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Symmetric Markovian Games of Commons with Potentially Sustainable Endogenous Growth ⋆

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Symmetric Markovian Games of Commons with Potentially Sustainable Endogenous Growth

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Abstract

Differential games of common resources that are governed by linear accumulation constraints have several applications. Examples include political rent-seeking groups expropriating public infrastructure, oligopolies expropriating common resources, industries using specific common infrastructure or equipment, capital-flight problems, pollution, etc. Most of the theoretical literature employs specific parametric examples of utility functions. For symmetric differential games with linear constraints and a general time-separable utility function depending only on the player's control variable, we provide an exact formula for interior symmetric Markovian-strategies. This exact solution, (a) serves as a guide for obtaining some new closed-form solutions and for characterizing multiple equilibria, and (b) implies that, if the utility function is an analytic function, then the Markovian strategies are analytic functions, too. This analyticity property facilitates the numerical computation of interior solutions of such games using polynomial projection methods and gives potential to computing modified game versions with corner solutions by employing a homotopy approach.

Keywords: differential games, endogenous growth, tragedy of the commons, Lagrange-d'Alembert equation, analytic functions

JEL classification: C73, C61, D74, E0, O40, O44

1. Introduction

Markovian differential games of common property resources have far-reaching applications. A substantial literature using such games with linear constraints, focusing on the question of how strategic interactions affect the growth rate of a common-property resource includes Tornell and Velasco (1992), Lane and Tornell (1996), Tornell (1997), Tornell and Lane (1999), Sorger (2005), and Long and Sorger (2006). This literature is surveyed and explained in Long (2010, pp. 130-136).[1] The questions examined by these models are corruption, rent seeking and cross-country capital flight. Similar applications include pollution problems and oligopolies exploiting a common resource.[2] The commons problems arising may lead to slow or negative growth of capital. Instead, resolving such commons problems may guarantee sustainable growth. While these models are useful, the literature restricts itself to parametric models with closed-form solutions. There is a need to develop further results that can serve as guides for developing well-grounded numerical solutions to such models, generalizing these parametric examples and taking the models to data.[3] Here, we first contribute to developing such results for the case of interior solutions of symmetric Markovian games with linear accumulation constraints of a common resource. Second, we further develop a characterization of the general solution that can serve as a guide for extending numerical solutions to addressing parameterizations with corner solutions.

[1] An earlier survey paper in differential games is Clemhout and Wan (1994). A recent paper by Kunieda and Nishimura (2018), extends the Tornell and Velasco (1992) model by introducing uncertainty and financial constraints. This study examines how commons problems are affected by imperfect financial markets and how the possibility for sustainable growth is affected by these commons problems. Although our model is deterministic, it can contribute to extending such analyses by using more general utility functions.

[2] An early application of Markovian differential games to pollution is Dockner and Long (1993).

[3] Typically, Markovian differential-game models require metric-space or other functional-analysis methods in order to prove that solutions exist, that they are well-behaved, or that they possess certain desirable functional properties. Such approaches are necessitated by the complexity of dynamic programming problems, especially if their constraints are nonlinear. Regarding the approximation-theory difficulties posed by dynamic-programming problems and an exposition of metric-space methods see, for example, Chow and Tsitsiklis (1989). Theoretical foundations of differential games are provided by Basar and Olsder (1999) and Dockner et al. (2000).

We achieve the first goal of the paper, the derivation of an analytical characterization, in two steps. In a first step, restricting attention to Markovian differential games of common property resources with linear accumulation constraints and interior solutions, we provide a full characterization of the interior solution for any time-separable utility function depending only on the player's control variable. We show that the Hamilton-Jacobi-Bellman equation of a player's dynamic problem can be reduced into a form of the Lagrange-d'Alembert differential equation. We show that this differential equation has an exact solution that involves an integration constant. In the case that this constant of integration is equal to zero, the Markovian strategy equals the indefinite integral of an expression involving the inverse function of marginal utility. This solution best characterizes an interior solution of the game, provided that this interior solution exists. For the case where the constant of integration is different from zero, the possibility of multiple equilibria arises.[4] We do not focus on characterizing these multiple equilibria here. Nevertheless, the differential equation that we provide can serve as a guide for either characterizing these equilibria analytically, or for obtaining them numerically.

This exact formula that we derive for the case where the integration constant is zero, allows us to achieve the second goal of this paper. We prove the analyticity of the Markovian exploitation strategies under the assumption that the utility function of players is an analytic function, subject to some weak requirements.

We demonstrate the usefulness of our exact solution through finding some closed-form interior solutions which, in some cases, are not listed in the literature.[5] In addition, we discuss the usefulness of the analyticity result for Markovian strategies. Analyticity can

[4] See, for example, Tsutsui and Mino (1990) and Dockner and Long (1993), who use a similar approach for characterizing multiple Markovian equilibria, but who are restricted to linear quadratic games.

[5] Such solutions can provide insights for other extensions of dynamic games of commons with piece-wise linear constraints such as Colombo and Labrecciosa (2015) or partly-linear/partly- non-linear constraints, such as Benchekroun (2008).

help in employing polynomial projection methods for computing interior solutions and for guiding the parameterization of games in order to guarantee interior solutions. Yet, despite that the scope of our paper is restricted to characterizing games with interior solutions, our results can be useful for pursuing interesting extensions to commons problems involving corner solutions, such as resource-depletion, exploitation quota policies, etc. Specifically, our exact interior solution can provide a starting point for homotopy approaches that lead to corner solutions after gradually changing the parameterization of the problem.

The homotopy computational approach, explained by Garcia and Zangwill (1981) and Eaves and Schmedders (1999), further adapted to dynamic games by, e.g., Borkovsky et al. (2010) and Besanko et al. (2010), starts from a well-behaved and well-characterized solution to a model for certain parameter values. By changing parameter values gradually, one can proceed to more complicated versions of the model. For the common-property applications we have in mind, some parameterizations can imply a well-behaved interior solution and some other parameterizations of the same model can imply a complicated corner solution that is difficult to compute recursively. A key contribution of our paper regarding such a homotopy approach is that it can provide ways to find a well-behaved solution that can serve as the starting point for this method.

The exact or numerical solutions of our setup can also be extended to studying stochastic games numerically. An early paper showing that stochastic Markovian games of common-property resource extraction have tractable continuity properties is Amir (1996).

2. Statement of the problem

There are N identical (symmetric) players consuming a common resource, k. Player $i \in \{1, ..., N\}$ consumes $c_i(t)$ units of $k(t)$ at time $t \geq 0$, and the evolution of the common resource, k, is driven by,

$$\dot{k}(t) = Ak(t) - \sum_{i=1}^{N} c_i(t) \ , \tag{1}$$

with $A > 0$.[6] Each player $i \in \{1, ..., N\}$ is infinitely-lived and maximizes the same utility time-separable utility function,

$$U\left((c_i(t))_{t\geq 0}\right) = \int_0^\infty e^{-\rho t} u(c_i(t))\, dt \ , \quad \text{for all } i \in \{1, ..., N\} \ , \tag{2}$$

with parameters $\rho > 0$ being the rate of time preference.

Assumption 1 *Function $u : \mathbb{C} \to \mathbb{R}$, $\mathbb{C} \subseteq \mathbb{R}_+$, is twice continuously differentiable and has $u'(c) > 0$ and $u''(c) < 0$ for all $c \in \mathbb{C}$.*

We state further assumptions on the momentary utility function, u, as we proceed with our analysis in order to intuitively justify them.[7] We focus on Markovian (memoryless) strategies, $\left(\{c_i(t) = C_i(k(t))\}_{i=1}^N\right)_{t\geq 0}$, i.e., on consumption strategies $\{C_i(k)\}_{i=1}^N$ that are time-invariant.

Definition 1 *A Markov perfect Nash equilibrium (MPNE) is a set of strategies $\{C_i^*(k)\}_{i=1}^N$ such that the corresponding consumption paths $\left(\{c_i^*(t) = C_i^*(k(t))\}_{i=1}^N\right)_{t\geq 0}$ simultaneously solve problems $\{\mathcal{P}_i\}_{i=1}^N$, with $\mathcal{P}_i$ being player i's problem for all*

[6] Notice that we exclude $A = 0$, which is games with non-renewable resources. We focus on games with potentially sustainable resource outcomes.

[7] For example, unlike in many papers, such as in Dockner and Sorger (1996, p. 213), an upper bound is imposed on the consumption level, c, and the resource-reproduction function is also bounded in their study. Here in some cases of sustainable growth, c can grow to infinity. In examples that we present in a later section we identify the cases where an upper bound must be placed on c and cases in which such a bound does not apply.

$i \in \{1, ..., N\}$, given by,

$$\mathcal{P}_i \begin{cases} \max_{(c_i(t),k(t))_{t\geq 0}} \int_0^\infty e^{-\rho t} u\left(c_i\left(t\right)\right) dt \\ \textit{subject to,} \\ \dot{k}\left(t\right) = Ak\left(t\right) - \sum_{j\neq i} C_j^*\left(k\left(t\right)\right) - c_i\left(t\right) \ , \\ c_i\left(t\right) \in \mathbb{C} \ , \ k\left(t\right) \in \mathbb{K} \subseteq \mathbb{R}_+ \\ \textit{given } k\left(0\right) = k_0 > 0 \ , \quad \lim_{t\to\infty} e^{-\rho t} J_i'\left(k\left(t\right)\right) k\left(t\right) = 0 \end{cases}$$

with $J_i(k)$ being the value function of problem $\mathcal{P}_i$, and with

$$\mathbb{K} = \left\{k \geq 0 \ \mid C_i^*\left(k\right) \in \mathbb{C} \ , \ i \in \{1, ..., N\}\right\} \ .$$

Definition 1 is equivalent to Definition 6.6 in Basar and Olsder (1999, Definition 6.6, p. 321) for the case of $T \to \infty$ therein. The Hamilton-Jacobi-Bellman (HJB) equation of problem $\mathcal{P}_i$ is,

$$\rho J_i\left(k\right) = \max_{c_i\in\mathbb{C}} \left\{u\left(c_i\right) + J_i'\left(k\right)\left[Ak - \sum_{j\neq i} C_j^*\left(k\right) - c_i\right]\right\} \quad \text{for all } k \in \mathbb{K} \ , \tag{3}$$

with first-order conditions

$$u'\left(c_i\right) = J_i'\left(k\right) \ . \tag{4}$$

The first concept we focus on is this of interiority of MPNE, given by Definition 2.

Definition 2 *An interior Markov perfect Nash equilibrium (IMPNE) is a set of strategies $\{C_i^*(k)\}_{i=1}^N$ described by Definition 1, with $\left(\{c_i^*(t) = C_i^*(k^*(t))\}_{i=1}^N\right)_{t\geq 0}$ and $(k^*(t))_{t\geq 0} = \left\{k\left(t\right) \in \mathbb{K} \;\middle|\; \dot{k}\left(t\right) = Ak\left(t\right) - \sum_{i=1}^N C_i^*\left(k\left(t\right)\right) \ , t \geq 0 \ , k\left(0\right) = k_0\right\}$, such that for all $t \geq 0$, $k^*(t) \in int(\mathbb{K})$ and $c_i^*(t) \in int(\mathbb{C})$ for all $i \in \{1, ..., N\}$, where $int(\cdot)$ denotes the interior of a set.*

Definition 2 leads to another assumption we make about the problem.

Assumption 2 *Function u is such that there a symmetric IMPNE defined as in Definition 2 is guaranteed.*

We make Assumption 2 in order to validate the main result of the paper. The conditions on function u guaranteeing that Assumption 2 holds must be examined for specific utility functions on a case-by-case basis.

