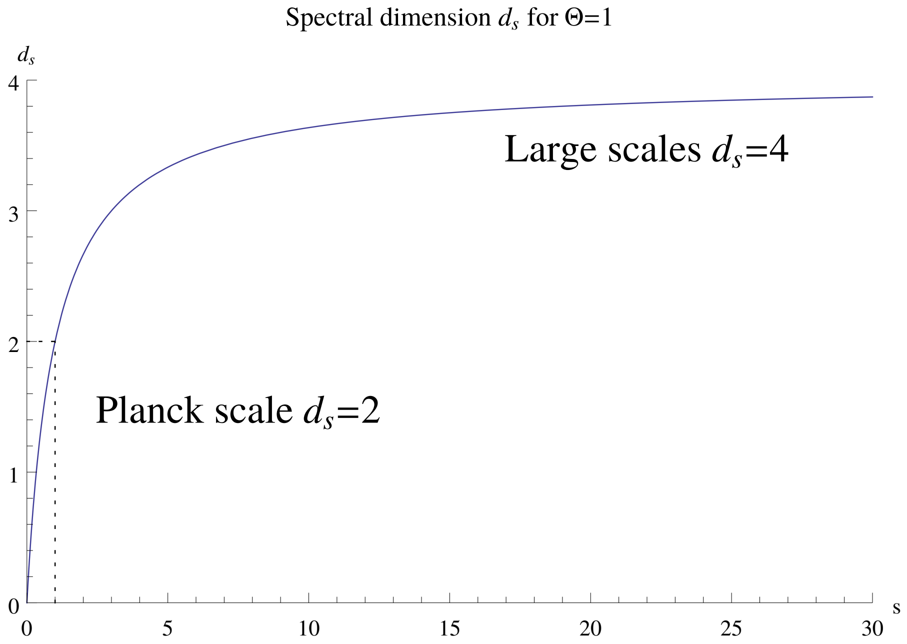


Figure 6: The spectral dimension as equation (116) of noncommutative geometry for $\ell^2 = \Theta = 1$.

UN-SPECTRAL DIMENSION: As an addition to the minimal length spectral dimension, the un-spectral dimension has been proposed [67]. It is the spectral dimension an *un-particle* [39] probe would measure. The key point of un-particles is that they are scale-invariant though they posses a nonzero mass. Usually mass breaks scale-invariance. Un-particles have a continuous, scale-invariant mass spectrum and introduce a non-integer particle number.

Un-particles are massive and scale invariant

The Green function for a scalar un-particle is [38]

$$G_U(x-y) = A_{d_U} \int_0^\infty d(m^2)\,(m^2)^{d_U-2} G(x-y;m^2)\,, \tag{117}$$

with G the standard Green function of a scalar particle, m the mass, d_U the scale dimension of the un-particle field and A_{d_U} some energy scale dependent factor whose exact form does not matter here. Usually $1 < d_U < 2$. Using $G(x-y;m^2) = \int_0^\infty ds\, K(x,y;s)$ with K the heat kernel, we have

$$G_U(x-y) = \int_0^\infty ds\, K_U(x,y;s) \tag{118}$$

$$= A_{d_U} \int_0^\infty d(m^2)\,(m^2)^{d_U-2} \int_0^\infty ds\, K(x,y;s)\,. \tag{119}$$

Upon exchanging the integrals in s and m^2 we get the heat kernel for scalar un-particles

$$K_U(x,y;s) = A_{d_U} \int_0^\infty d(m^2)\,(m^2)^{d_U-2} K(x,y;s)\,. \tag{120}$$

The diffusion equation for an un-field takes the form

$$\Delta_U K_U(x,y;s) = \frac{\partial}{\partial s} K_U(x,y;s)\,, \tag{121}$$

with the un-Laplacian $\Delta_U = \Delta - (d_U - 1)/s$. This is in analogy to a bar with a time dependent, spatially uniform "heat source" $(d_U - 1)/s$ that preserves the scale invariance [67].

Using $K(x,y;s) = (4\pi s)^{-D/2} \exp[-m^2 s + (x-y)^2/(4s)]$, where D is the spacetime dimension, the un-particle heat kernel (120) can be calculated and the return probability in flat spacetime is

$$K(x,x;s) = A_{d_U} \left(\frac{1}{4\pi s}\right)^{\frac{D}{2}} s^{1-d_U} \Gamma(d_U - 1)\,. \tag{122}$$

From the definition of the spectral dimension we see that the term s^{1-d_U} gives a contribution $2d_U - 2$ and the spectral dimension reads

$$d_{s,U} = D + 2d_U - 2\,. \tag{123}$$

This is consistent with the fact that un-particle corrections go away for $d_U = 1$ [40].

