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Generalized Uncertainty Principle inspired Schwarzschild Black Holes in Extra Dimensions

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Notation

In this thesis the following set of notations will be used:

Natural units

If not explicitly stated, all formulae are in natural units, which means:

$$\hbar = c = k_b = 1$$

Vectors/Matrices

For clearer distinction 3-vectors have an arrow and 4-vectors are in bold letters. Vectors are in lowercase and matrices (which are always bold) in uppercase letters. If the components of a vector are meant, this is stated by roman ($i = 1, \ldots, 3$) or greek ($\mu = 0, \ldots, 3$) indices.

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \lambda\mathbf{x} \\ \mathbf{x} &= x^\mu \mathbf{e}_\mu \\ \vec{x} &= x^i \vec{e}_i \end{aligned}$$

Here the metric $g_{\mu\nu}$ will be the (mostly positive) east coast metric $(-, +, +, +)$. Of course the *Einstein summation convention* is used.

For the higher dimensional case we use uppercase roman indices, e.g. g_{MN}. It will be clear from the context if they go from 0 to n or 1 to n, where n is the number of spatial dimensions, so the theory is essentially $n+1$–dimensional.

Contents

Chapter 1

Introduction

1.1 Motivation

One of the greatest challenges of theoretical physics is finding a theory of everything (TOE), a theory that combines the three forces of the *Standard Model* — electromagnetic, weak and strong interaction — with gravity. The first three can be expressed as gauge theories in quantum field theory (QFT) and the combination of the three of them in one single force is called a grand unified theory (GUT). Gravity is described in Einstein's theory of general relativity (GR) which is classical, i.e. non quantum mechanical.

Both theories, QFT and GR itself are very successful. They describe and predict experimental results up to an enormous degree of accuracy. QFT very successfully predicts the measurements at particle accelerators and just recently the *Higgs boson* has been measured at CERN, which is a big achievement for the Standard Model. GR on the other hand describes the universe at big scales so good that until now no deviations have been found. The change in the period time of the Hulse-Taylor pulsar (because of emission of gravitational waves) has been measured for almost 40 years and the GR prediction is so accurate that one could think it were a fit curve and not a theoretical prediction.

The problem is that on the quest of finding a TOE one faces many signals that standard GR and quantum mechanics (QM) (or better QFT) do not fit together. Gravity is very weak compared to other forces like the electromagnetic one, therefore the regime of small scales is governed by QFT. However, since there exist negative electric charges in the same amount as positive ones, on big scales they cancel out and this regime is almost entirely controlled by gravity (astrophysics/cosmology). Yet nobody has been able to unite those two regimes in one big TOE, e.g. one finds non-renormalizable infinities when trying to describe gravity by means of QFT. The most popular approaches of solving this problem are string theory, path-integral quantisation, M-theory and loop quantum gravity (LQG). Often such theories introduce two basic adjustments to well established theories: Additional space dimensions and the existence of a minimal length of ca. the *Plack length* $\ell_\mathrm{p} \approx 1.6 \cdot 10^{-36}\,\mathrm{m}$.

> "The Planck scale represents the *critical point* where quantum mechanics intersects general relativity" [SA11]

Black holes (BH) — especially microscopic black holes with masses $M_\mathrm{p} \approx 22\,\mu\mathrm{g} < M < 10^{24}\,\mathrm{g} \approx M_\mathrm{Moon}$ — are of great interest since they are at the scale where QFT and GR intersect. Since Hawking [Haw74] we strongly expect that black holes emit so called *Hawking radiation* (sometimes also called Hawking-Bekenstein radiation) with temperature T_HB. This is GR combined with quantum mechanical effects. The problem is that $T_{HB} \sim 1/M_\mathrm{BH}$ and with shrinking mass M_BH the black hole becomes hotter and hotter and we do not know what will happen in the end. We expect a quantum theory of gravity to be able to solve this problem and also to cure the singularities inside of black holes. One might not be interested in the curvature singularities inside of (Schwarzschild) black holes, since the *cosmic censorship conjecture* tells us that singularities in the solutions of the Einstein field equations (EFEs) are behind an event horizon and cannot be probed. From a deeper point of view we nevertheless want to find a theory that is able solve this problem, since singularities indicate the breakdown of our theory.

We know that either QM or GR or both of them have to be modified. Starting with the idea of a minimal length one can construct a generalized uncertainty principle (GUP), which encloses this feature into QM. The goal is to have at least an effective theory of quantum gravity (QG) from which we can learn something about a full theory of QG for some regime of validity. By implementing this modification into GR we expect to see changes in the temperature and singularities of black holes.

The authors of [ACS01] computed in a heuristic way GUP corrections to the Hawking temperature of a black hole. However this approach has some problems. First there is a sign ambiguity, the sign has to be chosen so that the result is reasonable and the other sign gives bad results, but that is not the biggest issue. Their calculations give black hole remnants (zero heat capacity) that still have a non-vanishing temperature. Furthermore, the temperature of a black hole is proportional to its surface gravity, but this temperature cannot be retraced to any surface gravity. We are going to fix all of those three problems by explicitly writing down the metric according to [IMN13].

Isi, Mureika and Nicolini (IMN) [IMN13] have calculated a GUP-inspired Schwarzschild metric and investigated its properties. As we will see the most important features are: "Smoothening" of the singularity at the center (but it is not removed) and a change in the mass dependence of the BH temperature such that cold black hole remnants exist. We will also see that we can treat the black hole remnants as a new phase of matter like quantum mechanical particles, which leads us to the self-completeness of our theory. This bachelor thesis will give an introduction to the "IMN-model" and generalize it to extra space dimensions. There we will see that for more than four space dimensions $n > 4$ the GUP no longer improves the solution, but it is basically as in normal GR. This leads to the question if the GUP is wrong. The GUP is a String theory effect and string theory needs extra dimensions. Is String theory then in trouble?

1.2 Minimal Length

Every ongoing physics student learns early in her/his education the *Heisenberg uncertainty principle*, which sets a minimal value for the product of the uncertainty in *position* Δx and *momentum* Δp

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \tag{1.1}$$

Here Planck's constant $\hbar$ has been written explicitly. A general rule of thumb is: If $\hbar$ is in it, it is quantum mechanics. Planck's quantum of action is very small ($\hbar \approx 10^{-34}$ Js), so it can be neglected as long as classical systems are the subject of our theory (classical limit: $\hbar \to 0$). For a system whose action is of the same order of magnitude as $\hbar$, one can no longer neglect it and thus we need QM. The Heisenberg uncertainty principle eq. (1.1) however does not imply a genuine *minimal length* and one can define position eigenstates $|x\rangle$, which are plain waves in momentum space ($\Delta x \to 0\,, \Delta p \to \infty$) and correspondingly momentum eigenstates $|p\rangle$, which are plain waves in position space ($\Delta p \to 0\,, \Delta x \to \infty$). There are however some gedankenexperiments that show that there might be an essentially minimal observable length, such that this is the smallest physically meaningful length. In fact, even Heisenberg in 1930 postulated a minimal length scale by introducing non-commuting position operators $[x^\mu, x^\nu] \neq 0$. He needed this relation in order to cure infinities in quantum electrodynamics (QED), which turned out to be solved otherwise (by *renormalization*), but the idea lived on and was picked up by lots of other people (noncommutative geometry NCG). For more on the history of minimal length scenarios and a broad overview about the whole topic see [Hos13]. In fact, the concept of a minimal length occurs in all above mentioned modern endeavors of constructing a QG such as path-integral quantisation [Pad85], string theory [GM88] and loop quantum gravity [RS95].

There are a lot of gedankenexperiments showing that at the Planck scale a minimal length should occur [Adl10; Hos13]. Here we give one example in order to show the limits of length measurements following [SW58; Hos13]:

We take a very simple method of measuring the distance between a clock and a mirror. The clock sends out a photon, which travels the distance D, gets reflected by the mirror and detected by the clock some time $T = 2D$ later, see figure 1.1. If the position of the clock at the beginning is known to an uncertainty of Δx then according to the *uncertainty relation* (1.1) the velocity of the clock with mass M is only known up to

$$\Delta p = M \Delta v = \frac{1}{2\Delta x}, \tag{1.2}$$

which means that while the photon is traveling the clock can move a distance $\Delta l = T\Delta v$ until the photon is measured. Adding those two uncertainties gives the uncertainty by which the distance D can be determined

$$\Delta x + \Delta l = \Delta x + \frac{T}{2M\Delta x} = \Delta x_{\text{new}}, \tag{1.3}$$

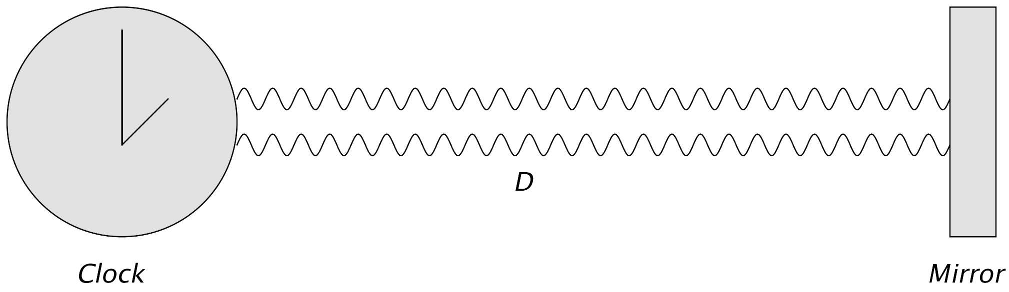


Figure 1.1: Gedanken experiment showing a minimal measurable length

The minimum of Δx_{new} is obtained by solving $\frac{\partial}{\partial(\Delta x)}\Delta x_{\text{new}} = 0$

$$\Delta x_{\min} = \sqrt{\frac{T}{2M}} \gtrsim \ell_{\text{Pl}} = \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \cdot 10^{-35}\text{m}\,, \tag{1.4}$$

where for the last step $T = 2D$ and the *hoop conjecture* (explained below) requiring $D > 2MG$ — since otherwise we would have a black hole — was used. This shows that even such a simple attempt of measuring a distance will always have a nonzero uncertainty of about the Planck length.

The *hoop conjecture* [Tho72] has been introduced by Kip Thorne in 1972 when he was studying the formation of black holes from different shapes of matter, especially an infinite long cylinder. He came up with the following conjecture (paraphrased):

> A collapsing object forms a black hole if and only if a hoop with the critical circumference $r_{\text{c}} = 2\pi R_{\text{S}} = \frac{4\pi GM}{c^2}$ (R_{S} is the Schwarzschild radius of the object) fits around the object and can be rotated arbitrarily.

That is, if an object is squeezed into a region whose extend in *every* direction is bounded by $2R_{\text{S}}$ then the object must be a black hole. In particular, an infinite long cylinder can never be a black hole.

Another gedankenexperiment that introduces a minimal length is the Heisenberg microscope with gravity effects. One tries to measure the position of a particle by shining light on it. The argument is that for smaller distances one needs photons with higher energies and those energies will eventually disturb spacetime enough to cause significant effects. When calculating this, one again ends up with a minimal length of the order of the Planck length [Adl10].

1.3 The Uncertainty Relation

First we want to do a recap of the uncertainty relation and how one gets from a commutator to such a relation [see Sha94, pp. 237–239].

