Nonlinear Dynamics in Classical and Quantum Systems

Dissertation
zur Erlangung des Doktorgrades
der Naturwissenschaften

vorgelegt beim Fachbereich Physik
der Johann Wolfgang Goethe-Universität
in Frankfurt am Main

von
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aus Mexiko-Stadt, Mexiko

Frankfurt am Main 2020
D30

vom Fachbereich Physik der
Johann Wolfgang Goethe-Universität als Dissertation angenommen.

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Datum der Disputation: 23.06.2020

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Mathematical Notations

$\varphi : M \to N$	mapping of a manifold M onto another manifold N
$\varphi : m \mapsto n = \varphi(m)$	effect of φ on $m \in M$
i	imaginary unit $\sqrt{-1}$
$\mathbb{R}^n$	Euclidian space of dimension n
$\mathbb{C}^n$	complex hyperplane of dimension $2n$
$\mathbb{RP}^1$	real projective line
$\mathbb{CP}$	complex projective space
$\mathbb{D}^2$	Poincaré disk of dimension 2
$\mathbb{HP}^2$	Siegel upper half plane of dimension 2
$\mathcal{H}$	Hilbert space
$f \in \mathfrak{F}(M)$	C^∞ functions on M
$X \in \mathfrak{X}(M)$	vector fields in M
$\theta \in \Omega^k(M)$	k-forms on M
TM	tangent bundle of M
T^*M	cotangent bundle of M
$U \subset M$	open submanifold or neighborhood in M
(U, φ), $\varphi : U \to V \subset \mathbb{R}^n$	local chart of a manifold of dimension n
$\wedge$	exterior product
i_X	inner product, contraction
$\otimes$	tensor product
d	exterior derivative
$\pounds_X$	Lie derivative with respect to (along) X
$\mathrm{Lin}(E, F)$	linear map from E onto F
ω	symplectic form
g	Riemannian metric
$\mathcal{D}$	distribution
H	Hamiltonian function
$\mathfrak{L}$	Lagrangian function
$\mathscr{H}$	contact Hamiltonian
$\mathscr{L}$	contact Lagrangian

Zusammenfassung

Zweck der Untersuchungen ist es, die statistisch-probabilistischen Inhalte der Quantenmechanik vom üblichen Hilbertraum-Formalismus auf eine allgemeinere differentialgeometrische Beschreibung zu transferrieren, was zu einer Formulierung der Quantendynamik mittels nichtlinearer komplexer Riccati-Gleichungen führt. Dies hat neben der formalen Verallgemeinerung nicht nur den Vorteil, dass wegen der quadratischen Nichtlinearität der Riccati-Gleichung die intrinsische Verknüpfung von Real- und Imaginärteil (bzw. Amplitude und Phase) der für die Quantenmechanik essentiellen Komplexen Größen zum Ausdruck kommt, sondern eröffnet auch neue Zugänge zur Beschreibung dissipativer Systeme.

In Kapitel 1 soll der Übergang von der linearen Schrödinger-Dynamik zu einer nichtlinearen Riccati-Dynamik zunächst anhand eines N-Niveau-Quantensystems demonstriert werden. Um einen solchen Übergang herbeizuführen, erinnern wir uns daran, dass ein reiner Zustand in der Quantenmechanik eine Äquivalenzklasse ist, das heisst eine vollständige Messung in der Quantenmechanik liefert uns keinen eindeutig definierten Vektor im Hilbert-Raum, sondern eine Äquivalenzklasse von Vektoren, erhalten durch Multiplikation mit einer komplexen Zahl. Das heisst auf dem Hilbert-Raum gibt es eine natürliche Wirkung der Abelschen Gruppe $\mathbb{C}_0 = \mathbb{C} - \{\,\mathbf{0}\,\}$, gegeben durch $|\psi\rangle \;\mapsto \lambda\,|\psi\rangle \;=\; \varrho\,\mathrm{e}^{\mathrm{i}\,\theta}\,|\psi\rangle$ mit $\varrho > 0$. Ein reiner Zustand ist ein Strahl im Hilbert-Raum.

Die Äquivalenzklasse kann infinitesimal durch zwei Vektorfelder in Involution beschrieben werden: Das Dilatationsvektorfeld Δ und das Vektorfeld Γ, verknüpft mit der Multiplikation mit einem Phasenfaktor. Diese Tatsache ermöglicht die Reduktion der Dynamik auf einen Raum mit niedrigerer Dimension, insbesondere auf den komplexen projektiven Raum $\mathbb{CP}$ des Hilbert-Raums, was ein Beispiel für eine Hilbert-Mannigfaltigkeit ist. Wenn wir nun konkret die Distribution $\mathcal{D}_2 = \{\Gamma, \Delta\}$ in Betracht ziehen, welche involutiv ist, entsteht eine Blätterung $\Phi^{\mathcal{D}_2}$, die regulär ist. Daraufhin lässt sich der Quotientenraum $\mathcal{H}_0$ zu $\mathbb{CP}(\mathcal{H}_0)$, mit $\mathcal{H}_0 = \mathcal{H} - \{\mathbf{0}\}$, definieren, der als komplexer projektiver Raum $\mathbb{CP}(\mathcal{H}_0)$ dem $\mathcal{H}_0$ zugeordnet ist, d.h. $\mathbb{CP}(\mathcal{H}_0) := \{\lambda\,|\psi\rangle|\;\lambda \in \mathbb{C}_0\,\}$. Die Projektion von $\mathcal{H}_0$ nach $\mathbb{CP}(\mathcal{H}_0)$ wird mit π und die Elemente von $\mathbb{CP}(\mathcal{H}_0)$ mit $[\psi]$ bezeichnet, wobei die $[\psi]$ mit den reinen Zuständen eines Quantensystems identifiziert werden können. Sie stehen Eins-zu-Eins in Korrespondenz mit Projektoren vom Rang-1, d.h. $[\psi] \mapsto \rho_\psi$. Demzufolge ist $\mathbb{CP}(\mathcal{H}_0)$ der Raum physikalischer Zustände, die normiert sind und in dem man die globale Phase beseitigt hat. Dies bedeutet, dass eine Beschreibung des Raumes reiner Zustände ohne Redundanzen angestrebt wird. Diese Beschreibung der Quantendynamik wird als komplexes projektives Bild von Quantensystemen bezeichnet. In dieser Arbeit werden wir uns an der Hilbert-Mannigfaltigkeit orientieren und somit an dem komplexen projektiven Raum, um seine intrinsische "Nichtlinearität" zu betonen.

Für die Analyse auf dem $\mathbb{CP}$ wird ausgenutzt, dass es möglich ist, den komplexen projektiven Raum mittels komplexer homogener Koordinaten vollständig zu beschreiben, so dass in diesen Koordinaten die Schrödinger-Dynamik in die Riccati-Dynamik projiziert wird. Darüber hinaus beweisen wir, dass nicht nur die Dynamik projizierbar ist, sondern auch wichtige geometrische Strukturen. Im komplexen projektiven Raum $\mathbb{CP}$ werden eine symplektische Form ω_{FS}, eine Riemannsche-Metrik g_{FS} und das $(1,1)$-Tensorfeld J_{FS} als komplexe Strukturen definiert. Folglich besitzt der komplexe projektive Raum eine Kähler-Struktur $(\omega_{\mathrm{FS}}, g_{\mathrm{FS}}, J_{\mathrm{FS}})$.

Die symplektische Form und die Riemannsche Metrik definieren ein Hamilton-Vektorfeld sowie ein Gradientenvektorfeld. Dies bedeutet, dass jeder Funktion $e_A \in \mathfrak{F}(\mathbb{CP})$ ein Gradientenvektorfeld und ein Hamilton-Vektorfeld mittels der intrinsischen Definitionen $g_{\mathrm{FS}}(Y_{e_A}, \cdot\,) = \mathrm{d}\,e_A$ und $\omega_{\mathrm{FS}}(X_{e_A}, \cdot\,) = \mathrm{d}\,e_A$ zugeordnet werden, so dass $J_{\mathrm{FS}}(X_{e_A}) = Y_{e_A}$ gilt. Insbesondere soll der Erwartungswerts des Hamilton-

Operators betrachtet werden, d. h.

$$\begin{aligned} e_H &= \frac{1}{1+\mathbf{z}^\dagger\,\mathbf{z}}\;(\,\mathbf{z}^\dagger\;,\;1\,)\begin{pmatrix}\mathbb{H}_1 & \mathbb{V}\\ \bar{\mathbb{V}}^\dagger & H_2\end{pmatrix}\begin{pmatrix}\mathbf{z}\\ 1\end{pmatrix}\\ &= \frac{1}{1+\mathbf{z}^\dagger\,\mathbf{z}}\left(\,\mathbf{z}^\dagger\mathbb{H}_1\mathbf{z}+\mathbf{z}^\dagger\mathbb{V}+\mathbb{V}^\dagger\mathbf{z}+H_2\,\right), \end{aligned} \tag{1}$$

wobei die Matrix $\mathbb{H}_1$ eine $(N-1)\times(N-1)$–dimensionale Matrix ist, $\mathbb{V}$ ein $(N-1)$–Komponenten Spaltenvektor und H_2 eine reelle Zahl. Konkret bedeutet dies, dass die von der Hamilton-Funktion induzierte Dynamik folgendermaßen dargestellt werden kann;

$$X_{e_H} = \frac{\mathrm{i}}{\hbar}\left(z^k\,\bar{V}_l\,z^l - |\mathbb{H}_1|^k_l\,z^l + H_2\,z^k - V^k\right)\frac{\partial}{\partial z^k} - \frac{\mathrm{i}}{\hbar}\left(\bar{z}_k\,V^l\,\bar{z}_l - |\mathbb{H}_1|^l_k\,\bar{z}_l + H_2\,\bar{z}_k - \bar{V}_k\right)X_{\bar{z}_k}\,\frac{\partial}{\partial\bar{z}_k}\,. \tag{2}$$

Somit sind die Integralkurven dieses Hamilton-Vektorfeldes eines N-Niveau Quantensystems durch die Lösungen der folgenden Matrix-Riccati-Gleichung gegeben

$$\dot{z}^k = \frac{\mathrm{i}}{\hbar}\left(z^k\,\bar{V}_l\,z^l - |\mathbb{H}_1|^k_l\,z^l + H_2\,z^k - V^k\right). \tag{3}$$

Ferner ist es möglich, die Poisson- und Jordan-Klammern für den komplexen projektiven Raum einzuführen. Für gegebene Erwartungswerte e_A und e_B, die den Observablen A und B zugeordnet sind, ergibt sich, dass

$$\{e_A,e_B\}_{\omega_{\mathrm{FS}}} = \omega_{\mathrm{FS}}(X_{e_A},X_{e_B})\,,\qquad \{e_A,e_B\}_{g_{\mathrm{FS}}} = g_{\mathrm{FS}}(Y_{e_A},Y_{e_B})\,. \tag{4}$$

Nach weiteren Rechenschritten kann gezeigt werden, dass die Poisson-Klammen und Jordan-Klammen wie folgt lauten:

$$\{e_A,e_B\}_{\omega_{\mathrm{FS}}} = e_{\frac{1}{\mathrm{i}\hbar}[A,B]_-}\qquad\text{und}\qquad \{e_A,e_B\}_{g_{\mathrm{FS}}} = -\frac{2}{\hbar}\left[\,\frac{1}{2}\,e_{[A,B]_+} - e_A\,e_B\,\right], \tag{5}$$

wobei $[A,B]_- = A\,B - B\,A$ und $[A,B]_+ = A\,B + B\,A$. Demzufolge besteht ein klarer Zusammenhang zwischen den Poisson-Klammern und dem Quantenkommutator. Andererseits sind die Jordan-Klammern mit der Dispersion und der Korrelation der Observablen verknüpft. Dies bedeutet, dass für jedes Observablen-Paar A und B die Unbestimmtheiten und Korrelationen jeweils gegeben sind durch

$$\sigma_A^2 = e_{A^2} - e_A^2 = -\frac{\hbar}{2}\{e_A,e_A\}_{g_{\mathrm{FS}}}\quad\text{und}\quad \sigma_{AB} = \frac{1}{2}\,e_{[A,B]_+} - e_A\,e_B = -\frac{\hbar}{2}\{e_A,e_B\}_{g_{\mathrm{FS}}}\,. \tag{6}$$

Demzufolge berücksichtigt die vom komplexen projektiven Raum getragene Riemannsche Metrik den probabilistischen Charakter der Quantenmechanik.

Außerdem muss man feststellen, dass in der nichtlinearen Formulierung die Evolution reiner Zustände durch Lösungen einer Riccati-Gleichung beschrieben wird, die, obwohl diese Gleichung nichtlinear ist, ihren Ursprung in der linearen Schrödinger-Gleichung mit linearem Superpositionsprinzip haben. Demnach muss es einen Weg geben, Lösungen der Riccati-Gleichung zu überlagern, was zu einem nichtlinearen Superpositionsprinzip in $\mathbb{CP}$ führt. Der überlagerte Zustand $|\Psi\rangle = p_1\,|\psi\rangle + p_2\,|\phi\rangle$ wird dann in den homogenen Koordinaten des $\mathbb{CP}$ durch den Vektorzustand $\mathbf{Z}$ mit Komponenten

$$Z^k = \frac{p_1\,z^k_\psi\,(z^k_0 - z^k_\phi) + p_2\,z^k_\phi\,(z^k_\psi - z^k_0)}{p_1(z^k_\phi - z^k_0) + p_2(z^k_\psi - z^k_0)} \tag{7}$$

mit $k=1,\ldots,n-1$ dargestellt, wobei $z^k_\psi = \frac{\psi^k}{\psi^n}$, $z^k_\phi = \frac{\phi^k}{\phi^n}$ gilt und z^k_0 die k-te Komponente einer beliebigen Lösung $\mathbf{z}_0$ ist.

Für die weitere Untersuchung der nichtlinearen Dynamik in der Quantenmechanik wechseln wir von endlichen zu unendlichen Quantensystem. Wir beschränken dabei unsere Studie in Kapitel 2 auf die

sogenannten generalisierten kohärenten Zustände. Daraufhin wird in diesem Kapitel gezeigt, dass die Dynamik der verallgemeinerten kohärenten Zustände eine inhärente nichtlineare Evolution aufweist. Hierfür bedienen wir uns der Tatsache, dass es in der Quantenmechanik möglich ist, eine "klassische" Mannigfaltigkeit in den Hilbert-Raum einzubettenn, so dass man die Zeitabhängigkeit der Wellenfunktion durch die Variation der Parameter der klassischen Mannigfaltigkeit parametrisieren kann.

Es lässt sich erkennen, dass es keine eindeutige Parametrisierung der verallgemeinerten kohärenten Zustände gibt. Tatsächlich findet man eine Parametrisierung durch die komplexe Zahl $\alpha \in \mathbb{C}$, eine Beschreibung, die üblicherweise als kohärenter Zustand bezeichnet wird. Ferner lässt sich in der Quantenoptik auch feststellen, dass die Gaußschen Wellenpakete durch den Squeezing-Parameter $\xi \in \mathbb{C}$ parametrisiert werden können, was auch als Gequetschte Zustände (squeezed states) bekannt ist, oder man parametrisiert sie mittels der Lösungen der nichtlinearen komplexen Riccati (oder reellen Ermakov) Gleichung. Aus physikalischer Sicht sind alle diese Darstellungen gültig, daher sollte man in der Lage sein, die Verbindungen zwischen diesen unterschiedlichen Beschreibungen herzustellen. In Kapitel zwei fokussieren wir uns deshalb darauf, dieses Problem aus geometrischer Perspektive zu analysieren. Dies bedeutet, dass wir den geometrischen Raum festlegen, in dem die verschiedenen Dynamiken stattfinden, zeigen, wie diese im Hilbert-Raum $\mathcal{H}$ eingebettet (immersed) sind sowie den Zusammenhang zwischen den verschiedenen Beschreibungen herstellen.

Der Einfachheit halber betrachten wir die verallgemeinerten kohärenten Zustände für ein eindimensionales Quantensystem; dann wird die Einbettung wie folgt konstruiert: Für einen gegebenen normierten Zustand $|0\rangle$ im Hilbert-Raum $\mathcal{H}$, wenden wir die stark kontinuierliche Abbildung $\hat{D}$ an, um die komplexe Ebene $\mathbb{C}$ in den Hilbert-Raum einzubetten, $i : \alpha \in \mathbb{C} \mapsto |\alpha\rangle \in \mathcal{H}$ mit $|\alpha\rangle := \hat{D}(\alpha)\,|0\rangle$. Der Operator $\hat{D}(\alpha)$ ist der bekannte Verschiebungsoperator (displacement operator). Betrachtet man nun den Hilbert-Raum der quadratintegrablen Funktionen $\mathfrak{L}^2(\mathbb{R}, \mathrm{d}q)$, so entspricht die eingebettete Untermannigfaltigkeit $i(\mathbb{C})$ den Gaußschen Wellenpaketen, das heisst wir erhalten die bekannte Einbettung $i : \mathbb{C} \to \mathfrak{L}^2(\mathbb{R}, \mathrm{d}q) : \alpha \mapsto \psi(\alpha, q)$.

Nachdem wir nun die kinematische Situation der verallgemeinerten kohärenten Zustände ermittelt haben, interessieren uns nun für die dynamischen Eigenschaften des Systems. Die Evolution eines reinen Zustands $|\alpha\rangle \in \mathcal{H}$ wird durch die Schrödinger-Gleichung festgelegt, die mit dem Hamilton-Operator $\hat{H}$ verknüpft ist. Daher sei $\hat{U}_t : \mathcal{H} \to \mathcal{H}$ die einparametrische Gruppe der Transformationen assoziiert mit dieser Schrödinger-Gleichung. In dieser Arbeit interessieren wir uns nur für Hamilton-Operatoren, die quadratisch in den Orts- und Impulsvariablen sind. Wenn in diesem Fall der Anfangszustand ein generalisierter kohärenter Zustand ist, ist der Endzustand olcher, d. h. $\hat{U}_t|\alpha_0\rangle = |\alpha(t)\rangle$. Dies impliziert, dass die Untermannigfaltigkeit $i(\mathbb{C}) \in \mathcal{H}$ invariant bezüglich des Flusses U_t ist.

Da $\mathbb{C}$ andererseits eine symplektische Struktur aufweist, ist sie mit einer symplektischen Form ω ausgestattet. Dementsprechend lässt sich jeder differenzierbaren Funktion mit Domäne in der komplexen Ebene, $H \in \mathfrak{F}(\mathbb{C})$, ein Hamilton-Vektorfeld X_H zuordnen, das intrinsisch durch $i_{X_H}\omega = -\mathrm{d}H$ definiert ist. Im Allgemeinen kann die Lösung der Hamiltonschen Bewegungsgleichung in $\mathbb{C}$ als $\alpha(t) = \Phi_t\alpha_0$ dargestellt werden, wobei $\Phi_t : \mathbb{C} \to \mathbb{C}$ eine einparametrische Gruppe symplektischer Transformationen ist. Daher wird die Verbindung zwischen der unitären Evolution $\hat{U}_t$ und der (kanonischen) symplektischen Evolution Φ_t durch Immersion erhalten; wir haben $i \circ \Phi_t = i \circ \hat{U}_t$.

Genauer gesagt beginnen wir unsere Untersuchung mit der Feststellung, dass die Evolution eines Gaußschen Wellenpakets von der Evolution der Erwartungswerte $(\langle\hat{q}\rangle, \langle\hat{p}\rangle)$ und der komplexen zeitabhängigen Funktionen (Q, P) abhängt, so dass diese Zustände folgendermaßen

$$\psi(\alpha, q) = \langle q|\hat{D}(\alpha)|0\rangle = \frac{1}{(\pi\,\hbar)^{1/4}}\frac{1}{\sqrt{Q}}\exp\left\{\frac{\mathrm{i}}{2\hbar}\frac{P}{Q}\,(q - \langle\hat{q}\rangle)^2 + \frac{\mathrm{i}}{\hbar}\langle\hat{p}\rangle(q - \langle\hat{q}\rangle) + \frac{\mathrm{i}}{2\hbar}\,\langle\hat{q}\rangle\,\langle\hat{p}\rangle\right\}, \qquad (8)$$

im Hilbert-Raum $\mathfrak{L}^2(\mathbb{R}, \mathrm{d}q)$ ausgedrückt werden können. Diese beiden Aspekte der kohärenten Zustände bewegen sich in unterschiedlichen Mannigfaltigkeiten. Einerseits befinden sich gemäß des Ehrenfest-Theorems die Erwartungswerte $(\langle\hat{q}\rangle, \langle\hat{p}\rangle)$ in einem euklidischen linearen Phasenraum $T^*\mathbb{R}^2$ mit Hamiltonscher Evolution. Andererseits befinden sich die Parameter (Q, P) in der Mannigfaltigkeit $M = \{(Q, P) \in \mathbb{C}^2 \,|\, \bar{Q}P - Q\bar{P} = 2\,\mathrm{i}\}$, die direkt mit der Dispersion und Korrelation $(\sigma_q, \sigma_p, \sigma_{qp})$ verbunden ist. Außerdem ist die Dynamik in TM Hamiltonisch, wobei Hamiltons Gleichungen die gleiche Form wie die klassischen Bewegungsgleichungen haben, jedoch mit komplexen Variablen.

Jedoch ist die zuvor erwähnte Parametrisierung der kohärenten Zustände nicht eindeutig. In der Quantenoptik ist es möglich, diese Zustände durch die sogenannten Squeezing-Parameter (τ, φ) zu beschreiben, wobei diese Parameter ein System von Koordinaten darstellen, das an das Hyperboloid $\mathbf{H}^2$ angepasst ist, und mit den Momenten $(\sigma_q, \sigma_p, \sigma_{qp})$ verbunden ist. Wir analysieren also die dynamischen Eigenschaften in $\mathbf{H}^2$ und zeigen, dass eine symplektische Struktur darauf definiert ist und eine Hamiltonsche Dynamik vorliegt, die außerdem noch nichtlinear ist.

Um den Zusammenhang zwischen der Beschreibungen in M und $\mathbf{H}^2$ zu zeigen, ist ein Zwischenschritt erforderlich. Hierfür zeigen wir zunächst die bestehende Verbindung zwischen M und dem Hyperboloid $\mathbf{H}^3$; es stellt sich heraus, dass die Verbindung zwischen den Hyperboloiden $\mathbf{H}^3$ und $\mathbf{H}^2$ nur die Konsequenz der Beziehung zwischen der speziellen linearen Gruppe $\mathrm{SL}(2,\mathbb{R})$ und ihrer Lie-Algebra $\mathfrak{sl}(2,\mathbb{R})$ ist und so die Projectionsabbildung (covering map) $\chi : \mathbf{H}^3 \to \mathbf{H}^2$ erzeugt.

Der letzte Teil des zweiten Kapitels ist der nichtlinearen Riccati-Dynamik gewidmet. Diesbezüglich beobachten wir, dass unter Berücksichtigung der Symmetrie der Dynamik $Y \in \mathfrak{X}(TM)$ bei Multiplikation mit einem globalen Phasenfaktor, M auf einen niedriger dimensionalen Raum reduziert werden kann, der als Siegel-upper-half-plane $\mathbb{HP}^2$ bekannt und als Raum komplexer Zahlen mit streng positivem Imaginärenteil definiert ist. Es kann nicht nur der Raum reduziert, sondern auch die Dynamik von M auf $\mathbb{HP}^2$ projiziert werden, wobei die projizierte Dynamik der nichtlinearen Riccati-Evolution entspricht, welche, wie bereits bewiesen, eine Hamilton-Dynamik ist. Zudem wird in diesem Reduktionsprozess eine völlig neue Parametrisierung des Gaußschen Wellenpakets gezeigt, was mittels des Punktes in der Poincaré-Scheibe $\mathbb{D}^2$ gelingt, der sich als stereografische Projektion von $\mathbf{H}^2$ auf die Ebene herausstellt und dessen Dynamik Hamiltonisch mit Riccati-Typ Evolution ist.

In Kapitel 3 wollen wir dissipative Phänomene in die nichtlineare Beschreibung physikalischer Systeme einbeziehen. Im Anschluss werden wir uns mit einer Unterklasse dieser Systeme befassen, uns zwar solche, die ein Zerlegungsprinzip (decomposition principle) besitzen. Dies bedeutet, dass die Dynamik in der Form $\Gamma = \Gamma_{\mathrm{R}} + \Gamma_{\mathrm{P}}$ ausgedrückt werden kann, wobei der erste Term Γ_{R} auch Referenzterm (Vergleichsterm) genannt wird und mindestens einen Lagrangian (Hamiltonian) besitzt, z. B. $\mathfrak{L}$. Der zweite Term Γ_{P} ist der Störungsterm (perturbation term), der das System dissipativ macht. Wir nennen ein System dissipativ, wenn Γ_{P} die Lagrange-Energie $\mathcal{E}_{\mathfrak{L}}$ des Referenzsystems dissipiert. Als Ergebnis dieser Überlegungen kann gezeigt werden, dass diese Unterklasse dissipativer Systeme, obwohl sie keinen Lagrange-Formalismus zulässt, dennoch eine Lagrange-(Hamilton-)Kontaktbeschreibung ermöglicht. Das heisst, wir berücksichtigen den Fall, in dem eine Beschreibung im Rahmen der Kontaktgeometrie für einige dissipative Kräfte möglich ist.

Es wird angenommen, dass der Raum für die Dynamik $TQ \times \mathbb{R}$ mit einer exakten Kontaktstruktur (η, ξ) versehen ist. Um die Dynamik zu definieren, kann man jeder glatten Funktion $\mathcal{E}_{\mathrm{C}}$ ein Vektorfeld Γ_{C} durch die Ausdrücke

$$i_{\Gamma_{\mathrm{C}}} \mathrm{d}\eta = \mathrm{d}\mathcal{E}_{\mathrm{C}} - (\pounds_{\xi}\mathcal{E}_{\mathrm{C}})\eta \qquad \text{und} \qquad i_{\Gamma_{\mathrm{C}}}\eta = -\mathcal{E}_{\mathrm{C}}\,, \tag{9}$$

zuordnen, wobei $\mathcal{E}_{\mathrm{C}}$ Kontakt-Lagrange-Energie genannt wird. Insbesondere nehmen wir an, dass die 1-Form η lokal wie folgt geschrieben werden kann:

$$\eta = \mathrm{d}S - \theta_{\mathfrak{L}} \quad \text{mit} \quad \theta_{\mathfrak{L}} = \mathrm{d}q_j \frac{\partial \mathfrak{L}}{\partial \dot{q}_j}\,, \tag{10}$$

wobei $(q_j,\, \dot{q}_j,\, S)$ lokale Koordinaten auf $TQ \times \mathbb{R}$ sind, $\mathfrak{L}$ stellt die Lagrange-Funktion des Referenzsystem dar und die Kontakt-Lagrange-Energie kann in der form $\mathcal{E}_{\mathrm{C}} = \mathcal{E}_{\mathfrak{L}} + h(S)$ geschrieben werden. Der erste Term $\mathcal{E}_{\mathfrak{L}}$ ist die Lagrange-Energie des konservativen Referenzsystems, der zweite Term $h(S)$ stellt eine "Störung" des Systems dar, die eine effektive Beschreibung der Wechselwirkung zwischen dem konservativen System und der Umgebung repräsentiert. Unter Berücksichtigung dieser Überlegungen erhält man die Kontakt-Euler-Lagrange-Gleichung und eine zusätzliche Gleichung für die Komponente des Vektorfeldes in Richtung des Reeb-Vektors, nämlich

$$\pounds_{\Gamma_{\mathrm{C}}}\theta_{\mathfrak{L}} - \mathrm{d}\mathfrak{L} = -\frac{\mathrm{d}h}{\mathrm{d}S}\,\theta_{\mathfrak{L}}\,, \quad \text{and} \quad \dot{S} = i_{\Gamma_{\mathrm{C}}}\theta_{\mathfrak{L}} - \mathcal{E}_{\mathfrak{L}}\,. \tag{11}$$

Entsprechend erhält man unter Verwendung der Definition für die 1-Form $\theta_{\mathfrak{L}}$ und Anwendung der Lie-

Ableitung auf $\theta_{\mathfrak{L}}$ den Koordinatenausdruck der Kontakt-Euler-Lagrange-Gleichungen

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathfrak{L}}{\partial \dot{q}^k} - \frac{\partial \mathfrak{L}}{\partial q^k} = -\frac{\mathrm{d}h}{\mathrm{d}S}\frac{\partial \mathfrak{L}}{\partial \dot{q}^k}\,, \tag{12}$$

die implizite Differentialgleichungen darstellen. Diese Systeme lassen sich als eine Art Verallgemeinerung des sogenannten dissipativen Caldirola–Kanai-Systems verstehen. Hier bleibt die Lagrange-Energie entlang der dynamischen Trajektorien nicht erhalten, so dass

$$\frac{\mathrm{d}\mathcal{E}_{\mathfrak{L}}}{\mathrm{d}t} = -\frac{\mathrm{d}h}{\mathrm{d}S}\frac{\partial \mathfrak{L}}{\partial \dot{q}^k}\,, \tag{13}$$

wobei der Ausdruck je nach Vorzeichen von $\frac{\mathrm{d}h}{\mathrm{d}S}$ sowohl positiv als auch negativ sein kann.

Es ist möglich, eine Kontakt-Mannigfaltigkeit im allgemeineren Sinne mittels einer globalen 1-Form η und einer globalen 2-Form $\omega_{\mathfrak{L}}$ näher zu definieren. Unsere Kontakt-Mannigfaltigkeit soll wieder $TQ \times \mathbb{R}$ sein. Dann definieren wir nun eine dynamische Evolution mittels eines Vektorfeldes mit zugeordneter glatter Function $\mathcal{E}_{\mathrm{C}}$ unter Verwendung der beiden Bedingungen

$$i_{\Gamma_{\mathrm{C}}}\omega_{\mathfrak{L}} = \mathrm{d}\mathcal{E}_{\mathrm{C}} - (\pounds_{\xi}\mathcal{E}_{\mathrm{C}})\eta \qquad \text{und} \qquad i_{\Gamma_{\mathrm{C}}}\eta = -\mathcal{E}_{\mathrm{C}}\,. \tag{14}$$

Ähnlich der Vorgehensweise bei den exakten Kontaktstrukturen nehmen wir hier an, η und $\omega_{\mathfrak{L}}$ seien durch $\eta = \mathrm{d}S - \alpha$ und $\omega_{\mathfrak{L}} = -\mathrm{d}\theta_{\mathfrak{L}}$ gegeben, wobei α eine 1-Form mit dem lokalen Ausdruck $\alpha = a_k\,\mathrm{d}q^k$ und $a^k \in \mathfrak{F}(TQ)$ ist. Da wir uns für die Charakterisierung dissipativer Systeme anhand eines Zerlegungsprinzips interessieren, gehen wir davon aus, dass die Lagrange-Kontaktenergie $\mathcal{E}_{\mathrm{C}} = \mathcal{E}_{\mathfrak{L}} + h(S.)$ entspricht. Genauer gesagt bekommt man die Bedingungen

$$\pounds_{\Gamma_{\mathrm{C}}}\theta_{\mathfrak{L}} - \mathrm{d}\mathfrak{L} = -\frac{\mathrm{d}h}{\mathrm{d}S}\,\alpha \qquad \text{and} \qquad \dot{S} = i_{\Gamma_{\mathrm{C}}}\alpha - \mathcal{E}_{\mathrm{C}}\,, \tag{15}$$

wobei die erste Bedingung Γ_{C} eindeutig bis auf ein Vektorfeld proportional zu ξ bestimmt und dieser zusätzliche Term durch die zweite Bedingung festgelegt wird. Indem wir Annahmen für α treffen, können einige spezielle Klassen dissipativer Systeme identifiziert werden. Zum Beispiel kann man $\alpha = \frac{\partial \mathcal{F}}{\partial \dot{q}^k}\mathrm{d}q^k$ setzen, wobei $\mathcal{F}$ eine beliebige Funktion auf TQ darstellt. Diese besondere Situation führt zu einer konformen Version der Rayleigh-Dissipation, die in Koordinaten wie folgt dargestellt werden kann

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathfrak{L}}{\partial \dot{q}_j} - \frac{\partial \mathfrak{L}}{\partial q_j} = -\frac{\mathrm{d}h}{\mathrm{d}S}\frac{\partial \mathcal{F}}{\partial \dot{q}_j}\,, \tag{16}$$

wobei die Änderungsrate der Lagrange-Energie gleich

$$\frac{\mathrm{d}\mathcal{E}_{\mathrm{C}}}{\mathrm{d}t} = -\frac{\mathrm{d}h}{\mathrm{d}S}\frac{\partial \mathcal{F}}{\partial \dot{q}_j} \tag{17}$$

ist.

Nun wird der Kontaktformalismus in dem sogenannten Kontaktphasenraum eingeführt, der als Mannigfaltigkeit $T^*Q \times \mathbb{R}$ mit ungerader Dimension definiert ist. Daraufhin soll die Verbindung zwischen den Kontakt-Mannigfaltigkeiten $TQ \times \mathbb{R}$ und $T^*Q \times \mathbb{R}$ hergestellt werden. Um dies zu erreichen, betrachten wir den Diffeomorphismus $F\mathscr{L}$, der mit $\mathscr{L} \in \mathfrak{F}(TQ \times \mathbb{R})$ verknüpft ist und in lokalen Koordinaten symbolisch in der Form

$$F\mathscr{L} : (q^k, \dot{q}^k, S) \mapsto \left(q^k, p_k = \frac{\partial \mathfrak{L}}{\partial \dot{q}^k}, S\right) \tag{18}$$

geschrieben werden kann.

Wir möchten darauf hinweisen, dass die als $p_k = \frac{\partial \mathfrak{L}}{\partial \dot{q}^k}$ definierte Komponente der üblichen Definition des Impulses entspricht. Unter einer solchen Transformation wird daher die Kontaktdynamik $\Gamma_{\mathrm{C}} \in \mathfrak{X}(TQ \times R)$ auf ein Vektorfeld $T^*Q \times \mathbb{R}$ durch $\Gamma_{\mathrm{C}} = (F\mathscr{L})^* X_{\mathrm{C}}$ transferiert. Darüber hinaus existiert die Kontaktform $\eta \in \Omega^1(T^*Q \times \mathbb{R})$, so dass $(F\mathscr{L})_*\eta = \eta_{\mathfrak{L}}$, für alle $\mathfrak{L} \in \mathfrak{F}(TQ)$. Das heißt, $\eta_{\mathfrak{L}}$ ist der

Rücktransport (pullback) bezüglich $F\mathscr{L}$ einer festgelegten Kontaktform η in $T^*Q \times \mathbb{R}$, der unabhängig von dem benutzten Lagrangian ist. Somit ist die Dynamik X_C im Kontaktphasenraum durch

$$i_{X_\mathrm{C}}\,\mathrm{d}\eta = \mathrm{d}\mathscr{H} - (\pounds_\xi \mathscr{H})\eta \qquad \text{und} \qquad i_{X_\mathrm{C}}\eta = -\mathscr{H}\,, \tag{19}$$

gegeben, das heisst die Dynamik hängt nur vom Kontakt-Hamiltonian $\mathscr{H} \in \mathfrak{F}(T^*Q \times \mathbb{R})$ ab. Die Kontaktform und das Reeb-Vektorfeld können in den lokalen Koordinaten (q^k, p_k, S) als

$$\eta = \mathrm{d}S - p_k \mathrm{d}q^k\,, \quad \xi = \frac{\partial}{\partial S} \tag{20}$$

dargestellt werden. Der Kontakt-Hamiltonian hat die Form $\mathscr{H} = H + h(S)$, wobei H den Hamiltonian des Referenzsystems darstellt. Außerdem kann das Kontakt-Hamilton-Vektorfeld X_C in diesen Koordinaten in der Form

$$X_\mathrm{C} = \frac{\partial H}{\partial p_k}\frac{\partial}{\partial q^k} - \left(\frac{\partial H}{\partial q^k} + p_k \frac{\mathrm{d}h}{\mathrm{d}S}\right)\frac{\partial}{\partial p_k} + \left(p_k \frac{\partial H}{\partial p_k} - \mathscr{H}\right)\frac{\partial}{\partial S} \tag{21}$$

ausgedrückt werden, deren Integralkurven durch die Lösungen des Differentialgleichungssystems

$$\dot{q}^k = \frac{\partial H}{\partial p_k}\,, \quad \dot{p}_k = -\frac{\partial H}{\partial q^k} - p_k \frac{\mathrm{d}h}{\mathrm{d}S}\,, \quad \dot{S} = p_k \frac{\partial H}{\partial p_k} - \mathscr{H}\,. \tag{22}$$

bestimmt sind. Der Kontakt Term führt die Dissipation in die Dynamik ein, wobei hier spezifisch der Hamiltonian des Referenzsystems dissipiert wird. Die Dissipationsrate des Hamiltonians ist somit gegeben durch

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -p_k \frac{\mathrm{d}h}{\mathrm{d}S}\frac{\partial H}{\partial p_k}\,. \tag{23}$$

Allgemein gilt, dass die Evolution für eine beliebige Funktion $\mathscr{F} \in C^\infty(T^* \times \mathbb{R})$ im Kontaktphasenraum durch

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = X_\mathrm{C}[\mathscr{F}] = -\mathscr{H}\frac{\partial \mathscr{F}}{\partial S} + p_k\left[\frac{\partial \mathscr{F}}{\partial S}\frac{\partial \mathscr{H}}{\partial p_k} - \frac{\partial \mathscr{F}}{\partial p_k}\frac{\partial \mathscr{H}}{\partial S}\right] + \frac{\partial \mathscr{F}}{\partial q^k}\frac{\partial \mathscr{H}}{\partial p_k} - \frac{\partial \mathscr{F}}{\partial p_k}\frac{\partial \mathscr{H}}{\partial q^k} \tag{24}$$

festgelegt wird.

Es wurden insgesamt sowohl zeitunabhängig als auch zeitabhängig Kontaktsysteme untersucht, wobei festgestellt werden konnte, dass in beiden Fällen die Transformationen (hier als Kontakttransformationen bezeichnet) die die Hamilton-Kontaktgleichungen unverändert lassen, zeigen, dass kanonische Transformationen ein Spezialfall einer symplektischen Dynamik sind.

Introduction

A minimalistic description of physical systems (quantum or classical) requires the identification of: i) states, ii) observables, iii) probability functions, iv) evolution equations and v) composition rules for elementary systems. So, it should be clear that the mathematics involved in each element of our physical model clearly belongs to different categories. The states (pure or mix) will have the structure of manifolds, the observables will be (part of) an associative algebra over the real or the complex numbers. The probability function: $P(\rho, A, E) \in [0, 1]$, with $E \in \mathbb{R}$, will represent the probability of finding a result in the Borelian set E when we measure the observable A, while the system is in state ρ; then, these aspects belong to measure theory. The evolution equations will be usually required to be differential equations. For example, in Classical Mechanics quoting V. I. Arnold: "Many mathematical methods are used in classical mechanics: differential equations and phase flows, smooth mappings and manifolds, Lie groups and Lie algebras, symplectic theory and ergodic theory. Many modern mathematical theories arose from problems in mechanics and only later acquired that axiomatic-abstract forms which make them so hard to study" [1]. This emphasizes the fact that the best math teacher is nature.

Then, for a deep understanding of physical systems a theoretical physicist should be prepared in different areas of modern mathematics, even one should be prepared to mix methods belonging to different mathematical fields, which is not a trivial task. Having this in mind, in the second part of the last century physicist and mathematicians started the study of Classical Mechanics from a modern mathematical perspective; some excellent literature can be found in references [1, 2, 3, 4, 5, 6]. This geometrical studies developed a set of techniques that transformed Classical Mechanics into a full-blown area of mathematics and it led to important advances in many areas of classical physics.

Having in mind that our best description of *the external world* is provided by quantum mechanics and the success of the geometrization of Classical Mechanics, the geometrization of the Quantum Mechanics is the natural extension of research for many mathematicians and physicist. Recently the development of geometrical description of Quantum Mechanics has an increasing interest and the literature is broad, for some examples see [6, 7, 8] and references therein. In this work we follow this new perspective in the study of quantum systems. In particular, it is very useful to consider the so-called *principle of analogy*, i.e. using the procedures and available structures from the classical setting to employ them in the quantum setting.

In this thesis we will take advantage of the *principle of analogy* to unveil ambiguities, and exhibit alternative nonlinear descriptions and additional mathematical structures in quantum theory. Partivularly in Chapter one, the subject of study is to show that an interesting application of the statistical-probabilistic contents of quantum mechanics is the nonlinear description of an N-level quantum system by means of the nonlinear Riccati equation, as an alternative description of quantum evolution [9]. This constitutes a transition from linear Schrödinger dynamics to a nonlinear Riccati equation. To introduce such a transition, we first recall that a pure state in quantum mechanics is an equivalence class, i.e. a ray in the Hilbert space, where the equivalence class may be described infinitesimally by two vector fields in involution: the dilation vector field Δ and the vector field Γ associated with the multiplication by a phase factor. In addition, it is straightforward to see that these vector fields are symmetries of the Schrödinger dynamics because Δ and Γ permute solutions of the Schrödinger equation [4]. This allows to reduce the dynamics to a lower dimensional space, specifically the complex projective space $\mathbb{CP}$ of the Hilbert space, an example of a Hilbert manifold.

Usually in quantum mechanics, pure states are considered to be rank-one projectors and therefore are studied as elements of the space of Hermitian, non-negative and trace one operators. However, in this work we will stick to the Hilbert manifold, i.e., the complex projective space, to stress its intrinsic "non-

linearity". For the analysis on $\mathbb{CP}$, it is used that it is possible to give a complete description of the complex projective space by means of complex homogeneous coordinates [7, 6], such that in those coordinates the Schödinger dynamics is projected into the Riccati dynamics. Furthermore, we prove that not only the dynamics is projectable but also important geometrical structures may be projected. In particular, the Khäler structure on the Hilbert space is also presented in the complex projective space [7, 5, 6]. This structure allows to show that the nonlinear Riccati evolution is actually Hamiltonian.

In addition, we notice that in the nonlinear description the evolution of pure states is described by solutions of a Riccati equation that, although it is nonlinear, originates from the linear Schrödinger equation with linear superposition rule, thus there must be a way to superimpose solutions of the Riccati equation, which leads to a nonlinear superposition rule in $\mathbb{CP}$ [10, 11, 12]. To conclude Chapter one, we apply the formalism developed in a concrete physical system, particularly in the study of the evolution of a 2-level atom irradiated by a laser source, i.e. the radiation is considered as a classical field [13].

Roughly speaking, we will start with a problem that, after the analysis of its symmetries, will be reduced onto a more abstract and fundamental description. In this process we always preserve the fundamental parts, i.e. we will preserve the physics behind the problem. So the natural question is: why we want to replace a mathematically simpler linear form of Quantum Mechanics with a nonlinear one? The answer is immediate: this is done in order to have a better and deeper understand of Quantum Mechanics. As we already know the description of pure states of a N-level quantum systems may be given in the *Schrödinger picture* in terms of wave functions or by means of the *Von Neumann picture* in terms of rank-one projectors; however, as we will show it is also possible by the *Complex Projective picture* in terms of the Hermitian manifold $\mathbb{CP}$. Moreover, we will prove that all these pictures are physically equivalent, in the sense that all of them leads to a complete description of the system.