6.4.3 *Spectral Dimension and Curved Spacetime*

The curvature of spacetime also has an effect on the heat equation and thus on the spectral dimension. It has been argued [25] that these contributions due to curvature should not be seen as fundamental and thus be left out. Otherwise one could not distinguish spurious curvature effects from genuine new Quantum Gravity effects [25]. This point is not very clear and nevertheless it is good to know how the curvature contributions behave, since there is always curvature when there is matter. The contribution of curvature to the noncommutative case has been calculated as follows [63]:

The return probability in a background metric g is

$$P_g(s) = \frac{\int d^d x \sqrt{\det g_{ab}}\, K(x,x;s)}{\int d^d x \sqrt{\det g_{ab}}}\,. \tag{124}$$

Then the Laplace operator gets a modification proportional to the Ricci scalar

$$\Delta \to \Delta_g \equiv \Delta - \xi_d R\,, \quad \text{with} \quad \xi_d = \frac{1}{4}\frac{d-2}{d-1}\,, \tag{125}$$

and the choice of ξ is called *conformal coupling*. The minimal length ℓ is an invariant scalar, but it can in general get a spacetime dependence $\ell(x)$. The heat kernel for this scenario has been calculated [81] and at $x = x'$ it is

$$K_\ell(x,x;s) = \frac{a_0 e^{\frac{s\ell^2(x)}{s+\ell^2(x)}\xi_d R} + e^{\frac{s\ell^2(x)}{s+\ell^2(x)}\Delta_g} \sum_{n=1}^{\infty} s^n a_n(x,x)}{[4\pi(s+\ell^2(x))]^{\frac{d}{2}}}\,. \tag{126}$$

Basically the expansion of the heat kernel in Seeley-De Witt coefficients has been used. Now the calculation of the spectral dimension is straightforward: Get the return probability by plugging the heat kernel (126) into (124) and then derive the spectral dimension according to (109). The result is to leading order

$$d_{s,g} \approx \int_R d_s(x) - 2s \int_R \left(\frac{\ell^2(x)}{\ell^2(x)+s} \right)^2 \xi_d R \tag{127}$$

$$+ \frac{2s}{a_0} \int_{\Delta_g} \left[a_1(x) + \left(\frac{\ell^2(x)}{s+\ell^2(x)} \right)^2 \Delta_g a_1(x) - \frac{s\frac{d}{2}}{s+\ell^2(x)} a_1 \right], \tag{128}$$

where

$$d_s(x) = d \frac{s}{s+\ell^2(x)}, \tag{129}$$

is the spectral dimension coming from noncommutative geometry with spacetime dependent minimal length $\ell(x)$ and the notion

$$\int_R f(x) \equiv \frac{\int d^d x \frac{\sqrt{\det g_{ab}}}{(s+\ell^2(x))^{d/2}} \, e^{\frac{s\ell^2(x)}{s+\ell^2(x)} \xi_d R} \, f(x)}{\int d^d x \frac{\sqrt{\det g_{ab}}}{(s+\ell^2(x))^{d/2}} \, e^{\frac{s\ell^2(x)}{s+\ell^2(x)} \xi_d R}}, \tag{130}$$

$$\int_{\Delta_g} f(x) \equiv \frac{\int d^d x \frac{\sqrt{\det g_{ab}}}{(s+\ell^2(x))^{d/2}} \, e^{\frac{s\ell^2(x)}{s+\ell^2(x)} \Delta_g} \, f(x)}{\int d^d x \frac{\sqrt{\det g_{ab}}}{(s+\ell^2(x))^{d/2}} \, e^{\frac{s\ell^2(x)}{s+\ell^2(x)} \xi_d R}}, \tag{131}$$

has been introduced. Again as in the flat noncommutative model $\lim_{s\to 0} d_{s,g}(s) = 0$ and $\lim_{s\to\infty} d_{s,g}(s) = d$. For $s = \ell^2$ the spectral dimension fluctuates around 2.

6.4.4 *Generalized Uncertainty Principle*

As already mentioned in equation (73), the *Generalized Uncertainty Principle* (GUP) introduces a minimal length by modifying the Heisenberg Uncertainty Principle to

$$\Delta x \Delta p \geqslant 1 + \beta(\Delta p)^2 \tag{132}$$

Thus the Dirac Delta must be smeared out. In the framework of the GUP this is not a Gaussian, but

$$\delta^{(D)}(\vec{x}) \to \rho_\beta(\vec{x}) = \frac{1}{(2\pi)^{\frac{D}{2}}} \int \frac{d^D p}{1+\beta p^2} e^{i\vec{x}\cdot\vec{p}} = \beta^{-\frac{D}{2}} \left(\frac{\sqrt{\beta}}{x} \right)^{\frac{D}{2}-1} K_{\frac{n}{2}-1}\left(\frac{x}{\sqrt{\beta}} \right) \tag{133}$$

with the modified Bessel function of the second kind $K_\alpha(x)$. In 3+1 dimensions ($D = 3$) this has the form

$$\rho_\beta^{(3)}(r) = \sqrt{\frac{\pi}{2}} \frac{e^{-\frac{x}{\sqrt{\beta}}}}{\beta x}, \tag{134}$$

