Nonlocal equations with fractional order dependence

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Remi Yvant Temgoua

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Johann Wolfgang Goethe-Universität als Dissertation angenommen.

Dekan: Prof. Dr. Martin Möller

Gutachter: Prof. Dr. Tobias Weth
Prof. Dr. Mouhamed Moustapha Fall
(African Institute for Mathematical Sciences, Senegal)

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Overview

1.1 Introduction

The study of Partial Differential Equations (which is based in general, on proving the existence of an unknown function and other properties of it) is related to the type of operators that lead the equation, and the nature of these operators depends on the type of phenomena they describe. Many real-life phenomena have a local nature in the sense that they are modeled by operators of a local type, while others have a nonlocal nature, that is, they are modeled by nonlocal type operators. In the former case, to understand the behavior of the operator around a given point, one only needs information on the value of the unknown function in a neighborhood of this point. One of the most widely studied operators of a local type in the past decades by the mathematic community is the Laplace operator $-\Delta$, which is the infinitesimal generator of the Brownian motion.

In the case of nonlocal phenomena, more information is needed. More precisely, in order to understand how a nonlocal operator acts on a function at a point, we need both the information on the value of the function in a neighborhood and far away from this point since the operator is mostly of integral form. This is in general observed when studying Lévy processes (the simplest Markovian models with jumps): it is a generalization of Brownian motion since here, stochastic continuity occurs instead of continuity of paths as in Brownian motion. It is worth noticing that operators of integral form enter in the class of the so-called *singular integral operators* and their general form is as follows (up to a normalization constant)

$$L_{K,\Omega}u(x) = P.V. \int_{\Omega} (u(x) - u(y))K(x - y)\; dy \tag{1.1.1}$$

where $\Omega \subseteqq \mathbb{R}^N$ and K is a kernel with certain growth, satisfying the properties

$$K(z) \geq 0 \qquad \text{and} \qquad K(-z) = K(z). \tag{1.1.2}$$

Moreover, we assume the following *Lévy type integrability condition* on K:

$$\int_{\mathbb{R}^N} \min\{1, |z|^2\}K(z)\; dz < \infty. \tag{1.1.3}$$

We recall that "*P.V.*" stands for the Cauchy principal value of the integral. The reason of having the principal value in definition (1.1.1) is that very often the kernel K is singular at x. As a consequence, to evaluate $L_{K,\Omega}u(x)$ pointwisely, one needs u to be sufficiently regular. The kernel we will be interested in throughout this thesis belongs in a particular class of kernels K satisfying

$$\frac{\lambda}{|z|^{N+2s}} \leq K(z) \leq \frac{\Lambda}{|z|^{N+2s}}, \tag{1.1.4}$$

where $0 < \lambda \leq \Lambda$ are positive constants and $s \in (0,1)$ is a positive parameter. Note that condition (1.1.4) is the nonlocal counterpart of ellipticity assumption on the coefficients of second-order differential operators in PDEs, namely, for example operators of the form $\sum_{i,j} \partial_i(a_{i,j}(x)\partial_j u(x))$ (divergence form) or of the form $\sum_{i,j} a_{i,j}(x)\partial^2_{i,j}u(x)$ (non-divergence form).

Before going further, it is important to comment on the operator $L_{K,\Omega}$. Consider a domain $\Omega \subset \mathbb{R}^N$ and $X_t,\ t \geq 0$ a Lévy process starting in Ω; here, one can think for example of a process described by the movement of a particle jumping randomly from one point to another with probability K. Then the particle may jump from one point in Ω to another or may also jump from one point in Ω to another point in the complement of Ω. If the first situation keeps running until the process is killed in Ω (censoring arises), then this means that particle will never exit the domain and in this case, we include the subscript "Ω" in the definition of $L_{K,\Omega}$ to indicate the restriction of the Lévy process to the domain Ω ($L_{K,\Omega}$ is a regional operator). In the non-censored case when arbitrary jumps of the particle are taken into account, the operator is of the form $L_{K,\mathbb{R}^N}$.

The aim of this thesis is to analyze equations driven by both $L_{K,\mathbb{R}^N}$ and $L_{K,\Omega}$. Precisely, operator of the type $L_{K,\mathbb{R}^N}$ is treated in paper [R1] while the one of type $L_{K,\Omega}$ (with $\Omega \subsetneqq \mathbb{R}^N$) is considered in papers [R2, R3, R4, R5, R6, R7].

Singular integral operators arise as appropriate operators for studying several phenomena in the world since models led by such operators are more accurate, more realistic, and therefore provide a better understanding of these phenomena. In recent years, these operators have been the subject of intensive study due to their large applications in many fields of mathematics. Just to name a few (the list of applications is far to be exhaustive), they can be used in finance [56,120,134], in crystal dislocation [123], in minimal surfaces [39], in anomalous diffusion [1,145], in thin obstacle problems (see [38]) and in image processing (see [31,32,90]). For further applications, we refer to [145] and the references therein.

1.1.1 The fractional Laplacian

In this subsection, we introduce a special case of singular integral operator of the form (1.1.1). Moreover, we also present some (non-exhaustive) properties related to it.

Coming back to (1.1.1), when $\Omega = \mathbb{R}^N$ and the kernel K is of the form $K(z) = c_{N,s}|z|^{-N-2s}$ for $z \in \mathbb{R}^N,\ z \neq 0$ with $s \in (0,1)$, then the operator $L_{K,\Omega}$ is the standard *fractional Laplacian*

$$(-\Delta)^s u(x) = c_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \quad x \in \mathbb{R}^N \tag{1.1.5}$$

where $c_{N,s}$ is a normalization constant which is explicitly given by (see [33, Theorem 3.5])

$$c_{N,s} := \left(\int_{\mathbb{R}^N} \frac{1-\cos(\zeta_1)}{|\zeta|^{N+2s}}\, d\zeta \right)^{-1} = s(1-s)\pi^{-N/2}2^{2s}\frac{\Gamma(\frac{N+2s}{2})}{\Gamma(2-s)}. \tag{1.1.6}$$

Here, Γ stands for the usual Gamma function. The definition (1.1.6) reveals that $c_{N,s}$ can be approximated in the limits $s \to 0^+$ and $s \to 1^-$ as follows:

$$c_{N,s} \sim s(1-s) \quad \text{as} \quad s \to 0^+ \quad \text{and} \quad s \to 1^-. \tag{1.1.7}$$

The constant $c_{N,s}$ play an important role in understanding the assymptotic behavior of $(-\Delta)^s$ as $s \to 0^+$ and $s \to 1^-$. In fact, by considering the approximation (1.1.7), it is proved that (see [65]) for u sufficiently regular and bounded,

$$\lim_{s\to 0^+}(-\Delta)^s u = u \qquad \text{and} \qquad \lim_{s\to 1^-}(-\Delta)^s u = -\Delta u. \tag{1.1.8}$$

The second limit in (1.1.8) shows in particular that local PDEs involving the Laplace operator are in general the limit case as $s \to 1^-$ of nonlocal equations involving the fractional Laplacian. This also follows from the definitions of $-\Delta$ and $(-\Delta)^s$ via Fourier transform. Indeed, it is known that $-\Delta$ can be expressed via Fourier transform as

$$\mathcal{F}[-\Delta u](\xi) = |\xi|^2\mathcal{F}[u](\xi) \quad \text{for} \quad u \in \mathcal{S}$$

showing that the classical Laplacian $-\Delta$ possesses Fourier multiplier (or symbol) of the form $|\xi|^2$.

On the other hand, the fractional Laplacian $(-\Delta)^s$ admits Fourier multiplier (or symbol) of the form $|\xi|^{2s}$; this follows from the definition of $(-\Delta)^s$ via Fourier transform (see e.g., [65]):

$$\mathcal{F}[(-\Delta)^s u](\xi) = |\xi|^{2s}\mathcal{F}[u](\xi) \qquad \text{for} \quad u \in \mathcal{S}. \tag{1.1.9}$$

We recall that the Fourier transform $\mathcal{F}[u]$ of $u \in \mathcal{S}$ is defined by

$$\mathcal{F}[u](\xi) = \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} u(x)e^{-2\pi i\xi\cdot x}\, dx, \qquad \xi \in \mathbb{R}^N$$

and $\mathcal{S}$ denotes the space of Schwartz functions, that is, the space of smooth and rapidly decreasing functions at infinity. From (1.1.9), it is deduced that $(-\Delta)^s$ is a pseudo-differential operator of order $2s$ and that $(-\Delta)^{s+t} = (-\Delta)^s \circ (-\Delta)^t = (-\Delta)^t \circ (-\Delta)^s$.

Notice that the definitions (1.1.5) and (1.1.9) are equivalent (see [140,141]). Besides, there exist other equivalent definitions of the fractional Laplacian $(-\Delta)^s$. We refer to [113] for a complete exposition.

From a probabilistic point of view, the fractional Laplacian represents the infinitesimal generator of a special class of Lévy processes called $2s$-*stable processes*. This means precisely that if X_t is a $2s$-stable Lévy process, then for every $u \in C^2(\mathbb{R}^N)$,

$$(-\Delta)^s u(x) = \lim_{t\to 0^+}\frac{\mathbb{E}[u(x+X_t)] - u(x)}{t}, \tag{1.1.10}$$

where, $\mathbb{E}[u(x+X_t)] := \mathbb{E}_x[u(X_t)]$ represents the expectation for the process starting from x.

The fractional Laplacian possesses some nice properties. Among other things, it is translation and rotation invariant and scales like λ^{2s}: if λ is a real parameter, u a function and by setting $u_\lambda(x) := u(\lambda x)$, then one has following scaling property

$$(-\Delta)^s u_\lambda(x) = \lambda^{2s}[(-\Delta)^s u](\lambda x) \quad \text{for} \quad x \in \mathbb{R}^N. \tag{1.1.11}$$

Moreover, this operator grows like $|x|^{-N-2s}$ at infinity. Indeed, for $\varphi \in \mathcal{S}$ it is shown that (see e.g., [25,77,144])

$$|(-\Delta)^s\varphi(x)| \le \frac{C_{\varphi,N,s}}{1+|x|^{N+2s}} \qquad \text{for} \qquad x \in \mathbb{R}^N. \tag{1.1.12}$$

Now, let $\mathcal{L}^1_s(\mathbb{R}^N)$ be the L^1-weighted space defined as

$$\mathcal{L}^1_s(\mathbb{R}^N) = \Big\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable} : \|u\|_{\mathcal{L}^1_s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \frac{|u(x)|}{1+|x|^{N+2s}}\, dx < \infty \Big\}.$$

From the above decay, we have that for any $u \in \mathcal{L}^1_s(\mathbb{R}^N)$, $(-\Delta)^s u$ is a distribution defined as

$$\langle (-\Delta)^s u, \varphi \rangle := \int_{\mathbb{R}^N} u(-\Delta)^s \varphi \, dx, \quad \forall \varphi \in C^\infty_c(\Omega).$$

One of the main difficulties in manipulating $(-\Delta)^s$ is its nonlocal nature. However, to bypass this difficulty in some cases, Caffarelli and Silvestre proposed, in their seminal work (see [42]), an interesting argument, the so-called Caffarelli-Silvestre extension formula. The idea behind this is to localize the fractional Laplacian with respect to a Dirichlet-to-Neumann operator. More precisely, let $\mathbb{R}^{N+1}_+ = \{(x,t) : x \in \mathbb{R}^N,\ t > 0\}$ be the upper half-space and consider $v : \mathbb{R}^{N+1}_+ \to \mathbb{R}$ be a solution of

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla v) = 0 & \text{in } \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0,+\infty) \\ v = u & \text{on } \partial\mathbb{R}^{N+1}_+ = \mathbb{R}^N, \end{cases}$$

with $u \in \mathcal{S}$ and

$$-\lim_{t\to 0^+} t^{1-2s} v_t = \frac{1}{\kappa_s}(-\Delta)^s u,$$

where $\kappa_s > 0$ is a positive constant given by

$$\kappa_s = 2^{2s-1}\frac{\Gamma(s)}{\Gamma(1-s)}.$$

The function v is called the *s-harmonic extension* of u. The main advantage of this argument relies on the fact that classical techniques for elliptic PDEs involving (weighted) divergence-type operators can be used. In [41], the authors used this argument to prove full regularity of solution to obstacle problem as well as regularity of the free boundary. Cabré and Sire [37] also used this technique to prove regularity, maximum principles and Hamiltonians estimates for solution of nonlinear equations for fractional Laplacian. The authors in [40] also use this extension technique to study variational problems with free boundaries for the fractional Laplacian. Other interesting references where such an argument has been used can be found in [36, 86, 87].

Throughout this thesis, we will not use this technique. We will analyze the nonlocal problem directly. We therefore expect that the approach used to prove the main results in [R1] can be adapted to a very large class of nonlocal operators.

1.1.2 The regional fractional Laplacian

We are now interested in a particular singular integral operator of the form (1.1.1) whose measure is restricted to a domain $\Omega \subset \mathbb{R}^N$. Roughly speaking, let us assume again that the kernel K in (1.1.1) is of the form $K(z) = c_{N,s}|z|^{-N-2s},\ z \in \Omega,\ z \neq 0$ with $c_{N,s}$ the constant define in (1.1.6). Then $L_{K,\Omega}$ agrees with the so-called *regional fractional Laplacian* (sometimes called *censored fractional Laplacian*)

$$(-\Delta)^s_\Omega u(x) = c_{N,s} P.V. \int_\Omega \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \quad x \in \Omega. \tag{1.1.13}$$

This operator is the infinitesimal generator of the so-called *censored* $2s$*-stable process*, that is, a stable Lévy process in which Lévy measure is restricted to the domain Ω: jumps outside the domain are forbidden. In other words, a censored stable process in an open domain Ω is a stable process forced to stay inside Ω. Similarly to (1.1.10), if Y_t is a censored $2s$-stable process, then for every $u \in C^2(\Omega)$,

$$(-\Delta)^s_\Omega u(x) = \lim_{t\to 0^+} \frac{\mathbb{E}[u(x+Y_t)] - u(x)}{t}. \tag{1.1.14}$$

The censored stable process can be obtained in particular through the Feynman-Kac transform as well as from the so-called Ikeda-Nagasawa-Watanabe piecing together procedure. We refer to [24] for a complete exposition. We point out that this process is strongly related to reflected stable processes in a bounded domain with killing within the domain, at its boundary, and eventually not approaching the boundary at all (see [99, 138]). We also refer to [24] where the authors proved that the censored $2s$-stable process is conservative and will never approach the boundary $\partial\Omega$ when $s \in (0, \frac{1}{2}]$ and for $s \in (\frac{1}{2}, 1)$ the process could approach the boundary $\partial\Omega$ in a finite time.

Similar to the fractional Laplacian, the regional fraction Laplacian admits also some nice properties. For instance, consider a bounded domain Ω of $\mathbb{R}^N$ and let $u \in C^2(\overline{\Omega})$. Then, $(-\Delta)^s_\Omega$ has the following pointwise asymptotic properties in Ω:

$$\lim_{s\to 0^+} (-\Delta)^s_\Omega u = 0 \qquad \text{and} \qquad \lim_{s\to 1^-} (-\Delta)^s_\Omega u = -\Delta u. \tag{1.1.15}$$

When Ω is unbounded, we notice that the first limit in (1.1.15) can be also obtained, provided that an additional assumption is satisfied. We refer to the Appendix 8.3 for more details. We notice also that the second limit in (1.1.15) can be found in [73, Section 8]. It is worth pointing out the difference between the first limit in (1.1.15) to that in (1.1.8). On the other hand, due to its domain dependence, the operator $(-\Delta)^s_\Omega$ is not translation and rotation invariant, and does not satisfy the scaling property (1.1.11). Moreover, $(-\Delta)^s_\Omega$ does not admit any symbol since it is not clear what should be its Fourier transform. The lack of these properties sometimes makes the regional fractional Laplacian difficult to study. However, in the case when Ω is unbounded, e.g., $\Omega = \mathbb{R}^N_+$ the upper half-space, then $(-\Delta)^s_{\mathbb{R}^N_+}$ shares with $(-\Delta)^s$ many properties such as the translation (parallel to the boundary) invariance, the scaling property (1.1.11) (with $\lambda > 0$) and the decay estimate (1.1.12). In contrast, we have the following asymptotics

$$\lim_{s\to 0^+} (-\Delta)^s_{\mathbb{R}^N_+} u = \frac{1}{2} u \qquad \text{and} \qquad \lim_{s\to 1^-} (-\Delta)^s_{\mathbb{R}^N_+} u = -\Delta u \tag{1.1.16}$$

for all $u \in C^2_c(\mathbb{R}^N_+)$. The first limit in (1.1.16) can be found in the Appendix 8.3. For the second limit, its proof follows exactly as in the case of $(-\Delta)^s$ given in [65].

1.1.3 Fractional Laplacian versus regional fractional Laplacian

Let Ω be an open set in $\mathbb{R}^N$. There is a nice relationship between the regional fractional Laplacian $(-\Delta)^s_\Omega$ and the fractional Laplacian $(-\Delta)^s$ restricted to the set Ω. For this let $u : \mathbb{R}^N \to \mathbb{R}$ be a sufficiently regular function with compact support in Ω. Then, trivially, $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$. By splitting the integral over $\mathbb{R}^N$ as the sum of that in Ω and in the complement $\mathbb{R}^N \setminus \Omega$, one arrives at the following identity

$$(-\Delta)^s_\Omega u(x) = (-\Delta)^s u(x) - \kappa_\Omega(x) u(x), \quad x \in \Omega \tag{1.1.17}$$

where

$$\kappa_\Omega(x) = c_{N,s} \int_{\mathbb{R}^N \setminus \Omega} \frac{1}{|x-y|^{N+2s}}\, dy. \tag{1.1.18}$$

κ_Ω is called the density function of the killing measure of $(-\Delta)^s$ restricted to Ω. In this way, one observes that the regional fractional Laplacian $(-\Delta)^s_\Omega$ can be identified with the Schrödinger operator $(-\Delta)^s - \kappa_\Omega$ for sufficiently regular functions u defined in the whole space and vanishing outside Ω, that is, in $\mathbb{R}^N \setminus \Omega$. For functions defined only on Ω we may consider the trivial extension of u in all of $\mathbb{R}^N$, that is, the unique extension $u : \mathbb{R}^N \to \mathbb{R}$ with $u = 0$ in $\mathbb{R}^N \setminus \Omega$. Then one obtains the identity (1.1.17).

In view of this, it is natural to ask whether it is advantageous to deal with the operator $(-\Delta)^s - \kappa_\Omega$ in place of $(-\Delta)^s_\Omega$. The answer depends on the particular problem. For instance, since κ_Ω is of class C^∞ in Ω, the Schrödinger operator $(-\Delta)^s - \kappa_\Omega$ appears as a good candidate for analyzing local properties (e.g., interior regularity) of solutions to problems involving the regional fractional Laplacian. On the other hand, regarding boundary properties, the operator $(-\Delta)^s - \kappa_\Omega$ fails to be a good candidate since the killing measure κ_Ω behaves like $\mathrm{dist}(x, \partial\Omega)^{-2s}$ (if Ω is bounded and Lipschitz) near the boundary (see [94, Eq. (1.3.2.12)]). Here, $\mathrm{dist}(x, \partial\Omega)$ denotes the distance from x to the boundary $\partial\Omega$ of Ω.

Remark 1.1.1. In spite of their apparent similarity, $(-\Delta)^s$ and $(-\Delta)^s_\Omega$ are in general two different operators and their difference κ_Ω is given in (1.1.17). From (1.1.8) and (1.1.15), we observe that for sufficiently regular function u with compact support in Ω, $\kappa_\Omega u \to u$ resp. $\kappa_\Omega u \to 0$ as $s \to 0^+$ and $s \to 1^-$ respectively.

Another structural difference between $(-\Delta)^s$ and $(-\Delta)^s_\Omega$ can be seen in their behavior on boundary value problems. In case of the fractional Laplacian, the well-posed Dirichlet problem is of the form (see [95, 104])

$$\begin{cases} (-\Delta)^s u = f & \text{in } \ \Omega \\ \qquad\ u = 0 & \text{in } \ \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.1.19}$$

for any $s \in (0, 1)$, while for the regional fractional Laplacian, the well-posed Dirichlet problem reads as (see [98])

$$\begin{cases} (-\Delta)^s_\Omega u = f & \text{in } \ \Omega \\ \qquad\ u = 0 & \text{on } \ \partial\Omega \end{cases} \tag{1.1.20}$$

provided that $s \in (\frac{1}{2}, 1)$.

It is important to point out that in (1.1.19), the Dirichlet condition is given on the complement $\mathbb{R}^N \setminus \Omega$ while in (1.1.20), it is defined on the boundary $\partial\Omega$ of Ω. Notice that, if in (1.1.19) we impose the Dirichlet condition on $\partial\Omega$ instead, then the problem will not be well-posed. We refer to [98, 104] and the references therein for more details.

The restriction $s \in (\frac{1}{2}, 1)$ in (1.1.20) is due to the fact that for $s \in (0, \frac{1}{2}]$ the boundary value problem (1.1.20) is not well defined, as it is shown in [24] from a probabilistic point of view. Recently, a similar result has been established in [54] from analytical point of view. More significant differences between the fractional Laplacian and the regional fractional Laplacian are discussed in [1, 69].

To conclude this section, we introduce the following notation and functional spaces that will be used throughout the thesis.

Notation and functional setting.

Let $s \in (0,1)$ and let $\mathcal{O} \subset \mathbb{R}^N$ be an open set. We denote by $H^s(\mathcal{O})$ the fractional Sobolev space consisting of all measurable functions $u : \mathcal{O} \to \mathbb{R}$ such that

$$\|u\|^2_{H^s(\mathcal{O})} := \|u\|^2_{L^2(\mathcal{O})} + [u]^2_{H^s(\mathcal{O})}$$

is finite. Here, $[u]_{H^s(\mathcal{O})} := \int_{\mathcal{O}}\int_{\mathcal{O}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\, dxdy$. It is a Hilbert space endowed with the norm $\|\cdot\|_{H^s(\mathcal{O})}$. We denote by $\mathcal{H}^s_0(\mathcal{O})$ the completion of $C^\infty_c(\mathcal{O})$ with respect to the norm $\|\cdot\|_{H^s(\mathbb{R}^N)}$. It is also a Hilbert space with the norm $\|\cdot\|_{H^s(\mathbb{R}^N)}$. If $\mathcal{O}$ is bounded with continuous boundary, then one has the characterization (see [94, Theorem 1.4.2.2])

$$\mathcal{H}^s_0(\mathcal{O}) = \{u \in H^s(\mathbb{R}^N) : u = 0 \ \text{ in } \ \mathbb{R}^N \setminus \mathcal{O}\}.$$

Next, we denote by $H^s_0(\mathcal{O})$ the completion of $C^\infty_c(\mathcal{O})$ under the norm $\|\cdot\|_{H^s(\mathcal{O})}$. This space admits also a characterization of the form (see [147, Theorem 4.5])

$$H^s_0(\mathcal{O}) = \{u \in H^s(\mathcal{O}) : u = 0 \ \text{ on } \ \partial\mathcal{O}\}.$$

Moreover, when $s \in (\frac{1}{2}, 1)$, then $H^s_0(\mathcal{O})$ is a Hilbert space equipped with the norm

$$\|u\|_{H^s_0(\mathcal{O})} = [u]_{H^s(\mathcal{O})}.$$

If $\mathcal{O}$ is bounded, then the above norm is equivalent to the usual one in $H^s(\mathcal{O})$ thanks to the Poincaré inequality.

Given a function u, then $u^+ := \max\{u, 0\}$ and $u^- := \max\{-u, 0\}$ represents respectively its positive and negative part. Moreover, if f and g are two functions, we write $f \asymp g$ to mean that there exists $C > 1$ such that $C^{-1}g \le f \le Cg$. The N-dimensional Lebesgue measure of the set $\mathcal{O}$ is denoted by $|\mathcal{O}|$. Also, the characteristic function of any set $A \subset \mathbb{R}^N$ is denoted by $\mathbb{1}_A$. Next, by $\mathbb{S}^{N-1}$ we denote the $(N-1)$-dimensional sphere of $\mathbb{R}^N$. Throughout the thesis, we will always use $B_r(x)$ as the ball centered at x with radius r, and $B^+_r(x) = \{y = (y', y_N) \in B_r(x) : y_N > 0\}$ the upper half ball. Also, we put $B_r = B_r(0)$. Now, for $x \in \mathcal{O}$, the distance from x to the boundary $\partial\mathcal{O}$ of $\mathcal{O}$ is defined as $\delta_{\mathcal{O}}(x) = \mathrm{dist}(x, \partial\mathcal{O})$. If u and v are two measurable functions defined on $E \subseteq \mathbb{R}^N$, we put

$$\mathcal{E}^E_s(u,v) := \frac{c_{N,s}}{2}\int_E\int_E \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy \tag{1.1.21}$$

whenever this integral is defined in Lebesgue sense.

1.2 A short introduction to the main topics of the thesis

The aim of this section is to put the main topics of the thesis into perspective. The first subject of this section is related to paper [R1] while the remainder to papers [R2, R3, R4, R5, R6, R7]. Topics may or may not have interconnections but we have tried to make them as self-contained as possible.

1.2.1 Morse index analysis for elliptic problems

The Morse index of solutions of nonlinear elliptic problems appears nowadays as a classical topic in the calculus of variations. It goes back to the so-called *Morse theory* introduced in 1934 by Marston Morse (see [121]) which relates the topology of a manifold with the critical points of a function defined over it.

Roughly speaking, the Morse index $m(u)$ of a solution u of a given equation on a Hilbert space is the number of negative eigenvalues (including multiplicity) of the corresponding linearized problem provided that the linearized operator is self adjoint and has no negative essential spectrum. Equivalently, it can also be defined as the maximal dimension of a subspace in which the corresponding quadratic form of the linearized operator at u is negative definite. More precisely, let consider the following elliptic semilinear Dirichlet problem

$$-\Delta u = f(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{1.2.1}$$

with a nonlinearity $f \in C^1(\mathbb{R})$. It is well-known that the classical Sobolev space $H_0^1(\Omega)$ is the appropriate framework to study problem (1.2.1). Next, by J, we denote the energy functional of (1.2.1) define as

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \; dx - \int_\Omega F(u) \; dx, \quad u \in H_0^1(\Omega)$$

where $F(t) = \int_0^t f(\tau) \; d\tau$ is the primitive of f. We recall that the functional J is well-defined and of class C^1 if f has subcritical growth. From critical point theory, it is known that (weak) solutions of (1.2.1) are also critical points to J. Now, given u a (weak) solution of (1.2.1) (or equivalently a critical point of J), the linearized operator L_u at u is defined as

$$L_u := -\Delta - f'(u)$$

where f' is the derivative of f. We assign by Q_{L_u} the corresponding energy functional to L_u define as

$$Q_{L_u}(v) = \frac{1}{2}\int_\Omega (|\nabla v|^2 - f'(u)v^2) \; dx, \quad v \in H_0^1(\Omega).$$

Then,

$$\begin{aligned} m(u) &:= \#\{\lambda < 0 : \lambda \in \sigma(L_u)\} \\ &:= \max\{\dim X,\ X \subset H_0^1(\Omega) : Q_{L_u} \text{ is negative definite on } X\}. \end{aligned}$$

Here, $\sigma(L_u)$ denotes the spectrum of L, that is, the set of all eigenvalues of L_u counted with their multiplicity. Also, by negative definiteness on X, we mean $\langle Q_{L_u}(v), v\rangle < 0$ for all $v \in X \setminus \{0\}$, where $\langle \cdot, \cdot \rangle$ represents the usual pairing on $H_0^1(\Omega)$.

In the past decades, the Morse index of solutions of semilinear elliptic equations (with suitable boundary conditions: Dirichlet, Neumann or Mixed) has been widely studied by many mathematicians due to its various applications in calculus of variations and partial differential equations. In fact, its estimate appears naturally in determining qualitative properties of solutions such as symmetry (see [58, 92, 124, 125]), symmetry breaking (see [2]) as well as nondegeneracy [3]. Also, bifurcation can be established through Morse index estimate see, e.g., [5, 6]. For more details (in a unified way) about the Morse index of solutions of (local) nonlinear elliptic equations, we refer to the recent book [59] by Lucio Damascelli and Filomena Pacella.

Problem (1.2.1) represents the classical version (the limit case as $s \to 1^-$) of the following nonlocal semilinear Dirichlet problem

$$(-\Delta)^s u = f(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \tag{1.2.2}$$

This problem has been extensively studied in the literature. However, in contrast to (1.2.1), the Morse index analysis of solution to (1.2.2) is much less understood due to the nonlocality of the problem. Very few references can be found in the literature. For instance, in [128] it is shown that under some mild assumptions on f, problem (1.2.2) has at least six nontrivial solutions: two with Morse index 0, another two with Morse index 1 and the last one with Morse index $d \geq 2$. In [85], it has been established that the ground state Q to the problem $(-\Delta)^s Q + Q - Q^{\alpha+1} = 0$ in $\mathbb{R}$ is nondegenerate and its Morse index equals 1. We recall that by ground state, we mean a nontrivial, nonnegative and radial function $Q(x) = Q(|x|)$ vanishing at infinity and satisfies (in distributional sense) the equation. Finally, in [60], the authors proved that ground state to the Choquard equation in $\mathbb{R}^N$ driven by the fractional Laplacian has Morse index 1.

In the first paper [R1], we estimate the Morse index of any radial sign changing solution to (1.2.2) in the case when Ω is the unit ball.

1.2.2 Asymptotic analysis for the s-power of the Laplacian with respect to s

In this subsection, we introduce the topics that will be treated in papers [R2] and [R3], namely small order asymptotic and s-derivative of solution u_s of an elliptic equation.

Small order asymptotic for nonlocal operators such as fractional Laplacian has received growing attention in recent years. This subject naturally arises in population dynamics [126], in optimal control [139], in fractional harmonic maps [10], and in image processing [9]. In [55], Chen and Weth introduced the so-called *Logarithmic Laplacian* L_Δ which arises as a formal derivative $\partial_s|_{s=0}(-\Delta)^s$ of the fractional Laplacian at $s = 0$ define as

$$L_\Delta u(x) = c_N \int_{\mathbb{R}^N} \frac{u(x)\mathbb{1}_{B_1(x)}(y) - u(y)}{|x-y|^N}\, dy + \rho_N u(x), \quad x \in \mathbb{R}^N. \tag{1.2.3}$$

This operator appears in the description of eigenvalues and eigenfunctions of the fractional Laplacian [55, 82], in the study of 0-fractional perimeter [43] as well as in the study of C^1-regularity of the map $(0,1) \to L^\infty(\Omega),\ s \mapsto u_s$ where u_s is the solution to the homogeneous Dirichlet problem $(-\Delta)^s u_s = f$ in Ω, $u_s = 0$ in $\mathbb{R}^N \setminus \Omega$, see e.g. [106]. Spectral properties of L_Δ have been recently investigated in [114]. An interesting analysis regarding small order asymptotics for nonlinear problems involving fractional Laplacian can be found in [102]. For more asymptotic result in s for the fractional Laplacian, we refer to [21, 101, 106].

It is natural to ask whether related asymptotic results hold for $(-\Delta)^s_\Omega$. This question is the main focus of [R2] and [R3].

1.2.3 Nonlinear problems involving regional fractional Laplacian: existence results

The purpose of this subsection is to briefly expose the topics treated in papers [R4] and [R7]. More precisely in [R4], we deal with nonlinear problems with critical growth while in [R7], problems with subcritical growth are considered.

The critical problem: existence of minimizers for fractional Sobolev inequality on domains

The topic considered here is related to [R4]. It is concerned about the existence of nonnegative extremals for the best Sobolev constant

$$S_{N,s}(\Omega) = \inf_{\substack{u \in H_0^s(\Omega) \\ u \not\equiv 0}} \frac{Q_{N,s,\Omega}(u)}{\|u\|^2_{L^{2_s^*}(\Omega)}} \tag{1.2.4}$$

where $s \in (\frac{1}{2}, 1)$, Ω is a C^1 domain of $\mathbb{R}^N$ $(N \geq 2)$ and $2_s^* = \frac{2N}{N-2s}$ is the so-called fractional critical Sobolev exponent and $Q_{N,s,\Omega}$ is a nonnegative quadratic form defined on $H_0^s(\Omega)$ by

$$Q_{N,s,\Omega}(u) = \frac{c_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \, dxdy.$$

Variationally, nonnegative minimizers for (1.2.4) are weak solutions to

$$(-\Delta)^s_\Omega u = u^{2_s^*-1} \quad \text{in } \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega. \tag{1.2.5}$$

Before going further, let us say a few words on minimization problems for the full fractional Laplacian. For that, we introduce the infimum

$$\mu_{N,s}(\Omega) := \inf \Big\{ Q_{N,s,\mathbb{R}^N}(u) : u \in C_c^\infty(\Omega), \int_\Omega |u|^{2_s^*} \, dx = 1 \Big\}.$$

The behavior of $\mu_{N,s}(\Omega)$ is well-understood in the literature. In 1983, Lieb [117] provides a classification result for all minimizers of $\mu_{N,s}(\mathbb{R}^N)$. He showed that minimizers do not vanish anywhere on $\mathbb{R}^N$ and therefore, the constant $\mu_{N,s}(\Omega)$ is never achieved unless $\Omega = \mathbb{R}^N$. In this case, he obtained that minimizers agree up to multiplications, dilations and translations with the function $(1 + |x|^2)^{\frac{2s-N}{2}}$.

In recent years, a lot of people have been investigating the minimization problem $\mu_{N,s}(\Omega)$ and solutions to

$$(-\Delta)^s u = \mu_{N,s}(\Omega) u^{2_s^*-1} \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \tag{1.2.6}$$

But, the problem (1.2.6) does not admit any solution whenever Ω is a star-shaped domain as a consequence of fractional Pohozaev identity. The references being not exhaustive, we refer to [77, 131, 132] for more details.

Now, coming back to the minimization problem (1.2.4), an interesting question that arises is whether the constant $S_{N,s}(\Omega)$ is attained or not. In other words, does the Sobolev constant $S_{N,s}(\Omega)$ behaves differently from $\mu_{N,s}(\Omega)$? As a first observation, in contrast with $(-\Delta)^s$, no Pohozaev type identity is available yet for $(-\Delta)^s_\Omega$. Second, a lack of compactness is observed in (1.2.5) and then standard argument of the calculus of variations cannot be applied to problem (1.2.5) in order to derive solutions. If $u \in C_c^\infty(\Omega)$, one observes that

$$\begin{aligned} Q_{N,s,\Omega}(u) &= Q_{N,s,\mathbb{R}^N}(u) - c_{N,s} \int_\Omega u(x)^2 \int_{\mathbb{R}^N \setminus \Omega} \frac{1}{|x-y|^{N+2s}} \, dydx \\ &= Q_{N,s,\mathbb{R}^N}(u) - \int_\Omega u(x)^2 \kappa_\Omega(x) dx \end{aligned} \tag{1.2.7}$$

$$\asymp Q_{N,s,\mathbb{R}^N}(u) - \int_\Omega u(x)^2 \delta_\Omega^{-2s}(x)dx.$$

From the above identity, we see that the negative quantity in (1.2.7) lowers the infimum $\mu_{N,s}(\Omega)$ and therefore, may produce minimizers for $S_{N,s}(\Omega)$. Such observation was pointed out by Brézis-Nirenberg [30] in the study of critical problems.

The minimization problem (1.2.4) was studied by Frank et al. [83]. They established the existence of minimizers for a particular class of domains Ω assuming $N \geq 4s$. For instance, in the case when $\Omega \subset \mathbb{R}^N_+$ is a C^1 bounded domain whose boundary possesses a flat part, that is Ω is a C^1 domain with the shape $B_r^+(z) \subset \Omega \subset \mathbb{R}^N_+$ for some $r > 0$ and $z \in \partial\mathbb{R}^N_+$, and such that $\mathbb{R}^N_+ \setminus \Omega$ has nonempty interior, they showed that $S_{N,s}(\Omega)$ is attained. They also prove that when $\Omega = \mathbb{R}^N_+$, the constant $S_{N,s}(\mathbb{R}^N_+)$ is attained for $s \neq \frac{1}{2}$.

The constant $S_{N,s}(\Omega)$ possesses some nice qualitative properties. For instance, $S_{N,s}(\Omega) > 0$ and $\underline{S}_{N,s} := \inf_\Omega S_{N,s}(\Omega) > 0$ for any open set Ω whenever $N \geq 2$ and $s > \frac{1}{2}$ see e.g. [71]; $S_{N,s}(\Omega) = 0$ for any sufficiently regular open set Ω with finite measure whenever $N \geq 1$ and $s < \frac{1}{2}$, see e.g. [83]. Moreover, provided that Ω is the complement of the closure of bounded Lipschitz domain or a domain above the graph of Lipschitz function, it follows from [70] that $S_{N,s}(\Omega) > 0$ for $N \geq 1$ and $s < \frac{1}{2}$.

It is worth remarking that the case $s = \frac{1}{2}$ was left open in [83]. As we will see later in [R4], for Ω bounded and Lipschitz, we obtain that $S_{N,\frac{1}{2}}(\Omega) = 0$ for $N \geq 2$. We will then exploit this behavior in order to obtain minimizers when s is close to $\frac{1}{2}$.

The subcritical problem: existence of mountain pass solutions

We now turn our attention to the subcritical counterpart to problem (1.2.5), that is to the homogeneous Dirichlet problem

$$(-\Delta)^s_\Omega u = f(u) \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega \tag{1.2.8}$$

where the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is of subcritical growth. This problem will be treated in [R7].

In the calculus of variations, solutions of mountain pass type are well-studied in the literature both for local and nonlocal operators (see e.g., [7, 127, 129, 136, 148]). These types of solutions are in general of nonminimal type in contrast with those obtained in (1.2.5). One of the most important ingredients for obtaining such solutions is the mountain pass Theorem proved in 1973 by Ambrosetti and Rabinowitz [7].

The basic idea of the mountain pass Theorem consists of showing that the corresponding energy functional $Q \in C^1(H, \mathbb{R})$ (H being a Hilbert space) for a boundary value problem possesses the geometric features of mountain pass type and satisfies the Palais-Smale compactness condition. We say that Q satisfies the Palais-Smale compactness condition if any sequence u_i such that $Q(u_i) \to c$ (for some $c \in \mathbb{R}$) and $\sup\{|\langle Q'(u_i), \varphi\rangle| : \varphi \in H, \|\varphi\|_H = 1\} \to 0$ as $i \to \infty$, possesses a convergence subsequence.

For more background on mountain pass solutions of nonlocal problems, see the recent book by Dipierro et al. [67].

1.2.4 Symmetry and monotonicity of positive solutions of nonlinear problems: the moving plane method

In the calculus of variations, one of the most important questions lies in the classification of solutions whenever they exist. Qualitative properties like symmetry and monotonicity appear themself as key ingredients in the classification of solutions. One of the most effective strategies to establish radial symmetry and monotonicity of positive solutions is the celebrated method of *moving planes*, which goes back to the work of Alexandrov [4], Serrin [135] and Gidas, Ni, and Nirenberg [89]. This method exploits the invariance of the equation with respect to reflections as well as maximum and comparison principles for uniformly elliptic operators.

The moving planes method naturally arises in overdetermined problems [135] and can also be applied to obtain non-existence results [50]. For more applications of the moving plane method, we refer to [18, 20, 68, 79, 89, 109, 135] and the references therein.

In [R6], we establish the symmetry and monotonicity of positive solutions of the Dirichlet problem

$$(-\Delta)^s_{\mathbb{R}^N_+} u = u^{2^*_s-1}, \quad u > 0 \quad \text{in } \mathbb{R}^N_+, \quad u = 0 \quad \text{on } \partial\mathbb{R}^N_+ = \mathbb{R}^{N-1} \tag{1.2.9}$$

via the moving plane method. Here, $s \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ and 2^*_s is as in Subsection 1.2.3. As pointed out in this subsection, solutions to (1.2.9) are positive minimizers to the best constant in Sobolev inequality $S_{N,s}(\mathbb{R}^N_+)$, see (1.2.4).

1.2.5 Hopf lemma

We introduce in this subsection, the topic which is treated in paper [R5], namely, a Hopf type lemma for the regional fractional Laplacian. In the calculus of variation, Hopf lemma appeared as early as the maximum principle for harmonic functions. In 1927 Eberhard Hopf established a strong maximum principle in the case of second order elliptic partial differential equations see [105]. Formally, this result reads as follows:

> *If a function u satisfies a second order elliptic partial differential inequality in a domain and attains its maximum in the interior of the domain, then u is constant.*

In recent years, Hopf lemma has been successfully extended to nonlocal operators such as the fractional Laplacian $(-\Delta)^s$ introduce in Subsection 1.1.1. The nonlocality of this operator produces new challenges while studying it since local techniques cannot be applied. A careful analysis is therefore needed. A non-exhaustive list of papers in which Hopf lemma for $(-\Delta)^s$ has been investigated is [75, 93, 111, 116]. The common point in these papers lies in the fact that as in the local case, an *interior ball condition* is assumed on the domain. We recall that a domain $\Omega \subset \mathbb{R}^N$ satisfies an *interior ball condition* at $x_0 \in \partial\Omega$ if there exists a ball $B_r \subset \Omega$ such that $x_0 \in \partial B_r$.

It is natural to ask whether Hopf's result can be also applied to the regional fractional Laplacian $(-\Delta)^s_\Omega$. We will give an affirmative answer to this question in [R5] by analyzing the super-solution to the equation

$$(-\Delta)^s_\Omega u = c(x)u \qquad \text{in } \Omega. \tag{1.2.10}$$

1.3 Results of the thesis

The purpose of this section is to give an overview of the papers constituting this thesis. Precisely, we briefly present the results obtained in Chapters 2, 3, 4, 5, 6, 7, and 8. In Chapter 2, we study the Morse index of solution to the Dirichlet problem for $(-\Delta)^s$. In Chapter 3, small order asymptotics of eigenvalues of $(-\Delta)^s_\Omega$ are treated. After this, in Chapter 4, we analyze the s-dependence for solution to the Poisson problem for $(-\Delta)^s_\Omega$. Existence result for nonlinear equations with critical nonlinearity is considered in Chapter 5. Hopf lemma for regional fractional Laplacian is proved in Chapter 6 while symmetry and monotonicity for solution of Dirichlet problem for $(-\Delta)^s_{\mathbb{R}^N_+}$ are studied in Chapter 7. Finally, in the last chapter, Chapter 8, we study the existence of mountain pass solutions for $(-\Delta)^s_\Omega$.

1.3.1 Paper 1. Morse index versus radial symmetry for fractional Dirichlet problems

In this paper, which is joint work with M. M. Fall, P. A. Feulefack and T. Weth, we obtain an estimate of the Morse index of any radial sign changing solution to the nonlocal semilinear Dirichlet problem

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \mathcal{B} \\ u = 0 & \text{in } \mathbb{R}^N \setminus \mathcal{B} \end{cases} \tag{1.3.1}$$

where $\mathcal{B}$ is the unit ball in $\mathbb{R}^N$ centered at the origin and $f \in C^1(\mathbb{R})$. The first main result that we establish is the following.

Theorem 1.3.1. *Let u be a radially symmetric sign changing solution of problem* (1.3.1)*, and suppose that one of the following additional conditions holds.*

(A1) $s \in (\frac{1}{2}, 1)$.

(A2) $s \in (0, \frac{1}{2}]$, *and*

$$\int_0^t f(\tau)d\tau > \frac{N-2s}{2N}\, t f(t) \qquad \text{for } t \in \mathbb{R} \setminus \{0\}. \tag{1.3.2}$$

Then u has Morse index greater than or equal to $N+1$.

This result is the fractional counterpart to that obtained in [2] when the underlying domain is a ball. In fact, in [2], the authors analyzed qualitative properties of sign changing solutions to the semilinear Dirichlet problem

$$-\Delta u = f(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.3.3}$$

where Ω is a ball or an annulus centered at zero and $f \in C^1(\mathbb{R})$. They proved that (see e.g. [2, Theorem 1.1]) any radial sign changing solutions of (1.3.3) has Morse index greater than or equal to $N+1$. As a consequence, they also deduced that any index 1 sign changing solution in the ball or the annulus is nonradial. In particular, every eigenfunction corresponding to the second eigenvalue of $-\Delta$ in the ball or the annulus is nonradial. A natural question is whether such a qualitative property holds in the case of fractional Laplacian. Let us first mention that the assumption (1.3.2) applies to nonlinearities with subcritical growth. In particular, it holds for $f(t) = \lambda t$ with $\lambda > 0$. This leads to our second main result which is concerned with the Dirichlet eigenvalue problem

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \mathcal{B} \\ u = 0 & \text{in } \mathbb{R}^N \setminus \mathcal{B}. \end{cases} \tag{1.3.4}$$

It reads as follows.

Theorem 1.3.2. *Let $N \geq 1$ and $0 < s < 1$, and let λ_2 be the second eigenvalue of problem* (1.3.4). *Then every eigenfunction u corresponding to λ_2 is antisymmetric, i.e., it satisfies $u(-x) = -u(x)$ for $x \in \mathcal{B}$.*

This result gives a complete positive answer to a conjecture by Bañuelos and Kulczycki [72]. Partial answers to this conjecture were obtained in [15,72,81,112]. Precisely, in [15], this result was proved for $N = 1,\ s = \frac{1}{2}$ and was extended in [112] to $N = 1,\ s \in [\frac{1}{2}, 1)$. In [72], the conjecture in the cases $N \leq 2,\ s \in (0,1)$ and $3 \leq N \leq 9,\ s = \frac{1}{2}$ is proved. Finally, in [81], the result is established for $N = 3,\ s \in (0,1)$.

We would like to point out that, after our result was published on arXiv.org, Benedikt et al. [19] also resolved this conjecture by following a different strategy to ours. Their approach is based on a polarization argument.

We prove Theorem 1.3.1 by following the same strategy as in [2]. The main idea is to use partial derivative of u to build test functions in order to estimate the Morse index of u. The nonlocality of our problem brings many difficulties in the argument, for instance, it is more delicate to control the oscillations of the radial derivative of u near the boundary. Notice that this radial derivative is given by $\partial_r u(x) = \nabla u(x) \cdot \frac{x}{|x|}$. Therefore, in order to study the oscillations of $\partial_r u$ close to the boundary, a gradient estimate due to Fall and Jarohs [74] is needed. More precisely, by [74] we have

$$\lim_{x \to z} \delta_{\mathcal{B}}^{1-s}(x) \partial_r u(x) = -s\Big(\frac{u}{\delta_{\mathcal{B}}^s}\Big)(z) \quad \text{for} \quad z \in \partial\mathcal{B},$$

and a careful analysis of the behavior of the ratio $\frac{u}{\delta_{\mathcal{B}}^s}$ near the boundary is also needed. Here, $\frac{u}{\delta_{\mathcal{B}}^s}$ is defined as a limit. In the case $s \in (\frac{1}{2}, 1)$, we use a regularity result by Grubb [96, Theorem 2.2] to complete the proof in the case where $\frac{u}{\delta_{\mathcal{B}}^s}$ vanishes on $\partial\mathcal{B}$. Moreover, in the case $s \in (0, \frac{1}{2}]$, we use the assumption (1.3.2) to ensure that $\frac{u}{\delta_{\mathcal{B}}^s}$ does not vanish on the boundary.

To prove Theorem 1.3.2, we use Theorem 1.3.1 together with the following observation due to Dyda et al. [72, p. 503]: *Either* (6.5.1) *admits a radially symmetric eigenfunction corresponding to the second eigenvalue λ_2, or every eigenfunction corresponding to λ_2 is a product of a linear and a radial function.* Since every such eigenfunction u is a sign changing solution of (1.3.1) with $t \mapsto f(t) = \lambda_2 t$ and has Morse index $1 < N+1$, it cannot be radially symmetric as a consequence of Theorem 1.3.1. Therefore, u must be a product of a linear and a radial function, and this implies that u is antisymmetric. This completes the proof of Theorem 1.3.2.

We now comment on the proof of Theorem 1.3.1. Let us introduce first some notation. We denote by

$$L_u := (-\Delta)^s - f'(u)$$

the linearized operator at u, a radial sign changing solution of (1.3.1). Moreover, let

$$(v, w) \mapsto \mathcal{E}_{s,L_u}(v, w) = \mathcal{E}_s^{\mathbb{R}^N}(v, w) - \int_{\mathcal{B}} f'(u) vw \, dx$$

be the corresponding bilinear form to L_u define on $\mathcal{H}^s_0(\mathcal{B})$. Here, $\mathcal{E}^{\mathbb{R}^N}_s(v,w)$ is defined in (1.1.21) with $E=\mathbb{R}^N$.

To complete the proof of Theorem 1.3.1, we need to construct an $(N+1)$-dimensional subspace X of $\mathcal{H}^s_0(\mathcal{B})$ where the quadratic form $\mathcal{E}_{s,L_u}$ is negative definite. To this end, as stated above, we build test functions via partial derivatives of u. For details, see Chapter 2.

1.3.2 Paper 2. The eigenvalue problem for the regional fractional Laplacian in the small order limit

In this work, joint with T. Weth, we study the asymptotic behavior of eigenvalues and eigenfunctions for the regional fractional Laplacian $(-\Delta)^s_\Omega$ as $s\to 0^+$. For this, we introduce and analyze an operator denoted by L^Ω_Δ, which arises as a formal derivative of $(-\Delta)^s_\Omega$ at $s=0$. This operator is called *regional Logarithmic Laplacian*, in similarity with the operator L_Δ introduce in [55].

Before we state the main results of this paper, we defined for every $s\in(0,1)$, the renormalized operator $\mathcal{D}^s_\Omega$ by

$$\mathcal{D}^s_\Omega u(x) := P.V.\int_\Omega \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy \tag{1.3.5}$$

so that

$$(-\Delta)^s_\Omega u(x) = c_{N,s}\mathcal{D}^s_\Omega u(x), \qquad x\in\Omega.$$

Recall that (1.3.5) is well defined for $u\in C^\alpha(\overline{\Omega})$ if $\alpha>s$.

Our first main result provides an expansion of $\mathcal{D}^s_\Omega$ in a convergence power series in the fractional order s at $s=0$.

Theorem 1.3.3. *Let Ω be a bounded open Lipschitz set in $\mathbb{R}^N$, and $\alpha\in(0,1)$. Then we have*

$$\mathcal{D}^s_\Omega u = \mathcal{D}^0_\Omega u + \sum_{k=1}^{\infty} s^k\mathcal{D}_k u \quad \text{for } u\in C^\alpha(\overline{\Omega}) \text{ and } s\in(0,\frac{\alpha}{2}), \tag{1.3.6}$$

where, for $k\in\mathbb{N}$, $\mathcal{D}_k u\in C(\overline{\Omega})$ is defined by

$$[\mathcal{D}_k u](x) = (-1)^k 2^k\int_\Omega \frac{u(x)-u(y)}{|x-y|^N}\log^k(|x-y|)\,dy.$$

Here the series on the RHS of (1.3.6) converges in $L^\infty(\Omega)$, and the convergence is uniform if s is taken from a compact subset of $[0,\frac{\alpha}{2})$ and u is taken from a bounded subset of $C^\alpha(\overline{\Omega})$.

By definition (1.1.6), we clearly have

$$c_{N,s} = sc_N + o(s) \quad \text{as} \quad s\to 0^+ \qquad \text{with} \quad c_N := \pi^{-\frac{N}{2}}\Gamma(\frac{N}{2}).$$

From this, we have as a direct consequence of Theorem 1.3.3 the following.

Corollary 1.3.4. *Let $\Omega\subset\mathbb{R}^N$ be an open bounded Lipschitz set and $\alpha\in(0,1)$. For $u\in C^\alpha(\overline{\Omega})$, we then have*

$$(-\Delta)^s_\Omega u = sL^\Omega_\Delta u + o(s) \quad \text{in } L^\infty(\Omega) \text{ as } s\to 0^+, \tag{1.3.7}$$

where

$$\left[L^\Omega_\Delta u\right](x) := c_N\mathcal{D}^0_\Omega u(x) = c_N\int_\Omega \frac{u(x)-u(y)}{|x-y|^N}\,dy, \quad x\in\Omega. \tag{1.3.8}$$

Moreover, the expansion in (1.3.7) is uniform in bounded subsets of $C^\alpha(\overline{\Omega})$.

We notice that an operator of the form (1.3.8) with Ω replaced by the N-dimensional sphere $\mathbb{S}^N$ of $\mathbb{R}^{N+1}$ has been recently studied in [84]. The authors in [84] classify all nonnegative solutions of an equation arising as the Euler-Lagrange equation of a conformally invariant logarithmic Sobolev inequality due to Beckner.

It is worth pointing out that in (1.3.8), the operator strongly depends on the domain. This domain dependence yields some difficulties when analyzing this operator. As we will see in the sequel, it plays a crucial role in analyzing the asymptotic behavior of eigenvalue and eigenfunction of $(-\Delta)^s_\Omega$ for small order s.

Our second main result on eigenvalues of $(-\Delta)^s_\Omega$ and related eigenfunctions is contained in the following.

Theorem 1.3.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open Lipschitz set, let $n \in \mathbb{N}$, and let $\mu^\Omega_{n,s}$ resp. $\mu^\Omega_{n,0}$ denote the n-th eigenvalues of the operators $(-\Delta)^s_\Omega$, L^Ω_Δ in increasing order, respectively. Then we have*

$$\mu^\Omega_{n,s} \to 0 \quad \textit{as } s \to 0^+ \qquad \textit{and} \qquad \frac{d}{ds}\Big|_{s=0} \mu^\Omega_{n,s} = \lim_{s\to 0^+} \frac{\mu^\Omega_{n,s}}{s} = \mu^\Omega_{n,0}.$$

Moreover, if, for some sequence $s_k \to 0^+$, $\{\xi_{n,s_k}\}_k$ is a sequence of L^2-normalized eigenfunctions of $(-\Delta)^{s_k}_\Omega$ corresponding to μ^Ω_{n,s_k}, then $\xi_{n,s_k} \in C(\overline{\Omega})$ for every $k \in \mathbb{N}$ and

$$\xi_{n,s_k} \to \xi_n \quad \textit{uniformly in } \overline{\Omega},$$

where ξ_n is an eigenfunction of L^Ω_Δ corresponding to $\mu^\Omega_{n,0}$.

We notice that related result has been obtained recently in [82] in the case of the fractional Laplacian. The proof of Theorem 1.3.5 requires first of all some uniform L^∞-estimates related to renormalized operator family $\mathcal{D}^s_\Omega$, $s \in [0,1)$. Next, one needs also uniform equicontinuity result in a more general setting, that applies to eigenfunctions. The main difficulty for proving Theorem 1.3.5 is, in contrast with the one in [82], the lack of boundedness and boundary regularity of the renormalized operator $\mathcal{D}^s_\Omega$ which are uniform in s.

However, exploiting the Lipschitz property of the domain, we obtain the following a priori uniform L^∞-estimate.

Proposition 1.3.6. *Let $s \in [0,1)$, let $\Omega \subset \mathbb{R}^N$ be a bounded open Lipschitz set, let $V, f \in L^\infty(\Omega)$, and let u be a weak solution of the problem*

$$\mathcal{D}^s_\Omega u + V(x)u = f \quad \textit{in} \quad \Omega. \tag{1.3.9}$$

Then $u \in L^\infty(\Omega)$, and there exists a constant $c_0 = c_0(N, \Omega, \|V\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(\Omega)}, \|u\|_{L^2(\Omega)}) > 0$ independent of s with the property that $\|u\|_{L^\infty(\Omega)} \le c_0$ in Ω.

Notice that a bounded open set with Lipschitz boundary satisfy the uniform cone property, see [94, Theorem 1.2.2.2]. This means that, there is h, θ such that for any $x \in \overline{\Omega}$, there exists a rotation $\mathcal{R}_x$ of $\mathbb{R}^N$ with $x + \mathcal{R}_x(\mathcal{C}_{h,\theta}) \subset \Omega$. Here, $\mathcal{C}_{h,\theta}$ denotes the cone with vertex 0, opening angle θ, and bounded with a sphere of radius h in $\mathbb{R}^N$. This property allows us to get the lower bound

$$\int_{\Omega\setminus B_r(x)} |x-y|^{-N-2s}\, dy \ge C \log\frac{h}{r} \quad \text{for all } \; r \in (0,h), \quad x \in \overline{\Omega}, \quad s \in [0,1) \tag{1.3.10}$$

which is crucial for the proof of Proposition 1.3.6. More details can be found in Chapter 3.