To continue our study of nonlinear dynamics in Quantum Mechanics we transit from finite dimensional quantum system to infinite ones. However, to analyze the problem in all its generality is quite difficult because, from the mathematical point of view, one has to deal with all the technical difficulties related to the differential geometry of infinit dimensions. For this reason in Chapter two we restrict our study to the so-called generalized coherent states. Then, in this Chapter we show that there is an inherent nonlinear evolution in the dynamics of the generalized coherent states. To show this, the fact that in Quantum Mechanics it is possible to immerse a "classical" manifold into the Hilbert space is employed, such that one may parametrize the time-dependence of the wave function through the variation of parameters in the classical manifold [14]. The immersion allows to consider Dirac's principle of analogy also for this kind of systems.

One of the traditional problems in quantum mechanics is the study of Gaussian wave packets as analytic solution of the Schrödinger equation for Hamiltonians that are at most quadratic (or bilinear) in position and momentum variables. This kind of systems has been widely analized since the seventies where there was a large activity in the semiclassical limit of the quantum mechanics. Even nowadays this topic has relevance in several branches of physics.

For this reason there are several methods in the literature for the description of Gaussian wave packets. For instance, one of the most effective and popular method is the one of so-called *linear invariant operators*, introduced by Lewis in ref. [16] and Malkin, Man'ko and Trifonov in Ref. [17]. In this method it is proved that all the dynamics is contained in a linear system of differential equations whose solutions parametrize the time-dependence of the Gaussian wave packets. On the other hand there are other important methods that involve nonlinear evolution, namely it has been proved in [18, 19] that the same Gaussian wave packets may be also parametrized by the solutions of a nonlinear Riccati equation.

Hence, from the before mention models of the generalized coherent states, one may realize that there is not a unique parametrization of these states. In fact, we have a parametrization by means of the complex number $\alpha \in \mathbb{C}$, a description usually called the coherent states, also we may find in quantum optics that the Gaussian wave packets are parametrized by the squeezing parameter $\xi \in \mathbb{C}$, known as the squezzing states, or one may parametrize by means of the solutions of the nonlinear Riccati (or Ermakov) equation. From a physical point of view all this representations are valid, then one should be able to establish the connections between these deferent description. The connection between the linear and the nonlinear evolution of the generalized coherent state has been already studied in [20, 19], where the transformations that connect the different descriptions of the time-dependent solutions of quadratic Hamiltonian are explicitly established.

In Chapter two, in contrast to the before mentioned approaches for the study of the generalized coherent states, we will analize this problem from a geometrical point of view. This means that we will establish the geometrical space where the different evolutions take place and show how they are *immersed* in the Hilbert space $\mathcal{H}$, as well as establish the connection among the different descriptions. Moreover, it is shown that each space has a symplectic structure such that the dynamics on these spaces are actually Hamiltonian. Then we not only deduce from a geometrical perspective the results previously mentioned, but also it is showed that there are several nonlinear descriptions involved in the dynamics of the generalized coherent states.

In Chapter three we are interested to include the dissipative phenomena in the nonlinear description of physical systems. There are several ways to address dissipative phenomena in physics, see e.g. [21] for a recent review. For example, one may introduce *stochastic dynamics* to model the effect of fluctuations due to the environment on the system of interest. This leads to stochastic equations of the Langevin or Fokker–Planck type with diffusion terms [22, 23]. A different although related approach is the *system-plus-reservoir* technique, in which the system of interest is coupled to an environment (usually modeled as a collection of harmonic oscillators). The system and the environment together are considered as an isolated Hamiltonian system and after averaging out the environmental degrees of freedom one obtains the equations of motion for the system of interest, including dissipative terms. This is the case for example in the Caldeira–Laggett formalism [24, 25, 26]. An alternative approach is to propose *effective Hamiltonians* with an explicit time-dependence that reproduce the correct Newtonian equation, including the dissipative forces. A famous example is the Caldirola–Kanai (CK) model [27, 28]. Another proposal based on a nonconservative action principle, allows for time-irreversible processes, such as dissipation, to be included at the level of the action [29]. Finally, a more geometrical attempt towards the description of dissipative systems is given by the so-called *bracket formulation* of dynamical systems [30]. Here one generalizes the standard Poisson bracket to a non-canonical Poisson bracket and exploits the algebraic properties of the latter to include dissipation. The literature on all these proposals is very extensive and it is not our purpose here to review them in detail. We refer the interested reader to the standard references cited above and references therein.

Even though there are several approaches to the descriptions of dissipative systems there is something in common, the fact that dissipation is a relational concept, i.e., first one has to declare the *reference* (*comparison*) system such that when it is considered a perturbation in this system as a result of its interaction with the environment, one may observe then what has been dissipated. Thus, we want to stress that if we want to declare a system to be dissipative, first we should say what is actually being dissipated. Indeed, it is clear that one may deal with physical systems for which it might make sense to say that the system is *dissipating* energy, mass, angular momentum or *probability*.

After realizing the relational nature of dissipation, we proceed to verbalize this concept in a mathematical language, i.e. to give a formal definition of dissipation for a dynamical system. Then, given a dynamical system, for example a second order vector field, is it possible to characterize its dynamics as *conservative* or *dissipative*? From the dynamical point of view this question does not make sense, because it is necessary to declare first what is dissipating, i.e. there is information missing. To establish the problem correctly let us consider the subclass of dynamical systems Γ that possesses the *decomposition principle*

$$\Gamma = \Gamma_{\text{R}} + \Gamma_{\text{P}} \tag{25}$$

where the first term Γ_{R} describe the reference (comparison) and the second term Γ_{P} is the perturbation term, which turn the system dissipative. Then, one may say that the system is dissipative in the sense that Γ_{P} is dissipating some declared structure of Γ_{R}.

Once the definition of dissipative dynamics is established, the problem is how to address this kind of systems. One way is to look for a Lagrangian (or Hamiltonian) description of the dynamics Γ, i.e. solve the the so-called *inverse problem in the calculus of variations* [31]; however, this is not an easy task and even for some systems one may prove that it is impossible to find a Lagrangian (Hamiltonian) description [31, 32]. So, when the system does not admit a Lagrangian description, we have to develop new strategies in order to characterize dissipation. To avoid this problem we take advantage of the decomposition principle of dissipative systems, so it is possible to think of the dynamical vector field as the sum of a *reference* or *comparison dynamics* plus a *perturbation term*, but the *reference dynamics* has to be selected in such a way that it admits at least one Lagrangian description, and the perturbation term

turns the system into a *dissipative* one, dissipating the mechanical energy of the comparison system. The advantage of this decomposition is that the system may be treated by means of the Lagrangian (Hamiltonian) contact formalism [33, 34], which is a natural extension of the usual symplectic Lagrangian (Hamiltonian) formalism.

Then in the third Chapter a detailed analysis of the Lagrangian and Hamiltonian contact formalism for the mechanical description of dissipative systems is pursued. This new proposal consists in extending the symplectic phase space of classical mechanics by adding an extra dimension, thus dealing with a contact manifold instead of a symplectic one. Note that contact geometry arises naturally in mechanics. First of all, in describing mechanical systems where the Hamiltonian function explicitly depends on time, one usually appeals to an extended phase space, the additional dimension being time, endowed with the Poincaré–Cartan 1-form, which defines a contact structure on the extended space [3, 1, 35]. Besides, the time-dependent Hamilton–Jacobi theory is naturally formulated in this extended phase space [36, 37]. So, here it is not assumed that the additional dimension is time, letting the additional dimension be represented by a non-trivial dynamical variable and hence one may derive the equations of motion for the system from *contact Hamiltonian dynamics*, which is the most natural extension of symplectic Hamiltonian dynamics [1]. Then contact formalism is a classical mechanical theory that not only contains all the advantages of the Hamiltonian formalism, but also it may take into account the effects of the environment on the system.

Chapter 1

Nonlinear Dynamics of Quantum Systems. N-level Systems

The nonlinear description of quantum phenomena has currently gained considerable interest [9, 20, 38, 19]. Not only because this constitutes an alternative description of quantum theory, but also because this evolution presents interesting properties allowing a better understanding of quantum theory itself.

There are several ways to introduce "nonlinearity" in quantum mechanics. For example, in the non-relativistic Quantum Mechanics the evolution of the wave function $\psi(\mathbf{q},t)$, in the Hilbert space $\mathcal{H}=\mathfrak{L}^2(\mathbb{R}^n,\mathrm{d}\mu)$, is given by the linear Schrödinger equations

$$\mathrm{i}\,\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi+V(\mathbf{q})\psi\,. \tag{1.1}$$

However, using the polar representation $\psi(\mathbf{q},t)=A(\mathbf{q},t)\mathrm{e}^{-\frac{i}{\hbar}S_{\mathrm{Q}}(\mathbf{q},t)}$, with A and S_{Q} real functions, the Schrödinger equation becomes a system of two coupled partial differential equations

$$\frac{\partial S_{\mathrm{Q}}}{\partial t}=\frac{1}{2m}(\nabla S_{\mathrm{Q}})^2+V(\mathbf{q})-\frac{\hbar^2}{2m}\frac{\nabla^2 A}{A} \tag{1.2}$$

$$\frac{\partial A}{\partial t}=\frac{1}{2m}(2\nabla A\cdot\nabla S_{\mathrm{Q}}+A\,\nabla^2 S_{\mathrm{Q}}) \tag{1.3}$$

which clearly is a nonlinear description of the quantum system. In fact, because the nonlinear change of coordinates $\psi=A\,\mathrm{e}^{-\frac{i}{\hbar}S_{\mathrm{Q}}}$ in the Hilbert space is performed, it should be clear that the superposition rule is not trivial.

Another way to generate a nonlinear dynamics is to construct a nonlinear Schrödinger equation. To see this, let us recalled that the evolution equation in (1.3) may be rewritten as the continuity equation for the probability density considering $\rho(\mathbf{q},t)=A^2(\mathbf{q},t)$, namely

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{j}=0 \tag{1.4}$$

expressing conservation of probability, with the current density $\mathbf{j}$ given by[1]

$$\mathbf{j}=\frac{\hbar}{2m\mathrm{i}}(\bar{\psi}\nabla\psi-\psi\nabla\bar{\psi})\,. \tag{1.5}$$

Then, from the hydrodynamical formulation of Quantum Mechanics it is possible to generalize the continuity equation by adding a diffusion term, which leads to the Fokker–Plank type equation

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{j}-D\,\nabla^2\rho=0\,, \tag{1.6}$$

[1] The overbar is used along this work to denote the complex conjugate quantity.

in position space also called Smoluchowski equation, where D is a real diffusion coefficient. So, it has been shown in Ref. [39] that the Schrödinger equation associated to this Fokker–Plank type equation has the form

$$\mathrm{i}\,\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{q})\psi \ + F[\psi,\bar{\psi}]\psi\,, \tag{1.7}$$

where $F[\psi,\bar{\psi}]$ is a nonlinear function of ψ and $\bar{\psi}$. For instance, in Ref. [40], considering the anzatz

$$D\frac{\nabla^2\rho}{\rho} = -\gamma(\ln\rho - \langle\ln\rho\rangle) \tag{1.8}$$

one can show that the associated nonlinear Schrödinger equation is

$$\mathrm{i}\,\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{q})\psi + \gamma\frac{\hbar}{\mathrm{i}}(\ln\psi - \langle\ln\psi\rangle)\psi\,. \tag{1.9}$$

This nonlinear equation is employed in the description of quantum systems corresponding to a classical system with a linear velocity dependent friction force [19]. Furthermore, in the same spirit of the quantum treatment of dissipative systems, there are several approaches of nonlinear Schrödinger equations. For instance, the ones proposed by Kostin, Albrecht, Süssmann and Hasse, all of them reviewed in Ref. [41]; however, different from the Schuch–Chung–Harmann and Doebner–Gisin approaches they do not add a diffusion term in the continuity equation, the nonlinearity comes from a nonlinear potential added suitably to the Schrödinger equation to obtain the correct classical equation of motion including the friction force for the mean values.

From the examples presented before, one could think that the nonlinearity is a consequence of performing nonlinear transformations in the Hilbert space or a consequence of interactions of the quantum system with a suitable environment. However, as we will see along this Chapter the nonlinear dynamics in Quantum Mechanics actually arises from a more fundamental aspect, namely from the statistical-probabilistic contents of this theory. This perspective has already been considered in [9] and also taken into account in spin Hamiltonian systems in [42].

1.1 Nonlinear dynamics from linear classical systems

Before the nonlinear dynamics in quantum mechanics is addressed, it is instructive to see first how one may obtain a nonlinear dynamics from a linear one in Classical Mechanics. It is well-known in Classical Mechanics that a nonlinear dynamics arises from a linear one by considering the homogeneity of the space, where the nonlinear dynamics is given by the so-called Riccati transformation, introduced by Riccati in 1720. So, let us consider a linear system of differential equations describing the dynamical evolution of a point in the linear phase space $T^*\mathbb{R}\approx\mathbb{R}^2$ with coordinates (q,p) of the euclidean plane

$$\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix} = \begin{pmatrix} b & c\\ a & -b\end{pmatrix}\begin{pmatrix} q\\ p\end{pmatrix}, \tag{1.10}$$

with a,b,c real and possibly time-dependent functions. Thus, the linear system is traceless and then belongs to the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, and the dynamics is given by the vector field

$$X = (b\,q + c\,p)\frac{\partial}{\partial p} + (a\,q - b\,p)\frac{\partial}{\partial p}\,. \tag{1.11}$$

Recall now, that linear systems are invariant under the action of the dilation group, associated to the homogeneity of the phase space, and whose infinitesimal generator is the vector field

$$\Delta = q\,\frac{\partial}{\partial q} + p\,\frac{\partial}{\partial p} \tag{1.12}$$

known as the *Euler–Liouville vector field* and it is straightforward to see that $[X,\Delta]=0$. So, because X commutes with Δ, the dynamics X is *projectable* with respect to a foliation obtained from Δ. The

result of such a projection is a reduction of the dynamics to a lower dimensional space, a review of the reduction procedure is in Appendix A.

To reduce the dynamics, let us notice that there is a foliation Φ^{Δ} of $\mathbb{R}^2$ obtained by the integral curves of the Euler–Liouville vector field. This integral curves are open half-lines starting at the origin of $\mathbb{R}^2$ and the origin itself, therefore the foliation is singular at the origin. In spite of having a singular foliation, we may define the equivalence relation

$$(q^1, p_1) \sim (q^2, p_2) \quad \text{iff} \quad q^1 p_2 - q^2 p_1 = 0\,, \tag{1.13}$$

for points in $\mathbb{R}^2 - \{\mathbf{0}\}$, i.e. two points in the same half-line are equivalent; then, we may define the quotient space $\mathbb{R}^2/\Phi^{\Delta}$ with respect to the equivalence relation defined by the foliation. This quotient space may be identifed with the circle

$$S^1 = \{(q,p) \in \mathbb{R}^2 \,|\, q^2 + p^2 = 1\}\,. \tag{1.14}$$

Also the quotient space $\mathbb{R}^2/\Phi^{\Delta}$ it is known in the literature as *real projective line* denoted by $\mathbb{RP}^1$. The connection between S^1 and $\mathbb{RP}^1$ is simply the stereographic projection of the circle onto the line. The description of the real projective line is complete by means of the homogeneous coordinates, i.e. let $V_j \subset \mathbb{RP}^1$, with $j = 1, 2$, we denotes the coordinate charts

$$\phi_1 : V_1 \to \mathbb{R} : [q,p] \mapsto z = \frac{p}{q}\,, \quad \text{for} \quad q \neq 0 \tag{1.15}$$

and

$$\phi_2 : V_2 \to \mathbb{R} : [q,p] \mapsto \zeta = \frac{q}{p}\,, \quad \text{for} \quad p \neq 0 \tag{1.16}$$

such that the set of (V_j, ϕ_j) constitutes an *atlas* for the real projective line. Therefore one has arrived at the Riccati transformations

$$\pi : \mathbb{R}^2 - \{\mathbf{0}\} \to \mathbb{RP}^1 : (q,p) \mapsto \frac{p}{q}\,, \quad \text{and} \quad \tilde{\pi} : \mathbb{R}^2 - \{\mathbf{0}\} \to \mathbb{RP}^1 : (q,p) \mapsto \frac{q}{p}\,, \tag{1.17}$$

where these transformations are actually the projection with respect to the equivalence relation described above, where the two charts ϕ_1 and ϕ_2 are covering $\mathbb{RP}^1$. In addition, the Riccati transformation provides a local expression of the vector field $Y \in \mathfrak{X}(\mathbf{RP}^1)$ induced by $X \in \mathfrak{X}(\mathbb{R}^2 - \{\mathbf{0}\})$, namely

$$Y_z = -(c\,z^2 + 2\,b\,z - a)\frac{\partial}{\partial z} \quad \text{and} \quad Y_\zeta = -(a\,\zeta^2 - 2\,b\,\zeta - c)\frac{\partial}{\partial \zeta} \tag{1.18}$$

whose integral curves are given by the solution of the nonlinear Riccati equations

$$\dot{z} + c\,z^2 + 2\,b\,z - a = 0\,, \quad \text{and} \quad \dot{\zeta} + a\,\zeta^2 - 2\,b\,\zeta - c = 0\,. \tag{1.19}$$

Therefore, the vector field X described by the linear system of differential equations has been projected onto a vector field Y_z, or Y_ζ, described by a Riccati equation, i.e. a nonlinear differential equation involving only one degree of freedom.

The same procedure may be considered in the quantum setting to obtain a nonlinear dynamics instead of the linear Schrödinger dynamics. However, the reduction procedure, as will be shown in the next section, comes from a completly different interpretation.

1.2 From Linear Schrödinger Equation to Nonlinear Evolution

For simplicity the study starts with a 2-level systems, also known as q-bit systems, and later the results will be extended to N-level systems. The quantum evolution for a 2-level system on the Hilbert space is obtained from the solutions of the Schrödinger equation

$$\mathrm{i}\hbar \begin{pmatrix} \dot{\psi}^1 \\ \dot{\psi}^2 \end{pmatrix} = \begin{pmatrix} H_1 & V \\ \bar{V} & H_2 \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}\,, \tag{1.20}$$

where $H_1, H_2 \in \mathbb{R}$ and $V \in \mathbb{C}$ are possibly time-dependent functions. Alternatively, defining the Hilbert space $\mathcal{H}_0 = \mathcal{H} - \{\,\mathbf{0}\,\}$, i.e. the Hilbert space $\mathcal{H}$ with the origin removed, one may introduce homogeneous coordinates

$$\pi : \mathcal{H}_0 \to \mathbb{C} : (\psi^1, \psi^2) \mapsto z = \frac{\psi^1}{\psi^2}\,, \tag{1.21}$$

which associates to each point in $\mathcal{H}_0$ a point in the complex plane. It is clear that the transformation π is not defined for states with $\psi_2 = 0$; however, in order to take into account those states one simply may consider a different chart, i.e., the transformation

$$\tilde{\pi} : \mathcal{H}_0 \to \mathbb{C} : (\psi^1, \psi^2) \mapsto \zeta = \frac{\psi^2}{\psi^1}\,. \tag{1.22}$$

Then, it is straightforward to prove that the dynamics in the Hilbert space $\mathcal{H}_0$, given by the Schrödinger equation (1.20), induces a nonlinear dynamics in the complex plane given by the Riccati equation

$$\dot{z} = \frac{\mathrm{i}}{\hbar}\left[\bar{V} z^2 - (H_1 - H_2) z - V\right]\,, \tag{1.23}$$

or equivalently for $\zeta = 1/z$ by the Riccati equation

$$\dot{\zeta} = \frac{\mathrm{i}}{\hbar}\left[V \zeta^2 + (H_1 - H_2)\zeta - \bar{V}\right]\,. \tag{1.24}$$

These equations constitute nonlinear evolution equations for quantum states.

Before adressing the dynamical properties of the Riccati equations (1.23) and (1.24), the geometrical nature of the transformations (1.21) and (1.22) it is established. For this purpose it is necessary to introduce some geometrical structures in quantum mechanics [7, 6].

It is well-known that a complete measurement in quantum mechanics does not provide us with a uniquely defined vector in the Hilbert space, but rather with an equivalence class of vectors, obtained by multiplication with a complex number, i.e. on the Hilbert space there is a natural action of the Abelian group $\mathbb{C}_0 = \mathbb{C} - \{\,\mathbf{0}\,\}$ given by

$$|\psi\rangle \;\mapsto \lambda\,|\psi\rangle \;=\; \varrho\,\mathrm{e}^{\mathrm{i}\,\theta}\,|\psi\rangle \quad \text{with} \quad \varrho > 0\,. \tag{1.25}$$

A pure state will be an equivalence class, i.e. a ray in the Hilbert space.

Thus, we consider an N-level quantum system with Hilbert space $\mathcal{H}_0$. Selecting an orthonormal basis $\{|e_k\rangle\}_{k=1,\ldots,n}$ in $\mathcal{H}_0$ we may introduce a Cartesian coordinate system $\{x^k, y^k\}_{k=1,\ldots,n}$ on $\mathcal{H}_0$, namely for any element $|\psi\rangle \in \mathcal{H}_0$ we have that

$$|\psi\rangle = \psi^k |e_k\rangle = (x^k + \mathrm{i}\,y^k)|e_k\rangle\,. \tag{1.26}$$

Then, the group action defining the equivalence class may be described infinitesimally by means of two commuting linear vector fields, given in Cartesian coordinates as

$$\Delta = \; x^k \frac{\partial}{\partial x^k} + y^k \frac{\partial}{\partial y^k} \quad \text{and} \quad \Gamma \;= y^k \frac{\partial}{\partial x^k} - x^k \frac{\partial}{\partial y^k}\,, \tag{1.27}$$

where Δ is the infinitesimal generator of dilations, while Γ is the infinitesimal generator of the multiplication by a global phase factor.

Then, on the space $\mathcal{H}_0$ there are two regular distributions related with Δ and Γ. The first one is $\mathcal{D} = \{\Delta\}$, and it is possible to see that the quotient space $\mathcal{H}_0/\Phi^{\Delta}$ may be represented by the unit sphere in $\mathcal{H}_0$, namely

$$S^{2n-1} \;:=\; \{|\psi\rangle \in \mathcal{H}_0 \,|\, \langle\psi|\psi\rangle \;=\; 1\}\,. \tag{1.28}$$

In the following, the projection from $\mathcal{H}_0$ to S^{2n-1} will be denoted by τ, while, an element of S^{2n-1} will be denoted as $|\psi)$, whereas $|\psi\rangle$ is a vector in $\mathcal{H}_0$.

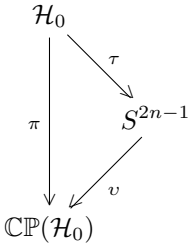
If one considers the distribution $\mathcal{D}_2 = \{\Gamma, \Delta\}$, which is involutive, it gives rise to a foliation $\Phi^{\mathcal{D}_2}$ which is regular. The quotient space $\mathcal{H}_0/\Phi^{\mathcal{D}_2}$ is known as *the complex projective space* $\mathbb{CP}(\mathcal{H}_0)$ associated with $\mathcal{H}_0$, namely

$$\mathbb{CP}(\mathcal{H}_0) \;:=\; \{\lambda\,|\psi\rangle |\, \lambda \in \mathbb{C}_0\,\}\,. \tag{1.29}$$

The projection from $\mathcal{H}_0$ to $\mathbb{CP}(\mathcal{H}_0)$ will be denoted by π and the elements of $\mathbb{CP}(\mathcal{H}_0)$, denoted by $[\psi]$ with $|\psi\rangle \in \mathcal{H}_0$, $[\psi]$ are identified with the pure states of a quantum system. They are in one-to-one correspondence with rank-one projectors, i.e.,

$$[\psi] \mapsto \rho_\psi := \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}\,. \tag{1.30}$$

Furthermore, because the vector field Γ is tangent to S^{2n-1}, it is possible to consider its restriction Γ_s to S^{2n-1} and build the distribution $\mathcal{D}_s$ associated with it. Clearly, the quotient space $S^{2n-1}/\Phi^{\mathcal{D}_s}$ will be precisely the complex projective space $\mathbb{CP}(\mathcal{H}_0)$, and the canonical projection from S^{2n-1} to $\mathbb{CP}(\mathcal{H}_0)$ will be denoted by υ, and hence it holds that $\pi = \upsilon \circ \tau$. Then, we arrive at the following diagramme



Therefore, $\mathbb{CP}(\mathcal{H}_0)$ is the space of physical states where they have been normalized and one has gotten rid of the global phase, i.e., we are aiming at a description on the space of pure states described without redundancies.

Q-bit system

To better visualize the situation, let us again consider a 2-level quantum system. In this case the Hilbert space is isomorphic to $\mathbb{C}^2$ where the vectors are given by

$$|\psi\rangle = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}, \tag{1.31}$$

with $\psi^k \in \mathbb{C}$. In this case the unit sphere $S^3 \in \mathbb{C}^2$ of normalized states is given by the normalized vectors

$$\tau(|\psi\rangle) = |\psi) = \frac{1}{\sqrt{\psi^1\bar{\psi}^1 + \psi^2\bar{\psi}^2}} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}. \tag{1.32}$$

On the other hand, the complex projective space may be thought of as the unit sphere $S^2 \in \mathbb{R}^3$. By using the covering map $\pi : \mathcal{H}_0 \to S^2$, we associate a pure state

$$\rho_\psi = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} = |\psi)(\psi| \tag{1.33}$$

with a vector $|\psi\rangle$. In coordinates we have that

$$\pi(|\psi\rangle) = \rho_\psi = \frac{1}{\psi^1\bar{\psi}^1 + \psi^2\bar{\psi}^2} \begin{pmatrix} \psi^1\bar{\psi}^1 & \psi^1\bar{\psi}^2 \\ \psi^2\bar{\psi}^1 & \psi^2\bar{\psi}^2 \end{pmatrix}. \tag{1.34}$$

In general, we may immerse the complex projective space for the q-bit in the space of 2×2 Hermitian matrices where a basis is provided by Pauli matrices and the identity matrix

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.35}$$

i.e. every density matrix may be expressed as

$$\rho = \frac{1}{2}(\mathbb{I} + x^j\sigma_j)\,. \tag{1.36}$$

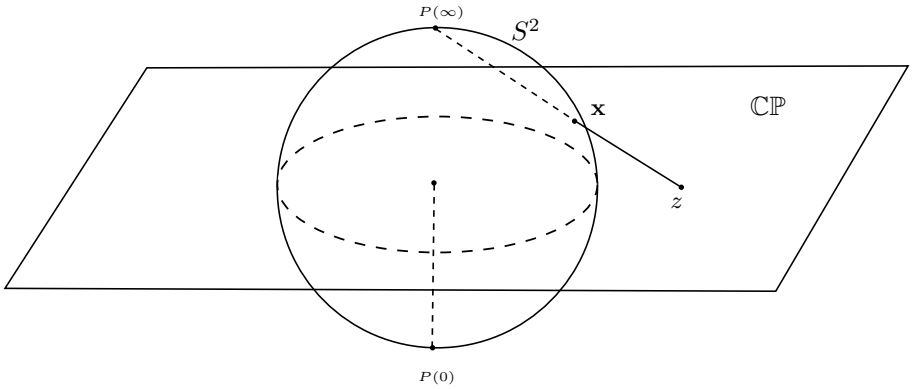


Figure 1.1: The stereographic projection from the north pole onto the equatorial plane, which maps every point $\mathbf{x} = (x^1, x^2, x^3) \in S^2$ into points in the complex plane $z \in \mathbb{CP}$.

This means that every quantum state is represented by the vector $(x^1, x^2, x^3) \in S^2$, such that $x^j = \mathrm{Tr}\{\sigma_j \rho\}$, and the purity condition $\rho^2 = \rho$ identifies the unit sphere

$$S^2 = \{(x^1, x^2, x^3) \in \mathbb{R}^3 \,|\, (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}\,. \tag{1.37}$$

However, the description of the complex projective space is complete by means of complex homogeneous coordinates, this is, let $U_j \subset \mathbb{CP}$, with $j = 1, 2$, denote the coordinate patch where $\psi^j \neq 0$ defined by

$$\phi_1 : [\psi^1, \psi^2] \mapsto z = \frac{\psi^1}{\psi^2} \quad \text{and} \quad \phi_2 : [\psi^1, \psi^2] \mapsto \zeta = \frac{\psi^2}{\psi^1}\,, \tag{1.38}$$

such that the set of (U_j, ϕ_j) constitutes an *atlas* for the complex projective space. Thus, we may consider the alternative expression of the vector states $|\psi) \in S^3$ and its image $\rho_\psi \in S^2$ by means of

$$|\psi) = \frac{\mathrm{e}^{\mathrm{i}\varphi}}{\sqrt{1 + z\bar{z}}} \begin{pmatrix} z \\ 1 \end{pmatrix}, \qquad \rho_\psi = |\psi)(\psi| = \frac{1}{1 + z\bar{z}} \begin{pmatrix} z\bar{z} & \bar{z} \\ z & 1 \end{pmatrix}, \tag{1.39}$$

where φ is a real and possibly time-dependent function. Then, we may compare this expression of the density matrix with the one given in Eq. (1.36) to obtain the transformation

$$x^1 = \frac{2z_{\mathrm{R}}}{1 + z\bar{z}}\,, \qquad x^2 = \frac{2z_{\mathrm{I}}}{1 + z\bar{z}}\,, \qquad x^3 = \frac{-1 + z\bar{z}}{1 + z\bar{z}}\,, \tag{1.40}$$

where $z = z_{\mathrm{R}} + \mathrm{i}\, z_{\mathrm{I}}$.

This transformation is simply the stereographic projection from the "north pole" of the sphere onto the equatorial plane, see Fig. 1.1. For the map $\zeta = \frac{\psi^2}{\psi^1}$, one obtains the stereographic projection from the "south pole" of the sphere onto the equatorial plane, i.e.

$$x^1 = \frac{2\zeta_{\mathrm{R}}}{1 + \zeta\bar{\zeta}}\,, \qquad x^2 = \frac{-2\zeta_{\mathrm{I}}}{1 + \zeta\bar{\zeta}}\,, \qquad x^3 = \frac{1 - \zeta\bar{\zeta}}{1 + \zeta\bar{\zeta}}\,, \tag{1.41}$$

with $\zeta = \zeta_{\mathrm{R}} + \mathrm{i}\,\zeta_{\mathrm{I}}$. Therefore, we have a complete covering of S^2. The obtained complex plane $\mathbb{CP}$ is known in quantum physics as the *ray space* description of the quantum system.

For the general case, i.e. for an N-level quantum system, it is straightforwards to adapt the complex homogeneous coordinates. This is, let $U_j \subset \mathbb{CP}$ denote the coordinates patch defined by

$$\phi_j : U_j \to \mathbb{C}^{n-1} : [\psi^1, \ldots, \psi^n] \mapsto \left(z^1, \ldots, z^{j-1}, z^{j+1}, \ldots, z^n\right), \quad \text{with} \quad z^k = \frac{\psi^k}{\psi^j}\,. \tag{1.42}$$

with $\psi^j \neq 0$, then the set of (U_j, ϕ_j), with $j = 1, \ldots, n$, constitutes an *atlas* for the complex projective space. Therefore the projection of $\mathcal{H}_0$ onto the complex projective space may be expressed as

$$\pi : \mathcal{H}_0 \to \mathbb{CP} : |\psi\rangle \mapsto |\psi) = \frac{1}{\sqrt{1 + \mathbf{z}^\dagger \mathbf{z}}} \begin{pmatrix} \mathbf{z} \\ 1 \end{pmatrix}, \tag{1.43}$$

where the chart ϕ_n has been employed, so $\mathbf{z}$ is an $(N-1)$-dimensional complex vector, with components $z^k = \frac{\psi^k}{\psi^n}$ for $k = 1, \ldots, n-1$.

Now that we have established the canonical projection π from the Hilbert space $\mathcal{H}_0$ to the complex projective space $\mathbb{CP}$, we may apply this results to the dynamics of the system. The Schrödinger dynamics defines a dynamical vector field X_H in the carrier space $T\mathcal{H} \approx \mathcal{H} \times \mathcal{H}$, given by

$$X_H = \frac{\mathrm{i}}{\hbar} \bar{\psi}_j \, H^j_k \, \frac{\partial}{\partial \bar{\psi}_k} - \frac{\mathrm{i}}{\hbar} \, H^k_j \, \psi^j \frac{\partial}{\partial \psi^k} \tag{1.44}$$

with H^j_k being an entry of the Hamiltonian matrix operator, where the upper index denotes the column and the lower the row. Therefore, a major question is whether there is a vector field which describes the Schrödinger motion on $\mathbb{CP}(\mathcal{H}_0)$, or as is said, whether X_H is projectable. In fact, a suficient condition for projectability (see reference [4] for the formal proof) is

$$[X_H, \Delta] = [X_H, \Gamma] = 0 \,. \tag{1.45}$$

It is interesting to mention that because the dynamics is projectable, it carries leaves of the foliation $\Phi^{\mathcal{D}_2}$, into leaves, i.e. the foliation is invariant under X_H [4]. In this sense the group action is a symmetry for the Schrödinger dynamics.

1.3 A Digression to the Schrödinger Picture

So far it has been shown that the Schrödinger dynamics is projected onto the complex projective space in a nonlinear Riccati equation; however, not only the dynamics is projectable but also important geometrical structures may be projected. In this section the geometrical structure of the Hilbert space will be introduced in the usual description of quantum systems, in order to project them later onto $\mathbb{CP}$.

Thus, an important mathematical property of the Hilbert space is the fact that there is an *inner product* defined on it. Then, from the system of coordinates introduced in (1.26), we define the complex coordinate functions

$$|\mathrm{d}\psi\rangle = \mathrm{d}\psi^k |e_k\rangle = (\mathrm{d}x^k + \mathrm{i}\,\mathrm{d}y^k)|e_k\rangle \,, \tag{1.46}$$

such that the Hermitian product acquires the form

$$\langle \mathrm{d}\psi | \mathrm{d}\psi \rangle = \hbar \, \delta^k_j (\mathrm{d}x_k \otimes \mathrm{d}x^j + \mathrm{d}y_k \otimes \mathrm{d}y^j) + \mathrm{i}\,\hbar \, \delta^k_j (\mathrm{d}x_k \otimes \mathrm{d}y^j - \mathrm{d}y_k \otimes \mathrm{d}x^j) \,, \tag{1.47}$$

which is decomposed into the real and the imaginary parts

$$g_{\mathcal{H}} = \hbar \, (\mathrm{d}x_k \otimes \mathrm{d}x^k + \mathrm{d}y_k \otimes \mathrm{d}y^k) \quad \text{and} \quad \omega_{\mathcal{H}} = \hbar \, (\mathrm{d}x_k \otimes \mathrm{d}y^k - \mathrm{d}y_k \otimes \mathrm{d}x^k) \,. \tag{1.48}$$

It is clear from the above definitions that $g_{\mathcal{H}}$ is a symmetrical tensor, a metric, and $\omega_{\mathcal{H}}$ is a skew-symmetrical tensor, a 2-form. Therefore, the Hilbert space may be seen as a real space $\mathcal{H}_{\mathbb{R}} \approx \mathbb{R}^{2n}$ with a metric tensor $g_{\mathcal{H}} = \mathrm{Re}\{\langle \phi | \psi \rangle\}$ and a symplectic structure $\omega_{\mathcal{H}} = \mathrm{Im}\{\langle \phi | \psi \rangle\}$ [7, 6].

As a further remark, we recall that the Hilbert space is endowed with the natural complex structure J defined simply by $J_{\mathcal{H}} : |\psi\rangle \mapsto |i\psi\rangle$, and then $J^2 = -1$. This endomorphism in local coordinates has the form

$$J_{\mathcal{H}} \;=\; \mathrm{d}y_k \otimes \frac{\partial}{\partial x^k} - \mathrm{d}x_k \otimes \frac{\partial}{\partial y^k} \,, \tag{1.49}$$

and then in the realification of the Hilbert space $J_{\mathcal{H}}$ connects the metric tensor and the symplectic structure as

$$g_{\mathcal{H}}(\phi, \psi) = -\omega_{\mathcal{H}}(J_{\mathcal{H}}\phi, \psi) \,, \quad \omega_{\mathcal{H}}(\phi, \psi) = g_{\mathcal{H}}(J_{\mathcal{H}}\phi, \psi) \,. \tag{1.50}$$

Moreover, the relations

$$g_{\mathcal{H}}(J_{\mathcal{H}}\phi, J_{\mathcal{H}}\psi) = g_{\mathcal{H}}(\phi,\psi)\,, \quad \omega_{\mathcal{H}}(J_{\mathcal{H}}\phi, J_{\mathcal{H}}\psi) = \omega_{\mathcal{H}}(\phi,\psi)\,, \tag{1.51}$$

show that $J_{\mathcal{H}}$ is an infinitesimal generator for transformations preserving the metric and the symplectic structures. All the results presented before summarize the fact that the Hilbert space has a Kähler structure [7, 6, 5].

The symplectic form $\omega_{\mathcal{H}}$ and the metric $g_{\mathcal{H}}$ define two relevant vector fields. Let f be an arbitrary function in $\mathcal{H}$, i.e. $f \in \mathfrak{F}(\mathcal{H})$, then *the gradient vector field* ∇f and *the Hamiltonian vector field* X_f are defined by

$$g_{\mathcal{H}}(\nabla f, \cdot\,) = \mathrm{d}f\,, \qquad \omega_{\mathcal{H}}(X_f, \cdot\,) = \mathrm{d}f\,, \tag{1.52}$$

such that $J(\nabla f) = X_f$.

Upon adopting the complex coordinates $\psi^k = x^k + \mathrm{i}\,y^k$ and $\bar{\psi}^k = x^k - \mathrm{i}\,y^k$ previously defined in Eq. (1.26) one may express

$$g_{\mathcal{H}} = \frac{\hbar}{2}(\mathrm{d}\psi^k \otimes \mathrm{d}\bar{\psi}_k + \mathrm{d}\bar{\psi}_k \otimes \mathrm{d}\psi^k) \quad \text{and} \quad \omega_{\mathcal{H}} = \frac{\mathrm{i}\,\hbar}{2}(\mathrm{d}\psi^k \otimes \mathrm{d}\bar{\psi}_k - \mathrm{d}\bar{\psi}_k \otimes \mathrm{d}\psi^k)\,, \tag{1.53}$$

and the complex structure

$$J_{\mathcal{H}} = \frac{1}{i}\left(\mathrm{d}\psi^k \otimes \frac{\partial}{\partial\psi^k} - \mathrm{d}\bar{\psi}_k \otimes \frac{\partial}{\partial\bar{\psi}_k}\right). \tag{1.54}$$

Hence, in this local coordinates and taking into account the definitions (1.52), it is possible to deduce the form of the Hamiltonain vector field and the gradient vector field, namely

$$X_f = \frac{2\,\mathrm{i}}{\hbar}\left(\frac{\partial f}{\partial\psi^k}\frac{\partial}{\partial\bar{\psi}_k} - \frac{\partial f}{\partial\bar{\psi}_k}\frac{\partial}{\partial\psi^k}\right) \tag{1.55}$$

and

$$\nabla f = \frac{2}{\hbar}\left(\frac{\partial f}{\partial\psi^k}\frac{\partial}{\partial\bar{\psi}_k} + \frac{\partial f}{\partial\bar{\psi}_k}\frac{\partial}{\partial\psi^k}\right), \tag{1.56}$$

respectively. In the Hilbert space one may describe observables in terms of bilinear functions $f_A = \frac{1}{2}\langle\psi, \mathbb{A}\,\psi\rangle$, which are proportional to the expectation values and in local coordinates has the form

$$f_A = \frac{1}{2}\bar{\psi}_k\,\mathbb{A}^k_j\,\psi^j\,. \tag{1.57}$$

From the definitions in Eq. (1.52) it is possible to obtain the gradient vector field $Y_{f_A} = \nabla f_A$ and the Hamiltonian vector field X_{f_A} associated to the observable. Namely

$$X_{f_A} = \frac{\mathrm{i}}{\hbar}\,\bar{\psi}_j\,A^j_k\,\frac{\partial}{\partial\bar{\psi}_k} - \frac{\mathrm{i}}{\hbar}\,A^k_j\,\psi^j\frac{\partial}{\partial\psi^k}\,, \qquad Y_{f_A} = \frac{1}{\hbar}\,\bar{\psi}_j\,A^j_k\,\frac{\partial}{\partial\bar{\psi}_k} + \frac{1}{\hbar}\,A^k_j\,\psi^j\frac{\partial}{\partial\psi^k}\,. \tag{1.58}$$

The Hamiltonian vector field introduces a dynamics in the Hilbert space whose integral curves are determined by the solution of the linear equation

$$\dot{\psi}^k = -\frac{\mathrm{i}}{\hbar}\,A^k_j\,\psi^j\,. \tag{1.59}$$

In particular, if the observable is the Hamiltonian then the before presented equation corresponds to the Schrödinger equation. In addition, it is possible to obtain the Poisson and the Jordan brackets, this is for given $f, h \in \mathfrak{F}(\mathcal{H})$ one has that

$$\{f,h\}_{\omega_{\mathcal{H}}} = \omega_{\mathcal{H}}(X_f, X_h)\,, \qquad \{f,h\}_{g_{\mathcal{H}}} = g_{\mathcal{H}}(Y_f, Y_h)\,. \tag{1.60}$$

Extending both these brackets to complex functions via complex linearity, we obtain the complex bracket

$$\{f,g\}_{\mathcal{H}} := \{f,h\}_{g_{\mathcal{H}}} + \mathrm{i}\,\{f,h\}_{\omega_{\mathcal{H}}}\,, \tag{1.61}$$

expressed in complex coordinates by

$$\{f,g\}_{\mathcal{H}} = \frac{4}{\hbar}\,\frac{\partial f}{\partial\bar{\psi}_k}\frac{\partial g}{\partial\psi^k}\,. \tag{1.62}$$

1.4 Nonlinear Evolution of N-level Quantum Systems

For the q-bit system it has been shown that the evolution of the states is, in homogeneous coordinates, given by a nonlinear Riccati equation. In this section we would like to generalize this result to any N-level system as well as to discuss, in some details, the structure of the complex projective space of the Hilbert space.

