Calculus of Variations

A fractional Hadamard formula and applications

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Abstract

We derive a shape derivative formula for the family of principal Dirichlet eigenvalues $\lambda_s(\Omega)$ of the fractional Laplacian $(-\Delta)^s$ associated with bounded open sets $\Omega \subset \mathbb{R}^N$ of class $C^{1,1}$. This extends, with a help of a new approach, a result in Dalibard and Gérard-Varet (Calc. Var. 19(4):976–1013, 2013) which was restricted to the case $s = \frac{1}{2}$. As an application, we consider the maximization problem for $\lambda_s(\Omega)$ among annular-shaped domains of fixed volume of the type $B \setminus \overline{B}'$, where B is a fixed ball and B' is ball whose position is varied within B. We prove that $\lambda_s(B \setminus \overline{B}')$ is maximal when the two balls are concentric. Our approach also allows to derive similar results for the fractional torsional rigidity. More generally, we will characterize one-sided shape derivatives for best constants of a family of subcritical fractional Sobolev embeddings.

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1 Introduction

Let $s \in (0, 1)$ and $\Omega \subset \mathbb{R}^N$ be a bounded open set. The present paper is devoted to the study of best constants $\lambda_{s,p}(\Omega)$ in the family of subcritical Sobolev inequalities

$$\lambda_{s,p}(\Omega)\|u\|^2_{L^p(\Omega)} \leq [u]^2_s \qquad \text{for all } u \in \mathcal{H}^s_0(\Omega), \tag{1.1}$$

where $p \in [1, \frac{2N}{N-2s})$ if $2s < N$ and $p \in [1, \infty)$ if $2s \geq N = 1$. Here, the Sobolev space $\mathcal{H}^s_0(\Omega)$ is given as completion of $C^\infty_c(\Omega)$ with respect to the norm $[\,\cdot\,]_s$ defined by

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$$[u]_s^2 = \frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} dx dy \qquad \text{with} \quad c_{N,s} = \pi^{-\frac{N}{2}} s 4^s \frac{\Gamma(\frac{N}{2} + s)}{\Gamma(1-s)}. \tag{1.2}$$

The normalization constant $c_{N,s}$ is chosen such that $[u]_s^2 = \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi$ for $u \in \mathcal{H}_0^s(\Omega)$, where $\hat{u}$ denotes the Fourier transform of u. The best (i.e., largest possible) constant in (1.1) is given by

$$\lambda_{s,p}(\Omega) := \inf \left\{ [u]_s^2 : u \in \mathcal{H}_0^s(\Omega), \quad \|u\|_{L^p(\Omega)} = 1 \right\}. \tag{1.3}$$

As a consequence of the subcriticality assumption on p and the boundedness of Ω, the space $\mathcal{H}_0^s(\Omega)$ compactly embeds into $L^p(\Omega)$. Therefore a direct minimization argument shows that $\lambda_{s,p}(\Omega)$ admits a nonnegative minimizer $u \in \mathcal{H}_0^s(\Omega)$ with $\|u\|_{L^p(\Omega)} = 1$. Moreover, every such minimizer solves, in the weak sense, the semilinear problem

$$(-\Delta)^s u = \lambda_{s,p}(\Omega) u^{p-1} \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \tag{1.4}$$

where $(-\Delta)^s$ stands for the fractional Laplacian. It therefore follows from regularity theory and the strong maximum principle for $(-\Delta)^s$ that u is strictly positive in Ω, see Lemma 2.3 below. We recall that, for functions $\varphi \in C_c^{1,1}(\mathbb{R}^N)$, the fractional Laplacian is given by

$$(-\Delta)^s \varphi(x) = c_{N,s} \, PV \int_{\mathbb{R}^N} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}} \, dy = \frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \frac{2\varphi(x) - \varphi(x+y) - \varphi(x-y)}{|y|^{N+2s}} \, dy.$$

Of particular interest are the cases $p = 1$ and $p = 2$ which correspond to the fractional torsion problem

$$(-\Delta)^s u = \lambda_{s,1}(\Omega) \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \tag{1.5}$$

and the eigenvalue problem

$$(-\Delta)^s u = \lambda_{s,2}(\Omega) u \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \tag{1.6}$$

associated with the first Dirichlet eigenvalue of the fractional Laplacian, respectively. In these cases, the minimization problem for $\lambda_{s,p}(\Omega)$ in (1.3) possesses a unique positive minimizer. Indeed, it is a well-known consequence of the fractional maximum principle that (1.5) admits a unique solution, and that (1.6) has a unique positive eigenfunction with $\|u\|_{L^2(\Omega)} = 1$. Incidentally, the uniqueness of positive minimizers extends to the full range $1 \leq p \leq 2$, as we shall show in Lemma A.1 in the appendix of this paper.

Our first goal in this paper is to analyze the dependence of the best constants on the underlying domain Ω. For this we shall derive a formula for a one-sided shape derivative of the map $\Omega \mapsto \lambda_{s,p}(\Omega)$. We assume from now on that $\Omega \subset \mathbb{R}^N$ is a bounded open set of class $C^{1,1}$, and we consider a family of deformations $\{\Phi_\varepsilon\}_{\varepsilon \in (-1,1)}$ with the following properties:

$$\begin{aligned} &\Phi_\varepsilon \in C^{1,1}(\mathbb{R}^N; \mathbb{R}^N) \text{ for } \varepsilon \in (-1,1), \ \Phi_0 = \mathrm{id}_{\mathbb{R}^N}, \text{ and} \\ &\text{the map } (-1,1) \to C^{0,1}(\mathbb{R}^N, \mathbb{R}^N), \varepsilon \to \Phi_\varepsilon \text{is of class } C^2. \end{aligned} \tag{1.7}$$

We note that (1.7) implies that $\Phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ is a global diffeomorphism if $|\varepsilon|$ is small enough, see e.g. [7, Chapter 4.1]. To clarify, we stress that we only need the C^2-dependence of Φ_ε on ε with respect to Lipschitz-norms, while Φ_ε is assumed to be a $C^{1,1}$-function for $\varepsilon \in (-1,1)$ to guarantee $C^{1,1}$-regularity of the perturbed domains $\Phi_\varepsilon(\Omega)$.

From the variational characterization of $\lambda_{s,p}(\Omega)$ it is not difficult to see that the map $\varepsilon \mapsto \lambda_{s,p}(\Phi_\varepsilon(\Omega))$ is continuous. However, since $\lambda_{s,p}(\Omega)$ may not have a unique positive

minimizer, we cannot expect this map to be differentiable. We therefore rely on determining the right derivative of $\varepsilon \mapsto \lambda_{s,p}(\Phi_\varepsilon(\Omega))$ from which we derive differentiability whenever $\lambda_{s,p}(\Omega)$ admits a unique positive minimizer, thereby extending the classical Hadamard shape derivative formula for the first Dirichlet eigenvalue of the Laplacian $-\Delta$.

Throughout this paper, we consider a fixed function $\delta \in C^{1,1}(\mathbb{R}^N)$ which coincides with the *signed* distance function $\text{dist}(\cdot, \mathbb{R}^N \setminus \Omega) - \text{dist}(\cdot, \Omega)$ in a neighborhood of the boundary $\partial\Omega$. We note here that, since we assume that Ω is of class $C^{1,1}$, the signed distance function is also of class $C^{1,1}$ in a neighborhood of $\partial\Omega$ but not globally on $\mathbb{R}^N$. We also suppose that δ is chosen with the property that δ is positive in Ω and negative in $\mathbb{R}^N \setminus \overline{\Omega}$, as it is the case for the signed distance function.

Our first main result is the following.

Theorem 1.1 *Let $\lambda_{s,p}(\Omega)$ be given by* (1.3) *and consider a family of deformations Φ_ε satisfying* (1.7). *Then the map $\varepsilon \mapsto \theta(\varepsilon) := \lambda_{s,p}(\Phi_\varepsilon(\Omega))$ is right differentiable at $\varepsilon = 0$. Moreover,*

$$\partial_+\theta(0) = \min\left\{\Gamma(1+s)^2 \int_{\partial\Omega} (u/\delta^s)^2 X \cdot \nu \, dx \; : \; u \in \mathcal{H}\right\}, \tag{1.8}$$

where ν denotes the interior unit normal on $\partial\Omega$, $\mathcal{H}$ is the set of positive minimizers for $\lambda_{s,p}(\Omega)$ and $X := \partial_\varepsilon\big|_{\varepsilon=0}\Phi_\varepsilon$.

Here the function u/δ^s is defined on $\partial\Omega$ as a limit. Namely, for $x_0 \in \partial\Omega$, the limit

$$\frac{u}{\delta^s}(x_0) = \lim_{\substack{x \to x_0 \\ x \in \Omega}} \frac{u}{\delta^s}(x) \tag{1.9}$$

exists, as the function u/δ^s extends to a function in $C^\alpha(\overline{\Omega})$ for some $\alpha > 0$, see [20]. In addition, the function $\delta^{1-s}\nabla u$ also admits a Hölder continuous extension on $\overline{\Omega}$ satisfying $\delta^{1-s}\nabla u \cdot \nu = su/\delta^s$ on $\partial\Omega$, see [8]. As a consequence, the expression u/δ^s, restricted on $\partial\Omega$, plays the role of an inner fractional normal derivative. Note that, for $s = 1$, the limit on the RHS of (1.9) coincides with the classical inner normal derivative of u at x_0.

We observe that the constant $\Gamma(1+s)^2$ appears also in the fractional Pohozaev identity, see e.g. [21]. This is, to some extend, not surprising at least in the classical case since Pohozaev's identity can be obtained using techniques of domain variation, see e.g. [23].

We also remark that one-sided derivatives naturally arise in the analysis of parameter-dependent minimization problems, see e.g. [7, Section 10.2.3] for an abstract result in this direction. Related to this, they also appear in the analysis of the domain dependence of eigenvalue problems with possible degeneracy, see e.g. [11] and the references therein.

A natural consequence of Theorem 1.1 is that the map $\varepsilon \mapsto \theta(\varepsilon) = \lambda_{s,p}(\Phi_\varepsilon(\Omega))$ is differentiable at $\varepsilon = 0$ whenever $\lambda_{s,p}(\Omega)$ admits a unique positive minimizer. Indeed, applying Theorem 1.1 to the map $\varepsilon \mapsto \widetilde{\theta}(\varepsilon) := \lambda_{s,p}(\Phi_{-\varepsilon}(\Omega))$ yields

$$\partial_-\theta(0) = -\partial_+\widetilde{\theta}(0) = \max\left\{\Gamma(1+s)^2 \int_{\partial\Omega} (u/\delta^s)^2 X \cdot \nu \, dx \; : \; u \in \mathcal{H}\right\},$$

where $\mathcal{H}$ is given as in Theorem 1.1. As a consequence, we obtain the following result.

Corollary 1.2 *Let $\lambda_{s,p}(\Omega)$ be given by* (1.3) *and consider a family of deformations Φ_ε satisfying* (1.7). *Suppose that $\lambda_{s,p}(\Omega)$ admits a unique positive minimizer $u \in \mathcal{H}_0^s(\Omega)$. Then the map $\varepsilon \mapsto \theta(\varepsilon) = \lambda_{s,p}(\Phi_\varepsilon(\Omega))$ is differentiable at $\varepsilon = 0$. Moreover*

$$\theta'(0) = \Gamma(1+s)^2 \int_{\partial\Omega} (u/\delta^s)^2 X \cdot \nu \, dx, \tag{1.10}$$

where $X := \partial_\varepsilon\big|_{\varepsilon=0} \Phi_\varepsilon$.

As mentioned earlier, $\lambda_{s,p}(\Omega)$ admits a unique positive minimizer $u \in \mathcal{H}_0^s(\Omega)$ for $1 \le p \le 2$, see Lemma A.1 in the appendix. Therefore Corollary 1.2 extends, in particular, the classical Hadamard formula, for the first Dirichlet eigenvalue $\lambda_{1,2}(\Omega)$ of $-\Delta$, to the fractional setting. We recall, see e.g. [15], that the classical Hadamard formula is given by

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \lambda_{1,2}(\Phi_\varepsilon(\Omega)) = \int_{\partial\Omega} |\nabla u|^2 X \cdot \nu \, dx. \tag{1.11}$$

An analogue of Corollary 1.2 for the case of the local r-Laplace operator was obtained in [2,12]. We also point out that, prior to this paper, a Hadamard formula in the fractional setting of the type (1.10) was obtained in [6] for the special case $p = 1$, $s = \frac{1}{2}$, $N = 2$ and Ω of class C^∞. We are not aware of any other previous work related to Theorem 1.1 or 1.2 in the fractional setting.

Our next result provides a characterization of constrained local minima of $\lambda_{s,p}$. Here and in the following, we call a bounded open subset Ω of class $C^{1,1}$ a constrained local minimum for $\lambda_{s,p}$ if for all families of deformations Φ_ε satisfying (1.7) and the volume invariance condition $|\Phi_\varepsilon(\Omega)| = |\Omega|$ for $\varepsilon \in (-1, 1)$, there exists $\varepsilon_0 \in (0, 1)$ with $\lambda_{s,p}(\Phi_\varepsilon(\Omega)) \ge \lambda_{s,p}(\Omega)$ for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Our classification result reads as follows.

Corollary 1.3 *Let $p \in \{1\} \cup [2, \infty)$. If an open subset Ω of $\mathbb{R}^N$ of class C^3 is a volume constrained local minimum for $\Omega \mapsto \lambda_{s,p}(\Omega)$, then Ω is a ball.*

Corollary 1.3 is a consequence of Theorem 1.1, from which we derive that if Ω is a constrained local minimum then any element $u \in \mathcal{H}$ satisfies the overdetermined condition $u/\delta^s \equiv constant$ on $\partial\Omega$. Therefore by the rigidity result in [9] we find that Ω must be a ball. We point out that we are not able to include the case $p \in (1, 2)$ in Corollary 1.3, since the rigidity result in [9] is based on the moving plane method and therefore requires the nonlinearity in (1.4) to be Lipschitz. The case $p \in (1, 2)$ therefore remains an open problem in Corollary 1.3.

We note that the authors in [6] considered the shape minimization problem for $\lambda_{s,p}(\Omega)$ in the case $p = 1$, $s = \frac{1}{2}$, $N = 2$ among domains Ω of class C^∞ of fixed volume. They showed in [6] that such minimizers are discs.

Next we consider the optimization problem of $\Omega \mapsto \lambda_{s,p}(\Omega)$ for $p \in \{1, 2\}$ and Ω a punctured ball, with the hole having the shape of ball. We show that, as the hole moves in Ω then $\lambda_{s,p}(\Omega)$ is maximal when the two balls are concentric. In the local case $s = 1$ and $N = 2$, this is a classical result by Hersch [16]. For subsequent generalizations in the case of the local problem, see [5,14,18].

Theorem 1.4 *Let $p \in \{1, 2\}$, $B_1(0)$ be the unit centered ball and $\tau \in (0, 1)$. Define*

$$\mathcal{A} := \{a \in B_1(0) \; : \; B_\tau(a) \subset B_1(0)\}.$$

Then the map $\mathcal{A} \to \mathbb{R}$, $a \mapsto \lambda_{s,p}(B_1(0) \setminus \overline{B_\tau(a)})$ takes its maximum at $a = 0$.

The proof of Theorem 1.4 is inspired by the argument given in [14,18] for the local case $s = 1$. It uses the fractional Hadamard formula in Corollary 1.2 and maximum principles for anti-symmetric functions. Our proof also shows that the map $a \mapsto \lambda_{s,p}(B_1(0) \setminus \overline{B_\tau(a)})$ takes its minimum when the boundary of the ball $B_\tau(a)$ touches the one of $B_1(0)$, see Sect. 5 below.