3. Exploiting properties of the Hamilton-Jacobi-Bellman equation

We focus on characterizing symmetric IMPNEs, as these are given by Definition 2, having $C_i^*(k) = C_j^*(k)$ for all $i, j \in \{1, ..., N\}$. Since u is strictly concave, u' is strictly decreasing and hence invertible. Therefore, (4) implies,

$$c_i = (u')^{-1}(J_i'(k)) \tag{5}$$

is a function of k, after assuming that $J'(k)$ is a well-defined strictly monotone function. We discuss conditions guaranteeing that $J'(k)$ is a well-defined strictly monotone function below.

By the symmetry of the problem,

$$C_i^*(k) = (u')^{-1}(J_i'(k)) \text{ for all } i \in \{1, ...N\} \ . \tag{6}$$

Therefore, we drop subscript i, and we use (6) in order to substitute $C_i^*(k)$ into (3). Moreover, substituting $c = (u')^{-1}(J'(k))$ into (3) we obtain *a special case of the Lagrange-d'Alembert first-order nonlinear differential equation* (cf. Polyanin and Zaitsev (2003)),

$$J(k) = \frac{A}{\rho} k J'(k) + f(J'(k)) \ . \tag{7}$$

in which,

$$f(J'(k)) \equiv \frac{1}{\rho}\left[u\left((u')^{-1}(J'(k))\right) - NJ'(k) \cdot (u')^{-1}(J'(k))\right] \ .$$

Before we proceed, we introduce the problem's Lagrange multiplier, λ, as,

$$\lambda = J'(k) > 0 \ , \tag{8}$$

which we will use throughout the next section. Notice that $\lambda(t) = J'(k(t)) > 0$ for all $t \geq 0$, an implication of (4) and Assumption 1.

In order to characterize and solve a Lagrange-d'Alembert equation such as this given by (7), we must examine two cases separately, distinguished by the relationship between parameters A and ρ: first, the case in which $A \neq \rho$ and second the case in which $A = \rho$. In this section we focus on the more general and more interesting case, this of $A \neq \rho$.

Differentiating both sides of (7) with respect to k we obtain,

$$\left(1 - \frac{A}{\rho}\right) \frac{J'(k)}{J''(k)} = \frac{A}{\rho} k + f'(J'(k)) \ . \tag{9}$$

Let $K(\cdot)$ be the inverse function of $J'(\cdot)$. Then

$$k = K(J'(k)) \ . \tag{10}$$

Notice that by differentiating both sides of (10) we obtain $K'(J'(k)) = 1/J''(k)$. Use again $\lambda = J'(k)$, the model's Lagrange multiplier, and substitute these terms into (9) in order to obtain,

$$K'(\lambda) = g(\lambda) \cdot K(\lambda) + h(\lambda) \ , \tag{11}$$

a first-order linear differential equation in $K(\lambda)$ with variable coefficients, in which,

$$g(\lambda) \equiv \frac{\frac{A}{\rho}}{1 - \frac{A}{\rho}} \frac{1}{\lambda} \quad \text{and} \quad h(\lambda) \equiv \frac{1}{1 - \frac{A}{\rho}} \frac{f'(\lambda)}{\lambda} \ .$$

The solution to (11) is obtained through an integrating factor and is of the form,

$$K(\lambda) = \omega e^{\int g(\lambda) d\lambda} + e^{\int g(\lambda) d\lambda} \cdot \int e^{-\int g(\lambda) d\lambda} h(\lambda) \, d\lambda \ , \tag{12}$$

in which $\omega \in \mathbb{R}$ is an *integration constant*. The integration constant, ω, is very important for specifying the class of equilibrium solutions we focus on in this paper.

3.1 Characterizing the inverse of the value function of a single player in a symmetric MPNE when $A \neq \rho$

Since $K(\cdot)$ is the inverse function of $J'(\cdot)$, we can set,

$$K(\lambda) = (J')^{-1}(\lambda) \ . \tag{13}$$

Therefore, by characterizing $K(\lambda)$, we characterize the inverse of the value function of a single player in a symmetric MPNE.

The integral $\int g(\lambda)\, d\lambda$ has an explicit solution, namely,

$$\int g(\lambda)\, d\lambda = \xi \ln(\lambda) \ . \tag{14}$$

Notice also that the expression for $f'(\lambda)$ can be simplified. Specifically,

$$f'(\lambda) = \frac{1}{\rho}\left\{\left((u')^{-1}\right)'(\lambda)\left[u'\left((u')^{-1}\right)(\lambda) - N\lambda\right] - N(u')^{-1}(\lambda)\right\} \ ,$$

and after utilizing the identity $u'\left((u')^{-1}\right)(\lambda) = \lambda$,

$$f'(\lambda) = -\frac{1}{\rho}\left[(N-1)\lambda\left((u')^{-1}\right)'(\lambda) + N(u')^{-1}(\lambda)\right] \ . \tag{15}$$

Therefore, equation (12) can be re-written as,

$$K(\lambda) = \omega\lambda^{\xi} - \frac{1}{A}\xi\lambda^{\xi}\int \lambda^{-\xi-1}\left[(N-1)\lambda\left((u')^{-1}\right)'(\lambda) + N(u')^{-1}(\lambda)\right] d\lambda \ . \tag{16}$$

In order calculate the derivative of the value function of player i, $J'(k) = \lambda$, we can rewrite equation (16) as,

$$K(\lambda) = \omega\lambda^{\xi} + \phi(\lambda) \ , \tag{17}$$

in which,

$$\xi \equiv -\frac{A}{A-\rho} \ ,$$

and,

$$\phi(\lambda) \equiv \frac{1}{A-\rho}\lambda^{\xi} \int \lambda^{-\xi-1} \left[(N-1)\lambda \left((u')^{-1} \right)' (\lambda) + N (u')^{-1}(\lambda) \right] d\lambda \ . \tag{18}$$

Equation (18) can be simplified, after making use of the result that

$$\int \lambda^{\zeta} h'(\lambda)\, d\lambda = \lambda^{\zeta} h(\lambda) - \zeta \int \lambda^{\zeta-1} h(\lambda)\, d\lambda \ ,$$

for some $\zeta \in R$, and for a function $h(\lambda)$ that is differentiable and integrable. Specifically, we set $h(\lambda) = (u')^{-1}(\lambda)$ in (18) to simplify $\phi(\lambda)$ and obtain,

$$\phi(\lambda) = \frac{1}{A-\rho} \left\{ (N-1)(u')^{-1}(\lambda) + [N + \xi(N-1)]\lambda^{\xi} \int \lambda^{-\xi-1} (u')^{-1}(\lambda)\, d\lambda \right\} \ . \tag{19}$$

3.2 The role of the integration constant ω when $A \neq \rho$: examining or eliminating multiple MPNEs

Based on equations (13) and (17),

$$(J')^{-1}(\lambda) = \omega\lambda^{\xi} + \phi(\lambda) \ . \tag{20}$$

The key to proceeding with characterizing the MPNE strategies implied by (20), is to either obtain the inverse of the right-hand side of equation (20), or to obtain $\lambda = J'(k)$ as an implicit function of (20). Yet, since parameter ω is an integration constant, equation (20) implies that there are potentially infinite MPNE strategies. This is the point made early on by Tsutsui and Mino (1990) and Dockner and Long (1993), who studied linear-quadratic games. Up to Tsutsui and Mino (1990), the literature of linear-quadratic games was focusing only on MPNEs with linear strategies, as the constant ω in equation (20) was considered to be only equal to 0. Specifically, if the utility function is quadratic, $\phi(\lambda)$ is an affine function of λ, which implies that after setting $\omega = 0$ to equation (20), the MPNE strategies, $C(k)$ are affine functions of k.[8] However, when one sets $\omega \neq 0$, non-linear strategies arise in the linear

[8] See Tsutsui and Mino (1990, p. 144) and Dockner and Long (1993, p. 22). We demonstrate this point in a later section of this paper, too.

quadratic game, too, giving rise to multiple equilibria. These multiple equilibria can exist, for example, because of an "incomplete transversality condition", perhaps best explained by Tsutsui and Mino (1990, p. 153), who demonstrate the indeterminacy of stationary steady states in the linear-quadratic game they examine. Several papers deal with the characterization of such multiple equilibria in settings with different resource reproduction functions than ours, such as Rincon-Zapatero et al. (1998), Dockner and Wagener (2014), and Colombo and Labrecciosa (2015).

An important note is that equation (20) generalizes and extends the literature on linear-quadratic games substantially. Specifically, while the resource-reproduction function in our paper is linear of the "Ak" type, the objective function u of the players is quite general. The key to our generalization is the differential equation (9) and the proposed transformation given by (10). For example, the key differential equation in Tsutsui and Mino (1990, eq. 4.4, p. 143) that is a transformed analogue of our equation (9), is restricted to linear-quadratic games only, which are just a special case of our analysis in this paper.

Equation (20) can help in characterizing multiple equilibria, beyond standard "guess and verify" approaches. Finding or characterizing the inverse of the right-hand side of equation (20), $\omega\lambda^{\xi} + \phi(\lambda)$, can be challenging, as there are two additively-separable terms involving λ. Nevertheless, characterizing the special case with integration constant $\omega = 0$ can serve as a starting point. By setting $\omega = 0$, one can focus on finding the inverse function $\phi^{-1}(\lambda)$, in order to obtain $J'(k)$. Once this mission is accomplished analytically, then multiple equilibria can be derived analytically or can be numerically computed. In general, equation (20) can serve as a guide for computing all equilibria, including the case of $\omega \neq 0$, numerically. In the next section we focus on deriving an explicit solution for the case $\omega = 0$. The rest of the paper focuses on characterizing this special case of $\omega = 0$.

4. An explicit solution for the case with integration constant $\omega = 0$

There are two cases, $A \neq \rho$ and $A = \rho$ that we will examine separately. In the case of $A \neq \rho$, we focus on characterizing interior solutions, i.e., the symmetric IMPNE (see Definition 2), for the case of setting $\omega = 0$, that provides an exact solution to the problem. In the case of $A = \rho$, the solution is implicit but straightforward to characterize.

4.1 Case 1: $A \neq \rho$

The Lagrange-d'Alembert first-order nonlinear differential equation given by (7) allows us to arrive at an exact solution for the Markovian strategies $\{C_i^*(k)\}_{i=1}^N$. This solution is given by Proposition 1.

Proposition 1 *Under Assumptions 1 and 2, with $A \neq \rho$, the symmetric interior Markov perfect Nash equilibrium, IMPNE (see Definition 2), $\{C_i^*(k)\}_{i=1}^N$, corresponding to the case of setting $\omega = 0$ in equation (20), is given by,*

$$C_i^*(k) = C(k) = (u')^{-1}\left(\phi^{-1}(k)\right) \ , \ \ i \in \{1, ..., N\} \ , \tag{21}$$

with $\phi(\lambda)$ given by equation (19), provided that function u is such that $\phi'(\lambda) \neq 0$, $\phi(\lambda)$ is invertible, and $\lim_{t\to\infty} e^{-\rho t}\phi^{-1}(k(t))\, k(t) = 0$.

Proof

Due to Assumption 2 we do not need to worry about corner solutions. Therefore, we can set $\omega = 0$ in equation (20) and use the initial condition, $k(0) = k_0$ in order to identify function $J'(k)$ by solving,

$$k_0 = K(\lambda_0)|_{\omega=0} = \phi(\lambda_0) \ . \tag{22}$$

Since problem $\mathcal{P}_i$ falls in the class of discounted dynamic-programming problems with interior solutions, any admissible value k can be treated as initial conditions and (22) implies that the identification of function $J'(k)$ can be obtained by the rule,

$$J'(k) = \phi^{-1}(k) \ , \tag{23}$$

provided that ϕ is invertible. Equation (23) leads to (21). □

Proposition 1 is one of the key results of this paper, leading to further characterizations of the solution that we provide below. Nevertheless, we cannot ignore the special case of $A = \rho$ that follows.

