Stability, Hyperbolicity, and Imaginary Projections of Polynomials

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Abstract

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Stability, Hyperbolicity, and Imaginary Projections of Polynomials

by Thorsten Jörgens

In this thesis we introduce the imaginary projection of (multivariate) polynomials as the projection of their variety onto its imaginary part,

$$\mathcal{I}(f) \;=\; \{\, \mathrm{Im}(z_1, \ldots, z_n) \,:\, f(z_1, \ldots, z_n) = 0 \,\}.$$

This induces a geometric viewpoint to stability, since a polynomial f is stable if and only if $\mathcal{I}(f) \cap (\mathbb{R}_{>0})^n = \emptyset$. Accordingly, the thesis is mainly motivated by the theory of stable polynomials.

Interested in the number and structure of components of the complement of imaginary projections, we show as a key result that there are only finitely many components which are all convex. This offers a connection to the theory of amoebas and coamoebas as well as to the theory of hyperbolic polynomials.

For hyperbolic polynomials, we show that hyperbolicity cones coincide with components of the complement of imaginary projections, which provides a strong structural relationship between these two sets. Based on this, we prove a tight upper bound for the number of hyperbolicity cones and, respectively, for the number of components of the complement in the case of homogeneous polynomials. Beside this, we investigate various aspects of imaginary projections and compute imaginary projections of several classes explicitly.

Finally, we initiate the study of a conic generalization of stability by considering polynomials whose roots have no imaginary part in the interior of a given proper cone $K \subset \mathbb{R}^n$. This appears to be very natural, since many statements known for univariate and multivariate stable polynomials can be transferred to the conic situation, like the Hermite-Biehler Theorem and the Hermite-Kakeya-Obreschkoff Theorem. When considering K to be the cone of positive semidefinite matrices, we prove a criterion for conic stability of determinantal polynomials.

The thesis is based on the preprints [58–60], which are a joint work with Thorsten Theobald. The first article was also written in collaboration with Timo de Wolff.

Contents

Chapter 1

Introduction and motivation

1.1 Introduction

A polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$, $n \geq 1$, is called stable if every root $(z_1, \ldots, z_n) \in \mathbb{C}^n$ satisfies $\mathrm{Im}(z_j) \leq 0$ for some j. Here, $\mathrm{Im}(z_j)$ denotes the imaginary part of the complex number z_j and we write $\mathrm{Im}(z_1, \ldots, z_n)$ for $\big(\mathrm{Im}(z_1), \ldots, \mathrm{Im}(z_n)\big)$. Moreover, if not stated otherwise we use bold letters for vectors of length n, e.g., $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$. This condition to the imaginary parts of the roots motivates to consider the set

$$\mathcal{I}(f) \ = \ \{ \mathrm{Im}(z_1, \ldots, z_n) \ : \ f(z_1, \ldots, z_n) = 0 \},$$

which is the projection of the variety of f onto its imaginary part. We call $\mathcal{I}(f)$ the imaginary projection of f. This induces a geometric viewpoint to stability, since a polynomial f is stable if and only if

$$\mathcal{I}(f) \cap (\mathbb{R}_{>0})^n = \emptyset. \tag{1.1}$$

Accordingly, the thesis is mainly motivated by the theory of stable polynomials. These attracted a lot of interest in diverse fields of mathematics, for example in graph theory and combinatorics, see [14, 22, 42, 125], and appeared at prominent places in recent literature. For instance, the proof of the Kadison-Singer Conjecture uses stable polynomials as well as the proof for the existence of infinite families of bipartite Ramanujan graphs and the verification of Johnson's Conjectures; see [8, 82, 83].

At the beginning of our study of imaginary projections we consider the number and structure of the components in the complement and we show as a key result that there are only finitely many connected components in the complement of the closure which are all convex. This offers a connection to the theory of amoebas and coamoebas, which shows the same characteristic, see [35, 41], and also to the theory of hyperbolic polynomials, whose hyperbolicity cones are convex; see [40, 45, 111]. A homogeneous polynomial $f \in \mathbb{R}[z_1, \ldots, z_n]$ is called hyperbolic in direction $\mathbf{e} \in \mathbb{R}^n$ if $f(\mathbf{e}) \neq 0$ and for every $\mathbf{x} \in \mathbb{R}^n$ the function $t \mapsto f(\mathbf{x} + t\mathbf{e})$ has only real roots. Geometrically speaking, for a polynomial f of degree d hyperbolicity means that any line in $\mathbb{R}^n$ with direction $\mathbf{e}$ intersects the real variety of f in exactly d points (counting multiplicities). The connected component in the complement of the real variety of f that contains $\mathbf{e}$ is called the hyperbolicity cone of f (with respect to $\mathbf{e}$). Moreover, f is hyperbolic with respect to any point in the hyperbolicity cone. Hyperbolic polynomials are of interest in real algebraic geometry and allow definite determinantal descriptions for certain convex sets; see, e.g., [76, 77, 97, 99, 107] as well as the survey [124]. But also convex optimization is interested in hyperbolic polynomials, since hyperbolic programs generalize semidefinite programs in a natural way and they can be solved numerically by using interior point methods; see, e.g., [45, 93, 95].

We are also interested in hyperbolic polynomials. Beside general aspects of imaginary projections, we study imaginary projections of hyperbolic polynomials. Here, we observe a strong connection between imaginary projections and hyperbolicity cones, which generalizes the well-known fact that a homogeneous polynomial is stable if and only if it is hyperbolic with respect to every point in the positive orthant; see, e.g., [71, 106]. Furthermore, our examinations lead to a tight upper bound for the maximal number of hyperbolicity cones.

Moreover, we generalize the usual stability notion to stability with respect to a proper cone $K \subset \mathbb{R}^n$, which is a full-dimensional, closed and pointed convex cone. We say a polynomial f is K-stable if it has no root $(z_1, \ldots, z_n) \in \mathbb{C}^n$ such that $\mathrm{Im}(z_1, \ldots, z_n) \in \operatorname{int} K$, where $\operatorname{int} K$ denotes the interior of K. This appears to be a very natural generalization, since many statements known for univariate and multivariate stable polynomials can be transferred to the conic situation. Moreover, $(\mathbb{R}_{\geq 0})^n$-stability coincides with the usual stability notion. For polynomials on complex symmetric matrix variables, we consider K to be the cone of positive semidefinite matrices, $\mathcal{S}^+$. Then, we say f is psd-stable if f is $\mathcal{S}^+$-stable. We prove a criterion for psd-stability of determinantal polynomials.

Chapter 1 surveys stable polynomials, hyperbolic polynomials as well as amoebas and coamoebas. For hyperbolic polynomials, we present the historical origin in partial differential equations, look at determinantal polynomials $f(z_1, \ldots, z_n) = \det(z_1 A_1 + \cdots + z_n A_n)$, and discuss their application in describing convex sets. In this context, we look at the well-known Lax Conjecture and some generalizations together with related results. Moreover, we mention the interest for hyperbolic polynomials in conic optimization.

For stable polynomials, we collect classical theorems known for univariate polynomials like the Hermite-Biehler Theorem, which characterizes stability of $g + if$ in terms of

g and f, and the Hermite-Kakeya-Obreschkoff Theorem, which considers stability of linear combinations of two polynomials. We also discuss more recent results concerning multivariate polynomials, for example regarding polynomials that are affine-linear in every variable. Stable polynomials are motivated by dynamical systems, where Hurwitz-polynomials play an important role. These are univariate polynomials whose roots have a negative real part and they provide asymptotical stability for linear and non-linear autonomous systems. The class of polynomials with the half-plane property are a multivariate generalization containing (multivariate) Hurwitz-polynomials and stable polynomials. These are polynomials whose roots lie in a closed half-space of $\mathbb{C}^n$. Stable polynomials and, more general, polynomials with the half-plane property, play an important role in different areas of mathematics; see, e.g., the survey [126]. Moreover, we give an overview on prominent results proven with stable polynomials. For instance, we look at the famous Kadison-Singer Conjecture, which was a problem in functional analysis and which was proven 2015 by Marcus, Spielman, and Srivastava using discrete methods and, in particular, univariate stable polynomials; see [83].
In Section 1.4, we provide a brief overview on the theory of amoebas and coamoebas. We mention results regarding the number and structure of components of the complement, the boundary as well as the behavior "at infinity". It will turn out during the thesis, that there are interesting analogies between amoebas and imaginary projections.

In Chapter 2, we study imaginary projections of polynomials. In the introductory Section 2.1 we prove basic properties as well as convexity of components of the complement. Since the imaginary projection of a polynomial is the image of an algebraic set under a projection, it is semi-algebraic and, thus, its complement has only finitely many components. The convexity is a result of Bochner's Tube Theorem in complex analysis, [7]. This result leads to first studies about the total number of components in the complement. For example, we prove via a construction that for an arbitrarily given positive integer K, there is a polynomial whose imaginary projection has at least K bounded and strictly convex components in the complement. Moreover, we can construct polynomials that have exactly K bounded components in the complement.
After that, in Section 2.2, we discuss the imaginary projection of special classes of polynomials. Most of them appear later in several situations, like in proofs or as examples. It turns out that already the imaginary projection of affine-linear polynomials behave subtle and very sensitive to ε-perturbations of the coefficients. For instance, their imaginary projection is a hyperplane whenever the coefficients are a complex multiple of a real vector and it equals the whole space otherwise. Also the behavior of imaginary projections of real quadratic polynomials is somehow surprising, since it contains various different classes. For example, the boundary of the imaginary projection can be a sphere or a 2-sheeted hyperboloid. For instance, in the

case of two variables, we have $\mathcal{I}(z_1^2 + z_2^2 + 1) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 \geq 1\}$ and $\mathcal{I}(z_1^2 - z_2^2 - 1) = \{(y_1, y_2) \in \mathbb{R}^2 : -1 \leq y_1^2 - y_2^2 < 0\} \cup \{0\}$. We present a complete description of the imaginary projections of real quadratic polynomials.

As an important tool for understanding imaginary projections in general, we study in Section 2.3 imaginary projections of hyperbolic polynomials. It turns out that their imaginary projection behaves nicely and they will become useful for the study of imaginary projections of inhomogeneous polynomials. As a fundamental result, we show that hyperbolicity cones coincide with components of the complement of imaginary projections. This provides a strong structural relationship between these two sets. Based on this, we prove a tight upper bound for the number of hyperbolicity cones and, respectively, for the number of components of the complement of the imaginary projection. The maximal number equals the number of cells in a hyperplane arrangement with hyperplanes passing through the origin. For a polynomial in n variables of degree d, this is 2^d for $d \leq n$ and $2\sum_{k=0}^{n-1}\binom{d-1}{k}$ for $d > n$. The constructive proof builds on a recent result by Kummer for the number of hyperbolicity cones of irreducible, homogeneous polynomials, [70]. The structure of the proof allows to characterize the set of polynomials for which the maximum is attained. These are all polynomials that are products of independent linear factors.

Beside this, we investigate in Section 2.4 the boundary of imaginary projections, where we distinguish the cases of homogeneous and of inhomogeneous polynomials. For homogeneous polynomials we understand the situation very well. Here, the boundary is a subset of the real variety and, thus, for an irreducible polynomial f the Zariski closure of the boundary of $\mathcal{I}(f)$ equals the variety of f. Moreover, we are able to characterize the situation where the boundary and the real variety coincide. For inhomogeneous polynomials the situation is more complicated. We give a necessary condition for boundary points that belong to the imaginary projection. These are the imaginary parts of so called critical points which are those points $\mathbf{z}$ in the variety of the polynomial f such that $\left(\frac{\partial f}{\partial z_1}(\mathbf{z}), \ldots, \frac{\partial f}{\partial z_n}(\mathbf{z})\right)$ is the complex multiple of a real vector.

In Section 2.5, we discuss the behavior "at infinity" of imaginary projections by considering a limit set, which is an analog of the logarithmic limit set for amoebas. For a given polynomial $f \in \mathbb{C}[\mathbf{z}]$, it is defined as

$$\mathcal{I}_\infty(f) \;=\; \lim_{r \to \infty} \left(\frac{1}{r}\mathcal{I}(f) \cap \mathbb{S}^{n-1}\right)$$

But in contrast to amoebas, this limit set is not always a spherical polyhedral complex. In the case of bivariate polynomials, we are able to describe the limit set completely and it turns out that either $\mathcal{I}_\infty(f) = \mathbb{S}^1$ or the cardinality of the limit set is twice the degree of f (counting multiplicities). This leads to further results for components of the complement which have a full-dimensional recession cone. For this purpose we consider the initial form in(f) of a polynomial f, which is defined as the sum of those terms in f that have maximal total degree. The initial form is homogenous and it turns out that there is a one-to-one correspondence between the hyperbolicity cones of in(f) and the components in the complement of $\mathcal{I}(f)$ that have full-dimensional recession cone. In particular, this provides the statement that $\mathcal{I}_\infty(f) \neq \mathbb{S}^{n-1}$ if and only if in(f) is hyperbolic.

Finally, in Chapter 3 we initiate the study of a conic generalization of stability by replacing the positive orthant in (1.1) by the interior of an arbitrary proper cone $K \subset \mathbb{R}^n$. Accordingly, a polynomial f is K-stable if and only if $\mathcal{I}(f) \cap \operatorname{int} K = \emptyset$. We prove a conic version of the Hermite-Biehler Theorem for the case of polyhedral and non-polyhedral cones as well as a conic version of the Hermite-Kakeya-Obreschkoff Theorem. Moreover, we consider K to be the cone of positive semidefinite matrices. This case is naturally related to the Siegel upper half-space in the theory of modular forms, which is for given degree g defined as $\mathcal{H}_g = \{A \in \mathbb{C}^{g\times g} \text{ symmetric} : \operatorname{Im}(A) \text{ is positive definite}\}$. This extends to usual stability, since a polynomial $f(z_1, \ldots, z_n)$ is stable if and only if $f(\operatorname{diag}(z_1, \ldots, z_n))$ is psd-stable. We prove that for a given Hermitian block matrix $A = (A_{ij})_{n\times n}$ with blocks of size $d \times d$ and a Hermitian $d \times d$ matrix B, the polynomial

$$Z = (z_{ij})_{n\times n} \mapsto f(Z) = \det(z_{11}A_{11} + z_{12}A_{12} + \cdots + z_{nn}A_{nn} + B)$$

on the complex symmetric matrix variable Z is psd-stable. The prove relies on a construction using the Kronecker product and the Khatri-Rao product.

The thesis is based on the preprints [58–60]. All of them are a joint work with Thorsten Theobald. The first article was also written in collaboration with Timo de Wolff. A previous version of [60] was accepted for a regular oral presentation at MEGA 2017 in Nice, France. [58] is going to appear in the Proceedings of the American Mathematical Society. [59] is published in Research of the Mathematical Sciences.
A large part of the contents that are reproduced from the articles [58–60] are quoted verbatim in this thesis. Several results are extensions or additional results that were not yet published.

1.1.1 Notations

We collect basic notations which we will use within this thesis.

Throughout the paper, we use bold letters for vectors, e.g., $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Unless stated otherwise, the dimension of these vectors is n. We write $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ for the set of non-negative and the set of strictly positive real numbers. We denote by $\operatorname{Re}(z)$

and $\mathrm{Im}(z)$ the real and the imaginary part of a point $z \in \mathbb{C}$, i.e., $z = \mathrm{Re}(z) + i\,\mathrm{Im}(z)$, and component-wise for points $\mathbf{z} \in \mathbb{C}^n$. For an arbitrary set $M \subseteq \mathbb{C}^n$ we understand $\mathrm{Re}(M)$ and $\mathrm{Im}(M)$ as the set of real parts and imaginary parts of all elements in M. Moreover, we use the notations $\mathcal{H}^n_{\mathbb{C}}$ for the set $\{\mathbf{z} \in \mathbb{C}^n : \mathrm{Im}(z_j) > 0,\, 1 \leq j \leq n\}$ and $\mathcal{H}^n_{\mathbb{R}} = \mathrm{Im}(\mathcal{H}^n_{\mathbb{C}})$, which is the positive orthant. We write $\mathbf{y} > 0$ if $\mathbf{y} \in \mathcal{H}^n_{\mathbb{R}}$.

For a polynomial $f \in \mathbb{C}[\mathbf{z}]$ we denote by $\mathcal{V}(f)$ the complex variety of a polynomial $f \in \mathbb{C}[\mathbf{z}]$ and by $\mathcal{V}_{\mathbb{R}}(f)$ the real variety of f. Moreover, we denote by $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ the real part and the imaginary part of f after the realification $\mathbf{z} = \mathbf{x} + i\mathbf{y} \in \mathbb{C}^n \mapsto (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2n}$, i.e., $f(\mathbf{x}, \mathbf{y}) = \mathrm{Re}\, f(\mathbf{x}, \mathbf{y}) + i\,\mathrm{Im}\, f(\mathbf{x}, \mathbf{y})$. Note that $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are real polynomials in $\mathbb{R}[\mathbf{x}, \mathbf{y}]$.

We use the notation $\deg f$ for the degree of f. If $f \in \mathbb{C}[\mathbf{z}]$ is of degree d, we denote by $f_h(z_0, \mathbf{z}) = z_0^d f(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}) \in \mathbb{C}[z_0, \mathbf{z}]$ its homogenization with respect to the new variable z_0. We call a polynomial f homogeneous of degree d if all monomials in f appear with degree d. The sum of those terms in f that have maximal total degree, is denoted by $\mathrm{in}(f)$ and is called the initial form of f. Note that $\mathrm{in}(f) = f_h(0, \mathbf{z})$, which is a homogeneous polynomial of degree $\deg f$.

For $M \subseteq \mathbb{C}^n$ we set $M^{\mathsf{c}} = \mathbb{C}^n \setminus M$ for the complement of M. We write $\mathrm{cone}\, M$ for the cone spanned by M, which is the set of all positive combinations of elements in M. Similarly, we write $\mathrm{cone}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ for the cone spanned by $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Furthermore, $\mathrm{conv}\, M$ denotes the convex hull of M, which is the set of all convex combinations of elements in M. Moreover, $\mathrm{int}\, M$ denotes the interior of M and $\overline{M}$ its topological closure.

For matrices, we use the notation $\mathcal{S}_d$ for the set of real symmetric $d \times d$ matrices. $\mathcal{S}_d^+$ and $\mathcal{S}_d^{++}$ denote the set of positive semidefinite and the set of positive definite matrices in $\mathcal{S}_d$. We use the usual notations $A \succeq 0$ for $A \in \mathcal{S}_d^+$ and $A \succ 0$ for $A \in \mathcal{S}_d^{++}$. We write I_d for the identity matrix of order d.

1.2 Hyperbolic polynomials

Originally motivated by partial differential equations, hyperbolic polynomials have become of interest in real algebraic geometry and optimization; see, e.g., [45, 76, 93, 99, 107, 111]. The real variety of hyperbolic polynomials shows interesting convexity properties and hyperbolic programs generalize semidefinite and second-order cone programs in a natural way. This section provides an introduction to hyperbolic polynomials and collects basic properties. Section 1.2.1 surveys the historical motivation. An important class of hyperbolic polynomials are developed by determinantal polynomials. They are considered in Section 1.2.1.1. In Section 1.2.1.2 we discuss hyperbolic programming and interior point methods.

In this thesis, we especially deal with hyperbolic polynomials in Section 2.3. There, we consider hyperbolic polynomials from the viewpoint of imaginary projections and derive connections between these two objects. Moreover, we prove a result regarding the maximal number of hyperbolicity cones in Section 2.3.1. Namely, the maximal number is regarded as the number of cells in a central hyperplane arrangement, where "central" expresses that all hyperplanes pass through the origin. Moreover, hyperbolicity cones provide helpful information when considering the imaginary projection of inhomogeneous polynomials. This leads to a statement about the maximal number of components of the complement with full-dimensional recession cone.

Definition 1.1. *A homogeneous polynomial $f \in \mathbb{R}[\mathbf{z}]$ is called hyperbolic in direction $\mathbf{e} \in \mathbb{R}^n$ if $f(\mathbf{e}) \neq 0$ and for every $\mathbf{x} \in \mathbb{R}^n$ the function $t \mapsto f(\mathbf{x} + t\mathbf{e})$ has only real roots. We call the vector $\mathbf{e}$ the hyperbolicity direction of f. Moreover, we say f is hyperbolic, if there is a vector $\mathbf{e} \in \mathbb{R}^n$ such that f is hyperbolic with respect to $\mathbf{e}$.*

Geometrically speaking, hyperbolicity means that any line in $\mathbb{R}^n$ with direction $\mathbf{e}$ intersects the real variety $\mathcal{V}_{\mathbb{R}}(f)$ in $\deg f$ many real points (counting multiplicities). For fixed $\mathbf{x} \in \mathbb{R}^n$, this implies that we can express the univariate polynomial $t \mapsto f(\mathbf{x} + t\mathbf{e})$ in terms of linear factors and obtain

$$f(\mathbf{x} + t\mathbf{e}) = f(\mathbf{e}) \prod_{j=1}^{\deg f} (t - \lambda_j) \ , \tag{1.2}$$

where $\lambda_1, \ldots, \lambda_{\deg f}$ are the real roots of $f(\mathbf{x} + t\mathbf{e})$. Clearly, the roots $\lambda_1, \ldots, \lambda_{\deg f}$ depend on the choice of $\mathbf{x}$ and $\mathbf{e}$. Sometimes, these roots are called the eigenvalues of f in direction $\mathbf{e}$ at $\mathbf{x}$. Example 1.2 provides a motivation for this term.

Example 1.2. *Let* $\det : \mathcal{S}_d \to \mathbb{C}$ *be the determinant of a real symmetric matrix. It can be considered as polynomial with $d(d+1)/2$ variables. It is well-known that the eigenvalues* *of a symmetric matrix are all real. Hence, the univariate polynomial* $t \mapsto \det(A + tI_d)$ *has for all* $A \in \mathcal{S}_d$ *only real roots. This implies that the determinant restricted to real symmetric matrices is hyperbolic with respect to* I_d*.*

The following example considers the first elementary symmetric polynomial. Although it looks simple it is still worth mentioning since it occurs at different places in literature, e.g., in [16, 114], and we come back to it several times within this thesis.

Example 1.3. *Let* $f(\mathbf{z}) = z_1 \cdots z_n$*, which is the first elementary symmetric polynomial. Writing* $\mathbf{x} = (x_1, \ldots, x_n)$ *and* $\mathbf{e} = (e_1, \ldots, e_n)$ *a direct computation shows that the eigenvalues of* f *with respect to* $\mathbf{e}$ *and* $\mathbf{x}$ *are* $-x_1/e_1, \ldots, -x_n/e_n$*. Hence,* f *is hyperbolic with respect to any point* $\mathbf{e} \in (\mathbb{R} \setminus \{0\})^n$*.*

Remark 1.4. Usually, when studying hyperbolicity, one considers polynomials with real coefficients. That is not a strong restriction. Assume $f \in \mathbb{C}[\mathbf{z}]$ is hyperbolic with respect to $\mathbf{e} \in \mathbb{R}^n$. By Equation (1.2), $t \mapsto f(\mathbf{x} + t\mathbf{e})/f(\mathbf{e})$ is a real polynomial whenever $\mathbf{x}$ is real. This implies that f has real coefficients up to a common complex factor. Keeping this in mind we consider in the following only hyperbolic polynomials with real coefficients.

Naturally connected to hyperbolicity is the question for the set of all hyperbolicity directions. It turns out that this set is a finite collection of open convex cones. We use the following definition.

Definition 1.5. *Let* f *be hyperbolic with respect to* $\mathbf{e}$*. Then*

$$C_f(\mathbf{e}) = \{\mathbf{x} \in \mathbb{R}^n : t \mapsto f(\mathbf{x} + t\mathbf{e}) = 0 \Rightarrow t < 0\}$$

is called the hyperbolicity cone of f *in direction* $\mathbf{e}$*. If the context is clear, we often write* $C(\mathbf{e})$*,* C_f *or* C *for short.*

More geometrically, we can view a hyperbolicity cone as that component of $\mathcal{V}_{\mathbb{R}}(f)^{\mathsf{c}} \subset \mathbb{R}^n$ which contains $\mathbf{e}$. Hyperbolicity cones are semi-algebraic sets; see [45, Theorem 5.3].

The roots of the univariate polynomial $t \mapsto f(\mathbf{x} + t\mathbf{e})$ depend continuously on the choice of $\mathbf{e}$. Due to the strict negativity of all roots, the set $C_f(\mathbf{e})$ is open and due to the homogeneity of f, it is a cone. Gårding showed in [40] that $C(\mathbf{e})$ is convex and f is hyperbolic with respect to every point $\mathbf{e}'$ in $C(\mathbf{e})$. By now, there are also proofs using different methods; see, e.g., [45, 111].

Note, that $0 \notin C(\mathbf{e})$ and $C(-\mathbf{e}) = -C(\mathbf{e})$ is a hyperbolicity cone for f as well. Since hyperbolicity cones occur pairwise (in this sense), we often speak about a pair of hyperbolicity cones, when we mean both. The lineality space of C_f, which is defined as

$\{\mathbf{x} \in \mathbb{R}^n : C_f + \mathbf{x} = C_f\}$ is $\{0\}$ if and only if f depends on all variables; see [40]. We remark that hyperbolicity is invariant under action of $GL_n(\mathbb{R})$ on the space of variables, since real invertible matrices preserve the structure of hyperbolicity cones.

Remark 1.6. If f_1 and f_2 are hyperbolic polynomials with hyperbolicity cones C_{f_1} and C_{f_2}, then $C_{f_1} \cap C_{f_2}$ is a hyperbolicity cone of the product $f_1 \cdot f_2$. This follows immediately from Definition 1.1.

We consider two prominent examples for hyperbolicity cones. The first one is an important generalization of Example 1.2.

Example 1.7. *Let* $f(\mathbf{z}) = \det(A_1 z_1 + \cdots + A_n z_n)$*, with* $A_1, \ldots, A_n \in \mathcal{S}_d$*. The two hyperbolicity cones are*

$$\{\mathbf{x} \in \mathbb{R}^n : A_1 x_1 + \cdots + A_n x_n \succ 0\} \tag{1.3}$$

as well as its negative; see [76, Prop. 2] for a proof. In Example 2.41 we give a proof using imaginary projections. The closure of (1.3)*, which is* $\{\mathbf{x} \in \mathbb{R}^n : A_1 x_1 + \cdots + A_n x_n \succeq 0\}$*, is called a spectrahedron and plays a central role in semidefinite programming. Namely, they are the set of feasible solutions of a semidefinite program (SDP).*
As a concrete non-trivial example, we remark that the elementary symmetric functions $\varepsilon_j(x_1, \ldots, x_n) = \sum_{i_1 < \cdots < i_j} x_{i_1} \cdots x_{i_j}$*,* $1 \leq j \leq n$*, have spectrahedral hyperbolicity cones; see [16, 114]. Here, we call an open cone spectrahedral if it is the interior of a spectrahedron.*

Example 1.8. *As a second example, let* $f(\mathbf{z}) = z_1^2 - \sum_{j=2}^n z_j^2$*,* $n > 2$*. It is known that* f *is hyperbolic with respect to any point* $\mathbf{e} = (e_1, \ldots, e_n) \in \mathbb{R}^n$ *satisfying* $e_1^2 - \sum_{j=2}^n e_j^2 > 0$*, see, e.g., [40, Example 1], and that the two hyperbolicity cones are the open second-order cone (or Lorentz cone or sometimes ice cream cone)*

$$\mathcal{L} = \{x \in \mathbb{R}^n \, : \, x_1^2 - \sum_{j=2}^n x_j^2 > 0, \, x_1 > 0\}$$

as well as its negative $-\mathcal{L}$*. In optimization, the second-order cone occurs as the set of feasible solutions of second-order cone programs (SOCP).*

As Example 1.3 indicates, there does not need to be only one pair of hyperbolicity cones. Namely, the hyperbolicity cones of the first elementary symmetric polynomials are all 2^n chambers in $(\mathbb{R} \setminus \{0\})^n$. The question for the maximal number of hyperbolicity cones was answered in [58], phrased here in Section 2.3. And indeed, 2^n is the maximal number for polynomials in n variables and of degree n. The proof builds on a result by Kummer,

[70], which states that there is at most one pair of hyperbolicity cones whenever the underlying polynomial is irreducible.

We close this section with a remark regarding the hyperbolicity cones of a polynomial and its derivative; see [111]. It is due to Rolle's Theorem which states that the roots of a polynomial and its derivative alternate.

Proposition 1.9. *Let f be hyperbolic with respect to $\mathbf{e}$ and with hyperbolicity cone $C_f(\mathbf{e})$. Then its directional derivative $D_{\mathbf{e}}(f)$ is also hyperbolic with respect to $\mathbf{e}$ and $C_f(\mathbf{e}) \subseteq C_{D_{\mathbf{e}}(f)}(\mathbf{e})$.*

1.2.1 Historical motivation

The interest in hyperbolic polynomials is motivated by the study of partial differential equations; hyperbolic polynomials guarantee the existence and the uniqueness of solutions for the Cauchy Problem for hyperbolic differential equations. This section gives a brief overview of this connection without going too much into the theory of partial differential equations.
In the following Section 1.2.1.1 we consider determinantal polynomials, which are an important class of hyperbolic polynomials. Moreover, we give an overview on the Lax Conjecture and some generalizations, which ask for certain determinantal representations of hyperbolic polynomials.
In Section 1.2.1.2, we look at hyperbolic programs, which are conic optimization problems generalizing semidefinite programming and second-order cone programming. They can be solved numerically by interior point methods.

Originally, in the theory of partial differential equations the term of hyperbolicity was not restricted to homogeneous polynomials and it reads as follows; see [53, Definition 12.3.3] and also [45, 103]. We recall, that $\mathrm{in}(f)$ denotes the sum of those terms in f, that have maximal total degree. The polynomial $\mathrm{in}(f)$ is homogeneous.

Definition 1.10. *A polynomial $f \in \mathbb{R}[\mathbf{z}]$ is called hyperbolic in direction $\mathbf{e} \in \mathbb{R}^n \setminus \{0\}$ if $\mathrm{in}(f)(\mathbf{e}) \neq 0$ and if there exists a real number t_0 such that for every $\mathbf{x} \in \mathbb{R}^n$ the function $t \mapsto f(\mathbf{x} + it\mathbf{e})$ has no root for $t < t_0$.*

As pointed out in [53, Chapter 12.4], if a polynomial is hyperbolic with respect to a vector $\mathbf{e}$, then its initial part $\mathrm{in}(f)$ is hyperbolic with respect to $\mathbf{e}$ as well. Moreover, for homogeneous polynomials, the Definitions 1.1 and 1.10 coincide.
In analogy to homogeneous, hyperbolic polynomials, we can consider hyperbolicity cones also in the inhomogeneous case. Again, if f is hyperbolic in direction $\mathbf{e}$, then the

corresponding hyperbolicity cone is that component of $\mathcal{V}_{\mathbb{R}}(f)^{\mathsf{c}} \subset \mathbb{R}^n$ that contains $\mathbf{e}$. It is convex and occurs pairwise, since $-\mathbf{e}$ is another hyperbolicity direction of f.

Hyperbolic polynomials appear in the theory of partial differential equations. We follow the notation of [53] and for a polynomial $q \in \mathbb{C}[\mathbf{z}]$ we write $D_q := q(-i\frac{\partial}{\partial \mathbf{x}})$, where $\frac{\partial}{\partial \mathbf{x}} := (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$. Moreover, for a vector $\mathbf{v} \in \mathbb{R}^d \setminus \{0\}$, we denote by $\mathcal{H}_{\mathbf{v}} := \{\mathbf{x} : \langle \mathbf{v}, \mathbf{x} \rangle = 0\}$ the hyperplane with normal vector $\mathbf{v}$ and by $\mathcal{H}_{\mathbf{v}}^+ := \{\mathbf{x} : \langle \mathbf{v}, \mathbf{x} \rangle \geq 0\}$ the “upper” half-space bounded by $\mathcal{H}_{\mathbf{v}}$. Here, we denote the region in $\mathbb{R}^n$ where f is non-vanishing as the support of f, $\operatorname{supp} f$.
The Cauchy Problem for the differential operator D_q is defined as

$$D_q(f) = g \ \text{ in } \mathbb{R}^n \ , \ \operatorname{supp} f \subset \mathcal{H}_{\mathbf{v}}^+ \ , \ \operatorname{supp} g \subset \mathcal{H}_{\mathbf{v}}^+ \ , \tag{1.4}$$

possibly together with boundary conditions involving f and its derivatives. It holds that $g \equiv 0$ implies $f \equiv 0$ if and only if $\operatorname{in}(q)(\mathbf{v}) \neq 0$. Moreover, the Cauchy Problem with homogeneous boundary conditions has a unique solution (in the sense of distributions) if and only if q is hyperbolic with respect to $\mathbf{v}$. We refer to Hörmander, [53, Chapter XII].

Example 1.11. *As an example, we consider the non-homogeneous wave equation in one space dimension, denoted by x. For given initial conditions g_1, g_2, the homogeneous wave equation reads as follows*

$$\frac{\partial^2 f}{\partial t^2}(t,x) - \frac{\partial^2 f}{\partial x^2}(t,x) = 0 \ , \ f(0,x) = g_1(x) \ , \ \frac{\partial f}{\partial x}(0,x) = g_2(x) \ \textit{for } x \in \mathbb{R}, \ t > 0. \tag{1.5}$$

The general solution of (1.5) *is*

$$f(t,x) = \frac{1}{2}\left(g_1(x-t) + g_1(x+t) + \int_{x-t}^{x+t} g_2(s)ds\right) \ ,$$

which is due to d'Alembert, 1747. For $s > 0$, let f be the solution for (1.5) *with $g_1 = 0$ and $g_2(x) = h(s,x)$. Then $u(s,x) = \int_0^s f(s-t,x)dt$ is a unique solution for*

$$\frac{\partial^2 u}{\partial s^2}(s,x) - \frac{\partial^2 u}{\partial x^2}(s,x) = h(s,x) \ , \ u(0,x) = 0 \ , \ \frac{\partial u}{\partial x}(0,x) = 0 \ \textit{for } x \in \mathbb{R}, \ s > 0. \tag{1.6}$$

Equation (1.6) *is a Cauchy Problem with homogeneous boundary condition in the sense of Hörmander. Stating* (1.6) *in terms of the wave operator $\Box := D_q$, the polynomial q has the form $q(s,x) = x^2 - s^2$. Moreover, $\mathcal{H}_{\mathbf{v}}^+ = \mathbb{R}_{>0} \times \mathbb{R} = \mathcal{H}_{(1,0)}^+$ and the homogeneous polynomial q is hyperbolic with respect to $(1,0)$. We refer to [29, Chapter 2.4].*

We remark, that there is a slightly different definition of hyperbolicity for inhomogeneous polynomials used, e.g., by Gårding in [39]. Using the notion there, hyperbolicity guarantees a solution for linear hyperbolic differential equations with constant coefficients. For an illustrative, but handwaving argument of this statement, we refer to [106, Part I].

There is a recent generalization of hyperbolicity of homogeneous polynomials to inhomogeneous polynomials. These are called real-zero polynomials and defined as follows; see, e.g., [51]. We remark that from a historical point of view, Definition 1.10 for inhomogeneous, hyperbolic polynomials is not a generalization of Definition 1.1.

Definition 1.12. *A polynomial $f \in \mathbb{R}[\mathbf{z}]$ is called real-zero polynomial if the univariate polynomial $t \mapsto f(t\mathbf{x})$ has for all $\mathbf{x} \in \mathbb{R}^n$ only real roots.*

It turns out, that real-zero polynomials fit well to determinantal descriptions of sets, as the following Example 1.13 shows. In particular, real-zero polynomials play a substantial role in the proof of the Lax Conjecture 1.15 in [51, 76], which concerns certain determinantal descriptions of hyperbolic polynomials. We will discuss determinantal descriptions in the subsequent Section 1.2.1.1 in more detail.

Example 1.13. *[51] Let $A_0, A_1, \ldots, A_n$ be positive definite matrices. Then, the polynomial*

$$f(\mathbf{x}) = \det(A_0 + A_1 x_1 + \cdots + A_n x_n)$$

is a real-zero polynomial.

We discuss determinantal polynomials in the section following. However, we focus on hyperbolic polynomials there. For recent literature to real-zero polynomials, we refer to [44, 49, 97, 99] and the references therein.

1.2.1.1 Determinantal polynomials

In real algebraic geometry, determinantal polynomials play an important role in optimization and for the description of convex sets. A determinantal polynomial is a polynomial of the form

$$f(\mathbf{z}) = \det(A_0 + A_1 z_1 + \cdots + A_n z_n) , \tag{1.7}$$

where $A_0, A_1, \ldots, A_n$ are matrices of the same order. We are interested in that cases where these matrices are real symmetric, Hermitian or positive semidefinite. We write $\mathcal{A}(\mathbf{x}) := A_0 + A_1 x_1 + \cdots + A_n x_n$. If there is a vector $\mathbf{e} \in \mathbb{R}^n$ such that $\mathcal{A}(\mathbf{e}) \succ 0$, then we say that (1.7) is a definite determinantal representation of f. If $\mathcal{A}(\mathbf{x})$ provides a determinantal representation for f, then $P\mathcal{A}(\mathbf{x})P^H$ provides for any invertible matrix

P an equivalent determinantal representation for f, where P^H denotes the Hermitian transpose of P. Deciding whether a determinantal description has an equivalent definite determinantal representation can be rather difficult. We refer to [77, 107, 123] and the references therein.

Now, consider f to be homogeneous, i.e., $A_0 = 0$, with real symmetric matrices $A_1, \ldots, A_n$. If f is hyperbolic with respect to $\mathbf{e} \in \mathbb{R}^n$ such that $\mathcal{A}(\mathbf{e}) \succ 0$, then f has the hyperbolicity cones $\{\mathbf{x} \in \mathbb{R}^n : \mathcal{A}(\mathbf{x}) \succ 0\}$ and $\{\mathbf{x} \in \mathbb{R}^n : \mathcal{A}(-\mathbf{x}) \succ 0\}$, where the first one contains $\mathbf{e}$; see Example 1.7. The closure $\{\mathbf{x} \in \mathbb{R}^n : \mathcal{A}(\mathbf{x}) \succeq 0\}$ is a spectrahedron and the condition $\mathcal{A}(\mathbf{x}) \succeq 0$ is often called linear matrix inequality (LMI). Spectrahedra are originally defined as the intersection of an affine space with the convex cone of real symmetric matrices. They are always convex and semi-algebraic. Hence, determinantal polynomials provide one way to describe convex sets. But on the one hand, it is not clear whether a convex set has an LMI description, and, on the other hand, it can be a hard problem finding an LMI description for a given convex set. We refer to [50, 51, 73, 77, 96, 107] and the references therein. Moreover, we refer to the survey by Vinnikov, [124].

Example 1.14. *The polynomial $f(\mathbf{z}) = z_1^2 - \sum_{j=2}^n z_j^2$, $n > 2$, is hyperbolic with hyperbolicity cones $\mathcal{L} = \{\mathbf{x} \in \mathbb{R}^n : x_1^2 - \sum_{j=2}^n x_j^2 > 0,\, x_1 > 0\}$ and $-\mathcal{L}$; see Example 1.8. The polynomial $z_1^{n-2} f(\mathbf{z})$ can be written as the determinant of the following $n \times n$-matrix, see, e.g., [71, Example 3.5]:*

$$\mathcal{A}(\mathbf{z}) = \begin{pmatrix} z_1 & -z_2 & \cdots & -z_n \\ -z_2 & z_1 & & 0 \\ \vdots & & \ddots & \vdots \\ -z_n & 0 & \cdots & z_1 \end{pmatrix}$$

This implies that the closure of the Lorentz cone can be regarded as a spectrahedron. Namely, it is $\overline{\mathcal{L}} = \{\mathbf{x} \in \mathbb{R}^n : x_1^2 - \sum_{j=2}^n x_j^2 \geq 0, x_1 \geq 0\} = \{\mathbf{x} \in \mathbb{R}^n : \mathcal{A}(\mathbf{x}) \succeq 0\}$.

Strongly connected to the question for definite determinantal representations is the question for the order of the matrices used. In general, this is is an open question, which was first captured by Lax in 1958, [74].

Conjecture 1.15 (Lax Conjecture; see [76])**.** Let $f \in \mathbb{R}[x_1, x_1, x_3]$ be a homogeneous polynomial of degree d. Then, f is hyperbolic with respect to $\mathbf{e} = (1, 0, 0)$ and satisfies $f(\mathbf{e}) = 1$ if and only if there are real symmetric $d \times d$-matrices A, B such that

$$f(x_1, x_2, x_3) = \det(x_1 I_d + x_2 A + x_3 B).$$

Here, I_d denotes the unit matrix of order d.