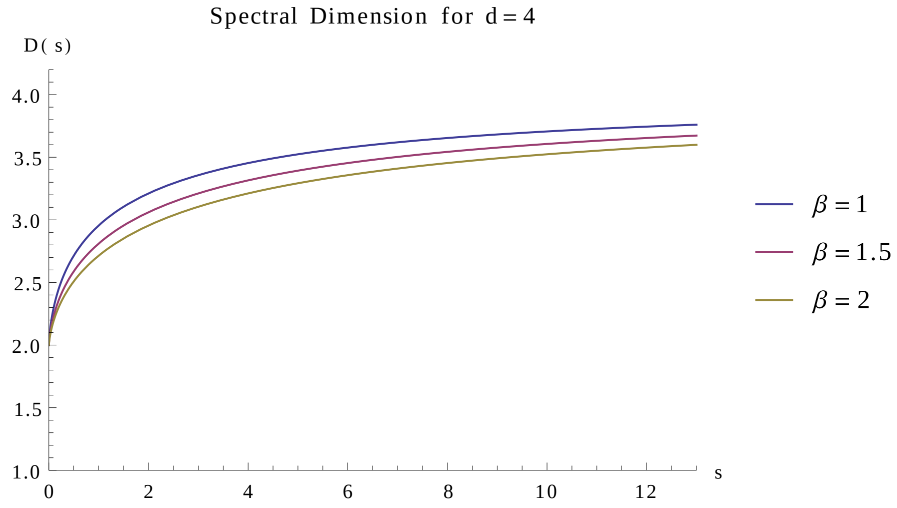


Figure 7: The spectral dimension for the GUP with $D = 4$ spacetime dimensions.

which is the energy density one would take to calculate e.g. the GUP modified Schwarzschild Black Hole [51].

For the spectral dimension in D spacetime dimensions one has to take the density (133) in D-dimensional Euclidean spacetime. This leads to

$$d_S = -\frac{1}{(2\pi)^{D/2}} \frac{2s}{\beta} \left(-1 + \frac{e^{-s/\beta} \left(\frac{s}{\beta}\right)^{-D/2}}{\Gamma\left(1 - \frac{D}{2}, \frac{s}{\beta}\right)} \right) , \tag{135}$$

running from $d_S(s \to 0) = 2$ in the UV to $d_S(s \to \infty) = 4$ in the IR. A plot of the spectral dimension for different β and $D = 4$ can be seen in figure 7. Actually the limit $s \to 0$ has no meaning, since there exists a minimal length $l_{min} = \sqrt{\beta}$ in the theory. For this limit we have $d_S(s \to l_{min}^2) > 2$. Interestingly GUP momentum-suppression $(1 + \beta p^2)^{-1}$ is not as strong as in the noncommutative case $\exp\left(-\frac{\theta}{2}p^2\right)$ which changes the UV spectral dimension. The noncommutative spectral dimension has $d_S(s \to 0) = 0$ and $d_S(s \to l_{min}^2) = 2$, while the GUP has $d_S(s \to l_{min}^2) > 2$.

6.4.5 *Effective Quantum Gravity*

Although Gravity is not renormalizable it is possible to analyze some effects in the framework of effective field theories and make predictions. The idea is to separate the scale of interest from the unknown physics at much higher energies [33] without unwarranted assumptions about the higher energies regime. In effective Quantum Gravity the resummed graviton propagator is given by [16]

$$iD^{\alpha\beta,\mu\nu}(q^2) = \frac{i\left(L^{\alpha\mu}L^{\beta\nu} + L^{\alpha\nu}L^{\beta\mu} - L^{\alpha\beta}L^{\mu\nu}\right)}{2q^2\left(1 - \frac{NG_N q^2}{120\pi}\log\left(-\frac{q^2}{\mu^2}\right)\right)}, \tag{136}$$

where

$$L^{\alpha\beta} = \eta^{\alpha\beta} - \frac{q^\alpha q^\beta}{q^2} \tag{137}$$

is a projector which is not of interest for our calculation. This propagator has additional poles and has *e.g.* been used to discuss the consequences of Effective QG on gravitational waves from astrophysical sources [27].