4.2 Case 2: $A = \rho$

Setting $A = \rho$, makes (7) to collapse into Clairaut's differential equation,[9]

$$J(k) = kJ'(k) + f(J'(k)) \ . \tag{24}$$

After differentiating both sides of (24), we arrive at,

$$0 = [k + f'(J')] \cdot J'' \ . \tag{25}$$

Equation (25) leads to an explicit characterization of the Markovian strategies $\{C_i^*(k)\}_{i=1}^N$. This characterization is given by Proposition 2.

Proposition 2 *Under Assumptions 1 and 2, with $A = \rho$, the symmetric interior Markov perfect Nash equilibrium, IMPNE (see Definition 2), $\{C_i^*(k) = C(k)\}_{i=1}^N$ with $C(k)$ being an implicit function derived from the expression,*

$$(N-1)\frac{u'(C(k))}{u''(C(k))} = \rho k - NC(k) \ , \tag{26}$$

provided that $\lim_{t\to\infty} e^{-\rho t} u'(C(k(t)))\, k(t) = 0$.

[9] See Clairaut (1734).

Proof

Equation (25) implies,

$$k = -f'\left(J'\left(k\right)\right) \ . \tag{27}$$

Equation (27) holds because $J''(k) = 0$ is ruled out. To see that $J''(k) = 0$ cannot hold, consider the contrary, namely that $J'(k) = b$ for some constant $b \in \mathbb{R}$, which can be substituted into (24) to obtain the general solution,

$$J\left(k\right) = bk + f\left(b\right) \ . \tag{28}$$

However, the solution suggested by (28) is not acceptable, because it violates the transversality condition. Specifically, if (28) were acceptable, then equation (5) implies that $c(t) = (u')^{-1}(b) = \xi$, a constant, for all $t \geq 0$. But then $k(t)$ should satisfy the linear equation $\dot{k}(t) = \rho k(t) - N\xi$ with solution $k(t) = N\xi/\rho + e^{\rho t}(k_0 - N\xi/\rho)$, which implies that the transversality condition is violated unless $b = 0$, since $\lim_{t\to\infty} e^{-\rho t} J'(k(t))\, k(t) = b(k_0 - N\xi/\rho)$, which would not be equal to 0 for some k_0 if $b \neq 0$. But if $b = 0$, then $u'(c) = J'(k) = 0$, a contradiction since $u'(c) > 0$ for all $c \geq 0$.

Equations (27) and (15) imply,

$$\rho k = (N-1)\, J'(k) \left((u')^{-1} \right)' \left(J'(k)\right) + N\, (u')^{-1} \left(J'(k)\right) \ . \tag{29}$$

From equation (5),

$$C\left(k\right) = (u')^{-1} \left(J'\left(k\right)\right) \ , \tag{30}$$

which implies,

$$C'\left(k\right) = \left((u')^{-1} \right)' \left(J'\left(k\right)\right) J''\left(k\right) \ . \tag{31}$$

Combining (30) and (31) with (29), we obtain,

$$\rho k - NC\left(k\right) = (N-1)\, \frac{J'\left(k\right)}{J''\left(k\right)} C'\left(k\right) \ . \tag{32}$$

From equation (5), $J'(k) = u'(C(k))$, which implies $J''(k) = u''(C(k))\,C'(k)$. Substituting these two last expressions into (32) proves equation (26). $\square$

Proposition 2 offers the ability to compute $C(k)$ using numerical or analytical methods when applicable. In the next section we combine Propositions 1 and 2 in order to show that the analyticity of the utility function implies the analyticity of $C(k)$.

However, before we proceed, we can note that, given the requirement of interior solutions (see Assumption 2), the strategies given by both Proposition 1 and Proposition 2 either are continuous or they can be constructed so as to be continuous. To see this, consider the function $\phi(\lambda)$, given by (19). Given Assumption 1, $\phi(\lambda)$ is derived from derivatives and integrals of the inverse of the twice continuously differentiable and strictly increasing function u. Therefore, $\phi(\lambda)$ is also a continuous function that we assume to be strictly monotonic. Therefore, $\phi^{-1}(\lambda)$ is also a continuous function and the strategy $C(k) = (u')^{-1}\left(\phi^{-1}(\lambda)\right)$ is continuous, too.[10] In addition, (26) yields the strategy, $C(k)$, as an implicit function. If we assume that u is thrice continuously differentiable, then $C(k)$ will also be continuously differentiable and, hence, continuous.[11]

5. Analytic utility functions

Propositions 1 and 2 give the opportunity to characterize the functional properties of Markovian strategies if the utility function of players is a real analytic function. For the definition of a real analytic function see Krantz and Parks (2002, Definition 1.1.5, p. 3). Specifically, a function $f : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$, is real analytic if for all $x_0 \in D$, the value $f(x)$ can be

[10] See, for example, Lang (1997, Theorem 4.2, p. 60), proving that the composition of continuous functions gives a continuous function.

[11] This property of continuity of strategies differs from Dockner and Sorger (1996, Theorem 1, p. 2015), where the strategies can be disconinuous functions.

written as a power series of the form,

$$f(x) = \sum_{i=0}^{\infty} \alpha_i (x - x_0)^i \ .$$

Examples of analytic functions include polynomial function, the exponential function, the logarithmic function, or the power function. Proposition 3 proves that if the utility function of players is real analytic, then the symmetric Markovian strategies are also real analytic.

Proposition 3 *Under Assumptions 1 and 2, if $u(c)$ is a real analytic function, then the function $C(k)$ characterizing the symmetric Markov perfect Nash equilibrium $\{C_i^*(k) = C(k)\}_{i=1}^N$ is also a real analytic function.*

Proof

The proof is straightforward through the use of known results regarding analytic functions. Specifically, in the case of $A \neq \rho$, the expression of $\phi(\lambda)$ given by equation (19) involves inverses, derivatives, and integrals of utility functions. In addition, the exponential function, λ^α for some α, is also analytic. By the definition of real analytic functions, it is immediate to prove that the products and sums of real analytic functions are also real analytic. That the derivative of a real analytic function is also real analytic is proved in Krantz and Parks (2002, Proposition 1.1.14, p. 9). That the inverse of a real analytic function is also real analytic, the proof is in Krantz and Parks (2002, Theorem 1.5.3, p. 22). That the indefinite integral of a real analytic function is also real analytic is proved in Krantz and Parks (2002, Proposition 2.2.3, p. 30). These Propositions and Theorems show that $\phi^{-1}(\lambda)$ is real analytic. The expression for the Markovian strategies, $C(k)$, is given by (6) which involves the composition of real analytic functions. That compositions of real analytic functions are also real analytic is proved in Krantz and Parks (2002, Proposition 1.4.2, p. 19). These arguments prove the proposition for the case of $A \neq \rho$.

For the case of $A = \rho$, the Markovian strategies, $C(k)$, are an implicit function of equation (26). The proof of the implicit function theorem for real analytic functions, stating that the implicit functions of real analytic functions are also real analytic is given by Krantz and Parks (2002, Theorem 2.3.5, p. 40). □

5.1 Usefulness of analyticity for interior solutions

Proposition 3 is an important result for solving the problem numerically. For example, by using polynomial approximations to value functions and Markovian strategies, Proposition 3 guarantees that the approximated functions may remain in the same space of approximating polynomials and be convergent. With the approximation error remaining bounded, a well-behaved computation is guaranteed. For computation one can use either the exact solution given by (19) and (6), or recursive methods on the Lagrange-d'Alembert differential equation given by (7).

5.2 Analyticity and extensions to corner solutions through homotopy approaches

The central assumption we have made in this paper is Assumption 2, namely that the utility function allows for an interior solution. In practice, if solving such a game requires a numerical approach, it is difficult to know in advance which combinations of parameter values of the utility function with A and ρ indeed deliver an interior solution. Yet, a trial-and-error approach can help in verifying whether the dynamics of k implied by the strategies based on Proposition 1 or Proposition 2 reconfirm that the solution is interior or not. In brief, the game can be solved using Proposition 1 or Proposition 2 under the working hypothesis that for some parameter values the problem has an interior solution. If the interiority of the solution is not reconfirmed, then parameters can be re-calibrated.[12]

In the literature of differential games modeling commons problems interesting applications involve studying the potential depletion of a common-property resource, or placing quotas on resource exploitation. Such applications involve corner solutions. Analytical results for differential games with corner solutions are more difficult to obtain. Therefore, using numerical approximations may be the only resort. Nevertheless, calibrating a Markovian differential game with corner solutions so as to achieve convergence using recursive methods may be challenging. A possibility is to employ a homotopy computational approach as in Eaves and Schmedders (1999).

The homotopy computational approach is the practice of starting from calibrated parameters of a well-behaved solution. Afterwards, changing parameter values in a gradual, step-by-step fashion, one arrives to the desired parameterization of the computational problem. This homotopy procedure is explained in detail by Garcia and Zangwill (1981). Examples of papers such as Borkovsky et al. (2010) and Besanko et al. (2010) adapt the homotopy approach to some classes of dynamic games.

For following such a homotopy procedure, our results in this paper can be proved very useful. Propositions 1 or 2 can provide well-behaved interior solutions and can guide through parameterizations that guarantee the interiority of solutions. In a next step, parameterizations leading to corner solutions of research usefulness can be pursued in a step-by-step manner.

[12]The next section, where we present several closed-form solutions, gives "hands-on" examples of how the choice of parameters affects whether a Markov-perfect Nash-equilibrium solution is interior or not.

6. Examples with closed-form solutions

Propositions 1 and 2 lead to immediate results in cases where the problem we study admits closed-form solutions in the special case of setting $\omega = 0$ in equation (20). We list these examples below, demonstrating the usefulness of Propositions 1 and 2. First, we list cases that are more or less known. These known examples use utility functions from the comprehensive class of functions guaranteeing linear aggregation in dynamic models, identified by Koulovatianos et al. (2019). The common feature of these examples is that resource exploitation strategies, $C(k)$, for the case of setting $\omega = 0$ in equation (20), are all linear functions in k. This common feature is essential for aggregation. In addition, it helps in deriving explicit dynamics for k, which helps in identifying parametric constraints guaranteeing that the solution is interior. Examining these known cases is useful, as it helps in demonstrating our solution method.[13]

At a second stage, after our method is demonstrated, we present a final example that, to the best of our knowledge, does not exist in the literature. For this particular new case the resource exploitation strategies, $C(k)$, for the case of setting $\omega = 0$ in equation (20), are non-linear.[14] Crucially, our suggested method is essential for identifying this closed-form solution. Therefore, we believe that this new example demonstrates the usefulness of our approach.

[13]Most of our examples, except the sligtly more generalized case with "Gorman preferences" and the case of constant-absolute-risk-averstion preferences, which we present below, have been thoroughly studied by Gaudet and Lohoues (2008), who go beyond the use of linear resource-reproduction functions, specifying the types of resource reproduction functions that allow for linear strategies. We thank Hassan Benchekroun for pointing this paper to us.

[14]In Tasneem, Engle-Warnick and Benchekroun (2017) there is experimental evidence that players may choose both linear and nonlinear strategies. The theoretical model employed in Tasneem, Engle-Warnick and Benchekroun (2017) allows for multiple equilibria, providing a clear distinction between linear and nonlinear equilibria. The evidence that non-linear strategies may be chosen by players, supports the usefulness of our new example. We are indebted to Hassan Benchekroun for making this point to us.