Clearly, this conjecture can be extended to further variables and one may ask whether a hyperbolic polynomial $f \in \mathbb{R}[\mathbf{z}]$ of degree d admits a definite determinantal representation of $d \times d$-matrices. If $\mathcal{A}(\mathbf{x})$ is homogeneous, positive definite and of order d, then the polynomial $f(\mathbf{x}) = \det(I_d + \mathcal{A}(\mathbf{x}))$ is a real-zero polynomial; see, e.g., [124]. Building upon a result of Helton and Vinnikov for real-zero polynomials [51], Lewis, Parrilo and Ramana showed in [76] the correctness of Conjecture 1.15. In general, for $n > 3$, the conjecture fails, as a comparison of the dimension of the set of hyperbolic polynomials with the dimension of the set of determinantal polynomials (1.7) shows; see again [76].

Example 1.16. *As an illustration of the Lax Conjecture, we return to Example 1.14. The order of matrices used there is n, which is independent of the degree. The Lax Conjecture states that there is a definite determinantal representation for f using 2×2-matrices. Indeed, for $n = 3$, it holds*

$$f(z_1, z_2, z_3) = z_1^2 - z_2^2 - z_3^2 = \det \begin{pmatrix} z_1 - z_2 & z_3 \\ z_3 & z_1 + z_2 \end{pmatrix}$$

and f is hyperbolic with respect to $(1, 0, 0)$. In particular, this implies

$$\mathcal{L} = \left\{ \mathbf{x} \in \mathbb{R}^n : \begin{pmatrix} x_1 - x_2 & x_3 \\ x_3 & x_1 + x_2 \end{pmatrix} \succ 0, x_1 > 0 \right\} .$$

For a higher number of variables, a determinantal representation using 2×2-matrices is impossible. Assume for $n > 3$ there would be a representation with a 2×2-matrix $\mathcal{A}(\mathbf{x})$ such that $\det \mathcal{A}(\mathbf{x}) = x_1^2 - \sum_{j=2}^n x_j^2$. Then, we can choose an $\mathbf{x} \in \mathbb{R}^n$, such that $x_1 = 0$ and the first row of $\mathcal{A}(\mathbf{x})$ vanishes. This implies $\det \mathcal{A}(\mathbf{x}) = 0$, but $x_1^2 - \sum_{j=2}^n x_j^2 < 0$.

The conjecture concerning definite determinantal representations of any size is known as the Generalized Lax Conjecture. For $n > 3$, it was disproven by Brändén in [15]. His smallest counter example is the bases generating polynomial of the Vámos matroid, a polynomial of degree 4 in 8 variables; see Example 1.38.

This negative result gave rise to further variations and generalizations (see, e.g., [8, Section 5] and [15]) and caused active research in real algebraic geometry. For instance, it was shown recently in [1, 17] that the bases generating polynomial of generalized Vámos matroids do not have a definite determinantal representation although they are hyperbolic. For an overview on Lax Conjecture and generalizations, we refer to [97, 124]. The currently best known generalization, which is also often called the Generalized Lax Conjecture, reads as follows. Here, we call an open cone spectrahedral if it is the interior of a spectrahedron.

Conjecture 1.17. Every hyperbolicity cone is a spectrahedral cone.

Note, that the converse is always true; a spectrahedral cone is always a hyperbolicity cone. This is due to Example 1.7.

An equivalent formulation of Conjecture 1.17 is the following; see [124].

Conjecture 1.18. Let $f \in \mathbb{R}[\mathbf{z}]$ be hyperbolic with hyperbolicity cone C_f. Then, there is a hyperbolic polynomial $g \in \mathbb{R}[\mathbf{z}]$ with hyperbolicity cone C_g, such that $C_f \subseteq C_g$ and $g \cdot f$ has a definite determinantal representation.

This generalization of the Lax Conjecture is known to be true for some special cases. Namely, for instance for $n = 3$, see [76], for elementary symmetric functions, see [16], and for quadratic polynomials that are real-zero, see [99]. Moreover, if the projective variety of f is smooth, there is always a hyperbolic polynomial g such that $g \cdot f$ has a definite determinantal representation, but with lack of the additional condition to the hyperbolicity cone of g, see [69]. Furthermore, every smooth hyperbolicity cone is the projection of a spectrahedral cone, see [98]. An example, where Conjecture 1.18 is valid, is provided by Example 1.14.
For recent results regarding definite determinantal representations of sets, we refer to [68, 99, 100, 107, 124] and the references therein.

1.2.1.2 Hyperbolic programming

Hyperbolic polynomials became of recent interest in optimization when studying hyperbolic programs. Hyperbolic programming is a natural generalization of semidefinite programming and second-order cone programming, see Examples 1.7 and 1.8, by replacing the set of feasible solutions by an affine-linear slice of the topological closure of the hyperbolicity cone of a (homogeneous) hyperbolic polynomial.

Being precise, a hyperbolic program has the form

$$\begin{aligned} h^* := \min\ & \mathbf{c}^T \mathbf{x} \\ \text{s.t. } & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \in \overline{C_f} \end{aligned}$$

where $\mathbf{c}, \mathbf{x} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$, and $\overline{C_f}$ denotes the closure of a hyperbolicity cone C_f of a given polynomial $f \in \mathbb{R}[z_1, \ldots, z_n]$, which is

$$\{\mathbf{x} \in \mathbb{R}^n : t \mapsto f(\mathbf{x} + t\mathbf{e}) = 0 \ \Rightarrow\ t \leq 0\}.$$

Note that $h^* < \infty$, since f is hyperbolic, which implies $C_f \neq \emptyset$, by definition. Hyperbolic programs are convex optimization problems. They can be solved numerically by interior point methods.

Interior point methods are an effective tool for solving convex optimization problems numerically. The idea of interior point methods is to consider a family of adapted convex optimizations problems $\{\mathcal{O}_\mu\}_{\mu>0}$ by using a barrier function depending on a parameter $\mu > 0$. Generally, a barrier function for a set M is a continuous function whose value increases to infinity when the argument approaches the boundary of M. Often, one uses logarithmic barrier functions that encode the defining constraints of M. A barrier function guarantees that a solution of $\mathcal{O}_\mu$ lies in the (relative) interior of the set of feasible solutions of the initial convex optimization program $\mathcal{O}$. For fixed μ, the optimization problem $\mathcal{O}_\mu$ is strictly convex. Hence, it has one unique solution $\mathbf{x}_\mu^*$, which can be computed for example by Newton's method. For $\mu \to 0$, the optimal values $\mathbf{x}_\mu^*$ tend to the optimal solution of $\mathcal{O}$. The family $\{(\mu, \mathbf{x}_\mu^*)\}_{\mu>0}$ is called the central path of $\{\mathcal{O}_\mu\}_{\mu>0}$. If $\mathbf{y}_\mu^*$ denotes the optimal solution of the dual problem to $\mathcal{O}_\mu$, then $\{(\mu, \mathbf{x}_\mu^*, \mathbf{y}_\mu^*)\}_{\mu>0}$ is called the primal-dual path and $\mathbf{y}_\mu^*$ converges for $\mu \to 0$ to the optimal solution of the dual to the initial problem.

Example 1.19. *Let $P := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_i^T\mathbf{x} \leq b_i, 1 \leq i \leq m\}$ be a full-dimensional polytope with $\operatorname{int} P \neq 0$. The function*

$$b_\mu(\mathbf{x}) := -\mu \sum_{i=1}^{m} \log(b_i - \mathbf{a}_i^T\mathbf{x}) \tag{1.8}$$

is for all $\mu > 0$ a barrier function for P. Clearly, it holds $b \to \infty$ whenever $\mathbf{x} \to \partial P$, $\mathbf{x} \in \operatorname{int} P$.

If $\max\{\, f(\mathbf{x}) : \mathbf{x} \in P\,\}$ is an optimization problem with an optimal solution $\mathbf{x}^*$, then for every $\mu > 0$ the optimal solution $\mathbf{x}_\mu^*$ of the adapted optimization problem

$$\mathcal{O}_\mu : \ \max\left\{ f(\mathbf{x}) - \mu \sum_{i=1}^{m} \log(b_i - \mathbf{a}_i^T\mathbf{x}) : \mathbf{x} \in \operatorname{int} P \right\}$$

fulfills:

1. $\mathbf{x}_\mu^* \in \operatorname{int} P$,
2. $\mathbf{x}_\mu^* \to \mathbf{x}^*$ *for* $\mu \to 0$.

The interest in interior point methods started in 1984, when Kamarkar presented in his seminal paper [63] an interior point algorithm for linear programs. Interior point methods are polynomial time algorithms. His algorithm was the first known polynomial time algorithm for linear programs beside the ellipsoid method of Khachiyan, [64], but which is slow in practice. Modern interior point methods are a real alternative to the simplex algorithm, which has theoretically exponential complexity due to an example by Klee and Minty; see [66]. Subsequent research improved the result of Kamarkar, for example by using Newton's method to follow the primal-dual path; see [110]. Following research led to improved computational complexity for interior point methods for example in linear, semidefinite and hyperbolic programming, and other classes of convex optimization problems.

In [94], Nesterov and Nemirovskii developed the theory of self-concordant barrier functions. These are barrier functions, whose third derivative is bounded, in a specific way, by its second derivative. This leads to a very well behavior for Newton's method, which optimizes them efficiently. Nesterov and Nemirovskii studied special classes of convex optimization problems and presented adapted barrier functions. Equation (1.8) shows a self-concordant barrier function for a polytope. Nesterov and Nemirovskii proved the existence of a self-concordant barrier function for any full-dimensional, closed, convex cone; see [94, Chapter 2]. Hence, the method of self-concordant barrier functions can be applied to any convex optimization problem, theoretically.

For a hyperbolic polynomial f with hyperbolicity cone C_f, the function $-\log f(x)$ is a barrier function to $\overline{C_f}$, since $\partial C_f \subset \mathcal{V}_{\mathbb{R}}(f)$, and it is self-concordant; see [45]. Consequently, the theory of interior point methods and self-concordant barrier functions can be applied to hyperbolic programs.

For hyperbolic programs, interior point methods with self-concordant barrier functions were first studied by Güler in 1997; see [45]. Subsequent research improves the convergence of the interior point algorithm or considers algorithms solving hyperbolic programming in exact arithmetic; see, e.g., [93, 95]. However, we remark that due to the unsolved generalized Lax Conjecture 1.17, it is open whether hyperbolic programming provides indeed more feasible sets than semidefinite programming.

For an introduction to hyperbolic programming, we refer to [45, 111] and for a general introduction to convex optimization and interior point methods we refer to [27, 94, 108].

1.3 Stable polynomials

In recent years, stable polynomials attracted a lot of interest; see, e.g., [8, 10, 83, 126] and the references therein. As prominent applications, Marcus, Spielman, and Srivastava employed (univariate) stable polynomials in the proof of the Kadison-Singer Conjecture, see [83], and in the existence proof of infinite families of bipartite Ramanujan graphs of every degree larger than two; see [82]. Stable polynomials have also been used by Borcea and Brändén to prove Johnson's Conjecture, see [8], and in Gurvits' simple proof of van der Waerden's Conjecture for permanents; see [46]. We discuss this more detailed in Section 1.3.3, where we give a historical motivation and show connections to different mathematical fields. Before that, in Section 1.3.1 and 1.3.2, we give an overview on classical results for univariate stable polynomials and on more recent results for multivariate stable polynomials.

Definition 1.20. *A polynomial $f \in \mathbb{C}[\mathbf{z}]$ is called stable if every root $\mathbf{z} = (z_1, \ldots, z_n)$ satisfies $\mathrm{Im}(z_j) \leq 0$ for some j. f is called real stable if f has real coefficients and is stable.*

Equivalently, we can say that a polynomial is stable if and only if there is no root of f with only positive imaginary parts. For univariate polynomials with real coefficients, stability is equivalent to being real-rooted. In that sense, stability is a generalization of real-rootedness. Moreover, there is a strong connection to hyperbolic polynomials and their hyperbolicity cones; see Proposition 1.28.
As a general source on stability of polynomials, we refer to [109, 126] and the references therein.

In Chapter 3, we generalize stability further to stability with respect to the interior of an arbitrary proper cone $K \subset \mathbb{R}^n$. We say, a polynomial f is K-stable if there is no root $\mathbf{z} \in \mathbb{R}^n$ with $\mathrm{Im}(\mathbf{z}) \in \mathrm{int}\, K$; see Definition 3.1. Then, Definition 1.20 coincides with $(R_{\geq 0})^n$-stability. We transfer important results for univariate and multivariate stable polynomials to the more general conic situation. In Section 3.1.2, we consider conic stability for the cone of positive semidefinite matrices as an application.

1.3.1 Stability of univariate polynomials

For univariate polynomials, stability is well-understood and has a long history. In this section, we give a short overview on some major results.

For univariate polynomials, real stability is equivalent to real-rootedness, since complex roots occur pairwise with its complex conjugate. This leads to the following equivalent viewpoints on real stability. The connection to stability of polynomials with complex coefficients is captured by the Hermite-Biehler Theorem 1.22.

For univariate, real stable polynomials $f, g \in \mathbb{R}[z]$, let $W(f,g) = f'g - g'f$ denote the Wronskian of f and g and write $f \ll g$ if $W(f,g) \leq 0$ on $\mathbb{R}$. In the context of univariate, real stable polynomials, the following concept of interlacing roots naturally appears.

Definition 1.21. *Let $f, g \in \mathbb{R}[z]$ be two univariate, real-rooted polynomials with roots $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{\deg f}$ and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_{\deg g}$. We say that f and g interlace if their roots alternate, i.e., $\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \cdots$ or $\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots$. If all inequalities are strict, we say f and g interlace strictly.*

We say that f interlaces g properly (or: f is a proper interlacing of g), if

- $\beta_{\deg g} \geq \alpha_{\deg f} \geq \beta_{\deg g-1} \geq \alpha_{\deg f-1} \geq \cdots$, *when the leading coefficients of f and g have the same sign,*
- $\alpha_{\deg f} \geq \beta_{\deg g} \geq \alpha_{\deg f-1} \geq \beta_{\deg g-1} \geq \cdots$, *when the leading coefficients of f and g have different signs.*

The sequences of signs of $\big(f(\beta_j)\big)_{j=1,\ldots,\deg g}$ and $\big(g(\beta_j)\big)_{j=1,\ldots,\deg f}$ alternate if and only if f and g interlace strictly. For interlacing polynomials f and g, the degrees of f and g can only differ by at most 1. Note that $f \ll g$ is equivalent to the fact that f is a proper interlacing of g. See Figure 1.1 for two proper interlacing polynomials.

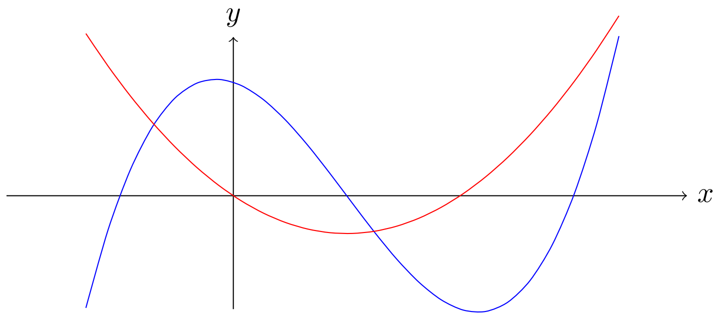

FIGURE 1.1: The polynomial $g(x) = x^2 - 2x$ (red) is a proper interlacing of $f(x) = x^3 - 3x^2 - x + 3$ (blue).

We collect two classical theorems on univariate stable polynomials; see [109, 126]. These results were extended by Borcea and Brändén in [11] to multivariate polynomials; see Section 1.3.2. In Chapter 3 we generalize these two theorems to conic stability.

Proposition 1.22 (Hermite-Biehler)**.** *For non-constant polynomials $f, g \in \mathbb{R}[z]$, the following are equivalent:*

1. *$g + if$ is stable.*
2. *f, g are real stable and $f \ll g$.*

3. *f, g are real stable and f is a proper interlacing of g.*

When extending the definition of $\ll$ and of interlacing to arbitrary $f, g \in \mathbb{R}[x]$ by requiring real stability of f and g, condition (2) can be written shortly as $f \ll g$ and (3) can be written shortly as: f is a proper interlacing of g.

The following theorem characterizes stability of linear combinations of polynomials and characterizes their stability.

Proposition 1.23 (Hermite-Kakeya-Obreschkoff (HKO, for short))**.** *Let $f, g \in \mathbb{R}[z]$. Then $\lambda f + \mu g$ is stable or the zero polynomial for all $\lambda, \mu \in \mathbb{R}$ if and only if f and g interlace or $f \equiv g \equiv 0$.*

We remark that due to the Hermite-Biehler Theorem 1.22 the "only if"-part is equivalent to

1. $f \ll g$ or $g \ll f$ or $f \equiv g \equiv 0$,
2. $g + if$ or $f + ig$ is stable or $f \equiv g \equiv 0$.

The following theorem considers convex combinations of stable polynomials and weakens the condition of interlacing; see, e.g., [26] or [32, Theorem 2'].

Proposition 1.24. *Let f and g be stable polynomials. Then $\lambda f + (1 - \lambda)g$ is stable or the zero polynomial for all $\lambda \in [0, 1]$ if and only if f and g have a common interlacing, i.e., there is a real-rooted polynomial which is an interlacing of f and g.*

Note that the degree of polynomials with a common interlacing can only differ by at most one. Figure 1.2 illustrates the concept of a common interlacing by showing an example.

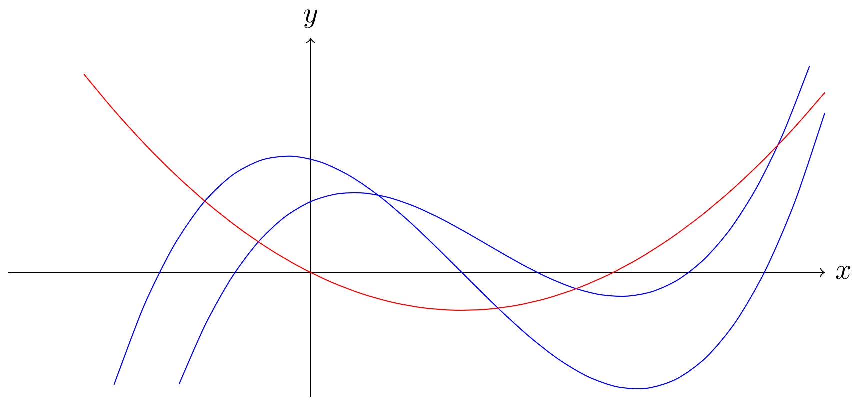

FIGURE 1.2: The two blue polynomials $f_1(x) = x^3 - 3x^2 - x + 3$ and $f_2(x) = x^3 - 3.5x^2 + 1.75x + 1.875$ have $g(x) = x^2 - 2x$ (red) as a common interlacing. The polynomials f_1 and f_2 do not interlace.

Polynomials with a common interlacing play an essential role in the proof of the Kadison-Singer Conjecture in [83] and the existence of bipartite Ramanujan graphs in [82]. For this, Marcus, Spielman, and Srivastava extended the concept of a common interlacing to families of univariate, real stable polynomials in the natural way. At the moment, a multivariate version of Proposition 1.24 does not exist.

1.3.2 Stability of multivariate polynomials

The theory of multivariate stable polynomials is much more recent than univariate stability. Since they provide interesting connections to matroid theory and other areas of mathematics, there is interest from several points of view; see, e.g., [8, 14, 15, 22, 71, 106] as well as the survey [126]. In Section 1.3.3, we discuss the context of stable polynomials. Here, we summarize important results for multivariate stable polynomials, which will be relevant later on.

Following [126], for multivariate polynomials $f, g \in \mathbb{R}[\mathbf{z}]$, one writes $f \ll g$ if $g + if$ is stable. This makes the multivariate Hermite-Biehler statement a definition rather than a theorem. It turns out that if $g + if$ is stable, then g and f are real stable or identically zero; see, e.g., [11, 126]. We remark, that in literature the zero polynomial is sometimes stable by definition; see, e.g., [126].

The multivariate version of the HKO Theorem then has the same format as the univariate version; see Proposition 1.25 below. The multivariate theorem was shown in [11, Theorem 1.6]; see also [9, Theorem 2.9], [126, Theorem 2.9]. It will turn out in Chapter 3 that the multivariate HKO Theorem 1.25 is a special case of the conic HKO Theorem 3.12.

Proposition 1.25 (Multivariate HKO of Borcea and Brändén)**.** *Let $f, g \in \mathbb{R}[\mathbf{z}]$. Then $\lambda f + \mu g$ is stable or the zero polynomial for all $\lambda, \mu \in \mathbb{R}$ if and only if $f \ll g$ or $g \ll f$ or $f \equiv g \equiv 0$.*

An important class of stable polynomials comes from determinantal representations; see [8, Theorem 2.4] and [15, 76].

Proposition 1.26 (Borcea, Brändén)**.** *Let $A_1, \ldots, A_n$ be positive semidefinite $d \times d$-matrices and B be a Hermitian $d \times d$-matrix, then*

$$f(\mathbf{x}) \;=\; \det\big(\sum_{j=1}^{n} x_j A_j + B\big)$$

is real stable or the zero polynomial.

Determinantal representations of this kind are relevant in the context of the Lax Conjecture proven by Lewis, Parrilo, and Ramana in [76]. In [11], Borcea and Brändén proved an analog of the Lax Conjecture for bivariate stable polynomials.

Stable polynomials $f \in \mathbb{C}[\mathbf{z}]$ were preserved by several operations. Due to the symmetry and the conic structure of $\mathcal{H}_{\mathbb{C}}^n$, stability is preserved obviously by permutation of variables, scaling variables with positive scalars, by diagonalization $f \mapsto f(\mathbf{z})|_{z_j=z_k}$ for $\{j,k\} \subseteq \{1,\ldots,n\}$, and by inversion $f \mapsto z_1^d f(-z_1^{-1}, z_2, \ldots, z_n)$, where $\deg_{z_1} f = d$ denotes the degree of f in z_1. We refer to [126, Lemma 2.4] for the proofs.

Moreover, the following holds.

Proposition 1.27. ([8, Prop. 2.2], [126, Lemma 2.4]) *For a stable polynomial $f \in \mathbb{C}[\mathbf{z}]$, the following polynomials are stable or identically zero:*

1. *the specialization $f(a, z_2, \ldots, z_n)$, $a \in \mathbb{C}$, $\mathrm{Im}(a) \geq 0$,*
2. *the partial derivative $\partial_{z_1} f(z_1, \ldots, z_n)$.*

There is a close connection between stable, homogeneous polynomials and hyperbolic polynomials; see, e.g., [71, 106].

Proposition 1.28. *Let $f \in \mathbb{R}[\mathbf{z}]$ be homogeneous. Then, f is stable if and only if f is hyperbolic with respect to every point in the positive orthant.*

In Theorem 2.37 we confirm this relation by proving a strengthening of this statement, which specializes easily to Proposition 1.28. So we omit its proof here. The lemma following gives a hint, where this connection comes from. Furthermore, it yields a possibility to reduce the stability of multivariate polynomials to the univariate situation. The univariate case is well-understood; an overview was given above in Section 1.3.1.

Lemma 1.29. *Let $f \in \mathbb{C}[\mathbf{z}]$ be a polynomial. f is stable if and only if for every $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} > 0$, the univariate polynomial $t \mapsto f(\mathbf{x} + t\mathbf{y})$ is stable.*

This statement is a consequence of the fact, that a root $\mathbf{x} + i\mathbf{y}$ of f with $\mathbf{y} > 0$ yields the imaginary root i of $t \mapsto f(\mathbf{x} + t\mathbf{y})$ and similarly vice versa.

As a special class of polynomials, multi-linear polynomials were considered. These are polynomials which are multi-affine-linear in every variable. Brändén characterized stability of multi-linear polynomials with real coefficients in [14].

Proposition 1.30. ([14, Theorem 5.6]) *Let $f \in \mathbb{R}[\mathbf{z}]$ be multi-linear. Then f is stable if and only if for all $1 \leq j, k \leq n$ the function*

$$\Delta_{jk}(f) \;=\; \frac{\partial f}{\partial z_j}\frac{\partial f}{\partial z_k} - \frac{\partial^2 f}{\partial z_j \partial z_k} \cdot f \tag{1.9}$$

is non-negative on $\mathbb{R}^n$.

Hence, the bivariate polynomial $f(z_1, z_2) = \alpha z_1 z_2 + \beta z_1 + \gamma z_2 + \delta$ with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ is real stable if and only if $\beta\gamma - \alpha\delta \geq 0$.

The non-multi-linear case can be reduced to the multi-linear case via the polarization $\mathcal{P}(f)$ of a multivariate polynomial f; see [14]. Denoting by d_j the degree of f in the variable z_j, $\mathcal{P}(f)$ is the unique polynomial in the variables z_{jk}, $1 \leq j \leq n$, $1 \leq k \leq d_j$, with the properties

1. $\mathcal{P}(f)$ is multi-linear,
2. $\mathcal{P}(f)$ is symmetric in the variables $z_{j1}, \ldots, z_{jd_j}$, $1 \leq j \leq n$,
3. applying the substitutions $z_{jk} = z_j$ for all j, k, then $\mathcal{P}(f)$ coincides with f.

According to the Grace-Walsh-Szegö Theorem, see [14, Cor. 5.8] or [109], $\mathcal{P}(f)$ is stable if and only if f is stable; see, e.g., [14, Cor. 5.9]. The application of the Grace-Walsh-Szegö Theorem makes essentially use of the fact that $\mathcal{H}_{\mathbb{C}}^n$ is bounded by a hyperplane.

By Proposition 1.30, deciding whether a multivariate, multi-linear polynomial f is stable is equivalent to the decision whether $\Delta_{jk}(f) \geq 0$ on $\mathbb{R}^n$ for all j, k. In [71], sum of squares-relaxations are considered to decide this question. It turns out that (1.9) is not always a sum of squares. A counterexample is provided by the bases generating polynomial of the Vámos matroid; see [71, Example 5.11].

The following theorem does not consider stable polynomials specifically. But setting $U = \mathcal{H}_{\mathbb{C}}^n$ we can apply Hurwitz' Theorem 1.31 to stable polynomials. This allows to build limits of stable polynomials at the costs of potentially getting the zero polynomial. We will apply this theorem several times in Chapter 3.

Proposition 1.31. (Hurwitz; see, e.g., [8, Theorem 2.3], [109, Theorem 1.3.8]) *Let $\{f_j\}_{j \in \mathbb{N}} \subset \mathbb{C}[\mathbf{z}]$ be a sequence of polynomials non-vanishing in a connected open set $U \subset \mathbb{C}^n$, and assume it converges to a function f uniformly on compact subsets of U. Then f is either non-vanishing on U or it is identically 0.*

1.3.3 Historical motivation, generalization and applications

The interest in stable polynomials dates back to the 19th century, when Hermite, Hurwitz, Routh and others started to consider (univariate) polynomials that only have roots in a certain region of the complex plane.

Widely known are polynomials whose roots only have a negative real part. Often, they were called Hurwitz-polynomials and they occur by studying dynamical systems in discrete or continuous time. Without going too much into details, Hurwitz-polynomials can be used to certify asymptotic stability of linear and non-linear autonomous systems. Let

$$\mathbf{y}' = A\,\mathbf{y},\ \mathbf{y}(t) \in \mathbb{R}^n \tag{1.10}$$

be a linear autonomous system, with a real or complex $n \times n$-matrix A. If α is a strict upper bound to the real parts of the eigenvalues of A, then there is a constant $\beta > 0$ such that $\|e^{At}\| \leq \beta e^{\alpha t}$ for all $t \geq 0$. Here, we use the matrix exponential $e^{At} := \sum_{j=0}^{\infty} A^j/j!$. Since every solution of (1.10) can be represented as $\mathbf{y}(t) = e^{At}\mathbf{y}(0)$, it follows that if and only if all eigenvalues of A have a negative real part, then $\mathbf{y} = 0$ is an asymptotically stable solution for (1.10). This means, that for any solution $\mathbf{y}$ of (1.10) holds

$$\lim_{t\to\infty} \|\mathbf{y}(t)\| = 0.$$

This implies that any solution of (1.10) behaves asymptotically like the steady state solution $\mathbf{y} = 0$. In general, asymptotic stability is important in the study of the long time behavior of a dynamical system.
Note that if all eigenvalues of a matrix have a negative real part, then its characteristic polynomial is a Hurwitz-polynomial.

More general, let

$$\mathbf{y} = f(\mathbf{y}),\ \mathbf{y}(t) \in \mathbb{R}^n \tag{1.11}$$

be a non-linear autonomous system with a continuously differentiable map $f : \mathbb{R}^n \to \mathbb{R}^n$, then $\mathbf{y} = 0$ is an asymptotically stable solution for (1.11) if and only if all eigenvalues of the Jacobian of f evaluated in 0 have negative real parts. We refer to [130, Chapter III.7]. Moreover, we refer to [87, Section 36] for an illustrative description of the appearance of Hurwitz-polynomials in dynamical systems by considering an exemplary system, which describes the movement of a particle in the two-dimensional plane.

The following criterion for Hurwitz-stability is commonly known and usually it is called the Routh-Hurwitz criterion. It is due to Hurwitz, [55]. More recent proofs using different methods are presented, for example, in [2, 19, 52, 88].

Proposition 1.32. *Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{R}[x]$. The univariate polynomial f is a Hurwitz-polynomial if and only if all leading principal minors of*

$$\begin{pmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ a_n & a_{n-2} & a_{n-4} & \cdots \\ 0 & a_{n-1} & a_{n-3} & \cdots \\ 0 & a_n & a_{n-2} & \cdots \\ 0 & 0 & a_{n-1} & \cdots \\ 0 & 0 & a_n & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \in \mathbb{R}^{n\times n}$$

(where $a_j = 0$ for $j < 0$) have a positive determinant.

For a version concerning polynomials with complex coefficients see, e.g., [87, Theorem 40.1] and [36]. For further criteria certifying Hurwitz-stability, we refer to [128, 131] and the references therein.

Beside many other multivariate generalizations of Hurwitz-stability (see, e.g., [65, Section 2.2]) the concept of half-plane stability is well-established.

Definition 1.33. *Let H be an open half-space of $\mathbb{C}^n$ containing the origin in its boundary. Then, a polynomial $f \in \mathbb{C}[\mathbf{z}]$ is called H-stable if it has no roots inside H. And we say f has the half-plane property if there is an open half-space H whose boundary contains the origin such that f is H-stable.*

If H is the right half-space $\{\mathbf{z} \in \mathbb{R}^n : \operatorname{Re} z > 0\}$, then f is called Hurwitz-stable; see, e.g., [14, 22, 127]. If H is the upper half-space $\mathcal{H}_{\mathbb{C}}^n = \{\mathbf{z} \in \mathbb{R}^n : \operatorname{Im} z > 0\}$, then f is stable in the sense of Definition 1.20. If a polynomial $f(\mathbf{z})$ is H-stable, then $f(O^T\mathbf{z})$ is OH-stable, where O is an orthogonal matrix. Hence, up to rotation, the different notions of half-plane stability coincide.

We give a brief and certainly not complete overview on the appearance of H-stable polynomials in different areas, starting with combinatorics and graph theory. We begin with the connection between stable polynomials and matroids. Namely, Choe, Oxley, Sokal, and Wagner proved in [22] that the support of an H-stable, multi-linear and homogeneous polynomial is the set of bases of a matroid. Since then, the relationship between polynomials with the half-plane property and matroid theory has been considered further, e.g., in [14, 21, 125]. Here, we call a polynomial multi-linear if it has degree at most one in each variable and the support of a polynomial $f(\mathbf{z}) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha \mathbf{z}^\alpha$ is defined as

$$\operatorname{supp}(f) = \{\alpha \in \mathbb{N}^n : a_\alpha \neq 0\}.$$

The connection to matroids will be shortly summarized in the following. We recall basic definitions in matroid theory.

Definition 1.34. *Let E be a finite set and let $\mathcal{M}$ be a collection of subsets of E, which are called independent sets. The pair $(\mathcal{M}, E)$ is called a matroid if it satisfies the following:*

1. $\emptyset \in \mathcal{M}$,
2. *if* $A \in \mathcal{M}$ *and* $B \subseteq A$, *then* $B \in \mathcal{M}$,
3. *if* $A, B \in \mathcal{M}$ *and* $|A| > |B|$, *then there is* $x \in A \setminus B$, *such that* $B \cup \{x\} \in \mathcal{M}$.

The sets of $\mathcal{M}$ that are maximal with respect to inclusion are called bases of the matroid $(\mathcal{M}, E)$. If $\mathcal{B}$ is the basis of a matroid $(\mathcal{M}, \{1, \dots, n\})$, then the polynomial

$$\sum_{B \in \mathcal{B}} \prod_{j \in B} x_j \ \in \ \mathbb{R}[x_1, \dots, x_n]$$

is called the bases generating polynomial of the matroid $(\mathcal{M}, E)$.

There are two important concepts related to matroids, the Δ-matroids and the jump systems. For an overview on Δ-matroids and jump systems, we refer to [13].

The notion of a Δ-matroid was introduced in [12] as a generalization of the independent sets and the set of bases of a matroid. It reads as follows; see [13].

Definition 1.35. *Let $\mathcal{M}$ be a family of subsets of a finite set E. Then the pair $(\mathcal{M}, E)$ is called a Δ-matroid if it satisfies the following symmetric exchange axiom:*
If $A_1, A_2 \in \mathcal{M}$ and $x \in A_1 \Delta A_2$, then there is an $x' \in A_1 \Delta A_2$ such that $A_1 \Delta \{x, x'\} \in \mathcal{M}$. Here, Δ denotes the symmetric difference of two sets.

As pointed out in [13], if $\mathcal{F}$ is the family of bases of a matroid $(\mathcal{M}, E)$, then $(\mathcal{F}, E)$ is a Δ-matroid. And if $(\mathcal{F}, E)$ is a Δ-matroid such that all elements in $\mathcal{F}$ have the same cardinality, then $\mathcal{F}$ is the set of bases of a matroid $(\mathcal{M}, E)$.

Jump systems were introduced by Bouchet [12]; see also [13]. They are defined as follows; see [14]. We use the notation $\mathrm{St}(x, x') := \{y \in \mathbb{Z}^n : \|y\|_1 = 1, \|x+y-x'\|_1 = \|x-x'\|_1 - 1\}$. This set is called the set of steps from x to x'.

Definition 1.36. *Let $\mathcal{M} \subset \mathbb{Z}^n$ be a collection of points. $\mathcal{M}$ is called a jump system if it satisfies the following two-step axiom:*
If $x, x' \in \mathcal{M}$, $y \in St(x, x')$ and $x + y \notin \mathcal{M}$, then there is a $y' \in St(x + y, x')$ such that $x + y + y' \in \mathcal{M}$.

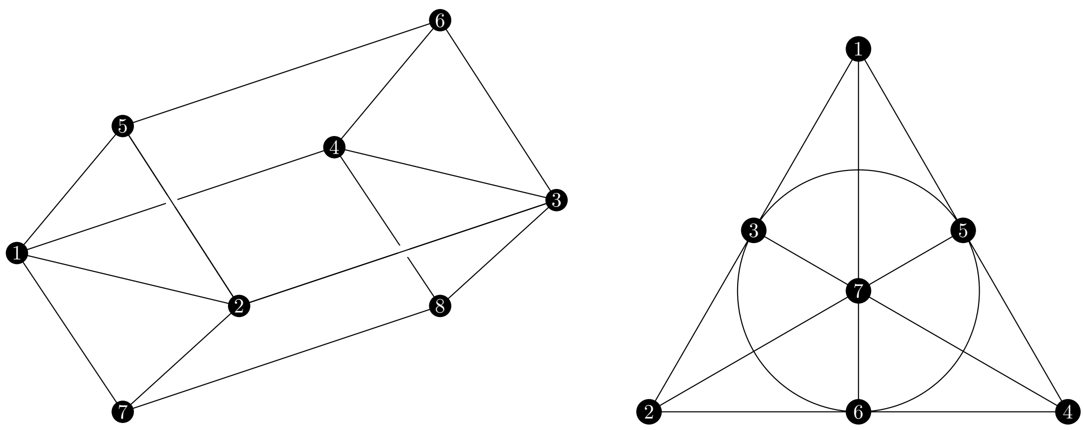

FIGURE 1.3: The left image illustrates the Vámos matroid. The surfaces bounded by four lines represent these subsets of $\{1, 2, \ldots, 8\}$ that are not basis sets. The right image illustrates the Fano matroid. Here, the lines represent these sets, which are no bases sets.

Δ-matroids are these jump systems for which $\mathcal{M} \subseteq \{0, 1\}^n$ for some positive integer n; see [14].

The following proposition captures the connection between stable polynomials and matroid theory.

Proposition 1.37. *Let f be a polynomial with half-plane property. Then the following holds:*

1. *The support of f is a jump system; see [14, Theorem 3.2].*
2. *If f is multi-linear, then the support of f is a Δ-matroid; see [14, Cor. 3.3].*
3. *If f is homogeneous and multi-linear, then the support of f is the set of bases of a matroid; see [14, Cor. 3.4] and [22].*

Example 1.38. *As a non-trivial example for Proposition 1.37, (3), we consider the Vámos matroid. The set of bases of the Vámos matroid are all subsets of* $\{1, 2, \ldots, 8\}$ *with cardinality* 4 *except the sets* $\{1, 2, 3, 4\}$, $\{1, 2, 5, 6\}$, $\{1, 2, 7, 8\}$, $\{3, 4, 5, 6\}$, *and* $\{5, 6, 7, 8\}$*; see Figure 1.3 for an illustration. It was proven in [127], that the bases generating polynomial of the Vámos matroid has the half-plane property. We refer to [127] for more examples.*

Bases generating polynomials of matroids are homogeneous, multi-linear polynomials. This motivates to consider the converse direction of Proposition 1.37, (3). Unfortunately, the converse is false in general.

Example 1.39. *A counterexample to the converse of Proposition 1.37, (3) is provided by the Fano matroid. For the Fano matroid the set of bases is the collection of all subsets* *of* $\{1,2,\ldots,7\}$ *with cardinality* 3*, that do not lie on a line. These are all subsets of cardinality* 3*, except* $\{1,2,3\}$*,* $\{3,4,5\}$*,* $\{1,5,6\}$*,* $\{1,4,7\}$*,* $\{2,5,7\}$*,* $\{3,6,7\}$*, and* $\{2,4,6\}$*; see Figure 1.3 for an illustration. It was proven in [14, Theorem 6.6] that there is no polynomial with half-plane property whose support is the Fano matroid. We refer to [1, 17] for more counterexamples.*

Stable polynomials also provide a connection to graph theory. The following polynomial is a prominent example.

Example 1.40. *Let* G *be a graph with* n *vertices and let* m_k *be the number of* k*-matchings. A* k*-matching is a set of* k *independent edges. The matching polynomial of* G *is defined as*

$$\mu_G(x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k m_k x^{n-2k} \ . \tag{1.12}$$

It was first proven by Heilmann, Lieb in [48] that μ_G *has only real roots. Hence, it is real stable. An independent proof was given by Godsil in [42] using the method of spanning trees. In [75] a multivariate generalization of the matching polynomial is considered and proven to be real stable as well.*
For an overview on matching polynomials, we refer to [31, 43].

We call a graph d-regular if every vertex has exactly d adjacent edges. A Ramanujan graph is a connected, d-regular graph whose non-trivial eigenvalues of the adjacency matrix lie in the interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$. An example for a Ramanujan graph is the Petersen graph; see, e.g., [67, Example 3.9]. Ramanujan graphs are of interest in theoretical computer sciences, in complexity theory, network theory and in coding theory, since they are good expander graphs. Expander graphs are sparse graphs with strong connectivity properties. In [80] a first construction of Ramanujan graphs was described and there are some more constructions known; see, e.g., [20, 92]. It was a question by Lubotzky whether for every positive integer d there exists an infinite class of d-regular, bipartite graphs that are Ramanujan; see [79, Problem 10.7.3]. This was recently proven by Marcus, Spielman, and Srivastava in [82]. Their proof substantially relies on univariate, real stable polynomials with common interlacing. They used a method which they called the "method of interlacing families" and which they also applied in subsequent articles; see [83, 85] as well as their 2015 paper [84] concerning bipartite Ramanujan graphs of arbitrary degree and arbitrary number of vertices. An interlacing family is a family of polynomials with one common interlacing. This structure provides control about the largest root of one of these polynomials in terms of the largest root of their sum.

Beyond that, stable polynomials were used to prove a lot of open conjectures in the last few years. In the following, we will present a selection, which is certainly not complete.

The most famous one is the Kadison-Singer Conjecture, stated in 1959; see [61]. Arising in quantum mechanics, it was a major open problem in operator theory until it was solved by Marcus, Spielman, and Srivastava in 2013, [83], by using discrete methods and, in particular, their "method of interlacing families". Until the verification, many authors, including Kadison and Singer, believed that the statement would be wrong.
Without going into the definitions, the Kadison-Singer Conjecture asks for the unique extension of pure states in the space of bounded diagonal operators on ℓ_2 to pure states in the space of bounded operators in ℓ_2.
We refer to [18] for an overview of the Kadison-Singer Conjecture and its influence on other mathematical areas and on lots of equivalent conjectures. Moreover, we refer to [47] for a 2-dimensional example and for the connection of the Kadison-Singer Conjecture to the equivalent Weaver Conjecture [129], which was actually proven by Marcus, Spielman, and Srivastava. Their proof uses discrete methods and relies essentially on univariate, real stable polynomials with a common interlacing. We refer to former lecture notes of Marcus and Srivastava, [86], for a more detailed proof of the Kadison-Singer Conjecture including annotations and examples.