The important part is that the propagator goes like

$$D(q^2) \sim \frac{1}{q^2\left(1 + Aq^2\log\left(+\frac{q^2}{\mu^2}\right)\right)}, \tag{138}$$

where we have switched to Euclidean spacetime in order to calculate the spectral dimension. This is the propagator in momentum space, but for the spectral dimension we need it in position space. Thus we do a Fourier transform (and do not care about prefactors). From now on $D = 4$.

$$D(x) \sim \int_0^\infty ds \int d^4q\, e^{iq\cdot x - sq^2\left[1 + Aq^2\log\left(-\frac{q^2}{\mu^2}\right)\right]} = \int_0^\infty ds\, K(x;s)\,. \tag{139}$$

The return probability in flat spacetime is $P(s) = K(x-x;s)$ and thus we leave out the x-term in equation (139)

$$P(s) = \int d^4q\, e^{-sq^2\left[1 + Aq^2\log\left(-\frac{q^2}{\mu^2}\right)\right]} \tag{140}$$

$$\sim \int dq\, q^3 e^{-sq^2\left[1 + Aq^2\log\left(-\frac{q^2}{\mu^2}\right)\right]}, \tag{141}$$

up to prefactors (volume of sphere). Numerical evaluation of the spectral dimension from this return probability equation (141) is shown in figure 8. Numerically some problems occur when going to $s \to 0$ even in the standard case $D \sim 1/p^2$, so in this regime the calculation cannot be trusted[8]. From looking at the numbers we get $d_S(s \to 0) \approx 1.65$, but in this region numerical errors become very big. In [25] the same propagator has been used with the same results. Analytical approximations are given and the limit $d_S(s \to 0) = 2$ is calculated although the plot given in the paper shows a number smaller than 2.

8 Smaller than the Planck length.

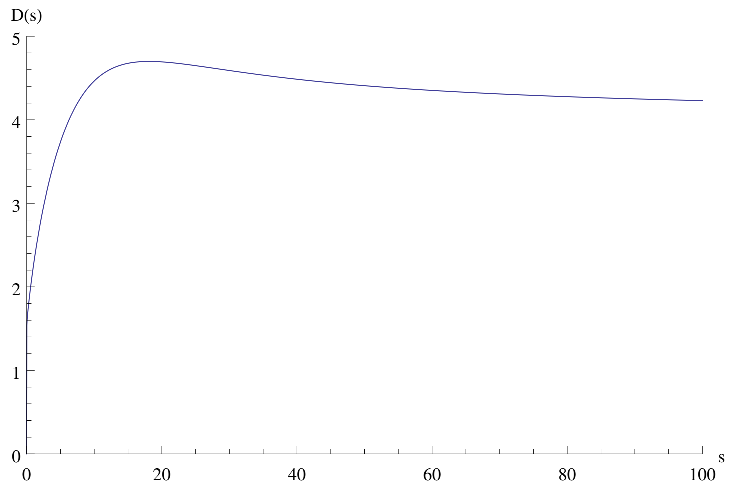


Figure 8: Plot of the spectral dimension from the effective Quantum Gravity propagator (136) calculated numerically with Mathematica.

6.4.6 *Strings and p-Branes*

CALCULATION OF THE SPECTRAL DIMENSION FROM THE HEAT KERNEL: In this section we are using the heat kernel as introduced in section 5.5 by using the quenched and minisuperspace approximation for a p-brane living in D dimensions. In Euclidean spacetime ($s \to -is$) the heat kernel of the propagator (83) is

$$K(x, x', \sigma, \sigma'; s) = \frac{1}{2M_0} e^{s\frac{M_0}{2}(p+1)} \left(\frac{\pi M_0}{s}\right)^{D/2} \tag{142}$$

$$e^{-\frac{M_0}{2s}(x-x_0)^2} \left(\frac{M_0}{\pi V_p^2 s}\right)^{\frac{1}{2}\binom{D}{p+1}} e^{-\frac{M_0(\sigma-\sigma')\cdot(\sigma-\sigma')}{(p+1)!2sV_p^2}} . \tag{143}$$

With the definition of the spectral dimension (109) we get

$$d_S = D + \binom{D}{p+1} - M_0(p+1)s . \tag{144}$$

The mass dependent term ($\sim M_0$) is set to zero, since one cannot probe the spectral dimension with massive particles/branes. Alternatively we could modify the heat equation to be suited for a massive state (see [67]), which would cancel the mass term in the spectral dimension. This leaves us with

$$d_S = D + \binom{D}{p+1} . \tag{145}$$

In table 1 the spectral dimension is calculated for different values of D and p. The case $p = -1$ is an instanton [43]. Note that the spectral dimension it is always bigger than D. This differs from what one would expect. Since a minimal length follows from string theory we

	D=3	D=4	D=5	D=6	D=7	D=8	D=9	D=10	D=11
p=-1	4	5	6	7	8	9	10	11	12
p=0	6	8	10	12	14	16	18	20	22
p=1	6	10	15	21	28	36	45	55	66
p=2	4	8	15	26	42	64	93	130	176
p=3	3	5	10	21	42	78	135	220	341
p=4		4	6	12	28	64	135	262	473

Table 1: Spectral dimension for different spacetime dimensions D and p-brane dimensions p calculated from equation (145). A p-brane in a $D < p$ dimensional space of course does not make much sense.

had expected that the spectral dimension would be modified similar to the previous examples where for s comparable to the Planck length one gets a spectral dimension of two. It turns out that our result is that a string ($p = 1$) sees a spectral dimension of $\mathbb{D} = 10$ for spacetime dimension $D = 4$. This number coincides with the spacetime dimension that is needed in the various superstring theories that are part of 11-dimensional M-theory. This way no extra dimensions are needed. We have 4 spacetime dimensions and the string sees 10 spectral dimensions.