On the other hand, by means of oscillation estimates and a contradiction argument, we also prove the following.

Theorem 1.3.7. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz set. Moreover, let $(s_k)_k$ be a sequence in $(0,1)$ with $s_k \to 0^+$, and let $\varphi_k \in C(\overline{\Omega})$, $k \in \mathbb{N}$ be functions with*

$$\|\varphi_k\|_{L^\infty(\Omega)} \le C \qquad \text{and} \qquad \Big|\int_\Omega \frac{\varphi_k(x)-\varphi_k(y)}{|x-y|^{N+2s_k}}dy\Big| \le C \quad \text{for all } x \in \overline{\Omega},\ k \in \mathbb{N} \tag{1.3.11}$$

with a constant $C > 0$. Then the sequence $(\varphi_k)_k$ is equicontinuous.

Now, Theorem 1.3.5 is then deduced from Proposition 1.3.6 and Theorem 1.3.7. For more details, see Chapter 3.

1.3.3 Paper 3. On the s-derivative of weak solutions of the Poisson problem for the regional fractional Laplacian

In the present paper, we analyze the regularity of the solution map $(0,1) \to L^2(\Omega),\ s \mapsto u_s$, where u_s is the unique weak solution to the Poisson problem

$$(-\Delta)^s_\Omega u_s = f \quad \text{in } \ \Omega \tag{1.3.12}$$

with $f \in L^\infty(\Omega)$ satisfying $\int_\Omega f\ dx = 0$. Precisely, as a first result, we prove the following theorem.

Theorem 1.3.8. *Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,1}$ domain and let u_s be the unique weak solution of* (1.3.12). *Then the map*

$$(0,1) \to L^2(\Omega), \qquad s \mapsto u_s$$

is of class C^1 and $w_s := \partial_s u_s$ uniquely solves in the weak sense the equation

$$(-\Delta)^s_\Omega w_s = M^s_\Omega u_s \quad \text{in } \ \Omega. \tag{1.3.13}$$

Here, for every $x \in \Omega$,

$$M^s_\Omega u(x) = -\frac{\partial_s c_{N,s}}{c_{N,s}} f(x) + 2c_{N,s} P.V. \int_\Omega \frac{(u(x)-u(y))}{|x-y|^{N+2s}} \log|x-y|\ dy. \tag{1.3.14}$$

We notice that our result is the regional counterpart to that obtained in [106] for the fractional Laplacian. In this paper, the authors obtained a characterization of the derivative in terms of the Green function. Mind that in our case, we do not have such a characterization since no boundary condition is prescribed in (1.3.12). In contrast, we have a much more explicit RHS in (1.3.13) given by (1.3.14).

As a first observation, due to the fact that a Logarithmic factor appears in (1.3.14), one needs higher Sobolev regularity of u_s of the form $H^{s+\varepsilon}(\Omega)$ ($\varepsilon > 0$). This is obtained by exploiting (uniform) boundary regularity of u_s established in [73]. In fact, very recently, Fall [73] obtained $C^{2s-\varepsilon}(\overline{\Omega})$ regularity of u_s with uniform estimates when $s \in [s_0, 1)$ for some $s_0 \in (0,1)$. This uniform boundary regularity together with an uniform L^∞-estimate of u_s with respect to s established in **Paper 2** (see Proposition 1.3.6) allow us to obtain our desired higher Sobolev regularity. It states the following.

Proposition 1.3.9. *Let $f \in L^\infty(\Omega)$ with $\int_\Omega f\ dx = 0$ and let $u_{s+h} \in H^{s+h}(\Omega) \cap L^\infty(\Omega)$ be the unique weak solution of problem* (1.3.12) *with s replaced by $s+h$. Then $u_{s+h} \in H^{s+\varepsilon}(\Omega)$ for some $\varepsilon > 0$ and*

$$\|u_{s+h}\|_{H^{s+\varepsilon}(\Omega)} \le K \qquad \text{for all } h \in (-h_0, h_0) \tag{1.3.15}$$

for some $h_0 > 0$.

An important piece of information that follows from Proposition 1.3.9 is the uniform boundedness of u_{s+h} in $H^{s+\varepsilon}(\Omega)$ with respect to h. This enables us to get a convergence in $H^{s+\varepsilon}(\Omega)$ which is capital for the results of this paper.

Now, we comment on the proof of Theorem 1.3.8. It suffices to check that the map $s \mapsto u_s$ satisfies the assumptions of [106, Lemma 6.6]. The continuity of the map $s \mapsto u_s$ is obtained by exploiting Proposition 1.3.9. In fact, from the uniform estimate (1.3.15), it follows that, after passing to a subsequence, $u_{s+h} \rightharpoonup \overline{u}_s$ in $H^{s+\varepsilon}(\Omega)$ (so in $H^s(\Omega)$) as $h \to 0$ for some $\overline{u}_s \in H^{s+\varepsilon}(\Omega)$. Since the embedding $H^s(\Omega) \hookrightarrow L^2(\Omega)$ is compact, then $u_{s+h} \to \overline{u}_s$ strongly in $L^2(\Omega)$. Now, by exploiting that $\overline{u}_s$ solves in the distributional sense the equation $(-\Delta)^s_\Omega \overline{u}_s = f$ in Ω, then by uniqueness, it follows that $u_s = \overline{u}_s$ as wanted.

Regarding the existence of $\partial_s^+ u_s$ in $L^2(\Omega)$ for all $s \in (0,1)$, the strategy is to obtain a uniform bound of the difference quotient

$$v_h := \frac{u_{s+h} - u_s}{h} \tag{1.3.16}$$

in the space $H^s(\Omega)$. To this end, one exploits Cauchy-Schwarz inequality, a Logarithmic decays together with Proposition 1.3.9. Next, utilizing that $H^s(\Omega)$ is a Hilbert space, then up to a subsequence, $v_h \rightharpoonup w_s$ weakly in $H^s(\Omega)$ as $h \to 0^+$ for some $w_s \in H^s(\Omega)$ satisfying (1.3.13). Moreover, from compactness, we also have $v_h \to w_s$ strongly in $L^2(\Omega)$. Now, utilizing the continuity of the map $s \mapsto u_s$, Proposition 1.3.9, and Dominated Convergence Theorem, we obtain that w_s is unique as the limit of v_h. Therefore, $\partial_s^+ u_s$ exists in $L^2(\Omega)$ and $\partial_s^+ u_s = w_s$.

To prove the last assumption in [106, Lemma 6.6], that is, the continuity of the map $s \mapsto w_s := \partial_s^+ u_s$, we argue as follows: we first prove that w_{s+h} is uniformly bounded in $H^s(\Omega)$. To do so, we again exploit Proposition 1.3.9. By compactness, we have that up to a subsequence, $w_{s+h} \to \overline{w}_s$ strongly in $L^2(\Omega)$ for some $\overline{w}_s \in H^s(\Omega)$. Now, from the fact that $\overline{w}_s$ solves in the distributional sense the equation $(-\Delta)^s_\Omega \overline{w}_s = M^s_\Omega u_s$ in Ω, we obtain by uniqueness that $w_s = \overline{w}_s$, as needed.

Theorem 1.3.8 is then a direct consequence of [106, Lemma 6.6]. For more details, we refer to Theorem 4.3.1 of Chapter 4.

In the second part of this paper, we also analyze the eigenvalue problem

$$(-\Delta)^s_\Omega u_s = \mu_s u_s \quad \text{in } \ \Omega. \tag{1.3.17}$$

The main goal of this part is to study the differentiability of the first nontrivial eigenvalue $\mu_{1,s}$ regarded as a function of s. We notice that this eigenvalue is not in general simple. This lack of simplicity causes differentiability breaking. In other words, $\mu_{1,s}$ is not in general differentiable regarded as a function of s. We will then investigate directional differentiabilities.

To alleviate the notation, we write μ_s in place of $\mu_{1,s}$. Also, by u_s we mean an L^2-normalized eigenfunction associated with μ_s. The second main result of this paper reads as follows.

Theorem 1.3.10. *Regarded as function of s, μ_s is right differentiable on $(0,1)$ and*

$$\partial_s^+ \mu_s := \lim_{h \to 0^+} \frac{\mu_{s+h} - \mu_s}{h} = \mu'_{s,+} := \inf\{J_s(u) : u \in \mathcal{M}_s\} \tag{1.3.18}$$

where

$$J_s(u) = \frac{\partial_s c_{N,s}}{c_{N,s}} \mu_s - c_{N,s} \int_\Omega \int_\Omega \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \log|x-y| \, dxdy$$

and $\mathcal{M}_s$ the set of L^2-normalized eigenfunctions of $(-\Delta)^s_\Omega$ corresponding to μ_s. Moreover, the infimum in (1.3.18) *is attained.*

We make the following remark. The left differentiability of μ_s as a function of s can be stated similarly. But however, as stated above, due to the non-simplicity of μ_s, the values $\partial_s^+\mu_s$ and $\partial_s^-\mu_s$ may be different.

In order to prove Theorem 1.3.10, the following is of key importance.

Lemma 1.3.11. *Let $k \geq 1$ and $\mu_{k,s}$ the k-th eigenvalue of $(-\Delta)^s_\Omega$ in Ω. Then, regarded as function of s, $\mu_{k,s}$ is continuous on $(0,1)$ for all $k \in \mathbb{N}$.*

This lemma is analog to Theorem 1.3 established in [61] for the fractional Laplacian $(-\Delta)^s$. By taking advantage of Lemma 1.3.11, it follows that Proposition 1.3.9 also applies to eigenfunctions. Having this lemma in mind, one can now briefly expose the idea behind the proof of Theorem 1.3.10.

As mentioned above, Proposition 1.3.9 implies that u_{s+h} is uniformly bounded in $H^s(\Omega)$ with respect to h. Then, after passing to a subsequence, there exists $\xi_s \in H^s(\Omega)$ such that $u_{s+h} \rightharpoonup \xi_s$ weakly in $H^s(\Omega)$ and $u_{s+h} \to \xi_s$ strongly in $L^2(\Omega)$ as $h \to 0^+$. Moreover, ξ_s is an L^2-normalized eigenfunction associated with μ_s. Now, since $u_{s+h} \in H^{s+h}(\Omega) \subset H^s(\Omega)$, one can then use u_{s+h} as an admissible test function for μ_s and Dominated Convergence Theorem to get

$$\begin{aligned}\liminf_{h\to 0^+} \frac{\mu_{s+h}-\mu_s}{h} &\geq \frac{\partial_s c_{N,s}}{c_{N,s}}\mu_s - c_{N,s}\int_\Omega\int_\Omega \frac{(\xi_s(x)-\xi_s(y))^2}{|x-y|^{N+2s}}\log|x-y|\,dxdy\\ &\geq \inf\{J_s(u) : u \in \mathcal{M}_s\}.\end{aligned} \tag{1.3.19}$$

Now for the reverse limit, that is limsup, we argue as follows: since from Proposition 1.3.9 one has $u_s \in H^{s+h}(\Omega)$, then by using it as an admissible test function for μ_{s+h}, we obtain the following

$$\begin{aligned}\limsup_{h\to 0^+} \frac{\mu_{s+h}-\mu_s}{h} &\leq \frac{\partial_s c_{N,s}}{c_{N,s}}\mu_s - c_{N,s}\int_\Omega\int_\Omega \frac{(u_s(x)-u_s(y))^2}{|x-y|^{N+2s}}\log|x-y|\,dxdy\\ &\leq \inf\{J_s(u) : u \in \mathcal{M}_s\}.\end{aligned} \tag{1.3.20}$$

In the latter, we used the arbitrariness of $u_s \in \mathcal{M}_s$. Now, Theorem 1.3.10 follows from (1.3.19) and (1.3.20). For more details, see Theorem 4.4.1 of Chapter 4.

1.3.4 Paper 4. Existence results for nonlocal problems governed by the regional fractional Laplacian

In this joint work with M. M. Fall, we study existence results of minimizers of the critical fractional Sobolev constant on bounded domains

$$S_{N,s}(\Omega) = \inf_{\substack{u\in H_0^s(\Omega)\\ u\neq 0}} \frac{Q_{N,s,\Omega}(u)}{\|u\|^2_{L^{2_s^*}(\Omega)}}. \tag{1.3.21}$$

Under some values of the fractional parameter, we show that the above best constant is achieved. Moreover, if the underlying domain is a ball, we obtain positive minimizers for all possible values of the fractional parameter in higher dimension, while we impose a positive mass condition in low dimension.

Our first main result reads as follows.

Theorem 1.3.12. *Let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^1 boundary. There exists $s_0 \in (\frac{1}{2}, 1)$ such that for all $s \in (\frac{1}{2}, s_0)$, any minimizing sequence for $S_{N,s}(\Omega)$, normalized in $H^s_0(\Omega)$ is relatively compact in $H^s_0(\Omega)$. In particular, the infimum is achieved.*

As already mentioned in Section 1.2 (see Subsection 1.2.3), the minimization problem (1.3.21) was studied in [83] in the case when a portion of the boundary $\partial\Omega$ of Ω lies on a hyperplane, assuming $N \geq 4s$. This restriction on the shape of the domain is crucial in establishing the strict inequality

$$S_{N,s}(\Omega) < S_{N,s}(\mathbb{R}^N_+) \tag{1.3.22}$$

which is the key ingredient for the existence of minimizers. The novelty in our result is that, in contrast to that in [83], we do not impose any restriction on the shape of the domain. However, we demand the fractional parameter to be close to $\frac{1}{2}$. This is due to the fact that in order to get the key inequality (1.3.22), we needed to study the asymptotic behavior of $S_{N,s}(\Omega)$ as s tends to $\frac{1}{2}$ and we got the following.

Proposition 1.3.13. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Then*

$$\lim_{s \searrow \frac{1}{2}} S_{N,s}(\Omega) = 0.$$

This proposition is a direct consequence of the following two lemmas.

Lemma 1.3.14. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Fix $\overline{s} \in (0,1)$. Then*

$$\limsup_{s \searrow \overline{s}} S_{N,s}(\Omega) \leq S_{N,\overline{s}}(\Omega).$$

Lemma 1.3.15. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Then $1 \in H^{\frac{1}{2}}(\Omega)$. In particular*

$$S_{N,\frac{1}{2}}(\Omega) = 0.$$

The result of Lemma 1.3.15 clarifies the behavior of the constant $S_{N,s}(\Omega)$ in the case when $s = \frac{1}{2}$ and hence produces complete qualitative properties (positivity and nullity) of the constant $S_{N,s}(\Omega)$ with respect to s. The strategy behind the proof of this result is to approximate constant functions with respect to the $H^{\frac{1}{2}}(\Omega)$-norm. Notice that our result in Lemma 1.3.15 is consistent with the fact that for bounded Lipschitz domains $\Omega \subset \mathbb{R}^N$, $C^\infty_c(\Omega)$ is dense in $H^s(\Omega)$ whenever $0 < s \leq \frac{1}{2}$ (see e.g. [94, Theorem 1.4.2.4]). Having the above two lemmas in mind, Proposition 1.3.13 therefore follows as

$$0 \leq \lim_{s \searrow \frac{1}{2}} S_{N,s}(\Omega) \leq \limsup_{s \searrow \frac{1}{2}} S_{N,s}(\Omega) \leq S_{N,\frac{1}{2}}(\Omega) = 0.$$

From this, and recalling that $S_{N,s}(\mathbb{R}^N_+) > 0$ for all $s \in (0,1)$ (see e.g. [71, Lemma 2.1]), we find some $s_0 \in (\frac{1}{2}, 1)$ such that

$$0 < S_{N,s}(\Omega) < S_{N,s}(\mathbb{R}^N_+) \quad \text{for all} \quad s \in (\frac{1}{2}, s_0). \tag{1.3.23}$$

Having the above inequality, we prove our first main result as in [83]. The method used for the proof of Theorem 1.3.12 is the so-called *missing mass method*. The strategy behind this method is to prove that a minimizing sequence for $S_{N,s}(\Omega)$ does not concentrate in the interior and on the boundary of the domain. To this end, one exploits the key inequality (1.3.23) together with the fact that $S_{N,s}(\mathbb{R}^N_+) \leq S_{N,s}(\mathbb{R}^N)$ to rule out such behaviors. This then means that the minimizing sequence weakly converges to a nontrivial limit. Moreover, by exploiting the nonlinear character of the minimization problem, one can upgrade this weak limit to a strong one. This completes the proof of Theorem 1.3.12.

The second and third main results of this paper are related to the radial minimization problem

$$S_{N,s,rad}(\mathcal{B}, h) = \inf_{\substack{u \in H^s_{0,rad}(\mathcal{B}) \\ u \not\equiv 0}} \frac{Q_{N,s,\mathcal{B}}(u) + \int_{\mathcal{B}} hu^2 \, dx}{\|u\|^2_{L^{2^*_s}(\mathcal{B})}} \tag{1.3.24}$$

where $h \in L^\infty(\Omega)$ belongs to the class of radial potentials for which

$$S_{N,s,rad}(\mathcal{B}, h) > 0. \tag{1.3.25}$$

We recall that $H^s_{0,rad}(\mathcal{B})$ is the space of radially symmetric functions of $H^s_0(\mathcal{B})$.

Our next result is valid for all $s \in (\frac{1}{2}, 1)$ and $N \geq 4s$. It reads as follows.

Theorem 1.3.16. *Let $s \in (\frac{1}{2}, 1)$ and $N \geq 4s$. Then any minimizing sequence for $S_{N,s,rad}(\mathcal{B}, 0)$, normalized in $H^s_{0,rad}(\mathcal{B})$ is relatively compact in $H^s_{0,rad}(\mathcal{B})$. In particular, the infimum is achieved.*

The main difference between minimization problems (1.3.21) and (1.3.24) is that in the latter case, concentration can only happens at the origin. In order to rule out this behavior, the following key inequality is crucial.

Lemma 1.3.17. *Let $s \in (\frac{1}{2}, 1)$ and $N \geq 4s$. Then*

$$S_{N,s,rad}(\mathcal{B}, 0) < S_{N,s}(\mathbb{R}^N). \tag{1.3.26}$$

The proof of the lemma above takes advantage of the fact that minimizers to $S_{N,s}(\mathbb{R}^N)$ are known explicitly and were fully classified by Lieb [117]. In fact, Lieb showed that up to dilations, translations, and multiplications, the fractional Sobolev constant $S_{N,s}(\mathbb{R}^N)$ is attained exactly by the function $(1 + |x|^2)^{\frac{2s-N}{2}}$. Therefore, for $\lambda > 0$, one considers the functions

$$U_\lambda(x) = \gamma_0 \Big(\frac{\lambda}{1 + \lambda^2 |x|^2} \Big)^{\frac{N-2s}{2}}$$

where γ_0 is a positive constant (independent of λ) such that $\|U_\lambda\|_{L^{2^*_s}(\mathbb{R}^N)} = 1$, which satisfy the Euler-Lagrange equation

$$(-\Delta)^s U_\lambda = S_{N,s}(\mathbb{R}^N) U_\lambda^{2^*_s - 1} \quad \text{in } \; \mathbb{R}^N.$$

Next, we cut them off in a small ball $B_r \subset\subset \mathcal{B}$ in order to construct an admissible test function

$$u_\lambda = \eta U_\lambda$$

for $S_{N,s,rad}(\mathcal{B}, 0)$. After some calculations, we obtain the following estimates

$$S_{N,s,rad}(\mathcal{B}, 0) \leq (1 - C\lambda^{-N}) \times \begin{cases} S_{N,s}(\mathbb{R}^N) - \dfrac{K}{C} \lambda^{-2s} + C\lambda^{-N+2s}, & \text{if } \; N > 4s, \\ S_{N,s}(\mathbb{R}^N) - \dfrac{K}{C} \lambda^{-2s} \log \lambda + C\lambda^{-N+2s}, & \text{if } \; N = 4s, \end{cases}$$

where K and C are positive constants independent on λ. Since the RHS of the inequality above is strictly less than $S_{N,s}(\mathbb{R}^N)$ for λ sufficiently large, Lemma 1.3.17 therefore follows.

In addition to (1.3.26), one also needs an approximate inequality of the form

$$(S_{N,s}(\mathbb{R}^N) - \varepsilon)\|u\|^2_{L^{2^*_s}(\mathcal{B})} \leq CQ_{N,s,\mathcal{B}}(u) + C_\varepsilon \|u\|^2_{L^2(\mathcal{B})}. \tag{1.3.27}$$

We recall that such inequality is obtained by exploiting the fractional version of the Strauss radial lemma as well as the Schur test principle and Young inequality.

Now, the proof of Theorem 1.3.16 follows in the same spirit as the one of Theorem 1.3.12. For more details, see Chaper 5.

The third main result of this paper is the existence of radial minimizers in low dimensions $2s < N < 4s$. In this case, we introduce the *mass* $\mathbf{k}$ of $\mathcal{B}$ associated to the fractional Schrödinger operator $(-\Delta)^s + h$ as

$$\mathbf{k}(x) = G(x, 0) - \mathcal{R}(x)$$

where $G(x, y)$ is the green function of $(-\Delta)^s + h$ and $\mathcal{R}(x)$ the Riesz potential of $(-\Delta)^s$ in $\mathbb{R}^N$. Our result is a theorem in the spirit of [88, 133] which relies on the positivity of *mass*.

Theorem 1.3.18. *Let $s \in (\frac{1}{2}, 1)$, $2s < N < 4s$ and $h \in L^\infty(\mathcal{B})$ such that* (1.3.25) *holds. Assume that $\mathbf{k}(0) > 0$. Then any minimizing sequence for $S_{N,s,rad}(\mathcal{B}, h)$, normalized in $H^s_{0,rad}(\mathcal{B})$ is relatively compact in $H^s_{0,rad}(\mathcal{B})$. In particular, the infimum is achieved.*

Notice that the assumption $\mathbf{k}(0) > 0$ is crucial in obtaining minimizers. It allows us to restore compactness as we can see in the next lemma.

Lemma 1.3.19. *Let $s \in (\frac{1}{2}, 1)$ and $2s < N < 4s$. Suppose that $\mathbf{k}(0) > 0$. Then*

$$S_{N,s,rad}(\mathcal{B}, h) < S_{N,s}(\mathbb{R}^N). \tag{1.3.28}$$

To prove this lemma, we use

$$v_\lambda(x) = \eta(x)U_\lambda(x) + c\lambda^{\frac{2s-N}{2}}\mathbf{k}(x)$$

as an admissible test function for $S_{N,s,rad}(\mathcal{B}, h)$. Here, η is a smooth radial cut-off function compactly supported in B_{2r} with $\eta = 1$ on B_r, and $c > 0$ a positive constant depending on N, s and γ_0. We then obtain an inequality of the form

$$S_{N,s,rad}(\mathcal{B}, h) \leq S_{N,s}(\mathbb{R}^N) - c^2\lambda^{2s-N}\mathbf{k}(0) + o(\lambda^{2s-N}) + O(\lambda^{2s-N})o_r(1).$$

Now by letting $\lambda \to \infty$ in the inequality above, we obtain (1.3.28) provided that $\mathbf{k}(0) > 0$. Once this lemma is proved, one can then prove Theorem 1.3.18 in the same manner as above. For more details, see Chapter 5.

1.3.5 Paper 5. A Hopf lemma for the regional fractional Laplacian

This paper, joint work with N. Abatangelo and M. M. Fall, is concerned with Hopf boundary lemma for regional fractional Laplacian $(-\Delta)^s_\Omega$ with being $\Omega \subset \mathbb{R}^N$ a bounded open set and $s \in (\frac{1}{2}, 1)$. Our results analyze the behavior of the ratio $\frac{u}{\delta_\Omega^{2s-1}}$ near the boundary $\partial\Omega$ of Ω. Here, the function u is either a pointwise or weak super-solution to the equation

$$(-\Delta)^s_\Omega u = c(x)u \quad \text{in} \ \ \Omega. \tag{1.3.29}$$

Our results strongly use the behavior of the solution to the torsion problem for the regional fractional Laplacian, that is the function u_{tor} satisfying

$$\begin{cases} (-\Delta)^s_\Omega u_{tor} = 1 & \text{in } \Omega \\ u_{tor} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3.30}$$

which, on smooth domains, satisfy the double-sided estimate

$$C^{-1}\delta_\Omega^{2s-1} \le u_{tor} \le C\delta_\Omega^{2s-1} \quad \text{in } \overline{\Omega} \tag{1.3.31}$$

for some $C > 1$, see [26, 44]. In the case of fractional Laplacian, the torsion solution is known explicitly in the case of the ball (see [34, Section 3.6]). Taking advantage of this, the Hopf lemma for $(-\Delta)^s$ follows in a standard way: the approach consists of proving first a Hopf lemma in the ball, and from an *interior ball condition*, the result is recovered on the whole domain. However, for the regional fractional Laplacian, there is no explicit formula for u_{tor}, even in the case when Ω is a ball. This function has been numerically studied in [69] in the one-dimensional case $\Omega = (-1, 1)$.

Now, for $n \in \mathbb{N}$, we define

$$v_n(x) = \frac{1}{n}u_{tor}(x) \qquad \text{and} \qquad w_n(x) = v_n(x) - u(x) \quad \text{for } x \in \overline{\Omega}.$$

Then, by definition and (1.3.31), by the boundedness of Ω, it follows that

$$v_n \to 0 \quad \text{uniformly in } \overline{\Omega}. \tag{1.3.32}$$

Before stating our main results, we recall the following definitions.

Definition 1.3.20. We say that a function $u : \Omega \to \mathbb{R}$ is a pointwise super-solution of (1.3.29) if $u \in C^{2s+\varepsilon}_{loc}(\Omega) \cap L^\infty(\Omega)$ for some $\varepsilon > 0$ and

$$(-\Delta)^s_\Omega u(x) \ge c(x)u(x) \qquad \text{for any } x \in \Omega.$$

Definition 1.3.21. We say that a function $u : \Omega \to \mathbb{R}$ is a weak super-solution of (1.3.29) if $u \in H^s(\Omega)$ and

$$\mathcal{E}^\Omega_s(u, \varphi) \ge \int_\Omega cu\varphi \qquad \text{for any } \varphi \in C^\infty_c(\Omega),\ \varphi \ge 0 \text{ in } \Omega.$$

Definition 1.3.22. We say that a function $u : \Omega \to \mathbb{R}$ is a distributional super-solution of (1.3.29) if $u \in L^1(\Omega)$ and

$$\int_\Omega u\,(-\Delta)^s_\Omega\varphi \ge \int_\Omega cu\varphi \qquad \text{for any } \varphi \in C^\infty_c(\Omega),\ \varphi \ge 0 \text{ in } \Omega.$$

Our first main result is concerned with Hopf boundary lemma for pointwise super-solutions to (1.3.29) and it reads as follows.

Theorem 1.3.23 (Hopf lemma for pointwise super-solutions)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with $C^{1,1}$ boundary and $s \in (1/2, 1)$. Let $c \in L^\infty(\Omega)$ and let $u : \overline{\Omega} \to \mathbb{R}$ be a lower semicontinuous super-solution (in the sense of Definition 1.3.20) of* (1.3.29).

(*i*) *If* $c \leq 0$ *in* Ω *and* $u \geq 0$ *on* $\partial\Omega$*, then either* u *vanishes identically in* Ω *or*

$$\liminf_{\Omega \ni x \to z} \frac{u(x)}{\delta_\Omega(x)^{2s-1}} > 0 \qquad \textit{for any } z \in \partial\Omega. \tag{1.3.33}$$

(*ii*) *If* $u \geq 0$ *in* $\overline{\Omega}$*, then either* u *vanishes identically in* Ω *or* (1.3.33) *holds true.*

Its proof requires the following strong maximum principle.

Lemma 1.3.24 (Strong maximum principle for pointwise super-solutions)**.** *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded open set. Let* $c \in L^\infty(\Omega)$ *and* $u : \overline{\Omega} \to \mathbb{R}$ *be a lower semicontinuous function super-solution (in the sense of Definition 1.3.20) of* (1.3.29)*.*

(*i*) *If* $c \leq 0$ *in* Ω *and* $u \geq 0$ *on* $\partial\Omega$*, then either* u *vanishes identically in* Ω*, or* $u > 0$ *in* Ω*.*

(*ii*) *If* $u \geq 0$ *in* $\overline{\Omega}$*, then either* u *vanishes identically in* Ω*, or* $u > 0$ *in* Ω*.*

We notice that under the hypotheses of assertion (i) one gets that $u \geq 0$ on $\overline{\Omega}$. Therefore, Lemma 1.3.24 follows if we only show that either u vanishes identically in Ω or $u > 0$ in Ω. Exploiting the lower semicontinuous property of u, we get from a contradiction argument, the desired strong maximum principle.

Once this lemma is established, one can then explain the main ideas of the proof of Theorem 1.3.23. The strategy is based on constructing a barrier of u from below with respect to u_{tor} assuming that u does not vanish identically in Ω. In other words, we have to show that there exist $n_0 \in \mathbb{N}$ such that

$$w_n \leq 0, \quad i.e. \;\; u \geq v_n \quad \text{on } \; \overline{\Omega}, \quad \text{for all } \; n \geq n_0. \tag{1.3.34}$$

To this end, we argue by contradiction assuming that for any $n \in \mathbb{N}$, the function w_n is positive somewhere in $\overline{\Omega}$. Exploiting the lower semicontinuity of u and the fact that $w_n \leq 0$ on $\partial\Omega = \overline{\Omega} \setminus \Omega$, then a positive maximum of w_n is attained in Ω, say at some x_n. Now, from Lemma 1.3.24 and (1.3.32), one gets $u(x_n) \to 0$, and then $x_n \to \partial\Omega$. Consequently, x_n keeps far from y whenever y runs in a compact subset of Ω. From this, and by using once again Lemma 1.3.24, the boundedness of c and (1.3.32), and recalling that $u(x_n) \to 0$, one gets

$$0 > (-\Delta)^s_\Omega u(x_n) \geq c(x_n) u(x_n) = 0 \quad \text{as } \; n \to \infty,$$

which is a contradiction. Therefore, (1.3.34) must be true and thereby proving Theorem 1.3.23. For more details, see Chapter 6.

Our second main result deals with Hopf boundary lemma for weak super-solutions to (1.3.29). We establish the following.

Theorem 1.3.25 (Hopf lemma for weak super-solutions)**.** *Let* $\Omega \subset \mathbb{R}^N$ *be an open bounded set with* $C^{1,1}$ *boundary and* $s \in (1/2, 1)$*. Let* $c : \Omega \to \mathbb{R}$ *be a measurable function and let* $u \in H^s(\Omega)$ *be a weak super-solution (in the sense of Definition 1.3.21) of* (1.3.29)*. Suppose that either*

$$c \in L^\infty(\Omega)$$

or

$$c \in L^q(\Omega), \;\; q > \frac{N}{2s}, \qquad \textit{and} \qquad u \in L^\infty_{loc}(\Omega),$$

hold.

(i) If $c \leq 0$ *in* Ω *and* $u \geq 0$ *on* $\partial\Omega$*, then either* u *vanishes identically in* Ω *or*

$$\text{there exists } \varepsilon_0 > 0 \quad \text{such that} \quad \frac{u(x)}{\delta_\Omega(x)^{2s-1}} > \varepsilon_0. \tag{1.3.35}$$

(ii) If $u \geq 0$ *in* Ω*, then either* u *vanishes identically in* Ω *or* (1.3.35) *holds true.*

The proof of Theorem 1.3.25 follows the same line of thought as the one of Theorem 1.3.23, although with some more technical difficulties due to the weak character of super-solutions involved. For example, when $c \in L^q(\Omega)$ the strong maximum principle involved in our strategy takes the following form.

Lemma 1.3.26 (Strong maximum principle for distributional super-solutions)**.** *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded open set and* $u \in L^\infty_{\text{loc}}(\Omega)$ *be a distributional super-solution (in the sense of Definition 1.3.22) of* (1.3.29) *with*

$$c \in L^q_{loc}(\Omega), \quad q > \frac{N}{2s}.$$

If $u \geq 0$ *in* Ω*, then*

$$\textit{either} \qquad u \equiv 0 \quad \textit{in } \Omega \qquad \textit{or} \qquad \operatorname{essinf}_K u > 0 \quad \textit{for any } K \subset\subset \Omega.$$

A more detailed exposition can be found in Chapter 6.

1.3.6 Paper 6. Qualitative properties of positive solutions for elliptic problem driven by the regional fractional Laplacian in the half-space

In this note, we study symmetry and monotonicity of weak solutions to the Dirichlet problem

$$\begin{cases} (-\Delta)^s_{\mathbb{R}^N_+} u = u^{2^*_s - 1} & \text{in} \quad \mathbb{R}^N_+ \\ u > 0 & \text{in} \quad \mathbb{R}^N_+ \\ u = 0 & \text{on} \quad \partial\mathbb{R}^N_+ = \mathbb{R}^{N-1} \end{cases} \tag{1.3.36}$$

where $s \in (0,1) \setminus \{\frac{1}{2}\}$ and $N > 2s$. We recall that from critical point theory, weak solutions to (1.3.36) correspond to positive critical points of the associated Euler-Lagrange functional $J : H^s_0(\mathbb{R}^N_+) \to \mathbb{R}$ defined by

$$J(u) := \frac{c_{N,s}}{4} \int_{\mathbb{R}^N_+} \int_{\mathbb{R}^N_+} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \, dxdy - \frac{1}{2^*_s} \int_{\mathbb{R}^N_+} u^{2^*_s} \, dx. \tag{1.3.37}$$

Notice also that positive minimizers of the fractional Sobolev constant $S_{N,s}(\mathbb{R}^N_+)$ are weak solutions of (1.3.36). As stated earlier, $S_{N,s}(\mathbb{R}^N_+)$ is attained for $s \in (0,1) \setminus \{\frac{1}{2}\}$ assuming $N \geq 4s$, see [83].

Putting $x = (x', x_N) \in \mathbb{R}^N_+$, we establish in this paper the symmetry and monotonicity of solutions of (1.3.36) with respect to the horizontal variable x' by exploiting the *moving plane method*. Concretely, we prove the following.

Theorem 1.3.27. *Let* $u \in H^s_0(\mathbb{R}^N_+)$ *be a weak solution to* (1.3.36)*. Then* u *is radially symmetric in* x' *and monotonic in the radial variable. In other words, there exists a monotonic function* $(0,\infty) \times (0,\infty) \ni (r, x_N) \mapsto v(r, x_N)$ *with respect to* r *such that*

$$u(x', x_N) = v(r, x_N) \quad \textit{with} \quad r = |x'|. \tag{1.3.38}$$

Before going further, an interesting question is whether Theorem 1.3.27 apply to solutions of the problem

$$(-\Delta)^s_B u = u^{2^*_s-1}, \quad u > 0 \ \text{ in } \ B, \quad u = 0 \ \text{ on } \ \partial B. \tag{1.3.39}$$

We recall that in (1.3.39), the fractional parameter s runs in $(\frac{1}{2}, 1)$. The results of **Paper 4** tell us that the Dirichlet problem (1.3.39) possesses symmetric solutions. However, since the operator $(-\Delta)^s_B$ depends on the domain, the moving plane method for the problem (1.3.39) remains a challenging question. The main difficulty in this setting is that, upon reflecting the domain, the operator changes as well. We think that a *symmetry breaking* can occur.

We now aim to state the key ingredient for the proof of Theorem 1.3.27. Before doing so, let us first introduce the following notations. For $\lambda \in \mathbb{R}$, we set $T_\lambda = \{x \in \mathbb{R}^N_+ : x_1 = \lambda\}$ and $\Sigma_\lambda = \{x \in \mathbb{R}^N_+ : x_1 < \lambda\}$. Moreover, we put $x_\lambda = (2\lambda - x_1, x_2, \dots, x_N)$ and $u_\lambda(x) = u(x_\lambda)$.

Our key ingredient concerns the strong maximum principle for odd functions and it is stated as follows.

Proposition 1.3.28 (Strong maximum principle)**.** *Let $s \in (0, 1/2) \cup (1/2, 1)$, $\lambda \in \mathbb{R}$ and $U \subset\subset \Sigma_\lambda$ be a bounded set. Let $v \in H^s_0(\mathbb{R}^N_+)$ be a continuous function on $\overline{U}$, satisfying*

$$(-\Delta)^s_{\mathbb{R}^N_+} v \geq 0 \quad \text{in} \quad U. \tag{1.3.40}$$

If v is nonnegative in Σ_λ and odd with respect to the hyperplane T_λ, then either $v \equiv 0$ in $\mathbb{R}^N_+$ or $v > 0$ in U.

We recall that a function $v \in H^s_0(\mathbb{R}^N_+)$ satisfying (1.3.40) is called weak superharmonic with respect to $(-\Delta)^s_{\mathbb{R}^N_+}$. Now, let us say a few words about the proof of the proposition above. For $s \in (0,1)\setminus\{\frac{1}{2}\}$ it is known that $H^s_0(\mathbb{R}^N_+)$ can be identified with $\mathcal{H}^s_0(\mathbb{R}^N_+)$ thanks to Hardy's inequality (see [94]). Taking advantage of this, we can then identify v with its trivial extension so that $v \in \mathcal{H}^s_0(\mathbb{R}^N_+)$. Exploiting the inequality (1.3.40) we see that $v \in \mathcal{H}^s_0(\mathbb{R}^N_+)$ is a continuous function on $\overline{U}$ satisfying

$$(-\Delta)^s v \geq 0 \quad \text{in} \ \ U.$$

One can then use a strong maximum principle result for $(-\Delta)^s$ (see e.g. [68, Proposition 3.1]) to complete the proof of Proposition 7.3.1.

In the sequel, we briefly present the idea of the proof of Theorem 1.3.27. Before doing so, let us mention that utilizing Moser's iteration method, one can prove that any weak solution to (1.3.36) is bounded and therefore continuous in $\mathbb{R}^N_+$ thanks to [122, Theorem D].

The main idea of the proof of Theorem 1.3.27 is to compare the values of u and u_λ along the $(N-1)$-variables $x_1, x_2, \dots, x_{N-1}$. For this, we exploit the nonlinear structure of the problem together with Proposition 1.3.28. For more details, see Chapter 7.

1.3.7 Paper 7. Mountain pass solutions for the regional fractional Laplacian

In this last paper, we obtain nontrivial mountain pass solutions to the nonlinear Dirichlet problem

$$\begin{cases} (-\Delta)^s_\Omega u = f(u) & \text{in} \ \ \Omega \\ \qquad u = 0 & \text{on} \ \ \partial\Omega \end{cases} \tag{1.3.41}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, $s \in (\frac{1}{2}, 1)$ and $f : \mathbb{R} \to \mathbb{R}$ a function satisfying some growth conditions. In this framework, the solutions are constructed with a variational method by a minimax procedure on the associated energy functional. In other words, in order to detect nontrivial solutions of mountain pass type, we analyze the existence of nontrivial critical points of the corresponding functional energy to (1.3.41) define by

$$J(u) = \frac{c_{N,s}}{4} \int_\Omega \int_\Omega \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}}\, dxdy - \int_\Omega F(u)\, dx, \quad u \in H_0^s(\Omega)$$

where $F(t) = \int_0^t f(\tau)\, d\tau$ is the primitive of f. To this end, one imposes some nice growth on the nonlinearity f. For our purposes, we assume that the nonlinearity f fulfills the conditions below.

(F_1) There is $C > 0$ and $p \in (2, 2_s^*)$ such that

$$|f(t)| \leq C(1 + |t|^{p-1});$$

(F_2) $\lim\limits_{t\to 0} \frac{f(t)}{t} \leq 0$;

(F_3) $\lim\limits_{|t|\to\infty} \frac{F(t)}{t^2} = +\infty$;

(F_4) Denote by $H(t) = tf(t) - 2F(t)$. Then there is $c_0 > 0$ such that

$$H(t_1) \leq H(t_2) + c_0$$

for all $0 < t_1 < t_2$ or $t_2 < t_1 < 0$.

It is important to point out that from the above assumptions, the nonlinearity f is of subcritical growth. Therefore, our result does not applied to the critical power nonlinearity considered in **Paper 4**.

Our main result reads as follows.

Theorem 1.3.29. *Let f be a function satisfying conditions (F_1)-(F_4). Then, there exists nontrivial Mountain Pass solution to the problem* (1.3.41)*.*

We mention that Theorem 1.3.29 remains valid if $f(u)$ is replaced by $f(x, u)$ provided that assumptions (F_2) and (F_3) hold uniformly in the first variable, that is, in x. In the case of fractional Laplacian with homogeneous exterior Dirichlet data, such type of existence result were obtained in [136, 148].

Wishing to apply the mountain pass Theorem to prove Theorem 1.3.29, the following lemmas are of key importance.

Lemma 1.3.30. *Under the condition (F_3), the functional J is unbounded from below.*

Lemma 1.3.31. *Under the conditions (F_1) and (F_2), there exist ρ, $\beta > 0$ such that for any $u \in H_0^s(\Omega)$ with $\|u\|_{H_0^s(\Omega)} = \rho$, it follows that $J(u) \geq \beta$.*

Lemma 1.3.32 (Palais-Smale condition)**.** *Under the conditions (F_1), (F_3) and (F_4), every Palais-Smale sequence for J strongly converges in $H_0^s(\Omega)$, up to a subsequence.*

Once we have the above lemmas, we can then prove Theorem 1.3.29 as follows.

Lemmas 1.3.30 and 1.3.31 reveal that the functional J possesses the mountain pass geometry. Moreover, it also satisfies the Palais-Smale condition, thanks to Lemma 1.3.32. Then by the mountain pass Theorem (see [7]), there exists a critical point $u \in H_0^s(\Omega)$ for J. Furthermore, since by Lemma 1.3.31, $J(u) \geq \beta > 0 = J(0)$, it follows that $u \not\equiv 0$, that is, u is a nontrivial mountain pass solution. This concludes the proof.

From the classical De Giorgi iteration method, we also obtain the following a priori L^∞ bounds.

Proposition 1.3.33. *Let f be a function satisfying conditions (F_1)-(F_4), and u be a solution of* (1.3.41)*. Then $u \in L^\infty(\Omega)$.*

More details can be found in Chapter 8.

Chapter 2

Morse index versus radial symmetry for fractional Dirichlet problems

The main focus of this chapter is to analyze the Morse index of a solution to a semilinear problem involving the fractional Laplacian with exterior Dirichlet data. The present content follows the original article [R1]. This paper is a collaboration with Mouhamed Moustapha Fall, Pierre Aime Feulefack, and Tobias Weth. The notation may be slightly different from those used in chapter 1.

2.1 Introduction and main result

The purpose of this paper is to estimate the Morse index of radial sign changing solutions of the problem

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \mathcal{B} \\ u = 0 & \text{in } \mathbb{R}^N \setminus \mathcal{B}, \end{cases} \tag{2.1.1}$$

where $s \in (0,1)$, $\mathcal{B} \subset \mathbb{R}^N$ is the unit ball centred at zero and where the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is of class C^1. The fractional Laplacian operator $(-\Delta)^s$ is defined for all $u \in C^2_c(\mathbb{R}^N)$ by

$$(-\Delta)^s u(x) = c(N,s) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy,$$

where $c(N,s) = 2^{2s}\pi^{-\frac{N}{2}} s \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(1-s)}$ is a normalization constant. The operator $(-\Delta)^s$ can be seen as the infinitesimal generator of an isotropic stable Lévy processes (see [12]), and it arises in specific mathematical models within several areas of physics, biology, chemistry and finance (see [12,13,34]). For basic properties of $(-\Delta)^s$ and associated function spaces, we refer to [65].

In recent years, the study of linear and nonlinear Dirichlet boundary value problems involving fractional Laplacian has attracted extensive and steadily growing attention, whereas, in contrast to the local case $s = 1$, even basic questions still remain largely unsolved up to now. Even in the linear case where $f(t) := \lambda t$, the structure of Dirichlet eigenvalues and eigenfunctions of the fractional Laplacian on the unit ball $\mathcal{B}$ is not completely understood. In particular, we mention a conjecture of Bañuelos and Kulczycki which states that every Dirichlet eigenfunction u of $(-\Delta)^s$ on $\mathcal{B}$ corresponding to the second Dirichlet eigenvalue is antisymmetric, i.e., it satisfies $u(-x) = -u(x)$ for $x \in \mathcal{B}$. So far, by the results in [15, 72, 81, 112], this conjecture has been verified in the special cases $N \leq 3$, $s \in (0,1)$ and $4 \leq N \leq 9$, $s = \frac{1}{2}$. In the present paper, we will derive the full conjecture essentially as a corollary of our main result on the semilinear Dirichlet problem (2.1.1), see Theorem 2.1.2 below.

Our main result on sign changing radial solutions of (2.1.1) is heavily inspired by the seminal work of Aftalion and Pacella [2], where the authors studied qualitative properties of sign changing solutions of the local semilinear elliptic problem

$$-\Delta u = f(u) \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{2.1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a ball or an annulus centered at zero and $f \in C^1(\mathbb{R})$. It is proved in [2, Theorem 1.1] that any radial sign changing solution of (2.1.2) has Morse index greater than or equal to $N+1$.

In the following, we present a nonlocal version of this result in the case where Ω is the unit ball in $\mathbb{R}^N$. We need to fix some notation first. Consider the function space

$$\mathcal{H}^s_0(\mathcal{B}) := \{u \in H^s(\mathbb{R}^N) : u \equiv 0 \text{ on } \mathbb{R}^N \setminus \mathcal{B}\} \subset H^s(\mathbb{R}^N). \tag{2.1.3}$$

By definition, a function $u \in \mathcal{H}^s_0(\mathcal{B}) \cap L^\infty(\mathcal{B})$ is a weak solution of (1.3.1) if

$$\mathcal{E}_s(u,v) = \int_{\mathcal{B}} f(u)v\,dx \qquad \text{for all } v \in \mathcal{H}^s_0(\mathcal{B}),$$

where

$$(v,w) \mapsto \mathcal{E}_s(v,w) := \frac{c(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^{N+2s}}\,dxdy. \tag{2.1.4}$$

is the bilinear form associated with $(-\Delta)^s$. By definition, the *Morse index* $m(u)$ of a weak solution $u \in \mathcal{H}^s_0(\mathcal{B}) \cap L^\infty(\mathcal{B})$ of (2.1.1) is the maximal dimension of a subspace $X \subset \mathcal{H}^s_0(\mathcal{B})$ where the quadratic form

$$(v,w) \mapsto \mathcal{E}_s(v,w) - \int_{\mathcal{B}} f'(u)vw\,dx \tag{2.1.5}$$

associated to the linearized operator $L := (-\Delta)^s - f'(u)$ is negative definite. Equivalently, $m(u)$ can be defined as the number of the negative Dirichlet eigenvalues of L counted with their multiplicity.

Our first main result reads as follows.

Theorem 2.1.1. *Let u be a radially symmetric sign changing solution of problem* (2.1.1)*, and suppose that one of the following additional conditions holds.*

(A1) $s \in (\frac{1}{2}, 1)$.

(A2) $s \in (0, \frac{1}{2}]$, *and*

$$\int_0^t f(\tau)d\tau > \frac{N-2s}{2N}\,tf(t) \qquad \text{for } t \in \mathbb{R}\setminus\{0\}. \tag{2.1.6}$$

Then u has Morse index greater than or equal to $N+1$.

We briefly comment on the inequality (2.1.6). In our proof of Theorem 2.1.1, this assumption arises when we use the Pohozaev identity for the fractional Laplacian, see [131, Theorem 1.1]. It is satisfied for homogeneous nonlinearities with subcritical growth, i.e., if

$$f(t) = \lambda|t|^{p-2}t \qquad \text{with } \lambda > 0 \text{ and } 2 \leq p < \frac{2N}{N-2s}.$$

We also note that, in the supercritical case where $\int_0^t f(\tau)d\tau < \frac{N-2s}{2N}\, tf(t)$ for $t \in \mathbb{R} \setminus \{0\}$, problem (2.1.1) does not admit any nontrivial weak solutions $u \in \mathcal{H}_0^s(\mathcal{B}) \cap L^\infty(\mathcal{B})$. This is a consequence of the Pohozaev identity stated in [131, Theorem 1.1].

In particular, assumption (2.1.6) is satisfied in the linear case $t \mapsto \lambda t$ with $\lambda > 0$. In fact, we can deduce the following result for the Dirichlet eigenvalue problem

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \mathcal{B} \\ \qquad\quad u = 0 & \text{in } \mathbb{R}^N \setminus \mathcal{B}, \end{cases} \tag{2.1.7}$$

from Theorem 2.1.1, thereby providing a complete positive answer to a conjecture by Bañuelos and Kulczycki (see [72]).

Theorem 2.1.2. *Let $N \geq 1$ and $0 < s < 1$, and let $\lambda_2 > 0$ be the second eigenvalue of problem (2.1.7). Then every eigenfunction u corresponding to λ_2 is antisymmetric, i.e. it satisfies $u(-x) = -u(x)$ for $x \in \mathcal{B}$.*

In recent years, partial results towards this conjecture have been obtained in [15, 72, 81, 112], covering the special cases $N \leq 3$, $s \in (0, 1)$ and $4 \leq N \leq 9$, $s = \frac{1}{2}$. More precisely, in [15, Theorem 5.3], Bañuelos and Kulczycki proved antisymmetry of second eigenfunctions in the special case $N = 1$, $s = \frac{1}{2}$. In [112], this result was extended to $N = 1$, $s \in [\frac{1}{2}, 1)$. Recently in [72], the conjecture was proved in the cases $N \leq 2$, $s \in (0, 1)$ and $3 \leq N \leq 9$, $s = \frac{1}{2}$. Moreover, in [81], the result has been proved for $N = 3$, $s \in (0, 1)$.

While the proofs in these papers are based on fine eigenvalue estimates, our proof of Theorem 2.1.2 is completely different: In addition to Theorem 2.1.1, we shall only use the following important alternative which is implicitely stated in [72, p. 503]: *Either (2.1.7) admits a radially symmetric eigenfunction corresponding to the second eigenvalue λ_2, or every eigenfunction corresponding to λ_2 is a product of a linear and a radial function.* Since every such eigenfunction u is a sign changing solution of (2.1.1) with $t \mapsto f(t) = \lambda_2 t$ and has Morse index $1 < N + 1$, it cannot be radially symmetric as a consequence of Theorem 2.1.1. Hence u must be a product of a linear and a radial function, and therefore u is antisymmetric. This completes the proof of Theorem 2.1.2. For a more detailed presentation of this argument and the underlying results from [72], see Section 2.5 below.

We briefly comment on the proof of Theorem 2.1.1. The general strategy, inspired by the paper [2] of Aftalion and Pacella for the local problem (2.1.2), is to use partial derivatives of u to construct suitable test functions which allow to estimate the Morse index of u. In the nonlocal case, several difficulties arise since local PDEs techniques do not apply. The most severe difficulty is related to the fact that weak solutions $u \in \mathcal{H}_0^s(\mathcal{B}) \cap L^\infty(\mathcal{B})$ of (2.1.1) have much less boundary regularity than solutions of (2.1.2), see Proposition 2.3.1 for details. Moreover, even though there exists a fractional version of the Hopf boundary lemma related to the fractional boundary derivative $\frac{u}{\delta^s}$ (see [75, Proposition 3.3]), it does not apply to sign changing solutions of (2.1.1) due to the nonlocality of the problem. We mention at this point that the classical Hopf boundary lemma is used in [2] together with an extra assumption on $f(0)$, but a slight change of the proof, exploiting the local character of the problem, allows to deal with solutions u having a vanishing derivative on the boundary; therefore [2, Theorem 1.1] extends to arbitrary nonlinearities $f \in C^1(\mathbb{R})$[1]. In the nonlocal case of radial solutions u of (2.1.1), it is more difficult to deal with possible oscillations of the radial derivative of u close to the boundary. In our proof of Theorem 2.1.1, we distinguish

[1]We wish to thank the referee for pointing out this fact.

two cases. In the case $s \in (\frac{1}{2}, 1)$, we use a regularity result of Grubb given in [96, Theorem 2.2] to complete the argument in the case where $\frac{u}{\delta^s}$ vanishes on $\partial\mathcal{B}$. Moreover, in the case $s \in (0, \frac{1}{2}]$, we use the extra assumption (2.1.6) to ensure that $\frac{u}{\delta^s}$ does not vanish on the boundary. Here we point out that (2.1.6) implies $f(0) = 0$, while no extra assumption on $f(0)$ is needed in the case $s \in (\frac{1}{2}, 1)$.

We point out that our proof of Theorem 2.1.1 does not use the extension method of Caffarelli and Silvestre [42], which allows to reformulate (2.1.1) as a boundary value problem where $(-\Delta)^s$ arises as a Dirichlet-to-Neumann type operator. We therefore expect that our approach applies to a more general class of nonlocal operators in place of $(-\Delta)^s$.

We wish to add some remarks on the role of Morse index estimates in the variational study of (2.1.1). In the case where $f \in C^1(\mathbb{R})$ has subcritical growth, weak solutions of (2.1.1) are precisely the critical points of the associated energy functional $J : \mathcal{H}^s_0(\mathcal{B}) \to \mathbb{R}$ defined by

$$J(u) = \frac{c(N,s)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}}\, dxdy - \int_{\mathcal{B}} F(u)\, dx,$$

where $F(t) = \int_0^t f(s)\, ds$. Moreover, J is of class C^2, and thus the behaviour of J near a critical point u is closely related to the Morse index $m(u)$. Typically, critical points detected via minimax principles lead to bounds on the Morse index. In combination with Theorem 2.1.1, this allows to show the non-radiality of certain classes of sign changing critical points. In this spirit, it is proved in [2] that, under suitable additional assumptions on f, least energy sign changing solutions of the local problem (2.1.2) are non-radial functions.

With regard to the existence of least energy sign changing solutions of the nonlocal problem (2.1.1), we refer to the recent paper [143]. For existence results for sign changing solutions to related nonlocal problems, see e.g. [118, 146] and the references therein.

The paper is organized as follows. In Section 2.2 we introduce preliminary notions and collect preliminary results on function spaces. In Section 2.3, we investigate radial solutions of (2.1.1) and properties of their partial derivatives. In Section 2.4 we complete the proof of Theorem 2.1.1. Finally, in Section 2.5, we complete the proof of Theorem 2.1.2.

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2.2 Preliminary definitions and results

In this section, we introduce some notation and state preliminary results to be used throughout this paper.

We first introduce and recall some notation related to sets and functions. If $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ are open subsets, we write $\Omega_1 \subset\subset \Omega_2$ if $\overline{\Omega}_1$ is compact and contained in Ω_2. We denote by $1_U : \mathbb{R}^N \to \mathbb{R}$ the characteristic function of a subset $U \subset \mathbb{R}^N$. For a function $u : \mathbb{R}^N \to \mathbb{R}$, we use $u^+ := \max\{u, 0\}$ and $u^- := -\min\{u, 0\}$ to denote the positive and negative part of u, respectively.

Next we recall some notation related to function spaces associated with the fractional power $s \in (0,1)$. We consider the space

$$\mathcal{L}^1_s := \left\{ u \in L^1_{loc}(\mathbb{R}^N) : \|u\|_{\mathcal{L}^1_s} < \infty \right\}, \quad \text{where} \quad \|u\|_{\mathcal{L}^1_s} := \int_{\mathbb{R}^N} \frac{|u(x)|}{1+|x|^{N+2s}}\, dx. \tag{2.2.1}$$

If $w \in \mathcal{L}^1_s$, then $(-\Delta)^s w$ is well defined as a distribution on $\mathbb{R}^N$ by setting

$$[(-\Delta)^s w](\varphi) = \int_{\mathbb{R}^N} w(-\Delta)^s \varphi\, dx \qquad \text{for } \varphi \in \mathcal{C}^\infty_c(\mathbb{R}^N).$$

Here and in the following, for an open subset $\Omega \subset \mathbb{R}^N$, we denote by $\mathcal{C}^\infty_c(\Omega)$ the space of smooth functions on $\mathbb{R}^N$ with compact support in Ω. We recall a maximum principle for the fractional Laplacian in distributional sense due to Silvestre.

Proposition 2.2.1. *[137, Proposition 2.17] Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, and let $w \in \mathcal{L}^1_s$ be a lower-semicontinuous function in $\overline{\Omega}$ such that $w \geq 0$ in $\mathbb{R}^N \setminus \Omega$ and $(-\Delta)^s w \geq 0$ in Ω in distributional sense, i.e.,*

$$\int_{\mathbb{R}^N} w(-\Delta)^s \varphi\, dx \geq 0 \qquad \text{for all nonnegative functions } \varphi \in \mathcal{C}^\infty_c(\Omega).$$

Then $w \geq 0$ in $\mathbb{R}^N$.