The proof of Theorem 1.1 is based on the use of test functions in the variational characterization of $\lambda_{s,p}(\Omega)$ and $\lambda_{s,p}(\Phi_\varepsilon(\Omega))$. The general strategy is inspired by the direct approach

in [11], which is related to a Neumann eigenvalue problem on manifolds. In the case of $\lambda_{s,p}(\Phi_\varepsilon(\Omega))$, it is important to make a change of variables so that $\lambda_{s,p}(\Phi_\varepsilon(\Omega))$ is determined by minimizing an ε-dependent family of seminorms among functions $u \in \mathcal{H}_0^s(\Omega)$, see Sect. 2 below. An obvious choice of test functions are minimizers u and v_ε for $\lambda_{s,p}(\Omega)$ and $\lambda_{s,p}(\Phi_\varepsilon(\Omega))$, respectively. However, due to the fact that u is only of class C^s up to the boundary, we cannot obtain a boundary integral term directly from the divergence theorem. In particular, the integration by parts formula given in [21, Theorem 1.9] does not apply to general vector fields X which appear in (1.8). Hence, we need to replace u with $\zeta_k u$, where ζ_k is a cut-off function vanishing in a $\frac{1}{k}$-neighborhood of $\partial\Omega$. This leads to upper and lower estimates of $\lambda_{s,p}(\Phi_\varepsilon(\Omega))$ up to order $o(\varepsilon)$, where the first order term is given by an integral involving $(-\Delta)^s(\zeta_k u)$ and $\nabla(\zeta_k u)$. We refer the reader to Sect. 4 below for more precise information. A highly nontrivial task is now to pass to the limit as $k \to \infty$ in order to get boundary integrals involving $\psi := u/\delta^s$. This is the most difficult part of the paper. We refer to Proposition 2.4 and Sect. 6 below for more details.

The paper is organized as follows. In Sect. 2, we provide preliminary results on convergence properties of integral functional, inner approximations of functions in $\mathcal{H}_0^s(\Omega)$ and on properties of minimizers of (1.3). In Sect. 3, we introduce notation related to domain deformations and related quantities. In Sect. 4 we establish a preliminary variant of Theorem 1.1, which is given in Proposition 4.1. In this variant, the constant $\Gamma(1+s)^2$ in (1.8) is replaced by an implicitly given value which still depends on cut-off data. The proofs of the main results, as stated in this introduction, are then completed in Sect. 5. Finally, Sect. 6 is devoted to the proof of the main technical ingredient of the paper, which is given by Proposition 2.4.

2 Notations and preliminary results

Throughout this section, we fix a bounded open set $\Omega \subset \mathbb{R}^N$. As noted in the introduction, we define the space $\mathcal{H}_0^s(\Omega)$ as completion of $C_c^\infty(\Omega)$ with respect to the norm $[\,\cdot\,]_s$ given in (1.2). Then $\mathcal{H}_0^s(\Omega)$ is a Hilbert space with scalar product

$$(u, v) \mapsto [u, v]_s = \frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy,$$

where $c_{N,s}$ is given in (1.2). It is well known and easy to see that $\mathcal{H}_0^s(\Omega)$ coincides with the closure of $C_c^\infty(\Omega)$ in the standard fractional Sobolev space $H^s(\mathbb{R}^N)$. Moreover, if Ω has a continuous boundary, then $\mathcal{H}_0^s(\Omega)$ admits the highly useful characterization

$$\mathcal{H}_0^s(\Omega) = \left\{ w \in L_{loc}^1(\mathbb{R}^N) \; : \; [w]_s^2 < \infty, \quad w \equiv 0 \text{ on } \mathbb{R}^N \setminus \Omega \right\}, \tag{2.1}$$

see e.g. [13, Theorem 1.4.2.2]. We start with an elementary but useful observation.

Lemma 2.1 *Let $\mu \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$, and let $(v_k)_k$ be a sequence in $\mathcal{H}_0^s(\Omega)$ with $v_k \to v$ in $\mathcal{H}_0^s(\Omega)$ as $k \to \infty$. Then we have*

$$\lim_{k \to \infty} \int_{\mathbb{R}^{2N}} \frac{(v_k(x) - v_k(y))^2 \mu(x, y)}{|x - y|^{N+2s}} dx dy = \int_{\mathbb{R}^{2N}} \frac{(v(x) - v(y))^2 \mu(x, y)}{|x - y|^{N+2s}} dx dy.$$

Proof We have

$$\left| \int_{\mathbb{R}^{2N}} \frac{(v_k(x) - v_k(y))^2 - (v(x) - v(y))^2 \mu(x, y)}{|x - y|^{N+2s}} dx dy \right|$$

$$\leq \|\mu\|_{L^\infty} \int_{\mathbb{R}^{2N}} \frac{|(v_k(x) - v_k(y))^2 - (v(x) - v(y))^2|}{|x - y|^{N+2s}} dxdy,$$

where

$$\int_{\mathbb{R}^{2N}} \frac{|(v_k(x) - v_k(y))^2 - (v(x) - v(y))^2|}{|x - y|^{N+2s}} dxdy$$

$$= \int_{\mathbb{R}^{2N}} \frac{|[(v_k(x) - v(x)) - (v_k(y) - v(y))][(v_k(x) + v(x)) - (v_k(y) + v(y))]|}{|x - y|^{N+2s}} dxdy$$

$$\leq \frac{2}{c_{N,s}} [v_k - v]_s [v_k + v]_s \rightarrow 0 \qquad \text{as } k \rightarrow \infty.$$

$\square$

Throughout the remainder of this paper, we fix $\rho \in C_c^\infty(-2, 2)$ with $0 \leq \rho \leq 1$, $\rho \equiv 1$ on $(-1, 1)$, and we define

$$\zeta \in C^\infty(\mathbb{R}), \qquad \zeta(t) = 1 - \rho(t). \tag{2.2}$$

Moreover, for $k \in \mathbb{N}$, we define the functions

$$\rho_k, \; \zeta_k \; \in \; C^{1,1}(\mathbb{R}^N), \qquad \rho_k(x) = \rho(k\delta(x)), \quad \zeta_k(x) = \zeta(k\delta(x)). \tag{2.3}$$

We note that the function ρ_k is supported in the $\frac{2}{k}$-neighborhood of the boundary, while the function ζ_k vanishes in the $\frac{1}{k}$-neighborhood of the boundary.

Lemma 2.2 *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and let $u \in \mathcal{H}_0^s(\Omega)$. Moreover, for $k \in \mathbb{N}$, let $u_k := u\zeta_k \in \mathcal{H}_0^s(\Omega)$ denote inner approximations of u. Then we have*

$$u_k \rightarrow u \qquad \text{in } \mathcal{H}_0^s(\Omega).$$

Proof In the following, the letter $C > 0$ stands for various constants independent of k. Since $\rho_k = 1 - \zeta_k$, it suffices to show that

$$u\rho_k \in \mathcal{H}_0^s(\Omega) \text{ for } k \text{ sufficiently large} \qquad \text{and} \qquad [u\rho_k]_s \rightarrow 0 \text{ as } k \rightarrow \infty. \tag{2.4}$$

For $\varepsilon > 0$, we put $A_\varepsilon = \{x \in \Omega \; : \; \delta(x) < \varepsilon\}$. Since $u\rho_k$ vanishes in $\mathbb{R}^N \setminus A_{\frac{2}{k}}$, $0 \leq \rho_k \leq 1$ on $\mathbb{R}^N$ and $|\rho_k(x) - \rho_k(y)| \leq C \min\{k|x - y|, 1\}$ for $x, y \in \mathbb{R}^N$, we observe that

$$\frac{1}{c_{N,s}} [\rho_k u]_s^2 = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[u(x)\rho_k(x) - u(y)\rho_k(y)]^2}{|x - y|^{N+2s}} dydx$$

$$= \frac{1}{2} \int_{A_{\frac{4}{k}}} \int_{A_{\frac{4}{k}}} \frac{[u(x)\rho_k(x) - u(y)\rho_k(y)]^2}{|x - y|^{N+2s}} dydx$$

$$+ \int_{A_{\frac{2}{k}}} u(x)^2 \rho_k(x)^2 \int_{\mathbb{R}^N \setminus A_{\frac{4}{k}}} |x - y|^{-N-2s} dydx$$

$$\leq \frac{1}{2} \int_{A_{\frac{4}{k}}} \int_{A_{\frac{4}{k}}} \frac{\left[u(x)\left(\rho_k(x) - \rho_k(y)\right) + \rho_k(y)\left(u(x) - u(y)\right)\right]^2}{|x - y|^{N+2s}} dydx$$

$$+ C \int_{A_{\frac{2}{k}}} u(x)^2 \text{dist}(x, \mathbb{R}^N \setminus A_{\frac{4}{k}})^{-2s} dx$$

$$\leq \int_{A_{\frac{4}{k}}} u^2(x) \int_{A_{\frac{4}{k}}} \frac{(\rho_k(x) - \rho_k(y))^2}{|x-y|^{N+2s}} \, dy dx + \int_{A_{\frac{4}{k}}} \int_{A_{\frac{4}{k}}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dy dx$$

$$+ C \int_{A_{\frac{2}{k}}} u(x)^2 \delta^{-2s}(x) dx$$

$$\leq Ck^2 \int_{A_{\frac{4}{k}}} u^2(x) \int_{B_{\frac{1}{k}}(x)} |x-y|^{2-2s-N} dy dx$$

$$+ C \int_{A_{\frac{4}{k}}} u^2(x) \int_{\mathbb{R}^N \setminus B_{\frac{1}{k}}(x)} |x-y|^{-N-2s} dy dx$$

$$+ \int_{A_{\frac{4}{k}}} \int_{A_{\frac{4}{k}}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dy dx + C \int_{A_{\frac{2}{k}}} u(x)^2 \delta^{-2s}(x) dx$$

$$\leq Ck^{2s} \int_{A_{\frac{4}{k}}} u^2(x) dx + \int_{A_{\frac{4}{k}}} \int_{A_{\frac{4}{k}}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dy dx + C \int_{A_{\frac{2}{k}}} u(x)^2 \delta^{-2s}(x) dx$$

$$\leq C \int_{A_{\frac{4}{k}}} u^2(x) \delta^{-2s}(x) dx + \int_{A_{\frac{4}{k}}} \int_{A_{\frac{4}{k}}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dy dx. \tag{2.5}$$

Now, since Ω has a Lipschitz boundary, using $\int_{\mathbb{R}^N \setminus \Omega} |x-y|^{-N-2s} \, dy \sim \delta^{-2s}(x)$ see e.g [3], we get

$$\int_{\Omega} u^2(x) \delta^{-2s}(x) dx \leq C \int_{\Omega} u^2(x) \int_{\mathbb{R}^N \setminus \Omega} |x-y|^{-N-2s} \, dy dx \leq C[u]_s^2,$$

and therefore

$$\int_{A_{\frac{4}{k}}} u^2(x) \delta^{-2s}(x) dx \to 0 \qquad \text{as } k \to \infty. \tag{2.6}$$

Moreover, since also

$$\int_{\Omega} \int_{\Omega} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dy dx \leq \frac{2}{c_{N,s}} [u]_s^2,$$

we have

$$\int_{A_{\frac{4}{k}}} \int_{A_{\frac{4}{k}}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dy dx \to 0 \qquad \text{as } k \to \infty. \tag{2.7}$$

Combining (2.5)–(2.7), we obtain (2.4), as required. □

From now on, we fix a bounded $C^{1,1}$-domain $\Omega \subset \mathbb{R}^N$. We also let

$$C_0^s(\overline{\Omega}) = \left\{ w \in C^s(\overline{\Omega}) : w = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\},$$

and we recall the following regularity and positivity properties of nonnegative minimizers for $\lambda_{s,p}(\Omega)$ as defined in (1.3).

Lemma 2.3 *Let* $u \in \mathcal{H}_0^s(\Omega)$ *be a nonnegative minimizer for* $\lambda_{s,p}(\Omega)$. *Then* $u \in C_0^s(\overline{\Omega}) \cap C_{loc}^\infty(\Omega)$. *Moreover,* $\psi := \frac{u}{\delta^s} \in C^\alpha(\overline{\Omega})$ *for some* $\alpha \in (0,1)$, *and there exists a constant* $c = c(N,s,\Omega,\alpha,p) > 0$ *with the property that*

$$\|\psi\|_{C^\alpha(\overline{\Omega})} \leq c \tag{2.8}$$

and

$$|\nabla \psi(x)| \leq c\delta^{\alpha-1}(x) \quad \text{for all } x \in \Omega. \tag{2.9}$$

Moreover, $\psi > 0$ on $\overline{\Omega}$, so in particular $u > 0$ in Ω.

Proof By standard arguments in the calculus of variations, u is a weak solution of (1.4). By [22, Proposition 1.3] we have that $u \in L^\infty(\Omega)$, and therefore the RHS of (1.4) is a function in $L^\infty(\Omega)$. Thus the regularity up to the boundary $u \in C_0^s(\overline{\Omega})$ is proved in [20], where also the C^α-bound (2.8) for the function $\psi = \frac{u}{\delta^s}$ is established for some $\alpha > 0$. Moreover, (2.9) is proved in [8]. It also follows from (1.4), the strong maximum principle and the Hopf lemma for the fractional Laplacian that ψ is a strictly positive function on $\overline{\Omega}$. In particular, $u > 0$ in Ω, Therefore $u \in C_{loc}^\infty(\Omega)$ follows by interior regularity theory (see e.g. [19]) and the fact that the function $t \mapsto t^{p-1}$ is of class C^∞ on $(0, \infty)$. □

The computation of one-sided shape derivatives as given in Theorem 1.1 will be carried out in Sect. 4, and it requires the following key technical proposition. Since its proof is long and quite involved, we postpone the proof to Sect. 6 below.

Proposition 2.4 *Let $X \in C^0(\overline{\Omega}, \mathbb{R}^N)$, let $u \in C_0^s(\overline{\Omega}) \cap C^1(\Omega)$, and assume that $\psi := \frac{u}{\delta^s}$ extends to a function on $\overline{\Omega}$ satisfying (2.8) and (2.9). Moreover, put $U_k := u\zeta_k \in C_c^{1,1}(\Omega)$, where ζ_k is defined in (2.3). Then*

$$\lim_{k\to\infty} \int_\Omega \nabla U_k \cdot X \left(u(-\Delta)^s \zeta_k - I(u, \zeta_k) \right) dx = -\kappa_s \int_{\partial\Omega} \psi^2 X \cdot \nu \, dx,$$

where

$$\kappa_s := -\int_\mathbb{R} h'(r)(-\Delta)^s h(r) \, dr \quad \text{with} \quad h(r) := r_+^s \zeta(r) = \max(r, 0)^s \zeta(r) \tag{2.10}$$

and ζ given in (2.2), and where we use the notation

$$I(u, v)(x) := c_{N,s} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dy \tag{2.11}$$

for $u \in C_c^s(\mathbb{R}^N)$, $v \in C^{0,1}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$.

Remark 2.5 The minus sign in the definition of the constant κ_s in (2.10) might appear a bit strange at first glance. We shall see later that, defined in this way, κ_s has a positive value. A priori it is not clear that the value of κ_s does not depend on the particular choice of the function ζ. This follows a posteriori once we have established in Proposition 4.1 below that this constant appears in Theorem 1.1. This will then allow us to show that $\kappa_s = \frac{\Gamma(1+s)^2}{2}$ by applying the resulting shape derivative formula to a one-parameter family of concentric balls, see Sect. 5 below. A more direct, but somewhat lengthy computation of κ_s is possible via the logarithmic Laplacian, which has been introduced in [4].