A new feature is that the spectral dimension is actually larger than the topological dimension and it does not depend on the scale. The interpretation is as follows: The propagator equation (83) contains a *center of mass* and a *volume* part in the kernel. The center of mass part gives exactly the same spectral dimension as a standard quantum mechanical particle: D. The volume part gives $\binom{D}{p+1}$. This means that in the approximation that was used more dimensions are seen by the brane, since it can probe spacetime as a particle and also as a p dimensional volume. The string/brane just has more degrees of freedom to move and wiggle in spacetime and this is measured by the spectral dimension.

Since all we were looking at is a quantum mechanical string/brane propagating on a classical background, there is no quantized spacetime and also no minimal length in spacetime. The minimal length would only be observed by scattering strings, but this is not happening here. Therefore it makes sense that no quantum gravitational running of the spectral dimension is observed.

This is in contrast to the thermal considerations in [14], where thermodynamic properties of a gas of strings suggest two dimensions in the high temperature regime. The differences can have two causes:

1. The propagator used in this work may be not suited to the spectral dimension. The propagator is calculated in loop space and so also the “diffusion” is in loop space.

2. The spectral dimension just measures degrees of freedom of the diffusion in Euclidean spacetime. Implementing Quantum Gravity effects in this spacetime leads to a running spectral dimension. As stated above our results have an easy and plausible explanation. The difference to the other works with two dimensions stems from not quantizing spacetime in String Theory. Nevertheless some properties like in [14] of String Theory imply a dimensional reduction to effectively two dimensions. It might be the case that just another definition of dimension is needed here.

THE POINT PARTICLE LIMIT PROBLEM: Looking at table 1, we immediately see one problem: The point particle limit $p \to 0$ gives 8 and not 4. In section 5.5 was noted that the kernel $K_{p=0} = \delta(0)K_{cm}$. Except for a diverging factor the kernel and propagator approach the correct limit, so why not the spectral dimension? There are two methods that do not give the same result:

1. First calculating the spectral dimension and then doing the limit $p \to 0$,

2. First doing the limit $p \to 0$ and then calculating the spectral dimension.

The heart of the problem lies in the volume heat kernel equation (86)

$$K_p(\sigma; s) = \left(\frac{M_0}{i\pi V_p^2 s}\right)^{\frac{1}{2}\binom{D}{p+1}} e^{\frac{i}{(p+1)!}\frac{M_0}{2sV_p^2}\sigma^{\mu_1\cdots\mu_{p+1}}\sigma_{\mu_1\cdots\mu_{p+1}}}. \tag{146}$$

The point particle limit is $p = 0$ and $V_p \to 0$. The delta distribution can be seen as the limit of the sequence of functions. These functions can be for example Gaussians

$$\delta_\epsilon(x)^{(d)} = \frac{1}{(2\pi\epsilon)^{\frac{d}{2}}} e^{-\frac{x^2}{2\epsilon}}, \tag{147}$$

There are also other such functions. Also replacing $\epsilon \to i\epsilon$ works and is called the Fresnel representation.

Thus in equation (146) making the replacements

$$\frac{\sigma^2}{(p+1)!} \to x^2, \quad -\frac{iM_0}{sV_p^2} \to \frac{1}{\epsilon}, \quad \binom{D}{p+1} = d, \tag{148}$$

as done in [12] (note that they had a factor 4 wrong in ϵ) leads to

$$K_p \to \frac{1}{(\pi\epsilon)^{\frac{d}{2}}} e^{-\frac{x^2}{2\epsilon}} \overset{\epsilon\to 0}{\longrightarrow} \delta^{(d)}(x) \tag{149}$$

which is $\delta(\sigma)$ (and not $\delta(\sigma^2)$ as in [12]). The "loop coordinate" σ is not defined for $p < 1$, but it is assumed to vanish for a point particle. So finally

$$K_p \overset{\epsilon\to 0}{\longrightarrow} \delta(0)\,, \tag{150}$$

where the δ is a delta distribution in $\binom{D}{1} = D$ dimensional loopspace with the problem that σ (which defines this space) is not defined for $p = 0$. This might rise some concern. Note the fact that $\epsilon \sim sV_p^2$. What happens for $s \to \infty$, namely in the IR limit? We plan to address these issues in the future.