The uncertainty $\Delta\Omega$ of an operator $\hat{\Omega}$ (hats on top like $\hat{a}, \hat{b}$ denote operators) is given by

$$\Delta\Omega = \left[\langle\psi|\,(\hat{\Omega} - \langle\hat{\Omega}\rangle)^2\,|\psi\rangle\right]^{\frac{1}{2}}\,, \tag{1.5}$$

where $\langle\hat{\Omega}\rangle = \langle\psi|\,\hat{\Omega}\,|\psi\rangle$ is the expectation value of $\hat{\Omega}$. By imposing a commutator between two hermitean operators $\hat{\Omega}$ and $\hat{\Lambda}$

$$[\hat{\Omega}, \hat{\Lambda}] = i\Gamma\,, \tag{1.6}$$

we will get the uncertainty relation as follows:

Since $\hat{\Omega} - \langle\hat{\Omega}\rangle$ is hermetian, we can write

$$\begin{aligned}
(\Delta\Omega)^2(\Delta\Lambda)^2 &= \langle\psi|\,(\hat{\Omega} - \langle\hat{\Omega}\rangle)^2\,|\psi\rangle\,\langle\psi|\,(\hat{\Lambda} - \langle\hat{\Lambda}\rangle)^2\,|\psi\rangle & (1.7)\\
&= \underbrace{\Big\langle(\hat{\Omega} - \langle\hat{\Omega}\rangle)\psi\,\Big|\,(\hat{\Omega} - \langle\hat{\Omega}\rangle)\psi\Big\rangle}_{|v_1|^2}\underbrace{\Big\langle(\hat{\Lambda} - \langle\hat{\Lambda}\rangle)\psi\,\Big|\,(\hat{\Lambda} - \langle\hat{\Lambda}\rangle)\psi\Big\rangle}_{|v_2|^2} & (1.8)\\
&= |v_1|^2|v_2|^2 \geq |\,\langle v_1\,|\,v_2\rangle\,|^2 = \Big|\langle\psi|\,(\hat{\Omega} - \langle\hat{\Omega}\rangle)(\hat{\Lambda} - \langle\hat{\Lambda}\rangle)\,|\psi\rangle\Big|^2\,, & (1.9)
\end{aligned}$$

where for the '$\geq$' the *Schwarz inequality* was used. The next step is to include the commutators: $[\cdot,\cdot]$ denotes the commutator and $\{\cdot,\cdot\}$ the anticommutator.

$$\begin{aligned}
(\Delta\Omega)^2(\Delta\Lambda)^2 &\geq \Big|\langle\psi|\,(\hat{\Omega} - \langle\hat{\Omega}\rangle)(\hat{\Lambda} - \langle\hat{\Lambda}\rangle)\,|\psi\rangle\Big|^2 & (1.10)\\
&= \left|\left\langle\psi\,\middle|\,\frac{1}{2}\left\{(\hat{\Omega} - \langle\hat{\Omega}\rangle), (\hat{\Lambda} - \langle\hat{\Lambda}\rangle)\right\} + \frac{1}{2}\left[(\hat{\Omega} - \langle\hat{\Omega}\rangle), (\hat{\Lambda} - \langle\hat{\Lambda}\rangle)\right]\,\middle|\,\psi\right\rangle\right|^2 & (1.11)\\
&\geq \frac{1}{4}\underbrace{\left\langle\psi\,\middle|\,\left\{(\hat{\Omega} - \langle\hat{\Omega}\rangle), (\hat{\Omega} - \langle\hat{\Omega}\rangle)\right\}\,\middle|\,\psi\right\rangle^2}_{\geq 0} + \frac{1}{4}\,\langle\psi\,|\,\Gamma\,|\,\psi\rangle^2\,. & (1.12)
\end{aligned}$$

This is the general form for the uncertainty when measuring two hermitean operators. Usually one forgets about the first term and by setting $\hat{\Omega} = \hat{x}$, $\hat{\Lambda} = \hat{p}_x$ and $\Gamma = {}^{[\hat{x},\hat{p}_x]}/_{i} = \hbar$ we get the *Heisenberg uncertainty relation*

$$[\hat{x}, \hat{p}_x] = i\hbar \longrightarrow \Delta x\Delta p_x \geq \frac{\hbar}{2}. \tag{1.13}$$

1.4 Generalization of the $[\hat{x}, \hat{p}]$ Commutator

In the generalized uncertainty principle (GUP) one extends the commutator eq.(1.13) to [KMM95]

$$[\hat{x}^i, \hat{p}_j] = i\hbar\delta^i_j \left(1 + \alpha\hat{\vec{x}}^2 + \beta\hat{\vec{p}}^2\right) . \tag{1.14}$$

This gives the uncertainty relation, which now is dependent on $\hat{\vec{x}}$ and $\hat{\vec{p}}$

$$\Delta x^i \Delta p_j \geq \frac{\hbar}{2}\delta^i_j \left[1 + \alpha(\Delta x_a)(\Delta x^a) + \beta(\Delta p_a)(\Delta p^a) + \gamma\right], \tag{1.15}$$

where $\gamma = \alpha \left\langle \hat{\vec{x}} \right\rangle^2 + \beta \left\langle \hat{\vec{p}} \right\rangle^2$ and $\beta, \alpha \geq 0$. Modifications like equation (1.15) are e.g. motivated by calculations of superstring collisions at planckian energies [ACV89]. This implies a nonzero uncertainty in position and in momentum. Nevertheless there exist *formal position eigenvectors*. These are of infinite energy and can no longer be represented through a series of physical states $|\psi_n\rangle$ with $\lim_{n\to\infty} \Delta x_{|\psi_n\rangle} = 0$. Those formal position eigenvectors will not be discussed here any further, they are covered in [KMM95].

Usually one restricts oneself to $\alpha = 0$, this is motivated since the α-part would lead to modifications in the large distances regime [BC05] and this is not where adjustments are needed. As a consequence eq.(1.15) leads to a minimal uncertainty in position, but not in momentum. Now we focus on one direction, e.g. the x_1 direction and we call $x_1 = x$, so $\Delta x \Delta p \geq \frac{\hbar}{2}(1 + \beta\Delta p^2 + \beta \langle \hat{p} \rangle^2)$. Imposing the "=" in this equation and solving for Δp yields

$$\Delta p = \frac{\Delta x}{\hbar\beta} \pm \sqrt{\left(\frac{\Delta x}{\hbar\beta}\right)^2 - \frac{1}{\beta} - \langle \hat{p} \rangle^2} . \tag{1.16}$$

By minimizing the term under the square root we get the smallest possible uncertainty for the position

$$\Delta x_0 \left(\langle \hat{p} \rangle\right) = \hbar\sqrt{\beta}\sqrt{1 + \beta \langle \hat{p} \rangle^2}, \tag{1.17}$$

and so the minimal position uncertainty $\Delta x_{\min} = \hbar\sqrt{\beta}$ for $\langle \hat{p} \rangle = 0$. This then is the fundamental smallest physically meaningful length, because for smaller distances the uncertainty is bigger than the length itself. Since the quantum mechanical uncertainty is not only a uncertainty in measurement, but really a feature of QM. We have to take it serious and interprete it such that smaller scales cannot be accessed. Therefore we call $\Delta x_{\min} \equiv x_{\min}$ the minimal length.

The combination $\beta(\Delta p)^2$ needs to be dimensionless and thus β has dimensions of $1/[p]^2$. In the following Planck units will be used and $x_{\min} = \sqrt{\beta}$ with β in units of ℓ_p^2. A visualisation of $\Delta x \Delta p \geq \frac{\hbar}{2}(1 + \beta(\Delta p)^2)$ can be seen in figure 1.2, where $\Delta x(\Delta p)$ is plotted for $\beta = 0.7\ell_\mathrm{p}$, $1\ell_\mathrm{p}$, $2\ell_\mathrm{p}$ and $\gamma = 0$. The parameter $\sqrt{\beta}$ can be chosen to be the parameter characterizing the particular theory. From a string theory perspective one might choose the string length ℓ_s to be this fundamental distance.

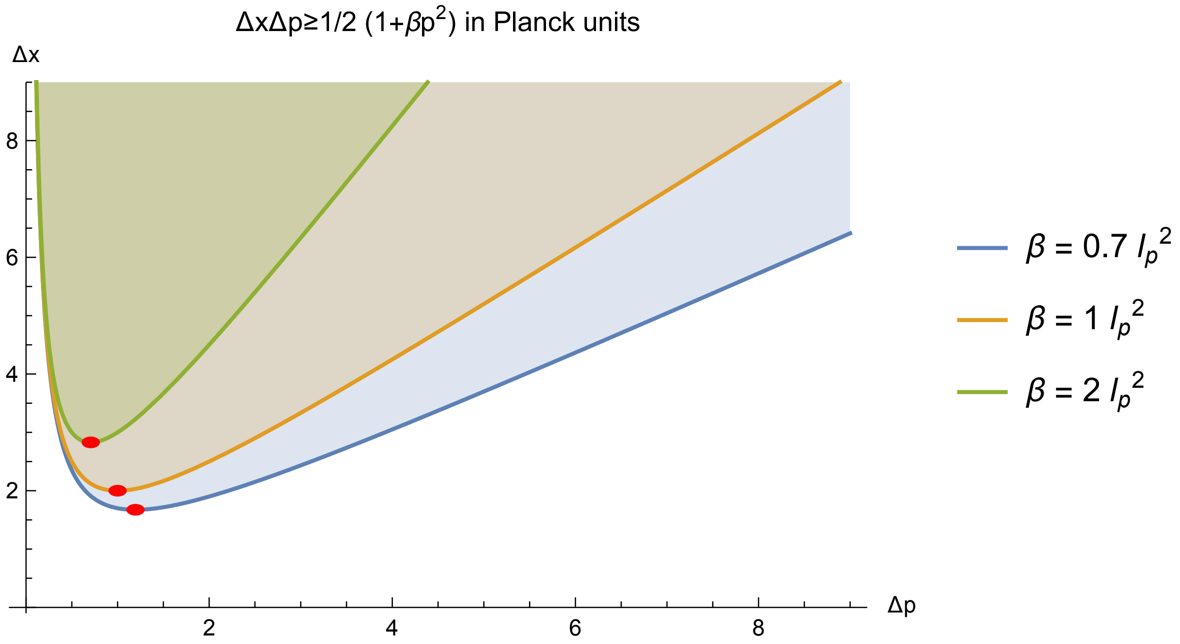


Figure 1.2: The uncertainty in position Δx as a function of Δp for different values of β and $\gamma = 0$. The red points show the respective minima. Only the region above the graphs is the physical meaningful.

Because of $x_{\min} = \sqrt{\beta}$ one has to work in momentum space, since position eigenstates do not make sense when they cannot have a vanishing uncertainty on their position. The minimal uncertainty in momentum still vanishes (we chose $\alpha = 0$), as a consequence momentum eigenstates still are meaningful, i.e. one can work in momentum space. As stated in [KMM95] the momentum and position operators in momentum space are

$$\hat{p}_i \psi(\vec{p}) = p_i \psi(\vec{p})\,, \tag{1.18}$$

$$\hat{x}^i \psi(\vec{p}) = i\hbar(1 + \beta \vec{p}^2)\partial_{p_i} \psi(\vec{p}). \tag{1.19}$$

Since the position operator is symmetric, it is ensured that all eigenvalues are real. Symmetry in $\hat{x}^i$ means that $(\langle\psi|\, \hat{x}^i)\, |\psi\rangle = \langle\psi|\, (\hat{x}^i\, |\psi\rangle)$. According to [KMM95] we have to change the integration measure to

$$\langle \psi \,|\, \phi \rangle = \int_{-\infty}^{\infty} \frac{\mathrm{d}^3 \vec{p}}{1 + \beta \vec{p}^2} \psi^*(\vec{p}) \phi(\vec{p}), \tag{1.20}$$

since this ensures symmetry.

$$\begin{aligned}
\langle \hat{x}^i \psi \,|\, \phi \rangle &= \int_{-\infty}^{\infty} \frac{\mathrm{d}^3\,\vec{p}}{1+\beta\vec{p}^2} \psi^*(\vec{p}) i\hbar(1+\beta\vec{p}^2)(\partial_{p_i}\phi) \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}^3\,\vec{p}}{1+\beta\vec{p}^2} (-1) i\hbar(1+\beta\vec{p}^2)(\partial_{p_i}\psi^*(\vec{p}))\phi(\vec{p}) + \cancelto{0}{\mathit{surface\ term}} \qquad (1.21) \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}^3\,\vec{p}}{1+\beta\vec{p}^2} \left\{ i\hbar(1+\beta\vec{p}^2)\partial_{p_i}\psi(\vec{p}) \right\}^* \phi(\vec{p}) \\
&= \langle \psi \,|\, \hat{x}^i \phi \rangle \,. \qquad (1.22)
\end{aligned}$$

Therefore the identity operator changes to

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}^3\vec{p}}{1+\beta\vec{p}^2} \left|\vec{p}\right\rangle \left\langle\vec{p}\right| = \mathbb{1} \qquad (1.23)$$

Note that [BJM10] used another position operator and another integration measure. Investigating this and its consequences for our calculation might be a reasonable future project.