3 Domain perturbation and the associated variational problem

Here and in the following, we define $\Omega_\varepsilon := \Phi_\varepsilon(\Omega)$. In order to study the dependence of $\lambda_{s,p}(\Omega_\varepsilon)$ on ε, it is convenient to pull back the problem on the fixed domain Ω via a change

of variables. For this we let $\text{Jac}_{\Phi_\varepsilon}$ denote the Jacobian determinant of the map $\Phi_\varepsilon \in C^{1,1}(\mathbb{R}^N)$, and we define the kernels

$$K_\varepsilon(x, y) := c_{N,s} \frac{\text{Jac}_{\Phi_\varepsilon}(x)\text{Jac}_{\Phi_\varepsilon}(y)}{|\Phi_\varepsilon(x) - \Phi_\varepsilon(y)|^{N+2s}} \quad \text{and} \quad K_0(x, y) = c_{N,s} \frac{1}{|x - y|^{N+2s}}. \quad (3.1)$$

Then (1.7) gives rise to the well known expansions

$$\text{Jac}_{\Phi_\varepsilon}(x) = 1 + \varepsilon \text{div} X(x) + O(\varepsilon^2), \qquad \partial_\varepsilon \text{Jac}_{\Phi_\varepsilon}(x) = \text{div} X(x) + O(\varepsilon) \qquad (3.2)$$

uniformly in $x \in \mathbb{R}^N$, where $X := \partial_\varepsilon\big|_{\varepsilon=0} \Phi_\varepsilon \in C^{0,1}(\mathbb{R}^N; \mathbb{R}^N)$ and therefore $\text{div} X$ is a.e. defined on $\mathbb{R}^N$. From (1.7), we also get

$$|\Phi_\varepsilon(x) - \Phi_\varepsilon(y)|^{-N-2s} = |x - y|^{-N-2s} \left(1 + 2\varepsilon \frac{x - y}{|x - y|} \cdot P_X(x, y) + O(\varepsilon^2)\right)^{-\frac{N+2s}{2}},$$

and

$$\partial_\varepsilon |\Phi_\varepsilon(x) - \Phi_\varepsilon(y)|^{-N-2s} = |x - y|^{-N-2s} \left(-(N + 2s) \frac{x - y}{|x - y|} \cdot P_X(x, y) + O(\varepsilon)\right),$$

uniformly in $x, y \in \mathbb{R}^N$, $x \neq y$ with

$$P_X \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N), \qquad P_X(x, y) = \frac{X(x) - X(y)}{|x - y|}.$$

Moreover by (3.2) and the fact that $\partial_\varepsilon \Phi_\varepsilon$, $X \in C^{0,1}(\mathbb{R}^N)$, we have that

$$K_\varepsilon(x, y) = K_0(x, y) + \varepsilon \partial_\varepsilon\big|_{\varepsilon=0} K_\varepsilon(x, y) + O(\varepsilon^2) K_0(x, y), \qquad (3.3)$$

and

$$\partial_\varepsilon K_\varepsilon(x, y) = \partial_\varepsilon\big|_{\varepsilon=0} K_\varepsilon(x, y) + O(\varepsilon) K_0(x, y), \qquad (3.4)$$

uniformly in $x, y \in \mathbb{R}^N$, $x \neq y$, where

$$\partial_\varepsilon\big|_{\varepsilon=0} K_\varepsilon(x, y) = -\left[(N + 2s) \frac{x - y}{|x - y|} \cdot P_X(x, y) - (\text{div} X(x) + \text{div} X(y))\right] K_0(x, y). \tag{3.5}$$

In particular, it follows from (3.3) and (3.5) that there exist $\varepsilon_0, C > 0$ with the property that

$$\frac{1}{C} K_0(x, y) \leq K_\varepsilon(x, y) \leq C K_0(x, y) \qquad \text{for all } x, y \in \mathbb{R}^N, x \neq y \text{ and } \varepsilon \in (-\varepsilon_0, \varepsilon_0). \tag{3.6}$$

For $v \in \mathcal{H}_0^s(\Omega)$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we now define

$$\mathcal{V}_v(\varepsilon) := \frac{1}{2} \int_{\mathbb{R}^{2N}} (v(x) - v(y))^2 K_\varepsilon(x, y) dx dy. \tag{3.7}$$

Then, by (1.3), (1.7) and a change of variables, we have the following variational characterization for $\lambda_{s,p}(\Omega_\varepsilon)$:

$$\lambda_{s,p}^\varepsilon := \lambda_{s,p}(\Omega_\varepsilon) = \inf\left\{[u]_s^2 \ : \ u \in \mathcal{H}_0^s(\Omega_\varepsilon), \ \int_{\Omega_\varepsilon} |u|^p \, dx = 1\right\}$$

$$= \inf\left\{\mathcal{V}_v(\varepsilon) \ : \ v \in \mathcal{H}_0^s(\Omega), \ \int_\Omega |v|^p \text{Jac}_{\Phi_\varepsilon}(x) \, dx = 1\right\} \quad \text{for } \varepsilon \in (-\varepsilon_0, \varepsilon_0). \quad (3.8)$$

As mentioned earlier, we prefer to use (3.8) from now on where the underlying domain is fixed and the integral terms depend on ε instead. It follows from (3.3) and (3.4) that, for given $v \in \mathcal{H}_0^s(\Omega)$, the function $\mathcal{V}_v : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ is of class C^1 with

$$\mathcal{V}_v'(0) = \frac{1}{2} \int_{\mathbb{R}^{2N}} (v(x) - v(y))^2 \partial_\varepsilon \big|_{\varepsilon=0} K_\varepsilon(x, y) dx dy, \tag{3.9}$$

where $\partial_\varepsilon \big|_{\varepsilon=0} K_\varepsilon(x, y)$ is given in (3.5),

$$|\mathcal{V}_v'(0)| \le C[v]_s^2 \quad \text{with a constant } C > 0 \tag{3.10}$$

and we have the expansions

$$\mathcal{V}_v(\varepsilon) = \mathcal{V}_v(0) + \varepsilon \mathcal{V}_v'(0) + O(\varepsilon^2)[v]_s^2, \quad \mathcal{V}_v'(\varepsilon) = \mathcal{V}_v'(0) + O(\varepsilon)[v]_s^2 \tag{3.11}$$

with $O(\varepsilon)$, $O(\varepsilon^2)$ independent of v. From (3.2), (3.6) and the variational characterization (3.8), it is easy to see that

$$\frac{1}{C} \le \lambda_{s,p}^\varepsilon \le C \quad \text{for all } \varepsilon \in (-\varepsilon_0, \varepsilon_0) \text{with some constant } C > 0.$$

Using this and (3.2), (3.6) once more, we can show that

$$\frac{1}{C} \le \|v_\varepsilon\|_{L^p(\Omega)} \le C \quad \text{and} \quad \frac{1}{C} \le [v_\varepsilon]_s \le C. \tag{3.12}$$

for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and every minimizer $v_\varepsilon \in \mathcal{H}_0^s(\Omega)$ for (3.8) with a constant $C > 0$.

The following lemma is essentially a corollary of Lemma 2.1.

Lemma 3.1 *Let $(v_k)_k$ be a sequence in $\mathcal{H}_0^s(\Omega)$ with $v_k \to v$ in $\mathcal{H}_0^s(\Omega)$. Then we have*

$$\lim_{k\to\infty} \mathcal{V}_{v_k}(0) = \mathcal{V}_v(0) \quad \text{and} \quad \lim_{k\to\infty} \mathcal{V}_{v_k}'(0) = \mathcal{V}_v'(0).$$

Proof The first limit is trivial since $\mathcal{V}_v(0) = [v]_s^2$ for $v \in \mathcal{H}_0^s(\Omega)$. The second limit follows from Lemma 2.1, (3.5) and (3.9) by noting that $\mu \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ for the function

$$\mu(x, y) = -(N + 2s) \frac{x - y}{|x - y|} \cdot P_X(x, y) + (\text{div} X(x) + \text{div} X(y)).$$

$\square$

4 One-sided shape derivative computations

We keep using the notation of the previous sections, and we recall in particular the variational characterization of $\lambda_{s,p}^\varepsilon = \lambda_{s,p}(\Omega_\varepsilon)$ given in (3.8). The aim of this section is to prove the following result.

Proposition 4.1 *We have*

$$\partial_\varepsilon^+ \big|_{\varepsilon=0} \lambda_{s,p}^\varepsilon = \min \left\{ 2\kappa_s \int_{\partial\Omega} (u/\delta^s)^2 X \cdot \nu \, dx \; : \; u \in \mathcal{H} \right\},$$

where $\mathcal{H}$ is the set of positive minimizers for $\lambda_{s,p}^0 := \lambda_{s,p}(\Omega)$, $X := \partial_\varepsilon \big|_{\varepsilon=0} \Phi_\varepsilon$ and κ_s is given by (2.10).

The proof of Proposition 4.1 requires several preliminary results. We start with a formula for the derivative of the function given by (3.7).

Lemma 4.2 *Let $U \in C_c^{1,1}(\Omega)$. Then*

$$\mathcal{V}_U'(0) = -2 \int_{\mathbb{R}^N} \nabla U \cdot X(-\Delta)^s U dx. \tag{4.1}$$

Proof By (3.5), (3.11) and Fubini's theorem, we have

$$
\begin{aligned}
\mathcal{V}_U'(0) &= \frac{-(N+2s)c_{N,s}}{2} \int_{\mathbb{R}^{2N}} (U(x) - U(y))^2 \frac{(x-y) \cdot (X(x) - X(y))}{|x-y|^{N+2s+2}} dx dy \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^{2N}} (U(x) - U(y))^2 K_0(x,y)(\mathrm{div}X(x) + \mathrm{div}X(y)) dx dy \\
&= \frac{-(N+2s)c_{N,s}}{2} \lim_{\mu \to 0} \int_{|x-y|>\mu} (U(x) - U(y))^2 \frac{(x-y) \cdot (X(x) - X(y))}{|x-y|^{N+2s+2}} dx dy \\
&\quad + \int_{\mathbb{R}^{2N}} (U(x) - U(y))^2 K_0(x,y)\mathrm{div}X(x) dx dy \\
&= -(N+2s)c_{N,s} \lim_{\mu \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N \setminus \overline{B_\mu(y)}} (U(x) - U(y))^2 \frac{(x-y) \cdot X(x)}{|x-y|^{N+2s+2}} dx dy \\
&\quad + \int_{\mathbb{R}^{2N}} (U(x) - U(y))^2 K_0(x,y)\mathrm{div}X(x) dx dy
\end{aligned}
$$

Applying, for fixed $y \in \mathbb{R}^N$ and $\mu > 0$, the divergence theorem in the domain $\{x \in \mathbb{R}^N : |x-y| > \mu\}$ and using that $\nabla_x |x-y|^{-N-2s} = -(N+2s)\frac{x-y}{|x-y|^{N+2s+2}}$, we obtain

$$
\begin{aligned}
\mathcal{V}_U'(0) &= c_{N,s} \lim_{\mu \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N \setminus \overline{B_\mu(y)}} (U(x) - U(y))^2 \nabla_x |x-y|^{-N-2s} \cdot X(x) dx dy \\
&\quad + \int_{\mathbb{R}^{2N}} (U(x) - U(y))^2 K_0(x,y)\mathrm{div}X(x) dx dy \\
&= - \lim_{\mu \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N \setminus \overline{B_\mu(y)}} (U(x) - U(y))^2 K_0(x,y)\mathrm{div}X(x) dx dy \\
&\quad - \lim_{\mu \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N \setminus \overline{B_\mu(y)}} (U(x) - U(y))\nabla U(x) \cdot X(x) K_0(x,y) dx dy \\
&\quad + \lim_{\mu \to 0} \int_{\mathbb{R}^N} \int_{\partial B_\mu(y)} (U(x) - U(y))^2 \frac{y-x}{|x-y|} \cdot X(x) K_0(x,y) \, d\sigma(y) \, dx \\
&\quad + \int_{\mathbb{R}^{2N}} (U(x) - U(y))^2 K_0(x,y)\mathrm{div}X(x) dx dy \\
&= - \lim_{\mu \to 0} \int_{|x-y|>\mu} (U(x) - U(y))\nabla U(x) \cdot X(x) K_0(x,y) d(x,y) \\
&\quad + \lim_{\mu \to 0} \mu^{-N-1-2s} \int_{|x-y|=\mu} (U(x) - U(y))^2 (y-x) \cdot X(x) \, d\sigma(x,y) \\
&= -\frac{c_{N,s}}{2} \lim_{\mu \to 0} \int_{\mathbb{R}^N} \nabla U(x) \cdot X(x) \int_{\mathbb{R}^N \setminus \overline{B_\mu(0)}} \frac{2U(x) - U(x+z) - U(x-z)}{|z|^{N+2s}} dz dx \\
&\quad + \frac{1}{2} \lim_{\mu \to 0} \mu^{-N-1-2s} \int_{|x-y|=\mu} (U(x) - U(y))^2 (y-x) \cdot (X(x) - X(y)) \, d\sigma(x,y)
\end{aligned}
$$

$$\tag{4.2}$$

Since $U \in C_c^{1,1}(\Omega)$, we have that

$$\frac{c_{N,s}}{2} \lim_{\mu \to 0} \int_{\mathbb{R}^N} \nabla U(x) \cdot X(x) \int_{\mathbb{R}^N \setminus \overline{B_\mu(0)}} \frac{2U(x) - U(x+z) - U(x-z)}{|z|^{N+2s}} dz dx$$

$$= \frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \nabla U(x) \cdot X(x) \int_{\mathbb{R}^N} \frac{2U(x) - U(x+z) - U(x-z)}{|z|^{N+2s}} dz dx$$

$$= \int_{\mathbb{R}^N} (-\Delta)^s U(x) \nabla U(x) \cdot X(x) dx. \tag{4.3}$$

Moreover, since U is compactly supported, we may fix $R > 0$ large enough such that $(U(x) - U(y))^2 = 0$ for all $x, y \in B_R(0)$ with $|x - y| < 1$. Setting $N_\mu := \{(x, y) \in B_R(0) \times B_R(0) : |x - y| = \mu\}$ for $0 < \mu < 1$ and using that $U, X \in C^{0,1}(\mathbb{R}^N)$, we thus deduce that

$$\mu^{-N-1-2s} \int_{|x-y|=\mu} (U(x) - U(y))^2 (y - x) \cdot (X(x) - X(y)) \, d\sigma(x, y)$$

$$= \mu^{-N-1-2s} \int_{N_\mu} (U(x) - U(y))^2 (y - x) \cdot (X(x) - X(y)) \, d\sigma(x, y) = O(\mu^{3-1-2s}) \to 0, \tag{4.4}$$

as $\mu \to 0$, since the $2N - 1$-dimensional measure of the set N_μ is of order $O(N - 1)$ as $\mu \to 0$. The claim now follows by combining (4.2)–(4.4). $\qquad\square$

We cannot apply Lemma 4.2 directly to minimizers $u \in \mathcal{H}_0^s(\Omega)$ of $\lambda_{s,p}(\Omega)$ since these are not contained in $C_c^{1,1}(\Omega)$. The aim is therefore to apply Lemma 4.2 to $U_k := u\zeta_k \in C_c^{1,1}(\Omega)$ with ζ_k given in (2.3), and to use Proposition 2.4. This leads to the following derivative formula which plays a key role in the proof of Proposition 4.1.