In 1989, Johnson presented a generalization of the determinant for pairs of matrices; see [57]. If A and B are $d \times d$-matrices, then the Johnson polynomial is defined as

$$\eta(A, B) = \sum_{M \subset \{1,\ldots,d\}} \det A[M] \det B[M^{\mathrm{c}}], \tag{1.13}$$

where $A[M]$ denotes the $|M| \times |M|$-principle sub-matrix of A whose rows and columns are indexed by M. By convention, it is $A[\emptyset] = B[\emptyset] = 1$. It holds $\eta(zI_d, -B) = \det(zI_d - B)$, which is the characteristic polynomial of B. In that way, (1.13) generalizes the characteristic polynomial of a matrix.
It is classically known that the characteristic polynomial of Hermitian matrices is real-rooted and that the eigenvalues of a Hermitian matrix and the eigenvalues of any of its $(d-1) \times (d-1)$ principal sub-matrices interlace, which is known as the Cauchy-Poincaré Theorem.
Johnson conjectured that these two properties as well as a generalization of Sylvester's law of inertia still hold for $\eta(zA, -B)$ whenever $A \succeq 0$. We refer to [57] and [8, Conjectures 1.1 – 1.3] for an exact formulation of Johnson's conjectures. For almost two decades there were only a few positive answers for very special classes of matrices known. We refer to [3, 24, 113] for some of these results, like for tridiagonal matrices. In 2008 Johnson's conjectures were proven by Borcea and Brändén in [8]. The proof relies on

the construction of an adapted multivariate real-stable polynomial, such that a suitable application of operations preserving real stability yields Johnson's polynomial.

The van der Waerden Conjecture from 1928 concerns a lower bound for the permanent of a matrix. If Ω_d denotes the set of doubly stochastic $d \times d$-matrices and $\mathrm{per}(A)$ denotes the permanent of a matrix A, the conjecture states that

$$\min_{A \in \Omega_d} \mathrm{per}(A) = \frac{d!}{d^d} ,$$

where the minimum is only attained for the diagonal matrix $\frac{1}{d} \cdot I_d$. Beside the first proofs [28, 30], there is a more simple proof given by Gurvits in 2008 using polynomials with the half-plane property; see [46]. Gurvits confirms the van der Waerden Conjecture and derived immediately the so called Schrijver-bound and the Bapat-bound for permanents; see [46, Section 1] for their definition. His proof relies on the observation that for a homogeneous Hurwitz-stable polynomial p with positive coefficients it holds

$$\frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0, \dots, 0) \geq \frac{n!}{n^n} \mathrm{Cap}(p), \tag{1.14}$$

where

$$\mathrm{Cap}(p) \coloneqq \inf_{\substack{x_j > 0 \\ 1 \leq j \leq n}} \frac{p(x_1, \dots, x_n)}{x_1 \cdots x_n}$$

denotes the capacity of p. Moreover, if p is doubly stochastic (i.e., it is homogeneous, has positive coefficients and satisfies $\frac{\partial p}{\partial x_j}(1, \dots, 1) = 1$ for $1 \leq j \leq n$), then $\mathrm{Cap}(p) = 1$. Gurvits constructed in [46] a doubly stochastic polynomial p with the half-plane property such that the left side in (1.14) equals the permanent. In analogy to Proposition 1.27, the left side of (1.14) has the half-plane property as well.

1.4 Amoebas and coamoebas

Beside stable and hyperbolic polynomials, the study of imaginary projections is motivated by the theory of amoebas and coamoebas. For amoebas, important structural results as well as their occurrences in a broad spectrum of mathematical disciplines have been intensively studied; see [90, 104, 105] as well as the recent survey [25]. For coamoebas, investigations are much more recent [34, 35, 102]. A prominent result states that the complement of an amoeba as well as the complement of the closure of a coamoeba consists of finitely many convex components; see [35, 41]. This holds true for imaginary projections as well; see Theorem 2.10. See Figure 1.4 for an example of an amoeba and a coamoeba. While there are important analogies among amoebas, coamoebas, and imaginary projections, there are also fundamental differences between these structures. Within the thesis, we point out similarities between these structures and remark differences.

This section collects some known facts about amoebas and coamoebas.

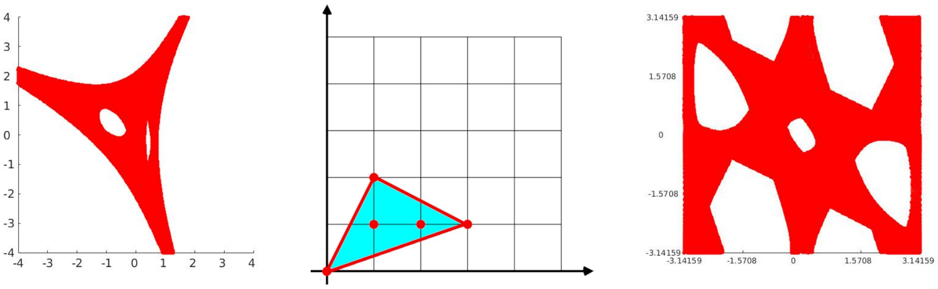

FIGURE 1.4: An approximation of the amoeba and the coamoeba of the Laurent polynomial $f(z_1, z_2) = 2z_1^3 z_2 + z_1 z_2^2 - 4z_1 z_2 - 2.5 \cdot e^{0.7\cdot\pi\cdot i} z_1^2 z_2 + 1$ together with its corresponding Newton polytope in the middle.

The theory of amoebas builds upon algebraic varieties in the complex torus $(\mathbb{C}^*)^n = (\mathbb{C} \setminus \{0\})^n$. For a Laurent polynomial $f \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$, define the semi-algebraic amoeba $\mathcal{U}(f)$ (also known as unlog amoeba) by

$$\mathcal{U}(f) \;=\; \{|\mathbf{z}| = (|z_1|, \ldots, |z_n|) \in \mathbb{R}^n : \mathbf{z} \in \mathcal{V}(f) \cap (\mathbb{C}^*)^n\}, \tag{1.15}$$

and the amoeba $\mathcal{A}(f)$ by

$$\mathcal{A}(f) \;=\; \{\log|\mathbf{z}| = (\log|z_1|, \ldots, \log|z_n|) \in \mathbb{R}^n : \mathbf{z} \in \mathcal{V}(f) \cap (\mathbb{C}^*)^n\}\,. \tag{1.16}$$

Amoebas were first introduced and studied by Gelfand, Kapranov and Zelevinsky in [41]. Computations and approximations of amoebas were studied, e.g., in [120, 121].
We recall some basic statements about amoebas; see [33, 41, 121]. For a Laurent polynomial $f \in \mathbb{C}[\mathbf{z}]$, the amoeba $\mathcal{A}(f)$ is a closed set. The complement of $\mathcal{A}(f)$ consists

of finitely many convex regions, and these regions are in bijective correspondence with the different Laurent series expansions of the rational function $1/f$. The number of components in the complement of an amoeba is bounded from above by the number of lattice points in the Newton polytope of f and bounded from below by the number of vertices of the Newton polytope of f.

For amoebas, the behavior "at infinity" can be described by the logarithmic limit set

$$\mathcal{A}_\infty(f) \;=\; \lim_{r\to\infty}\left(\frac{1}{r}\mathcal{A}(f)\cap\mathbb{S}^{n-1}\right)$$

and it is known that this is a spherical polyhedral complex. A polyhedral complex is a finite collection of polyhedra such that their pairwise intersection is either empty or a face of both. The logarithmic limit set provides one way of defining a tropical hypersurface; see, e.g., [81, Section 1.4].

It is hard to describe the boundary of amoebas explicitly. Only for linear polynomials there is an explicit description known; see [33]. In [89, 91], Mikhalkin characterizes critical points as a necessary criterion for boundary points. Define the logarithmic Gauß map, introduced by Kapranov [62], as

$$\gamma : \mathcal{V}(f) \to \mathbb{P}^{n-1}_{\mathbb{C}}, \mathbf{z} = (z_1,\ldots,z_n) \mapsto \left(z_1\cdot\frac{\partial f}{\partial z_1}(\mathbf{z}) : \cdots : z_n\cdot\frac{\partial f}{\partial z_n}(\mathbf{z})\right).$$

Then, the set of critical points is

$$\mathcal{S}(f) \coloneqq \left\{\mathbf{z}\in\mathcal{V}(f)\cap(\mathbb{C}^*)^n : \gamma(\mathbf{z})\in\mathbb{P}^{n-1}_{\mathbb{R}}\subset\mathbb{P}^{n-1}_{\mathbb{C}}\right\}.$$

Thus, the critical points are that points in the variety of f whose image under the logarithmic Gauß map is projectively real in $\mathbb{P}^{n-1}_{\mathbb{C}}$, i.e., a complex multiple of a real point. This criterion is necessary, but not sufficient. It holds $\partial\mathcal{A}(f)\subseteq\log|\mathcal{S}(f)| = \{\log|\mathbf{z}| : \mathbf{z}\in\mathcal{S}(f)\}$; see, e.g., [91, 115]. In [115], this concept was considered further and the boundary was characterized more explicitly.

Similarly to amoebas, the coamoeba of f is defined as

$$co\mathcal{A}(f) \;=\; \{\arg(\mathbf{z}) = (\arg(z_1),\ldots,\arg(z_n)) : \mathbf{z}\in\mathcal{V}(f)\cap(\mathbb{C}^*)^n\} \;\subseteq\; \mathbb{T}^n\,, \qquad (1.17)$$

where arg denotes the argument of a complex number and $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$.

For coamoebas, it has been conjectured that the number of connected components of the closure of the complement of $co\mathcal{A}(f)$ is bounded by $n!\mathrm{vol}\,\mathrm{New}(f)$ where vol denotes the volume; see [34] for more background information as well as a proof for the special case $n = 2$. For a complex plane curve, the area of the coamoeba in the complex torus is bounded by $2\pi^2\mathrm{Area}\,\mathrm{New}(f)$, where Area denotes the area of a surface; see [34].

Moreover, it was proven in [34] that the complement of the closure of $co\mathcal{A}(f)$ consists of convex components. Nisse and Sottile have introduced a variant of the logarithmic limit set for coamoebas, which are defined on a torus, by considering accumulation points of arguments of sequences with unbounded logarithm [102].

If $\log_{\mathbb{C}}$ denotes the complex logarithm, the following relations hold

$$\mathcal{A}(f) = \mathrm{Re} \circ \log_{\mathbb{C}} \mathcal{V}(f) \quad \text{and} \quad co\mathcal{A}(f) = \mathrm{Im} \circ \log_{\mathbb{C}} \mathcal{V}(f)\,,$$

where all maps are understood component-wise. Hence, the amoeba and the coamoeba can be viewed as a projection of the deformed variety. But in contrast to the fibers of imaginary projections and coamoebas, the fibers of the log-absolute maps underlying amoebas are compact.

Chapter 2

Imaginary projections

In Chapter 1, we looked at stable polynomials in particular. By definition, stable polynomials fulfill restrictions to the imaginary parts of their roots. Namely, there are no roots $(z_1, \ldots, z_n) \in \mathbb{C}^n$ with $\mathrm{Im}(z_j) > 0$ for all $j = 1, \ldots, n$; independently of the behavior of the real parts of the roots. This motivates to consider the projection of the variety of a polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ onto its imaginary part. Accordingly, we define the set

$$\mathcal{I}(f) \ = \ \{\mathrm{Im}(\mathbf{z}) \, : \, \mathbf{z} \in \mathcal{V}(f)\} \ \subseteq \ \mathbb{R}^n \,,$$

which we call the imaginary projection of f. This leads to a geometric viewpoint on stability.

In this Chapter, we study the imaginary projection of polynomials from various aspects. We start with basic properties in Section 2.1 and we prove the key result that all components of the complement of the closure of $\mathcal{I}(f)$ are convex and that there are only finitely many.
This motivates to investigate components of the complement in more detail. In particular, we are interested in the maximal number of components. For example, we show in Theorem 2.19 that for any positive integer K there is a polynomial whose imaginary projection has at least K bounded and strictly convex components in the complement.
When studying imaginary projections of hyperbolic polynomials in Section 2.3, it turns out that there is a one-to-one correspondence between hyperbolicity cones and components of the complement of imaginary projections. This motivates to determine the maximal number of hyperbolicity cones. For a polynomial in n variables of degree d, this is 2^d for $d \leq n$ and $2\sum_{k=0}^{n-1}\binom{d-1}{k}$ for $d > n$. The maximum is attained for polynomials that are products of independent linear polynomials.
Whereas hyperbolicity captures only homogeneous polynomials, hyperbolicity cones provide information also for imaginary projections of inhomogeneous polynomials, as we will

see in Section 2.5.1. There, the hyperbolicity cones of the initial form $\mathrm{in}(f)$ are in bijection with the components in the complement of $\mathcal{I}(f)$ which have full-dimensional recession cone.

As a helpful tool and for the sake of illustrations, we compute in Section 2.2 the imaginary projection of several classes of polynomials. We will apply these results in proofs and use them as examples. Moreover, these classification results show that imaginary projections can behave subtle and sensitive to ε-perturbations of the coefficients. Furthermore, already for quadratic polynomials, there appear various qualitatively different cases.

In Section 2.3, we study hyperbolic polynomials from the viewpoint of imaginary projections. As a strong sturctural relationship, it turns out that hyperbolicity cones coincide with components in the complement of the imaginary projection. Interested in the maximal number of components, we proof the maximal number of hyperbolicity cones for a given polynomial of fixed degree.

Beside this, we investigate in Section 2.4 the boundary of imaginary projections. Because of the connection between hyperbolicity cones and imaginary projections, we understand the homogeneous situation very well, since in that situation the boundary is a subset of the real variety. In contrast to this, the inhomogeneous situation behaves more complicated. We provide a necessary condition for boundary points that belong to the imaginary projection.

In Section 2.5, we deal with the behavior "at infinity". For that, we consider the limit set

$$\mathcal{I}_\infty(f) \;=\; \lim_{r\to\infty}\left(\frac{1}{r}\mathcal{I}(f)\cap\mathbb{S}^{n-1}\right) \;,$$

which is an analog of the logarithmic limit set for amoebas. Besides other results, we prove a complete description of the limit set for bivariate polynomials. Again, the connection to hyperbolicity cones helps and we derive a criterion for deciding whether $\mathcal{I}_\infty(f) = \mathbb{S}^{n-1}$.

The content of this chapter is a joint work with Thorsten Theobald and Timo de Wolff and it merges [58] and [60]. Most content of the Sections 2.1 and 2.2 as well as parts of the Sections 2.5 and 2.6 can be found in [60]. The sections concerning homogeneous and hyperbolic polynomials are mostly covered by [58]. These are the Sections 2.3, 2.5.1 as well as parts of the Sections 2.4 and 2.6. Moreover, Theorem 2.19 in Section 2.1.2 appears in [58].

2.1 Introduction and basic properties

We investigate the structure of imaginary projections in order to enrich algebraic properties like stability or hyperbolicity by a geometric perspective. This section initiates the studies of the underlying projection on the imaginary part from geometric and combinatorial points of view. We start with basic properties and easy examples, which will be considered in greater generality later on. As a key result we show in Theorem 2.10 that the complement of the closure of the imaginary projection of a polynomial consists of finitely many convex components. Moreover, we prove first results regarding the maximal number of components in the complement. For example, we show that there are polynomials with an arbitrarily given number of bounded components.

Definition 2.1. *Let $f \in \mathbb{C}[\mathbf{z}]$ be a polynomial. Then, we call the set*

$$\mathcal{I}(f) \;=\; \{\mathrm{Im}(\mathbf{z}) : \mathbf{z} \in \mathcal{V}(f)\} \;\subseteq\; \mathbb{R}^n$$

the imaginary projection of f.

The imaginary projection of a polynomial is a semi-algebraic set. Namely, writing $z_j = x_j + iy_j$ with real variables x_j, y_j, we regard $\mathcal{I}(f)$ as image of a real algebraic variety under the projection

$$\mathbb{R}^{2n} \to \mathbb{R}^n, \quad (x_1, y_1, \ldots, x_n, y_n) \mapsto (y_1, \ldots, y_n), \tag{2.1}$$

and thus $\mathcal{I}(f)$ is semi-algebraic. Hence, it is possible to express imaginary projections in terms of quantifier free formulas, e.g., by using quantifier elimination. Since the map (2.1) is continuous, the imaginary projection of an irreducible polynomial f is connected. Further note that $0 \notin \mathcal{I}(f)$ if and only if f has no real roots. See Figure 2.1 for an example.

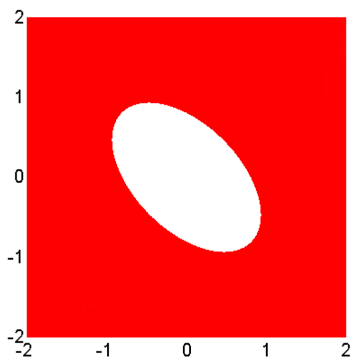

FIGURE 2.1: The imaginary projections of $f(z_1, z_2) = z_1^2 + z_2^2 + z_1 z_2 + z_1 + z_2 + 1$.

If we use the notations $\mathcal{H}_{\mathbb{C}}^n$ for the set $\{\mathbf{z} \in \mathbb{C}^n : \operatorname{Im}(z_j) > 0,\, 1 \leq j \leq n\}$ and $\mathcal{H}_{\mathbb{R}}^n = \operatorname{Im} \mathcal{H}_{\mathbb{C}}^n = (\mathbb{R}_{>0})^n$, then we obtain immediately the following characterization of stable polynomials:

Theorem 2.2. *A polynomial $f \in \mathbb{C}[\mathbf{z}]$ is stable if and only if $\mathcal{I}(f) \cap \mathcal{H}_{\mathbb{R}}^n = \emptyset$.*

In this section and the sections following, we consider the imaginary projection of polynomials from diverse viewpoints. Starting with two examples for illustration, we show in Proposition 2.6 the unboundedness of imaginary projections of multivariate polynomials. In Section 2.1.1 we prove there are only finitely many components in the complement which are all convex (Theorem 2.10) and we consider immediate consequences. Moreover, we show in Section 2.1.2 bounds on the number of components in the complement and that there are polynomials with an arbitrary large number of strictly convex complement components.

In order to illustrate imaginary projections, we start with two simple examples. Both of them will reappear in Section 2.2 in greater generality.

Example 2.3. *Let $f(\mathbf{z}) = a_0 + \sum_{j=1}^n a_j z_j$ be an affine-linear polynomial with $0 \neq (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $a_0 \in \mathbb{C}$. Writing $z_j = x_j + iy_j$ for $j = 1, \ldots, n$, we obtain*

$$\begin{aligned} \mathcal{I}(f) &= \left\{ \mathbf{y} \in \mathbb{R}^n : \exists \mathbf{x} \in \mathbb{R}^n \ \operatorname{Re}(a_0) + \sum_{j=1}^n a_j x_j = 0 \textit{ and } \operatorname{Im}(a_0) + \sum_{j=1}^n a_j y_j = 0 \right\} \\ &= \left\{ \mathbf{y} \in \mathbb{R}^n : \operatorname{Im}(a_0) + \sum_{j=1}^n a_j y_j = 0 \right\} \\ &= \mathcal{V}_{\mathbb{R}}(\operatorname{Im} f)\,, \end{aligned}$$

which is an affine hyperplane. If we allow complex coefficients beside a_0, the situation can change drastically; see Theorem 2.23.

Example 2.4. *Let $f(z_1, z_2) = z_1^2 + z_2^2 + 1$. Again, writing $z_j = x_j + iy_j$, the equation $f(z_1, z_2) = 0$ becomes the system*

$$\begin{aligned} \operatorname{Re} f &= x_1^2 - y_1^2 + x_2^2 - y_2^2 + 1 = 0\,, \\ \operatorname{Im} f &= 2x_1 y_2 + 2x_2 y_1 = 0\,. \end{aligned}$$

Whereas the second equation has for arbitrary $y_1, y_2 \in \mathbb{R}$ always a real solution in x_1, x_2, the first equation is unfulfillable for $y_1^2 + y_2^2 < 1$. Hence, the open unit-disk is (the only) component in the complement of $\mathcal{I}(f)$; see Figure 2.2.

Remark 2.5. If f has real coefficients, then the zeros of f come in conjugated pairs. In that situation, $\mathcal{I}(f)$ is symmetric with respect to the origin.

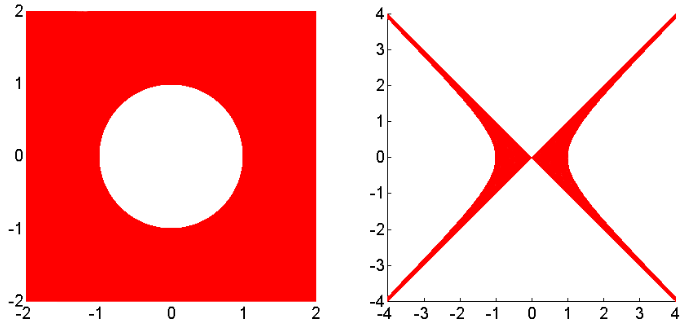

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FIGURE 2.2: The imaginary projections of $f(z_1, z_2) = z_1^2 + z_2^2 + 1$ and $f(z_1, z_2) = z_1^2 - z_2^2 - 1$.

The set $\mathcal{I}(f)$ is not always a closed set. Indeed, already in the quadratic setting all the following cases can occur; see Figure 2.2 for an illustration.

1. $\mathcal{I}(f)$ is open for $f(z_1, z_2) = z_1^2 + z_2^2 - 1$. In fact, $\mathcal{I}(f) = \mathbb{R}^n$.
2. $\mathcal{I}(f)$ is closed for $f(z_1, z_2) = z_1^2 + z_2^2 + 1$.
3. $\mathcal{I}(f)$ is neither open nor closed for $f(z_1, z_2) = z_1^2 - z_2^2 - 1$. The hyperbolic curve belongs to $\mathcal{I}(f)$, but, except the origin, the asymptotes do not belong to $\mathcal{I}(f)$.

For further details on the specific examples we refer to the discussion of quadratic polynomials in Section 2.2.3. Note, that this changes completely for homogeneous polynomials, where $\mathcal{I}(f)$ is always closed; see Corollary 2.38.

Proposition 2.6. *Unless $f \in \mathbb{C}[\mathbf{z}]$ is a univariate or a non-zero constant polynomial, $\mathcal{I}(f)$ is unbounded.*

Proof. For $n = 1$, the imaginary projection is a finite collection of points and, thus, bounded. Assume now $n \geq 2$. Let $R > 0$ and $(a_1, \ldots, a_{n-1}) \in \mathbb{C}^{n-1}$ such that $\|(\mathrm{Im}(a_1), \ldots, \mathrm{Im}(a_{n-1}))\|_2 > R$ and $f(a_1, \ldots, a_{n-1}, z_n) \in \mathbb{C}[z_n]$ is not a non-zero constant. Then there exists a $z_n \in \mathbb{C}$ with $f(a_1, \ldots, a_{n-1}, z_n) = 0$. Sending $R \to \infty$, the statement follows. $\square$

We close this section with three basic lemmas, which we need in what follows. Lemma 2.7 allows to reduce the case of reducible polynomials to the case of irreducible polynomials. Lemmas 2.8 and 2.9 describe the transformation of imaginary projections when applying real transformations to the space of variables. Note that a complex transformation would rotate the variety in complex directions. The induced action on $\mathcal{I}(f)$ cannot be regarded as a continuous real deformation in $\mathbb{R}^n$ in general.

Lemma 2.7. *For $f_1, f_2 \in \mathbb{C}[\mathbf{z}]$, we have $\mathcal{I}(f_1 \cdot f_2) = \mathcal{I}(f_1) \cup \mathcal{I}(f_2)$.*

Proof. By definition of the imaginary projection we have

$$\mathcal{I}(f_1 \cdot f_2) = \operatorname{Im} \mathcal{V}(f_1 \cdot f_2) = \operatorname{Im}(\mathcal{V}(f_1) \cup \mathcal{V}(f_2)) = \operatorname{Im} \mathcal{V}(f_1) \cup \operatorname{Im} \mathcal{V}(f_2) = \mathcal{I}(f_1) \cup \mathcal{I}(f_2).$$

□

Lemma 2.8. *Let $f \in \mathbb{C}[\mathbf{z}]$ and $A \in \mathbb{R}^{n \times n}$ be an invertible matrix. Then, $\mathcal{I}(f(A\mathbf{z})) = A^{-1}\mathcal{I}(f(\mathbf{z}))$.*

Proof. Writing $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, the matrix A operates separately on $\mathbf{x}$ and $\mathbf{y}$. Hence,

$$\begin{aligned} \mathcal{I}(f(A\mathbf{z})) &= \{\mathbf{y} : \exists \mathbf{x} \in \mathbb{R}^n \ f(A(\mathbf{x} + i\mathbf{y})) = 0\} = \{A^{-1}\mathbf{y}' : \exists \mathbf{x}' \in \mathbb{R}^n \ f(\mathbf{x}' + i\mathbf{y}') = 0\} \\ &= A^{-1}\mathcal{I}(f(\mathbf{z})). \end{aligned}$$

□

Lemma 2.9. *A real translation $\mathbf{z} \mapsto \mathbf{z} + \mathbf{a}$, $\mathbf{a} \in \mathbb{R}^n$, does not change the imaginary projection of a polynomial. An imaginary translation $\mathbf{z} \mapsto \mathbf{z} + i\mathbf{a}$, $\mathbf{a} \in \mathbb{R}^n$, shifts an imaginary projection in direction $-\mathbf{a}$.*

Proof. The statement holds, since the first kind of transformation just translates the real part of the variables and the second one shifts the imaginary parts of the solutions of $f(\mathbf{z}) = 0$ in direction $-\mathbf{a}$. □

2.1.1 Convexity of components of the complement

Similar to amoebas and coamoebas, the complement of an imaginary projection can have several connected components. As pointed out in Section 1.4, it is an important property of amoebas and coamoebas that the components of the complement are convex. As a key property of imaginary projections, we show that its complement shows the same convexity phenomenon. But in contrast to amoebas, already quadratic polynomials can lead to bounded components in the complement. Indeed, the complement of the imaginary projection of $f(z_1, z_2) = z_1^2 + z_2^2 + 1$ has one bounded component; see Example 2.4.

Theorem 2.10. *For every polynomial $f \in \mathbb{C}[\mathbf{z}]$, the number of components in the complement of $\mathcal{I}(f)$ is finite. Moreover, all components of $\overline{\mathcal{I}(f)}^{\mathsf{c}}$ are convex.*

Proof. Let C be a component of the complement of $\overline{\mathcal{I}(f)}$. Define the holomorphic map

$$\psi : \mathbb{C}^n \to \mathbb{C}^n, \quad \mathbf{z} \mapsto -i\,\mathbf{z},$$

which is equivalent to $\mathbf{x} + i\mathbf{y} \mapsto \mathbf{y} - i\mathbf{x}$. Furthermore, let

$$C_\psi = \psi(\mathbb{R}^n + iC) = C - i\mathbb{R}^n = C + i\mathbb{R}^n.$$

We observe that C_ψ is a tubular region, that is, for any $\mathbf{y} \in C_\psi \cap \mathbb{R}^n$ we have $\mathbf{y} + i\mathbf{x} \in C_\psi$ for all $\mathbf{x} \in \mathbb{R}^n$. Moreover, the function

$$g : C_\psi \to \mathbb{C}, \quad \mathbf{w} \mapsto \frac{1}{f(\psi(\mathbf{w}))}$$

is holomorphic on C_ψ, and C_ψ is the maximal tube with this property. By Bochner's Tube Theorem [7], g is holomorphic on the convex hull of C_ψ (considered as set in $\mathbb{R}^{2n} \cong \mathbb{C}^n$). Due to the maximality of C_ψ, this implies the convexity of C_ψ. Since $C_\psi = \psi(\mathbb{R}^n + iC) = C + i\mathbb{R}^n$, we obtain the convexity of C.

As $\mathcal{I}(f)$ is a semi-algebraic set, $\mathcal{I}(f)$ is given by a Boolean combination of polynomial equations and polynomial inequalities, and thus the number of components in the complement is finite. $\square$

Strongly related to convex sets are their recession cones; see, e.g., [112, Chapter 8].

Definition 2.11. *Let $A \neq \emptyset$ be a convex set in $\mathbb{R}^n$. Then, the recession cone is defined as*

$$\operatorname{rec}(A) \coloneqq \{\mathbf{a} \in \mathbb{R}^n : \mathbf{a} + A \subset A\} = \{\mathbf{y} \in \mathbb{R}^n : \mathbf{x} + \lambda\mathbf{y} \in A \ \forall \lambda \geq 0, \ \forall \mathbf{x} \in A\}.$$

More descriptive: The recession cone $\operatorname{rec}(A)$ of A is the maximal cone (with apex in the origin) such that $\mathbf{a} + \operatorname{rec}(A) \subset A$ for all $\mathbf{a} \in A$. It is $\operatorname{rec}(A) = \{0\}$ if and only if A is bounded. Since recession cones are convex sets, we can define its dimension as the dimension of its affine hull.

Proposition 2.12. ([112, Cor. 8.3.1]) *Let $A \neq \emptyset$ be a convex set in $\mathbb{R}^n$ and denote by* $\operatorname{relint}(A)$ *its relative interior and $\overline{A}$ its (topological) closure. Then,* $\operatorname{rec}(\overline{A}) = \operatorname{rec}(\operatorname{relint}(A))$.

As a natural consequence of the preceding Theorem 2.10, we obtain the following statement on the unbounded components of the complement.

Corollary 2.13. *Every unbounded component in the complement of the closure of an imaginary projection includes a ray.*

Proof. Let C be an unbounded component of the complement of $\overline{\mathcal{I}(f)}$. Then the convex set C is at least one-dimensional. By Proposition 2.12, the relative interior of C has a recession cone which coincides with the recession cone of the closure of C, and that recession cone contains a non-zero vector. Hence, C contains a ray. $\square$

If the complement component is higher dimensional, one even finds a higher dimensional convex cone inside. The recession cone of components of $\overline{\mathcal{I}(f)}^{\mathsf{c}}$ provides one way to consider the behavior "at infinity" of the imaginary projection. We discuss this in Section 2.5.1, where we study components of the complement whose recession cones are full-dimensional. In particular, we show a one-to-one correspondence between components with full-dimensional recession cone and hyperbolicity cones of the initial form of the underlying polynomial.

The left picture in Figure 2.3 shows the imaginary projection of the polynomial $f(z_1, z_2) = -3z_1z_2^2 + z_2^3 - z_2^2 - 3z_1 + z_2 - 4$ with its 6 convex components in the complement. As the two right pictures in Figure 2.3 show, it is possible that an imaginary projection contains both bounded and unbounded components in the complement. We consider the number of components in the complement further in Section 2.1.2. Later, in Section 2.3, we determine a tight upper bound for the number of components in the complement of the imaginary projection of a homogeneous polynomial.

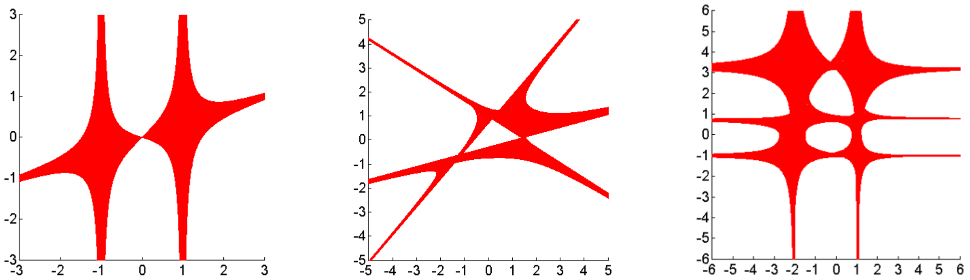

FIGURE 2.3: The imaginary projection of $f(z_1, z_2) = -3z_1z_2^2 + z_2^3 - z_2^2 - 3z_1 + z_2 - 4$, $f(z_1, z_2) = 5z_1^3 - 4z_1^2z_2 - 3z_1z_2^2 + z_2^3 - 7iz_1^2 - 2iz_2^2 + 5iz_1 - 5z_2 - 6i$ and of $f(z_1, z_2) = z_1^3z_2^2 + iz_1^3z_2 - 3iz_1^2z_2^2 + (2+i)z_1^3 + 3z_1^2z_2 + (1+3i)z_1z_2^2 - 8iz_1^2 + iz_1z_2 + (1-3i)z_2^2 + (3-2i)z_1 + (3+1)z_2 - 5i$.

Moreover, Lemmas 2.8 and 2.9 together with Corollary 2.13 provide a statement about the existence of unbounded components in $\mathcal{I}(f)^{\mathsf{c}}$.

Theorem 2.14. *Let* $f \in \mathbb{C}[\mathbf{z}]$.

1. *The complement of* $\mathcal{I}(f)$ *contains the non-negative* y_1*-axis* $\mathbb{R}_{\geq 0} \times \{0\}^{n-1}$ *if and only if the polynomial* $f(z_1 + ir, z_2, \ldots, z_n)$ *has no real root in* $\mathbf{z}$ *for any* $r \geq 0$.
2. $\overline{\mathcal{I}(f)}$ *has an unbounded component in the complement if and only if there is an affine transformation* $\mathbf{z} \mapsto A\mathbf{z} + i\mathbf{b}$ *with a real matrix* A *and a real vector* $\mathbf{b}$ *such that condition* (1) *is satisfied.*

Proof. The first statement immediately follows from the definition of $\mathcal{I}(f)$. For the second statement, Corollary 2.13 implies that the existence of an unbounded component in the complement of the closure is equivalent to the existence of a ray in the complement. By the Lemmas 2.8 and 2.9, the affine transformation reduces the situation to (1). $\square$

2.1.2 Number of components in the complement

As seen in Theorem 2.10, there are only finitely many components in $\mathcal{I}(f)^{\mathsf{c}}$. At the moment the maximal number of components in the complement is unknown for arbitrary polynomials. In Section 2.3, we show a sharp upper bound for homogeneous polynomials (Theorem 2.45), which is attained by products of independent linear polynomials. For non-homogeneous polynomials, we claim that the maximal number is attained for products of affine-linear polynomials. In Section 2.5.1, we derive the maximal number of components in the complement with full-dimensional recession cone (Theorem 2.68).

Here, we begin with the following observation about the product of polynomials in different variables. In the statements following, we collect assertions regarding a lower bound for the maximal number of components in $\mathcal{I}(f)^{\mathsf{c}}$. Namely, we show that there can occur an arbitrary number of (bounded) components in the complement. After that, Theorem 2.19 extends these observations to strictly convex components in the complement. Most of the proofs are constructive.

Proposition 2.15. *Let $f(\mathbf{z}_1, \mathbf{z}_2) = f_1(\mathbf{z}_1)f_2(\mathbf{z}_2)$ with $\mathbf{z}_j \in \mathbb{R}^{n_j}$, $j = 1, 2$.*

1. *If and only if C_j is a component of $\mathcal{I}(f_j)^{\mathsf{c}}$ for $j = 1, 2$, then $C_1 \times C_2$ is a component of $\mathcal{I}(f)^{\mathsf{c}}$.*
2. *If $\mathcal{I}(f_j)^{\mathsf{c}}$ consists of exactly k_j components, then $\mathcal{I}(f)^{\mathsf{c}}$ consists of exactly k_1k_2 components.*

Proof. For the first statement, it suffice to observe that

$$\mathcal{I}(f) \;=\; \{(\mathbf{y}_1, \mathbf{y}_2) \in \mathbb{R}^{n_1+n_2} : \exists(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}^{n_1+n_2} \quad f_1(\mathbf{x}_1 + i\mathbf{y}_1)f_2(\mathbf{x}_2 + i\mathbf{y}_2) = 0\}\,.$$

As a consequence, the Cartesian product of the components of $\mathcal{I}(f_1)^{\mathsf{c}}$ and of $\mathcal{I}(f_2)^{\mathsf{c}}$ provides components of $\mathcal{I}(f)^{\mathsf{c}}$. And conversely, if $C = C_1 \times C_2$ is a component of $\mathcal{I}(f)^{\mathsf{c}}$, then C_j is a component of $\mathcal{I}(f_j)^{\mathsf{c}}$, $j = 1, 2$. Calculating the number of possible combinations shows the second claim. □

Proposition 2.16. *For any integers $n > 0$ and $t > 0$, there exists a polynomial $f \in \mathbb{C}[\mathbf{z}]$ such that the complement of $\mathcal{I}(f)$ has exactly t bounded components.*

Proof. We choose an arrangement $\mathcal{H}$ of k affine hyperplanes $H_1, \ldots, H_k \subseteq \mathbb{R}^n$ such that $\mathcal{H}$ has t bounded components of the complement. Using Example 2.3, each of the hyperplanes is the imaginary projection of an affine-linear polynomial $f_1, \ldots, f_k \in \mathbb{C}[\mathbf{z}]$. By Lemma 2.7, the imaginary projection of the product of these polynomials gives exactly the hyperplane arrangement. □

The proof of Proposition 2.16 constructs f as a product of linear polynomials. In the following, we investigate the imaginary projection of products of linear polynomials in more detail.

Proposition 2.17. *Let $f \in \mathbb{C}[\mathbf{z}]$ be a product of m affine-linear polynomials in n variables. Then the complement of $\mathcal{I}(f)$ consists of at most $\sum_{k=0}^{n} \binom{m}{k}$ components, and this bound is tight.*

Proof. As seen in Example 2.3, the imaginary projection of an affine-linear polynomial with real coefficients is always a hyperplane. In the general situation, which is treated in Theorem 2.23, the imaginary projection of an affine-linear polynomial with complex coefficients is either a hyperplane or the whole space $\mathbb{R}^n$. We can assume here that the first case holds for every affine-linear factor in f. Then the imaginary projection of the product defines a hyperplane arrangement in $\mathbb{R}^n$. If the hyperplanes are in general position, then they decompose the ambient space into exactly $\sum_{k=0}^{n} \binom{m}{k}$ many regions; see [118, Prop. 2.4]. $\square$

Proposition 2.17 implies for polynomials f of total degree d in n variables the following lower bound for the maximal number of components in $\mathcal{I}(f)^{\mathsf{c}}$.

Corollary 2.18. *There exists a polynomial f of total degree d in n variables such that the complement of $\mathcal{I}(f)$ consists of exactly $\sum_{k=0}^{n} \binom{d}{k}$ components.*

As we will see in Section 2.5.1, there are always at most $2\sum_{k=0}^{n-1} \binom{d-1}{k}$ components in the complement whose recession cone is full-dimensional.

Now, we show the following theorem which states that for $n \geq 2$ there can appear arbitrary large numbers of strictly convex components in $\mathcal{I}(f)^{\mathsf{c}}$.

Theorem 2.19. *Let $n \geq 2$. For any integer $K > 0$ there exists a polynomial $f \in \mathbb{R}[\mathbf{z}]$ such that f has at least K strictly convex and bounded components in the complement of $\mathcal{I}(f)$.*

Note that for $n = 1$, every convex set is strictly convex. Hence, for distinct real numbers $\lambda_1, \ldots, \lambda_{K+1}$, the polynomial $f(z) = \prod_{j=1}^{K+1}(z - i\lambda_j)$ has exactly K strictly convex, bounded components in $\mathcal{I}(f)^{\mathsf{c}}$. For $n \geq 2$, the proof is also constructive. We make use of the classification results in the Theorems 2.25 and 2.27 for imaginary projections of real quadratic polynomials; see Section 2.2.3. We remark, that their proofs are independent of Theorem 2.19.

Proof. Given $K \in \mathbb{N}$, we construct a polynomial $p_{K,n}$ in n variables with at least K strictly convex complement components. For $\varphi \in \mathbb{R}$, denote by $R^\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ the linear mapping rotating a given point $\mathbf{x} \in \mathbb{R}^2$ by an angle φ around the origin. R^φ has a real representation matrix and can also be viewed as a linear map $\mathbb{C}^2 \to \mathbb{C}^2$. Due to Lemma 2.8, R^φ also provides a rotation of the imaginary projection.