The spectral dimension is defined in equation (109)

$$d_S = -2\frac{\partial \ln K(x,x;s)}{\partial \ln s} = -2s\frac{1}{K(x,x;s)}\frac{\partial K(x,x;s)}{\partial s}\,. \tag{151}$$

For the volume heat kernel equation (146) one obtains

$$d_{S,p} = \binom{D}{p+1} \overset{p\to 0}{\longrightarrow} D\,, \tag{152}$$

but if one first does the point particle limit, the dependence on s in the volume kernel goes away and $d_{S,p} = 0$. Obviously the two limits (point particle limit and the derivative in d_S) do not commute. One could also argue that since $\epsilon \sim sV_p^2$ the limit $V_p \to 0$ includes or is equivalent to $s \to 0$.

A way out would be to manually set $d_S = D$ in the point particle limit and leave the rest of the table the same. Then we end up with table 2.

	D=3	D=4	D=5	D=6	D=7	D=8	D=9	D=10	D=11
p=-1	4	5	6	7	8	9	10	11	12
p=0	3	4	5	6	7	8	9	10	11
p=1	6	10	15	21	28	36	45	55	66
p=2	4	8	15	26	42	64	93	130	176
p=3	3	5	10	21	42	78	135	220	341
p=4		4	6	12	28	64	135	262	473

Table 2: p-brane spectral dimension with $d_S = D$ manually set, since the limits do not commute.

6.4.7 *Problems Interpreting the Spectral Dimension and New QFT Interpretation*

It seems that most Quantum Gravity motivated deformations of the heat (or diffusion) equation (106) lead to a spectral dimension that asymptotes the spacetime dimension in the IR and a lower value like

2 or 3 in the UV. For the string or a p-brane this is not the case, see equation (145).

The discrepancy between the spectral dimension from String Theory and from Noncommutative Geometry or the Generalized Uncertainty Principle can be interpreted as a big hit for the spectral dimension. Independent of this result the spectral dimension has been criticized before [9, 25]. We will give a summary of the critics followed by a introduction to a newly proposed dimension, the "thermal dimension" [9].

This section will be about the problems of the interpretation of the spectral dimension. A problem arising for $f(E^2 - p^2)$-theories and off-shell modes will be mentioned in the next section when the *thermal dimension* will be discussed.

PROBLEMS WITH THE SPECTRAL DIMENSION: The biggest two struggles with the spectral dimension can be called the "diffusion-time problem" and the "negative-probabilities problem" [25].

- *The diffusion-time problem:* This addresses the very approach to the spectral dimension. The *heat kernel* fulfills a diffusion equation and the time variable s in it. What is the physical meaning of s, $P(s)$ and ultimately the spectral dimension? The diffusion equation itself comes from the Schwinger representation of the propagator. It is not clear what exactly s tells us, since it is a fictitious time of a diffusion equation in D-dimensional Euclidean spacetime. It is not clear how is can be interpreted physically.
- *The negative-probabilities problem:* The heat kernel $K(x, x'; s)$ is usually interpreted as the probability density for a diffusing particle in Euclidean spacetime with fictitious diffusion time s. From $K(x, x'; s)$ the spectral dimension is calculated, but $K(x, x'; s)$ is not in all theories positive definite [24] and thus the interpretation as a probability density is in trouble.

There has been an attempt at solving those interpretation problems by reinterpreting the spectral dimension as a "quantum spectral dimension in QFT" [25] and what follows is a quick overview of this. It is important to note that here we are solely talking about the interpretation. The way of calculating the spectral dimension is unchanged.

PROPOSED SOLUTION TO THE INTERPRETATION PROBLEMS: We completely forget about the diffusion interpretation and look at the spectral dimension from a different perspective: Take $|G|^2$ and not $K(x, x'; s)$ as a *probability density*. In this picture s is not a diffusion time parameter, but a *resolution scale*. The return probability $P(s)$ also gets a new meaning. If we look at *vacuum-to-vacuum diagrams* the term

$$-\frac{1}{V}\int d^D x\; G(0) = -G(0) = \int_0^\infty ds\; P(s) \qquad (153)$$

describes the one-loop bubble diagram (for G the tree-level propagator). Thus $P(s)$ is the bubble contribution of scale s to the vacuum-energy of the particle. The spectral dimension is how this contribution scales with s assuming a massless particle.

Spectral dimension as scaling of bubble contribution

Since there is no probability interpretation of $K(x, x'; s)$ any more, there is no negative-probabilities problem. On the other hand the probability for the propagation of a particle from x' to x is proportional to $|G(x - x')|^2$ and

$$\int d^D x \, |G(x - x')|^2 = \int_0^\infty ds \; sP(s) \, . \tag{154}$$

Not $G(x, x'; s)$ needs to be positive definite, but the return probability $P(s)$. This is fulfilled in all known examples [25].