Lemma 4.3 *Let $u \in \mathcal{H}_0^s(\Omega)$ be a solution to (1.4). Then we have*

$$\mathcal{V}_u'(0) = \frac{2\lambda_{s,p}(\Omega)}{p} \int_\Omega u^p \, div \, X \, dx + 2\kappa_s \int_{\partial\Omega} (u/\delta^s)^2 X \cdot \nu \, dx.$$

Proof By Lemma 2.3 and since Ω is of class $C^{1,1}$, we have $U_k := u\zeta_k \in C_c^{1,1}(\Omega) \subset \mathcal{H}_0^s(\Omega)$ for $k \in \mathbb{N}$, and $U_k \to u$ in $\mathcal{H}_0^s(\Omega)$ by Lemma 2.2. Consequently, $\mathcal{V}_u'(0) = \lim_{k \to \infty} \mathcal{V}_{U_k}'(0)$ by Corollary 3.1, so it remains to show that

$$\lim_{k \to \infty} \mathcal{V}_{U_k}'(0) = \frac{2\lambda_{s,p}(\Omega)}{p} \int_\Omega u^p \mathrm{div} X \, dx + 2\kappa_s \int_{\partial\Omega} (u/\delta^s)^2 X \cdot \nu \, dx. \tag{4.5}$$

Applying Lemma 4.2 to U_k, we find that

$$\mathcal{V}_{U_k}'(0) = -2 \int_{\mathbb{R}^N} \nabla U_k \cdot X (-\Delta)^s U_k dx \quad \text{for } k \in \mathbb{N}.$$

By the standard product rule for the fractional Laplacian, we have $(-\Delta)^s U_k = u(-\Delta)^s \zeta_k + \zeta_k(-\Delta)^s u - I(u, \zeta_k)$ with $I(u, \zeta_k)$ given by (2.11). We thus obtain

$$\mathcal{V}_{U_k}'(0) = -2 \int_{\mathbb{R}^N} \nabla U_k \cdot X \zeta_k (-\Delta)^s u \, dx - 2 \int_{\mathbb{R}^N} [\nabla U_k \cdot X] u(-\Delta)^s \zeta_k \, dx$$

$$+ 2 \int_{\mathbb{R}^N} \nabla U_k \cdot X I(u, \zeta_k) \, dx$$

$$= -2\lambda_{s,p}(\Omega) \int_\Omega \nabla U_k \cdot X \zeta_k u^{p-1} \, dx - 2 \int_{\mathbb{R}^N} \nabla U_k \cdot X \Big(u(-\Delta)^s \zeta_k - I(u, \zeta_k) \Big) dx, \tag{4.6}$$

where we used that $(-\Delta)^s u = \lambda_{s,p}(\Omega) u^{p-1}$ in Ω. Consequently, Proposition 2.4 yields that

$$\lim_{k\to\infty} \mathcal{V}'_{U_k}(0) = -2\lambda_{s,p}(\Omega) \lim_{k\to\infty} \int_\Omega \nabla U_k \cdot X \zeta_k u^{p-1}\, dx + 2\kappa_s \int_{\partial\Omega} \psi^2 X \cdot v\, dx. \quad (4.7)$$

Moreover, integrating by parts, we obtain, for $k \in \mathbb{N}$,

$$\int_\Omega [\nabla U_k \cdot X]\zeta_k u^{p-1}\, dx = \frac{1}{p} \int_\Omega [\nabla u^p \cdot X]\zeta_k^2\, dx + \int_\Omega [\nabla \zeta_k \cdot X]\zeta_k u^p\, dx$$

$$= -\frac{1}{p} \int_\Omega u^p \operatorname{div} X \zeta_k^2\, dx - \frac{2}{p} \int_\Omega u^p \zeta_k [X \cdot \nabla \zeta_k]\, dx + \int_\Omega u^p \zeta_k [X \cdot \nabla \zeta_k]\, dx. \quad (4.8)$$

Since $u^p \in C_0^s(\overline{\Omega})$ by Lemma 2.3, it is easy to see from the definition of ζ_k that the last two terms in (4.8) tend to zero as $k \to \infty$, whereas

$$\lim_{k\to\infty} \int_\Omega u^p \operatorname{div} X \zeta_k^2\, dx = \int_\Omega u^p \operatorname{div} X\, dx.$$

Hence

$$\lim_{k\to\infty} \int_\Omega \nabla U_k \cdot X \zeta_k u^{p-1}\, dx = -\frac{1}{p} \int_\Omega u^p \operatorname{div} X\, dx.$$

Plugging this into (4.7), we obtain (4.5), as required. □

Our next lemma provides an upper estimate for $\partial_\varepsilon^+\big|_{\varepsilon=0} \lambda_{s,p}^\varepsilon$.

Lemma 4.4 *Let $u \in \mathcal{H}$ be a positive minimizer for $\lambda_{s,p}^0 = \lambda_{s,p}(\Omega)$. Then*

$$\limsup_{\varepsilon\to 0^+} \frac{\lambda_{s,p}^\varepsilon - \lambda_{s,p}^0}{\varepsilon} \leq 2\kappa_s \int_{\partial\Omega} (u/\delta^s)^2 X \cdot v\, dx. \quad (4.9)$$

Proof For $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we define

$$j(\varepsilon) := \frac{\mathcal{V}_u(\varepsilon)}{\tau(\varepsilon)} \quad \text{for } k \in \mathbb{N} \text{with} \quad \tau(\varepsilon) := \left(\int_\Omega |u|^p \operatorname{Jac}_{\Phi_\varepsilon}(x)\, dx \right)^{2/p}.$$

By (3.8), we then have $\lambda_{s,p}^\varepsilon \leq j(\varepsilon)$ for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Moreover,

$$\tau(0) = \|u\|_{L^p(\Omega)}^{2/p} = 1, \quad \mathcal{V}_u(0) = [u]_s^2 = \lambda_{s,p}(\Omega) \quad \text{and} \quad j(0) = \frac{\mathcal{V}_u(0)}{\tau(0)} = \lambda_{s,p}^0,$$

which implies that

$$\partial_\varepsilon^+\big|_{\varepsilon=0} \lambda_{s,p}^\varepsilon \leq j'(0) = 2\kappa_s \int_{\partial\Omega} (u/\delta^s)^2 X \cdot v\, dx,$$

by Lemma 4.3 and (3.2), as claimed. □

Next, we shall prove a lower estimate for $\partial_\varepsilon^+\big|_{\varepsilon=0} \lambda_{s,p}^\varepsilon$.

Lemma 4.5 *We have*

$$\liminf_{\varepsilon\searrow 0^+} \frac{\lambda_{s,p}^\varepsilon - \lambda_{s,p}^0}{\varepsilon} \geq \inf \left\{ 2\kappa_s \int_{\partial\Omega} (u/\delta^s)^2 X \cdot v\, dx \; : \; u \in \mathcal{H} \right\}.$$

Proof Let $(\varepsilon_n)_n$ be a sequence of positive numbers converging to zero and with the property that

$$\lim_{n \to \infty} \frac{\lambda_{s,p}^{\varepsilon_n} - \lambda_{s,p}^0}{\varepsilon_n} = \liminf_{\varepsilon \searrow 0^+} \frac{\lambda_{s,p}^{\varepsilon} - \lambda_{s,p}^0}{\varepsilon}. \tag{4.10}$$

For $n \in \mathbb{N}$, we let v_{ε_n} be a positive minimizer corresponding to the variational characterization of $\lambda_{s,p}^{\varepsilon_n}$ given in (3.8), i.e. we have

$$\lambda_{s,p}^{\varepsilon_n} = \mathcal{V}_{v_{\varepsilon_n}}(\varepsilon_n) \quad \text{and} \quad \int_{\Omega} v_{\varepsilon_n}^p \mathrm{Jac}_{\Phi_{\varepsilon_n}} dx = 1. \tag{4.11}$$

Since v_{ε_n} remains bounded in $\mathcal{H}_0^s(\Omega)$ by (3.12), we may pass to a sub-sequence with the property that $v_{\varepsilon_n} \rightharpoonup u$ in $\mathcal{H}_0^s(\Omega)$ for some $u \in \mathcal{H}_0^s(\Omega)$. Moreover, $v_{\varepsilon_n} \to u$ in $L^p(\Omega)$ as $n \to \infty$ since the embedding $\mathcal{H}_0^s(\Omega) \to L^p(\Omega)$ is compact. In the following, to keep the notation simple, we write ε in place of ε_n. By (3.10), (3.11) and (4.11), we have

$$\mathcal{V}_{v_\varepsilon}(0) = \mathcal{V}_{v_\varepsilon}(\varepsilon) - \varepsilon \mathcal{V}'_{v_\varepsilon}(0) + O(\varepsilon^2)[v_\varepsilon]_s^2 = \lambda_{s,p}^\varepsilon - \varepsilon \mathcal{V}'_{v_\varepsilon}(0) + O(\varepsilon^2) = \lambda_{s,p}^\varepsilon + O(\varepsilon) \tag{4.12}$$

and therefore

$$\mathcal{V}_u(0) = [u]_s^2 \leq \liminf_{\varepsilon \to 0}[v_\varepsilon]_s^2 = \liminf_{\varepsilon \to 0} \mathcal{V}_{v_\varepsilon}(0) \leq \limsup_{\varepsilon \to 0} \lambda_{s,p}^\varepsilon \leq \lambda_{s,p}^0, \tag{4.13}$$

where the last inequality follows from Lemma 4.4. In view of (3.2) and the strong convergence $v_\varepsilon \to u$ in $L^p(\Omega)$, we see that

$$1 = \int_{\Omega} v_\varepsilon^p \mathrm{Jac}_{\Phi_\varepsilon} dx = \int_{\Omega} v_\varepsilon^p (1 + \varepsilon \mathrm{div} X) dx + O(\varepsilon^2) = \int_{\Omega} u^p \, dx + o(1) \tag{4.14}$$

as $\varepsilon \to 0$, and hence $\|u\|_{L^p(\Omega)} = 1$. Combining this with (4.13), we see that $u \in \mathcal{H}$ is a minimizer for $\lambda_{s,p}^0$, and that equality must hold in all inequalities of (4.13). From this we deduce that

$$v_\varepsilon \to u \text{ strongly in } \mathcal{H}_0^s(\Omega). \tag{4.15}$$

Now (4.12) and the variational characterization of $\lambda_{s,p}^0$ imply that

$$\lambda_{s,p}^0 \left(\int_{\Omega} v_\varepsilon^p dx \right)^{2/p} \leq \mathcal{V}_{v_\varepsilon}(0) = \lambda_{s,p}(\Omega_\varepsilon) - \varepsilon \mathcal{V}'_{v_\varepsilon}(0) + O(\varepsilon^2) \tag{4.16}$$

whereas by (4.14) we have

$$\int_{\Omega} v_\varepsilon^p \, dx = 1 - \varepsilon \int_{\Omega} v_\varepsilon^p \mathrm{div} X dx + O(\varepsilon^2) = 1 - \varepsilon \int_{\Omega} u^p \mathrm{div} X dx + o(\varepsilon)$$

and therefore

$$\left(\int_{\Omega} v_\varepsilon^p dx \right)^{2/p} = 1 - \frac{2\varepsilon}{p} \int_{\Omega} u^p \mathrm{div} X dx + o(\varepsilon). \tag{4.17}$$

Plugging this into (4.16), we get the inequality

$$\lambda_{s,p}^\varepsilon \geq \left(1 - \frac{2\varepsilon}{p} \int_{\Omega} u^p \mathrm{div} X dx \right) \lambda_{s,p}^0 + \varepsilon \mathcal{V}'_{v_\varepsilon}(0) + o(\varepsilon).$$

Since, moreover, $\mathcal{V}'_{v_\varepsilon}(0) \to \mathcal{V}'_u(0)$ as $\varepsilon \to 0$ by Lemma 3.1 and (4.15), it follows that

$$\lambda^\varepsilon_{s,p} - \lambda^0_{s,p} \geq \varepsilon\left(\mathcal{V}'_u(0) - \frac{2\lambda^0_{s,p}}{p}\int_\Omega u^p \mathrm{div} X dx\right) + o(\varepsilon)$$

and therefore

$$\lambda^\varepsilon_{s,p} - \lambda^0_{s,p} \geq 2\varepsilon\kappa_s \int_{\partial\Omega}(u/\delta^s)^2 X \cdot \nu\, dx + o(\varepsilon)$$

by Lemma 4.3. We thus conclude that

$$\lim_{\varepsilon\to 0^+} \frac{\lambda^\varepsilon_{s,p} - \lambda^0_{s,p}}{\varepsilon} \geq 2\kappa_s \int_{\partial\Omega}(u/\delta^s)^2 X \cdot \nu\, dx.$$

Taking the infimum over $u \in \mathcal{H}$ in the RHS of this inequality and using (4.10), we get the result. $\qquad\square$

Proof of Proposition 4.1 (completed) Proposition 4.1 is a consequence of Lemmas 4.4 and 4.5. Indeed, let

$$A_{s,p}(\Omega) := \inf\left\{2\kappa_s \int_{\partial\Omega}(u/\delta^s)^2 X \cdot \nu\, dx \,:\, u \in \mathcal{H}\right\}.$$

Thanks to (2.8) the infimum $A_{s,p}(\Omega)$ is attained.

Finally by Lemma 4.4 and Lemma 4.5 we get

$$A_{s,p}(\Omega) \geq \partial^+_\varepsilon\Big|_{\varepsilon=0}\lambda^\varepsilon_{s,p} \geq \liminf_{\varepsilon\searrow 0}\frac{\lambda^\varepsilon_{s,p} - \lambda^0_{s,p}}{\varepsilon} \geq A_{s,p}(\Omega).$$

$\qquad\square$

5 Proof of the main results

In this section we complete the proofs of the main results stated in the introduction.

Proof of Theorem 1.1 (completed) In view of Proposition 4.1, the proof of Theorem 1.1 is complete once we show that

$$2\kappa_s = \Gamma(1+s)^2, \tag{5.1}$$

where Γ is the usual Gamma function. In view of (2.10), the constant κ_s does not depend on N, p and Ω, we consider the case $N = p = 1$ and the family of diffeomorphisms Φ_ε on $\mathbb{R}^N$ given by $\Phi_\varepsilon(x) = (1+\varepsilon)x$, $\varepsilon \in (-1, 1)$, so that $X := \partial_\varepsilon|_{\varepsilon=0}\Phi_\varepsilon$ is simply given by $X(x) = x$. Letting $\Omega_0 := (-1, 1)$, we define $\Omega_\varepsilon = \Phi_\varepsilon(\Omega_0) = (-1-\varepsilon, 1+\varepsilon)$. Moreover, we consider $w_\varepsilon \in \mathcal{H}^s_0(\Omega_\varepsilon) \cap C^s_0([-1-\varepsilon, 1+\varepsilon])$ given by

$$w_\varepsilon(x) = \ell_s((1+\varepsilon)^2 - |x|^2)^s_+ \quad \text{with} \quad \ell_s := \frac{2^{-2s}\Gamma(1/2)}{\Gamma(s+1/2)\Gamma(1+s)}. \tag{5.2}$$

It is well known that w_ε is the unique solution of the problem

$$(-\Delta)^s w_\varepsilon = 1 \quad \text{in } \Omega_\varepsilon, \qquad w_\varepsilon \equiv 0 \quad \text{on } \mathbb{R}^N \setminus \Omega_\varepsilon,$$

see e.g. [21] or [9]. Recalling (1.4), we thus deduce that $u_\varepsilon = \lambda_{s,1}(\Omega_\varepsilon)w_\varepsilon$ is the unique positive minimizer corresponding to (1.3) in the case $N = p = 1$, which implies that $\|u_\varepsilon\|_{L^1(\mathbb{R})} = 1$ and therefore

$$\lambda_{s,1}(\Omega_\varepsilon) = \|w_\varepsilon\|_{L^1(\mathbb{R})}^{-1} = (1+\varepsilon)^{-(2s+1)}\|w_0\|_{L^1(\mathbb{R})}^{-1}. \tag{5.3}$$

Moreover, by standard properties of the Gamma function,

$$\|w_0\|_{L^1(\mathbb{R})} = \ell_s \int_{-1}^{1} (1 - |x|^2)^s \, dx = 2\ell_s \int_0^1 (1 - r^2)^s \, dr = \ell_s \int_0^1 t^{-1/2}(1-t)^s \, dt$$

$$= \ell_s \frac{\Gamma(1/2)\Gamma(s+1)}{\Gamma(s+3/2)} = \ell_s \frac{\Gamma(1/2)\Gamma(s+1)}{(s+1/2)\Gamma(s+1/2)} = \frac{2^{2s}\,\ell_s^2\,\Gamma(s+1)^2}{s+1/2}.$$

By differentiating (5.3), we get

$$\partial_\varepsilon\Big|_{\varepsilon=0}\lambda_{s,1}(\Omega_\varepsilon) = -\frac{2s+1}{\|w_0\|_{L^1(\mathbb{R})}}. \tag{5.4}$$

On the other hand, by Proposition 4.1 and the fact that u_0 is the unique positive minimizer for $\lambda_{s,1}$, we deduce that

$$\partial_\varepsilon^+\Big|_{\varepsilon=0}\lambda_{s,1}(\Omega_\varepsilon) = -2\kappa_s[(u_0/\delta^s)^2(1) + (u_0/\delta^s)^2(-1)]$$

$$= -2^{2+2s}\kappa_s\,\ell_s^2\,\lambda_{s,1}(\Omega_0)^2 = -\frac{2^{2+2s}\kappa_s\,\ell_s^2}{\|w_0\|_{L^1(\mathbb{R})}^2}.$$