6.1 Gorman preferences

Let's consider preferences as in Gorman (1961), given by,

$$u(c) = \frac{(c+\chi)^{1-\theta}}{1-\theta} \ , \tag{33}$$

in which $\theta > 0$ and $\chi \in \mathbb{R}$. Based on (18), the corresponding function ϕ is

$$\phi(\lambda) = \frac{1-(1-\theta)N}{\rho-(1-\theta)A}\lambda^{-\frac{1}{\theta}} - N\frac{\chi}{A} \ . \tag{34}$$

Therefore, if parameters ρ, A, θ, and N are such that,

$$\frac{1-(1-\theta)N}{\rho-(1-\theta)A} > 0 \ , \tag{35}$$

then $\phi' < 0$, implying that ϕ^{-1} exists. In particular equation (23) gives,

$$J'(k) = \phi^{-1}(k) = \left[\frac{\rho-(1-\theta)A}{1-(1-\theta)N}\right]^{-\theta}\left(k + N\frac{\chi}{A}\right)^{-\theta} \ , \tag{36}$$

which implies,

$$J(k) = \left[\frac{\rho-(1-\theta)A}{1-(1-\theta)N}\right]^{-\theta}\frac{\left(k+N\frac{\chi}{A}\right)^{1-\theta}}{1-\theta} \ .$$

Moreover, (6) gives,

$$c^* = C(k) = \frac{\rho-(1-\theta)A}{1-(1-\theta)N}k - \frac{\chi}{A}\cdot\frac{A-N\rho}{1-(1-\theta)N} \ . \tag{37}$$

Substituting (37) into the constraint $\dot{k} = Ak - NC(k)$, and solving the resulting linear differential equation, we obtain the explicit dynamics of $k(t)$, namely,

$$k(t) = -\frac{N\chi}{A} + e^{\frac{A-N\rho}{1-(1-\theta)N}t}\left[k(0) + \frac{N\chi}{A}\right] \ . \tag{38}$$

Equation (38) implies that once $k(0) > -N\chi/A$, $k(t) > -N\chi/A$ for all $t \geq 0$, making $J(k(t))$ be well-defined for all $t \geq 0$. Therefore,

$$k(0) > -\frac{N\chi}{A} \ , \tag{39}$$

is one of the necessary parametric constraints of this problem.[15] In order to examine the parametric constraints for guaranteeing that the transversality condition is met, we use (38) and (36) to obtain,

$$J'(k(t)) = \left[\frac{\rho - (1-\theta)A}{1-(1-\theta)N}\right]^{-\theta}\left[k(0) + N\frac{\chi}{A}\right]^{-\theta} e^{-\theta\frac{A-N\rho}{1-(1-\theta)N}t} . \tag{40}$$

Combining (40) and (38) shows that $\lim_{t\to\infty} e^{-\rho t} J_i'(k(t))\, k(t) = 0$ is equivalent to,

$$\lim_{t\to\infty}\left\{-\frac{N\chi}{A}e^{-\left[\rho+\frac{\theta(A-N\rho)}{1-(1-\theta)N}\right]t} + \left[k(0) + \frac{N\chi}{A}\right] e^{-\left[\rho-\frac{(1-\theta)(A-N\rho)}{1-(1-\theta)N}\right]t}\right\} = 0 . \tag{41}$$

After some algebra, the requirement implied by (41) that $\rho + \theta(A - N\rho)/[1-(1-\theta)N] > 0$, is simplified to,

$$\frac{\rho(1-N) + \theta A}{1-(1-\theta)N} > 0 . \tag{42}$$

The condition $\rho - (1-\theta)(A - N\rho)/[1-(1-\theta)N] > 0$, which is the other requirement implied by (41), is equivalent to (35). This equivalence can be verified after some algebra. Therefore, conditions (35), (39) and (42) are the three parametric requirements that guarantee a well-behaved solution. An important observation from (38) is that these three conditions do not rule out the possibility of sustainable perpetual growth, i.e. $k(t) \to \infty$. Obviously from (38), sustainable growth occurs if $(A - N\rho)/[1-(1-\theta)N] > 0$.

A crucial observation is that equation (37) holds also for the case where $A = \rho$. Specifically, after setting $A = \rho$ in equation (37), the resulting strategy $C(k)|_{A=\rho}$ satisfies the condition given by (26). We are not aware of any paper in the literature solving this problem for $\chi \neq 0$. Therefore, equation (37) is a novelty of this paper.

Finally, notice that by adding the constant $-1/(1-\theta)$ to the utility function given by (33), a modification that does not affect optimization, we can consider the case where $\theta = 1$,

[15] Apparently, combining (39), (38), and (37) is necessary in order to identify parametric restrictions guaranteeing that $C(k(t)) > 0$ for all $t \geq 0$, consistently with an interior solution.

which leads to having logarithmic utility, since, after using L'Hôpital's rule,

$$\lim_{\theta \to 1} \frac{(c+\chi)^{1-\theta} - 1}{1-\theta} = \ln(c+\chi) \ .$$

6.1.1 Special case: CRRA preferences ($\chi = 0$)

A familiar example in the literature (see Lane and Tornell, 1996) is this of CRRA preferences,

$$u(c) = \frac{c^{1-\theta}}{1-\theta} \ ,$$

which is the case of setting $\chi = 0$ in (33). Setting $\chi = 0$ in (37) gives

$$c^* = C(k) = \frac{\rho - (1-\theta)A}{1-(1-\theta)N} k \ ,$$

which coincides with the solution in Lane and Tornell (1996, eq. 17, p. 221).

6.2 Constant absolute risk aversion (CARA) preferences

Let a utility function representing CARA preferences, namely,

$$u(c) = -e^{-\frac{1}{\beta}c} \ , \tag{43}$$

in which $\beta > 0$. Based on (18), the corresponding function ϕ is

$$\phi(\lambda) = -\frac{\beta N}{A}\left[\ln(\beta) + \frac{N + \xi(N-1)}{\xi N} + \ln(\lambda)\right] \ . \tag{44}$$

Equation (44) implies that $\phi'(\lambda) < 0$ for all $\lambda > 0$. Therefore, ϕ^{-1} exists, with equation (23) implying,

$$J'(k) = \phi^{-1}(k) = \frac{e^{\frac{A-\rho}{A} - \frac{N-1}{N}}}{\beta N} e^{-\frac{A}{\beta N}k} \ , \tag{45}$$

leading to,

$$J(k) = -\frac{e^{\frac{A-\rho}{A} - \frac{N-1}{N}}}{A} e^{-\frac{A}{\beta N}k} \ .$$

In turn, (6) gives,

$$c^* = C(k) = \frac{A}{N}k - \frac{\beta(A-\rho)}{A} + \beta\frac{N-1}{N} \ . \tag{46}$$

After inserting (46) into the constraint $\dot{k} = Ak - NC(k)$, we obtain $\dot{k}(t) = \beta(A - \rho N)/A$, which has the obvious solution,

$$k(t) = k(0) + \beta\frac{A-\rho N}{A}t \ . \tag{47}$$

Equations (47) and (46) reveal the parametric constraint that guarantee an interior solution. Specifically, equations (47) shows that,

$$A - \rho N \geq 0 \ , \tag{48}$$

guarantees $k(t) \geq k(0) > 0$ for all $t \geq 0$. Equation (46) reveals that placing a constraint on the initial conditions $k(0)$, namely,

$$k(0) \geq \frac{\beta(A-\rho)}{A} - \beta\frac{N-1}{N} \ , \tag{49}$$

guarantees $c(t) \geq c(0) > 0$ for all $t \geq 0$ if (48) also holds. In summary, inequalities (48) and (49) guarantee the interiority of the solution given by (46).

Combining (47) with (45) leads to a tractable expression for the transversality condition. Specifically,

$$\lim_{t\to\infty} e^{-\rho t} J_i'(k(t))\, k(t) = \frac{e^{\frac{A-\rho}{A} - \frac{N-1}{N}}}{A} e^{-\frac{A}{\beta N}k(0)} \lim_{t\to\infty}\left[k(0)\, e^{-\frac{A}{N}t} + \beta\frac{A-\rho N}{A}\frac{t}{e^{\frac{A}{N}t}}\right] \ . \tag{50}$$

Equation (50) reveals that, in the case of CARA preferences, no parametric restrictions on A, ρ, β, N, and $k(0)$ beyond inequalities (48) and (49) are needed in order to ensure that the transversality condition is met. Importantly, equation (47) reveals that sustainable growth is possible. Finally, exactly as in the case of Gorman preferences, equation (46) holds also for the case where $A = \rho$, with the strategy $C(k)|_{A=\rho}$ satisfying condition (26).

6.3 Quadratic preferences

Let the utility function be,

$$u(c) = -\frac{1}{2}(\chi - c)^2 \ , \tag{51}$$

with $0 \leq c < \chi$. It is broadly known that linear quadratic differential games have linear strategies as solutions.[16] Combining (51) with (18), the corresponding function ϕ is

$$\phi(\lambda) = -\frac{2N-1}{2A-\rho}\lambda + N\frac{\chi}{A} \ .$$

Therefore, ϕ^{-1} exists and equation (23) implies,

$$J'(k) = \phi^{-1}(k) = \frac{2A-\rho}{2N-1}\left(N\frac{\chi}{A} - k\right) \ , \tag{52}$$

leading to,

$$J(k) = -\frac{1}{2}\frac{2A-\rho}{2N-1}\left(N\frac{\chi}{A} - k\right)^2 \ .$$

Equation Moreover, (6) combined with (52) give the formula of the Markovian strategy,

$$c^* = C(k) = \frac{2A-\rho}{2N-1}k + \frac{\chi}{A}\frac{\rho N - A}{2N-1} \ . \tag{53}$$

Important in this example is to restrict parameters so that $2A - \rho > 0$, and to use (53) in order to select N, A, ρ, χ, $k(0)$ so that the dynamics of k imply $k(t) < N\chi/A$ for all $t \geq 0$, guaranteeing that $J'(k(t)) > 0$ for all $t \geq 0$, and that the solution is interior. These conditions can be found after we substitute the strategies given by (53) into $\dot{k}(t) = Ak(t) - NC(k(t))$, and after solving for the explicit dynamics of $k(t)$. Specifically, these

[16] A study explaining that non-linear strategies can also exist is Tsutsui and Mino (1990). Nevertheless, focusing on interior solutions is important on whether such non-linear strategies can exist or not in linear quadratic games.

dynamics are given by,

$$k(t) = N\frac{\chi}{A} + e^{-\frac{A-\rho N}{2N-1}t}\left[k(0) - N\frac{\chi}{A}\right] \ . \tag{54}$$

Equation (54) tells us that while $k(0) < N\chi/A$, the parametric constraint needed for guaranteeing that $k(t) < N\chi/A$ for all $t \geq 0$ is,

$$A > \rho N \ . \tag{55}$$

Notice that (55) implies $A > \rho/2$, which is a necessary condition guaranteeing $J'(k(t)) > 0$ for all $t \geq 0$. In addition, using (54) and (52), we can verify that the transversality condition holds if (55) holds, since $\lim_{t\to\infty} e^{-\rho t} J_i'(k(t))\, k(t) = 0$ is equivalent to,

$$\lim_{t\to\infty} \frac{N\chi}{A} e^{-\left[\rho + \frac{A-\rho N}{2N-1}\right]t} - \left[N\frac{\chi}{A} - k(0)\right] e^{-\left[\rho + 2\frac{A-\rho N}{2N-1}\right]t} = 0 \ .$$

As we saw above, similar parametric constraints are needed in the case of Gorman preferences in order to guarantee that $J'(k(t)) > 0$ for all $t \geq 0$ and that the transversality condition is met.[17] Nevertheless, these parametric constraints in the case of Gorman preferences do not rule out the possibility of sustainable growth. On the contrary, in the case of quadratic preferences, because the utility function has a bliss point, the growth of $k(t)$ must be bounded above. This feature of linear quadratic games motivates the main purpose of this paper, which is to discover further solutions for Markovian games of commons with linear constraints.