For the case $n = 2$, let

$$g(z_1, z_2) = (-z_1^2 + z_2^2 - 1)(z_1^2 - z_2^2 - 1),$$

and

$$p_{K,2}(\mathbf{z}) = \left(z_1^2 + z_2^2 + r^2\right) \cdot \prod_{j=0}^{m-1} g(R_1^{2\pi j/m}(z_1, z_2), R_2^{2\pi j/m}(z_1, z_2)) \tag{2.2}$$

where $m = \left\lceil \frac{K}{4} \right\rceil$ and $r > 0$ sufficiently large. In analog to Example 2.4, $\mathcal{I}(z_1^2 + z_2^2 + r^2)^{\mathsf{c}}$ is the open disk with radius r centered at the origin. The boundary of $-z_1^2 + z_2^2 - 1$ and $z_1^2 - z_2^2 - 1$ will be computed as part of Theorem 2.25; see Figures 2.2 and 2.5 for an illustration. The boundaries of the two-dimensional components of $\mathcal{I}(g)^{\mathsf{c}}$ are given by four hyperbolas. Since the convex components of $\mathcal{I}(g)^{\mathsf{c}}$ and of $\mathcal{I}(z_1^2+z_2^2+r^2)^{\mathsf{c}}$ are strictly convex, the components of $\mathcal{I}(p_{K,2})^{\mathsf{c}}$ are strictly convex. Figure 2.4 depicts $\mathcal{I}(p_{4,2})$.

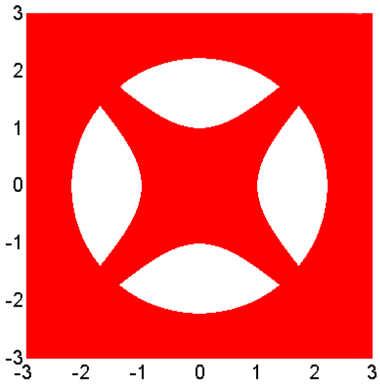

FIGURE 2.4: The imaginary projections of $p_{4,2}$, with $r = 5$.

The expressions $(R_1^{2\pi j/m}(z_1, z_2), R_2^{2\pi j/m}(z_1, z_2))$ in the arguments of g provide a rotation of its imaginary projection by an angle of $-2\pi j/m$; see Lemma 2.8. Choosing r large enough guarantees that the complement component of $\mathcal{I}(z_1^2 + z_2^2 + r^2)$ is not completely covered by the imaginary projections of the $g(R_1^{2\pi j/m}(z_1, z_2), R_2^{2\pi j/m}(z_1, z_2))$. Altogether, $\mathcal{I}(p_{K,2})^{\mathsf{c}}$ has $4m \geq K$ bounded and strictly convex, two-dimensional components.

Note that the asymptotes of the hyperbolas do not belong to the imaginary projection of g, except the origin. Therefore, $\mathcal{I}(p_{K,2})^{\mathsf{c}}$ has in total $8m$ bounded components.

The case $n \geq 3$ follows by a suitable modification of (2.2). Namely, set

$$g(\mathbf{z}) \;=\; (rz_1)^2 - \Big(\sum_{j=2}^{n} z_j^2\Big) + 1 \;=\; r^2 z_1^2 - \Big(\sum_{j=2}^{n} z_j^2\Big) + 1$$

and

$$p_{K,n}(\mathbf{z}) \;=\; \left(\sum_{j=1}^{n} z_j^2 + 1\right) \cdot \prod_{j=0}^{m-1} g(R_1^{2\pi j/m}(z_1, z_2), R_2^{2\pi j/m}(z_1, z_2), z_3, \ldots, z_n)\,,$$

where $m = \lceil \frac{K}{2} \rceil$. By the classification result in Theorem 2.27, $\mathcal{I}(\sum_{j=1}^{n} z_j^2 + 1)^{\mathsf{c}}$ is the open ball in $\mathbb{R}^n$ with radius 1 centered at the origin, and the boundaries of the two convex components of $\mathcal{I}(g)^{\mathsf{c}}$ are given by $B_1 \coloneqq \{\mathbf{y} \in \mathbb{R}^n \,:\, y_1 \geq 1/r \text{ and } r^2 y_1^2 - \sum_{j=2}^{n} y_j^2 = 1\}$ and $B_2 \coloneqq \{\mathbf{y} \in \mathbb{R}^n \,:\, y_1 \leq -1/r \text{ and } r^2 y_1^2 - \sum_{j=2}^{n} y_j^2 = 1\}$. Since $B_1 \cup B_2$ is a two-sheeted n-dimensional hyperboloid, B_1 and B_2 are the boundaries of strictly convex sets. Note that for $r \to \infty$, the set $\mathcal{I}(g)$ converges to the y_1-hyperplane on all compact regions of $\mathbb{R}^n$.

Again, since the rotation $(R_1^{2\pi j/m}(z_1, z_2), R_2^{2\pi j/m}(z_1, z_2))$ in the arguments of g induces a rotation of its imaginary projection by an angle of $-2\pi j/m$ with respect to the $y_1 y_2$-plane, choosing r large enough gives $2m \geq K$ bounded and strictly convex components.

$\square$

2.2 Imaginary projection of special classes of polynomials

Before we continue to derive further results for imaginary projections in general, we study imaginary projections of special classes of polynomials. Most of them will reappear throughout this thesis as an example or in a proof.

For the sake of completeness, we begin with univariate polynomials. Imaginary projections of polynomials in one single variable differ significantly from multivariate polynomials. Only in this situation, imaginary projections are compact and a collection of finitely many points. Section 2.2.1 collects some more properties following from this. But the main focus of this thesis lies on imaginary projections of multivariate polynomials. So, Section 2.2.1 provides only a short excerpt of the geometry of roots of univariate polynomials. In the sections following, we focus on multivariate polynomials.

In Section 2.2.2, we investigate affine-linear polynomials $\sum_{j=1}^{n} a_j z_j + a_0$, which we already know in a simplified version from Example 2.3. In Theorem 2.23, we consider the complete general setting. It turns out that their imaginary projection is either an affine hyperplane or the whole space. Afterwards, we characterize the set of stable affine-linear polynomials in Corollary 2.24.

In contrast to the affine-linear situation, the imaginary projections of quadratic polynomials show a large variety of different types. There can occur bounded components in the complement and imaginary projections can be neither open nor closed. In Section 2.2.3 we deal with real quadratic polynomials and describe their imaginary projection by reducing them to their normal form (Theorems 2.25 and 2.27). This makes it possible to give a criterion for real stable quadrics in Corollary 2.31.

The subsequent Section 2.2.4 considers polynomials that are affine-linear in one variable. We describe the imaginary projection of bivariate multi-affine linear polynomials (Theorem 2.32) and give a less explicit description for the imaginary projection of polynomials of the form $f(\mathbf{z}) + z_{n+1}g(\mathbf{z}) \in \mathbb{C}[\mathbf{z}, z_{n+1}]$ in Corollary 2.34.

2.2.1 Univariate polynomials

As a first class of polynomials, we consider polynomials in one single variable. Over the complex numbers, an univariate polynomial always decomposes into linear factors and, counting multiplicities, the total number of roots equals its degree. Thus, the imaginary projection of a non-constant polynomial is a finite collection of points. If m denotes the number of roots with a different imaginary part, the imaginary projection cuts the real line into $m + 1$ open intervals such that $m - 1$ of them are bounded. In particular, the imaginary projection of a non-constant univariate polynomial is compact. Due to this fact, the imaginary projection of an univariate polynomial is significantly easier to understand than in the multivariate situation.

For the geometry of roots of univariate polynomials, there are many classical results known. Some of them are already presented in Section 1.3.1, like Hermite-Biehler Theorem 1.22 and Hermite-Kakeya-Obreschkoff Theorem 1.23. Here, we collect some additional statements and consider their application to imaginary projections.
As a general source on the geometry of roots of a univariate polynomials, we refer to [87, 109], and we refer to [4] for a more algorithmic point of view.

Only for univariate polynomials, the imaginary projection is bounded. We can give some direct bounds to the absolute values of points in the imaginary projection by using $|\operatorname{Im}(z)| \leq |z|$. The following classical theorem is due to Cauchy. It describes circles containing all roots.

Proposition 2.20. (Cauchy; see [87, Theorems 27.1 and 27.2]) *The absolute values of the roots of $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ are at most as large as the positive root of*

$$|a_0| + |a_1| z + \cdots + |a_{n-1}| z^{n-1} - |a_n| z^n$$

and strictly smaller than

$$1 + \max_{j=0,\ldots,n-1} \left| \frac{a_j}{a_n} \right| .$$

There are lots of results counting the number of roots in a certain area of the complex plane. But lower bounds to the absolute values are not preserved under imaginary projection. Moreover, we refer to [109, Chapter 11.3] for statements about the number of roots in the upper half-plane. Clearly, these provide upper bounds for the number of positive points in the imaginary projection.

Moreover, we remark the following property, which will be useful later within this thesis. From Rouché's Theorem it follows that the roots of a univariate polynomial depend continuously on the coefficients; see [87, Theorem 1.4]. This allows to approximate polynomials with multiple roots by polynomials with all single roots. Furthermore, this implies that the eigenvalues of a matrix depend continuously on the matrix entries. This allows to approximate positive semidefinite matrices by positive definite matrices.

For multivariate polynomials, the roots do not depend continuously on the coefficients anymore. Hence, the class of univariate polynomials is the only class of polynomials where we can safely approximate imaginary projections by approximating the underlying polynomials.

The following classical theorem states that the roots of a derivative f' are expressible by convex combinations of the roots of f. It generalizes Rolle's Theorem which implies that the zeros of a real-rooted polynomial and its derivative interlace, unless one of them is constant.

Proposition 2.21. (Gauß-Lucas; see [87, Theorem 6.1]) *Let K be a convex set containing the roots of $f \in \mathbb{C}[z]$. Then, K contains all roots of the derivative f'.*

Hence, $\operatorname{conv} \mathcal{I}(f') \subset \operatorname{conv} \mathcal{I}(f)$. It follows immediately that derivatives of univariate stable polynomials remain stable.

Example 2.22. *Consider $f(z) = z^d - 1$ for a fixed integer $d > 0$. The roots of f are the d roots of unity. Hence,*

$$\mathcal{I}(f) = \left\{ \sin\left(\frac{2\pi k}{d}\right) : k = 0, \dots, d-1 \right\} .$$

Moreover, $\mathcal{I}(f') = \left\{ \frac{1}{\sqrt{d}} \sin\left(\frac{2\pi k}{d-1}\right) : k = 0, \dots, d-2 \right\}$ *and* $\operatorname{conv} \mathcal{I}(f') \subset \operatorname{conv} \mathcal{I}(f)$.

Throughout the following sections, we consider multivariate polynomials unless we mention this explicitly.

2.2.2 Affine-linear polynomials

We begin the study of multivariate polynomials by considering affine-linear polynomials $f(\mathbf{z}) = a_0 + \sum_{j=1}^{n} a_j z_j$, $(a_1, \dots, a_n) \neq 0$ with $n \geq 0$. Polynomials of this type appear at different places throughout this thesis, for example in Sections 2.3 and 2.4. In Example 2.3, we already looked at real affine-linear polynomials. For real coefficients $a_0, a_1, \dots, a_n$ it turns out that $\mathcal{I}(f) = \mathcal{V}_{\mathbb{R}}(f - a_0)$, which is always a linear hyperplane in $\mathbb{R}^n$. For complex coefficients, the imaginary projection can behave differently. The following theorem treats the case of affine-linear polynomials in the general setting.

Theorem 2.23. *For every affine-linear polynomial $f(\mathbf{z}) = a_0 + \sum_{j=1}^{n} a_j z_j \in \mathbb{C}[\mathbf{z}]$ with $(a_1, \dots, a_n) \neq 0$ the imaginary projection is given as follows:*

$$\mathcal{I}(f) = \begin{cases} \mathcal{V}_{\mathbb{R}}(\operatorname{Im}(a_0 e^{-i\varphi}) + \sum\limits_{j=1}^{n} a_j e^{-i\varphi} y_j) \, , & (a_1, \dots, a_n) \in e^{i\varphi} \cdot \mathbb{R}^n \textit{ for some } \varphi \in [0, 2\pi) \, , \\ \mathbb{R}^n \, , & \textit{otherwise.} \end{cases}$$

Proof. If all coefficients $a_1, \ldots, a_n$ are real, then

$$\begin{aligned}\mathcal{I}(f) &= \left\{ \mathbf{y} \in \mathbb{R}^n \,:\, \exists \mathbf{x} \in \mathbb{R}^n \ \operatorname{Re}(a_0) + \sum_{j=1}^{n} a_j x_j = 0 \text{ and } \operatorname{Im}(a_0) + \sum_{j=1}^{n} a_j y_j = 0 \right\} \\ &= \left\{ \mathbf{y} \in \mathbb{R}^n \,:\, \operatorname{Im}(a_0) + \sum_{j=1}^{n} a_j y_j = 0 \right\} ,\end{aligned}$$

in analog to Example 2.3, and in the situation $(a_1, \ldots, a_n) \in e^{i\varphi} \cdot \mathbb{R}^n$, apply the real case to $e^{-i\varphi} f$.

Now assume that $(a_1, \ldots, a_n)$ is not a complex multiple of a real vector. That is, the real matrix $\begin{pmatrix} \operatorname{Re}(a_1) & \cdots & \operatorname{Re}(a_n) \\ \operatorname{Im}(a_1) & \cdots & \operatorname{Im}(a_n) \end{pmatrix}$ has rank 2. By possibly changing the order of the coefficients a_j, we can assume that the matrix $A = \begin{pmatrix} \operatorname{Re}(a_1) & \operatorname{Re}(a_2) \\ \operatorname{Im}(a_1) & \operatorname{Im}(a_2) \end{pmatrix}$ is invertible. In order to show $\mathcal{I}(f) = \mathbb{R}^n$, consider a fixed $\mathbf{y} \in \mathbb{R}^n$ and choose arbitrary $x_3, \ldots, x_n \in \mathbb{R}$. Then the conditions $\operatorname{Re} f(\mathbf{x} + i\mathbf{y}) = 0$ and $\operatorname{Im} f(\mathbf{x} + i\mathbf{y}) = 0$ yield a system of two real linear equations in x_1, x_2 with coefficient matrix A,

$$\begin{aligned} 0 &= \operatorname{Re} f(\mathbf{x} + i\mathbf{y}) = \operatorname{Re} a_0 + \sum_{j=1}^{n} \operatorname{Re}(a_j) x_j - \sum_{j=1}^{n} \operatorname{Im}(a_j) y_j , \\ 0 &= \operatorname{Im} f(\mathbf{x} + i\mathbf{y}) = \operatorname{Im} a_0 + \sum_{j=1}^{n} \operatorname{Im}(a_j) x_j + \sum_{j=1}^{n} \operatorname{Re}(a_j) y_j . \end{aligned}$$

Since A is invertible and $x_3, \ldots, x_n$ are fixed, there exists a solution $x_1, x_2 \in \mathbb{R}$, and thus $\mathbf{y} \in \mathcal{I}(f)$. This completes the proof. $\square$

Note, that the affine situation is very sensitive to perturbations of the coefficients. In particular, a complex ε-perturbation of some coefficients can change the situation from a hyperplane to the whole space. And even a real ε-perturbation of some complex coefficients can yield this effect. This can be seen as an example, why Lemma 2.8 requires a real and regular transformation.
Due to Lemma 2.8, for real polynomials a real ε-perturbation preserves the structure.

The following corollary of Theorem 2.23 characterizes the set of affine-linear stable polynomials.

Corollary 2.24. *Let $f(\mathbf{z}) = a_0 + \sum_{j=1}^{n} a_j z_j \in \mathbb{C}[\mathbf{z}]$ with $(a_1, \ldots, a_n) \neq 0$. Then, f is stable if and only if there is a $\varphi \in [0, 2\pi)$ such that the following three properties hold*

- $(a_1, \ldots, a_n) \in e^{i\varphi} \cdot \mathbb{R}^n$,

- *either* $e^{-i\varphi}a_j \leq 0$ *for all* $1 \leq j \leq n$ *or* $e^{-i\varphi}a_j \geq 0$ *for all* $1 \leq j \leq n$*,*
- $\mathrm{Im}(a_0 e^{-i\varphi}) \geq 0$*.*

If all coefficients are real, f *is stable if and only if* $a_1, \ldots, a_n \geq 0$ *or* $a_1, \ldots, a_n \leq 0$*.*

Proof. If there is a $\varphi \in [0, 2\pi)$ such that all three properties hold, the stability of f follows by the characterization in Theorem 2.23.

Now, assume f to be stable. The first condition is clear, since otherwise the imaginary projection would be $\mathbb{R}^n$. So, assume $(a_1, \ldots, a_n) \in e^{i\varphi} \cdot \mathbb{R}^n$ for one $\varphi \in [0, 2\pi)$. Then, the imaginary projection is given by a hyperplane H with normal vector $n = e^{-i\varphi}(a_1, \ldots, a_n)$ and with translation vector $\mathrm{Im}(a_0 e^{-i\varphi})$. For stability it is necessary and sufficient that $n \in (\mathbb{R}_{>0})^n$ or $-n \in (\mathbb{R}_{>0})^n$ and that $\mathrm{Im}(a_0 e^{-i\varphi}) \geq 0$, since otherwise $H \cap (\mathbb{R}_{>0})^n \neq \emptyset$. These are the remaining two conditions.

If all coefficients are real, there remains only the second condition, which becomes the stated sign constraint. □

2.2.3 Quadratic polynomials

This section deals with polynomials of degree 2 and characterizes the imaginary projections of quadratic polynomials with real coefficients. For that, we distinguish the two cases $n = 2$ (Theorem 2.25) and $n \geq 3$ (Theorem 2.27). These polynomials provide a rich class of examples for different situations and we return to them at many occasions.

By Lemmas 2.8 and 2.9, we can reduce real quadratic polynomials to a normal form in order to compute their imaginary projection. Doing so, we discuss the imaginary projection of quadratic polynomials in $n = 2$ and $n \geq 3$ variables. It turns out that the case $n \geq 3$ differs significantly from the case $n = 2$. Nevertheless, we are able to observe interesting behavior of the imaginary projection already for $n = 2$, such as an imaginary projection with only one bounded component in the complement (Theorem 2.25, type (i)) or an imaginary projection which is neither open nor closed (Theorem 2.25, type (ii)). Indeed, the classification results in this section are somewhat unexpected, since it involves various qualitatively different cases. And the behavior differs from the behavior of amoebas; see Section 1.4. There, bounded components in the complement can only occur for polynomials with degree at least three. Moreover, amoebas are always closed and have always unbounded components in the complement.
As a corollary, we obtain a characterization of the normal forms of real stable quadratic polynomials.

The imaginary projections of some quadratic polynomials are shown in Figure 2.5.

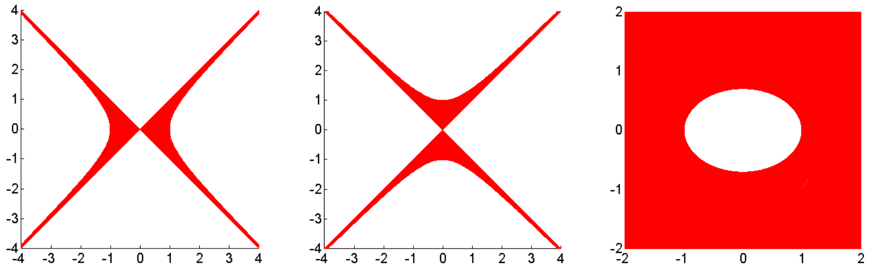

FIGURE 2.5: The imaginary projections of $f(z_1, z_2) = z_1^2 - z_2^2 - 1$, $f(z_1, z_2) = -z_1^2 + z_2^2 - 1$, and $f(z_1, z_2) = 2z_1^2 + z_2^2 + 1$.

By Lemmas 2.8 and 2.9, it suffices to study the imaginary projections of polynomials in a normal form. For $n = 2$ this yields that every real bivariate quadratic polynomial is affinely equivalent to a quadric given by one of the following nine polynomials. The names come from the conic sections considering these polynomials as real polynomials. We call these the normal forms of real bivariate quadratic polynomials.

(*i*) $z_1^2 + z_2^2 - 1$ (ellipse),

(*ii*) $z_1^2 - z_2^2 - 1$ (hyperbola),

(*iii*) $z_1^2 + z_2$ (parabola),

(*iv*) $z_1^2 + z_2^2 + 1$ (empty set),

or one of the special cases (*v*) $z_1^2 - z_2^2$ (pair of crossing lines), (*vi*) $z_1^2 - 1$ (parallel lines, or a single line z_1^2), (*vii*) $z_1^2 + z_2^2$ (isolated point), (*viii*) $z_1^2 + 1$ (empty set).

In the following theorem, we characterize the imaginary projections of these quadratic polynomials.

Theorem 2.25. *For a quadratic polynomial $f \in \mathbb{R}[z_1, z_2]$, we have*

$$\mathcal{I}(f) = \begin{cases} \mathbb{R}^2 & \text{if } f \text{ is of type } (i), \\ \{-1 \le y_1^2 - y_2^2 < 0\} \cup \{0\} & \text{if } f \text{ is of type } (ii), \\ \mathbb{R}^2 \setminus \{(0, y_2) : y_2 \neq 0\} & \text{if } f \text{ is of type } (iii), \\ \{y_1^2 + y_2^2 - 1 \ge 0\} & \text{if } f \text{ is of type } (iv). \end{cases}$$

In the cases (v) – (viii), we have $\mathcal{I}(f) = \{\mathbf{y} \in \mathbb{R}^2 : y_1^2 - y_2^2 = 0\}$, $\mathcal{I}(f) = \{\mathbf{y} \in \mathbb{R}^2 : y_1 = 0\}$, $\mathcal{I}(f) = \mathbb{R}^2$, and $\mathcal{I}(f) = \{\mathbf{y} \in \mathbb{R}^2 : y_1 = \pm 1\}$.

Already these bivariate real quadratic polynomials provide a variety of interesting phenomena:

1. The imaginary projection of type (*ii*) is neither open nor closed.

2. The complement of the imaginary projection of type (iii) has two components whose recession cones are both one-dimensional. This means, these components are not open.
3. The imaginary projection of type (iv) has one bounded component of the complement and it is dense outside the unit ball.

Proof. For cases (i)–(ii) and (iv), we consider a polynomial $f = \alpha z_1^2 + \beta z_2^2 + \gamma$ with $\alpha, \beta, \gamma \in \mathbb{R}$. Decomposing $f(\mathbf{z}) = 0$ into the real and imaginary parts gives $\alpha x_1^2 - \alpha y_1^2 + \beta x_2^2 - \beta y_2^2 + \gamma = 0$ and $\alpha x_1 y_1 + \beta x_2 y_2 = 0$. For $y_1 \neq 0$, eliminating x_1 shows that

$$\mathcal{I}(f) \ = \ \{\mathbf{y} \in \mathbb{R}^n \, : \, x_2^2 \left(\alpha\beta y_1^2 + \beta^2 y_2^2\right) = \alpha y_1^2(\alpha y_1^2 + \beta y_2^2 - \gamma) \text{ has a real solution } x_2\}\,. \tag{2.3}$$

In case (i), we have $\alpha = \beta = 1$, $\gamma = -1$, which altogether gives $\mathcal{I}(f) = \mathbb{R}^2$. In case (ii), we have $\alpha = 1$, $\beta = \gamma = -1$. For $y_1 \neq 0$, real solutions for x_2 in (2.3) exist for $0 < y_2^2 - y_1^2 < 1$ as well as in the special case $y_1^2 - y_2^2 + 1 = 0$. And for $y_1 = 0$, we obtain $\mathbf{y} \in \mathcal{I}(f)$ if and only if $y_2^2 \leq 1$.

Case (iv) can be treated similarly; see Example 2.4. Case (iii) is linear in z_2, so that the equations for the real and imaginary part can be solved directly for x_2 and y_2. Finally, the cases (v)–$(viii)$ are reducible over the complex numbers. Hence, their imaginary projection follow from the classification result of affine-linear polynomials in Theorem 2.23 together with Lemma 2.7. □

We now deal with quadrics in n-dimensional space with $n \geq 3$. After permutation of the variables, every quadric in $\mathbb{R}^n$ is affinely equivalent to a quadric given by one of the following polynomials, which are the normal forms for $n \geq 3$. (See, e.g., [5] as a general background reference for real quadrics.)

$$\begin{array}{lll}
\text{(I)} & \sum_{j=1}^{p} z_j^2 - \sum_{j=p+1}^{r} z_j^2 & (1 \leq p \leq r,\, r \geq 3,\, p \geq \frac{r}{2})\,, \\
\text{(II)} & \sum_{j=1}^{p} z_j^2 - \sum_{j=p+1}^{r} z_j^2 + 1 & (0 \leq p \leq r,\, r \geq 3)\,, \\
\text{(III)} & \sum_{j=1}^{p_1} z_j^2 - \sum_{j=p_1+1}^{r-1} z_j^2 + z_n & (1 \leq p_1 \leq r,\, r \geq 3,\, p_1 \geq \frac{r}{2})\,.
\end{array}$$

Hence, it suffices to discuss these cases. For the sake of higher generality, the normal forms are assumed to be independent of some input-variables $z_{r+1}, \ldots, z_p$. If $r+1 \leq p$, the imaginary projection becomes cylindric as the following lemma shows.

Lemma 2.26. *Let $f \in \mathbb{C}[\mathbf{z}]$ be independent of some variables $z_{r+1}, \ldots, z_n$, $1 < r < n$, i.e., there is a polynomial $p \in \mathbb{C}[z_1, \ldots, z_r]$ such that $f(\mathbf{z}) = p(z_1, \ldots, z_r)$ for all $\mathbf{z} \in \mathbb{C}^n$. Then,*

$$\mathcal{I}(f) = \mathcal{I}(p) \times \mathbb{R}^{n-r}.$$

Due to this observation, we assume without loss of generality $r = n$.

Theorem 2.27. *Let $n \geq 3$ and $f \in \mathbb{R}[\mathbf{z}]$ be a quadratic polynomial.*

1. *If f is of type* (I)*, then*

$$\mathcal{I}(f) \;=\; \begin{cases} \mathbb{R}^n & \text{if } \frac{n}{2} \leq p < n-1 \text{ or } p = n\,, \\ \{\mathbf{y} \in \mathbb{R}^n \;:\; -\sum_{j=1}^{n-1} y_j^2 + y_n^2 \leq 0\} & \text{if } p = n-1\,, \end{cases} \tag{2.4}$$

2. *If f is of type* (II)*, then*

$$\mathcal{I}(f) \;=\; \begin{cases} \mathbb{R}^n & \text{if } p = 0 \text{ or } 1 < p < n-1\,, \\ \{\mathbf{y} \in \mathbb{R}^n \;:\; y_1^2 - \sum_{j=2}^{n} y_j^2 \leq 1\} & \text{if } p = 1\,, \\ \{\mathbf{y} \in \mathbb{R}^n \;:\; \sum_{j=1}^{n-1} y_j^2 - y_n^2 \geq 0\} & \text{if } p = n-1\,, \\ \{\mathbf{y} \in \mathbb{R}^n \;:\; \sum_{j=1}^{n} y_j^2 \geq 1\} & \text{if } p = n\,. \end{cases} \tag{2.5}$$

3. *If f is of type* (III)*, then*

$$\mathcal{I}(f) \;=\; \mathbb{R}^n \setminus \{(0,\ldots,0,y_n) \;:\; y_n \neq 0\}.$$

Note that the case $n \geq 3$ differs significantly from $n = 2$. The proof of Theorem 2.27 is given in the Lemmas 2.28–2.30.

Lemma 2.28. *Let $n \geq 3$. If $f(\mathbf{z}) = \sum_{j=1}^{p} z_j^2 - \sum_{j=p+1}^{n} z_j^2$ with $\frac{n}{2} < p \leq n$, then $\mathcal{I}(f)$ is given by* (2.4).

Proof. Splitting the problem into the real and imaginary part yields

$$\sum_{j=1}^{p} x_j^2 - \sum_{j=p+1}^{n} x_j^2 - \sum_{j=1}^{p} y_j^2 + \sum_{j=p+1}^{n} y_j^2 \;=\; 0\,, \tag{2.6}$$

$$\sum_{j=1}^{p} x_j y_j - \sum_{j=p+1}^{n} x_j y_j \;=\; 0\,. \tag{2.7}$$

Consider a fixed $\mathbf{y} \in \mathbb{R}^n$. If $-\sum_{j=1}^{p} y_j^2 + \sum_{j=p+1}^{n} y_j^2 = 0$, then $x_1 = \cdots = x_n = 0$ gives a solution to (2.6) and (2.7). Therefore, we can assume $-\sum_{j=1}^{p} y_j^2 + \sum_{j=p+1}^{n} y_j^2 \neq 0$.

In the case $p = n$, by reordering the indices, we can assume that $y_1^2 + y_2^2 \neq 0$, and choose $\mathbf{x} = \frac{\|\mathbf{y}\|_2}{(y_1^2+y_2^2)^{1/2}}(-y_2, y_1, 0, \ldots, 0)$ to obtain a solution for (2.6) and (2.7).

In the case $p = n-1$, the n-dimensional hyperboloid (2.6) in the x-variables is one-sheeted for $-\sum_{j=1}^{n-1} y_j^2 + y_n^2 < 0$ and two-sheeted for $-\sum_{j=1}^{n-1} y_j^2 + y_n^2 > 0$. In case of a one-sheeted hyperboloid, its intersection with the hyperplane (2.7) is never empty.

Namely, choosing $x_3 = \cdots = x_{n-1} = 0$, gives the hyperboloid $x_1^2+x_2^2-x_n^2 = \sum_{j=1}^{n-1} y_j^2 - y_n$ in x_1, x_2, x_n, that contains the origin in the inner component of its complement.

Now consider the case where the hyperboloid consists of two sheets. For any $\alpha > 0$, the sets $\{\mathbf{y} \in \mathbb{R}^n \,:\, -\sum_{j=1}^{n-1} y_j^2 + y_n^2 = \alpha > 0\}$ and $\{\mathbf{x} \in \mathbb{R}^n \,:\, \sum_{j=1}^{n-1} x_j^2 - x_n^2 = -\alpha\}$ coincide. Furthermore, after a coordinate transformation we can assume $\alpha = 1$ and set $H = \{\mathbf{x} \in \mathbb{R}^n \,:\, -\sum_{j=1}^{n-1} x_j^2 + x_n^2 = 1\}$.

We claim that the intersection of H with the hyperplane (2.7) is always empty. Due to the symmetry of H with respect to all the coordinate hyperplanes $x_k = 0$ for $1 \leq k \leq n-1$, it suffices by (2.7) to show that the hyperboloid H does not contain two distinct points, whose position vectors are orthogonal to each other with respect to the Euclidean scalar product. Because of the rotational symmetry of H with regard to the x_n-axis and the invariance of scalar products under orthogonal transformations, by applying an orthogonal transformation it suffices to consider the situation $x_2 = \cdots = x_{n-1} = 0$. The resulting hyperbola $-x_1^2 + x_n^2 = 1$ in the x_1-x_n-plane has no two orthogonal position vectors. Namely, the asymptotes $x_1 = \pm x_n$ divide the plane into four quarters, and the hyperbola is contained in the strict interiors of two opposite quarters.

Now consider the case $\frac{n}{2} < p < n-1$. By our initial considerations in the proof, we have already covered the $\mathbf{y} = 0$. In the case $\mathbf{y} \neq 0$, by changing the coordinates we can assume that (y_1, y_2, y_{p+1}) is not the zero vector. Choose $(x_{p+2}, \ldots, x_n) \in \mathbb{R}^{n-p-1}$ such that $-\sum_{j=p+2}^{n} x_j^2 - \sum_{j=1}^{p} y_j^2 + \sum_{j=p+1}^{n} y_j^2 =: \alpha < 0$. Then, since $\mathbf{y}$ is fixed, (2.6) becomes a hyperboloid of one sheet and (2.7) becomes an affine hyperplane. The intersection of these two hypersurfaces is non-empty. Namely, choosing $x_3 = \cdots = x_p = 0$, gives the one-sheeted hyperboloid $\{(x_1, x_2, x_{p+1}) \in \mathbb{R}^3 \,:\, x_1^2 + x_2^2 - x_{p+1}^2 = -\alpha > 0\}$, which intersects the affine hyperplane with normal vector $(y_1, y_2, -y_{p+1})$ and constant term $-\sum_{j=p+2}^{n} x_j y_j$. Hence, there exists an $\mathbf{x} \in \mathbb{R}^n$ satisfying (2.6) and (2.7). $\square$

Lemma 2.29. *Let $n \geq 3$ and $f(\mathbf{z}) = \sum_{j=1}^{p} z_j^2 - \sum_{j=p+1}^{n} z_j^2 + 1$ with $0 \leq p \leq n$.*

1. *If $p = 0$, then $\mathcal{I}(f) = \mathbb{R}^n$.*
2. *If $p = 1$, then $\mathcal{I}(f) = \{\mathbf{y} \in \mathbb{R}^n \,:\, y_1^2 - \sum_{j=2}^{n} y_j^2 \leq 1\}$.*
3. *If $1 < p < n-1$, then $\mathcal{I}(f) = \mathbb{R}^n$.*
4. *If $p = n-1$ then $\mathcal{I}(f) = \{\mathbf{y} \in \mathbb{R}^n \,:\, \sum_{j=1}^{n-1} y_j^2 \geq y_n^2\}$.*
5. *If $p = n$, then $\mathcal{I}(f) = \{\mathbf{y} \in \mathbb{R}^n \,:\, \sum_{j=1}^{n} y_j^2 \geq 1\}$.*

Note that for $n = 3$ the case (3) cannot occur.

Proof. Similar to the proof of Lemma 2.28, we split the problem into the real and imaginary part,

$$\sum_{j=1}^{p} x_j^2 - \sum_{j=p+1}^{n} x_j^2 - \sum_{j=1}^{p} y_j^2 + \sum_{j=p+1}^{n} y_j^2 + 1 = 0\,, \tag{2.8}$$

$$\sum_{j=1}^{p} x_j y_j - \sum_{j=p+1}^{n} x_n y_n = 0\,. \tag{2.9}$$

Consider a fixed $\mathbf{y} \in \mathbb{R}^n$.

In the case $p = 0$, we obtain the two equations $\sum_{j=1}^n x_j^2 = \sum_{j=1}^n y_j^2 + 1$ and $\sum_{j=1}^n x_j y_j = 0$. Setting $\mathbf{x} = \big(\frac{\|\mathbf{y}\|_2^2+1}{y_1^2+y_2^2}\big)^{1/2}(-y_2, y_1, 0, \ldots, 0)$ gives a solution.

In the case $p = 1$, set $\alpha = -y_1^2 + \sum_{j=2}^n y_j^2 + 1$. Then the statement follows identically as in Lemma 2.28 in the cases $\alpha = 0$, $\alpha < 0$ and $\alpha > 0$. For $\alpha = 0$, the point $\mathbf{x} = \mathbf{0}$ is a solution for $f(\mathbf{z}) = f(\mathbf{x} + i\mathbf{y}) = 0$. For $\alpha > 0$, (2.8) is a one-sheeted hyperboloid and (2.9) is a hyperplane. Their intersection is non-empty. For $\alpha < 0$, the formula for α and (2.8) both define two-sheeted hyperboloids. We consider the hyperboloids $H_1 := \{\mathbf{y} \in \mathbb{R}^n \,:\, y_1^2 - \sum_{j=2}^n y_j^2 = 1 - \alpha\}$ and $H_2 := \{\mathbf{x} \in \mathbb{R}^n \,:\, x_1^2 - \sum_{j=2}^n x_j^2 = -\alpha\}$. Via the transformations $\mathbf{y} \mapsto \mathbf{y}/\sqrt{1-\alpha}$ and $\mathbf{x} \mapsto \mathbf{x}/\sqrt{-\alpha}$ these sets are transformed into the same set $C = \{\mathbf{x} \in \mathbb{R}^n \,:\, x_1^2 - \sum_{j=2}^n x_j^2 = 1\}$. We know by the proof of Lemma 2.28 that there is no pair of orthogonal position vectors on C. Therefore, there are no orthogonal position vectors in H_1 and H_2. Hence, for $\alpha < 0$ the equation $f(\mathbf{z}) = 0$ has no solution in $\mathbf{x}$.

The case $p = n - 1$ is similar. Define $\alpha = -\sum_{j=1}^{n-1} y_j^2 + y_n^2 + 1$. For $\mathbf{y} = 0$, the vector $\mathbf{x} = (0, \ldots, 0, 1)$ is a solution for $f(\mathbf{x} + i\mathbf{y}) = 0$, and for $\alpha = 0$, the vector $\mathbf{x} = \mathbf{0}$ solves (2.8) and (2.9). Consider $\alpha < 0$. Then, there is an index $j \in \{1, \ldots, n\}$ such that $y_j \neq 0$. After possibly permuting the variables, we can assume $j = 1$. Then, $\mathbf{x} = \sqrt{\frac{-\alpha y_1^2}{y_1^2+y_2^2}}\big(-\frac{y_2}{y_1}, 1, 0, \ldots, 0\big)$ is a solution for (2.8) and (2.9). Now, assume $\alpha > 0$ and consider the hyperboloids $H_1 := \{\mathbf{y} \in \mathbb{R}^n \,:\, y_n^2 - \sum_{j=1}^{n-1} y_j^2 = \alpha - 1\}$ and $H_2 := \{\mathbf{x} \in \mathbb{R}^n \,:\, x_n^2 - \sum_{j=1}^{n-1} x_j^2 = \alpha\}$. Let $0 < \alpha < 1$. Due to the rotational symmetry of H_1 with respect to the y_n-axis and the invariance of scalarproducts under orthogonal transformations, we can consider the case $y_2 = \ldots = y_{n-1} = 0$ and $y_1 \neq 0$. Then, $\mathbf{x} = \sqrt{1 + y_n^2}(\frac{1}{y_1}, 1, 0, \ldots, 0, 1)$ is a solution for $f(\mathbf{x}+i\mathbf{y}) = 0$. For $\alpha > 1$, the hyperboloids H_1 and H_2 are both 2-sheeted. Hence, there is no real solution for $\mathbf{x}$ as seen above in the case $p = 1$, $\alpha < 0$. Similarly, it follows for $\alpha = 1$, $y \neq 0$, that there is no solution.

In the case $1 < p < r - 1$, the statement follows as in Lemma 2.28.

In the case $p = n$, there exists an $\mathbf{x}$ satisfying (2.8) and (2.9) if and only if $\sum_{j=1}^{n} y_j^2 - 1 \geq 0$.

$\square$

Lemma 2.30. *Let $n \geq 3$. If $f(\mathbf{z}) = \sum_{j=1}^{p} z_j^2 - \sum_{j=p+1}^{n-1} z_j^2 + z_n$ with $1 \leq p \leq n$, $p \geq \frac{n}{2}$, then*

$$\mathcal{I}(f) = \mathbb{R}^n \setminus \{(0, \ldots, 0, y_n) \, : \, y_n \neq 0\} \, .$$

Proof. In the system for the real and the imaginary parts

$$\sum_{j=1}^{p} x_j^2 - \sum_{j=p+1}^{n-1} x_j^2 - \sum_{j=1}^{p} y_j^2 + \sum_{j=p+1}^{n-1} y_j^2 + x_n = 0 \, , \tag{2.10}$$

$$2 \sum_{j=1}^{p} x_j y_j - 2 \sum_{j=p+1}^{n-1} x_j y_j + y_n = 0 \, , \tag{2.11}$$

consider a fixed $(y_1, \ldots, y_n) \in \mathbb{R}^n$.
If $(y_1, \ldots, y_{n-1}) \neq 0$, then we can choose $(x_1, \ldots, x_{n-1}) \in \mathbb{R}^n$ such that (2.11) is satisfied. Since (2.10) is linear in x_n, it has a real solution for x_n. In the special case $(y_1, \ldots, y_{n-1}) = 0$, we see that $\mathbf{y} = (y_1, \ldots, y_n) \in \mathcal{I}(f)$ if and only if $y_n = 0$. $\square$

As a consequence of Theorems 2.25 and 2.27 we obtain the following corollary which characterizes the normal forms of real stable quadrics.

Corollary 2.31. *Let $f \in \mathbb{R}[\mathbf{z}]$ be a quadratic polynomial. If f is real stable, its normal form is*

- *for $n = 2$ either a hyperbola or two lines (maybe identical),*
- *for $n > 2$ the quadric $\sum_{j=1}^{n-1} z_j^2 - z_n^2$ or $\sum_{j=1}^{n-1} z_j^2 - z_n^2 + 1$.*

Clearly, this criterion is not sufficient. If $\mathcal{I}(f)$ is the imaginary projection of a real stable polynomial, then there is an orthogonal matrix O, such that the rotation of $\mathcal{I}(f)$ provided by O intersects the positive orthant $\mathcal{H}_{\mathbb{R}}^n$. By Lemma 2.8, it is $O\mathcal{I}(f(\mathbf{z}) = \mathcal{I}(f(O^T\mathbf{z}))$. Moreover, the normal forms of $f(\mathbf{z})$ and $f(O^T\mathbf{z})$ coincide.

Proof. If f is stable, the complement of its imaginary projection contains the positive orthant. Hence, the imaginary projections of its normal form need to have a complement component with full-dimensional recession cone (see Definition 2.11).
That implies that the only suitable normal forms for $n = 2$ are the hyperbola (ii) and the lines (v) and (vi). For $n > 2$ the potential normal forms are type (I) with $p = n - 1$, which is the quadric $\sum_{j=1}^{n-1} z_j^2 - z_n^2$, or type (II) with $p = 1$ or $p = n - 1$.