For an open subset $\Omega \subset \mathbb{R}^N$, we now consider the fractional Sobolev space

$$H^s(\Omega) = \left\{ u \in L^2(\Omega) \ : \ \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy < \infty \right\}. \tag{2.2.2}$$

Setting

$$[u]_{s,\Omega} := \Big(\frac{1}{2} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy \Big)^{\frac{1}{2}} \qquad \text{for } u \in H^s(\Omega),$$

we note that $H^s(\Omega)$ is a Hilbert space whose norm can be written as

$$\|u\|_{H^s(\Omega)} = \Big(\|u\|^2_{L^2(\Omega)} + [u]^2_{s,\Omega} \Big)^{\frac{1}{2}} \tag{2.2.3}$$

We will also use the local fractional Sobolev space $H^s_{loc}(\Omega)$ defined as the space of functions $\psi \in L^2_{loc}(\Omega)$ with $\psi \in H^s(\Omega')$ for every $\Omega' \subset\subset \Omega$.

For a *bounded* open subset $\Omega \subset \mathbb{R}^N$, we let $\mathcal{H}^s_0(\Omega)$ denote the closure of $C^\infty_c(\Omega)$ in $H^s(\mathbb{R}^N)$. Then $\mathcal{H}^s_0(\Omega)$ is a Hilbert space with scalar product

$$(u,v) \mapsto \mathcal{E}_s(u,v) := \langle u, v \rangle_{\mathcal{H}^s_0(\Omega)} = \frac{c(N,s)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{N+2s}}\, dxdy$$

and corresponding norm

$$\|u\|_{\mathcal{H}^s_0(\Omega)} = \sqrt{\mathcal{E}_s(u,u)} = \sqrt{c(N,s)}\,[u]_{s,\mathbb{R}^N}.$$

This is a consequence of the fact that

$$\inf\{\mathcal{E}_s(u,u) \ : \ u \in \mathcal{H}^s_0(\Omega),\ \|u\|_{L^2(\Omega)} = 1\} \ > \ 0,$$

which in turn follows from the fractional Sobolev inequality (see e.g. [65, Theorem 6.5]) and the boundedness of Ω. In particular, $\mathcal{H}_0^s(\Omega)$ embeds into $L^2(\Omega)$. We also note that, by definition,

$$\mathcal{H}_0^s(\tilde{\Omega}) \subset \mathcal{H}_0^s(\Omega) \qquad \text{for bounded open sets } \Omega, \tilde{\Omega} \text{ with } \tilde{\Omega} \subset \Omega. \tag{2.2.4}$$

We also recall the following property, see e.g. [94, Theorem 1.4.2.2]:

$$\begin{aligned} &\textit{For any bounded domain } \Omega \textit{ with continuous boundary,} \\ &\textit{we have } \mathcal{H}_0^s(\Omega) = \{u \in H^s(\mathbb{R}^N) : u \equiv 0 \;\; \textit{on} \;\; \mathbb{R}^N \setminus \Omega\}. \end{aligned} \tag{2.2.5}$$

Consequently, the definition of $\mathcal{H}_0^s(\Omega)$ is consistent with (2.1.3).

For the remainder of this section, we fix a bounded open subset $\Omega \subset \mathbb{R}^N$. The following lemma is known, but we include a short proof for the convenience of the reader.

Lemma 2.2.2. *Let $\varphi \in H^s_{loc}(\Omega)$ be compactly supported in Ω. Then $\varphi \in \mathcal{H}_0^s(\Omega)$.*

Here and in the following, we identify φ with its trivial extension to $\mathbb{R}^N$.

Proof. Without loss of generality, we may assume that Ω has a continuous boundary, since otherwise we may use (2.2.4) after replacing Ω by a bounded open subset $\tilde{\Omega}$ with continuous boundary containing the support of φ.

Let $\Omega' \subset\subset \Omega$ be an open subset of Ω which contains the support K of φ. Then we have

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{N+2s}}\,dxdy = [\varphi]^2_{s,\Omega'} + \int_{\Omega'}\int_{\mathbb{R}^N\setminus\Omega'} \frac{|\varphi(x)|^2}{|x-y|^{N+2s}}\,dydx, \tag{2.2.6}$$

where $[\varphi]^2_{s,\Omega'} < \infty$ since $\varphi \in H^s_{loc}(\Omega)$. Moreover,

$$\begin{aligned} \int_{\Omega'}\int_{\mathbb{R}^N\setminus\Omega'} \frac{|\varphi(x)|^2}{|x-y|^{N+2s}}\,dydx &= \int_K |\varphi(x)|^2 \int_{\mathbb{R}^N\setminus\Omega'} \frac{dy}{|x-y|^{N+2s}}dx \\ &\leq \|\varphi\|^2_{L^2(K)} \sup_{x\in K}\int_{\mathbb{R}^N\setminus\Omega'} \frac{dy}{|x-y|^{N+2s}} < \infty \end{aligned}$$

since $\operatorname{dist}(K, \mathbb{R}^N \setminus \Omega') > 0$. Since Ω has a continuous boundary and $\varphi \equiv 0$ in $\mathbb{R}^N \setminus \Omega$, we conclude that $\varphi \in \mathcal{H}_0^s(\Omega)$ as a consequence of (2.2.5). $\square$

We also need the following lemma.

Lemma 2.2.3. *Let $v \in \mathcal{L}^1_s \cap H^s_{loc}(\Omega)$, and let $\varphi \in H^s_{loc}(\Omega)$ be a function with compact support. Then the integral*

$$\mathcal{E}_s(v,\varphi) = \frac{c(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big(v(x)-v(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+2s}}\,dxdy$$

is well defined in Lebesgue sense. More precisely, for any choice of open subsets

$$\Omega' \subset\subset \Omega'' \subset\subset \Omega$$

with $\operatorname{supp}\varphi \subset \Omega'$, *there exist constants c_1, c_2 – depending only on Ω', Ω'', N and s but not on v and φ —such that*

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big|v(x)-v(y)\big|\big|\varphi(x)-\varphi(y)\big|}{|x-y|^{N+2s}}\,dxdy \\ &\qquad\leq [v]_{s,\Omega''}[\varphi]_{s,\Omega''} + c_1\|v\|_{L^2(\Omega')}\|\varphi\|_{L^2(\Omega')} + c_2\|\varphi\|_{L^1(\Omega')}\|v\|_{\mathcal{L}^1_s}. \end{aligned} \tag{2.2.7}$$

Proof. We put $\mathbf{k}(z) = |z|^{-N-2s}$. Since $\operatorname{supp}\varphi \subset \Omega'$, we see that

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(x)-v(y)||\varphi(x)-\varphi(y)|\mathbf{k}(x-y)\ dxdy =$$
$$\frac{1}{2}\int_{\Omega''}\int_{\Omega''} \frac{\big|v(x)-v(y)\big|\big|\varphi(x)-\varphi(y)\big|}{|x-y|^{N+2s}}\ dxdy + \int_{\Omega'}\int_{\mathbb{R}^N\setminus\Omega''} \frac{\big|v(x)-v(y)\big|\big|\varphi(x)\big|}{|x-y|^{N+2s}}\ dydx$$
$$\leq [v]_{s,\Omega''}[\varphi]_{s,\Omega''} + \int_{\Omega'}|\varphi(x)|\int_{\mathbb{R}^N\setminus\Omega''}|v(x)-v(y)|\mathbf{k}(x-y)\ dydx,$$

where

$$\begin{aligned}\int_{\Omega'}|\varphi(x)|&\int_{\mathbb{R}^N\setminus\Omega''}|v(x)-v(y)|\mathbf{k}(x-y)\ dydx\\ &\leq \int_{\Omega'}|\varphi(x)||v(x)|\kappa_{\Omega''}(x)\,dx + \int_{\Omega'}|\varphi(x)|\int_{\mathbb{R}^N\setminus\Omega''}|v(y)|\mathbf{k}(x-y)dydx\\ &\leq c_1\|\varphi\|_{L^2(\Omega')}\|v\|_{L^2(\Omega')} + c_2\|\varphi\|_{L^1(\Omega')}\|v\|_{\mathcal{L}^1_s}\end{aligned}$$

with

$$\kappa_{\Omega''}(x) = \int_{\mathbb{R}^N\setminus\Omega''}\mathbf{k}(x-y)\ dy, \qquad x\in\Omega'$$

and

$$c_1 := \sup_{x\in\Omega'}\kappa_{\Omega''}(x), \qquad c_2 := \sup_{x\in\Omega', y\in\mathbb{R}^N\setminus\Omega''}\mathbf{k}(x-y)(1+|y|)^{N+2s}.$$

Note that the values c_1 and c_2 are finite since $\Omega' \subset\subset \Omega''$. It thus follows that $\mathcal{E}_s(u,v)$ is well-defined in Lebesgue sense and that (2.2.7) holds. □

Corollary 2.2.4. *Let $v \in \mathcal{L}^1_s \cap H^s_{loc}(\Omega)$. If $\Omega' \subset\subset \Omega$ and $(\varphi_n)_n$ is a sequence in $H^s_{loc}(\Omega)$ with* $\operatorname{supp}\varphi$, $\operatorname{supp}\varphi_n \subset \Omega'$ *for all $n\in\mathbb{N}$ and $\varphi_n \to \varphi$ in $H^s_{loc}(\Omega)$, then we have*

$$\mathcal{E}_s(v,\varphi_n) \to \mathcal{E}_s(v,\varphi) \qquad \text{as } n\to\infty.$$

Proof. By Lemma 2.2.3,

$$|\mathcal{E}_s(v,\varphi_n-\varphi)| \leq$$
$$c(N,s)[v]_{s,\Omega'}[\varphi_n-\varphi]_{s,\Omega'} + \mathbf{C}_1\|v\|_{L^2(\Omega')}\|\varphi_n-\varphi\|_{L^2(\Omega')} + \mathbf{C}_2\|\varphi_n-\varphi\|_{L^1(\Omega')}\|v\|_{\mathcal{L}^1_s},$$

where $\mathbf{C}_1$ and $\mathbf{C}_2$ are positive constants. Thanks to the embeddings $H^s_{loc}(\Omega) \hookrightarrow L^2_{loc}(\Omega) \hookrightarrow L^1_{loc}(\Omega)$, we conclude that $\mathcal{E}_s(v,\varphi_n-\varphi) \to 0$ as $n\to\infty$. □

2.3 Properties of radial solutions and their partial derivatives

In the following, we restrict our attention to the case $\Omega = \mathcal{B}$ and to bounded weak solutions of equation (2.1.1). Here and in the following, we fix a nonlinearity $f:\mathbb{R}\to\mathbb{R}$ of class C^1, and we call a function $u \in \mathcal{H}^s_0(\mathcal{B}) \cap L^\infty(\mathcal{B})$ a weak solution of (2.1.1) if

$$\mathcal{E}_s(u,\varphi) = \int_{\mathcal{B}} f(u)\varphi\,dx \qquad \text{for all } \varphi \in \mathcal{H}^s_0(\mathcal{B}).$$

We note the following regularity properties for weak solutions of (2.1.1). For this we consider the distance function to the boundary

$$\delta : \overline{\mathcal{B}} \to \mathbb{R}, \qquad \delta(x) = \operatorname{dist}(x, \partial\mathcal{B}) = 1 - |x|.$$

Proposition 2.3.1. *(cf. [74, 96, 130, 137])*
Let $u \in \mathcal{H}_0^s(\mathcal{B}) \cap L^\infty(\mathcal{B})$ be a weak solution of (2.1.1). Then $u \in C^{2,s}_{loc}(\mathcal{B}) \cap C^s_0(\overline{\mathcal{B}})$. Moreover,

$$\psi := \frac{u}{\delta^s} \in C^\alpha(\overline{\mathcal{B}}) \qquad \text{for some } \alpha \in (0,1), \tag{2.3.1}$$

and the following properties hold with some constant $c > 0$:

(i) $|\nabla u(x)| \le c\delta^{s-1}(x)$ for all $x \in \mathcal{B}$.

(ii) $|\nabla \psi(x)| \le c\delta^{\alpha-1}(x)$ for all $x \in \mathcal{B}$.

(iii) For every $x_0 \in \partial\mathcal{B}$, we have $\lim\limits_{x \to x_0} \delta^{1-s}(x)\partial_r u(x) = -s\psi(x_0)$, where $\partial_r u(x) = \nabla u(x) \cdot \frac{x}{|x|}$ denotes the radial derivative of u at x.

(iv) If $s \in (\frac{1}{2}, 1)$, then $\psi \in C^1(\overline{\mathcal{B}})$.

Proof. Since $u \in L^\infty(\mathcal{B})$ and f is of class C^1, we have $f(u(\cdot)) \in L^\infty(\mathcal{B})$. Hence the regularity theory for the fractional Dirichlet-Possion problem developed in [130] shows that $u \in C^s_0(\mathcal{B})$, and that (i) holds. It is also shown in [130] that $\psi := \frac{u}{\delta^s} \in C^\alpha(\overline{\mathcal{B}})$ for some $\alpha \in (0,1)$. Moreover, (ii) and (iii) are proved in [74].
Finally, noting that $f(u(\cdot)) \in C^s(\mathcal{B})$ since $u \in C^s_0(\mathcal{B})$, it follows from interior regularity (see e.g. [137]) that $u \in C^{2,s}_{loc}(\mathcal{B})$. Moreover, if $s \in (\frac{1}{2}, 1)$ we have $\psi \in C^{2s}(\overline{\mathcal{B}}) \subset C^1(\overline{\mathcal{B}})$ by [96, Theorem 2.2]. □

The regularity estimates above allow to apply the following simple integration by parts formula to weak solutions of (2.1.1).

Lemma 2.3.2. *Let $u \in C^0(\overline{\mathcal{B}}) \cap C^1_{loc}(\mathcal{B})$ be a function satisfying $u \equiv 0$ on $\partial\mathcal{B}$ and $|\nabla u| \in L^1(\mathcal{B})$. Then*

$$\int_{\mathcal{B}} (\partial_j u)\varphi \, dx = -\int_{\mathcal{B}} u \partial_j \varphi \, dx \qquad \text{for } \varphi \in C^1(\overline{\mathcal{B}}),\ j = 1, \dots, N. \tag{2.3.2}$$

Proof. Let $\varphi \in C^1(\overline{\mathcal{B}})$, and let $\Omega_n := B_{1-\frac{1}{n}}(0) \subset \mathcal{B}$ for $n \in \mathbb{N}$. Then $u \in C^1(\overline{\Omega}_n)$ for $n \in \mathbb{N}$ since $u \in C^1_{loc}(\mathcal{B})$. Integrating by parts over Ω_n and using a change of variables, we find that

$$\int_{\Omega_n} \big((\partial_j u)\varphi + u\partial_j\varphi\big) \, dx = \int_{\partial\Omega_n} u\varphi\nu_j \, d\sigma = (1 - \frac{1}{n})^{N-1} \int_{\partial\mathcal{B}} u((1-\frac{1}{n})\sigma)\varphi((1-\frac{1}{n})\sigma)\nu_j \, d\sigma,$$

where ν_j is the j-th component of the unit outward normal to $\partial\mathcal{B}$ at x. Since $u \in C^0(\overline{\mathcal{B}})$, $u = 0$ on $\partial\mathcal{B}$, $\Omega_n \uparrow \mathcal{B}$ and $\varphi \in C^1(\overline{\mathcal{B}})$, we can apply the Lebesgue dominated convergence theorem to both sides of the equation above to deduce (2.3.2). □

In the following, we fix a *radial* solution $u \in \mathcal{H}_0^s(\mathcal{B}) \cap L^\infty(\mathcal{B})$ of (2.1.1), and we consider the function ψ defined in (2.3.1) which is also radial. Hence we write

$$\psi(x) = \psi_0(r) \text{ for } r = |x| \text{ with a function } \psi_0 : [0,1] \to \mathbb{R} \tag{2.3.3}$$

which is of class C^α for some $\alpha > 0$ by Proposition 2.3.1. Moreover, by Proposition 2.3.1 we have

$$\psi_0(1) = \lim_{|x|\to 1} \frac{u(|x|)}{(1-|x|)^s} = -\frac{1}{s} \lim_{|x|\to 1} (1-|x|)^{1-s} \partial_r u(x). \tag{2.3.4}$$

By the Pohozaev type identity given in [131, Theorem 1.1], this value also satisfies

$$\psi_0^2(1) = \frac{1}{|S^{N-1}|\Gamma(1+s)^2} \int_{\mathcal{B}} \Big[(2s-N)uf(u) + 2NF(u)\Big]\,dx. \tag{2.3.5}$$

Here $F : \mathbb{R} \to \mathbb{R}$ is given by $F(t) = \int_0^t f(\tau)\,d\tau$.

The aim of this section is to construct test functions related to partial derivatives of u, which allow to estimate Dirichlet eigenvalues of the linearized operator

$$L := (-\Delta)^s - f'(u). \tag{2.3.6}$$

For $j \in \{1,\dots,N\}$, we consider the partial derivatives of u given by

$$v^j : \mathbb{R}^N \to \mathbb{R}, \qquad v^j(x) = \begin{cases} \partial_j u(x) = \dfrac{\partial u}{\partial x_j}(x), & x \in \mathcal{B}, \\ 0, & x \in \mathbb{R}^N \setminus \mathcal{B}, \end{cases} \qquad j = 1,\dots,N.$$

From Proposition 2.3.1, it then follows that

$$v^j \in \mathcal{L}^1_s \cap H^s_{loc}(\mathcal{B}) \qquad \text{for } j \in \{1,\dots,N\}. \tag{2.3.7}$$

Hence $\mathcal{E}_s(v^j,\varphi)$ is well defined for every $\varphi \in \mathcal{H}^s_0(\mathcal{B})$ with compact support by Lemma 2.2.3. We have the following key lemma.

Lemma 2.3.3. *For any $j \in \{1,\dots,N\}$, we have $Lv^j = (-\Delta)^s v^j - f'(u)v^j = 0$ in distributional sense in $\mathcal{B}$, i.e.*

$$\int_{\mathcal{B}} v^j (-\Delta)^s \varphi\,dx = \mathcal{E}_s(v^j,\varphi) = \int_{\mathcal{B}} f'(u)v^j\varphi\,dx \qquad \textit{for all } \varphi \in \mathcal{C}^\infty_c(\mathcal{B}). \tag{2.3.8}$$

Moreover, if $\varphi \in \mathcal{H}^s_0(\mathcal{B})$ has compact support in $\mathcal{B}$, then we have

$$\mathcal{E}_s(v^j,\varphi) = \int_{\mathcal{B}} f'(u)v^j\varphi\,dx. \tag{2.3.9}$$

Furthermore, if $v^j \in \mathcal{H}^s_0(\mathcal{B})$, then (2.3.9) is true for all $\varphi \in \mathcal{H}^s_0(\mathcal{B})$.

Proof. Since $u \in C^{2,s}_{loc}(\mathcal{B})$ by Proposition 2.3.1, we have $v^j \in C^{1,s}_{loc}(\mathcal{B}) \subset H^s_{loc}(\mathcal{B})$. Let $\varphi \in \mathcal{C}^\infty_c(\mathcal{B}) \subset \mathcal{C}^\infty_c(\mathbb{R}^N)$. Then

$$\partial_j\varphi \in \mathcal{C}^\infty_c(\mathcal{B}), \quad (-\Delta)^s\varphi \in C^\infty(\mathbb{R}^N), \qquad \text{and} \qquad \partial_j(-\Delta)^s\varphi = (-\Delta)^s\partial_j\varphi \quad \text{on } \mathbb{R}^N.$$

Consequently, since u satisfies the assumptions of Lemma 2.3.2, (2.3.2) implies that

$$\int_{\mathcal{B}} v^j(-\Delta)^s\varphi\,dx = -\int_{\mathcal{B}} u\partial_j(-\Delta)^s\varphi\,dx = -\int_{\mathcal{B}} u(-\Delta)^s\partial_j\varphi\,dx$$

$$= -\mathcal{E}_s(u, \partial_j \varphi) = -\int_{\mathcal{B}} f(u)\partial_j \varphi \; dx = \int_{\mathcal{B}} \partial_j f(u) \varphi \; dx = \int_{\mathcal{B}} f'(u) v^j \varphi \; dx.$$

Hence v^j solves $Lv^j = (-\Delta)^s v^j - f'(u)v^j = 0$ in distributional sense. Next we show that

$$\mathcal{E}_s(v^j, \varphi) = \int_{\mathcal{B}} f'(u) v^j \varphi \; dx \qquad \text{for all } \varphi \in \mathcal{C}_c^\infty(\mathcal{B}). \tag{2.3.10}$$

Since $v^j \in \mathcal{L}_s^1 \cap H_{loc}^s(\mathcal{B})$, the integral

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\big|v^j(x) - v^j(y)\big|\big|\varphi(x) - \varphi(y)\big|}{|x-y|^{N+2s}} \; dxdy$$

exists by Lemma 2.2.3, and therefore we have, by Lebesgue's Theorem,

$$\begin{aligned}
\mathcal{E}_s(v^j, \varphi) &= \frac{c(N,s)}{2} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \int_{|x-y| \geq \varepsilon} \frac{\big(v^j(x) - v^j(y)\big)\big(\varphi(x) - \varphi(y)\big)}{|x-y|^{N+2s}} \; dxdy \\
&= c(N,s) \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} v^j(x) \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}} \; dydx \\
&= c(N,s) \int_{\mathbb{R}^N} v^j(x) \; \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}} \; dydx \\
&= \int_{\mathbb{R}^N} v^j (-\Delta)^s \varphi \; dx = \int_{\mathcal{B}} v^j (-\Delta)^s \varphi \; dx = \int_{\mathcal{B}} f'(u) v^j \varphi \; dx.
\end{aligned}$$

Next, let $\varphi \in \mathcal{H}_0^s(\mathcal{B})$ with compact support in $\mathcal{B}$, and choose an open subset $\Omega' \subset\subset \mathcal{B}$ such that $\operatorname{supp} \varphi \subset \Omega'$. By definition of $\mathcal{H}_0^s(\Omega')$, there exists a sequence $(\varphi_n)_n$ in $\mathcal{C}_c^\infty(\Omega') \subset \mathcal{C}_c^\infty(\mathcal{B})$ with $\varphi_n \to \varphi$ in $\mathcal{H}_0^s(\Omega')$, hence also $\varphi_n \to \varphi$ in $\mathcal{H}_0^s(\mathcal{B})$. Then Corollary 2.2.4 and (2.3.10) imply that

$$\mathcal{E}_s(v^j, \varphi) = \lim_{n \to \infty} \mathcal{E}_s(v^j, \varphi_n) = \lim_{n \to \infty} \int_{\mathcal{B}} f'(u) v^j \varphi_n \; dx = \int_{\mathcal{B}} f'(u) v^j \varphi \; dx, \tag{2.3.11}$$

and thus (2.3.9) holds.

Finally, assume that $v^j \in \mathcal{H}_0^s(\mathcal{B})$, let $\varphi \in \mathcal{H}_0^s(\mathcal{B})$, and let $(\varphi_n)_n$ be a sequence in $\mathcal{C}_c^\infty(\mathcal{B})$ with $\varphi_n \to \varphi$ in $\mathcal{H}_0^s(\mathcal{B})$. Then (2.3.11) holds again by the continuity of the quadratic form $\mathcal{E}_s$ on $\mathcal{H}_0^s(\mathcal{B})$, as claimed. □

We now have all the tools to build suitable test functions from partial derivatives in order to estimate the Morse index of u as a solution of (2.1.1). As remarked before, the construction is inspired by [2].

Definition 2.3.4. Let ψ_0 be the function defined in (2.3.3). For $j = 1, \dots, N$, we define the open half spaces

$$H_\pm^j := \{x \in \mathbb{R}^N \; : \; \pm x_j > 0\} \tag{2.3.12}$$

and the functions $d_j : \mathbb{R}^N \to \mathbb{R}$ by

$$d_j := \begin{cases} (v^j)^+ 1_{H_+^j} - (v^j)^- 1_{H_-^j} & \text{if } \psi_0(1) \geq 0; \\ (v^j)^+ 1_{H_-^j} - (v^j)^- 1_{H_+^j} & \text{if } \psi_0(1) < 0. \end{cases}$$

We note that, for $j = 1, \dots, N$, the function d_j is odd with respect to the reflection

$$\sigma_j : \mathbb{R}^N \to \mathbb{R}^N, \qquad x = (x_1, \dots, x_j, \dots, x_N) \mapsto \sigma_j(x) = (x_1, \dots, -x_j, \dots, x_N)$$

at the hyperplane $\{x_j = 0\}$ since the function v^j is odd.

Lemma 2.3.5. *$d_j \in H^s_{loc}(\mathcal{B})$ for $j = 1, \dots, N$.*

Proof. By definition of d_j, it suffices to show that

$$(v^j)^\pm \, 1_{H^j_\pm} \;\in\; H^s_{loc}(\mathcal{B}). \tag{2.3.13}$$

We only consider the function $(v^j)^+ \, 1_{H^j_+}$, the proof for the other functions is essentially the same. As noted in (2.3.7), we have $v^j \in H^s_{loc}(\mathcal{B})$, and therefore also $(v^j)^+ \in H^s_{loc}(\mathcal{B})$ by a standard estimate. To abbreviate, we now put $\chi = 1_{H^j_+}$, $v := (v^j)^+$, and we let $\Omega' \subset\subset \mathcal{B}$ be an open subset of $\mathcal{B}$. Making Ω' larger if necessary, we may assume that Ω' is symmetric with respect to the reflection σ_j. To show that $v\chi \in H^s_{loc}(\Omega')$, we write

$$\begin{aligned}
[v\chi]^2_{s,\Omega'} &= [v]^2_{s,\Omega' \cap H^j_+} + \int_{\Omega' \cap H^j_+} |v(x)|^2 \int_{\Omega' \cap H^j_-} |x-y|^{-N-2s} \, dydx \\
&\le [v]^2_{s,\Omega'} + \int_{\Omega' \cap H^j_+} |v(x)|^2 \int_{\{y \in \mathbb{R}^N, |y-x| \ge |x_j|\}} |x-y|^{-N-2s} dydx \\
&= [v]^2_{s,\Omega'} + \int_{\Omega' \cap H^j_+} |v(x)|^2 \int_{\{z \in \mathbb{R}^N, |z| \ge |x_j|\}} |z|^{-N-2s} dzdx \\
&= [v]^2_{s,\Omega'} + \frac{|S^{N-1}|}{2s} \int_{\Omega' \cap H^j_+} |v(x)|^2 |x_j|^{-2s} dx.
\end{aligned}$$

Since $v = (v^j)^+ \in C^s_{loc}(\mathcal{B})$ by Proposition 2.3.1 and $v \equiv 0$ on $\{x_j = 0\}$, we have $|v(x)| \le C|x_j|^s$ for $x \in \Omega' \cap H^j_+$. Therefore, the latter integral is finite, and $(v^j)^+ \, 1_{H^j_+} = v\chi \in H^s_{loc}(\mathcal{B})$. $\square$

The next lemma is of key importance for the proof of Theorem 2.1.1.

Lemma 2.3.6. *Let $j = 1, \dots, N$.*

(i) If $\psi_0(1) \neq 0$, we have $d_j \in \mathcal{H}^s_0(\mathcal{B})$, and d_j has compact support in $\mathcal{B}$.

(ii) If $s \in (\frac{1}{2}, 1)$ and $\psi_0(1) = 0$, then we have $v^j \in \mathcal{H}^s_0(\mathcal{B})$ and $d_j \in \mathcal{H}^s_0(\mathcal{B})$.

Proof. (i) By Lemma 2.2.2 and Lemma 2.3.5, it suffices to show that d_j has compact support in $\mathcal{B}$. We now distinguish the cases $\psi_0(1) > 0$ and $\psi_0(1) < 0$.

If $\psi_0(1) > 0$, we have $\partial_r u(x) \le 0$ in $\mathcal{B} \setminus B_{r_*}(0)$ for some $r_* \in (0,1)$ by (2.3.4), and therefore

$$v^j(x) = \partial_j u(x) = \frac{x_j}{|x|} \partial_r u(x) \le 0 \qquad \text{for } x \in \mathcal{B} \setminus B_{r_*}(0) \text{ with } x_j \ge 0.$$

Consequently, $d_j(x) = (v^j)^+(x) = 0$ for $x \in \mathcal{B} \setminus B_{r_*}(0)$ with $x_j \ge 0$. Since d_j is odd with respect to the reflection σ_j it follows that $\operatorname{supp} d_j \subset \overline{B_{r_*}(0)}$, so d_j is compactly supported in $\mathcal{B}$.

If $\psi_0(1) < 0$, we have $\partial_r u(x) \ge 0$ in $\mathcal{B} \setminus B_{r_*}(0)$ for some $r_* \in (0,1)$ by (2.3.4), which in this case, similarly as above, implies that $d_j(x) = -(v^j)^-(x) = 0$ for $x \in \mathcal{B} \setminus B_{r_*}(0)$ with $x_j \ge 0$. Again

we conclude that d_j is compactly supported in $\mathcal{B}$ since it is odd with respect to the reflection σ_j.
(ii) Since $s \in (\frac{1}{2}, 1)$, it follows from Proposition 2.3.1(iv) that $\psi \in C^1(\overline{\mathcal{B}})$ and therefore $\psi_0 \in C^1([0,1])$, whereas $\psi_0(1) = 0$ by assumption. Consequently, $\psi(x)\delta^{s-1}(x) \to 0$ as $|x| \to 1$, and therefore

$$\nabla u(x) = \delta^s(x)\nabla\psi(x) + s\psi(x)\delta^{s-1}(x)\nabla\delta(x) \to 0 \qquad \text{as } |x| \to 1.$$

It thus follows that $u \in C^1(\mathbb{R}^N)$ with $u \equiv 0$ on $\mathbb{R}^N \setminus \mathcal{B}$, and therefore $v^j \in C^0(\mathbb{R}^N)$ with $v^j \equiv 0$ in $\mathbb{R}^N \setminus \mathcal{B}$. To see that $v^j \in \mathcal{H}^s_0(\mathcal{B})$, we shall use Proposition 2.2.1 as follows: Since the function $f'(u)v^j$ is continuous and therefore bounded in $\overline{\mathcal{B}}$, there exists a unique weak solution $w \in \mathcal{H}^s_0(\mathcal{B})$ to the Poisson problem

$$(-\Delta)^s w = f'(u)v^j \quad \text{in } \mathcal{B}, \qquad\qquad w = 0 \quad \text{in } \mathbb{R}^N \setminus \mathcal{B} \tag{2.3.14}$$

which satisfies $w \in C^s_0(\mathcal{B})$ by [130, Proposition 1.1]. By setting $V := w - v^j$, it follows that $V \in C^0(\mathbb{R}^N)$ with $V \equiv 0$ in $\mathbb{R}^N \setminus \mathcal{B}$. Moreover, by Lemma 2.3.3 the function V satisfies the equation $(-\Delta)^s V = 0$ in $\mathcal{B}$ in the sense of distributions. Since V is continuous, Proposition 2.2.1 – applied to $\pm V$ – implies that $V \equiv 0$ in $\mathbb{R}^N$, i.e.,

$$v^j = w \in \mathcal{H}^s_0(\mathcal{B}) \cap C^s_0(\mathcal{B}). \tag{2.3.15}$$

By a similar argument as in the proof of Lemma 2.3.5, we will now see that $d_j \in \mathcal{H}^s_0(\mathcal{B})$. For the convenience of the reader, we give the details. It is clearly sufficient to show that

$$(v^j)^{\pm}\, 1_{H^j_\pm} \;\in\; \mathcal{H}^s_0(\mathcal{B}). \tag{2.3.16}$$

We only consider the function $(v^j)^+\, 1_{H^j_+}$, the proof for the other functions is the same. Since $v^j \in \mathcal{H}^s_0(\mathcal{B})$, we also have $(v^j)^{\pm} \in \mathcal{H}^s_0(\mathcal{B})$ by a standard estimate. To abbreviate, we now put $\chi = 1_{H^j_+}$ and $v := (v^j)^+$. To show that $v\chi \in \mathcal{H}^s_0(\mathcal{B})$, we note that $v\chi \equiv 0$ in $\mathbb{R}^N \setminus \mathcal{B}$, and we estimate

$$\begin{aligned}
[v\chi]^2_{s,\mathbb{R}^N} &= [v]^2_{s,H^j_+} + \int_{H^j_+\cap\mathcal{B}} |v(x)|^2 \int_{H^j_-} |x-y|^{-N-2s}\; dydx \\
&\le [v]^2_{s,\mathbb{R}^N} + \int_{H^j_+\cap\mathcal{B}} |v(x)|^2 \int_{\{z\in\mathbb{R}^N, |z|\ge|x_j|\}} |z|^{-N-2s} dzdx \\
&= [v]^2_{s,\mathbb{R}^N} + \frac{|S^{N-1}|}{2s} \int_{H^j_+\cap\mathcal{B}} |v(x)|^2 |x_j|^{-2s} dx.
\end{aligned}$$

Since $v = (v^j)^+ \in C^s(\overline{\mathcal{B}})$ by (2.3.15) and $v \equiv 0$ on $\{x_j = 0\}$, we have $|v(x)| \le C|x_j|^s$ for $x \in H^j_+ \cap \mathcal{B}$. Therefore, the latter integral is finite, and $(v^j)^+\, 1_{H^j_+} = v\chi \in \mathcal{H}^s_0(\mathcal{B})$. $\square$

Corollary 2.3.7. *If $\psi_0(1) \neq 0$ or $s \in (\frac{1}{2}, 1)$, then the values $\mathcal{E}_s(d_j, d_k)$ and $\mathcal{E}_s(v^j, d_k)$ are well-defined and satisfy*

$$\mathcal{E}_s(v^j, d_k) = \int_{\mathcal{B}} f'(u) v^j d_k \; dx \qquad \text{for } j,k = 1,\dots,N.$$

Proof. This follows from Lemma 2.2.3, Lemma 2.3.3 and Lemma 2.3.6. $\square$

2.4 Proof of Theorem 2.1.1

In this section we complete the proof of Theorem 2.1.1. As before, we consider a fixed radial weak solution $u \in \mathcal{H}_0^s(\mathcal{B}) \cap L^\infty(\mathcal{B})$ of (2.1.1), and we will continue using the notation related to u as introduced in Section 2.3. Moreover, in accordance with the assumptions of Theorem 2.1.1, we assume that u changes sign, which implies that

$$(v^j)^\pm 1_{H_+^j} \not\equiv 0 \quad \text{and} \quad (v^j)^\pm 1_{H_-^j} \not\equiv 0 \qquad \text{for } j = 1, \dots, N, \tag{2.4.1}$$

where the half spaces $H_\pm^j$ are defined in (2.3.12). We first note that, under the assumptions of Theorem 2.1.1, we have

$$\psi_0(1) \neq 0 \qquad \text{or} \qquad s \in (\frac{1}{2}, 1). \tag{2.4.2}$$

Indeed, if $s \in (0, \frac{1}{2}]$, then $\psi_0^2(1) > 0$ by (2.1.6) and (2.3.5).

Next we recall that the n-th Dirichlet eigenvalue $\lambda_{n,L}$ of the linearized operator L defined in (2.3.6) admits the variational characterization

$$\lambda_{n,L} = \min_{V \in \mathcal{V}_n} \max_{v \in S_V} \mathcal{E}_{s,L}(v, v) \tag{2.4.3}$$

where

$$(v, w) \mapsto \mathcal{E}_{s,L}(v, w) := \mathcal{E}_s(v, w) - \int_{\mathcal{B}} f'(u) vw \; dx \tag{2.4.4}$$

is the bilinear form associated to L, $\mathcal{V}_n$ denotes the family of n-dimensional subspaces of $\mathcal{H}_0^s(\mathcal{B})$ and $S_V := \{v \in V \; : \; \|v\|_{L^2(\mathcal{B})} = 1\}$ for $V \in \mathcal{V}_n$.

To estimate $\lambda_{n,L}$ from above, we wish to build test function spaces V by using the functions d_j introduced in Definition 2.3.4. By Lemma 2.3.6 and (2.4.2), we have $d_j \in \mathcal{H}_0^s(\Omega)$ for $j = 1, \dots, N$. Moreover, as a consequence of Corollary 2.3.7, the values $\mathcal{E}_s(v^j, d_k)$ are well-defined and satisfy

$$\mathcal{E}_{s,L}(v^j, d_k) = 0 \qquad \text{for } j, k = 1, \dots, N. \tag{2.4.5}$$

We need the following key inequality.

Lemma 2.4.1. *For $j \in \{1, \dots, N\}$ we have $\mathcal{E}_{s,L}(d_j, d_j) < 0$.*

Proof. To simplify notation, we put $k(z) = c(N, s)|z|^{-N-2s}$ for $z \in \mathbb{R}^N \setminus \{0\}$. Since $v^j d_j = d_j^2$ in $\mathbb{R}^N$ by definition of d_j and therefore

$$\int_{\mathcal{B}} f'(u) v^j d_j \; dx = \int_{\mathcal{B}} f'(u) d_j^2 \; dx,$$

we have, by (2.4.5),

$$\begin{aligned}
\mathcal{E}_{s,L}(d_j, d_j) &= \mathcal{E}_{s,L}(d_j - v^j, d_j) \\
&= \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Big(\big(d_j(x) - v^j(x) - (d_j(y) - v^j(y))\big)\big(d_j(x) - d_j(y)\big) \Big) k(x-y) dx dy \\
&= \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Big(v^j(x) d_j(y) + v^j(y) d_j(x) - 2 d_j(x) d_j(y) \Big) k(x-y) \; dx dy
\end{aligned}$$

In the following, we put

$$\ell_j(x,y) := k(x-y) - k(\sigma_j(x)-y) \qquad \text{for } x,y \in \mathbb{R}^N,\ x \neq y.$$

Using the oddness of the functions v^j and d_j with respect to the reflection σ_j, we deduce that

$$\begin{aligned}
\mathcal{E}_{s,L}(d_j,d_j) &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{H_+^j}\Big(v^j(x)d_j(y)+v^j(y)d_j(x)-2d_j(x)d_j(y)\Big)\ell_j(x,y)\;dxdy \\
&= \frac{1}{2}\int_{H_+^j}\int_{H_+^j}\Big(v^j(x)d_j(y)+v^j(y)d_j(x)-2d_j(x)d_j(y)\Big)\big(\ell_j(x,y)-\ell_j(x,\sigma_j(y))\big)\;dxdy \\
&= \int_{H_+^j}\int_{H_+^j}\Big(v^j(x)d_j(y)+v^j(y)d_j(x)-2d_j(x)d_j(y)\Big)\ell_j(x,y)dxdy. \qquad (2.4.6)
\end{aligned}$$

Here we used in the last step that

$$k(\sigma_j(x)-\sigma_j(y)) = k(x-y) \qquad \text{and} \qquad k(\sigma_j(x)-y) = k(x-\sigma_j(y))$$

for $x,y \in \mathbb{R}^N$, $x \neq y$ and therefore

$$\begin{aligned}
\ell_j(x,y)-\ell_j(x,\sigma_j(y)) &= k(x-y)-k(\sigma_j(x)-y)-\big(k(x-\sigma_j(y))-k(\sigma_j(x)-\sigma_j(y))\big) \\
&= 2\ell_j(x,y).
\end{aligned}$$

Next, we note that

$$\ell_j(x,y) = k(x-y)-k(\sigma_j(x)-y) > 0 \qquad \text{for } x,y \in H_+^j. \qquad (2.4.7)$$

Moreover, we claim that the function

$$\begin{aligned}
(x,y) \mapsto h_j(x,y) &= v^j(x)d_j(y)+v^j(y)d_j(x)-2d_j(x)d_j(y) \\
&= (v^j(x)-d_j(x))d_j(y)+(v^j(y)-d^j(y))d_j(x)
\end{aligned}$$

satisfies

$$h_j \leq 0 \quad \text{and} \quad h_j \not\equiv 0 \qquad \text{on } H_+^j \times H_+^j. \qquad (2.4.8)$$

Indeed, if $\psi_0(1) \geq 0$, we have $d_j = (v^j)^+$ and therefore $v^j - d_j = -(v^j)^-$ on H_+^j. Hence (2.4.8) follows from (2.4.1). Moreover, if $\psi_0(1) < 0$, we have $d_j = -(v^j)^-$ and therefore $v^j - d_j = (v^j)^+$ on H_+^j. Again (2.4.8) follows from (2.4.1). The claim now follows by combining (2.4.6), (2.4.7) and (2.4.8). □

Lemma 2.4.2. *Let $\alpha = (\alpha_1,\dots,\alpha_N) \in \mathbb{R}^N$ and $d = \sum_{j=1}^N \alpha_j d_j$. Then we have*

$$\mathcal{E}_{s,L}(d,d) = \sum_{j=1}^N \alpha_j^2 \mathcal{E}_L(d_j,d_j) \leq 0.$$

Moreover,

$$\mathcal{E}_{s,L}(d,d) < 0 \qquad \text{if and only if} \qquad \alpha \neq 0, \qquad (2.4.9)$$

and therefore the functions $d_1,\dots,d_N$ are linearly independent.

Proof. We first note that

$$\mathcal{E}_{s,L}(d_j, d_k) = 0 \qquad \text{for } j,k \in \{1,\dots,N\},\ j \neq k. \tag{2.4.10}$$

Indeed, since u is radially symmetric, the function d_j is odd with respect to the reflection σ_j and even with respect to the reflection σ_k for $k \neq j$. Hence, by a change of variable,

$$\begin{aligned}
\mathcal{E}_{s,L}(d_j, d_k) &= \frac{c(N,s)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\Big(d_j(\sigma_j(x)) - d_j(\sigma_j(y))\Big)\Big(d_k(\sigma_j(x)) - d_k(\sigma_j(y))\Big)}{|\sigma_j(x) - \sigma_j(y)|^{N+2s}}\, dxdy \\
&\qquad - \int_{\mathcal{B}} f'(u(\sigma_j(x))) d_j(\sigma_j(x)) d_k(\sigma_j(x))\ dx \\
&= \frac{c(N,s)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\Big(d_j(y) - d_j(x)\Big)\Big(d_k(x) - d_k(y)\Big)}{|x-y|^{N+2s}}\, dxdy + \int_{\mathcal{B}} f'(u(x)) d_j(x) d_k(x)\ dx \\
&= -\mathcal{E}_{s,L}(d_j, d_k).
\end{aligned}$$

Hence (2.4.10) is true. Now, for $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$ and $d = \sum\limits_{j=1}^{N} \alpha_j d_j$, we have

$$\mathcal{E}_{s,L}(d,d) = \sum_{j=1}^{N} \alpha_j^2 \mathcal{E}_{s,L}(d_j, d_j) + \sum_{\substack{j,k=1 \\ j \neq k}}^{N} \alpha_j \alpha_k \mathcal{E}_{s,L}(d_j, d_k) = \sum_{j=1}^{N} \alpha_j^2 \mathcal{E}_{s,L}(d_j, d_j) \leq 0$$

by (2.4.10) and Lemma 2.4.1. Moreover, if $\alpha \neq 0$, it follows from Lemma 2.4.1 that $\mathcal{E}_{s,L}(d,d) < 0$, which in particular implies that $d \neq 0$. Consequently, the functions $d_1, \dots, d_N$ are linearly independent, as claimed. □

Lemma 2.4.3. *The first eigenvalue $\lambda_{1,L}$ of the operator $L = (-\Delta)^s - f'(u)$ is simple, and the corresponding eigenspace is spanned by radially symmetric eigenfunction $\varphi_{1,L}$. Furthermore,*

$$\mathcal{E}_{s,L}(d_j, \varphi_{1,L}) = 0 \quad \textit{for } j = 1, 2, \dots, N \qquad \textit{and} \qquad \lambda_{1,L} = \mathcal{E}_{s,L}(\varphi_{1,L}, \varphi_{1,L}) < 0.$$

Proof. The simplicity of $\lambda_{1,L}$ and the radial symmetry of $\varphi_{1,L}$ are well known, but we recall the proof for the convenience of the reader. The variational characterization of $\lambda_{1,L}$ is given by

$$\lambda_{1,L} = \inf_{v \in \mathcal{H}_0^s(\mathcal{B}) \setminus \{0\}} \frac{\mathcal{E}_{s,L}(v,v)}{\|v\|^2_{L^2(\mathcal{B})}} = \inf_{M} \mathcal{E}_{s,L}(v,v) \quad \text{with} \quad M = \{v \in \mathcal{H}_0^s(\mathcal{B}) : \|v\|_{L^2(\mathcal{B})} = 1\},$$

and the associated minimizers $\varphi \in M$ are precisely the L^2-normalized eigenfunctions of L corresponding ot $\lambda_{1,L}$, i.e., the L^2-normalized (weak) solutions of

$$L\varphi = \lambda_{1,L}\varphi \quad \text{in } \mathcal{B}, \qquad \varphi \equiv 0 \quad \text{in } \mathbb{R}^N \setminus \mathcal{B}. \tag{2.4.11}$$

Moreover, if $\varphi \in M$ is such a minimizer, then also $|\varphi| \in M$ and

$$\lambda_{1,L} = \mathcal{E}_{s,L}(\varphi, \varphi) \geq \mathcal{E}_{s,L}(|\varphi|, |\varphi|) \geq \inf_{M} \mathcal{E}_{s,L}(v,v) = \lambda_{1,L},$$

which implies that $|\varphi|$ is also a minimizer and therefore a weak solution of (2.4.11). By the strong maximum principle for nonlocal operators (see e.g. [25, p.312–313] or [108]), $|\varphi|$ is strictly positive in $\mathcal{B}$. Consequently, every eigenfunction φ of L is either strictly positive or strictly negative in $\mathcal{B}$. Consequently, $\lambda_{1,L}$ does not admit two L^2-orthogonal eigenfunctions, and therefore $\lambda_{1,L}$ is simple.

Next we note that, by a simple change of variable, if φ is an eigenfunction of L corresponding to $\lambda_{1,L}$, then also $\varphi \circ \mathcal{R}$ is an eigenfunction for every rotation $\mathcal{R} \in O(N)$. Consequently, the simplicity of $\lambda_{1,L}$ implies that the associated eigenspace is spanned by a radially symmetric eigenfunction $\varphi_{1,L}$.

Next, using the radially symmetry of u and $\varphi_{1,L}$ and the oddness of d_j with respect to the reflection σ_j, we find, by a change of variable, that

$$\begin{aligned}\mathcal{E}_{s,L}(d_j,\varphi_{1,L}) &= \frac{c(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(d_j(\sigma_j(x))-d_j(\sigma_j(y)))(\varphi_{1,L}(\sigma_j(x))-\varphi_{1,L}(\sigma_j(x)))}{|x-y|^{N+2s}}dxdy\\ &\qquad - \int_{\mathcal{B}} f'(u(\sigma_j(x)))d_j(\sigma_j(x))\varphi_{1,L}(\sigma_j(x))\;dx\\ &= \frac{c(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(d_j(y)-d_j(x))(\varphi_{1,L}(x)-\varphi_{1,L}(y))}{|x-y|^{N+2s}}\;dxdy + \int_{\mathcal{B}} f'(u(x))d_j(x)\varphi_{1,L}(x)dx\\ &= -\mathcal{E}_{s,L}(d_j,\varphi_{1,L})\end{aligned}$$

and therefore $\mathcal{E}_{s,L}(d_j,\varphi_{1,L}) = 0$ for $j = 1,\dots,N$. Finally, by Lemma 2.4.1 and the variational characterization of $\lambda_{1,L}$, we have $\lambda_{1,L} = \mathcal{E}_{s,L}(\varphi_{1,L},\varphi_{1,L}) < 0$, as claimed. □

Proof of Theorem 2.1.1(completed). Let $\varphi_{1,L} \in \mathcal{H}^s_0(\mathcal{B})$ be an eigenfunction of L corresponding to the first eigenvalue $\lambda_{1,L}$ as given in Lemma 2.4.3. We consider the subspace $V =$ span$\{\varphi_{1,L}, d_1, \dots, d_N\}$. For $\alpha \in \mathbb{R}^{N+1}\setminus\{0\}$ and $d = \alpha_0\varphi_{1,L} + \sum\limits_{j=1}^{N}\alpha_j d_j \in V$, we then have, by Lemma 2.4.2 and Lemma 2.4.3,

$$\mathcal{E}_{s,L}(d,d) = \alpha_0^2\,\mathcal{E}_{s,L}(\varphi_{1,L},\varphi_{1,L}) + \mathcal{E}_{s,L}(\sum_{j=1}^{N}\alpha_j d_j, \sum_{j=1}^{N}\alpha_j d_j) < 0.$$

In particular, it follows that the functions $\varphi_{1,L}, d_1, \dots, d_N$ are linearly independent and therefore V is $N+1$-dimensional. By (2.4.3) and the compactness of $S_V = \{v \in V \;:\; \|v\|_{L^2(\mathcal{B})} = 1\}$, it then follows that $\lambda_{N+1,L} < 0$, which means that u has Morse index greater than or equal to $N+1 \geq 2$, as claimed. □

2.5 The linear case

In this section we discuss the linear eigenvalue problem (2.1.7) and complete the proof of Theorem 2.1.2. In particular, we wish to recall a useful characterization of eigenvalues and eigenfunctions of (2.1.7) derived in [72]. For this we need to consider the following radially symmetric version of (2.1.7) in general dimensions $d \in \mathbb{N}$:

$$\begin{cases}(-\Delta)^s u = \lambda u & \text{in } \;\mathcal{B} \subset \mathbb{R}^d\\ u \in \mathcal{H}^s_0(\mathcal{B}), & u \text{ radially symmetric.}\end{cases} \tag{2.5.1}$$

In the following, we let $\lambda_{d,0} < \lambda_{d,1} \leq \dots$ denote the increasing sequence of eigenvalues of this problem (counted with multiplicity).

The following characterization is essentially a reformulation of [72, Proposition 1.1].

Proposition 2.5.1. *The eigenvalues of (2.1.7) in $\mathcal{B} \subset \mathbb{R}^N$ are of the form $\lambda = \lambda_{N+2\ell,n}$ with integers $\ell, n \geq 0$. Moreover, if*

$$Z_\lambda := \{(\ell, n) \, : \, \lambda_{N+2\ell,n} = \lambda\},$$

then the eigenspace corresponding to λ is spanned by functions of the form $u(x) = V_\ell(x)\varphi_{N+2\ell,n}(|x|)$, where $(\ell, n) \in Z_\lambda$, V_ℓ is a solid harmonic polynomial of degree ℓ and $x \mapsto \varphi_{N+2\ell,n}(|x|)$ is a (radial) eigenfunction of the problem (2.5.1) in dimension $d = N + 2\ell$ corresponding to the eigenvalue $\lambda_{N+2\ell,n}$.

Here and in the following, a solid harmonic polynomial V of degree ℓ is a function of the form $V(x) = |x|^\ell Y(\frac{x}{|x|})$, where Y is a spherical harmonic of degree ℓ. Hence $V : \mathbb{R}^N \to \mathbb{R}$ is a homogenous polynomial of degree ℓ satisfying $\Delta V = 0$.

Regarding the eigenvalues $\lambda_{d,n}$ of (2.5.1), it is also proved in [72, Section 3] that

$$\text{the sequence } (\lambda_{d,0})_d \text{ is strictly increasing in } d \geq 1. \tag{2.5.2}$$

Moreover,

$$\lambda_{d,n} > \lambda_{d,0} \qquad \text{for every } d, n \geq 1 \tag{2.5.3}$$

by the simplicity of the first eigenvalue of (2.5.1). Consequently, the first eigenvalue λ_1 of (2.1.7) equals $\lambda_{N,0}$, whereas the second eigenvalue λ_2 of (2.1.7) is given as the minimum of $\lambda_{N+2,0}$ and $\lambda_{N,1}$.

Theorem 2.1.2 is now a direct consequence of the following result, which we will derive from Theorem 2.1.1 and from the observations above.

Theorem 2.5.2. *We have $\lambda_{N+2,0} < \lambda_{N,1}$. Consequently, the second eigenvalue λ_2 of (2.1.7) is given by $\lambda_{N+2,0}$, and every corresponding eigenfunction u is antisymmetric, i.e., it satisfies $u(-x) = -u(x)$ for every $x \in \mathcal{B}$.*

Proof. Suppose by contradiction that $\lambda_2 = \lambda_{N,1} \leq \lambda_{N+2,0}$. Then, noting that the only solid harmonic polynomials of degree zero are the constants, it follows from Proposition 2.5.1 that (2.1.7) admits a radially symmetric eigenfunction corresponding to λ_2. But then u is a radially symmetric sign changing solution of (2.1.1) with $t \mapsto f(t) = \lambda_2 t$, so it must have Morse index greater than or equal to $N + 1$. This contradicts the fact that λ_2 is the second eigenvalue.
We thus conclude that $\lambda_2 = \lambda_{N+2,0} < \lambda_{N,1}$. Combining this inequality with (2.5.2) and (2.5.3), we then deduce that $Z_{\lambda_2} = \{(1, 0)\}$, and therefore the eigenspace corresponding to λ_2 is spanned by functions of the form $x \mapsto V_1(x)\varphi_{N+2,0}(|x|)$, where V_1 is a solid harmonic polynomial of degree one, hence a linear function, and $x \mapsto \varphi_{N+2,0}(|x|)$ is an eigenfunction of the problem (2.5.1) in dimension $d = N + 2$ corresponding to the eigenvalue $\lambda_{N+2,0}$. Since every such function is antisymmetric, the claim follows. ☐

Chapter 3

The eigenvalue problem for the regional fractional Laplacian in the small order limit

In this chapter, we try to understand the behavior of eigenvalues of the regional fractional Laplacian as the fractional parameter is close to zero. The presentation below obeys the original paper [R2]. This is a collaboration with Tobias Weth. We notice that the notation may be different from the one of the previous chapters.

3.1 Introduction and main results

In recent decades, the study of nonlocal operators has been an active area of research in different branches of mathematics. In particular, these operators are used to model problems in which different length scales are involved. In this work, we study the regional fractional Laplace operator of order s, which we will denote by $(-\Delta)^s_\Omega$, where, here and in the following, $\Omega \subset \mathbb{R}^N$ is a bounded open set with Lipschitz boundary. This operator is known to be the infinitesimal generator of the so-called censored stable Lévy processes and has received extensive attention in this context in recent years, see e.g. see [24, 48, 97–99] and the references therein. The censored stable process is a jump process restricted to the underlying open set Ω, so it only involves jumps from points in Ω to points in Ω. From the point of view of partial differential equations, equations involving the regional fractional Laplacian arise as nonlocal, lower order variants of elliptic second order equations on Ω with homogeneous Neumann boundary conditions, see e.g. [8] and [76, Theorem 1.1].

If the underlying open set Ω equals $\mathbb{R}^N$, then $(-\Delta)^s_\Omega$ coincides with the standard fractional Laplacian $(-\Delta)^s$. Recently, Chen and the second author [55] have studied Dirichlet problems for the Logarithmic Laplacian operator L_Δ, which arises as formal derivative $\partial_s\big|_{s=0}(-\Delta)^s$. In particular, they provide a relationship between the first non-zero Dirichlet eigenvalue of $(-\Delta)^s$ on Ω with that of L_Δ. More precisely, denoting by $\lambda_1^s(\Omega)$ resp. $\lambda_1^L(\Omega)$ the first non-zero Dirichlet eigenvalue of $(-\Delta)^s$ with corresponding L^2-normalized eigenfunction u_s and L_Δ with corresponding L^2-normalized eigenfunction ξ_1, respectively, they have shown that $\lambda_1^L(\Omega) = \frac{d}{ds}|_{s=0}\lambda_1^s(\Omega)$ and $u_s \to \xi_1$ in $L^2(\Omega)$ as $s \to 0^+$. Related results for higher eigenvalues and eigenfunctions, including refined uniform regularity results and uniform convergence estimates, have been obtained more recently in [82]. The main aim of this work is to establish analogous results in the case of the regional fractional Laplacian. As a motivation, we mention order-dependent optimization problems arising e.g. in image processing [9] and population dynamics [126, 139]. In many of these problems the optimal order s is small. Hence the small order limit $s \to 0^+$ in s-dependent operator equations arises as a natural object of interest and has even been studied even in the framework of nonlinear problems recently [102].