It is well-known that the Hilbert space $\mathcal{H}_0$ is a Kähler manifold [7, 6], i.e., there are a symplectic form $\omega_{\mathcal{H}}$, a Riemannian metric tensor $g_{\mathcal{H}}$ and a complex structure $J_{\mathcal{H}}$ such that $g_{\mathcal{H}} = J_{\mathcal{H}} \circ \omega_{\mathcal{H}}$. Hence, it is expected that the same structures are defined on the complex projective space $\mathbb{CP}$. In fact, considering the homogeneous coordinates in Eq. (1.43) we proceed to introduce the 1-form

$$\begin{aligned}\theta_{\text{FS}} &= \frac{\hbar}{i}\frac{1}{\sqrt{1+\mathbf{z}\,\bar{\mathbf{z}}}}(\,\bar{\mathbf{z}}\,,\,1\,)\left[\frac{1}{\sqrt{1+\mathbf{z}\,\bar{\mathbf{z}}}}\begin{pmatrix}\mathrm{d}\mathbf{z}\\0\end{pmatrix}+\begin{pmatrix}\mathbf{z}\\1\end{pmatrix}\mathrm{d}\left(\frac{1}{\sqrt{1+\mathbf{z}\,\bar{\mathbf{z}}}}\right)\right]\\ &= \frac{\hbar}{2\,\mathrm{i}}\frac{\bar{z}_k\,\mathrm{d}z^k - z^k\,\mathrm{d}\bar{z}_k}{1+\bar{\mathbf{z}}\,\mathbf{z}}\,.\end{aligned} \tag{1.63}$$

Thus, the symplectic form ω_{FS} on $\mathbb{CP}$ is then defined as

$$\begin{aligned}\omega_{\text{FS}} = \mathrm{d}\theta_{\text{FS}} &= \frac{\hbar}{\mathrm{i}}(\mathrm{d}\psi\,|\wedge|\,\mathrm{d}\psi)\\ &= \frac{-\mathrm{i}\,\hbar}{(1+\bar{\mathbf{z}}\,\mathbf{z})^2}\left[(1+\bar{\mathbf{z}}\,\mathbf{z})\,\mathrm{d}\bar{z}_k\wedge\mathrm{d}z^k - \frac{1}{2}\,(\bar{z}_l\,\mathrm{d}z^l + z^l\,\mathrm{d}\bar{z}_l)\wedge(\bar{z}_k\,\mathrm{d}z^k - z^k\,\mathrm{d}\bar{z}_k)\right].\end{aligned} \tag{1.64}$$

Furthermore, by means of the map π in (1.43), we may look at the pullback of $\omega_{\mathcal{H}}$ to $\mathcal{H}_0$ and see that $\omega_{\mathcal{H}} = \pi^*\,\omega_{\text{FS}}$. The same is also true for the metric, i.e., $g_{\mathcal{H}} = \pi^*\,g_{\text{FS}}$, where g_{FS} is the so-called Fubini–Study metric [7], which in homogeneous coordinates has the form

$$\begin{aligned}g_{\text{FS}} &= (\mathrm{d}\psi\,|\otimes_{\text{S}}|\,\mathrm{d}\psi)\\ &= \frac{-\hbar}{(1+\mathbf{z}\,\bar{\mathbf{z}})^2}\left[(1+\mathbf{z}\,\bar{\mathbf{z}})\,\mathrm{d}\bar{z}_k\otimes_{\text{S}}\mathrm{d}z^k + \frac{1}{2}\,(\bar{z}_k\,\mathrm{d}z^k - z^k\,\mathrm{d}\bar{z}_k)\otimes(\bar{z}_l\,\mathrm{d}z^l - z^l\,\mathrm{d}\bar{z}_l)\right.\\ &\left. - \frac{1}{2}\,(\bar{z}_k\,\mathrm{d}z^k + z^k\,\mathrm{d}\bar{z}_k)\otimes(\bar{z}_l\,\mathrm{d}z^l + z^l\,\mathrm{d}\bar{z}_l)\right],\end{aligned} \tag{1.65}$$

where $\mathrm{d}\bar{z}_k\otimes_{\text{S}}\mathrm{d}z^k = \mathrm{d}\bar{z}_k\,\mathrm{d}z^k + \mathrm{d}z^k\,\mathrm{d}\bar{z}_k$. On the other hand, the complex structure on the complex projective space may be obtained by a $(1,1)$-tensor which contains vector fields and forms. While vector fields may be projected (if projectable), forms cannot be projected. Therefore one may speak of "related" $(1,1)$-tensor fields by a related $(1,1)$-tensor field

$$J_{\text{FS}} = \frac{1}{\mathrm{i}}\left(\mathrm{d}z^k\otimes\frac{\partial}{\partial z^k} - \mathrm{d}\bar{z}_k\otimes\frac{\partial}{\partial\bar{z}_k}\right). \tag{1.66}$$

Therefore, the complex projective space has a Kähler structure $(\omega_{\text{FS}}, g_{\text{FS}}, J_{\text{FS}})$ [7, 5].

The symplectic form and the Riemannian metric define a Hamiltonian vector field and a gradient vector field, respectively. This means to any function $e_A \in \mathfrak{F}(\mathbb{CP})$ a gradient and a Hamiltonian vector fields is associated by means of the intrinsic definitions

$$g_{\text{FS}}(Y_{e_A},\,\cdot\,) = \mathrm{d}\,e_A\,, \qquad \omega_{\text{FS}}(X_{e_A},\,\cdot\,) = \mathrm{d}\,e_A\,, \tag{1.67}$$

such that

$$J_{\text{FS}}(X_{e_A}) = Y_{e_A}\,. \tag{1.68}$$

To introduce the definitions in Eq. (1.67) in complex homogeneous coordinates on $\mathbb{CP}$, then in these

coordinates the expectation value of an arbitrary observable A may be expressed in general as

$$\begin{aligned} e_A &= (\psi|A|\psi) \\ &= \frac{1}{1+\mathbf{z}^\dagger\,\mathbf{z}}\,(\,\mathbf{z}^\dagger\;,\;1\,)\begin{pmatrix}\mathbb{A}_1 & \mathbb{V}\\ \mathbb{V}^\dagger & A_2\end{pmatrix}\begin{pmatrix}\mathbf{z}\\ 1\end{pmatrix} \\ &= \frac{1}{1+\mathbf{z}^\dagger\,\mathbf{z}}\left(\,\mathbf{z}^\dagger\mathbb{A}_1\mathbf{z}+\mathbf{z}^\dagger\mathbb{V}+\mathbb{V}^\dagger\mathbf{z}+A_2\,\right), \end{aligned} \tag{1.69}$$

where the matrix $\mathbb{A}_1$ is an $(N-1)\times(N-1)$–dimensional matrix, $\mathbb{V}$ is an $(N-1)$–component column vector and A_2 a real quantity. In particular we can consider the Hamiltonian as the observable with representative matrix

$$\mathbb{H} = \begin{pmatrix}\mathbb{H}_1 & \mathbb{V}\\ \bar{\mathbb{V}}^\dagger & H_2\end{pmatrix}. \tag{1.70}$$

Thus, the Hamiltonian vector field may be obtained from the right-hand expression in Eq. (1.67), taking into account the expression in coordinates of the symplectic form in Eq. (1.64). Namely, we have that the dynamics induced by the Hamiltonian corresponds to

$$X_{e_H} = X_{z^k}\,\frac{\partial}{\partial z^k} + X_{\bar{z}_k}\,\frac{\partial}{\partial \bar{z}_k}\,, \tag{1.71}$$

where the component $X_{\bar{z}_k}$ is the complex conjugated of X_{z^k}, and X_{z^k} is given by

$$X_{z^k} = \frac{\mathrm{i}}{\hbar}\left(z^k\,\bar{V}_l\,z^l - |\mathbb{H}_1|^k_l\,z^l + H_2\,z^k - V^k\right). \tag{1.72}$$

Therefore, the integral curves of this Hamiltonian vector field are provided by the Hamiltonian equations of motion

$$\begin{aligned} \dot{z}^k &= -\frac{\mathrm{i}}{\hbar}\,(1+\mathbf{z}^\dagger\,\mathbf{z})\left(\frac{\partial e_H}{\partial \bar{z}_k} + z^k\,\bar{z}_l\,\frac{\partial e_H}{\partial \bar{z}_l}\right) = \frac{\mathrm{i}}{\hbar}\left(z^k\,\bar{V}_l\,z^l - |\mathbb{H}_1|^k_l\,z^l + H_2\,z^k - V^k\right), \\ \dot{\bar{z}}_k &= \frac{\mathrm{i}}{\hbar}\,(1+\mathbf{z}^\dagger\,\mathbf{z})\left(\frac{\partial e_H}{\partial z^k} + \bar{z}_k\,z^l\,\frac{\partial e_H}{\partial z^l}\right) = -\frac{\mathrm{i}}{\hbar}\left(\bar{z}_l\,V^l\,\bar{z}_k - \bar{z}_l\,|\mathbb{H}_1|^l_k + H_2\,\bar{z}_k - \bar{V}_k\right). \end{aligned} \tag{1.73}$$

Therefore the equation of motion of the N-level quantum system is given by the well-known *matrix Riccati equation* [9]

$$\dot{z}^k = \frac{\mathrm{i}}{\hbar}\left(z^k\,\bar{V}_l\,z^l - |\mathbb{H}_1|^k_l\,z^l + H_2\,z^k - V^k\right). \tag{1.74}$$

Hence, the matrix Riccati equation is simply the coordinate expression of the Schrödinger equation on the complex-projective space. Moreover, we have proved that the nonlinear Riccati evolution is actually a Hamiltonian dynamics.

The gradient vector field can be obtained similarly, i.e. considering now the left-hand expression in Eq. (1.67) and the expression in coordinates of the Riemannian metric in Eq. (1.65). However, because we have already determined the Hamiltonian vector field and we also know the complex structure J_{FS} in Eq. (1.66), so it is easier to obtain the gradient vector field from the property in Eq. (1.68). Hence the gradient vector field is simply

$$\begin{aligned} Y_{e_H} &= J_{\text{FS}}(X_{e_H}) \\ &= \frac{1}{\hbar}\left[z^k\,\bar{V}_l\,z^l - |\mathbb{H}_1|^k_l\,z^l + H_2\,z^k - V^k\right]\frac{\partial}{\partial z^k} + \frac{1}{\hbar}\left[\bar{z}_l\,V^l\,\bar{z}_k - \bar{z}_l\,|\mathbb{H}_1|^l_k + H_2\,\bar{z}_k - \bar{V}_k\right]\frac{\partial}{\partial \bar{z}_k}\,. \end{aligned} \tag{1.75}$$

In addition, we may introduce the Poisson and Jordan brackets for the complex projective space [6, 7]. So, given the expectation values e_A and e_B associated to the observables A and B

$$\{e_A, e_B\}_{\omega_{\text{FS}}} = \omega_{\text{FS}}(X_{e_A}, X_{e_B})\,, \qquad \{e_A, e_B\}_{g_{\text{FS}}} = g_{\text{FS}}(Y_{e_A}, Y_{e_B})\,, \tag{1.76}$$

where in complex homogeneous coordinates these brackets have the form

$$\{e_A, e_B\}_{\omega_{\rm FS}} = -\frac{\mathrm{i}}{\hbar}(1+\mathbf{z}^\dagger\,\mathbf{z})\left[\left(\frac{\partial e_A}{\partial z^k}\,\frac{\partial e_B}{\partial \bar z_k} - \frac{\partial e_A}{\partial \bar z_k}\,\frac{\partial e_B}{\partial z^k}\right) + \left(z^k\,\frac{\partial e_A}{\partial z^k}\,\bar z_l\frac{\partial e_B}{\partial \bar z_l} - \bar z_l\,\frac{\partial e_A}{\partial \bar z_l}\,z^k\frac{\partial e_B}{\partial z^k}\right)\right] \tag{1.77}$$

and

$$\{e_A, e_B\}_{g_{\rm FS}} = -\frac{1}{\hbar}(1+\mathbf{z}^\dagger\,\mathbf{z})\left[\left(\frac{\partial e_A}{\partial z^k}\,\frac{\partial e_B}{\partial \bar z_k} + \frac{\partial e_A}{\partial \bar z_k}\,\frac{\partial e_B}{\partial z^k}\right) + \left(z^k\,\frac{\partial e_A}{\partial z^k}\,\bar z_l\frac{\partial e_B}{\partial \bar z_l} + \bar z_l\,\frac{\partial e_A}{\partial \bar z_l}\,z^k\frac{\partial e_B}{\partial z^k}\right)\right]. \tag{1.78}$$

In addition, after some calculations it is possible to prove that the Poisson and the Jordan brackets are such that

$$\{e_A, e_B\}_{\omega_{\rm FS}} = e_{\frac{1}{\mathrm{i}\hbar}[A,B]_-} \qquad \text{and} \qquad \{e_A, e_B\}_{g_{\rm FS}} = -\frac{2}{\hbar}\left[\frac{1}{2}\,e_{[A,B]_+} - e_A\,e_B\right], \tag{1.79}$$

where $[A,B]_- = A\,B - B\,A$ and $[A,B]_+ = A\,B + B\,A$. Therefore, we have a clear connection between the Poisson brackets and the quantum commutator. In particular, considering the Hamiltonian of the system H with expectation value e_H, then the evolution of the expectation value e_A of an arbitrary observable A is given by

$$\frac{\mathrm{d}e_A}{\mathrm{d}t} = \{e_H, e_A\}_{\omega_{\rm FS}} = e_{\frac{1}{\mathrm{i}\hbar}[H,A]_-}\,. \tag{1.80}$$

This result implies immediately that for the time-independent case e_H is a first integral of the flow, i.e., the expectation value of the Hamiltonian is conserved. In addition, the expectation value of any observable commuting with H is also a first integral.

On the other hand, the Jordan bracket is connected with the dispersion and the correlation of the observables. This is, for every couple of observables A and B their uncertainties and correlations are given by

$$\sigma_A^2 = e_{A^2} - e_A^2 = -\frac{\hbar}{2}\{e_A, e_A\}_{g_{\rm FS}} \tag{1.81}$$

and

$$\sigma_{AB} = \frac{1}{2}\,e_{[A,B]_+} - e_A\,e_B = -\frac{\hbar}{2}\{e_A, e_B\}_{g_{\rm FS}}\,, \tag{1.82}$$

respectively. Thus, the Riemannian metric carried by the complex projective space takes into account the probabilistic character of quantum mechanics [43].

Q-bit system

For the q-bit example the expectation value of the Hamiltonian is expressed in homogeneous coordinates by

$$\begin{aligned} e_H &= (\psi|H|\psi) \\ &= \frac{1}{1+z\bar z}\,(\,\bar z\ ,\ 1\,)\begin{pmatrix} H_1 & V \\ \bar V & H_2 \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix} \\ &= \frac{1}{1+z\bar z}\left(\,\bar z H_1 z + \bar z V + \bar V z + H_2\,\right). \end{aligned} \tag{1.83}$$

The Hamiltonian dynamics intrinsically defined in (1.67) may be found explicitly employing Hamilton's equations in Eq. (1.73). Then, Hamiliton's equations of motion are given by

$$\begin{aligned} \dot z &= -\frac{\mathrm{i}}{\hbar}\,(1+z\bar z)^2\,\frac{\partial e_H}{\partial \bar z} = \frac{i}{\hbar}\left[\bar V z^2 - (H_1 - H_2)z - V\right] \\ \dot{\bar z} &= \frac{\mathrm{i}}{\hbar}\,(1+z\bar z)^2\,\frac{\partial e_H}{\partial z} = -\frac{i}{\hbar}\left[V\bar z^2 - (H_1 - H_2)\bar z - \bar V\right], \end{aligned} \tag{1.84}$$

and are identical with the Riccati equation (1.23). Therefore the Hamiltonian vector field has the form

$$X_{e_H} = \frac{\mathrm{i}}{\hbar}\left[\bar{V}z^2 - (H_1 - H_2)z - V\right]\frac{\partial}{\partial z} - \frac{\mathrm{i}}{\hbar}\left[\bar{z}^2 V - (H_1 - H_2)\bar{z} - \bar{V}\right]\frac{\partial}{\partial \bar{z}}\,. \tag{1.85}$$

Furthermore, with the help of the complex structure we may obtain the gradient vectorfield, i.e.

$$\begin{aligned} Y_{e_H} &= J_{\mathrm{FS}}(X_{e_H}) && (1.86)\\ &= \frac{1}{\hbar}[\bar{V}z^2 - (H_1 - H_2)z - V]\frac{\partial}{\partial z} + \frac{1}{\hbar}[\bar{V}\bar{z}^2 - (H_1 - H_2)\bar{z} - \bar{V}]\frac{\partial}{\partial \bar{z}}\,, && (1.87)\end{aligned}$$

which is orthogonal to the Hamiltonian vector field.

Time-dependent Hamiltonian systems

For the N-level quantum system whose Hamiltonian depends explicitly on time the definition (1.67) for the Hamiltonian vector field is no longer valid, because the differential of the expectation value of the Hamiltonian depends on time. In order to deal with time-dependent systems, as in classical Hamiltonian theory [1], one usually extends the space with an extra dimension representing time. The extended space $\mathbb{CP}^{\mathrm{E}} = \mathbb{CP} \times \mathbb{R}$ is a manifold endowed with the 1-form

$$\theta^{\mathrm{E}}_{\mathrm{FS}} = \theta_{\mathrm{FS}} + e_H\,\mathrm{d}t\,, \tag{1.88}$$

where θ_{FS} is the one-form defined in (1.63) and the expectation value e_H may depend explicitly on time. We may notice that in this manner one is dealing with a *contact Hilbert manifold* [33, 32]. Then one proceeds to define a dynamics on $\mathbb{CP}^{\mathrm{E}}$ that correctly extends the Hamiltonian dynamics. The dynamics $X^{\mathrm{E}}_{\mathrm{e}_H} \in \mathfrak{X}(\mathbb{CP}^{\mathrm{E}})$ is now defined by the intrinsic condition

$$\mathrm{d}\theta^{\mathrm{E}}(X^{\mathrm{E}}_{\mathrm{e}_H}, \cdot\,) = 0\,, \tag{1.89}$$

which is satisfied if and only if the extended vector field $X^{\mathrm{E}}_{\mathrm{e}_H}$ takes the form

$$X^{\mathrm{E}}_{\mathrm{e}_H} = X_{\mathrm{e}_H} + \frac{\partial}{\partial t}\,, \tag{1.90}$$

where X_{e_H} is given by Eq. (1.71). Therefore, one obtains the Hamiltonian equations (1.73), augmented with the trivial equation $\dot{t} = 1$. It follows that the evolution of an arbitrary time-dependent expectation value of e_A is given by

$$\frac{\mathrm{d}e_A}{\mathrm{d}t} = X^{\mathrm{E}}_{e_H}[e_A] = \{e_H, e_A\}_{\omega_{\mathrm{FS}}} + \frac{\partial e_A}{\partial t}\,, \tag{1.91}$$

with the Poisson bracket $\{e_H, e_A\}_{\omega_{\mathrm{FS}}}$ defined in Eq. (1.77). Consequently for time-dependent Hamiltonian systems the expectation value of its Hamiltonian is not any longer conserved.

1.5 Nonlinear Superposition Rule

In the Hilbert space, given two solutions of the Schrödinger equation $|\psi\rangle$ and $|\phi\rangle$, the state given by linear superposition

$$|\Psi\rangle = p_1\,|\psi\rangle + p_2\,|\phi\rangle\,, \tag{1.92}$$

with $p_1, p_2 \in \mathbb{R}$, is also a solution of the Schrödinger equation. Physically, quoting Dirac, this means that: "Any state may be considered as the result of a superposition of two or more other states, and indeed in an infinite number of ways. Conversely any two or more states may be superposed to give a new state" [44]. To obtain such a superposition rule in $\mathbb{CP}$, we notice that in the nonlinear description the evolution of pure states is describe by solutions of a Riccati equation, that although it is nonlinear, originates from the linear Schrödinger equation with linear superposition rule, thus there must be a way to superimpose its solutions.

Let us first establish the superposition rule for the q-bit system. So, employing the superposition principle on the Hilbert space then in the complex projective space the superimposed solution $|\Psi\rangle$ in homogeneous coordinates (1.42) is given by

$$Z = \frac{p_1\psi^1 + p_2\,\phi^1}{p_1\psi^2 + p_2\,\phi^2}\,, \tag{1.93}$$

which by construction is a solution of the Riccati equation (1.23). It is not difficult to realize that it is impossible to express Z just in terms of the particular solutions $z_\psi = \frac{\psi^1}{\psi^2}$ and $z_\phi = \frac{\phi^1}{\phi^2}$. However, introducing the auxiliarly solution

$$z_0 = \frac{\psi^1 + \phi^1}{\psi^2 + \phi^2}\,, \tag{1.94}$$

after some algebra the superposed solution can be expressed as

$$Z = \frac{p_1\,z_\psi\,(z_0 - z_\phi) + p_2\,z_\phi\,(z_\psi - z_0)}{p_1(z_\phi - z_0) + p_2(z_\psi - z_0)}\,. \tag{1.95}$$

This relation between solutions of the Riccati equation is known in the literature as *the nonlinear superposition rule* [10, 11, 12]. Note that in order to obtain the expression (1.95) we have chosen the particular solution (1.94); however, the nonlinear superposition principle establishes that the general solution of a Riccati equation may be expressed as a function $Z = \Phi(z_0, z_\psi, z_\phi, \kappa)$ of three particular solutions and an arbitrary constant κ, thus any arbitrary solution z_0 may be employed to obtain the relation (1.95), for a formal proof see [12].

Therefore, we have proven that the linear superposition principle on the Hilbert space (1.92) is translated into the complex projective space as a nonlinear superposition rule (1.95). This means, given three particular solutions $\{z_0, z_\psi, z_\phi\}$ of the Riccati equation (1.23) the general solution can be written as

$$\frac{(Z - z_\psi)(z_0 - z_\phi)}{(Z - z_\phi)(z_0 - z_\psi)} = \kappa\,, \tag{1.96}$$

where κ is a constant determined by the initial conditions [11, 6].

The use of an auxiliary solutions in order to obtain a superposition principle of pure states is not new. In fact, in terms of the density matrix description, the rule to add two pure states, by their density operators ρ_ψ and ρ_ϕ, involves the use of a fiducial projector ρ_0 such that the superimposed operator is

$$\rho_\Psi = \frac{1}{N}\left(p_1\rho_\psi + p_2\rho_\phi + \frac{\sqrt{p_1p_2}(\rho_\psi\,\rho_0\,\rho_\phi + \text{h.c.})}{\sqrt{\text{Tr}(\rho_\psi\,\rho_0\,\rho_\phi\,\rho_0)}}\right) \tag{1.97}$$

with N a normalization constant, for details see [45]. From this point of view, one should not be surprised that the nonlinear superposition principle involves an auxiliary solution.

Finally, it is straightforwards to generalize the nonlinear superposition rule to N-level systems. The superimposed state $|\Psi\rangle = p_1\,|\psi\rangle + p_2\,|\phi\rangle$ is then represented in the homogeneous coordinates of $\mathbb{CP}(\mathcal{H}_0)$ by the vector state $\mathbf{Z}$ with components

$$Z^k = \frac{p_1\,z^k_\psi\,(z^k_0 - z^k_\phi) + p_2\,z^k_\phi\,(z^k_\psi - z^k_0)}{p_1(z^k_\phi - z^k_0) + p_2(z^k_\psi - z^k_0)} \tag{1.98}$$

with $k = 1, \ldots, n-1$, where $z^k_\psi = \frac{\psi^k}{\psi^n}$, $z^k_\phi = \frac{\phi^k}{\phi^n}$ and where z^k_0 is the k-th component of an arbitrary solution $\mathbf{z}_0$. For example, one may consider $\mathbf{z}_0$ with components

$$z^k_0 = \frac{\psi^k + \phi^k}{\psi^n + \phi^n}\,. \tag{1.99}$$

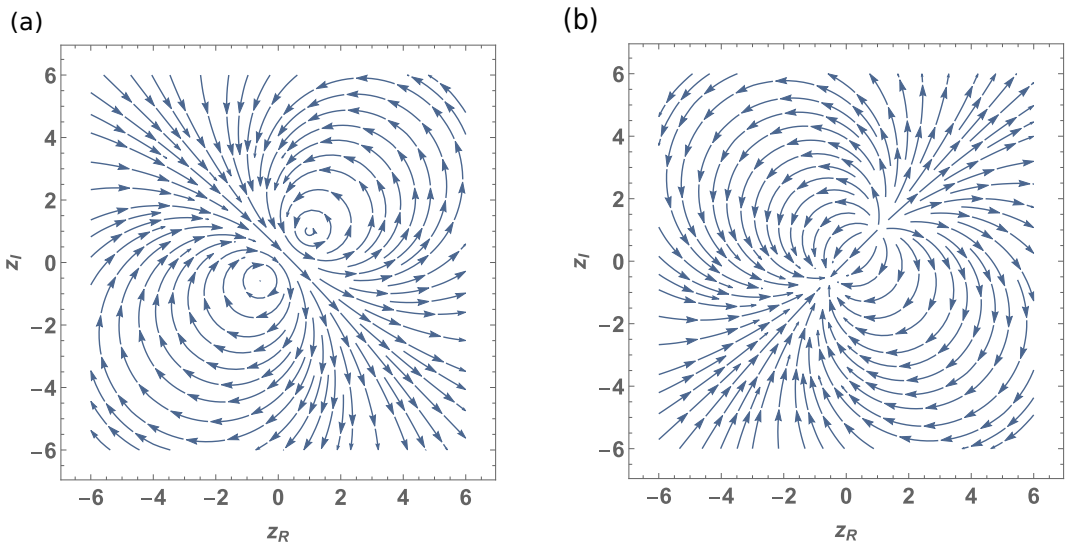


Figure 1.2: Phase portrait of (a) the Hamiltonian vector field and (b) the gradient vector field associated to the time-independent case $\omega = 0$ and parameters: $\omega_a = 1$, $\alpha = 1 + \mathrm{i}$.

1.6 Semiclassical Jaynes–Cummings model

As an example of nonlinear evolution of quantum systems, let us consider the Jaynes–Cummings model of quantum optics. This model consists of a single two-level atom interacting with a single quantized cavity mode of the electromagnetic field [13]. Employing the rotating wave approximation, this interaction is described by the Hamiltonian

$$H = H_{\mathrm{F}} + H_{\mathrm{A}} + H_{\mathrm{I}} = \hbar\,\omega\,\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\hbar\,\omega_a\sigma_3 + 2\,\hbar\,g(\sigma_+\hat{a} + \sigma_-\hat{a}^{\dagger})\,, \tag{1.100}$$

where $\{\hat{a}^{\dagger}, \hat{a}\}$ represent the photon creation and annihilation operators, and $\{\sigma_+, \sigma_-\}$ are the transition operators acting on the atomic states $\{E_1, E_2\}$. In terms of the Pauli matices $\{\sigma_1, \sigma_2, \sigma_3\}$ the transition operators have the form

$$\sigma_+ = \frac{1}{2}(\sigma_1 + \mathrm{i}\,\sigma_2)\,, \qquad \sigma_- = \frac{1}{2}(\sigma_1 - \mathrm{i}\,\sigma_2)\,. \tag{1.101}$$

Besides, ω represents the frequency of the electromagnetic field, $\hbar\,\omega_a$ is the difference of energy between the two states of the atom, i.e. $\hbar\,\omega_a = E_2 - E_1$, and g is the atom-field coupling constant.

Then, let us consider an *effective description* of the 2-level atom (reference system) in iteraction with the electromagnetic field (environment). By effective description we mean that given a conservative (reference) system we want to take into account the coupling with the environment without the consideration of additional degrees of freedom with respect to those possessed by the system. Hence we may consider the expectation value of the Jaynes–Cumming Hamiltonian (1.100) with respect to the coherent states of the quantum oscillator system H_{F}, i.e. with respect to the states $|\alpha\,e^{-\mathrm{i}\omega t}\rangle$ such that it satisfies the eigenvalue equation $\hat{a}|\alpha\,e^{-\mathrm{i}\,\omega\,t}\rangle = \alpha\,e^{-\mathrm{i}\,\omega\,t}|\alpha\,e^{-\mathrm{i}\,\omega\,t}\rangle$, with $\alpha \in \mathbb{C}$. It is not difficult to show that these states are solutions of the Schrödinger equation associated to H_{F}. Hence, taking this expectation value, the effective Hamiltonian

$$H_{\mathrm{Eff}}(t) = \frac{1}{2}\,\hbar\,\omega_a\,\sigma_3 + \hbar\,[\,\xi_{\mathrm{R}}(t)\,\sigma_1 - \xi_{\mathrm{I}}(t)\,\sigma_2\,] \tag{1.102}$$

is obtained, where we have defined the complex number $\xi(t) = g\,\alpha\,e^{-\mathrm{i}\,\omega\,t}$ and ignored constant terms. In physical terms, we found the Hamiltonian that describes a 2-level atom irradiated by a laser source, i.e. the radiation is considered as a classical field [13].

The expectation value of the Hamiltonian (1.102) defined on the complex projective space $\mathbb{CP}$ is given

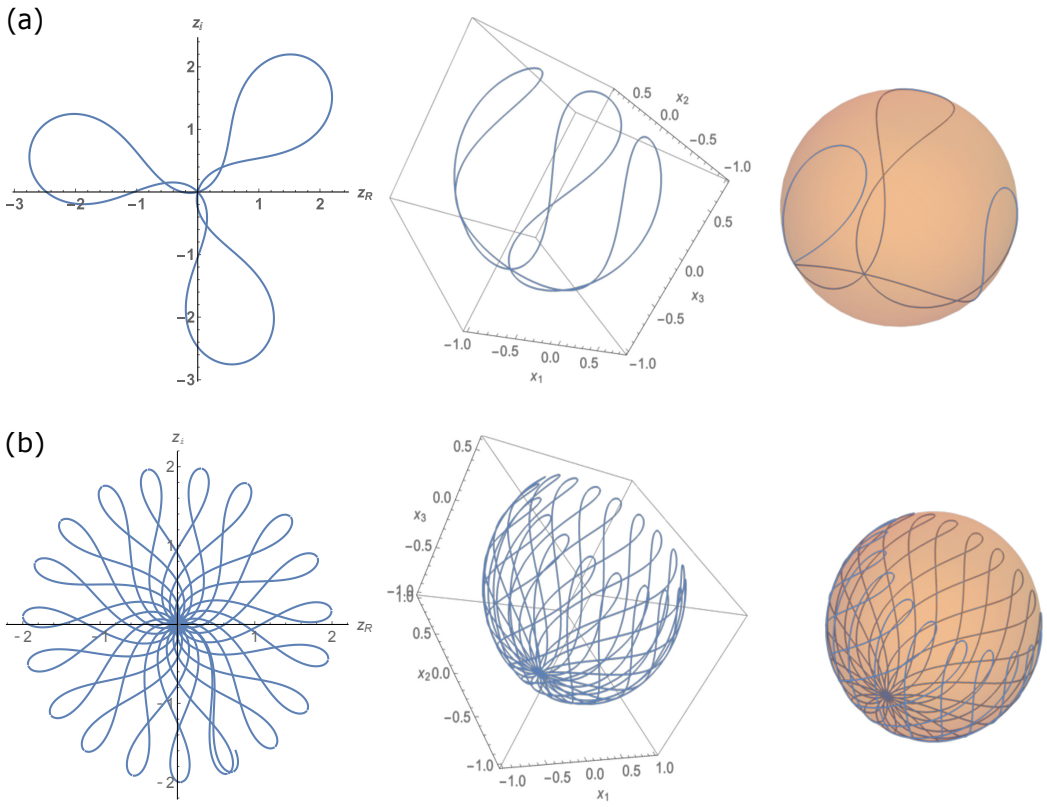


Figure 1.3: Time evolution of the two-level atom in interaction with a classical source of radiation with initial condition $z_0 = 0$ and parameters: (a) $\omega_a = 1$, $\omega = 1$, $g = 1$, $\beta = 1 + \mathrm{i}$ and (b) $\omega_a = 1$, $\omega = 1$, $g = 1$ and $\beta = 1$.

by

$$\begin{aligned} e_H &= (\psi|H_{\text{Eff}}|\psi) \\ &= \frac{1}{1+z\bar{z}} \begin{pmatrix} \bar{z} & , & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}\omega_a & \xi(t) \\ \bar{\xi}(t) & -\frac{1}{2}\omega_a \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} \\ &= \frac{1}{1+z\bar{z}} \left(\frac{1}{2}\omega_a \bar{z} z + \xi(t)\bar{z} + \bar{\xi}(t) z - \frac{1}{2}\omega_a \right) , \end{aligned} \tag{1.103}$$

and hence the Hamiltonian equations of motion (1.84) lead, as it should be, to the Riccati equation

$$\dot{z} = \frac{\mathrm{i}}{\hbar} \left[\bar{\xi}(t)\, z^2 - \omega_a\, z - \xi(t) \right] . \tag{1.104}$$

For the time-independent case, i.e. $\omega = 0$ it is possible obtain the phase portrait of the Hamiltonian vector field and the gradient vector field. The phase portraits of both vector fields are plotted in Fig. 1.2a and Fig. 1.2b, respectively. Here we can see that the solutions of the Hamilton vector field for this case are periodic curves except for the two singular solutions where the vector field vanishes. These singular solutions for the gradient vector fields become an attractive and a repulsive singular point.

For the time-dependent case, i.e. $\omega \neq 0$, the solution of the Riccati equation is well-known, see for example [10, 11, 20, 38]. These solutions are Möbius transformations in $\mathbb{CP}$, namely

$$\Phi(U, z_0) \mapsto z(t) = \frac{a(t)z_0 + b(t)}{c(t)z_0 + d(t)} \quad \text{with} \quad U = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \tag{1.105}$$

and z_0 being the initial condition. For our case of interest we have the matrix

$$U(t) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\omega t/2}\left(\cos\Omega t - \frac{i\Delta}{2\Omega}\sin\Omega t\right) & -\frac{\mathrm{i}\,g\,\beta}{\Omega}\,\mathrm{e}^{-\mathrm{i}\omega t/2}\sin\Omega t \\ -\frac{\mathrm{i}}{g\,\beta}\,\mathrm{e}^{\mathrm{i}\omega t/2}\left(\Omega - \frac{\Delta^2}{4\Omega}\right)\sin\Omega t & \mathrm{e}^{\mathrm{i}\omega t/2}\left(\cos\Omega t + \frac{i\Delta}{2\Omega}\sin\Omega t\right) \end{pmatrix}, \tag{1.106}$$

where $\Delta = \omega_a - \omega$ is the difference between frequencies and $\Omega^2 = \frac{\Delta^2}{4} + \xi\bar{\xi}$ is a time-independent frequency. Important solutions of the Riccati equation are the periodic ones, i.e. the solutions such that $z(t) = z(t+T)$ with the period T. This condition of periodicity is reflected by $U(t) = U(t+T)$, which is satisfied under the constraint

$$\omega_a = \omega \pm \sqrt{\omega^2 - 4\xi\bar{\xi}}\,, \tag{1.107}$$

with period $T = \frac{n\pi}{\Omega}$. From this constraint we see that not all physical systems have periodic solutions, because it is necessary that $\omega^2 \geq 4\,\xi\,\bar{\xi}$ to obtain a real frequency ω_a. Some examples of the quantum evolution in $\mathbb{CP}$ and in S^2 are plotted in Fig. 1.3. For the case in Fig. 1.3a, we have a periodic orbit with period $T = 2\pi$ and initial condition $z_0 = 0$. On the other hand, in Fig. 1.3b we consider the same initial condition but, due to the choice of the parameters, one does not obtain a closed curve, but as the time progresses the curve becomes dense on the sphere.

Chapter 2

Nonlinear Dynamics of Quantum Systems. Generalized Coherent States

One of the traditional problem in quantum mechanics is the study of Gaussian wave packets as analytic solution of the Schrödinger equation for Hamiltonians that are at most quadratic (or bilinear) in position and momentum variables, including problems whose Hamiltonian depends explicitly on time.

The rise of the Gaussian wave packets goes back to the seventies, where there was a large activity in the semiclassical limit of quantum mechanics; for example, many aspects of the molecular quantum dynamics fall into the semiclassical domain [46, 47]. The study of such a domain goes from the expansion in $\hbar$, by introducing the description of the wave function in terms of phase and amplitude real functions [46], to the Feynman path integral approach [47] or equivalently the linear time-dependent invariants, i.e., time-dependent Gaussian wave packets [16, 17, 48]. In addition, it has been found, for narrow wave packets and smooth potentials, that the wave packet feels only the first terms of the Taylor series expansion of the potential around its center [48]. A review of Hamiltonian models which have integrable solutions within the framework of the time-dependent Schrödinger equation is presented in [49]. The before mentioned works have had impact on studies of several fields in the domain of time; for example, in matter wave optics the changes in the energy spectrum of ultra-cold neutrons [50], in atom optics experiments using a time-modulated mirror [51] or studies of diffraction of a Bose–Einstein condensate in the time domain [52].

As we have already mention, there are several methods in the literature to study Gaussian wave packets. One of the most popular is the use of the so-called *linear invariant operators* introduced by Lewis in ref. [16] and Malkin, Man'ko and Trifonov in Ref. [17]. This approach is based on the construction of operators $\hat{I}$ such that their total derivative with respect time is equal to zero and they are linear in the position and momentum variables, i.e.

$$\hat{I} = a(t)\hat{q} + b(t)\hat{p}\,. \tag{2.1}$$

The time-dependent functions $a(t)$ and $b(t)$ are determined from the invariance of the operator, because this invariance defines a system of linear differential equation for these functions, which together with the initial conditions one fixes the units of the functions $a(t)$ and $b(t)$, further details are given in [17]. In addition, one may demonstrate that these linear time dependent invariants are obtained from the Hamiltonian formulation of Nöther's theorem. The variations are related to the time dependent functions $a(t)$ and $b(t)$ [53].

In this approach, the solution of the Schrödinger equation is obtained from the fact that an invariant operator $\hat{I}$ is an operator that transforms every solution $|\psi\rangle$ of the Schrödinger equation into a solution $\hat{I}|\psi\rangle$ of the same equation, this is clear from the fact that $\hat{I}$ satisfies the equation

$$\left[\left(\mathrm{i}\hbar\frac{\partial}{\partial t} - \hat{H}\right), \hat{I}\right]|\Psi, t\rangle = 0\,. \tag{2.2}$$

Therefore, with a single solution of the Schrödinger equation it is possible to find a family of them. In particular, it is possible to construct the invariant creation and annihilation operators $\hat{A}$ and $\hat{A}^\dagger$ with the usual commutation relation $[\hat{A},\hat{A}^\dagger]=1$. Thus we define the eigenvalue equation for the annihilation operator

$$\hat{A}|\alpha_{\text{Inv}}\rangle=\alpha_{\text{Inv}}|\alpha_{\text{Inv}}\rangle\,, \tag{2.3}$$

where the eigenvalue $\alpha_{\text{Inv}}=\langle\alpha_{\text{Inv}}|\hat{A}|\alpha_{\text{Inv}}\rangle$ is a complex constant of motion and the eigenvector $|\alpha_{\text{Inv}}\rangle$ is a solution of the Schrödinger equation. To prove that the state $|\alpha_{\text{Inv}}\rangle$ is a solution of the Schödinger equation, let us consider the fiducial state $|0\rangle$ that is not only a solution of the Schrödinger equation but also satisfies that $\hat{A}|0\rangle=0$; thus, we may define the state

$$|\alpha_{\text{Inv}}\rangle=e^{\alpha_{\text{Inv}}\hat{A}^\dagger-\bar{\alpha}_{\text{Inv}}\hat{A}}|0\rangle\,. \tag{2.4}$$

Then, because invariants transform solution of the Schrödinger equation into a solution of the same equation, as we mentioned before, from the expression (2.4) it should be clear that $|\alpha_{\text{Inv}}\rangle$ is a solution of the the Schrödinger equation. For example, for the one-dimensional system with Hamiltonian operator

$$\hat{H}=\frac{1}{2}\left(\,\hat{q}\;,\;\hat{p}\,\right)\begin{pmatrix}H_1 & V\\ V & H_2\end{pmatrix}\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix}\,, \tag{2.5}$$

where H_1, H_2 and V are possibly time-dependent functions, the bosonic linear invariant operators are

$$\hat{A}=\frac{\mathrm{i}}{\sqrt{2\hbar}}\left(P\,\hat{q}-Q\,\hat{p}\right)\quad\text{and}\quad\hat{A}^\dagger=\frac{-\mathrm{i}}{\sqrt{2\hbar}}\left(\bar{P}\,\hat{q}-\bar{Q}\,\hat{p}\right)\,. \tag{2.6}$$

Now, in order to fulfill $[\hat{A},\hat{A}^\dagger]=1$ the time-dependent complex functions Q and P are constrained to satisfy

$$\bar{Q}P-Q\bar{P}=2\,\mathrm{i}\,. \tag{2.7}$$

On the other hand, from the invariance of the operators one may prove that these functions obey a linear system of differential equations, i.e.

$$\begin{pmatrix}\dot{Q}\\ \dot{P}\end{pmatrix}=\begin{pmatrix}V & H_2\\ -H_1 & -V\end{pmatrix}\begin{pmatrix}Q\\ P\end{pmatrix}\,. \tag{2.8}$$

Then, the Gaussian solution of the Schrödinger equation may be obtained with the help of the linear invariant operators and establishing the definition (2.4) in the position representation,

$$\langle q\,|\,\alpha_{\text{Inv}}\rangle=\frac{1}{(\pi\,\hbar)^{1/4}}\frac{1}{\sqrt{Q}}\exp\left\{\frac{\mathrm{i}}{2\hbar}\frac{P}{Q}\,(q-\langle\hat{q}\rangle)^2+\frac{\mathrm{i}}{\hbar}\langle\hat{p}\rangle(q-\langle\hat{q}\rangle)+\frac{\mathrm{i}}{2\hbar}\,\langle\hat{q}\rangle\,\langle\hat{p}\rangle\right\}\,. \tag{2.9}$$

The state $|\alpha_{\text{Inv}}\rangle$ defined by the condition (2.3) is called in the literature *the generalized coherent state* [54], where at difference to the coherent state the correlation between $\hat{p}$ and $\hat{q}$ may be different from zero, and hence these states are such that they minimize the so-called *Robertson–Schrödinger uncertainty relations*, i.e.

$$\sigma_q\sigma_p-\sigma_{qp}^2=\frac{\hbar^2}{4}\,, \tag{2.10}$$

where

$$\sigma_q=\langle\hat{q}^2\rangle-\langle\hat{q}\rangle^2\,,\quad\sigma_p=\langle\hat{p}^2\rangle-\langle\hat{p}\rangle^2\quad\text{and}\quad\sigma_{qp}=\frac{1}{2}\langle\hat{q}\hat{p}+\hat{p}\hat{q}\rangle-\langle\hat{q}\rangle\langle\hat{p}\rangle\,. \tag{2.11}$$

The linear invariant approach for the study of Gaussian wave packets depends on the solution of the linear equation of motions defined for the time-dependent functions $a(t)$ and $b(t)$, which may be complex or real functions. However, there are different approaches that involve non-linear differential equations, specifically an evolution through the Ermakov–Lewis and Riccati solutions [16, 55, 20, 38, 19].