1.5 Schwarzschild Black Holes in 4 and Higher Dimensions

Here only *Schwarzschild black holes* are discussed, i.e. spherically symmetric, static, non-charged, non-rotating black holes. The reason for this will be clear soon.

In 1974 Hawking [Haw74] first noticed that black holes have to emit thermal radiation which now is called *Hawking radiation*, more in section 2.5. We are interested in the quantum mechanical regime of black holes and their radiation, therefore focusing on small (micro) black holes that could for example be produced in particle accelerators such as the Large Hadron Collider (LHC). If in a collision such a micro black hole was formed, it will at first be rotating rather fast and have an asymmetric form. The following four phases are reasonable for modeling the black hole evolution [GT02; Ble+11]:

- **Balding phase**: In this phase the black hole radiates away any asymmetries through *gravitational waves* leaving it axisymmetric and still rotating. Also any (gauge field) charge the black hole could have from the particles it was formed of, is emitted primarily through the *Schwinger pair production mechanism* [Sch51].
- **Spin-down phase**: Through *Hawking* and *Unruh-Starobinskiĭ radiation* [Unr74; Sta73] the spinning non-charged black hole loses some mass and all of its angular momentum.
- **Schwarzschild phase**: It continues to radiate Hawking radiation, decreasing its mass and according to the standard formula increasing the temperature $T_{\mathrm{H}} \sim {}^{1}\!/_{M_{\mathrm{BH}}}$.

- **Planck phase**: This is the phase where our description breaks down. When the mass and/or the temperature reaches the planck scale $T_{\mathrm{H}} \approx T_{\mathrm{p}}$, $M_{\mathrm{BH}} \approx M_{\mathrm{p}}$ our semi-classical description no longer holds and a theory of quantum gravity is needed.

Since we are interested in QG-effects it is sufficient to ignore the first two phases and investigate the Schwarzschild phase and the transition to the Planck phase in terms of the GUP.

The well known 3+1-dimensional Schwarzschild metric is a solution to the EFEs, which describe how the spacetime is deformed by matter [Car03, p. 159]

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \,. \tag{1.24}$$

The *Einstein tensor* $G_{\mu\nu}$ which is just the trace reversed *Ricci tensor* $R_{\mu\nu}$ describes the geometry of spacetime and up to some factors is equal to the *energy-momentum tensor* $T_{\mu\nu}$ that is determined by the matter distribution.

By imposing vacuum and spherical symmetry in eq.(1.24), the Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \, \mathrm{d}\Omega^2 \,, \tag{1.25}$$

$$\mathrm{d}\Omega^2 = \mathrm{d}\Theta^2 + \sin^2\Theta \; \mathrm{d}\phi^2 \tag{1.26}$$

can be obtained [Car03, p. 193]. We will talk about the problems with the derivation of this metric more in section 1.6. In 1963 Tangherlini [Tan63] generalized this metric for large extra dimensions and found

$$\begin{aligned} \mathrm{d}s^2 &= -f_{N-1}(r) \; \mathrm{d}t^2 + f_{N-1}^{-1}(r) \; \mathrm{d}r^2 + r^2 \, \mathrm{d}\Omega_{N-1}^2 \,, \\ f_{N-1}(r) &= 1 - \frac{16\pi G}{(N-1) A_{N-1}} \frac{M}{r^{N-2}}, \end{aligned} \tag{1.27}$$

where A_{N-1} is the surface of the N-dimensional unit sphere

$$A_{N-1} = \frac{2\pi^{N/2}}{\Gamma\left(N/2\right)}. \tag{1.28}$$

The angular part $\mathrm{d}\Omega_{N-1}^2$ is the N-dimensional version of (1.26) and $\Gamma(t) = \int\limits_0^\infty x^{t-1} e^{-x} \, \mathrm{d}x$ is the usual Gamma-function, the extention of the factorial function and $\Gamma(N) = (N-1)!$ for $N \in \mathbb{Z}_{\neq 0}$.

1.6 All is not well with Schwarzschild

The Schwarzschild solution (1.25) of the EFEs (1.24) is probably the most known and best understood black hole metric. Usually one makes their first encounter with black hole physics through the Schwarzschild metric, but that does not mean that *all is well* with this solution.

First of all, the standard approach [see e.g. Car03, p. 196] for deriving such a geometry is by assuming a *vacuum* configuration as a source, solving the vacuum EFEs. Upon spherical symmetry one can interpret the equations and later find an integration constant to be the mass of the black hole [CMN14] by considering the Newton limit. One ignores the center of the black hole where all the mass must lie in a point-like energy density. This is still mathematically reasonable but physically not consistent. Gravity and curvature are the response of spacetime to energy (mass) and a δ-source is nothing that we see in reality.

The singularity of the Schwarzschild metric at the *event horizon* R_S is well understood and turns out to be a coordinate singularity. One just has to transform to *Kruskal-Szekeres coordinates* and all is well behaved at the horizon. The horizon is still interesting, because no particle at a distance $r < R_\mathrm{S}$ from the center can ever escape the black hole and combined with QM the horizon is responsible for Hawking radiation, but there is no curvature singularity. There are however other problems regarding the horizon, like the *information loss problem* [Pre92] and with it the *firewall problem* [Alm+13], but they are beyond the scope of this thesis.

The singularity at the origin on the other hand is a *curvature singularity*, this can be seen by looking at contractions of the curvature tensor, e.g. the Ricci scalar diverges. Should this really be a surprise? By starting with a delta distribution for the energy density one can only end up with a singularity in the curvature. Inside a black hole there will be huge energy densities and we are at a region, which needs to be described quantum mechanically and not only with GR. A full theory of QG will probably — or rather is supposed to — smoothen the singularity and explain what happens at the center of a black hole. It is expected to change the source term to an improved, UV-finite energy density.

In [CO13] a *regular metric*, i.e. a metric without curvature singularities, is found by interpreting black holes as a *condensate of gravitons*, but the same metric can also be found in terms of *non-commutative geometry* [NSS06; CMN14]. Here we will follow another path in the search for a similar metric by assuming the GUP. We will present the derivation of the IMN-metric for GUP-inspired Schwarzschild black holes.

Chapter 2

Review of the IMN-Model

In [IMN13] Isi, Mureika and Nicolini have found a GUP-inspired Schwarzschild metric (the IMN-metric) in $3+1$ dimensions, which has some interesting features. We present this solution and its discussion in this chapter. In chapter 3 we propose the generalization of the IMN-model to ADD large extra dimensions.

2.1 Effect of the GUP on the Einstein Field Equations

As seen in section 1.4 the GUP changes the integration measure according to eq. (1.20), i.e. $\int \frac{\mathrm{d}^N p}{1+\beta p^2} \left|p\right\rangle \left\langle p\right| = \mathbb{1}$. This deformation is non-local since $\|\vec{p}\|$ and $\|\vec{p}\|^2$ are no Lorentz scalars. It corresponds to the UV-suppression and can be implemented into an action S which comes from non-local gravity. Such a modified Einstein Hilbert action has been formulated and studied in [Mod12]. By functional variation of the action, $\delta S = 0$, modified Einstein Field equations can be derived [Mof11; MMN11]. The derivation is beyond the scope of this thesis and therefore we just give the results. The non-local modified version of the EFEs is [Nic12]

$$\mathcal{A}^2(\Box)\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 8\pi G T_{\mu\nu} \ , \tag{2.1}$$

or by applying $\mathcal{A}^{-2}$ on both sides we get the it in the form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G \mathcal{T}_{\mu\nu} \ . \tag{2.2}$$

Here the following has been used:

- As usual $R_{\mu\nu}$ is the *Ricci tensor* with its contraction R, the *Ricci scalar.*
- By casting $\mathcal{A}^{-2}$ on the *energy momentum tensor* $T_{\mu\nu}$ one gets $\mathcal{T}_{\mu\nu} = \mathcal{A}^{-2}T_{\mu\nu}$.
- The D'Alembertian operator is used in a generally covariant form $\Box = \ell^2 g_{\mu\nu}\nabla^\mu\nabla^\nu$, where $\ell \equiv 1/\Lambda$ and Λ is the energy scale of the theory.
- The function $\mathcal{A}(\Box)$ is a "non-polynomial" entire function.

Equation 2.1 contains a classical source term $T_{\mu\nu}$ on the rhs and modified gravity on the lhs. Doing the calculation like this would be very hard and so we applied $\mathcal{A}^{-2}$ on both sides. Equation (2.2) looks just like the EFEs, with the one difference that the source has been changed to a modified source term $\mathcal{T}_{\mu\nu}$. For now this is just non-local gravity, next the GUP effects have to be implemented.

Therefore we start with a function $\mathcal{A}(p^2)$ which mimics the GUP effects, i.e. it suppresses higher momenta á la GUP [Nic12]. Here the entire function

$$\mathcal{A}(\hat{p}^2) = \left(1 + \beta\hat{p}^2\right)^{1/2} \tag{2.3}$$

does the job, where the cut-off is $\beta = \ell^2$. As usual $\hat{\vec{p}} = -i\nabla$ and so

$$\beta\hat{\vec{p}}^2 = \ell^2(-i)^2\nabla^2 = -\Box. \tag{2.4}$$

In the last step the fact that $\mathcal{A}$ acts on a time-independent function and therefore $\partial_t = 0$ has been used. So equation (2.3) in position space has the form

$$\mathcal{A}(\Box) = (1 - \Box)^{1/2}. \tag{2.5}$$

As a source we use $\delta^{(3)}(\vec{x})$, so we have to calculate

$$\mathcal{A}^{-2}(\Box)\delta^{(3)}(\vec{x}) = \underbrace{\int_0^\infty \mathrm{d}s\, e^{-s(1-\Box)}}_{\mathcal{A}^{-2}(\Box)\text{ Schwinger representation}} \underbrace{\frac{1}{(2\pi)^3}\int \mathrm{d}^3\vec{p}\, e^{i\vec{x}\cdot\vec{p}}}_{\delta^{(3)}(\vec{x})}, \tag{2.6}$$

where the *Schwinger representation* of a differential operator $\hat{\Delta}$ has been used , i.e.

$$\hat{\Delta}^\alpha = \frac{1}{\Gamma(-\alpha)}\int_0^\infty \mathrm{d}s\ s^{-\alpha-1}\ e^{-s\hat{\Delta}}\ . \tag{2.7}$$

By letting $\Box$ operate on $e^{i\vec{x}\cdot\vec{p}}$ it becomes $\Box \to \beta(i\vec{p})^2$ and after switching the integration order (as always in physics supposing Fubini's theorem holds) and reversing the Schwinger representation we end up with

$$\mathcal{A}^{-2}(\Box)\delta^{(3)}(\vec{x}) = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\vec{p}}{1+\beta p^2} e^{i\vec{x}\cdot\vec{p}}. \tag{2.8}$$

This is just the Fourier transform of a delta distribution, but with the modified measure $\frac{d^3\vec{p}}{1+\beta p^2}$. So the UV-suppression of the GUP "flattens" the delta source. A few words have to be said about this proceeding: In eq.(2.6) the Fourier transform of $\delta^{(3)}(\vec{x})$ has been used and eq.(2.8) accordingly also is a Fourier transform. The problem is that in curved space one cannot classify positive and negative modes as one can in Minkowski space. This is the basis behind the *Unruh effect* and *Hawking radiation* which are effects of quantum field theory in curved spacetime. In particular, we cannot perform a Fourier transform in curved space, every observer has their own Fourier transform. So what do we do? We switch to *free-falling, Cartesian like coordinates*. Here we can do the Fourier transform, but we must remember that now a preferred coordinate system has been chosen.