We thus conclude that

$$2\kappa_s = \frac{(2s+1)\|w_0\|_{L^1(\mathbb{R})}}{2^{1+2s}\ell_s^2} = \Gamma(s+1)^2.$$

Thus, by Proposition 4.1, we get the result as stated in the theorem. □

Proof of Corollary 1.3 Let $h \in C^3(\partial\Omega)$, with $\int_{\partial\Omega} h \, dx = 0$. Then it is well known (see e.g. [11, Lemma 2.2]) that there exists a family of diffeomorphisms $\Phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$, $\varepsilon \in (-1, 1)$ satisfying (1.7) and having the following properties:

$$|\Phi_\varepsilon(\Omega)| = |\Omega| \text{ for } \varepsilon \in (-1, 1), \text{ and } X := \partial_\varepsilon\Big|_{\varepsilon=0}\Phi_\varepsilon \text{ equals } h\nu \text{ on } \partial\Omega. \tag{5.5}$$

By assumption, there exists $\varepsilon_0 \in (0, 1)$ with $\lambda_{s,p}(\Phi_\varepsilon(\Omega)) \geq \lambda_{s,p}(\Omega)$ for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Applying Theorem 1.1 and noting that $X \cdot \nu \equiv h$ on $\partial\Omega$ by (5.5), we get

$$\min\left\{\Gamma(1+s)^2 \int_{\partial\Omega} (u/\delta^s)^2 h \, dx \; : \; u \in \mathcal{H}\right\} \geq 0.$$

By the same argument applied to $-h$, we get

$$\max\left\{\Gamma(1+s)^2 \int_{\partial\Omega} (u/\delta^s)^2 h \, dx \; : \; u \in \mathcal{H}\right\} \leq 0. \tag{5.6}$$

We thus conclude that

$$\int_{\partial\Omega} (u/\delta^s)^2 h \, dx = 0 \quad \text{for every } u \in \mathcal{H} \text{ and for all } h \in C^3(\partial\Omega), \text{ with } \int_{\partial\Omega} h \, dx = 0.$$

By a standard argument, this implies that u/δ^s is constant on $\partial\Omega$. Now, since u solves (1.4) and $p \in \{1\} \cup [2, \infty)$, we deduce from [9, Theorem 1.2] that Ω is a ball. □

Proof of Theorem 1.4 Consider the unit centered ball $B_1 = B_1(0)$. For $\tau \in (0, 1)$ and $t \in (\tau - 1, 1 - \tau)$, we define $B^t := B_\tau(te_1)$, where e_1 is the first coordinate direction. To prove Theorem 1.4, we can take advantage of the invariance under rotations of the problem and may restrict our attention to domains of the form $\Omega(t) = B_1 \setminus \overline{B^t}$. We define

$$\theta : (\tau - 1, 1 - \tau) \to \mathbb{R}, \qquad \theta(t) := \lambda_{s,p}(\Omega(t)). \tag{5.7}$$

We claim that θ is differentiable and satisfies

$$\theta'(t) < 0 \qquad \text{for } t \in (0, 1 - \tau). \tag{5.8}$$

For this we fix $t \in (\tau - 1, 1 - \tau)$ and a vector field $X : \mathbb{R}^N \to \mathbb{R}^N$ given by $X(x) = \rho(x)e_1$, where $\rho \in C_c^\infty(B_1)$ satisfies $\rho \equiv 1$ in a neighborhood of B^t. For $\varepsilon \in (-1, 1)$, we then define $\Phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ by $\Phi_\varepsilon(x) = x + \beta\varepsilon X(x)$, where $\beta > 0$ is chosen sufficiently small to guarantee that $\Phi_\varepsilon, \varepsilon \in (-1, 1)$ is a family of diffeomorphisms satisfying (1.7) and satisfying $\Phi_\varepsilon(B_1) = B_1$ for $\varepsilon \in (-1, 1)$. Then, by construction, we have

$$\Phi_\varepsilon(\Omega(t)) = \Phi_\varepsilon\left(B_1 \setminus \overline{B^t}\right) = B_1 \setminus \overline{\Phi_\varepsilon(B^t)} = B_1 \setminus \overline{B^{t+\beta\varepsilon}} = \Omega(t + \beta\varepsilon). \tag{5.9}$$

Next we recall that, since $p \in \{1, 2\}$, there exists a unique positive minimizer $u \in \mathcal{H}_0^s(\Omega(t))$ corresponding to the variational characterization (1.3) of $\lambda_{s,p}(\Omega(t))$. Hence, by Corollary 1.2, the map $\varepsilon \mapsto \lambda_{s,p}(\Phi_\varepsilon(\Omega(t)))$ is differentiable at $\varepsilon = 0$. In view of (5.9), we thus find that the map θ in (5.7) is differentiable at t, and

$$\theta'(t) = \frac{1}{\beta}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\lambda_{s,p}(\Phi_\varepsilon(\Omega(t))) = \Gamma(1+s)^2 \int_{\partial\Omega(t)} \left(\frac{u}{\delta^s}\right)^2 X \cdot \nu\, dx$$

$$= \Gamma(1+s)^2 \int_{\partial B^t} \left(\frac{u}{\delta^s}\right)^2 \nu_1\, dx \tag{5.10}$$

by (1.10). Here ν denotes the interior unit normal on $\partial\Omega(t)$ which coincides with the exterior unit normal to B^t on ∂B^t, and we used that

$$X \equiv e_1 \quad \text{on } \partial B^t, \qquad X \equiv 0 \quad \text{on } \partial B_1 = \partial\Omega(t) \setminus \partial B^t$$

to get the last equality in (5.10). Next, for fixed $t \in (0, 1 - \tau)$, let H be the half space defined by $H = \{x \in \mathbb{R}^N : x \cdot e_1 > t\}$ and let $\Theta = H \cap \Omega(t)$. We also let $r_H : \mathbb{R}^N \to \mathbb{R}^N$ be the reflection map with respect to he hyperplane $\partial H := \{x \in \mathbb{R}^N : x \cdot e_1 = t\}$. For $x \in \mathbb{R}^N$, we denote $\bar{x} := r_H(x)$, $\overline{u}(x) := u(\overline{x})$. Using these notations, we have

$$\theta'(t) = \Gamma(1+s)^2 \int_{\partial B^t} \left(\frac{u}{\delta^s}\right)^2 \nu_1\, dx$$

$$= \Gamma(1+s)^2 \int_{\partial B^t \cap \Theta} \left(\left(\frac{u}{\delta^s}\right)^2(x) - \left(\frac{\overline{u}}{\delta^s}\right)^2(x)\right) \nu_1\, dx. \tag{5.11}$$

Let $w = \overline{u} - u \in H^s(\mathbb{R}^N)$. Then w is a (weak) solution of the problem

$$(-\Delta)^s w = \lambda_{s,p}(\Omega(t))\overline{u}^{p-1} - \lambda_{s,p}(\Omega(t))u^{p-1} = c_p w \qquad \text{in } \Theta, \tag{5.12}$$

where

$$\begin{cases} c_p := \lambda_{s,p}(\Omega(t)) & \text{for } p = 2, \\ c_p = 0 & \text{for } p = 1. \end{cases}$$

Moreover, by definition, $w \equiv \overline{u} \geq 0$ in $H \setminus \overline{\Theta}$, and $w \equiv \overline{u} > 0$ in the subset $[r_H(B_1) \cap H] \setminus \overline{\Theta}$ which has positive measure since $t > 0$. Using that w is anti-symmetric with respect to H and the fact that $\lambda_{s,p}(\Theta) > c_p$ (which follows since Θ is a proper subdomain of $\Omega(t)$), we can apply the weak maximum principle for antisymmetric functions (see [9, Proposition 3.1] or [17, Proposition 3.5]) to deduce that $w \geq 0$ in Θ. Moreover, since $w \not\equiv 0$ in $\mathbb{R}^N$, it follows from the strong maximum principle for antisymmetric functions given in [17, Proposition 3.6] that $w > 0$ in Θ. Now by the fractional Hopf lemma for antisymmetric functions (see [9, Proposition 3.3]) we conclude that

$$0 < \frac{w}{\delta^s} = \frac{\overline{u}}{\delta^s} - \frac{u}{\delta^s} \quad \text{and therefore} \quad \frac{\overline{u}}{\delta^s} > \frac{u}{\delta^s} \geq 0 \qquad \text{on } \partial B^t \cap \Theta.$$

From this and (5.11) we get (5.8), since $\nu_1 > 0$ on $\partial B^t \cap \Theta$.

To conclude, we observe that the function $t \mapsto \lambda_{s,p}(t) = \lambda_{s,p}(\Omega(t))$ is even, thanks to the invariance of the problem under rotations. Therefore the function θ attains its maximum uniquely at $t = 0$. $\qquad\square$

6 Proof of Proposition 2.4

The aim of this section is to prove Proposition 2.4. For the readers convenience, we repeat the statement here.

Proposition 6.1 *Let $X \in C^0(\overline{\Omega}, \mathbb{R}^N)$, let $u \in C_0^s(\overline{\Omega}) \cap C^1(\Omega)$, and assume that $\psi := \frac{u}{\delta^s}$ extends to a function on $\overline{\Omega}$ satisfying (2.8) and (2.9). Moreover, put $U_k := u\zeta_k \in C_c^{1,1}(\Omega)$, where ζ_k is defined in (2.3). Then*

$$\lim_{k \to \infty} \int_\Omega \nabla U_k \cdot X \left(u(-\Delta)^s \zeta_k - I(u, \zeta_k) \right) dx = -\kappa_s \int_{\partial\Omega} \psi^2 X \cdot \nu \, dx, \qquad (6.1)$$

where

$$\kappa_s := -\int_{\mathbb{R}} h'(r)(-\Delta)^s h(r) \, dr \quad \text{with} \quad h(r) := r_+^s \zeta(r) \qquad (6.2)$$

and ζ given in (2.2), and where we use the notation

$$I(u, v)(x) := \int_{\mathbb{R}^N} (u(x) - u(y))(v(x) - v(y)) K_0(x, y) \, dy \qquad (6.3)$$

for $u \in C_c^s(\mathbb{R}^N)$, $v \in C^{0,1}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$.

The remainder of this section is devoted to the proof of this proposition. For $k \in \mathbb{N}$, we define

$$g_k := \nabla U_k \cdot X \left(u(-\Delta)^s \zeta_k - I(u, \zeta_k) \right) \quad : \quad \Omega \to \mathbb{R}. \qquad (6.4)$$

For $\varepsilon > 0$, we put

$$\Omega^\varepsilon = \{x \in \mathbb{R}^N \ : \ |\delta(x)| < \varepsilon\} \quad \text{and}$$
$$\Omega_+^\varepsilon = \{x \in \mathbb{R}^N \ : \ 0 < \delta(x) < \varepsilon\} = \{x \in \Omega \ : \ \delta(x) < \varepsilon\}.$$

For every $\varepsilon > 0$, we then have

$$\lim_{k \to \infty} \int_{\Omega \setminus \Omega^\varepsilon} g_k \, dx = 0. \qquad (6.5)$$

To see this, we first note that $\zeta_k \to 1$ pointwise on $\mathbb{R}^N \setminus \partial\Omega$, and therefore a.e. on $\mathbb{R}^N$. Moreover, choosing a compact neighborhood $K \subset \Omega$ of $\Omega \setminus \Omega^\varepsilon$, we have

$$(-\Delta)^s \zeta_k(x) = c_{N,s} \int_{\mathbb{R}^N \setminus K} \frac{1 - \zeta_k(y)}{|x - y|^{N+2s}} dy \qquad \text{for } x \in \Omega \setminus \Omega^\varepsilon \text{ and } k \text{ sufficiently large,}$$

where $\frac{|1-\zeta_k(y)|}{|x-y|^{N+2s}} \le \frac{C}{1+|y|^{N+2s}}$ for $x \in \Omega \setminus \Omega^\varepsilon$, $y \in \mathbb{R}^N \setminus K$ and $C > 0$ independent of x and y. Consequently, $\|(-\Delta)^s \zeta_k\|_{L^\infty(\Omega \setminus \Omega^\varepsilon)}$ remains bounded independently of k and $(-\Delta)^s \zeta_k \to 0$ pointwise on $\Omega \setminus \Omega^\varepsilon$ by the dominated convergence theorem. Similarly, we see that $\|I(u, \zeta_k)\|_{L^\infty(\Omega \setminus \Omega^\varepsilon)}$ remains bounded independently of k and $I(u, \zeta_k) \to 0$ pointwise on $\Omega \setminus \Omega^\varepsilon$. Consequently, we find that

$$\|g_k\|_{L^\infty(\Omega \setminus \Omega^\varepsilon)} \text{ is bounded independently of } k \text{ and } g_k \to 0 \text{ pointwise on } \Omega \setminus \Omega^\varepsilon.$$

Hence (6.5) follows again by the dominated convergence theorem. As a consequence,

$$\lim_{k \to \infty} \int_\Omega g_k(x)\, dx = \lim_{k \to \infty} \int_{\Omega_+^\varepsilon} g_k(x)\, dx \qquad \text{for every } \varepsilon > 0. \tag{6.6}$$

Let, as before, $\nu : \partial\Omega \to \mathbb{R}^N$ denotes the unit interior normal vector field on Ω. Since we assume that $\partial\Omega$ is of class $C^{1,1}$, the map ν is Lipschitz, which means that the derivative $d\nu : T\partial\Omega \to \mathbb{R}^N$ is a.e. well defined and bounded. Moreover, we may fix $\varepsilon > 0$ from now on with the property that the map

$$\Psi : \partial\Omega \times (-\varepsilon, \varepsilon) \to \Omega^\varepsilon, \qquad (\sigma, r) \mapsto \Psi(\sigma, r) = \sigma + r\nu(\sigma) \tag{6.7}$$

is a bi-Lipschitz map with $\Psi(\partial\Omega \times (0, \varepsilon)) = \Omega_+^\varepsilon$. In particular, Ψ is a.e. differentiable, and the variable r is precisely the signed distance of the point $\Psi(\sigma, r)$ to the boundary $\partial\Omega$, i.e.,

$$\delta(\Psi(\sigma, r)) = r \qquad \text{for } \sigma \in \partial\Omega, \ 0 \le r < \varepsilon. \tag{6.8}$$

Moreover, for $0 < \varepsilon' \le \varepsilon$, it follows from (6.6) that

$$\lim_{k \to \infty} \int_\Omega g_k\, dx = \lim_{k \to \infty} \int_{\Omega_+^{\varepsilon'}} g_k\, dx = \lim_{k \to \infty} \int_{\partial\Omega} \int_0^{\varepsilon'} \text{Jac}_\Psi(\sigma, r) g_k(\Psi(\sigma, r))\, dr d\sigma$$

$$= \lim_{k \to \infty} \frac{1}{k} \int_{\partial\Omega} \int_0^{k\varepsilon'} j_k(\sigma, r) G_k(\sigma, r)\, dr d\sigma, \tag{6.9}$$

where we define

$$j_k(\sigma, r) = \text{Jac}_\Psi\left(\sigma, \frac{r}{k}\right) \quad \text{and} \quad G_k(\sigma, r) = g_k\left(\Psi\left(\sigma, \frac{r}{k}\right)\right) \qquad \text{for a.e. } \sigma \in \partial\Omega, \ 0 \le r < k\varepsilon. \tag{6.10}$$

We note that

$$\|j_k\|_{L^\infty(\partial\Omega \times [0, k\varepsilon))} \le \|\text{Jac}_\Psi\|_{L^\infty(\Omega_\varepsilon)} < \infty \quad \text{for all } k, \text{ and}$$
$$\lim_{k \to \infty} j_k(\sigma, r) = \text{Jac}_\Psi(\sigma, 0) = 1 \quad \text{for a.e. } \sigma \in \partial\Omega, r > 0. \tag{6.11}$$

By definition of the functions g_k in (6.4), we may write

$$G_k(\sigma, r) = G_k^0(\sigma, r)[G_k^1(\sigma, r) - G_k^2(\sigma, r)] \qquad \text{for } \sigma \in \partial\Omega, \ 0 \le r < k\varepsilon \tag{6.12}$$

with

$$G_k^0(\sigma, r) = [\nabla U_k \cdot X](\Psi(\sigma, \frac{r}{k}))$$

$$G_k^1(\sigma, r) = [u(-\Delta)^s \zeta_k](\Psi(\sigma, \frac{r}{k})) \qquad \text{and} \qquad (6.13)$$

$$G_k^2(\sigma, r) = I(u, \zeta_k)(\Psi(\sigma, \frac{r}{k})).$$

In order to analyze the limit in (6.9) for suitable $\varepsilon' \in (0, \varepsilon]$, we provide estimates for the functions G_k^0, G_k^1, G_k^2 separately in the following. We start with an estimate for G_k^0 given by the following lemma.