For example, according to *the Wei–Norman method* [56] employed in [55] there are complex functions $\mathcal{C}_k(t)$, with $k = 1, 2, 3$ and $\mathcal{C}_k(0) = 0$, such that the unitary evolution operator may be expressed as

$$\begin{aligned}\hat{U}(t) &= \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}\mathcal{C}_1(t)\hat{q}^2}\mathrm{e}^{\frac{\mathrm{i}}{2\hbar}\mathcal{C}_2(t)[\hat{q},\hat{p}]_+}\ \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}\mathcal{C}_3(t)\hat{p}^2}\\ &= \hat{U}_{\mathcal{C}_1}(t)\hat{U}_{\mathcal{C}_2}(t)\hat{U}_{\mathcal{C}_3}(t)\,,\end{aligned} \tag{2.12}$$

con The time-dependent functions $\mathcal{C}_k$ are determined by direct substitution of the evolution operator into the equation of motion

$$\mathrm{i}\hbar\frac{\mathrm{d}\hat{U}}{\mathrm{d}t} = \hat{H}\hat{U}\,, \tag{2.13}$$

with initial condition $\hat{U}(0) = \hat{1}$. For the Hamiltonian given in (2.5) one may prove that the complex functions $\mathcal{C}_k$ obey the system of differential equations

$$\dot{\mathcal{C}}_1 = -H_2\,\mathcal{C}_1^2 - 2V\,\mathcal{C}_1 - H_1\,, \tag{2.14}$$

$$\dot{\mathcal{C}}_2 = -H_2\,\mathcal{C}_1 - V\,, \tag{2.15}$$

$$\dot{\mathcal{C}}_3 = -\mathrm{e}^{2\mathcal{C}_2}\,H_2\,. \tag{2.16}$$

Notice that the evolution operator may be constructed with different orders of the operators, $\hat{U}_{\mathcal{C}_k}(t)$, given rise to different Riccati differential equations; however, all of them are connected by Möbius transformations, for the details see Ref. [55]. Furthermore, according to reference [19], the Gaussian wave function solution of the Schrödinger equation may be expressed as

$$\psi(q) = \frac{1}{(\pi\,\hbar)^{1/4}}\exp\left\{\frac{\mathrm{i}}{2\hbar}\left(\mathcal{C}\,(q-\langle\hat{q}\rangle)^2 + 2\langle\hat{p}\rangle(q-\langle\hat{q}\rangle) + \langle\hat{q}\rangle\,\langle\hat{p}\rangle\right) - \frac{1}{2}\int[H_2\,\mathcal{C}(\tau)+V]\mathrm{d}\tau\right\}. \tag{2.17}$$

where $\mathcal{C}$ is a complex time-dependent function with imaginary part strictly positive and different from zero. In addition, this function obeys the nonlinear Riccati equation

$$\dot{\mathcal{C}} + H_2\,\mathcal{C}^2 + 2V\,\mathcal{C} + H_1 = 0\,, \tag{2.18}$$

which is the same Riccati equation given in Eq. (2.14) for the determination of the unitary evolution operator.

Hence, from the before mention models of the generalized coherent states, one may realize that there is not a unique parametrization of these states. In fact, we have a parametrization by means of the complex number $\alpha \in \mathbb{C}$, a description usually called the coherent states, also we may find in quantum optics that the Gaussian wave packets are parametrized by the squeezing parameter $\xi \in \mathbb{C}$, known as the squeezed states, or one may parametrize by means of the solutions of the nonlinear Riccati (or Ermakov) equation. From a physical point of view all this representations are valid, then one should be able to establish the connections between these deferent description. In this Chapter, different from the before mentioned approaches for the study of the generalized coherent states, this problem will be analized from a geometrical point of view. This means that the geometrical space where the different evolutions take place will be established and it is shown how they are *immersed* in the Hilbert space $\mathcal{H}$, as well as establish the connection among the different descriptions will be established. Moreover, it is shown that each space has a symplectic structure such that the dynamics on these spaces are actually Hamiltonian. Then here the results previously mentioned are not only deduced from a geometrical perspective but also it is shown that there are several nonlinear descriptions involved in the dynamics of the generalized coherent states.

2.1 Linear Description of Generalized Coherent States

To start our study we establish some geometrical aspects of the usual description of the generalized coherent states, for a more complete review see [14, 57, 6]. So, in quantum mechanics it is possible to immerse a manifold $M \subset \mathbb{R}^n$ in the Hilbert space $\mathcal{H}$ of a physical system, i.e.,

$$\varphi : M \to \mathcal{H} : \mathbf{x} \mapsto |\psi(\mathbf{x})\rangle\,, \tag{2.19}$$

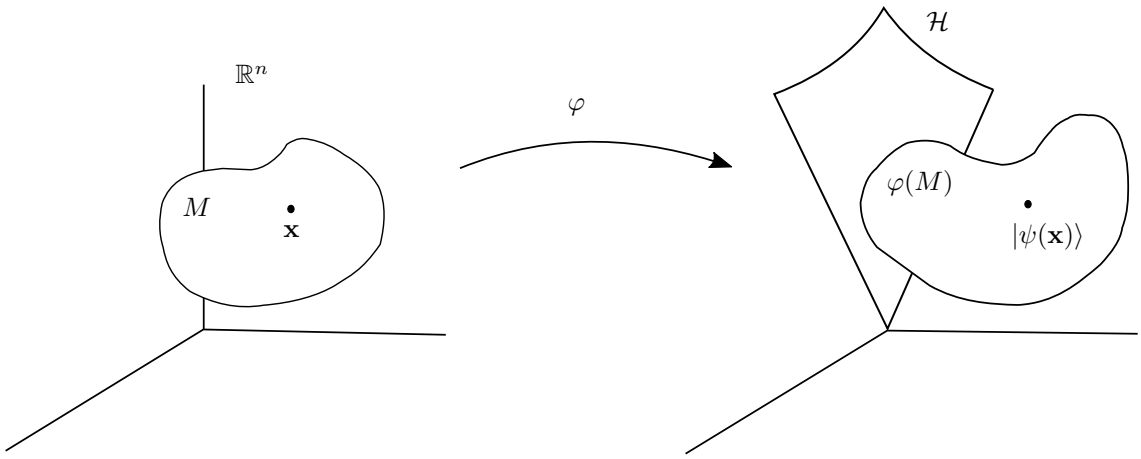


Figure 2.1: Representative picture of the immersion φ of the submanifold $M \in \mathbb{R}^n$ into the Hilbert space $\mathcal{H}$, leading to the submanifold $\varphi(M) \subset \mathcal{H}$.

such that $\varphi(M)$ is a submanifold of $\mathcal{H}$ [58], see Fig. 2.1. Recall that an *immersion* φ is a differentiable map in which $T\varphi$ is injective, i.e. $T_{\mathbf{x}}\varphi : TM_{\mathbf{x}} \to T\mathcal{H}_{|\psi\rangle}$ is an injective function at every point $\mathbf{x} \in M$. Hence, we may parametrize the time dependence of the wave function $|\psi\rangle$ through the variation of the parameters $\mathbf{x}$, whose physical significance will depend on the problem at hand.

Let us now show for the case of the generalized coherent state that we may define an immersion. For simplicity, we consider the case of the generalized coherent states for a one-dimensional quantum system, the generalization is straightforward; then, the immersion is constructed as follows [14, 58].

Given a fiducial normalized state $|0\rangle$ in the Hilbert space $\mathcal{H}$, we employ *the strongly continuous map* $\hat{D}$ to immerse the complex plane $\mathbb{C}$ in the Hilbert space,

$$i : \alpha \in \mathbb{C} \mapsto |\alpha\rangle \in \mathcal{H} \quad \text{with} \quad |\alpha\rangle := \hat{D}(\alpha)\,|0\rangle. \tag{2.20}$$

The operator $\hat{D}(\alpha)$ is the well-known *displacement operator* given by

$$\hat{D}(\alpha) = \mathrm{e}^{\alpha\,\hat{a}^\dagger - \bar{\alpha}\,\hat{a}}\,, \tag{2.21}$$

where $\hat{a}$ the annihilation and $\hat{a}^\dagger$ the creation operators with α and $\bar{\alpha}$ their respective expectation values. Now, considering the Hilbert space of quadratically integrable function $\mathfrak{L}^2(\mathbb{R}, \mathrm{d}q)$, the immersed submanifold $i(\mathbb{C})$ corresponds to the *Gaussian wave packets*, i.e. one have the well-known immersion [58, 15]

$$i : \mathbb{C} \to \mathfrak{L}^2(\mathbb{R}, \mathrm{d}q) : \alpha \mapsto \psi(\alpha, q)\,. \tag{2.22}$$

It should be clear that this immersion corresponds to a pure state with a fixed dependence on the parameter α. So, the superposition of generalized coherent states does not belong to $i(\mathbb{C})$, because such superposition is not a Gaussian wave packet. An important consequence of this is that the subset $i(\mathbb{C})$ of the Hilbert space is a nonlinear space. As a final remark, it is not difficult to see from the definition of the displacement operator that

$$\hat{D}^\dagger(\alpha)\,\hat{D}^\dagger(\beta)\,\hat{D}(\alpha)\,\hat{D}(\beta) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\,\omega(\alpha,\beta)}\,, \tag{2.23}$$

where $\omega(\alpha, \beta)$ is the standard symplectic form $\omega = \mathrm{d}x \wedge \mathrm{d}y$ in $\mathbb{C}$, which is represented by the matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.24}$$

Thus, being $\alpha = x + \mathrm{i}\,y$ and $\beta = x' - \mathrm{i}\,y'$ we have that

$$\omega(\alpha, \beta) = (x\,,\,y) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = x\,y' - x'\,y\,. \tag{2.25}$$

The relation (2.23) is known as the *Weyl form* of the commutation relations and together with the immersion (2.20) allows to see that we are actually dealing with a *Weyl system* [59, 7], i.e., a projective unitary representation of the linear manifold $\mathbb{C}$ in the Hilbert space. An important consequence of dealing with a Weyl system is the fact that we are avoiding domain problems of the Hermitian operators, that is, the domain of the creation and annihilation operator are well defined in the submanifold $i(\mathbb{C}) \subset \mathcal{H}$.

Now that the kinematical situation involved in the generalized coherent states has been established, we are now interested in the dynamical properties of the system. So, the evolution of a pure state $|\alpha\rangle \in \mathcal{H}$ is given by the Schrödinger equation

$$\mathrm{i}\,\hbar \frac{\partial\,|\alpha\rangle}{\partial\, t} = \hat{H}\,|\alpha\rangle\,, \tag{2.26}$$

with $\hat{H}$ the Hamiltonian operator. Thus assuming that the solution of the system can be extended to the whole time, let be $\hat{U}_t : \mathcal{H} \to \mathcal{H}$ the one-parameter group of unitary transformations associated to the Schrödinger equation (2.26). In this work only Hamiltonian operators quadratic in the position and momentum variables are considered. Then for these cases, it has been shown in [60, 61] that if the initial state is a generalized coherent state, the final state is also a generalized coherent state,

$$\hat{U}_t|\alpha_0\rangle = |\alpha(t)\rangle\,, \tag{2.27}$$

this implies that the submanifold $i(\mathbb{C}) \subset \mathcal{H}$ is invariant with respect to the flow U_t.

On the other hand, because $\mathbb{C}$ has a symplectic structure, i.e. it is naturally endowed with a symplectic form ω. Then, one may associate to every differentiable function with domain in the complex plane, i.e., $H \in \mathfrak{F}(\mathbb{C})$, the vector field X_H, which is described by the internal product i_{X_H} acting on the symplectic form ω, that is,

$$i_{X_H}\omega = \omega(X_H, \cdot) = -\mathrm{d}H\,. \tag{2.28}$$

The *Hamiltonian vector field* X_H defines the dynamics of the system, and its integral curves are fixed by the solutions of the Hamiltonian equations of motion. In addition, it is possible to introduce the Poisson brackets, namely given the functions $F, G \in \mathfrak{F}(\mathbb{C})$ with associated vector fields X_F and X_G, so one has that

$$\{F, G\} = \omega(X_F, X_G)\,. \tag{2.29}$$

In general, the solution of the Hamiltonian equations of motion in $\mathbb{C}$ may be expressed as

$$\alpha(t) = \Phi_t \alpha_0\,, \tag{2.30}$$

where $\Phi_t : \mathbb{C} \to \mathbb{C}$ is the one-parameter group of symplectic transformations, i.e. $\Phi_t \in \mathrm{Sp}(2n, M)$[1]. Therefore, the connection between the unitary evolution $\hat{U}_t$ and the (canonical) symplectic evolution Φ_t is obtained by means of the immersion (2.20), namely

$$i \circ \Phi_t = i \circ \hat{U}_t\,. \tag{2.33}$$

Quantum parametric oscillator

To exemplify the situation, we consider the Schrödinger equation of a Hamiltonian operator quadratic in the position and the momentum variables given in Eq. (2.5). This Hamiltonian can be equivalently expressed in terms of bosonic operators

$$\hat{a} = \frac{\mathrm{i}}{\sqrt{2\hbar}}\,(P\,\hat{q} - Q\,\hat{p}) \quad \text{and} \quad \hat{a}^\dagger = \frac{-\mathrm{i}}{\sqrt{2\hbar}}\,\left(\bar{P}\,\hat{q} - \bar{Q}\,\hat{p}\right)\,, \tag{2.34}$$

[1] The $\mathrm{Sp}(2n, M)$ is the symplectic group of degree $2n$ over M defined as

$$\mathrm{Sp}(2n, M) = \{S \in \mathbb{M}_{2n\times 2n} \mid S^{\mathrm{T}}\, J\, S = J\}\,, \tag{2.31}$$

where J is the symplectic matrix

$$J = \begin{pmatrix} 0 & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & 0 \end{pmatrix}\,. \tag{2.32}$$

where in order to have $[\hat{a}, \hat{a}^\dagger] = 1$, (Q, P) are points in the manifold M defined as

$$M = \{(Q, P) \in \mathbb{C}^2 \,|\, \bar{Q}P - Q\bar{P} = 2\,\mathrm{i}\}\,. \tag{2.35}$$

Thus in terms of the creation and annihilation operators we have the equivalent Hamiltonian

$$\hat{H} = \frac{\hbar}{4}\,(\,\hat{a}\;\;,\;\;\hat{a}^\dagger\,)\begin{pmatrix} \bar{G} & W \\ W & G \end{pmatrix}\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}, \tag{2.36}$$

where the time-dependent functions G and W are connected with H_1, H_2 and V by

$$G = H_1 Q^2 + 2VQP + H_2 P^2\,, \tag{2.37}$$

$$W = H_1|Q|^2 + V(Q\bar{P} + P\bar{Q}) + H_2|P|^2\,. \tag{2.38}$$

Let us now explicitly establish the immersion (2.22). So, the normalized state $|0\rangle$ is defined by the usual condition: $\hat{a}|0\rangle = 0$. Thus, the wave function of the ground state is obtained by means of the relation $\langle q|\hat{a}|0\rangle = 0$, which in the position representation defines a partial differential equation for the ground state wave function, whose normalized solution is given by

$$\psi(0, q) = \frac{1}{(\pi\,\hbar)^{1/4}}\frac{1}{\sqrt{Q}}\,\exp\left\{\frac{\mathrm{i}}{2\hbar}\,\frac{P}{Q}\,q^2\right\}. \tag{2.39}$$

Now, the immersion defined in (2.22) allows to establish a correspondence between the complex number

$$\alpha = \langle\alpha|\hat{a}|\alpha\rangle = \frac{\mathrm{i}}{\sqrt{2\hbar}}\,(P\,\langle\hat{q}\rangle - \,Q\,\langle\hat{p}\rangle) \tag{2.40}$$

and the Gaussian wave packet

$$\begin{aligned}\psi(\alpha, q) &= \langle q|\hat{D}(\alpha)|0\rangle \\ &= \frac{1}{(\pi\,\hbar)^{1/4}}\frac{1}{\sqrt{Q}}\,\exp\left\{\frac{\mathrm{i}}{2\hbar}\,\frac{P}{Q}\,(q - \langle\hat{q}\rangle)^2 + \frac{\mathrm{i}}{\hbar}\langle\hat{p}\rangle(q - \langle\hat{q}\rangle) + \frac{\mathrm{i}}{2\hbar}\,\langle\hat{q}\rangle\,\langle\hat{p}\rangle\right\}\end{aligned} \tag{2.41}$$

in the Hilbert space $\mathfrak{L}^2(\mathbb{R}, \mathrm{d}q)$. Furthermore, it should be clear that one may also consider the immersion

$$i : \mathbb{C} \to \mathfrak{L}^2(\mathbb{R}, \mathrm{d}p) : \alpha \mapsto \tilde{\psi}(\alpha, p)\,. \tag{2.42}$$

where for this case the Gaussian wave packet in the momentum representation is given by

$$\begin{aligned}\tilde{\psi}(\alpha, p) &= \langle p|\hat{D}(\alpha)|0\rangle \\ &= \frac{1}{(\pi\,\hbar)^{1/4}}\sqrt{\frac{\mathrm{i}}{P}}\,\exp\left\{-\frac{\mathrm{i}}{2\hbar}\,\frac{Q}{P}\,(p - \langle\hat{p}\rangle)^2 - \frac{\mathrm{i}}{\hbar}\langle\hat{q}\rangle(p - \langle\hat{p}\rangle) - \frac{\mathrm{i}}{2\hbar}\,\langle\hat{q}\rangle\,\langle\hat{p}\rangle\right\}.\end{aligned} \tag{2.43}$$

With the help of the Gaussian wave function $\psi(\alpha, q)$ (or its Fourier transform $\tilde{\psi}(\alpha, p)$), one may prove that there is a direct connection between the uncertainties σ_q, σ_p and their correlation σ_{qp}, all of them defined in (2.11), with the complex quantities (Q, P) by

$$\sigma_q = \frac{\hbar}{2}|Q|^2\,,\quad \sigma_p = \frac{\hbar}{2}|P|^2 \quad \text{and} \quad \sigma_{qp} = \frac{\hbar}{4}(P\,\bar{Q} + Q\,\bar{P})\,. \tag{2.44}$$

Establishing the covariance matrix Σ, one has that it may be factorized as

$$\Sigma = \frac{2}{\hbar}\begin{pmatrix} \sigma_{qp} & \sigma_p \\ -\sigma_q & -\sigma_{qp} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \bar{P} & P \\ -\bar{Q} & -Q \end{pmatrix}\begin{pmatrix} Q & P \\ \bar{Q} & \bar{P} \end{pmatrix}. \tag{2.45}$$

Therefore, the quantities $(\sigma_q, \sigma_p, \sigma_{qp})$, along with its constraint (2.10) determine the values of (Q, P). Note that the minimization of the Robertson–Schrödinger uncertainty directly introduce the constraint in Eq. (2.7) for (Q, P).

Let us now address the dynamical properties of this example. Considering the unitary evolution in the Hilbert space $\mathfrak{L}^2(\mathbb{R}, \mathrm{d}q)$ generated by the Hamiltonian operator (2.5) (or its complexification the Hamiltonian (2.36)). One has then that the associated symplectic evolution in $\mathbb{C}$ is determined by the Hamiltonain dynamics $X \in \mathfrak{X}(T\mathbb{C})$ defined by

$$i_X \omega = -\,\mathrm{d} e_{\hat{H}}\,, \tag{2.46}$$

where ω is the symplectic form, which in terms of the complex coordinates given in Eq. (2.40), has the form

$$\omega = \mathrm{d}\langle\hat{p}\rangle \wedge \mathrm{d}\langle\hat{q}\rangle = \mathrm{i}\,\hbar\,\mathrm{d}\bar{\alpha} \wedge \mathrm{d}\alpha\,, \tag{2.47}$$

and the Hamiltonian function $e_{\hat{H}}$ corresponds to the expectation value of the Hamiltonian, i.e.,

$$e_{\hat{H}} = \langle\alpha|\hat{H}|\alpha\rangle = \frac{\hbar}{4}\,(\,\alpha\;,\;\bar{\alpha}\,)\begin{pmatrix} \bar{G} & W \\ W & G \end{pmatrix}\begin{pmatrix} \alpha \\ \bar{\alpha} \end{pmatrix}. \tag{2.48}$$

So, inserting the expression in coordinates of the symplectic form and the expectation value into the definition (2.46), one may prove that

$$X = \frac{\mathrm{i}}{2}\,\left(\bar{G}\,\alpha + W\,\bar{\alpha}\right)\frac{\partial}{\partial\bar{\alpha}} - \frac{\mathrm{i}}{2}\,\left(G\,\bar{\alpha} + W\,\alpha\right)\frac{\partial}{\partial\alpha}\,, \tag{2.49}$$

whose integral curves are given by the solutions of the Hamiltonian equations of motion

$$\begin{aligned} \dot{\alpha} &= -\frac{\mathrm{i}}{\hbar}\,\frac{\partial\, e_{\hat{H}}}{\partial\bar{\alpha}} = -\frac{\mathrm{i}}{2}\,\left(G\,\bar{\alpha} + W\,\alpha\right)\,, \\ \dot{\bar{\alpha}} &= \frac{\mathrm{i}}{\hbar}\,\frac{\partial\, e_{\hat{H}}}{\partial\alpha} = \frac{\mathrm{i}}{2}\,\left(\bar{G}\,\alpha + W\,\bar{\alpha}\right)\,. \end{aligned} \tag{2.50}$$

Moreover, one may use the Poisson brackets defined by the relation (2.29), thus given the expectation values $e_A \in \mathfrak{F}(\mathbb{C})$ and $e_B \in \mathfrak{F}(\mathbb{C})$ associated with the observables $\hat{A}$ and $\hat{B}$. The corresponding Poisson bracket is given by

$$\{e_A, e_B\} = \frac{\mathrm{i}}{\hbar}\,\left[\frac{\partial e_A}{\partial\alpha}\frac{\partial e_B}{\partial\bar{\alpha}} - \frac{\partial e_A}{\partial\bar{\alpha}}\frac{\partial e_B}{\partial\alpha}\right]. \tag{2.51}$$

This definition allows to introduce the evolution of an arbitrary function in $\mathbb{C}$, because of the evolution of the expectation value e_A of an arbitrary observable $\hat{A}$ is simply given by

$$\frac{\mathrm{d}\, e_A}{\mathrm{d}t} = \{e_H, e_A\}\,. \tag{2.52}$$

Note that the formalism presented may be extended to the time-dependent case, i.e., for the evolution of functions which explicitly depend on time. For such cases, as usual, one simply considers the extended space $T\mathbb{C} \times \mathbb{R}$ as a carrier space, such that the evolution of any function in this space is given by

$$\frac{\mathrm{d}\, e_A}{\mathrm{d}t} = \{e_H, e_A\} + \frac{\partial e_A}{\partial t}\,. \tag{2.53}$$

See that until here there is nothing new, because the Hamiltonian equations (2.50) are simply the complexification of the well-known *Ehrenfest theorem*, which establishes that the maximum of the Gaussian wave packet follows the classical trajectories of motion

$$\begin{aligned} \dot{\langle q\rangle} &= \frac{\partial\, e_{\hat{H}}}{\partial\langle p\rangle} = V\langle q\rangle + H_2\langle p\rangle\,, \\ \dot{\langle p\rangle} &= -\frac{\partial\, e_{\hat{H}}}{\partial\langle q\rangle} = -H_1\langle q\rangle - V\langle p\rangle\,. \end{aligned} \tag{2.54}$$

In addition, by construction the complex quantities (Q, P) are constant parameters; thus, from the relations in Eq. (2.44), it is clear that we have considered Gaussian wave packets which preserve the values of the second momenta $(\sigma_q, \sigma_p, \sigma_{qp})$ during their evolution.

In order to consider the general situation, i.e., to have Gaussian wave packets with time dependent second moments, it is necessary to introduce the dynamics in the manifold M, defined in Eq. (2.35). To define such dynamics we use the vector field Y, which lives in the tangent bundle of the manifold, $Y \in \mathfrak{X}(TM)$. As the manifold M has a symplectic structure of the form

$$\omega_M = \mathrm{d}\bar{P} \wedge \mathrm{d}Q + \mathrm{d}P \wedge \mathrm{d}\bar{Q}\,, \tag{2.55}$$

then the dynamics is symplectic, $i_Y \omega_M = -\mathrm{d}H_M$, where the Hamiltonian function corresponds to

$$H_M = \begin{pmatrix} \bar{Q} & , & \bar{P} \end{pmatrix} \begin{pmatrix} H_1 & V \\ V & H_2 \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}. \tag{2.56}$$

Therefore the vector field is given by

$$X_M = (VQ + H_2 P)\frac{\partial}{\partial Q} - (H_1 Q + V P)\frac{\partial}{\partial P} + (V\bar{Q} + H_2\bar{P})\frac{\partial}{\partial \bar{Q}} - (H_1\bar{Q} + V\bar{P})\frac{\partial}{\partial \bar{P}}\,, \tag{2.57}$$

with linear Hamilton's equations of motion

$$\begin{pmatrix} \dot{Q} \\ \dot{P} \end{pmatrix} = \begin{pmatrix} V & H_2 \\ -H_1 & -V \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}. \tag{2.58}$$

Note that these are the equations of motion obtained in the linear invariant approach, see Eq. (2.8). This is not a mere coincidence as it is seen next. Taking into account the evolution of the complex quantities (Q, P) into the definition (2.40), we obtain the time-dependent functions

$$\alpha_{\text{Inv}}(t) = \frac{\mathrm{i}}{\sqrt{2\hbar}}\left(P(t)\,\langle \hat{q}\rangle - Q(t)\,\langle \hat{p}\rangle\right), \tag{2.59}$$

which obey that

$$\frac{\mathrm{d}\alpha_{\text{Inv}}}{\mathrm{d}t} = \{e_H, \alpha_{\text{Inv}}\} + \frac{\partial \alpha_{\text{Inv}}}{\partial t} = 0\,, \tag{2.60}$$

i.e., $\alpha_{\text{Inv}}(t)$ is a constant of motion of the dynamics defined by the Hamiltonian function e_H in Eq (2.48).

The bosonic creation $\hat{A}^\dagger$ and annihilation $\hat{A}$ operators for the time-dependent case are given by

$$\hat{A} = \frac{\mathrm{i}}{\sqrt{2\hbar}}\left(P(t)\,\hat{q} - Q(t)\,\hat{p}\right) \quad \text{and} \quad \hat{A}^\dagger = \frac{-\mathrm{i}}{\sqrt{2\hbar}}\left(\bar{P}(t)\,\hat{q} - \bar{Q}(t)\,\hat{p}\right), \tag{2.61}$$

where (Q, P) are solutions of the linear system of equations (2.58). These operators are actually the linear invariant operators defined in Eq. (2.6), they are such that

$$\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} = \mathrm{i}\hbar\,[\hat{A}, \hat{H}] + \frac{\partial \hat{A}}{\partial t} = 0\,. \tag{2.62}$$

Therefore in the most general case, one has the immersion of the constant of motion $\alpha_{\text{Inv}} = \langle \hat{A}\rangle$ into the Hilbert space $\mathfrak{L}^2(\mathbb{R}, \mathrm{d}q)$ (or $\mathfrak{L}^2(\mathbb{R}, \mathrm{d}p)$), i.e.

$$i : \alpha_{\text{Inv}} \mapsto \psi(\alpha_{\text{Inv}}, q)\,. \tag{2.63}$$

Such immersion gives rise to the Gaussian wave packet Eq. (2.41) in the position representation (or (2.43) in the momentum representation) as a solution of the Schrödinger equation, where the maximum of the packet follows the classical Hamiltonian equations of motion in Eq. (2.54), whereas the evolution of the uncertainties and their correlations may be obtained by the Hamiltonian equations of motion in Eq. (2.58).

2.2 From Linear to Nonlinear Dynamics

So far it has been shown that the evolution of the second moments $(\sigma_q, \sigma_p, \sigma_{qp})$ may be determined by means of the solution of the equation of motion (2.58) under the constraint (2.7). This section is devoted to show that this evolution is connected directly to a nonlinear one, which has important applications in quantum optics.

Let us start studying the geometrical implications involved in the relation (2.45). The action of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ in $\mathbb{R}^2$ consists of the matrices Σ,

$$\tilde{\varphi} : \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{Lin}(\mathbb{R}^2, \mathbb{R}^2) : a \mapsto \Sigma\,, \tag{2.64}$$

such that $\mathrm{Tr}\,\Sigma = 0$ and $\det\Sigma = 1$. Then, it is immediate that the covariance matrix Σ, defined in Eq. (2.45), is an element of $\mathfrak{sl}(2,\mathbb{R})$. Thus a general element $\Sigma \in \mathfrak{sl}(2,\mathbb{R})$ may be expressed in terms of the basis

$$e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.65}$$

as

$$\Sigma = y^k e_k\,. \tag{2.66}$$

This expression directly establishes a one-to-one correspondence between real traceless matrices and vectors in $\mathbb{R}^3$ by

$$\begin{pmatrix} y^2 & y^3 + y^1 \\ y^3 - y^1 & -y^2 \end{pmatrix} \mapsto (y^1, y^2, y^3)\,. \tag{2.67}$$

Moreover, the constraint $\det\Sigma = 1$ defines the hyperboloid of two sheets $\mathbf{H}^2 \subset \mathbb{R}^3$ as the manifold[2]

$$\mathbf{H}^2 = \{(y^1, y^2, y^3) \in \mathbb{R}^3 \,|\, (y^1)^2 - (y^2)^2 - (y^3)^2 = 1\}\,. \tag{2.68}$$

Then considering only the upper sheet, see Fig. 2.2, one may see that each point in this manifold represent a generalized coherent state. For example, a *coherent state*, characterized by $\sigma_{qp} = 0$, corresponds to the points with $y^2 = 0$ of the hyperboloid, i.e. with the points in the curve

$$\mathbf{H}^1 = \{(y^1, y^3) \in \mathbb{R}^3 \,|\, (y^1)^2 - (y^3)^2 = 1\}\,. \tag{2.69}$$

This curve is plotted in Fig. 2.2 in a red-dashed line. In particular, the point $(1,0,0) \in \mathbf{H}^2$ corresponds to the state with equal uncertainties $\sigma_q = \sigma_p$; then, all point on the hyperboloid different from this minimum characterize a squeezed state. Another example of interest is to consider states that preserve the correlation between the position and momentum, these are represented in Fig. 2.2 as green-dotted lines, where all of them are curves parallel to $\mathbf{H}^1$.

In consequence the upper sheet of $\mathbf{H}^2$ plays an important role in the description of the *Squeezed states* [62]. To connect with the usual squeezing parameters (τ, φ), we describe the coordinates y^k by means of the Hyperbolic coordinates, this is considering the atlas

$$\phi : \mathbf{H}^2 \to \mathbb{R}^2 : (y^1, y^2, y^3) \mapsto (\tau, \varphi) \tag{2.70}$$

defined by

$$y^1 = \cosh\tau\,, \quad y^2 = \sinh\tau\cos\varphi\,, \quad y^3 = \sinh\tau\sin\varphi\,. \tag{2.71}$$

The system of coordinates (τ, φ) is connected with the *Squeezed operator* $\hat{S}$, which allows to immerse $\mathbf{H}^2$ into the Hilbert space $\mathcal{H}$ as follows. Given the fiducial normalized state $|\,w\rangle$ in the Hilbert state $\mathcal{H}$, one employs the operator $\hat{S}$ to immerse $\mathbf{H}^2$ in the Hilbert space by

$$s : \xi \in \mathbf{H}^2 \to |\xi\rangle \in \mathcal{H} \quad \text{given by} \quad |\xi\rangle = \hat{S}(\xi)|w\rangle\,, \tag{2.72}$$

[2]This description of the quantum system is also known as the "Pseudosphere" [63].

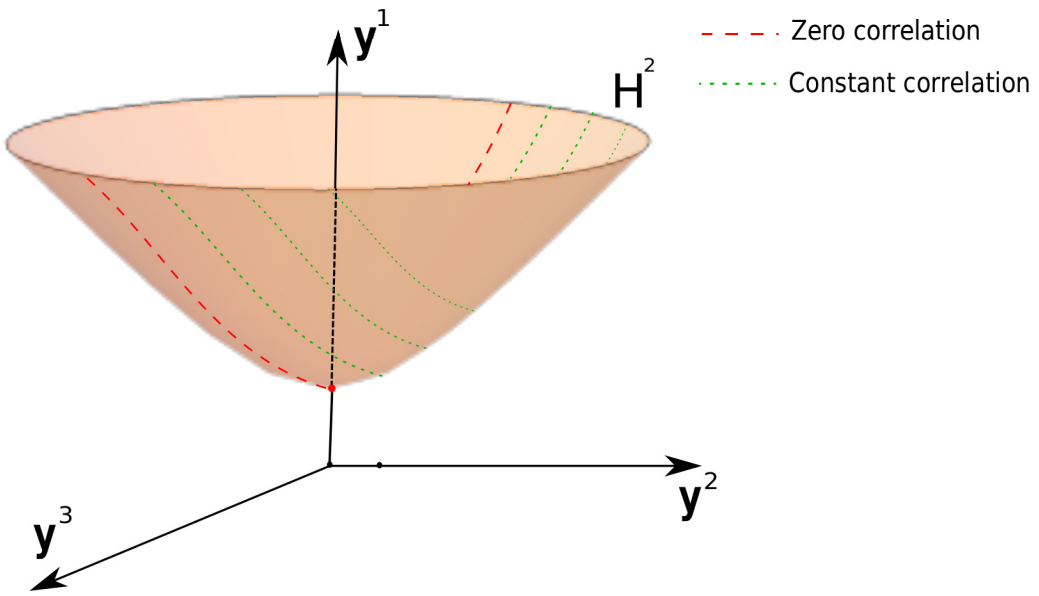


Figure 2.2: Upper sheet of hyperboloid $\mathbf{H}^2$, where each point on it represent a generalized coherent state. In particular, in the dashed red line we have plotted the coherent states and in green dotted lines the states with constant correlation.

where $\xi = 2\,\tau\,\mathrm{e}^{\mathrm{i}\,\varphi}$ [3]. The operator $\hat{S}(\xi)$ is defined as

$$\hat{S}(\xi) = \mathrm{e}^{\bar{\xi}\,\hat{K}_- - \xi\,\hat{K}_+} \tag{2.73}$$

where the operators $\hat{K}_\pm$ are elements of the $\mathfrak{su}(1,1)$ algebra, which is defined in terms of the creation and annihilation operators by

$$\hat{K}_- = \frac{1}{2}\,\hat{a}^2\,, \quad \hat{K}_+ = \frac{1}{2}\,(\hat{a}^\dagger)^2\,, \quad \hat{K}_0 = \frac{1}{4}\,(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})\,, \tag{2.74}$$

with the commutation relations

$$[\hat{K}_+, \hat{K}_-] = -2\hat{K}_0\,, \quad [\hat{K}_0, \hat{K}_\pm] = \pm\hat{K}_\pm\,. \tag{2.75}$$

Let us be more explicit in the immersion (2.72). The fiducial normalized state $|w\rangle$ is usually defined by means of the condition $K_-|w\rangle = 0$, this condition is satisfied by the Fock states $|0\rangle$ and $|1\rangle$. Considering the Hilbert space of quadratically integrable functions $\mathfrak{L}^2(\mathbb{R}, \mathrm{d}q)$, the immersed submanifold $s(\mathbf{H}^2) \subset \mathcal{H}$ depends on the choice of the fiducial state; then, considering the state $|0\rangle$ the immersed submanifold in $\mathcal{H}$ corresponds to the Gaussian wave packets $\psi(\xi, q) = \langle q|\hat{S}(\xi)|0\rangle$. These states are known in quantum optics as the *squeezed vacuum states*, whose expectation values are $(\langle\hat{q}\rangle, \langle\hat{p}\rangle) = (0, 0)$. On the other hand, considering the state $|1\rangle$ the immersed submanifold clearly does not belong to the space of Gaussian wave functions; therefore, it will not be considered further in this work.

In quantum optics the generalized coherent state is also known as *the squeezed coherent state* [62] and is denoted by $|\alpha, \xi\rangle$ being the result of the following immersion

$$h : \mathbf{H}^2 \times \mathbb{C} \to \mathcal{H} \quad \text{defined as} \quad \hat{D}(\alpha)\hat{S}(\xi)|\,0\rangle = |\alpha, \xi\rangle\,. \tag{2.76}$$

From this definition we see that the parametrization of the generalized coherent state is composed by two different parametrizations. The immersion of the complex plane $\mathbb{C}$ parametrizes the expectation values, whereas the immersion of the hyperboloid $\mathbf{H}^2$ parametrizes the second moments. For instance, in the

[3] The coordinate $\xi \in \mathbb{C}$ is simply a complexification of $(\tau, \phi) \in \mathbb{R}^2$.

immersion defined before, i.e. in Eq. (2.76), the action of the operator $\hat{S}(\xi)$ on $|0\rangle$ fixes the point in the hyperboloid, later the action of the operator $\hat{D}(\alpha)$ fixes the point in the complex plane.

As it has been seen, an important consequence of immersing a "classical" submanifold in the Hilbert space is the fact that the time-dependence of the wave function is completely parametrized by the evolution on such a submanifold. The manifold $\mathbf{H}^2$ is embedded with the symplectic form

$$\begin{aligned}\omega_{\mathbf{H}^2} &= \frac{1}{1+y^1}\left[(1+y^1)\mathrm{d}y^3\wedge\mathrm{d}y^2 + y^3\mathrm{d}y^3\wedge\mathrm{d}y^2 + y^2\mathrm{d}y^3\wedge\mathrm{d}y^2\right]\\ &= \sinh\tau\,\mathrm{d}\phi\wedge\mathrm{d}\tau\,.\end{aligned} \tag{2.77}$$

Thus there is a Hamiltonian dynamics, $X_{\mathbf{H}^2}\in\mathfrak{X}(T\mathbf{H}^2)$, which is defined by

$$i_{X_{\mathbf{H}^2}}\omega_{\mathbf{H}^2} = -\mathrm{d}e_H\,.$$

To construct the Hamiltonian function $e_H\in\mathfrak{F}(\mathbf{H}^2)$, let us notice that in quantum mechanics the special linear Lie algebra $\mathfrak{sl}(2)$ is given by the operators

$$\hat{L}_1 = \frac{1}{4}\left(\hat{p}^2-\hat{q}^2\right)\,,\quad \hat{L}_2 = \frac{1}{4}\left(\hat{q}\hat{p}+\hat{p}\hat{q}\right)\,,\quad\text{and}\quad \hat{L}_0 = \frac{1}{4}\left(\hat{p}^2+\hat{q}^2\right) \tag{2.78}$$

with the commutation relations[4]

$$[\hat{L}_1,\hat{L}_2] = -\mathrm{i}\,\hbar\,\hat{L}_0\,,\quad [\hat{L}_0,\hat{L}_1] = \mathrm{i}\,\hbar\,\hat{L}_2\quad\text{and}\quad [\hat{L}_0,\hat{L}_2] = -\mathrm{i}\,\hbar\,\hat{L}_1\,. \tag{2.80}$$

In terms of these operators the quadratic Hamiltonian operator (2.5) has the form

$$\hat{H} = (H_2+H_1)\,\hat{L}_0 + 2V\,\hat{L}_2 + (H_2-H_1)\,\hat{L}_1\,, \tag{2.81}$$

and hence the Hamiltonian function in the hyperboloid is simply the expectation value

$$\begin{aligned}e_H &= \langle\,\xi\,|\hat{H}|\,\xi\,\rangle\\ &= (H_2+H_1)\,y^1 + 2V\,y^2 + (H_2-H_1)\,y^3\\ &= (H_2+H_1)\,\cosh\tau + 2V\,\sinh\tau\cos\phi + (H_2-H_1)\,\sinh\tau\sin\phi\,.\end{aligned} \tag{2.82}$$

Finally, from the definition of the symplectic evolution, it is not difficult to find that the Hamiltonian dynamics has the form

$$X_{\mathbf{H}^2} = -(2V\sin\phi + (H_1-H_2)\cos\phi)\frac{\partial}{\partial\tau} - \left[H_1+H_2+(2V\cos\phi+(H_2-H_1)\sin\phi)\coth\tau\right]\frac{\partial}{\partial\phi}\,, \tag{2.83}$$

with the Hamiltonian equations of motion

$$\begin{aligned}\dot{\tau} &= \frac{1}{\sinh\tau}\frac{\partial e_H}{\partial\phi} = -2V\sin\phi - (H_1-H_2)\cos\phi\\ \dot{\phi} &= -\frac{1}{\sinh\tau}\frac{\partial e_H}{\partial\tau} = -\left[2V\cos\phi-(H_1-H_2)\sin\phi\right]\coth\tau - (H_1+H_2)\,,\end{aligned} \tag{2.84}$$

which clearly is a nonlinear dynamics.

Now, the connection between the manifold M defined in Eq. (2.35) and the hyperboloid $\mathbf{H}^2$ in Eq. (2.68) is established. To do that, recall that we may represent an element of the group $\mathrm{SU}(2,\mathbb{R})$ by

$$\mathbf{s} = x^\mu e_\mu\,, \tag{2.85}$$

[4]The connection between the elements of $\mathfrak{sl}(2,\mathbb{R})$ and the elements of $\mathfrak{su}(1,1)$ is

$$\hat{K}_- = \frac{1}{\hbar}\left(\hat{L}_1-\mathrm{i}\,\hat{L}_2\right)\,,\quad \hat{K}_+ = \frac{1}{\hbar}\left(\hat{L}_1+\mathrm{i}\,\hat{L}_2\right)\,,\quad \hat{K}_0 = \frac{1}{\hbar}\,\hat{L}_0 \tag{2.79}$$

where $\mu = 0, 1, 2, 3$, being e_0 the identity matrix and e_k given in (2.65). Hence, one may introduce the map

$$\begin{pmatrix} x^0 + x^2 & x^3 + x^1 \\ x^3 - x^1 & x^0 - x^2 \end{pmatrix} \mapsto (x^0, x^1, x^2, x^3)\,. \tag{2.86}$$

Moreover, the coordinates x^μ define

$$\mathbf{H}^3 = \{(x^0, x^1, x^2, x^3) \in \mathbb{R}^4 \,|\, (x^0)^2 + (x^1)^2 - (x^2)^2 - (x^3)^2 = 1\}\,, \tag{2.87}$$

from the constraint $\det \mathbf{s} = 1$.