At this point the solution of the integral (2.8) will not be retraced, since the steps of the N+1–dimensional calculation in chapter 3 will also hold in 3+1 dimensions. As a result we get the energy density $\mathcal{T}_0^0$

$$\mathcal{T}_0^0(\vec{x}) = M\mathcal{A}^{-2}(\Box)\delta^{(3)}(\vec{x}) = \frac{M}{\beta}\frac{e^{-|\vec{x}|/\sqrt{\beta}}}{4\pi|\vec{x}|} = \mathcal{T}_0^0(r) \ , \tag{2.9}$$

which is spherically symmetric as it should be, it only depends on the radial distance $r = |\vec{x}|$. We obtain the mass term $\mathcal{M}(r)$ by integrating the energy density $\mathcal{T}_0^0$ in spherical coordinates from 0 to r like in flat space (no factor $\sqrt{-\det g}$)

$$\begin{aligned} \frac{\mathcal{M}(r)}{M} &= \frac{1}{M}\int_0^r \mathrm{d}\xi \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\, \xi^2 \sin\theta\, \mathcal{T}_0^0(\xi) \\ &= 1 - e^{-r/\sqrt{\beta}} - \frac{re^{-r/\sqrt{\beta}}}{\sqrt{\beta}} = \gamma(2; r/\sqrt{\beta}) \ , \end{aligned} \tag{2.10}$$

where for notational convenience the *lower incomplete gamma function* $\gamma(s;x) = \int\limits_0^x t^{s-1}e^{-t}\ \mathrm{d}t$ has been used instead of the term with the exponential functions.

2.2 Derivation of the GUP Schwarzschild Metric

Since we have a spread out source term the GUP inspired Schwarzschild metric solving the modified EFEs (2.2) looks like the inner Schwarzschild solution

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + f^{-1}(r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 \, , \tag{2.11}$$

$$f(r) = 1 - \frac{2G\mathcal{M}(r)}{r} \ , \tag{2.12}$$

where we have already calculated $\mathcal{M}(r)$ in equation (2.10). So the metric we get by mimicking GUP effects is given by (here including factors c)

$$\boxed{\mathrm{d}s^2 = -\left(1 - 2\frac{GM}{c^2 r}\gamma(2; r/\sqrt{\beta})\right) c^2\,\mathrm{d}t^2 + \left(1 - 2\frac{GM}{c^2 r}\gamma(2; r/\sqrt{\beta})\right)^{-1} \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 \ .} \tag{2.13}$$

We can check that this metric indeed has the property one would hope and expect for the GUP, namely:

- For large distances the minimal length as in eq. (1.17) $x_{\min} = \hbar\sqrt{\beta}$ is irrelevant. Looking at eq. (2.10) we see that

$$\lim_{r\to\infty} \mathcal{M}(r) = M \qquad (\beta \neq \infty) \tag{2.14}$$

  so the metric is the Schwarzschild metric (1.25) at large scales.

- If we let the minimal length $\hbar\sqrt{\beta}$ vanish we get

$$\lim_{\hbar\sqrt{\beta}\to 0} \mathcal{M}(r) = M \ , \tag{2.15}$$

  so eq. (2.13) also becomes the usual Schwarzschild metric (1.25).

- In other words we have

$$\mathcal{M}(r) \to M \quad \text{for} \quad r \gg \hbar\sqrt{\beta} = x_{\min} \ , \tag{2.16}$$

  and so the GUP at large scales is just ordinary GR. Not that these limits are also fulfilled when considering the Newton limit $\hbar \to 0$, i.e. when we turn off quantum mechanics we also obtain standard GR.

2.3 Curvature and Singularity

The GUP fails to cure the curvature singularity at $r = 0$. The reason is, that it is represented by an entire function (2.8) of order lower than $1/2$, and as shown in [Nic12] order higher than $1/2$ would be needed. In figure 2.1 the curvature scalar is shown and it diverges for $r \to 0$.

Plotting $f(r)$ in spherical coordinates gives a nice picture of the black hole, see figures 2.2(a) and 2.2(b). The spike at the origin is responsible for the curvature singularity.

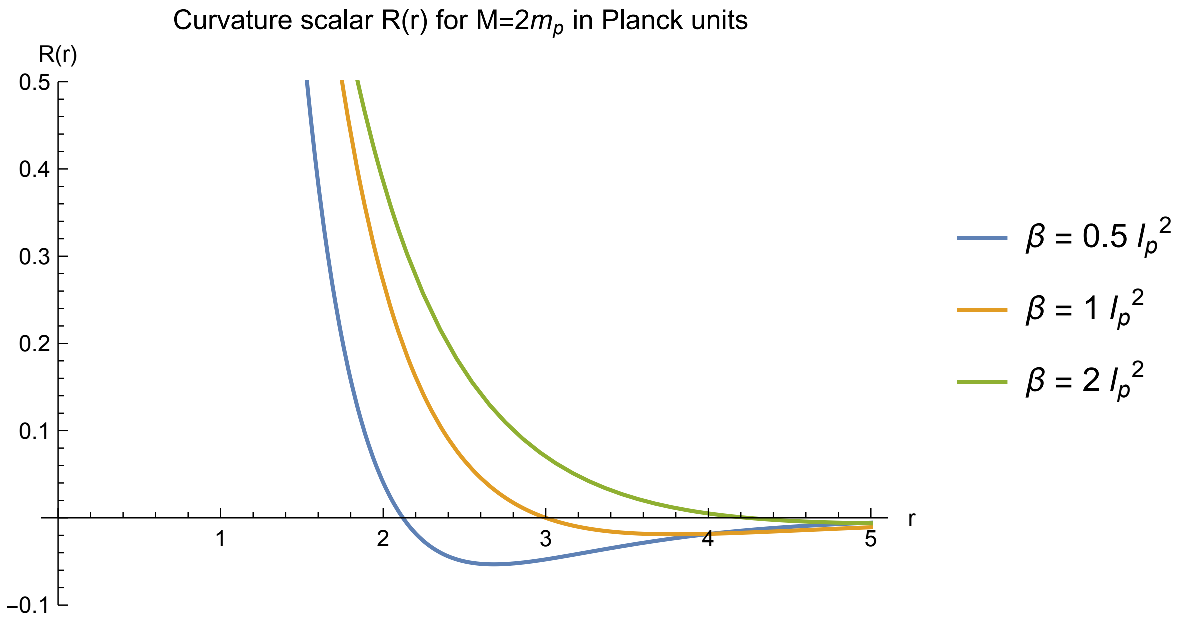
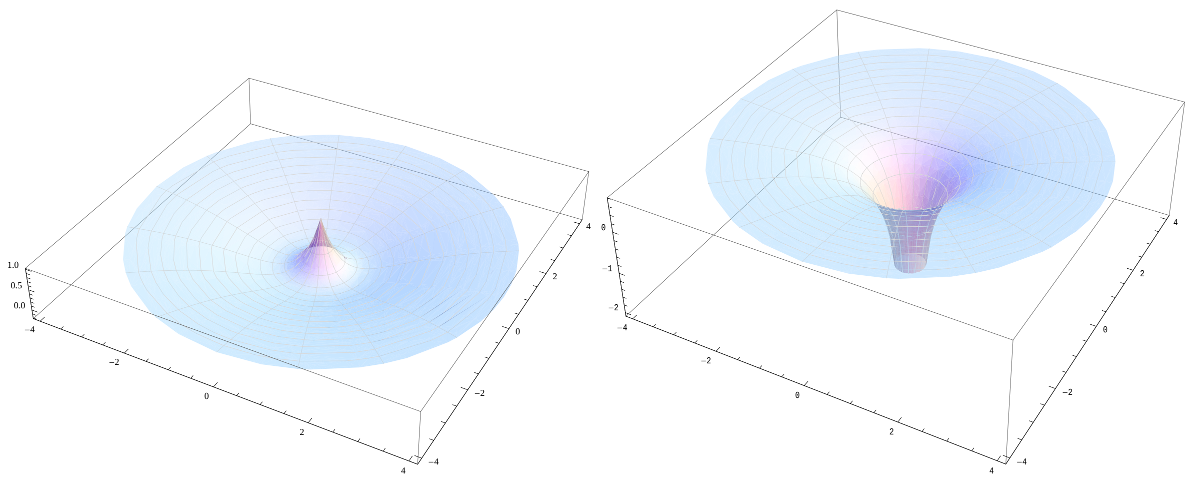


Figure 2.1: The curvature scalar $R(r)$ for the GUP inspired metric



(a) GUP

(b) Schwarzschild

Figure 2.2: Plots of $f(r)$ in spherical coordinates

2.4 Horizon Structure of the GUP Inspired Schwarzschild Metric

By finding the roots of $g_{00}(r_{\mathrm{H}}) \equiv f(r_{\mathrm{H}}) = 0$ one gets the horizon radii r_{H}. For this metric (2.13) we have the following horizon structure [IMN13], as can be seen in figure 2.3:

We distinguish three cases depending on the mass M of the black hole relative to M_0 which is supposed to be of the order of the Planck mass, we call it the "extremal mass". Note that $M_0(\beta)$ generally depends on the variable β of our theory. Let us consider the case $\beta = 1\ell_{\mathrm{p}}^2$ and denote $M_0 \equiv M_0(\beta = 1\ell_{\mathrm{p}}^2)$. For larger values of β the extremal mass $M_0(\beta)$ will be bigger.

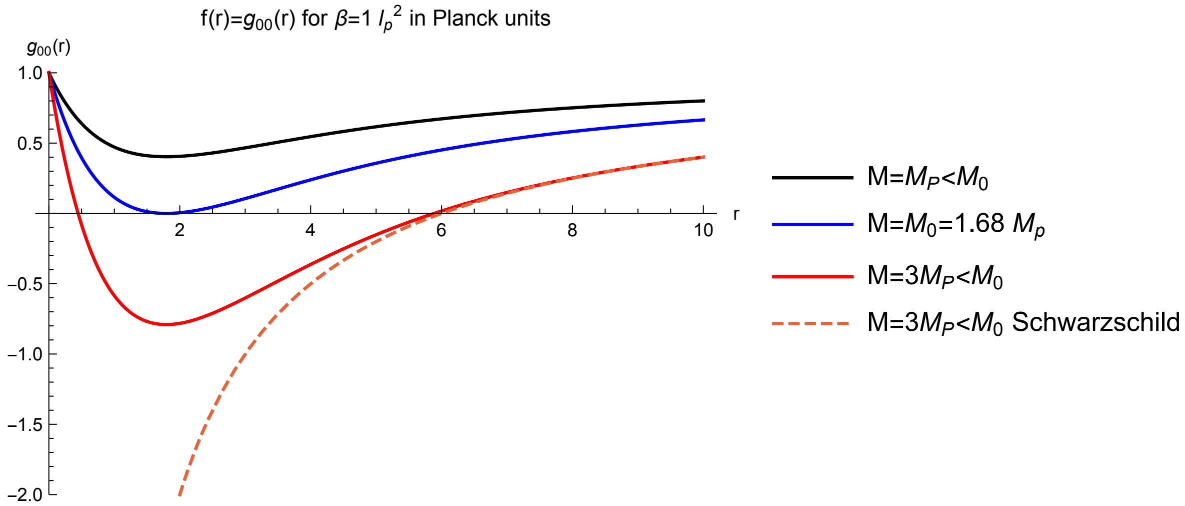


Figure 2.3: The three cases for horizons of the GUP metric 2.13. Here $f(r) \equiv g_{00}(r)$ is shown for $\beta = 1\ell_{\mathrm{p}}^2$.