To state our main results, we need to introduce some notation. Let $s \in (0,1)$. The regional fractional Laplacian $(-\Delta)^s_\Omega u$ of a function $u \in L^1(\Omega)$ is defined at a point $x \in \Omega$ by

$$(-\Delta)^s_\Omega u(x) = c_{N,s}\mathcal{D}^s_\Omega u(x) \tag{3.1.1}$$

with

$$\mathcal{D}^s_\Omega u(x) = \text{P.V.}\int_\Omega \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy = \lim_{\varepsilon\to 0^+}\int_{\Omega\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy, \tag{3.1.2}$$

provided that the limit exists. Here the normalization constant $c_{N,s}$ coincides with the one of the fractional Laplacian and is given by

$$c_{N,s} := \frac{s4^s\Gamma(\frac{N+2s}{2})}{\pi^{\frac{N}{2}}\Gamma(1-s)} = \frac{s(1-s)4^s\Gamma(\frac{N+2s}{2})}{\pi^{\frac{N}{2}}\Gamma(2-s)}. \tag{3.1.3}$$

As a consequence, we have

$$(-\Delta)^s_\Omega u(x) = (-\Delta)^s u(x) - \kappa_{\Omega,s}(x)u(x) \qquad \text{with} \qquad \kappa_{\Omega,s}(x) = c_{N,s}\int_{\mathbb{R}^N\setminus\Omega}|x-y|^{-N-2s}\,dy \tag{3.1.4}$$

for $u \in L^1(\Omega)$ and $x \in \Omega$ whenever the limit in (3.1.2) exists. Here we identify u with its trivial extension on $\mathbb{R}^N$ to compute $(-\Delta)^s u(x)$.

It is important to note here that the definition of the renormalized operator $\mathcal{D}^s_\Omega$ in (3.1.2) extends to the case $s = 0$. More importantly, we shall see in our first preliminary result that the family of operators $\mathcal{D}^s_\Omega$, $s \in [0,1)$ can be expanded, in a suitable strong sense, as a convergent power series in the fractional order s at $s = 0$.

Theorem 3.1.1. *Let Ω be a bounded open Lipschitz set in $\mathbb{R}^N$, and $\alpha \in (0,1)$. Then we have*

$$\mathcal{D}^s_\Omega u = \mathcal{D}^0_\Omega u + \sum_{k=1}^\infty s^k\mathcal{D}_k u \quad \text{for } u \in C^\alpha(\overline{\Omega}) \text{ and } s \in (0,\frac{\alpha}{2}), \tag{3.1.5}$$

where, for $k \in \mathbb{N}$, $\mathcal{D}_k u \in C(\overline{\Omega})$ is defined by

$$[\mathcal{D}_k u](x) = (-1)^k 2^k \int_\Omega \frac{u(x)-u(y)}{|x-y|^N}\log^k(|x-y|)\,dy. \tag{3.1.6}$$

Here the series on the RHS of (3.1.5) converges in $L^\infty(\Omega)$, and the convergence is uniform if s is taken from a compact subset of $[0,\frac{\alpha}{2})$ and u is taken from a bounded subset of $C^\alpha(\overline{\Omega})$.

Since

$$c_{N,s} := sc_N + o(s) \quad \text{as} \quad s \to 0^+, \qquad \text{with} \quad c_N := \pi^{-\frac{N}{2}}\Gamma(\frac{N}{2}), \tag{3.1.7}$$

the following is a direct corollary of Theorem 3.1.1.

Corollary 3.1.2. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded Lipschitz set and $\alpha \in (0,1)$. For $u \in C^\alpha(\overline{\Omega})$, we then have*

$$(-\Delta)^s_\Omega u = sL^\Omega_\Delta u + o(s) \quad \text{in } L^\infty(\Omega) \text{ as } s \to 0^+, \tag{3.1.8}$$

where

$$\left[L^\Omega_\Delta u\right](x) := c_N \mathcal{D}^0_\Omega u(x) = c_N \int_\Omega \frac{u(x)-u(y)}{|x-y|^N}\, dy, \quad x \in \Omega. \tag{3.1.9}$$

Moreover, the expansion in (3.1.8) is uniform in bounded subsets of $C^\alpha(\overline{\Omega})$.

In analogy to the work [55], we call $L^\Omega_\Delta = c_N \mathcal{D}^0_\Omega$ the *regional logarithmic Laplacian* on Ω. So Corollary 3.1.2 states that the nonlocal operator L^Ω_Δ arises as formal derivative $\partial_s\big|_{s=0}(-\Delta)^s_\Omega$ of regional fractional Laplacians at $s = 0$. As we shall see now in our second main result, this operator arises naturally when studying the asymptotic behavior of eigenvalues and eigenfunctions of $(-\Delta)^s_\Omega$ for s close to 0.

Theorem 3.1.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open Lipschitz set, let $n \in \mathbb{N}$, and let $\mu^\Omega_{n,s}$ resp. $\mu^\Omega_{n,0}$ denote the n-th eigenvalues of the operators $(-\Delta)^s_\Omega$, L^Ω_Δ in increasing order, respectively. Then we have*

$$\mu^\Omega_{n,s} \to 0 \quad \text{as } s \to 0^+ \qquad \text{and} \qquad \frac{d}{ds}\Big|_{s=0} \mu^\Omega_{n,s} = \lim_{s\to 0^+} \frac{\mu^\Omega_{n,s}}{s} = \mu^\Omega_{n,0}.$$

Moreover, if, for some sequence $s_k \to 0^+$, $\{\xi_{n,s_k}\}_k$ is a sequence of L^2-normalized eigenfunctions of $(-\Delta)^{s_k}_\Omega$ corresponding to μ^Ω_{n,s_k}, then $\xi_{n,s_k} \in C(\overline{\Omega})$ for every $k \in \mathbb{N}$ and

$$\xi_{n,s_k} \to \xi_n \quad \text{uniformly in } \overline{\Omega},$$

where ξ_n is an eigenfunction of L^Ω_Δ corresponding to $\mu^\Omega_{n,0}$.

We stress that, here and in the following, an open bounded set $\Omega \subset \mathbb{R}^N$ will be called a Lipschitz set if every point $p \in \partial\Omega$ has an open neighborhood $N_p \subset \mathbb{R}^N$ with the property that $\partial\Omega \cap N_p$ can be written as the graph of a Lipschitz function after a suitable rotation. In the literature, this is sometimes called a strongly Lipschitz set.

The main difficulty in the proof of Theorem 3.1.3 is the lack of boundedness and regularity estimates for the renormalized regional fractional Laplacian $\mathcal{D}^s_\Omega$ which are *uniform in $s \in (0,1)$*. In fact, even for fixed $s \in (0,1)$, the elliptic boundary regularity theory for this operator has only been developed very recently with regularity estimates containing s-dependent constants, see [14,73,76]. For the proof of Theorem 3.1.3, we need to consider uniform L^∞-estimates related to the operator family $\mathcal{D}^s_\Omega$, $s \in [0,1)$ first. In this context, we note the following result of possible independent interest.

Theorem 3.1.4. *Let $s \in [0,1)$, let $\Omega \subset \mathbb{R}^N$ be a bounded open Lipschitz set, let $V, f \in L^\infty(\Omega)$, and let u be a weak solution of the problem*

$$\mathcal{D}^s_\Omega u + V(x)u = f \quad \text{in} \quad \Omega. \tag{3.1.10}$$

Then $u \in L^\infty(\Omega)$, and there exists a constant $c_0 = c_0(N, \Omega, \|V\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(\Omega)}, \|u\|_{L^2(\Omega)}) > 0$ independent of s with the property that $\|u\|_{L^\infty(\Omega)} \le c_0$ in Ω.

For the notion of weak solution, see Section 3.3. While the uniform boundedness of the sequence $(\xi_{n,s_k})_k$ in Theorem 3.1.3 follows rather directly from Theorem 3.1.4, it is more difficult to see that this sequence is equicontinous on $\overline{\Omega}$. We shall prove this fact in Theorem 3.5.5 below based on a series of relative oscillation estimates and a contradiction argument.

In view of Theorem 3.1.3, it is natural to ask for upper and lower bounds for the eigenvalues of L_Δ^Ω depending on Ω. This remains an open problem. In the case of the standard fractional Laplacian, upper and lower bounds have been obtained recently in [114] by means of Fourier analysis and Faber-Krahn type estimates. We believe that different methods have to be developed to tackle the problem for the regional logarithmic Laplacian.

The article is organized as follows. In Section 3.2, we introduce some notation and give the proof of Theorem 3.1.1. In Section 3.3, we present the functional analytic framework for Poisson problem for the operator family $\mathcal{D}_\Omega^s$ and the associated eigenvalue problem. In Section 3.4, we first derive a one-sided uniform estimate for subsolutions of equations of the type $\mathcal{D}_\Omega^s u + V(x)u = f$ in Ω with potential $V \in L^\infty(\Omega)$ and source function $f \in L^\infty(\Omega)$. As a corollary of this uniform estimate, we then derive Theorem 3.1.4. Finally, in Section 3.5, we complete the proof of Theorem 3.1.3.

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3.2 Preliminaries and proof of Theorem 3.1.1

In this section, we first introduce some notation. After that, we will give the proof of Theorem 3.1.1.

For an arbitrary subset $A \subset \mathbb{R}^N$, we denote by $|A|$ resp. χ_A the N-dimensional Lebesgue measure and the characteristic function of A, respectively. Moreover, we let $d_A := \sup\{|x-y| : x, y \in A\}$ denote the diameter of A. For $x \in \mathbb{R}^N$, $r > 0$, $B_r(x)$ denotes the open ball centered at x with radius r, and $B_r := B_r(0)$. Given a function $u : A \to \mathbb{R}$, $A \subset \mathbb{R}^N$, we denote by $u^+ := \max\{u, 0\}$ resp. $u^- := -\min\{u, 0\}$ the positive and negative part of u, respectively.

Throughout the remainder of the paper, $\Omega \subset \mathbb{R}^N$ always denotes a bounded open Lipschitz set. For a function $u \in C^\alpha(\overline{\Omega})$, we put

$$[u]_{\alpha,x} := \sup_{y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \qquad \text{for } x \in \Omega$$

and

$$[u]_\alpha := \sup_{x\in\Omega} [u]_{\alpha,x}, \qquad \|u\|_{C^\alpha} := \|u\|_{L^\infty(\Omega)} + [u]_\alpha.$$

We may now give the

Proof of Theorem 3.1.1. We first note that

$$r^{-2s} = e^{-2s\ln r} = \sum_{k=0}^{\infty} \frac{(-2\ln r)^k}{k!} s^k \qquad \text{for } r > 0 \tag{3.2.1}$$

We now fix $u \in C^\alpha(\overline{\Omega})$, $x \in \Omega$ and $s \in (0, \frac{\alpha}{2})$. Moreover we define, for $s \in (0,1)$,

$$f : \Omega \setminus \{x\} \to \mathbb{R}, \qquad f(y) := \frac{u(x)-u(y)}{|x-y|^{N+2s}}. \tag{3.2.2}$$

By (3.2.1), we have $f(y) = \sum\limits_{k=0}^{\infty} s^k f_k(y)$ for $y \in \Omega \setminus \{x\}$ with

$$f_k : \Omega \setminus \{x\} \to \mathbb{R}, \qquad f_k(y) := \frac{2^k}{k!}(u(x) - u(y))(-\ln|x-y|)^k |x-y|^{-N}.$$

Next we choose $R > 0$ such that $\Omega \subset B_R(x)$ for every $x \in \Omega$, and we note that

$$\begin{aligned}
&\int_\Omega |f_k(y)|\,dy \le \frac{2^k}{k!}[u]_\alpha \int_\Omega |x-y|^{\alpha-N} \big| \log^k |x-y| \big| \,dy \\
&\le \frac{2^k}{k!}[u]_\alpha \Big((-1)^k \int_{B_1} |z|^{\alpha-N} \log^k |z|\,dz + \int_{B_R \setminus B_1} |z|^{\alpha-N} \log^k |z|\,dz \Big) \\
&\le \frac{2^k}{k!}[u]_\alpha |S^{N-1}| \Big((-1)^k \int_0^1 r^{\alpha-1} \log^k r\,dr + R^\alpha \log^k R \Big)
\end{aligned}$$

Since

$$(-1)^k \int_0^1 r^{\alpha-1} \log^k r\,dr = \int_0^\infty t^k e^{-\alpha t} dt = \alpha^{-k-1} \int_0^\infty t^k e^{-t}\,dt = \frac{k!}{\alpha^{k+1}},$$

we thus find that

$$\int_\Omega |f_k(y)|\,dy \le [u]_\alpha\, c_k \qquad \text{with} \quad c_k = \frac{2^k}{\alpha^{k+1}} + \frac{R^\alpha (2\log R)^k}{k!}.$$

Since $\limsup\limits_{k\to\infty} (c_k)^{\frac{1}{k}} = \frac{2}{\alpha} < \frac{1}{s}$ by assumption, we conclude that

$$\sum_{k=j}^{\infty} \Big(\int_\Omega |f_k(y)|\,dy \Big) s^k \le [u]_\alpha d_j(s) \qquad \text{with} \quad d_j(s) := \sum_{k=j}^{\infty} c_k s^k < \infty$$

for $j \in \mathbb{N}$. Hence the function $g := \sum\limits_{k=0}^{\infty} s^k |f_k|$ is integrable on Ω. Since

$$\Big| \sum_{k=0}^{j} f_k \Big| \le g \qquad \text{in } \Omega \setminus \{x\} \text{ for every } j \in \mathbb{N},$$

it thus follows from the dominated convergence theorem that

$$\mathcal{D}^s_\Omega u(x) = \int_\Omega \Big(\sum_{k=0}^{\infty} s^k f_k(y) \Big) dy = \sum_{k=0}^{\infty} s^k \int_\Omega f_k(y)\,dy = \mathcal{D}^0_\Omega u(x) + \sum_{k=1}^{\infty} s^k [\mathcal{D}_k u](x),$$

where $[\mathcal{D}_k u](x)$ is defined in (3.1.6). This holds for every $x \in \Omega$. Moreover,

$$\Big| \mathcal{D}^s_\Omega u(x) - \mathcal{D}^0_\Omega u(x) - \sum_{k=1}^{j-1} s^k [\mathcal{D}_k u](x) \Big| \le \sum_{k=j}^{\infty} \Big(\int_\Omega |f_k(y)|\,dy \Big) s^k \le [u]_\alpha\, d_j(s)$$

for $x \in \Omega$, $u \in C^\alpha(\overline{\Omega})$, where $d_j(s) \to 0$ as $j \to \infty$. Consequently, the series expansion holds in $L^\infty(\Omega)$ and uniformly for u taken from a bounded subset of $C^\alpha(\overline{\Omega})$. $\square$

3.3 Functional setting for the Poisson problem and the eigenvalue problem

In this section, we discuss the variational framework for the study of weak solutions to the Poisson problems

$$\mathcal{D}^s_\Omega u = f \quad \text{in} \quad \Omega, \tag{3.3.1}$$

related to the operator family $\mathcal{D}^s_\Omega$ for $s \in [0,1)$ and $f \in L^2(\Omega)$. Here and throughout this section, $\Omega \subset \mathbb{R}^N$ is a bounded open Lipschitz set. The variational framework for this problem is well-known for $s \in (0,1)$, and some aspects have also been studied recently in a setting related to the case $s = 0$, see e.g. [57]. Since we need additional properties which are not addressed in the present literature, we give a unified account for general $s \in [0,1)$ in the following.

Let us denote by $\langle \cdot,\cdot \rangle_2$ the usual scalar product in $L^2(\Omega)$, i.e. $\langle u,v \rangle_2 = \int_\Omega uv\, dx$ for $u,v \in L^2(\Omega)$. We define the space $L^2_0(\Omega)$ consisting of functions $u \in L^2(\Omega)$ with zero average over Ω, i.e.

$$L^2_0(\Omega) := \Big\{ u \in L^2(\Omega) : \int_\Omega u\, dx = 0 \Big\}.$$

Moreover, we put

$$\mathbb{H}^s(\Omega) := \Big\{ u \in L^2(\Omega) \ : \ \int_\Omega \int_\Omega \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\, dxdy < \infty \Big\}.$$

Then

$$\mathcal{E}_s(u,v) := \frac{1}{2} \int_\Omega \int_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy \tag{3.3.2}$$

is well-defined for functions $u, v \in \mathbb{H}^s(\Omega)$. We have the following.

Proposition 3.3.1. *Let $s \in [0,1)$.*
(i) $\mathbb{H}^s(\Omega)$ is a Hilbert space with inner product

$$\langle u,v \rangle_{\mathbb{H}^s(\Omega)} := \langle u,v \rangle_2 + \mathcal{E}_s(u,v);$$

(ii) Moreover, $\mathbb{H}^s(\Omega)$ is compactly embedded into $L^2(\Omega)$.

Before given the proof of this Proposition, we first recall that, for $s \in (0,1)$, the space $\mathbb{H}^s(\Omega)$ coincides, by definition, with the usual fractional Sobolev space $H^s(\Omega)$. For $s \in (0, \frac{1}{2})$ this space can be identified, by trivial extension, with the space $\mathcal{H}^s_0(\Omega)$ of all functions $u \in H^s(\mathbb{R}^N)$ with $u \equiv 0$ on $\mathbb{R}^N \setminus \Omega$, see e.g. [94, Chapter 1]. This is a consequence of the fractional boundary Hardy inequality. For the case $s = 0$, we have the following related property.

Lemma 3.3.2. *Let $\mathcal{H}(\Omega)$ be the space of all measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ with $u \equiv 0$ on $\mathbb{R}^N \setminus \Omega$ and*

$$\iint_{\substack{x,y \in \mathbb{R}^N \\ |x-y| \le 1}} \frac{(u(x)-u(y))^2}{|x-y|^N}\, dxdy < \infty,$$

endowed with the norm

$$\|u\|_{\mathcal{H}(\Omega)} = \Big(\frac{1}{2} \iint_{\substack{x,y \in \mathbb{R}^N \\ |x-y| \le 1}} \frac{(u(x)-u(y))^2}{|x-y|^N}\, dxdy \Big)^{\frac{1}{2}},$$

Then, by trivial extension, the space $\mathbb{H}^0(\Omega)$ is isomorphic to $\mathcal{H}(\Omega)$, so there exists a constant $C > 0$ with

$$\frac{1}{C}\|u\|_{\mathcal{H}(\Omega)} \leq \|u\|_{\mathbb{H}^0(\Omega)} \leq C\|u\|_{\mathcal{H}(\Omega)} \qquad \textit{for } u \in \mathbb{H}^0(\Omega),$$

where we identify a function u on Ω with its trivial extension to $\mathbb{R}^N$.

We note that the space $\mathcal{H}(\Omega)$ has been introduced in [55] as the form domain for Dirichlet problems for the logarithmic Laplacian.

Proof. Let $u \in \mathbb{H}^0(\Omega)$. In the following, $C > 0$ stands for a constant which may change its value from line to line but does not depend on u. We first note that

$$\begin{aligned}
\mathcal{E}_0(u,u) &= \frac{1}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))^2}{|x-y|^N}\,dxdy \\
&\leq \frac{1}{2}\iint_{\substack{x,y\in\mathbb{R}^N\\|x-y|\leq 1}} \frac{(u(x)-u(y))^2}{|x-y|^N}\,dxdy + \frac{1}{2}\iint_{\substack{x,y\in\Omega\\|x-y|>1}} \frac{(u(x)-u(y))^2}{|x-y|^N}\,dxdy \\
&\leq \|u\|^2_{\mathcal{H}(\Omega)} + \kappa_{max}\|u\|^2_{L^2(\Omega)} \qquad \text{with} \quad \kappa_{max} := 2\max_{x\in\overline{\Omega}} \int_{\Omega\setminus B_1(x)} |x-y|^{-N}\,dy.
\end{aligned}$$

Since $\|u\|_{L^2(\Omega)} \leq C\|u\|_{\mathcal{H}(\Omega)}$ for all $u \in \mathcal{H}(\Omega)$ e.g. by [80, Lemma 2.7], we conclude that

$$\|u\|^2_{\mathbb{H}^0(\Omega)} = \mathcal{E}_0(u,u) + \|u\|^2_{L^2(\Omega)} \leq C\|u\|^2_{\mathcal{H}(\Omega)}.$$

The opposite inequality will be derived from the logarithmic boundary Hardy inequality given in [55, Corollary 6.2.], which states that there exists a constant $C(\Omega) > 0$ with the property that

$$\int_\Omega c_\Omega(x)u^2(x)\,dx \leq C\Big(\frac{1}{2}\int_\Omega\int_\Omega (u(x)-u(y))^2 J(x-y)\,dxdy + \|u\|^2_{L^2(\Omega)}\Big) \qquad \text{for } u \in \mathcal{H}(\Omega) \quad (3.3.3)$$

with the kernel J given by $J(z) := c_N\chi_{B_1}(z)|z|^{-N}$ for $z \in \mathbb{R}^N \setminus \{0\}$ and

$$c_\Omega(x) = \int_{B_1(x)\setminus\Omega} |x-y|^{-N}\,dy.$$

It follows from (3.3.3) that

$$\begin{aligned}
\|u\|^2_{\mathcal{H}(\Omega)} &\leq \mathcal{E}_0(u,u) + \int_\Omega c_\Omega(x)u^2(x)\,dx \\
&\leq \mathcal{E}_0(u,u) + C\Big(\frac{1}{2}\int_\Omega\int_\Omega (u(x)-u(y))^2 J(x-y)\,dxdy + \|u\|^2_{L^2(\Omega)}\Big) \\
&\leq C\Big(\mathcal{E}_0(u,u) + \|u\|^2_{L^2(\Omega)}\Big) \leq C\|u\|^2_{\mathbb{H}^0(\Omega)}.
\end{aligned}$$

The proof is thus finished. $\square$

We may now complete the

Proof of Proposition 3.3.1. The proof is well-known for $s > 0$, so we restrict our attention to the case $s = 0$ in the following.

(i) Obviously, $\langle \cdot, \cdot \rangle_{\mathbb{H}^0(\Omega)}$ is a scalar product in $\mathbb{H}^0(\Omega)$. In the following, we prove that $\mathbb{H}^0(\Omega)$ is complete for the norm $\| \cdot \|_{\mathbb{H}^0(\Omega)} := \sqrt{\langle \cdot, \cdot \rangle_{\mathbb{H}^0(\Omega)}}$. Let $\{u_n\}_n$ be a Cauchy sequence with respect to this norm, and set

$$v_n(x, y) := \frac{1}{\sqrt{2}}(u_n(x) - u_n(y))|x - y|^{-\frac{N}{2}}.$$

Since $L^2(\Omega)$ is complete, $u_n \to u$ in $L^2(\Omega)$. Passing to a subsequence, we may thus assume that u_n converges a.e. to u on Ω and therefore v_n converges a.e. on $\Omega \times \Omega$ to the function

$$v(x, y) = (u(x) - u(y))|x - y|^{-\frac{N}{2}}.$$

Now, by Fatou's lemma, we have that

$$\int_\Omega \int_\Omega |u(x) - u(y)|^2 |x - y|^{-N} \, dxdy \leq \liminf_{n \to \infty} \int_\Omega \int_\Omega |v_n(x, y)|^2 \, dxdy = \liminf_{n \to \infty} \|u_n\|^2_{\mathbb{H}^0(\Omega)} < \infty,$$

since the sequence $(u_n)_n$ is bounded in $\mathbb{H}^0(\Omega)$. Hence $u \in \mathbb{H}^0(\Omega)$. Applying again Fatou's lemma, we find that

$$\begin{aligned} \mathcal{E}_0(u_n - u, u_n - u) &= \int_\Omega \int_\Omega |v_n(x, y) - v(x, y)|^2 \, dxdy \leq \liminf_{m \to \infty} \int_\Omega \int_\Omega |v_n(x, y) - v_m(x, y)|^2 \, dxdy \\ &= \liminf_{m \to \infty} \|u_n - u_m\|^2_{\mathbb{H}^0(\Omega)} \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Since we have already seen that $u_n \to u$ in $L^2(\Omega)$, it follows that $u_n \to u$ in $\mathbb{H}^0(\Omega)$. Hence, we infer that $\mathbb{H}^0(\Omega)$ is complete and therefore is a Hilbert space.
(ii) This merely follows from the fact that, as noted in Lemma 3.3.2, the space $\mathbb{H}^0(\Omega)$ is isomorphic to $\mathcal{H}(\Omega)$ by trivial extension, and the space $\mathcal{H}(\Omega)$ is compactly embedded into $L^2(\Omega)$ by [57, Theorem 2.1.].

□

Remark 3.3.3. (i) The space $C_c^\infty(\Omega)$ is dense in $\mathbb{H}^s(\Omega)$ for $s \in (0, \frac{1}{2}]$. For $s \in (0, \frac{1}{2}]$, this is proved e.g. in [64, Corollary 2.71.]. Moreover, for $s = 0$, it follows from Lemma 3.3.2 and [55, Theorem 3.1.].

(ii) We have $C^2(\overline{\Omega}) \subset \mathbb{H}^s(\Omega)$ for $s \in [0, 1)$ and

$$\int_\Omega [\mathcal{D}^s_\Omega u] v \, dx = \mathcal{E}_s(u, v) \quad \text{for all} \quad u \in C^2(\overline{\Omega}), v \in \mathbb{H}^s(\Omega). \tag{3.3.4}$$

Moreover, integrating the Poisson problem (3.3.1) over Ω and using (3.3.4) with $v \equiv 1 \in C^1(\overline{\Omega})$, we see that $f \in L^2_0(\Omega)$ is a necessary condition for the existence of a solution of (3.3.1).

For $s \in [0, 1)$, we consider the closed subspace

$$\mathbb{X}^s(\Omega) := \left\{ u \in \mathbb{H}^s(\Omega) : \int_\Omega u \, dx = 0 \right\} \subset \mathbb{H}^s(\Omega).$$

By Proposition 3.3.1, the embedding $\mathbb{X}^s(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Furthermore, the following uniform Poincaré-type inequality holds with a constant $C_\Omega > 0$:

$$\|u\|^2_{L^2(\Omega)} \leq C_\Omega \mathcal{E}_s(u, u) \qquad \text{for } s \in [0, 1) \text{ and } u \in \mathbb{X}^s(\Omega). \tag{3.3.5}$$

Indeed, for $u \in \mathbb{X}^s(\Omega)$ we have $u_\Omega := \frac{1}{|\Omega|}\int_\Omega u \; dy = 0$ and therefore, by Jensen's inequality,

$$\begin{aligned}\int_\Omega u^2 \; dx &= \int_\Omega |u(x) - u_\Omega|^2 \; dx = \int_\Omega \Big|\frac{1}{|\Omega|}\int_\Omega (u(x) - u(y)) \; dy\Big|^2 \; dx \\ &\leq \frac{1}{|\Omega|}\int_\Omega\int_\Omega (u(x)-u(y))^2 \; dydx = \frac{1}{|\Omega|}\int_\Omega\int_\Omega \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} \cdot |x-y|^{N+2s} \; dydx \\ &\leq C_\Omega \mathcal{E}_s(u,u) \qquad \text{with} \quad C_\Omega := 2\frac{\max\{d_\Omega^N, d_\Omega^{N+2}\}}{|\Omega|}.\end{aligned}$$

We note that, thanks to Proposition 3.3.1 and (3.3.5), $\mathbb{X}^s(\Omega)$ is a Hilbert space with scalar product given by the bilinear form $(u,v) \mapsto \mathcal{E}_s(u,v)$.

Definition 3.3.4. Let $f \in L^2(\Omega)$. We say that a function $u \in \mathbb{H}^s(\Omega)$ is a weak solution of (3.3.1) if

$$\mathcal{E}_s(u,v) = \int_\Omega fv \; dx, \quad \text{for all} \quad v \in \mathbb{H}^s(\Omega). \tag{3.3.6}$$

Proposition 3.3.5. *For $s \in [0,1)$ and $f \in L_0^2(\Omega)$, there exists a unique weak solution $u \in \mathbb{X}^s(\Omega)$ of* (3.3.1).

Proof. Let $f \in L_0^2(\Omega)$. Since $\mathbb{X}^s(\Omega)$ is a Hilbert space with scalar product $\mathcal{E}_s$, the Riesz representation theorem implies that there exists $u \in \mathbb{X}^s(\Omega)$ with

$$\mathcal{E}_s(u,v) = \int_\Omega fv \; dx, \quad \text{for all} \quad v \in \mathbb{X}^s(\Omega).$$

Moreover, since $f \in L_0^2(\Omega)$, it follows that (3.3.6) also holds for constant functions $v \in \mathbb{H}^s(\Omega)$. Hence (3.3.6) holds for every $v \in \mathbb{H}^s(\Omega)$, and thus u is a weak solution of (3.3.1). □

Our next aim is to study, for $s \in [0,1)$, the eigenvalue problem related to $\mathcal{D}_\Omega^s$, that is the problem

$$\mathcal{D}_\Omega^s u = \lambda u \quad \text{in} \quad \Omega. \tag{3.3.7}$$

We consider corresponding eigenfunctions in weak sense i.e., a weak solution of (3.3.1) with $f = \lambda u$.

Proposition 3.3.6. *For every $s \in [0,1)$, the problem (3.3.7) admits a sequence of eigenvalues*

$$0 = \lambda_{0,s}^\Omega < \lambda_{1,s}^\Omega \leq \lambda_{2,s}^\Omega \leq \cdots \leq \lambda_{k,s}^\Omega \leq \cdots \to \infty \tag{3.3.8}$$

counted with multiplicity and a corresponding sequence of eigenfunctions which forms an orthonormal basis of $L^2(\Omega)$. Moreover, we have:

(i) The eigenspace corresponding to $\lambda_{0,s}^\Omega = 0$ is one-dimensional and consists of constant functions.

(ii) The first non-zero eigenvalue of $\mathcal{D}_\Omega^s$ in Ω is characterized by

$$\lambda_{1,s}^\Omega := \inf\Big\{\frac{\mathcal{E}_s(u,u)}{\|u\|_{L^2(\Omega)}^2} : u \in \mathbb{X}^s(\Omega) \setminus \{0\}\Big\} = \inf\Big\{\mathcal{E}_s(u,u) : u \in \mathbb{X}^s(\Omega), \|u\|_{L^2(\Omega)} = 1\Big\}. \tag{3.3.9}$$

For $s \in (0,1)$, the proof of the characterization (3.3.9) can be found in [63, Theorem 3.1.]. For the reader's convenience, we briefly sketch a proof which covers the case $s = 0$.

Proof. We first note that it follows in a standard way from Proposition 3.3.1 and the nonnegativity and symmetry of the quadratic form $\mathcal{E}_s$ that (3.3.7) admits a sequence of eigenvalues

$$0 \le \lambda_{0,s}^{\Omega} \le \lambda_{1,s}^{\Omega} \le \lambda_{2,s}^{\Omega} \le \cdots \le \lambda_{k,s}^{\Omega} \le \cdots \to \infty$$

Moreover, by definition, a function $u \in \mathbb{H}^s(\Omega)$ is an eigenfunction of (3.3.7) corresponding to the eigenvalue $\lambda = 0$ if and only if $\mathcal{E}_s(u,v) = 0$ for every $v \in \mathbb{H}^s(\Omega)$, and this is true if and only if u is constant. Hence we have $\lambda_{0,s}^{\Omega} = 0$ with a one-dimensional eigenspace consisting of constant functions, and thus $\lambda_{1,s}^{\Omega} > 0$. To prove (3.3.9), we first note that

$$\lambda_{1,s}^{\Omega} \ge \inf \Big\{ \mathcal{E}_s(u,u) : u \in \mathbb{X}^s(\Omega), \|u\|_{L^2(\Omega)} = 1 \Big\} \tag{3.3.10}$$

since every eigenfunction u corresponding to $\lambda_{1,s}^{\Omega} > 0$ is L^2-orthogonal to constant functions and therefore contained in $\mathbb{X}^s(\Omega)$, whereas $\mathcal{E}_s(u,u) = \lambda_{1,s}^{\Omega}$ if $\|u\|_{L^2(\Omega)} = 1$.

Moreover, it follows from the compactness of the embedding $\mathbb{H}^s(\Omega) \hookrightarrow L^2(\Omega)$ and the weak lower semicontinity of the functional $u \mapsto \mathcal{E}_s(u,u)$ on $\mathbb{H}^s(\Omega)$ that the infimum on the RHS of (3.3.10) is attained by a function $u \in \mathbb{X}^s(\Omega)$ with $\|u\|_{L^2(\Omega)} = 1$. By Lagrange multiplier rule, we can thus find $\lambda \in \mathbb{R}$ such that

$$\mathcal{E}_s(u,v) = \lambda \int_{\Omega} uv \, dx \quad \text{for all} \quad v \in \mathbb{X}^s(\Omega).$$

As in the proof of Proposition 3.3.5, it then follows that u weakly solves $\mathcal{D}_{\Omega}^s u = \lambda u$, which implies that $\lambda = \lambda \|u\|_{L^2(\Omega)}^2 = \mathcal{E}_s(u,u) \le \lambda_{1,s}^{\Omega}$ by (3.3.10). Moreover, $\lambda > 0$ since u is non-constant. Since $\lambda_{1,s}^{\Omega}$ is the smallest positive eigenvalue by definition, it thus follows that $\lambda = \lambda_{1,s}^{\Omega}$, and hence we have equality in (3.3.10). □

Remark 3.3.7. In a standard way, it can also be shown that, for $s \in [0,1)$ the higher eigenvalues $\lambda_{n,s}^{\Omega}$, $n \in \mathbb{N}$ are variationally characterized as

$$\lambda_{n,s}^{\Omega} = \inf_{V \in V_n^s} \sup_{u \in S_V} \mathcal{E}_s(u,u). \tag{3.3.11}$$

Here V_n^s denotes the family of n-dimensional subspaces of $\mathbb{X}^s(\Omega)$ and $S_V := \{u \in V : \|u\|_{L^2(\Omega)} = 1\}$ for $V \in V_n^s$.

3.4 Uniform bounds for weak subsolutions

In this section we establish uniform boundedness of weak solutions of problem (3.3.1) in the case when $f \in L^\infty(\Omega)$. Since we are also interested in uniform bounds on L^2-normalized eigenfunctions of $\mathcal{D}_{\Omega}^s$ independent of $s \in [0,1)$, it is in fact necessary to consider a generalization of (3.3.1) involving L^∞-potentials V. This is the content of Theorem 3.1.4, which we shall derive in this section as an immediate consequence of the following more general result on subsolutions.

Theorem 3.4.1. *Let $s \in [0,1)$, let $\Omega \subset \mathbb{R}^N$ be a bounded open Lipschitz set, let $V, f \in L^\infty(\Omega)$, and let $u \in \mathbb{H}^s(\Omega)$ be a weak subsolution of the problem*

$$\mathcal{D}^s_\Omega u + V(x)u = f \quad \text{in} \quad \Omega, \tag{3.4.1}$$

i.e., we have

$$\mathcal{E}^s(u,\varphi) + \int_\Omega V(x)u\varphi\, dx \le \int_\Omega f\varphi\, dx \qquad \text{for all } \varphi \in \mathbb{H}^s(\Omega),\ \varphi \ge 0. \tag{3.4.2}$$

Then there exists a constant $c_0 = c_0(N, \Omega, \|V\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(\Omega)}, \|u^+\|_{L^2(\Omega)}) > 0$ independent of s with the property that $u \le c_0$ in Ω.

As noted above, Theorem 3.1.4 immediately follows by applying Theorem 3.4.1 to u and $-u$, noting that $-u$ is a weak subsolution of the equation (3.4.1) with f replaced by $-f$. For the proof of Theorem 3.4.1, we need the following preliminary estimate.

Lemma 3.4.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open Lipschitz set. Then there exist constants $C_0 = C_0(N,\Omega) > 0$ and $\delta_0 = \delta_0(N,\Omega) \in (0,1)$ with the property that*

$$\int_{\Omega\setminus B_\delta(x)} |x-y|^{-N-2s}\, dy \ge C_0 \log\frac{\delta_0}{\delta} \qquad \text{for all } \ \delta \in (0,\delta_0), x \in \overline{\Omega}, s \in [0,1). \tag{3.4.3}$$

Proof. Since the boundary $\partial\Omega$ is Lipschitz, then Ω has the uniform cone property (see for instance [94, Theorem 1.2.2.2]). Therefore, there exist a cone segment

$$\mathcal{C}_{\alpha,\delta_0} := \{z \in \mathbb{R}^N : 0 < |z| \le \delta_0,\ \frac{z}{|z|}\cdot e_N < \alpha\}$$

for some $\delta_0 \in (0,1)$, $\alpha \in (0, \frac{\pi}{2}]$ with the property that for every $x \in \overline{\Omega}$ there exists a rotation $\mathcal{R}_x \in O(N)$ with

$$x + \mathcal{R}_x(\mathcal{C}_{\alpha,\delta_0}) \subset \Omega.$$

Setting $S_\alpha := \{z \in S^{N-1} \ : \ z\cdot e_N < \alpha\}$, we thus have

$$\begin{aligned}\int_{\Omega\setminus B_\delta(x)} |x-y|^{-N-2s}dy &\ge \int_{(x+\mathcal{R}_x(\mathcal{C}_{\alpha,\delta_0}))\setminus B_\delta(x)} |x-y|^{-N-2s}dy = \int_{\mathcal{C}_{\alpha,\delta_0}\setminus B_\delta(0)} |z|^{-N-2s}dz\\ &\ge \int_{\mathcal{C}_{\alpha,\delta_0}\setminus B_\delta(0)} |z|^{-N}\, dz \ge \mathcal{H}^{N-1}(S_\alpha)\int_\delta^{\delta_0} \rho^{-1}d\rho = \mathcal{H}^{N-1}(S_\alpha)\log\frac{\delta_0}{\delta},\end{aligned}$$

where $\mathcal{H}^{N-1}(S_\alpha)$ is the surface measure of the set $S_\alpha \subset S^{N-1}$. Hence (3.4.3) holds with $C_0 := \mathcal{H}^{N-1}(S_\alpha)$. □

Proof of Theorem 3.4.1. In the following, we let C_0 and $\delta_0 > 0$ be given by Lemma 3.4.2. For $\delta \in (0,\delta_0)$ and $s \in [0,1)$, we consider the kernel function

$$z \mapsto j_{\delta,s}(z) = \chi_{B_\delta(0)}(z)|z|^{-N-2s}$$

and the corresponding quadratic form defined by

$$\mathcal{E}^\delta_s(v,\varphi) = \frac{1}{2}\int_\Omega\int_\Omega (v(x)-v(y))(\varphi(x)-\varphi(y))j_{\delta,s}(x-y)\, dydx \quad \text{for} \quad v,\varphi \in \mathbb{H}^s(\Omega). \tag{3.4.4}$$

Since $u \in \mathbb{H}^s(\Omega)$ satisfies (3.4.2), we have

$$\begin{aligned}\int_\Omega f\varphi\,dx &\geq \mathcal{E}_s(u,\varphi) + \int_\Omega V(x)u(x)\varphi(x)\,dx \qquad (3.4.5)\\ &= \mathcal{E}_s^\delta(u,\varphi) + \int_\Omega V(x)u(x)\varphi\,dx + \frac{1}{2}\iint_{\substack{x,y\in\Omega\\|x-y|\geq\delta}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dxdy\\ &= \mathcal{E}_s^\delta(u,\varphi) + \int_\Omega (\gamma_{s,\delta}(x)+V(x))u(x)\varphi(x)\,dx - \int_\Omega \kappa_{s,\delta,u}(x)\varphi(x)\,dx \qquad (3.4.6)\\ &\geq \mathcal{E}_s^\delta(u,\varphi) + \int_\Omega (\gamma_{s,\delta}(x)+V(x))u(x)\varphi(x)\,dx - \int_\Omega \kappa_{s,\delta,u^+}(x)\varphi(x)\,dx\end{aligned}$$

for $\varphi \in \mathbb{H}^s(\Omega)$, $\varphi \geq 0$ with

$$\gamma_{s,\delta}(x) = \int_{\Omega\setminus B_\delta(x)} |x-y|^{-N-2s}\;dy, \qquad \kappa_{s,\delta,u}(x) := \int_{\Omega\setminus B_\delta(x)} u(y)|x-y|^{-N-2s}\,dy$$

and

$$\kappa_{s,\delta,u^+}(x) := \int_{\Omega\setminus B_\delta(x)} u^+(y)|x-y|^{-N-2s}\,dy.$$

We note that

$$\inf_{x\in\Omega} \gamma_{s,\delta}(x) \geq C_0 \log\frac{\delta_0}{\delta} \qquad \text{for } \delta\in(0,\delta_0),\; s\in[0,1) \tag{3.4.7}$$

by Lemma 3.4.2. Next we fix $c > 0$ and apply (3.4.5) to $\varphi_c := (u-c)^+$, which is easily seen to be a function in $\mathbb{H}^s(\Omega)$. Since $u\varphi_c \geq c\varphi_c$ in Ω, (3.4.5) and (3.4.7) give

$$\int_\Omega f\varphi_c\,dx \geq \mathcal{E}_s^\delta(u,\varphi_c) + \Big(\big(C_0\log\frac{\delta_0}{\delta} - \|V\|_{L^\infty(\Omega)}\big)c - \|\kappa_{s,\delta,u^+}\|_{L^\infty(\Omega)}\Big)\int_\Omega \varphi_c\,dx \tag{3.4.8}$$

where

$$\mathcal{E}_s^\delta(u,\varphi_c) = \mathcal{E}_s^\delta(u-c,(u-c)^+) = \mathcal{E}_s^\delta(\varphi_c,\varphi_c) - \mathcal{E}_s^\delta((u-c)^-,(u-c)^+) \geq \mathcal{E}_s^\delta(\varphi_c,\varphi_c) \geq 0.$$

Consequently, (3.4.8) implies that

$$\Big(\|f\|_{L^\infty(\Omega)} + \|\kappa_{s,\delta,u^+}\|_{L^\infty(\Omega)} - \big(C_0\log\frac{\delta_0}{\delta} - \|V\|_{L^\infty(\Omega)}\big)c\Big)\int_\Omega \varphi_c\,dx \geq 0. \tag{3.4.9}$$

Next, we fix $\delta\in(0,\delta_0)$ with the property that $C_0\log\frac{\delta_0}{\delta} - \|V\|_{L^\infty(\Omega)} \geq 1$, so that (3.4.9) reduces to

$$\Big(\|f\|_{L^\infty(\Omega)} + \|\kappa_{s,\delta,u^+}\|_{L^\infty(\Omega)} - c\Big)\int_\Omega \varphi_c\,dx \geq 0. \tag{3.4.10}$$

If $c > \|f\|_{L^\infty(\Omega)} + \|\kappa_{s,\delta,u^+}\|_{L^\infty(\Omega)}$, (3.4.10) implies that $\int_\Omega \varphi_c\,dx = 0$ and therefore $u \leq c$ in Ω. We thus conclude that $u \leq c_0$ with

$$c_0 := \|f\|_{L^\infty(\Omega)} + \|\kappa_{s,\delta,u^+}\|_{L^\infty(\Omega)}.$$

Since

$$\begin{aligned}0 \leq \kappa_{s,\delta,u^+}(x) &= \int_{\Omega\setminus B_\delta(x)} u^+(y)|x-y|^{-N-2s}\,dy\\ &\leq \delta^{-N-2s}\int_\Omega u^+(y)\,dy \leq \delta^{-N-2}\sqrt{|\Omega|}\,\|u^+\|_{L^2(\Omega)}\end{aligned}$$

for $x\in\Omega$, it follows that c_0 only depends on $N, \Omega, \|V\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(\Omega)}$ and $\|u^+\|_{L^2(\Omega)}$, as claimed. $\square$

3.5 Uniform estimates for convergence of eigenvalues and eigenfunctions of $\mathcal{D}^s_\Omega$

In this section we first prove global bounds on eigenvalues and eigenfunctions of the operator family $\mathcal{D}^s_\Omega$. Then we shall prove convergence of eigenvalues and eigenfunctions in the limit $s \to 0^+$.

The first result of this section is the following.

Proposition 3.5.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open Lipschitz set. For every $n \in \mathbb{N}$, $s_0 \in (0,1)$ we have*

$$\Lambda^\Omega_{n,s_0} := \sup_{s\in[0,s_0]} \lambda^\Omega_{n,s} < \infty.$$

Proof. Fix $n \in \mathbb{N}$, $s_0 \in (0,1)$. To estimate $\lambda^\Omega_{n,s}$ for $s \in [0,s_0]$, we use the variational characterization (3.3.11) and let V be a fixed n-dimensional subspace of $C^1_*(\overline{\Omega}) = \big\{u \in C^1(\overline{\Omega}) \,:\, \int_\Omega u\,dx = 0\big\}$. For all $u \in V$, we then have

$$\begin{aligned}\mathcal{E}_s(u,u) &= \frac{1}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dxdy \le \frac{\|\nabla u\|^2_{L^\infty(\Omega)}}{2}\int_\Omega\int_\Omega |x-y|^{2-N-2s}\,dxdy\\ &\le \frac{\|u\|^2_{C^1(\overline{\Omega})}}{2}\int_\Omega\int_{B_{d_\Omega}(x)} |x-y|^{2-N-2s}\,dydx \le \frac{\|u\|^2_{C^1(\overline{\Omega})}}{4(1-s)}|\Omega|\mathcal{H}^{N-1}(S^{N-1})d_\Omega^{2(1-s)}. \qquad (3.5.1)\end{aligned}$$

Moreover, since the norms $\|\cdot\|_{C^2}$ and $\|\cdot\|_{L^2}$ are equivalent on V, there exists $C_V = C(V) > 0$ such that

$$\|u\|_{C^1(\overline{\Omega})} \le C_V\|u\|_{L^2(\Omega)} \qquad \text{for every } u \in V. \qquad (3.5.2)$$

Combining (3.5.1) and (3.5.2), we deduce that

$$\mathcal{E}_s(u,u) \le \frac{C_V}{4(1-s_0)}|\Omega|\mathcal{H}^{N-1}(S^{N-1})\max\{1,d^2_\Omega\} \quad \text{for } u \in V \text{ with } \|u\|_{L^2(\Omega)} = 1.$$

It thus follows from (3.3.11) that $\sup\limits_{s\in[0,s_0]} \lambda^\Omega_{n,s} < \infty$, as claimed. □

Combining Theorem 3.4.1 and Proposition 3.5.1, we obtain the following uniform bound on eigenfunctions.

Theorem 3.5.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open Lipschitz set, let $n \in \mathbb{N}$, and let $s_0 \in (0,1)$. Then there exists a constant $C = C(N,\Omega,n,s_0) > 0$ with the property that for every $s \in [0,s_0]$ and every eigenfunction $\xi \in \mathbb{X}^s(\Omega)$ of the eigenvalue problem (3.3.7) corresponding to the eigenvalue $\lambda^\Omega_{n,s}$ we have*

$$\xi \in L^\infty(\Omega) \qquad \textit{and} \qquad \|\xi\|_{L^\infty(\Omega)} \le C\|\xi\|_{L^2(\Omega)}.$$

Proof. By homogeneity, it suffices to consider eigenfunctions $\xi \in \mathbb{X}^s(\Omega)$ with $\|\xi\|_{L^2(\Omega)} = 1$. The result then follows by applying Theorem 3.1.4 to $V \equiv -\lambda^\Omega_{n,s}$ and $f \equiv 0$, noting that $\|V\|_{L^\infty} = \lambda^\Omega_{n,s}$ is uniformly bounded independently of $s \in [0,s_0]$ by Proposition 3.5.1. □

In the remainder of this section, we study the transition from the fractional case $s > 0$ to the logarithmic case $s = 0$ with regard to the eigenvalues $\lambda^\Omega_{n,s}$ and corresponding eigenfunctions. For simplicity, we first consider the case $n = 1$, that is the first positive eigenvalue $\lambda^\Omega_{1,s}$.

Theorem 3.5.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open Lipschitz set. Then*

$$\lambda_{1,s}^{\Omega} \to \lambda_{1,0}^{\Omega} \qquad \text{as} \quad s \to 0^+. \tag{3.5.3}$$

Moreover, if, for some sequence $s_k \to 0^+$, $\{\xi_{1,s_k}\}_k$ is a sequence of L^2-normalized eigenfunctions of $\mathcal{D}_\Omega^{s_k}$ corresponding to λ_{1,s_k}^{Ω}, we have that, after passing to a subsequence,

$$\xi_{1,s_k} \to \xi_1 \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad k \to \infty, \tag{3.5.4}$$

where ξ_1 is an eigenfunction of $\mathcal{D}_\Omega^0$ corresponding to $\lambda_{1,0}^{\Omega}$.

Proof. It is convenient to introduce the subspace $C_*^2(\overline{\Omega}) := \{u \in C^2(\overline{\Omega}) : \int_\Omega u\, dx = 0\}$. Let $u \in C_*^2(\overline{\Omega})$ such that $\|u\|_{L^2(\Omega)} = 1$. Then Theorem 3.1.1 together with (3.3.4) yields

$$\limsup_{s\to 0^+} \lambda_{1,s}^{\Omega} \le \limsup_{s\to 0^+} \mathcal{E}_s(u,u) = \lim_{s\to 0^+} \langle \mathcal{D}_\Omega^s u, u\rangle_2 = \langle \mathcal{D}_\Omega^0 u, u\rangle_2 = \mathcal{E}_0(u,u).$$

Using the fact that, by Remark 3.3.3, $C_*^2(\overline{\Omega})$ is dense in $\mathbb{X}^0(\Omega)$, we get

$$\limsup_{s\to 0^+} \lambda_{1,s}^{\Omega} \le \inf_{\substack{u\in \mathbb{X}^0(\Omega)\\ \|u\|_{L^2(\Omega)}=1}} \mathcal{E}_0(u,u) = \lambda_{1,0}^{\Omega}. \tag{3.5.5}$$

Next we consider

$$\lambda_* := \liminf_{s\to 0^+} \lambda_{1,s}^{\Omega} \qquad \in [0, \lambda_{1,0}^{\Omega}],$$

and we let $\{s_k\}_{k\in\mathbb{N}} \subset (0,1)$ be a sequence with $s_k \to 0^+$ as $k \to \infty$ and such that $\lim\limits_{k\to\infty} \lambda_{1,s_k}^{\Omega} = \lambda_*$. Moreover, we let ξ_{1,s_k} be an eigenfunction associated to λ_{1,s_k}^{Ω} with $\|\xi_{1,s_k}\|_{L^2(\Omega)} = 1$. We claim that

$$\limsup_{k\to\infty} \mathcal{E}_0(\xi_{1,s_k}, \xi_{1,s_k}) \le \lambda_{1,0}^{\Omega}. \tag{3.5.6}$$

Indeed, from (3.5.5) we have, with

$$A_\Omega := \{(x,y) \in \Omega \times \Omega \,:\, |x-y| \le 1\} \quad \text{and} \quad B_\Omega := \{(x,y) \in \Omega\times\Omega \,:\, |x-y| > 1\},$$

the estimate

$$\begin{aligned}
\lambda_{1,0}^{\Omega} + o(1) \ge \lambda_{1,s_k}^{\Omega} &= \mathcal{E}_{s_k}(\xi_{1,s_k}, \xi_{1,s_k}) = \frac{1}{2}\int_\Omega\int_\Omega \frac{(\xi_{1,s_k}(x) - \xi_{1,s_k}(y))^2}{|x-y|^{N+2s_k}}\, dxdy \\
&= \frac{1}{2}\Big(\iint_{A_\Omega} \frac{(\xi_{1,s_k}(x) - \xi_{1,s_k}(y))^2}{|x-y|^{N+2s_k}}\, dxdy + \iint_{B_\Omega} \frac{(\xi_{1,s_k}(x) - \xi_{1,s_k}(y))^2}{|x-y|^{N+2s_k}}\, dxdy\Big) \\
&\ge \frac{1}{2}\Big(\iint_{A_\Omega} \frac{(\xi_{1,s_k}(x) - \xi_{1,s_k}(y))^2}{|x-y|^{N}}\, dxdy + d_\Omega^{-2s_k}\iint_{B_\Omega} \frac{(\xi_{1,s_k}(x) - \xi_{1,s_k}(y))^2}{|x-y|^{N}}\, dxdy\Big) \\
&= \mathcal{E}_0(\xi_{1,s_k}, \xi_{1,s_k}) + \frac{d_\Omega^{-2s_k} - 1}{2}\iint_{B_\Omega} \frac{(\xi_{1,s_k}(x) - \xi_{1,s_k}(y))^2}{|x-y|^{N}}\, dxdy \\
&\ge \mathcal{E}_0(\xi_{1,s_k}, \xi_{1,s_k}) + d_\Omega^{-N}\frac{d_\Omega^{-2s_k} - 1}{2}\iint_{B_\Omega} (\xi_{1,s_k}(x) - \xi_{1,s_k}(y))^2\, dxdy.
\end{aligned}$$

If $d_\Omega \le 1$, we infer that $\mathcal{E}_0(\xi_{1,s_k}, \xi_{1,s_k}) \le \lambda_{1,0}^\Omega + o(1)$ and therefore (3.5.6) already follows. If $d_\Omega > 1$, we estimate

$$\iint_{B_\Omega} (\xi_{1,s_k}(x) - \xi_{1,s_k}(y))^2 \, dxdy \le 2 \iint_{\Omega\times\Omega} (\xi_{1,s_k}^2(x) + \xi_{1,s_k}^2(y)) dxdy \le 4|\Omega| \|\xi_{1,s_k}\|_{L^2(\Omega)}^2 = 4|\Omega|$$

which yields

$$\lambda_{1,0}^\Omega + o(1) \ge \mathcal{E}_0(\xi_{1,s_k}, \xi_{1,s_k}) + 2|\Omega| d_\Omega^{-N} (d_\Omega^{-2s_k} - 1) = \mathcal{E}_0(\xi_{1,s_k}, \xi_{1,s_k}) + o(1).$$

Hence (3.5.6) also follows in this case.

As a consequence of (3.5.6), the sequence ξ_{1,s_k} is uniformly bounded in $\mathbb{H}^0(\Omega)$. So, after passing to a subsequence, there exists $\xi_1 \in \mathbb{H}^0(\Omega)$ such that $\xi_{1,s_k} \rightharpoonup \xi_1$ in $\mathbb{H}^0(\Omega)$, which by Proposition 3.3.1 implies that $\xi_{1,s_k} \to \xi_1$ in $L^2(\Omega)$. Consequently, $\|\xi_1\|_{L^2(\Omega)} = 1$ and $\int_\Omega \xi_1 \, dx = 0$, so in particular $\xi_1 \in \mathbb{X}^0(\Omega)$.

Next, from Theorem 3.1.1 and Remark 3.3.3, we have that for all $\varphi \in C_*^2(\overline{\Omega})$,

$$\lim_{k\to\infty} \lambda_{1,s_k}^\Omega \langle \xi_{1,s_k}, \varphi \rangle_2 = \lim_{k\to\infty} \mathcal{E}_{s_k}(\xi_{1,s_k}, \varphi) = \lim_{k\to\infty} \langle \xi_{1,s_k}, \mathcal{D}_\Omega^{s_k} \varphi \rangle_2 = \langle \xi_1, \mathcal{D}_\Omega^0 \varphi \rangle_2 = \mathcal{E}_0(\xi_1, \varphi) \tag{3.5.7}$$

Since also $\langle \xi_{1,s_k}, \varphi \rangle_2 \to \langle \xi_1, \varphi \rangle_2$ for all $\varphi \in C_*^2(\overline{\Omega})$ as $k \to \infty$, it follows from (3.5.7) that

$$\mathcal{E}_0(\xi_1, \varphi) = \lambda_* \langle \xi_1, \varphi \rangle_2 \quad \text{for all} \quad \varphi \in C_*^2(\overline{\Omega}).$$

By density, we get

$$\mathcal{E}_0(\xi_1, \varphi) = \lambda_* \langle \xi_1, \varphi \rangle_2 \quad \text{for all} \quad \varphi \in \mathbb{X}^0(\Omega).$$

Since $\xi_1 \in \mathbb{X}^0(\Omega) \setminus \{0\}$, we then deduce that $\lambda_* \in (0, \lambda_{1,0}^\Omega]$ is an eigenvalue of $\mathcal{D}_\Omega^0$ with corresponding eigenfunction ξ_1. Since $\lambda_{1,0}^\Omega$ is the smallest positive eigenvalue of $\mathcal{D}_\Omega^0$ by definition, we conclude that $\lambda_* = \lambda_{1,0}^\Omega$. Combining this equality with (3.5.5), we conclude that $\lambda_{1,s}^\Omega \to \lambda_{1,0}^\Omega$ as $s \to 0^+$, as claimed in (3.5.3). Moreover, we have already proved above that if, for some sequence $s_k \to 0^+$, $\{\xi_{1,s_k}\}_k$ is a sequence of L^2-normalized eigenfunctions of $\mathcal{D}_\Omega^{s_k}$ corresponding to λ_{1,s_k}^Ω, we have that $\xi_{1,s_k} \to \xi_1$ in $L^2(\Omega)$ after passing to a subsequence, where ξ_1 is an eigenfunction of $\mathcal{D}_\Omega^0$ corresponding to $\lambda_{1,0}^\Omega$. The proof is thus finished. □

Next, we now consider the case of higher eigenvalues. We have the following.