6.4 New Example with non-linear exploitation strategy: demonstrating our method

Here we consider a utility function that does not fall in the class of preferences that lead to aggregation (see Koulovatianos et al., 2019, p. 172, Theorem 1). Specifically, the resource

[17]Specifically, in the case of Gorman preferences, the parametric restrictions on N, A, ρ, χ, θ, $k(0)$, given by conditions (35), (39) and (42), are needed in order to guarantee that $k(t) > -N\chi/A$ for all $t \geq 0$.

exploitation strategies, $C(k)$, are not linear in k. The utility function is,

$$u(c) = c - \kappa c^{\frac{3}{2}} , \tag{56}$$

where $\kappa > 0$. Notice that for $u'(c) = 1 - 3/2\kappa c^{1/2} > 0$ we need to place an upper bound on c. The requirement $u'(c) > 0$ holds if and only if,

$$0 \leq c < \frac{4}{9\kappa^2} \equiv \bar{c} . \tag{57}$$

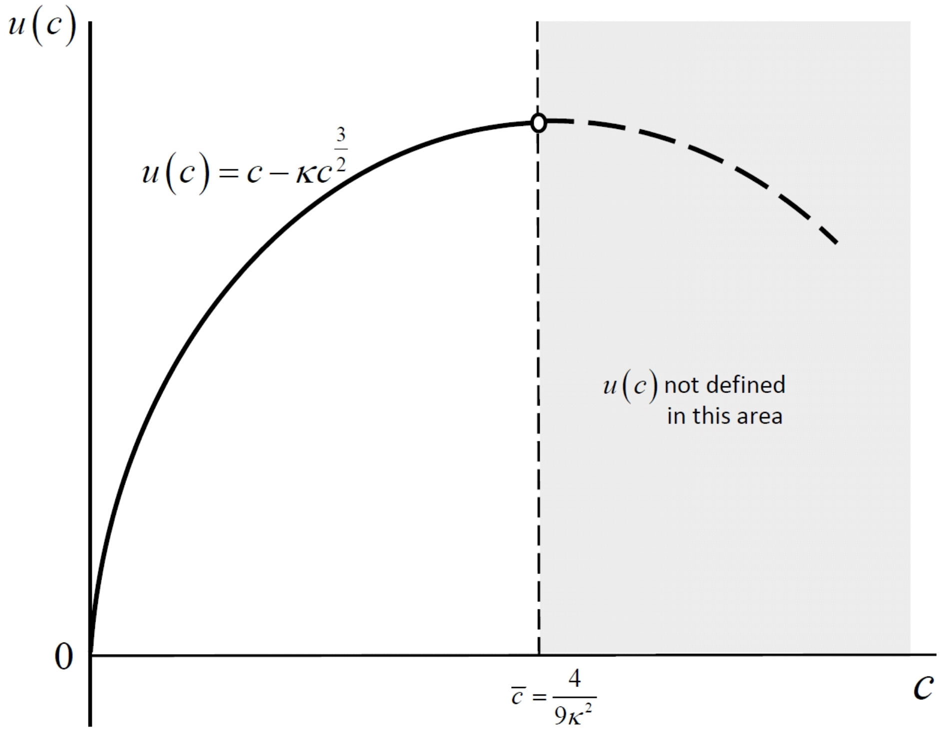

Figure 1 The utility function $u(c) = c - \kappa c^{3/2}$ is not defined in the shaded area.

Beyond the value of $\bar{c} = 4/(9\kappa^2)$ for c, $u(c)$ becomes downward-sloping, as shown by Figure 1. As marginal utility becomes negative for $c > \bar{c}$, one can view $\bar{c}$ as the bliss point of this utility function. Think, for example, that c is the consumption of a renewable resource such as fish, and that $\bar{c}$ is the maximum instantaneous flow of a country's fish consumption, as the residents of a country become instantaneously satiated by consuming fish. As for the linear technology of the fish reproduction stock, k, driven by Ak, think of Ak as a production function of fish through sustainable fish-farming.[18]

Within the range of values for c given by (57), the inverse function of $u'(c)$ is given by,

$$(u')^{-1}(\lambda) = \frac{4}{9\kappa^2}(1-\lambda)^2 \ . \tag{58}$$

After introducing (58) into (18) and after some algebra, the corresponding function ϕ is,[19]

$$\phi(\lambda) = \eta(\lambda-\theta)^2 + \psi \ , \tag{59}$$

where,

$$\psi = \frac{4}{9\kappa^2}\left[\frac{N}{A} - \left(\frac{2N-1}{2A-\rho}\right)^2 \frac{3A-2\rho}{3N-2}\right] \ , \tag{60}$$

$$\theta = \frac{3A-2\rho}{2A-\rho}\frac{2N-1}{3N-2} \ , \tag{61}$$

and

$$\eta = \frac{4}{9\kappa^2}\frac{3N-2}{3A-2\rho} \ . \tag{62}$$

Inverting $\phi(\lambda)$ seems straightforward, but one must pay attention to one feature. Specifically, the procedure for inverting $\phi(\lambda)$ is setting $k = \phi(\lambda)$, and then using equation (59) in order to solve for variable λ. During this function inversion process, a step is given by,

$$|\lambda - \theta| = \frac{1}{\eta^{\frac{1}{2}}}(k-\psi)^{\frac{1}{2}} \ . \tag{63}$$

Notice a first parametric constraint implied by (63), that

$$k(t) \geq \max\{\psi, 0\}\ , \text{ for all } t \geq 0 \ . \tag{64}$$

[18] Fish reproduction is the application in Sorger (2005), who also uses a linear, constant-reproduction rate, Ak. Alternative interpretations would include exogenously supplied infrastructure by governments to users, such as public roads, assuming that users have an upper capacity of usage, $\bar{c}$.
[19] For the derivation of $\phi(\lambda)$ see the Appendix.

A second parametric constraint implied by (63) comes from the requirement that $\eta^{1/2}$ exists and that it is different from 0, i.e., that $\eta > 0$. Since $3N - 2 > 0$ for all $N \in \{1, 2, ...\}$, $\eta > 0$ if and only if,

$$A > \frac{2}{3}\rho \; . \tag{65}$$

Condition (65) implies that,[20]

$$0 < \theta < \frac{3}{2} \tag{66}$$

which means,

$$(1 - \theta)^2 < 1 \; . \tag{67}$$

We will examine conditions that guarantee (64) below. There are two possibilities for the left-hand side of equation (63). The first is to have $\lambda - \theta \geq 0$, which leads to, a value function, $J(k)$ that is strictly convex and which implies dynamics that violate the property that the solution is interior.[21] The second possibility, of $\lambda - \theta \leq 0$, is admissible, but it still remains to identify parameter restrictions guaranteeing that u is such that the solution is interior.

Let,

$$\lambda - \theta \leq 0 \; , \tag{68}$$

hold. Combining (68) with (63) implies,

$$\lambda = J'(k) = \phi^{-1}(k) = \theta - \frac{1}{\eta^{\frac{1}{2}}}(k - \psi)^{\frac{1}{2}} \;\; , \;\text{if } \lambda \leq \theta \; . \tag{69}$$

Equation (69) gives the variable part of the formula for $J(k)$, namely,

$$J(k) = \theta k - \frac{2}{3\eta^{\frac{1}{2}}}(k - \psi)^{\frac{3}{2}} \;\; , \tag{70}$$

[20]For a proof of this result, see the Appendix.

[21]See the Appendix for details on this point.

where a constant of integration can be added, specified by the HJB equation of the problem of each player. Importantly, this value function is concave, since (69) implies,

$$J''(k) = -\frac{1}{2\eta^{\frac{1}{2}}}(k-\psi)^{-\frac{1}{2}} < 0 \ . \tag{71}$$

Based on (21), (58) together with (69) reveal the formula of the optimal symmetric strategy, which is given by,

$$C(k) = \frac{4}{9\kappa^2}\left[\frac{1}{\eta^{\frac{1}{2}}}(k-\psi)^{\frac{1}{2}} + 1 - \theta\right]^2 \ . \tag{72}$$

Given (57), it should be that $0 \leq C(k) < 4/(9\kappa^2) = \bar{c}$, an inequality that leads to,

$$k \in [\underline{k}, \bar{k}) \ , \text{ with } \ \bar{k} = \eta\theta^2 + \psi = \frac{4N}{9\kappa^2 A} \ , \text{ and } \ \underline{k} = \max\left\{0, \eta(1-\theta)^2 + \psi\right\} \ . \tag{73}$$

with,

$$\eta(1-\theta)^2 + \psi = \bar{k} - (2\theta - 1)\eta \ . \tag{74}$$

To ensure that the interval $[\underline{k}, \bar{k})$ is non-empty, the formulas given by (73) and (74) indicate that $2\theta - 1 > 0$, as (62) together with condition (65) imply that $\eta > 0$. After some algebra we can prove that $2\theta - 1 > 0$ is equivalent to having $A > (5N-2)\rho/[2(3N-1)]$. Therefore, a sufficient condition to guarantee that $[\underline{k}, \bar{k})$ is non-empty, is given by,

$$A > \frac{5}{6}\rho \ , \tag{75}$$

and notice that the parametric constraint given by (75), implies the parametric constraint given by (65).[22]

To examine the dynamics of this game, we first examine the monotonicity of the symmetric strategy $C(k)$. Based on the first-order condition given by (5),

$$u'(C(k)) = J'(k) \ , \tag{76}$$

[22]To see why (75) implies $A > (5N-2)\rho/[2(3N-1)]$ for all $N \geq 1$, define $H(N) = (5N-2)\rho/[2(3N-1)]$. Notice that $H(1) = 3\rho/4$, with $H'(N) = \rho/\left[2(3N-1)^2\right] > 0$, and with $\lim_{N\to\infty} H(N) = 5\rho/6$.

differentiating both sides of (76) implies,

$$J''(k) = u''(C(k))\,C'(k) \ . \tag{77}$$

Because $u''(c) < 0$ for all c complying with (57), and because of (71),

$$J''(k) < 0 \ \text{ combined with (77) imply that } C'(k) > 0 \ . \tag{78}$$

After some algebra, we can verify that $C'(k) > 0$ is equivalent to having $\eta^{-1/2}(k-\psi)^{1/2} + 1 - \theta > 0$.

Introducing strategies $C(k)$ into (1) implies,

$$\dot{k} = Ak - NC(k) \ . \tag{79}$$

Differentiating (79) with respect to k we obtain,

$$\frac{\partial \dot{k}}{\partial k} = A - NC'(k) \ . \tag{80}$$

The monotonicity of $C(k)$ given by (78), together with (80), jointly imply that it is possible to have stable dynamics toward a 0-growth steady state of k. Such a 0-growth steady state of k can either lie within the domain of admissible interior strategies of the game, the interval $\left[\underline{k}, \bar{k}\right)$, or it can be the supremum of the interval $\left[\underline{k}, \bar{k}\right)$, which is $\bar{k}$. We explore parametric conditions that allow for this possibility.