6.5 OTHER PROPOSED DEFINITIONS OF DIMENSION

In 2016 the "thermal dimension" [9] has been proposed and applied to generalized Horava-Lifshitz scenarios as well as to theories where the d'Alembertian is modified to a function of itself $E^2 - p^2 \to f(E^2 - p^2)$. The definition is via a standard calculation in thermal field theory. From the thermodynamic partition function

$$\log Q = -\frac{2V}{(2\pi)^3} \int dE d^3p \; \delta(\Omega)\Theta(E) 2E \log\left(1 - e^{-\beta E}\right) \, , \tag{155}$$

where $\Omega = \Omega(E, p)$ is the eventually modified d'Alembertian and $\beta = \frac{1}{k_B T}$ is the inverse temperature. From equation (155) the energy density ρ and pressure P can be calculated

$$\rho \equiv -\frac{1}{V}\frac{\partial}{\partial \beta} \log Q \, , \tag{156}$$

$$P \equiv \frac{1}{\beta}\frac{\partial}{\partial V} \log Q \, . \tag{157}$$

The temperature scaling of ρ defines the *thermal dimension* $\rho \sim T^{d_T}$. For a gas of radiation in classical spacetime with $3+1$ topological spacetime dimensions the Stefan Boltzmann law states that the internal energy U is proportional to T^4 or in $D+1$ dimensions

$$U \sim T^{D+1} \, . \tag{158}$$

This means that for the standard d'Alembertian the thermal dimension and the topological one agree.

Also the equation of state parameter $\omega \equiv \frac{P}{\rho}$ with the pressure P is in $D+1$ dimensions $\omega = 1/D = 1/(d_T - 1)$. Modifications to the d'Alembertian will introduce modifications to ω and to ρ and thus to the corresponding dimensions. In summary the thermal dimension is given by

$$\rho \sim T^{d_T} \, , \quad \text{and} \quad \omega = \frac{1}{d_T - 1} \tag{159}$$

and both definitions must agree to make sense.

HOŘAVA–LIFSHITZ GRAVITY: In [9] this procedure has been carried out for the generalized Horava-Lifshitz scenario in D = 3 dimensions. The effects are basically encoded in a modified d'Alembertian

$$\Omega_{\gamma_t\gamma_x}(E,p) = E^2 - p^2 + \ell_t^{2\gamma_t} E^{2(1+\gamma_t)} - \ell_x^{2\gamma_x} p^{2(1+\gamma_x)} . \quad (160)$$

The calculation of $\rho_{\gamma_t\gamma_x}$ and $P_{\gamma_t\gamma_x}$ gives for both the consistent result for the thermal dimension in the high temperature limit

$$d_T = 1 + 3\frac{1+\gamma_t}{1+\gamma_x} . \quad (161)$$

The UV (→ hight temperature) spectral dimension for the same model is

$$d_S(0) = \frac{1}{1+\gamma_t} + \frac{D}{1+\gamma_x} . \quad (162)$$

Spectral and thermal dimension coincide for $\gamma_t = 0$.

$f(E^2 - p^2)$ D'ALEMBERTIAN: Taking the modification

$$\Omega_\gamma(E,p) = E^2 - p^2 - \ell^{2\gamma}(E^2 - p^2)^{1+\gamma} , \quad (163)$$

the UV spectral dimension is in $D = 3$

$$d_S = \frac{4}{1+\gamma} , \quad (164)$$

but this has a problem. Equation (163) states that in the UV γ has no effect on the on-shell properties of the massless theory, since then $E^2 = p^2$. This is a problem for the spectral dimension d_S, it depends on γ and thus on the off-shell modes. The high temperature (=UV) thermal dimension in this case $d_T = 4$ is in agreement with the discussion above. There is still dimensional flow, but only close to the Planck temperature, the plot is given in figure 9. This running of the dimension only at about the Planck temperature is something that might be worth more investigation. Also the thermal dimension has some better properties concerning active diffeomorphisms in momentum space, the interested reader can find the discussion in [9].

As a conclusion the thermal dimension can be a better candidate in some scenarios. Also its definition on the scaling of ρ with T is more related to actual measurements than the interpretation of the spectral dimension as mentioned in the last section.

Nevertheless the spectral dimension's definition does not depend on thermodynamics and the notion of a temperature or a pressure. To date no other paper on the thermal dimension has been published.