Theorem 3.5.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open Lipschitz set. Then*

$$\lambda_{n,s}^\Omega \to \lambda_{n,0}^\Omega \qquad \text{as} \quad s \to 0^+. \tag{3.5.8}$$

Moreover, if, for some sequence $s_k \to 0^+$, $\{\xi_{n,s_k}\}_k$ is a sequence of L^2-normalized eigenfunctions of $\mathcal{D}_\Omega^{s_k}$ corresponding to λ_{n,s_k}^Ω, we have that, after passing to a subsequence,

$$\xi_{n,s_k} \to \xi_n \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad k \to \infty, \tag{3.5.9}$$

where ξ_n is an eigenfunction of $\mathcal{D}_\Omega^0$ corresponding to $\lambda_{n,0}^\Omega$.

The proof of this theorem is similar to the one of Theorem 3.5.3 but somewhat more involved technically.

Proof of Theorem 3.5.4. Similarly as in the proof of Theorem 3.5.3, we first show that

$$\limsup_{s\to 0^+} \lambda_{n,s}^{\Omega} \le \lambda_{n,0}^{\Omega}. \tag{3.5.10}$$

For this we consider again the subspace $C_*^2(\overline{\Omega}) \subset \mathbb{X}^s(\Omega)$, and we fix an n-dimensional subspace $V \subset C_*^2(\overline{\Omega})$. Then $S_V := \{u \in V \,:\, \|u\|_{L^2(\Omega)} = 1\}$ is bounded in $C_*^2(\overline{\Omega})$ since the L^2-norm and the C^2-norm are equivalent on V. Thus Theorem 3.1.1 gives, together with (3.3.4) and (3.3.11), the estimate

$$\begin{aligned}\limsup_{s\to 0^+} \lambda_{n,s}^{\Omega} \le \limsup_{s\to 0^+} \sup_{u\in S_V} \mathcal{E}_s(u,u) = \lim_{s\to 0^+} \sup_{u\in S_V} \langle \mathcal{D}_\Omega^s u, u\rangle_2 &= \sup_{u\in S_V} \langle \mathcal{D}_\Omega^0 u, u\rangle_2 \\ &= \sup_{u\in S_V} \mathcal{E}_0(u,u).\end{aligned}$$

Using again the fact that, by Remark 3.3.3, $C_*^2(\overline{\Omega})$ is dense in $\mathbb{X}^0(\Omega)$ and that

$$\lambda_{n,0}^{\Omega} = \inf_{V\in V_n^0} \sup_{u\in S_V} \mathcal{E}_0(u,u),$$

by (3.3.11), where V_n^0 denotes the family of n-dimensional subspaces of $\mathbb{X}^0(\Omega)$, we deduce (3.5.10).

Next we show the corresponding liminf inequality. For this, we fix $n \in \mathbb{N}$ and set

$$\lambda_j^* := \liminf_{s\to 0^+} \lambda_{j,s}^{\Omega} \qquad \text{for } j = 1,\dots,n,$$

noting that

$$\lambda_j^* \le \lambda_n^* \qquad \text{for } j = 1,\dots,n \tag{3.5.11}$$

since the sequence of numbers $\lambda_{j,s}^{\Omega}$ is increasing in j for every $s \in (0,1)$. Moreover, we choose a sequence of numbers $s_k \in (0,1)$, $k \in \mathbb{N}$ with $s_k \to 0^+$ and $\lambda_{n,s_k}^{\Omega} \to \lambda_n^*$ as $k \to \infty$. We then choose, for every $k \in \mathbb{N}$, a system of L^2-orthonormal eigenfunctions $\xi_{1,s_k},\dots,\xi_{n,s_k}$ associated to the eigenvalues $\lambda_{1,s_k}^{\Omega},\dots,\lambda_{n,s_k}^{\Omega}$.

Proceeding precisely as in the proof of Theorem 3.5.3, we find that ξ_{j,s_k} is uniformly bounded in $\mathbb{H}^0(\Omega)$ for $j = 1,\dots,n$. Therefore, after passing to a subsequence, there exists $\xi_j \in \mathbb{H}^0(\Omega)$ such that $\xi_{j,s_k} \rightharpoonup \xi_j$ in $\mathbb{H}^0(\Omega)$ for $j = 1,\dots,n$, which by Proposition 3.3.1 implies that $\xi_{j,s_k} \to \xi_j$ in $L^2(\Omega)$ for $j = 1,\dots,n$.

The L^2-convergence implies that the functions $\xi_1,\dots,\xi_n$ are also L^2-orthonormal. Moreover, for $j = 1,\cdots,n$, we have, by Theorem 3.1.1 and Remark 3.3.3,

$$\begin{aligned}\lambda_j^* \langle \xi_j, \varphi\rangle_2 &= \lim_{k\to\infty} \lambda_{j,s_k}^{\Omega} \langle \xi_{j,s_k}, \varphi\rangle_2 = \lim_{k\to\infty} \mathcal{E}_{s_k}(\xi_{j,s_k}, \varphi) \\ &= \lim_{k\to\infty} \langle \xi_{j,s_k}, \mathcal{D}_\Omega^{s_k}\varphi\rangle_2 = \langle \xi_j, \mathcal{D}_\Omega^0 \varphi\rangle_2 = \mathcal{E}_0(\xi_j,\varphi) \qquad \text{for } \varphi \in C_*^2(\overline{\Omega}).\end{aligned} \tag{3.5.12}$$

By density of $C_*^2(\overline{\Omega})$ in $\mathbb{X}^0(\Omega)$, we thus have

$$\mathcal{E}_0(\xi_j,\varphi) = \lambda_j^* \langle \xi_j, \varphi\rangle_2 \;\text{ for all }\; \varphi \in \mathbb{X}^0(\Omega),\; j = 1,\dots,n.$$

Therefore, λ_j^* is an eigenvalue of $\mathcal{D}_\Omega^0$ with corresponding eigenfunction ξ_j for $j = 1,\dots,n$. Now, by considering in particular the n-dimensional subspace $V = \operatorname{span}\{\xi_1,\xi_2,\dots,\xi_n\}$ of $\mathbb{X}^0(\Omega)$ in (3.3.11), it follows that

$$\lambda_{n,0}^{\Omega} \le \sup_{u\in S_V} \mathcal{E}_0(u,u). \tag{3.5.13}$$

Moreover, every $u \in S_V$ writes as $u = \sum_{j=1}^{n} c_j \xi_j$ with $c_j \in \mathbb{R}$ satisfying $\sum_{j=1}^{n} c_j^2 = 1$, so we have

$$\mathcal{E}_0(u,u) = \mathcal{E}_0\Big(\sum_{j=1}^{n} c_j \xi_j, \sum_{j=1}^{n} c_j \xi_j\Big) = \sum_{i,j=1}^{n} c_i c_j \lambda_i^* \langle \xi_i, \xi_j \rangle_2 = \sum_{i=1}^{n} c_i^2 \lambda_i^* \leq \lambda_n^* \sum_{i=1}^{n} c_i^2 = \lambda_n^*$$

by (3.5.11). Hence (3.5.13) yields that

$$\lambda_{n,0}^{\Omega} \leq \lambda_n^* = \liminf_{s \to 0^+} \lambda_{n,s}^{\Omega} \tag{3.5.14}$$

Combining (3.5.10) and (3.5.14) now shows that $\lambda_{n,s}^{\Omega} \to \lambda_{n,0}^{\Omega}$ as $s \to 0^+$, as claimed in (3.5.8). The rest of the proof follows exactly as in the case of Theorem 3.5.3. □

Next, we wish to study the uniform convergence of sequences of eigenfunctions of $\mathcal{D}_{\Omega}^{s_k}$ associated with a sequence $s_k \to 0^+$. We first state a uniform equicontinuity result in a somewhat more general setting.

Theorem 3.5.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz set. Moreover, let $(s_k)_k$ be a sequence in $(0,1)$ with $s_k \to 0^+$, and let $\varphi_k \in C(\overline{\Omega})$, $k \in \mathbb{N}$ be functions with*

$$\|\varphi_k\|_{L^\infty(\Omega)} \leq C \qquad \text{and} \qquad \Big| \int_{\Omega} \frac{\varphi_k(x) - \varphi_k(y)}{|x-y|^{N+2s_k}} dy \Big| \leq C \quad \text{for all } x \in \overline{\Omega},\ k \in \mathbb{N} \tag{3.5.15}$$

with a constant $C > 0$. Then the sequence $(\varphi_k)_k$ is equicontinuous.

Proof. Since $s_k \to 0^+$, we may assume, without loss of generality, that $s_k \in (0, \frac{1}{4})$ for every $k \in \mathbb{N}$. Moreover, relabeling the functions φ_k if necessary, we may assume that the sequence s_k is monotone decreasing. Arguing by contradiction, we assume that there exists a point $x_0 \in \overline{\Omega}$ such that the sequence $(\varphi_k)_k$ is not equicontinuous at x_0, which means that

$$\lim_{t \to 0^+} \sup_{k \in \mathbb{N}} \operatorname*{osc}_{B_t(x_0) \cap \overline{\Omega}} \varphi_k = \varepsilon > 0. \tag{3.5.16}$$

This limit exists since the function

$$(0,\infty) \to [0,\infty), \qquad t \mapsto \sup_{k \in \mathbb{N}} \operatorname*{osc}_{B_t(x_0) \cap \overline{\Omega}} \varphi_k$$

is bounded by assumption and nondecreasing. Without loss of generality, to simplify the notation, we may assume that $x_0 = 0 \in \overline{\Omega}$. We first choose $\delta > 0$ sufficiently small so that

$$\frac{\varepsilon - \delta}{2^{N+2}} - 2 \cdot 3^N \delta > 0. \tag{3.5.17}$$

We then choose $t_0 \in (0,1)$ sufficiently small so that

$$\varepsilon \leq \sup_{k \in \mathbb{N}} \operatorname*{osc}_{B_t \cap \overline{\Omega}} \varphi_k \leq \varepsilon + \delta \qquad \text{for } 0 < t \leq 2t_0. \tag{3.5.18}$$

From (3.5.15) and the assumption that the sequence $(\varphi_k)_k$ is uniformly bounded in $\overline{\Omega}$, it follows that there exists a constant $C_1 = C_1(t_0) > 0$ with

$$\Big| \int_{B_{t_0}(x) \cap \Omega} \frac{\varphi_k(x) - \varphi_k(y)}{|x-y|^{N+2s_k}} dy \Big| \leq C_1 \qquad \text{for all } x \in \overline{\Omega},\ k \in \mathbb{N}. \tag{3.5.19}$$

Next, we choose a sequence of numbers $t_k \in (0, \frac{t_0}{5})$ with $t_k \to 0^+$ and

$$C_2 := \inf_{k \in \mathbb{N}} t_k^{s_k} > 0. \tag{3.5.20}$$

We then define a strictly increasing sequence of numbers σ_k, $k \in \mathbb{N}$ inductively with the property that

$$\operatorname*{osc}_{B_{t_k} \cap \overline{\Omega}} \varphi_{\sigma_k} \geq \varepsilon - \delta \qquad \text{for all } k \in \mathbb{N}. \tag{3.5.21}$$

For this, we first note that (3.5.18) implies that there exists some $\sigma_1 \in \mathbb{N}$ with

$$\operatorname*{osc}_{B_{t_1} \cap \overline{\Omega}} \varphi_{\sigma_1} \geq \varepsilon - \delta.$$

Next, suppose that $\sigma_1 < \cdots < \sigma_k$ are already defined for some $k \in \mathbb{N}$. Since the finite set of functions $\{\varphi_{\sigma_1}, \ldots, \varphi_{\sigma_k}\}$ is equicontinuous on $\overline{\Omega}$ by assumption, there exists $t' \in (0, t_{k+1})$ with the property that

$$\operatorname*{osc}_{B_{t'} \cap \overline{\Omega}} \varphi_\ell < \varepsilon - \delta \qquad \text{for } \ell = \sigma_1, \ldots, \sigma_k.$$

Hence, by (3.5.18), there exists some $\sigma_{k+1} \in \mathbb{N}$, $\sigma_{k+1} > \sigma_k$ with

$$\varepsilon - \delta \leq \operatorname*{osc}_{B_{t'} \cap \overline{\Omega}} \varphi_{\sigma_{k+1}} \leq \operatorname*{osc}_{B_{t_{k+1}} \cap \overline{\Omega}} \varphi_{\sigma_{k+1}}.$$

With this inductive choice, (3.5.21) holds for all $k \in \mathbb{N}$. Moreover, since $\sigma_k \geq k$ and therefore $s_{\sigma_k} \leq s_k$, we have $t_k^{s_{\sigma_k}} \geq t_k^{s_k} \geq C_2$ for every $k \in \mathbb{N}$ by (3.5.20) and since $t_k \in (0,1)$. Hence we may pass of a subsequence, replacing s_k by s_{σ_k} and φ_k by φ_{σ_k} in the following, with the property that (3.5.20) still holds and

$$\varepsilon - \delta \leq \operatorname*{osc}_{B_{t_k} \cap \overline{\Omega}} \varphi_k \leq \varepsilon + \delta \qquad \text{for all } k \in \mathbb{N}. \tag{3.5.22}$$

By (3.5.22), we may write

$$\varphi_k(\overline{B_{t_k} \cap \Omega}) = [d_k - r_k, d_k + r_k] \qquad \text{for } k \in \mathbb{N} \text{ with some } d_k \in \mathbb{R},\ r_k \geq \frac{\varepsilon - \delta}{2}. \tag{3.5.23}$$

Together with (3.5.18) and the fact that $\overline{B_{t_k} \cap \Omega} \subset \overline{B_{2t_0} \cap \Omega}$, we deduce that

$$\varphi_k(\overline{B_{2t_0} \cap \Omega}) \subset [d_k - \frac{\varepsilon + 3\delta}{2}\ ,\ d_k + \frac{\varepsilon + 3\delta}{2}]. \tag{3.5.24}$$

Moreover, we let

$$c_k := \int_{\Omega \cap (B_{t_0} \setminus B_{3t_k})} |y|^{-N-2s_k}\, dy \qquad \text{for } k \in \mathbb{N},$$

and we note that

$$c_k \to \infty \qquad \text{as } k \to \infty \tag{3.5.25}$$

by Lemma 3.4.2. We now set

$$A_+^k := \{y \in \Omega \cap (B_{t_0} \setminus B_{3t_k}) \ :\ \varphi_k(y) \geq d_k\} \quad \text{and} \quad A_-^k := \{y \in \Omega \cap (B_{t_0} \setminus B_{3t_k}) \ :\ \varphi_k(y) \leq d_k\}.$$

Since

$$c_k \leq \int_{A_+^k} |y|^{-N-2s_k}\, dy + \int_{A_-^k} |y|^{-N-2s_k}\, dy \qquad \text{for all } k \in \mathbb{N},$$

we may again pass to a subsequence such that

$$\int_{A_+^k} |y|^{-N-2s_k}\, dy \geq \frac{c_k}{2} \quad \text{for all } k \in \mathbb{N} \qquad \text{or} \qquad \int_{A_-^k} |y|^{-N-2s_k}\, dy \geq \frac{c_k}{2} \quad \text{for all } k \in \mathbb{N}.$$

Without loss of generality, we may assume that the second case holds (otherwise we may replace φ_k by $-\varphi_k$ and d_k by $-d_k$). We then define the Lipschitz function $\psi_k \in C_c(\mathbb{R}^N)$ by

$$\psi_k(x) = \begin{cases} 2\delta, & |x| \leq t_k \\ 0, & |x| \geq 2t_k \\ \dfrac{2\delta}{t_k}(2t_k - |x|), & t_k \leq |x| \leq 2t_k. \end{cases}$$

We also define, for $k \in \mathbb{N}$,

$$\tau_k : \overline{\Omega} \to \mathbb{R}, \qquad \tau_k(x) = \varphi_k(x) + \psi_k(x)$$

By (3.5.24), we have

$$\tau_k = \varphi_k \leq d_k + \frac{\varepsilon + 3\delta}{2} \leq d_k + r_k + 2\delta \qquad \text{in } \overline{\Omega \cap (B_{2t_0} \setminus B_{2t_k})}.$$

Moreover, since $d_k + r_k \in \varphi_k(\overline{B_{t_k} \cap \Omega})$ by (3.5.23), we have

$$d_k + r_k + 2\delta \in \tau_k(\overline{B_{t_k} \cap \Omega}) \subset \tau_k(B_{2t_k} \cap \overline{\Omega}).$$

Consequently, $\max\limits_{\overline{B_{2t_0} \cap \Omega}} \tau_k$ is attained at a point $x_k \in B_{2t_k} \cap \overline{\Omega}$ with

$$\tau_k(x_k) \geq d_k + r_k + 2\delta$$

which implies that

$$\varphi_k(x_k) \geq d_k + r_k \geq d_k + \frac{\varepsilon - \delta}{2}. \tag{3.5.26}$$

By (3.5.19) and since $B_{3t_k} \cap \Omega \subset B_{t_0}(x_k) \cap \Omega$ for $k \in \mathbb{N}$ by construction, we have that

$$\begin{aligned} C_1 &\geq \int_{B_{t_0}(x_k) \cap \Omega} \frac{\varphi_k(x_k) - \varphi_k(y)}{|x_k - y|^{N+2s_k}} dy \\ &= \int_{B_{3t_k} \cap \Omega} \frac{\varphi_k(x_k) - \varphi_k(y)}{|x_k - y|^{N+2s_k}} dy + \int_{\Omega \cap (B_{t_0}(x_k) \setminus B_{3t_k})} \frac{\varphi_k(x_k) - \varphi_k(y)}{|x_k - y|^{N+2s_k}} dy. \end{aligned} \tag{3.5.27}$$

To estimate the first integral, we note that, by definition of the function ψ_k,

$$|\psi_k(x) - \psi_k(y)| \leq \frac{2\delta}{t_k}|x - y| \qquad \text{for all } x, z \in \mathbb{R}^N.$$

Moreover, by the choice of x_k we have $\tau_k(x_k) \geq \tau_k(y)$ for all $y \in B_{3t_k} \cap \Omega$. Consequently,

$$\begin{aligned}
&\int_{B_{3t_k}\cap\Omega} \frac{\varphi_k(x_k)-\varphi_k(y)}{|x_k-y|^{N+2s_k}}dy = \int_{B_{3t_k}\cap\Omega} \frac{\tau_k(x_k)-\tau_k(y)}{|x_k-y|^{N+2s_k}}dy - \int_{B_{3t_k}\cap\Omega} \frac{\psi_k(x_k)-\psi_k(y)}{|x_k-y|^{N+2s_k}}dy \\
&\geq -\int_{B_{3t_k}\cap\Omega} \frac{\psi(x_k)-\psi(y)}{|x_k-y|^{N+2s_k}}dy \geq -\frac{2\delta}{t_k}\int_{B_{3t_k}} |x_k-y|^{1-N-2s_k}dy \geq -\frac{2\delta}{t_k}\int_{B_{3t_k}} |y|^{1-N-2s_k}dy \\
&= -\frac{3^{1-2s_k}\omega_{N-1}2\delta t_k^{-2s_k}}{1-2s_k} \geq -12\omega_{N-1}\delta t_k^{-2s_k} \geq -C_3 \qquad (3.5.28)
\end{aligned}$$

with a constant $C_3 > 0$ independent of k. Here we used (3.5.20) and the standard estimate

$$\int_{B_t} |x-z|^{\rho-N}\,dz \leq \int_{B_t} |z|^{\rho-N}\,dz = \frac{\omega_{N-1}t^\rho}{\rho} \qquad \text{for every } t>0,\ \rho\in(0,N) \text{ and } x\in\mathbb{R}^N.$$

To estimate the second integral in (3.5.27) we first note, since $x_k \in B_{2t_k}$, we have that

$$2|y| \geq |y-x_k| \geq \frac{|y|}{3} \qquad \text{for every } k\in\mathbb{N} \text{ and } y\in\mathbb{R}^N\setminus B_{3t_k}.$$

Moreover, by (3.5.18), (3.5.24), and (3.5.26) we have

$$\varepsilon+\delta \geq \varphi_k(x_k)-\varphi_k(y) \geq d_k + \frac{\varepsilon-\delta}{2} - \varphi_k(y) \geq -2\delta$$

for $y \in B_{t_0}(x_k)\cap\Omega \subset B_{2t_0}\cap\Omega$. Consequently, combining (3.5.27) and (3.5.28), using again (3.5.26) and the fact that $x_k \in B_{2t_k}$, we may estimate as follows:

$$\begin{aligned}
C_1+C_3 &\geq \int_{(B_{t_0}(x_k)\setminus B_{3t_k})\cap\Omega} \frac{\varphi_k(x_k)-\varphi_k(y)}{|y-x_k|^{N+2s_k}}dy \\
&\geq \int_{(B_{t_0}(x_k)\setminus B_{3t_k})\cap\Omega} \frac{[\varphi_k(x_k)-\varphi_k]^+(y)}{|y-x_k|^{N+2s_k}}dy - 2\delta\int_{(B_{t_0}(x_k)\setminus B_{3t_k})\cap\Omega} |y-x_k|^{-N-2s_k}dy \\
&\geq \frac{1}{2^{N+2s_k}}\int_{(B_{t_0}(x_k)\setminus B_{3t_k})\cap\Omega} \frac{[\varphi_k(x_k)-\varphi_k]^+(y)}{|x_k-y|^{N+2s_k}}dy - 2\cdot 3^{N+2s_k}\delta\int_{(B_{t_0}(x_k)\setminus B_{3t_k})\cap\Omega} |y|^{-N-2s_k}dy \\
&\geq \frac{1}{2^{N+2s_k}}\Big(\int_{(B_{t_0}\setminus B_{3t_k})\cap\Omega} \frac{[\varphi_k(x_k)-\varphi_k]^+(y)}{|y|^{N+2s_k}}dy - \int_{(B_{t_0}\setminus B_{t_0}(x_k))\cap\Omega} \frac{[\varphi_k(x_k)-\varphi_k]^+(y)}{|y|^{N+2s_k}}dy\Big) \\
&\qquad - 2\cdot 3^{N+2s_k}\delta\Big(\int_{(B_{t_0}\setminus B_{3t_k})\cap\Omega} |y|^{-N-2s_k}dy + \int_{(B_{t_0}(x_k)\setminus B_{t_0})} |y|^{-N-2s_k}dy\Big) \\
&\geq \frac{1}{2^{N+2s_k}}\Big(r_k\int_{A_k^-} |y|^{-N-2s_k}dy - (\varepsilon+\delta)\int_{B_{t_0}\setminus B_{t_0}(x_k)} |y|^{-N-2s_k}dy\Big) \\
&\qquad - 2\cdot 3^{N+2s_k}\delta\Big(c_k + \int_{B_{t_0}(x_k)\setminus B_{t_0}} |y|^{-N-2s_k}dy\Big) \\
&\geq \Big(\frac{r_k}{2\cdot 2^{N+2s_k}} - 2\cdot 3^{N+2s_k}\delta\Big)c_k \\
&\qquad - \frac{(\varepsilon+\delta)}{2^{N+2s_k}}\int_{B_{t_0}\setminus B_{t_0-2t_k}} |y|^{-N-2s_k}dy \ - \ 2\cdot 3^{N+2s_k}\delta\int_{B_{t_0+2t_k}\setminus B_{t_0}} |y|^{-N-2s_k}dy
\end{aligned}$$

$$\geq \Big(\frac{\varepsilon-\delta}{2^{N+2+2s_k}} - 2\cdot 3^{N+2s_k}\delta\Big)c_k - o(1) = \Big(\frac{\varepsilon-\delta}{2^{N+2}} - 2\cdot 3^N\delta + o(1)\Big)c_k - o(1)$$

as $k\to\infty$, where we used (3.5.23). By our choice of $\delta>0$ satisfying (3.5.17), we arrive at a contradiction to (3.5.25). The proof is thus finished. □

Finally, we complete the

Proof of Theorem 3.1.3. Since $c_{N,s} := sc_N + o(s)$ as $s\to 0^+$ with $c_N = \pi^{-\frac{N}{2}}\Gamma(\frac{N}{2})$ and $L_\Delta^\Omega = c_N\mathcal{D}_\Omega^0$, then the first part of Theorem 3.1.3 is just a reformulation of Theorems 3.5.3 and 3.5.4.

To see the second part, we first note that $\xi_{n,s_k}\in C(\overline{\Omega})$ for every $k\in\mathbb{N}$ by [14, Theorem 1.3, see also Theorem 4.7]. We may then apply Theorem 3.5.5 to the sequence $(\xi_{n,s_k})_k$ in place of $(\varphi_k)_k$, noting that assumption (3.5.15) is satisfied by Proposition 3.5.1 and Theorem 3.5.2. Consequently, the sequence $(\xi_{n,s_k})_k$ is both bounded in $C(\overline{\Omega})$ and equicontinuous on $\overline{\Omega}$, so it is relatively compact in $C(\overline{\Omega})$ by the Arzelà-Ascoli Theorem. Combining this fact with the convergence property $\xi_{n,s_k}\to\xi_n$ in $L^2(\Omega)$ stated in Theorem 3.5.4, it follows that $\xi_{n,s_k}\to\xi_n$ in $C(\overline{\Omega})$. □

Chapter 4

On the s-derivative of weak solutions of the Poisson problem for the regional fractional Laplacian

In this chapter, we study the differentiability of a solution u_s (regarded as a function of $s \in (0,1)$) to a Poisson equation governed by the regional fractional Laplacian. We also analyze the differentiability of the first nontrivial eigenvalue $\lambda_{1,s}$ of $(-\Delta)^s_\Omega$ regarded as a function of s. The presentation of this note is the same as the original paper [R3]. The notation may be slightly different from those in the previous chapters.

4.1 Introduction

In a bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ with $C^{1,1}$ boundary, we consider the following nonlocal Poisson problem

$$(-\Delta)^s_\Omega u_s = f \quad \text{in} \quad \Omega, \tag{4.1.1}$$

where, $s \in (0,1)$ and $f \in L^\infty(\Omega)$ with $\int_\Omega f\, dx = 0$. Here, $(-\Delta)^s_\Omega$ stands for the regional fractional Laplacian define for all $u \in C^2(\overline{\Omega})$ by

$$(-\Delta)^s_\Omega u(x) = C_{N,s} P.V. \int_\Omega \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \qquad x \in \Omega, \tag{4.1.2}$$

where $P.V.$ is a commonly used abbreviation for "in the principal value sense" and the normalized constant $C_{N,s}$ is defined by

$$C_{N,s} = s(1-s)\pi^{-\frac{N}{2}} 2^{2s} \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(2-s)} \in \Big(0, 4\Gamma\Big(\frac{N}{2}+1\Big)\Big], \tag{4.1.3}$$

Γ being the usual Gamma function. The bounds in (4.1.3) can be found in [144, page 8]. As known, $(-\Delta)^s_\Omega$ represents the infinitesimal generator of the so-called *censored stable Lévy process*, that is, a stable process in which the jumps between Ω and its complement are forbidden (see e.g. [8, 24, 44, 97, 99, 147] and the references therein).

The study of the s-dependence of the solution of the Poisson problem involving nonlocal operators has recently received quite some interest. This kind of study allows a well understanding of the asymptotic behavior of the solution at $s \in (0,1)$. For instance, in [21], the authors analyzed the limit behavior as $s \to 1^-$ of the solution to the fractional Poisson equation $(-\Delta)^s u_s = f_s$, $x \in \Omega$ with homogeneous Dirichlet boundary conditions $u_s \equiv 0$, $x \in \mathbb{R}^N \setminus \Omega$ and provided continuity in a weak setting. We also refer to [61] where the equicontinuity of eigenfunctions and eigenvalues of $(-\Delta)^s$ in Ω was studied for s belonging in a compact subset of $(0,1)$. Small order asymptotics of eigenvalue problems for the operators $(-\Delta)^s$ and $(-\Delta)^s_\Omega$ has been studied recently in [55, 82, 142]. We notice that this type of optimization in the small order-dependent appears in population dynamics [126], in optimal control [11, 139], in fractional harmonic maps [10], and in image processing [9].

Recently, Burkovska and Gunzburger [35, section 5] studied the s-regularity of solutions to Dirichlet problems driven by the fractional peridynamic operator. Precisely, they proved that the solution map $(0,1) \to L^2(\Omega),\ s \mapsto u_s$ is of class C^∞. Their argument is based on differentiating the corresponding bilinear form of the problem with respect to s. In [106], Jarohs, Saldaña, and Weth established the C^1-regularity of the map $(0,1) \to L^\infty(\Omega)$, $s \mapsto u_s$, where u_s is given as the unique weak solution to the fractional Poisson problem $(-\Delta)^s u_s = f$, $x \in \Omega$ with the homogeneous Dirichlet boundary data $u_s \equiv 0$, $x \in \mathbb{R}^N \setminus \Omega$, where Ω is a bounded domain with C^2 boundary and $f \in C^\alpha(\overline{\Omega})$ for some $\alpha > 0$. The main advantage in their analysis relies on the representation formula of the solution u_s by mean of Green function which allows obtaining several important and powerful estimates.

The purpose of the present paper is to analyze the C^1-regularity of the map $(0,1) \to L^2(\Omega)$, $s \mapsto u_s$, where u_s is the unique solution of the fractional Poisson problem (4.1.1). The major difficulty in our development stems from the fact that, contrary to [106], we do not have an explicit representation of the solution u_s in terms of Green function for every $s \in (0,1)$. We note that for $s \in (0, \frac{1}{2}]$, the fractional Poisson problem $(-\Delta)^s_\Omega u_s = f$, $x \in \Omega$ with Dirichlet boundary conditions $u_s \equiv g$, $x \in \partial\Omega$ remains ill-posed since the space $C_c^\infty(\Omega)$ is dense in the fractional Sobolev space $H^s(\Omega)$. We refer to [24] and the references therein for the probabilistic interpretation and to Chen and Wei [54] for recent results from a purely analytic point of view.

Throughout the paper, we consider (4.1.1) as a free Poisson problem, so without Dirichlet boundary condition.

In the following, we present the main results of the present paper. The first main result deals with the differentiability of the solution map $(0,1) \to L^2(\Omega),\ s \mapsto u_s$. It reads as follows.

Theorem 4.1.1. *Let $f \in L^\infty(\Omega)$ with $\int_\Omega f\, dx = 0$ and let u_s be the unique weak solution of* (4.1.1) *(see Section 4.2 below for the definition of weak solution). Then the map*

$$(0,1) \to L^2(\Omega), \quad s \mapsto u_s$$

is of class C^1 and $w_s := \partial_s u_s$ uniquely solves in the weak sense the equation

$$(-\Delta)^s_\Omega w_s = M^s_\Omega u_s \quad in \quad \Omega, \tag{4.1.4}$$

where for every $x \in \Omega$,

$$M^s_\Omega u(x) = -\frac{\partial_s C_{N,s}}{C_{N,s}} f(x) + 2C_{N,s} P.V. \int_\Omega \frac{(u(x) - u(y))}{|x-y|^{N+2s}} \log|x-y|\; dy.$$

We now consider the problem (4.1.1) with $f \equiv \lambda_s u_s$ i.e., the eigenvalue problem

$$(-\Delta)^s_\Omega u_s = \lambda_s u_s \quad \text{in} \quad \Omega.$$

Our second main result is concerned with the one-sided differentiability of the map $s \mapsto \lambda_{1,s}$ where $\lambda_{1,s}$ is the first nontrivial eigenvalue of $(-\Delta)^s_\Omega$ in Ω (see Section 4.4 below for the definition of $\lambda_{1,s}$). It reads as follows.

Theorem 4.1.2. *Regarded as function of s, $\lambda_{1,s}$ is right differentiable on $(0,1)$ and*

$$\partial_s^+ \lambda_{1,s} := \lim_{\sigma \to 0^+} \frac{\lambda_{1,s+\sigma} - \lambda_{1,s}}{\sigma} = \inf\{J_s(u) : u \in \mathcal{M}_s\} \tag{4.1.5}$$

where

$$J_s(u) = \frac{\partial_s C_{N,s}}{C_{N,s}} \lambda_{1,s} - C_{N,s} \int_\Omega \int_\Omega \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} \log|x-y| \; dxdy \tag{4.1.6}$$

and $\mathcal{M}_s$ the set of L^2-normalized eigenfunctions of $(-\Delta)^s_\Omega$ corresponding to $\lambda_{1,s}$. Moreover, the infimum in (4.1.5) is attained.

The stategy of the proof of Theorems 4.1.1 and 4.1.2 is based on differentiating the quadratic form $\mathcal{E}_s$ (see (4.2.2)) with respect to s. To this end, higher Sobolev regularity of any weak solution u_s of (4.1.1) of the form $s + \varepsilon$ is needed, see Proposition 4.2.2 below. The latter is obtained by exploiting boundary regularity result by Fall [73]. Let us also mention that Proposition 4.2.2 plays a crucial role in getting uniform $H^s(\Omega)$-estimates of $u_{s+\sigma}$ and the difference quotient $\frac{u_{s+\sigma}-u_s}{\sigma}$ with respect to σ.

The paper is organized as follows. In Section 4.2 we present some preliminaries that will be useful throughout this article. In Section 4.3, we prove Theorem 4.1.1. Finally, Section 4.4 is devoted to the proof of Theorem 4.1.2.

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4.2 Preliminary and functional setting

In this section, we introduce some preliminary properties that will be useful in this work. First of all, throughout the end of the paper, $d_A := \sup\{|x-y| : x, y \in A\}$ is the diameter of $A \subset \mathbb{R}^N$ and $B_r(x)$ denotes the open ball centered at x with radius r. We also denote by $|A|$ the N-dimensional Lebesgue measure of every set $A \subset \mathbb{R}^N$.
Now, for all $s \in (0,1)$ the usual fractional Sobolev space $H^s(\Omega)$ is defined by

$$H^s(\Omega) = \{u \in L^2(\Omega) : |u|^2_{H^s(\Omega)} < \infty\},$$

where

$$|u|_{H^s(\Omega)} := \Big(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \; dxdy \Big)^{1/2},$$

is the so-called Gagliardo seminorm of u. Moreover, $H^s(\Omega)$ is a Hilbert space endowed with the norm

$$\|u\|_{H^s(\Omega)} := (\|u\|^2_{L^2(\Omega)} + |u|^2_{H^s(\Omega)})^{1/2}.$$

Also, we notice the following continous embedding $H^t(\Omega) \hookrightarrow H^s(\Omega)$ for $t > s$, see e.g. [65, Proposition 2.1]. It is also useful to recall that the space $C^\infty(\overline{\Omega})$ is dense in $H^s(\Omega)$ (see [64, Corollary 2.71]). We recall that $C^\infty(\overline{\Omega})$ denotes the restriction of all $C^\infty(\mathbb{R}^N)$ functions on $\overline{\Omega}$. Moreover, we define the space $\mathbb{X}^s(\Omega)$ consists of functions in $H^s(\Omega)$ orthogonal to constants i.e.,

$$\mathbb{X}^s(\Omega) := \Big\{u \in H^s(\Omega) : \int_\Omega u \; dx = 0\Big\}.$$

Clearly, $\mathbb{X}^s(\Omega)$ is a Hilbert space (with the norm $\|\cdot\|_{\mathbb{X}^s(\Omega)} := |\cdot|_{H^s(\Omega)}$ equivalent to the usual one in $H^s(\Omega)$) for which every function $u \in \mathbb{X}^s(\Omega)$ satisfies the following fractional Poincaré inequality

$$\|u\|^2_{L^2(\Omega)} \leq \gamma_{N,s,\Omega} |u|^2_{H^s(\Omega)} \quad \text{with} \quad \gamma_{N,s,\Omega} = |\Omega|^{-1} d_\Omega^{N+2s}. \tag{4.2.1}$$

We notice also that the space of functions $\varphi \in C^\infty(\overline{\Omega})$ with $\int_\Omega \varphi \; dx = 0$ is dense in $\mathbb{X}^s(\Omega)$. For simplicity, we set $C_0^\infty(\overline{\Omega}) := \{u \in C^\infty(\overline{\Omega}) : \int_\Omega \varphi \; dx = 0\}$. The inner product and the norm in $L^2(\Omega)$ will be denoted by $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ and $\|\cdot\|_{L^2(\Omega)}$ respectively.

Now, let $\mathcal{E}_s$ be the quadratic form define on $H^s(\Omega)$ by

$$(u, v) \mapsto \mathcal{E}_s(u, v) = \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \; dxdy. \tag{4.2.2}$$

We have the following.

Definition 4.2.1. Let $f \in L^\infty(\Omega)$ with $\int_\Omega f \; dx = 0$. We say that $u_s \in \mathbb{X}^s(\Omega)$ is a weak solution of (4.1.1) if

$$\mathcal{E}_s(u_s, \varphi) = \int_\Omega f\varphi \; dx, \quad \forall \varphi \in \mathbb{X}^s(\Omega). \tag{4.2.3}$$

The existence and uniqueness of weak solution of the Poisson problem (4.1.1) in $\mathbb{X}^s(\Omega)$ is guaranteed by Riesz representation theorem.

Let $u_s \in \mathbb{X}^s(\Omega)$ be the unique weak solution of (4.1.1). Then, thanks to [142], there exists a constant $c_1 > 0$ independent of s such that

$$\|u_s\|_{L^\infty(\Omega)} \leq c_1. \tag{4.2.4}$$

Very recently, Fall [73] established boundary regularity for any weak solution to (4.1.1). Presicely, among other results, he proved that $u_s \in C^\beta(\overline{\Omega})$ and

$$\|u_s\|_{C^\beta(\overline{\Omega})} \leq C(\|u_s\|_{L^2(\Omega)} + \|f\|_{L^p(\Omega)}) \tag{4.2.5}$$

with

$$\beta := 2s - \frac{N}{p}$$

for every $s \in (0, 1)$ and $p > \frac{N}{2s}$. Moreover if $s \in (\frac{1}{2}, 1)$ so that $\beta = 2s - \frac{N}{p} > 1$, he also obtained boundary Hölder regularity for the gradient of the form $\nabla u_s \in C^{\beta-1}(\overline{\Omega})$ with

$$\|\nabla u_s\|_{C^{\beta-1}(\overline{\Omega})} \leq C(\|u_s\|_{L^2(\Omega)} + \|f\|_{L^p(\Omega)}). \tag{4.2.6}$$

From now on and without loss of generality, we fix p such that $p > \frac{N}{s}$. Moreover, the constant C appearing in (4.2.5) is continuous at $s \in [s_0, 1)$ for some $s_0 \in (0,1)$ see [73, Theorem 1.1]. The same conclusion holds for (4.2.6) provided that $s \in [s_0, 1)$ for some $s_0 \in (\frac{1}{2}, 1)$ see [73, Remark 1.4]. Hence, by taking into account (4.2.4) we obtain from (4.2.5) and (4.2.6) uniform bound with respect to s on C^β and $C^{\beta-1}$ norm of u_s and ∇u_s respectively as follows

$$\|u_s\|_{C^\beta(\overline{\Omega})} \le c_2 \qquad \text{and} \qquad \|\nabla u_s\|_{C^{\beta-1}(\overline{\Omega})} \le c_3. \tag{4.2.7}$$

As a direct advantage of the above boundary regularity, we derive in the next proposition, higher Sobolev regularity for solution of (4.1.1).

Proposition 4.2.2. *Let $f \in L^\infty(\Omega)$ with $\int_\Omega f\, dx = 0$ and let $u_{s+\sigma} \in H^{s+\sigma}(\Omega) \cap L^\infty(\Omega)$ be the unique weak solution of problem (4.1.1) with s replaced by $s+\sigma$. Then $u_{s+\sigma} \in H^{s+\varepsilon}(\Omega)$ for some $\varepsilon > 0$ and*

$$\|u_{s+\sigma}\|_{H^{s+\varepsilon}(\Omega)} \le K \qquad \textit{for all } \sigma \in (-s_0, s_0) \tag{4.2.8}$$

for some $s_0 > 0$.

Proof. Let $u_{s+\sigma} \in H^{s+\sigma}(\Omega) \cap L^\infty(\Omega)$ be the unique weak solution of (4.1.1) with s replaced by $s+\sigma$ for all $\sigma \in (-s_0, s_0)$ for some $s_0 > 0$. Then,

(i) If $2s \le 1$ then from (4.2.5) we have that

$$\begin{aligned}
\int_\Omega \int_\Omega \frac{|u_{s+\sigma}(x) - u_{s+\sigma}(y)|^2}{|x-y|^{N+2(s+\varepsilon)}}\, dxdy &\le \|u_s\|^2_{C^{2(s+\sigma)-\frac{N}{p}}(\overline{\Omega})} \int_\Omega \int_\Omega |x-y|^{4(s+\sigma)-\frac{2N}{p}-N-2(s+\varepsilon)}\, dxdy \\
&\le d_\Omega^{4\sigma} \|u_s\|^2_{C^{2s-\frac{N}{p}}(\overline{\Omega})} \int_\Omega \int_{B_{d_\Omega}(x)} |x-y|^{2s-\frac{2N}{p}-N-2\varepsilon}\, dydx \\
&\le \frac{\max\{d_\Omega^{-4s_0}, d_\Omega^{4s_0}\} |S^{N-1}| |\Omega| \|u_s\|^2_{C^{2s-\frac{N}{p}}(\overline{\Omega})}}{2(s-\varepsilon-\frac{N}{p})} d_\Omega^{2(s-\varepsilon-\frac{N}{p})} < \infty
\end{aligned}$$

provided that $0 < \varepsilon < s - \frac{N}{p}$. This shows that $u_{s+\sigma} \in H^{s+\varepsilon}(\Omega)$ with uniform bound in $\sigma \in (-s_0, s_0)$ provided that $0 < \varepsilon < s - \frac{N}{p}$.

(ii) If $2s > 1$ then from (4.2.6) we have that $\nabla u_{s+\sigma} \in L^\infty(\overline{\Omega}) \subset L^\infty(\Omega)$ and therefore $\nabla u_{s+\sigma} \in L^2(\Omega)$. Since also $u_{s+\sigma} \in L^2(\Omega)$, we deduce that $u_{s+\sigma} \in H^1(\Omega)$. Hence,

$$u_{s+\sigma} \in H^s(\Omega) \cap H^1(\Omega) \qquad \text{for all } \sigma \in [0, s_0) \tag{4.2.9}$$

$$u_{s+\sigma} \in H^{\alpha(s)}(\Omega) \cap H^1(\Omega) \qquad \text{for all } \sigma \in (-s_0, 0] \tag{4.2.10}$$

for some $\alpha(s) << s$ depending only on s. Applying the well-known Gagliardo-Nirenberg interpolation inequality (see e.g. [29, Theorem 1]), we find that $u_{s+\sigma} \in H^r(\Omega)$ with

(a) $r = \theta s + (1-\theta) \cdot 1$ for all $\theta \in (0,1)$ in the case (4.2.9), and

$$\|u_{s+\sigma}\|_{H^r(\Omega)} \le C(\theta, s, \Omega) \|u_{s+\sigma}\|^\theta_{H^s(\Omega)} \|u_{s+\sigma}\|^{1-\theta}_{H^1(\Omega)}. \tag{4.2.11}$$

(b) $r = \theta\alpha(s) + (1-\theta) \cdot 1$ for all $\theta \in (0,1)$ in the case (4.2.10), and

$$\|u_{s+\sigma}\|_{H^r(\Omega)} \le C(\theta, s, \Omega) \|u_{s+\sigma}\|^\theta_{H^{\alpha(s)}(\Omega)} \|u_{s+\sigma}\|^{1-\theta}_{H^1(\Omega)}. \tag{4.2.12}$$

Let us focus on the situation (a). By choosing in particular $\theta = \frac{1}{2}$, then $r = \frac{s}{2} + \frac{1}{2} = s + \frac{1-s}{2}$ and we have that $u_{s+\sigma} \in H^{s+\frac{1-s}{2}}(\Omega)$. From this, we conclude that $u_{s+\sigma} \in H^{s+\varepsilon}(\Omega)$ for every $0 < \varepsilon < \frac{1-s}{2}$. To complete the proof, it remains to show that the RHS of (4.2.11) is uniform for σ sufficiently small.

From (4.2.4) and (4.2.7) we have that

$$\|u_{s+\sigma}\|_{H^1(\Omega)} \le C_1 \qquad \text{for all } \ \sigma \ \text{ sufficiently small.} \tag{4.2.13}$$

On the other hand, since $s < s+\sigma$, then from [65, Proposition 2.1] there exists $c > 0$ depending only on s and N such that

$$|u_{s+\sigma}|_{H^s(\Omega)} \le c|u_{s+\sigma}|_{H^{s+\sigma}(\Omega)}. \tag{4.2.14}$$

Using now $u_{s+\sigma}$ as a test function in (4.1.1) with s replace by $s+\sigma$ and integrating over Ω, one has

$$|u_{s+\sigma}|^2_{H^{s+\sigma}(\Omega)} = \frac{2}{C_{N,s+\sigma}} \int_\Omega f u_{s+\sigma}\, dx \le \frac{2|\Omega|}{C_{N,s+\sigma}} \|f\|_{L^\infty(\Omega)} \|u_{s+\sigma}\|_{L^\infty(\Omega)} \le c \quad \text{as} \ \ \sigma \to 0^+, \tag{4.2.15}$$

thanks to (4.2.4) and the continuity of the map $s \mapsto C_{N,s}$. This, together with (4.2.14) yield

$$\|u_{s+\sigma}\|_{H^s(\Omega)} \le C_2 \qquad \text{for } \ \sigma \ \text{ sufficiently small.} \tag{4.2.16}$$

From this, one gets $u_{s+\sigma} \in H^{s+\varepsilon}(\Omega)$ with uniform bound in $\sigma \in [0, s_0)$.

In situation (b), a similar argument as above yields $u_{s+\sigma} \in H^{s+\tilde{\varepsilon}}(\Omega)$ for some $\tilde{\varepsilon} > 0$ depending on s.

Now, by combining (i) and (ii), we conclude the proof. $\square$

This higher Sobolev regularity will be of a capital interest in the rest of the paper.
Next, we recall the following decay estimate regarding the logarithmic function. For all $r, \varepsilon_0 > 0$, there holds that

$$|\log|z|| \le \frac{1}{e\varepsilon_0}|z|^{-\varepsilon_0} \ \text{ if } \ |z| \le r \quad \text{ and } \quad |\log|z|| \le \frac{1}{e\varepsilon_0}|z|^{\varepsilon_0} \ \text{ if } \ |z| \ge r. \tag{4.2.17}$$

We end this section by recalling the following.

Proposition 4.2.3. *([106, Lemma 6.6]) Let $I \subset \mathbb{R}$ be an open interval, E be a Banach space and $\gamma : I \to E$ be a curve with the following properties*

(i) γ is continuous.

(ii) $\partial_s^+\gamma(s) := \lim\limits_{\sigma\to 0^+} \frac{\gamma(s+\sigma)-\gamma(s)}{\sigma}$ exists in E for all $s \in I$.

(iii) The map $I \to E$, $s \mapsto \partial_s^+\gamma(s)$ is continuous.

Then γ is continuously differentiable with $\partial_s\gamma = \partial_s^+\gamma$.

4.3 Differentiability of the solution map in $(0,1)$

In this section, we are concerned with the regularity of the map $(0,1) \to L^2(\Omega), s \mapsto u_s$, with being u_s the unique weak solution of (4.1.1). In order to obtain the regularity of the solution u_s, regarded as function of s, our strategy consist to bound uniformly the difference quotient $\frac{u_{s+\sigma}-u_s}{\sigma}$ in the Hilbert space $H^s(\Omega)$ with respect to σ, after what, due to compactness, we therefore reach our goal.

We restate Theorem 4.1.1 from the introduction for the reader's convenience.

Theorem 4.3.1. *Let $f \in L^\infty(\Omega)$ with $\int_\Omega f\, dx = 0$ and let $u_s \in \mathbb{X}^s(\Omega)$ be the unique weak solution of* (4.1.1)*. Then the map*

$$(0,1) \to L^2(\Omega), \quad s \mapsto u_s$$

is of class C^1 and $w_s := \partial_s u_s$ uniquely solves in the weak sense the equation

$$(-\Delta)^s_\Omega w_s = M^s_\Omega u_s \quad in \quad \Omega, \tag{4.3.1}$$

where for every $x \in \Omega$,

$$M^s_\Omega u(x) = -\frac{\partial_s C_{N,s}}{C_{N,s}} f(x) + 2C_{N,s} P.V. \int_\Omega \frac{(u(x)-u(y))}{|x-y|^{N+2s}} \log|x-y|\; dy.$$

Proof. The proof is devided into three steps.

Step 1. We prove that the solution map $(0,1) \to L^2(\Omega), s \mapsto u_s$ is continuous.

Fix $s_0 \in (0,1)$. Let $\delta \in (0, s_0)$ and $(s_n)_n \subset (s_0-\delta, 1)$ be a sequence such that $s_n \to s_0$. We want to show that

$$u_{s_n} \to u_{s_0} \quad \text{in} \quad L^2(\Omega) \quad \text{as} \qquad n \to \infty. \tag{4.3.2}$$

Put $s' := \inf_{n\in\mathbb{N}} s_n > s_0 - \delta$. Then,

$$\begin{aligned} |u_{s_n}|^2_{H^{s'}(\Omega)} &= \int_\Omega\int_\Omega \frac{(u_{s_n}(x)-u_{s_n}(y))^2}{|x-y|^{N+2s'}}\; dxdy \\ &= \int_\Omega\int_\Omega \frac{(u_{s_n}(x)-u_{s_n}(y))^2}{|x-y|^{N+2s_n}} |x-y|^{2(s_n-s')}\; dxdy \\ &\le d_\Omega^{2(s_n-s')} |u_{s_n}|^2_{H^{s_n}(\Omega)} \le c|u_{s_n}|^2_{H^{s_n}(\Omega)}. \end{aligned} \tag{4.3.3}$$

Now, since u_{s_n} is the unique weak solution to $(-\Delta)^{s_n}_\Omega u_{s_n} = f$ in Ω, then it follows that (we use u_{s_n} as a test function)

$$|u_{s_n}|^2_{H^{s_n}(\Omega)} = \frac{2}{C_{N,s_n}} \int_\Omega f u_{s_n}\; dx \le \frac{2}{C_{N,s_n}} \|f\|_{L^2(\Omega)} \|u_{s_n}\|_{L^2(\Omega)} \le \frac{2c_{s'}}{C_{N,s_n}} \|f\|_{L^2(\Omega)} |u_{s_n}|_{H^{s'}(\Omega)} \tag{4.3.4}$$

thanks to fractional Poincaré inequality (4.2.1). Using that $s \mapsto C_{N,s}$ is continuous, then it follows from (4.3.4) and (4.3.3) that

$$|u_{s_n}|_{H^{s'}(\Omega)} \le c \qquad \text{as} \quad n \to \infty. \tag{4.3.5}$$

This means that $(u_{s_n})_n$ is uniformly bounded in $H^{s'}(\Omega)$. Then there is $u_* \in H^{s'}(\Omega)$ such that after passing to a subsequence,

$$\begin{aligned} u_{s_n} &\rightharpoonup u_* \quad \text{weakly in} \quad H^{s'}(\Omega), \\ u_{s_n} &\to u_* \quad \text{strongly in} \quad L^2(\Omega), \\ u_{s_n} &\to u_* \quad \text{a.e. in} \quad \Omega. \end{aligned} \tag{4.3.6}$$

In particular, $\int_\Omega u_* \, dx = 0$. We wish now to show that $u_{s_0} \equiv u_*$. To this end, we first prove that $u_* \in H^{s_0}(\Omega)$.

By Fatou's Lemma, we have

$$|u_*|^2_{H^{s_0}(\Omega)} = \int_\Omega \int_\Omega \frac{(u_*(x) - u_*(y))^2}{|x-y|^{N+2s_0}} \, dxdy \le \liminf_{n\to\infty} \int_\Omega \int_\Omega \frac{(u_{s_n}(x) - u_{s_n}(y))^2}{|x-y|^{N+2s_n}} \, dxdy$$
$$= \liminf_{n\to\infty} |u_{s_n}|^2_{H^{s_n}(\Omega)} = \frac{2}{C_{N,s_0}} \|u_*\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)} < \infty. \tag{4.3.7}$$

This implies that $u_* \in H^{s_0}(\Omega)$. We recall that in (4.3.7), we have used (4.3.4) and (4.3.6).

On the other hand, for all $\varphi \in C_0^\infty(\overline{\Omega})$, we have

$$\langle f, \varphi \rangle_{L^2(\Omega)} = \int_\Omega f\varphi \, dx = \lim_{n\to\infty} \mathcal{E}_{s_n}(u_{s_n}, \varphi) = \lim_{n\to\infty} \int_\Omega u_{s_n} (-\Delta)^{s_n}_\Omega \varphi \, dx$$
$$= \int_\Omega u_* (-\Delta)^{s_0}_\Omega \varphi \, dx = \mathcal{E}_{s_0}(u_*, \varphi).$$

This shows that $u_* \in H^{s_0}(\Omega)$ with $\int_\Omega u_* \, dx = 0$ distributionaly solves the Poisson problem $(-\Delta)^{s_0}_\Omega u_* = f$ in Ω. Recalling that $u_{s_0} \in H^{s_0}(\Omega)$ with $\int_\Omega u_{s_0} \, dx = 0$ is the unique weak (distributional) solution to the Poisson problem $(-\Delta)^{s_0}_\Omega u_{s_0} = f$ in Ω, we find that $u_* \equiv u_{s_0}$, as wanted.

Step 2. We show that the solution map $(0,1) \to L^2(\Omega)$, $s \mapsto u_s$ is right differentiable.

Fix $s \in (0,1)$ and define

$$v_\sigma = \frac{u_{s+\sigma} - u_s}{\sigma}. \tag{4.3.8}$$

Here, $u_{s+\sigma}$ is the unique weak solution of (4.1.1) with s replaced by $s+\sigma$. We wish first to study the asymptotic behavior of v_σ as $\sigma \to 0^+$.