Now, expressing the complex coordinates Q and P in terms of their real and imaginary parts, that is, $Q = Q_{\mathrm{R}} + \mathrm{i}\, Q_{\mathrm{I}}$ and $P = P_{\mathrm{R}} + \mathrm{i}\, P_{\mathrm{I}}$, such that

$$\begin{aligned} Q_{\mathrm{R}} &= x^1 - x^3\,, \qquad Q_{\mathrm{I}} = x^2 - x^0\,, \\ P_{\mathrm{R}} &= x^2 + x^0\,, \qquad P_{\mathrm{I}} = x^1 + x^3\,. \end{aligned} \tag{2.88}$$

So, the constraint (2.7) in this real variables reproduces the condition in definition (2.87), i.e., one has the transformation $\nu : M \to \mathbf{H}^3$ and the Hamiltonian dynamics $X_{\mathbf{H}^3} \in T\mathbf{H}^3$ is defined by $i_{X_{\mathbf{H}^3}} \omega_{\mathbf{H}^3} = -\,\mathrm{d} H_{\mathbf{H}^3}$ with

$$H_{\mathbf{H}^3} = \nu \circ H_M\,, \quad \omega_M = \nu^* \omega_{\mathbf{H}^3} \quad \text{and} \quad X_{\mathbf{H}^3} = \nu_* X_M\,. \tag{2.89}$$

Explicitly the components of the vector field $X_{\mathbf{H}^3}$ define the Hamiltonian equations of motion

$$\begin{aligned} \dot{x}^0 &= -\frac{1}{2}(H_2 + H_1)x^1 - \frac{1}{2}(H_2 - H_1)x^3 - V x^2\,, \\ \dot{x}^1 &= \frac{1}{2}(H_2 + H_1)x^0 + \frac{1}{2}(H_2 - H_1)x^4 - V x^3\,, \\ \dot{x}^2 &= \frac{1}{2}(H_2 + H_1)x^3 + \frac{1}{2}(H_2 - H_1)x^1 - V x^0\,, \\ \dot{x}^3 &= -\frac{1}{2}(H_2 + H_1)x^2 - \frac{1}{2}(H_2 - H_1)x^0 - V x^1\,. \end{aligned} \tag{2.90}$$

Now, to obtain the connection between the hyperboloids $\mathbf{H}^3$ and $\mathbf{H}^2$, notice that

$$\mathrm{SL}(2, \mathbb{R}) \to \mathfrak{sl}(2, \mathbb{R}) : \mathbf{s} \mapsto \mathbf{s}\, e_1\, \mathbf{s}^{-1} \equiv y^k e_k\,, \tag{2.91}$$

where, as one may prove the matrix $\mathbf{s}\, e_1\, \mathbf{s}^{-1}$ is a traceless matrix. To show that $y^k e_k$ is an element of $\mathbf{H}^2$, it is enough to compute

$$\det\big(y^k e_k\big) = \det\big(\mathbf{s}\, e_1\, \mathbf{s}^{-1}\big) = 1. \tag{2.92}$$

To show the way $(y^1, y^2, y^3) \in \mathbf{H}^2$ depends on $(x^0, x^1, x^2, x^3) \in \mathbf{H}^3$, one simply uses the explicit form of the matrices $\mathbf{s}$ and e_1 to obtain

$$\begin{aligned} y^1 &= (x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2\,, \\ y^2 &= 2\,(x^1 x^2 - x^0 x^3)\,, \\ y^3 &= 2\,(x^1 x^3 + x^0 x^2)\,. \end{aligned} \tag{2.93}$$

Then one arrives at the covering map

$$\chi : \mathbf{H}^3 \to \mathbf{H}^2 : (x^0, x^1, x^2, x^3) \mapsto (y^1, y^2, y^3)\,. \tag{2.94}$$

Note that the generalization of the results in this sections for more degrees of freedom is not simple, because the connection between the Lie group $\mathrm{SL}(2n, \mathbb{R})$ and its Lie algebra $\mathfrak{sl}(2n, \mathbb{R})$, for $n > 1$, is more complex.

2.3 Nonlinear Riccati Evolution

In the last section it has been shown that for the one-dimensional parametric oscillator system it is possible to associate a nonlinear dynamics with the evolution of the Gaussian wave packets; however, this nonlinear description appears to be more difficult for more degrees of freedom. Here we will show that there is another nonlinear description of the quadratic Hamiltonian systems independently of the degrees of freedom of the system; such a nonlinear description is the Riccati evolution. The nonlinear Riccati evolution has currently gained considerable interest and has been widely studied and applied to quantum systems [20, 38, 19].

Let us introduce the Riccati evolution for one-dimensional quantum systems by means of the transformation

$$\pi : M \to \mathbb{HP}^2 : (Q, P) \mapsto \mathcal{C} = \frac{P}{Q}\,, \tag{2.95}$$

where the manifold M has been defined in (2.35), whereas the space $\mathbb{HP}^2$ is known as the *Siegel upper half plane* [64, 5] and is defined as

$$\mathbb{HP}^2 = \{\mathcal{C} \in \mathbb{C} \,|\, \mathcal{C}_{\rm I} > 0\}\,, \tag{2.96}$$

considering $\mathcal{C} = \mathcal{C}_{\rm R} + {\rm i}\,\mathcal{C}_{\rm I}$. Then, by means of the linear equations (2.58) follows that the dynamics on the tangent space of the manifold TM induces a nonlinear dynamics in the space $T\,\mathbb{HP}^2$ such that the integral curves of this dynamics are the solution of the Riccati equation (associated with the Hamiltonian defined in Eq. (2.5))

$$\dot{\mathcal{C}} + H_2\,\mathcal{C}^2 + 2V\,\mathcal{C} + H_1 = 0\,. \tag{2.97}$$

In fact, one may express the Gaussian wave packet (2.41) in terms of the Riccati variables $\mathcal{C} \in \mathbb{HP}^2$ as[5]

$$\psi(\alpha, \mathcal{C}, q) = \frac{1}{(\pi\,\hbar)^{1/4}} \exp\left\{ \frac{\rm i}{2\hbar}\,\mathcal{C}\,(q - \langle\hat{q}\rangle)^2 + \frac{\rm i}{\hbar}\langle\hat{p}\rangle(q - \langle\hat{q}\rangle) + \frac{\rm i}{2\hbar}\,\langle\hat{q}\rangle\,\langle\hat{p}\rangle - \frac{1}{2}\int [H_2\,\mathcal{C}(\tau) + V]{\rm d}\tau \right\}\,. \tag{2.98}$$

On the other hand, it is also posible to consider the coordinates

$$\tilde{\pi} : M \to \mathbb{HP} : (Q, P) \mapsto \tilde{\mathcal{C}} = \frac{Q}{P}\,, \tag{2.99}$$

where for this case we may express the generalized coherent state in the momentum representation as

$$\tilde{\psi}(\alpha, \tilde{\mathcal{C}}, p) = \frac{\sqrt{\rm i}}{(\pi\,\hbar)^{1/4}} \exp\left\{ -\frac{\rm i}{2\hbar}\,\tilde{\mathcal{C}}\,(p - \langle\hat{p}\rangle)^2 - \frac{\rm i}{\hbar}\langle\hat{q}\rangle(p - \langle\hat{p}\rangle) - \frac{\rm i}{2\hbar}\,\langle\hat{q}\rangle\,\langle\hat{p}\rangle + \frac{1}{2}\int [H_1\,\tilde{\mathcal{C}}(\tau) + V]{\rm d}\tau \right\}\,, \tag{2.100}$$

where now the complex time dependent function $\tilde{\mathcal{C}}(t)$ obeys the nonlinear Riccati equation

$$-\dot{\tilde{\mathcal{C}}} + H_1\,\tilde{\mathcal{C}}^2 + 2V\,\tilde{\mathcal{C}} + H_2 = 0\,. \tag{2.101}$$

Therefore, these nonlinear Riccati equations are closely connected to the evolution of the generalized coherent states.

The geometrical nature of the transformations (2.95) and (2.99) is the following. As one can see the linear system of equations (2.58) together with its constraint (2.7) is invariant under the multiplication by a global phase factor, i.e. it is invariant under the tranformation

$$(Q, P) \mapsto {\rm e}^{i\phi}(Q, P)\,. \tag{2.102}$$

This invariance allows to reduce the dynamics into a manifold of a lower dimension for the dynamical system, for a complete review about reduction procedure see Ref. [4]. The multiplication by the global phase factor is the group action of $U(1)$, which may be described infinitesimally by means of the linear vector field

$$\Gamma = P_{\rm R}\,\frac{\partial}{\partial Q_{\rm R}} - Q_{\rm R}\,\frac{\partial}{\partial P_{\rm R}} + P_{\rm I}\,\frac{\partial}{\partial Q_{\rm I}} - Q_{\rm I}\,\frac{\partial}{\partial P_{\rm I}}\,. \tag{2.103}$$

[5] Remember that the parameter α is given in terms of the expectation values of $\hat{p}$ and $\hat{q}$, see Eq. (2.40).

Then, on the carrier space TM there is defined a regular distribution $\mathcal{D} = \{\Gamma\}$, whose integral curves

$$\ell_{(r_1,r_2)} = \{(Q_{\rm R}, P_{\rm R}, Q_{\rm I}, P_{\rm I}) \in \mathbb{R}^4 \,|\, P_{\rm R}^2 + Q_{\rm R}^2 = r_1^2 \cup P_{\rm I}^2 + Q_{\rm I}^2 = r_2^2 \,\} \tag{2.104}$$

are a family $\{\ell_{(r_1,r_2)}\}$ of disjoint subsets which foliates the manifold M, with r_1 and r_2 real constant quantities. Furthermore, the foliation defines the equivalence relation Φ^Γ as

$$(Q_{\rm R}, P_{\rm R}, Q_{\rm I}, P_{\rm I}) \in \Phi^\Gamma \quad \text{iff} \quad (Q_{\rm R}, P_{\rm R}, Q_{\rm I}, P_{\rm I}) \in \ell_{(r_1,r_2)} \,. \tag{2.105}$$

Thus, it is possible to define the quotient space M/Φ^Γ with respect to the equivalence relation defined by the foliation, which is identified with the Siegel upper half plane $\mathbb{HP}^2$ [64, 5].

Now that the canonical projection π from the manifold M to the Siegel upper half plane $\mathbb{HP}^2$ has been established, one may apply this results to the dynamics of the system. Then, the dynamical vector field $X_M \in \mathfrak{X}(TM)$, given in (2.57), is *projectable* onto a dynamics $X_{\mathbb{HP}^2} \in \mathfrak{X}(T\mathbb{HP}^2)$, iff

$$[X_H, \Gamma] = 0 \,, \tag{2.106}$$

for a formal proof of this result see reference [4]. Then, because the dynamics is projectable, it carries leaves of the foliation Φ^Γ, into leaves, i.e. the foliation is invariant under X_H [4]. In this sense the group action of $U(1)$ is a symmetry for the dynamics. Therefore, $\mathbb{HP}^2$ is the space that results after taking into account the symmetry of the multiplication by a global phase factor in the linear equation of motion (2.58), which gives rise to a nonlinear evolution.

To have a closed picture of our study, now the connection between the upper sheet of the hyperboloid with the Siegel upper half plane is established. So, it is well-known that every point on the hyperboloid $\mathbf{H^2}$, defined in Eq. (2.68), may be projectable onto the *Poincaré disk* or in the Siegel upper half plane. In the first case, one considers the projection point $(-1, 0, 0) \in \mathbb{R}^3$, such that a point $(y^1, y^2, y^3) \in \mathbf{H}^2$ is projected onto the plane $y^1 = 0$, see Fig. 2.3a, given by the map

$$\zeta = \frac{y^2 + \mathrm{i}\, y^3}{1 + y^1} \,, \tag{2.107}$$

i.e., the result is the projection

$$v : \mathbf{H}^2 \to \mathbb{D}^2 : (y^1, y^2, y^3) \in \mathbb{R}^3 \mapsto \zeta \in \mathbb{C} \,, \tag{2.108}$$

where the complex number $\zeta = \zeta_{\rm R} + \mathrm{i}\,\zeta_{\rm I}$ is an element of the Poincaré disk defined as the open disk

$$\mathbb{D}^2 = \{\zeta \in \mathbb{C} \,|\, |\zeta| < 1\} \,. \tag{2.109}$$

This projection is displayed in Fig. 2.3a, where in addition it is possible to see that the curves for zero correlation and constant correlation are mapped in parallel lines on the Poincaré disk. It is interesting to mention that the evolution in the hyperboloid has induced a nonlinear evolution in $\mathbb{D}^2$ given by the Riccati equation

$$\dot{\zeta} - \frac{1}{2}(H_1 - H_2 - 2\,\mathrm{i}\,V)\zeta^2 + \mathrm{i}\,(H_2 + H_1)\zeta + \frac{1}{2}(H_2 - H_1 + 2\,\mathrm{i}\,V) = 0 \,. \tag{2.110}$$

Finally, it is possible obtain the upper half Siegel plane from the Poincaré disk as follow. First, we consider the stereographic projection of the Poincaré disk onto the"north hemisphere" of a sphere, employing as a projection point the "south pole" of the sphere, see Fig 2.3b. Later, the hemisphere points are projected onto the tangent plane to the sphere by means of another stereographic projection, see Fig. 2.3b. In Fig. 2.3b it is plotted the projection on the sphere and in $\mathbb{HP}^2$ the curves for $\sigma_{qp} = 0$ (red lines) and $\sigma_{qp} = \text{cte}$ (green lines).

Therefore, the open disk $\mathbb{D}^2$ is mapped bijectively to the half plane $\mathbb{HP}^2$ by the map

$$u : \mathbb{D}^2 \to \mathbb{HP}^2 : \zeta \in \mathbb{C} \mapsto \mathcal{C} \in \mathbb{C} \tag{2.111}$$

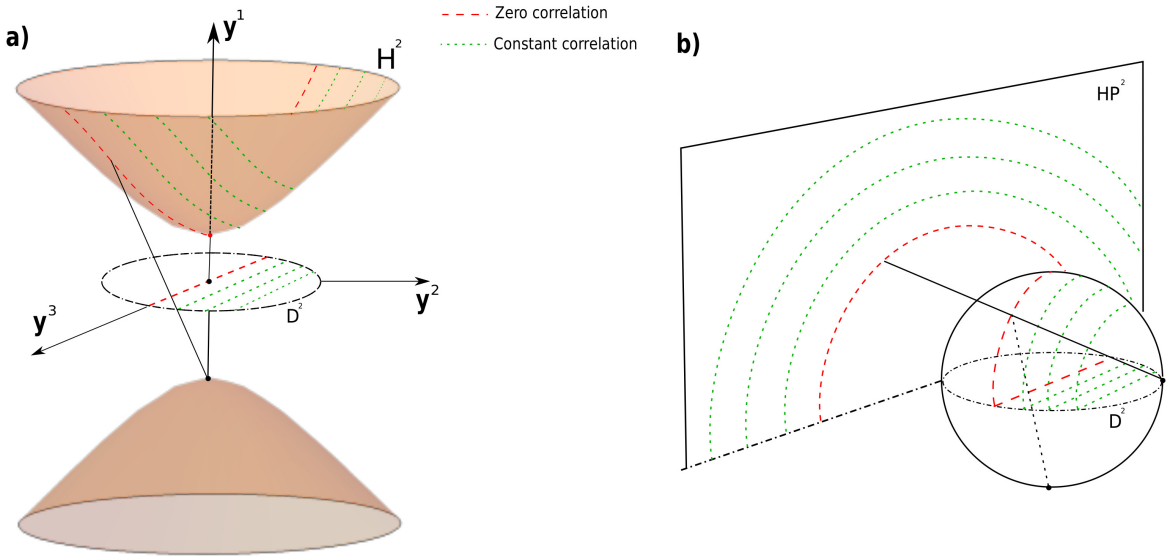


Figure 2.3: a) Projection of the upper sheet of the hyperboloid $\mathbf{H}^2$ onto the Poincaré disk $\mathbb{D}^2$ with the projection point $(-1,0,0)$. b) Stereographic projection of the Poincaré disk onto the "north hemisphere" of the sphere, later the hemisphere points are projected onto the Siegel upper half plane $\mathbb{HP}^2$.

by means of the Möbious transformation

$$\mathcal{C} = \frac{\zeta + \mathrm{i}}{\mathrm{i}\zeta + 1}\,. \tag{2.112}$$

Consequently we may construct the map $u \circ v : \mathbf{H}^2 \to \mathbb{HP}^2$ to obtain the relations

$$y^1 = \frac{1 + \mathcal{C}\bar{\mathcal{C}}}{2\,\mathcal{C}_{\mathrm{I}}}\,, \qquad y^2 = \frac{\mathcal{C}_{\mathrm{R}}}{\mathcal{C}_{\mathrm{I}}}\,, \qquad y^3 = \frac{-1 + \mathcal{C}\bar{\mathcal{C}}}{2\,\mathcal{C}_{\mathrm{I}}}\,, \tag{2.113}$$

or directly from the uncertainties and the correlation we have that

$$\mathcal{C} = \frac{\sigma_{qp}}{\sigma_q^2} + \frac{\mathrm{i}\,\hbar}{2}\,\frac{1}{\sigma_q^2}\,. \tag{2.114}$$

We have obtained two different new immersions for the generalized coherent states

$$d : \mathbb{D}^2 \times \mathbb{C} \to \mathcal{H} : (\zeta, \alpha) \mapsto |\zeta, \alpha\rangle \quad \text{and} \quad g : \mathbb{HP}^2 \times \mathbb{C} \to \mathcal{H} : (\mathcal{C}, \alpha) \mapsto |\mathcal{C}, \alpha\rangle\,. \tag{2.115}$$

Finally, we may summarize all the connections in the following diagramme

$$\begin{array}{ccc} M & \xrightarrow[\nu]{} & \mathbf{H}^3 \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \chi} \\ & & \mathbf{H}^2 \\ & & \Big\downarrow{\scriptstyle v} \\ \mathbb{HP}^2 & \xleftarrow[u]{} & \mathbb{D}^2 \end{array}$$

After the kinematical picture has been completed, the dynamical properties of the Riccati descriptions are now the subject of interest. Even though the dynamics on $\mathbb{HP}^2$ is nonlinear, it is possible to prove that it is Hamiltonian. To see this, notice that the upper half plane is endowed with the symplectic form ω_{WP} (the Weil–Petersson symplectic form), defined in coordinates as

$$\omega_{\mathrm{WP}} = \frac{-2\,\mathrm{i}}{(\mathcal{C} - \bar{\mathcal{C}})^2}\,\mathrm{d}\bar{\mathcal{C}} \wedge \mathrm{d}\mathcal{C}\,. \tag{2.116}$$

Furthermore, by means of the map in (2.95), one may look at the pullback of ω_M in Eq. (2.55) to $\mathbb{C}^2$ and see that $\omega_M = \pi^* \omega_{\rm WP}$. Hence, with the help of this symplectic form the evolution in $\mathbb{HP}^2$ is given by the Hamiltonian vector field $X_{\mathbb{HP}} \in T\,\mathbb{HP}^2$ defined intrinsically by

$$i_{X_{\mathbb{HP}}} \omega_{\rm WP} = -\,\mathrm{d}\, H_{\mathbb{HP}}\,, \tag{2.117}$$

where the Hamiltonian function in the coordinates $\mathcal{C} \in \mathbb{HP}^2$ has the form

$$H_{\mathbb{HP}} = \frac{2\,\mathrm{i}}{\mathcal{C} - \bar{\mathcal{C}}} \begin{pmatrix} 1 \ , & \bar{\mathcal{C}} \end{pmatrix} \begin{pmatrix} H_1 & V \\ V & H_2 \end{pmatrix} \begin{pmatrix} 1 \\ \mathcal{C} \end{pmatrix}. \tag{2.118}$$

Using the symplectic form (2.116) and the Hamiltonian function in (2.118), it is straightforward to obtain the Hamiltonian vector field in the explicit form

$$X_{\mathbb{HP}} = X_{\mathcal{C}}\, \frac{\partial}{\partial \mathcal{C}} + X_{\bar{\mathcal{C}}}\, \frac{\partial}{\partial \bar{\mathcal{C}}}\,, \tag{2.119}$$

where $X_{\mathcal{C}}$ is the complex conjugated of $X_{\bar{\mathcal{C}}}$ and it is given by

$$X_{\mathcal{C}} = -H_2\, \mathcal{C}^2 - 2\, V\, \mathcal{C} - H_1\,. \tag{2.120}$$

Hence, the integral curves of this Hamiltonian vector field are given by the solutions of the Hamiltonian equations of motion

$$\begin{aligned} \dot{\mathcal{C}} &= -\frac{(\mathcal{C} - \bar{\mathcal{C}})^2}{2\,\mathrm{i}}\, \frac{\partial H}{\partial \bar{\mathcal{C}}} = -H_2\, \mathcal{C}^2 - 2\, V\, \mathcal{C} - H_1\,, \\ \dot{\bar{\mathcal{C}}} &= \frac{(\mathcal{C} - \bar{\mathcal{C}})^2}{2\,\mathrm{i}}\, \frac{\partial H}{\partial \mathcal{C}} = -H_2\, \bar{\mathcal{C}}^2 - 2\, V\, \bar{\mathcal{C}} - H_1\,. \end{aligned} \tag{2.121}$$

These equations are identical with the Riccati equation presented in Eq. (2.97) as it should be. Moreover, we may introduce the Poisson brackets for the upper Siegel half plane space. So, given the functions $A, B \in \mathfrak{F}(\mathbb{HP}^2)$ the Poisson bracket is given by

$$\{A, B\}_{\rm WP} = \frac{(\mathcal{C} - \bar{\mathcal{C}})^2}{2\,\mathrm{i}} \left[\frac{\partial A}{\partial \mathcal{C}}\, \frac{\partial B}{\partial \bar{\mathcal{C}}} - \frac{\partial A}{\partial \bar{\mathcal{C}}}\, \frac{\partial B}{\partial \mathcal{C}} \right]. \tag{2.122}$$

In particular, the time evolution of any (not explicitly time-dependent) function on the Siegel half plane $A \in \mathfrak{F}(\mathbb{HP}^2)$ is given by

$$\frac{\mathrm{d}A}{\mathrm{d}t} = X_H[A] = \{H, A\}_{\rm WP}\,. \tag{2.123}$$

As it has pointed out, one may also describe the evolution on the Poincaré disk, with a Riccati evolution. Furthermore, now we are able to prove that the dynamics is also Hamiltonian, where the symplectic structure in $\mathbb{D}^2$ is obtained via the pullback $u^*(\omega_{\rm WP}) = \omega_{\mathbb{D}}$, which in coordinates has the form

$$\omega_{\mathbb{D}} = \frac{2\,\mathrm{i}}{(1 - \zeta\bar{\zeta})^2}\, \mathrm{d}\bar{\zeta} \wedge \mathrm{d}\zeta \tag{2.124}$$

such that the Hamiltonian dynamics given by $i_X \omega_{\mathbb{D}} = -\mathrm{d}H_{\mathbb{D}}$ has the Hamiltonian function $H_{\mathbb{D}} = u^{-1}(H_{\mathbb{HP}})$, i.e.,

$$H_{\mathbb{D}} = \frac{1}{1 - \bar{\zeta}\zeta} \begin{pmatrix} -\mathrm{i}\bar{\zeta} + 1 \ , & \bar{\zeta} - \mathrm{i} \end{pmatrix} \begin{pmatrix} H_1 & V \\ V & H_2 \end{pmatrix} \begin{pmatrix} \mathrm{i}\zeta + 1 \\ \zeta + \mathrm{i} \end{pmatrix}. \tag{2.125}$$

Then the Hamiltonian equations of motion are given by

$$\begin{aligned} \dot{\zeta} &= -\frac{(1 - \zeta\bar{\zeta})^2}{2\,\mathrm{i}}\, \frac{\partial H_{\mathbb{D}}}{\partial \bar{\zeta}} = \frac{1}{2}(H_1 - H_2 - 2\,\mathrm{i}\,V)\zeta^2 - \mathrm{i}\,(H_2 + H_1)\zeta - \frac{1}{2}(H_2 - H_1 + 2\,\mathrm{i}\,V)\,, \\ \dot{\bar{\zeta}} &= \frac{(1 - \zeta\bar{\zeta})2}{2\,\mathrm{i}}\, \frac{\partial H_{\mathbb{D}}}{\partial \zeta} = \frac{1}{2}(H_1 - H_2 + 2\,\mathrm{i}\,V)\bar{\zeta}^2 + \mathrm{i}\,(H_2 + H_1)\bar{\zeta} - \frac{1}{2}(H_2 - H_1 - 2\,\mathrm{i}\,V)\,, \end{aligned} \tag{2.126}$$

and hence the evolution of an arbitrary time-independent function $F \in \mathfrak{F}(\mathbb{D}^2)$ is given by

$$\frac{\mathrm{d}F}{\mathrm{d}t} = X_{\mathbb{D}^2}[F] = \{H, F\}_{\mathbb{D}^2}\,, \tag{2.127}$$

where $\{\,\cdot\,,\,\cdot\,\}_{\mathbb{D}^2}$ is the Poisson bracket defined as

$$\{H, F\}_{\mathbb{D}^2} = \frac{(1-\zeta\bar{\zeta})^2}{2\,\mathrm{i}}\left[\frac{\partial H}{\partial \zeta}\frac{\partial F}{\partial \bar{\zeta}} - \frac{\partial H}{\partial \bar{\zeta}}\frac{\partial F}{\partial \zeta}\right]. \tag{2.128}$$

2.4 Examples

2.4.1 Degenerate parametric amplification

In order to apply all the formalism previously developed to a concrete physical system, in this Section the study of the *degenerate parametric amplifier* [62] is considered . In quantum optics the parametric amplifier is an optical device in which there is a coupling of three modes of the electromagnetic field in a nonlinear optical crystal. The frequencies involved are ω_{p} (pump), ω_{i} (idler) and ω_{s} (signal) and they are such that $\omega_{\mathrm{p}} = \omega_{\mathrm{s}} + \omega_{\mathrm{i}}$, where for the case of degenerate parametric amplification the idler and the signal frequencies coincide, i.e. $\omega = \omega_{\mathrm{s}} = \omega_{\mathrm{i}}$ and then $\omega_{\mathrm{p}} = 2\omega$. The Hamiltonian describing this device is

$$\hat{H} = \hbar\,\omega\,\hat{a}^\dagger\hat{a} + 2\,\hbar\,\omega\,\hat{b}^\dagger\hat{b} + \hbar\,\kappa\,[\,(\hat{a}^\dagger)^2\,\hat{b} + \hat{a}^2\,\hat{b}^\dagger\,]\,. \tag{2.129}$$

Here $\hat{b}$ and $\hat{a}$ are the annihilation operators for the pump and the signal (idler) modes, respectively, κ is the coupling constant that depends on the properties of the nonlinear crystal. Because, we are interested in the *parametric approximation*, i.e. when the pump field is treated as a classical field[6], then the expectation value of the Hamiltonian (2.129) with respect the coherent state $|\beta\,\mathrm{e}^{-\mathrm{i}\omega t}\rangle$ of the pump field it is considered, such that $\hat{b}\,|\beta\,\mathrm{e}^{-\mathrm{i}\omega t}\rangle = \beta\,\mathrm{e}^{-\mathrm{i}\omega t}|\beta\,\mathrm{e}^{-\mathrm{i}\omega t}\rangle$, thus one has the effective Hamiltonian

$$\begin{aligned}\hat{H}_{\mathrm{Eff}} &= \frac{\hbar\,\omega}{2}\,[\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger] + \hbar\,\kappa\,[\,(\hat{a}^\dagger)^2\,\beta\,\mathrm{e}^{-\mathrm{i}\omega t} + \hat{a}^2\,\bar{\beta}\,\mathrm{e}^{\mathrm{i}\omega t}\,] \\ &= \frac{\hbar}{4}\,(\,\hat{a}\;,\;\hat{a}^\dagger\,)\begin{pmatrix} \bar{\xi}\,\mathrm{e}^{\mathrm{i}\omega t} & 2\,\omega \\ 2\,\omega & \xi\,\mathrm{e}^{-\mathrm{i}\omega t}\end{pmatrix}\begin{pmatrix}\hat{a}\\ \hat{a}^\dagger\end{pmatrix},\end{aligned} \tag{2.130}$$

where constant terms has been ignored and $\xi = 4\,\kappa\,\beta$ is considered. Then, the dynamics of the complex parameter $\alpha = \langle\alpha|\hat{a}|\alpha\rangle$ is given by the Hamiltonian equations of motion (2.50), which, for our case of interest, have the form

$$\begin{aligned}\dot{\alpha} &= -\frac{\mathrm{i}}{2}(\xi\,\mathrm{e}^{-\mathrm{i}\omega t}\bar{\alpha} + 2\omega\alpha)\,, \\ \dot{\bar{\alpha}} &= \frac{\mathrm{i}}{2}(\bar{\xi}\,\mathrm{e}^{\mathrm{i}\omega t}\alpha + 2\omega\bar{\alpha})\,.\end{aligned} \tag{2.131}$$

The integral curves of this system of differential equations are given by

$$\alpha(t) = \begin{cases} \left(\frac{1}{\Omega}[2\dot{\alpha}_0 + \mathrm{i}\,\omega\,\alpha_0]\sin\frac{\Omega t}{2} + \alpha_0\cos\frac{\Omega t}{2}\right)\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t} & \text{for} \quad \Omega = \sqrt{\omega^2 - |\xi|^2}\,, \\ \left(\left[\dot{\alpha}_0 + \frac{\mathrm{i}\,\omega}{2}\alpha_0\right]t + \alpha_0\right)\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t} & \text{for} \quad \Omega = 0\,, \\ \frac{1}{\tilde{\Omega}}\left([2\dot{\alpha}_0 + \mathrm{i}\,\omega\,\alpha_0]\sinh\frac{\tilde{\Omega}t}{2} + \alpha_0\cosh\frac{\tilde{\Omega}t}{2}\right)\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t} & \text{for} \quad \tilde{\Omega} = \sqrt{|\xi|^2 - \omega^2}\,, \end{cases} \tag{2.132}$$

[6] The interaction of different radiation modes through nonlinear crystals allow the generation of interesting states of light. Most of the theoretical analysis refers to situations where one mode is placed in a high amplitude coherent state. This is called the *parametric approximation*.

with α_0 and $\dot{\alpha}_0$ the initial conditions. Thus, the expectation value of the quadratures of the field $(\hat{q}, \hat{p})$ may be obtained by the expression

$$\alpha = \frac{1}{\sqrt{2\hbar}} \left(\sqrt{\omega}\, \langle q \rangle + \frac{\mathrm{i}}{\sqrt{\omega}} \langle p \rangle \right) . \tag{2.133}$$

The immersion of the complex number α, given in Eq. (2.132), into the Hilbert space $\mathfrak{L}^2(\mathbb{R}, \mathrm{d}q)$ gives rise to the evolution of the coherent states with constant second moments, namely

$$\sigma_q = \frac{\hbar\omega}{2} , \quad \sigma_p = \frac{\hbar}{2\omega} \quad \text{and} \quad \sigma_{qp} = 0 . \tag{2.134}$$

Because the aim is not to solve this system in all its generality, let us characterize the integral curves $\alpha(t)$ in the complex plane only for the first case $\Omega = \sqrt{\omega^2 - |\xi|^2}$. So, we may express the solution as

$$\alpha(t) = \left(\frac{1}{2} \left[1 + \frac{\omega}{\Omega} \right] \alpha_0 - \frac{\mathrm{i}\,\dot{\alpha}_0}{\Omega} \right) \mathrm{e}^{\frac{\mathrm{i}}{2}(\Omega - \omega)t} + \left(\frac{1}{2} \left[1 - \frac{\omega}{\Omega} \right] \alpha_0 + \frac{\mathrm{i}\,\dot{\alpha}_0}{\Omega} \right) \mathrm{e}^{-\frac{\mathrm{i}}{2}(\Omega + \omega)t} , \tag{2.135}$$

and considering the polar forms

$$\frac{1}{2} \left[1 + \frac{\omega}{\Omega} \right] \alpha_0 - \frac{\mathrm{i}\,\dot{\alpha}_0}{\Omega} = r_1 e^{\mathrm{i}\varphi_1} \quad \text{and} \quad \frac{1}{2} \left[1 - \frac{\omega}{\Omega} \right] \alpha_0 + \frac{\mathrm{i}\,\dot{\alpha}_0}{\Omega} = r_2 e^{\mathrm{i}\varphi_2} \tag{2.136}$$

one arrives at the final form

$$\alpha(t) = r_1 \mathrm{e}^{\frac{\mathrm{i}}{2}(\Omega - \omega)(t + \frac{2\varphi_1}{\Omega - \omega})} + r_2 \mathrm{e}^{-\frac{\mathrm{i}}{2}(\Omega + \omega)(t - \frac{2\varphi_2}{\Omega + \omega})} . \tag{2.137}$$

On the other hand, in complex calculus it is well-known that every curve defined by a parametrization of the form

$$C(\theta) = (a + b)\mathrm{e}^{\mathrm{i}\,\mu(\theta - \theta_1)} + d\mathrm{e}^{\mathrm{i}\,\nu(\theta - \theta_2)} , \tag{2.138}$$

where θ_1, θ_2, a, b and d are real arbitrary constants while the real constants μ and ν are such that $\mu\,\nu > 0$, are denominated as *epicycloid* if $|d| = |b|$ and *epitrochoid* for any other case. Geometrically this curves are traced by a point attached to a circle of radius b rolling around the outside of a fixed circle of radius a, where the point is at a distance d from the center of the exterior circle. Moreover, the curves defined before will be closed and periodic iff the quotient $\nu/\mu \in \mathbb{Q}$ is a rational number. Comparing the expression of the integral curves in Eq. (2.137) with the curves in Eq. (2.138), it is straightforward that the curves $\alpha(t)$ are epicycloids or epitrochoids in the complex plane with the identification

$$a + b = r_1 , \quad d = r_2 , \quad \theta_1 = -\frac{2\varphi_1}{\Omega - \omega} , \quad \theta_2 = \frac{2\varphi_2}{\Omega + \omega} , \quad \mu = \frac{1}{2}(\Omega - \omega) , \quad \nu = -\frac{1}{2}(\Omega + \omega) . \tag{2.139}$$

Therefore we have a periodic and closed curve iff

$$\frac{\nu}{\mu} = \frac{\omega + \Omega}{\omega - \Omega} \in \mathbb{Q} . \tag{2.140}$$

It is clear that this condition is fulfiled when $\Omega^2 + |\xi|^2 = \omega^2$, i.e. when $(\Omega, |\xi|, \omega)$ are *Pythagorean numbers*. Equivalently, we have a periodic and close curve iff we consider the parameters $(\Omega, |\xi|, \omega)$, which are such that

$$\left(\frac{\Omega}{\omega} \right)^2 + \left(\frac{|\xi|}{\omega} \right)^2 = 1 . \tag{2.141}$$

Some examples of these periodic cases are displayed in Fig. 2.4.

For this system, the auxiliary variables (Q, P) evolve according to the linear system of equations

$$\begin{pmatrix} \dot{Q} \\ \dot{P} \end{pmatrix} = \begin{pmatrix} 2\kappa \left(\beta_{\mathrm{I}} \cos\omega t - \beta_{\mathrm{R}} \sin\omega t \right) & 1 - 2\frac{\kappa}{\omega} \left(\beta_{\mathrm{R}} \cos\omega t + \beta_{\mathrm{I}} \sin\omega t \right) \\ -\omega^2 - 2\kappa\,\omega \left(\beta_{\mathrm{R}} \cos\omega t + \beta_{\mathrm{I}} \sin\omega t \right) & -2\kappa \left(\beta_{\mathrm{I}} \cos\omega t - \beta_{\mathrm{R}} \sin\omega t \right) \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix} , \tag{2.142}$$

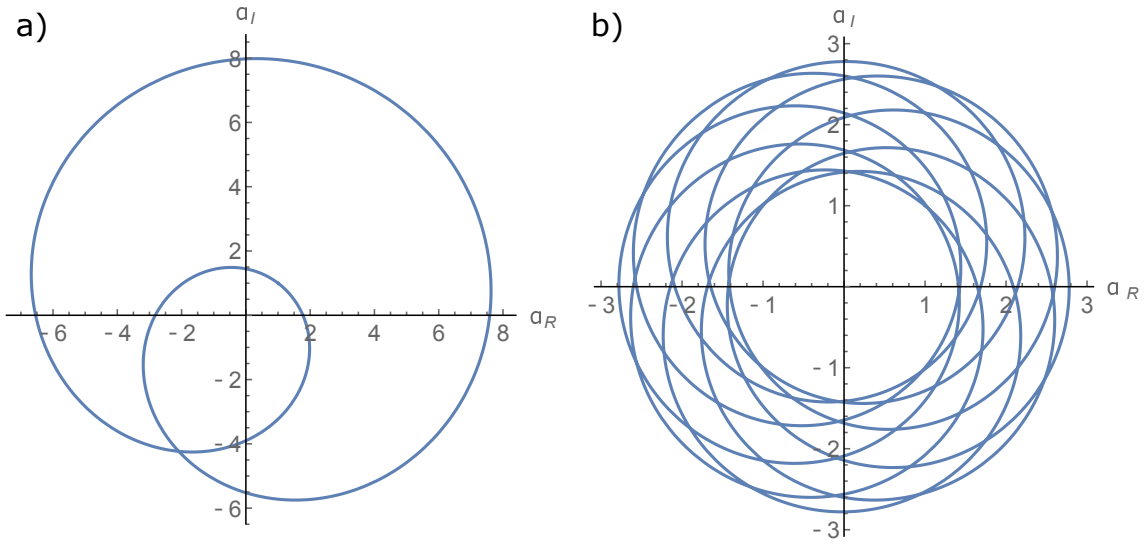


Figure 2.4: Time evolution on the complex plane of solutions $\alpha(t)$ with initial conditions $\alpha_0 = 1 + \mathrm{i}$ and parameters a) $(\omega, |\xi|, \Omega) = (6, 4\sqrt{2}, 2)$ and b) $(\omega, |\xi|, \Omega) = (5, 3, 4)$.

where the solutions are such that they must obey the constraint $\bar{Q}(t)P(t) - Q(t)\bar{P}(t) = 2\mathrm{i}$. One may prove by direct substitution that the solution of the linear system of equations (2.142) is given by

$$\begin{aligned} Q(t) &= a(t)Q_0 + b(t)P_0 \\ &= \frac{Q_0}{2}\left[\left(1-\frac{\omega}{\Omega}\right)\cos\left\{\frac{\Omega-\omega}{2}t\right\} + \left(1+\frac{\omega}{\Omega}\right)\cos\left\{\frac{\Omega+\omega}{2}t\right\} + \frac{4\kappa\varrho}{\Omega}\left(\cos\left\{-\frac{\Omega+\omega}{2}t+\theta\right\}\right.\right. \\ &\quad \left.\left. - \cos\left\{\frac{\Omega-\omega}{2}t+\theta\right\}\right)\right] + \frac{P_0}{2\omega}\left[\left(1+\frac{\omega}{\Omega}\right)\sin\left\{\frac{\Omega+\omega}{2}t\right\} - \left(1-\frac{\omega}{\Omega}\right)\sin\left\{\frac{\Omega-\omega}{2}t\right\}\right. \\ &\quad \left. + \frac{4\kappa\varrho}{\Omega}\left(\sin\left\{-\frac{\Omega+\omega}{2}t+\theta\right\} - \sin\left\{\frac{\Omega-\omega}{2}t+\theta\right\}\right)\right] \end{aligned} \tag{2.143}$$

and similarly

$$\begin{aligned} P(t) &= c(t)Q_0 + d(t)P_0 \\ &= \frac{\omega\, Q_0}{2}\left[\left(1-\frac{\omega}{\Omega}\right)\sin\left\{\frac{\Omega-\omega}{2}t\right\} - \left(1+\frac{\omega}{\Omega}\right)\sin\left\{\frac{\Omega+\omega}{2}t\right\} + \frac{4\kappa\varrho}{\Omega}\left(\sin\left\{-\frac{\Omega+\omega}{2}t+\theta\right\}\right.\right. \\ &\quad \left.\left. - \sin\left\{\frac{\Omega-\omega}{2}t+\theta\right\}\right)\right] + \frac{P_0}{2}\left[\left(1-\frac{\omega}{\Omega}\right)\cos\left\{\frac{\Omega-\omega}{2}t\right\} + \left(1+\frac{\omega}{\Omega}\right)\cos\left\{\frac{\Omega+\omega}{2}t\right\}\right. \\ &\quad \left. + \frac{4\kappa\varrho}{\Omega}\left(\cos\left\{\frac{\Omega-\omega}{2}t+\theta\right\} - \cos\left\{-\frac{\Omega+\omega}{2}t+\theta\right\}\right)\right]. \end{aligned} \tag{2.144}$$

where in the last expression $\beta = \varrho\, \mathrm{e}^{\mathrm{i}\theta}$ and the initial conditions are (Q_0, P_0). Besides, one may prove that these solutions satisfy

$$\bar{Q}(t)P(t) - Q(t)\bar{P}(t) = \bar{Q}_0 P_0 - Q_0\bar{P}_0\,, \tag{2.145}$$

then one has to choose the initial conditions (Q_0, P_0) such that our solutions satisfy the constraint $\bar{Q}(t)P(t) - Q(t)\bar{P}(t) = 2\,\mathrm{i}$.

With the help of the auxiliary variables (Q, P), one is able to obtain the nonlinear dynamical descriptions of the coherent states. To obtain the evolution in the Hyperboloid $\mathbf{H}^2$ considering the connections in Eqs. (2.88) and (2.93), some examples of the evolution on the hyperboloid are displayed in the first

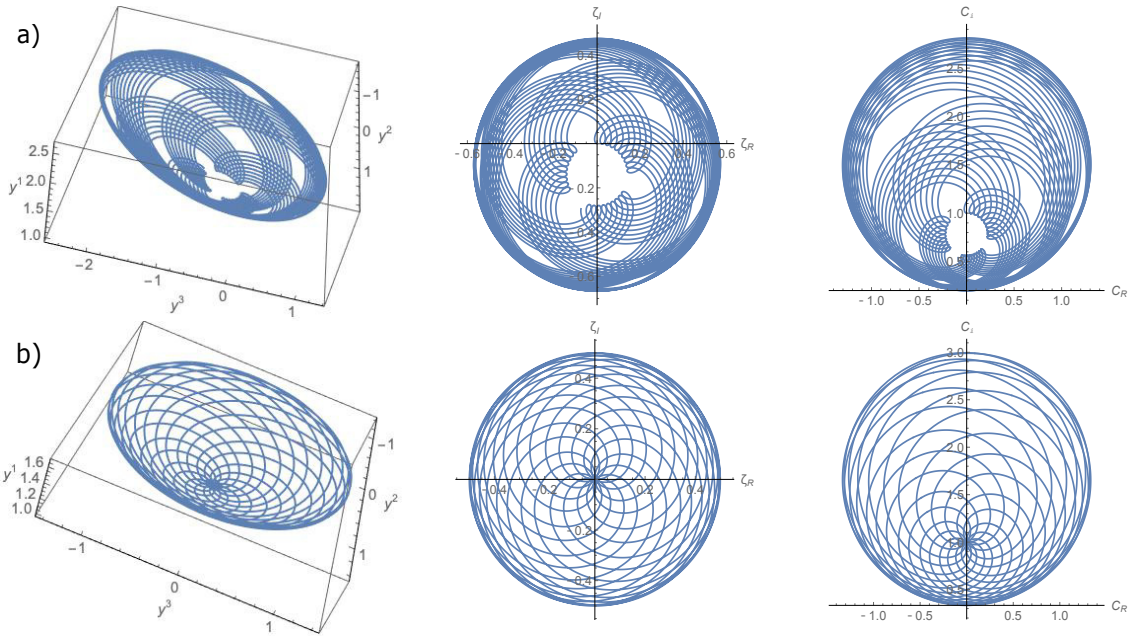


Figure 2.5: Time evolution in the hyperboloid $\mathbf{H}^2$ (first column), in the Poincaré disk $\mathbb{D}^2$ (second column) and in the Siegel upper half $\mathbb{HP}^2$ (third column), respectively. In both cases the initial conditions $(Q_0, P_0) = (1, \mathrm{i})$ are considered, but in a) the parameters are $(\omega, |\xi|, \Omega) = (3/4, 1/2, \sqrt{5}/4)$, while in b) the parameters are $(\omega, |\xi|, \Omega) = (1, 1/2, \sqrt{3}/4)$.

column in Fig. 2.5. In addition to the evolution of the coordinates (y^1, y^2, y^3), its projection in the Poincaré disk is obtained by the transformation in Eq. (2.107). In the second column of Fig. 2.5, one may notice the evolution in the Poincaré disk. Finally, in general the solution of the associated Riccati equation is a Möbius transformation of the form

$$\Phi(S, \mathcal{C}_0) \mapsto \mathcal{C}(t) = \frac{a(t)\mathcal{C}_0 + b(t)}{c(t)\mathcal{C}_0 + d(t)} \quad \text{with} \quad S = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}, \tag{2.146}$$

being $\mathcal{C}_0 = \frac{P_0}{Q_0}$ the initial condition and the time-dependent entries $a(t)$, $b(t)$, $c(t)$ and $d(t)$ of the symplectic matrix S are given in Eqs. (2.143) and (2.144). Some examples of the solutions of the Riccati equation are displayed in the third column of Fig 2.5.