After expanding the quadratic term in (72), the law of motion (79) becomes,

$$\dot{k} = Ak - \frac{4N}{9\kappa^2\eta}(k-\psi) - \frac{8N(1-\theta)}{9\kappa^2\eta^{\frac{1}{2}}}(k-\psi)^{\frac{1}{2}} - \frac{4N(1-\theta)^2}{9\kappa^2} \ . \tag{81}$$

Given the nature of (81), it is useful to introduce a function,

$$z(k) \equiv (k-\psi)^{\frac{1}{2}} \ . \tag{82}$$

In the Appendix we show that (81) can be re-written as,

$$\dot{k} = A\left(1 - \frac{\bar{k}}{\eta}\right) z(k)^2 - 2A\frac{\bar{k}(1-\theta)}{\eta^{\frac{1}{2}}} z(k) + A\left[\bar{k}(2-\theta) - \eta\theta\right]\theta \tag{83}$$

The right-hand side of (83) can be seen as a quadratic polynomial in terms of the function $z(k)$, with discriminant, Δ, given by,

$$\Delta = 4A^2 \left\{ \frac{\left[\bar{k}(1-\theta)\right]^2}{\eta} - \left(1 - \frac{\bar{k}}{\eta}\right)\left[\bar{k}(2-\theta) - \eta\theta\right]\theta \right\} . \tag{84}$$

The discriminant given by (84) can inform us on whether real roots of the quadratic polynomial exist. In order to achieve this goal, we use the formulas given by (62) and (61). After some algebra, we can show that,[23]

$$A \gtreqless \rho N \Leftrightarrow \theta \gtreqless 1 \Leftrightarrow \bar{k} \gtreqless \eta \Leftrightarrow \bar{k}(2-\theta) - \eta\theta \gtreqless 0 \text{ , for all } N \geq 2 \text{ and all } A > \frac{5}{6}\rho . \tag{85}$$

The equivalence given by (85) serves as a guide, indicating that we must focus on the relationship between A and ρN, keeping in mind that condition (75) should always hold. A first implication of (85) is that,

$$\left(1 - \frac{\bar{k}}{\eta}\right)\left[\bar{k}(2-\theta) - \eta\theta\right] \leq 0 \text{ , for all } A > \frac{5}{6}\rho\text{, with equality iff } A = \rho N . \tag{86}$$

Based on (66), $\theta > 0$, and (86) implies that,

$$\Delta \geq 0 \text{ , for all } A > \frac{5}{6}\rho\text{, with equality iff } A = \rho N . \tag{87}$$

Therefore, as long as $A > 5\rho/6$ and $A \neq \rho N$, there are always two real roots to the quadratic polynomial in $z(k)$ given by the right-hand side of (83). Let's call these two real roots r_1 and r_2. The first root, r_1, is easy to identify using (73) and (72) and introducing them into (79), namely,

$$\left.\dot{k}\right|_{k=\bar{k}} = A\bar{k} - NC\left(\bar{k}\right) = 0 . \tag{88}$$

[23]See the Appendix for a proof of this statement.

The result in (88) implies that,

$$r_1 = z\left(\bar{k}\right) = \left(\bar{k} - \psi\right)^{\frac{1}{2}} = \eta^{\frac{1}{2}}\theta > 0 \ . \tag{89}$$

The right-hand side of (83) implies,

$$r_1 r_2 = \frac{\left[\bar{k}\left(2-\theta\right) - \eta\theta\right]\theta}{1 - \frac{\bar{k}}{\eta}} \ , \ \text{ and } \ r_1 + r_2 = 2\frac{\bar{k}\left(1-\theta\right)}{\eta^{\frac{1}{2}}\left(1 - \frac{\bar{k}}{\eta}\right)} \ . \tag{90}$$

It is verifiable that all three equations in (89) and (90) comply with,

$$r_2 = \frac{\bar{k}\left(2-\theta\right) - \eta\theta}{\eta^{\frac{1}{2}}\left(1 - \frac{\bar{k}}{\eta}\right)} < 0 \ . \tag{91}$$

To see why r_2 is strictly negative, observe from (90) and (86) that for all $A > 5\rho/6$ with $A \neq \rho N$, $r_1 r_2 < 0$. Since $r_1 > 0$, it follows that $r_2 < 0$.

In brief, (83) can be re-written as,

$$\dot{k} = A\left(1 - \frac{\bar{k}}{\eta}\right)\left[z\left(k\right) - \eta^{\frac{1}{2}}\theta\right]\left[z\left(k\right) - \frac{\bar{k}\left(2-\theta\right) - \eta\theta}{\eta^{\frac{1}{2}}\left(1 - \frac{\bar{k}}{\eta}\right)}\right] \ . \tag{92}$$

Since $z\left(k\right) \geq 0$ and since, according to (91), $r_2 < 0$, the last term of (92), $z\left(k\right) - r_2 > 0$. In addition, since $z'\left(k\right) = 1/2\left(k - \psi\right)^{-1/2} > 0$, and since $k < \bar{k}$, $z\left(k\right) < z\left(\bar{k}\right) = \eta^{1/2}\theta$, i.e., the penultimate term of (92), $z\left(k\right) - r_1 < 0$. Given these two observations, (92) and (85) imply,

$$\dot{k} \gtreqqless 0 \Leftrightarrow A \gtreqqless \rho N \ , \ \text{ for all } N \geq 2 \text{ and all } A > \frac{5}{6}\rho \ . \tag{93}$$

Therefore, (93) implies that the parametric constraint

$$A > \rho N \ , \tag{94}$$

guarantees an interior solution where $k \rightarrow \bar{k}$ asymptotically, as $t \rightarrow \infty$, as is depicted by Figure 2.

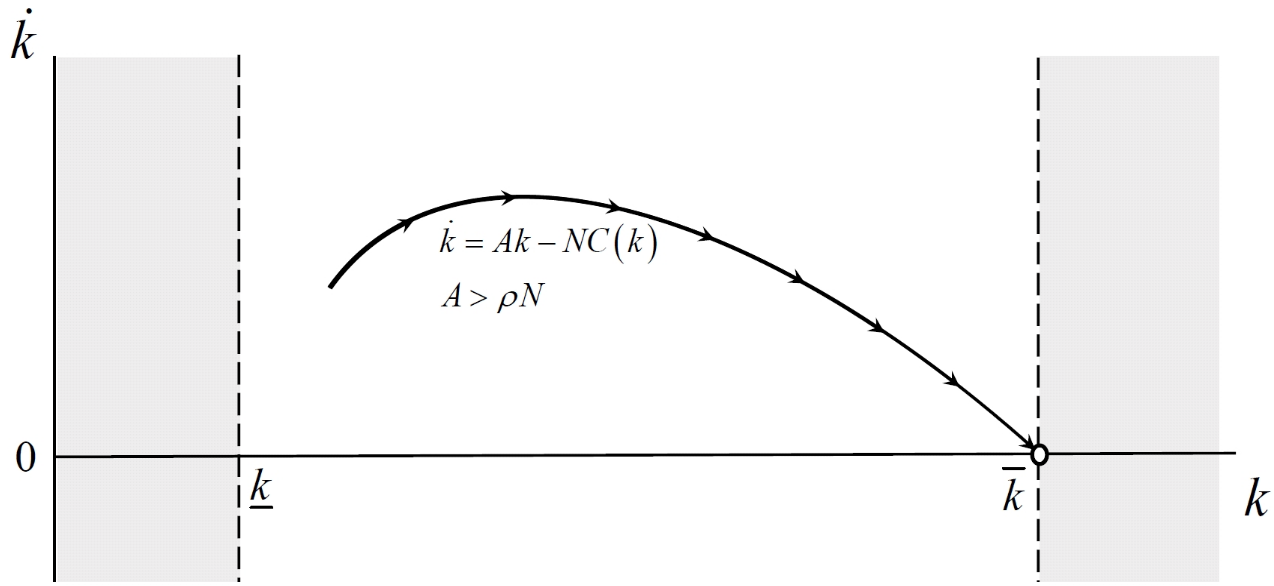

Figure 2 Dynamics of k toward the supremum of the $[\underline{k}, \bar{k})$ interval, $\bar{k}$. The resource, k, converges asymptotically to $\bar{k}$ from any $k(0) \in [\underline{k}, \bar{k})$, guaranteeing an interior solution.

This asymptotic-convergence property guarantees that the transversality condition holds as well. Specifically, from (69) we can see that,

$$\lim_{t\to\infty} e^{-\rho t} J'(k(t))\, k(t) = \left[\theta - \frac{1}{\eta^{\frac{1}{2}}}\left(\bar{k} - \psi\right)^{\frac{1}{2}}\right] \bar{k} \lim_{t\to\infty} e^{-\rho t} = 0 \ .$$

We conclude proving the uniqueness of the optimal response to symmetric strategies played by other players for the case of setting $\omega = 0$ in equation (20) for this novel example presented in this section, by demonstrating that $C(k)$ given by (72) is the unique maximizer of $\mathcal{P}_i$, for all $i \in \{1, ..., N\}$ (see Definition 1 and set $C(k) = C^*(k)$). First, notice that, under the parametric constraint (94),

$$C''(k) = \frac{-2(1-\theta)}{9\kappa^2\eta^{\frac{1}{2}}}(k-\psi)^{-\frac{3}{2}} > 0 \ , \tag{95}$$

which is an implication of (85). For a symmetric equilibrium in $C(k)$, a player's Hamiltonian is,

$$\mathcal{H} = u(c) + \lambda\left[Ak - (N-1)\,C(k) - c\right] \ .$$

The Hessian matrix of this Hamiltonian with respect to variables (c, k) is,

$$H_{\mathcal{H}} = \begin{bmatrix} u''(c) & 0 \\ 0 & -\lambda (N-1) C''(k) \end{bmatrix} .$$

Given that $u''(c) < 0$, (95) implies that $H_{\mathcal{H}}$ is negative definite. Therefore, given Mangasarian's theorem (see, for example, Sydsaeter et al., 2008, p. 330, Theorem 9.7.1), the optimal strategy, $C(k)$, that solves the individual problem of player $i \in \{1, ..., N\}$, is a unique maximizer in response to the symmetric strategies, $C(k)$, played by the other $N-1$ players.

An alternative interpretation and application of this game is the case where N monopolists, co-exploit a common-property resource each supplying to a different market at zero cost, facing a constant interest rate equal to ρ.[24] The HJB equation of player $i \in \{1, ..., N\}$ in such a setup is,

$$\rho J_i(k) = \max_{q_i \geq 0} \left\{ p(q_i) q_i + J_i'(k) \left[Ak - \sum_{j \neq i} Q_j^*(k) - q_i \right] \right\} ,$$

in which q_i is the extracted and supplied quantity of the common resource by monopolist i and $Q_j^*(k)$ is the optimal Markovian strategy of oligopolist $j \neq i$. The first-order conditions of this problem are,

$$p'(q_i) q_i + p(q_i) = J_i'(k) \ . \tag{96}$$

Finding the inverse of the derivative of the revenue function in (96), where $R(q_i) = p(q_i) q_i$ is the revenue function and $R'(q_i) = p'(q_i) q_i + p(q_i)$, can be challenging to do analytically,

[24]Think, for example, of a railroad that is provided exogenously by a government, with railway companies utilizing this railroad infrastructure in a rivalrous and non-excludable, manner, at no cost. This infrastructure, k, can depreciate with utilization, i.e., by the number of passengers of each company, q_i, according to an endogenous depreciation function that is linear in q_i, say, $\delta(q_i) = \psi k_i$, and parameter A in the law of motion of k is normalized so as to set $\psi = 1$. A discrete-time version of this setup is given, for example, in Koulovatianos and Mirman (2007, p. 203).

unless we express the inverse demand function in the form,

$$p(q_i) = r(q_i)\, q_i^{-1} \ .$$

In this case, the revenue function, $R(q_i) = p(q_i)\, q_i = r(q_i)$, and (96) becomes $r'(q_i) = J'(k)$, exactly as in (4), making this game identical to the one examined in this paper, after setting $r(q) = u(q)$. An advantage of this interpretation is that much of the literature focuses on linear-demand functions, making the revenue function quadratic, while the method we propose can lead to other functional forms for the inverse-demand function.[25] In the example examined here, the inverse-demand function would be,

$$p(q_i) = 1 - \kappa q_i^{\frac{1}{2}} \ .$$

7. Conclusion

We have provided a thorough and comprehensive characterization of the interior solution to the class of symmetric Markovian differential games of commons problems with linear constraints. For a broad class of time separable utility functions that depend only on the player's control variable and that allow for interior solutions, we have provided an exact interior solution to the problem when the coefficient of the linear resource reproduction function differs from the rate of time preference ($A \neq \rho$). The solution to the special case where the rate of time preference equals the coefficient of the linear resource reproduction function is given as an implicit function of a simple expression. In the more interesting and more common case of $A \neq \rho$, our analytical approach involves a differential equation with an explicit solution involving an integration constant. When we give this constant the value of zero, we obtain an analytical for the Markovian strategies. This particular case with the zero-integration constant, quickly leads to the verification of closed-form solutions.

[25]We thank an anonymous referee for pointing this interpretation to us.