1. **Two horizons**: For $M > M_0$ there are two horizons. When considering large black holes $M \gg M_0$ the outer radius r_+ converges to the standard Schwarzschild radius $R_{SBH} = 2GM/c^2$ and the inner one goes to zero.

2. **Extremal solution**: For $M = M_0$ the two horizons coincide and give a degenerate horizon at $r_+ = r_- = r_0$. This is an extremal black hole and will be important in the following.

3. **No horizon** For $M < M_0$ the function $f(r)$ has no roots and so there is no horizon. This would correspond to a naked singularity at the origin, but — as will be shown below — this case will never happen when starting with a black hole with mass $M > M_0$ and letting evaporate the black hole through *Hawking Radiation*.

2.5 GUP Black Hole Thermodynamics

Since nothing can escape from inside a black hole, it might be surprising to talk about the *temperature of a black hole*. People started thinking about it when they were considering rotating black holes described by the Kerr metric. One can derive (c.f. [Car03, pp. 267–272]) the equation

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega_{\mathrm{H}}\delta J\,, \tag{2.17}$$

with

- the mass M of the black hole,
- the surface gravity $\kappa = \left.\frac{\mathrm{d}\,f(r)}{\mathrm{d}r}\right|_{r=r_{\mathrm{H}}}$,
- the horizon area A,
- the horizon angular velocity Ω,
- and the angular momentum J.

Comparing eq.(2.17) with the first law of thermodynamics

$$\mathrm{d}E = T\,\mathrm{d}S - p\,\mathrm{d}V\,, \tag{2.18}$$

one can make the identifications

$$E = M\,, \tag{2.19}$$

$$S = \frac{A}{4G}\,, \tag{2.20}$$

$$T = \frac{\kappa}{2\pi}\,. \tag{2.21}$$

The last term in eqs.(2.17) and (2.18) can be related to the *Penrose effect*, but are of no further interest to us. Stephen Hawking [Haw74] derived the same temperature by doing Quantum Field Theory in curved spacetime, it is also called *Hawking Bekenstein temperature*, after Jacob Bekenstein. One interpretation is that in the vacuum surrounding the black hole, virtual pairs of particles emerge and after a very short time annihilate themselves. If one of the virtual particles falls into the black hole, the other one becomes real and escapes, flying away as Hawking radiation.

So the black hole temperature is proportional to the surface gravity and a calculation gives

$$T(r_+) = \frac{\kappa}{2\pi} = \frac{1}{4\pi}\left.\frac{\mathrm{d}\,f(r)}{\mathrm{d}r}\right|_{r=r_+}\,, \qquad f(r) = g_{\mathrm{rr}}^{-1} = g_{00}\,,$$

$$T(r_+) = \frac{\hbar c}{4\pi r_+}\left(1 - \frac{r_+^2}{\beta}\,\frac{e^{-r_+/\sqrt{\beta}}}{\gamma(2; r/\sqrt{\beta})}\right). \tag{2.22}$$

The 1 in the brackets in equation (2.22) corresponds to the standard Hawking temperature and the second term is the GUP correction. Note that the temperature of the GUP black hole reaches zero for the black hole radius $r_{+,0} \approx 1.793\sqrt{\beta}$ (see figure 2.4, blue dot), i.e. the black hole only radiates until it approaches this size. This corresponds to the extremal black hole mentioned in the previous section and shown in figure 2.3 (blue line). Here we see a first sign that GUP-inspired Schwarzschild black holes are self complete, i.e. even though the GUP fails to cure the singularity at the origin, this point cannot be probed by watching a black hole evaporate.

The resulting *cold black hole remnants* of about the Planck mass $m_\mathrm{P} \approx 22\,\mu\mathrm{g}$ could be possible candidates for *dark matter*. Dark matter is one of the biggest challenges for modern astrophysics and cosmology. One of the best approaches for them is the concept of *weakly interacting massive particles* (WIMPs) [JKG96]. Those particles would only interact through the weak coupling and gravity. Zero temperature remnants are candidates for WIMPs.

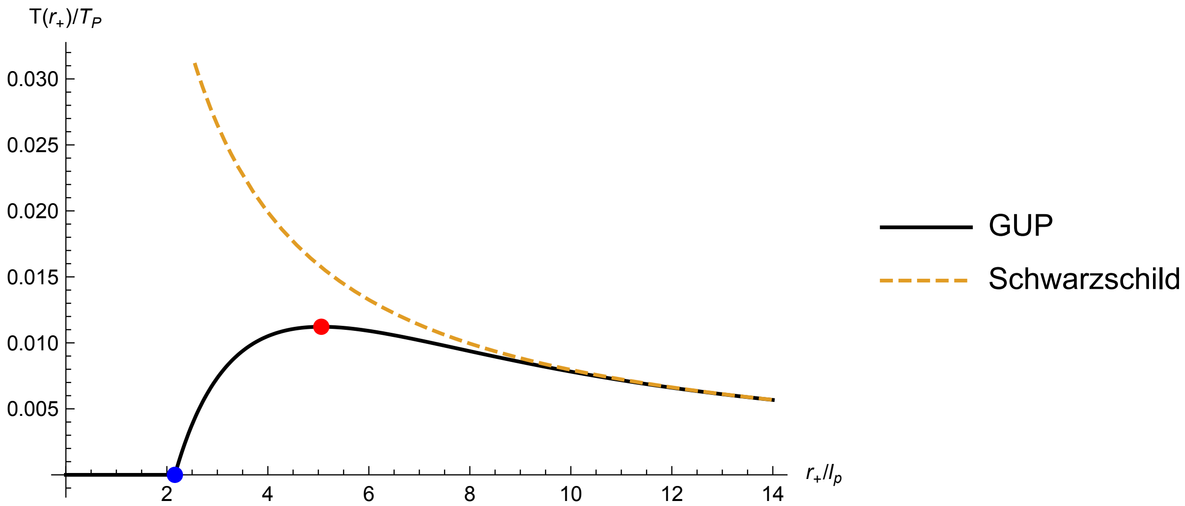


Figure 2.4: Temperature of the GUP black hole in $n = 3$ spatial dimensions as a function of the outer horizon radius according to eq. (2.22) and for comparison the Hawking temperature (dashed). The blue dot marks the cold remnant at $r_{+,0} \approx 1.793\sqrt{\beta}$ and the red dot marks the maximum of the temperature at $r_{+,\max} \approx 4.201\sqrt{\beta}$ and $T_{\max} \approx 9.34 \cdot 10^{-3} T_P$

2.6 Black Holes as New Phase of Matter and Self-Completeness

As shown in section 2.4 in the GUP-inspired Schwarzschild case a black hole evaporates until it stops and becomes a cold remnant of about the Planck mass and length. While evaporating the two horizons get closer and finally merge into one degenerate horizon corresponding to the extremal case with vanishing temperature. But when inspecting the *self-completeness* of our solution we have to also consider the case where $M < M_0$, i.e. the mass for which there is no solution to the equation $g_{00}(r_H) = 0$. By self-completeness we mean, that the ultraviolet regime where the curvature singularity arises and the (standard) black hole temperature diverges, is protected by QG-effects and cannot be accessed.

The basic idea here is to distinguish between quantum mechanical particles and black holes [SA11]. The *Compton wavelength* $\lambda_\mathrm{C} = {}^{2\pi\hbar}/_{cM_{QM}}$ is a good candidate for the 'size' of a QM particle, while a black hole is characterized by its *horizon radius* $r_H = {}^{2GM_{BH}}/_{c^2}$ (ordinary Schwarzschild). Basically one could probe down to arbitrarily small lengths with QM particles by just increasing its energy. If λ gets too small (energy too big) the *hoop conjecture* as mentioned in section 1.2 applies and the particle collapses into a black hole, because it is squeezed into a region whose extend in every direction is bounded by $2R_\mathrm{S}$. Giving the black hole more energy or mass just increases its radius, so we end up with a minimal length when the transition from particle to black hole appears. Setting $\lambda = r_H$ and $M_{QM} = M_{BH}$ we arrive at the Plack length (up to a factor or order $\mathcal{O}(1)$) for this minimal length.

For the GUP case the same can be done [IMN13]. First the *mass parameter* $M \equiv M_{BH}(r_+)$ is defined and the extremal configuration occurs at r_0 when its derivate vanishes

$$\left.\frac{\mathrm{d}M(r_+)}{\mathrm{d}r_+}\right|_{r_+=r_0} = 0\,. \tag{2.23}$$

Equaling the minimal Schwarzschild radius with the Compton wavelength

$$r_0 \overset{!}{=} \lambda_\mathrm{C} = \frac{2\pi\hbar}{cM_{BH}(r_0)} \tag{2.24}$$

identifies the transition point. The next step is to switch to dimensionless quantities $m_0 \equiv M_0 G/\sqrt{\beta}c^2$ and $x_0 \equiv r_0/\sqrt{\beta}$. Then one solves equation (2.23) and requires (2.24).

The numerical results are [IMN13]

$$\begin{aligned} &x_0 \approx 1.79, \quad m_0 \approx 1.68, \quad \sqrt{\beta} \approx 1.45\ell_p\,, \\ &r_0 \approx 2.59\ell_p, \quad M_0 \approx 2.42 M_p\,. \end{aligned} \tag{2.25}$$

Looking at figure 2.5 we see r_0 and M_0 and realize that the GUP is self-complete since the gray area and with it the singularity are excluded from any measurements.

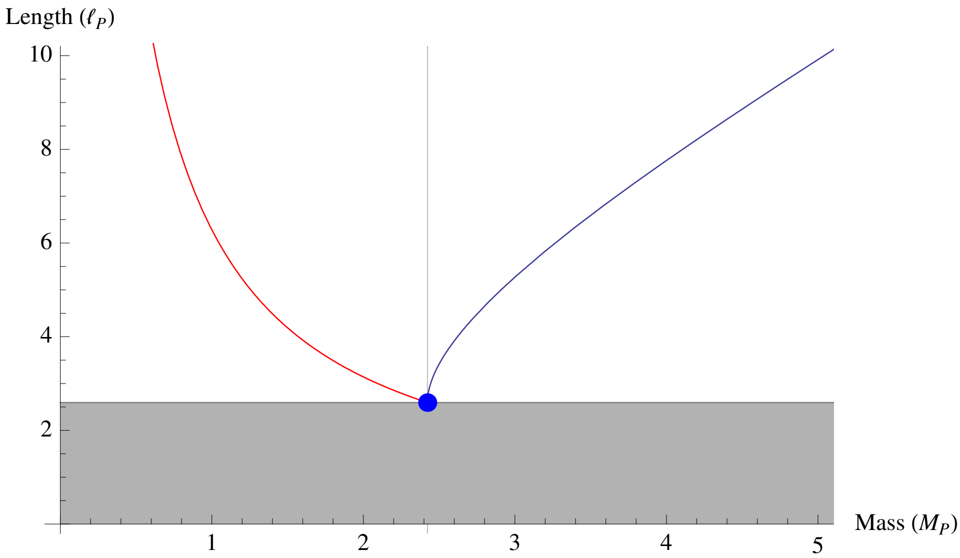


Figure 2.5: Plot of length as a function of mass ($\sqrt{\beta} = 1.45\ell_p$) showing the self-completeness of the GUP. At the blue dot (M_0,r_0) the transition between QM particle (Compton wavelength, red) and black hole (horizon radius, blue) takes place. The gray area can never be probed by experiment and so the singularity, i.e. GUP is self-complete (plot taken from [IMN13]).