2.4.2 Parametric oscillator with frequency $\omega = \frac{1}{at+b}$

Another interesting example with analytic solutions is the parametric oscillator

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2(t)\,\hat{q}^2 \tag{2.147}$$

with time dependent frequency

$$\omega(t) = \frac{1}{a\,t + b}\,, \tag{2.148}$$

with a and b arbitrary real constants. Then, for this system there are three different cases for the solutions of the classical equations of motion (2.54). Namely, for $a > 2$ the solutions have the form

$$\begin{aligned}
\langle\hat{q}\rangle(t) &= a_1(t)\langle\hat{q}\rangle_0 + b_1(t)\langle\hat{p}\rangle_0 \\
&= \frac{a\,\langle\hat{q}\rangle_0}{\sqrt{a^2-4}}\left[\lambda_1 b^{\lambda_1-1}(at+b)^{\lambda_2} - \lambda_2 b^{\lambda_2-1}(at+b)^{\lambda_1}\right] + \frac{\langle\hat{p}\rangle_0}{\sqrt{a^2-4}}\left[b^{\lambda_2}(at+b)^{\lambda_1} - b^{\lambda_1}(at+b)^{\lambda_2}\right] \\
\langle\hat{p}\rangle(t) &= c_1(t)\langle\hat{q}\rangle_0 + d_1(t)\langle\hat{p}\rangle_0 \\
&= \frac{a^2\lambda_1\lambda_2\langle\hat{q}\rangle_0}{\sqrt{a^2-4}}\left[b^{\lambda_1-1}(at+b)^{\lambda_2-1} - b^{\lambda_2-1}(at+b)^{\lambda_1-1}\right] + \frac{a\langle\hat{p}\rangle_0}{\sqrt{a^2-4}}\left[\lambda_1 b^{\lambda_2}(at+b)^{\lambda_1-1} - \lambda_2 b^{\lambda_1}(at+b)^{\lambda_2-1}\right],
\end{aligned} \tag{2.149}$$

with parameters

$$\lambda_1 = \frac{a+\sqrt{a^2-4}}{2a} \quad \text{and} \quad \lambda_2 = \frac{a-\sqrt{a^2-4}}{2a}. \tag{2.150}$$

For the case $a = 2$ one has

$$\begin{aligned}
\langle\hat{q}\rangle(t) &= a_2(t)\langle\hat{q}\rangle_0 + b_2(t)\langle\hat{p}\rangle_0 \\
&= \sqrt{\frac{at+b}{b}}\left[1 + \ln|b|^{1/2} - \ln|at+b|^{1/2}\right]\langle\hat{q}\rangle_0 + \frac{\sqrt{b(at+b)}}{a}\left[\ln|at+b| - \ln|b|\right]\langle\hat{p}\rangle_0 \\
\langle\hat{p}\rangle(t) &= c_2(t)\langle\hat{q}\rangle_0 + d_2(t)\langle\hat{p}\rangle_0 \\
&= \frac{a}{2}\frac{\langle\hat{q}\rangle_0}{\sqrt{b(at+b)}}\left[\ln|b|^{1/2} - \ln|at+b|^{1/2}\right] + \sqrt{\frac{b}{at+b}}\left[\ln|at+b|^{1/2} + 1 - \ln|b|^{1/2}\right]\langle\hat{p}\rangle_0 .
\end{aligned} \tag{2.151}$$

Finally, for the case $a < 2$ one has the set of solutions

$$\begin{aligned}
\langle\hat{q}\rangle(t) &= a_3(t)\langle\hat{q}\rangle_0 + b_3(t)\langle\hat{p}\rangle_0 \\
&= \frac{a\,\langle\hat{q}\rangle_0}{\mathrm{i}\sqrt{4-a^2}}\left[\tilde{\lambda}_1 b^{\tilde{\lambda}_1-1}(at+b)^{\tilde{\lambda}_2} - \tilde{\lambda}_2 b^{\tilde{\lambda}_2-1}(at+b)^{\tilde{\lambda}_1}\right] + \frac{\langle\hat{p}\rangle_0}{\mathrm{i}\sqrt{4-a^2}}\left[b^{\tilde{\lambda}_2}(at+b)^{\tilde{\lambda}_1} - b^{\tilde{\lambda}_1}(at+b)^{\tilde{\lambda}_2}\right] \\
\langle\hat{p}\rangle(t) &= c_3(t)\langle\hat{q}\rangle_0 + d_3(t)\langle\hat{p}\rangle_0 \\
&= \frac{a^2\tilde{\lambda}_1\tilde{\lambda}_2\langle\hat{q}\rangle_0}{\mathrm{i}\sqrt{4-a^2}}\left[b^{\tilde{\lambda}_1-1}(at+b)^{\tilde{\lambda}_2-1} - b^{\tilde{\lambda}_2-1}(at+b)^{\tilde{\lambda}_1-1}\right] + \frac{a\langle\hat{p}\rangle_0}{\mathrm{i}\sqrt{4-a^2}}\left[\tilde{\lambda}_1 b^{\tilde{\lambda}_2}(at+b)^{\tilde{\lambda}_1-1} - \tilde{\lambda}_2 b^{\tilde{\lambda}_1}(at+b)^{\tilde{\lambda}_2-1}\right],
\end{aligned} \tag{2.152}$$

where now the parameters are

$$\tilde{\lambda}_1 = \frac{a+\mathrm{i}\sqrt{4-a^2}}{2a} \quad \text{and} \quad \tilde{\lambda}_2 = \frac{a-\mathrm{i}\sqrt{4-a^2}}{2a}. \tag{2.153}$$

Examples of the behaviour of these solutions in the phase space $(\langle\hat{q}\rangle, \langle\hat{p}\rangle)$ are displayed in Fig. 2.6, where Fig. 2.6a corresponds to the case $a < 2$, Fig. 2.6b corresponds to $a = 2$ and Fig. 2.6c corresponds to $a > 2$.

Now, as has been shown in this Chapter, to obtain a complete description of the dynamics it is also necessary to obtain the solutions of the set of equations in (2.58) associated with our system of interest. The solution of the equations in (2.58) for the auxiliar variables (Q, P) are simply

$$\begin{pmatrix} Q(t) \\ \\ P(t) \end{pmatrix} = \begin{pmatrix} a_i(t) & b_i(t) \\ \\ c_i(t) & d_i(t) \end{pmatrix} \begin{pmatrix} Q_0 \\ \\ P_0 \end{pmatrix}, \tag{2.154}$$

where the set of time-dependent functions $\{a_i(t), b_i(t), c_i(t), d_i(t)\}_{i=1,2,3}$ are defined in the solutions (2.149), (2.151) and (2.152), depending on the case that one wants to consider and the initial conditons (Q_0, P_0) must be chosen such that the constraint

$$\bar{Q}(t)P(t) - Q(t)\bar{P}(t) = \bar{Q}_0 P_0 - Q_0\bar{P}_0 = 2\mathrm{i}. \tag{2.155}$$

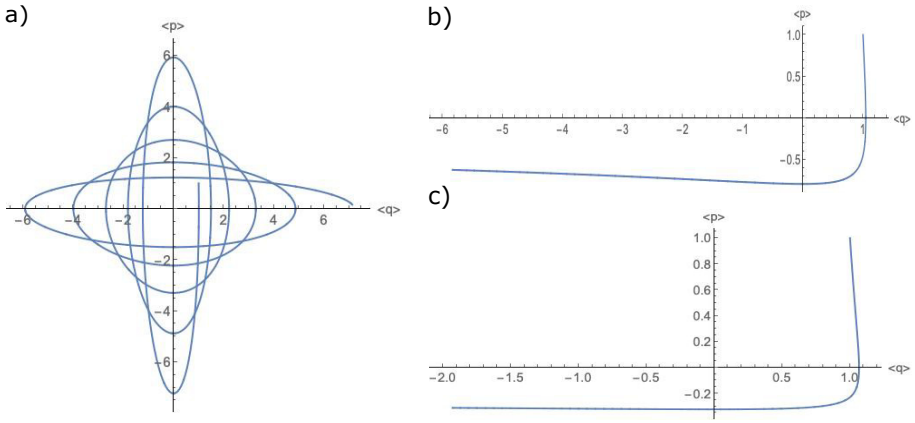


Figure 2.6: Time evolution in the phase space $(\langle\hat{q}\rangle, \langle\hat{p}\rangle)$ with initial conditions $(\langle\hat{q}\rangle_0, \langle\hat{p}\rangle_0) = (1, 1)$ with parameters: a) $(a, b) = (1/8, 1/8)$, b) $(a, b) = (2, 1/4)$ and c) $(a, b) = (3, 1/4)$.

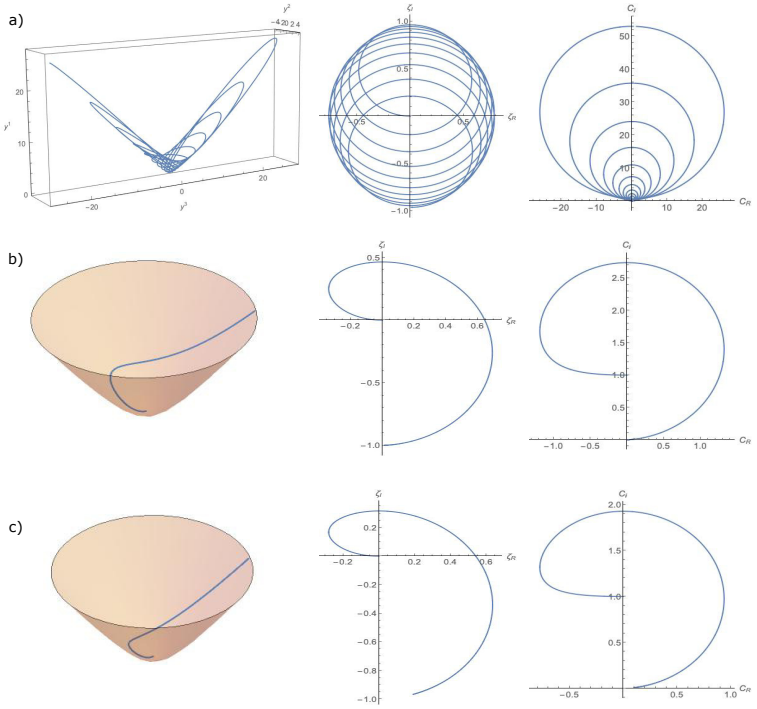


Figure 2.7: Time evolution in the hyperboloid $\mathbf{H}^2$ (first column), in the Poincaré disk $\mathbb{D}^2$(second column) and in the Siegel upper half $\mathbb{HP}^2$ (third column), respectively. In all the cases the initial conditions $(Q_0, P_0) = (1, \mathrm{i})$ are considered, with the parameters: a) $(a, b) = (1/8, 1/8)$, b) $(a, b) = (2, 1/4)$ and c) $(a, b) = (3, 1/4)$.

is satisfied. Alternatively, one may look for the solution of the nonlinear Riccati equation (2.121) associated with the system. This solution corresponds to the Möbius transformation

$$\Phi(S, \mathcal{C}_0) \mapsto C(t) = \frac{a_i(t)\mathcal{C}_0 + b_i(t)}{c_i(t)\mathcal{C}_0 + d_i(t)} \quad \text{with} \quad S = \begin{pmatrix} a_i(t) & b_i(t) \\ c_i(t) & d_i(t) \end{pmatrix}, \tag{2.156}$$

where $\mathcal{C}_0$ is the initial condition. From this example it is clear that it is sufficient to know the solutions of the classical equation of motion to obtain the solutions (Q, P) or $\mathcal{C}$, which contain implicitly the information of the uncertainties of position and momentum variables as well as their correlation.

Some examples of the evolution in the Hyperboloid, connected with the squeezing phenomena, the evolution in the Poincare Disk and the evolution in the Siegel half planes are displayed in Fig. 2.7. More specifically, in Fig. 2.7a the evolution in the different spaces it is ploted for the case $a < 2$, in Fig. 2.7b for the case $a = 2$, whereas in Fig. 2.7c for the case $a > 2$.

Chapter 3

Contact Hamiltonian Mechanics and Dissipation

So far, in this thesis we have consider classical and quantum dynamics without introducing the concept of dissipation. Dissipation is playing an increasing role in the description of dynamical systems. This is in part due to the interest in describing open quantum systems in relation with many physical applications like quantum computing, quantum information and quantum thermodynamics. While in classical physics the "coupling" with the environment may be idealized to be negligible in the consideration of fundamental aspects, in the quantum realm the situation is more complex because of the probabilistic-statistical interpretation which gives rise to indetermination relations and other inequalities.

In the Newtonian picture, the interaction of the system with an environment is usually expressed by means of forces which in general are not *conservative*. What is called *dissipative forces* is usually an effective way to take into account the coupling with the environment without considerig additional degrees of freedom with respect to those possessed by the system. For example, the introduction of explicitly *time-dependent forces* is a way to take into account the effect of the environment and possible interactions with it [38].

In this classical setting, dissipation is usually understood as the fact that energy is not conserved along a given dynamical trajectory. However, if one wants to declare a system to be dissipative, first one should say what is actually being dissipated. Indeed, one may deal with physical systems for which it might make sense to say that the system is *dissipating* mass, angular momentum or *probability*. For instance, in the Quantum Mechanics context the phase-damping evolution for a finite-level quantum system may be thought of as a dynamical evolution which is dissipating the off-diagonal terms of the quantum states with respect to a fixed orthonormal basis depending on the dynamical evolution itself [58].

On the other hand, given a system whose dynamics is described by means of a vector field on a suitable carrier space, one must realize that it does not make sense to say that the system is *conservative* or *dissipative per se* because dissipation is a relational concept. For instance, in the Newtonian setting dynamical vector fields Γ are second-order vector fields on the tangent bundle TQ of some configuration space Q, one may be interested in systems dissipating for example energy, and in order to analyze such dissipation, first of all one must define and fix what we mean by *energy of the system*. Therefore, a major question is whether it is possible to characterize a dynamics as *conservative* or *dissipative*.

For this purpose, let us consider a Newtonian dynamical system, i.e. a dynamical system for which the dynamics is expressed by means of a second-order vector field on the tangent bundle TQ of some configuration space Q. Roughly speaking, a second order vector field means that there is a coordinate system $(q^k, \dot{q}^k)$ on TQ such that the equation of motion has the form

$$\ddot{q}^k = f^k(\mathbf{q}, \dot{\mathbf{q}})\,. \tag{3.1}$$

Given a Newtonian dynamical system, it makes sense to ask if the "forces" generating the motion are derived from a potential because in this way we would find that the kinetic energy plus the potential one would be conserved. A covariant way to look for a potential is the search for a Lagrangian description.

Thus, in order to address the characterization of a given second order dynamical system as *conservative* or *dissipative*, we may proceed as follows. First of all, one may look for a possible Lagrangian description of the given second order vector field, i.e. solve the so-called *inverse problem in the calculus of variations* [31]. It may happen one finds none, one or many solutions for this problem, that is, either one does not find a Lagrangian description, or one finds a "unique"[1] Lagrangian description, or one finds different alternative Lagrangian descriptions. Then, if a Lagrangian description exists, in the time independent case we would say that the system preserves the Lagrangian energy function

$$\mathcal{E}_{\mathfrak{L}} = \dot{q}_j \frac{\partial \mathfrak{L}}{\partial \dot{q}_j} - \mathfrak{L}\,, \tag{3.2}$$

associated with the Lagrangian $\mathfrak{L}$ for the second order vector field Γ. There will be an associated "Lagrangian energy" for each alternative Lagrangian description of the second order vector field. However, the actual Lagrangian energy $\mathcal{E}_{\mathfrak{L}}$ need not to coincide with what one may want to call the "physical energy" E of the system, therefore, it is possible to qualify the system to be dissipative even if it admits a description by means of a Lagrangian.

To clarify what we mean let us give an example. Consider a second order differential equation on $TQ = \mathbb{R} \times \mathbb{R}$ with a friction force proportional to the velocity with a friction coefficient γ, i.e.

$$\ddot{q} + \gamma\, \dot{q} \;=\; 0\,. \tag{3.3}$$

A possible (local) Lagrangian for this system with its corresponding Lagrangian energy are

$$\mathfrak{L} = \dot{q}\,(\ln \dot{q}) - \gamma\, q \quad \text{and} \quad \mathcal{E}_{\mathfrak{L}} = \dot{q} + \gamma\, q\,, \tag{3.4}$$

respectively. Then, the system allows for a Lagrangian description, but the Lagrangian energy does not coincide with the mechanical energy

$$E = \frac{m}{2}\dot{q}^2\,, \tag{3.5}$$

and one may look at this Lagrangian dynamical system as being dissipative with respect to the mechanical energy E, although it preserves its Lagrangian energy, i.e. $\dot{\mathcal{E}}_{\mathfrak{L}} = 0$. Namely, the rate of dissipation of the mechanical energy is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\,\gamma\, \dot{q}^2\,. \tag{3.6}$$

When the system does not admit a Lagrangian description, it is necessary to develop new strategies in order to characterize dissipation. In order to explain such a strategy, let us consider the physically relevant situation represented by linear dynamical systems.

In several instances, either because of approximations or because of specific requirements, the equations of motion may be given in a linear form

$$m^{jk}\ddot{q}_k + \gamma^{jk}\dot{q}_k + \omega^{jk} q_k = 0 \tag{3.7}$$

where $||m^{jk}||$, $||\gamma^{jk}||$ and $||\omega^{jk}||$ are numerical matrices. Notice that this equation is in general an implicit differential equation because the matrix $||m^{jk}||$ could be degenerate. When $||m^{jk}||$ is non-degenerate, the differential equation defines a second order vector field. In this case, there is a necessary and sufficient condition for this vector field to admit a Hamiltonian (or Lagrangian) description in terms of a constant Poisson structure on $T\mathbb{R}^n$ [65, 6].

Consider the linear vector field Γ associated with the equations of motion (3.7) on the linear manifold $T\mathbb{R}^n$, with representative matrix $||G^j_k||$ defined by

$$\Gamma \;=\; G^j_k\, \xi^k\, \frac{\partial}{\partial \xi^j}\,, \tag{3.8}$$

[1] As it is well-known, we may always add a global time derivative to every Lagrangian, or multiply the Lagrangian by a real number without altering the explicit form for the equation of motions, hence, uniqueness here has always to be understood modulo the addition of a global time derivative or multiplication by a real number.

where $\{\xi^j\}_{j=1,\ldots,2n}$ is a collective Cartesian coordinates system on $T\mathbb{R}^n$ with the first n coordinates representing position variables. The system may be given a description by means of a constant Poisson structure Λ on $T\mathbb{R}^n$, represented in the coordinate system $\{\xi^j\}_{j=1,\ldots,2n}$ by an antisymmetric numerical matrix $||\Lambda^{jk}||$, and a quadratic Hamiltonian function:

$$H = \frac{1}{2} H_{jk} \xi^j \xi^k \,, \tag{3.9}$$

if and only if the representative matrix $||G^j_k||$ is traceless with all its odd powers, namely,

$$\mathrm{Tr}\{G^{2k+1}\} = 0 \quad \text{for all } k \,. \tag{3.10}$$

It is also posible to see that this condition also implies that the representative matrix G may be expressed as the product of a skew-symmetric matrix times a symmetric one, i.e. $G^j_k = -\Lambda^{jl} \cdot H_{lk}$. Even powers G^{2k} will act as non-canonical symmetries of the dynamics which take from one Hamiltonian description to an alternative one.

For instance, let us consider the following system of two coupled differential equations representing coupled oscillations with different frequencies ω_k, different damping coefficients γ_k and coupling constants κ and δ,

$$\begin{aligned} &\ddot{q}_1 + \gamma_1 \dot{q}_1 + \omega_1^2 q_1 + \kappa q_2 + \delta \dot{q}_2 = 0 \\ &\ddot{q}_2 + \gamma_2 \dot{q}_2 + \omega_2^2 q_2 + \kappa q_1 + \delta \dot{q}_1 = 0 \,. \end{aligned} \tag{3.11}$$

Then, the representative matrix is

$$G = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega_1^2 & -\kappa & -\gamma_1 & -\delta \\ -\kappa & -\omega_2^2 & -\delta & -\gamma_2 \end{pmatrix} . \tag{3.12}$$

This matrix cannot be expressed as the product of a skew-symmetric matrix times a symmetric one because its trace is different from zero, and hence according to the before presented result this system does not allow for a Hamiltonian description. Actually, in full generality, one may prove (what will be shown in the next section) that this system does not allow for any Lagrangian description even if one allows the Lagrangian to be any function of position and velocity. If the system does not admit a Lagrangian description, one may manipulate the system of differential equations in (3.11) in order to obtain a particularly useful decomposition of the dynamical system. For instance, the representative matrix of the system given in Eq. (3.12) can be decomposed as

$$G^j_k = A^j_k + D^j_k \,, \tag{3.13}$$

where A^j_k is a traceless matrix. Clearly, this decomposition is arbitrary, but once a choice has been made, it is possible to think of the dynamical vector field as the sum of a *reference* or *comparison dynamics* plus a *perturbation term*. Then, one selects the *reference dynamics* in such a way that it admits at least one Lagrangian description, and the perturbation term turns our system into a *dissipative* system which is dissipating the Lagrangian energy of the comparison system. For the system described by the flow in Eq. (3.11), the linear vector field Γ may be decomposed as

$$\Gamma \;=\; A^j_k \, \xi^k \, \frac{\partial}{\partial \xi^j} + D^j_k \, \xi^k \, \frac{\partial}{\partial \xi^j} \,, \tag{3.14}$$

where the first term on the right-hand side plays the role of the comparison dynamics admitting a Hamiltonian description, while the second term may be thought of as the perturbation term responsible for the dissipation of the Hamiltonian energy function associated with the comparison dynamics. This analysis can be extended to the general linear systems in Eq. (3.7); however, it is important to remark that the decomposition is not going to be unique and thus we will have many alternative energy functions which are going to be dissipated by the same perturbation.

The established *decomposition principle* has a natural meaning in the contact formalism, where a dissipative system may be described in terms of a *contact Hamiltonian formalism* [33, 34, 32]. However, before the description of dissipative systems by a contact structure is studied, some notions on Lagrangian symplectic forms and contact manifolds in geometry are introduced in the next sections.

3.1 A Digression to Lagrangian symplectic forms

Before introducing the contact Lagrangian theory it is necessary first to review some aspects of the Lagrangian formalism and its connection with symplectic forms to generalize these concepts in the next section. So, it is well-known that it is easy to construct one-forms out of functions on any manifolds by means of the exterior derivative map $\mathrm{d} : \mathfrak{F}(M) \to \Omega^1(M) : \mathcal{F} \mapsto \mathrm{d}\,\mathcal{F}$. However, this construction cannot be applied twice, because $\mathrm{d}^2 = 0$, and then one needs to found another way to construct two-forms. For the purpose to introduce a two-form in TQ it is convenient to consider the C^∞ mapping

$$\mathrm{d}_{\mathcal{S}} : \Omega^k(TQ) \to \Omega^{k+1}(TQ)\,, \tag{3.15}$$

which is called *the vertical derivative*. In all generality the vertical derivative is discussed in references [4, 6], here we are just interested in its application on functions. The map $\mathrm{d}_{\mathcal{S}}$ is the exterior derivative associated with the $(1,1)$-tensor $\mathcal{S} = \frac{\partial}{\partial \dot{q}^k} \otimes \mathrm{d}\,q^k$ and locally

$$\mathrm{d}_{\mathcal{S}}\mathcal{F} \coloneqq \mathcal{S}[\mathrm{d}\mathcal{F}] = \frac{\partial \mathcal{F}}{\partial \dot{q}^k}\mathrm{d}q^k\,. \tag{3.16}$$

Then, the exterior derivative d and the vertical derivative $\mathrm{d}_{\mathcal{S}}$ are used to defined one-forms and two-forms in terms of a function $\mathcal{F}$ throught

$$\theta_{\mathcal{F}} = \mathrm{d}_{\mathcal{S}}\mathcal{F} \quad \text{and} \quad \omega_{\mathcal{F}} = -\mathrm{d}\theta_{\mathcal{F}}\,, \tag{3.17}$$

respectively. In local coordinates the two-form is given by

$$\omega_{\mathcal{F}} = \frac{\partial^2 \mathcal{F}}{\partial \dot{q}^k \partial \dot{q}^j}\mathrm{d}q^k \wedge \mathrm{d}\dot{q}^j + \frac{\partial^2 \mathcal{F}}{\partial \dot{q}^k \partial q^j}\mathrm{d}q^k \wedge \mathrm{d}q^j\,. \tag{3.18}$$

One may prove straightforwardly that the two-form $\omega_{\mathcal{F}}$ is *closed*, recalling that a close form is a differential form whose exterior derivative is zero. To see that $\omega_{\mathfrak{L}}$ is nondegenerate let us consider Γ as a vector field in TQ, having a local expression

$$\Gamma = \Gamma_{q^k}\frac{\partial}{\partial q^k} + \Gamma_{\dot{q}^k}\frac{\partial}{\partial \dot{q}^k}\,, \tag{3.19}$$

then the interior product of Γ on the 2-form $\omega_{\mathcal{F}}$ is

$$i_\Gamma \omega_{\mathcal{F}} = \frac{\partial^2 \mathcal{F}}{\partial \dot{q}^k \partial \dot{q}^j}\,\Gamma_{q^k}\,\mathrm{d}\dot{q}^j + \left[\left(\frac{\partial^2 \mathcal{F}}{\partial \dot{q}^k \partial q^j} - \frac{\partial^2 \mathcal{F}}{\partial \dot{q}^j \partial q^k}\right)\Gamma_{q^k} - \frac{\partial^2 \mathcal{F}}{\partial \dot{q}^j \partial \dot{q}^k}\Gamma_{\dot{q}^k}\right]\mathrm{d}q^j\,. \tag{3.20}$$

Hence $\omega_{\mathcal{F}}$ will be nondegenerate (i.e., $i_\Gamma \omega_{\mathcal{F}} = 0$ iff $\Gamma = 0$) iff the determinant of the *Hessian matrix* is different from zero,

$$\left|\frac{\partial^2 \mathcal{F}}{\partial \dot{q}^k \partial \dot{q}^j}\right| \neq 0\,, \tag{3.21}$$

i.e., iff $\mathcal{F}$ is a *regular* function. Therefore, one has constructed a closed and non degenerate two-form for the tangent bundle TQ, i.e. $\omega_{\mathcal{F}}$ is a symplectic form on TQ. In addition, it is said that a dynamics $\Gamma \in \mathfrak{X}(TQ)$ is $\omega_{\mathcal{F}}$–Hamiltonian iff there is a function $\mathcal{E}_{\mathcal{F}} \in \mathfrak{F}(TQ)$ such that

$$i_\Gamma\, \omega_{\mathcal{F}} = \mathrm{d}\,\mathcal{E}_{\mathcal{F}}\,. \tag{3.22}$$

This definition introduces a Hamilonian dynamics on TQ.

Let us now connect the Hamiltonian formalism on TQ with the well-known Lagrangian formalism. Employing *Cartan's identity*: $\pounds_\Gamma \theta = \mathrm{d}i_\Gamma\theta + i_\Gamma \mathrm{d}\theta$ into definition (3.22), one has that

$$\mathrm{d}\mathcal{E}_{\mathcal{F}} = i_\Gamma \omega_{\mathcal{F}} = -i_\Gamma \mathrm{d}\theta_{\mathcal{F}} = -\pounds_\Gamma \theta_{\mathcal{F}} + \mathrm{d}i_\Gamma \theta_{\mathcal{F}}\,, \tag{3.23}$$

which is equivalent to the definitions

$$\pounds_\Gamma \theta_{\mathcal{F}} = \mathrm{d}\mathfrak{L}\,, \quad \text{where} \quad \mathfrak{L} = i_\Gamma \theta_{\mathcal{F}} - \mathcal{E}_{\mathcal{F}}\,, \tag{3.24}$$

where the *Lagrangian* $\mathfrak{L}$ and the energy $\mathcal{E}_{\mathcal{F}}$ depend on the function $\mathcal{F}$ and the dynamics Γ. Therefore it has been shown, under the condition that there is a $\theta_{\mathcal{F}}$ such that $\omega_{\mathcal{F}} = -\mathrm{d}\theta_{\mathcal{F}}$, that the equation (3.22) is equivalent to (3.24). Then if Eq. (3.22) is the equation for the Hamiltonian formalism, Eq. (3.24) is called the equation for the Lagrangian formalism [4].

To obtain the conventional Lagrangian formalism, where the vector field is recover form a single function, the Lagrangian $\mathfrak{L}$, one considers the special case in which Γ is a second order vector field, i.e. in local coordinates the vector field has the form

$$\Gamma = \dot{q}^k \frac{\partial}{\partial q^k} + f^k(\mathbf{q}, \dot{\mathbf{q}}) \frac{\partial}{\partial \dot{q}^k}\,, \tag{3.25}$$

which is associated with the second order differential (Newtonian) equation $\ddot{q}^k = f^k(q^k, \dot{q}^k)$. Then for these vector fields the Lagrangian function is

$$\mathfrak{L} = \dot{q}^k \frac{\partial \mathcal{F}}{\partial \dot{q}^k} - \mathcal{E}_{\mathcal{F}}\,, \tag{3.26}$$

which is independent of Γ. In fact for the case of the second order vector field it is possible to prove that $\theta_{\mathcal{F}} = \theta_{\mathfrak{L}}$. To show this, according to definitions (3.17) and (3.26)

$$\theta_{\mathfrak{L}} = \frac{\partial \mathfrak{L}}{\partial \dot{q}^k} \mathrm{d}q^k = \left[\frac{\partial^2 \mathcal{F}}{\partial \dot{q}^k \partial q^j} \mathrm{d}\dot{q}^j + \frac{\partial \mathcal{F}}{\partial \dot{q}^k} - \frac{\partial \mathcal{E}_{\mathfrak{L}}}{\partial \dot{q}^k} \right] \mathrm{d}q^k\,, \tag{3.27}$$

but from the definition (3.22) and the expression (3.20) in coordinates one has that

$$\frac{\partial \mathcal{E}_{\mathfrak{L}}}{\partial \dot{q}^k} = \frac{\partial^2 \mathcal{F}}{\partial \dot{q}^k \partial q^j} \mathrm{d}\dot{q}^j\,, \tag{3.28}$$

and then

$$\theta_{\mathfrak{L}} = \frac{\partial \mathcal{F}}{\partial \dot{q}^k} \mathrm{d}q^k = \theta_{\mathcal{F}}\,. \tag{3.29}$$

Therefore, for a second order dynamics the Lagrangian function alone can be used to recover the dynamical vector field and hence the Eq. (3.24) for the Lagrangian formalism becomes

$$\pounds_{\Gamma} \theta_{\mathfrak{L}} = \mathrm{d}\mathfrak{L}\,. \tag{3.30}$$

Besides Γ is $\omega_{\mathfrak{L}}$-Hamiltonian on the symplectic manifold TQ; then, the Eq. (3.22) for the Hamiltonian formalism now has the form

$$i_{\Gamma} \omega_{\mathfrak{L}} = \mathrm{d}\mathcal{E}_{\mathfrak{L}}\,, \tag{3.31}$$

where the Lagrangian $\mathfrak{L}$ and the *Lagrangian energy* $\mathcal{E}_{\mathfrak{L}}$ are related by

$$\mathcal{E}_{\mathfrak{L}} = i_{\Gamma} \theta_{\mathfrak{L}} - \mathfrak{L}\,. \tag{3.32}$$

Therefore, the Lagrangian function alone may be used to recover the dynamical vector field Γ, as long as Γ is both second order and $\omega_{\mathfrak{L}}$-Hamiltonian.

In local coordinates Eq. (3.30) leads to the *Euler–Lagrange* equations, because the part in $\mathrm{d}\dot{q}^j$ is an identity, and the $\mathrm{d}q^j$ part yields

$$\frac{\partial \mathfrak{L}}{\partial q^j} = \frac{\partial^2 \mathfrak{L}}{\partial \dot{q}^j \partial q^k} \dot{q}^k + \frac{\partial^2 \mathfrak{L}}{\partial \dot{q}^j \partial \dot{q}^k} f^k = \frac{\partial^2 \mathfrak{L}}{\partial \dot{q}^j \partial q^k} \dot{q}^k + \frac{\partial^2 \mathfrak{L}}{\partial \dot{q}^j \partial \dot{q}^k} \frac{\mathrm{d}\dot{q}^k}{\mathrm{d}t}\,, \tag{3.33}$$

and then one has the usual Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathfrak{L}}{\partial \dot{q}^j} - \frac{\partial \mathfrak{L}}{\partial q^j} = 0\,, \tag{3.34}$$

which in general is an *implicit* differential equation on the tangent bundle.

Notice that there are two conditions for arriving at Eq. (3.34): i) Γ is a second order dynamics and ii) Γ has to be Hamiltonian with respect to the symplectic form $\omega_{\mathfrak{L}}$ obtained from a Lagrangian function. This means that the existence of the Lagrangian function has been always assumed. However, one may formulate the problem in reverse, i.e., given a second order dynamics Γ, is it possible to find a Lagrangian function such that Eq. (3.30) (or equivalently Eq. (3.31)) holds? In other words, one may look for a solution of Eq. (3.33), thought of as a partial differential equation for $\mathfrak{L}$ when the forces $f^k(\mathbf{q},\dot{\mathbf{q}})$ are given, with the constraint that the function $\mathfrak{L}$ being a regular function, namely

$$\left|\frac{\partial^2\mathfrak{L}}{\partial\dot{q}^k\partial\dot{q}^j}\right| \neq 0\,. \tag{3.35}$$

This problem, i.e. to discuss the possibility of a Lagrangian description for a given set of Newtonian equations is called the *inverse problem in the calculus of variations*, which has been quite extensively studied both for finite dimensional systems and fields. Here we will not deal with this problem in complete generality, for this we refer to existing literature, see for example [31] and references therein.

Then, to formulate the inverse problem there is an important result obtained from the Lagrangian symplectic forms developed before. In particular, it should be clear that instead of seeking for the Lagrangian, one may look for the Lagrangian symplectic form $\omega_{\mathfrak{L}}$, because they are univocally connected. Hence, if $\omega_{\mathfrak{L}}$ is unique also the Lagrangian is *unique*, or if one can show that there are many Lagrangian symplectic forms hence there are many alternative Lagrangians. All the before mention results are contained in the following theorem on Lagrangian symplectic forms [31].

A two-form ω on TQ is derivable from a Lagrangian function (at least at the local level) if and only if

i) $\left\langle \omega \,|\, \frac{\partial}{\partial\dot{q}^i}\wedge\frac{\partial}{\partial\dot{q}^j}\right\rangle = 0$, for all i,j ,

ii) $\pounds_\Gamma\,\omega = 0$, for some second order vector field Γ ,

iii) $\mathrm{d}\,\omega = 0$,

where $\langle\,\cdot\,|\,\cdot\,\rangle$ stands for the fully-contacted multiplication of ω with the bivector $\frac{\partial}{\partial\dot{q}^i}\wedge\frac{\partial}{\partial\dot{q}^j}$, such that

$$\omega(X,Y) = \frac{1}{2}\langle\,\omega\,|X\wedge Y\rangle\,,\quad \text{where}\quad X,Y\in\mathfrak{X}(TQ)\,. \tag{3.36}$$

Taking successive Lie derivatives of i) with respect to Γ, and taking into account ii), one obtains after iterating k times, the set of constrains

$$\left\langle \omega\middle|(\pounds_\Gamma)^k\left(\frac{\partial}{\partial\dot{q}^i}\wedge\frac{\partial}{\partial\dot{q}^j}\right)\right\rangle = 0\,. \tag{3.37}$$

If n is the dimension of the configuration space Q, the dimension of the space of bivectors is the same as that of 2-forms, i.e. $n(2n-1)$, and hence this is the maximal number of independent constrains that can be generated. It is important to observe that if for certain k the $k+1$-th bivector depends on the previous ones, then the same will be true for the subsequent ones, i.e. k will be the maximum of independent constrains. In particular, if $k = n(2n-1)$, the bivector generated for a basis, which implies $\omega = 0$, and the inverse problem has no solution [6]. As a matter of fact, the codimention of the set of bivectors will be the maximum number of possible alternative Lagrangians [6].

To better visualize the situation let us consider the system of two coupled oscillators in $T\mathbb{R}^2$, i.e. $n=2$, with coordinates $(q^1,q^2,\dot{q}^1,\dot{q}^2)$ and second order vector field

$$\Gamma = \dot{q}^1\frac{\partial}{\partial q^1} + \dot{q}^2\frac{\partial}{\partial q^2} - (\gamma_1\,\dot{q}^1 + \omega_1^2\,q^1 + \kappa\,q^2 + \delta\,\dot{q}^2)\frac{\partial}{\partial\dot{q}^1} - (\gamma_2\,\dot{q}^2 + \omega_2^2\,q^2 + \kappa\,q^1 + \delta\,\dot{q}^1)\frac{\partial}{\partial\dot{q}^2}\,, \tag{3.38}$$

whose integral curves are given by the solutions of the system of equations in (3.11). Let us now take the successive Lie derivatives of the bivector with respect this vector field. However, at each step one will obtain a bivector which gives zero upon contraction with ω. Then, two bivectors are *equivalent*, and

we will use the symbol $\simeq$ for such equivalent relation, if they differ by a vector which gives zero upon contraction. Under this consideration one has that

$$\pounds_\Gamma\left(\frac{\partial}{\partial\dot{q}^i}\wedge\frac{\partial}{\partial\dot{q}^j}\right)\simeq\frac{\partial}{\partial q^1}\wedge\frac{\partial}{\partial\dot{q}^2}+\frac{\partial}{\partial\dot{q}^1}\wedge\frac{\partial}{\partial q^2} \tag{3.39}$$

$$(\pounds_\Gamma)^2\left(\frac{\partial}{\partial\dot{q}^i}\wedge\frac{\partial}{\partial\dot{q}^j}\right)\simeq 2\frac{\partial}{\partial q^1}\wedge\frac{\partial}{\partial q^2}+\delta\left(\frac{\partial}{\partial q^1}\wedge\frac{\partial}{\partial\dot{q}^1}+\frac{\partial}{\partial\dot{q}^2}\wedge\frac{\partial}{\partial q^2}\right) \tag{3.40}$$

$$(\pounds_\Gamma)^3\left(\frac{\partial}{\partial\dot{q}^i}\wedge\frac{\partial}{\partial\dot{q}^j}\right)\simeq 0\,, \tag{3.41}$$

and hence it is clear that the maximal number of independent bivectors is given by

$$\begin{aligned} X_0&=\frac{\partial}{\partial\dot{q}_1}\wedge\frac{\partial}{\partial\dot{q}_2} & X_1&=\frac{\partial}{\partial q_1}\wedge\frac{\partial}{\partial q_2}\\ X_2&=\frac{\partial}{\partial q_1}\wedge\frac{\partial}{\partial\dot{q}_1} & X_3&=\frac{\partial}{\partial q_1}\wedge\frac{\partial}{\partial\dot{q}_2}\\ X_4&=\frac{\partial}{\partial q_2}\wedge\frac{\partial}{\partial\dot{q}_1} & X_5&=\frac{\partial}{\partial q_2}\wedge\frac{\partial}{\partial\dot{q}_2}\,. \end{aligned} \tag{3.42}$$

Therefore, the dimension is maximal and the Lagrangian does not exist. As is proven next, it is possible to describe this systems by means of the contac Lagrangian theory, but before introducing this formalism it is important to introduce the notion of contact structure on manifolds.

3.2 A Digression to Contact Manifolds in Geometry

The aim of this section is to introduce the basic notion of contact structures on a manifold and the definition of contact Hamiltonian dynamics, for its further application in the description of dissipative systems. The results mentioned here are given without proof, all the proofs may be found in Ref. [66].

The contact geometry is the odd-dimensional counterpart of the symplectic geometry. A $(2n+1)$-dimensional manifold M is said to be an *exact contact manifold* or to carry a contact structure if it carries a global differentiable 1-form η such that

$$\eta\wedge(\mathrm{d}\eta)^n\neq 0\,, \tag{3.43}$$

everywhere on M, η is called a *contact form*. In addition, the left hand side of (3.43) provides the *standard volume form* on M, i.e. the contact manifold in this sense is orientable.

Most geometrically, a contact structure on a manifold may be thought of in term of a *hyperplane field* on M, i.e. by means of a $2n$-dimensional sub-bundle $\mathcal{D}$ of the tangent bundle TM. The hyperplane is often called a *Pfaffian equation*. A Pfaffian equation on M is a codimension one distribution $\mathcal{D}: M\to TM$. The subspace $\mathcal{D}^\perp\subset T^*M$, defined by

$$\mathcal{D}^\perp=\{\eta\in T^*M\} \tag{3.44}$$

is one dimensional and locally it can be represented by a 1-form η. The class of $\mathcal{D}$ at $m\in M$ is the integer $C(m)=\max\{2p+1|\alpha\wedge(\mathrm{d}\alpha)^p\neq 0 \text{ at } m\}$. A point $m\in M$ is said to be a singular point of $\mathcal{D}$ if $C(m)<n$, n being the dimention of the manifold M. Therefore a contact structure in the wider sense is a codimension one distribution without singularities.

The condition (3.43) implies that $\mathcal{D}$ is not integrable, actually it is as far from integrable as possible. $\mathcal{D}$ is also called *the contact distribution*. Moreover, the orientability of M and the regularity of $\mathcal{D}$ implies that the line bundle $TM/\mathcal{D}$ admits a cross section s on which $\langle\eta|s\rangle=1$. Thus M admits a global non-vanishing vector field, say ξ, such that

$$i_\xi\eta=1,\quad i_\xi\mathrm{d}\eta=0\,, \tag{3.45}$$

ξ is called the characteristic vector field of the contact structure (or also the Reeb vector field). Two basic properties of ξ are the invariance of η and $\mathrm{d}\eta$ under its 1-parameter group, i.e.

$$\pounds_\xi\eta=0\quad\text{and}\quad\pounds_\xi\mathrm{d}\eta=0\,. \tag{3.46}$$

Finally, it is said that a contact structure is regular if ξ defines a regular distribution, i.e. $M \xrightarrow{\pi_\xi} M/\{\xi\}$ is a smooth projection onto the quotient $M/\{\xi\}$.