It remains to show that a polynomial f, whose normal form f_n is of type (II) with $p = 1$ or $p = n-1$, cannot be stable.
The imaginary projection of f_n is symmetric with respect to the origin and contains the unit ball centered at the origin. Since the imaginary projection of f shows the same symmetry, the affine transformation A mapping f to f_n need to preserve it. Thus, since f is stable, A must retract the unit ball to a zero-dimensional point. But this is impossible since A is assumed to be invertible. $\square$

We refer to Figure 2.2 for the illustration of the imaginary projection of the polynomial $f(z_1, z_2) = z_1 z_2 + z_1 + z_2 - 1$, whose normal form is the hyperbola $z_1^2 - z_2^2 - 1$.

For complex polynomials, the situation is significantly more complicated. If $f \in \mathbb{C}[\mathbf{z}]$ is a quadratic polynomial with complex coefficients, say $f(\mathbf{z}) = \mathbf{z}^T A \mathbf{z} + \mathbf{b}^T \mathbf{z} + c$, the coefficient matrix A is complex symmetric and not Hermitian. Therefore, A is not necessarily orthogonal diagonalizable, which is essential to bring real quadrics to a normal form. However, there is a unitary matrix U such that $U^T A U$ is a real diagonal matrix whose entries are the eigenvalues of $A^T A$; see [119]. But due to the fact, that these unitary matrices are complex, Lemma 2.8 does not work for our purpose and we cannot describe the action of U as a continuous deformation of $\mathcal{I}(f)$.

In [101], Newstead derived a classification of homogeneous, complex quadrics in three variables by only using real affine-linear transformations. He distinguishes 10 different classes of irreducible and 9 different classes of reducible polynomials; some of them contain additional parameters. Remembering to the sometimes involved computations for real quadrics, it would go beyond the scope of this thesis to classify the imaginary projections of all the cases. However, Figure 2.6 illustrates the imaginary projection of two complex normal forms.

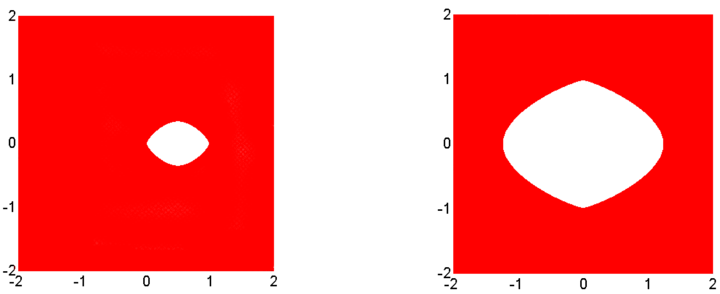

FIGURE 2.6: The imaginary projections of $f_1(z_1, z_2) = z_1^2 + z_2 + iz_2^2$ (left) and of $f_2(z_1, z_2) = z_1^2 + 0.25iz_2^2 + (1 - 0.25i)$ (right). The polynomial f_1 is the affine version of (B_+) in [101] and f_2 is the affine version of $(F_0)_\beta$, with $\beta = 0.25i$.

2.2.4 Polynomials affine-linear in one variable

Now, we study the imaginary projection of polynomials that are affine-linear in one variable. In the special case of polynomials, which are affine-linear in every variable, they are called multi-affine-linear (or short: multi-linear). Brändén's stability result for this class was shown in Theorem 1.30.

The following statement describes the imaginary projection of bivariate multi-linear polynomials.

Theorem 2.32. *Let $f(z_1, z_2) = z_1 z_2 + \beta z_1 + \gamma z_2 + \delta$ be a multi-linear polynomial with $\beta, \gamma, \delta \in \mathbb{R}$. Then*

$$\mathcal{I}(f) \;=\; \left\{\mathbf{y} \in \mathbb{R}^2 \,:\, 0 < \frac{y_1 y_2}{\delta - \beta\gamma} \leq 1\right\} \cup \{0\} \qquad \textit{for } \delta - \beta\gamma \neq 0\,.$$

In the special case $\delta = \beta\gamma$, the multi-linear polynomial is reducible and thus $\mathcal{I}(f) = \mathcal{I}(z_1 + \gamma) \cup \mathcal{I}(z_2 + \beta) = \mathcal{I}(z_1) \cup \mathcal{I}(z_2) = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$.

As a consequence, we rediscover that the multi-linear polynomial f is stable if and only if $\beta\gamma - \delta \geq 0$; see Theorem 1.30. See Figure 2.7 for an illustration.

Proof. Since f can be written as $f(z_1, z_2) = (z_1 + \gamma)(z_2 + \beta) + \delta - \beta\gamma$, Lemma 2.9 implies that $\mathcal{I}(f) = \mathcal{I}(g)$ where $g(z_1, z_2) = z_1 z_2 + \delta - \beta\gamma$. Substituting $z_1 = z_1' + z_2'$ and $z_2 = (\beta\gamma - \delta)(z_1' - z_2')$, we can express g as $g(z_1', z_2') = (\beta\gamma - \delta)(z_1^2 - z_2^2 - 1)$, and by Theorem 2.25, the imaginary projection of g with respect to the $\mathbf{z}'$-variables is

$$\{\mathbf{y}' \in \mathbb{R}^2 \,:\, -1 \leq (y_1')^2 - (y_2')^2 < 0\} \cup \{0\}\,.$$

Using $\begin{pmatrix} 1 & 1 \\ \beta\gamma - \delta & -(\beta\gamma - \delta) \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1/(\beta\gamma - \delta) \\ 1 & -1/(\beta\gamma - \delta) \end{pmatrix}$, transforming back to the $\mathbf{z}$-variables with Lemma 2.8 yields the claim. $\square$

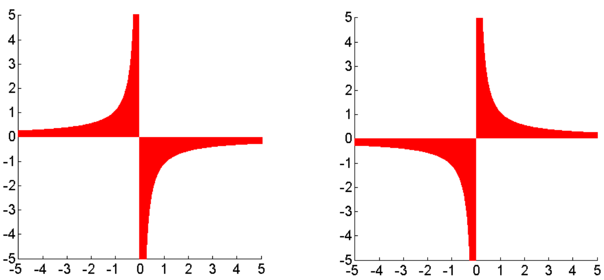

FIGURE 2.7: The imaginary projections of $f(z_1, z_2) = z_1 z_2 + z_1 + 2z_2 + 3$ and $f(z_1, z_2) = z_1 z_2 + z_1 + 2z_2 + 1$.

For polynomials of the form $f = g + z_{n+1}h \in \mathbb{C}[\mathbf{z}, z_{n+1}]$ with $g, h \in \mathbb{C}[\mathbf{z}]$, we provide the subsequent, less explicit, characterization of the imaginary projection. We illustrate these results in Examples 2.35 and 2.36.

Lemma 2.33. *Let $f = g + z_{n+1}h \in \mathbb{C}[\mathbf{z}, z_{n+1}]$ and $v \in \mathbb{R}$. A point $(\mathbf{z}, z_{n+1})$ with $\operatorname{Im}(z_{n+1}) = v$ and $h(\mathbf{z}) \neq 0$ is contained in $\mathcal{V}(f)$ if and only if the determinant*

$$\det \begin{pmatrix} \operatorname{Re} g - v \operatorname{Im} h & \operatorname{Re} h \\ \operatorname{Im} g + v \operatorname{Re} h & \operatorname{Im} h \end{pmatrix} \tag{2.12}$$

vanishes in $\mathbf{z}$.

Proof. Writing $z_{n+1} = u + iv$, the conditions $\operatorname{Re} f = 0$ and $\operatorname{Im} f = 0$ give

$$\begin{pmatrix} \operatorname{Re} g \\ \operatorname{Im} g \end{pmatrix} + \begin{pmatrix} \operatorname{Re} h & -\operatorname{Im} h \\ \operatorname{Im} h & \operatorname{Re} h \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \operatorname{Re} g - v \operatorname{Im} h \\ \operatorname{Im} g + v \operatorname{Re} h \end{pmatrix} + u \begin{pmatrix} \operatorname{Re} h \\ \operatorname{Im} h \end{pmatrix} = 0.$$

Considering this equation as a linear equation in u shows that there exists a solution if and only if the coefficient vector and the constant vector are linearly dependent, that is, if and only the determinant (2.12) vanishes. □

We obtain the following corollary.

Corollary 2.34. *Let $f = g + z_{n+1}h \in \mathbb{C}[\mathbf{z}, z_{n+1}]$. Then, writing $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, the sets $\mathcal{I}(g + z_{n+1}h)$ and*

$$\left\{ (\mathbf{y}, v) \in \mathbb{R}^{n+1} \,:\, \exists \mathbf{x} \in \mathbb{R}^n \text{ with } \det \begin{pmatrix} \operatorname{Re} g - v \operatorname{Im} h & \operatorname{Re} h \\ \operatorname{Im} g + v \operatorname{Re} h & \operatorname{Im} h \end{pmatrix} = 0 \right\} \tag{2.13}$$

coincide outside of the exceptional set $E = \{\operatorname{Im}((\mathbf{z}, z_{n+1})) \,:\, h(\mathbf{z}) = 0 \text{ and } g(\mathbf{z}) \neq 0\}$.

We observe that the determinantal conditions in (2.12) and (2.13) give a linear condition in v. For a multi-linear polynomial of the form $f = g + z_{n+1}h \in \mathbb{R}[\mathbf{z}, z_{n+1}]$ with g and h multi-linear, the condition is quadratic in any of the variables $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$.

Example 2.35. *We revisit the multi-linear polynomial $f(z_1, z_2) = z_1 z_2 + \delta$, $\delta \in \mathbb{R}\backslash\{0\}$ to illustrate Corollary 2.34; see Theorem 2.32. Setting $g = \delta$ and $h = z_1$, the determinantal condition* (2.12) *gives (where we write y_2 instead of v)*

$$\delta y_1 - y_1^2 y_2 - x_1^2 y_2 \;=\; 0\,.$$

For $y_2 \neq 0$, there exists a real solution for x_1 if and only if $\frac{y_1}{y_2}(\delta - y_1 y_2) \geq 0$. Taking into account the exceptional set $E = \{0\} \times \mathbb{R}$, we obtain $\mathcal{I}(f) = \{\mathbf{y} \in \mathbb{R}^2 : 0 < \frac{y_1 y_2}{\delta} \leq 1\} \cup \{0\}$, in accordance with Theorem 2.32. See the left picture in Figure 2.8 for an illustration.

Example 2.36. *We consider the non-multi-linear polynomial $f(z_1, z_2) = 1 + z_2 z_1^2$, which is of the form $f = g + z_2 h$ with $g = 1$ and $h = z_1^2$. Corollary 2.34 gives the quartic condition in the variable x_1*

$$- y_2 x_1^4 - 2 y_1^2 y_2 x_1^2 + 2 y_1 x_1 - y_2 y_1^4 \; = \; 0\,. \tag{2.14}$$

Recall that the discriminant of a general polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$ is given by $\mathrm{Disc}(p) = (-1)^{\frac{1}{2}n(n-1)} \frac{1}{a_n} \mathrm{Res}(p, p')$, *where* Res *denotes the resultant. For a quartic, a positive discriminant corresponds to zero or four real roots, while a negative discriminant corresponds to two real roots. Moreover, with the notation*

$$H \; = \; 8a_2a_4 - 3a_3^2\,, \quad I \; = \; 12a_0a_4 - 3a_1a_3 + a_2^2\,,$$

the case of four real roots corresponds to $H \leq 0$ and $H^2 - 16a_4^2 I \geq 0$, while the case of four complex roots corresponds to $H > 0$ or $H^2 - 16a_4^2 I < 0$; see, e.g., [23, Prop. 7]. In our situation, $p = p(x_1)$ is the polynomial in (2.14), $\mathrm{Disc}(p) = 16(64y_1^4y_2^2 - 27)y_1^4y_2^2$, *$H = 16y_1^2y_2^2$ and $H^2 - 16a_4^2 I = 0$. The set of points $\mathbf{y} \in \mathbb{R}^2$, where* (2.14) *has at least two real solutions in x_1, is given by $64y_1^4y_2^2 \leq 27$. Taking into account the exceptional set $E = \{0\} \times \mathbb{R}$ gives*

$$\mathcal{I}(f) \; = \; \{\mathbf{y} \in \mathbb{R}^2 \, : \, 0 < 64y_1^4y_2^2 \leq 27\} \cup (\mathbb{R} \times \{0\})\,.$$

See the right picture in Figure 2.8 for an illustration.

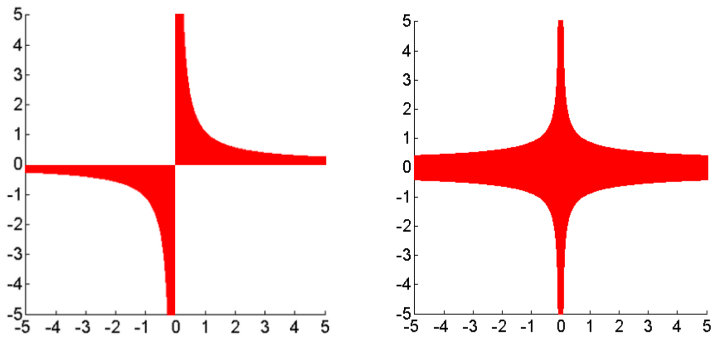

FIGURE 2.8: The imaginary projections of $f(z_1, z_2) = z_1 z_2 + 1$ and $f(z_1, z_2) = 1 + z_2 z_1^2$.

2.3 The imaginary projection of hyperbolic polynomials

Hyperbolic polynomials are of great interest in real algebraic geometry, since they show interesting convexity phenomena and provide a natural generalization of semidefinite programming. We introduced hyperbolic polynomials in Section 1.2. Building on these fundamentals we study now imaginary projections of homogeneous and, in particular, hyperbolic polynomials. Hyperbolicity is a phenomenon of the real variety of a polynomial. By results of Helton and Vinnikov [51], the real variety of a smooth and hyperbolic polynomial consists of nested ovals (and a pseudo-line in case of odd degree) in the projective space. The most inner oval bounds the convex hyperbolicity cone.

In Theorem 2.37, we show a strong connection between hyperbolicity cones and imaginary projections. Namely, for a given homogeneous polynomial its hyperbolicity cones and the components of the complement of its imaginary projection coincide. This generalizes the well-known relation between homogeneous real stable polynomials and hyperbolicity cones, which is phrased in Proposition 1.28. In the hyperbolic situation, the components of the complement of the imaginary projection have a manageable structure and we can determine the total number of components by counting hyperbolicity cones. This gives reason to study the total number of hyperbolicity cones in Section 2.3.1. Building on a result by Kummer [70], it turns out that the maximal number equals the maximal count of cells in a central hyperplane arrangement; see Theorem 2.43. This bound is sharp and it is attained by polynomials that are products of independent linear factors. In Proposition 2.47, we show via a constructive proof that there are polynomials with the same number of strictly convex hyperbolicity cones.
Due to Remark 1.4, we restrict ourselves to polynomials with real coefficients.

Note that for a homogeneous polynomial f, the imaginary projection $\mathcal{I}(f)$ can be regarded as a (non-convex) cone, i.e., for any $\mathbf{y} \in \mathcal{I}(f)$ and $\lambda \geq 0$ we have $\lambda \mathbf{y} \in \mathcal{I}(f)$. Thus, in particular, $0 \in \mathcal{I}(f)$.

Theorem 2.37. *Let $f \in \mathbb{R}[\mathbf{z}]$ be homogeneous. Then the hyperbolicity cones of f coincide with the components of $\mathcal{I}(f)^{\mathsf{c}}$.*

Hence, imaginary projections offer a new viewpoint on the collection of all hyperbolicity cones of a given polynomial. Moreover, if a polynomial f is homogeneous, it holds $\mathcal{I}(f) \neq \mathbb{R}^n$ if and only if f is hyperbolic.

Proof. We show the following two properties:

1. If f is hyperbolic with respect to $\mathbf{e} \in \mathbb{R}^n$, then the hyperbolicity cone $C(\mathbf{e})$ satisfies $C(\mathbf{e}) \subseteq \mathcal{I}(f)^{\mathsf{c}}$.

2. If there is a convex cone C with $C \subseteq \mathcal{I}(f)^{\mathsf{c}}$, then f is hyperbolic with respect to every point in C, i.e., C is contained in that hyperbolicity cone of f.

Assume first, that f is hyperbolic with respect to $\mathbf{e} \in \mathbb{R}^n$, and let $\mathbf{e}' \in C(\mathbf{e})$. Then $\mathbf{e}'$ cannot be the imaginary part of a root $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, since otherwise i would be a non-real zero of the univariate function $t \mapsto f(\mathbf{x} + t\mathbf{e}')$.

Assume now that there is a convex cone C with $C \subseteq \mathcal{I}(f)^{\mathsf{c}}$. The homogeneity of f implies $-C \subseteq \mathcal{I}(f)^{\mathsf{c}}$. For $\mathbf{e} \in \pm C$, we have $f(\mathbf{x} + i\mathbf{e}) \neq 0$ for all $\mathbf{x} \in \mathbb{R}^n$, which gives in particular

$$f(\mathbf{e}) = (1+i)^{-\deg f} f((1+i)\mathbf{e}) = (1+i)^{-\deg f} f(\mathbf{e} + i\mathbf{e}) \neq 0,$$

where $\deg(f)$ denotes the degree of the homogeneous polynomial f. Furthermore, if there were an $\mathbf{x} \in \mathbb{R}^n$ such that $t \mapsto f(\mathbf{x} + t\mathbf{e})$ has a non-real solution $a + ib$, $b \neq 0$, then

$$f(\mathbf{x} + a\mathbf{e} + ib\mathbf{e}) = 0$$

in contradiction to $b\mathbf{e} \in \pm C \subseteq \mathcal{I}(f)^{\mathsf{c}}$. $\square$

As an immediate consequence, we obtain the following description of the imaginary projection of a homogeneous polynomial.

Corollary 2.38. *If $f \in \mathbb{C}[\mathbf{z}]$ is homogeneous, then its imaginary projection is a closed cone (in general non-convex). The components $C_1, \ldots, C_t$ of $\mathcal{I}(f)^{\mathsf{c}}$ are hyperbolicity cones of f and occur pairwise, with $C_{i_1} = -C_{i_2}$. In particular, the imaginary projection of a homogeneous polynomial has no bounded components in its complement.*

Proof. By Theorem 2.37, the components $C_1, \ldots, C_t$ of $\mathcal{I}(f)^{\mathsf{c}}$ are the hyperbolicity cones of f, which occur pairwise. Since hyperbolicity cones are open and since $\mathcal{I}(f)$ has only finitely many of these conic components in the complement, $\mathcal{I}(f)$ is closed. And since $\mathcal{I}(f)$ is a cone, there are no bounded components in the complement. $\square$

The examples following illustrate the connection stated in Theorem 2.37 in well-known cases.

Example 2.39. *Let $f(\mathbf{z}) = z_1 \cdots z_n$. Then f is hyperbolic with respect to every point $\mathbf{e} \in (\mathbb{R} \setminus \{0\})^n$; see Example 1.3. Setting $z_j = x_j + iy_j$, we obtain*

$$\mathcal{I}(f) \;=\; \{\mathbf{y} \in \mathbb{R}^n \,:\, \prod_{j=1}^{n}(x_j + iy_j) = 0 \textit{ for some } \mathbf{x} \in \mathbb{R}^n\} \;=\; \bigcup_{j=1}^{n} \{\mathbf{y} \in \mathbb{R}^n : y_j = 0\}.$$

Example 2.40. *Let* $f(\mathbf{z}) = z_1^2 - \sum_{j=2}^n z_j^2$*,* $n > 2$*. Its two hyperbolicity cones are* $\mathcal{L} = \{\mathbf{x} \in \mathbb{R}^n : x_1^2 - \sum_{j=2}^n x_j^2 > 0,\, x_1 > 0\}$ *and its negative* $-\mathcal{L}$*; see Example 1.8. Likewise, the imaginary projection of* f *is* $\mathcal{I}(f) = \{\mathbf{y} \in \mathbb{R}^n : y_1^2 - \sum_{j=2}^n y_j^2 \leq 0\} = \mathbb{R}^n \setminus (\mathcal{L} \cup -\mathcal{L})$*, which was computed as part of Theorem 2.27. This illustrates Theorem 2.37.*

Furthermore, if a real, homogeneous, and quadratic polynomial $f \in \mathbb{R}[\mathbf{z}]$ is hyperbolic, then its hyperbolicity cone is the image of the second-order cone under a regular linear transformation; that property follows from the classification of the imaginary projections of real quadratic polynomials in Section 2.2.3.

Example 2.41. *Let* $f(\mathbf{z}) = \det(z_1 A_1 + \cdots + z_n A_n)$*, where* $A_1, \ldots, A_n$ *are Hermitian* $d \times d$*-matrices. It is well-known, that* f *has the spectrahedral hyperbolicity cone*

$$C \ = \{\mathbf{x} \in \mathbb{R}^n : x_1 A_1 + \cdots + x_n A_n \succ 0\};$$

see [76, Prop. 2] and also Example 1.7. This implies that C *and* $-C$ *are components of* $\mathcal{I}(f)^{\mathsf{c}}$*. Namely, given some* $\mathbf{y} \in \mathbb{R}^n$ *with* $A(\mathbf{y}) := y_1 A_1 + \cdots + y_n A_n \succ 0$*, we have*

$$f(\mathbf{x}+i\mathbf{y}) = \det\left(\sum_{j=1}^n x_j A_j + iA(\mathbf{y})\right) = \det(A(\mathbf{y})) \cdot \det\left(\sum_{j=1}^n x_j A(\mathbf{y})^{-1/2} A_j A(\mathbf{y})^{-1/2} + iI\right), \tag{2.15}$$

where $A(\mathbf{y})^{-1/2}$ *is the unique matrix with* $A(\mathbf{y})^{-1/2} \cdot A(\mathbf{y})^{-1/2} = A(\mathbf{y})^{-1}$*. If* $f(\mathbf{x} + i\mathbf{y})$ *vanished for some* $\mathbf{x} \in \mathbb{R}^n$*, then the Hermitian matrix* $\sum x_j A(\mathbf{y})^{-1/2} A_j A(\mathbf{y})^{-1/2}$ *would have the eigenvalue* $-i$*. But this is impossible, since Hermitian matrices have only real eigenvalues. Hence,* $\mathbf{y} \notin \mathcal{I}(f)$*.*

Conversely, let $f(\mathbf{x} + i\mathbf{y}) = 0$*. Assuming* $A(\mathbf{y}) \succ 0$*, the right hand side of* (2.15) *vanishes, which again gives the contradiction that* $-i$ *is an eigenvalue of the Hermitian matrix.*

In the following, we consider the number and structure of hyperbolicity cones of a homogeneous polynomial. In order to see that there can appear many hyperbolicity cones, consider polynomials of the form $f(\mathbf{z}) = \det(A_1 z_1 + \cdots + A_n z_n)$ with real diagonal $d \times d$-matrices. This is a special case of Example 2.41, where the spectrahedral hyperbolicity cone becomes a polyhedron. In that case, it becomes profitable to use the viewpoint of imaginary projections to describe exactly the hyperbolicity cones. Namely, $\mathcal{I}(f)$ is an algebraic variety here, whereas the hyperbolicity cones are semi-algebraic. Theorem 2.49 in Section 2.4 will characterize all the homogeneous polynomials for which the imaginary projection is algebraic.

Theorem 2.42. *Let $f(\mathbf{z}) = \det(A_1 z_1 + \cdots + A_n z_n)$, where $A_1, \ldots, A_n$ are $d \times d$ real diagonal matrices, $A_j = \operatorname{diag}(a_1^{(j)}, \ldots, a_n^{(j)})$. Then $\mathcal{I}(f)$ is the hyperplane arrangement*

$$\mathcal{I}(f) = \bigcup_{l=1}^{d} \{\mathbf{y} \in \mathbb{R}^n : \sum_{j=1}^{n} a_l^{(j)} y_j = 0\}. \tag{2.16}$$

Lemma 2.45 will show that if d' is the number of distinct hyperplanes in (2.16), then the number of complement components is at most $2^{d'}$ for $d' \leq n$ and at most $2 \cdot \sum_{k=0}^{n-1} \binom{d'-1}{k}$ for $d' > n$.

Clearly, Theorem 2.42 extends to affine-linear polynomials by substituting $z_n = 1$. Then, the central hyperplane arrangement in (2.16) becomes an affine hyperplane arrangement, which was considered in Proposition 2.17 in Section 2.1.2. Furthermore, it is immediate that the statement holds true for upper (lower) triangular matrices. We can capture the case of complex coefficients by using Theorem 2.23. The statement does not change unless there appear linear factors whose imaginary projections are the whole $\mathbb{R}^n$. Then the set in (2.16) equals $\mathbb{R}^n$ as well.

Proof. We have

$$\det(A_1 z_1 + \cdots + A_n z_n) = \prod_{l=1}^{d} \left(\sum_{j=1}^{n} a_l^{(j)} x_j + i \sum_{j=1}^{n} a_l^{(j)} y_j \right). \tag{2.17}$$

Assume that there is some $\mathbf{y} \in \mathbb{R}^n$ such that $\sum_{j=1}^{n} a_l^{(j)} y_j = 0$ for an $l \in \{1, \ldots, d\}$. Then, choosing $\mathbf{x} = \mathbf{y}$, we have $f(\mathbf{x} + i\mathbf{y}) = 0$.
Assume now $\sum_{j=1}^{n} a_l^{(j)} y_j \neq 0$ for all $1 \leq l \leq d$. Since (2.17) vanishes if and only if at least one factor vanishes, we have $f(\mathbf{x} + i\mathbf{y}) \neq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. $\square$

2.3.1 The maximal number of hyperbolicity cones

By Theorem 2.37, components of the complement of $\mathcal{I}(f)$ coincide with hyperbolicity cones. Hence, one way for counting the number of complement components for imaginary projections of homogeneous polynomials is to determine the number of hyperbolicity cones. In this section, we prove the maximal number of hyperbolicity cones for homogeneous polynomials of degree d. We show that this number is the maximal number of cells of a central hyperplane arrangement, where “central” expresses that all hyperplanes pass through the origin. Moreover, Theorem 2.43 will show that the maximal number is attained by polynomials that are products of independent linear factors. In that situation, the hyperbolicity cones are convex, but not strictly convex.

In Proposition 2.47, we show via a construction that there are polynomials of degree $2d$ with the same number of strictly convex hyperbolicity cones as in the maximal situation of Theorem 2.43.

Theorem 2.43. *Let $f \in \mathbb{R}[\mathbf{z}]$ be homogeneous of degree d. Then the number of hyperbolicity cones of f and thus the number of components in the complement of $\mathcal{I}(f)$ is at most*

$$\begin{cases} 2^d & \text{for } d \leq n\,, \\ 2\sum_{k=0}^{n-1}\binom{d-1}{k} & \text{for } d > n\,. \end{cases}$$

The maximum is attained if and only if f is a product of independent linear polynomials in the sense that any n of them are linearly independent.

An illustration, where this number is attained, is given by Theorem 2.42.

The proof of this theorem relies on the construction of a polynomial with the stated maximal number of hyperbolicity cones. We start with an arbitrary polynomial of degree d and replace iteratively irreducible factors by linear factors without reducing the total number of hyperbolicity cones. One key ingredient is the fact that irreducible homogeneous polynomials have at most one pair of hyperbolicity cones. This is a result by Kummer, [70]:

Proposition 2.44. *Let $f \in \mathbb{R}[\mathbf{z}]$ be an irreducible homogeneous polynomial. Then f has at most two hyperbolicity cones (i.e., one pair) and thus at most two components in the complement of $\mathcal{I}(f)$.*

In order to collect here some insights into the underlying geometry, we provide an alternative proof for the weaker situation of irreducible, homogeneous and smooth polynomials by using a geometric argument (where "smooth" indicates the smoothness of the variety).

Proof for the case of irreducible, homogeneous and smooth polynomials. Let f be an irreducible, homogeneous polynomial, which is smooth. If f is not hyperbolic, the statement is trivial. Therefore, we can assume f is hyperbolic with at least one pair of hyperbolicity cones.

Assume, there are two different hyperbolicity cones $C_1 \neq \pm C_2$. Hyperbolicity of f with respect to some vector $\mathbf{e}$ implies hyperbolicity of f with respect to every point $\mathbf{e}'$ in its hyperbolicity cone and it is $C(\mathbf{e}) = C(\mathbf{e}')$. Hence, we have $C_1 \cap C_2 = \emptyset$.

Since the statement is clear for linear polynomials, we can assume that f is of degree at least 2. First, we deal with the case that C_1 and C_2 are both pointed.

Consider a smooth boundary point $\mathbf{p}$ of C_1 and ℓ be a line tangent to C_1 in $\mathbf{p}$ such that ℓ intersects the open cone C_2. Further let H be the tangent hyperplane of C_1 at $\mathbf{p}$. Since $\mathcal{V}(f)$ has only finitely many real components, we can also enforce that ℓ is not tangent to any other component of $\mathcal{V}(f)$ (since the tangent condition for any of the components only excludes a lower-dimensional set of lines).
Since hyperbolicity cones are open, by infinitesimally translating ℓ in the inner normal direction of H, the resulting line $\ell' = \mathbf{p}' + \mathbb{R}\mathbf{w}$ intersects the open cone C_1, the function $t \mapsto f(\mathbf{p}' + t\mathbf{w})$ has only real roots, and ℓ' still intersects C_2. Translating ℓ infinitesimally in the opposite direction give lines ℓ'', which still intersect C_2, but which do not intersect C_1. Since all other intersection multiplicities remain locally invariant, this contradicts the property that f is hyperbolic with all points in C_2.

Now, assume that C_1 or C_2 have a lineality space different from $\{0\}$. By [40, Theorem 3], the lineality spaces L_1 and L_2 of C_1 and C_2 coincide. Factoring this lineality space, we can continue with the argument from above. □

Hence, high numbers of hyperbolicity cones arise from different factors in the polynomial. The following lemma describes the total number of hyperbolicity cones in the case of a polynomial that splits completely in linear factors.

Lemma 2.45. *Let $f(\mathbf{z}) = p_1(\mathbf{z}) \cdots p_d(\mathbf{z}) \in \mathbb{C}[\mathbf{z}]$ be a product of d linear polynomials $p_1, \ldots, p_d$. Unless $\mathcal{I}(f) = \mathbb{R}^n$, the number of hyperbolicity cones of f is positive and at most*

1. *2^d for $1 \leq d \leq n$,*
2. *$2\sum_{k=0}^{n-1} \binom{d-1}{k}$ for $d > n$.*

Proof. Let $f(\mathbf{z}) = p_1(\mathbf{z}) \cdots p_d(\mathbf{z})$ be a product of d linear polynomials $p_1, \ldots, p_d$ and $\mathcal{I}(f) \neq \mathbb{R}^n$. Since $\mathcal{I}(p_j)$ is a hyperplane for all j (Theorem 2.23), the imaginary projection $\mathcal{I}(f)$ defines a central hyperplane arrangement in $\mathbb{R}^n$. We can assume that the hyperplanes are in general position, since otherwise the number of hyperbolicity cones may only become smaller.

By Zaslavsky's results [132] (see also [118, Prop. 2.4]), the number of chambers in an affine hyperplane arrangement of d affine hyperplanes in general position is $\sum_{k=0}^{n} \binom{d}{k}$, out of which $\binom{d-1}{n}$ chambers are bounded. Determining the number of chambers in a central hyperplane arrangement of d affine hyperplanes in general position can be reduced to an affine hyperplane arrangement in $\mathbb{R}^{n-1}$ and gives

$$\sum_{k=0}^{n-1} \binom{d}{k} + \binom{d-1}{n-1} = 2\sum_{k=0}^{n-1} \binom{d-1}{k}.$$

For $1 \leq d \leq n$, the Binomial Formula specializes this to the expression given. $\square$

The following technical result describes the number of hyperbolicity cones when multiplying two polynomials. We can assume that one of them is irreducible. Clearly, this extends to arbitrary number of factors. Hence, it can be used to describe the number of hyperbolicity cones for products of arbitrary polynomials.

Lemma 2.46. *Let $f_1, f_2 \in \mathbb{C}[\mathbf{z}]$ be homogeneous and f_1 be irreducible. Then the number of hyperbolicity cones of $f_1 \cdot f_2$ is at most twice the number of hyperbolicity cones of f_2.*

Proof. First recall that any hyperbolicity cone C of $f_1 \cdot f_2$ is of the form $C = C_1 \cap C_2$ with hyperbolicity cones C_1 and C_2 of f_1 and f_2; see Remark 1.6.
We can assume that f_1 and f_2 are hyperbolic. Then, by Theorem 2.44, f_1 has at most one pair of hyperbolicity cones. Intersecting these two cones with the hyperbolicity cones of f_2 gives the bound. $\square$

Since the lemma inductively extends to an arbitrary number of factors, two or more pairs of hyperbolicity cones only arise from different factors in the polynomial f. This fact is captured explicitly by Theorem 2.43, whose proof is now given.

Proof of Theorem 2.43. Since the case $n = 1$ is trivial, we can assume $n \geq 2$. Let $f = p_1 \cdots p_k$ be a homogeneous polynomial of degree d, where $p_1, \ldots, p_k$ are irreducible. Hence, $d = \deg(p_1) + \cdots + \deg(p_k)$. We construct a polynomial $g = q_1 \cdots q_k$ with linear polynomials q_i such that g has at least as many hyperbolicity cones as f.

By Lemma 2.46, the number of hyperbolicity cones of f is at most twice the number of hyperbolicity cones of $p_2 \cdots p_k$. Since the irreducible polynomial p_1 has at most two hyperbolicity cones, there exists some hyperplane H separating these two (open) convex cones. Set q_1 to be a linear polynomial whose zero set is H. The set of hyperbolicity cones of f injects to the set of hyperbolicity cones of $f^* = q_1 p_2 \cdots p_k$. Repeating this process for $p_2, \ldots, p_k$ provides a polynomial $g = q_1 \cdots q_k$ whose number of hyperbolicity cones is at least the number of hyperbolicity cones of f.

Hence, the number of hyperbolicity cones is maximized if f is a product of linear polynomials. Since replacing any non-linear polynomial p_i by a linear polynomial q_i decreases the total degree of the overall product, the maximum number of hyperbolicity cones of a degree d polynomial cannot be attained if f has a non-linear irreducible factor p_i.

Now, the stated numbers follows from Lemma 2.45. $\square$

Theorem 2.43 describes the maximal number of hyperbolicity cones for fixed degree d. In the extremal situation, where the maximum is attained, the underlying polynomial splits completely in linear factors and its hyperbolicity cones are the cells of a central hyperplane arrangement. Hence, the cones are convex, but not strictly convex.

The following proposition illustrates that there are polynomials of degree $2d$, that have 2^d for $1 \leq d \leq n$ and $2\sum_{k=0}^{n-1}\binom{d-1}{k}$ for $d > n$ strictly convex hyperbolicity cones. The proof is constructive.

Proposition 2.47. *Let $d \in \mathbb{N}$ and $n > 2$. Then, there is a polynomial f of degree $2d$ such that f has exactly*

- 2^d *for* $1 \leq d \leq n$,
- $2\sum_{k=0}^{n-1}\binom{d-1}{k}$ *for* $d > n$.

strictly convex hyperbolicity cones.

Proof. Let $\ell_1, \ldots, \ell_d$ be a central hyperplane arrangement with d independent linear hyperplanes, which has the maximal number of cells. This is 2^d for $1 \leq d \leq n$ and $2\sum_{k=0}^{n-1}\binom{d-1}{k}$ for $d > n$; see the proof of Lemma 2.45. Denote this number by N_d. For the proof, we construct a polynomial with d quadratic factors and with N_d strictly convex hyperbolicity cones.

For $1 \leq j \leq d$, let $\ell_j = \langle \mathbf{n}_j, \mathbf{x} \rangle$ with $\|\mathbf{n}_j\|_2 = 1$. For every normal vector $\mathbf{n}_j$, we are going to replace the corresponding hyperplane ℓ_j by a suitable rotated and flattened pair of second-order cones. Without loss of generality, we can assume $\mathbf{n}_1 = \mathbf{e}_1 = (1, 0, \ldots, 0)$. Then for $\alpha \in \mathbb{N}$ the flattened second-order cone

$$\mathcal{L}_{\mathbf{e}_1,\alpha} := \{\mathbf{x} \in \mathbb{R}^n : \alpha x_1^2 - \sum_{j=2}^{n} x_j^2 > 0, x_1 > 0\}$$

contains $\mathbf{e}_1$. Similar to Example 1.8, the polynomial $\alpha x_1^2 - \sum_{j=2}^{n} x_j^2$ has one pair of strictly convex hyperbolicity cones, which is $\mathcal{L}_{\mathbf{e}_1,\alpha}$ and $-\mathcal{L}_{\mathbf{e}_1,\alpha}$.

The remaining normal vectors $\mathbf{n}_j$, $1 < j \leq d$, are a rotation of the unit vector $\mathbf{e}_1$, say $\mathbf{n}_j = O_j \mathbf{e}_1$ with an orthogonal matrix O_j. Then it follows

$$\mathbf{n}_j \in \mathcal{L}_{\mathbf{n}_j,\alpha} := \{\mathbf{x} \in \mathbb{R}^n : O_j^T \mathbf{x} \in \mathcal{L}_{\mathbf{e}_1,\alpha}\}.$$

The hyperplane ℓ_j separates $\mathcal{L}_{\mathbf{n}_j,\alpha}$ and $-\mathcal{L}_{\mathbf{n}_j,\alpha}$ strictly.

Since $\alpha \in \mathbb{N}$ can be arbitrary large it is ensured that there is an $\alpha_0 \in \mathbb{N}$ such that $\mathcal{L}_{\mathbf{n}_i,\alpha_0} \cap \mathcal{L}_{\mathbf{n}_j,\alpha_0} \neq \emptyset$ and $\mathcal{L}_{\mathbf{n}_i,\alpha_0} \cap -\mathcal{L}_{\mathbf{n}_j,\alpha_0} \neq \emptyset$ for $1 \leq i < j \leq d$.

The polynomial $f_{\mathbf{n}_j,\alpha_0}(\mathbf{x}) \coloneqq \alpha_0(x_1^{(j)})^2 - \sum_{j=2}^{n}(x_j^{(j)})^2$, with $((x_1^{(j)}),\ldots,(x_n^{(j)})) \coloneqq O_j\mathbf{x}$, is hyperbolic and its two hyperbolicity cones are $\mathcal{L}_{\mathbf{n}_j,\alpha_0}$ and $-\mathcal{L}_{\mathbf{n}_j,\alpha_0}$. By Lemma 2.46, the number of strictly convex hyperbolicity cones of $f_{\mathbf{n}_1,\alpha_0}\cdots f_{\mathbf{n}_d,\alpha_0}$ equals N_d. □

We give an example for the construction.

Example 2.48. *Let $f(\mathbf{z}) = z_1 \cdots z_n$. By Example 2.39, f has 2^n hyperbolicity cones, which is the maximal number for a polynomial of degree n. We replace the factor $z_1 = \langle \mathbf{e}_1, \mathbf{z}\rangle$ by $f_{\mathbf{e}_1,\alpha}(\mathbf{z}) = \alpha z_1^2 - \sum_{j=2}^{n} z_j^2$. In analog to Example 1.8, $f_{\mathbf{e}_1,\alpha}$ has two hyperbolicity cones. The remaining factors $z_2,\ldots,z_n$ were replaced by the polynomials $f_{\mathbf{e}_j,\alpha}(\mathbf{z}) = \alpha z_j^2 - \sum_{k\neq j} z_k^2$, $j = 2,\ldots,n$. For $\alpha \geq n+1$, their hyperbolicity cones intersect pairwise. Thus, for $\alpha_0 \geq n+1$ the polynomial $f_{\mathbf{e}_1,\alpha_0}\cdots f_{\mathbf{e}_n,\alpha_0}$ has 2^n strictly convex hyperbolicity cones. The 2^n vectors $(\pm 1,\ldots,\pm 1)$ identify each of the different hyperbolicity cones.*

2.4 The boundary of imaginary projections

Beside number and structure of components in the complement of imaginary projections we are also interested in describing the boundary. Difficulties arise from the fact that the imaginary projection of a polynomial does not need to be closed. So, there can occur boundary points belonging to the imaginary projection and boundary points belonging to the complement.

However, for homogeneous polynomials the imaginary projection is always closed, since the complement is either empty or a finite collection of open hyperbolicity cones; see Corollary 2.38. In particular, the boundary is a subset of the real variety. This offers a way to describe the boundary in terms of the variety. Moreover, we characterize all homogeneous polynomials f for which $\mathcal{I}(f) = \mathcal{V}_{\mathbb{R}}(f)$.