Lemma 6.2 *Let* $\alpha \in (0, 1)$ *be given by Lemma 2.3. Then we have*

$$k^{s-1}|G_k^0(\sigma, r)| \leq C(r^{s-1} + r^{s-1+\alpha}) \quad \text{for } k \in \mathbb{N}, 0 \leq r < k\varepsilon \qquad (6.14)$$

with a constant $C > 0$, *and*

$$\lim_{k \to \infty} k^{s-1} G_k^0(\sigma, r) = h'(r)\psi(\sigma)[X(\sigma) \cdot \nu(\sigma)] \quad \text{for } \sigma \in \partial\Omega, r > 0 \qquad (6.15)$$

with the function $r \mapsto h(r) = r_+^s \zeta(r)$ *given in* (6.2).

Proof Since $u = \psi\delta^s$, we have

$$\nabla u = s\delta^{s-1}\psi\nabla\delta + \delta^s\nabla\psi = s\delta^{s-1}\psi\nabla\delta + O(\delta^{s-1+\alpha}) \quad \text{in } \Omega$$

by Lemma 2.3, and therefore, since $\zeta_k = \zeta \circ (k\delta)$ by (2.3),

$$\nabla U_k = \nabla\Big(u\zeta_k\Big) = \Big(s\zeta \circ (k\delta) + k\delta\zeta' \circ (k\delta)\Big)\psi\delta^{s-1}\nabla\delta$$
$$+ O(\delta^{s-1+\alpha}) \quad \text{in } \Omega.$$

Consequently, by (6.8) we have

$$[(\nabla U_k) \circ \Psi](\sigma, \frac{r}{k}) = \Big(s\zeta(r) + r\zeta'(r)\Big)\psi(\sigma + \frac{r}{k}\nu(\sigma))(\frac{r}{k})^{s-1}\nabla\delta(\sigma + \frac{r}{k}\nu(\sigma))$$
$$+ O\Big((\frac{r}{k})^{s-1+\alpha}\Big)$$

for $\sigma \in \partial\Omega, 0 \leq r < \varepsilon$ with $O(r^{s-1+\alpha})$ independent of k, and therefore

$$G_k^0(\sigma, r) = \Big(s\zeta(r) + r\zeta'(r)\Big)\psi(\sigma + \frac{r}{k}\nu(\sigma))\nabla\delta(\sigma + \frac{r}{k}\nu(\sigma)) \cdot X(\sigma + \frac{r}{k}\nu(\sigma))k^{1-s}r^{s-1}$$
$$+ k^{1-s-\alpha}O(r^{s-1+\alpha}) \quad \text{for } \sigma \in \partial\Omega, 0 \leq r < k\varepsilon.$$

Since $\alpha > 0$, we deduce that

$$k^{s-1}G_k^0(\sigma, r) \to \Big(s\zeta(r) + r\zeta'(r)\Big)\psi(\sigma)\nabla\delta(\sigma) \cdot X(\sigma)r^{s-1}$$
$$= h'(r)\psi(\sigma)X(\sigma) \cdot \nu(\sigma) \quad \text{as } k \to \infty$$

for $\sigma \in \partial\Omega, r > 0$, while

$$k^{s-1}|G_k^0(\sigma, r)| \leq C(r^{s-1} + r^{s-1+\alpha}) \quad \text{for } k \in \mathbb{N}, 0 \leq r < k\varepsilon$$

with a constant $C > 0$ independent of k and r, as claimed. $\qquad\square$

Next we consider the functions G_k^1 defined in (6.13), and we first state the following estimate.

Proposition 6.3 *There exists $\varepsilon' > 0$ with the property that*

$$|k^{-2s}(-\Delta)^s \zeta_k(\Psi(\sigma, \frac{r}{k}))| \leq \frac{C}{1+r^{1+2s}} \quad \text{for } k \in \mathbb{N}, 0 \leq r < k\varepsilon' \tag{6.16}$$

with a constant $C > 0$. Moreover,

$$\lim_{k \to \infty} k^{-2s}(-\Delta)^s \zeta_k(\Psi(\sigma, \frac{r}{k})) = (-\Delta)^s \zeta(r) \quad \text{for } \sigma \in \partial\Omega, r > 0. \tag{6.17}$$

Before giving the somewhat lengthy proof of this proposition, we infer the following corollary related to the functions G_k^1.

Corollary 6.4 *There exists $\varepsilon' > 0$ with the property that*

$$|k^{-s} G_k^1(\sigma, r)| \leq \frac{Cr^s}{1+r^{1+2s}} \quad \text{for } k \in \mathbb{N}, 0 \leq r < k\varepsilon' \tag{6.18}$$

with a constant $C > 0$. Moreover,

$$\lim_{k \to \infty} k^{-s} G_k^1(\sigma, r) = \psi(\sigma) r^s (-\Delta)^s \zeta(r) \quad \text{for } \sigma \in \partial\Omega, r > 0. \tag{6.19}$$

Proof Since $u = \psi \delta^s$ we have $u(\Psi(\sigma, \frac{r}{k})) = k^{-s} \psi(\sigma + \frac{r}{k}\nu(\sigma)) r^s$ for $k \in \mathbb{N}, 0 \leq r < k\varepsilon$, and

$$\lim_{k \to \infty} k^s u(\Psi(\sigma, \frac{r}{k})) = \psi(\sigma) r^s \quad \text{for } \sigma \in \partial\Omega, r > 0.$$

Since moreover $\|\psi\|_{L^\infty(\Omega_\varepsilon)} < \infty$, the claim now follows from Proposition 6.3 by recalling the definition in G_k^1 in (6.13). $\qquad\square$

We now turn to the proof of Proposition 6.3, and we need some preliminary considerations. Since $\partial\Omega$ is of class $C^{1,1}$ by assumption, there exists an open ball $B \subset \mathbb{R}^{N-1}$ centered at the origin and, for every $\sigma \in \partial\Omega$, a parametrization $f_\sigma : B \to \partial\Omega$ of class $C^{1,1}$ with the property that $f_\sigma(0) = \sigma$ and $df_\sigma(0) : \mathbb{R}^{N-1} \to \mathbb{R}^N$ is a linear isometry. For $z \in B$ we then have

$$f_\sigma(z) - f_\sigma(0) = df_\sigma(0)z + O(|z|^2)$$

and therefore

$$|f_\sigma(0) - f_\sigma(z)|^2 = |df_\sigma(0)z|^2 + O(|z|^3) = |z|^2 + O(|z|^3), \tag{6.20}$$

$$(f_\sigma(0) - f_\sigma(z)) \cdot \nu(\sigma) = -df_\sigma(0)z \cdot \nu(\sigma) + O(|z|^2) = O(|z|^2), \tag{6.21}$$

where we used in (6.21) that $df_\sigma(0)z$ belongs to the tangent space $T_\sigma \partial\Omega = \{\nu(\sigma)\}^\perp$. Here and in the following, the term $\mathcal{O}(\tau)$ stands for a function depending on τ and possibly other quantities but satisfying $|\mathcal{O}(\tau)| \leq C\tau$ with a constant $C > 0$.

Recalling the definition of the map Ψ in (6.7) and writing $\nu_\sigma(z) := \nu(f_\sigma(z))$ for $z \in B$, we now define

$$\Psi_\sigma : (-\varepsilon, \varepsilon) \times B \to \Omega^\varepsilon, \quad \Psi_\sigma(r, z) = \Psi(f_\sigma(z), r) = f_\sigma(z) + r\nu_\sigma(z). \tag{6.22}$$

Then Ψ_σ is a bi-Lipschitz map which maps $(-\varepsilon, \varepsilon) \times B$ onto a neighborhood of σ. Consequently, there exists $\varepsilon' \in (0, \frac{\varepsilon}{2})$ with the property that

$$|\sigma - y| \geq 3\varepsilon' \quad \text{for all } y \in \mathbb{R}^N \setminus \Psi_\sigma((-\varepsilon, \varepsilon) \times B). \tag{6.23}$$

Moreover, ε' can be chosen independently of $\sigma \in \partial\Omega$.

Coming back to the proof of Proposition 6.3, we now write, for $\sigma \in \partial \Omega$ and $r \in [0, k\varepsilon')$,

$$(-\Delta)^s \zeta_k(\Psi(\sigma, \frac{r}{k})) = c_{N,s} \left(A_k(\sigma, r) + B_k(\sigma, r) \right) \tag{6.24}$$

with

$$A_k(\sigma, r) := \int_{\Psi_\sigma((-\varepsilon,\varepsilon) \times B)} \frac{\zeta(r) - \zeta_k(y)}{|\Psi(\sigma, \frac{r}{k}) - y|^{N+2s}} \, dy$$

and

$$B_k(\sigma, r) := \int_{\mathbb{R}^N \setminus \Psi_\sigma((-\varepsilon,\varepsilon) \times B)} \frac{\zeta(r) - \zeta_k(y)}{|\Psi(\sigma, \frac{r}{k}) - y|^{N+2s}} \, dy.$$

Here we used that $\zeta_k(\Psi(\sigma, \frac{r}{k})) = \zeta(r)$ for $\sigma \in \partial\Omega$, $r \in [0, k\varepsilon')$ by (6.8) and the definition of ζ_k. We first provide a rather straightforward estimate for the functions B_k.

Lemma 6.5 *We have*

$$k^{-2s} |B_k(\sigma, r)| \leq \frac{C}{1 + r^{1+2s}} \qquad \text{for } k \in \mathbb{N}, 0 \leq r < k\varepsilon', \sigma \in \partial\Omega \tag{6.25}$$

with a constant $C > 0$ and

$$\lim_{k \to \infty} k^{-2s} |B_k(\sigma, r)| = 0 \quad \text{for every } \sigma \in \Omega, r \geq 0. \tag{6.26}$$

Proof By (6.23) and since $r < k\varepsilon'$, we have

$$|\Psi(\sigma, \frac{r}{k}) - y| = |\sigma - y + \frac{r}{k} \nu(\sigma)| \geq |\sigma - y| - \frac{r}{k}$$

$$\geq \frac{|\sigma - y|}{3} + \varepsilon' \quad \text{for } y \in \mathbb{R}^N \setminus \Psi_\sigma((-\varepsilon, \varepsilon) \times B).$$

Recalling that $\zeta = 1 - \rho$, $\zeta_k = 1 - \rho_k$ and that ρ_k is supported in $\Omega^{\frac{2}{k}}$, we thus estimate

$$|B_k(\sigma, r)| \leq \int_{\mathbb{R}^N \setminus \Psi_\sigma((-\varepsilon,\varepsilon) \times B)} \frac{|\rho(r) - \rho_k(y)|}{|\Psi(\sigma, \frac{r}{k}) - y|^{N+2s}} \, dy$$

$$\leq 3^{N+2s} |\rho(r)| \int_{\mathbb{R}^N} \left(|\sigma - y| + 3\varepsilon' \right)^{-N-2s} dy + (\varepsilon')^{-N-2s} \int_{\mathbb{R}^N} |\rho_k(y)| \, dy$$

$$\leq C \left(|\rho(r)| + |\Omega^{\frac{2}{k}}| \right) \leq C \left(|\rho(r)| + k^{-1} \right).$$

Here and in the following, the letter C stands for various positive constants. This estimate readily yields (6.26). Moreover,

$$k^{-2s} |B_k(\sigma, r)| \leq C k^{-2s} \left(|\rho(r)| + k^{-1} \right) \leq \frac{C}{1 + r^{1+2s}} + k^{-1-2s} \leq \frac{C}{1 + r^{1+2s}}$$

for $k \in \mathbb{N}$, $0 \leq r < k\varepsilon'$, $\sigma \in \partial\Omega$, as claimed in (6.25). □

To complete the proof of Proposition 6.3, it thus remains to consider the functions A_k in the following. For this, we need the following additional estimates for the maps $\Psi_\sigma, \sigma \in \partial\Omega$. We note here that Ψ_σ is a.e. differentiable since it is Lipschitz, so the Jacobian determinant Jac_{Ψ_σ} is a.e. well-defined on $(-\varepsilon, \varepsilon) \times B$.

Lemma 6.6 *There exists a constant C_0 with the property that for every $\sigma \in \partial\Omega$ we have the following estimates:*

(i) $|\mathrm{Jac}_{\Psi_\sigma}(r, z)| \le C_0$ for a.e. $r \in (-\varepsilon, \varepsilon)$, $z \in B$;

(ii) $|\mathrm{Jac}_{\Psi_\sigma}(r, z) - 1| \le C_0(|r| + |z|)$ for a.e. $r \in (-\varepsilon, \varepsilon)$, $z \in B$;

(iii) $|\mathrm{Jac}_{\Psi_\sigma}(r + t, z) - \mathrm{Jac}_{\Psi_\sigma}(r - t, z)| \le C_0|t|$ for a.e. $r \in (-\varepsilon, \varepsilon)$, $z \in B$, $t \in (-\varepsilon - r, \varepsilon - r)$.

Moreover, for $\sigma \in \partial\Omega$, $r \in (-\varepsilon, \varepsilon)$, $z \in B$, $t \in (-\varepsilon - r, \varepsilon - r)$ we have

(iv) $\frac{1}{C_0}(t^2 + |z|^2)^{\frac{1}{2}} \le |\Psi_\sigma(r, 0) - \Psi_\sigma(r + t, z)| \le C_0(t^2 + |z|^2)^{\frac{1}{2}}$,

and for $\sigma \in \partial\Omega$, $r \in (-\varepsilon, \varepsilon)$, $t \in (-\varepsilon - r, \varepsilon - r) \setminus \{0\}$ and $z \in \frac{1}{|t|}B$ we have

(v) $\left| \frac{|\Psi_\sigma(r, 0) - \Psi_\sigma(r + t, |t|z)|^2}{t^2} - (1 + |z|^2) \right| \le C_0(|t| + |r| + |tz|)|z|^2$;

(vi)

$$\left| \left| \Psi_\sigma(r, 0) - \Psi_\sigma(r + t, |t|z) \right|^{-N-2s} - \left| \Psi_\sigma(r, 0) - \Psi_\sigma(r - t, |t|z) \right|^{-N-2s} \right|$$

$$\le C_0|t|^{1-N-2s}(1 + |z|^2)^{-\frac{N+2s}{2}}.$$

Proof The inequalities (i) and (iv) are direct consequences of the fact that Ψ_σ is bi-Lipschitz. In particular, if C_0 is a Lipschitz constant for Ψ_σ^{-1}, we have

$$(t^2 + |z|^2)^{\frac{1}{2}} = |(-t, z)| = |(r, 0) - (r + t, z)| \le C_0|\Psi_\sigma(r, 0) - \Psi_\sigma(r + t, z)|$$

for $\sigma \in \partial\Omega$, $r \in (-\varepsilon, \varepsilon)$, $z \in B$ and $t \in (-\varepsilon - r, \varepsilon - r)$, so the first inequality in (iv) follows. By making C_0 larger if necessary so that it is also a Lipschitz constant for Ψ_σ, we then deduce the second inequality in (iv).