Form the definition of the contact structure by means of a contact distribution we can define a contact manifold in a more general sense. In general, a contact manifold is defined as an odd-dimensional differential manifold admitting a global 1-form η and a global 2-form ω such that

$$\eta \wedge \omega^n \neq 0 \tag{3.47}$$

everywhere. Similarly to what happens in the case of exact contact manifolds, the $(2n+1)$-form $\eta \wedge \omega^n$ provides a volume form. It is clear that the exact contact manifold is a special case whit $\omega = \mathrm{d}\eta$.

In the contact manifold one may define a dynamical evolution in terms of a vector field associated with a smooth function $\mathscr{F}$ by means of the intrinsic relation:

$$i_\Gamma (\eta \wedge \omega^n) = n\,(\mathrm{d}\mathscr{F} \wedge \eta) \wedge \omega^{n-1} + \mathscr{F}\,\omega^n\,. \tag{3.48}$$

Notice that for $\mathscr{F} = 1$ one obtains that Γ corresponds to the Reeb vector field. Furthermore, it is proven in Ref. [66] that definition (3.48) is equivalent to the two intrinsic conditions

$$i_\Gamma \mathrm{d}\eta = (\pounds_\xi \mathscr{F})\eta - \mathrm{d}\mathscr{F} \qquad \text{and} \qquad i_\Gamma \eta = \mathscr{F}\,. \tag{3.49}$$

Now given a manifold with contact structure (η, ω, ξ) it is possible define a Lie algebra structure of the space of functions by means of

$$[\mathscr{F}, \mathscr{G}]\,(\eta \wedge \omega^n) = (n-1)\,(\mathrm{d}\mathscr{F} \wedge \mathrm{d}\mathscr{G} \wedge \eta) \wedge \omega^{n-1} + (\mathscr{F}\,\mathrm{d}\mathscr{G} - \mathscr{G}\,\mathrm{d}\mathscr{F})\,\omega^n \tag{3.50}$$

which is called the Jacobi brackets. Although it defines a Lie algebra, i.e., antisymmetry and Jacobi identity are satisfied, the Leibniz rule of derivation is not satisfied in general. In a more standard form the Jacobi brackets may be expressed as

$$[\mathscr{F}, \mathscr{G}] = \mathscr{F}(\pounds_\xi \mathscr{G}) + \mathscr{G}(\pounds_\xi \mathscr{G}) + \Lambda(\mathrm{d}\mathscr{F}, \mathrm{d}\mathscr{G}) \tag{3.51}$$

where Λ is the bivector associated to the two-form ω. Therefore, in general, one may define the contact Hamiltonian vector field Γ associated to the Hamiltonian function $\mathscr{F}$ by means of

$$\Gamma = \mathscr{F}\xi + \Lambda(\mathrm{d}\mathscr{F}, \cdot\,)\,, \tag{3.52}$$

and this association is a homomorphism of a Lie algebra, i.e.

$$[\Gamma_{\mathscr{F}}, \Gamma_{\mathscr{G}}] = \Gamma_{[\mathscr{F},\mathscr{G}]}\,. \tag{3.53}$$

3.3 Lagrangian contact forms and dissipation

So far it has been shown that given a physical system with dynamics described by a second vector field Γ in a suitable carrier space, the condition that exist a Lagrangian (or Hamiltonian) formalism is that there is a *Lagrangian* function $\mathfrak{L}$ for Γ with respect to $\omega_{\mathfrak{L}}$, such that one can define the equation of the Lagrangian formalism (3.30) with Lagrangian energy defined in Eq. (3.32). However, it has been also observed that there is the possibility that the second order dynamics does not allow a Lagrangian description, even for the linear case. Then, one may deal with a subclass of such systems, the one that possesses a *decomposition principle*. This means, the dynamics that can be expressed in the form

$$\Gamma = \Gamma_{\mathrm{R}} + \Gamma_{\mathrm{P}} \tag{3.54}$$

where the firs terms Γ_{R} is known as the reference (comparison) term and it has at least one Lagrangian (Hamiltoanian), say $\mathfrak{L}$, and the second term Γ_{P} is the perturbation term, which turns the system dissipative. It is said that the system is dissipative in the sense that Γ_{P} is dissipating the Lagrangian energy $\mathcal{E}_{\mathfrak{L}}$ of the reference system. After these consideration, in this section it is shown that this subclass of

dissipative systems, even though it does not admit a Lagrangian formalism, it accepts a Lagrangian (Hamiltonian) contact description.

Once a "decomposition principle" for the system has been defined, it is possible to go to a contact formalism to show that a "dissipative" system may be described in terms of a "contact Hamiltonian formalism" [33, 34]. Here, the case is considered where a description in terms of contact geometry is possible for some dissipative forces.

Now, assuming that the carrier space for the dynamics is $TQ \times \mathbb{R}$ with an exact contact structure (η, ξ), in order to define the dynamics on $TQ \times \mathbb{R}$ one may associate with every smooth function $\mathcal{E}_{\mathrm{C}}$ a vector field Γ_{C} on $TQ \times \mathbb{R}$ by means of:

$$i_{\Gamma_{\mathrm{C}}} \mathrm{d}\eta = \mathrm{d}\mathcal{E}_{\mathrm{C}} - (\pounds_{\xi}\mathcal{E}_{\mathrm{C}})\eta \qquad \text{and} \qquad i_{\Gamma_{\mathrm{C}}}\eta = -\mathcal{E}_{\mathrm{C}}\,, \tag{3.55}$$

where $\mathcal{E}_{\mathrm{C}}$ is called the *contact Lagrangian energy*.

In particular, we assume that, locally, the 1-form η can be written as:

$$\eta = \mathrm{d}S - \theta_{\mathfrak{L}} \quad \text{with} \quad \theta_{\mathfrak{L}} = \mathrm{d}q_j \frac{\partial \mathfrak{L}}{\partial \dot{q}_j}\,, \tag{3.56}$$

where $(q_j,\, \dot{q}_j,\, S)$ are local coordinates on $TQ \times \mathbb{R}$, $\mathfrak{L}$ is the Lagrangian function of the *comparison system*, and the contact Lagrangian energy may be written as:

$$\mathcal{E}_{\mathrm{C}} = \mathcal{E}_{\mathfrak{L}} + h(S)\,. \tag{3.57}$$

The first term $\mathcal{E}_{\mathfrak{L}}$ is the Lagrangian energy in Eq. (3.2) of the conservative "comparison system", while the second term $h(S)$ is a "perturbation" of the system giving an effective characterization of the interaction between the conservative system and the environment. The system is thus decomposed into a "conservative (Lagrangian) comparison dynamics" and a "dissipative term", and what is being dissipated is the Lagrangian energy of the conservative comparison dynamics.

Introducing definitions (3.56) and (3.57) into the conditions in (3.55) one obtains *the contact Euler–Lagrange equation* and the equation for the component of the vector field in the direction of the Reeb vector, namely

$$\pounds_{\Gamma_{\mathrm{C}}}\theta_{\mathfrak{L}} - \mathrm{d}\mathfrak{L} = -\frac{\mathrm{d}h}{\mathrm{d}S}\,\theta_{\mathfrak{L}}\,, \tag{3.58}$$

$$\dot{S} = i_{\Gamma_{\mathrm{C}}}\theta_{\mathfrak{L}} - \mathcal{E}_{\mathfrak{L}}\,. \tag{3.59}$$

Equivalently, using the definition for the 1-form $\theta_{\mathfrak{L}}$ in Eq. (3.56) and applying the Lie derivative and employing Cartian's identity one obtains on $\theta_{\mathfrak{L}}$, one obtains the coordinate expression of the contact Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathfrak{L}}{\partial \dot{q}^k} - \frac{\partial \mathfrak{L}}{\partial q^k} = -\frac{\mathrm{d}h}{\mathrm{d}S}\frac{\partial \mathfrak{L}}{\partial \dot{q}^k}\,, \tag{3.60}$$

which in general are implicit differential equations. In addition, one has in coordinates that

$$\dot{S} = \mathfrak{L} - h(S)\,. \tag{3.61}$$

One may look at these systems as a sort of generalization of the so-called Caldirola–Kanai dissipative systems. Here, the Lagrangian energy is not preserved along the dynamical trajectories, indeed

$$\frac{\mathrm{d}\mathcal{E}_{\mathfrak{L}}}{\mathrm{d}t} = -\frac{\mathrm{d}h}{\mathrm{d}S}\frac{\partial \mathfrak{L}}{\partial \dot{q}^k}\,, \tag{3.62}$$

which may be positive or negative according to the sign of $\frac{\mathrm{d}h}{\mathrm{d}S}$. Additionally, the vector field associated with the dynamical system can be easily decomposed into a comparison dynamics admitting a Lagrangian description and a perturbation term (contact corrections).

Defined the contact dynamics $\Gamma_{\mathrm{C}} \in \mathfrak{X}(TQ \times \mathbb{R})$ a natural question is whether it is possible to project such dynamics onto a second order vector field $\Gamma \in \mathfrak{X}(TQ)$ to relate such dynamics with a second order dynamics in the original variables. This means that one gets an *effective* equation of motion in the

original variables $(\mathbf{q}, \dot{\mathbf{q}})$ which is *decoupled* from the equation on the auxiliary variable S. Then, it is clear from the contact Euler–Lagrange Eq. (3.60), that it reproduces a second order dynamics on TQ if $h(S)$ is proportional to S.

Formally to see when the contact dynamics is projectable into a second order dynamics, recall from the last subsection that the Reeb vector field of a regular contact manifold defines a regular one-dimesional distribution $\mathcal{D}^{\xi}$. Then, the integral curves of this Reeb distribution introduce a foliation Φ^{ξ} and hence an equivalence relation with respect to it, which defines a quotient set $TQ \times \mathbb{R}/\sim_{\mathcal{D}^{\xi}} \approx TQ$, where

$$\pi : TQ \times \mathbb{R} \to TQ \tag{3.63}$$

is the canonical projection. Therefore, as one can see in Appendix A, the contact dynamics Γ_{C} is projectable with respect to the foliation generated by the Reeb distribution iff

$$[\Gamma_{\text{C}}, \xi] \wedge \xi = 0\,. \tag{3.64}$$

In addition, this directly implies that the contact energy $\mathcal{E}_{\mathfrak{L}}$, associated with the contact vector field, is linear in S.

Concrete examples of second order linear systems which have a clear physical interpretation are provided by electric circuits. For instance, for the description of the basic circuit in Fig. 3.1a we can consider the Lagrangian and the perturbation term as

$$\mathfrak{L} = \frac{1}{2} L\,\dot{I}^2 - \frac{1}{2\,C} I^2 \quad \text{and} \quad h(S) = R\,S\,, \tag{3.65}$$

respectively, using the notation indicated in the figure. Thus, the contact Euler–Lagrange equation (3.60) for this case corresponds to the usual RLC equation

$$L\,\ddot{I} + R\dot{I} + \frac{1}{C} I = 0\,, \tag{3.66}$$

together with the equation

$$\dot{S} = \frac{1}{2} L\,\dot{I}^2 - \frac{1}{2\,C} I^2 - R\,S\,. \tag{3.67}$$

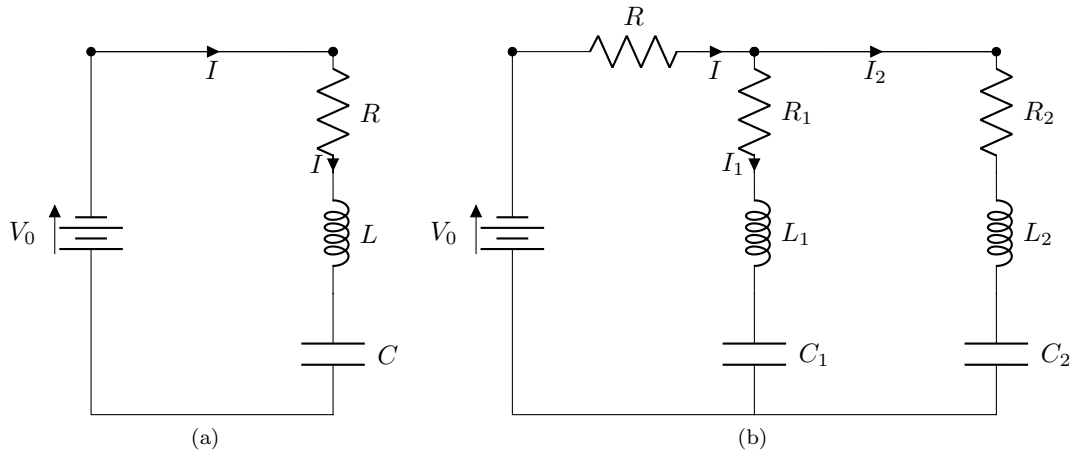


Figure 3.1: (a) Diagram of a basic RLC circuit. (b) Diagram of the coupling of two RLC circuits connected in parallel by means of the resistance R.

It is clear that Eq. (3.66) is the generalization of the Caldirola–Kanai dissipation allowing to describe dissipative systems which are "linear in the velocity". However, in order to handle more general situations like the one considered in Fig. 3.1b, it is necessary to extend the formalism and consider a more general definition of contact structure.

As has been shown in the last section, it is possible to define in a more general sense a contact manifold by means of a global 1-form η and a global 2-form $\omega_{\mathfrak{L}}$. The contact manifold is assumed to be again $TQ \times \mathbb{R}$; then, one defines now a dynamical evolution in terms of a vector field associated with a smooth function $\mathcal{E}_{\mathrm{C}}$ by means of the two conditions

$$i_{\Gamma_{\mathrm{C}}}\omega_{\mathfrak{L}} = \mathrm{d}\mathcal{E}_{\mathrm{C}} - (\pounds_{\xi}\mathcal{E}_{\mathrm{C}})\eta \qquad \text{and} \qquad i_{\Gamma_{\mathrm{C}}}\eta = -\mathcal{E}_{\mathrm{C}}\,. \tag{3.68}$$

Then, similarly to what has been done in the case of exact contact structures on $TQ \times \mathbb{R}$, here it is assumed that η and $\omega_{\mathfrak{L}}$ are given by

$$\eta = \mathrm{d}S - \alpha \qquad \text{and} \qquad \omega_{\mathfrak{L}} = -\,\mathrm{d}\theta_{\mathfrak{L}}\,, \tag{3.69}$$

where α is a semi-basic 1-form with the following local expression[2]

$$\alpha = a_k\,\mathrm{d}q^k\,, \tag{3.72}$$

with $a^k \in \mathfrak{F}(TQ)$. Because we are interested in characterizing dissipative systems in terms of a *decomposition principle*, one shall assume that the Lagrangian contact energy $\mathcal{E}_{\mathrm{C}}$ has the form defined in (3.57). Inserting these definitions into equation (3.68) one gets the conditions

$$\pounds_{\Gamma_{\mathrm{C}}}\theta_{\mathfrak{L}} - \mathrm{d}\mathfrak{L} = -\frac{\mathrm{d}h}{\mathrm{d}S}\,\alpha \qquad \text{and} \qquad \dot{S} = i_{\Gamma_{\mathrm{C}}}\alpha - \mathcal{E}_{\mathrm{C}}\,. \tag{3.73}$$

The first condition in (3.73) determines uniquely Γ_{C} up to a vector field proportional to ξ, and this additional term is fixed by the second requirement in (3.73). Because Γ_{C} should be projectable onto a second order dynamics $\Gamma \in \mathfrak{X}(TQ)$, it is necessary and sufficient that $h(S)$ is linear in S and besides α must be a semi-basic 1-form independent of S. Therefore, one may associate the contact dynamics with a second order differential equation with respect to the q variables in TQ.

One may identify some special classes of dissipative systems by making specific choices for α. For instance, one may take

$$\alpha = \mathrm{d}_{S}\mathcal{F}\,. \tag{3.74}$$

where $\mathcal{F}$ is an arbitrary function on TQ, and the operator d_S as defined in Eq. (3.16). This particular situation gives rise to a conformal version of the Rayleigh dissipation that in coordinates has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathfrak{L}}{\partial\dot{q}_j} - \frac{\partial\mathfrak{L}}{\partial q_j} = -\frac{\mathrm{d}h}{\mathrm{d}S}\frac{\partial\mathcal{F}}{\partial\dot{q}_j}\,, \tag{3.75}$$

and where the rate of change of the Lagrangian energy is

$$\frac{\mathrm{d}\mathcal{E}_{\mathrm{C}}}{\mathrm{d}t} = -\frac{\mathrm{d}h}{\mathrm{d}S}\frac{\partial\mathcal{F}}{\partial\dot{q}_j}\,. \tag{3.76}$$

One may now employ the conformal version of the Rayleigh dissipation for the description of the circuit in Fig. 3.1b. This consists in the composition of two RLC circuits connected in parallel, where the coupling is carried out by a resistance R. Then, the dynamics of the system is determined by

$$\mathcal{L} = \frac{1}{2}L^{jk}\dot{I}_k\dot{I}_j - \frac{1}{2}C^{jk}I_kI_j\,, \quad h(S) = S \quad \text{and} \quad \mathcal{F} = -\frac{1}{2}R^{jk}\dot{I}_k\dot{I}_j\,, \tag{3.77}$$

with $j,k = 1,2$ and where here I_1 and I_2 denotes the currents in the branches and according to the notation in Fig. 3.1b the matrices are

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}\,, \quad C = \begin{pmatrix} 1/C_1 & 0 \\ 0 & 1/C_2 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} R_1 & R \\ R & R_2 \end{pmatrix}\,. \tag{3.78}$$

[2]Semi-basic forms allow to introduce forces that do not admit a Lagrangian description. To see this one may consider the usual notion of dissipative systems

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathfrak{L}}{\partial\dot{q}^k} - \frac{\partial\mathfrak{L}}{\partial q^k} = Q_k\,, \tag{3.70}$$

where Q_k does not admit a Lagrangian description. This intrinsically is expressed as

$$\pounds_{\Gamma_{\mathrm{C}}}\theta_{\mathfrak{L}} - \mathrm{d}\mathfrak{L} = \alpha\,, \tag{3.71}$$

where locally $\alpha = Q_k\,\mathrm{d}q^k$ is a one-form.

Therefore, the Euler–Lagrange equations associated with this system correspond to

$$L^{jk}\ddot{I}_k + R^{jk}\dot{I}_k + C^{jk}I_k = 0\,, \tag{3.79}$$

which are in agreement with the equations obtained from Kirchhoff's circuit laws. In addition, it is clear that if the coupling parameter $R \to 0$ the system is reduced to two non-interacting RLC circuits.

3.4 Contact Hamiltonian Mechanics

In the last section we introduced the contact Hamiltonian formalism in the carrier space $TQ\times\mathbb{R}$. Here we are interested in introducing the contact formalism in the so-called *contact phase space*, which is defined as the odd-dimensional manifold $T^*Q \times \mathbb{R}$ [33].

To introduce the transition from $TQ \times \mathbb{R}$ to $T^*Q \times \mathbb{R}$, it is illustrative to recalled how one performs this transition in the usual transition from TQ to T^*Q. Consider a second order dynamical system Γ with Hamiltonian representation on TQ, i.e. Γ has associated the functions $\mathfrak{L}$ and $\mathcal{E}_\mathfrak{L}$ by Eqs. (3.24). Moreover, let us assume that $\mathfrak{L}$ is *hyperregular* such that one transforms the vector field $\Gamma \in \mathfrak{X}(TQ)$ into a vector field $X \in \mathfrak{X}(T^*Q)$ by means of

$$X = (F\mathfrak{L})_*\Gamma\,, \tag{3.80}$$

such that our dynamical system will be represented on T^*Q by the family of equivalent vector fields on TQ through the diffeomorphism $F\mathfrak{L}$ known in the literature as the *fiber derivative* of $\mathfrak{L}$ [3], and it is defined by its action on any element $v_q \in T_qQ$ by

$$F\mathfrak{L} : v_q \mapsto D\mathfrak{L}_q(v_q) \in \mathrm{Lin}(T_qQ, \mathbb{R}) \equiv T_q^*Q\,, \tag{3.81}$$

where $D\mathfrak{L}_q(v_q)$ is by definition a linear map from T_qQ to $\mathbb{R}$, thus $F\mathfrak{L}$, maps each element $v_q \in T_qQ$ to an element in $\mathrm{Lin}(T_qQ, \mathbb{R})$, or to an element of T_q^*Q. To put this definition in local coordinates, let $(q^k, \dot{q}^k)$ be the natural coordinates in TQ, and let the coordinates of $v_q \in T_qQ$ be $\dot{q}^j = v^j$. Then, the components of $D\mathfrak{L}_q(v_q)$ are denoted by p_j and are equal to the derivatives $\frac{\partial\mathfrak{L}}{\partial\dot{q}^j}$ evaluated at v^j. Namely, if w_q with components w^j is a general point in T_qQ, then $D\mathfrak{L}_q(v_q)$ maps it according to

$$D\mathfrak{L}_q(v_q) : w_q \mapsto p_j w^j = w^j \frac{\partial\mathfrak{L}}{\partial\dot{q}^j}\bigg|_{v_q}\,. \tag{3.82}$$

Consequently, for an arbitrary point in TQ we may write this definition as

$$F\mathfrak{L} : (q^k, \dot{q}^k) \to \left(q^k, \frac{\partial\mathfrak{L}}{\partial\dot{q}^k}\right)\,, \tag{3.83}$$

where the component $p_k = \frac{\partial\mathfrak{L}}{\partial\dot{q}^k}$ corresponds to the definition of the momentum as given usually in Classical Mechanics.

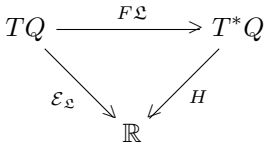
An important property of the fiber derivative of the Lagrangian function is that there is a unique symplectic form $\omega \in \Omega^2(T^*Q)$ such that

$$(F\mathfrak{L})^*\omega = \omega_\mathfrak{L}\,, \tag{3.84}$$

independently of the Lagrangian function used [3]. Hence $\omega_\mathfrak{L}$ is the pullback with respect to $F\mathfrak{L}$ of the symplectic form ω in T^*Q. The advantage of this is that a Lagrangian $\mathfrak{L}$ dynamical system in TQ is mapped over T^*Q to always yield a dynamical system Hamiltonian with respect to the same symplectic structure ω on T^*Q, given in local coordinates by

$$\omega = \mathrm{d}p_k \wedge \mathrm{d}q^k\,. \tag{3.85}$$

The Hamiltonain function H on T^*Q may be defined by means of the diagramme



where one may see that

$$H = \mathcal{E}_{\mathfrak{L}} \circ (F\mathfrak{L})^{-1} \equiv (F\mathfrak{L})^* \mathcal{E}_{\mathfrak{L}} \,. \tag{3.86}$$

This may be written in local coordinates taking into account the definition of $\mathcal{E}_{\mathfrak{L}}$ in Eq. (3.32)

$$H = \dot{q}^k \circ (F\mathfrak{L})^{-1} p_k - \mathfrak{L} \circ (F\mathfrak{L})^{-1} \,, \tag{3.87}$$

which is the well-known expression of the *Legendre transformation* for the Hamiltonian, except that it is usually written without $(F\mathfrak{L})^{-1}$.

After this short digression to fiber derivatives as the diffeomorphism form TQ to T^*Q, let us now establish the connection between the contact manifolds $TQ \times \mathbb{R}$ and $T^*Q \times \mathbb{R}$. To do that we consider the natural projections $\pi : TQ \times \mathbb{R} \to TQ$ and $\tilde{\pi} : T^*Q \times \mathbb{R} \to T^*Q$, such that according to the diagramme

$$\begin{array}{ccc} TQ \times \mathbb{R} & \xrightarrow{F\mathscr{L}} & T^*Q \times \mathbb{R} \\ & {\scriptstyle \mathcal{E}_C} \searrow \quad \swarrow {\scriptstyle \mathscr{H}} & \\ {\scriptstyle \pi} \downarrow & \mathbb{R} & \downarrow {\scriptstyle \tilde{\pi}} \\ TQ & \xrightarrow{F\mathfrak{L}} & T^*Q \end{array}$$

one has a well defined diffeomorphism $F\mathscr{L}$ associated to $\mathscr{L} \in \mathfrak{F}(TQ \times \mathbb{R})$ and given by

$$F\mathscr{L} = (\tilde{\pi})^{-1} \circ F\mathfrak{L} \circ \pi \,. \tag{3.88}$$

In local coordinates, it is possible to write this symbolically in the form

$$F\mathscr{L} : (q^k, \dot{q}^k, S) \mapsto \left(q^k, p_k = \frac{\partial \mathfrak{L}}{\partial \dot{q}^k}, S \right) . \tag{3.89}$$

We emphasize that the component defined as $p_k = \frac{\partial \mathfrak{L}}{\partial \dot{q}^k}$ corresponds to the usual definition of the momentum. Thus, under such map a contact dynamics $\Gamma_{\rm C} \in \mathfrak{X}(TQ \times \mathbb{R})$ will be carried over a vector field on $T^*Q \times \mathbb{R}$ by

$$\Gamma_{\rm C} = (F\mathscr{L})^* X_{\rm C} \,. \tag{3.90}$$

Recall that the definition of contact manifold and the dynamics on it is closely related to the contact form $\eta_{\mathfrak{L}}$ defined in Eq. (3.56). So there is a contact form $\eta \in \Omega^1(T^*Q \times \mathbb{R})$ such that

$$(F\mathscr{L})_* \eta = \eta_{\mathfrak{L}} \,. \tag{3.91}$$

for all $\mathfrak{L} \in \mathfrak{F}(TQ)$. This is, $\eta_{\mathfrak{L}}$ is the pullback with respect to $F\mathscr{L}$ of the fixed contact form η in $T^*Q \times \mathbb{R}$, which is independent of the Lagrangian used. Thus the dynamics $X_{\rm C}$ on the contact phase space is given by the conditions

$$i_{X_{\rm C}} \mathrm{d}\eta = \mathrm{d}\mathscr{H} - (\pounds_\xi \mathscr{H})\eta \qquad \text{and} \qquad i_{X_{\rm C}} \eta = -\mathscr{H} \,, \tag{3.92}$$

i.e. the dynamics only depends on the *contact Hamiltonian* function defined form the contact Lagrangian formulation as

$$\mathscr{H} = \mathcal{E}_{\rm C} \circ (F\mathscr{L})^{-1} \,. \tag{3.93}$$

In addition, the vector field ξ in the left condition of (3.92) is the Reeb vector field defined intrinsically by

$$i_\xi \, \eta = 1 \,, \quad \text{and} \quad i_\xi \, \mathrm{d}\eta = 0 \,. \tag{3.94}$$

In the local coordinates (q^k, p_k, S) induced by (3.89) the contact form and the Reeb vector field may be written as

$$\eta = \mathrm{d}S - p_k \mathrm{d}q^k \,, \quad \xi = \frac{\partial}{\partial S} \,, \tag{3.95}$$

and the *contact Hamiltonian* given by (3.93) has the form

$$\mathscr{H} = H + h(S)\,, \tag{3.96}$$

with H the Hamiltonian of the reference system. Besides in this coordinates, the contact Hamiltonian vector fields X_{C} has the form

$$X_{\mathrm{C}} = \frac{\partial H}{\partial p_k}\frac{\partial}{\partial q^k} - \left(\frac{\partial H}{\partial q^k} + p_k\frac{\mathrm{d}h}{\mathrm{d}S}\right)\frac{\partial}{\partial p_k} + \left(p_k\frac{\partial H}{\partial p_k} - \mathscr{H}\right)\frac{\partial}{\partial S} \tag{3.97}$$

whose integral curves are determined by the solutions of the system of differential equations

$$\begin{aligned}
\dot{q}^k &= \frac{\partial H}{\partial p_k}\,,\\
\dot{p}_k &= -\frac{\partial H}{\partial q^k} - p_k\frac{\mathrm{d}h}{\mathrm{d}S}\,,\\
\dot{S} &= p_k\frac{\partial H}{\partial p_k} - \mathscr{H}\,.
\end{aligned} \tag{3.98}$$

It is clear that the contact dynamics accepts a decomposition principle, because it may be expressed as

$$X_{\mathrm{C}} = X_H + X_{\mathrm{p}} \tag{3.99}$$

where

$$X_H = \frac{\partial H}{\partial p_k}\frac{\partial}{\partial q^k} - \frac{\partial H}{\partial q^k}\frac{\partial}{\partial p_k}\,, \tag{3.100}$$

corresponds to the reference dynamics with Hamiltonian description, whereas the perturbative term corresponds to

$$X_{\mathrm{p}} = -p_k\frac{\mathrm{d}h}{\mathrm{d}S}\frac{\partial}{\partial p_k} + \left(p_k\frac{\partial H}{\partial p_k} - \mathscr{H}\right)\frac{\partial}{\partial S}\,. \tag{3.101}$$

Then it is clear that the perturbation term becomes the dissipative dynamics, where what is being dissipated is the Hamiltonian of the comparison dynamics. In fact, the rate of dissipation of the Hamiltonian is

$$\frac{\mathrm{d}H}{\mathrm{d}t} = X_{\mathrm{p}}(H) = -p_k\frac{\mathrm{d}h}{\mathrm{d}S}\frac{\partial H}{\partial p_k}\,. \tag{3.102}$$

In general, given any function in the contact phase space $\mathscr{F} \in C^\infty(\mathcal{T})$, its evolution according to Eq. (3.98) is given by

$$\begin{aligned}
\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} &= X_{\mathrm{C}}[\mathscr{F}]\\
&= -\mathscr{H}\frac{\partial\mathscr{F}}{\partial S} + p_k\left[\frac{\partial\mathscr{F}}{\partial S}\frac{\partial\mathscr{H}}{\partial p_k} - \frac{\partial\mathscr{F}}{\partial p_k}\frac{\partial\mathscr{H}}{\partial S}\right] + \frac{\partial\mathscr{F}}{\partial q^k}\frac{\partial\mathscr{H}}{\partial p_k} - \frac{\partial\mathscr{F}}{\partial p_k}\frac{\partial\mathscr{H}}{\partial q^k}\\
&= -\mathscr{H}\frac{\partial\mathscr{F}}{\partial S} + p_k\left\{\mathscr{F},\mathscr{H}\right\}_{(S,p_k)} + \left\{\mathscr{F},\mathscr{H}\right\}_{(q^k,p_k)}\,,
\end{aligned} \tag{3.103}$$

where $\{\,\cdot\,,\,\cdot\,\}_{(q^k,p_k)}$ is the standard Poisson bracket and the remaining terms are contact corrections. We point out that the bracket $\{\,\cdot\,,\,\cdot\,\}_{(S,p_k)}$ is just a shorthand notation and we do not provide any intrinsic definition for it. We say that a function $\mathscr{F} \in C^\infty(\mathcal{T})$ is a *first integral* (or *invariant*) of the contact dynamics given by X_{C} if $\mathscr{F}$ is constant along the flow of X_{C}, that is if $X_{\mathrm{C}}[\mathscr{F}] = 0$.

As an example, given the (1-dimensional) contact Hamiltonian system

$$\mathscr{H}_S = \frac{p^2}{2m} + V(q) + \gamma\, S = H_{\mathrm{mec}} + \gamma\, S\,, \tag{3.104}$$

where $V(q)$ is the mechanical potential and γ is a constant parameter, the equations of motion in (3.98) take the form

$$\dot{q} = \frac{p}{m}\,, \tag{3.105}$$

$$\dot{p} = -\frac{\partial V(q)}{\partial q} - \gamma\, p\,, \tag{3.106}$$

$$\dot{S} = \frac{p^2}{2m} - V(q) - \gamma\, S\,. \tag{3.107}$$

From (3.105) and (3.106) one can derive the damped Newtonian equation

$$\ddot{q} + \gamma\,\dot{q} + \frac{1}{m}\frac{\partial V(q)}{\partial q} = 0\,, \tag{3.108}$$

which describes all systems with a friction force that depends linearly on the velocity. Besides, we may compute the rate of dissipation of the mechanical energy given by

$$\frac{\mathrm{d}H_{\mathrm{mec}}}{\mathrm{d}t} = -m\,\gamma\,\dot{q}^2\,, \tag{3.109}$$

which agrees with standard results based on Rayleigh's dissipation function [35].

3.4.1 Contact transformations and Liouville's theorem

In the preceding sections we have introduced *the contact phase space* for time-independent mechanical systems, equipped with the local coordinates (q^k, p_k, S), called *contact coordinates*. In these variables the equations of motion are expressed in terms of *the contact Hamiltonian equations* (3.98) and the contact form is expressed as in (3.95). As in the symplectic case, we are now interested in introducing those transformations that leave the contact structure unchanged, which are known as *contact transformations* [1]. Here we consider only time-independent contact transformations and in the next subsection we introduce the time-dependent case.

A *contact transformation* is a transformation that leaves the contact form invariant up to multiplication by a conformal factor [36, 37], that is

$$\tilde{\eta} = f\,\eta\,. \tag{3.110}$$

From (3.110), an arbitrary transformation of coordinates from (q^k, p_k, S) to $(\tilde{Q}^k, \tilde{P}_k, \tilde{S})$ is a contact transformation if

$$f(\mathrm{d}S - p_k \mathrm{d}q^k) = \mathrm{d}\tilde{S} - \tilde{P}_k \mathrm{d}\tilde{Q}^k\,, \tag{3.111}$$

which is equivalent to

$$f = \frac{\partial \tilde{S}}{\partial S} - \tilde{P}_k \frac{\partial \tilde{Q}^k}{\partial S} \tag{3.112}$$

$$-f p_i = \frac{\partial \tilde{S}}{\partial q^i} - \tilde{P}_k \frac{\partial \tilde{Q}^k}{\partial q^i} \tag{3.113}$$

$$0 = \frac{\partial \tilde{S}}{\partial p_i} - \tilde{P}_k \frac{\partial \tilde{Q}^k}{\partial p_i}\,. \tag{3.114}$$

As in the standard symplectic theory, it is possible to obtain the generating function of a contact transformation. Assuming that the coordinates $(q^k, \tilde{Q}^k, S)$ are independent, we compute the differential of the generating function $\tilde{S}(q^k, \tilde{Q}^k, S)$, namely

$$\mathrm{d}\tilde{S} = \frac{\partial \tilde{S}}{\partial S}\mathrm{d}S + \frac{\partial \tilde{S}}{\partial q^k}\mathrm{d}q^k + \frac{\partial \tilde{S}}{\partial \tilde{Q}^k}\mathrm{d}\tilde{Q}^k\,. \tag{3.115}$$

Substituting (3.115) into (3.111) one obtains the following conditions for $\tilde{S}$

$$f = \frac{\partial \tilde{S}}{\partial S}\,, \quad f\,p_k = -\frac{\partial \tilde{S}}{\partial q^k}\,, \quad \tilde{P}_k = \frac{\partial \tilde{S}}{\partial \tilde{Q}^k}\,. \tag{3.116}$$

In particular, for contact transformations with $f = 1$ the conditions in (3.116) imply that the generating function has the form

$$\tilde{S} = S - F_1(q^k, \tilde{Q}^k)\,, \tag{3.117}$$

where $F_1(q^k, \tilde{Q}^k)$ is the generating function of a symplectic canonical transformation. This result is remarkable, since it implies that all canonical transformations are a special case of contact transformations corresponding to $f = 1$.

While canonical transformations preserve the symplectic volume form, we show now that contact transformations induce a re-scaling of the contact volume form $\eta \wedge (\mathrm{d}\eta)^n$. Let us assume that we have a transformation that induces the change $\tilde{\eta} = f\,\eta$; then, $\mathrm{d}\tilde{\eta} = \mathrm{d}f \wedge \eta + f\,\mathrm{d}\eta$. It follows that

$$\tilde{\eta} \wedge (\mathrm{d}\tilde{\eta})^n = f^{n+1}\eta \wedge (\mathrm{d}\eta)^n\ , \tag{3.118}$$

i.e. the volume form is rescaled by a term f^{n+1}, with f given in general in (3.112). Note that canonical transformations are a special case with $f = 1$ and therefore they preserve the contact volume form.

Finally, applying the contact Hamiltonian vector field X_{C} to η, one sees from (3.92) that

$$\pounds_{X_{\mathrm{C}}}\eta = -\frac{\partial \mathscr{H}}{\partial S}\eta\,. \tag{3.119}$$

Comparing (3.119) with (3.110), one can conclud that contact Hamiltonian vector fields are the infinitesimal generators of contact transformations [36, 37]. Again, this is the analogue of the fact that symplectic Hamiltonian vector fields are the infinitesimal generators of canonical transformations. Moreover, equation (3.119) also implies that the volume element contracts (or expands) along the contact Hamiltonian flow according to [67]

$$\pounds_{X_{\mathrm{C}}}\left(\eta \wedge (\mathrm{d}\eta)^n\right) = -(n+1)\frac{\partial \mathscr{H}}{\partial S}\left(\eta \wedge (\mathrm{d}\eta)^n\right)\ , \tag{3.120}$$

which means that the contact flow has a non-zero divergence

$$\mathrm{div}(X_{\mathrm{C}}) = -(n+1)\frac{\partial \mathscr{H}}{\partial S} \tag{3.121}$$

and therefore Liouville's theorem does not hold. However, an analogous statement of Liouville's theorem for contact flows has been proved in [67]. In fact, although the volume element $\eta \wedge (\mathrm{d}\eta)^n$ is not preserved along the contact Hamiltonian flow, nevertheless a unique invariant measure depending only on $\mathscr{H}$ can be found whenever $\mathscr{H} \neq 0$, given by

$$\mathrm{d}\mu = |\mathscr{H}|^{-(n+1)}\left(\eta \wedge (\mathrm{d}\eta)^n\right)\ , \tag{3.122}$$

where the absolute value $|\cdot|$ has been introduced in order to ensure that the probability distribution is positive. As it provides an invariant measure for the flow, this is the analogue of Liouville's theorem for contact Hamiltonian flows.

3.4.2 Time-dependent contact Hamiltonian systems

In the preceding sections it has been seen that contact Hamiltonian mechanics may account for the dynamics of mechanical systems with dissipation and it has been proven some results that extend the symplectic formalism to the contact case. However, so far we have considered only time-independent systems. Now we introduce contact Hamiltonian systems that explicitly depend on time.

To begin, let us extend the contact phase space by adding the time variable to it. Therefore we have an extended manifold $\mathcal{T}^{\mathrm{E}} = T^*Q \times \mathbb{R} \times \mathbb{R}$ with natural coordinates derived from contact coordinates as (q^k, p_k, S, t). Then we extend the contact 1-form (3.95) to the 1-form

$$\eta^{\mathrm{E}} = \mathrm{d}S - p_k\mathrm{d}q^k + \mathscr{H}\mathrm{d}t\,, \tag{3.123}$$

where $\mathscr{H}$ is the contact Hamiltonian, that in this case is allowed to depend on t too. Notice that whenever $\mathscr{H}$ depends on S, $\mathrm{d}\eta^{\mathrm{E}}$ is non-degenerate (and closed) and therefore $(\mathcal{T}^{\mathrm{E}}, \mathrm{d}\eta^{\mathrm{E}})$ is a symplectification

of $(\mathcal{T},\eta)$. However, such a symplectification is not the standard (natural) one defined e.g. in [1]. The proposed symplectification depends on the Hamiltonian of the system as it is clear from equation (3.123). Besides, the coordinates (q^k,p_k,S,t) are non-canonical coordinates for $\mathrm{d}\eta^{\mathrm{E}}$, as it is easy to check. An interesting set of canonical coordinates for $\mathrm{d}\eta^{\mathrm{E}}$ is given by $(q^k,p_k,\mathscr{H},t)$. This shows that $\mathscr{H}$ is the canonically conjugated variable to t, which, together with the fact that $\mathscr{H}$ contains a 'dissipation potential' through its dependence on S, is calling for an interpretation of $\mathscr{H}$ in terms of the total energy of the system (conservative plus dissipative).

Now we want to define the dynamics on $\mathcal{T}^{\mathrm{E}}$. To do so, we set the two (intrinsic) simultaneous conditions

$$\pounds_{X_{\mathrm{C}}^{\mathrm{E}}}\eta^{\mathrm{E}}=g_{\mathscr{H}}\,\eta^{\mathrm{E}}\qquad\text{and}\qquad\eta^{\mathrm{E}}\left(X_{\mathrm{C}}^{\mathrm{E}}\right)=0\,, \tag{3.124}$$

with $g_{\mathscr{H}}\in C^\infty(\mathcal{T}^{\mathrm{E}})$ a function depending on $\mathscr{H}$ to be fixed below, cf. equation (3.129). Notice that (3.124) is the natural extension of (3.92) to $\mathcal{T}^{\mathrm{E}}$. We argue that these two conditions define a vector field $X_{\mathrm{C}}^{\mathrm{E}}$ on $\mathcal{T}^{\mathrm{E}}$ which is completely equivalent to the contact Hamiltonian flow (3.97). To prove this, let us first use Cartan's identity to re-write (3.124) as

$$\mathrm{d}\eta^{\mathrm{E}}(X_{\mathrm{C}}^{\mathrm{E}})=g_{\mathscr{H}}\eta^{\mathrm{E}}\qquad\text{and}\qquad\eta^{\mathrm{E}}(X_{\mathrm{C}}^{\mathrm{E}})=0\,. \tag{3.125}$$

Then, using the second condition in (3.125) it can be written in local coordinates as

$$\left(\mathrm{d}S-p_k\mathrm{d}q^k+\mathscr{H}\mathrm{d}t\right)\left(X^S\frac{\partial}{\partial S}+X^{q^k}\frac{\partial}{\partial q^k}+X^{p_k}\frac{\partial}{\partial p_k}+X^t\frac{\partial}{\partial t}\right)=0\,, \tag{3.126}$$

where the X^i are the general components of the vector field $X_{\mathrm{C}}^{\mathrm{E}}$ in these coordinates. One is free to fix a normalization for $X_{\mathrm{C}}^{\mathrm{E}}$ such that $X^t=1$. Now condition (3.126) yields

$$X^S=p_kX^{q^k}-\mathscr{H}\,. \tag{3.127}$$

Using (3.127) one may write the first condition in (3.125) as

$$\mathrm{d}\eta^{\mathrm{E}}\left([p_k\,X^{q^k}-\mathscr{H}]\frac{\partial}{\partial S}+X^{q^k}\frac{\partial}{\partial q^k}+X^{p_k}\frac{\partial}{\partial p_k}+\frac{\partial}{\partial t}\right)=g_{\mathscr{H}}\eta^{\mathrm{E}}\,, \tag{3.128}$$

and, after a direct calculation, one arrives at

$$g_{\mathscr{H}}=-\frac{\partial\mathscr{H}}{\partial S}\,,\quad X^{q^k}=\frac{\partial\mathscr{H}}{\partial p_k}\,,\quad X^{p_k}=-\frac{\partial\mathscr{H}}{\partial q^k}-p_k\frac{\partial\mathscr{H}}{\partial S}\,. \tag{3.129}$$

Finally, considering all the above conditions, it is possible to write the resulting vector field $X_{\mathrm{C}}^{\mathrm{E}}$ satisfying both conditions in (3.125) in its general form as

$$X_{\mathrm{C}}^{\mathrm{E}}=X_{\mathrm{C}}+\frac{\partial}{\partial t}\,, \tag{3.130}$$

with X_{C} given by (3.97). From this it is immediate to recognize that the equations of motion given by such field on $\mathcal{T}^{\mathrm{E}}$ are the same as those of the contact Hamiltonian vector field (3.97), with the addition of the trivial equation $\dot{t}=1$. We call a system defined by a contact Hamiltonian $\mathscr{H}(q^k,p_k,S,t)$ and by the vector field $X_{\mathrm{C}}^{\mathrm{E}}$ of the form (3.130) a *time-dependent contact Hamiltonian system*. From (3.130) and (3.103) it follows that the evolution of any function $\mathscr{F}\in C^\infty(\mathcal{T}^{\mathrm{E}})$ under the dynamics given by a time-dependent contact Hamiltonian system reads

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t}=-\mathscr{H}\frac{\partial\mathscr{F}}{\partial S}+p_k\left\{\mathscr{F},\mathscr{H}\right\}_{(S,p_k)}+\left\{\mathscr{F},\mathscr{H}\right\}_{(q^k,p_k)}+\frac{\partial\mathscr{F}}{\partial t}\,. \tag{3.131}$$

Now that a formal prescription to write the equations of motion for time-dependent contact Hamiltonian systems has been found , let us discuss time-dependent contact transformations and their generating functions. *Time-dependent contact transformations* are transformation of coordinates

$$(q^k,p_k,S,t)\mapsto(\tilde{Q}^k,\tilde{P}_k,\tilde{S},t)\,, \tag{3.132}$$

that leave the equations of motion, i.e. the vector field $X_{\mathrm{C}}^{\mathrm{E}}$, invariant. By definition, this amounts at finding a transformation that leaves both conditions in (3.125) unchanged. To find such a transformation, we start with the second condition and write the invariance as the fact that the transformed extended 1-form must have the same form as the original one up to multiplication by a non-zero function f, that is

$$f\left(\mathrm{d}S - p_k \mathrm{d}q^k + \mathscr{H}\mathrm{d}t\right) = \mathrm{d}\tilde{S} - \tilde{P}_k \mathrm{d}\tilde{Q}^k + \mathscr{K}\,\mathrm{d}t\,, \tag{3.133}$$

where $\mathscr{K}$ is a function on $\mathcal{T}^{\mathrm{E}}$ which is going to be the new contact Hamiltonian in the transformed coordinates. This condition provides a way to check whether a transformation of the type (3.132) is a time-dependent contact transformation. Indeed, inserting the differentials of $\tilde{Q}^k$ and $\tilde{S}$ into (3.133) one obtains the standard conditions (3.112)-(3.114) for a time-independent contact transformation, together with the following rule for the transformation of the Hamiltonians

$$f\,\mathscr{H} = \frac{\partial \tilde{S}}{\partial t} - \tilde{P}_k \frac{\partial \tilde{Q}^k}{\partial t} + \mathscr{K}\,. \tag{3.134}$$

As in the time-independent case, in order to find the conditions on the generating function $\tilde{S}(q^k, \tilde{Q}^k, S, t)$ we assume that the coordinates $(q^k, \tilde{Q}^k, S, t)$ are independent. Thus, from (3.133) one finds that $\tilde{S}$ must satisfy (3.116) and the additional constraint

$$f\mathscr{H} = \frac{\partial \tilde{S}}{\partial t} + \mathscr{K}\,, \tag{3.135}$$

which defines the new contact Hamiltonian for the new coordinates. In the special case $f = 1$ the generating function reduces to $\tilde{S} = S - F_1(q^k, \tilde{Q}^k, t)$, where $F_1(q^k, \tilde{Q}^k, t)$ is the generating function of the time-dependent canonical transformation.