For inhomogeneous polynomials the boundary behaves more subtle. We focus on boundary points belonging to the imaginary projection and we prove a necessary condition for boundary points in Theorem 2.54. In analogy to the set of critical points of the logarithmic Gauß map for amoebas, the set of critical points for the imaginary projection are these points $\mathbf{z} \in \mathcal{V}(f)$, such that $\left(\frac{\partial f}{\partial z_1}(\mathbf{z}), \ldots, \frac{\partial f}{\partial z_n}(\mathbf{z})\right)$ is the complex multiple of a real vector. It will turn out in Theorem 2.54 that all points in $\partial\mathcal{I}(f) \cap \mathcal{I}(f)$ are the imaginary parts of critical points. We illustrate in two examples, that this criterion is necessary, but not sufficient.

2.4.1 The homogeneous situation

For homogeneous polynomials f, we know by Theorem 2.37 that the components in $\mathcal{I}(f)^{\mathsf{c}}$ equal the open hyperbolicity cones of f. This allows to characterize the boundary in terms of the variety $\mathcal{V}(f)$. Moreover, due to the homogeneity of f it holds $\mathcal{V}_{\mathbb{R}}(f) \subseteq \mathcal{I}(f)$. Using the uniqueness statement in Proposition 2.44 we obtain a necessary and sufficient condition that imaginary projections are algebraic.

Theorem 2.49. *Let $f \in \mathbb{C}[\mathbf{z}]$ be homogeneous. Then*

1. *$\mathcal{V}_{\mathbb{R}}(f) \subseteq \mathcal{I}(f)$, with equality if and only if $e^{i\varphi} f$ is a product of real linear polynomials for some $\varphi \in [0, 2\pi)$.*
2. *If f is hyperbolic and irreducible, then the Zariski closure of the boundary of $\mathcal{I}(f)^{\mathsf{c}}$ equals $\mathcal{V}(f)$,*
$$\overline{\partial\mathcal{I}(f)^{\mathsf{c}}}^{Z} = \mathcal{V}(f).$$

Proof. By homogeneity, if $\mathbf{z}$ is a root of f, then $i\mathbf{z}$ is a root of f as well. Hence, if $\mathbf{x} \in \mathcal{V}_{\mathbb{R}}(f)$, then $\mathbf{x} \in \mathcal{I}(f)$.

Let $e^{i\varphi}f$ be a linear polynomial with real coefficients. By Theorem 2.23, the imaginary projection of $e^{i\varphi}f$ and thus of f is exactly $\mathcal{V}_{\mathbb{R}}(f)$ (notice that $\mathcal{I}(f) \neq \mathbb{R}^n$). Hence, the statement holds for products of linear polynomials as well.

For the converse direction, let $\mathcal{I}(f) = \mathcal{V}_{\mathbb{R}}(f)$. Assume first that f is irreducible. We observe that f must be hyperbolic, since otherwise $\mathcal{I}(f) = \mathbb{R}^n$, which would imply $f \equiv 0$. By Proposition 2.44, f has exactly one pair of hyperbolicity cones. It corresponds to the two convex, open components C and $-C$ of $\mathcal{I}(f)^{\mathsf{c}}$. By assumption, $\mathcal{I}(f)$ is a real algebraic set, and hence $\mathcal{I}(f) = \partial\mathcal{I}(f) = \partial C = \partial(-C)$. Thus, $\mathcal{I}(f) = \overline{C} \cap \overline{-C}$ is a convex set, where $\overline{C}$ denotes the topological closure of C. Since for any two points $\mathbf{a}, \mathbf{b} \in \mathcal{I}(f)$, $\mathbf{a} \neq \mathbf{b}$, their convex combination is contained in $\mathcal{I}(f) = \mathcal{V}_{\mathbb{R}}(f)$, and hence the underlying polynomial must be linear. Since $\mathcal{I}(f) \neq \mathbb{R}^n$, the classification of linear polynomials in Theorem 2.23 provides that f is of the form $e^{i\varphi}f$.
If f is a product of non-constant irreducible polynomials, we can consider the imaginary projection of each factor and obtain the overall statement.

For the second statement, let f be hyperbolic with respect to $\mathbf{e}$ and irreducible. By Theorem 2.37, the hyperbolicity cone $C = C(\mathbf{e})$ is a component of $\mathcal{I}(f)^{\mathsf{c}}$. Since C is the connected component in the complement of $\mathcal{V}(f)$ containing $\mathbf{e}$, it is bounded by some subset of its real variety. And since f is irreducible, its Zariski closure is $\mathcal{V}(f)$. $\square$

Example 2.50. *As an example for the first part of Theorem 2.49, we refer to Example 2.39. The imaginary projection of the first elementary symmetric polynomial* $f(\mathbf{z}) = z_1 \cdots z_n$ *is the hyperplane arrangement*

$$\mathcal{I}(f) = \bigcup_{j=1}^{n} \{\mathbf{y} \in \mathbb{R}^n : y_j = 0\}.$$

$\mathcal{I}(f)$ *is algebraic and* f *is a product of linear polynomials.*

Example 2.51. *As an example for the second part, we refer to the Lorentz polynomial* $f(\mathbf{z}) = z_1^2 - \sum_{j=2}^{n} z_j^2$, $n > 2$, *in Example 2.40. The complement of its imaginary projection equals its two hyperbolicity cones* $\mathcal{L}$ *and* $-\mathcal{L}$. *Hence, the boundary of the components in the complement are the boundary of* $\mathcal{L}$ *and* $-\mathcal{L}$, *which is* $\mathcal{V}_{\mathbb{R}}(f)$. *Hence, the Zariski closure of* $\partial\mathcal{I}(f)$ *is* $\mathcal{V}(f)$

The situation changes completely for inhomogeneous polynomials. The proof of part (1) fails through a lack of an inhomogeneous analog to the uniqueness statement in Proposition 2.44. And the second part of Theorem 2.49 is not valid for inhomogeneous polynomials in general as the following example shows.

Example 2.52. *Let* $f(z_1, z_2) = z_1^2 + z_2^2 + 1$. *Then* $\mathcal{I}(f) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 - 1 \geq 0\}$, *which was computed as part of Theorem 2.25; see Figure 2.2 for an illustration. The* *boundary of $\mathcal{I}(f)$ is $\mathcal{V}_{\mathbb{R}}(z_1^2 + z_2^2 - 1)$. Hence,*

$$\overline{\partial\mathcal{I}(f)^{\mathsf{c}}}^{Z} = \mathcal{V}(z_1^2 + z_2^2 - 1) \neq \mathcal{V}(z_1^2 + z_2^2 + 1) \ .$$

2.4.2 The inhomogeneous situation

In the general situation the boundary behaves more complicated than in the homogeneous case. The following two different types of boundary points can occur:

1. boundary points that belong to the imaginary projection,
2. boundary points that do not belong to the imaginary projection.

The hyperbola $f(z_1, z_2) = z_1^2 - z_2^2 - 1$ provides an example, where both types of boundary points appear; see Theorem 2.25, type *(ii)*, and see Figure 2.2 for an illustration. Boundary points of the first type can be handled, since for $\mathbf{y} \in \partial\mathcal{I}(f) \cap \mathcal{I}(f)$ there is a point $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x} + i\mathbf{y} \in \mathcal{V}(f)$. If the polynomial is smooth, we can consider the tangent space at this point and obtain a necessary criterion. We follow that idea in Theorem 2.54. For points of type (2), the real part of the variety shows an asymptotic behavior which makes it harder to control. Namely, for any sequence $(\mathbf{y}_n) \subset \mathcal{I}(f)$ with $\mathbf{y}_n \to \mathbf{y} \in \partial\mathcal{I}(f) \cap \mathcal{I}(f)^{\mathsf{c}}$ and for every point $\mathbf{x}_n \in \mathbb{R}^n$ such that $\mathbf{x}_n + i\mathbf{y}_n \in \mathcal{V}(f)$, the sequence of absolute values of $\mathbf{x}_n$ tends to infinity for $n \to \infty$.

Since the main focus of Chapter 2 lies on geometric and combinatorial properties of imaginary projections, we do not consider every aspect of the boundary here. So, in this section we concentrate on boundary points of type (1). We start with the definition of critical points. It will turn out in Theorem 2.54 that only critical points can be boundary points of type (1). In the proof, we make essential use of the characterization result for the imaginary projection of affine-linear polynomials in Theorem 2.23.

Definition 2.53. *Let $f \in \mathbb{C}[\mathbf{z}]$ be a polynomial. A point $\mathbf{z}_0 \in \mathcal{V}(f) \subset \mathbb{C}^n$ is called a critical point of $\mathcal{I}(f)$ if there is a $\varphi \in [0, 2\pi)$ such that*

$$\left(\frac{\partial f}{\partial z_1}(\mathbf{z}_0), \ldots, \frac{\partial f}{\partial z_n}(\mathbf{z}_0)\right) \in e^{i\varphi} \cdot \mathbb{R}^n \ ,$$

i.e., $\left(\frac{\partial f}{\partial z_1}(\mathbf{z}_0), \ldots, \frac{\partial f}{\partial z_n}(\mathbf{z}_0)\right)$ is a complex multiple of a real vector. We denote the set of critical points by $\mathcal{C}(f)$.

We remark the similarity between this set of critical points and the set of critical points of the logarithmic Gauß map for amoebas; see Section 1.4. Indeed, for the first statement in the subsequent Theorem 2.54 there is an exact analog for amoebas: If $\mathcal{S}(f)$ denotes the critical points of the logarithmic Gauß map, then $\partial\mathcal{A}(f) \subseteq \log|\mathcal{S}(f)|$; see, e.g., [91, 115]. This implies an analog for the first statement of Corollary 2.55, as well. We remark that, in contrast to imaginary projections, amoebas are always closed.

The following theorem describes a necessary condition for boundary points of type (1). Moreover, it provides one way for computing these boundary points.

Theorem 2.54. *Let $f \in \mathbb{C}[\mathbf{z}]$ be smooth. Then,*

$$\partial\mathcal{I}(f) \cap \mathcal{I}(f) \subseteq \operatorname{Im}\big(\mathcal{C}(f)\big).$$

Furthermore, $\mathcal{C}(f)$ equals the solution set of the following system of equations:

$$f(\mathbf{x},\mathbf{y}) = 0,\ D_{jk} = \det\begin{pmatrix} \frac{\partial \operatorname{Re} f}{\partial y_j}(\mathbf{x},\mathbf{y}) & \frac{\partial \operatorname{Re} f}{\partial y_k}(\mathbf{x},\mathbf{y}) \\ \frac{\partial \operatorname{Im} f}{\partial y_j}(\mathbf{x},\mathbf{y}) & \frac{\partial \operatorname{Im} f}{\partial y_k}(\mathbf{x},\mathbf{y}) \end{pmatrix} = 0\ ,\ 1 \le j < k \le n. \tag{2.18}$$

The second part allows to describe the set of critical points as quantifier free formula, e.g., by applying quantifier elimination. Note that for $n = 2$ there are only two equations in (2.18).

Proof. Let $\mathbf{y}_0 \in \partial\mathcal{I}(f) \cap \mathcal{I}(f)$ and denote by C a component in $\mathcal{I}(f)^{\mathsf{c}}$ such that $\mathbf{y}_0 \in \overline{C}$, where $\overline{C}$ denotes the topological closure of C. Let H be a supporting hyperplanes for the convex set $\overline{C}$ in $\mathbf{y}_0$. By assumption to $\mathbf{y}_0$, it holds $(\mathbb{R}^n + iH) \cap \mathcal{V}(f) \neq \emptyset$. Choose $\mathbf{z} = \mathbf{x} + i\mathbf{y}_0 \in (\mathbb{R}^n + iH) \cap \mathcal{V}(f)$. Since f is smooth, there is a tangent hyperplane T to $\mathcal{V}(f)$ in $\mathbf{z} = (z_1, \ldots, z_n)$, which is

$$T = \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : \sum_{j=1}^{n} (\lambda_j - z_j) \frac{\partial f}{\partial z_j}(\mathbf{z}) = 0 \right\}.$$

By Theorem 2.23, the imaginary projection of a non-constant affine-linear polynomial $\sum_{j=1}^n a_j z_j + a_0$ is

$$\begin{cases} \mathcal{V}_{\mathbb{R}}\big(\operatorname{Im}(a_0 e^{-i\varphi}) + \sum_{j=1}^n a_j e^{-i\varphi} y_j\big) & \text{if } (a_1, \ldots, a_n) \in e^{i\varphi} \cdot \mathbb{R}^n \text{ for one } \varphi \in [0, 2\pi), \\ \mathbb{R}^n & \text{otherwise.} \end{cases}$$

Since $\operatorname{Im}(T) = H$ is a hyperplane in $\mathbb{R}^n$, the vector $\big(\frac{\partial f}{\partial z_1}(\mathbf{z}), \ldots, \frac{\partial f}{\partial z_n}(\mathbf{z})\big)$ need to be a complex multiple of a real vector. This is $\mathbf{z} \in \mathcal{C}(f)$.

For the second part of the statement, we recall that for complex derivatives the following holds; see, e.g., [37, Chapter 5]:

$$\frac{\partial f}{\partial z_j} = \frac{\partial \operatorname{Im} f}{\partial y_j} - i\frac{\partial \operatorname{Re} f}{\partial y_j} \,.$$

Let $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ be critical. By the first part, it holds $\left(\frac{\partial f}{\partial z_1}(\mathbf{z}), \ldots, \frac{\partial f}{\partial z_n}(\mathbf{z})\right) \in e^{i\varphi} \cdot \mathbb{R}^n$ for one $\varphi \in [0, 2\pi)$. After realification $\mathbf{x} + i\mathbf{y} \mapsto (\mathbf{x}, \mathbf{y})$, we can write this as

$$\begin{pmatrix} -\frac{\partial \operatorname{Re} f}{\partial y_1}(\mathbf{x},\mathbf{y}) & \cdots & -\frac{\partial \operatorname{Re} f}{\partial y_n}(\mathbf{x},\mathbf{y}) \\ \frac{\partial \operatorname{Im} f}{\partial y_1}(\mathbf{x},\mathbf{y}) & \cdots & \frac{\partial \operatorname{Im} f}{\partial y_n}(\mathbf{x},\mathbf{y}) \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \cdot \mathbf{v},$$

where $\mathbf{v} \in \mathbb{R}^n$. Hence, the $2 \times n$-matrix

$$\begin{pmatrix} -\frac{\partial \operatorname{Re} f}{\partial y_1}(\mathbf{x},\mathbf{y}) & \cdots & -\frac{\partial \operatorname{Re} f}{\partial y_n}(\mathbf{x},\mathbf{y}) \\ \frac{\partial \operatorname{Im} f}{\partial y_1}(\mathbf{x},\mathbf{y}) & \cdots & \frac{\partial \operatorname{Im} f}{\partial y_n}(\mathbf{x},\mathbf{y}) \end{pmatrix} \tag{2.19}$$

has rank 1. The rank of a matrix is the largest non-vanishing subdeterminant. Hence, in (2.19) the determinants of all 2×2-submatrices vanish, i.e.,

$$D_{jk} = \det \begin{pmatrix} -\frac{\partial \operatorname{Re} f}{\partial y_j}(\mathbf{x},\mathbf{y}) & -\frac{\partial \operatorname{Re} f}{\partial y_k}(\mathbf{x},\mathbf{y}) \\ \frac{\partial \operatorname{Im} f}{\partial y_j}(\mathbf{x},\mathbf{y}) & \frac{\partial \operatorname{Im} f}{\partial y_k}(\mathbf{x},\mathbf{y}) \end{pmatrix} = 0$$

for all $1 \leq j < k \leq n$. Due to the multilinearity of the determinant, one can forget about the minus-sign in the first row. □

The condition stated in Theorem 2.54 is necessary, but not sufficient. We illustrate this in Example 2.57 by using the following Corollary 2.55. Moreover, Example 2.56 considers a quadratic polynomial, where the set of critical points equals the boundary of the imaginary projection.

Corollary 2.55. *If $f \in \mathbb{R}[\mathbf{z}]$, then*

$$\mathbf{x} \in \mathcal{V}_{\mathbb{R}}(f) \Rightarrow \mathbf{x} \in \mathcal{C}(f),$$

i.e., $\mathcal{V}_{\mathbb{R}}(f) \subseteq \mathcal{C}(f)$. If f is in addition a homogeneous polynomial, then

$$\mathcal{C}(f) = \left\{\mathbf{z} \in \mathbb{C}^n \mid \exists \varphi \in [0, 2\pi) : \; e^{i\varphi}\mathbf{z} \in \mathcal{V}_{\mathbb{R}}(f)\right\}.$$

Proof. If $f \in \mathbb{R}[\mathbf{z}]$, then $\left(\frac{\partial f}{\partial z_1}(\mathbf{z}), \ldots, \frac{\partial f}{\partial z_n}(\mathbf{z})\right)$ is a real vector for every $\mathbf{z} \in \mathbb{R}^n$. This implies $\mathcal{V}_{\mathbb{R}}(f) \subseteq \mathcal{C}(f)$.

If f is in addition homogeneous, then all partial derivatives are homogeneous as well. Let $\mathbf{z} \in \mathcal{C}(f)$, i.e., $f(\mathbf{z}) = 0$ and $\left(\frac{\partial f}{\partial z_1}(\mathbf{z}), \ldots, \frac{\partial f}{\partial z_n}(\mathbf{z})\right) \in e^{i\varphi} \cdot \mathbb{R}^n$ for one $\varphi \in [0, 2\pi)$. Then,

$\left(\frac{\partial f}{\partial z_1}(e^{-i\varphi}\mathbf{z}), \ldots, \frac{\partial f}{\partial z_n}(e^{-i\varphi}\mathbf{z})\right) \in \mathbb{R}^n$ and $f(e^{-i\varphi}\mathbf{z}) = e^{-i\varphi \deg f} f(\mathbf{z}) = 0$. This implies, $e^{-i\varphi}\mathbf{z} \in \mathcal{V}_{\mathbb{R}}(f)$. □

Example 2.56. *Let $f(z_1, z_2, z_3) = z_1^2 - z_2^2 - z_3^2 + 1$. The critical points of f are these points in the variety, such that $(z_1, -z_2, -z_3) \in e^{i\varphi} \cdot \mathbb{R}^3$ for one $\varphi \in [0, 2\pi)$. For these points, we write $(z_1, z_2, z_3) = e^{i\varphi} \cdot (x_1, x_2, x_3)$ with $(x_1, x_2, x_3) \in \mathbb{R}^3$. Then, it holds*

$$(e^{i\varphi}x_1)^2 - (e^{i\varphi}x_2)^2 - (e^{i\varphi}x_3)^2 + 1 = 0,$$

which is equivalent to

$$x_1^2 - x_2^2 - x_3^2 + e^{-2i\varphi} = 0. \tag{2.20}$$

Since $x_1, x_2, x_3 \in \mathbb{R}$, there is only for $\varphi = \frac{\pi}{2}$ and $\varphi = -\frac{\pi}{2}$ a solution in (2.20). This implies $(z_1, z_2, z_3) \in i\mathbb{R}^3$ or $-(z_1, z_2, z_3) \in i\mathbb{R}^3$. Hence, the critical points are

$$\mathcal{C}(f) = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : -z_1^2 + z_2^2 + z_3^2 + 1 = 0\}.$$

The equation $-z_1^2 + z_2^2 + z_3^2 + 1 = 0$ describes exactly the 2-sheeted hyperboloid from Theorem 2.27, type (II) with $p = 1$. Altogether, this implies $\partial\mathcal{I}(f) = \mathcal{C}(f)$.

Example 2.57. *Let $f(z_1, z_2) = 144(z_1^4 + z_2^4) - 225(z_1^2 + z_2^2) + 350z_1^2z_2^2 + 81$. Its real variety is the so called Trott curve. See Figure 2.9 for the real variety and for the imaginary projection of f. By Corollary 2.55, the real variety is a subset of the set of critical*

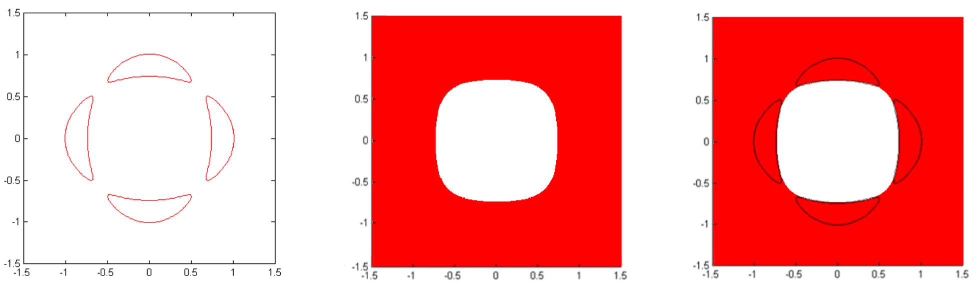

FIGURE 2.9: The real variety of $f(z_1, z_2) = 144(z_1^4 + z_2^4) - 225(z_1^2 + z_2^2) + 350z_1^2z_2^2 + 81$ (left picture) and the imaginary projection of f (middle picture). The right picture shows an overlay of the left and the middle picture.

points. Here, there are critical points that are not real and there are real critical points that are no boundary points. This becomes evident, since the four bounded components in the real variety are not convex. But the right picture in Figure 2.9 suggests that at least parts of the real variety provide boundary points.

The Trott curve belongs to the class of Harnack curves, since it has four ovals in its real variety, which is the maximal number for degree four polynomials. For amoebas, it *is known that (simple) Harnack curves, like the Trott curve, maximize the number of components in the complement of an amoeba; see [90].*

2.5 The limit set of imaginary projections

For the amoeba $\mathcal{A}(f)$ of a polynomial f it is well-known that the logarithmic limit set

$$\mathcal{A}_\infty(f) \;=\; \lim_{r\to\infty}\left(\frac{1}{r}\mathcal{A}(f)\cap\mathbb{S}^{n-1}\right)$$

is a spherical polyhedral complex; see Section 1.4. For imaginary projections, the situation is different from amoebas. As shown by the following counterexample, for $f\in\mathbb{C}[\mathbf{z}]$, the limit set

$$\mathcal{I}_\infty(f) \;=\; \lim_{r\to\infty}\left(\frac{1}{r}\mathcal{I}(f)\cap\mathbb{S}^{n-1}\right)$$

describing the behavior of the imaginary projection "at infinity" is not a spherical polyhedral complex in general.

Example 2.58. *Let $f(\mathbf{z}) = z_1^2 - \sum_{j=2}^n z_j^2 + 1$ with $n\geq 3$. Then, by Theorem 2.27,*

$$\mathcal{I}(f) \;= \{\mathbf{y}\in\mathbb{R}^n \,:\, y_1^2 - \sum_{j=2}^n y_j^2 \leq 1\}\,.$$

Therefore, $\mathcal{I}_\infty(f) = \lim_{r\to\infty}\left\{\mathbf{y}\in\mathbb{S}^{n-1} \,:\, (ry_1)^2 - \sum_{j=2}^n (ry_j)^2 \leq 1\right\}$ can be written as

$$\mathcal{I}_\infty(f) \;=\; \left\{\mathbf{y}\in\mathbb{S}^{n-1} \,:\, y_1^2 \leq \sum_{j=2}^n y_j^2\right\} \;=\; \left\{\mathbf{y}\in\mathbb{S}^{n-1} \,:\, y_1^2 \leq \frac{1}{2}\right\}.$$

Since $n\geq 3$, this cannot be written as the intersection of $\mathbb{S}^{n-1}$ with a polyhedral fan. Hence, $\mathcal{I}_\infty(f)$ is not a spherical polyhedral complex, and since $\mathcal{I}_\infty(f)$ is already closed, this persists under taking the closure.

Nevertheless, the limit set provides interesting information about the behavior "at infinity" of imaginary projections. This motivates studying this kind of set in this section.

Definition 2.59. *Let $f\in\mathbb{C}[\mathbf{z}]$. We call a point $p\in\mathbb{R}^n$ a limit direction of the imaginary projection of f if $p\in\mathcal{I}_\infty(f)$.*

Note that $\mathcal{I}_\infty(f_1\cdot f_2) = \mathcal{I}_\infty(f_1)\cup\mathcal{I}_\infty(f_2)$ by applying Lemma 2.7.

We start with a general criterion for one-dimensional families of limit directions in Theorem 2.60. This implies for $n=2$ that every point on $\mathbb{S}^1$ is a limit direction of the imaginary projection. After that, we consider limit directions of special classes of polynomials. First, we deal with the limit directions of multi-linear polynomials in Theorem 2.62. After that, we consider bivariate polynomials and characterize their limit directions in Corollary 2.63. As a consequence, we derive a result for the limit directions of bivariate determinantal polynomials.

In Section 2.5.1, we obtain a connection between limit directions and components of $\mathcal{I}(f)^{\mathsf{c}}$, which have full-dimensional recession cone. This will lead to a further relationship with hyperbolic polynomials. Moreover, we derive a certificate to decide whether all points on the sphere are limit directions.

Theorem 2.60. *Let $f \in \mathbb{C}[\mathbf{z}]$ be a non-constant polynomial. If its homogenization $f_h \in \mathbb{C}[z_0, \mathbf{z}]$ has a zero $\mathbf{p}_h = (0 : p) = (0 : p_1 : \cdots : p_n) \in \mathbb{P}^n_{\mathbb{C}}$, then every point in the intersection $\mathbb{S}^{n-1} \cap \mathcal{H}$ is a limit direction, where $\mathcal{H} = \{\lambda \operatorname{Re}(\mathbf{p}) + \mu \operatorname{Im}(\mathbf{p}) : \lambda, \mu \in \mathbb{R}\}$ and $\mathbf{p} = (p_1, \ldots, p_n)$.*

Proof. Let $\mathbf{p}_h = (0 : p_1 : \cdots : p_n)$ be a zero at infinity of f_h. Since $i\mathbf{p}_h$ is also a point at infinity for f_h, we can assume that $\sum_{j=1}^n \operatorname{Im}(p_j)^2 \neq 0$. Since f is non-constant, there exists a sequence $(\mathbf{p}^{(k)}) = (p_1^{(k)}, \ldots, p_n^{(k)})$ of points in $\mathcal{V}(f)$ such that $(1 : p_1^{(k)} : \cdots : p_n^{(k)})$ converges to $\mathbf{p}_h$. Hence,

$$\frac{1}{(\sum_{j=1}^n \operatorname{Im}(p_j)^2)^{1/2}} (\operatorname{Im} p_1, \ldots, \operatorname{Im} p_n)$$

is a limit direction. Multiplying $\mathbf{p}_h$ with a complex number $\mu + i\lambda$, $\lambda, \mu \in \mathbb{R}$ keeps $\mathbf{p}_h$ invariant, and under the imaginary projection it leads to a projected point $\operatorname{Im}((\mu + i\lambda) \cdot \mathbf{p}) = \mu \operatorname{Im}(\mathbf{p}) + \lambda \operatorname{Re}(\mathbf{p})$. Considering all complex numbers $\mu + i\lambda \in \mathbb{C}$, these points form the subspace $\mathcal{H}$. $\square$

Example 2.61. *We revisit the polynomial $f(z_1, z_2) = z_1^2 - z_2^2 - 1$; see Figure 2.2 for its imaginary projection. Its homogenization is $f_h(z_0, z_1, z_2) = z_1^2 - z_2^2 - z_3^2$ whose zeros at infinity are given by the equation $z_1^2 - z_2^2 = (z_1 + z_2)(z_1 - z_2) = 0$. For the points $\mathbf{p}_h = (0 : 1 : \pm 1)$, Theorem 2.60 provides the two one-dimensional lines $\mathcal{H}_{1,2} = \{\lambda(1, \pm 1) : \lambda \in \mathbb{R}\}$. We obtain the intersection $\mathbb{S}^1 \cap \mathcal{H}_{1,2} = \{(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}})\}$. Indeed, we know by Theorem 2.25 that $\mathcal{I}(f) = \{\mathbf{y} \in \mathbb{R}^2 : -1 \leq y_1^2 - y_2^2 < 0\} \cap \{0\}$, which confirms $\mathcal{I}_\infty(f) = \{(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}})\}$.*

For multi-linear polynomials, there is one situation which is easy to understand. If the monomial $z_1 \cdots z_n$ appears in a multi-linear polynomial f, then $z_1 \cdots z_n$ determine the limit directions of f. This suggests a relationship between the limit direction of f and the limit direction of its initial form, which is the sum of those terms which have maximal total degree. We consider this further in Section 2.5.1.

Theorem 2.62. *Let $f \in \mathbb{C}[\mathbf{z}]$ be a multi-linear polynomial, and assume that the monomial $z_1 \cdots z_n$ appears in f, i.e., $\deg(f) = n$. Then the limit directions of $\mathcal{I}(f)$ are given* *by* $\mathbb{S}^{n-1} \cap \mathcal{H}$, *where* $\mathcal{H}$ *is the union of the* n *coordinate hyperplanes* $\{\mathbf{y} \in \mathbb{R}^n : y_j = 0\}$, $1 \leq j \leq n$.

Proof. Homogenizing f to $f_h(z_0, z_1, \ldots, z_n)$, the homogeneous polynomial f_h has a zero at infinity, i.e., $(0, z_1, \ldots, z_n) \in \mathcal{V}(f_h)$, if and only if $z_1 \cdots z_n = 0$. Hence, the set of limit points of points in $\frac{1}{r}\mathcal{V}(f)$, $r \to \infty$, is $\mathcal{V}(z_1 \cdots z_n)$. The imaginary projections of the n hyperplanes $\{\mathbf{z} \in \mathbb{C}^n : z_j = 0\}$ then imply the claim. □

For bivariate polynomials Theorem 2.60 implies the following statement about the limit directions:

Corollary 2.63. *Let* $f \in \mathbb{C}[z_1, z_2]$ *be of total degree* d *and assume its homogenization* f_h *has the zeros at infinity* $(0 : 1 : a_j)$, $j = 1, \ldots, d$. *Then,*

$$\mathcal{I}_\infty(f) = \begin{cases} \bigcup_{j=1}^{d} \left\{ \pm \frac{1}{\sqrt{1+a_j^2}}(1, a_j) \right\} & \textit{if all } a_j \textit{ are real,} \\ \mathbb{S}^1 & \textit{otherwise.} \end{cases}$$

Note that by changing coordinates, zeros of the form $(0 : 0 : b_j)$ with $b_j \neq 0$ are covered by the statement as well. For bivariate homogeneous polynomials, this corollary implies that the imaginary projection is either a finite collection of lines or the whole plane. This is due to their conic structure, which implies $\mathcal{I}_\infty(f) = \mathcal{I}(f) \cap \mathbb{S}^{n-1}$.

Proof. Assume first that there is an a_j with $\mathrm{Im}(a_j) \neq 0$, say a_1. Then, by Theorem 2.60, the subspace $\mathcal{H} = \{\lambda(1, \mathrm{Re}(a_1)) + \mu(0, \mathrm{Im}(a_1)) : \lambda, \mu \in \mathbb{R}\}$ is two-dimensional and thus the set of limit directions is $\mathcal{H} \cap \mathbb{S}^1 = \mathbb{S}^1$.

If all a_j are real, then all the subspaces $\mathcal{H}_j$ corresponding to the points $(0 : 1 : a_j)$ are one-dimensional. The intersection $\mathcal{H}_j \cap \mathbb{S}^1$ contains the points $\pm\frac{1}{\sqrt{1+a_j^2}}(1, a_j)$.

In order to show that there are no further limit directions, let $(\mathbf{p}^{(n)})_{n\in\mathbb{N}}$ be a sequence of points in $\mathcal{V}(f)$ with $\|\mathrm{Im}(\mathbf{p}^{(n)})\|_2 \to \infty$. Since the curve $\mathcal{V}(f)$ has only a finite number of points in the plane at infinity, namely d, the sequence $(\mathbf{p}^{(n)})_{n\in\mathbb{N}}$ can be decomposed into d disjoint subsequences $(\mathbf{q}_1^{(n)}), \ldots, (\mathbf{q}_l^{(n)})$ (some of them possibly contain only finitely many elements) such that any infinite sequence $(\mathbf{q}_j^{(n)})$ converges to the projective point $(0 : 1 : a_j)$. □

Example 2.64. *Let* $f(z_1, z_2) = z_1^2 + z_2^2 + 1$. *Then the zeros of* f_h *at infinity are determined by the equation* $z_1^2 + z_2^2 = 0$, *giving the two zeros* $(0 : 1 : \pm i)$. *Since the third coordinate is purely imaginary, any point on* $\mathbb{S}^1$ *is a limit direction, as already visualized in the left picture of Figure 2.2.*

Another example is provided by Example 2.61 above.

Moreover, for illustration, we consider bivariate determinantal polynomials $f(z_1, z_2) = \det(A_1z_1 + A_2z_2 + B)$. The corollary following shows that for positive definite matrix A_1 the limit directions are determined by the signs of the eigenvalues of A_2. We denote by $\operatorname{sgn}(a)$ the sign of a real number a.

Corollary 2.65. *Let $f(z_1, z_2) = \det(A_1z_1+A_2z_2+B)$ with real $d\times d$-matrices A_1, A_2, B and let A_1 be positive definite. If A_2 is symmetric with real eigenvalues $\lambda_1, \dots, \lambda_d$, then there are real numbers $\mu_1, \dots, \mu_d$ with $\operatorname{sgn}(\mu_j) = \operatorname{sgn}(\lambda_j)$ for $j = 1, \dots, d$ such that*

$$\mathcal{I}_\infty(f) = \bigcup_{j=1}^{d} \left\{ \pm\frac{1}{\sqrt{1+\mu_j^2}}(-\mu_j, 1) \right\} .$$

If A_2 is not symmetric, then $\mathcal{I}_\infty(f) = \mathbb{S}^1$.

We remark that this statement is independent of the choice of B.

Proof. Due to Corollary 2.63, it is enough to consider $g(z) := f_h(0, z, 1) = \det(A_1z+A_2)$. Let A_1 be positive definite. Hence, A_1 has a unique Cholesky decomposition $A_1 = PP^T$. P is a lower triangular matrix with real and positive diagonal entries. Moreover, P is invertible and $(P^T)^{-1} = (P^{-1})^T$. It follows

$$g(z) = \det(A_1z + A_2) = \det(A_1)\det(I_dz + P^{-1}A_2(P^{-1})^T).$$

Hence, the roots of g are the negatives of the eigenvalues of $P^{-1}A_2(P^{-1})^T =: Q$. The matrix Q is symmetric if and only if A_2 is. Hence, the roots of g are real if and only if A_2 is symmetric.

First, we consider A_2 to be symmetric. Assume $A_1 = I_d$, i.e., $Q = A_2$. If $\lambda_1, \dots, \lambda_d$ are the eigenvalues of A_2, the limit directions of $\det(I_dz + A_2)$ are $\frac{1}{\sqrt{1+\lambda_j^2}}(-\lambda_j, 1)$ and $-\frac{1}{\sqrt{1+\lambda_j^2}}(-\lambda_j, 1)$, $1 \le j \le d$, which is due to Corollary 2.63.

Now, let $A_1 \neq I_d$. Although the eigenvalues of A_2 and Q might differ, the number of positive, negative and zero eigenvalues coincide by Sylvester's law of inertia. Hence, there are real numbers $\mu_1, \dots, \mu_d$ with $\operatorname{sgn}(\mu_j) = \operatorname{sgn}(\lambda_j)$ for $1 \le j \le d$, such that $\det(-I_d\mu_j + Q) = 0$ for all $1 \le j \le d$. This implies

$$\mathcal{I}_\infty(f) = \bigcup_{j=1}^{d} \left\{ \pm\frac{1}{\sqrt{1+\mu_j^2}}(-\mu_j, 1) \right\} .$$

If A_2 is not symmetric, there are complex eigenvalues of Q. Due to Corollary 2.63, the limit directions of f equal the whole sphere $\mathbb{S}^1$. $\square$

If the polynomial f in the theorem is homogeneous, then the imaginary projection consists of a finite collection of lines or equals the whole plane. Moreover, by applying Hurwitz' Theorem 1.31, we obtain a similar statement for positive semidefinite matrices, but which the exceptional case that the polynomial f is potentially identically zero. However, this implies that f is not stable, whenever A_1 and A_2 are not positive semidefinite. Furthermore, if A_1 and A_2 are positive definite, then f is stable. This changes if one assume f to be inhomogeneous, as the following example shows.

Example 2.66. *Let*

$$\begin{aligned} f(z_1, z_2) =& \det(A_1 z_1 + A_2 z_2 + B) \\ =& \det\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} z_1 + \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} z_2 + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix}\right) \\ =& 2z_1^2 + 8z_1^2 + z_2 + z_1^2 + 8z_1 z_2^2 + z_1 z_2 + 2z_2^3 - z_2^2 - z_2 - 3. \end{aligned}$$

Both matrices A_1 and A_2 are positive definite. The eigenvalues of A_2 are 2, $\frac{1}{2}(3+\sqrt{5})$ and $\frac{1}{2}(3-\sqrt{5})$. But f is not stable, since its imaginary projection intersects the positive orthant; see Figure 2.10 for an illustration.

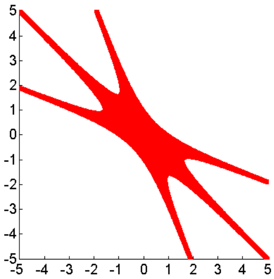

FIGURE 2.10: The imaginary projection of $f(z_1, z_2) = 2z_1^2 + 8z_1^2 + z_2 + z_1^2 + 8z_1 z_2^2 + z_1 z_2 + 2z_2^3 - z_2^2 - z_2 - 3$.

2.5.1 Connection between components in the complement and limit directions

In this section, we deal with the components in the complement for non-homogeneous polynomials as well as with homogenization. Namely, we consider for inhomogeneous polynomials the relationship between limit directions and components in the complement of imaginary projections. The hyperbolicity cones of the initial form, which is a homogeneous polynomial, will provide us helpful information. In particular, we show that for a non-homogeneous polynomial $f \in \mathbb{C}[\mathbf{z}]$ there is a bijection between the set of unbounded components of $\mathcal{I}(f)^{\mathsf{c}}$ with full-dimensional recession cone and the hyperbolicity cones of $\mathrm{in}(f)$; see Theorem 2.68. This leads to a certificate to decide whether all points on the sphere are limit directions.

For homogeneous polynomials, the set of limit direction is easier to understand than in the inhomogeneous situation. Namely, due to the conic structure of their imaginary projection, it holds for homogeneous polynomials $f \in \mathbb{R}[\mathbf{z}]$ that

$$\mathcal{I}_\infty(f) = \mathcal{I}(f) \cap \mathbb{S}^{n-1}. \tag{2.21}$$

By Theorem 2.37, the complement of the imaginary projection of a given homogeneous polynomial f coincides with the collection of its hyperbolicity cones. Hence, it suffices to understand the behavior of the hyperbolicity cones in order to understand the behavior of $\mathcal{I}_\infty(f)$. In particular, the number of complement components equals the number of hyperbolicity cones and it holds $\mathcal{I}_\infty(f) = \mathbb{S}^{n-1}$ if and only if f is not hyperbolic.

Denote by $f_h = f_h(z_0, \mathbf{z})$ the homogenization of f with respect to the variable z_0. The following statement captures the connection between the imaginary projection of f and the imaginary projection of its homogenization.

Theorem 2.67. *If $f \in \mathbb{C}[\mathbf{z}]$, then $\mathcal{I}(f_h) \cap \{(y_0, \mathbf{y}) \in \mathbb{R}^{n+1} : y_0 = 0\} = \{0\} \times \mathrm{cone}\,\overline{\mathcal{I}(f)}$.*

Proof. If $\mathbf{y}$ is a non-zero point in $\mathrm{cone}\,\mathcal{I}(f)$, we have $\lambda\mathbf{y} \in \mathcal{I}(f)$ for some $\lambda \geq 0$. Hence, there exists an $\mathbf{x} \in \mathbb{R}^n$ with $f_h((1, \mathbf{x} + i\lambda\mathbf{y})) = 0$. By homogeneity of f_h, this also gives $(0, \mathbf{y}) \in \mathcal{I}(f_h) \cap \{(y_0, \mathbf{y}) \in \mathbb{R}^{n+1} : y_0 = 0\}$. Taking the closure on both sides preserves the inclusion. Note that the left side is a closed set, since f_h is homogeneous.