Moreover, our solution gives an immediate result regarding analytic functions. If the utility function is analytic, then the resulting Markovian strategies are also analytic functions. This analyticity property can facilitate the numerical computation of such games using, e.g. polynomial approximations for value functions and Markovian strategies. Additionally, the analyticity property can be useful in numerically approximating commons problems with corner solutions. For the cases where the integration constant is not equal to zero, multiple Markovian strategies can arise. This case can be intractable analytically, but it can help in either characterizing these multiple equilibria, or in numerically computing them. An interesting extension of our findings is to study conditions for the sustainability of cooperation as Jørgensen, Martin-Herran and Zaccour (2005) have done for linear-quadratic games. Finally, a future extension could be to characterize this class of games for a finite horizon.

8. Appendix

Derivation of function $\phi(\lambda)$ in equation (59)

Substituting (58) into (19) leads to,

$$\phi(\lambda) = \frac{1}{A-\rho}\left\{\frac{4(N-1)}{9\kappa^2}(1-\lambda)^2 + [N+\xi(N-1)]\frac{4\lambda^{\xi}}{9\kappa^2}\int \lambda^{-\xi-1}(1-\lambda)^2\, d\lambda\right\} . \quad (97)$$

To calculate the integral in (97) we expand the quadratic form, namely,

$$\int \lambda^{-\xi-1}(1-\lambda)^2\, d\lambda = \int \left(\lambda^{-\xi-1} - 2\lambda^{-\xi} + \lambda^{-\xi+1}\right) d\lambda ,$$

which leads to,

$$\int \lambda^{-\xi-1}(1-\lambda)^2\, d\lambda = \frac{1}{-\xi+2}\lambda^{-\xi}\left(\lambda^2 - 2\frac{-\xi+2}{-\xi+1}\lambda + \frac{-\xi+2}{-\xi}\right) . \quad (98)$$

Combining (98) with (97) gives,

$$\phi(\lambda) = \alpha\cdot\left(\lambda^2 - 2\lambda + 1\right) + \beta\cdot\left(\lambda^2 - 2\frac{-\xi+2}{-\xi+1}\lambda + \frac{-\xi+2}{-\xi}\right) , \quad (99)$$

where

$$\alpha \equiv \frac{4(N-1)}{9\kappa^2(A-\rho)} \quad \text{and} \quad \beta \equiv \frac{4[N+\xi(N-1)]}{9\kappa^2(A-\rho)(-\xi+2)} . \quad (100)$$

Collecting terms in (99) leads to,

$$\phi(\lambda) = (\alpha+\beta)\left[\lambda^2 - 2\frac{\alpha+\zeta\beta}{\alpha+\beta}\lambda + \left(\frac{\alpha+\zeta\beta}{\alpha+\beta}\right)^2\right] + \alpha + \beta\frac{-\xi+2}{-\xi} - (\alpha+\beta)\left(\frac{\alpha+\zeta\beta}{\alpha+\beta}\right)^2 ,$$

or,

$$\phi(\lambda) = (\alpha+\beta)\left(\lambda - \frac{\alpha+\zeta\beta}{\alpha+\beta}\right)^2 + \alpha + \beta\frac{-\xi+2}{-\xi} - (\alpha+\beta)\left(\frac{\alpha+\zeta\beta}{\alpha+\beta}\right)^2 , \quad (101)$$

where,

$$\zeta \equiv \frac{-\xi+2}{-\xi+1} . \quad (102)$$

Substituting the expressions for α, β, and ζ given by (100) and (102) into (101), gives equation (59), together with the expressions given by (60), (61) and (62). □

Proof of inequality (66)

Fix any value of ρ and observe that

$$\theta = F(A)\,G(N) \ , \tag{103}$$

where,

$$F(A) = \frac{3A - 2\rho}{2A - \rho} \ , \tag{104}$$

and,

$$G(N) = \frac{2N - 1}{3N - 2} \ . \tag{105}$$

Notice that, according to (65), $A > 2/3\rho > 1/2\rho$, and therefore, $F(A) > 0$. Moreover,

$$F'(A) = \frac{\rho}{(2A - \rho)^2} > 0 \ , \tag{106}$$

and

$$\lim_{A \downarrow \frac{2}{3}\rho} F(A) = 0 \ , \quad \text{and} \ \lim_{A \to \infty} F(A) = \frac{3}{2} \ . \tag{107}$$

Combining (107) with (106) gives,

$$0 < F(A) < \frac{3}{2} \quad \text{for all } A \text{, given any } \rho \text{ complying with (65)} \tag{108}$$

Similarly, notice that,

$$G'(N) = \frac{-1}{(3N - 2)^2} < 0 \ , \tag{109}$$

while,

$$G(1) = 1 \ \text{ and } \ \lim_{N \to \infty} G(N) = \frac{2}{3} \ . \tag{110}$$

Therefore, (103), (109) and (110) imply,

$$\frac{2}{3} < G(N) \leq 1 \text{ , for all } N \in \{1, 2, ...\} \ . \tag{111}$$

Combining, (103), (109) and (111) proves inequality (66). □

Why the case of $\lambda - \theta \geq 0$ in equation (63) is not admissible

Substituting $\lambda - \theta \geq 0$ into (63) gives,

$$\lambda = \phi^{-1}(k) = \frac{1}{\eta^{\frac{1}{2}}}(k - \psi)^{\frac{1}{2}} + \theta \ , \ \text{ if } \lambda \geq \theta \ . \tag{112}$$

Recall that $\lambda = J'(k)$. Differentiating the right-hand side of (112) we can see that $J''(k) = 1/\left(2\eta^{1/2}\right)(k - \psi)^{-1/2} > 0$ for all $k \geq \max\{0, \psi\}$. Yet, $J''(k) > 0$ is not a property of the value function that complies with the transversality condition. To see this, consider the first-order condition given by (5), which implies,

$$u'(C(k)) = J'(k) \ . \tag{113}$$

Differentiating both sides of (113) implies,

$$J''(k) = u''(C(k))\, C'(k) \ . \tag{114}$$

Because $u''(c) < 0$ for all c complying with (57),

$$J''(k) > 0 \ \text{ combined with (114) imply that } C'(k) < 0 \ . \tag{115}$$

Yet, remember that the budget constraint given by (1) implies,

$$\dot{k}(t) = Ak(t) - NC(k(t)) \ . \tag{116}$$

Combining (116) with $C'(k) < 0$ means that the right-hand side of equation (116) is upward-sloping in $k(t)$. Based on (21), we can combine (112) with (58) to obtain the explicit formula for $C(k)$, namely,

$$C(k) = \frac{4}{9\kappa^2}\left[1-\theta-\frac{1}{\eta^{\frac{1}{2}}}(k-\psi)^{\frac{1}{2}}\right]^2 . \tag{117}$$

Using (117) we derive the first and second derivatives of the strategies $C(k)$, i.e.,

$$C'(k) = \frac{-4}{9\kappa^2\eta^{\frac{1}{2}}}\left[(1-\theta)(k-\psi)^{-\frac{1}{2}}-\frac{1}{\eta^{\frac{1}{2}}}\right] . \tag{118}$$

Notice that (118) combined with (115) implies that

$$C'(k) < 0 \quad \text{holds if} \quad k < \psi + \eta(1-\theta)^2 . \tag{119}$$

In addition, (118) implies,

$$C''(k) = \frac{2}{9\kappa^2\eta^{\frac{1}{2}}}(1-\theta)(k-\psi)^{-\frac{3}{2}} > 0 . \tag{120}$$

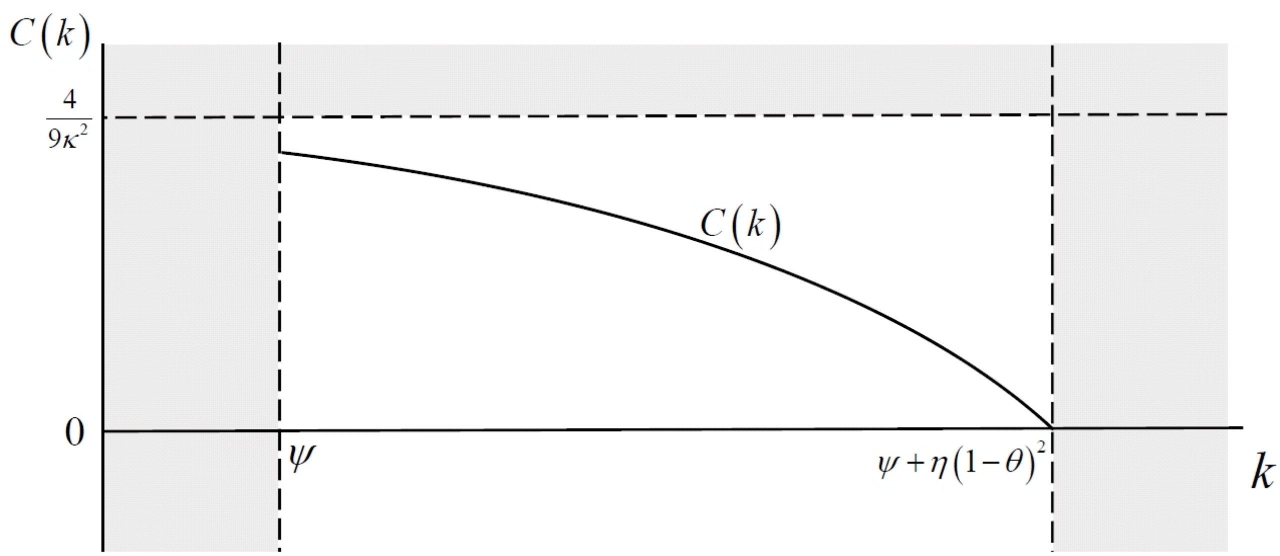

Figure A.1 Properties of the decision rule if $\lambda - \theta \geq 0$

All properties of $C(k)$ described by (57), (117), (118), (119), and (120) are depicted by Figure A.1, where the shaded areas indicate value regions where the strategies $C(k)$ are not defined.

Without loss of generality, Figure A.1 depicts a case where $\psi > 0$. The case of $\psi \leq 0$ would simply depict a picture with $C(k)$ exhibiting the same properties for $k \in \left[0, \psi + \eta(1-\theta)^2\right]$.

Introducing strategies $C(k)$ into (1) we obtain,

$$\dot{k} = Ak - NC(k) \ . \tag{121}$$

Differentiating (121) with respect to k we obtain,

$$\frac{\partial \dot{k}}{\partial k} = A - NC'(k) > 0 \ , \quad \text{for all } k \in \left[\max\{0, \psi\}, \psi + \eta(1-\theta)^2\right] \ . \tag{122}$$

Equation (122) is a consequence of equation (119). The key message of (122) it implies dynamics of k. These unstable dynamics of k imply a violation of the feature that the solution is interior. In the absence of an interior solution, Proposition 1 does not apply and, therefore, the closed form solution of the strategies, $C(k)$, given by (117), is invalid.

Figures A.2 and A.3 depict (121) and the dynamics of k, based on all parametric cases. Specifically, we distinguish cases of parametric values of A such that $\psi > 0$ and otherwise. Based on equation (60), after some algebra, and making use of the parametric constraint given by (65), we can show that,

$$\psi > 0 \Leftrightarrow \left(A - \frac{3N-2}{4N-3}\rho\right)(A - \rho N) > 0 \Leftrightarrow A \in \left(\frac{2}{3}\rho \ , \ \frac{3N-2}{4N-3}\rho\right) \cup (\rho N \ , \ \infty) \ , \tag{123}$$

$$\psi = 0 \Leftrightarrow A = \rho N \ \text{ or } A = \frac{3N-2}{4N-3}\rho \ , \tag{124}$$

and

$$\psi < 0 \Leftrightarrow A \in \left(\frac{3N-2}{4N-3}\rho \ , \ \rho N\right) \ . \tag{125}$$

A common feature between Figures A.2 and A.3 is that when $k = \psi + \eta(1-\theta)^2$, which is the upper bound of k for which $C(k)$ is admissible in this case of $\lambda - \theta \geq 0$, $\dot{k} > 0$. To see this, insert $k = \psi + \eta(1-\theta)^2$ into (121) to obtain,

$$\left.\dot{k}\right|_{k=\psi+\eta(1-\theta)^2} = A\left[\psi + \eta(1-\theta)^2\right] > 0 \ . \tag{126}$$

Inequality (126) justifies why in both Figures A.2 and A.3 the curve depicting the law of motion for k is above the 0-line.