Chapter 3

Higher Dimensional GUP

3.1 Higher Dimensional GUP-Inspired Schwarzschild Metric

We choose the ADD [ADD98] model for extra dimensions which is an endeavour to explain the hierachy problem, i.e. the large gap between the electroweak scale and the scale of gravity. This is done by lowering the effective Planck mass by allowing gravity to "leak" into the $n+1$ dimensional "bulk". The Standard Model fields must stay on a $3+1$ dimensional manifold ("the brane"). The spatial extra dimensions are compactified, but they are supposed to be big enough that for our purposes space in all directions looks symmetric and $n+1$ dimensional. For more details see [ADD98] and for a pedagogical introduction [Shi10].

In order to find the $n+1$-dimensional version of the GUP-inspired Schwarzschild metric (2.13) we follow basically the same steps as before (remember that n is the actual number of spatial dimensions, so we have $n+1$ dimensions and $n-3$ large extra dimensions):

The EFEs will essentially look the same as in equation (2.1) with the modification that the indices now are uppercase (g_{MN}) and go from 0 to n. So we will be able to get the metric by generalizing equations (2.11) and (2.12). That means in the Schwarzschild-Tangherlini metric (1.27) the mass M will be replaced by by the mass function $\mathcal{M}(r)$. The integration measure becomes [KMM95]

$$1 = \int \frac{\mathrm{d}^n \vec{p}}{1+\beta \vec{p}^2} \left|p\right\rangle \left\langle p\right| , \tag{3.1}$$

and similarly $\mathcal{A}^{-2}(\Box)$ acting on the delta-source will give as in equation (2.8)

$$\mathcal{A}^{-2}(\Box)\,\delta(\vec{x}) = \frac{1}{(2\pi)^n} \int \frac{\mathrm{d}^n \vec{p}}{1+\beta \vec{p}^2} e^{i\vec{x}\cdot\vec{p}} \,. \tag{3.2}$$

We calculate the energy density $\mathcal{T}_0^0$

$$\begin{aligned}\mathcal{T}_0^0 &= M\frac{1}{(2\pi)^n}\int\frac{\mathrm{d}^n\vec{p}\,e^{i\vec{x}\cdot\vec{p}}}{1+\beta\vec{p}^2} = M\frac{1}{(2\pi)^n}\int\frac{\mathrm{d}^n\vec{p}\,e^{-i\vec{x}\cdot\vec{p}}}{1+\beta\vec{p}^2}\\ &= M\frac{1}{(2\pi)^n}\int\mathrm{d}^n\vec{p}\int_0^\infty\mathrm{d}s\,e^{-s-s\beta\vec{p}^2}e^{-i\vec{x}\cdot\vec{p}}\\ &= M\frac{1}{(2\pi)^n}\int_0^\infty\mathrm{d}s\,e^{-s}\int\mathrm{d}^n\vec{p}\exp\left(-s\beta\vec{p}^2-i\vec{x}\cdot\vec{p}\right).\end{aligned}\tag{3.3}$$

At this point a common integral in QFT comes in handy [PS95, pp. 286–287]

$$\int\mathrm{d}^n\vec{x}\exp\left(-\frac{1}{2}\vec{x}\mathbf{A}\vec{x}-\vec{b}\cdot\vec{x}\right) = \left(\det\frac{\mathbf{A}}{2\pi}\right)^{-1/2}\exp\left(\frac{1}{2}\vec{b}\mathbf{A}^{-1}\vec{b}\right),\tag{3.4}$$

where $\mathbf{A}$ is a non-singular, symmetrical $n\times n$ matrix (here $\mathbf{A}=2s\beta\mathbb{1}_{n\times n}$) and $\vec{b}$ is a vector (here $\vec{b}=i\vec{x}$). Using this formula for equation (3.3) we get

$$\mathcal{T}_0^0 = M\frac{1}{(2\pi)^n}\left(\frac{\pi}{s\beta}\right)^{n/2}\int_0^\infty\mathrm{d}s\,s^{-n/2}\exp\left(-\frac{\vec{x}^2}{4\beta s}-s\right).\tag{3.5}$$

Here we use the integral [GR07, p. 370]

$$\int_0^\infty\mathrm{d}s\,s^{-n/2}\exp\left(-\frac{\vec{x}^2}{4\beta s}-s\right) = 2^{n/2}\left(\frac{\beta}{\vec{x}^2}\right)^{\frac{n}{4}-\frac{1}{2}}\cdot K_{\frac{n}{2}-1}\left(\frac{\|\vec{x}\|}{\sqrt{\beta}}\right),\tag{3.6}$$

with the *modified Bessel function of the second kind* K [Bro+07, p. 509] (sometimes called *Basset function* or *Macdonald function*). So the energy density for a delta-source in the GUP is given by

$$\mathcal{T}_0^0(r) = M\frac{1}{(2\pi)^{n/2}}\beta^{-\frac{n}{4}-\frac{1}{2}}r^{1-\frac{n}{2}}\cdot K_{\frac{n}{2}-1}\left(\frac{r}{\sqrt{\beta}}\right).\tag{3.7}$$

This is of course sphericaly symmetric, so in order to calculate the mass term $\mathcal{M}$ we use n-dimensional spherical coordinates

$$\begin{aligned}\frac{\mathcal{M}(r)}{M} &= \frac{1}{M}\int\mathrm{d}^n\vec{x}\,\mathcal{T}_0^0(\vec{x}) = \frac{1}{M}A_{n-1}\int_0^r\mathrm{d}\xi\,\xi^{n-1}\mathcal{T}_0^0(\xi)\\ &= \frac{2\pi^{n/2}}{\Gamma(n/2)}\frac{1}{(2\pi)^{n/2}}\beta^{-\frac{n}{4}-\frac{1}{2}}\int_0^r\mathrm{d}\xi\,K_{n/2}\left(\frac{\xi}{\sqrt{\beta}}\right)\xi^{\frac{n}{2}}.\end{aligned}\tag{3.8}$$

For this integral we use the formula [GR07, p. 676]

$$\int_0^r\mathrm{d}\xi\,K_{n/2}\left(\frac{\xi}{\sqrt{\beta}}\right)\xi^{\frac{n}{2}} = -\sqrt{\beta}\,r^{n/2}K_{n/2}\left(\frac{r}{\sqrt{\beta}}\right)+2^{\frac{n}{2}-1}\beta^{\frac{n}{4}+\frac{1}{2}}\Gamma(n/2),\tag{3.9}$$

and end up with

$$\frac{\mathcal{M}(r)}{M} = 1 - \frac{2^{1-\frac{n}{2}}}{\Gamma(n/2)} \beta^{-n/4} r^{n/2} K_{n/2}\left(\frac{r}{\sqrt{\beta}}\right) . \tag{3.10}$$

The n-dimensional GUP inspired Schwarzschild-Tangherlini metric thus is given by the line element

$$\boxed{\begin{aligned} \mathrm{d}s^2 &= -f(r)\,\mathrm{d}t^2 + f^{-1}(r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 \,, \\ f(r) &= 1 - \frac{16\pi G}{(n-1)A_{n-1}} \frac{M}{r^{n-2}} \left[1 - \frac{2^{1-\frac{n}{2}}}{\Gamma(n/2)} \beta^{-n/4} r^{n/2} K_{n/2}\left(\frac{r}{\sqrt{\beta}}\right)\right] . \end{aligned}} \tag{3.11}$$

3.2 Checking the Limits for β and $n = 3$

In order to test if the above results make sense, it is a good idea to check the limits:

- The limit $n = 3$ should give the same results as in chapter 2.
- The limit $\beta \to 0$ should give the Schwarzschild-Tangherlini metric (1.27).

If we set $n = 3$ and put these two identities

$$K_{3/2}\left(\frac{r}{\sqrt{\beta}}\right) = \sqrt{\frac{\pi}{2}} \frac{\beta^{1/4} e^{-r/\sqrt{\beta}} \left(1 + \frac{\sqrt{\beta}}{r}\right)}{\sqrt{r}} , \tag{3.12}$$

$$\Gamma(3/2) = \frac{\sqrt{\pi}}{2} , \tag{3.13}$$

into formula (3.10) we get

$$\begin{aligned} \frac{\mathcal{M}(r)}{M}(n=3) &= 1 - \frac{2^{1-3/2}}{\sqrt{\pi}/2} \frac{e^{-r/\sqrt{\beta}}}{\sqrt{r}} \left(1 + \frac{\sqrt{\beta}}{r}\right) \beta^{-3/4} r^{3/2} \sqrt{\frac{\pi}{2}} \beta^{1/4} \\ &= 1 - r\frac{e^{-r/\sqrt{\beta}}}{\sqrt{\beta}} - e^{-r/\sqrt{\beta}} = \gamma(2; r/\sqrt{\beta}), \end{aligned} \tag{3.14}$$

which is in agreement with the IMN-model in chapter 2, see equation (2.10).

For the limit $\beta \to 0$ we need to calculate

$$\lim_{\beta \to 0} \beta^{-n/4} K_{n/2}\left(\frac{r}{\sqrt{\beta}}\right) . \tag{3.15}$$

Here the following approximation [Bro+07, p. 511] for large arguments of K can be used

$$K_\alpha(z) \approx \sqrt{\frac{\pi}{2z}}\, e^{-z} \qquad \text{for large } z\,. \tag{3.16}$$

If we put this into equation (3.15), we get the result

$$\lim_{\beta\to 0} \beta^{-n/4} K_{n/2}\left(\frac{r}{\sqrt{\beta}}\right) = \lim_{\beta\to 0} \beta^{\frac{1}{4}-\frac{n}{4}} e^{-r/\sqrt{\beta}} = 0 \tag{3.17}$$

and therefore equation (3.10) becomes

$$\lim_{\beta\to 0} \frac{\mathcal{M}(r)}{M} = 1\,. \tag{3.18}$$

This is what we want, since it means, that for the limit $\beta \to 0$ or for large distances from the black hole our theory reduces to the standard GR black hole.