For all $\varphi \in C_0^\infty(\overline{\Omega})$,

$$\mathcal{E}_s(u_s, \varphi) = \int_\Omega f\varphi \, dx = \mathcal{E}_{s+\sigma}(u_{s+\sigma}, \varphi) = \mathcal{E}_s(u_s - u_{s+\sigma}, \varphi) + \mathcal{E}_s(u_{s+\sigma}, \varphi)$$

that is

$$\mathcal{E}_s(u_s - u_{s+\sigma}, \varphi) = \mathcal{E}_{s+\sigma}(u_{s+\sigma}, \varphi) - \mathcal{E}_s(u_{s+\sigma}, \varphi). \tag{4.3.9}$$

Now,

$$\begin{aligned}
&\mathcal{E}_{s+\sigma}(u_{s+\sigma}, \varphi) - \mathcal{E}_s(u_{s+\sigma}, \varphi) \\
&= \frac{C_{N,s+\sigma}}{2} \int_\Omega \int_\Omega \frac{(u_{s+\sigma}(x) - u_{s+\sigma}(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s+2\sigma}} \, dxdy \\
&\qquad - \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u_{s+\sigma}(x) - u_{s+\sigma}(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dxdy \\
&= \frac{C_{N,s+\sigma} - C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u_{s+\sigma}(x) - u_{s+\sigma}(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s+2\sigma}} \, dxdy \\
&+ \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \Big(\frac{1}{|x-y|^{N+2s+2\sigma}} - \frac{1}{|x-y|^{N+2s}} \Big) (u_{s+\sigma}(x) - u_{s+\sigma}(y))(\varphi(x) - \varphi(y)) \, dxdy
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{C_{N,s+\sigma}} \times (C_{N,s+\sigma} - C_{N,s})\mathcal{E}_{s+\sigma}(u_{s+\sigma}, \varphi) \\
&\quad + \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \Big(|x-y|^{-2\sigma} - 1 \Big) \frac{(u_{s+\sigma}(x) - u_{s+\sigma}(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dxdy \\
&= \frac{1}{C_{N,s+\sigma}} \times (C_{N,s+\sigma} - C_{N,s}) \int_\Omega f\varphi \, dx \\
&\quad + \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \Big(|x-y|^{-2\sigma} - 1 \Big) \frac{(u_{s+\sigma}(x) - u_{s+\sigma}(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dxdy. \qquad (4.3.10)
\end{aligned}
$$

Next, we write

$$
\begin{aligned}
|x-y|^{-2\sigma} - 1 &= \exp(-2\sigma \log|x-y|) - 1 = -2\sigma \log|x-y| \int_0^1 \exp(-2t\sigma \log|x-y|) \, dt \\
&= -2\sigma \psi_\sigma(x,y) \log|x-y| \quad \text{with} \quad \psi_\sigma(x,y) = \int_0^1 \exp(-2t\sigma \log|x-y|) \, dt. \qquad (4.3.11)
\end{aligned}
$$

Plugging (4.3.11) into (4.3.10), we get

$$
\begin{aligned}
&\mathcal{E}_{s+\sigma}(u_{s+\sigma}, \varphi) - \mathcal{E}_s(u_{s+\sigma}, \varphi) \\
&= \frac{1}{C_{N,s+\sigma}} \times (C_{N,s+\sigma} - C_{N,s}) \int_\Omega f\varphi \, dx \\
&- \sigma C_{N,s} \int_\Omega \int_\Omega \frac{(u_{s+\sigma}(x) - u_{s+\sigma}(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \psi_\sigma(x,y) \log|x-y| \, dxdy. \qquad (4.3.12)
\end{aligned}
$$

Equations (4.3.9) and (4.3.12) yield

$$
\begin{aligned}
\mathcal{E}_s(v_\sigma, \varphi) &= -\frac{1}{C_{N,s+\sigma}} \times \frac{C_{N,s+\sigma} - C_{N,s}}{\sigma} \int_\Omega f\varphi \, dx \\
&\quad + C_{N,s} \int_\Omega \int_\Omega \frac{(u_{s+\sigma}(x) - u_{s+\sigma}(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \psi_\sigma(x,y) \log|x-y| \, dxdy \qquad (4.3.13)
\end{aligned}
$$

for all $\varphi \in C_0^\infty(\overline{\Omega})$. Now, by density, there is $\varphi_n \in C_0^\infty(\overline{\Omega})$ such that $\varphi_n \to v_\sigma$ in $H^{s+\varepsilon}(\Omega)$ for $\varepsilon > 0$. Moreover, from (4.3.12),

$$
\begin{aligned}
\mathcal{E}_s(v_\sigma, \varphi_n) &= -\frac{1}{C_{N,s+\sigma}} \times \frac{C_{N,s+\sigma} - C_{N,s}}{\sigma} \int_\Omega f\varphi_n \, dx \\
&\quad + C_{N,s} \int_\Omega \int_\Omega \frac{(u_{s+\sigma}(x) - u_{s+\sigma}(y))(\varphi_n(x) - \varphi_n(y))}{|x-y|^{N+2s}} \psi_\sigma(x,y) \log|x-y| \, dxdy. \qquad (4.3.14)
\end{aligned}
$$

Using Cauchy-Schwarz inequality,

$$
\begin{aligned}
&|\mathcal{E}_s(v_\sigma, \varphi_n) - \mathcal{E}_s(v_\sigma, v_\sigma)| \\
&\le \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|v_\sigma(x) - v_\sigma(y)||(\varphi_n(x) - \varphi_n(y)) - (v_\sigma(x) - v_\sigma(y))|}{|x-y|^{N+2s}} \, dxdy \\
&\le \frac{C_{N,s}}{2} |v_\sigma|_{H^s(\Omega)} |\varphi_n - v_\sigma|_{H^s(\Omega)} \to 0 \qquad \text{as} \quad n \to \infty. \qquad (4.3.15)
\end{aligned}
$$

This implies that

$$
\mathcal{E}_s(v_\sigma, \varphi_n) \to \mathcal{E}_s(v_\sigma, v_\sigma) \qquad \text{as} \quad n \to \infty. \qquad (4.3.16)
$$

Since also $\varphi_n \to v_\sigma$ in $L^2(\Omega)$, thanks to Poincaré ineqality, then

$$\int_\Omega f\varphi_n \, dx \to \int_\Omega f v_\sigma \, dx. \tag{4.3.17}$$

On the other hand, using that

$$|\psi_\sigma(x,y)| \leq \max\{1, \exp(-2\sigma \log|x-y|)\} \tag{4.3.18}$$

then applying again Cauchy-Schwarz inequality, one can also show that

$$\begin{aligned}
&\int_\Omega\int_\Omega \frac{(u_{s+\sigma}(x)-u_{s+\sigma}(y))(\varphi_n(x)-\varphi_n(y))}{|x-y|^{N+2s}}\psi_\sigma(x,y)\log|x-y| \, dxdy \\
&\to \int_\Omega\int_\Omega \frac{(u_{s+\sigma}(x)-u_{s+\sigma}(y))(v_\sigma(x)-v_\sigma(y))}{|x-y|^{N+2s}}\psi_\sigma(x,y)\log|x-y| \, dxdy
\end{aligned} \tag{4.3.19}$$

as $n \to \infty$.

Combining (4.3.16), (4.3.17) and (4.3.19), then from (4.3.14) it follows that

$$\begin{aligned}
\mathcal{E}_s(v_\sigma, v_\sigma) = &-\frac{1}{C_{N,s+\sigma}} \times \frac{C_{N,s+\sigma}-C_{N,s}}{\sigma}\int_\Omega f v_\sigma \, dx \\
&+ C_{N,s}\int_\Omega\int_\Omega \frac{(u_{s+\sigma}(x)-u_{s+\sigma}(y))(v_\sigma(x)-v_\sigma(y))}{|x-y|^{N+2s}}\psi_\sigma(x,y)\log|x-y| \, dxdy.
\end{aligned} \tag{4.3.20}$$

From (4.3.20), we write

$$\begin{aligned}
\frac{C_{N,s}}{2}|v_\sigma|^2_{H^s(\Omega)} = \mathcal{E}_s(v_\sigma, v_\sigma) \leq &\frac{1}{C_{N,s+\sigma}}\Big|\frac{C_{N,s+\sigma}-C_{N,s}}{\sigma}\Big|\int_\Omega |f||v_\sigma| \, dx \\
&+ C_{N,s}\int_\Omega\int_\Omega \frac{|u_{s+\sigma}(x)-u_{s+\sigma}(y)||v_\sigma(x)-v_\sigma(y)|}{|x-y|^{N+2s}}|\psi_\sigma(x,y)||\log|x-y|| \, dxdy.
\end{aligned} \tag{4.3.21}$$

Since the map $(0,1) \ni s \mapsto C_{N,s}$ is of class C^1, then

$$\frac{1}{C_{N,s+\sigma}}\Big|\frac{C_{N,s+\sigma}-C_{N,s}}{\sigma}\Big| \leq \frac{|\partial_s C_{N,s}|}{C_{N,s}} + o(1) \qquad \text{as} \;\; \sigma \to 0^+. \tag{4.3.22}$$

Now, Hölder inequality and Poincaré inequality (see (4.2.1)) yield

$$\int_\Omega |f||v_\sigma| \, dx \leq \|f\|_{L^2(\Omega)}\|v_\sigma\|_{L^2(\Omega)} \leq C(N,s,\Omega,\|f\|_{L^2(\Omega)})|v_\sigma|_{H^s(\Omega)}. \tag{4.3.23}$$

On the other hand, from (4.3.18), we have

$$\begin{aligned}
&\int_\Omega\int_\Omega \frac{|u_{s+\sigma}(x)-u_{s+\sigma}(y)||v_\sigma(x)-v_\sigma(y)|}{|x-y|^{N+2s}}|\psi_\sigma(x,y)||\log|x-y|| \, dxdy \\
&\leq \int_\Omega\int_\Omega \frac{|u_{s+\sigma}(x)-u_{s+\sigma}(y)||v_\sigma(x)-v_\sigma(y)|}{|x-y|^{N+2s}}|\log|x-y|| \, dxdy \\
&\quad + \int_\Omega\int_\Omega \frac{|u_{s+\sigma}(x)-u_{s+\sigma}(y)||v_\sigma(x)-v_\sigma(y)|}{|x-y|^{N+2s+2\sigma}}|\log|x-y|| \, dxdy.
\end{aligned}$$

To estimate the first term on the RHS of the above inequality, we use Cauchy-Schwarz inequality together with the Logarithmic decay (4.2.17):

$$
\begin{aligned}
&\int_\Omega\int_\Omega \frac{|u_{s+\sigma}(x)-u_{s+\sigma}(y)||v_\sigma(x)-v_\sigma(y)|}{|x-y|^{N+2s}}|\log|x-y||\,dxdy\\
&\le \Big(\int_\Omega\int_\Omega \frac{|u_{s+\sigma}(x)-u_{s+\sigma}(y)|^2}{|x-y|^{N+2s}}|\log|x-y||^2\,dxdy\Big)^{1/2}|v_\sigma|_{H^s(\Omega)}\\
&\le \left(\frac{1}{(e\varepsilon_0)^2}\int_\Omega\int_\Omega \frac{|u_{s+\sigma}(x)-u_{s+\sigma}(y)|^2}{|x-y|^{N+2s}}(|x-y|^{-2\varepsilon_0}+|x-y|^{2\varepsilon_0})\,dxdy\right)^{1/2}|v_\sigma|_{H^s(\Omega)}\\
&\le c(\varepsilon_0)\left(\int_\Omega\int_\Omega \frac{|u_{s+\sigma}(x)-u_{s+\sigma}(y)|^2}{|x-y|^{N+2s+2\varepsilon_0}}\,dxdy + d_\Omega^{2\varepsilon_0}|u_{s+\sigma}|^2_{H^s(\Omega)}\right)^{1/2}|v_\sigma|_{H^s(\Omega)}\\
&= c(\varepsilon_0)\Big(|u_{s+\sigma}|^2_{H^{s+\varepsilon_0}(\Omega)} + d_\Omega^{2\varepsilon_0}|u_{s+\sigma}|^2_{H^s(\Omega)}\Big)^{1/2}|v_\sigma|_{H^s(\Omega)}. \qquad (4.3.24)
\end{aligned}
$$

By Proposition 4.2.2 there exist $K_1, K_2 > 0$ independent on σ such that

$$|u_{s+\sigma}|_{H^{s+\varepsilon_0}(\Omega)} \le K_1 \quad \text{and} \quad |u_{s+\sigma}|_{H^s(\Omega)} \le K_2 \quad \text{for } \sigma \text{ sufficiently small.} \qquad (4.3.25)$$

Combining this with (4.3.24), we obtain that

$$\int_\Omega\int_\Omega \frac{|u_{s+\sigma}(x)-u_{s+\sigma}(y)||v_\sigma(x)-v_\sigma(y)|}{|x-y|^{N+2s}}|\log|x-y||\,dxdy \le c|v_\sigma|_{H^s(\Omega)}. \qquad (4.3.26)$$

By a similar argument as above, we also obtain the following bound

$$\int_\Omega\int_\Omega \frac{|u_{s+\sigma}(x)-u_{s+\sigma}(y)||v_\sigma(x)-v_\sigma(y)|}{|x-y|^{N+2s+2\sigma}}|\log|x-y||\,dxdy \le c|v_\sigma|_{H^s(\Omega)}. \qquad (4.3.27)$$

Combining (4.3.22), (4.3.23), (4.3.26), and (4.3.27), we find from (4.3.21) that

$$|v_\sigma|_{H^s(\Omega)} \le c. \qquad (4.3.28)$$

In other words, v_σ is uniformly bounded in $H^s(\Omega)$ with respect to σ. Therefore, after passing to a subsequence, there is $w_s \in H^s(\Omega)$ such that

$$
\begin{aligned}
&v_\sigma \rightharpoonup w_s \quad \text{weakly in } \ H^s(\Omega),\\
&v_\sigma \to w_s \quad \text{strongly in } \ L^2(\Omega), \qquad (4.3.29)\\
&v_\sigma \to w_s \quad \text{a.e. in } \ \Omega.
\end{aligned}
$$

In particular, $\int_\Omega w_s\,dx = 0$ since does v_σ.

To obtain the right-differentiability of the solution map $s \mapsto u_s$, it suffices to show that w_s is unique as a limit of the whole sequence v_σ.

First of all, from (4.3.13), thanks to Proposition 4.2.2 and Dominated Convergence Theorem, we deduce that w_s solves

$$\mathcal{E}_s(w_s,\varphi) = -\frac{\partial_s C_{N,s}}{C_{N,s}}\int_\Omega f\varphi\,dx$$

$$+ C_{N,s} \int_\Omega \int_\Omega \frac{(u_s(x) - u_s(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \log|x-y| \ dxdy \tag{4.3.30}$$

for all $\varphi \in C_0^\infty(\overline{\Omega})$.

Let denote by v_{σ_k} the corresponding subsequence of v_σ for which (4.3.29) holds. Consider now another subsequence v_{σ_i} with $v_{\sigma_i} \to \overline{w}_s$ for some $\overline{w}_s \in H^s(\Omega)$ with $\int_\Omega \overline{w}_s \ dx = 0$. We wish to prove that prove that $w_s = \overline{w}_s$. Let set $W_s = w_s - \overline{w}_s$. Then, in particular, $\int_\Omega W_s \ dx = 0$. Moreover, for all $\varphi \in C_0^\infty(\overline{\Omega})$,

$$\begin{aligned} \mathcal{E}_s(W_s, \varphi) &= \mathcal{E}_s(w_s - \overline{w}_s, \varphi) = \mathcal{E}_s(w_s, \varphi) - \mathcal{E}_s(\overline{w}_s, \varphi) = \mathcal{E}_s(w_s, \varphi) - \lim_{i\to\infty} \mathcal{E}_s(v_{\sigma_i}, \varphi) \\ &= \mathcal{E}_s(w_s, \varphi) - \lim_{i\to\infty} \mathcal{E}_s(v_{\sigma_i} - w_s + w_s, \varphi) = - \lim_{i\to\infty} \mathcal{E}_s(v_{\sigma_i} - w_s, \varphi) \\ &= - \lim_{i\to\infty} \lim_{k\to\infty} \mathcal{E}_s(v_{\sigma_i} - v_{\sigma_k}, \varphi). \end{aligned} \tag{4.3.31}$$

From (4.3.13), one gets

$$\begin{aligned} \mathcal{E}_s(v_{\sigma_i} - v_{\sigma_k}, \varphi) = &\left(- \frac{1}{C_{N,s+\sigma_i}} \times \frac{C_{N,s+\sigma_i} - C_{N,s}}{\sigma_i} + \frac{1}{C_{N,s+\sigma_k}} \times \frac{C_{N,s+\sigma_k} - C_{N,s}}{\sigma_k} \right) \int_\Omega f\varphi \ dx \\ &+ C_{N,s} \int_\Omega \int_\Omega \frac{(u_{s+\sigma_i}(x) - u_{s+\sigma_i}(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \psi_{\sigma_i}(x,y) \log|x-y| \ dxdy \\ &- C_{N,s} \int_\Omega \int_\Omega \frac{(u_{s+\sigma_k}(x) - u_{s+\sigma_k}(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \psi_{\sigma_k}(x,y) \log|x-y| \ dxdy. \end{aligned}$$

Using that $s \mapsto C_{N,s}$ is of class C^1, the fact that $s \mapsto u_s$ is continuous and Proposition 4.2.2, the Dominated Convergence Theorem yields

$$\lim_{i\to\infty} \lim_{k\to\infty} \mathcal{E}_s(v_{\sigma_i} - v_{\sigma_k}, \varphi) = 0. \tag{4.3.32}$$

Consequently, one gets from (4.3.31) that

$$\mathcal{E}_s(W_s, \varphi) = 0. \tag{4.3.33}$$

By density, (4.3.33) also holds with φ replaced by W_s, that is,

$$\mathcal{E}_s(W_s, W_s) = 0. \tag{4.3.34}$$

This implies that $W_s = const$. Morever, since also $\int_\Omega W_s \ dx = 0$, then we get that $W_s = 0$, that is, $w_s = \overline{w}_s$ as wanted. In conclusion, the solution map $s \mapsto u_s$ is right-differentiable with $\partial_s^+ u_s = w_s$, solving uniquely (4.3.30).

Step 3. We establishe the continuity of the map $s \mapsto \partial_s^+ u_s = w_s$. The proof of this is similar to that in **Step 1** and we include it for the sake of completeness. Fix again $s_0 \in (0,1)$. Let $\delta \in (0, s_0)$ and $(s_n)_n \subset (s_0 - \delta, 1)$ be a sequence such that $s_n \to s_0$. By putting also $s' := \inf_{n\in\mathbb{N}} s_n > s_0 - \delta$, then as in (4.3.3) one get that

$$|w_{s_n}|^2_{H^{s'}(\Omega)} \le c |w_{s_n}|^2_{H^{s_n}(\Omega)}. \tag{4.3.35}$$

Now, from (4.3.30), it follows that

$$\mathcal{E}_{s_n}(w_{s_n}, w_{s_n}) = - \frac{\partial_{s_n} C_{N,s_n}}{C_{N,s_n}} \int_\Omega f w_{s_n} \ dx$$

$$+ C_{N,s_n} \int_\Omega \int_\Omega \frac{(u_{s_n}(x) - u_{s_n}(y))(w_{s_n}(x) - w_{s_n}(y))}{|x-y|^{N+2s_n}} \log|x-y| \, dxdy, \tag{4.3.36}$$

that is,

$$\begin{aligned} |w_{s_n}|^2_{H^{s_n}(\Omega)} \le & \Big|\frac{2\partial_{s_n} C_{N,s_n}}{C^2_{N,s_n}}\Big| \int_\Omega |f||w_{s_n}| \, dx \\ & + 2 \int_\Omega \int_\Omega \frac{|u_{s_n}(x) - u_{s_n}(y)||w_{s_n}(x) - w_{s_n}(y)|}{|x-y|^{N+2s_n}} |\log|x-y|| \, dxdy. \end{aligned} \tag{4.3.37}$$

Now, using that $s \mapsto C_{N,s}$ is of class C^1, then

$$\Big|\frac{2\partial_{s_n} C_{N,s_n}}{C^2_{N,s_n}}\Big| \le \Big|\frac{2\partial_{s_0} C_{N,s_0}}{C^2_{N,s_0}}\Big| + o(1) \quad \text{as} \quad n \to \infty. \tag{4.3.38}$$

By Hölder inequality and Poincaré inequality (4.2.1), we get

$$\int_\Omega |f||w_{s_n}| \, dx \le c|w_{s_n}|_{H^{s_n}(\Omega)} \tag{4.3.39}$$

On the other hand, Cauchy-Schwarz inequality together with (5.3.5) yield

$$\begin{aligned} & \int_\Omega \int_\Omega \frac{|u_{s_n}(x) - u_{s_n}(y)||w_{s_n}(x) - w_{s_n}(y)|}{|x-y|^{N+2s_n}} |\log|x-y|| \, dxdy \\ & \le \Big(c_{\varepsilon_0}|u_{s_n}|^2_{H^{s_n+\varepsilon_0}(\Omega)} + \tilde{c}_{\varepsilon_0}|u_{s_n}|^2_{H^{s_n}(\Omega)}\Big)|w_{s_n}|_{H^{s_n}(\Omega)}. \end{aligned} \tag{4.3.40}$$

By Proposition 4.2.2, we find that

$$|u_{s_n}|_{H^{s_n+\varepsilon_0}(\Omega)} \le C_1 \qquad \text{and} \qquad |u_{s_n}|_{H^{s_n}(\Omega)} \le C_2 \quad \text{as} \quad n \to \infty. \tag{4.3.41}$$

Taking this into account, we get from (4.3.40) that

$$\int_\Omega \int_\Omega \frac{|u_{s_n}(x) - u_{s_n}(y)||w_{s_n}(x) - w_{s_n}(y)|}{|x-y|^{N+2s_n}} |\log|x-y|| \, dxdy \le C|w_{s_n}|_{H^{s_n}(\Omega)} \tag{4.3.42}$$

for n sufficiently large.

It follows from (4.3.38), (4.3.39), (4.3.42) and (4.3.35) that

$$|w_{s_n}|_{H^{s'}(\Omega)} \le |w_{s_n}|_{H^{s_n}(\Omega)} \le c \quad \text{for} \quad n \quad \text{sufficiently large.} \tag{4.3.43}$$

This means that w_{s_n} is uniformly bounded in $H^{s'}(\Omega)$ with respect to n. Therefore, up to a subsequence, there is $w_* \in H^{s'}(\Omega)$ such that

$$\begin{aligned} & w_{s_n} \rightharpoonup w_* \quad \text{weakly in} \quad H^{s'}(\Omega), \\ & w_{s_n} \to w_* \quad \text{strongly in} \quad L^2(\Omega), \\ & w_{s_n} \to w_* \quad \text{a.e. in} \quad \Omega. \end{aligned} \tag{4.3.44}$$

In particular, we have $\int_\Omega w_* \, dx = 0$. Next, we show that $w_* \equiv w_{s_0}$.

By Fatou's Lemma,

$$
\begin{aligned}
|w_*|_{H^{s_0}(\Omega)} &= \Big(\int_\Omega\int_\Omega \frac{(w_*(x)-w_*(y))^2}{|x-y|^{N+2s_0}}\,dxdy\Big)^{1/2}\\
&\le \liminf_{n\to\infty}\Big(\int_\Omega\int_\Omega \frac{(w_{s_n}(x)-w_{s_n}(y))^2}{|x-y|^{N+2s_n}}\,dxdy\Big)^{1/2}\\
&= \liminf_{n\to\infty}|w_{s_n}|_{H^{s_n}(\Omega)} \le C<\infty.
\end{aligned}
$$

This implies that $w_*\in H^{s_0}(\Omega)$. Notice that we have used (4.3.37), (4.3.38), (4.3.39), (4.3.40) and Proposition 4.2.2.

On the other hand, for all $\varphi\in C_0^\infty(\overline{\Omega})$, we have

$$
\begin{aligned}
\mathcal{E}_{s_0}(w_*,\varphi) &= \int_\Omega w_*(-\Delta)^{s_0}_\Omega\varphi\,dx = \lim_{n\to\infty}\int_\Omega w_{s_n}(-\Delta)^{s_n}_\Omega\varphi\,dx = \lim_{n\to\infty}\mathcal{E}_{s_n}(w_{s_n},\varphi)\\
&= -\lim_{n\to\infty}\frac{\partial_{s_n}C_{N,s_n}}{C_{N,s_n}}\int_\Omega f\varphi\,dx + \lim_{n\to\infty}C_{N,s_n}\int_\Omega\int_\Omega\frac{(u_{s_n}(x)-u_{s_n}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s_n}}\log|x-y|\,dxdy.
\end{aligned}
\tag{4.3.45}
$$

In (4.3.45), we have used (4.3.30). Now, by **Step 1** and Proposition 4.2.2, one obtains from (4.3.45) that

$$
\mathcal{E}_{s_0}(w_*,\varphi) = -\frac{\partial_{s_0}C_{N,s_0}}{C_{N,s_0}}\int_\Omega f\varphi\,dx + C_{N,s_0}\int_\Omega\int_\Omega\frac{(u_{s_0}(x)-u_{s_0}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s_0}}\log|x-y|\,dxdy
\tag{4.3.46}
$$

for all $\varphi\in C_0^\infty(\overline{\Omega})$.

Since by **Step 2** $w_{s_0}\in H^{s_0}(\Omega)$ with $\int_\Omega w_{s_0}\,dx=0$ is the unique solution to (4.3.46), then one finds that $w_*\equiv w_{s_0}$. This yields the continuity of the map $s\mapsto w_s$ and this concudes the proof of **Step 3**.

In summary, from **Steps 1, 2** and **3**, we have shown that

(i) $s\mapsto u_s$ is continuous;

(ii) $\partial_s^+u_s$ exists in $L^2(\Omega)$ for all $s\in(0,1)$;

(iii) The map $(0,1)\to L^2(\Omega),\ s\mapsto\partial_s^+u_s$ is continuous.

Therefore, by Proposition 4.2.3, we conclude that the solution map $(0,1)\to L^2(\Omega),\ s\mapsto\partial_s^+u_s$ is continuously differentiable with $\partial_su_s=\partial_s^+u_s$. Moreover, from (4.3.30), we have that $w_s=\partial_su_s$ solves in weak sense the equation

$$
(-\Delta)^s_\Omega w_s = M^s_\Omega u_s \qquad \text{in } \Omega
\tag{4.3.47}
$$

with

$$
M^s_\Omega u(x) = -\frac{\partial_sC_{N,s}}{C_{N,s}}f(x) + 2C_{N,s}P.V.\int_\Omega\frac{(u(x)-u(y))}{|x-y|^{N+2s}}\log|x-y|\,dy,\quad x\in\Omega.
$$

$\square$

4.4 Eigenvalues problem case

The aim of this section is to study (4.1.1) when $f = \lambda_s u_s$ i.e., the eigenvalues problem

$$(-\Delta)^s_\Omega u_s = \lambda_s u_s \quad \text{in} \quad \Omega. \tag{4.4.1}$$

More precisely, we discuss the s-dependence of the map $s \mapsto \lambda_{1,s}$ where $\lambda_{1,s}$ is the first nontrivial eigenvalue of $(-\Delta)^s_\Omega$. We notice that equation (4.4.1) is understood in weak sense. Here and throughout the end of this section, we fix Ω as a bounded domain with $C^{1,1}$ boundary.

Let

$$0 < \lambda_{1,s} \leq \lambda_{2,s} \leq \cdots \leq \lambda_{k,s} \leq \cdots , \tag{4.4.2}$$

be the sequence of eigenvalues (counted with multiplicity) of $(-\Delta)^s_\Omega$ in Ω with corresponding eigenfunctions

$$\varphi_{1,s}, \varphi_{2,s}, \ldots, \varphi_{k,s}, \ldots .$$

It is known that the system $\{\varphi_{i,s}\}_i$ form an L^2-orthonormal basis. Variationnaly, we have

$$\lambda_{k,s} := \inf_{V \in V^s_k} \sup_{\varphi \in S_V} \mathcal{E}_s(\varphi, \varphi), \tag{4.4.3}$$

where $V^s_k := \{V \subset \mathbb{X}^s(\Omega) : \dim V = k\}$ and $S_V := \{\varphi \in V : \|\varphi\|_{L^2(\Omega)} = 1\}$ for all $V \in V^s_k$. However, when $k = 1$ then $\lambda_{1,s}$ is simply characterized by (see e.g., [63, Theorem 3.1])

$$\lambda_{1,s} := \inf\{\mathcal{E}_s(\varphi, \varphi) : \varphi \in \mathbb{X}^s(\Omega), \|\varphi\|_{L^2(\Omega)} = 1\}. \tag{4.4.4}$$

In this section, we wish to study the differentiability of the map $(0, 1) \ni s \mapsto \lambda_{1,s}$. As first remark, we know that the first nontrivial eigenvalue of $(-\Delta)^s_\Omega$ is not in general *simple*. Therefore, the main focus here is one-sided differentiability.

In what follows, we discuss the right differentiability of the map $s \mapsto \lambda_{1,s}$ stated in Theorem 4.1.2. For the reader's convenience, we restate it in the following. Here and throughout the end of this Section, we use respectively λ_s and u_s for $\lambda_{1,s}$ and $u_{1,s}$ to alleviate the notation.

Theorem 4.4.1. *Regarded as function of s, λ_s is right differentiable on $(0, 1)$ and*

$$\partial^+_s \lambda_s := \lim_{\sigma \to 0^+} \frac{\lambda_{s+\sigma} - \lambda_s}{\sigma} = \inf\{J_s(u) : u \in \mathcal{M}_s\} \tag{4.4.5}$$

where

$$J_s(u) = \frac{\partial_s C_{N,s}}{C_{N,s}} \lambda_s - C_{N,s} \int_\Omega \int_\Omega \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} \log|x - y| \, dxdy \tag{4.4.6}$$

and $\mathcal{M}_s$ the set of L^2-normalized eigenfunctions of $(-\Delta)^s_\Omega$ corresponding to λ_s. Moreover, the infimum in (4.4.5) *is attained.*

We now collect some partial results needed for the proof of Theorem 4.4.1. In the sequel, we prove the following two lemmas in the same spirit as Theorem 1.3 and Lemma 2.1 in [61].

Lemma 4.4.2. *Let $\varphi, \psi \in C^\infty(\overline{\Omega})$. Then, regarded as function of s,*

$$\mathcal{E}_s(\varphi, \psi) : (0, 1) \to \mathbb{R}$$

is continuous on $(0, 1)$.

Proof. It suffices to show that

$$\lim_{\alpha\to s}\mathcal{E}_\alpha(\varphi,\psi)=\mathcal{E}_s(\varphi,\psi).$$

Let $\alpha\in(s-\delta,s+\delta)$ where $\delta=\frac{1}{4}\min\{1-s,s\}$. Then,

$$\begin{aligned}\frac{(\varphi(x)-\varphi(y))(\psi(x)-\psi(y))}{|x-y|^{N+2\alpha}}&\le\frac{|\varphi(x)-\varphi(y)||\psi(x)-\psi(y)|}{|x-y|^{N+2\alpha}}\\&\le\|\nabla\varphi\|_{L^\infty(\Omega)}\|\nabla\psi\|_{L^\infty(\Omega)}\frac{1}{|x-y|^{N+2\alpha-2}}\\&\le C(\varphi,\psi)\max\Big\{\frac{1}{|x-y|^{N+2(s-\delta)-2}},\frac{1}{|x-y|^{N+2(s+\delta)-2}}\Big\}\\&=:g_{s,\delta}(x,y).\end{aligned}$$

Using polar coordinates, it is not difficult to see that $g_{s,\delta}$ is integrable on $\Omega\times\Omega$ with

$$\int_\Omega\int_\Omega|g_{s,\delta}(x,y)|\,dxdy\le C(\varphi,\psi)\Big(\frac{|\Omega||S^{N-1}|}{2(1-s+\delta)}d_\Omega^{2(1-s+\delta)}+\frac{|\Omega||S^{N-1}|}{2(1-s-\delta)}d_\Omega^{2(1-s-\delta)}\Big),$$

and therefore, applying Lebesgue's Dominated Convergence Theorem, we get that

$$\lim_{\alpha\to s}\mathcal{E}_\alpha(\varphi,\psi)=\mathcal{E}_s(\varphi,\psi),$$

as needed. □

Lemma 4.4.3. *Let $k\ge1$ and $\lambda_{k,s}$ the k-th eigenvalue of $(-\Delta)^s_\Omega$ in Ω. Then, regarded as function of s, $\lambda_{k,s}$ is continuous on $(0,1)$ for all $k\in\mathbb{N}$.*

Proof. The proof is divided in two steps. First one shows the limsup inequality. The second step is to obtain the reverse inequality i.e., the liminf inequality.

Step 1. We show that

$$\limsup_{\alpha\to s}\lambda_{k,\alpha}\le\lambda_{k,s}.\tag{4.4.7}$$

Let $\varepsilon>0$ and $k\ge1$. Using that $C_0^\infty(\overline{\Omega})$ is dense in $\mathbb{X}^s(\Omega)$, there exist $\varphi_1,\dots,\varphi_k\in C_0^\infty(\overline{\Omega})$ such that

$$|\langle\varphi_{i,s},\varphi_{j,s}\rangle_{L^2(\Omega)}-\langle\varphi_i,\varphi_j\rangle_{L^2(\Omega)}|\le\frac{\varepsilon}{8k^2}\quad\text{and}\quad|\mathcal{E}_s(\varphi_{i,s},\varphi_{j,s})-\mathcal{E}_s(\varphi_i,\varphi_j)|\le\frac{\varepsilon}{8k^2},\tag{4.4.8}$$

for all $1\le i,j\le k$. Now, from Lemma 4.4.2, there is $\beta_0>0$ such that for all $\alpha\in(s-\beta_0,s+\beta_0)$,

$$|\mathcal{E}_\alpha(\varphi_i,\varphi_j)-\mathcal{E}_s(\varphi_i,\varphi_j)|\le\frac{\varepsilon}{8k^2}.\tag{4.4.9}$$

According to (4.4.8), we also have

$$|\langle\varphi_i,\varphi_j\rangle_{L^2(\Omega)}|<\frac{\varepsilon}{8k^2}\ (i\ne j)\quad\text{and}\quad1-\frac{\varepsilon}{8k^2}<\|\varphi_i\|^2_{L^2(\Omega)}<1+\frac{\varepsilon}{8k^2},$$

and therefore as in [61, Section 2], the familly $\{\varphi_i\}_{i=1,\dots,k}$ is linearily independent. As a consequence, we have by setting in particular $V=\operatorname{span}\{\varphi_1,\dots,\varphi_k\}$ that

$$\lambda_{k,\alpha}\le\sup_{\varphi\in S_V}\mathcal{E}_\alpha(\varphi,\varphi)\le\mathcal{E}_\alpha(\varphi,\varphi)+\frac{\varepsilon}{4}.\tag{4.4.10}$$

Now, for $\varphi \in S_V$, there is a sequence of real numbers $\{a_i\}_{i=1,\dots,k} \subset \mathbb{R}$ satisfying $\sum_{i=1}^k a_i^2 = 1$ such that $\varphi = \sum_{i=1}^k a_i\varphi_i$. Using this and (4.4.9), we get

$$|\mathcal{E}_\alpha(\varphi,\varphi) - \mathcal{E}_s(\varphi,\varphi)| \le \sum_{i=1}^k\sum_{j=1}^k |a_i||a_j||\mathcal{E}_\alpha(\varphi_i,\varphi_j) - \mathcal{E}_s(\varphi_i,\varphi_j)| \le \frac{\varepsilon}{4},$$

i.e.,

$$\mathcal{E}_\alpha(\varphi,\varphi) \le \mathcal{E}_s(\varphi,\varphi) + \frac{\varepsilon}{4}.$$

Consequently, we have with (4.4.10) that

$$\lambda_{k,\alpha} \le \mathcal{E}_s(\varphi,\varphi) + \frac{\varepsilon}{2}. \tag{4.4.11}$$

Now, by letting $\psi = \sum_{i=1}^k a_i\varphi_{i,s}$ and by using (4.4.8), we can follows the argument above to show that

$$|\mathcal{E}_s(\psi,\psi) - \mathcal{E}_s(\varphi,\varphi)| < \frac{\varepsilon}{4} \tag{4.4.12}$$

i.e.,

$$\mathcal{E}_s(\varphi,\varphi) < \mathcal{E}_s(\psi,\psi) + \frac{\varepsilon}{4}. \tag{4.4.13}$$

Combining this with (4.4.11) and by using also the monotonicity of $\{\lambda_{i,s}\}_i$, we see that

$$\begin{aligned}\lambda_{k,\alpha} &\le \mathcal{E}_\alpha(\varphi,\varphi) + \frac{\varepsilon}{2} \le \mathcal{E}_s(\psi,\psi) + \frac{3\varepsilon}{4}\\ &\le \sum_{i=1}^k a_i^2\lambda_{i,s} + \frac{3\varepsilon}{4} \le \lambda_{k,s}\sum_{i=1}^k a_i^2 + \frac{3\varepsilon}{4} = \lambda_{k,s} + \frac{3\varepsilon}{4}.\end{aligned}$$

Since ε was chosen arbitrarily, we therefore have

$$\limsup_{\alpha\to s} \lambda_{k,\alpha} \le \lambda_{k,s},$$

as claimed.

Step 2. We show that

$$\liminf_{\alpha\to s} \lambda_{k,\alpha} \ge \lambda_{k,s}. \tag{4.4.14}$$

To this end, we set $\lambda^*_{k,s} := \liminf_{\alpha\to s}\lambda_{k,\alpha}$ and let $\alpha_n \in (0,1)$ be a sequence such that $\alpha_n \to s$ and $\lambda_{k,\alpha_n} \to \lambda^*_{k,s}$ as $n\to\infty$. We now choose a system of L^2-orthonormal eigenfunctions $\varphi_{1,\alpha_n},\dots,\varphi_{k,\alpha_n}$ associated to $\lambda_{1,\alpha_n},\dots,\lambda_{k,\alpha_n}$.

By Proposition 4.2.2, we have that for n sufficiently large,

$$\varphi_{j,\alpha_n} \quad \text{is uniformly bounded in} \quad H^s(\Omega) \quad \text{for} \quad j=1,\dots,k. \tag{4.4.15}$$

Therefore, after passing to a subsequence, there exists $e_{j,s} \in H^s(\Omega)$ such that

$$\begin{aligned}&\varphi_{j,\alpha_n} \rightharpoonup e_{j,s} \quad \text{weakly in} \quad H^s(\Omega),\\ &\varphi_{j,\alpha_n} \to e_{j,s} \quad \text{strongly in} \quad L^2(\Omega), \quad \text{for} \quad j=1,\dots,k,\\ &\varphi_{j,\alpha_n} \to e_{j,s} \quad \text{a.e. in} \quad \Omega,\end{aligned}$$

which therefore imply that $\int_\Omega e_{j,s}\,dx = 0$. Thus, $e_{j,s} \in \mathbb{X}^s(\Omega)$. Furthermore, by strong convergence in $L^2(\Omega)$, it follows also that $e_{1,s}, \dots, e_{k,s}$ form an L^2-orthonormal system.

Moreover, for every $j = 1, \dots, k$, we have

$$\begin{aligned}\lambda_{j,s}^* \langle e_{j,s}, \varphi\rangle_{L^2(\Omega)} &= \lim_{n\to\infty} \lambda_{j,\alpha_n} \langle \varphi_{j,\alpha_n}, \varphi\rangle_{L^2(\Omega)} = \lim_{n\to\infty} \langle \varphi_{j,\alpha_n}, (-\Delta)_\Omega^{\alpha_n}\varphi\rangle_{L^2(\Omega)} \\ &= \langle e_{j,s}, (-\Delta)_\Omega^s \varphi\rangle_{L^2(\Omega)} = \mathcal{E}_s(e_{j,s}, \varphi)\end{aligned}$$

i.e.,

$$\mathcal{E}_s(e_{j,s}, \varphi) = \lambda_{j,s}^* \langle e_j, \varphi\rangle_{L^2(\Omega)} \quad \text{for all} \quad \varphi \in C_0^\infty(\overline{\Omega}),$$

and by density,

$$\mathcal{E}_s(e_{j,s}, \varphi) = \lambda_{j,s}^* \langle e_{j,s}, \varphi\rangle_{L^2(\Omega)} \quad \text{for all} \quad \varphi \in \mathbb{X}^s(\Omega).$$

Therefore, $(\lambda_{j,s}^*)_{j\in\{1,\dots,k\}}$ is an increasing sequence of eigenvalues of $(-\Delta)_\Omega^s$ with corresponding eigenfunctions $(e_{j,s})_{j\in\{1,\dots,k\}}$. Now, by choosing in particular $V = \mathrm{span}\{e_{1,s}, e_{2,s}, \dots, e_{k,s}\}$, we have from (4.4.3) that

$$\lambda_{k,s} \le \sup_{\varphi\in S_V} \mathcal{E}_s(\varphi, \varphi). \tag{4.4.16}$$

Moreover, for all $\varphi \in S_V$, there exists a family of numbers $(c_j)_{j\in\{1,\cdots,k\}} \subset \mathbb{R}$ satisfying $\sum_{j=1}^k c_j^2 = 1$ such that $\varphi = \sum_{j=1}^k c_j e_{j,s}$. From this, we get that

$$\begin{aligned}\mathcal{E}_s(\varphi, \varphi) = \mathcal{E}_s\Big(\sum_{j=1}^k c_j e_{j,s}, \sum_{j=1}^k c_j e_{j,s}\Big) &= \sum_{i,j=1}^k c_i c_j \lambda_{j,s}^* \langle e_{i,s}, e_{j,s}\rangle_{L^2(\Omega)} \\ = \sum_{j=1}^k c_j^2 \lambda_{j,s}^* \le \max_{j\in\{1,\dots,k\}} \lambda_{j,s}^* \sum_{j=1}^k c_j^2 &= \max_{j\in\{1,\dots,k\}} \lambda_{j,s}^*.\end{aligned}$$

Hence, from (4.4.16), we have that

$$\lambda_{k,s} \le \max_{j\in\{1,\dots,k\}} \lambda_{j,s}^* \le \lambda_{k,s}^*,$$

which therefore implies that

$$\liminf_{\alpha\to s} \lambda_{k,\alpha} = \lambda_{k,s}^* \ge \lambda_{k,s}.$$

Combining both ***Steps 1*** and ***2*** we conclude that

$$\lim_{\alpha\to s} \lambda_{k,\alpha} = \lambda_{k,s} \tag{4.4.17}$$

as wanted. □

Below, we now give the proof of Theorem 4.4.1.

Proof of Theorem 4.4.1. By Lemma 4.4.3 and Proposition 4.2.2, we deduce that the function $u_{s+\sigma}$ is uniformly bounded in $H^s(\Omega)$ with respect to σ. Therefore after passing to a subsequence, there is $w_s \in H^s(\Omega)$ such that

$$\begin{aligned} u_{s+\sigma} &\rightharpoonup w_s \quad \text{weakly in} \quad H^s(\Omega), \\ u_{s+\sigma} &\to w_s \quad \text{strongly in} \quad L^2(\Omega), \\ u_{s+\sigma} &\to w_s \quad \text{a.e. in} \quad \Omega. \end{aligned} \tag{4.4.18}$$

We wish now to show that w_s is also an eigenfunction corresponding to λ_s. First of all, from (4.4.18), we have in particular that $\|w_s\|_{L^2(\Omega)} = 1$ and $\int_\Omega w_s \, dx = 0$.

Next, we claim that

(a) $\mathcal{E}_{s+\sigma}(u_{s+\sigma}, \varphi) \to \mathcal{E}_s(w_s, \varphi)$

(b) $\lambda_{s+\sigma} \int_\Omega u_{s+\sigma}\varphi \, dx \to \lambda_s \int_\Omega w_s \varphi \, dx$

as $\sigma \to 0^+$ for all $\varphi \in C_0^\infty(\overline{\Omega})$.

We start by proving (b). We write

$$\int_\Omega (\lambda_{s+\sigma} u_{s+\sigma} - \lambda_s w_s)\varphi \, dx = \lambda_s \int_\Omega (u_{s+\sigma} - w_s)\varphi \, dx + (\lambda_{s+\sigma} - \lambda_s) \int_\Omega u_{s+\sigma}\varphi \, dx.$$

From the above decomposition and thanks to (4.4.18) and Lemma 4.4.3, we deduce claim (b).

Regarding (a), we have

$$\begin{aligned}
&|\mathcal{E}_{s+\sigma}(u_{s+\sigma}, \varphi) - \mathcal{E}_s(w_s, \varphi)| \\
&\leq \Big|\frac{C_{N,s+\sigma} - C_{N,s}}{2}\Big| \Big| \int_\Omega \int_\Omega \frac{(w_s(x) - w_s(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dxdy\Big| \\
&+ \frac{C_{N,s+\sigma}}{2} \Big| \int_\Omega \int_\Omega \frac{((u_{s+\sigma}(x) - u_{s+\sigma}(y)) - (w_s(x) - w_s(y)))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dxdy\Big| \\
&+ \frac{C_{N,s+\sigma}}{2} \Big| \int_\Omega \int_\Omega \Big(|x-y|^{-2\sigma} - 1\Big) \frac{(u_{s+\sigma}(x) - u_{s+\sigma}(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dxdy\Big| \\
&=: I_\sigma + II_\sigma + III_\sigma.
\end{aligned} \tag{4.4.19}$$

Since $w_s, \varphi \in H^s(\Omega)$ and $s \mapsto C_{N,s}$ is of class C^1, then from Cauchy-Schwarz inequality, we get that

$$I_\sigma \leq c|C_{N,s+\sigma} - C_{N,s}| \to 0 \qquad \text{as } \ \sigma \to 0^+. \tag{4.4.20}$$

Now, using (4.1.3) and the fact that $u_{s+\sigma} \rightharpoonup w_s$ weakly in $H^s(\Omega)$, one gets

$$II_\sigma \to 0 \qquad \text{as } \ \sigma \to 0^+. \tag{4.4.21}$$

On the other hand, recalling (4.1.3), (4.3.11) and (4.3.18), we have

$$\begin{aligned}
III_\sigma \leq \sigma c \int_\Omega \int_\Omega & \frac{|u_{s+\sigma}(x) - u_{s+\sigma}(y)||\varphi(x) - \varphi(y)|}{|x-y|^{N+2s}} |\log|x-y|| \, dxdy \\
+ \sigma c \int_\Omega \int_\Omega & \frac{|u_{s+\sigma}(x) - u_{s+\sigma}(y)||\varphi(x) - \varphi(y)|}{|x-y|^{N+2s+2\sigma}} |\log|x-y|| \, dxdy.
\end{aligned}$$

Arguing as in Section 4.3, one obtains

$$III_\sigma \leq \sigma c \to 0 \qquad \text{as } \ \sigma \to 0^+. \tag{4.4.22}$$

From (4.4.20), (4.4.21) and (4.4.22), it follows from (4.4.19) that

$$\mathcal{E}_{s+\sigma}(u_{s+\sigma}, \varphi) \to \mathcal{E}_s(w_s, \varphi) \qquad \text{as } \ \sigma \to 0^+, \tag{4.4.23}$$

yielding claim (a).

Finally, using that $u_{s+\sigma}$ is solution to

$$\mathcal{E}_{s+\sigma}(u_{s+\sigma},\varphi)=\lambda_{s+\sigma}\int_\Omega u_{s+\sigma}\varphi\ dx \tag{4.4.24}$$

for all $\varphi\in C_0^\infty(\overline{\Omega})$, one deduces from claims (a) and (b) that w_s is solution to

$$\mathcal{E}_s(w_s,\varphi)=\lambda_s\int_\Omega w_s\varphi\ dx \tag{4.4.25}$$

and from this, one concludes that w_s with $\|w_s\|_{L^2(\Omega)}=1, \int_\Omega w_s\ dx=0$ is an eigenfunction corresponding to λ_s.

Coming back to the proof of (4.4.5), since $u_{s+\sigma}\in H^{s+\sigma}(\Omega)\subset H^s(\Omega)$, one can use it as an admissible function in the definition of λ_s to get

$$\lambda_s\le\mathcal{E}_s(u_{s+\sigma},u_{s+\sigma})=\frac{C_{N,s}}{2}\int_\Omega\int_\Omega\frac{(u_{s+\sigma}(x)-u_{s+\sigma}(y))^2}{|x-y|^{N+2s}}\ dxdy. \tag{4.4.26}$$

Now, from (4.4.26), we have

$$\begin{aligned}
\lambda_{s+\sigma}-\lambda_s&=\mathcal{E}_{s+\sigma}(u_{s+\sigma},u_{s+\sigma})-\lambda_s\\
&\ge\frac{C_{N,s+\sigma}}{2}\int_\Omega\int_\Omega\frac{(u_{s+\sigma}(x)-u_{s+\sigma}(y))^2}{|x-y|^{N+2(s+\sigma)}}\ dxdy-\frac{C_{N,s}}{2}\int_\Omega\int_\Omega\frac{(u_{s+\sigma}(x)-u_{s+\sigma}(y))^2}{|x-y|^{N+2s}}\ dxdy\\
&=\frac{C_{N,s+\sigma}-C_{N,s}}{2}\int_\Omega\int_\Omega\frac{(u_{s+\sigma}(x)-u_{s+\sigma}(y))^2}{|x-y|^{N+2s+2\sigma}}\ dxdy\\
&\qquad+\frac{C_{N,s}}{2}\int_\Omega\int_\Omega(|x-y|^{-2\sigma}-1)\frac{(u_{s+\sigma}(x)-u_{s+\sigma}(y))^2}{|x-y|^{N+2s}}\ dxdy\\
&=\frac{C_{N,s+\sigma}-C_{N,s}}{C_{N,s+\sigma}}\lambda_{s+\sigma}+\frac{C_{N,s}}{2}\int_\Omega\int_\Omega(|x-y|^{-2\sigma}-1)\frac{(u_{s+\sigma}(x)-u_{s+\sigma}(y))^2}{|x-y|^{N+2s}}\ dxdy
\end{aligned}$$

Hence,

$$\begin{aligned}
\liminf_{\sigma\to0^+}\frac{\lambda_{s+\sigma}-\lambda_s}{\sigma}&\ge\lim_{\sigma\to0^+}\frac{C_{N,s+\sigma}-C_{N,s}}{\sigma}\frac{\lambda_{s+\sigma}}{C_{N,s+\sigma}}\\
&\qquad+\lim_{\sigma\to0^+}\frac{C_{N,s+\sigma}}{2}\int_\Omega\int_\Omega\frac{(u_{s+\sigma}(x)-u_{s+\sigma}(y))^2}{|x-y|^{N+2s}}\frac{|x-y|^{-2\sigma}-1}{\sigma}\ dxdy.
\end{aligned} \tag{4.4.27}$$

Next, from (4.3.11)

$$\frac{|x-y|^{-2\sigma}-1}{\sigma}=\frac{\exp(-2\sigma\log|x-y|)-1}{\sigma}=-2\log|x-y|\int_0^1\exp(-2\sigma t\log|x-y|)\ dt.$$

Therefore,

$$\frac{|x-y|^{-2\sigma}-1}{\sigma}\to-2\log|x-y|\quad\text{as}\quad\sigma\to0^+. \tag{4.4.28}$$

Using this and recalling (5.3.5), we apply Lebesgue's Dominated Convergence Theorem in (4.4.27), thanks to Proposition 4.2.2, to get that

$$\liminf_{\sigma\to0^+}\frac{\lambda_{s+\sigma}-\lambda_s}{\sigma}\ge\frac{\partial_sC_{N,s}}{C_{N,s}}\lambda_s-C_{N,s}\int_\Omega\int_\Omega\frac{(w_s(x)-w_s(y))^2}{|x-y|^{N+2s}}\log|x-y|\ dxdy. \tag{4.4.29}$$

that is

$$\liminf_{\sigma\to 0^+} \frac{\lambda_{s+\sigma}-\lambda_s}{\sigma} \geq J_s(w_s) \geq \inf\{J_s(u) : u \in \mathcal{M}_s\}. \tag{4.4.30}$$

We now show the reverse inequality i.e.,

$$\limsup_{\sigma\to 0^+} \frac{\lambda_{s+\sigma}-\lambda_s}{\sigma} \leq \inf\{J_s(u) : u \in \mathcal{M}_s\}. \tag{4.4.31}$$

Thanks to Proposition 4.2.2, we have that $u_s \in H^{s+\sigma}(\Omega)$ for σ sufficiently small. Combining this with $\int_\Omega u_s\, dx = 0$ and $\|u_s\|_{L^2(\Omega)} = 1$, we can use u_s as an admissible function for $\lambda_{s+\sigma}$ to get

$$\begin{aligned}
\frac{\lambda_{s+\sigma}-\lambda_s}{\sigma} &\leq \frac{\mathcal{E}_{s+\sigma}(u_s,u_s)-\mathcal{E}_s(u_s,u_s)}{\sigma} \\
&= \frac{C_{N,s+\sigma}}{2\sigma}\int_\Omega\int_\Omega \frac{(u_s(x)-u_s(y))^2}{|x-y|^{N+2(s+\sigma)}}\, dxdy - \frac{C_{N,s}}{2\sigma}\int_\Omega\int_\Omega \frac{(u_s(x)-u_s(y))^2}{|x-y|^{N+2s}}\, dxdy \\
&= \frac{C_{N,s+\sigma}-C_{N,s}}{2\sigma}\int_\Omega\int_\Omega \frac{(u_s(x)-u_s(y))^2}{|x-y|^{N+2s}}\, dxdy \\
&\quad + \frac{C_{N,s+\sigma}}{2}\int_\Omega\int_\Omega \frac{(u_s(x)-u_s(y))^2}{|x-y|^{N+2s}}\frac{|x-y|^{-2\sigma}-1}{\sigma}\, dxdy.
\end{aligned}$$

By letting $\sigma \to 0^+$ and applying Lebesgue's Dominated Convergence Theorem and considering once again (4.4.28) and Proposition 4.2.2, we have that

$$\limsup_{\sigma\to 0^+} \frac{\lambda_{s+\sigma}-\lambda_s}{\sigma} \leq J_s(u_s).$$

Since the above inequality does not depends on the choice of $u_s \in \mathcal{M}_s$, we have that

$$\limsup_{\sigma\to 0^+} \frac{\lambda_{s+\sigma}-\lambda_s}{\sigma} \leq \inf\{J_s(u) : u \in \mathcal{M}_s\}. \tag{4.4.32}$$

Putting together (4.4.30) and (4.4.32) we infer that

$$\lim_{\sigma\to 0^+} \frac{\lambda_{s+\sigma}-\lambda_s}{\sigma} = \inf\{J_s(u) : u \in \mathcal{M}_s\} \equiv \partial_s^+\lambda_s. \tag{4.4.33}$$

Finally, from Proposition 4.2.2 we easily conclude that the infimum in (4.4.5) is achieved. □

Remark 4.4.4. By a similar argument as above, one can also prove that the map $(0,1) \ni s \mapsto \lambda_{1,s}$ is left differentiable. However, due to the non-simplicity of $\lambda_{1,s}$, the right and left derivative $\partial_s^+\lambda_{1,s}$ and $\partial_s^-\lambda_{1,s}$ might not be equal.

Chapter 5

Existence results for nonlocal problems governed by the regional fractional Laplacian

In this chapter, we analyze the fractional Sobolev constant on domains. Precisely, we prove that such a constant is achieved as well as its radial counterpart whenever the underline domain is a ball. The presentation of this chapter is the same as the original paper [R4], based on joint work with Mouhamed Moustapha Fall. The notation may slightly differ from those in the previous chapters.

5.1 Introduction and main results

Let Ω be a Lipschitz open set of $\mathbb{R}^N$, $s \in (1/2, 1)$ and $N > 2s$. The purpose of this paper is to study the existence of minimizers to the best Sobolev critical constant

$$S_{N,s}(\Omega) = \inf_{\substack{u \in H_0^s(\Omega) \\ u \neq 0}} \frac{Q_{N,s,\Omega}(u)}{\|u\|^2_{L^{2^*_s}(\Omega)}}, \tag{5.1.1}$$

where $H_0^s(\Omega)$ is the completion of $C_c^\infty(\Omega)$ with respect to the $H^s(\Omega)$-norm, $2^*_s := \frac{2N}{N-2s}$ is the so-called fractional critical Sobolev exponent and $Q_{N,s,\Omega}(\cdot)$ is a nonnegative quadratic form defined on $H_0^s(\Omega)$ by

$$Q_{N,s,\Omega}(u) := \frac{c_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}}\, dxdy.$$

We notice that for $s \in (0, 1/2]$ and Ω bounded, the constant function 1 belongs to $H_0^s(\Omega)$, and thus, the above Sobolev constant is zero in this case. We refer the reader to Appendix 5.6 below for more details and the definition of Lipschitz domains in this paper.

We recall that nonnegative minimizers of the constant $S_{N,s}(\Omega)$ are weak solutions to nonlinear Dirichlet problem

$$\begin{cases} (-\Delta)^s_\Omega u = u^{2^*_s - 1} & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{5.1.2}$$

where $(-\Delta)^s_\Omega$ is the *regional fractional Laplacian* defined as

$$(-\Delta)^s_\Omega u(x) = c_{N,s} P.V. \int_\Omega \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy, \quad x \in \Omega.$$

Here, $c_{N,s}$ is the usual positive normalization constant of $(-\Delta)^s$ and $P.V.$ stands for the principal value of the integral.

In the theory of partial differential equations, the existence of solutions of nonlinear equations appears as a natural question. This strongly depends on the type of nonlinearities that are considered. For instance, nonlinear equations involving subcritical power nonlinearities, say $f(t) = |t|^{p-1}$ with $p < 2_s^*$, are quite well-understood and due to compactness, the existence of solutions can be easily established by using for example the Mountain Pass theorem. One can also study the corresponding minimization problem and prove that a minimizer exists. Besides, at the critical exponent $p = 2_s^*$ we lose compactness and therefore standard argument of calculus of variation cannot be applied to derive the existence of solutions. As a typical example, when Ω is a star-shaped bounded domain, it has been proved that the Dirichlet problem

$$(-\Delta)^s u = u^{2_s^*-1}, \qquad u > 0 \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega \tag{5.1.3}$$

does not admit a solution. Such a nonexistrence result was first proved in [77] and later in [131,132] by means of a fractional Pohozaev type identity. However, (5.1.2) can have a solution even if Ω is star-shaped and smooth. It is therefore interesting to understand the type of domains and exponents for which (5.1.2) does not admit a solution.