To see (ii) and (iii), we note that $d\Psi_\sigma$ is a.e. given by

$$d\Psi_\sigma(r, z)(r', z') = [df_\sigma(z) + rdv_\sigma(z)]z' + r'v_\sigma(z)$$

for $(r, z) \in (-\varepsilon, \varepsilon) \times B$, $(r', z') \in \mathbb{R} \times \mathbb{R}^{N-1}$, which implies that

$$[d\Psi_\sigma(r, z) - d\Psi_\sigma(0, 0)](r', z') = [df_\sigma(z) - df_\sigma(0)]z' + rdv_\sigma(z) + r'\big(v_\sigma(z) - v_\sigma(0)\big)$$

and

$$[d\Psi_\sigma(r + t, z) - d\Psi_\sigma(r - t, z)](r', z') = 2tdv_\sigma(z)z'.$$

Since df_σ, v_σ are Lipschitz functions on B, dv_σ is a bounded function on B and the determinant is a locally Lipschitz continuous function on the space of linear endomorphisms of $\mathbb{R}^N$, it follows that

$$|\mathrm{Jac}_{\Psi_\sigma}(r, z) - \mathrm{Jac}_{\Psi_\sigma}(0, 0)| \le C_0(|r| + |z|) \quad \text{and} \quad |\mathrm{Jac}_{\Psi_\sigma}(r + t, z)$$
$$- \mathrm{Jac}_{\Psi_\sigma}(r - t, z)| \le C_0|t|$$

for a.e. $r \in (-\varepsilon, \varepsilon)$, $z \in B$, $t \in (-\varepsilon - r, \varepsilon - r)$. Moreover, $\mathrm{Jac}_{\Psi_\sigma}(0, 0) = 1$ since the map

$$\mathbb{R} \times \mathbb{R}^{N-1} \to \mathbb{R}^N, \qquad (r', z') \mapsto d\Psi_\sigma(0, 0)(r', z') = df_\sigma(0)z' + r'v_\sigma(0)$$

is an isometry. Hence (ii) and (iii) follow.

To see (v) and (vi), we note that by definition of Ψ_σ we have

$$\Psi_\sigma(r, 0) - \Psi_\sigma(r + t, 0) = f_\sigma(0) - f_\sigma(z) - tv_\sigma(0) + (r + t)(v_\sigma(0) - v_\sigma(z))$$

for $z \in B$, $r \in (0, \varepsilon')$ and $t \in (-\varepsilon - r, \varepsilon - r)$. Using moreover that $(v_\sigma(0) - v_\sigma(z)) \cdot v_\sigma(0) = \frac{1}{2}|v_\sigma(0) - v_\sigma(z)|^2$, we get

$$|\Psi_\sigma(r, 0) - \Psi_\sigma(r + t, z)|^2 = t^2 + |f_\sigma(0) - f_\sigma(z)|^2 + (r + t)^2|v_\sigma(0) - v_\sigma(z)|^2$$

$$
\begin{aligned}
&- 2t(f_\sigma(0) - f_\sigma(z)) \cdot v_\sigma(0) - t(r+t)|v_\sigma(0) - v_\sigma(z)|^2 \\
&+ 2(r+t)(f_\sigma(0) - f_\sigma(z)) \cdot (v_\sigma(0) - v_\sigma(z)) \\
&= t^2 + |f_\sigma(0) - f_\sigma(z)|^2 + r(r+t)|v_\sigma(0) - v_\sigma(z)|^2 \\
&- 2t(f_\sigma(0) - f_\sigma(z)) \cdot v_\sigma(0) + 2(r+t)(f_\sigma(0) - f_\sigma(z)) \cdot (v_\sigma(0) - v_\sigma(z)) \\
&= t^2 + |z|^2 + \big[|z|m_\sigma(z) + r(r+t)n_\sigma(z) - 2tp_\sigma(z) + 2(r+t)q_\sigma(z)\big]|z|^2 \quad (6.27)
\end{aligned}
$$

for $z \in B$, $r \in (-\varepsilon, \varepsilon)$ and $t \in (-\varepsilon - r, \varepsilon - r)$ with the functions

$$
m_\sigma(z) = \frac{|f_\sigma(0) - f_\sigma(z)|^2 - |z|^2}{|z|^3}, \quad n_\sigma(z) = \frac{|v_\sigma(0) - v_\sigma(z)|^2}{|z|^2},
$$

$$
p_\sigma(z) = \frac{(f_\sigma(0) - f_\sigma(z)) \cdot v_\sigma(0)}{|z|^2}
$$

and

$$
q_\sigma(z) = \frac{(f_\sigma(0) - f_\sigma(z)) \cdot (v_\sigma(0) - v_\sigma(z))}{|z|^2}, \quad z \in B \setminus \{0\},
$$

which are all bounded as a consequence of the Lipschitz continuity of f_σ and v_σ and of (6.20) and (6.21). We deduce that

$$
\begin{aligned}
&\left| \frac{|\Psi_\sigma(r,0) - \Psi_\sigma(r+t, |t|z)|^2}{t^2} - (1 + |z|^2) \right| \\
&= \left| |tz|m_\sigma(|t|z) + r(r+t)n_\sigma(|t|z) - 2tp_\sigma(|t|z) + 2(r+t)q_\sigma(|t|z) \right| |z|^2 \\
&\le C_0(|tz| + |r| + |t|)|z|^2
\end{aligned}
$$

for $\sigma \in \partial\Omega$, $r \in (-\varepsilon, \varepsilon)$, $t \in (-\varepsilon - r, \varepsilon - r) \setminus \{0\}$ and $z \in \frac{1}{|t|}B$ if C_0 is chosen sufficiently large, as claimed in (v).

For the proof of (vi), we now set $w_\sigma(r,t,z) := \frac{1}{t^2}|\Psi_\sigma(r,0) - \Psi_\sigma(r+t, |t|z)|^2$, and we note that

$$
w_\sigma(r,t,z) \ge \frac{1 + |z|^2}{C_0^2} \quad \text{for } \sigma \in \partial\Omega,\, r \in (-\varepsilon, \varepsilon),\, t \in (-\varepsilon - r, \varepsilon - r) \setminus \{0\},\, z \in \frac{1}{|t|}B
$$

by (iv). Moreover, from (6.27) we infer that

$$
\left| w_\sigma(r,t,z) - w_\sigma(r,-t,z) \right| = \left| 2rtn_\sigma(|t|z) + 4t\big(q_\sigma(|t|z) - p_\sigma(|t|z)\big) \right| |z|^2 \le C_0|t||z|^2
$$

for $\sigma \in \partial\Omega$, $r \in (-\varepsilon, \varepsilon)$, $t \in (-\varepsilon - r, \varepsilon - r) \setminus \{0\}$ and $z \in \frac{1}{|t|}B$ if C_0 is made larger if necessary. Using these estimates together with the mean value theorem, we get that, for some $\tau = \tau(\sigma, r, t, z)$ with $-t < \tau < t$,

$$
\begin{aligned}
&\left| |\Psi_\sigma(r,0) - \Psi_\sigma(r+t, |t|z)|^{-N-2s} - |\Psi_\sigma(r,0) - \Psi_\sigma(r-t, |t|z)|^{-N-2s} \right| \\
&= |t|^{-N-2s} \left| w_\sigma(r,t,z)^{-\frac{N+2s}{2}} - w_\sigma(r,-t,z)^{-\frac{N+2s}{2}} \right| \\
&= \frac{(N+2s)|t|^{-N-2s}}{2} w_\sigma(r,\tau,z)^{-\frac{N+2s+2}{2}} \left| w_\sigma(r,t,z) - w_\sigma(r,-t,z) \right| \\
&\le C_0|t|^{1-N-2s}(1 + |z|^2)^{-\frac{N+2s+2}{2}}|z|^2 \le C_0|t|^{1-N-2s}(1 + |z|^2)^{-\frac{N+2s}{2}}
\end{aligned}
$$

for $z \in B$, $r \in (0, \varepsilon')$ and $t \in (-\varepsilon + r, \varepsilon - r)$ after making C_0 larger if necessary, as claimed in (vi). $\qquad\square$

We now have all the tools to study the quantity $A_k(\sigma, r)$ in (6.24).

Lemma 6.7 *We have*

$$k^{-2s}|A_k(\sigma, r)| \leq \frac{C}{1 + r^{1+2s}} \qquad for \ k \in \mathbb{N}, 0 \leq r < k\varepsilon', \sigma \in \partial\Omega \tag{6.28}$$

with a constant $C > 0$ and

$$\lim_{k \to \infty} k^{-2s} A_k(\sigma, r) = \frac{(-\Delta)^s \zeta(r)}{c_{N,s}} \qquad for \ every \ \sigma \in \Omega, r \geq 0. \tag{6.29}$$

Proof For $\sigma \in \partial\Omega$ and $0 < r < k\varepsilon'$, we write, with a change of variables,

$$
\begin{aligned}
&A_k(\sigma, r) \\
&= \int_{\Psi_\sigma((-\varepsilon,\varepsilon)\times B)} \frac{\zeta(r) - \zeta_k(y)}{|\Psi(\sigma, \frac{r}{k}) - y|^{N+2s}} \, dy \\
&= \int_{-\varepsilon}^{\varepsilon} \int_B \mathrm{Jac}_{\Psi_\sigma}(\tilde{r}, z) \frac{\zeta(r) - \zeta(k\tilde{r})}{|\Psi_\sigma(\frac{r}{k}, 0) - \Psi_\sigma(\tilde{r}, z)|^{N+2s}} \, dz d\tilde{r} \\
&= \frac{1}{k} \int_{-k\varepsilon - r}^{k\varepsilon - r} \int_B \mathrm{Jac}_{\Psi_\sigma}\left(\frac{r+t}{k}, z\right) \frac{\zeta(r) - \zeta(r+t)}{|\Psi_\sigma(\frac{r}{k}, 0) - \Psi_\sigma(\frac{r+t}{k}, z)|^{N+2s}} \, dz dt \\
&= \int_{-k\varepsilon - r}^{k\varepsilon - r} \frac{|t|^{N-1}}{k^N} \int_{\frac{k}{|t|}B} \mathrm{Jac}_{\Psi_\sigma}\left(\frac{r+t}{k}, \frac{|t|z}{k}\right) \frac{\zeta(r) - \zeta(r+t)}{|\Psi_\sigma(\frac{r}{k}, 0) - \Psi_\sigma(\frac{r+t}{k}, \frac{|t|z}{k})|^{N+2s}} \, dz dt \\
&= k^{2s} \int_{\mathbb{R}} \frac{\zeta(r) - \zeta(r+t)}{|t|^{1+2s}} \mathcal{K}_k(r, t) dt \tag{6.30}
\end{aligned}
$$

with the kernels $\mathcal{K}_k : (0, k\varepsilon') \times \mathbb{R} \to \mathbb{R}$ defined by

$$
\mathcal{K}_k(r, t) =
\begin{cases}
\left(\frac{|t|}{k}\right)^{N+2s} \displaystyle\int_{\frac{k}{|t|}B} \frac{\mathrm{Jac}_{\Psi_\sigma}(\frac{r+t}{k}, \frac{|t|z}{k})}{\left|\Psi_\sigma(\frac{r}{k}, 0) - \Psi_\sigma(\frac{r+t}{k}, \frac{|t|z}{k})\right|^{N+2s}} dz, & t \in (-k\varepsilon - r, k\varepsilon - r), \\
0, & t \notin (-k\varepsilon - r, k\varepsilon - r).
\end{cases}
$$

Consequently,

$$A_k(\sigma, r) = k^{2s}\left(J_k^1(\sigma, r) + J_k^2(\sigma, r)\right) \tag{6.31}$$

with

$$J_k^1(\sigma, r) := \frac{1}{4} \int_{\mathbb{R}} \frac{2\zeta(r) - \zeta(r+t) - \zeta(r-t)}{|t|^{1+2s}} \left(\mathcal{K}_k(r, t) + \mathcal{K}_k(r, -t)\right) dt$$

and

$$J_k^2(\sigma, r) := -\frac{1}{4} \int_{\mathbb{R}} \frac{\zeta(r+t) - \zeta(r-t)}{|t|^{2s}} \frac{\mathcal{K}_k(r, t) - \mathcal{K}_k(r, -t)}{|t|} dt.$$

By Lemma 6.6(i), (iv) and the definition of $\mathcal{K}_k$, we have

$$|\mathcal{K}_k(r, t)| \leq C_0^{N+2s+1} \int_{\frac{k}{|t|}B} \left(1 + |z|^2\right)^{-\frac{N+2s}{2}} dz \leq C_0^{N+2s+1} a_{N,s} \tag{6.32}$$

for $r \in (-k\varepsilon', k\varepsilon')$ and $t \in \mathbb{R} \setminus \{0\}$ with

$$a_{N,s} := \int_{\mathbb{R}^{N-1}} (1 + |z|^2)^{-\frac{N+2s}{2}} dz < \infty. \tag{6.33}$$

Moreover, by Lemma 6.6(i)(ii),(iv),(v) and the dominated convergence theorem, we have

$$\lim_{k \to \infty} \mathcal{K}_k(r, t) = \int_{\mathbb{R}^{N-1}} (1 + |z|^2)^{-\frac{N+2s}{2}} \, dz = a_{N,s} \qquad \text{for every } r \geq 0, t \in \mathbb{R} \setminus \{0\}.$$

$$(6.34)$$

Using (6.32) and the fact that $\rho = 1 - \zeta \in C_c^\infty(\mathbb{R})$, we obtain the estimate

$$|J_k^1(\sigma, r)| \leq C \int_{\mathbb{R}} \frac{|2\zeta(r) - \zeta(r+t) - \zeta(r-t)|}{|t|^{1+2s}} \, dt$$

$$= C \int_{\mathbb{R}} \frac{|2\rho(r) - \rho(r+t) - \rho(r-t)|}{|t|^{1+2s}} \, dt \leq \frac{C}{1 + r^{1+2s}} \qquad (6.35)$$

for $k \in \mathbb{N}$, $r \in (0, k\varepsilon')$ and $\sigma \in \partial\Omega$. Here and in the following, the letter $C > 0$ stands for different positive constants. Moreover, by (6.32), (6.34) and the dominated convergence theorem, we find that

$$\lim_{k \to \infty} J_k^1(\sigma, r) = \frac{a_{N,s}}{2} \int_{\mathbb{R}} \frac{2\zeta(r) - \zeta(r+t) - \zeta(r-t)}{|t|^{1+2s}} \, dt$$

$$= \frac{a_{N,s}}{c_{1,s}} (-\Delta)^s \zeta(r) = \frac{(-\Delta)^s \zeta(r)}{c_{N,s}}. \qquad (6.36)$$

Here we have used the fact that

$$c_{N,s} a_{N,s} = c_{1,s}, \qquad (6.37)$$

see e.g. [10].