Conversely, if $(0, \mathbf{y})$ is a non-zero point in $\mathcal{I}(f_h) \cap \{(y_0, \mathbf{y}) \in \mathbb{R}^{n+1} : y_0 = 0\}$, then there exists some $\mathbf{x} \in \mathbb{R}^n$ and some $c \in \mathbb{R}$ such that $f_h((c, \mathbf{x} + i\mathbf{y})) = 0$. For $c \neq 0$, $\frac{1}{c}(\mathbf{x} + i\mathbf{y})$ is a zero of f, and therefore $\mathbf{y} \in \mathrm{cone}\,\mathcal{I}(f)$. Assume $c = 0$. This implies that $\mathrm{in}(f)(\mathbf{x} + i\mathbf{y}) = 0$, which is $\mathbf{y} \in \mathcal{I}_\infty(\mathrm{in}(f))$. Since $\mathrm{in}(f)$ is homogeneous, we can assume $\|\mathbf{y}\|_2 = 1$. Hence, $\mathbf{y} \in \mathcal{I}_\infty(\mathrm{in}(f))$. By the subsequent Lemma 2.69, 1., which is proven independently from this theorem, it holds $\mathcal{I}(\mathrm{in}(f)) = \mathcal{I}_\infty(f)$. This implies that there is a sequence $(\mathbf{y}_r)_{r\in\mathbb{N}} \subset \mathcal{I}(f)$ with $\|\mathbf{y}_r\|_2 = r$ such that $\frac{1}{r}\mathbf{y}_r \to \mathbf{y}$ for $r \to \infty$. Since $\frac{1}{r}\mathbf{y}_r \in \mathrm{cone}\,\mathcal{I}(f)$, it holds $\mathbf{y} \in \overline{\mathrm{cone}\,\mathcal{I}(f)} = \mathrm{cone}\,\overline{\mathcal{I}(f)}$. $\square$

By Theorem 2.67, bounded components in the complement vanish under homogenization, and only conic components with apex at the origin remain. Concerning dehomogenization, note that the intersection of the imaginary projection of a homogeneous polynomial $f \in \mathbb{C}[z_0, \mathbf{z}] = \mathbb{C}[z_0, \ldots, z_n]$ with a fixed hyperplane $\{(y_0, \mathbf{y}) \in \mathbb{R}^{n+1} : y_0 = \beta\}$,

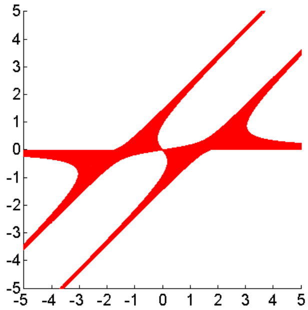

FIGURE 2.11: $f(z_1, z_2) = z_1^3 - 2z_1^2 z_2 + z_1 z_2^2 + z_1 + z_2 + 1$. We have $\mathrm{in}(f) = z_1(z_1 - z_2)^2$, and any boundary point of the complement of $\mathcal{I}(\mathrm{in}(f))$ satisfies $z_1 = 0$ or $z_1 = z_2$. Altogether, $\mathcal{I}(f)^c$ has six components.

$\beta \neq 0$ is

$$\mathcal{I}(f_h) \cap \{(y_0, \mathbf{y}) \in \mathbb{R}^{n+1} \, : \, y_0 = \beta\} = \bigcup_{\alpha \in \mathbb{R}} \mathcal{I}(f_h(\alpha + i\beta, \mathbf{z})).$$

Recall that the recession cone of a convex set $A \subseteq \mathbb{R}^n$ is $\mathrm{rec}(A) = \{\mathbf{a} \in A \, : \, \mathbf{a} + \mathbf{x} \in A \text{ for all } \mathbf{x} \text{ in } A\}$; see Definition 2.11. Whenever A is closed then $\mathrm{rec}(A)$ is closed. For a polynomial f, we can characterize the components of $\mathcal{I}(f)^c$ with full-dimensional recession cones in terms of the hyperbolicity cones of $\mathrm{in}(f)$.

Theorem 2.68. *For $f \in \mathbb{C}[\mathbf{z}]$, there is a bijection between the set of unbounded components of $\mathcal{I}(f)^c$ with full-dimensional recession cone and the hyperbolicity cones of* $\mathrm{in}(f)$.

Hence, there are at least as many unbounded components in $\mathcal{I}(f)^c$ as components in $\mathcal{I}(\mathrm{in}(f))^c$. This implies that the total number of components with full-dimensional recession cone satisfies the bound for the number of hyperbolicity cones, which is stated in Theorem 2.43. Moreover, if $\mathrm{in}(f)$ is hyperbolic, $\mathcal{I}(f)^c$ has at least two (full-dimensional) components. Note that for a polynomial f, the terms of lower degree can cause some unbounded components in the complement that have lower-dimensional recession cones. See Figure 2.11 for an example.

Moreover, hyperbolicity of $\mathrm{in}(f)$ provides a certificate whether all points on the sphere are limit directions. If the initial form is hyperbolic, their hyperbolicity cones cause components of $\mathcal{I}(f)^c$ with full-dimensional recession cone. It follows from the proof of statement 2 in the following Lemma 2.69, that only the components with full-dimensional recession cone cause components in $\mathcal{I}_\infty(f)^c$. Hence, $\mathcal{I}_\infty(f) \neq \mathbb{S}^{n-1}$ if and only if $\mathrm{in}(f)$ is hyperbolic.

In order to prove Theorem 2.68, we show the following lemma, where int denotes the interior of a set.

Lemma 2.69. *For $f \in \mathbb{C}[\mathbf{z}]$, the following statements hold.*

1. *The sets of limit directions* $\mathcal{I}_\infty(f)$ *and* $\mathcal{I}_\infty(\mathrm{in}(f))$ *coincide.*
2. *If* C *is an unbounded component of* $\overline{\mathcal{I}(f)}^{\mathsf{c}}$ *with recession cone* C'*, then* $\operatorname{int} C'$ *is a component of* $\mathcal{I}(\mathrm{in}(f))^{\mathsf{c}}$ *if and only if* $\dim C' = n$*.*
3. *If* C *is a component of* $\mathcal{I}(\mathrm{in}(f))^{\mathsf{c}}$*, then there is a* $\mathbf{y}_0 \in \mathbb{R}^n$ *such that* $\mathbf{y}_0 + C$ *lies in a component of* $\mathcal{I}(f)^{\mathsf{c}}$ *and* C *equals the interior of the recession cone of that complement component.*

Proof. 1. The homogenization f_h has a zero at infinity, i.e., $(0, z_1, \ldots, z_n) \subseteq \mathcal{V}(f_h)$, if and only if $\mathrm{in}(f) = f_h(0, z_1, \ldots, z_n) = 0$. Hence, the limit directions of f and $\mathrm{in}(f)$ coincide.

2. Since hyperbolicity cones are open, $\mathcal{I}(\mathrm{in}(f))$ is closed and thus $\mathcal{I}_\infty(f) = \mathcal{I}_\infty(\mathrm{in}(f))$ is closed as well.

Let C be an unbounded component of $\overline{\mathcal{I}(f)}^{\mathsf{c}}$ with recession cone C'. If $\dim C' < n$, then $\operatorname{int} C' = \emptyset$, hence C' is not a hyperbolicity cone of the homogeneous polynomial $\mathrm{in}(f)$. Conversely, if $\dim C = n$, then let $\mathbf{y}_0 \in \mathbb{R}^n$ with $\mathbf{y}_0 + C' \subseteq C$. For all $r > 0$ we have

$$\frac{1}{r}\big((\mathbf{y}_0 + C') \cap \mathcal{I}(f)\big) \cap \mathbb{S}^{n-1} = \emptyset.$$

Under taking the limit $r \to \infty$, we obtain that no interior point of the set of limit points

$$\lim_{r\to\infty} \frac{1}{r}(\mathbf{y}_0 + C') \cap \mathbb{S}^{n-1} \tag{2.22}$$

is a limit direction of $\mathcal{I}(f)$. By (1), these interior points are not limit directions of $\mathcal{I}(\mathrm{in}(f))$ either. As a consequence, $\operatorname{int} C'$ is a component of $\mathcal{I}(\mathrm{in}(f))^{\mathsf{c}}$.

3. Let C' be a component of $\mathcal{I}(\mathrm{in}(f))^{\mathsf{c}}$. Set $U = C' \cap \mathbb{S}^{n-1}$ and note that $\operatorname{cone} U$, i.e., the cone spanned by U, satisfies $\operatorname{cone} U = C'$. Since $\mathcal{I}(\mathrm{in}(f))$ is a cone, we have $U \subseteq \mathcal{I}_\infty(\mathrm{in}(f)) = \mathcal{I}_\infty(f)$. Hence, there is a $\mathbf{y}_0 \in \mathcal{I}(f)^{\mathsf{c}}$ such that $\mathbf{y}_0 + \operatorname{cone} U$ is contained in a component $\mathcal{I}(f)^{\mathsf{c}}$.
Denote by C'' the recession cone of the component of $\mathcal{I}(f)^{\mathsf{c}}$ that contains $\mathbf{y}_0 + C'$. Clearly, $C' \subseteq C''$. Using (2), it follows that $\operatorname{int} C'' = C'$. □

Theorem 2.68 is a consequence of Lemma 2.69.

Proof of Theorem 2.68. If the recession cone C' of C is full-dimensional, then, by Lemma 2.69 (2), $\operatorname{int} C'$ is a component of $\mathcal{I}(\mathrm{in}(f))^{\mathsf{c}}$, i.e., $\operatorname{int} C'$ is a hyperbolicity cone of $\mathrm{in}(f)$.

Conversely, if the recession cone C' of C is a hyperbolicity cone of $\mathrm{in}(f)$, then, by Lemma 2.69 (3), it is open and thus full-dimensional. □

2.6 Further questions

In this chapter, we have introduced imaginary projections as the projection of the variety of a polynomial onto its imaginary part and we have developed results for various aspects of imaginary projections. Motivated by the convexity of components of the complement, we spent much effort in understanding the complement. Doing so, we provided quantitative and convex-geometric results for the components of the complement of imaginary projections and for the hyperbolicity cones of hyperbolic polynomials. But whereas we understand the situation for homogeneous polynomials, we have to leave open problems for the inhomogeneous situation. In particular, we have no uniqueness statement (up to sign) like Proposition 2.44 for inhomogeneous polynomials. For example, the polynomial $f(z_1, z_2) = z_1^2 - z_2^2 - 1$ is irreducible, but there are four components in $\mathcal{I}(f)^{\mathsf{c}}$; see Figure 2.5. In Section 2.1.2, we derived that for an n-variate polynomial f of degree d the maximal number of components in $\mathcal{I}(f)^{\mathsf{c}}$ is at least $\sum_{k=0}^{n} \binom{d}{k}$. Although we received a result for the components with full-dimensional recession cone in Section 2.5.1, we have to leave the question for the maximal total number of components open. We claim, that the maximal total number for an n-variate polynomial of degree d is indeed $\sum_{k=0}^{n} \binom{d}{k}$ and that this number is attained for products of independent affine-linear polynomials. Moreover, for a given n-variate polynomial of degree d or with Newton-polytope Δ, we do not know the exact number of components in the complement.

For amoebas, there is an order map known, which distinguishes the different components of the complement; see [33]. For homogeneous polynomials, we can distinguish components of the complement of the imaginary projection by equivalence classes of hyperbolicity directions. And for unbounded components in the case of non-homogeneous polynomials, Theorem 2.68 establishes a connection via the initial form. It is an open question, whether a variant or generalization of this also holds for components in the complement whose recession cone is not full-dimensional.

Recently, Shamovich and Vinnikov [116] studied a generalization of hyperbolic polynomials in terms of hyperbolic varieties; see also [72]. They call a k-dimensional subvariety $X \subset \mathbb{P}^n_{\mathbb{R}}$ hyperbolic with respect to a linear subspace $E \subset \mathbb{P}^n_{\mathbb{R}}$ of dimension $n-k-1$ if $E \cap X = \emptyset$ and for every linear subspace $L \subset \mathbb{P}^n_{\mathbb{R}}$ with $E \subset L$ and of dimension $n-k$, the intersection $L \cap X$ contains only real points. It could be interesting to extend our results to that setting.

Chapter 3

A conic generalization of stability

We have seen in Chapter 2, that imaginary projections offer a geometric viewpoint to stability. Namely, a polynomial f is stable if and only if its imaginary projection do not intersect the positive orthant, $\mathcal{I}(f) \cap (\mathbb{R}_{>0})^n = \emptyset$. We can extend this concept in a natural way by replacing the positive orthant with a more general cone. Given a proper cone K, we say a polynomial f is K-stable if $\mathcal{I}(f) \cap \operatorname{int} K = \emptyset$. Here, $\operatorname{int} K$ denotes the interior of K. For $K = (\mathbb{R}_{\geq 0})^n$, this notion coincides with the usual stability in sense of Definition 1.20. More general, setting $\Omega = \mathbb{R}^n + iK$, K-stability falls into the class of stability notions which forbid zeros in an arbitrarily given complex set $\Omega \subset \mathbb{C}^n$. However, as pointed out in [126, p. 81], little can be said on a class of that generality.

In the introductory Section 1.3, we summarized statements known for univariate and multivariate stable polynomials. Some of them can be transferred to conic stability. For example, we generalize the Hermite-Kakeya-Obreschkoff Theorem 1.23, which considers stable linear combinations of real polynomials. In Theorem 3.13 we prove a conic generalization of the Hermite-Biehler Theorem 1.22 for polyhedral and non-polyhedral cones in terms of a directional Wronskian. Moreover, we view the determinantal polynomial

$$f(\mathbf{z}) = \det(z_1 A_1 + \cdots + z_n A_n + B)$$

from Proposition 1.26 as a polynomial on complex symmetric matrix variables

$$Z = (z_{ij})_{n \times n} \mapsto f(Z) = \det(z_{11} A_{11} + z_{12} A_{12} + \cdots + z_{nn} A_{nn} + B)$$

and we give in Theorem 3.16 a necessary condition for stability with respect to the cone of positive semidefinite matrices. This generalizes Proposition 1.26.

While the study of conic stability was mainly motivated by the intrinsic relevance and the structure of stable polynomials, we note that the case $K = \mathcal{S}_n^+$ is naturally related to the Siegel upper half-spaces in the theory of modular forms. The Siegel upper half-space (or Siegel upper half-plane) $\mathcal{H}_g$ of degree g (or genus g) is defined as

$$\mathcal{H}_g \ = \{A \in \mathbb{C}^{g\times g} \text{ symmetric } : \operatorname{Im}(A) \text{ is positive definite}\},$$

where $\operatorname{Im}(A) = (\operatorname{Im}(a_{ij}))_{g\times g}$; see [117] and also, e.g., [122, Chapter 2]. The Siegel upper half-space constitutes the domain on which the Siegel theta functions are defined. It can be used to parameterize polarized varieties, see also, e.g., [56, Vol. 1, Chapter 3.I], and for the use in elliptic curve cryptography; see [38, Chapter 5.1]).
We remark, that the Siegel upper half-plane requires complex symmetric matrices, which is $A^T = A$ for a complex matrix A. This allows to apply the usual notion of positive (semi-)definiteness to the real symmetric matrix $\operatorname{Im}(A)$. Note that a complex symmetric matrix is not Hermitian in general.

Throughout this chapter, we use results and concepts known for univariate stable polynomials as Hermite-Biehler Theorem 1.22, Hermite-Kakeya-Obreschkoff Theorem 1.23 and the concept of interlacing polynomials; see Section 1.3.1. Moreover, we make use of Hurwitz' Theorem 1.31.

The content of this chapter is a joint work with Thorsten Theobald. Large parts are contained in [59].

3.1 Conic stability

In this section, we introduce the concept of conic stability and we define stability with respect to a proper cone $K \subset \mathbb{R}^n$, that is a full-dimensional, closed and pointed convex cone. This assumption is not a hard restriction. In particular, the assumption to the convexity is quite natural and follows from the convexity of components of the complement of imaginary projections. We will discuss this in Proposition 3.9. Moreover, in Theorem 3.5, we present a connection between conic stability and hyperbolicity. And we introduce the notion of psd-stability, which is stability with respect to the cone of positive semidefinite matrices.

Definition 3.1. *Let K be a proper cone in $\mathbb{R}^n$. A polynomial $f \in \mathbb{C}[\mathbf{z}]$ is called K-stable if $f(\mathbf{z}) \neq 0$ whenever $\operatorname{Im}(\mathbf{z}) \in \operatorname{int} K$.*
If $f \in \mathbb{C}[Z]$ on the symmetric matrix variables $Z = (z_{ij})_{n\times n}$ is $\mathcal{S}_n^+$-stable, then f is called positive semidefinite-stable (or psd-stable, for short).

It follows immediately that a polynomial f is K-stable if and only if $\mathcal{I}(f) \cap \operatorname{int} K = \emptyset$. Equivalently, a polynomial $f \in \mathbb{C}[Z]$ on the symmetric matrix variables $Z = (z_{ij})_{n\times n}$ is psd-stable if there does not exist a matrix Z in the Siegel upper half-space $\mathcal{H}_n$ with $f(Z) = 0$. Note that psd-stability generalizes the usual stability in the sense that a polynomial $f(z_1, \dots, z_n)$ is stable if and only if $f(\operatorname{diag}(z_1, \dots, z_n))$ is psd-stable.

Example 3.2. *(i) Let $f \in \mathbb{R}[\mathbf{z}]$ be given by $f(\mathbf{z}) = \mathbf{a}^T\mathbf{z} + b$, where $\mathbf{a}$ is a real n-dimensional vector and $b \in \mathbb{R}$. Then f is K-stable if and only if $\mathbf{a} \in \operatorname{int} K^*$ or $-\mathbf{a} \in \operatorname{int} K^*$, where $K^* = \{\mathbf{y} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{y}\rangle \geq 0$ for all $\mathbf{x} \in K\}$ denotes the dual cone of K and $\langle \cdot, \cdot \rangle$ is the Euclidean dot product on $\mathbb{R}^n$.*

Namely, if $\mathbf{a} \in \operatorname{int} K^*$ or $-\mathbf{a} \in \operatorname{int} K^*$, say, $\mathbf{a} \in \operatorname{int} K^*$, then for any $\mathbf{z} = \mathbf{x} + i\mathbf{y} \in \mathbb{C}^n$ with $\mathbf{y} \in \operatorname{int} K$ we have

$$f(\mathbf{z}) \;=\; \langle \mathbf{a}, \mathbf{x}\rangle + i\langle \mathbf{a}, \mathbf{y}\rangle + b,$$

because $\mathbf{a}$ is real. Hence $\operatorname{Im} f(\mathbf{z}) \neq 0$, and thus f is K-stable.
Conversely, let f be K-stable. Assuming $\mathbf{a} \notin \pm \operatorname{int} K^*$, there exists $\mathbf{y}' \in \operatorname{int} K$ with $\langle \mathbf{a}, \mathbf{y}'\rangle \leq 0$ and $\mathbf{y}'' \in \operatorname{int} K$ with $\langle \mathbf{a}, \mathbf{y}''\rangle \geq 0$. Hence, there exists some $\mathbf{y} \in \operatorname{int} K$ with $\langle \mathbf{a}, \mathbf{y}\rangle = 0$. Choosing $\mathbf{x} \in \mathbb{R}^n$ with $\langle \mathbf{a}, \mathbf{x}\rangle + b = 0$ gives a contradiction to the stability of f.

For usual stability, this implies the well-known statement that $f = \mathbf{a}^T\mathbf{z} + b$ is stable if and only if $\mathbf{a} \in (\mathbb{R}_{>0})^n$ or $-\mathbf{a} \in (\mathbb{R}_{>0})^n$. For psd-stability, this implies that $f(Z) = \langle Z, A\rangle + b$ with $A \in \mathcal{S}_n$ is psd-stable if and only if $A \succ 0$ or $-A \succ 0$; here, the scalar product is $\langle Z, A\rangle = \operatorname{tr}(A^H Z) = \operatorname{tr}(AZ)$ and A^H is the Hermitian transpose of A.

(ii) As an example for psd-stability, the polynomial $f(Z) = \det Z$ on the set of (complex) symmetric $n \times n$-matrices is psd-stable. We postpone the proof to Example 3.7 below.

Example 3.3. *For the polynomial*

$$\begin{aligned} f(z_1, z_2, z_3) \;&=\; \det\left(\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} z_1 + \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} z_2 + \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} z_3\right) \\ &=\; (z_1 + z_3)^2 - z_2^2 \\ &=\; (z_1 + z_3 - z_2)(z_1 + z_3 + z_2)\,, \end{aligned}$$

Theorem 2.23 implies that the imaginary projection is

$$\mathcal{I}(f) \;=\; \{\mathbf{y} \in \mathbb{R}^3 \,:\, y_1 - y_2 + y_3 = 0\} \cup \{\mathbf{y} \in \mathbb{R}^3 \,:\, y_1 + y_2 + y_3 = 0\}\,.$$

Since, for example, $(\frac{1}{2}, 1, \frac{1}{2}) \in \mathcal{I}(f) \cap \mathbb{R}^3_{>0}$, f is not stable. In contrast to this, setting $Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}$, the polynomial $f(Z) = f(z_1, z_2, z_3)$ is psd-stable. Namely, for $\mathbf{y} \in \mathcal{I}(f)$, *we have*

$$\det \begin{pmatrix} y_1 & \pm(y_1+y_3) \\ \pm(y_1+y_3) & y_3 \end{pmatrix} = y_1 y_3 - (y_1+y_3)^2 \leq 0$$

as a consequence of the arithmetic-geometric mean inequality, hence $\mathbf{y} \notin \operatorname{int} \mathcal{S}_2^+$.

The following lemma allows to reduce multivariate K-stability to univariate stable polynomials. It makes essentially use of the conic structure of the "forbidden area" $\operatorname{int} K$.

Lemma 3.4. *A non-zero polynomial $f \in \mathbb{C}[\mathbf{z}]$ is K-stable if and only if for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with $\mathbf{y} \in \operatorname{int} K$ the univariate polynomial $t \mapsto f(\mathbf{x} + t\mathbf{y})$ is stable.*

Proof. If f is not K-stable, then there exists $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \operatorname{int} K$ with $f(\mathbf{x} + i\mathbf{y}) = 0$. Hence, i is a zero of the univariate polynomial $t \mapsto f(\mathbf{x} + t\mathbf{y})$ and thus that univariate polynomial is not stable.

Conversely, if $t \mapsto f(\mathbf{x} + t\mathbf{y})$ is not stable for $\mathbf{y} \in \operatorname{int} K$, then there is some $\alpha + i\beta \in \mathbb{C}$ with $\beta > 0$ and $0 = f(\mathbf{x} + (\alpha + i\beta)\mathbf{y}) = f(\mathbf{x} + \alpha\mathbf{y} + i\beta\mathbf{y})$. Since $\beta\mathbf{y} \in \operatorname{int} K$, f is not K-stable. $\square$

As reviewed in Section 2.3, for a homogeneous polynomial f, every component in the complement of its imaginary projection $\mathcal{I}(f)$ is a hyperbolicity cone. In particular, f is stable if and only if f is hyperbolic with respect to every point in the positive orthant; see Proposition 1.28. This generalizes as follows.

Theorem 3.5. *Let $f \in \mathbb{C}[\mathbf{z}]$ be homogeneous. Then the following are equivalent:*

1. *f is K-stable.*
2. *$\mathcal{I}(f) \cap \operatorname{int} K = \emptyset$.*
3. *f is hyperbolic with respect to every point in* $\operatorname{int} K$.

The theorem implies, that a homogeneous polynomial f is K-stable if and only if $\operatorname{int} K$ is contained in a hyperbolicity cone of f.

Proof. The equivalence "*1.* $\Leftrightarrow$ *2.*" is clear.

"*3.* $\Rightarrow$ *1.*" If f is not K-stable, then there exists $\mathbf{x} \in \mathbb{R}$ and $\mathbf{e} \in \operatorname{int} K$ with $f(\mathbf{x} + i\mathbf{e}) = 0$. Hence, i is a root of $t \mapsto f(\mathbf{x} + t\mathbf{e})$, so that f is not hyperbolic with respect to $\mathbf{e}$.

"*1.* $\Rightarrow$ *3.*" Assume $t \mapsto f(\mathbf{x} + t\mathbf{e})$ is not hyperbolic for $\mathbf{e} \in \operatorname{int} K$. In case $f(\mathbf{e}) = 0$, the point $i\mathbf{e}$ is a root of the homogeneous polynomial f as well, so that f is not hyperbolic then. Hence, $f(\mathbf{e}) \neq 0$ and there is $\mathbf{x} \in \mathbb{R}$ and $\alpha + i\beta \in \mathbb{C}$ with $\beta \neq 0$ and $f(\mathbf{x} + (\alpha + i\beta)\mathbf{e}) = 0$. We can assume that $\beta > 0$, since $-\mathbf{x} - (\alpha + i\beta)\mathbf{e}$ is a zero of f, too. Hence, $0 = f(\mathbf{x} + \alpha\mathbf{e} + i\beta\mathbf{e})$ and $\beta\mathbf{e} \in \operatorname{int} K$, so that f is not K-stable. $\square$

Moreover, we observe the following statement, which is for homogeneous polynomials in analogy to Remark 1.6.

Proposition 3.6. *Let $f_1, f_2 \in \mathbb{C}[\mathbf{z}]$ be two polynomials. If f_j is K_j-stable for $j = 1, 2$, then $f_1 \cdot f_2$ is $(K_1 \cap K_2)$-stable unless $\operatorname{int}(K_1 \cap K_2) = \emptyset$.*

Proof. By assumption to f_1 and f_2 it holds for $j = 1, 2$ that $\operatorname{int} K_j \cap \mathcal{I}(f_j) = \emptyset$. Hence, by Lemma 2.7 it follows $\operatorname{int}(K_1 \cap K_2) \cap \mathcal{I}(f_1 \cdot f_2) = \operatorname{int} K_1 \cap \operatorname{int} K_2 \cap \big(\mathcal{I}(f_1) \cup \mathcal{I}(f_2)\big) = \emptyset$. Unless $\operatorname{int}(K_1 \cap K_2) = \emptyset$, the intersection $K_1 \cap K_2$ is a proper cone. This implies $(K_1 \cap K_2)$-stability for $f_1 \cdot f_2$ whenever $\operatorname{int}(K_1 \cap K_2) \neq \emptyset$. □

Example 3.7. *We complete Example 3.2 and show that $f(A) = \det A$ on the space of (complex) symmetric matrices is psd-stable.*

Let $B \in \mathcal{S}_n$ be positive definite and consider the univariate polynomial $t \mapsto f(A+tB)$. Its roots are the eigenvalues of the symmetric matrix $-B^{-1/2}AB^{-1/2}$, where $B^{-1/2}$ denotes the unique square root of B^{-1}. Hence, it is real-rooted. Thus, f is hyperbolic with respect to any positive definite matrix. By Theorem 3.5, f is psd-stable.

The following proposition generalizes the specialization property of stable polynomials; see Proposition 1.27. We will use it for the special case $K_1 = \mathbb{R}_{\geq 0}$.

Proposition 3.8. *Let $K = K_1 \times K_2 \subset \mathbb{R}^n \times \mathbb{R}^m$ be a cone. If $f(\mathbf{z}_1, \mathbf{z}_2)$ is K-stable, then $f(\mathbf{a} + i\mathbf{b}, \mathbf{z}_2)$ is K_2-stable for any $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \operatorname{int} K_1$.*

Proof. Let $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \operatorname{int} K_1$ and assume $f(\mathbf{a} + i\mathbf{b}, \mathbf{z}_2)$ is not K_2-stable. Then, there is a complex number $\mathbf{a}' + i\mathbf{b}'$ with $\mathbf{b}' \in \operatorname{int} K_2$ such that $f(\mathbf{a} + i\mathbf{b}, \mathbf{a}' + i\mathbf{b}') = 0$. This implies that f is not K-stable. □

We now discuss the assumptions to the cone K. Proper cones are common objects in optimization, since they provide useful structure. For example, the dual of a proper cone is proper itself and proper cones induce a partial ordering on the ambient space, namely $\mathbf{a} \geq_K \mathbf{b} :\Leftrightarrow \mathbf{a} - \mathbf{b} \in K$. The positive orthant, the cone of semidefinite matrices and the second-order cone are important examples for proper cones.
In our situation, the assumption to be proper are reasonable and natural. The assumption that K is full-dimensional is caused by Hurwitz' Theorem 1.31 which needs open subsets of $\mathbb{C}^n$. We apply Hurwitz' Theorem in several proofs. The assumption to be pointed is not essential and all of our statements can easily be transferred to non-pointed cones. The most crucial assumptions are that K is required to be closed and convex. Surprisingly, the convexity is not an assumption. As the following Proposition 3.9 will

show, it is a consequence of the connection between K-stability and imaginary projections. Moreover, the proposition covers the case of a non-closed cone. We remark, that in the general setting of the proposition, it is not necessarily $\overline{\operatorname{int} K} = \overline{K}$, since K may have some lower-dimensional pieces. As always, $\overline{M}$ denotes the topological closure of a set $M \subset \mathbb{R}^n$.

Proposition 3.9. *Let K be a non-convex cone K with non-empty, connected interior, If $f \in \mathbb{C}[\mathbf{z}]$ has no root $\mathbf{z} \in \mathbb{C}^n$ such that $\operatorname{Im}(\mathbf{z}) \in \operatorname{int} K$, then f is $\overline{\operatorname{conv}(\operatorname{int}(K))}$-stable.*

Proof. By assumption to f, it holds $\mathcal{I}(f) \cap \operatorname{int} K = \emptyset$, that is, $\operatorname{int} K \subseteq \mathcal{I}(f)^{\mathsf{c}}$. Since $\operatorname{int} K$ is connected, it is contained in one of the connected components of $\overline{\mathcal{I}(f)}^{\mathsf{c}}$. We denote this component by C. The convexity of any component in $\overline{\mathcal{I}(f)}^{\mathsf{c}}$ implies that for $K' := \overline{\operatorname{conv}(\operatorname{int}(K))}$, we have $\operatorname{int} K' \subseteq \operatorname{conv}(\operatorname{int}(K)) \subseteq C$. Since, $C \subseteq \mathcal{I}(f)^{\mathsf{c}}$, f is K'-stable. □

3.1.1 A conic generalization of the Hermite-Biehler and the HKO Theorem

In the previous section, we introduced conic stability and proved basic properties. Now, we are going to consider conic generalizations of the Hermite-Biehler Theorem 1.22 and the Hermite-Kakeya-Obreschkoff Theorem 1.23 (HKO, for short). These results were generalized to the multivariate situation by Borcea and Brändén; see Section 1.3. We show that the Theorem of Hermite-Kakeya-Obreschkoff as given in Proposition 1.23 can be generalized also to conic stability. Moreover, we show a conic version of Hermite-Biehler Theorem 1.22 for polyhedral and non-polyhedral cones. The conic version of Hermite-Biehler makes use of a directional Wronskian, which generalizes the univariate Wronskian in a natural way.
We use statements for univariate stable polynomials from Section 1.3.1 as well as interlacing polynomials.

First we prove the following auxiliary result.

Theorem 3.10. *Let f and g be real polynomials in $\mathbf{z} = (z_1, \ldots, z_n)$. Then $g + if$ is K-stable if and only if $g + wf \in \mathbb{R}[\mathbf{z}, w]$ is K'-stable, where $K' = K \times \mathbb{R}_{\geq 0}$.*

Proof. "$\Leftarrow$" This follows from Proposition 3.8, setting $w = i$.

"$\Rightarrow$" Let $g + if$ be K-stable. By Lemma 3.4, the univariate polynomial

$$t \mapsto g(\mathbf{x} + t\mathbf{y}) + if(\mathbf{x} + t\mathbf{y})$$

is stable for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with $\mathbf{y} \in \operatorname{int} K$. For fixed $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with $\mathbf{y} \in \operatorname{int} K$, we write $\tilde{f}(t) = f(\mathbf{x} + t\mathbf{y})$ and $\tilde{g}(t) = g(\mathbf{x} + t\mathbf{y})$ as polynomials in $\mathbb{R}[t]$. By the univariate Hermite-Biehler Theorem 1.22, $\tilde{f}$ interlaces $\tilde{g}$ properly, and in particular, $\tilde{f}$ and $\tilde{g}$ are real stable. Let $w = \alpha + i\beta$ with $\alpha \in \mathbb{R}$ and $\beta > 0$. By Lemma 3.4, we have to show that the univariate polynomial

$$t \ \mapsto \ \tilde{g} + \alpha\tilde{f} + i\beta\tilde{f} \ = \ \tilde{g} + (\alpha + i\beta)\tilde{f} \tag{3.1}$$

is stable.

Assume first, that $\tilde{f}$ and $\tilde{g}$ are strictly interlacing and denote the roots of $\tilde{f}$ by $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_d$. By the univariate Hermite-Kakeya-Obreschkoff Theorem 1.23, the linear combination $\tilde{g} + \alpha\tilde{f}$ is real stable. Moreover, the sequence of signs of $\big(\tilde{g}(\alpha_j) + \alpha\tilde{f}(\alpha_j)\big)_{j=1,\dots,d}$ has the same sign pattern as $\big(\tilde{g}(\alpha_j)\big)_{j=1,\dots,d}$. Hence, the polynomial $\beta\tilde{f}$ interlaces the polynomial $\tilde{g} + \alpha\tilde{f}$ strictly and properly.
If $\tilde{f}$ and $\tilde{g}$ do not interlace strictly, they can be approximated by strictly interlacing polynomials and it follows again, that $\beta\tilde{f}$ is a proper interlacing of $\tilde{g} + \alpha\tilde{f}$

Now, invoking again the univariate Hermite-Biehler Theorem 1.22 shows that the univariate polynomial (3.1) is stable. This completes the proof. □

Proposition 3.11. *For every K-stable polynomial $h = g + if$ with $g, f \in \mathbb{R}[\mathbf{z}]$ the polynomials f and g are K-stable or identically zero.*

Proof. By Theorem 3.10, a non-zero polynomial $g + if$ is stable if and only if $g + yf$ is K'-stable. Using Hurwitz's Theorem 1.31, sending $y \to 0$ and $y \to \infty$ respectively, gives that g and f are K-stable polynomials or identically zero. □

Now we show the following HKO generalization for K-stability.

Theorem 3.12 (Conic HKO Theorem)**.** *Let $f, g \in \mathbb{R}[\mathbf{z}]$. Then $\lambda f + \mu g$ is either K-stable or the zero polynomial for all $\lambda, \mu \in \mathbb{R}$ if and only if $f + ig$ or $g + if$ is K-stable or $f \equiv g \equiv 0$.*

Proof. "$\Leftarrow$" Let $g + if$ be K-stable and let $\lambda, \mu \in \mathbb{R}$ (the case $f + ig$ can be treated analogously). By Proposition 3.11, we can assume $\mu \neq 0$, and hence, by factoring μ, it suffices to consider $g + \lambda f$.
By Theorem 3.10, the polynomial $g + yf$ is $K \times \mathbb{R}_{\geq 0}$-stable. Using Proposition 3.8, we set $y = \lambda + i$, which gives the K-stable polynomial $(g + \lambda f) + if$. With Proposition 3.11, the K-stability of $g + \lambda f$ follows.

"$\Rightarrow$" Assume that $\lambda f + \mu g$ is either K-stable or the zero polynomial for all $\lambda, \mu \in \mathbb{R}$. Let $\mathbf{x} + i\mathbf{y} \in \mathbb{C}^n$ with $\mathbf{y} \in \operatorname{int} K$. We write $\tilde{f}(t) = f(\mathbf{x} + t\mathbf{y})$ and $\tilde{g}(t) = g(\mathbf{x} + t\mathbf{y})$. Due to Lemma 3.4, the univariate polynomial $\lambda\tilde{f} + \mu\tilde{g}$ is stable. The univariate HKO Theorem 1.23 implies that $\tilde{f}$ and $\tilde{g}$ interlace.

First, assume that $\tilde{f}$ interlaces $\tilde{g}$ properly for all $\mathbf{x} + i\mathbf{y} \in \mathbb{C}^n$ with $\mathbf{y} \in \operatorname{int} K$. By the Hermite-Biehler Theorem 1.22, $\tilde{g} + i\tilde{f}$ is stable for all $\mathbf{x} + i\mathbf{y} \in \mathbb{C}^n$ with $\mathbf{y} \in \operatorname{int} K$, which implies K-stability by Lemma 3.4. The case where $\tilde{g}$ interlaces $\tilde{f}$ properly for all $\mathbf{x} + i\mathbf{y} \in \mathbb{C}^n$ with $\mathbf{y} \in \operatorname{int} K$ is treated analogously.

It remains the case where $\tilde{f}$ interlaces $\tilde{g}$ properly for one $\mathbf{x}_1 + i\mathbf{y}_1 \in \mathbb{C}^n$ with $\mathbf{y}_1 \in \operatorname{int} K$ and $\tilde{g}$ interlaces $\tilde{f}$ properly for another $\mathbf{x}_2 + i\mathbf{y}_2 \in \mathbb{C}^n$ with $\mathbf{y}_2 \in \operatorname{int} K$. For $0 \leq \tau \leq 1$, we consider the homotopies

$$\mathbf{x}_\tau = \tau\mathbf{x}_1 + (1-\tau)\mathbf{x}_2, \quad \mathbf{y}_\tau = \tau\mathbf{y}_1 + (1-\tau)\mathbf{y}_2.$$

The roots of $\tilde{f}$ and $\tilde{g}$ vary continuously with τ. Since $\tilde{f}$ and $\tilde{g}$ interlace for all $x+iy \in \mathbb{C}^n$ with $y \in \operatorname{int} K$, there must be some $\tau \in [0,1]$ such that the roots of $f(\mathbf{x}_\tau + t\mathbf{y}_\tau)$ and the roots of $g(\mathbf{x}_\tau + t\mathbf{y}_\tau)$ coincide. Hence, there is a $c \in \mathbb{R}$ such that $cf(\mathbf{x}_\tau + t\mathbf{y}_\tau) \equiv g(\mathbf{x}_\tau + t\mathbf{y}_\tau)$. Let $h = cf - g$. Then $h(\mathbf{x}_\tau + t\mathbf{y}_\tau) \equiv 0$, which implies in particular $h(\mathbf{x}_\tau + i\mathbf{y}_\tau) = 0$. Due to the initial hypothesis, the polynomial $h = cf - g$ is either K-stable or the zero polynomial. Since the point $\mathbf{x}_\tau + i\mathbf{y}_\tau \in \mathbb{C}^n$ is a root of the polynomial h with $\mathbf{y}_\tau \in \operatorname{int} K$, h must be the zero polynomial. This implies $cf \equiv g$. Since by assumption, f and g are K-stable, and since K-stable polynomials remain K-stable under multiplication with a non-negative complex scalar, $f + ig$ and $g + if$ are K-stable as well or $f \equiv g \equiv 0$. □

For $f, g \in \mathbb{C}[\mathbf{z}]$, we denote by $W_\mathbf{v}(f,g) := \partial_\mathbf{v} f \cdot g - f \cdot \partial_\mathbf{v} g$ the $\mathbf{v}$-Wronskian of f and g, where $\partial_\mathbf{v}$ denotes the directional derivative with respect to $\mathbf{v}$. This allows to give the following conic generalization of Hermite-Biehler Theorem 1.22 for polyhedral and non-polyhedral cones in terms of the directional $\mathbf{v}$-Wronskian.

Theorem 3.13 (Conic Hermite-Biehler Theorem). *For $f, g \in \mathbb{R}[\mathbf{z}]$, the following are equivalent.*

1. *$g + if$ is K-stable or the zero polynomial.*
2. *$g + yf$ is $K \times \mathbb{R}_{\geq 0}$-stable or the zero polynomial.*
3. *f and g are K-stable or the zero polynomial and $W_\mathbf{v}(f,g) \leq 0$ on $\mathbb{R}^n$ for all $\mathbf{v} \in \operatorname{int} K$.*

If K is a polyhedral cone $K = \operatorname{cone}(\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(k)})$, the statements are also equivalent to

4. *f and g are K-stable or the zero polynomial and $W_{\mathbf{v}^{(j)}}(f,g) \leq 0$ on $\mathbb{R}^n$ for all $j = 1, \ldots, k$.*

Proof. "*1.* ⇔ *2.*" follows by Theorem 3.10.

"*1.* ⇒ *3.*" The first part follows by Proposition 3.11. For the second part, let $\mathbf{x}+i\mathbf{v} \in \mathbb{C}^n$ with $\mathbf{v} \in \operatorname{int} K$. By Lemma 3.4, the univariate restriction

$$t \mapsto g(\mathbf{x}+t\mathbf{v}) + if(\mathbf{x}+t\mathbf{v})$$

is stable. The univariate Hermite-Biehler Theorem 1.22 implies $f(\mathbf{x}+t\mathbf{v}) \ll g(\mathbf{x}+t\mathbf{v})$, i.e.,

$$0 \ \geq \ W(f(\mathbf{x}+t\mathbf{v}), g(\mathbf{x}+t\mathbf{v})) \ = \ g(\mathbf{x}+t\mathbf{v})^2 \frac{d}{dt}\left(\frac{f(\mathbf{x}+t\mathbf{v})}{g(\mathbf{x}+t\mathbf{v})}\right)$$

for all $t \in \mathbb{R}$. Now the claim follows from

$$W_{\mathbf{v}}(f,g)(\mathbf{x}) \ = \ W\big(f(\mathbf{x}+t\mathbf{v}), g(\mathbf{x}+t\mathbf{v})\big)|_{t=0} \ \leq \ 0\,.$$

"*3.* ⇒ *1.*" We can assume that neither f nor g is the zero polynomial. By Lemma 3.4 the univariate real polynomials $\tilde{f}(t) = f(\mathbf{x}+t\mathbf{v})$ and $\tilde{g}(t) = g(\mathbf{x}+t\mathbf{v})$ are stable for all for all $\mathbf{x}+i\mathbf{v} \in \mathbb{C}^n$ with $\mathbf{v} \in \operatorname{int} K$. Moreover, the elementary rule $\frac{d}{dt}h(\mathbf{x}+t\mathbf{v}) = \frac{\partial}{\partial \mathbf{v}}h(\mathbf{z})|_{\mathbf{z}=\mathbf{x}+t\mathbf{v}}$ gives

$$W(\tilde{f}(t), \tilde{g}(t)) \ = \ W_{\mathbf{v}}(f(\mathbf{z}), g(\mathbf{z}))|_{\mathbf{z}=\mathbf{x}+t\mathbf{v}} \ \leq \ 0$$

by assumption. Hence, the univariate restrictions

$$t \mapsto g(\mathbf{x}+t\mathbf{v}) + if(\mathbf{x}+t\mathbf{v})$$

are stable by the univariate Hermite-Biehler Theorem 1.22. Thus, by Lemma 3.4, $g+if$ is K-stable.