3.3 Metric Behavior

In figure 3.1 a plot of the function $f(r)$ is given for different n. For $n = 4$ the limit $\lim_{r\to 0} f(r)$ has another value than for $n = 3$, but is still finite. Plots for $n > 6$ look qualitatively the same as for $n = 6$, so they are not shown in this figure. The problem is,

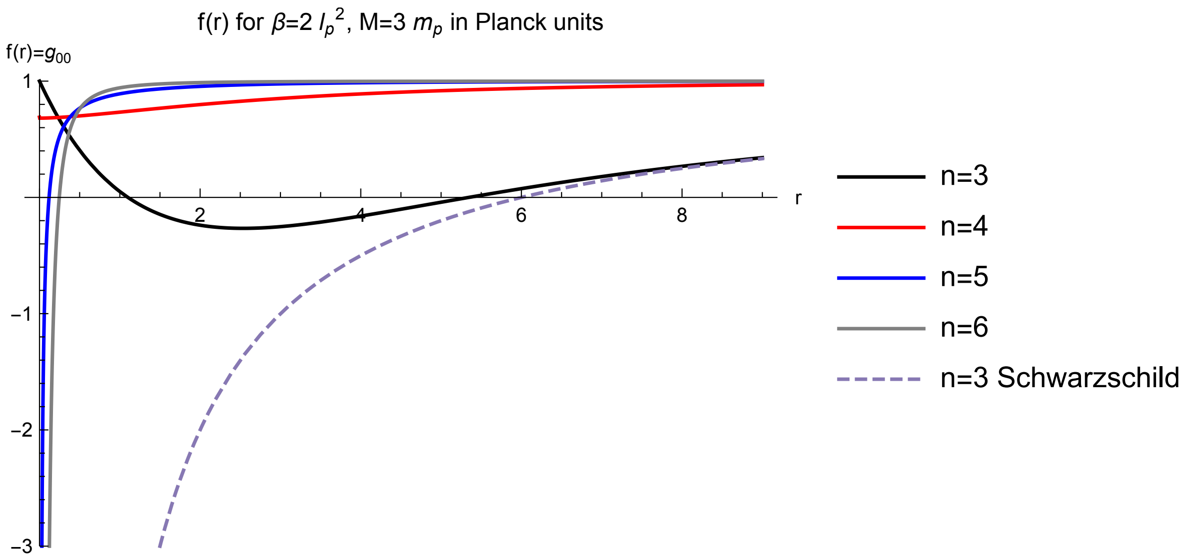


Figure 3.1: The function $f(r) \equiv g_{00}(r)$ for different spatial dimensions n

that for $n > 4$ the function $f(r)$ diverges for vanishing r like in the Schwarzschild case. As can be seen in figure 3.2 for the Mass $\mathcal{M}(r)$ it is still OK, but then when we go to $f(r)$ we get another factor r^{1-n} which causes the divergence. To see why this happens, we have to go back to equation (3.2), namely the action of $\mathcal{A}^{-2}(\Box)$ on the delta source $\mathcal{A}^{-2}(\Box)\,\delta(\vec{x}) = (2\pi)^{-n} \int \frac{\mathrm{d}^n\vec{p}}{1+\beta\vec{p}^2} e^{i\vec{x}\cdot\vec{p}}$. Roughly speaking the factor $\vec{p}^2$ in the denominator is not big enough to suppress the $\mathrm{d}^n\vec{p}$ as much as needed.

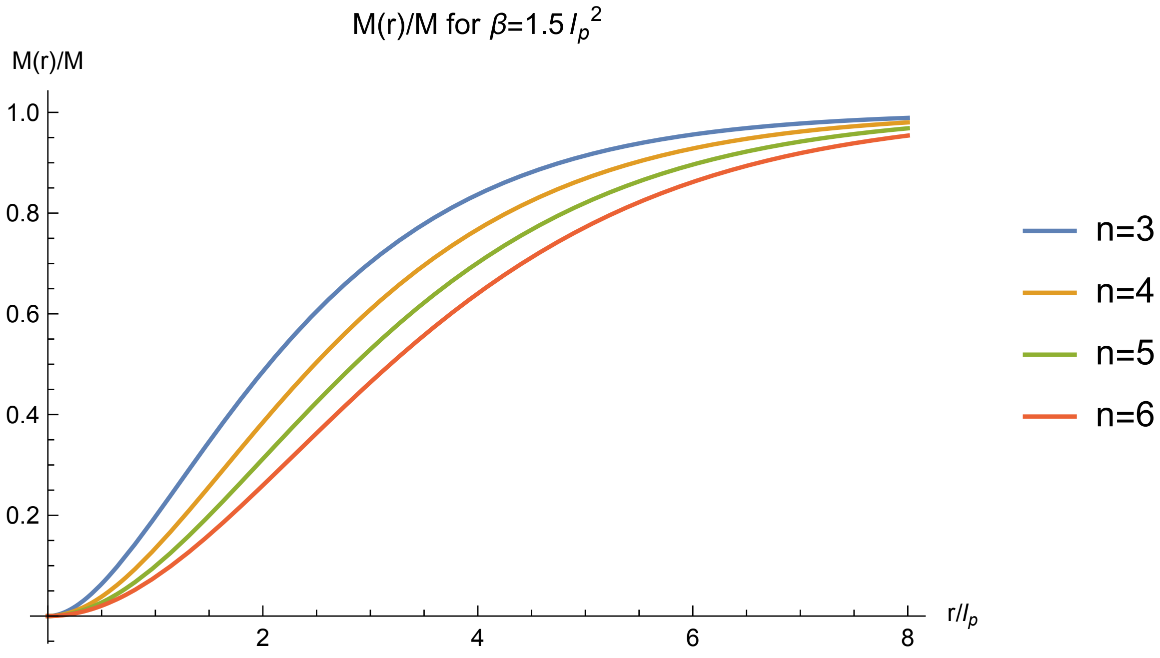


Figure 3.2: The mass $\mathcal{M}(r)$ inside r for different spatial dimensions n

We can also see the divergence by a Taylor expansion of $f(r)$ around $r = 0$

$$f(r) = r^{-n}\mathcal{O}(r)^4 + 1 + \frac{n2^{3-n}\pi^{1-\frac{n}{2}}r^2\beta^{-n/2}\Gamma\left(-\frac{n}{2}\right)}{n-1} + \mathcal{O}\left(r^4\right). \tag{3.19}$$

The first term $r^{-n}\mathcal{O}(r)^4$ is the one that diverges for $r \to 0$ when we put $n > 4$. For $n = 3$ the r-terms go to zero and we end up with 1. In figure 3.1 the $n = 4$ line does not go to 1 for vanishing r, since $r^{-4}\mathcal{O}(r^4) = \mathcal{O}(1)$ which gives an offset. The $n = 4$ case will be discussed in detail later.

3.4 Black Hole Temperature and Self-Completeness

From the surface gravity

$$\kappa = \frac{1}{2}\left.\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right|_{r=r_\mathrm{H}}, \quad f(r) = g_{\mathrm{rr}}^{-1}, \tag{3.20}$$

the *Black Hole temperature* $T_{BH} = \frac{\kappa}{2\pi}$ due to the *Hawking effect* can be obtained. The temperature is

$$T_n(r) = \frac{1}{4\pi}\frac{\mathrm{d}f(r)}{\mathrm{d}r}\bigg|_{r=r_\mathrm{H}} =$$

$$= \frac{1}{4\pi r}\left\{(n-2) - \frac{r^{\frac{n}{2}+1}}{\Gamma\left(\frac{n}{2}\right)2^{\frac{n}{2}}\beta^{\frac{n}{4}}}\cdot\frac{-\frac{n}{r}K_{n/2}\left(\frac{r}{\sqrt{\beta}}\right)+\beta^{-\frac{1}{2}}\left[K_{n/2-1}\left(\frac{r}{\sqrt{\beta}}\right)+K_{n/2+1}\left(\frac{r}{\sqrt{\beta}}\right)\right]}{1-\frac{2^{1-\frac{n}{2}}r^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)\beta^{\frac{n}{4}}}K_{n/2}\left(\frac{r}{\sqrt{\beta}}\right)}\right\}. \tag{3.21}$$

For n=3 the denominator is the incomplete gamma function

$$1 - \frac{2^{1-n/2}r^{n/2}}{\Gamma\left(\frac{n}{2}\right)\beta^{n/4}}K_{n/2}\left(\frac{r}{\sqrt{\beta}}\right)\bigg|_{n=3} = \gamma\left(2;\frac{r}{\sqrt{\beta}}\right) \tag{3.22}$$

and the numerator and the prefactor become $\frac{r^2}{\sqrt{\beta}}e^{-r/\sqrt{\beta}}$, so we get the same as in the $3+1$ dimensional case in eq. (2.22). In figure 3.3 a plot of the temperature $T_n(r_H)$ can be seen for different spatial dimensions n.

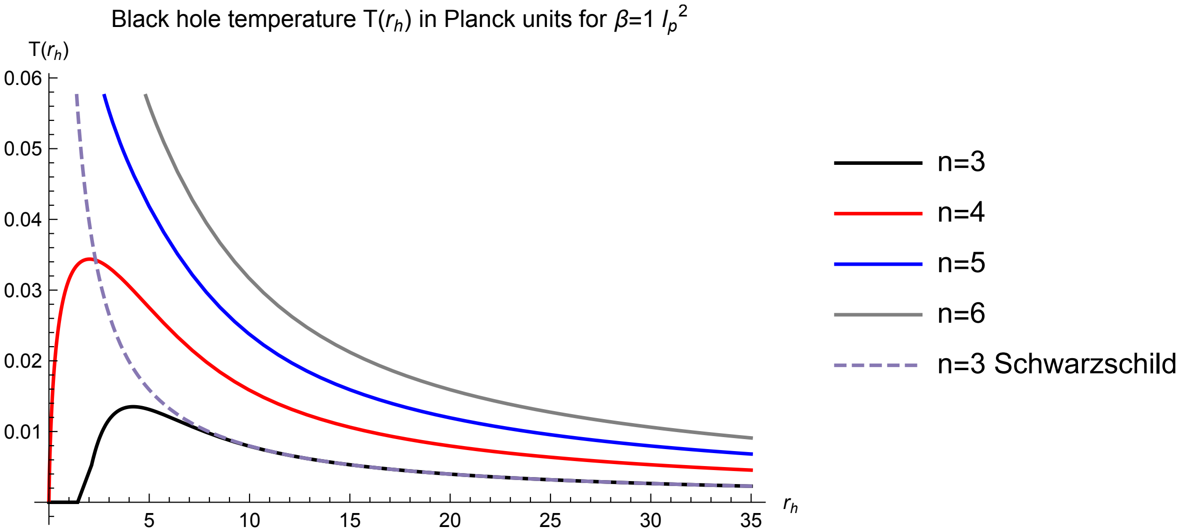


Figure 3.3: Black hole temperature as a function of the schwarzschild radius r_H

Depending on n the horizon structure changes:

- For $n = 3$ the horizon structure and self-completeness are as in section 2.4.
- For $n = 4$ the structure is different and will be discusses separately below.

- For $n \geq 4$ the function g_{00} as well as the temperature T do not look very different from the 3+1-dimensional Schwarzschild case. This behavior is exactly what was supposed to be cured by the GUP. Apparently this fails for higher dimensions. The metric no longer admits remnants and also it not self-complete.

The case n=4 The plot of the temperature, see the red line in figure 3.3, has the same form as in the 3+1-dim case, but with two differences: $T(r_{\mathrm{H}})$ becomes zero only for $r_{\mathrm{H}} = 0$, this is independent of β. The second difference is that now the maximum occurs at $r_{max} \approx 2.03\sqrt{\beta}$ (3+1-dim: $r_{max} \approx 4.20\sqrt{\beta}$). There exists a black hole remnant, because the temperature vanishes at one point. The horizon structure is different, too, see figures 3.4 and 3.4: Depending on β and M there are three cases:

1. **No horizon**: The function $f(r)$ stays positive and there is no root, see the blue lines in figures 3.4 and 3.5.
2. **One horizon**: There is one root corresponding to the green lines in the above mentioned figures. Note that $f(r)$ is negative inside the horizon, including at the origin. In figure 3.6 the horizon radius has been calculated numerically as a function of M for different values of β.
3. **"Extremal case"**: The blue lines show what we might call the "extremal case", but this one is different from the usual extremal case. It is when $f(0) = 0$, i.e. the horizon is at the origin $r_{\mathrm{H}} = 0$, corresponding to the vanishing temperature mentioned above.