To illustrate the formalism developed so far, we consider an important application of the time-dependent contact transformation. Systems with explicit time dependence are used for the effective description of dissipative systems within the symplectic-Hamiltonian formalism. The idea is to introduce a convenient time dependence into the Hamiltonian so that it reproduces the phenomenological equations of motion with energy dissipation. As an example, let us consider the approach by Caldirola [27] and Kanai [28] for a 1-dimensional dissipative system with a friction force linear in the velocity. This model considers the time-dependent Hamiltonian

$$H_{\mathrm{CK}} = e^{-\gamma t}\,\frac{p_{\mathrm{CK}}^2}{2m} + e^{\gamma t}\,V(q_{\mathrm{CK}})\,, \tag{3.136}$$

where p_{CK} and q_{CK} are the canonical coordinates in phase space, which are related to the physical positions and momenta by the non-canonical transformation

$$p_{\mathrm{CK}} = e^{\gamma t} p\,, \quad q_{\mathrm{CK}} = q. \tag{3.137}$$

It is easy to show that Hamilton's equations for H_{CK} as in (3.136) give the correct equation of motion for the position including the friction force, i.e. the damped Newton equation

$$\ddot{q} + \gamma\,\dot{q} + \frac{1}{m}\frac{\partial V(q)}{\partial q} = 0\,. \tag{3.138}$$

However, although this model reproduces the correct phenomenological equation of motion, it has the drawback that in order to describe dissipative systems one needs to take into account the non-canonical relationship (3.137) between canonical and physical quantities. As a consequence, at the quantum level this model has generated quite a dispute on whether it can describe a dissipative system without violating the Heisenberg uncertainty principle; we refer to e.g. the discussion in [68, 18, 20, 38] and references therein.

We may prove that the Caldirola–Kanai Hamiltonian (3.136) and the contact Hamiltonian (3.104), which both give the same damped Newtonian equation, are related by a time-dependent contact transformation with $f = e^{\gamma t}$. To do so, let us consider the Caldirola–Kanai Hamiltonian H_{CK} as a function on the

extended contact phase space $\mathcal{T}^{\rm E}$ written in the coordinates $(q_{\rm CK}, p_{\rm CK}, S_{\rm CK}, t)$ and the contact Hamiltonian $\mathscr{H}_S$ as a function on $\mathcal{T}^{\rm E}$ written in the coordinates (q, p, S, t). Defining the change of coordinates

$$(q, p, S, t) \to (q_{\rm CK} = q, p_{\rm CK} = e^{\gamma t} p, S_{\rm CK} = e^{\gamma t} S, t)\,, \tag{3.139}$$

which is the connection between canonical coordiantes and physical coordinate previously given in (3.137). Hence, it is easy to check that the conditions (3.112)-(3.114) and (3.134) are satisfied and therefore (3.139) is a time-dependent contact transformation.

3.4.3 The damped parametric oscillator

Here an important example is provided, which enables us to show the usefulness of the formalism. The example considered here is the one-dimensional damped parametric oscillator with mass m and time-dependent frequency $\omega(t)$, whose contact Hamiltonian is

$$\mathscr{H} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2(t) q^2 + \gamma\, S\,. \tag{3.140}$$

Clearly the damped harmonic oscillator is obtained for $\omega(t) = \omega_0$ and the damped free particle is recovered when $\omega(t) = 0$. The dynamics of the system is given by the contact Hamiltonian equations (3.105)-(3.107), with the time-dependent potential $V = \frac{1}{2} m\omega^2(t) q^2$. The aim is to use the tools of contact geometry to solve the dynamics. To find the solution of this contact dynamics the method of invariants of motion is employed.

As in the standard symplectic theory, an important tool to solve the contact Hamiltonian equations are the invariants (or first integrals) of the system, which are functions of the (extended) contact phase space that do not vary along the flow.

Let us then prove that $\mathscr{I}(q, p, t)$ and $\mathscr{G}(q, p, S, t)$ given in equations (3.157) and (3.158) are two invariants of the damped parametric oscillator defined by the contact Hamiltonian (3.140). An invariant is a function $\mathscr{F}$ of the (extended) contact phase space that satisfies the partial differential equation

$$-\mathscr{H}\frac{\partial \mathscr{F}}{\partial S} + p_a \left\{\mathscr{F}, \mathscr{H}\right\}_{(S, p_a)} + \left\{\mathscr{F}, \mathscr{H}\right\}_{(q^a, p_a)} = -\frac{\partial \mathscr{F}}{\partial t}\,, \tag{3.141}$$

where the same notation as in (3.103) has been used. To find a solution, one may propose the ansatz

$$\mathscr{F}(q, p, S, t) = \beta(t) p^2 - 2\xi(t) qp + \eta(t) q^2 + \zeta(t) S\,. \tag{3.142}$$

Inserting (3.142) into (3.141), one gets the system of ordinary differential equations

$$\dot{\beta} = \frac{2}{m}\xi + 2\gamma\beta - \frac{1}{2m}\zeta\,, \tag{3.143}$$

$$\dot{\eta} = -2m\omega^2\xi + \frac{1}{2} m\omega^2 \zeta\,, \tag{3.144}$$

$$\dot{\xi} = \frac{1}{m}\eta + \gamma\xi - m\omega^2\beta\,, \tag{3.145}$$

$$\dot{\zeta} = \gamma\,\zeta\,. \tag{3.146}$$

Then clearly

$$\zeta(t) = \zeta_0 e^{\gamma t}\,, \tag{3.147}$$

and one is left with the problem of solving the system (3.143)-(3.145). To do so, one considers the change of variables $\tilde{\beta}(t) = e^{-\gamma t}\beta(t)$, $\tilde{\eta}(t) = e^{-\gamma t}\eta(t)$ and $\tilde{\xi}(t) = e^{-\gamma t}\xi(t)$, which yields the equivalent system

$$\dot{\tilde{\beta}} = \frac{2}{m}\tilde{\xi} + \gamma\tilde{\beta} - \frac{\zeta_0}{2m}\,, \tag{3.148}$$

$$\dot{\tilde{\eta}} = -2m\omega\tilde{\xi} - \gamma\tilde{\eta} + \frac{\zeta_0}{2} m\omega^2\,, \tag{3.149}$$

$$\dot{\tilde{\xi}} = \frac{1}{m}\tilde{\eta} - m\omega^2\tilde{\beta}\,. \tag{3.150}$$

To solve this system, we re-write it as a third-order ordinary differential equation for $\tilde{\beta}(t)$

$$\dddot{\tilde{\beta}} + 4\,\Omega^2\,\dot{\tilde{\beta}} + 4\Omega\,\dot{\Omega}\,\tilde{\beta} = 0\,, \tag{3.151}$$

where for simplicity $\Omega^2 = \omega^2 - \frac{\gamma^2}{4}$ is defined. The above equation is known as *the normal form* of a third order equation of maximal symmetry [69].

Now, using the further change of variable

$$\tilde{\beta}(t) = \frac{1}{2m}\alpha^2(t) \tag{3.152}$$

in (3.151), one obtains that $\tilde{\beta}(t)$ is a solution of (3.151) if and only if $\alpha(t)$ is a solution of the *Ermakov equation*

$$\ddot{\alpha} + \left(\omega^2(t) - \frac{\gamma^2}{4}\right)\alpha = \frac{1}{\alpha^3}\,. \tag{3.153}$$

Moreover, from (3.152) one may re-write the remaining two equations as

$$\tilde{\eta}(t) = \frac{m}{2}\left(\left[\dot{\alpha}(t) - \frac{\gamma}{2}\alpha(t)\right]^2 + \frac{1}{\alpha^2(t)}\right)\,, \tag{3.154}$$

$$\tilde{\xi}(t) = \frac{\alpha(t)}{2}\left(\dot{\alpha}(t) - \frac{\gamma}{2}\alpha(t)\right) + \frac{1}{4}\,. \tag{3.155}$$

Finally, using (3.147), (3.152)-(3.155) and $\beta(t) = e^{\gamma t}\tilde{\beta}(t)$, $\eta(t) = e^{\gamma t}\tilde{\eta}(t)$, $\xi(t) = e^{\gamma t}\tilde{\xi}(t)$ into the ansatz (3.142), we find that

$$\mathscr{F}(q,p,S,t) = \mathscr{I}(q,p,t) + \zeta_0\mathscr{G}(q,p,S,t)\,, \tag{3.156}$$

with

$$\mathscr{I}(q,p,t) = \frac{m\,e^{\gamma\,t}}{2}\left[\left(\alpha(t)\frac{p}{m} - \left[\dot{\alpha}(t) - \frac{\gamma}{2}\alpha(t)\right]q\right)^2 + \left(\frac{q}{\alpha(t)}\right)^2\right]\,, \tag{3.157}$$

and

$$\mathscr{G}(q,p,S,t) = e^{\gamma t}\left[S - \frac{q(t)p(t)}{2}\right]\,. \tag{3.158}$$

Because $\mathscr{F}(q,p,S,t)$ is an invariant for any choice of the initial conditions and because ζ_0 only depends on the initial conditions, it follows that $\mathscr{I}(q,p,t)$ and $\mathscr{G}(q,p,S,t)$ separately are invariants of the system. As a final remark let us notice that the invariant $\mathscr{I}(q,p,t)$ is a generalization of the canonical invariant found by H. R. Lewis Jr. for the parametric oscillator[3] [16], which is recovered when $\gamma \to 0$. Besides, the invariant $\mathscr{G}$ is completely new.

With the help of these invariant we may find the solution of the contact Hamilton's equation of motion for the damped parametric oscillator as follows. To solve the equations of motion of the system (3.140) in the general case, we use the invariants $\mathscr{I}$ and $\mathscr{G}$ to define the time-dependent contact transformation

$$\tilde{Q} = \arctan\left(\alpha\left[\dot{\alpha} - \frac{\gamma}{2}\alpha\right] - \alpha^2\frac{p}{m\,q}\right)\,, \tag{3.159}$$

$$\tilde{P} = \mathscr{I}(q,p,t)\,, \tag{3.160}$$

$$\tilde{S} = \mathscr{G}(q,p,S,t)\,, \tag{3.161}$$

$$t = t\,. \tag{3.162}$$

The conformal factor in equation (3.133) for this transformation is $f = e^{\gamma t}$ and the new contact Hamiltonian, from equation (3.134), takes the simple form

$$\mathscr{K} = \frac{\mathscr{I}}{\alpha^2}\,. \tag{3.163}$$

[3]It is interesting to mention that this invariant is also found in Quantum Mechanics in the descriprion of the dissipative parametric oscillator using a logarithmic nonlinear Schrödinger equation [19].

Thus, as $\mathscr{K}$ does not involve the variables $\tilde{Q}$ and $\tilde{S}$, the new contact Hamiltonian equations have the trivial form

$$\dot{\tilde{Q}}^a = \frac{1}{\alpha^2}\,, \quad \dot{\tilde{P}}_a = 0\,, \quad \dot{\tilde{S}} = 0\,, \tag{3.164}$$

with solutions

$$\tilde{Q}(t) = \int^t \frac{\mathrm{d}\tau}{\alpha^2(\tau)}\,, \quad \tilde{P}(t) = \mathscr{I} \quad \text{and} \quad \tilde{S}(t) = \mathscr{G}\,. \tag{3.165}$$

Now, inverting the transformation (3.159)-(3.161) and using (3.165), one obtains the solutions in the original (physical) coordinates, namely

$$q(t) = \sqrt{\frac{2\mathscr{I}}{m}}\, e^{\gamma t}\alpha(t)\cos\phi(t)\,, \tag{3.166}$$

$$p(t) = \sqrt{2\,m\mathscr{I}}\, e^{\gamma t}\left[\left(\dot{\alpha} - \frac{\gamma}{2}\alpha\right)\cos\phi(t) - \frac{1}{\alpha}\sin\phi(t)\right]\,, \tag{3.167}$$

$$S(t) = \mathrm{e}^{-\gamma t}\,\mathscr{G} + \frac{q(t)p(t)}{2}\,, \tag{3.168}$$

where $\phi(t) = \tilde{Q}(t)$ and the values of the constants $\mathscr{I}$ and $\mathscr{G}$ are determined by the initial conditions. Therefore, we have derived here the solutions of the equations of motion of the damped parametric oscillator using the invariants of the contact Hamiltonian system and a proper contact transformation. From (3.166)-(3.168) one can see that all the dynamics of the system is encoded in the Ermakov equation (3.153).

Chapter 4

Conclusions and Perspectives

In this thesis we have been dealing with the major problem to show that the nonlinear dynamics is intrinsically related with the description of quantum system. The nonlinearity of the theory goes beyond performing nonlinear transformations in the Hilbert space or considering suitably an environment by introducing nonlinear terms into the Schrödinger equation. Here it has been shown that the nonlinearity of Quantum Mechanics is a fundamental part of it.

At the beginning of the analysis we started with the linear Schrödinger dynamics in the $2n$–dimensional Hilbert space $\mathcal{H}_0$. However, after taking into account the symmetry of the dynamics under the action of the dilation group, the Hilbert space has been reduced to the odd-dimensional sphere S^{2n-1} of normalized states. Moreover, the symmetry of the Schrödinger dynamics under the multiplication by a phase factor has been considered, which leads to the $(2n-2)$–dimensional complex projective space $\mathbb{CP}(\mathcal{H}_0)$, also know in the physics literature as the ray space. Then, it was been see that the complex projective space is a Hilbert space of states which are normalized and one has gotten rid of the global phase, i.e. one has a description on the space of pure states described without redundancies. For the analysis of the quantum systems in $\mathbb{CP}$ the complex homogeneous coordinates have been employed, such that in those coordinates the Schrödinger dynamics is projected onto the nonlinear Riccati dynamics.

Later, it was not only proved that the Schrödinger dynamics is projectable onto $\mathbb{CP}$, but also it has been possible to see that the Kähler structure of the Hilbert space is preserved and defined by $(\omega_{\text{FS}}, g_{\text{FS}}, J_{\text{FS}})$. This structure allows to define a Hamiltonian dynamics on $\mathbb{CP}$ whose integral curves are defined by the solutions of the matrix Riccati equation. Furthermore, the Poisson and Jordan brackets have been introduced, where the former brackets allows to obtain the evolution of the expectation values of the observables, while the second brackets are connected to the dispersion and the correlation of the observables.

Finally, even though one has a nonlinear dynamics, it has been shown that it is possible to introduce a nonlinear superposition rule on $\mathbb{CP}$, which is consistent with the linear superposition rule defined in the Hilbert space $\mathcal{H}_0$. This nonlinear superposition rule is also a consequence of working with a Lie-Scheffer system [12, 11]. Therefore it has been shown that the description of a finite quantum system is complete in the complex projective space. So, the description in this space will be denominated as *the Complex Projective picture*, in addition to the well-known pictures in Quantum Mechanics.

To continue the study of the nonlinear description of quantum dynamics, in the second Chapter of this thesis infinite dimensional quantum systems have been considered, particularly generalized coherent states evolving under the action of Hamiltonian operators quadratic in position and momentum variables. To perform the study of this kind of systems the fact has been used that in Quantum Mechanics it is possible to immerse a "classical" manifold into the Hilbert space, such that the evolution of the wave function is parametrized by the variation of points in this manifold. Besides, the motion of the wave function has been restricted to lie in a certain predetermined region of the Hilbert space. The advantage of this immersion is that there is an interplay between quantum and classical concepts that allows to consider the procedures and structures available in the classical theory to be employed in Quantum Mechanics.

The study has been started realizing that the evolution of Gaussian wave packets is encoded in the evolution of the expectation values $(\langle\hat{q}\rangle, \langle\hat{p}\rangle)$ and the complex time-dependent functions (Q, P), such that these states may be expressed as in (2.41) (or (2.43)). So, these two aspects of the generalized coherent states reside in different manifolds. On the one side the expectation values $(\langle\hat{q}\rangle, \langle\hat{p}\rangle)$ live in a Euclidean linear phase space $T^*\mathbb{R}^2$ with Hamiltonian evolution, according to the Ehrenfest theorem. On the other hand, the parameters (Q, P) live in the manifold M defined in Eq. (2.35) and are directly connected with the dispersions and correlation $(\sigma_q, \sigma_p, \sigma_{qp})$ as is shown in Eq. (2.44); besides, the dynamics in TM is Hamiltonian where the Hamiltonian equations have the same form as the classical equations of motion, but with complex variables, see Eq. (2.58).

However, the before mentioned paremetrization of the generalized coherent states is not unique. Recall that in Quantum Optics it is possible to describe these states by the so-called Squeezing parameters (τ, φ), where such parameters are the system of coordinates adapted in the hyperboloid $\mathbf{H}^2$, which is connected with $(\sigma_q, \sigma_p, \sigma_{qp})$ by Eq. (2.66). The dynamical properties in $\mathbf{H}^2$ have also been analized, showing that there is a symplectic structure defined on it and allows to see that one has a Hamiltonian dynamics which in addition is nonlinear.

To show the connection between the descriptions in M and in $\mathbf{H}^2$ an intermediate step was necessary. This is, first one has to show the connection between M and the hyperboloid $\mathbf{H}^3$ by Eq. (2.88); thus, the connection between the hyperboloids $\mathbf{H}^3$ and $\mathbf{H}^2$ turns out to be a mere consequence of the relation between the special linear group $\mathrm{SL}(2, \mathbb{R})$ and its Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, giving rise to the covering map in Eq. (2.93).

Finally, the last part of Chapter two was devoted to the nonlinear Riccati dynamics. Then, taking into account the symmetry of the dynamics $Y \in \mathfrak{X}(TM)$ under the multiplication by a global phase factor one may reduce M to a lower dimensional space known as the Siegel upper half plane $\mathbb{HP}^2$ defined as the space of complex numbers with imaginary part strictly positive. So, one not only is able to reduce the space but also it is possible to project the dynamics from M onto $\mathbb{HP}^2$, where the projected dynamics is the nonlinear Riccati evolution, which is a Hamiltonian dynamics. In this process of reduction also one may show a completely new parametrization of the Gaussian wave packets by means of the points in the Poincaré disc $\mathbb{D}^2$, which turns out to be the stereographic projection of $\mathbf{H}^2$ onto the plane, see Fig. 2.3a., and whose dynamics is also Hamiltonian with Riccati-type evolution. All the analized parametrizations and their connections are summarized in the following diagramme

$$\begin{array}{ccc} M & \xrightarrow[\nu]{} & \mathbf{H}^3 \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \chi} \\ & & \mathbf{H}^2 \\ & & \Big\downarrow{\scriptstyle v} \\ \mathbb{HP}^2 & \xleftarrow[u]{} & \mathbb{D}^2 \end{array}$$

The generalization for more degrees freedom of the reduction $M \to \mathbb{HP}$ has been considered in Ref. [71]. Given the Schrödinger equation

$$\mathrm{i}\,\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar}{2m}\nabla^2\Psi + V(\mathbf{q})\psi \tag{4.1}$$

where $\mathbf{q} \in \mathbb{R}^n$ and ∇^2 stands for the Laplacian in $\mathbb{R}^n$, it has been proven in Ref. [72] that the Gaussian wave packet solution has the parametrization

$$\psi(\mathbf{q}, t) = \frac{1}{(\pi\hbar)^{n/4}}\frac{1}{\sqrt{\det\mathbf{Q}}}\exp\left\{\frac{\mathrm{i}}{2\hbar}(\mathbf{q}-\langle\hat{\mathbf{q}}\rangle)^{\mathrm{T}}\frac{\mathbf{P}}{\mathbf{Q}}(\mathbf{q}-\langle\hat{\mathbf{q}}\rangle) + \frac{\mathrm{i}}{\hbar}\langle\hat{\mathbf{p}}\rangle\cdot(\mathbf{q}-\langle\hat{\mathbf{q}}\rangle) + \frac{\mathrm{i}}{\hbar}S(t)\right\}, \tag{4.2}$$

where $(\mathbf{Q}, \mathbf{P})$ are complex $(n \times n)$-dimensional matrices in $\mathbf{M}$ defined as

$$\mathbf{M} = \{(\mathbf{Q}, \mathbf{P}) | \bar{\mathbf{Q}}\mathbf{P} - \bar{\mathbf{P}}\mathbf{Q} = 2\,\mathrm{i}\,\mathbf{I}\}\,. \tag{4.3}$$

In order $\psi(\mathbf{q},t)$ to be solution of the Schorödinger equation, these matrices must satisfy the system of equations

$$\begin{aligned}\dot{\mathbf{Q}} &= \frac{\mathbf{P}}{m}\,,\\ \dot{\mathbf{P}} &= -\nabla^2 V\left(\mathbf{Q}\right),\end{aligned} \tag{4.4}$$

and the time dependent function $S(t)$ stands for the classical action

$$S(t) = \int_{t_0}^{t}\left(\frac{\mathbf{p}^2(\tau)}{2m} - V(\mathbf{q}(\tau))\right)\mathrm{d}\tau\,. \tag{4.5}$$

On the other hand, it has been proven in Refs. [48, 72, 71] that it is also possible to parametrize the Gaussian wave packet as

$$\psi(\mathbf{q},t) = \left(\frac{\det \mathcal{C}_{\mathrm{I}}}{(\pi\hbar)^n}\right)^{1/4}\exp\left\{\frac{\mathrm{i}}{2\hbar}(\mathbf{q}-\langle\hat{\mathbf{q}}\rangle)^{\mathrm{T}}\mathcal{C}(\mathbf{q}-\langle\hat{\mathbf{q}}\rangle) + \frac{\mathrm{i}}{\hbar}\langle\hat{\mathbf{p}}\rangle\cdot(\mathbf{q}-\langle\hat{\mathbf{q}}\rangle) + \frac{\mathrm{i}}{\hbar}\phi(t)\right\} \tag{4.6}$$

where in this generalization the $(n\times n)$-dimensional matrix $\mathcal{C} = \mathcal{C}_{\mathrm{R}} + \mathrm{i}\,\mathcal{C}_{\mathrm{I}}$ is an element of the *Siegel upper half space* [64, 5] defined as

$$\mathbb{HP} = \{\mathcal{C}|\mathcal{C}^{\mathrm{T}} = \mathcal{C}, \mathcal{C}_{\mathrm{I}} > 0\}\,, \tag{4.7}$$

which satisfies the matrix Riccati equation

$$\dot{\mathcal{C}} = -\frac{1}{m}\mathcal{C}^2 - \nabla^2 V(\mathbf{q}) \tag{4.8}$$

and the time-dependent phase is given by

$$\phi(t) = S(t) - \int_{t_0}^{t}\mathrm{tr}\{\mathcal{C}_{\mathrm{I}}(\tau)\}\mathrm{d}\tau\,. \tag{4.9}$$

The connection between the symplectic spaces $\mathbf{M}$ and $\mathbb{HP}$ was already established by Siegel in 1943 [64] and is given by the submersion

$$\pi : \mathbf{M} \to \mathbb{HP} : (\mathbf{Q},\mathbf{P}) \mapsto \mathcal{C} = \frac{\mathbf{P}}{\mathbf{Q}}\,, \tag{4.10}$$

which generalizes the Riccati transformation to more degrees of freedom. From a geometrical point of view it is provn in Ref. [71] that the submersion π is actually a projection that allows to reduce the dynamics in $\mathbf{M}$ onto the dynamics in $\mathbb{HP}$; further details will be presented in a future work.

In Chapter three a new geometric perspective for the Lagrangian and Hamiltonian description of mechanical systems has been proposed. The defining features of this formulation are that the carrier space of the mechanical system is assumed to be a contact manifold and that the evolution equations are given in an implicit way by the contact Euler–Lagrange equation (3.75) or explicitly given by the contact Hamiltonian equations (3.98). Furthermore, it has been shown that contact dynamics on the one hand recovers all the results of standard symplectic dynamics when the contact Lagrangian $\mathscr{L}$ and the contact Hamiltonian $\mathscr{H}$ do not explicitly depend on S and on the other hand can account for the evolution of systems with different types of dissipation in the more general case in which $\mathscr{L}$ or $\mathscr{H}$ depend on S. In this work we only consider cases linear in the S variable, but recently more general cases have been considered, see Ref. [73].

In this Chapter both time-independent and time-dependent contact systems have been considered and in both cases the transformations (called contact transformations) that leave the contact Hamiltonian equations invariant have been found, showing that canonical transformations of symplectic dynamics are a special case. To show the usefulness of contact transformations, an explicit example has been provided (the Caldirola–Kanai model for systems with linear dissipation) in which a non-canonical but contact transformation (3.139) allows to move from the usual time-dependent canonical description in terms of non-physical variables to a contact description in terms of the physical variables.

The contact description of Classical Mechanics is still far from being complete. For instance, to have a complete picture of the contact theory one still has to establish the *contact Hamilton–Jacobi* description; some first attempts have already been considered in Ref. [33]. On the other hand, given the importance of the symplectic perspective in the quantum mechanics of conservative systems, one may wonder if the contact perspective could play a similar role in Quantum Mechanics of dissipative systems, i.e. a relevant question is whether a quantization of this formalism is possible.

So, if one agrees that the most fundamental description of our external world is Quantum Mechanics, it makes no sense to go from classical contact theory to a quantum one. The best option is to construct a contact description from inside Quantum Mechanics. Actually, as has been pointed out in Ref. [32], there is already a contact manifold defined in quantum theory. The contact manifold we are referring to is the $(2n-1)$-dimensional sphere

$$S^{2n-1} = \{\, |\psi\rangle \in \mathcal{H} \,|\, \langle\psi|\psi\rangle = 1 \,\} \tag{4.11}$$

of normalized vectors in $\mathcal{H}$ and the dynamical system is generated by a contact Hamiltonian system as has been proven in [32]. However, one could proceed in a different way, i.e. constructing first a contact Hilbert manifold; let us sketch briefly such a possibility. As has been seen in the first Chapter, it is possible to give a description of quantum states in the complex projective space, in addition this picture of Quantum Mechanics has a Kähler structure defined on it $(\omega_{\text{FS}}, g_{\text{FS}}, J_{\text{FS}})$. Using this description as a reference system one could enlarge this space adding an extra dimension obtaining a *contact Hermitian manifold* and the dynamics may be described by a contact Hamiltonian system. This description of dissipation in Quantum Mechanics will be completely new and is presented in future work.

The contact Hamiltonian theory has been well received in the physics community and has now been applied in classical theories as Thermodynamics [74, 75, 76], Fluid Theory [77, 78], Astrophysics [79, 80], Relativity [81, 82, 83] and Information Geometry [84] to mention some examples.

Appendix A

Reduction of Dynamical Systems

The *reduction of a dynamical* system is employed in all the branches of theoretical physics. In fact, this procedure is often used in the physics literature without always having been mentioned explicitly and sometimes without a careful definition. The purpose of this Appendix is to give an introduction of the formal idea of the reduction procedure and how to deal with projections based on the results in references [4, 5, 6].

Let us consider a time independent vector field Γ on a differential manifold M of dimension μ. Then, given a dynamical system one attempts to integrate it, i.e. to find the integral curves of Γ. To integrate Γ one could go from Γ to a *reduced* vector field $\tilde{\Gamma}$ defined on a manifold N of dimension $\nu < \mu$. Then in some way one has partially solved the problem of integration of the system and one may hope that the reduced dynamics $\tilde{\Gamma}$ is easier to integrate.

Roughly speaking, to applied the reduction procedure one proceeds as follows. One attempts to find a foliation Φ on the carrier space M on which Γ lives. Then, assuming that the quotient set $N = M/\Phi$ has a differential manifold structure and that Γ is *projectable* onto $\tilde{\Gamma}$ with respect to the projection $\pi : M \to N$, one then tries to integrate lower dimensional dynamics $\tilde{\Gamma}$. In the next sections every step of this procedure is discussed in detail.

A.1 Foliations

A foliation of a differential manifold M is a family $\{\ell_\alpha\}$ of disjoint subsets ℓ_α (the *leaves* of the foliation) one of them is passing through each point $m \in M$, on each of which a differential structure may be given. So, the natural injection $i : \ell_\alpha \to M$ is an embedding[1]. Moreover, it shall be required that all of the ℓ_α be of the same dimension, that they all be connected, and that as submanifold they all be regular.[2]

For example, the solutions of a differential equation constitute a foliation by curves. A foliation by curves is a geometric object, whose associated differential equation is considered the infinitesimal expression of such an object. To show that the solutions of the differential equation constitute a foliation, it is sufficient to recalled that from the theorem of existence and uniqueness of solutions of analytic differential equations for each point $m \in M$ only one integral curve is passing through it; then, the set of integral curves constitutes an infinite family of analytic curves whose intersection is empty.

In general to obtain a foliation on a manifold is not an easy task; however, in physical systems there are three ways to generate foliations: i) submersions, ii) distributions and iii) differential forms.

[1] An embedding is an injective mapping φ from a connected manifold, for which $T\varphi$ is also injective.

[2] All these assumptions are not strictly necessary but we are considering them for simplicity, and so we will not get tied up in details. A more general discussion of foliation can be found in [4].

Submersions

Because the leaves of the foliation are submanifolds, submersions are helpful in understanding foliations. A submersion is a mapping $\varphi : M \to N$, with N being a differential manifold and $\dim(N) = \nu \leq \dim(M) = \mu$, such that $T\varphi$ is surjective. It follows that $\varphi^{-1}(n)$, $n \in N$, is a *regular* submanifold of M of dimension $\mu - \nu$. Then, the family of these submanifolds, as n runs through N, is a foliation of M whose codimentsion is ν. This foliation is said to be *generated* by the submersion.

To see how one generates a foliation by submertion in physics, let us recalled that to integrate a physical system one may employ the so-called *constants of motion*. Then, the submersion is defined as the function $F = (f_1, f_2, \ldots, f_\nu)$, where each $\{f_k\}_{k=1} \in \mathfrak{F}(M)$ is a constants of motion. This introduces an invariant submanifold that serves as a carrier manifold of lower dimension for the dynamical system.

Distribution

A *distribution* on M is a set of vector fields, all of the same dimension κ, defined as follows: at each point $m \in M$ the distribution $\mathcal{D}$ consists of a subspace of $T_m M$ such that there is a neighbourhood U around m in which it is possible to find κ independent vector fields $X_j \in \mathfrak{X}(U)$, with the property that $X_j(m')$, with $j = 1, \ldots, \kappa$, span $\mathcal{D}_{m'}$ at all points $m' \in U$. One may thus think of $\mathcal{D}$ as the set of linear spaces formed by the X_κ with coefficients in $\mathfrak{F}(U)$, then local vector fields can be said to belong or not to a given distribution.

Foliations Φ generate distributions, namely for each $m \in M$ one may chose $T_m \ell_\alpha$ to be a subspace of TM_m which defines $\mathcal{D}_\Phi$ at that point, i.e.

$$T_m \ell_\alpha = \mathcal{D}_\Phi(m)\,. \tag{A.1}$$

However, the converse is not true, because not every distribution can be integrated to form a foliation. When for a given distribution $\mathcal{D}$ Eq. (A.1) can be solved for the ℓ_α, then ℓ_α is called an *integral manifold* of $\mathcal{D}$, and $\mathcal{D}$ is said to be *integrable*. The Frobenius theorem tells us that a necessary and sufficient condition in order that $\mathcal{D}$ be integrable is that it be *involutive* [5], i.e. that

$$X, Y \in \mathcal{D} \quad \text{then} \quad [X, Y] \in \mathcal{D}\,. \tag{A.2}$$

Therefore every involutive distribution is also integrable.

Differential forms

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A third consideration to generate foliations is by *differential forms*. Let ω be a p-form of M. Its kernel, consisting of all $X \in \mathfrak{X}(M)$ such that $i_X \omega = 0$, may then form a distribution $\mathcal{D}$ if the dimension of span $\{X(m)\}$ is the same for every $m \in M$ and for all $X \in \ker \omega$. But even if $\mathcal{D}$ exists, it may not be involutive. The condition on ω, which is implied by the Frobenius theorem, is somewhat more complicated than the condition (A.2) on the distribution itself. It may be stated in the following way. Let X, $Y \in \ker \omega$, then

$$i_{[X,Y]}\omega = (\pounds_X i_Y - i_Y \pounds_X)\omega = -i_Y i_X \mathrm{d}\omega\,. \tag{A.3}$$

and thus

$$i_Y i_X \mathrm{d}\omega = 0 \qquad \text{for all} \quad X, Y \in \ker \omega \tag{A.4}$$

is a necessary and sufficient condition for involutivity of $\ker \omega$. It is sufficient, for example, that $\mathrm{d}\omega = 0$ or, less stringently, that there exists a one-form λ such that $\mathrm{d}\omega = \lambda \wedge \omega$. It follows, therefore, that closed p-forms through their kernels generate foliations.

When a foliation is used in analyzing a dynamical system it is important to understand how the system moves from one leaf to another, i.e. how the index α of the ℓ_α changes with time. This requires an understanding of the *quotient* of M with respect to the foliation.

Let $\Phi = \{\ell_\alpha\}$ be a foliation of M. Then Φ defines an equivalence relation r_Φ on M according to

$$(m_1, m_2) \in r_\Phi \quad \text{iff} \quad m_1,\, m_2 \in \ell_\alpha \quad \text{for some } \alpha\,. \tag{A.5}$$

Then the foliation defines the quotient set M/r_Φ with respect to the equivalence. We write simply M/Φ for the quotient set M/r_Φ. The point here is that because the leaves of the foliation need not be invariant submanifolds of the motion, the integral curves will in general intersect more than one leaf (the dynamical point will move from leaf to leaf). This motion from leaf to leaf is in fact the projected motion, and it takes place on M/Φ. Let

$$\pi : M \to M/\Phi : m \to [m] \tag{A.6}$$

be the canonical projection, where $[m]$ is the equivalence class of m. What will actually be required is that $M/\Phi = N$ be a quotient manifold, i.e. that a differential structure on M/Φ can be found such that π is a submersion. For the case in which the foliation is given by a submersion π, the submersion and the projection are the same map.

It is opportune at this point to add a few words about symplectic manifolds. A submersion given by a function F of the type already mentioned can sometimes be used to obtain more than one foliation on a symplectic manifold. Each function f_i in F uniquely defines a Hamiltonian vector field X_j through

$$i_{X_j}\omega = \mathrm{d}f_j\,, \tag{A.7}$$

(here ω is the symplectic form on M), and under certain conditions the X_j so defined form an involutive distribution $\mathcal{D}^X$. When this is so, one has two foliations: the first, which may be called Φ^F, is defined by the submersion F, the second, which may be called Φ^X, is defined by the distribution $\mathcal{D}^X$. These two foliations intersect and they have interesting and important properties for the reductions of dynamical systems on symplectic manifolds. The condition on the f_j, incidentally, in order that the foliation Φ^F exists (i.e. that the distribution $\mathcal{D}^X$ is involutive) is that they form a *function group*. This means that the Poisson brackets of any two functions is not independent of the function in the set [4, 5]. A converse procedure is also possible on a symplectic manifold. If an involutive distribution $\mathcal{D}^X$ has the property that it is spanned by a set of Hamiltonian vector fields, then their Hamiltonian function can also generate a second foliation.

A.2 Projectability

As already mentioned, the point of the foliation is to study not the entire motion of the system, but how it moves from leaf to leaf of the foliation M, i.e. how it moves in $N = M/\Phi = \pi(M)$. The first question is whether there exists a vector in $\mathfrak{X}(N)$ which describes the motion, or as is said, whether Γ is *projectable*.

Let Φ a foliation of the differential manifold M and assume that $M/\Phi \equiv N$ can be provided with a manifold structure. Let $\pi : M \to N$ be the projection with respect to the foliation. Then a vector field $Y \in \mathfrak{X}(M)$ is said to be *projectable with respect* to π (or *with respect to* Φ) iff a vector field $\tilde{Y} \in \mathfrak{X}(N)$ exists such that the following diagram commutes:

$$\begin{array}{ccc} TM & \xrightarrow{T\pi} & TN \\ \downarrow{\scriptstyle Y} & & \downarrow{\scriptstyle \tilde{Y}} \\ M & \xrightarrow{\pi} & N \end{array}$$

When $\tilde{Y}$ exists, it is called the projection of Y.

The second definition is the following. Let $g \in \mathfrak{F}(M)$ be any function constant on the leaves of Φ. Then there exists a unique $\tilde{g} \in \mathfrak{F}(N)$ such that

$$g = \tilde{g} \circ \pi\,. \tag{A.8}$$

Conversely, for every $\tilde{g} \in \mathfrak{F}(N)$ the function g defined by (A.8) is unique and constant on the leaves of Φ. In terms of such functions we have the following definition of projectability. A vector field $Y \in \mathfrak{X}(M)$ is projectable with respect to Φ iff for every $\tilde{g} \in \mathfrak{F}(N)$ there exists a $\tilde{g}' \in \mathfrak{F}(N)$ such that

$$g' \equiv \pounds_Y g = \pounds_Y(\tilde{g} \circ \pi) = \tilde{g}' \circ \pi \,. \tag{A.9}$$

In other words, the Lie derivative with respect to Y of every function constant on leaves is itself a function constant on the leaves. The equivalence of the commutative diagram and (A.9) can be demostrated by noticing first that (A.9) defines a derivation and hence the vector field $Y' \in \mathfrak{X}(M)$, for further details see reference [4].

If what we want to project is the dynamical vector field, one must have a way of finding a foliation with respect to which that vector field is projectable. Because a foliation will be generated by submersions, distributions or differential forms, what must be found are relations between these geometrical objects and the vector field of interest which will guarantee that the vector field is projectable with respect to the foliation generated.

Projection by submersions

As before, we consider first submersion in its simple functional form. Let us assume that the regular foliation Φ is generated by a submersion $\varphi : M \to N$. Then essentially everything that was said about projection remains valid, except that π must be replaced everywhere by φ.

Projection by distribution

Let us consider now distributions. If Φ is a foliation generated by a involutive distribution $\mathcal{D}$, then any function g constant on leaves has the property that $\pounds_X g = 0$ for all $X \in \mathcal{D}$, and conversely. Therefore Y is projectable with respect to Φ iff:

$$\pounds_X g = 0 \quad \text{for all} \quad X \in \mathcal{D} \quad \text{then} \quad \pounds_X \pounds_Y g = 0 \quad \text{for all} \quad X \in \mathcal{D} \,. \tag{A.10}$$

A necessary and sufficient conditions for (A.10) is that $[X, Y] \in \mathcal{D}$ for all $X \in \mathcal{D}$. Therefore a vector field Y is projectable with respect to a foliation generated by an involutive distribution D iff:

$$[Y, \mathcal{D}] = \pounds_Y \mathcal{D} \subset \mathcal{D} \,, \tag{A.11}$$

or, as may be said, iff $\mathcal{D}$ is invariant under Y.

Projection by differential forms

Finally, suppose that the foliation is generated by a differential form $\omega \in \Omega^p(M)$ through the involutive distribution $\mathcal{D} \in \ker \omega$ (of course we assume that ω satisfies the requirements of the Frobenius theorem). A vector field Y is projectable with respect to the foliation so generated if (A.11) is satisfied. But, since $\mathcal{D}$ is given by the kernel of ω, this means that for any $X \in \mathcal{D}$, the vector field $[X, Y]$ must be also in $\ker \omega$, or that:

$$0 = i_{[X,Y]}\omega = \pounds_Y i_X \omega - i_x \pounds_Y \omega = -i_X \pounds_Y \omega \quad \forall\, X \in \mathcal{D} \,. \tag{A.12}$$

In other words, $\ker \omega \subset \ker \pounds_Y \omega$.

As before, when dealing with differential forms the conditions become less transparent, and we therefore state a sufficient condition. If ω is what may be called *conformally invariant* under Y, i.e. if there exists an $f \in \mathfrak{F}(M)$ such that:

$$L_Y \omega = f\omega \,, \tag{A.13}$$

then (A.12) is satisfied. Therefore a conformally invariant, closed (this is a sort of double sufficient condition) differential form generates a foliation with respect to which the vector field is projectable.

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