In the case where $\Omega = \mathbb{R}^N$ or $\Omega = \mathbb{R}^N_+$, the infinimum $S_{N,s}(\Omega) > 0$ for all $s \in (0,1)$. Moreover, see e.g. [117] all minimizers of $S_{N,s}(\mathbb{R}^N)$ are of the form

$$u(x) = a\Big(\frac{1}{b^2 + |x - x_0|^2}\Big)^{\frac{N-2s}{2}}, \qquad x \in \mathbb{R}^N \tag{5.1.4}$$

where a, b are positive constants and $x_0 \in \mathbb{R}^N$.

Problem of type (5.1.2) is less understood in contrast with (5.1.3). The only paper investigating it is [83]. Precisely, the authors in [83] considered the equivalent minimization problem and obtain existence of minimizers under some assumptions on Ω and the range of the parameter s. In particular, it is proved in [83] that if a portion of $\partial\Omega$ lies on a hyperplane and $N \geq 4s$, then $S_{N,s}(\Omega)$ is achieved.
Our first main result removes this assumption on Ω provided s is close to 1/2.

Theorem 5.1.1. *Let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a bounded C^1 open set. Then there exists $s_0 \in (1/2, 1)$ such that for all $s \in (1/2, s_0)$, the infimum $S_{N,s}(\Omega)$ is achieved.*

The main ingredient to prove Theorem 5.1.1 is to show that $S_{N,s}(\Omega) < S_{N,s}(\mathbb{R}^N_+)$ for s close to 1/2. We achieve this by showing that $S_{N,1/2}(\Omega) = 0$ provided Ω is a bounded Lipschitz open set. We notice here that our notion of Lipschitz open set is that $\partial\Omega$ is locally given by the restriction of a bi-Lipschitz map. This is strictly weaker than the *strongly* Lipschitz property, meaning that $\partial\Omega$ is locally given by a graph of a Lipschitz function, see Definiton 5.6.2 and Remark 5.6.3 below.

Next, let $\mathcal{B}$ denote the unit centered ball in $\mathbb{R}^N$. We consider the minimization problem (5.1.1) on the space $H^s_{0,rad}(\mathcal{B})$, the completion of the space of radial functions belonging to $C_c^\infty(\mathcal{B})$ with respect to the norm $H^s_0(\mathcal{B})$. More precisely, we consider the infimum problem, for $h \in L^\infty(\mathcal{B})$ being radial,

$$S_{N,s,rad}(\mathcal{B}, h) = \inf_{\substack{u \in H^s_{0,rad}(\mathcal{B}) \\ u \neq 0}} \frac{Q_{N,s,\mathcal{B}}(u) + \int_{\mathcal{B}} hu^2 dx}{\|u\|^2_{L^{2_s^*}(\mathcal{B})}}. \tag{5.1.5}$$

Our next result is related to the existence of minimizers for the infimum $S_{N,s,rad}(\mathcal{B},0)$ in high dimension $N\geq 4s$. Our second main result is the following.

Theorem 5.1.2. *Let $s\in(1/2,1)$ and $N\geq 4s$. Then the infimum*

$$S_{N,s,rad}(\mathcal{B},0)=\inf_{\substack{u\in H^s_{0,rad}(\mathcal{B})\\ u\neq 0}}\frac{Q_{N,s,\mathcal{B}}(u)}{\|u\|^2_{L^{2^*_s}(\mathcal{B})}} \tag{5.1.6}$$

is achieved by a positive function $u\in H^s_{0,rad}(\mathcal{B})$, satisfying

$$(-\Delta)^s_{\mathcal{B}}u=u^{2^*_s-1}\quad in\quad \mathcal{B},\qquad u=0\quad on\quad \partial\mathcal{B}.$$

We now turn our attention to the minimization problem $S_{N,s,rad}(\mathcal{B},h)$ in low dimension $N<4s$. This Sobolev constant is related to the Schrödinger operator $(-\Delta)^s_{\mathcal{B}}+h$. As a necessary condition for the existence of positive minimizers, it is important to assume that $(-\Delta)^s_{\mathcal{B}}+h$ defines a coercive bilinear form on $H^s_{0,rad}(\mathcal{B})$.

Before stated our third main result, we need to introduce the mass of $\mathcal{B}$ at 0 associated to the Schrödinger operator $(-\Delta)^s+h$, where $(-\Delta)^s$ is the standard fractional Laplacian. Indeed, let $G(x,y)$ be the Green function of the operator $(-\Delta)^s+h$ on $\mathcal{B}$ and $\mathcal{R}$ be the Riesz potential of $(-\Delta)^s$ on $\mathbb{R}^N$. Then the function $x\mapsto \mathbf{k}(x)=G(x,0)-\mathcal{R}(x)$ is continuous in $\mathcal{B}$. The *mass* of the operator $(-\Delta)^s+h$ at 0 is given by $\mathbf{k}(0)$. Our next result is a "positive mass theorem" in the spirit of [88, 133].

Theorem 5.1.3. *Let $s\in(1/2,1)$, $2\leq N<4s$, $h\in L^\infty_{rad}(\mathcal{B})$ and suppose that $S_{N,s,rad}(\mathcal{B},h)>0$. Assume that $\mathbf{k}(0)>0$. Then $S_{N,s,rad}(\mathcal{B},h)$ is achieved by a positive function $u\in H^s_{0,rad}(\mathcal{B})$, satisfying*

$$(-\Delta)^s_{\mathcal{B}}u+hu=u^{2^*_s-1}\quad in\quad \mathcal{B},\qquad u=0\quad on\quad \partial\mathcal{B}.$$

The role of the mass in proving the existence of minimizers (for Sobolev constant) in low dimensions is very crucial. As we will see later, it helps us to restore the compactness. Indeed, the strict positivity $\mathbf{k}(0)>0$ implies that the Sobolev constant in $\mathcal{B}$ is strictly less than that of $\mathbb{R}^N$, and thereby produces the existence of minimizers.

An interesting question that arises is whether symmetry breaking occurs? More generally, for $p\geq 1$, is every positive solution to $u\in H^s_0(\mathcal{B})$ to

$$(-\Delta)^s_{\mathcal{B}}u=u^p\quad \text{in}\quad \mathcal{B},\qquad u=0\quad \text{on}\quad \partial\mathcal{B},$$

is radial? We conjecture that that the answer to this question is no.

In Proposition 5.2.3 we obtain a priori L^∞-bounds of minimizers. Hence, by the ineterior regularity theory and standard boostrap arguments, they belong to $C^\infty(\Omega)$, provided $h\in C^\infty(\Omega)$. In addition, the boundary regularity result in [44, 73] implies that minimizers are actually $C^{2s-1}(\overline{\Omega})$.

The rest of the paper is organized as follows. in Section 5.2 we give some preliminaries that will be useful throughout this paper. In Section 5.3 we prove Theorems 5.1.1 whereas in Section 5.5 we establish Theorems 5.1.2 and 5.1.3. Finally in the Appendix 5.6 we prove that the constant function 1 belongs to $H^s_0(\Omega)$ for $s\in(0,1/2]$.

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5.2 Preliminary

In this section, we introduce some preliminary properties which will be useful in this work. For all $s \in (0,1)$, the fractional Sobolev space $H^s(\Omega)$ is defined as the set of all measurable functions u such that

$$[u]^2_{H^s(\Omega)} := \frac{c_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \, dxdy$$

is finite. It is a Hilbert space endowed with the norm

$$\|u\|^2_{H^s(\Omega)} = \|u\|^2_{L^2(\Omega)} + [u]^2_{H^s(\Omega)}.$$

We refer to [65] for more details on this fractional Sobolev spaces. Next, we denote by $H^s_0(\Omega)$ the completion of $C^\infty_c(\Omega)$ under the norm $\|\cdot\|_{H^s(\Omega)}$. Moreover, for $s \in (1/2, 1)$, $H^s_0(\Omega)$ is a Hilbert space equipped with the norm

$$\|u\|^2_{H^s_0(\Omega)} = \frac{c_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \, dxdy$$

which is equivalent to the usual one in $H^s(\Omega)$ thanks to Poincaré inequality. We define the Hilbert space

$$\mathcal{H}^s_0(\Omega) = \{u \in H^s(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}$$

endowed with the norm $\|\cdot\|_{H^s(\mathbb{R}^N)}$, which is the completion of $C^\infty_c(\Omega)$ with respect to the norm $\|\cdot\|_{H^s(\mathbb{R}^N)}$. In the sequel, $H^s_{0,rad}(\Omega)$ and $\mathcal{H}^s_{0,rad}(\Omega)$ are respectively the space of radially symmetric functions of $H^s_0(\Omega)$ and $\mathcal{H}^s_0(\Omega)$.

Given $x \in \Omega$ and $r > 0$, we denote by $B_r(x)$ the open ball centered at x with radius r. When the center is not specified, we will understand that it's the origin, e.g. $B_2(0) = B_2$. The upper half-ball centered at x with radius r is denoted by $B^+_r(x)$. We will always use $\delta_\Omega(x) = \text{dist}(x, \partial\Omega)$ for the distance from x to the boundary. For every set $A \subset \mathbb{R}^N$, we denote by $\mathbb{1}_A$ its characteristic function.

Proposition 5.2.1 (see [63,65])**.** *The embedding $H^s_0(\Omega) \hookrightarrow L^p(\Omega)$ is continuous for any $p \in [2, 2^*_s]$, and compact for any $p \in [2, 2^*_s)$.*

The next proposition gives an elementary result regarding the role of convex functions applied to $(-\Delta)^s_\Omega$.

Proposition 5.2.2. *Assume that $\varphi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz convex function such that $\varphi(0) = 0$. Then if $u \in H^s_0(\Omega)$ we have*

$$(-\Delta)^s_\Omega \varphi(u) \leq \varphi'(u)(-\Delta)^s_\Omega u \quad \textit{weakly in} \quad \Omega. \tag{5.2.1}$$

Proof. The proof of the above lemma is standard. In fact, using that every convex φ satisfies $\varphi(a) - \varphi(b) \leq \varphi'(a)(a-b)$ for all $a, b \in \mathbb{R}$, the proof follows. □

We conclude this section showing in proposition below, the boundedness of any nonnegative solution of (5.1.2). The argument uses Moser's iteration method. A similar result has been established in [16] for the case of fractional Laplacian.

Proposition 5.2.3. *Let $u \in H^s_0(\Omega)$ be a nonnegative solution to problem* (5.1.2)*. Then $u \in L^\infty(\Omega)$.*

Proof. For $\beta \geq 1$ and $T > 0$ large, we define the following convex function

$$\varphi_{T,\beta}(t) = \begin{cases} 0, & \text{if } \ t \leq 0 \\ t^\beta, & \text{if } \ 0 < t < T \\ \beta T^{\beta-1}(t-T) + T^\beta, & \text{if } \ t \geq T. \end{cases}$$

Throughout the proof, we will use $\varphi_{T,\beta} =: \varphi$ for the sake of simplicity. Since φ is Lipschitz, with constant $\Lambda_\varphi = \beta T^{\beta-1}$, and $\varphi(0) = 0$, then $\varphi(u) \in H_0^s(\Omega)$ and by the convexity of φ, we have, according to Proposition 5.2.2 that

$$(-\Delta)^s_\Omega \varphi(u) \leq \varphi'(u)(-\Delta)^s_\Omega u. \tag{5.2.2}$$

By Proposition 5.2.1 and inequality (5.2.2) we have that

$$\begin{aligned} \|\varphi(u)\|^2_{L^{2^*_s}(\Omega)} \leq C\|\varphi(u)\|^2_{H^s_0(\Omega)} &= C\int_\Omega \varphi(u)(-\Delta)^s_\Omega \varphi(u)\ dx \\ &\leq C\int_\Omega \varphi(u)\varphi'(u)(-\Delta)^s_\Omega u\ dx \\ &= C\int_\Omega \varphi(u)\varphi'(u)u^{2^*_s-1}\ dx. \end{aligned}$$

Moreover, since $u\varphi'(u) \leq \beta\varphi(u)$, we have that

$$\|\varphi(u)\|^2_{L^{2^*_s}(\Omega)} \leq C\beta\int_\Omega (\varphi(u))^2 u^{2^*_s-2}\ dx. \tag{5.2.3}$$

We point out that the integral on the right-hand side of the above inequality is finite. Indeed, using that $\beta \geq 1$ and $\varphi(u)$ is linear when $u \geq T$, we have from a quick computation that

$$\begin{aligned} \int_\Omega (\varphi(u))^2 u^{2^*_s-2}\ dx &= \int_{\{u\leq T\}} (\varphi(u))^2 u^{2^*_s-2}\ dx + \int_{\{u>T\}} (\varphi(u))^2 u^{2^*_s-2}\ dx \\ &\leq T^{2\beta-2}\int_\Omega u^{2^*_s}\ dx + C\int_\Omega u^{2^*_s}\ dx < \infty. \end{aligned}$$

We now choose β in (5.2.3) so that $2\beta - 1 = 2^*_s$. Denoting by β_1 such a value, then we can equivalently write

$$\beta_1 := \frac{2^*_s + 1}{2}. \tag{5.2.4}$$

Let $K > 0$ be a positive number whose value will be fixed later on. Then applying Hölder's inequality with exponents $q := 2^*_s/2$ and $q' := 2^*_s/(2^*_s - 2)$ in the integral on the right-hand side of inequality (7.3.59), we find that

$$\begin{aligned} &\int_\Omega (\varphi(u))^2 u^{2^*_s-2}\ dx = \int_{\{u\leq K\}} (\varphi(u))^2 u^{2^*_s-2}\ dx + \int_{\{u>K\}} (\varphi(u))^2 u^{2^*_s-2}\ dx \\ &\leq \int_{\{u\leq K\}} \frac{(\varphi(u))^2}{u} K^{2^*_s-1}\ dx + \left(\int_\Omega (\varphi(u))^{2^*_s}\ dx\right)^{2/2^*_s} \left(\int_{\{u>K\}} u^{2^*_s}\ dx\right)^{\frac{2^*_s-2}{2^*_s}}. \end{aligned} \tag{5.2.5}$$

Now, thanks to Monotone Convergence Theorem, we can choose K as big as we wish so that

$$\left(\int_{\{u>K\}} u^{2_s^*}\, dx\right)^{\frac{2_s^*-2}{2_s^*}} \leq \frac{1}{2C\beta_1}, \tag{5.2.6}$$

where C is the positive constant appearing in (5.2.3). Therefore, by taking into account (5.2.6) in (5.2.5) and by using also (5.2.4), we deduce from (5.2.3) that

$$\|\varphi(u)\|^2_{L^{2_s^*}(\Omega)} \leq 2C\beta_1\left(K^{2_s^*-1}\int_\Omega \frac{(\varphi(u))^2}{u}\, dx\right).$$

Since $\varphi(u) \leq u^{\beta_1}$ and recalling (5.2.4), and by letting $T \to \infty$, we get that

$$\left(\int_\Omega u^{2_s^*\beta_1}\, dx\right)^{2/2_s^*} \leq 2C\beta_1\left(K^{2_s^*-1}\int_\Omega u^{2_s^*}\, dx\right) < \infty,$$

and therefore

$$u \in L^{2_s^*\beta_1}(\Omega). \tag{5.2.7}$$

Suppose now that $\beta > \beta_1$. Thus, using that $\varphi(u) \leq u^\beta$ in the right hand side of (5.2.3) and letting $T \to \infty$ we get

$$\left(\int_\Omega u^{2_s^*\beta}\, dx\right)^{2/2_s^*} \leq C\beta\left(\int_\Omega u^{2\beta+2_s^*-2}\, dx\right). \tag{5.2.8}$$

Therefore,

$$\left(\int_\Omega u^{2_s^*\beta}\, dx\right)^{\frac{1}{2_s^*(\beta-1)}} \leq (C\beta)^{\frac{1}{2(\beta-1)}}\left(\int_\Omega u^{2\beta+2_s^*-2}\, dx\right)^{\frac{1}{2(\beta-1)}}. \tag{5.2.9}$$

We are now in position to use an iterative argument as in [16, Proposition 2.2]. For that, we define inductively the sequence $\beta_{m+1},\ m \geq 1$ by

$$2\beta_{m+1} + 2_s^* - 2 = 2_s^*\beta_m,$$

from which we deduce that,

$$\beta_{m+1} - 1 = \left(\frac{2_s^*}{2}\right)^m(\beta_1 - 1).$$

Now by using β_{m+1} in place of β, in (5.2.9), it follows that

$$\left(\int_\Omega u^{2_s^*\beta_{m+1}}\, dx\right)^{\frac{1}{2_s^*(\beta_{m+1}-1)}} \leq (C\beta_{m+1})^{\frac{1}{2(\beta_{m+1}-1)}}\left(\int_\Omega u^{2_s^*\beta_m}\, dx\right)^{\frac{1}{2_s^*(\beta_m-1)}}.$$

For the sake of clarity, we set

$$C_{m+1} := (C\beta_{m+1})^{\frac{1}{2(\beta_{m+1}-1)}} \quad \text{and} \quad A_m := \left(\int_\Omega u^{2_s^*\beta_m}\, dx\right)^{\frac{1}{2_s^*(\beta_m-1)}}$$

so that

$$A_{m+1} \leq C_{m+1} A_m, \quad m \geq 1. \tag{5.2.10}$$

Then iterating the above inequality, we find that

$$A_{m+1} \leq \prod_{i=2}^{m+1} C_i A_1,$$

which implies that

$$\begin{aligned}\log A_{m+1} &\leq \sum_{i=2}^{m+1} \log C_i + \log A_1 \\ &\leq \sum_{i=2}^{\infty} \log C_i + \log A_1.\end{aligned}$$

Since $\beta_{m+1} = (\beta_1 - 1/2)^m(\beta_1 - 1) + 1$ then the serie $\sum_{i=2}^{\infty} \log C_i$ converges. Also, since $u \in L^{2_s^*\beta_1}(\Omega)$ (see (5.2.7)), then $A_1 \leq C$. From this, we find that

$$\log A_{m+1} \leq C_0 \tag{5.2.11}$$

with being $C_0 > 0$ a positive constant independent of m. By letting $m \to \infty$, it follows that

$$\|u\|_{L^\infty(\Omega)} \leq C_0' < \infty.$$

This completes the proof. □

5.3 Existence of minimizers for s close to $1/2$

We aim to study the existence of nontrivial solutions of (5.1.2). As pointed point out in the introduction the embedding $H_0^s(\Omega) \hookrightarrow L^{2_s^*}(\Omega)$ fails to be compact and due to this, the functional energy associated to (5.1.2) does not satisfy the Palais-Smale compactness condition. Hence finding the critical points by standard variational methods become a very tough task. Therefore, a natural question arises:

($\mathcal{Q}$) *Does problem* (5.1.2) *admits a nontrivial solution?*

In other words, we are looking at whether the quantity

$$S_{N,s}(\Omega) = \inf_{\substack{u \in H_0^s(\Omega) \\ u \neq 0}} \frac{Q_{N,s,\Omega}(u)}{\|u\|^2_{L^{2_s^*}(\Omega)}} \tag{5.3.1}$$

is attained or not. Here $Q_{N,s,\Omega}(\cdot)$ is a nonnegative quadratic form define on $H_0^s(\Omega)$ by

$$Q_{N,s,\Omega}(u) := \frac{c_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}}\, dxdy.$$

As a quick comment on the above question, Frank et al. [83, Theorem 4] gave a positive answer in the special case of a class of C^1 open sets whose boundary has a flat part, that is C^1 domains Ω with the shape $B_r^+(z) \subset \Omega \subset \mathbb{R}^N_+$ for some $r > 0$ and $z \in \partial\mathbb{R}^N_+$, and such that $\mathbb{R}^N_+ \setminus \Omega$ has nonempty interior. This flatness assumption on the boundary of Ω allows the authors in [83] to obtain the strict inequality $S_{N,s}(\Omega) < S_{N,s}(\mathbb{R}^N_+)$, which is the crucial ingredient for the proof of Theorem 4 in there. Notice that in [83], the question remains open for a larger class of sets.

In the sequel, we give a positive affirmation to the above question in the case of arbitrary open sets with C^1 boundary, provided that s is close to $1/2$. As a consequence, one has in contrast with the fractional Laplacian that the above question has a positive answer even if Ω is convex and of class C^∞.

For the reader's convenience, we restate our main result in the following.

Theorem 5.3.1. *Let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz open set. There exists $s_0 \in (1/2, 1)$ such that for all $s \in (1/2, s_0)$, any minimizing sequence for $S_{N,s}(\Omega)$, normalized in $H^s_0(\Omega)$ is relatively compact in $H^s_0(\Omega)$. In particular, the infimum is achieved.*

The proof of the above main theorem is a direct consequence of the key proposition below (see Proposition 5.3.2), in which we examine the asymptotic behavior of the Sobolev critical constant $S_{N,s}(\Omega)$ as s tends to $1/2^+$, by showing that the latter goes to zero. The proof of this only requires the domain to be Lipschitz. Our key proposition is stated as follows.

Proposition 5.3.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz open set. Then*

$$\lim_{s \searrow 1/2} S_{N,s}(\Omega) = 0. \tag{5.3.2}$$

We now collect some interesting results that are needed to complete the proof of Proposition 5.3.2 above. Let us start with the following upper semicontinuous lemma.

Lemma 5.3.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz open set. Fix $s_0 \in [1/2, 1)$. Then*

$$\limsup_{s \searrow s_0} S_{N,s}(\Omega) \leq S_{N,s_0}(\Omega). \tag{5.3.3}$$

Proof. For $t \in \mathbb{R}$, we recall the elementary inequality

$$|e^t - 1| \leq \sum_{k=1}^{+\infty} \frac{|t|^k}{k!} \leq \sum_{k=1}^{+\infty} \frac{|t|^k}{(k-1)!} \leq |t| e^{|t|}. \tag{5.3.4}$$

For all $r, \gamma > 0$, we also recall the following growth regarding the logarithmic function:

$$|\log|z|| \leq \frac{1}{e\gamma}|z|^{-\gamma} \text{ if } |z| \leq r \quad \text{and} \quad |\log|z|| \leq \frac{1}{e\gamma}|z|^{\gamma} \text{ if } |z| \geq r. \tag{5.3.5}$$

Let $\varepsilon > 0$ and let $u_\varepsilon \in C^\infty_c(\Omega)$ such that $\|u_\varepsilon\|_{L^{2^*_s}(\Omega)} = 1$ and $Q_{N,s_0,\Omega}(u_\varepsilon) \leq S_{N,s_0}(\Omega) + \varepsilon$. Then $S_{N,s}(\Omega) \leq Q_{N,s,\Omega}(u_\varepsilon)$. From this, we obtain that

$$S_{N,s}(\Omega) - S_{N,s_0}(\Omega) \leq Q_{N,s,\Omega}(u_\varepsilon) - Q_{N,s_0,\Omega}(u_\varepsilon) + \varepsilon. \tag{5.3.6}$$

On the other hand,

$$\begin{aligned}
&|Q_{N,s,\Omega}(u_\varepsilon) - Q_{N,s_0,\Omega}(u_\varepsilon)| \\
&\leq \frac{1}{2}|c_{N,s} - c_{N,s_0}| \int_\Omega\int_\Omega \frac{(u_\varepsilon(x) - u_\varepsilon(y))^2}{|x-y|^{N+2s_0}}\ dxdy \\
&\qquad + \frac{c_{N,s}}{2}\int_\Omega\int_\Omega \frac{(u_\varepsilon(x) - u_\varepsilon(y))^2}{|x-y|^{N+2s_0}}||x-y|^{2(s_0-s)} - 1|\ dxdy \\
&\leq \frac{1}{c_{N,s_0}}(S_{N,s_0}(\Omega) + \varepsilon)|c_{N,s} - c_{N,s_0}| \\
&\qquad + \frac{c_{N,s}}{2}\int_\Omega\int_\Omega \frac{(u_\varepsilon(x) - u_\varepsilon(y))^2}{|x-y|^{N+2s_0}}||x-y|^{2(s_0-s)} - 1|\ dxdy.
\end{aligned}$$

Next, from (5.3.4) we have that

$$\begin{aligned}
||x-y|^{2(s_0-s)} - 1| = |e^{2(s_0-s)\log|x-y|} - 1| &\leq 2|s_0 - s||\log|x-y||e^{2|s_0-s||\log|x-y||} \\
&= 2|s_0 - s||\log|x-y|||x-y|^{2|s_0-s|}.
\end{aligned}$$

Taking this into account and using the regularity of u_ε and the property (5.3.5), we find that

$$\begin{aligned}
&|Q_{N,s,\Omega}(u_\varepsilon) - Q_{N,s_0,\Omega}(u_\varepsilon)| \\
&\leq \frac{1}{c_{N,s_0}}(S_{N,s_0}(\Omega) + \varepsilon)|c_{N,s} - c_{N,s_0}| + Cc_{N,s}\mathrm{diam}(\Omega)^{2|s_0-s|}|s_0 - s| + \varepsilon
\end{aligned} \tag{5.3.7}$$

where $\mathrm{diam}(\Omega) = \sup\{|x-y| : x, y \in \Omega\}$ is the diameter of Ω and $C = C(N, s_0, \gamma, \Omega, u_\varepsilon) > 0$ is a positive constant. Now, by letting $s \searrow s_0$ in (5.3.7) we obtain that

$$\limsup_{s\searrow s_0} |Q_{N,s,\Omega}(u_\varepsilon) - Q_{N,s_0,\Omega}(u_\varepsilon)| \leq \varepsilon.$$

Since ε can be chosen arbitrarily small, it follows that

$$\limsup_{s\searrow s_0} |Q_{N,s,\Omega}(u_\varepsilon) - Q_{N,s_0,\Omega}(u_\varepsilon)| = 0$$

and therefore, we deduce from (5.3.6) that

$$\limsup_{s\searrow s_0} S_{N,s}(\Omega) \leq S_{N,s_0}(\Omega), \tag{5.3.8}$$

as desired. □

We have the following proposition. Its proof is given in the Appendix 5.6.

Proposition 5.3.4. *Let Ω be a bounded Lipschitz open set of $\mathbb{R}^N$. Then*

$$S_{N,1/2}(\Omega) = 0. \tag{5.3.9}$$

We can now give the proof of our key proposition.

Proof of Proposition 5.3.2. Since $S_{N,s}(\Omega) > 0$ then if follows that

$$\liminf_{s \searrow 1/2} S_{N,s}(\Omega) \geq 0. \tag{5.3.10}$$

On the other hand, applying Lemma 5.3.3 together with Proposition 5.3.4, we have that

$$\limsup_{s \searrow 1/2} S_{N,s}(\Omega) \leq S_{N,1/2}(\Omega) = 0, \tag{5.3.11}$$

Now, from (5.3.10) and (5.3.11) we deduce (5.3.2), and this ends the proof of Proposition 5.3.2. □

Having the above key tools in mind, we can now give the proof of Theorem 5.3.1.

Proof of Theorem 5.3.1. Let $s \in (1/2, 1)$ with s close to $1/2$. Then by Proposition 5.3.2, we have that $S_{N,s}(\Omega) \to 0$ as $s \searrow 1/2$. Consequently, for s close to $1/2$, and since $S_{N,s}(\mathbb{R}^N_+) > 0$ for all $s \in (0,1)$ (see e.g. [71, Lemma 2.1]), we deduce that

$$0 < S_{N,s}(\Omega) < S_{N,s}(\mathbb{R}^N_+) \quad \text{for all} \quad s \in (1/2, s_0) \tag{5.3.12}$$

for some $s_0 \in (1/2, 1)$. With the above key inequality, we complete the proof by following closely the argument developed by Frank et al. [83] for the proof of Theorem 4 in there. □

Remark 5.3.5. Since $Q_{N,s,\Omega}(|u|) \leq Q_{N,s,\Omega}(u)$ then the minimizer in (5.3.1), or equivalently, the solution of (5.1.2) can be assumed nonnegative.

5.4 The radial problem

In the present section, we consider the existence of minimizers to quotient

$$S_{N,s,rad}(\mathcal{B}, h) := \inf_{u \in C^\infty_{c,rad}(\mathcal{B})} \frac{[u]^2_{H^s(\mathcal{B})} + \int_{\mathcal{B}} hu^2 dx}{\|u\|^2_{L^{2^*_s}(\mathcal{B})}}. \tag{5.4.1}$$

Here and in the following, we consider the class of radial potentials $h \in L^\infty(\mathcal{B})$ such that

$$S_{N,s,rad}(\mathcal{B}, h) > 0. \tag{5.4.2}$$

We observe that if $h(x) \equiv -\lambda$ with $\lambda < \lambda_1(\mathcal{B})$, the first eigenvalue of $(-\Delta)^s_{\mathcal{B}}$, then (5.4.2) holds. The aim of this section is to provide situations in which $S_{N,s,rad}(\mathcal{B}, h) < S_{N,s}(\mathbb{R}^N)$.

Remark 5.4.1. We observe that if h satisfies (5.4.2), then if $u \in H^s_0(\mathcal{B})$ satisfies, weakly, $(-\Delta)^s_{\mathcal{B}} u + hu = f$ in $\mathcal{B}$ with $f \in L^p(\mathcal{B})$, for some $p > \frac{N}{2s}$, then $u \in C(\mathcal{B}) \cap L^\infty(\mathcal{B})$. This follows from the argument of Proposition 5.2.3 and the interior regularity.

We start recalling the following result from [83].

Proposition 5.4.2. *([83, Proposition 7]) Let $s \in (1/2, 1)$ and $N \geq 4s$. Then*

$$S_{N,s,rad}(\mathcal{B}, 0) < S_{N,s}(\mathbb{R}^N). \tag{5.4.3}$$

The following result plays a crucial role for the existence theorems.

Proposition 5.4.3. *Let $1/2 < s < 1$ and $N \geq 2$. Then there is a constant $C = C(N, s) > 0$ such that for all $u \in H^s_{0,rad}(\mathcal{B})$,*

$$Q_{N,s,\mathcal{B}}(u) \geq S_{N,s}(\mathbb{R}^N)\|u\|^2_{L^{2^*_s}(\mathcal{B})} - C_{\mathcal{B}}\|u\|^2_{L^2(\mathcal{B})}. \tag{5.4.4}$$

For this, we need the following two lemmas.

Lemma 5.4.4. *For every $\rho \in (0, 1)$, there exists $K_\rho > 0$ with the property that*

$$Q_{N,s,\mathcal{B}}(u) \geq S_{N,s}(\mathbb{R}^N)\|u\|^2_{L^{2^*_s}(\mathcal{B})} - K_\rho\|u\|^2_{L^2(\mathcal{B})} \quad \textit{for every } u \in H^s_{0,rad}(\mathcal{B}) \textit{ with } \operatorname{supp} u \subset B_\rho.$$

Proof. Let $u \in H^s_{0,rad}(\mathcal{B})$ with $\operatorname{supp} u \subset B_\rho$. We have

$$Q_{N,s,\mathcal{B}}(u) = Q_{N,s,\mathbb{R}^N}(u) - \int_{\mathcal{B}} \kappa_{\mathcal{B}}(x)u(x)^2\, dx \geq S_{N,s}(\mathbb{R}^N)\|u\|^2_{L^{2^*_s}(\mathcal{B})} - \int_{\mathcal{B}} \kappa_{\mathcal{B}}(x)u(x)^2\, dx,$$

with being $\kappa_{\mathcal{B}}$ the killing measure for $\mathcal{B}$ define as $\kappa_{\mathcal{B}}(x) = c_{N,s}\int_{\mathbb{R}^N\setminus\mathcal{B}} \frac{1}{|x-y|^{N+2s}}\, dy,\ x \in \mathcal{B}$. On the other hand, since $\operatorname{supp} u \subset B_\rho$, then

$$\int_{\mathcal{B}} \kappa_{\mathcal{B}}(x)u(x)^2\, dx = \int_{B_\rho} \kappa_{\mathcal{B}}(x)u(x)^2\, dx$$

and for every $x \in B_\rho$,

$$\kappa_{\mathcal{B}}(x) = c_{N,s}\int_{\mathbb{R}^N\setminus\mathcal{B}} \frac{dy}{|x-y|^{N+2s}} \leq c_{N,s}\int_{|z|\geq 1-\rho} |z|^{-N-2s}\, dz = a_{N,s}(1-\rho)^{-2s}.$$

Taking this into account, we find that

$$\int_{\mathcal{B}} \kappa_{\mathcal{B}}(x)u(x)^2\, dx \leq a_{N,s}(1-\rho)^{-2s}\int_{B_\rho} u(x)^2\, dx \leq K_\rho\|u\|^2_{L^2(B_\rho)} \leq K_\rho\|u\|^2_{L^2(\mathcal{B})},$$

with $K_\rho = a_{N,s}(1-\rho)^{-2s}$. From this, we get that

$$Q_{N,s,\mathcal{B}}(u) \geq S_{N,s}(\mathbb{R}^N)\|u\|^2_{L^{2^*_s}(\mathcal{B})} - K_\rho\|u\|^2_{L^2(\mathcal{B})},$$

concluding the proof. $\square$

Lemma 5.4.5. *For every $M, \rho > 0$ there exists $C_{\rho,M} > 0$ with*

$$Q_{N,s,\mathcal{B}}(u) \geq M\|u\|^2_{L^{2^*_s}(\mathcal{B})} - C_{\rho,M}\|u\|^2_{L^2(\mathcal{B})} \quad \textit{for every } u \in H^s_{0,rad}(\mathcal{B}) \textit{ with } u \equiv 0 \textit{ in } B_\rho.$$

Proof. We first recall that for $s \in (1/2, 1)$, $H^s_0(\mathcal{B}) = \mathcal{H}^s_0(\mathcal{B})$. Therefore, for every $u \in H^s_{0,rad}(\mathcal{B}) \subset H^s_0(\mathcal{B}) = \mathcal{H}^s_0(\mathcal{B})$, we have $u \in \mathcal{H}^s_{0,rad}(\mathcal{B})$. Thus, combining the fractional version of the Strauss radial lemma (see [66, Lemma 2.5]) and the Hardy inequality (see [70]) we get that

$$\begin{aligned}
|u(x)|^2 &\leq \gamma_{N,s}|x|^{-(N-2s)}Q_{N,s,\mathbb{R}^N}(u) = \gamma_{N,s}|x|^{-(N-2s)}\left(Q_{N,s,\mathcal{B}}(u) + \int_{\mathcal{B}} \kappa_{\mathcal{B}}(x)u(x)^2\, dx\right) \\
&\leq \gamma_{N,s}|x|^{-(N-2s)}\left(Q_{N,s,\mathcal{B}}(u) + \gamma_{N,s,\mathcal{B}}\int_{\mathcal{B}} \delta_{\mathcal{B}}(x)^{-2s}u(x)^2\, dx\right)
\end{aligned}$$

$$\leq d_{N,s,\mathcal{B}}|x|^{-(N-2s)}Q_{N,s,\mathcal{B}}(u), \tag{5.4.5}$$

which implies that

$$\|u\|^2_{L^\infty(\mathcal{B}\setminus B_\rho)} \leq d_{N,s,\mathcal{B}}\rho^{-(N-2s)}Q_{N,s,\mathcal{B}}(u) \quad \text{for every } u \in H^s_{0,rad}(\mathcal{B}) \text{ with } u \equiv 0 \text{ in } B_\rho. \tag{5.4.6}$$

Consequently, using interpolation and Young's inequality with exponents $p = 2/\alpha$ and $p' = 2/(2-\alpha)$, we find that, for all $M > 0$,

$$\begin{aligned}\|u\|^2_{L^{2^*_s}(\mathcal{B}\setminus B_\rho)} &\leq C\|u\|^\alpha_{L^2(\mathcal{B}\setminus B_\rho)}\|u\|^{2-\alpha}_{L^\infty(\mathcal{B}\setminus B_\rho)}\\ &\leq \frac{1}{Md_{N,s,\mathcal{B}}\rho^{-(N-2s)}}\|u\|^2_{L^\infty(\mathcal{B}\setminus B_\rho)} + \frac{C_{\rho,M}}{M}\|u\|^2_{L^2(\mathcal{B}\setminus B_\rho)}\end{aligned}$$

with suitable constants $\alpha \in (0,2)$ and $C_{\rho,M} > 0$, and hence

$$\begin{aligned}M\|u\|^2_{L^{2^*_s}(\mathcal{B}\setminus B_\rho)} &\leq \frac{1}{d_{N,s,\mathcal{B}}\rho^{-(N-2s)}}\|u\|^2_{L^\infty(\mathcal{B}\setminus B_\rho)} + C_{\rho,M}\|u\|^2_{L^2(\mathcal{B}\setminus B_\rho)}\\ &\leq Q_{N,s,\mathcal{B}}(u) + C_{\rho,M}\|u\|^2_{L^2(\mathcal{B})}\end{aligned}$$

for every $u \in H^s_{0,rad}(\mathcal{B})$ with $u \equiv 0$ in B_ρ. The claim follows. □

In the following, we give the

Proof of Proposition 5.4.3. We choose $0 < \rho_2 < \rho_1 < 1$. Moreover, let $\chi_1, \chi_2 \in C^\infty_c(\mathbb{R}^N)$ with $0 \leq \chi_i \leq 1$, $\chi_1^2 + \chi_2^2 \equiv 1$ in $\mathcal{B}$ and $\text{supp}\chi_1 \subset B_{\rho_1}$, $\text{supp}\chi_2 \subset \mathbb{R}^N \setminus \overline{B_{\rho_2}}$. Then we can write $u = \chi_1^2 u + \chi_2^2 u$ in $\mathcal{B}$.
Applying $Q_{N,s,\mathcal{B}}(\cdot)$ to $u = \sum_{i=1}^2 \chi_i^2 u$, we easily find that

$$Q_{N,s,\mathcal{B}}(u) = \sum_{i=1}^2 Q_{N,s,\mathcal{B}}(\chi_i u) - \frac{c_{N,s}}{2}\sum_{i=1}^2 \int_{\mathcal{B}}\int_{\mathcal{B}} \frac{(\chi_i(x)-\chi_i(y))^2}{|x-y|^{N+2s}}u(x)u(y)\;dxdy. \tag{5.4.7}$$

By the regularity of χ_i, we observe that there is no singularity in the double integral and therefore it follows from the Schur test that there exists a positive constant $C > 0$ such that

$$\sum_{i=1}^2 \int_{\mathcal{B}}\int_{\mathcal{B}} \frac{(\chi_i(x)-\chi_i(y))^2}{|x-y|^{N+2s}}u(x)u(y)\;dxdy \leq C\int_{\mathcal{B}} u^2\;dx. \tag{5.4.8}$$

In fact, we can write

$$\int_{\mathcal{B}}\int_{\mathcal{B}} \frac{(\chi_i(x)-\chi_i(y))^2}{|x-y|^{N+2s}}u(x)u(y)\;dxdy \leq C\int_{\mathcal{B}}\int_{\mathcal{B}} K(x,y)u(x)u(y)\;dxdy \tag{5.4.9}$$

$$= C\int_{\mathcal{B}} Tu(x)u(x)\;dx \tag{5.4.10}$$

where

$$Tu(x) = \int_{\mathcal{B}} K(x,y)u(y)\;dy \qquad \text{with} \qquad K(x,y) = |x-y|^{2-N-2s}.$$

Moreover, by Hölder inequality,

$$\int_{\mathcal{B}} Tu(x)u(x)\ dx \leq \|Tu\|_{L^2(\mathcal{B})}\|u\|_{L^2(\mathcal{B})}. \tag{5.4.11}$$

Now, the Schur test implies that there is $C > 0$ such that

$$\|Tu\|_{L^2(\mathcal{B})} \leq C\|u\|_{L^2(\mathcal{B})}. \tag{5.4.12}$$

Therefore, inequality (5.4.8) follows by combining (5.4.9), (5.4.11) and (5.4.12).
On the other hand, by Lemmas 5.4.4 and 5.4.5, there exists a positive constant $C > 0$, depending on ρ_1 and ρ_2 with the property that

$$Q_{N,s,\mathcal{B}}(\chi_i u) \geq S_{N,s}(\mathbb{R}^N)\|\chi_i u\|^2_{L^{2^*_s}(\mathcal{B})} - C\|\chi_i u\|^2_{L^2(\mathcal{B})}. \tag{5.4.13}$$

Plugging (5.4.8) and (5.4.13) into (5.4.7), we find that

$$Q_{N,s,\mathcal{B}}(u) \geq S_{N,s}(\mathbb{R}^N)\sum_{i=1}^{2}\|\chi_i u\|^2_{L^{2^*_s}(\mathcal{B})} - C\sum_{i=1}^{2}\|\chi_i u\|^2_{L^2(\mathcal{B})}. \tag{5.4.14}$$

Next, since $\sum_{i=1}^2 \chi_i^2 = 1$, we have

$$\begin{aligned}\sum_{i=1}^{2}\|\chi_i u\|^2_{L^{2^*_s}(\mathcal{B})} &= \sum_{i=1}^{2}\left\|\chi_i^2 u^2\right\|_{L^{\frac{N}{N-2s}}(\mathcal{B})} \geq \left\|\sum_{i=1}^{2}\chi_i^2 u^2\right\|_{L^{\frac{N}{N-2s}}(\mathcal{B})} \\ &= \|u^2\|_{L^{\frac{N}{N-2s}}(\mathcal{B})} = \|u\|^2_{L^{2^*_s}(\mathcal{B})}.\end{aligned}$$

Using this in (5.4.13), it follows that

$$Q_{N,s,\mathcal{B}}(u) \geq S_{N,s}(\mathbb{R}^N)\|u\|^2_{L^{2^*_s}(\mathcal{B})} - C\|u\|^2_{L^2(\mathcal{B})},$$

completing the proof. □

5.4.1 The case $2s < N < 4s$

We now let $G(x,y)$ be the Green function of $(-\Delta)^s + h$, with zero exterior Dirichlet boundary data. Letting $G(x) = G(x,0)$, we have that

$$\begin{cases} (-\Delta)^s G(x) + h(x)G(x) = \delta_0(x) & \text{in } \mathcal{B} \\ G(x) = 0 & \text{in } \mathbb{R}^N \setminus \mathcal{B}, \end{cases} \tag{5.4.15}$$

where δ_0 is the Dirac mass at 0. We recall that G is a radial function. In fact this follows from the construction and uniqueness of Green function. We let $\mathcal{R}(x) = t_{N,s}|x|^{2s-N}$ be the Riesz potential of $(-\Delta)^s$ on $\mathbb{R}^N$. It satisfies

$$(-\Delta)^s\mathcal{R}(x) = \delta_0(x), \tag{5.4.16}$$

where $t_{N,s} := \pi^{-\frac{N}{2}}2^{-s}\frac{\Gamma((N-s)/2)}{\Gamma(s/2)}$. We now define $\overline{\mathbf{k}} \in L^1(\mathcal{B})$, by

$$\overline{\mathbf{k}}(x) := G(x) - \mathcal{R}(x). \tag{5.4.17}$$

It then follows, from (5.4.15), that

$$(-\Delta)^s \overline{\mathbf{k}}(x) + h(x)\overline{\mathbf{k}}(x) = -h(x)\mathcal{R}(x). \tag{5.4.18}$$

Since $N < 4s$, we have that $\overline{\mathbf{k}} \in L^2(\mathcal{B})$ and $h\mathcal{R} \in L^p(\mathcal{B}) \cap L^2(\mathcal{B})$, for some $p > \frac{N}{2s}$. Therefore, by regularity theory, $\overline{\mathbf{k}} \in C(\overline{\mathcal{B}})$. Recall that $\overline{\mathbf{k}}(y)$ is the *mass* of $\mathcal{B}$ associated to the operator $\mathcal{L}_{\mathbb{R}^N} := (-\Delta)^s + h(x)$. We remark that if $\chi \in C_c^\infty(\mathcal{B})$, with $\chi = 1$ in a neighborhood of 0, then letting

$$\mathbf{k}(x) := G(x) - \chi(x)\mathcal{R}(x),$$

then, by continuity, $\mathbf{k}(y) = \overline{\mathbf{k}}(y)$, for all $y \in \mathcal{B}$. This follows from the fact that $(-\Delta)^s\mathbf{k} + h\mathbf{k} \in L^p(\mathcal{B})$, for some $p > \frac{N}{2s}$ and thus $\mathbf{k} \in C(\mathcal{B})$.

Remark 5.4.6. It would be interesting to find potential h for which $\mathbf{k}(0) > 0$.

First, for $\varepsilon > 0$ we set

$$u_\varepsilon(x) = \gamma_0\Big(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\Big)^{\frac{N-2s}{2}},$$

where γ_0 is a positive constant (independent of ε) such that $\|u_\varepsilon\|_{L^{2_s^*}(\mathbb{R}^N)} = 1$. It is known that u_ε satisfies the Euler-Lagrange equation

$$(-\Delta)^s u_\varepsilon = S_{N,s} u_\varepsilon^{2_s^*-1} \quad \text{in} \quad \mathbb{R}^N. \tag{5.4.19}$$

Our next result shows that in low dimension $N < 4s$, the positive mass implies existence of minimizers.

Lemma 5.4.7. *Suppose that $2s < N < 4s$. Suppose that $\mathbf{k}(0) > 0$. Then*

$$S_{N,s,rad}(\mathcal{B}, h) < S_{N,s} := S_{N,s}(\mathbb{R}^N). \tag{5.4.20}$$

Proof. For $r \in (0, 1/4)$, we let $\eta \in C_c^\infty(B_{2r})$ be radial, with $\eta = 1$ on B_r. We define the test function $v_\varepsilon \in H^s_{0,rad}(\mathcal{B})$ given by

$$\begin{aligned} v_\varepsilon(x) &= \eta(x)u_\varepsilon(x) + \varepsilon^{\frac{N-2s}{2}} \frac{\gamma_0}{t_{N,s}} \left(G(x) - \eta(x)\mathcal{R}(x)\right) \\ &= \eta(x)u_\varepsilon(x) + \varepsilon^{\frac{N-2s}{2}} \frac{\gamma_0}{t_{N,s}} \mathbf{k}(x). \end{aligned} \tag{5.4.21}$$

We define $W_\varepsilon := \eta u_\varepsilon - \varepsilon^{\frac{N-2s}{2}} \frac{\gamma_0}{t_{N,s}} \eta\mathcal{R}$ and $a_s := \frac{\gamma_0}{t_{N,s}}$.

Note that $\varepsilon^{-\frac{N-2s}{2}} W_\varepsilon \to 0 \in C_{loc}(\mathbb{R}^N \setminus \{0\}) \cap L^1(\mathcal{B})$ and $|\varepsilon^{-\frac{N-2s}{2}} u_\varepsilon(x)| \le \gamma_0 |x|^{2s-N}$. Hence, since $N < 4s$, we deduce that $|x|^{2(2s-N)} \in L^1_{loc}(\mathbb{R}^N)$ and thus by the dominated convergence theorem,

$$\int_{\mathcal{B}} u_\varepsilon(x)h(x)W_\varepsilon(x)\,dx = o(\varepsilon^{N-2s}). \tag{5.4.22}$$

We then have

$$\begin{aligned} &[v_\varepsilon]^2_{H^s(\mathcal{B})} + \int_{\mathcal{B}} h v_\varepsilon^2\,dx \le [v_\varepsilon]^2_{H^s(\mathbb{R}^N)} + \int_{\mathcal{B}} h v_\varepsilon^2\,dx = \int_{\mathcal{B}} v_\varepsilon(x)\mathcal{L}_{\mathbb{R}^N} v_\varepsilon(x)\,dx \\ &\le \varepsilon^{\frac{N-2s}{2}} a_s \int_{\mathcal{B}} v_\varepsilon(x)\mathcal{L}_{\mathbb{R}^N} G(x)\,dx + \int_{\mathcal{B}} v_\varepsilon(x)\mathcal{L}_{\mathbb{R}^N} W_\varepsilon(x)\,dx \end{aligned}$$

$$
\begin{aligned}
&\le \varepsilon^{\frac{N-2s}{2}} a_s u_\varepsilon(0) + \varepsilon^{N-2s} a_s^2 \mathbf{k}(0) + \int_{\mathcal{B}} \eta u_\varepsilon(x)(-\Delta)^s W_\varepsilon(x)\,dx \\
&+ \varepsilon^{\frac{N-2s}{2}} a_s \int_{\mathcal{B}} \mathbf{k}(x)\mathcal{L}_{\mathbb{R}^N} W_\varepsilon(x)\,dx + o(\varepsilon^{N-2s}) \\
&\le \varepsilon^{\frac{N-2s}{2}} a_s u_\varepsilon(0) + \varepsilon^{N-2s} a_s^2 \mathbf{k}(0) + \int_{\mathcal{B}} \eta u_\varepsilon(x)(-\Delta)^s (\eta u_\varepsilon)(x)\,dx \\
&- \varepsilon^{\frac{N-2s}{2}} a_s \int_{\mathcal{B}} \eta u_\varepsilon(x)(-\Delta)^s(\eta\mathcal{R})(x)\,dx + \varepsilon^{\frac{N-2s}{2}} a_s \int_{\mathcal{B}} \mathbf{k}(x)\mathcal{L}_{\mathbb{R}^N} W_\varepsilon(x)\,dx + o(\varepsilon^{N-2s}) \\
&\le \varepsilon^{\frac{N-2s}{2}} a_s u_\varepsilon(0) + \varepsilon^{N-2s} a_s^2 \mathbf{k}(0) \\
&+ \int_{\mathbb{R}^N} \eta u_\varepsilon(x)(-\Delta)^s(\eta u_\varepsilon)(x)\,dx - \varepsilon^{\frac{N-2s}{2}} a_s \int_{\mathbb{R}^N} \eta u_\varepsilon(x)(-\Delta)^s(\eta\mathcal{R})(x)\,dx \\
&+ \varepsilon^{\frac{N-2s}{2}} a_s \int_{\mathbb{R}^N} \mathbf{k}(x)\mathcal{L}_{\mathbb{R}^N} W_\varepsilon(x)\,dx + o(\varepsilon^{N-2s}).
\end{aligned}
$$

Letting $\overline{W}_\varepsilon = u_\varepsilon - \varepsilon^{\frac{N-2s}{2}} a_s \mathcal{R}(x)$, since $N < 4s$, we have that

$$
\varepsilon^{-\frac{N-2s}{2}} \overline{W}_\varepsilon \to 0 \qquad \text{in } C^1_{loc}(\mathbb{R}^N \setminus \{0\}) \cap \mathcal{L}^1_s \cap L^2_{loc}(\mathbb{R}^N). \tag{5.4.23}
$$

Therefore, using that $(-\Delta)^s\mathcal{R} = \delta_0$ and $(-\Delta)^s u_\varepsilon = S_{N,s} u_\varepsilon^{2_s^*-1}$, we get

$$
\begin{aligned}
&\varepsilon^{\frac{N-2s}{2}} a_s u_\varepsilon(0) + \int_{\mathbb{R}^N} \eta u_\varepsilon(x)(-\Delta)^s(\eta u_\varepsilon)(x)\,dx - \varepsilon^{\frac{N-2s}{2}} a_s \int_{\mathbb{R}^N} \eta u_\varepsilon(x)(-\Delta)^s(\eta\mathcal{R})(x)\,dx \\
&= \varepsilon^{\frac{N-2s}{2}} a_s u_\varepsilon(0) + \int_{\mathbb{R}^N} \eta^2 u_\varepsilon(x)(-\Delta)^s u_\varepsilon(x)\,dx - \varepsilon^{\frac{N-2s}{2}} a_s \int_{\mathbb{R}^N} \eta u_\varepsilon(x)(-\Delta)^s \mathcal{R}(x)\,dx \\
&+ \int_{\mathbb{R}^N} \eta u_\varepsilon(x)\overline{W}_\varepsilon(x)(-\Delta)^s\eta(x)\,dx - \int_{B_{2r}} \eta u_\varepsilon(x) J_\varepsilon(x)dx \\
&= S_{N,s}\int_{\mathbb{R}^N} \eta^2 u_\varepsilon^{2_s^*} + \int_{\mathbb{R}^N} \eta u_\varepsilon(x)\overline{W}_\varepsilon(x)(-\Delta)^s\eta(x)\,dx - \int_{B_{2r}} \eta u_\varepsilon(x) J_\varepsilon(x)dx \\
&= S_{N,s}\int_{\mathbb{R}^N} \eta^2 u_\varepsilon^{2_s^*} + o(\varepsilon^{N-2s}) - \int_{B_{2r}} \eta u_\varepsilon(x) J_\varepsilon(x)dx,
\end{aligned}
$$

where $J_\varepsilon(x) := c_{N,s}\int_{\mathbb{R}^N} \frac{(\overline{W}_\varepsilon(x)-\overline{W}_\varepsilon(y))(\eta(x)-\eta(y))}{|x-y|^{N+2s}}\,dy$. To estimate J_ε, we consider first $x \in B_{r/2}$ and thus

$$
J_\varepsilon(x) = c_{N,s}\int_{|y|>r} \frac{(\overline{W}_\varepsilon(x)-\overline{W}_\varepsilon(y))(\eta(x)-\eta(y))}{|x-y|^{N+2s}}\,dy = o(\varepsilon^{\frac{N-2s}{2}})O(|x|^{\frac{N-2s}{2}}).
$$

If now $|x| \ge r/2$, we estimate

$$
\begin{aligned}
|J_\varepsilon(x)| &\le c_{N,s}\int_{|y|<r/4} \frac{|(\overline{W}_\varepsilon(x)-\overline{W}_\varepsilon(y))(\eta(x)-\eta(y))|}{|x-y|^{N+2s}}\,dy \\
&\qquad + c_{N,s}\int_{|y|>r/4} \frac{|(\overline{W}_\varepsilon(x)-\overline{W}_\varepsilon(y))(\eta(x)-\eta(y))|}{|x-y|^{N+2s}}\,dy \\
&\le o(\varepsilon^{\frac{N-2s}{2}}) + \|\nabla\eta\|_{L^\infty(\mathbb{R}^N)} \int_{4r>|y|>r/4} \frac{\sup_{t\in[0,1]} |\nabla\overline{W}_\varepsilon(\gamma_{x,y}(t))||\gamma'_{x,y}(t)|}{|x-y|^{N+2s-1}}\,dy
\end{aligned}
$$

$$= o(\varepsilon^{\frac{N-2s}{2}})$$

where $\gamma_{x,y} : [0,1] \to B_{r/2} \setminus B_{r/4}$ is the C^1 shortest curve satisfying $\gamma_{x,y}(0) = x$, $\gamma_{x,y}(1) = y$ and $\sup_{t\in[0,1]} |\gamma'_{x,y}(t)| \leq C|x-y|$. Since $N < 4s$, by (5.4.18) and (5.4.23), we have

$$\left| \int_{\mathbb{R}^N} \mathbf{k}(x) \mathcal{L}_{\mathbb{R}^N} W_\varepsilon(x)\, dx \right| \leq \left| \int_{B_{2r}} |\mathcal{L}_{\mathbb{R}^N} \mathbf{k}(x)| |W_\varepsilon(x)|\, dx \right| = o(\varepsilon^{\frac{N-2s}{2}}).$$

We thus conclude that

$$\begin{aligned} [v_\varepsilon]^2_{H^s(\mathcal{B})} + \int_{\mathcal{B}} h v_\varepsilon^2\, dx &\leq S_{N,s} \int_{\mathbb{R}^N} \eta^2 u_\varepsilon^{2_s^*} + \varepsilon^{N-2s} a_s^2 \mathbf{k}(0) + o(\varepsilon^{N-2s}) + O(\varepsilon^{N-2s}) o_r(1) \\ &\leq S_{N,s} + \varepsilon^{N-2s} a_s^2 \mathbf{k}(0) + o(\varepsilon^{N-2s}) + O(r^{4s-N} \varepsilon^{N-2s}). \end{aligned} \tag{5.4.24}$$

Since $2_s^* > 2$, there exists a positive constant $C(N,s)$ such that

$$||a+b|^{2_s^*} - |a|^{2_s^*} - 2_s^* ab|a|^{2_s^*-2}| \leq C(N,s) \left(|a|^{2_s^*-2} b^2 + |b|^{2_s^*} \right) \qquad \text{for all } a, b \in \mathbb{R}.$$

As a consequence, with $a = \eta(x) u_\varepsilon(x)$ and $b = \varepsilon^{\frac{N-2s}{2}} a_s \mathbf{k}(x)$, we obtain

$$\begin{aligned} &\int_{\mathcal{B}} v_\varepsilon^{2_s^*} - \int_{\mathbb{R}^N} (\eta u_\varepsilon)^{2_s^*} = 2_s^* \varepsilon^{\frac{N-2s}{2}} a_s \int_{\mathcal{B}} (\eta u_\varepsilon)^{2_s^*-1} \mathbf{k}(x)\, dx \\ &+ o(\varepsilon^{N-2s}) + O\left(\varepsilon^{N-2s} \int_{\mathbb{R}^N} |\eta(x) u_\varepsilon(x)|^{2_s^*-2} \mathbf{k}^2(x) dx \right) \\ &= 2_s^* \varepsilon^{\frac{N-2s}{2}} \frac{a_s}{S_{N,s}} \int_{\mathcal{B}} \eta^{2_s^*-1} \mathbf{k}(x) (-\Delta)^s u_\varepsilon\, dx + o(\varepsilon^{N-2s}) + \varepsilon^{N-2s} O\left(\|\eta u_\varepsilon\|^{2_s^*-2}_{L^{2_s^*}(B_{2r})} \|\mathbf{k}\|^2_{L^{2_s^*}(B_{2r})} \right). \\ &= 2_s^* \varepsilon^{\frac{N-2s}{2}} \frac{a_s}{S_{N,s}} \int_{\mathcal{B}} \mathbf{k}(x) (-\Delta)^s \overline{W}_\varepsilon\, dx + 2_s^* \varepsilon^{\frac{N-2s}{2}} \frac{a_s}{S_{N,s}} \int_{\mathcal{B}} (\eta^{2_s^*-1} - 1) \mathbf{k}(x) (-\Delta)^s \overline{W}_\varepsilon\, dx \\ &+ 2_s^* \varepsilon^{N-2s} \frac{a_s^2}{S_{N,s}} \mathbf{k}(0) + o(\varepsilon^{N-2s}) + O(\varepsilon^{N-2s} r^{N-2s}) \\ &= 2_s^* \varepsilon^{\frac{N-2s}{2}} \frac{a_s}{S_{N,s}} \int_{\mathcal{B}} \overline{W}_\varepsilon(x) \mathcal{L}_{\mathbb{R}^N} \mathbf{k}(x)\, dx + 2_s^* \varepsilon^{\frac{N-2s}{2}} \frac{a_s}{S_{N,s}} \int_{\mathcal{B}} (\eta^{2_s^*-1} - 1) \mathbf{k}(x) (-\Delta)^s \overline{W}_\varepsilon\, dx \\ &+ 2_s^* \varepsilon^{N-2s} \frac{a_s^2}{S_{N,s}} \mathbf{k}(0) + o(\varepsilon^{N-2s}) + O(\varepsilon^{N-2s} r^{N-2s}) \\ &= 2_s^* \varepsilon^{N-2s} \frac{a_s^2}{S_{N,s}} O\left(\int_{|x|<2r} |x|^{2s-N} \left(\frac{1}{(\varepsilon^2 + |x|^2)^{\frac{N-2s}{2}}} - \frac{1}{|x|^{N-2s}} \right) dx \right) \\ &+ 2_s^* \varepsilon^{\frac{N-2s}{2}} \frac{a_s}{S_{N,s}} \int_{\mathcal{B}} (\eta^{2_s^*-1} - 1) \mathbf{k}(x) (-\Delta)^s \overline{W}_\varepsilon\, dx + 2_s^* \varepsilon^{N-2s} \frac{a_s^2}{S_{N,s}} \mathbf{k}(0) + o(\varepsilon^{N-2s}) + O(\varepsilon^{N-2s}) o_r(1). \end{aligned}$$

We estimate

$$\begin{aligned} &\int_{\mathcal{B}} (\eta^{2_s^*-1} - 1) \mathbf{k}(x) (-\Delta)^s \overline{W}_\varepsilon\, dx = \int_{\mathcal{B}} (\eta^{2_s^*-1} - 1) \mathbf{k}(x) (-\Delta)^s (\eta_{r/4} \overline{W}_\varepsilon)\, dx + o(\varepsilon^{\frac{N-2s}{2}}) \\ &= c_{N,s} \int_{|x|\geq r} (1 - \eta^{2_s^*-1}(x)) \mathbf{k}(x) \int_{|y|<r/2} \frac{\eta_{r/4}(y) \overline{W}_\varepsilon(y)\, dy}{|x-y|^{N+2s}}\, dy + o(\varepsilon^{\frac{N-2s}{2}}) = o(\varepsilon^{\frac{N-2s}{2}}). \end{aligned}$$

Here, from the definition of η, we define $\eta_{r/4} \in C_c^\infty(B_{r/2})$ with $\eta_{r/4} = 1$ on $B_{r/4}$. From the above estimates, we then obtain

$$\begin{aligned}\int_{\mathcal{B}} v_\varepsilon^{2_s^*} &= \int_{\mathbb{R}^N} (\eta u_\varepsilon)^{2_s^*} + 2_s^* \varepsilon^{N-2s} \frac{a_s^2}{S_{N,s}} \mathbf{k}(0) + o(\varepsilon^{N-2s}) + O(\varepsilon^{N-2s}) o_r(1) \\ &= 1 + 2_s^* \varepsilon^{N-2s} \frac{a_s^2}{S_{N,s}} \mathbf{k}(0) + o(\varepsilon^{N-2s}) + O(\varepsilon^{N-2s}) o_r(1).\end{aligned}$$

Combining this with (5.4.24), we finally get

$$\frac{[v_\varepsilon]^2_{H^s(\mathcal{B})} + \int_{\mathcal{B}} h v_\varepsilon^2 \, dx}{\|v_\varepsilon\|^2_{L^{2_s^*}(\mathcal{B})}} \le S_{N,s} - \varepsilon^{N-2s} a_s^2 \mathbf{k}(0) + o(\varepsilon^{N-2s}) + O(\varepsilon^{N-2s}) o_r(1).$$

This finishes the proof. □

5.5 Existence of radial minimizers

The goal of this section is to investigate the existence of a radial solution of problem (5.1.2) in the case when $\Omega = \mathcal{B}$ is the unit ball of $\mathbb{R}^N$, $N > 2s$. More precisely, we aim to analyze the attainability of the following radial critical level

$$S_{N,s,rad}(\mathcal{B}, h) = \inf_{\substack{u \in H^s_{0,rad}(\mathcal{B}) \\ u \neq 0}} \frac{Q_{N,s,\mathcal{B}}(u) + \int_{\mathcal{B}} h u^2 \, dx}{\|u\|^2_{L^{2_s^*}(\mathcal{B})}}. \tag{5.5.1}$$

To this end, we make use of the method of missing mass as in [83]. The idea is to prove that a minimizing sequence for $S_{N,s,rad}(\mathcal{B}, h)$ does not concentrate at the origin. For that, we will exploit inequalities (5.4.3) and (5.4.20) respectively for high ($N \ge 4s$) and low ($2s < N < 4s$) dimensions.