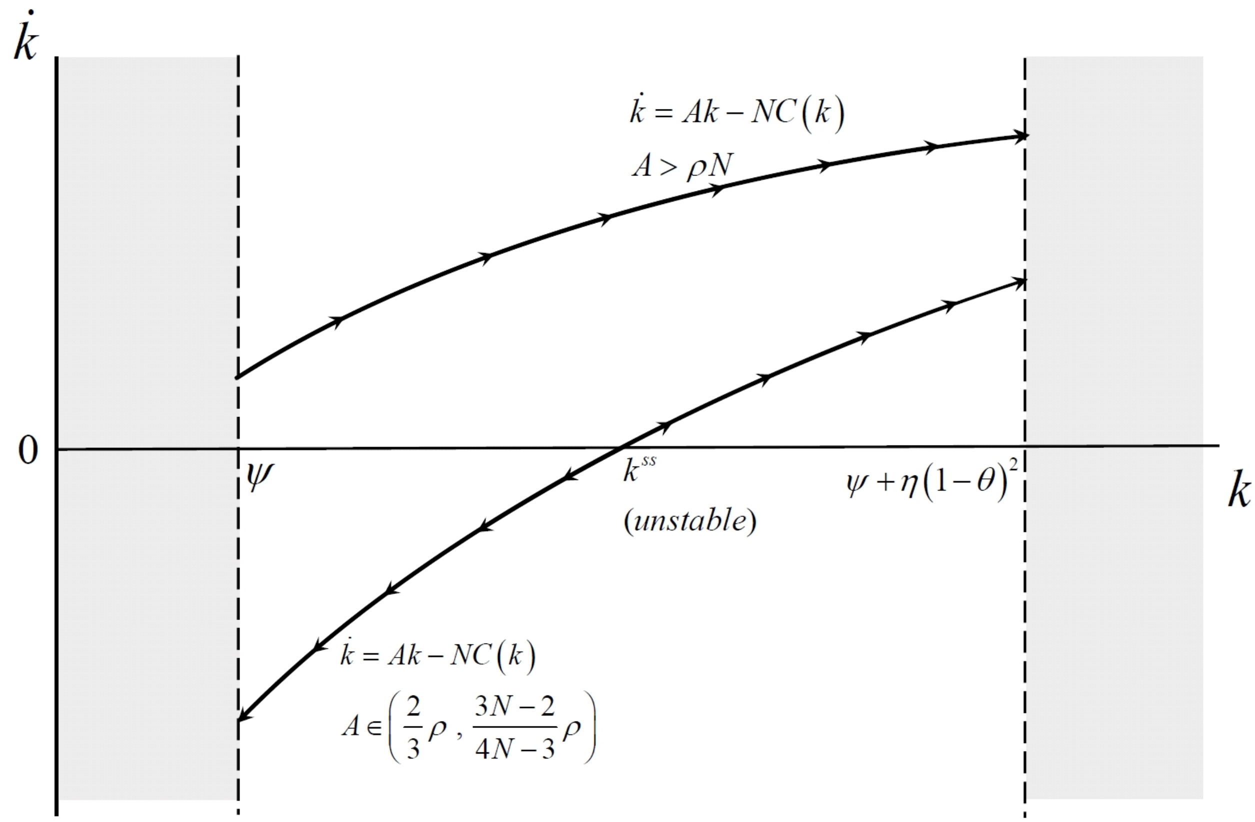

Figure A.2 Resource dynamics in the case where A is such that $\psi > 0$.

To understand why there are two curves depicting (121) in Figure A.2, which focuses on parameter values implying $\psi > 0$, consider the equivalence given by (123) and focus on the specific value of k, $k = \psi$. By inserting $k = \psi$ into (121),

$$\dot{k}\Big|_{k=\psi} = A\psi - \frac{4}{9\kappa^2}(1-\theta)^2 > 0 \Leftrightarrow (N-1)(A-\rho N) > 0 \ . \tag{127}$$

In the trivial case of $N = 1$, $\dot{k}\Big|_{k=\psi} = 0$. Yet, this does not correspond to an interior solution with free initial conditions. We therefore focus on cases with $N \geq 2$. When $N \geq 2$, the equivalence given by (127) implies that,

$$\dot{k}\Big|_{k=\psi} \gtreqless 0 \Leftrightarrow A \gtreqless \rho N \ . \tag{128}$$

Given that,

$$\frac{3N-2}{4N-3} \in \left(\frac{3}{4}, 1\right) \quad \text{for all } N \in \{2, 3, ...\} \, ,$$

the two curves depicting (121) in Figure A.2 are justified. The equivalence implied by (123) implies that, in the case where $A \in (2/3\rho \ , \ (3N-2)\,/\,(4N-3)\,\rho)$, there is a value k^{ss} for which $\dot{k}\big|_{k=k^{ss}} = 0$. Yet, this unstable 0-growth value does not correspond to an interior solution with free initial conditions for the problem.

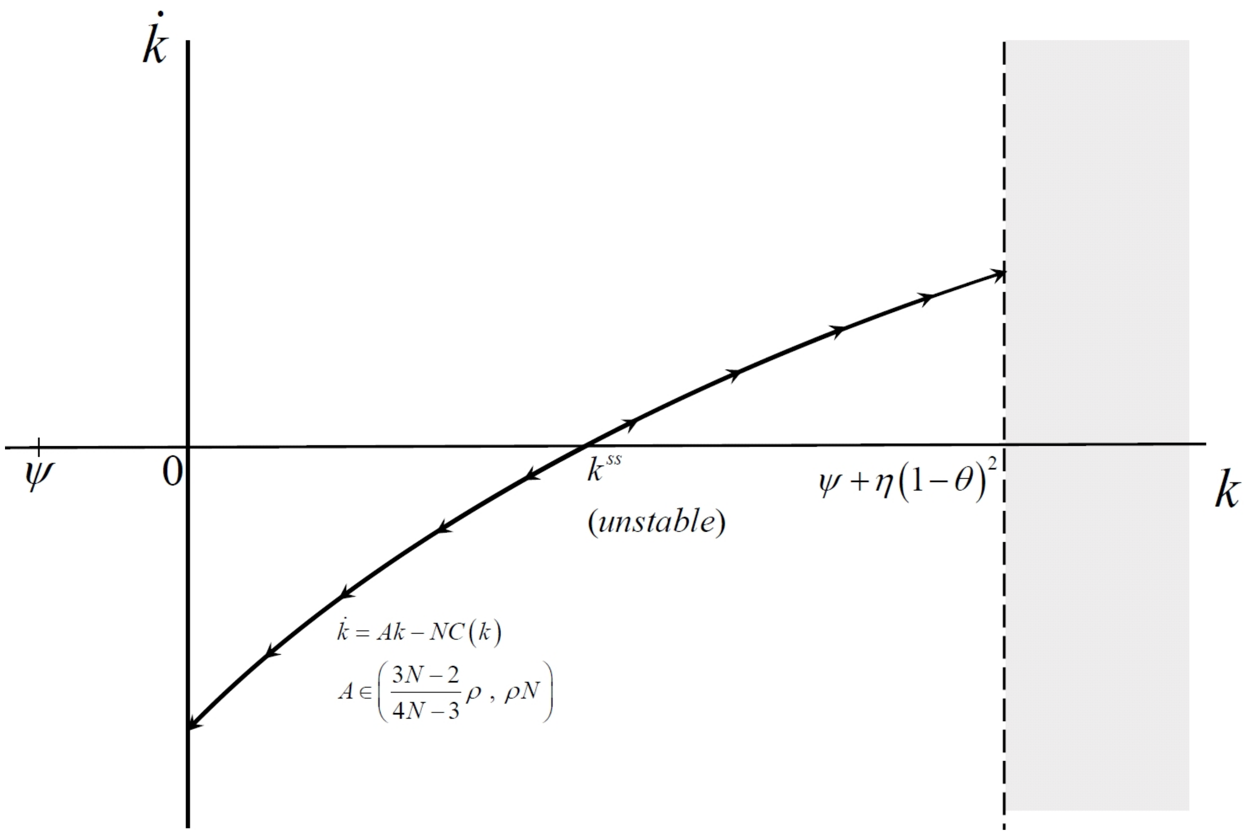

Figure A.3 Resource dynamics in the case where A is such that $\psi < 0$.

Figure A.3 focuses on the case implied by (125). Because of (128), in Figure A.3 we have once more a value k^{ss} for which $\dot{k}\big|_{k=k^{ss}} = 0$. Again, this unstable 0-growth value does not correspond to an interior solution with free initial conditions for the problem. The same problem arises for the two specific values of A given by (124), for which $\psi = 0$.

In summary, the case of $\lambda - \theta \geq 0$ does not correspond to an interior solution and it should, therefore, be discarded. □

Proof of equivalence (85)

To prove that $\theta \gtreqless 1 \Leftrightarrow A \gtreqless \rho N$, use (61) to obtain,

$$\theta \gtreqless 1 \Leftrightarrow \frac{3A - 2\rho}{2A - \rho} \frac{2N - 1}{3N - 2} \gtreqless 1 \ . \tag{129}$$

Based on the parametric constraint given by (75) numerators and denominators in the fractions appearing in (129) are strictly positive. This feature leads to verifying that

$$\frac{3A - 2\rho}{2A - \rho} \frac{2N - 1}{3N - 2} \gtreqless 1 \Leftrightarrow A \gtreqless \rho N \ ,$$

which confirms the first part of (85), that $\theta \gtreqless 1 \Leftrightarrow A \gtreqless \rho N$.

For proving the second part of (85), that $\bar{k} \gtreqless \eta \Leftrightarrow A \gtreqless \rho N$, observe that (62) and (73) imply,

$$\frac{\bar{k}}{\eta} = \frac{(3A - 2\rho) N}{3AN - 2A}. \tag{130}$$

Using the parametric constraint given by (75), which also implies $\eta > 0$, together we can show that

$$\frac{\bar{k}}{\eta} \gtreqless 1 \Leftrightarrow A \gtreqless \rho N \ ,$$

which proves the second part of (85), that $\bar{k} \gtreqless \eta \Leftrightarrow A \gtreqless \rho N$.

Finally, for proving that $\bar{k}\,(2 - \theta) - \eta\theta \gtreqless 0 \Leftrightarrow A \gtreqless \rho N$, use (61) and (62) to see that,

$$\bar{k}\,(2 - \theta) - \eta\theta \gtreqless 0 \Leftrightarrow \frac{N}{A}\left(2 - \frac{6AN - 3A - 4\rho N + 2\rho}{6AN - 4A - 3\rho N + 2\rho}\right) \gtreqless \frac{2N - 1}{2A - \rho} \Leftrightarrow$$

$$\Leftrightarrow \frac{6AN - 5A - 2\rho N + 2\rho}{2A - \rho} \gtreqless \frac{A}{N} \frac{2N - 1}{2A - \rho} \Leftrightarrow$$

$$\Leftrightarrow 2\left(3A - \rho\right) N^2 - \left(5A - 2\rho\right) N \gtreqless A\left(6N^2 - 7N + 2\right) \Leftrightarrow$$

$$\Leftrightarrow \left(N - 1\right)\left(A - \rho N\right) \gtreqless 0 \ ,$$

confirming that $\bar{k}\left(2 - \theta\right) - \eta\theta \gtreqless 0 \Leftrightarrow A \gtreqless \rho N$ for all $N \geq 2$. □

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