"*3.* ⇒ *4.*" Since $K = \operatorname{cone}(\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(k)})$, this implication follows immediately from a continuity argument.

"*4.* ⇒ *1.*" Let $\mathbf{x}+i\mathbf{y} \in \mathbb{C}^n$ with $\mathbf{y} \in \operatorname{int} K$. We can assume that $\mathbf{y} \in \operatorname{cone}(\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)})$ with linearly independent vectors $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)}$. Hence, there are $\lambda_1, \ldots, \lambda_n \geq 0$ such that $\mathbf{y} = \sum_{j=1}^n \lambda_j \mathbf{v}^{(j)}$. We can assume that neither f nor g is the zero polynomial. Then, by assumption, f and g are K-stable. By Lemma 3.4, the univariate restriction to $t \mapsto \mathbf{x}+t\mathbf{y}$ is stable. Its Wronskian fulfills

$$W\big(f(\mathbf{x}+\mathbf{y}t), g(\mathbf{x}+\mathbf{y}t)\big) = g(\mathbf{x}+t\mathbf{v})^2 \frac{d}{dt}\left(\frac{f(\mathbf{x}+\mathbf{y}t)}{g(\mathbf{x}+\mathbf{y}t)}\right)\,.$$

Expressing this via $\frac{d}{dt}h(\mathbf{x}+\mathbf{y}t) = \sum_{j=1}^{n} \lambda_j \frac{\partial}{\partial \mathbf{v}^{(j)}} h(\mathbf{z})\Big|_{\mathbf{z}=\mathbf{x}+\mathbf{y}t}$ in terms of directional derivatives, we obtain

$$\begin{aligned} W\big(f(\mathbf{x}+\mathbf{y}t), g(\mathbf{x}+\mathbf{y}t)\big) &= g(\mathbf{x}+t\mathbf{v})^2 \sum_{j=1}^{n} \lambda_j \frac{\partial}{\partial \mathbf{v}^{(j)}} \left(\frac{f(\mathbf{z})}{g(\mathbf{z})}\right)\Big|_{\mathbf{z}=\mathbf{x}+\mathbf{y}t} \\ &= \sum_{j=1}^{n} \lambda_j W_{\mathbf{v}^{(j)}}(f,g)(\mathbf{x}+\mathbf{y}t) \ \leq 0 \ . \end{aligned}$$

Hence, $f(\mathbf{x}+\mathbf{y}t) \ll g(\mathbf{x}+\mathbf{y}t)$, and thus by the Hermite-Biehler Theorem 1.22, $g(\mathbf{x}+\mathbf{y}t) + if(\mathbf{x}+\mathbf{y}t)$ is stable. By Lemma 3.4, $g+if$ is K-stable. $\square$

By using the Conic HKO-Theorem 3.12, the condition to the K-stability of f and g in statements 3. and 4. of Theorem 3.13 can be replaced by the assumption that every real linear combination $\lambda f + \mu g$ is either K-stable or identically zero. This yields an exact analog of [9, Theorem 2.9]; see also [126, Cor. 2.10].

3.1.2 psd-Stability

In this section we consider the cone $K = \mathcal{S}_n^+$ of positive semidefinite matrices. In many settings, this cone provides a natural generalization of the non-negative cone; see, e.g., [6]. As pointed out in Chapter 1, the polynomial

$$f(\mathbf{z}) = \det(z_1 A_1 + \cdots + z_n A_n + B) \tag{3.2}$$

is for positive semidefinite matrices $A_1, \ldots, A_n$ and a Hermitian matrix B stable or identically zero. Polynomials of this type are an interesting way to describe convex sets; see Section 1.2.1.1. In this section, we view the polynomial (3.2) in terms of a complex symmetric matrix variable

$$Z = (z_{ij})_{n\times n} \mapsto f(Z) = \det(z_{11}A_{11} + z_{12}A_{12} + \cdots + z_{nn}A_{nn} + B). \tag{3.3}$$

In Theorem 3.16, we prove a criterion for (3.3) to be psd-stable. Since a polynomial $f(z_1, \ldots, z_n)$ is stable if and only if $f(\operatorname{diag}(z_1, \ldots, z_n))$ is psd-stable, Theorem 3.16 will be an exact generalization of Proposition 1.26. Accordingly, we provide a conic version of the Generalized Lax Conjecture for homogeneous psd-stable polynomials in Section 3.2. We make use of the Kronecker product and the Khatri-Rao product of matrices, which we recall first.

For two matrices, $A = (a_{ij})_{n_1\times n_2}$ and $B = (b_{ij})_{k_1\times k_2}$, the Kronecker product (or tensor product) $A \otimes B$ is the $n_1k_1 \times n_2k_2$ block matrix $C = (C_{ij})_{n_1\times n_2}$ with blocks $C_{ij} = a_{ij}B$.

A generalization of the Kronecker product is the Khatri-Rao product, which is defined as follows.

Definition 3.14. *Let $A = (A_{ij})_{n_1 \times n_2}$ and $B = (B_{ij})_{n_1 \times n_2}$ be block matrices with $n_1 \times n_2$ blocks of size $p_1 \times p_2$ and $q_1 \times q_2$, respectively. The Khatri-Rao product of A and B is defined as*

$$A * B \ = \ (A_{ij} \otimes B_{ij})_{n_1 \times n_2},$$

which is a block matrix with $n_1 \times n_2$ blocks of size $p_1 q_1 \times p_2 q_2$.

Note that in the case of $p_1 = p_2 = 1$, the Khatri-Rao product provides a scalar multiplication of the blocks B_{ij} by the scalars a_{ij}. And in the case $n_1 = n_2 = 1$, A and B only consist of a single block and $A * B$ gives the usual Kronecker product. Moreover, we remark that in the special situation $p_1 = p_2 = q_1 = q_2 = 1$ the Khatri-Rao product coincides with the Hadamard product, which is the elementwise multiplication of two matrices: $(a_{ij})_{n_1 \times n_2} \circ (b_{ij})_{n_1 \times n_2} = (a_{ij} \cdot b_{ij})_{n_1 \times n_2}$.

While it is classically known that the Kronecker product of positive semidefinite matrices is positive semidefinite, see [54], the following result on the Khatri-Rao product will be relevant in our situation; see [78].

Proposition 3.15. (Liu, [78, Theorems 5 and 6]) *Let $A = (A_{ij})_{n_1 \times n_2}$ and $B = (B_{ij})_{n_1 \times n_2}$ be block matrices with the same block structure $n_1 \times n_2$. If A and B are positive semidefinite, then $A * B$ is positive semidefinite. If A is positive semidefinite with positive definite blocks on the diagonal and B is positive definite, then $A * B$ is positive definite.*

Note that the positive semidefiniteness of A implies that its blocks satisfy $A_{ij} = A_{ji}^H$, where A_{ij}^H denotes the Hermitian transpose of A_{ij}. Now we show the following generalization of Proposition 1.26 to psd-stability.

Theorem 3.16. *Let $A = (A_{ij})_{n \times n}$ be a block matrix with $n \times n$ blocks of size $d \times d$. If A is positive semidefinite and B is a Hermitian $d \times d$-matrix, then the polynomial $f(Z) = \det(\sum_{i,j=1}^{n} A_{ij} z_{ij} + B)$ on the set of symmetric $n \times n$-matrices is psd-stable or identically zero.*

In particular, the theorem implies that the diagonal blocks $A_{11}, \ldots, A_{nn}$ need to be positive semidefinite. The case $A_{ij} = 0$ for $i \neq j$ captures the setting $f(\mathrm{diag}(z_1, \ldots, z_n))$. In that situation, usual stability coincides with psd-stability. So, it is natural that in this special setting Theorem 3.16 describes the situation of Proposition 1.26.

Proof. We write I_d for the $d \times d$ identity matrix and $\mathbb{1}_{m_1 \times m_2}$ for the all one matrix of size $m_1 \times m_2$. First consider the case where A is positive semidefinite with positive definite blocks on the diagonal.

Let $X, Y \in \mathcal{S}_n$ with $Y \succ 0$. In view of Lemma 3.4, we have to show that the univariate polynomial $t \mapsto f(X + tY)$ has only real roots.

We can interpret Y as block matrix with blocks of size 1×1. Using the Khatri-Rao product, $Y * A$ is a block matrix whose (i, j)-th block is $y_{ij} A_{ij}$, and we obtain the identity

$$\sum_{i,j=1}^{n} y_{ij} A_{ij} \;=\; (\mathbb{1}_{1\times n} \otimes I_d) \cdot \big(Y * A\big) \cdot (\mathbb{1}_{n\times 1} \otimes I_d). \tag{3.4}$$

Note that the multiplication by the matrices from left and right in (3.4) provides a block-wise summation of all the blocks in $Y * A$.

By Proposition 3.15, $Y * A$ is positive definite. Hence, for $\mathbf{v} \in \mathbb{R}^d \setminus \{0\}$, we have

$$\begin{aligned} & \mathbf{v}^T \left((\mathbb{1}_{1\times n} \otimes I_d) \cdot \big(Y * A\big) \cdot (\mathbb{1}_{n\times 1} \otimes I_d)\right) \mathbf{v} \\ = \;& (\mathbf{v}^T \cdots \mathbf{v}^T)\big(Y * A\big)(\mathbf{v}^T \cdots \mathbf{v}^T)^T \;>\; 0. \end{aligned}$$

This implies $Q := \sum_{i,j=1}^{n} A_{ij} y_{ij} \succ 0$.

The positive definite matrix Q has a square root $Q^{1/2}$. Set $H = \sum_{i,j=1}^{n} A_{ij} x_{ij} + B$. Then, for any real symmetric $d \times d$-matrix X, the univariate polynomial

$$\begin{aligned} t \;\mapsto\; f(X + tY) \;&=\; \det\Big(\sum_{i,j=1}^{n} A_{ij}(x_{ij} + t y_{ij}) + B\Big) \\ &=\; \det(H + tQ) \\ &=\; \det(Q)\det(Q^{-1/2} H Q^{-1/2} + t I_d) \end{aligned}$$

has only real roots, since they are the negatives of the eigenvalues of the Hermitian matrix $Q^{-1/2} H Q^{-1/2}$.

Now, for the general case, let A be a positive semidefinite matrix. Let $A^{(k)} = (A_{ij}^{(k)})_{n\times n}$ be a sequence of positive semidefinite block matrices with positive definite blocks on the diagonal, which approximate A. Then the polynomials $f^{(k)}(Z) = \det(\sum_{i,j=1}^{n} A_{ij}^{(k)} z_{ij} + B)$ are psd-stable and hence have no root in the (open) Siegel upper half-plane. Due to Hurwitz' Theorem, the limit polynomial f is either identically zero or also non-vanishing on the Siegel upper half-plane, i.e., it is psd-stable. $\square$

For an example of Theorem 3.16, observe that choosing A as the block matrix with 2×2 blocks of size 2×2,

$$A_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_{12} = A_{21} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $B = 0$ results in Example 3.3. Since $A = (A_{ij})$ has the double eigenvalues $1/2$ and $3/2$, it is positive semidefinite, so that Theorem 3.16 implies the psd-stability of

$$f(Z) \;=\; \det(A_{11}z_{11} + 2A_{12}z_{12} + A_{22}z_{22})\,, \quad \text{where } Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix}.$$

The criterion stated in Theorem 3.16 is sufficient, but not necessary. The following example gives a counterexample.

Example 3.17. *Let $Z = (z_{ij})_{2\times 2}$ be symmetric and*

$$f(Z) = \det\left(\sum_{i,j=1}^{2} A_{ij}z_{ij}\right) = \det\left(\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} z_{11} + 2\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} z_{12} + \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} z_{22}\right). \quad (3.5)$$

We claim that f is psd-stable. Namely, for a real matrix $Y = (y_{ij})_{2\times 2} \succ 0$, we have

$$f(Y) \;=\; (y_{11} + 5y_{22})(5y_{11} + y_{22}) - 16y_{12}^2 \;>\; 5(y_{11}^2 + y_{22}^2) + \big(26 - 16\big)y_{11}y_{22} \;>\; 0\,,$$

since $y_{11}, y_{22} > 0$. Hence, $\sum_{i,j=1}^{2} A_{ij}y_{ij} \succ 0$, and thus, by Example 3.2 (ii), f is psd-stable. However, the matrix

$$A \;=\; (A_{ij}) \;=\; \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 5 & 2 & 0 \\ 0 & 2 & 5 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}$$

is not positive semidefinite, since already the 2×2-minor with indices $(1, 4)$ is negative.

We remark that the determinantal representation in (3.5) is definite. However, it is open whether f has an equivalent determinantal representation $\det(P(\sum_{j=1}^{n} A_j z_j + A_0)P^H)$, such that $A' = (PA_{ij}P^H)_{n\times n}$ is positive semidefinite; see Conjecture 3.19 in Section 3.2.

We note that already the most simple case of a 2×2-matrix Z and diagonal coefficient matrices A_{ij} provides nonlinear conditions as the following statement shows.

Proposition 3.18. *Let $A_{ij} = \operatorname{diag}(a_1^{(ij)}, \ldots, a_d^{(ij)})$ be diagonal $d \times d$-matrices, $1 \le i, j \le n$. Then the block matrix $A = (A_{ij})$ is positive semidefinite if and only if for every $k \in \{1, \ldots, d\}$ the matrix $(a_k^{(ij)})_{1 \le i,j \le n}$ is positive semidefinite.*

For $n = 2$ and a Hermitian block matrix $A = (A_{ij})$, the criterion becomes

$$a_k^{(11)}, a_k^{(22)} \ge 0, \text{ and } a_k^{(11)} a_k^{(22)} - |a_k^{(12)}|^2 \ \ge\ 0\ , \quad k = 1, \ldots, d.$$

Proof. Since the blocks of A are diagonal matrices, every row of A contains at most n non-zero entries. Namely, in the k-th row of the i-th block row, these are the elements $a_k^{(ij)}$, $j = 1, \ldots, n$. By reordering the rows and columns of A using a permutation matrix P, the resulting matrix $P^T AP$ has block diagonal structure with blocks $A_k := (a_k^{(ij)})_{i,j}$ of size $n \times n$. Thus, A is positive semidefinite if and only if each block A_k is positive semidefinite.

For $n = 2$, these blocks are of size 2. Hence, the minors of A consist of factors of the form $a_k^{(11)} a_k^{(22)} - a_k^{(12)} a_k^{(21)}$ together with diagonal elements. □

3.2 Further questions

We have introduced the concept of conic stability for multivariate polynomials in $\mathbb{C}[\mathbf{z}]$ and showed generalizations of some important classical statements for stable polynomials to conic stability. These positive results also show that the conic generalization of the stability notion appears to be very natural. In particular, this raises the question for further conic generalizations of statements known for stable polynomials, for example for stability preserving operations.

In Section 1.2.1.1, we discussed the Generalized Lax Conjecture which asks for definite determinantal representations of hyperbolic polynomials. Phrasing Theorem 3.5 in terms of psd-stability, a homogeneous polynomial is psd-stable if and only if it is hyperbolic with respect to every point in the cone of definite matrices. Hence, whenever the Generalized Lax Conjecture 1.17 is true, there is a definite determinantal representation for psd-stable polynomials. Example 3.17 provides an illustration for the verified case $n = 3$. But it is open whether there is an equivalent determinantal representation whose block matrix $A = (A_{ij})$ of coefficients is positive semidefinite. So, one may ask whether for every psd-stable polynomial there exists such a determinantal representation.
However, stability of $f(\mathbf{z})$ coincides with psd-stability of $f(\operatorname{diag}(z_1, \ldots, z_n))$. Hence, the bases generating polynomial of the Vámos matroid, which is known to be stable by [127], i.e., hyperbolic by Proposition 1.27, but which has no definite determinantal representation by [15], provides an example for a psd-stable polynomial without

definite determinantal representation. In particular, this implies that for all equivalent determinantal representations the matrix $A = (A_{ij})$ of coefficients is not positive semidefinite. Nevertheless, Theorem 3.16 motivates to ask for classes of polynomials where psd-stability implies a determinantal representation whose matrix $A = (A_{ij})$ of coefficients is positive semidefinite. Moreover, we can ask for a conic Lax Conjecture for psd-stable polynomials, which reads as follows.

Conjecture 3.19. Let $f \in \mathbb{R}[Z]$ be homogeneous and psd-stable. Then, there is a homogeneous K-stable polynomial $g \in \mathbb{R}[Z]$ with $S_d^+ \subseteq K$ and a positive semidefinite block matrix $A = (A_{ij})_{n\times n}$ such that

$$Z = (z_{ij})_{n\times n} \;\mapsto\; g(Z) \cdot f(Z) = \det\left(\sum_{i,j=1}^{n} A_{ij} z_{ij}\right).$$

If this conjecture is valid, then the Generalized Lax Conjecture 1.17 would be true. To see this, consider $f(z_{11}, \ldots, z_{nn})$ to be stable, which is equivalent to psd-stability of $f(\mathrm{diag}(z_{11}, \ldots, z_{nn}))$. If Conjecture 3.19 is true, there is a K-stable polynomial $g(Z)$ with $S_d^+ \subseteq K$,

$$P(Z) \coloneqq g(Z) f(\mathrm{diag}(z_{11}, \ldots, z_{nn}) = \det\left(\sum_{i,j=1}^{n} A_{ij} z_{ij}\right)$$

and $A = (A_{ij})_{n\times n} \succeq 0$. But then, the restriction $Q(z_{11}, \ldots, z_{nn}) \coloneqq P(Z)|_{z_{ij}=0, i\neq j}$ is still psd-stable and the block diagonal matrix $\mathrm{diag}(A_{11}, \ldots, A_{nn})$ is positive semidefinite. By Theorem 3.5, Q is hyperbolic with respect to every positive definite diagonal matrix and its hyperbolicity cone equals $\{\mathrm{diag}(e_1, \ldots, e_n) \in \mathbb{R}^{n\times n} : A_{11}e_1 + \cdots + A_{nn}e_n \succ 0\}$, which is due to Example 1.7. Hence, Q has a definite determinantal representation. Translating psd-stability of $f(\mathrm{diag}(z_{11}, \ldots, z_{nn}))$ and $g(\mathrm{diag}(z_{11}, \ldots, z_{nn}))$ back to stability of $f(z_{11}, \ldots, z_{nn})$ and $g(z_{11}, \ldots, z_{nn})$ yields the equivalent formulation of the Generalized Lax Conjecture 1.18.

Another open end concerns the region of stability. In most of our proofs regarding conic stability, we made use of Lemma 3.4, which allows to consider univariate stability and to apply classical univariate results. Lemma 3.4 itself relies substantially on the conic structure of the set K. However, this leads to the question whether the theorems for conic stability can be further extended to even more general types of stability regions.

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Deutsche Zusammenfassung

Ein Polynom $f \in \mathbb{C}[z_1, \ldots, z_n]$, $n \geq 1$, heißt stabil, wenn jede Nullstelle $(z_1, \ldots, z_n) \in \mathbb{C}^n$ die Bedingung erfüllt, dass der Imaginärteil mindestens einer Koordinate nichtpositiv ist, d.h. $\mathrm{Im}(z_j) \leq 0$ für ein $j \in \{1, \ldots, n\}$ gilt. Wir bezeichnen dabei mit $\mathrm{Im}(z_j)$ den Imaginärteil der komplexen Zahl z_j und wir schreiben $\mathrm{Im}(z_1, \ldots, z_n)$ für den Vektor $\big(\mathrm{Im}(z_1), \ldots, \mathrm{Im}(z_n)\big) \in \mathbb{R}^n$. Ferner verwenden wir im folgenden fett gedruckte Buchstaben für Vektoren der Länge n, zum Beispiel $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Durch diese Bedingung an die Imaginärteile der Nullstellen wird die Untersuchung der Menge

$$\mathcal{I}(f) \;=\; \{\, \mathrm{Im}(z_1, \ldots, z_n) \,:\, f(z_1, \ldots, z_n) = 0 \,\}$$

motiviert. Die Menge $\mathcal{I}(f)$ ist die Projektion der Nullstellenmenge von f auf ihren Imaginärteil. Wir nennen sie die Imaginärprojektion von f. Diese induziert eine geometrische Sichtweise auf die Stabilität von Polynomen, da ein Polynom f genau dann stabil ist, wenn

$$\mathcal{I}(f) \cap (\mathbb{R}_{>0})^n = \emptyset \tag{6}$$

gilt. Dementsprechend ist diese Dissertation sehr stark durch die Theorie stabiler Polynome motiviert. Stabile Polynome sind in verschiedenen mathematischen Teilbereichen interessant, beispielsweise in der Graphentheorie und der Kombinatorik, siehe [14, 22, 42, 125], und sie tauchen in der aktuellen Literatur an prominenten Stellen auf. Exemplarisch seien der Beweis der Kadison-Singer-Vermutung, der Beweis über die Existenz unendlicher Familien bipartiter Ramanujan-Graphen sowie der Beweis der Johnson-Vermutungen genannt; siehe [8, 82, 83].

Wir untersuchen zunächst die Struktur der Imaginärprojektion und ihrer Komplementkomponenten. Dabei zeigen wir die zentrale Eigenschaft, dass das Komplement des Abschlusses einer Imaginärprojektion stets aus konvexen Zusammenhangskomponenten besteht und dass die Anzahl an Komplementkomponenten endlich ist. Dies führt auf Beziehungen zu Amöben und Coamöben, welche ebenfalls nur endlich viele konvexe Komplentkomponenten besitzen, siehe [35, 41], aber auch zur Theorie von hyperbolischen Polynomen, deren Hyperbolizitätskegel konvex sind; siehe [40, 45, 111]. Dabei heißt ein homogenes Polynom $f \in \mathbb{R}[\mathbf{z}]$ hyperbolisch in Richtung $\mathbf{e} \in \mathbb{R}^n$, wenn $f(\mathbf{e}) \neq 0$ und wenn für jeden Vektor $\mathbf{x} \in \mathbb{R}^n$ die univariate Funktion $t \mapsto f(\mathbf{x} + t\mathbf{e})$ nur reelle Nullstellen besitzt. Geometrisch gesprochen bedeutet dies, dass jede Gerade mit Richtungsvektor $\mathbf{e}$ die reelle Nullstellenmenge eines Polynoms f vom Grad d in genau d Punkten schneidet (Vielfachheiten mit eingerechnet). Die Zusammenhangskomponente des Komplements der reellen Nullstellenmenge von f, welche $\mathbf{e}$ enthält, heißt Hyperbolizitätskegel von f (bezüglich $\mathbf{e}$) und ist konvex. Ferner ist f bezüglich jeden Punktes

im Hyperbolizitätskegel hyperbolisch. Hyperbolische Polynome sind in der reell algebraischen Geometrie von großem Interesse und ermöglichen beispielsweise positiv definite Determinantendarstellungen bestimmter konvexer Mengen; siehe [76, 97, 99, 107, 124]. Aber sie sind auch im Bereich der konvexen Optimierung interessant, da hyperbolische Programme in natürlicher Weise semidefinite Programme verallgemeinern und sich ebenfalls durch Innere-Punkte-Verfahren numerisch lösen lassen; siehe beispielsweise [45, 93, 95]. Auch wir interessieren uns für hyperbolische Polynome. Neben allgemeinen Gesichtspunkten betrachten wir deswegen insbesondere auch die Imaginärprojektionen hyperbolischer Polynome. Dabei beobachten wir eine starke Beziehung zu Hyperbolizitätskegeln, welche die bekannte Aussage verallgemeinert, dass ein homogenes Polynom genau dann stabil ist, wenn es bezüglich jeden Punktes im positiven Orthanten hyperbolisch ist; siehe beispielsweise [71, 106]. Unsere Untersuchungen führen zu einer scharfen oberen Schranke für die maximale Anzahl an Hyperbolizitätskegeln.
Darüber hinaus verallgemeinern wir den Stabilitätsbegriff auf Stabilität bezüglich eines echten Kegels $K \subset \mathbb{R}^n$. Ein echter Kegel ist ein volldimensionaler konvexer Kegel, der abgeschlossen und spitz ist und der ein nichtleeres Inneres besitzt. Wir sagen, dass ein Polynom f K-stabil ist, wenn es keine Nullstelle $(z_1, \ldots, z_n) \in \mathbb{C}^n$ gibt, sodass $\mathrm{Im}(z_1, \ldots, z_n) \in \mathrm{int}(K)$, wobei $\mathrm{int}(K)$ das Innere von K bezeichnet. Es stellt sich heraus, dass dies eine sehr natürliche Verallgemeinerung des bisherigen Stabilitätsbegriffs darstellt, da viele wohlbekannte Aussagen für univariate und multivariate stabile Polynome auf konische Stabilität übertragen werden können. Ferner fällt $(\mathbb{R}_{\geq 0})^n$-Stabilität mit dem gewöhnlichen Stabilitätsbegriff zusammen. Für Polynome in komplex-symmetrischen Matrixvariablen betrachten wir den Kegel der positiv semidefiniten Matrizen, bezeichnet als $\mathcal{S}^+$. Wir sprechen von psd-Stabilität, wenn $\mathcal{S}^+$-Stabilität vorliegt und wir beweisen ein Kriterium für psd-Stabilität für Determinantenpolynome.

Kapitel 1 gibt einen Überblick über stabile Polynome, hyperbolische Polynome sowie über Amöben und Coamöben. Dabei widmen wir uns zunächst den hyperbolischen Polynomen. Wir fassen hier den historischen Hintergrund im Bereich der partiellen Differentialgleichungen zusammen, betrachten Determinantenpolynome $f(z_1, \ldots, z_n) = \det(z_1 A_1 + \cdots + z_n A_n)$ und besprechen deren Anwendung beim Beschreiben konvexer Mengen. In diesem Zusammenhang werfen wir einen Blick auf die Lax-Vermutung zusammen mit Verallgemeinerungen und verwandten Resultaten. Ferner beleuchten wir das Interesse für hyperbolische Polynome in der konvexen Optimierung.
Wir geben auch einen Überblick über stabile Polynome. Wir führen klassische Resultate für univariate Polynome auf, wie das Hermite-Biehler Theorem, welches die Stabilität eines Polynoms $g+if$ in Abhängigkeit der Nullstellen von g und f charakterisiert, und das Hermite-Kakeya-Obreschkoff Theorem, das eine Aussage über die Stabilität von Linearkombinationen zweier Polynome trifft. Anschließend erwähnen wir aktuellere Resultate,

welche multivariate Polynome betreffen und beispielsweise die Stabilität von Polynomen beschreiben, welche in jeder Variable affin linear sind. Stabile Polynome sind durch die Theorie der dynamischen Systeme motiviert, bei denen sogenannte Hurwitz-Polynome relevant sind, welche die asymptotische Stabilität von linearen und nicht-linearen autonomen System garantieren. Univariate Hurwitz-Polynome sind dadurch definiert, dass alle Nullstellen einen negativen Realteil aufweisen. Eine multivariate Verallgemeinerung, welche (multivariate) Hurwitz-Polynome und stabile Polynome umfasst, bildet die Klasse der Polynome mit der Halbebenen-Eigenschaft. Das sind diejenigen Polynome, bei denen die Nullstellen alle in einem abgeschlossenen Halbraum des $\mathbb{C}^n$ liegen. Stabile Polynome und, allgemeiner, Polynome mit der Halbebenen-Eigenschaft sind in vielen verschiedenen Bereichen der Mathematik von Interesse; siehe hierzu beispielsweise die Übersichtsarbeit [126]. Wir geben ferner einen Überblick über wichtige Vermutungen aus verschiedenen mathematischen Disziplinen, die mithilfe stabiler Polynome bewiesen wurden. Beispielsweise schauen wir auf die Kadison-Singer Vermutung aus dem Bereich der Funktionalanalysis, die 2015 von Marcus, Spielman und Srivastava mithilfe diskreter Methoden und insbesondere mithilfe univariater, stabiler Polynome bewiesen wurde; siehe [83].
Im anschließenden Abschnitt 1.4 stellen wir wichtige Ergebnisse zu Amöben und Coamöben zusammen. Diese umfassen beispielsweise Anzahl und Struktur der Komplementmengen, aber auch den Rand und das Verhalten „im Unendlichen“. Wie sich im Verlauf der Dissertation herausstellen wird, gibt es einige interessante Analogien insbesondere zwischen Amöben und Imaginärprojektionen.

In Kapitel 2 widmen wir uns der Untersuchung von Imaginärprojektionen. Im einführenden Abschnitt 2.1 beweisen wir grundlegende Eigenschaften und zeigen die Konvexität der endlich vielen Komponenten im Komplement des Abschlusses. Die endliche Kardinalität folgt aus der Tatsache, dass die Imaginärprojektion als Projektion einer algebraischen Menge semialgebraisch ist. Die Konvexität folgt aus einem Resultat von Bochner aus dem Bereich der komplexen Analysis, [7]. Diese Konvexitätsaussage führt zu ersten Betrachtungen bezüglich der maximalen Anzahl an Komplementkomponenten. Dabei zeigen wir insbesondere anhand einer Konstruktion, dass es zu einer beliebig vorgegebenen positiven, ganzen Zahl K ein Polynom gibt, dessen Imaginärprojektion mindestens K strikt konvexe und beschränkte Komplementkomponenten besitzt. Ferner kann man Polynome konstruieren, welche genau K beschränkte Komplementkomponenten besitzen.
Anschließend diskutieren wir in Abschnitt 2.2 die Imaginärprojektionen von ausgewählten Polynomklassen. Die meisten dieser Polynome werden im Verlauf der Arbeit wiederholt in Beweisen oder als Beispiel auftauchen. Es stellt sich dabei heraus, dass sich bereits

die Imaginärprojektionen von affin linearen Polynomen sehr empfindlich gegenüber ε-Perturbationen von Koeffizienten verhalten und in Abhängigkeit von den Koeffizienten entweder, falls alle Koeffizienten das komplexe Vielfache eines reellen Vektors sind, eine Hyperebene beschreiben, oder andernfalls den gesamten Raum umfassen. Auch das Verhalten der Imaginärprojektionen reeller, quadratischer Polynome ist überraschend, da sie viele verschiedene Klassen umfassen. Beispielsweise kann der Rand der Imaginärprojektion eine Kugeloberfläche oder einen 2-schaligen Hyperboloid beschreiben. Konkret für den Fall zweier Variablen ist beispielsweise $\mathcal{I}(z_1^2 + z_2^2 + 1) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 \geq 1\}$ und $\mathcal{I}(z_1^2 - z_2^2 - 1) = \{-1 \leq y_1^2 - y_2^2 < 0\} \cup \{0\}$. Wir geben eine komplette Beschreibung der Imaginärprojektionen reeller, quadratischer Polynome an.
In Abschnitt 2.3 betrachten wir hyperbolische Polynome, deren Imaginärprojektion nicht nur eine angenehme Struktur aufweisen, sondern die sich auch als wichtiges Hilfsmittel bei der Untersuchung von Imaginärprojektionen inhomogener Polynome herausstellen. Als grundlegende Aussage beweisen wir, dass Hyperbolizitätskegel mit Komplementkomponenten von Imaginärprojektionen übereinstimmen. Dies beschreibt einen wichtigen strukturellen Zusammenhang dieser beiden Mengen und darauf aufbauend können wir eine scharfe obere Schranke für die Anzahl von Hyperbolizitätskegeln beziehungsweise für die Anzahl an Komplementkomponenten der Imaginärprojektion homogener Polynome beweisen. Die Maximalzahl entspricht der Anzahl der Zellen in einem Hyperebenenarrangement linearer Hyperebenen und beträgt für ein Polynom vom Grad d in n Variablen genau 2^d für $d \leq n$ und $2\sum_{k=0}^{n-1}\binom{d-1}{k}$ für $d > n$. Der Beweis beruht auf einer Aussage über die Anzahl der Hyperbolizitätskegeln irreduzibler, homogener Polynome von Kummer, [70], und funktioniert konstruktiv. Dies ermöglicht uns insbesondere, die Menge an Polynomen zu charakterisieren, welche die maximale Anzahl an Hyperbolizitätskegeln besitzen. Dies sind gerade diejenigen Polynome, welche aus unabhängigen, linearen Faktoren bestehen.
Darüber hinaus betrachten wir in Abschnitt 2.4 den Rand von Imaginärprojektionen. Dabei unterscheiden wir homogene und inhomogene Polynome. Für homogene Polynome verstehen wir die Situation sehr gut. Hier ist der Rand eine Teilmenge der reellen Nullstellenmenge und damit ergibt sich unmittelbar, dass für ein irreduzibles Polynom der Zariski-Abschluss des Randes der Imaginärprojektion genau der Varietät des Polynoms entspricht. Ferner können wir charakterisieren, wann der Rand mit der reellen Nullstellenmenge übereinstimmt. Für inhomogene Polynome verhält sich der Rand der Imaginärprojektion deutlich komplizierter. Wir geben hier ein notwendiges Kriterium für Randpunkte an, welche Teil der Imaginärprojektion sind. Diese sind Teil einer Menge sogenannter kritischer Punkte, welche aus denjenigen Nullstellen $\mathbf{z}$ von f bestehen, für die $\left(\frac{\partial f}{\partial z_1}(\mathbf{z}), \ldots, \frac{\partial f}{\partial z_n}(\mathbf{z})\right)$ das komplexe Vielfache eines reellen Vektors ist.
In Abschnitt 2.5 betrachten wir für Imaginärprojektionen das Verhalten „im Unendlichen“. Hierfür untersuchen wir die Menge der Häufungspunkte im Unendlichen, welches

für ein Polynom $f \in \mathbb{C}[\mathbf{z}]$ definiert ist als

$$\mathcal{I}_\infty(f) \;=\; \lim_{r\to\infty} \left(\frac{1}{r}\mathcal{I}(f) \cap \mathbb{S}^{n-1}\right) \; .$$

Diese Menge steht in Analogie zum logarithmic limit set für Amöben, ist jedoch im Gegensatz dazu im Allgemeinen kein sphärischer, polyedrischer Komplex. Für die Klasse der bivariaten Polynome können wir die Häufungspunkte im Unendlichen vollständig beschreiben und stellen insbesondere fest, dass entweder $\mathcal{I}_\infty(f) = \mathbb{S}^1$ oder die Kardinalität genau das Doppelte des Polynomgrads beträgt (Vielfachheiten mitgezählt). Die Resultate führen zu Aussagen über Komplementkompontenten, welche einen volldimensionalen Rezessionskegel besitzen. Bei diesen Überlegungen spielen auch wieder hyperbolische Polynome eine wichtige Rolle. Dabei betrachten wir die Initialform eines Polynoms f, bezeichnet als $\mathrm{in}(f)$, welche definiert ist als die Summe derjenigen Terme in f, welche den höchsten Totalgrad besitzen. Die Initialform ist homogen und es stellt sich heraus, dass die Hyperbolizitätskegel von $\mathrm{in}(f)$ in Bijektion zu den Komplementkomponenten mit volldimensionalem Rezessionskegel von $\mathcal{I}(f)$ stehen. Dies liefert insbesondere auch die Aussage, dass genau dann $\mathcal{I}_\infty(f) \neq \mathbb{S}^{n-1}$ gilt, wenn $\mathrm{in}(f)$ hyperbolisch ist.

In Kapitel 3 initiieren wir die Verallgemeinerung von „gewöhnlicher“ Stabilität auf konische Stabilität. Dabei ersetzen wir in (6) den positiven Orthanten durch das Innere eines echten Kegels $K \subset \mathbb{R}^n$. Ein Polynom f ist dementsprechend genau dann K-stabil, wenn $\mathcal{I}(f) \cap \mathrm{int}\, K = \emptyset$. Wir beweisen eine konische Version des Hermite-Biehler Theorems für den Fall von polyedrischen und nicht-polyedrischen Kegeln sowie eine konische Version des Hermite-Kakeya-Obreschkoff Theorems. Darüber hinaus betrachten wir den Fall, dass K der Kegel der positiv semidefiniten Matrizen ist. Dieser Fall ist in natürlicher Weise mit dem Siegel'schen Halbraum für Modulformen verwandt, welcher für gegebenen Grad g definiert ist als $\mathcal{H}_g = \{A \in \mathbb{C}^{g\times g} \text{ symmetrisch} : \mathrm{Im}(A) \text{ ist positiv definit}\}$. Und es gilt, dass ein Polynom $f(z_1, \ldots, z_n)$ genau dann stabil ist, wenn $f(\mathrm{diag}(z_1, \ldots, z_n))$ psd-stabil ist. Wir zeigen, dass für eine Hermitesche Blockmatrix $A = (A_{ij})_{n\times n}$ mit Blöcken der Größe $d \times d$ und für eine Hermitesche $d \times d$-Matrix B das Polynom

$$Z = (z_{ij})_{n\times n} \mapsto f(Z) = \det(z_{11}A_{11} + z_{12}A_{12} + \cdots + z_{nn}A_{nn} + B)$$

in der komplex-symmetrischen Matrix Z psd-stabil ist. Der Beweis beruht wesentlich auf einer Konstruktion mithilfe des Kronecker-Produkts und des Khatri-Rao Produkts für Blockmatrizen.

Die Dissertation basiert auf den Preprints [58–60]. Diese sind in gemeinsamer Arbeit mit Thorsten Theobald entstanden. Der erste Artikel wurde ferner in Zusammenarbeit

mit Timo de Wolff geschrieben. Eine frühere Version von [60] wurde für einen Vortrag bei der MEGA 2017 in Nizza, Frankreich, angenommen. [58] wurde von Proceedings of the American Mathematical Society akzeptiert. [59] ist in Research of the Mathematical Sciences veröffentlicht.

Die Inhalte dieser drei Artikel werden in dieser Arbeit zu großen Teilen wörtlich wiedergegeben. Andere Resultate stellen Erweiterungen oder zusätzliche Ergebnisse dar, die noch nicht publiziert wurden.

Curriculum Vitae

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Talk: SAGE in der Studieneingangsphase und E-Learning-Aspekte.

2017 Oberseminar Reelle Geometrie und Algebra, Konstanz (Germany)
December 1, 2017.
Talk: Conic Stability of Polynomials.

FOCM, Barcelona (Spain)
July 10 - 19, 2017.
Poster: Imaginary Projections of (Homogeneous) Polynomials.

MEGA, Nice (France)
June 12 – 16, 2017.
Talk: Imaginary Projections of Polynomials.

2016 Dagstuhl Seminar: Algorithms and Effectivity in Tropical Mathematics and Beyond, Schloss Dagstuhl (Germany)
November 27 - December 2, 2016.
Talk: Imaginary Projections of Polynomials.

Symposium Diskrete Mathematik, Berlin (Germany)
July 15 - 16, 2016.

A Darmstadt Frankfurt Afternoon on Optimization, Darmstadt (Germany)
June 10, 2016.

DMV-GAMM Jahrestagung, Braunschweig (Germany)
March 07 - 11, 2017.
Talk: Imaginary Projections of Polynomials.

2015 7. Thüringer Geometrietag, Jena (Germany)
December 04 - 05, 2015.

	Summer School Convex Geometry, Berlin (Germany) June 29 - July 10, 2015.
	MEGA 2015, Trento (Italy) June 15 - 19, 2015.
	Arbeitsgemeinschaft The Kadison-Singer Conjecture, Oberwolfach (Germany) 29 March – 4 April, 2015. Talk: Kadison-Singer via Real Stability.
2014	Colloquium in Honor of the 60th Birthday of Peter Gritzmann, München (Germany) Dezember 18 - 19, 2014.
	Workshop Solving Polynomial Equations, Berkeley (USA) October 13 - 16, 2014.

Teaching

Summerterm 2018	Tutorenschulung für Mathematik-Tutoren
Winterterm 2017	Seminar über algorithmische Mathematik with Thorsten Theobald
	Tutorenschulung für Mathematik-Tutoren
Summerterm 2017	Tutorenschulung für Mathematik-Tutoren
Winterterm 2016	Seminar über stabile Polynome with Thorsten Theobald
	Tutorenschulung für Mathematik-Tutoren
Summerterm 2016	Tutorenschulung für Mathematik-Tutoren