Thus the evaporation process in this case would be the following:

For every β a mass M_0 exists such that for $M > M_0$ there is exactly one horizon. Starting with $M > M_0$, a black hole with one horizon, the black hole emits Hawking radiation according to equation (3.21) and loses mass. It does this until it reaches zero temperature and a vanishing horizon radius, namely the "extremal case". It still has mass (see figure 3.5) and the mass is distributed according to equation (3.10) and figure 3.2. So while radiating its mass away, at some point the horizon of the black hole will be smaller than the planck length l_{p}, again something we cannot measure and we do not know if distances below this length are physically meaningful.

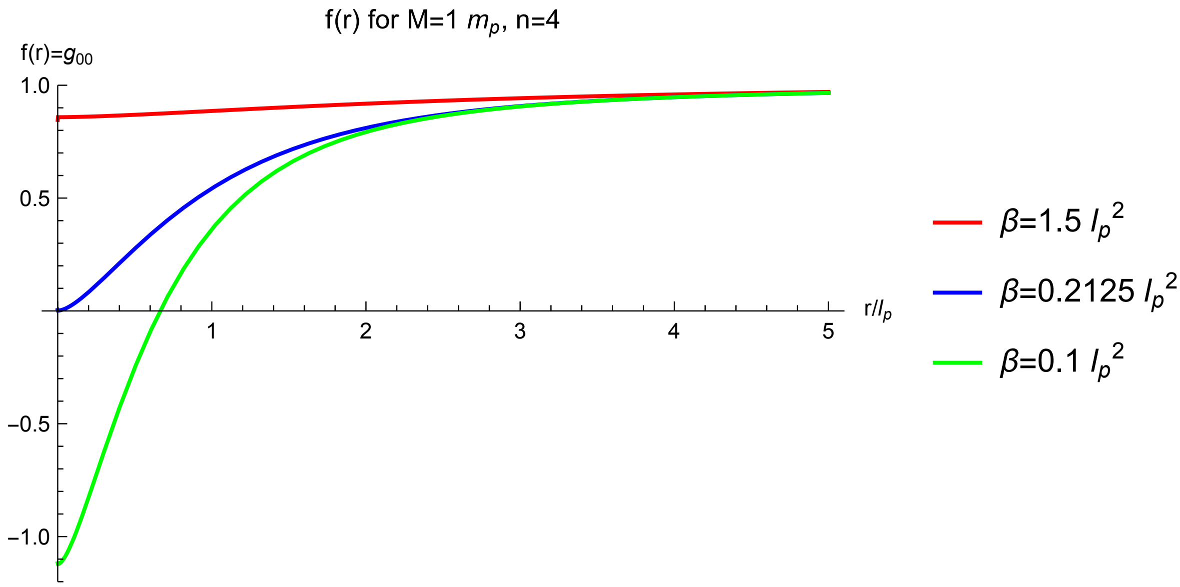


Figure 3.4: The function $f(r)$ for $n = 4$ and different values of β and $M = 1m_\mathrm{p}$

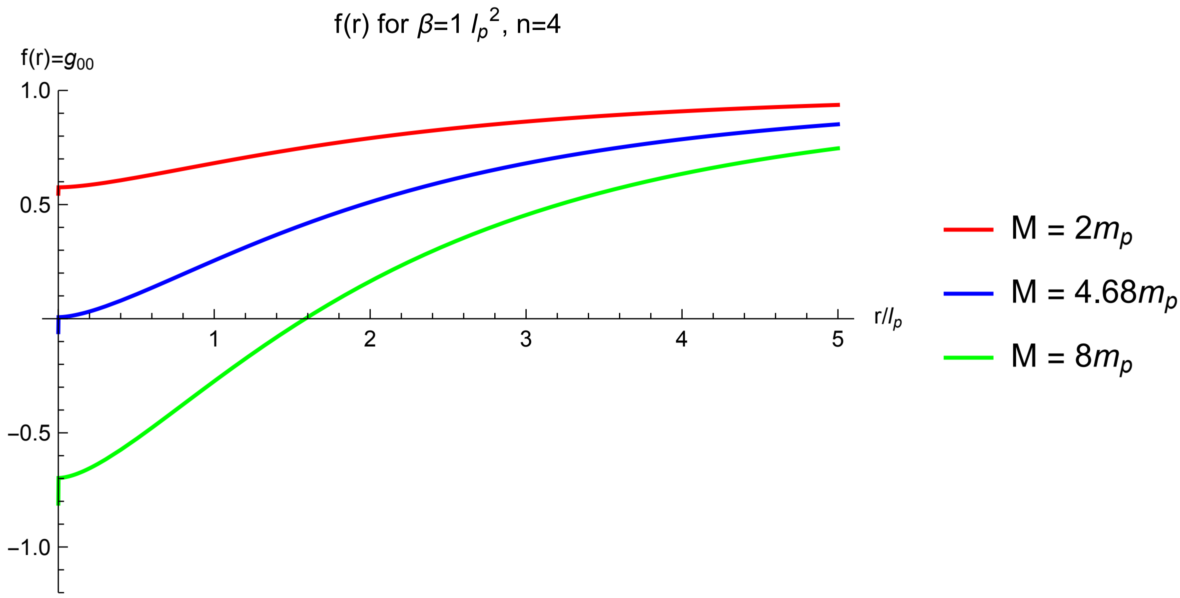


Figure 3.5: The function $f(r)$ for $n = 4$ and different values of M (in units of the Planck mass) and $\beta = 1\ell_\mathrm{p}^2$

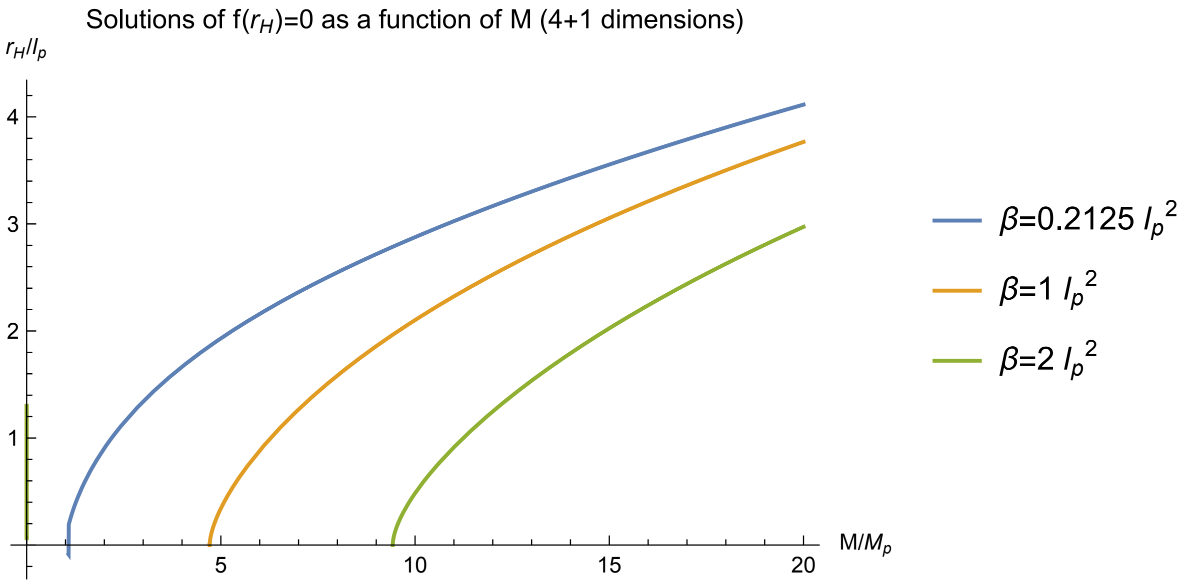


Figure 3.6: Plots for different β of the numerical calculation of the horizon radius $r_{\mathrm{H}}(M)$ solving the equation $f(r_{\mathrm{H}}) = 0$.

Chapter 4

Conclusion and Outlook

We have shown that in ordinary $3+1$ dimensional spacetime the generalized uncertainty principle has the following features:

- **Not quite UV-finite**: Although the curvature singularity at the origin is not removed it is smoothened such that g_{00} no longer diverges, but the curvature is still infinite.

- **Regular thermodynamics**: The Hawking temperature no longer goes to infinity for vanishing black hole masses. The black hole even stops evaporating when it reaches the extremal case and forms remnants, which can be possible candidates for WIMPs.

- **Self-completeness**: Dividing matter into two phases — quantum mechanical particles and black holes — provides us with the property that the GUP is self-complete, i.e. it forbids experimental measurements of the singularity that still exists at the origin.

Our results greatly improve the *hot black hole remnants* by [ACS01]. The GUP-modified black hole temperature arises from a surface gravity which can be computed from the metric eq.(2.13) and this temperature vanishes for the remnant.

As shown in chapter 3.3 the GUP in n dimensions does not give the nice results as in 3 space dimensions. The problem arises when performing the modified Fourier transform of the delta source, which gives the energy density. In order to derive $\mathcal{M}(r)$ one has to integrate this energy density to a distance r from the center. Even this $\mathcal{M}$ looks fine, but the prefactor it gets in $f(r)$ causes divergences for vanishing r and $n > 4$. Also the black hole thermodynamics no longer is regular, the temperature diverges in $n > 4$ for small black holes, which is a very bad sign and tells us that something is not right.

An ansatz for solving this problem is to adjust the GUP in higher dimensions such that equation (3.2)

$$\mathcal{A}^{-2}(\Box)\,\delta(\vec{x}) = (2\pi)^{-n} \int \frac{\mathrm{d}^n\vec{p}}{1+\beta\|\vec{p}\|^2} e^{i\vec{x}\cdot\vec{p}} \tag{4.1}$$

changes to

$$\mathcal{A}^{-2}(\Box)\,\delta(\vec{x}) = (2\pi)^{-n}\int \frac{\mathrm{d}^n\vec{p}}{1+\beta^{\frac{n-1}{2}}\|\vec{p}\|^{n-1}} e^{i\vec{x}\cdot\vec{p}}\,. \tag{4.2}$$

This gives the same in 3 space dimension, but changes the behavior in higher dimensions. It basically means that the commutator (1.14) is changed from $[\hat{x}^i,\hat{p}_j] = i\hbar\delta^i_j\left(1+\beta\vec{p}^2\right)$ to

$$[\hat{x}^i,\hat{p}_j] = i\hbar\delta^i_j\left(1+\beta^{\frac{n-1}{2}}\vec{p}^{n-1}\right)\,. \tag{4.3}$$

This form of higher dimensional GUP is current work and going to be published soon in a forthcoming paper [Isi+ss]. Is this modifiaction of the GUP compatible with string theory? One could also check the cases for black holes with charge and/or angular momentum.

The generalized uncertainty principle is a way to "extrapolate" our theories of gravity and quantum mechanics to a theory of quantum gravity. By implementing the non-commutativity of $\hat{x}$ and $\hat{p}$ and with it a minimal length we already see significant changes in black hole physics. These changes indicate what the correct solution to problems like "what happens at the end stage of black hole evaporation" could be. Keep in mind that we do not possess a *theory of everything* and only this theory can give a final answer. The GUP is not a theory with radical new concepts like string theory or loop quantum gravity, but it is motivated by them. Of course, finally the GUP is only a reasonable theory if its predictions can be verified by experiment. Possible observation windows are: Black hole remnants [BN14], GUP corrections to the Casimir effect [FP12] and neutrino oscillation [SNB12], as well as quantum optics [Pik+12].

List of Figures

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