Next we deal with $J_k^2(\sigma, r)$, and for this we have to estimate the kernel differences $|\mathcal{K}_k(r, t) - \mathcal{K}_k(r, -t)|$. By Lemma 6.6(i), (iii), (iv) and (vi), we have

$$\left| \frac{\text{Jac}_{\Psi_\sigma}(\frac{r+t}{k}, \frac{|t|z}{k})}{\left| \Psi_\sigma(\frac{r}{k}, 0) - \Psi_\sigma(\frac{r+t}{k}, \frac{|t|z}{k}) \right|^{N+2s}} - \frac{\text{Jac}_{\Psi_\sigma}(\frac{r-t}{k}, \frac{|t|z}{k})}{\left| \Psi_\sigma(\frac{r}{k}, 0) - \Psi_\sigma(\frac{r-t}{k}, \frac{|t|z}{k}) \right|^{N+2s}} \right|$$

$$\leq C \left(\frac{|t|}{k} \right)^{1-N-2s} (1 + |z|^2)^{-\frac{N+2s}{2}}$$

for $z \in \frac{k}{|t|} B$, $r \in (0, k\varepsilon')$ and $t \in (-k\varepsilon + r, k\varepsilon - r)$ and therefore

$$\frac{|\mathcal{K}_k(r, t) - \mathcal{K}_k(r, -t)|}{|t|} \leq \frac{C}{k} \int_{\frac{k}{|t|} B} (1 + |z|^2)^{-\frac{N+2s}{2}} \, dz$$

$$\leq \frac{C}{k} \int_{\mathbb{R}^{N-1}} (1 + |z|^2)^{-\frac{N+2s}{2}} \, dz \leq \frac{C}{k} \qquad (6.38)$$

for $r \in (0, k\varepsilon')$ and $t \in (-k\varepsilon + r, k\varepsilon - r)$. Moreover, by definition we have

$$|\mathcal{K}_k(r, t) - \mathcal{K}_k(r, -t)| = 0 \qquad \text{for } t \in \mathbb{R} \setminus (-k\varepsilon - r, k\varepsilon + r), \qquad (6.39)$$

while for $t \in (-k\varepsilon - r, -k\varepsilon + r) \cup (k\varepsilon - r, k\varepsilon + r)$ we have $|t| \geq k\varepsilon - \varepsilon' \geq \frac{k\varepsilon}{2}$ and therefore, similarly as in (6.32),

$$\frac{|\mathcal{K}_k(r, t)|}{|t|} \leq \frac{C}{|t|} \int_{\frac{k}{|t|} B} (1 + |z|^2)^{-\frac{N+2s}{2}} \, dz \leq \frac{C}{k} \int_{\frac{2}{\varepsilon} B} (1 + |z|^2)^{-\frac{N+2s}{2}} \, dz \leq \frac{C}{k}. \quad (6.40)$$

Note here that the constant $C > 0$ on the RHS depends on ε, but this is not a problem. Combining (6.38)–(6.40) and using that $\rho = 1 - \zeta \in C_c^\infty(\mathbb{R})$, we get

$$
|J_k^2(\sigma, r)| \leq \frac{1}{4} \int_\mathbb{R} \frac{|\zeta(r+t) - \zeta(r-t)|}{|t|^{2s}} \frac{|\mathcal{K}_k(r,t) - \mathcal{K}_k(r,-t)|}{|t|} dt
$$
$$
\leq \frac{C}{k} \int_\mathbb{R} \frac{|\zeta(r+t) - \zeta(r-t)|}{t^{2s}} dt = \frac{C}{k} \int_\mathbb{R} \frac{|\rho(r+t) - \rho(r-t)|}{t^{2s}} dt \leq \frac{C(1+r)^{-2s}}{k}
$$

for $k \in \mathbb{N}$, $\sigma \in \partial\Omega$ and $0 \leq r < k\varepsilon'$. Hence

$$
|J_k^2(\sigma, r)| \leq \frac{C}{1 + r^{1+2s}} \qquad \text{for } k \in \mathbb{N}, 0 \leq r < k\varepsilon' \tag{6.41}
$$

and

$$
\lim_{k\to\infty} |J_k^2(\sigma, r)| = 0 \qquad \text{for all } r \geq 0. \tag{6.42}
$$

Now (6.28) follows by combining (6.31), (6.35) and (6.41). Moreover, (6.29) follows by combining (6.31), (6.36) and (6.42). $\qquad\square$

Proof of Proposition 6.3 The proof is completed by combining (6.24) with Lemmas 6.5 and 6.7. $\qquad\square$

It finally remains to estimate the function G_k^2 in (6.12).

Lemma 6.8 *There exists $\varepsilon' > 0$ with the property that the function G_k^2 defined in (6.13) satisfies*

$$
|k^{-s} G_k^2(\sigma, r)| \leq \frac{C}{1 + r^{1+s}} \qquad \text{for } k \in \mathbb{N}, 0 \leq r < k\varepsilon', \sigma \in \partial\Omega \tag{6.43}
$$

with a constant $C > 0$. Moreover,

$$
\lim_{k\to\infty} k^{-s} G_k^2(\sigma, r) = \psi(\sigma)\tilde{I}(r) \tag{6.44}
$$

with

$$
\tilde{I}(r) = c_{1,s} \int_\mathbb{R} \frac{(r_+^s - (r+t)_+^s)(\zeta(r) - \zeta(r+t))}{|t|^{1+2s}} dt.
$$

Proof The proof is similar to the one of Proposition 6.3, but there are some differences we need to deal with. First, as in the proof of Proposition 6.3, we choose $\varepsilon' \in (0, \frac{\varepsilon}{2})$ small enough, so that (6.23) holds. Similarly as in (6.24) we can then write

$$
G_k^2(\sigma, r) = c_{N,s}\left(\tilde{A}_k(\sigma, r) + \tilde{B}_k(\sigma, r)\right) \tag{6.45}
$$

with

$$
\tilde{A}_k(\sigma, r) := \int_{\Psi_\sigma((-\varepsilon,\varepsilon)\times B)} \frac{(u(\Psi(\sigma, \frac{r}{k})) - u(y))(\zeta(r) - \zeta_k(y))}{|\Psi(\sigma, \frac{r}{k}) - y|^{N+2s}} dy
$$

and

$$
\tilde{B}_k(\sigma, r) = \int_{\mathbb{R}^N \setminus \Psi_\sigma((-\varepsilon,\varepsilon)\times B)} \frac{(u(\Psi(\sigma, \frac{r}{k})) - u(y))(\zeta(r) - \zeta_k(y))}{|\Psi(\sigma, \frac{r}{k}) - y|^{N+2s}} dy.
$$

As noted in the proof of Lemma 6.5, we have

$$|\Psi(\sigma, \frac{r}{k}) - y| \geq \frac{|\sigma - y|}{3} + \varepsilon' \quad \text{for } y \in \mathbb{R}^N \setminus \Psi_\sigma((-\varepsilon, \varepsilon) \times B), 0 < r < k\varepsilon'.$$

Therefore, since $u \in L^\infty(\mathbb{R}^N)$, we may estimate as in the proof of Lemma 6.5 to get

$$|\widetilde{B}_k(\sigma, r)| \leq 2\|u\|_{L^\infty} \int_{\mathbb{R}^N \setminus \Psi_\sigma((-\varepsilon,\varepsilon)\times B)} \frac{|\rho(r) - \rho_k(y)|}{|\Psi(\sigma, \frac{r}{k}) - y|^{N+2s}} dy \leq C\left(|\rho(r)| + k^{-1}\right).$$

Here, as before, the letter C stands for various positive constants. Consequently,

$$\lim_{k\to\infty} k^{-s}|\widetilde{B}_k(\sigma, r)| = 0 \quad \text{for every } \sigma \in \Omega, r \geq 0, \tag{6.46}$$

since ρ has compact support in $\mathbb{R}$, and

$$k^{-s}|\widetilde{B}_k(\sigma, r)| \leq Ck^{-s}\left(|\rho(r)| + k^{-1}\right) \leq \frac{C}{1 + r^{1+s}} \quad \text{for } k \in \mathbb{N}, 0 \leq r < k\varepsilon', \sigma \in \partial\Omega. \tag{6.47}$$

Hence it remains to estimate $\widetilde{A}_k(\sigma, r)$. For this we note that, by the same change of variables as in (6.30), we have

$$\widetilde{A}_k(\sigma, r) = \int_{-\varepsilon}^{\varepsilon} \int_B \text{Jac}_{\Psi_\sigma}(z, \tilde{r}) \frac{(u(\Psi(\frac{r}{k}, 0)) - u(\Psi_\sigma(\tilde{r}, z)))(\zeta(r) - \zeta(k\tilde{r}))}{|\Psi_\sigma(\frac{r}{k}, 0) - \Psi_\sigma(\tilde{r}, z)|^{N+2s}} dz d\tilde{r}$$

$$= k^s \int_{\mathbb{R}} \frac{\zeta(r) - \zeta(r+t)}{|t|^{1+s}} \widetilde{\mathcal{K}}_k(r, t) dt \tag{6.48}$$

with the kernel

$$\widetilde{\mathcal{K}}_k(r, t)$$
$$= \begin{cases} \left(\frac{|t|}{k}\right)^{N+s} \int_{\frac{k}{|t|}B} \frac{\left(u(\Psi_\sigma(\frac{r}{k}, 0)) - u(\Psi_\sigma(\frac{r+t}{k}, \frac{|t|}{k}z))\right)\text{Jac}_{\Psi_\sigma}(\frac{r+t}{k}, \frac{|t|z}{k})}{\left|\Psi_\sigma(\frac{r}{k}, 0) - \Psi_\sigma(\frac{r+t}{k}, \frac{|t|z}{k})\right|^{N+2s}} dz, & t \in (-k\varepsilon - r, k\varepsilon - r), \\ 0, & t \notin (-k\varepsilon - r, k\varepsilon - r). \end{cases}$$

Since $u \in C^s(\mathbb{R}^N)$ and Ψ_σ is Lipschitz, we have

$$\left|u(\Psi_\sigma(\frac{r}{k}, 0)) - u(\Psi_\sigma(\frac{r+t}{k}, \frac{|t|}{k}z))\right| \leq C\left(\left(\frac{|t|}{k}\right)^2 + \left(\frac{|tz|}{k}\right)^2\right)^{\frac{s}{2}} \leq C\left(\frac{|t|}{k}\right)^s(1 + |z|^s),$$

for $\sigma \in \partial\Omega$, $r \in (-k\varepsilon, k\varepsilon)$, $t \in (-k\varepsilon - r, k\varepsilon - r) \setminus \{0\}$ and $z \in \frac{k}{|t|}B$. Therefore, by using Lemma 6.6(i), (iv) as in (6.32),

$$|\widetilde{\mathcal{K}}_k(r, t)| \leq C\int_{\mathbb{R}^{N-1}}(1 + |z|^s)(1 + |z|^2)^{-\frac{N+2s}{2}} dz \leq C\int_{\mathbb{R}^{N-1}}(1 + |z|)^{-N-s} dz < \infty. \tag{6.49}$$

Inserting this estimate in (6.48), we conclude that

$$k^{-s}|\widetilde{A}_k(\sigma, r)| \leq C\int_{\mathbb{R}} \frac{|\zeta(r) - \zeta(r+t)|}{|t|^{1+s}} dt = C\int_{\mathbb{R}} \frac{|\rho(r) - \rho(r+t)|}{|t|^{1+s}} dt \leq \frac{C}{1 + r^{1+s}}.$$

for $k \in \mathbb{N}$, $0 \leq r < k\varepsilon'$, $\sigma \in \partial\Omega$. Combining this inequality with (6.47), we obtain (6.43). Moreover, since $u \in C_0^s(\overline{\Omega})$ and $\psi = \frac{u}{\delta^s} \in C^0(\overline{\Omega})$, we have

$$\lim_{k\to\infty} k^s\left[u(\Psi_\sigma(\frac{r}{k}, 0)) - u(\Psi_\sigma(\frac{r+t}{k}, \frac{|t|}{k}z))\right] = \psi(\sigma)(r_+^s - (r+t)_+^s) \tag{6.50}$$

for $\sigma \in \partial\Omega$, $r > 0$ and $t \in \mathbb{R}$ and $z \in \mathbb{R}^{N-1}$. Consequently, arguing as for (6.34) with Lemma 6.6(i)(ii),(iv),(v) and the dominated convergence theorem, we find that

$$\lim_{k\to\infty} \widetilde{\mathcal{K}}_k(r,t) = \psi(\sigma)\frac{(r_+^s - (r+t)_+^s)}{|t|^s}\int_{\mathbb{R}^{N-1}}(1+|z|^2)^{-\frac{N+2s}{2}}dz$$
$$= a_{N,s}\psi(\sigma)\frac{(r_+^s - (r+t)_+^s)}{|t|^s} \tag{6.51}$$

for $\sigma \in \partial\Omega$, $r > 0$ and $t \in \mathbb{R}$ with $a_{N,s}$ given in (6.33). Hence, by (6.48), (6.49), (6.51) and the dominated convergence theorem,

$$\lim_{k\to\infty} k^{-s}\widetilde{A}_k(\sigma,r) = a_{N,s}\psi(\sigma)\int_{\mathbb{R}}\frac{(r_+^s - (r+t)_+^s)(\zeta(r)-\zeta(r+t))}{|t|^{1+2s}}dt$$
$$= \frac{a_{N,s}}{c_{1,s}}\psi(\sigma)\tilde{I}(r) = \frac{\psi(\sigma)\tilde{I}(r)}{c_{N,s}},$$

where we used again (6.37) for the last equality. Combining this with (6.45) and (6.46), we obtain (6.44). □

We are now ready to complete the

Proof of Proposition 6.1 Combining (6.14), (6.18) and (6.43), we see that there exists $\varepsilon' > 0$ with the property that the functions G_k defined in (6.4) satisfy

$$\frac{G_k(\sigma,r)}{k} \le C\frac{r^{s-1}+r^{s-1+\alpha}}{1+r^{1+s}} \qquad \text{for } k\in\mathbb{N}, 0\le r < k\varepsilon' \tag{6.52}$$

with a constant $C > 0$ independent of k and r. Since $s, \alpha \in (0,1)$, the RHS of this inequality is integrable over $[0,\infty)$. Moreover, by (6.15), (6.19) and (6.44),

$$\frac{1}{k}G_k(\sigma,r) \to [X(\sigma)\cdot\nu(\sigma)]\psi^2(\sigma)h'(r)\big(r^s(-\Delta)^s\zeta(r) - \tilde{I}(r)\big) \tag{6.53}$$

for every $r > 0$, $\sigma \in \partial\Omega$ as $k \to \infty$. Next we note that, by a standard computation,

$$(-\Delta)^s h(r) = (-\Delta)^s[r_+^s\zeta(r)] = \zeta(r)(-\Delta)^s r_+^s + r_+^s(-\Delta)^s\zeta(r) - \tilde{I}(r)$$
$$= r_+^s(-\Delta)^s\zeta(r) - \tilde{I}(r) \tag{6.54}$$

for $r > 0$ since r_+^s is an s-harmonic function on $(0,\infty)$ see e.g [1]. Hence, by (6.9), (6.9), (6.52), (6.53), (6.54) and the dominated convergence theorem, we conclude that

$$\lim_{k\to\infty}\int_\Omega g_k dx = \int_0^\infty h'(r)(-\Delta)^s h(r)dr\int_{\partial\Omega}[X(\sigma)\cdot\nu(\sigma)]\psi^2(\sigma)d\sigma$$
$$= \int_{\mathbb{R}} h'(r)(-\Delta)^s h(r)dr\int_{\partial\Omega}[X(\sigma)\cdot\nu(\sigma)]\psi^2(\sigma)d\sigma,$$

as claimed in (6.1). □

Appendix A

Here we give a short proof of the uniqueness of positive minimizers of the problem (1.3) for $1 \leq p \leq 2$.

Lemma A.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set of class $C^{1,1}$, let $p \in [1, 2]$, and let u_1 and u_2 be two positive minimizers of (1.3). Then $u_1 = u_2$.*

Proof Suppose by contradiction that there are two different positive minimizers u_1, u_2 for the minimization problem. Then, since $\|u_1\|_{L^p(\Omega)} = \|u_2\|_{L^p(\Omega)} = 1$, the difference $u_1 - u_2$ changes sign. Since moreover $\frac{u_1}{\delta^s}$ and $\frac{u_2}{\delta^s}$ are continuous positive functions on $\overline{\Omega}$ by Lemma 2.3, there exists a maximal $\tau \in (0, 1)$ with

$$\tau u_1 \leq u_2 \qquad \text{on } \overline{\Omega}.$$

Moreover, $\tau u_1 \not\equiv u_2$ since $u_1 - u_2$ changes sign. Consequently, $v := u_2 - \tau u_1$ satisfies $v \geq 0$ on $\overline{\Omega}$ and $v \not\equiv 0$. Moreover, using that $p - 1 \in [0, 1]$ and $\tau \in (0, 1)$, we find that

$$(-\Delta)^s v = \lambda \big(u_2^{p-1} - \tau u_1^{p-1} \big) \geq \lambda \big(u_2^{p-1} - (\tau u_1)^{p-1} \big) \geq 0 \quad \text{in } \Omega, \qquad v = 0 \qquad \text{in } \mathbb{R}^N \setminus \Omega$$

with $\lambda := \lambda_{s,p}(\Omega) > 0$. Now the strong maximum principle for the fractional Laplacian and the fractional Hopf lemma implies that $v = u_2 - \tau u_1$ is strictly positive in Ω and $\frac{v}{\delta^s} > 0$ on $\partial\Omega$. This contradicts the maximality of τ. Hence uniqueness holds. □

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