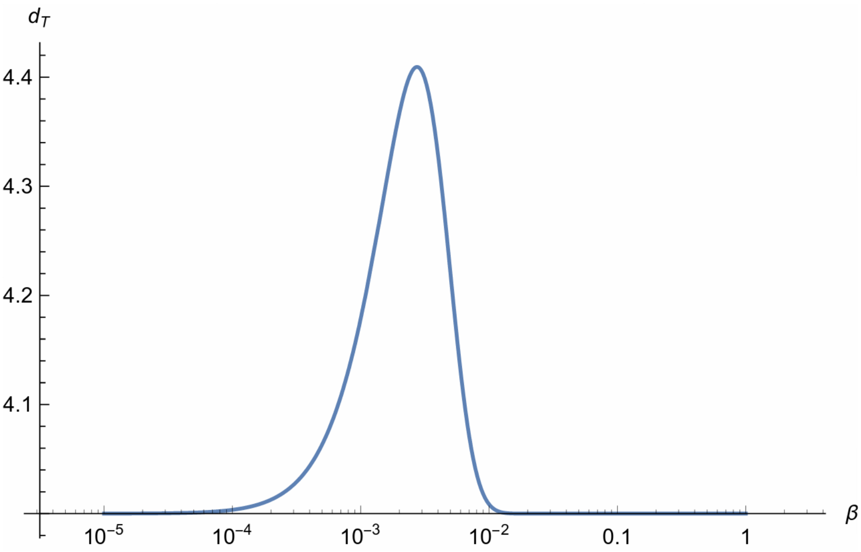


Figure 9: The thermal dimension for the d'Alembertian as in equation (163) as a function of the inverse temperature β. Plot from [9].

7

DISCUSSION AND CONCLUSIONS

In this thesis I have addressed the issue of the very small scale structure of spacetime. At the Planck scale quantum mechanical fluctuations of spacetime become so strong, that the very notion of a smooth spacetime breaks down and the term "spacetime foam" has been introduced for this effect. A smooth Riemannian geometry cannot describe this fuzzy geometry of the Quantum Gravity regime and should be augmented to some "quantum geometry". The title page shows this as an artist imagined it.

This thesis has been an overview of the combination of General Relativity and Quantum Field Theory. First I have reviewed GR and QFT, followed by a chapter on how to quantize fields in a curved background without backreaction of the field to the background. I have shown the arguments why GR and QFT do not fit together. However, there is always Gravity as long as there is matter. Thus Gravity and Quantum Mechanics do need to interact and Gravity at a fundamental level needs to be quantum. If one tries to quantize GR with the means of QFT the theory is nonrenormalizable in 4 spacetime dimensions, which is why there is still no full theory of Quantum Gravity. The main approaches to Quantum Gravity have been described followed by a brief introduction to String Theory where emphasis was given to the Generalized Uncertainty Principle and Noncommutative Geometry. I have also reviewed the loop space formalism for String Theory, which is a completely different formalism than usually used and it allows the calculation of an exact propagator for closed bosonic strings. There is also an approximate propagator for p-branes. The main part of this thesis was about *dimensional flow* and in particular the spectral dimension. Very general arguments show that dimensional flow occurs at very small scales in Quantum Gravity. I have reviewed the spectral dimension in several Quantum Gravity candidates and done new calculations regarding the Generalized Uncertainty Principle and strings/p-branes in the loop space formalism. The results of the former match with the other candidates, whereas String Theory in this formulation has no dimensional flow and a spectral dimension of 10.

The introduction of the spectral dimension allows two ways out of the nonrenormalizability of Gravity:

1. It seems to be a rather general feature of theories of Quantum Gravity that the spectral dimension flows from 4 in the IR to 2 in the UV. GR is renormalizable in 2 dimensions, which is why this dimensional flow to 2 spacetime dimensions is so interesting.

2. In this thesis I have shown that at least for the exact propagator calculated in the formalism of loop coordinates for a closed bosonic string, the spectral dimension is 10 for topological dimension 4. The reason that there is no dimensional reduction lies in how the propagator is constructed and that there is no directly quantized spacetime in this model. The number 10 for the spectral dimension is interesting, since Super String Theory requires 10 dimensions for internal consistency.

One could argue that the spectral dimension might be the more physical quantity than the topological dimension of spacetime, whose notion breaks down in the Planck regime, because there the smooth Riemannian manifold can no longer describe the quantum manifold. Then the problem of unifying Gravity and Quantum Theory certainly will be closely connected to the spectral dimension or other notions of dimension. I have also reviewed other proposed notions of dimension and regarding them some aspects of dimensional flow. The field of dimensional flow is still young and diffuse, but it could turn out to be the key ingredient to Quantum Gravity,

There are a lot of topics for further research in the area of dimensional reduction in Quantum Gravity. The proposed *Planckion* [15] has not yet been investigated in the literature and intrinsically contains a variable dimension and thus fits well with the notion of dimensional flow. *Scale Relativity* [69] combines Quantum Mechanics with the fractal properties of spacetime and one could investigate the connection to dimensional flow and to minimal length theories. Also calculations of the *thermal* [9] and *Unruh* [4] *dimension* in terms of noncommutative geometry and the GUP are an option. *Multifractional spacetimes* [23] intrinsically contain a changing dimension of spacetime. One could try to implement the *dimensional flow* of the various theories into QFT as an energy dependent dimension $d(E)$. It got suggested to have $d(E)$ in the integral for the propagator in order to see what happens with divergencies and how the renormalizability of Gravity is affected. Since the spectral dimension is deeply connected to the propagator a mathematical study of Impulse Response Functions in Nonlocal Gravity can be of great interest.

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Frankfurt am Main, 29.06.2017

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