For the reader's convenience, we restate the main result of this subsection in the following.

Theorem 5.5.1. *Let $s \in (1/2, 1)$, $N > 2s$ and $h \in L^\infty(\mathcal{B})$ be a radial function. Suppose that $0 < S_{N,s,rad}(\mathcal{B}, h) < S_{N,s}(\mathbb{R}^N)$. Then any minimizing sequence for $S_{N,s,rad}(\mathcal{B}, h)$, normalized in $H^s_{0,rad}(\mathcal{B})$ is relatively compact in $H^s_{0,rad}(\mathcal{B})$. In particular, the infimum is achieved.*

To prove the above theorem, we first collect some useful results. Let's introduce

$$S^*_{N,s,rad}(\mathcal{B}) := \inf \Big\{ \liminf_{k \to \infty} \|u_k\|^{-2}_{L^{2_s^*}(\mathcal{B})} : Q_{N,s,\mathcal{B}}(u_k) = 1, \; u_k \rightharpoonup 0 \text{ in } H^s_{0,rad}(\mathcal{B}) \Big\}. \tag{5.5.2}$$

We have the following interesting one-sided inequality.

Proposition 5.5.2. *Let $1/2 < s < 1$ and $N \ge 2$. Then*

$$S^*_{N,s,rad}(\mathcal{B}) \ge S_{N,s}(\mathbb{R}^N). \tag{5.5.3}$$

Proof. Let $(u_k) \subset H^s_{0,rad}(\mathcal{B})$ with $Q_{N,s,\mathcal{B}}(u_k) = 1$ and $u_k \rightharpoonup 0$ in $H^s_{0,rad}(\mathcal{B})$. Then by Proposition 5.4.3 there is $C_{\mathcal{B}} > 0$ such that

$$Q_{N,s,\mathcal{B}}(u_k) \ge S_{N,s}(\mathbb{R}^N) \|u_k\|^2_{L^{2_s^*}(\mathcal{B})} - C_{\mathcal{B}} \|u_k\|^2_{L^2(\mathcal{B})}.$$

By the compact embedding $H^s_{0,rad}(\mathcal{B}) \hookrightarrow L^2(\mathcal{B})$, we have $u_k \to 0$ in $L^2(\mathcal{B})$. Using this and by passing to the limit in the above inequality, we find that

$$1 \geq S_{N,s}(\mathbb{R}^N) \limsup_{k\to\infty} \|u_k\|^2_{L^{2^*_s}(\mathcal{B})},$$

that is,

$$\liminf_{k\to\infty} \|u_k\|^{-2}_{L^{2^*_s}(\mathcal{B})} \geq S_{N,s}(\mathbb{R}^N).$$

From the above inequality, we conclude the proof. □

Having collected the above results, we are ready to prove our main result.

Proof of Theorem 5.5.1. Let (u_k) be a minimizing sequence for $S_{N,s,rad}(\mathcal{B}, h)$, which is normalized in $H^s_{0,rad}(\mathcal{B})$. Then after passing to a subsequence, there is $u \in H^s_{0,rad}(\mathcal{B})$ such that

$$\begin{aligned} u_k &\rightharpoonup u \quad \text{weakly in } \; H^s_{0,rad}(\mathcal{B}) \\ u_k &\to u \quad \text{strongly in } \; L^2(\mathcal{B}) \\ u_k &\to u \quad \text{a.e. in } \; \mathcal{B}. \end{aligned} \tag{5.5.4}$$

Now, by setting $w_k = u_k - u$, it follows that $w_k \rightharpoonup 0$ weakly in $H^s_{0,rad}(\mathcal{B})$. Using this, we have that

$$1 = Q_{N,s,\mathcal{B},h}(u_k) := Q_{N,s,\mathcal{B}}(u_k) + \int_{\mathcal{B}} h u_k^2\, dx = Q_{N,s,\mathcal{B},h}(u) + Q_{N,s,\mathcal{B}}(w_k) + o(1), \tag{5.5.5}$$

where $Q_{N,s,\mathcal{B},h}(u) := Q_{N,s,\mathcal{B}}(u) + \int_{\mathcal{B}} hu^2\, dx$. From the above identities, we see that $Q_{N,s,\mathcal{B}}(w_k)$ converges, say, to R_1, which satisfies according to the above equality,

$$1 = Q_{N,s,\mathcal{B},h}(u) + R_1. \tag{5.5.6}$$

Moreover, using that $u_k \to u$ a.e. in $\mathcal{B}$ and the Brézis-Lieb lemma [28], we get that

$$S_{N,s,rad}(\mathcal{B}, h)^{-\frac{N}{N-2s}} + o(1) = \|u_k\|^{\frac{2N}{N-2s}}_{L^{2^*_s}(\mathcal{B})} = \|u\|^{\frac{2N}{N-2s}}_{L^{2^*_s}(\mathcal{B})} + \|w_k\|^{\frac{2N}{N-2s}}_{L^{2^*_s}(\mathcal{B})} + o(1), \tag{5.5.7}$$

from which we deduce that $\int_{\mathcal{B}} |w_k|^{\frac{2N}{N-2s}}\, dx$ converges, say, to R_2 satisfying

$$S_{N,s,rad}(\mathcal{B}, h)^{-\frac{N}{N-2s}} = \|u\|^{\frac{2N}{N-2s}}_{L^{2^*_s}(\mathcal{B})} + R_2. \tag{5.5.8}$$

Now by Proposition 5.5.2 we easily see that

$$R_1 \geq S_{N,s}(\mathbb{R}^N) R_2^{\frac{N-2s}{N}}. \tag{5.5.9}$$

The above inequality follows immediately if $R_2 = 0$. Otherwise, if $R_2 > 0$, then it suffices to use $\tilde{w}_k := w_k / Q_{N,s,\mathcal{B}}(w_k)^{1/2}$ in the definition of $S^*_{N,s,rad}(\mathcal{B})$ since $\tilde{w}_k \rightharpoonup 0$ weakly in $H^s_{0,rad}(\mathcal{B})$ and $Q_{N,s,\mathcal{B}}(\tilde{w}_k) = 1$ as well.

From (5.5.6), (5.5.8), (5.5.9) and by using the elementary inequality [1]

$$(a-b)^\alpha \geq a^\alpha - b^\alpha \quad \text{for } 0 \leq \alpha \leq 1,\ a \geq b \geq 0 \tag{5.5.10}$$

with $\alpha = (N-2s)/N$, we find that

$$\begin{aligned}
1 &= Q_{N,s,\mathcal{B},h}(u) + R_1 \\
&\geq Q_{N,s,\mathcal{B},h}(u) + S_{N,s}(\mathbb{R}^N) R_2^{\frac{N-2s}{N}} \\
&= Q_{N,s,\mathcal{B},h}(u) + (S_{N,s}(\mathbb{R}^N) - S_{N,s,rad}(\mathcal{B},h)) R_2^{\frac{N-2s}{N}} \\
&\qquad + S_{N,s,rad}(\mathcal{B}) \Big(S_{N,s,rad}(\mathcal{B},h)^{-\frac{N}{N-2s}} - \|u\|_{L^{2_s^*}(\mathcal{B})}^{\frac{2N}{N-2s}} \Big)^{\frac{N-2s}{N}} \\
&\geq Q_{N,s,\mathcal{B},h}(u) + (S_{N,s}(\mathbb{R}^N) - S_{N,s,rad}(\mathcal{B},h)) R_2^{\frac{N-2s}{N}} \\
&\qquad + S_{N,s,rad}(\mathcal{B},h) \Big(S_{N,s,rad}(\mathcal{B},h)^{-1} - \|u\|_{L^{2_s^*}(\mathcal{B})}^{2} \Big) \\
&= Q_{N,s,\mathcal{B},h}(u) + (S_{N,s}(\mathbb{R}^N) - S_{N,s,rad}(\mathcal{B},h)) R_2^{\frac{N-2s}{N}} + 1 - S_{N,s,rad}(\mathcal{B},h) \|u\|_{L^{2_s^*}(\mathcal{B})}^{2}.
\end{aligned}$$

Thus,

$$Q_{N,s,\mathcal{B},h}(u) - S_{N,s,rad}(\mathcal{B},h) \|u\|_{L^{2_s^*}(\mathcal{B})}^{2} + (S_{N,s}(\mathbb{R}^N) - S_{N,s,rad}(\mathcal{B},h)) R_2^{\frac{N-2s}{N}} \leq 0. \tag{5.5.11}$$

Since $Q_{N,s,\mathcal{B},h}(u) \geq S_{N,s,rad}(\mathcal{B},h)\|u\|_{L^{2_s^*}(\mathcal{B})}^{2}$ and $S_{N,s}(\mathbb{R}^N) > S_{N,s,rad}(\mathcal{B},h)$ by assumption, it follows from (5.5.11) that $R_2 = 0$ which implies that $u \not\equiv 0$ thanks to (5.5.8). Therefore,

$$Q_{N,s,\mathcal{B},h}(u) \leq S_{N,s,rad}(\mathcal{B},h) \|u\|_{L^{2_s^*}(\mathcal{B})}^{2},$$

which implies that u is an optimizer. Therefore, instead of the inequality (5.5.9), we have equality, yielding $R_1 = 0$. This implies that $Q_{N,s,\mathcal{B},h}(u) = 1$ and from this, we conclude that (u_k) converges strongly in $H^s_{0,rad}(\mathcal{B})$. The proof is therefore finished. □

Proof of Theorem 5.1.2 and Theorem 5.1.3 (completed). The proof of Theorem 5.1.2 and Theorem 5.1.3 are immediate consequences of Theorem 5.5.1, Lemma 5.4.7 and Proposition 5.4.2. □

5.6 Appendix

In this section, we prove that the constant function 1 belongs to $H^s_0(\Omega)$ for $s \in (0, 1/2]$. By Sobolev embedding, it is enough to treat the case $s = 1/2$.

For every $k \in \mathbb{N}$, we define $\chi_k \in C^{0,1}(\mathbb{R}_+)$ by

$$\chi_k(t) = \begin{cases} 0 & \text{if } \ t \leq \dfrac{1}{k^2}, \\[2mm] \dfrac{\log k^2 t}{|\log 1/k|} & \text{if } \ \dfrac{1}{k^2} \leq t \leq \dfrac{1}{k}, \\[2mm] 1 & \text{if } \ t \geq \dfrac{1}{k}. \end{cases} \tag{5.6.1}$$

[1] $0 \leq b \leq a \Rightarrow 0 \leq b/a \leq 1$ and then $0 \leq b/a \leq (b/a)^\alpha \leq 1$ for all $0 \leq \alpha \leq 1$. Hence,

$$\frac{a^\alpha - b^\alpha}{(a-b)^\alpha} = \frac{1-(b/a)^\alpha}{(1-(b/a))^\alpha} \leq \frac{1-(b/a)}{(1-(b/a))^\alpha} \leq 1.$$

We wish now to approximate the constant function 1 with respect to the $H^{1/2}(\Omega)$-norm. The general strategy is to build an approximation sequence with χ_k together with a partition of unity. Before going further in our analysis, we need first of all a *one*-dimensional approximation argument.

Lemma 5.6.1. *We have*

$$\chi_k \to 1 \qquad in\ H^{1/2}(\mathbb{R}_+) \quad as \quad k \to \infty. \tag{5.6.2}$$

Proof. Clearly, by definition $\chi_k \to 1$ a.e. in $\mathbb{R}_+$. The goal is to show that

$$\|\chi_k - 1\|_{H^{1/2}(\mathbb{R}_+)} \to 0 \qquad \text{as} \quad k \to \infty. \tag{5.6.3}$$

We start by proving that

$$\|\chi_k - 1\|_{L^2(\mathbb{R}_+)} \to 0 \qquad \text{as} \quad k \to \infty. \tag{5.6.4}$$

We have

$$\begin{aligned}
\|\chi_k - 1\|^2_{L^2(\mathbb{R}_+)} &= \int_0^\infty (\chi_k - 1)^2\ dt = \int_0^{1/k^2} (\chi_k - 1)^2\ dt + \int_{1/k^2}^{1/k} (\chi_k - 1)^2\ dt \\
&= \frac{1}{k^2} + \int_{1/k^2}^{1/k} \Big(\frac{\log k^2 t}{\log k} - 1\Big)^2\ dt = \frac{1}{k^2} + \frac{1}{k^2}\int_1^k \Big(\frac{\log t}{\log k} - 1\Big)^2\ dt \\
&= \frac{1}{k^2} + \frac{1}{k \log^2 k}\int_{1/k}^1 \log^2 t\ dt = \frac{1}{k^2} + \frac{1}{k^2 \log^2 k}\Big(2 - \frac{\log^2 k}{k} - \frac{2\log k}{k} - \frac{2}{k}\Big).
\end{aligned}$$

From the estimate above, (5.6.4) follows.

Next, we also prove that

$$[\chi_k - 1]_{H^{1/2}(\mathbb{R}_+)} \to 0 \qquad \text{as} \quad k \to \infty. \tag{5.6.5}$$

We have

$$\begin{aligned}
[\chi_k - 1]^2_{H^{1/2}(\mathbb{R}_+)} &= \frac{c_{1,1/2}}{2}\int_0^\infty \int_0^\infty \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2}\ dxdy \\
&= c\Bigg(\int_0^{1/k}\int_0^{1/k} \cdots + 2\int_0^{1/k}\int_{1/k}^\infty \cdots + \int_{1/k}^\infty \int_{1/k}^\infty \cdots\Bigg) \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2}\ dxdy.
\end{aligned}$$

Since $\chi_k(x) = \chi_k(y) = 1$ for $(x, y) \in (1/k, \infty) \times (1/k, \infty)$ then the third integral in the above equality vanishes. Therefore,

$$[\chi_k - 1]^2_{H^{1/2}(\mathbb{R}_+)} = c\int_0^\infty \int_0^\infty \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2}\ dxdy = c(I_k + J_k)$$

where

$$I_k := \int_0^{1/k}\int_0^{1/k} \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2}\ dxdy \quad \text{and} \quad J_k := 2\int_0^{1/k}\int_{1/k}^\infty \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2}\ dxdy.$$

Estimate of J_k. We have

$$\int_0^{1/k}\int_{1/k}^\infty \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2}\ dxdy$$

$$
\begin{aligned}
&= \left(\int_0^{1/k^2} \int_{1/k}^{\infty} \dots + \int_{1/k^2}^{1/k} \int_{1/k}^{\infty} \dots \right) \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2} \, dxdy \\
&= J_k^1 + J_k^2
\end{aligned}
$$

where

$$
J_k^1 := \int_0^{1/k^2} \int_{1/k}^{\infty} \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2} \, dxdy \quad \text{and} \quad J_k^2 := \int_{1/k^2}^{1/k} \int_{1/k}^{\infty} \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2} \, dxdy.
$$

Regarding J_k^1, we have from the definition of χ_k that

$$
\begin{aligned}
J_k^1 &= \int_0^{1/k^2} \int_{1/k}^{\infty} \frac{1}{(x-y)^2} \, dxdy \overset{\tau = \frac{x}{y}}{=} \int_0^{1/k^2} \frac{1}{y} \int_{1/ky}^{\infty} \frac{1}{(\tau - 1)^2} \, d\tau dy \\
&= \int_0^{1/k^2} \frac{k}{1 - ky} \, dy = -\log\Big(1 - \frac{1}{k}\Big). \qquad (5.6.6)
\end{aligned}
$$

For J_k^2, we also use the definition of χ_k to see that

$$
\begin{aligned}
J_k^2 &= \int_{1/k^2}^{1/k} \int_{1/k}^{\infty} \frac{\Big(1 - \frac{\log k^2 x}{\log k}\Big)^2}{(x-y)^2} \, dxdy = \frac{1}{\log^2 k} \int_{1/k^2}^{1/k} \int_{1/k}^{\infty} \frac{(\log k - \log k^2 x)^2}{(x-y)} \, dxdy \\
&= \frac{1}{\log^2 k} \int_{1/k^2}^{1/k} \int_{1/k}^{\infty} \frac{(\log kx)^2}{(x-y)^2} \, dxdy \overset{\substack{\tau = kx \\ t = ky}}{=} \frac{1}{\log^2 k} \int_{1/k}^{1} \int_1^{\infty} \frac{\log^2 \tau}{(\tau - t)^2} \, d\tau dt \\
&= \frac{1}{\log^2 k} \int_1^{\infty} \Big(\frac{1}{(\tau - \frac{1}{k})} - \frac{1}{(\tau - 1)} \Big) \log^2 \tau \, d\tau \\
&= \frac{1}{\log^2 k} \int_1^{\infty} \frac{\frac{1}{k} - 1}{(\tau - \frac{1}{k})(\tau - 1)} \log^2 \tau \, d\tau. \qquad (5.6.7)
\end{aligned}
$$

Using that $\log \tau \sim \tau - 1$ as $\tau \to 1$ and $\frac{\log^2 \tau}{(\tau - \frac{1}{k})(\tau - 1)} \sim \frac{\log^2 \tau}{\tau^2} \le \frac{c}{\tau^{2-\varepsilon}}$ as $\tau \to \infty$, for every $\varepsilon > 0$, then the above integral is convergence for k sufficiently large. This implies that

$$
J_k^2 = o(1) \qquad \text{as} \ \ k \to \infty. \qquad (5.6.8)
$$

Combining (5.6.6) and (5.6.7), and by using (5.6.8), we find that

$$
\begin{aligned}
J_k &= 2\bigg(-\log\Big(1 - \frac{1}{k}\Big) + \frac{1}{\log^2 k} \int_1^{\infty} \frac{\frac{1}{k} - 1}{(\tau - \frac{1}{k})(\tau - 1)} \log^2 \tau \, d\tau \bigg) \\
&\to 0 \qquad \text{as} \ \ k \to \infty. \qquad (5.6.9)
\end{aligned}
$$

Estimate of I_k. We have

$$
I_k = \bigg(\int_0^{1/k^2} \int_0^{2/k^2} \dots + \int_0^{1/k^2} \int_{2/k^2}^{1/k} \dots
$$

$$
\begin{aligned}
&\quad + \int_{1/k^2}^{1/k}\int_0^{2/k^2}\cdots + \int_{1/k^2}^{1/k}\int_{2/k^2}^{1/k}\cdots\Bigg)\frac{(\chi_k(x)-\chi_k(y))^2}{(x-y)^2}\,dxdy\\
&= I_k^1 + I_k^2 + I_k^3
\end{aligned}
$$

where

$$
I_k^1 := \int_0^{1/k^2}\int_0^{2/k^2}\frac{(\chi_k(x)-\chi_k(y))^2}{(x-y)^2}\,dxdy,\quad I_k^2 := \int_{1/k^2}^{1/k}\int_{2/k^2}^{1/k}\frac{(\chi_k(x)-\chi_k(y))^2}{(x-y)^2}\,dxdy
$$

and

$$
I_k^3 := \Bigg(\int_0^{1/k^2}\int_{2/k^2}^{1/k}\cdots + \int_{1/k^2}^{1/k}\int_0^{2/k^2}\cdots\Bigg)\frac{(\chi_k(x)-\chi_k(y))^2}{(x-y)^2}\,dxdy.
$$

It now suffices to estimate I_k^1, I_k^2 and I_k^3.
Concerning I_k^1, we have

$$
\begin{aligned}
I_k^1 &= \int_0^{1/k^2}\int_{1/k^2}^{2/k^2}\frac{\chi_k(x)^2}{(x-y)^2}\,dxdy = \frac{1}{\log^2 k}\int_0^{1/k^2}\int_{1/k^2}^{2/k^2}\frac{(\log k^2x)^2}{(x-y)^2}\,dxdy\\
&\overset{\substack{\tau=k^2x\\ t=k^2y}}{=} \frac{1}{\log^2 k}\int_0^1\int_1^2\frac{\log^2\tau}{(\tau-t)^2}\,d\tau dt = \frac{1}{\log^2 k}\int_0^1\int_1^2\frac{(\log\tau-\log 1)^2}{(\tau-t)^2}\,d\tau dt\\
&\le \frac{c}{\log^2 k}\int_0^1\int_1^2\frac{(\tau-1)^2}{(\tau-t)^2}\,d\tau dt = \frac{c}{\log^2 k}\int_1^2\int_0^1\frac{(\tau-1)^2}{(\tau-t)^2}\,dtd\tau\\
&= \frac{c}{\log^2 k}\int_1^2(\tau-1)^2\Big(\frac{1}{\tau-1}-\frac{1}{\tau}\Big) = \frac{c'}{\log^2 k}.
\end{aligned}
\tag{5.6.10}
$$

Next, as regards I_k^2, the change of variables $\tau = k^2x$ and $t = k^2y$ gives

$$
\begin{aligned}
I_k^2 &= \int_{1/k^2}^{1/k}\int_{2/k^2}^{1/k}\frac{(\log k^2x-\log k^2y)^2}{(x-y)^2}\,dxdy = \frac{1}{\log^2 k}\int_1^k\int_2^k\frac{(\log\tau-\log t)^2}{(\tau-t)^2}\,d\tau dt\\
&= \frac{1}{\log^2 k}\int_1^k\int_2^k\frac{(\log(\tau/t))^2}{(\tau-t)^2}\,d\tau dt \overset{r=\tau/t}{=} \frac{1}{\log^2 k}\int_1^k\frac{1}{t}\int_{2/t}^{k/t}\frac{\log^2 r}{(r-1)^2}\,drdt\\
&\le \frac{1}{\log^2 k}\int_1^k\frac{dt}{t}\int_0^\infty\frac{\log^2 r}{(r-1)^2}\,dr = \frac{c}{\log k}.
\end{aligned}
\tag{5.6.11}
$$

For I_k^3, we have

$$
\begin{aligned}
I_k^3 &\le 2\int_0^{2/k^2}\int_{1/k^2}^{1/k}\frac{(\chi_k(x)-\chi_k(y))^2}{(x-y)^2}\,dxdy = 2\int_0^{2/k^2}\int_{1/k^2}^{1/k}\frac{(\chi_k(x)-\chi_k(y))^2}{(x-y)^2}\,dxdy\\
&= 2\int_0^{1/k^2}\int_{1/k^2}^{1/k}\frac{(\chi_k(x)-\chi_k(y))^2}{(x-y)^2}\,dxdy + 2\int_{1/k^2}^{2/k^2}\int_{1/k^2}^{1/k}\frac{(\chi_k(x)-\chi_k(y))^2}{(x-y)^2}\,dxdy.
\end{aligned}
$$

Now,

$$
\int_0^{1/k^2}\int_{1/k^2}^{1/k}\frac{(\chi_k(x)-\chi_k(y))^2}{(x-y)^2}\,dxdy = \frac{1}{\log^2 k}\int_0^{1/k^2}\int_{1/k^2}^{1/k}\frac{(\log k^2x)^2}{(x-y)^2}\,dxdy
$$

$$
\begin{aligned}
&\overset{\substack{\tau=k^2x\\ t=k^2y}}{=} \frac{1}{\log^2 k}\int_0^1\int_1^k \frac{\log^2\tau}{(\tau-t)^2}\,d\tau dt = \frac{1}{\log^2 k}\int_1^k \Big(\frac{1}{(\tau-1)^2}-\frac{1}{\tau^2}\Big)\log^2\tau\, d\tau\\
&\le \frac{1}{\log^2 k}\int_1^\infty \Big(\frac{1}{(\tau-1)^2}-\frac{1}{\tau^2}\Big)\log^2\tau\, d\tau = \frac{c}{\log^2 k}. \qquad (5.6.12)
\end{aligned}
$$

Arguing as in the case of I_k^2, we have that

$$
\begin{aligned}
&\int_{1/k^2}^{2/k^2}\int_{1/k^2}^{1/k} \frac{(\chi_k(x)-\chi_k(y))^2}{(x-y)^2}\,dxdy = \frac{1}{\log^2 k}\int_1^k\int_1^2 \frac{(\log t-\log\tau)^2}{(t-\tau)^2}\,dtd\tau\\
&\overset{r=t/\tau}{=} \frac{1}{\log^2 k}\int_1^k \frac{d\tau}{\tau}\int_{1/\tau}^{2/\tau}\frac{\log^2 r}{(r-1)^2}\,dr \le \frac{1}{\log^2 k}\int_1^k \frac{d\tau}{\tau}\int_1^\infty \frac{\log^2 r}{(r-1)^2}\,dr\\
&= \frac{c}{\log k}. \qquad (5.6.13)
\end{aligned}
$$

Putting together (5.6.10), (5.6.11), (5.6.12) and (5.6.13), we find that

$$
I_k \le \frac{c}{\log^2 k} + \frac{c}{\log k} \to 0 \qquad \text{as } \ k\to\infty. \qquad (5.6.14)
$$

From (5.6.9) and (5.6.14), we conclude that

$$
[\chi_k - 1]_{H^{1/2}(\mathbb{R}_+)} \to 0 \qquad \text{as } \ k\to\infty. \qquad (5.6.15)
$$

Now, (5.6.3) follows by combining (5.6.4) and (5.6.15). As wanted. □

Definition 5.6.2. We say that an open subset Ω of $\mathbb{R}^N$ is Lipschitz if for each $q\in\partial\Omega$, there exist a tangent hyperplane H_q, a normal N_q of H_q, $r_q>0$, open r_q-balls $B_{r_q}\subset H_q$ and a function $\Phi_q : B_{r_q}\times I\to\mathbb{R}^N$ such that

(*i*) $\Phi_q(B_{r_q}\cap H_q^+)\subset\Omega$

(*ii*) $\Phi_q(B_{r_q}\cap\partial H_q^+)\subset\partial\Omega$

(*iii*) $C^{-1}|x-y|\le|\Phi_q(x)-\Phi_q(y)|\le C|x-y|, \quad C>1, \quad x,y\in B_{r_q}\times I, \quad I\subset\mathbb{R}.$

Here, H_q^+ is the upper half-tangent hyperplane containing N_q. Put $Q_q := B_{r_q}\times(-r_q,r_q)$ and we recall that B_{r_q} is a $(N-1)$-ball.

Remark 5.6.3. We would like to make the following observation. It is well-known that a domain Ω is said to be *strongly* Lipschitz if its boundary can be seen as a local graph of a Lipschitz function $\varphi:\mathbb{R}^{N-1}\to\mathbb{R}$. Moreover, by mean of a vectorfield η (with $|\eta|=1$ on $\partial\Omega$) which is globally transversal [2] to $\partial\Omega$, one can construct a bi-Lipschitz mapping via φ. In particular, Ω fulfills properties (*i*)-(*iii*). However, every Lipschitz domain in the sense of definition (*i*)-(*iii*) is not necessarily a local graph of a Lipschitz function. This clearly shows that *strongly* Lipschitz domain is also a Lipschitz domain. But the converse is not true. This is consistent with the fact that *strongly* Lipschitz domains are not stable under bi-Lipschitz map. See [103] for more details.

[2] η is said to be globally tranversal to $\partial\Omega$ if there is $\kappa>0$ such that $\eta\cdot\nu\ge\kappa$ a.e. on $\partial\Omega$. Here ν is the unit normal vector to $\partial\Omega$.

Clearly, there exists $\beta > 0$ such that

$$\overline{\Omega_\beta} := \{0 \le \delta_\Omega(x) \le \beta\} \subset \cup_{q\in\partial\Omega}\Phi_q(Q_q). \tag{5.6.16}$$

We recall that Ω_β is the so-called *inner tubular neighbourhood* of Ω. By compactness, there exists $m \in \mathbb{N}$ such that

$$\overline{\Omega_\beta} := \{0 \le \delta_\Omega(x) \le \beta\} \subset \cup_{j=1}^m \Phi_{q_j}(Q_{q_j}). \tag{5.6.17}$$

We will write j in the place of q_j provided there is no ambiguity. For $j = 1, \dots, m$, let u_k^j be a sequence define by

$$u_k^j(\Phi_j(x)) = \chi_k(x_N), \quad \forall x \in Q_j,$$

where χ_k is defined in (5.6.1). Equivalently, u_k^j can be defined as

$$u_k^j(x) = \chi_k(\Phi_j^{-1}(x)\cdot N_j), \quad \forall x \in \Omega. \tag{5.6.18}$$

Define $\mathcal{O}_j := \Phi_j(Q_j)$ and $\mathcal{O}_{m+1} = \Omega \setminus \overline{\Omega_\beta}$. We also write $Q_j^+ := B_{r_j} \times (0, r_j)$.

We have the following.

Lemma 5.6.4. *For all $j = 1, \dots, m$ there exists a positive constant $C > 0$ depending only on j, m, Ω and N such that*

$$\|u_k^j - \mathbb{1}_\Omega\|_{H^{1/2}(\mathcal{O}_j\cap\Omega)} \le C\|\chi_k - 1\|_{H^{1/2}(0,r_j)}. \tag{5.6.19}$$

Proof. For $j = 1, \dots, m$, by using the change of variables $x = \Phi_j(z)$ and $y = \Phi_j(\overline{z})$, we get

$$\begin{aligned}
&\int_{\mathcal{O}_j\cap\Omega}\int_{\mathcal{O}_j\cap\Omega} \frac{(u_k(x)-u_k(y))^2}{|x-y|^{N+1}}\,dxdy = \int_{Q_j^+}\int_{Q_j^+} \frac{(u_k(\Phi_j(z))-u_k(\Phi_j(\overline{z})))^2}{|\Phi_j(z)-\Phi_j(\overline{z})|^{N+1}}\,dzd\overline{z}\\
&= \int_{Q_j^+}\int_{Q_j^+} \frac{(\chi_k(z_N)-\chi_k(\overline{z}_N))^2}{|\Phi_j(z)-\Phi_j(\overline{z})|^{N+1}}\,dzd\overline{z} \le C\int_{Q_j^+}\int_{Q_j^+} \frac{(\chi_k(z_N)-\chi_k(\overline{z}_N))^2}{|z-\overline{z}|^{N+1}}\,dzd\overline{z}\\
&\le C\int_{B_{r_j}}\int_{B_{r_j}}\int_0^{r_j}\int_0^{r_j} \frac{(\chi_k(z_N)-\chi_k(\overline{z}_N))^2}{|z-\overline{z}|^{N+1}}\,dzd\overline{z}\\
&\le C\int_{B_{r_j}} dz' \int_{H_j} d\overline{z}' \int_0^{r_j}\int_0^{r_j} \frac{(\chi_k(z_N)-\chi_k(\overline{z}_N))^2}{(|z'-\overline{z}'|^2+|z_N-\overline{z}_N|^2)^{\frac{N+1}{2}}}\,dz_N d\overline{z}_N.
\end{aligned} \tag{5.6.20}$$

By translation and rotation, we have

$$\begin{aligned}
&\int_{B_{r_j}} dz' \int_{H_j} d\overline{z}' \int_0^{r_j}\int_0^{r_j} \frac{(\chi_k(z_N)-\chi_k(\overline{z}_N))^2}{(|z'-\overline{z}'|^2+|z_N-\overline{z}_N|^2)^{\frac{N+1}{2}}}\,dz_N d\overline{z}_N\\
&= \int_{B_{r_j}} dz' \int_{\mathbb{R}^{N-1}} d\overline{z}' \int_0^{r_j}\int_0^{r_j} \frac{(\chi_k(z_N)-\chi_k(\overline{z}_N))^2}{(|z'-\overline{z}'|^2+|z_N-\overline{z}_N|^2)^{\frac{N+1}{2}}}\,dz_N d\overline{z}_N\\
&\le CA\int_0^{r_j}\int_0^{r_j} \frac{(\chi_k(z_N)-\chi_k(\overline{z}_N))^2}{|z_N-\overline{z}_N|^2}\,dz_N d\overline{z}_N,
\end{aligned}$$

where $A = \int_{\mathbb{R}^{N-1}} \frac{dl}{(1+|l|^2)^{(N+1)/2}} \le C$ and B_{r_j} is a bounded open subset of $\mathbb{R}^{N-1}$. Therefore, since the estimate of the L^2 norm follows easily, this and (5.6.20) give (5.6.19), concluding the proof. $\square$

Consider $0 \leq \psi_j \in C_c^\infty(\mathcal{O}_j)$ a partitioning of unity subordinated to $\{\mathcal{O}_j\}_{j=1,\dots,m+1}$. Define

$$u_k := \sum_{j=1}^{m+1} \psi_j u_k^j \in C_c^{0,1}(\Omega), \tag{5.6.21}$$

where $u_k^{m+1} \equiv 1$ on Ω. We have the following approximation.

Lemma 5.6.5. *There holds*

$$\|u_k - \mathbb{1}_\Omega\|_{H^{1/2}(\Omega)} \to 0 \qquad \textit{as} \;\; k \to \infty. \tag{5.6.22}$$

Proof. We estimate

$$\begin{aligned}
[u_k - \mathbb{1}_\Omega]^2_{H^{1/2}(\Omega)} &\leq \left(\sum_{j=1}^{m+1} [\psi_j u_k^j - \psi_j]_{H^{1/2}(\Omega)} \right)^2 \leq m \sum_{j=1}^{m} [\psi_j u_k^j - \psi_j]^2_{H^{1/2}(\Omega)} \\
&\leq C \sum_{j=1}^{m} \int_{\mathcal{O}_j \cap \Omega \times \mathcal{O}_j \cap \Omega} \dots \, dxdy + C \sum_{j=1}^{m} \int_{\Omega \setminus \mathcal{O}_j \times \Omega \cap \mathcal{O}_j} \dots \, dxdy \\
&=: C I_1(k) + C I_2(k).
\end{aligned}$$

We now estimate $I_1(k)$ and $I_2(k)$. Let us start with $I_2(k)$.

We have

$$\begin{aligned}
I_2(k) =& \sum_{j=1}^{m} \int_{\Omega \setminus \mathcal{O}_j \times \Omega \cap \mathcal{O}_j} \frac{[(\psi_j u_k^j - \psi_j)(x) - (\psi_j u_k^j - \psi_j)(y)]^2}{|x-y|^{N+1}} dxdy \\
&= \sum_{j=1}^{m} \int_{\Omega \setminus \mathcal{O}_j} \frac{dx}{|x-y|^{N+1}} \int_{\Omega \cap \mathrm{Supp}\psi_j} (\psi_j u_k^j - \psi_j)(y)^2 dy \\
&\leq C \sum_{j=1}^{m} \mathrm{dist}(\mathrm{Supp}\psi_j, \partial \mathcal{O}_j)^{-N-1} \int_{\Omega \cap \mathcal{O}_j} \psi_j^2 |u_k^j(y) - 1|^2 dy \\
&\leq C(N) \max_{1 \leq j \leq m} \mathrm{dist}(\mathrm{Supp}\psi_j, \partial \mathcal{O}_j)^{-N-1} \sum_{j=1}^{m} \|u_k^j - \mathbb{1}_\Omega\|^2_{L^2(\Omega \cap \mathcal{O}_j)}.
\end{aligned} \tag{5.6.23}$$

Now regarding $I_1(k)$, we have

$$\begin{aligned}
I_1(k) &= \sum_{j=1}^{m} \int_{\mathcal{O}_j \cap \Omega} \int_{\mathcal{O}_j \cap \Omega} \frac{[\psi_j(x)(u_k^j(x) - 1) - \psi_j(y)(u_k^j(y) - 1)]^2}{|x-y|^{N+1}} \, dxdy \\
&= \sum_{j=1}^{m} \int_{\mathcal{O}_j \cap \Omega} \int_{\mathcal{O}_j \cap \Omega} \frac{[\psi_j(x)((u_k^j(x) - 1) - (u_k^j(y) - 1)) + (\psi_j(x) - \psi_j(y))(u_k^j(y) - 1)]^2}{|x-y|^{N+1}} \, dxdy \\
&\leq 2 \sum_{j=1}^{m} \int_{\mathcal{O}_j \cap \Omega} \int_{\mathcal{O}_j \cap \Omega} \frac{\psi_j(x)^2 [(u_k^j(x) - 1) - (u_k^j(y) - 1)]^2}{|x-y|^{N+1}} \, dxdy
\end{aligned}$$

$$+ 2\sum_{j=1}^{m}\int_{\mathcal{O}_j\cap\Omega}\int_{\mathcal{O}_j\cap\Omega}\frac{(\psi_j(x)-\psi_j(y))^2(u_k^j(y)-1)^2}{|x-y|^{N+1}}\,dxdy$$
$$= I_1^1(k) + I_1^2(k),$$

where

$$\begin{aligned} I_1^1(k) &= 2\sum_{j=1}^{m}\int_{\mathcal{O}_j\cap\Omega}\int_{\mathcal{O}_j\cap\Omega}\frac{\psi_j(x)^2[(u_k^j(x)-1)-(u_k^j(y)-1)]^2}{|x-y|^{N+1}}\,dxdy \\ &\le 2\sum_{j=1}^{m}\int_{\mathcal{O}_j\cap\Omega}\int_{\mathcal{O}_j\cap\Omega}\frac{[(u_k^j(x)-1)-(u_k^j(y)-1)]^2}{|x-y|^{N+1}}\,dxdy \qquad \text{(since } 0\le\psi_j\le 1\text{)} \\ &= c\sum_{j=1}^{m}[u^j-\mathbb{1}_\Omega]^2_{H^{1/2}(\mathcal{O}_j\cap\Omega)} \end{aligned} \tag{5.6.24}$$

and

$$I_1^2(k) = 2\sum_{j=1}^{m}\int_{\mathcal{O}_j\cap\Omega}\int_{\mathcal{O}_j\cap\Omega}\frac{(\psi_j(x)-\psi_j(y))^2(u_k^j(y)-1)^2}{|x-y|^{N+1}}\,dxdy.$$

Using that ψ_j is Lipschitz, we get

$$\begin{aligned} &2\int_{\mathcal{O}_j\cap\Omega}\int_{\mathcal{O}_j\cap\Omega}\frac{(\psi_j(x)-\psi_j(y))^2(u_k^j(y)-1)^2}{|x-y|^{N+1}}\,dxdy \\ &\le c(j)^2\iint_{|x-y|<1}\frac{(u_k^j(y)-1)^2|x-y|^2}{|x-y|^{N+1}}\,dxdy + 8\iint_{|x-y|\ge 1}\frac{(u_k^j(y)-1)^2}{|x-y|^{N+1}}\,dxdy \\ &\le \tilde{c}(j)\|u_k^j-\mathbb{1}_\Omega\|^2_{L^2(\mathcal{O}_j\cap\Omega)} \end{aligned}$$

which implies that

$$I_1^2(k) \le \max_{1\le j\le m}\tilde{c}(j)\sum_{j=1}^{m}\|u_k^j-\mathbb{1}_\Omega\|^2_{L^2(\mathcal{O}_j\cap\Omega)}. \tag{5.6.25}$$

Finally, (5.6.23), (5.6.24) and (5.6.25) yield

$$\begin{aligned} \|u_k-\mathbb{1}_\Omega\|^2_{H^{1/2}(\Omega)} &= \|u_k-\mathbb{1}_\Omega\|^2_{L^2(\Omega)} + [u_k-\mathbb{1}_\Omega]^2_{H^{1/2}(\Omega)} \\ &\le c\sum_{j=1}^{m}\|u_k^j-\mathbb{1}_\Omega\|^2_{L^2(\mathcal{O}_j\cap\Omega)} + CI_1(k) + CI_2(k) \\ &= \tilde{c}\sum_{j=1}^{m}\|u_k^j-\mathbb{1}_\Omega\|^2_{H^{1/2}(\mathcal{O}_j\cap\Omega)} \le C(N,m)\sum_{j=1}^{m}\|\chi_k-1\|^2_{H^{1/2}(0,r_j)}. \end{aligned} \tag{5.6.26}$$

In the latter inequality, we used Lemma 5.6.4. Now, since from Lemma 5.6.1 there holds $\|\chi_k - 1\|^2_{H^{1/2}(0,r_j)} \to 0$ as $k\to\infty$, we complete the proof by letting $k\to\infty$ in the inequality (5.6.26). □

As a direct consequence of the above approximation results, we have the following.

Proposition 5.6.6. *Let $N\ge 2$, $s\in(0,1/2]$ and let $\Omega\subset\mathbb{R}^N$ be a bounded Lipschitz domain. Then*

$$S_{N,s}(\Omega) = 0. \tag{5.6.27}$$

Before proving the proposition above, we mention that our result extends to $s = 1/2$ the one obtained in [83, Lemma 16]. Below, we give the

Proof of Proposition 5.6.6. By definition

$$S_{N,s}(\Omega) = \inf_{\substack{u \in H_0^s(\Omega) \\ u \neq 0}} \frac{Q_{N,s,\Omega}(u)}{\|u\|^2_{L^{2^*_s}(\Omega)}} = \inf_{\substack{u \in C_c^{0,1}(\Omega) \\ u \neq 0}} \frac{Q_{N,s,\Omega}(u)}{\|u\|^2_{L^{2^*_s}(\Omega)}}, \tag{5.6.28}$$

where $C_c^{0,1}(\Omega)$ is the space of Lipschitz functions with compact support. Now by Lemma 5.6.5, we get

$$0 \leq S_{N,s}(\Omega) \leq \frac{Q_{N,s,\Omega}(u_k)}{\|u_k\|^2_{L^{2^*_s}(\Omega)}} \leq C(N,s) \frac{Q_{N,1/2,\Omega}(u_k)}{\|u_k\|^2_{L^{2^*_s}(\Omega)}} = C(N,s) \frac{[u_k - \mathbb{1}_\Omega]_{H^{1/2}(\Omega)}}{\|u_k\|^2_{L^{2^*_s}(\Omega)}} \to 0, \tag{5.6.29}$$

where u_k is defined by (5.6.21), which satisfies $\liminf_{k \to \infty} \|u_k\|^2_{L^{2^*_s}(\Omega)} > 0$. $\square$

Chapter 6

A Hopf lemma for the regional fractional Laplacian

In this chapter, we analyze the behavior near the boundary of the boundary Neumann derivative (for functions vanishing on the boundary) for the regional fractional Laplacian. The presentation of this chapter agrees with the original paper [R5], a collaboration with Nicola Abatangelo and Mouhamed Moustapha Fall. The notation may slightly differ from those in the previous chapters.

6.1 Introduction and main results

Let $s \in (1/2, 1)$ and let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with $C^{1,1}$ boundary. The regional fractional Laplacian $(-\Delta)^s_\Omega$ of a function $u : \Omega \to \mathbb{R}$ is defined as

$$(-\Delta)^s_\Omega u(x) = c_{N,s} P.V. \int_\Omega \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy = c_{N,s} \lim_{\varepsilon \to 0^+} \int_{\Omega \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy, \tag{6.1.1}$$

provided that the limit exists. We recall that "$P.V.$" stands for the Cauchy principal value and that the normalization constant $c_{N,s}$ is explicitly given by

$$c_{N,s} := \left(\int_{\mathbb{R}^N} \frac{1 - \cos(\zeta_1)}{|\zeta|^{N+2s}}\, d\zeta \right)^{-1} = s(1-s) \frac{2^{2s}\, \Gamma(\frac{N+2s}{2})}{\pi^{N/2}\, \Gamma(2-s)}.$$

For functions u belonging to $C^{2s+\varepsilon}_{loc}(\Omega) \cap L^\infty(\Omega)$ for some $\varepsilon > 0$, the integral in (6.1.1) is finite. In this way then, we say that (6.1.1) is defined *pointwisely* in Ω.

The study of the regional fractional Laplacian has received some growing attention in recent years. However, in contrast to that of the[1] fractional Laplacian

$$(-\Delta)^s u(x) = c_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy, \tag{6.1.2}$$

the theory of elliptic problems driven by the regional fractional Laplacian is less developed in spite of some known results. We are concerned here in particular with the Hopf boundary lemma, which is a powerful tool for the study of qualitative properties of solutions like, for example, their monotonicity and symmetry, also via moving plane arguments.

[1] Sometimes it is also called *restricted* fractional Laplacian.

In [93], the authors obtained a Hopf lemma for pointwise super-solutions for an elliptic equation involving the fractional Laplacian $(-\Delta)^s$ under the assumption that an interior ball condition holds. For the Hopf boundary lemma for weak super-solutions related to the fractional p-Laplacian, we refer to [62] and references therein. Other references on the Hopf boundary lemma for fractional Laplacian can be found in [23, 40, 53, 75, 111, 116]. However, to the best of our knowledge, an analogue result for the regional fractional Laplacian has not been investigated before. Let us mention here that while the Hopf lemma is usually used to run a moving plane method in the case of the fractional Laplacian, as recalled above, this does not seem to be the case for the regional fractional Laplacian. The moving plane method for $(-\Delta)^s_\Omega$ remains indeed a challenging question: the main difficulty relies on the fact that the operator depends on the domain and therefore, upon scaling the domain, the operator changes as well. We expect a symmetry breaking in the case of the regional fractional Laplacian defined on bounded domains.

Here, we investigate the validity of a suitable Hopf-type lemma for super-solutions of the equation

$$(-\Delta)^s_\Omega u = c(x)u \quad \text{in} \;\; \Omega. \tag{6.1.3}$$

We analyse this both for the case of *pointwise* and *weak* super-solutions. Moreover, we also study a strong maximum principle for *distributional* super-solutions to (6.1.3). So, before stating our main results, let us recall the following definitions.

Definition 6.1.1. We say that a function $u : \Omega \to \mathbb{R}$ is a pointwise super-solution of (6.1.3) if $u \in C^{2s+\varepsilon}_{loc}(\Omega) \cap L^\infty(\Omega)$ for some $\varepsilon > 0$ and

$$(-\Delta)^s_\Omega u(x) \geq c(x)u(x) \qquad \text{for any } x \in \Omega.$$

Definition 6.1.2. We say that a function $u : \Omega \to \mathbb{R}$ is a weak super-solution of (6.1.3) if $u \in H^s(\Omega)$ and

$$\mathcal{E}(u, \varphi) \geq \int_\Omega cu\varphi \qquad \text{for any } \varphi \in C^\infty_c(\Omega), \; \varphi \geq 0 \text{ in } \Omega.$$

Definition 6.1.3. We say that a function $u : \Omega \to \mathbb{R}$ is a distributional super-solution of (6.1.3) if $u \in L^1(\Omega)$ and

$$\int_\Omega u\,(-\Delta)^s_\Omega \varphi \geq \int_\Omega cu\varphi \qquad \text{for any } \varphi \in C^\infty_c(\Omega), \; \varphi \geq 0 \text{ in } \Omega.$$

In this case, we briefly write

$$(-\Delta)^s_\Omega u \geq c(x)u \qquad \text{in } \mathcal{D}'(\Omega).$$

Remark 6.1.4. Sub-solutions can be defined in similar ways as in Definitions 6.1.1, 6.1.2, and 6.1.3. Also, in the case of Definition 6.1.2, by density the test function φ can be chosen in $H^s_0(\Omega)_+$ if c is somewhat well-behaved (see Lemma 6.4.1 below for more details).

We are going to denote by $\delta_\Omega(x) = \inf\{|x - \theta| : \theta \in \partial\Omega\}$ for $x \in \Omega$. The main results of the paper are the following.

Theorem 6.1.5 (Hopf lemma for pointwise super-solutions). *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with $C^{1,1}$ boundary and $s \in (1/2, 1)$. Let $c \in L^\infty(\Omega)$ and let $u : \overline{\Omega} \to \mathbb{R}$ be a lower semicontinuous super-solution (in the sense of Definition 6.1.1) of* (6.1.3).

(*i*) *If* $c \leq 0$ *in* Ω *and* $u \geq 0$ *on* $\partial\Omega$, *then either* u *vanishes identically in* Ω *or*

$$\liminf_{\Omega \ni x \to z} \frac{u(x)}{\delta_\Omega(x)^{2s-1}} > 0 \qquad \text{for any } z \in \partial\Omega. \tag{6.1.4}$$

(*ii*) *If* $u \geq 0$ *in* $\overline{\Omega}$, *then either* u *vanishes identically in* Ω *or* (6.1.4) *holds true.*

Theorem 6.1.6 (Hopf lemma for weak super-solutions)**.** *Let* $\Omega \subset \mathbb{R}^N$ *be an open bounded set with* $C^{1,1}$ *boundary and* $s \in (1/2, 1)$. *Let* $c : \Omega \to \mathbb{R}$ *be a measurable function and let* $u \in H^s(\Omega)$ *be a weak super-solution (in the sense of Definition 6.1.2) of* (6.1.3). *Suppose that either*

$$c \in L^\infty(\Omega) \tag{6.1.5}$$

or

$$c \in L^q(\Omega), \; q > \frac{N}{2s}, \qquad \text{and} \qquad u \in L^\infty_{loc}(\Omega), \tag{6.1.6}$$

hold.

(*i*) *If* $c \leq 0$ *in* Ω *and* $u \geq 0$ *on* $\partial\Omega$, *then either* u *vanishes identically in* Ω *or*

$$\text{there exists } \varepsilon_0 > 0 \quad \text{such that} \quad \frac{u(x)}{\delta_\Omega(x)^{2s-1}} > \varepsilon_0. \tag{6.1.7}$$

(*ii*) *If* $u \geq 0$ *in* Ω, *then either* u *vanishes identically in* Ω *or* (6.1.7) *holds true.*

Let us first comment on the proof of Theorem 6.1.5. Starting with a strong maximum principle, we obtain the strict positivity of non-trivial super-solutions of (6.1.3): this is where the lower semicontinuity of u is needed. In a next step, we construct a barrier from below for u in terms of the torsion function u_{tor}, *i.e.*, the solution to the boundary value problem

$$\begin{cases} (-\Delta)^s_\Omega u_{tor} = 1 & \text{in } \Omega, \\ u_{tor} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1.8}$$

This function is known to satisfy, on smooth domains, the double-sided estimate

$$C^{-1}\delta_\Omega^{2s-1} \leq u_{tor} \leq C\delta_\Omega^{2s-1} \qquad \text{in } \Omega \tag{6.1.9}$$

for some $C > 1$, see [26, 44] which are based on some estimates in [24, 47, 99]. Intuitively, (6.1.9) gives that the boundary behaviour of super-solutions described by (6.1.4) and (6.1.7) is optimal. We notice that, in contrast to what happens for the fractional Laplacian, there are no explicit examples of torsion functions for the regional fractional Laplacian, even in the case when Ω is a ball. In [69], a numerical analysis is performed in the one-dimensional case $\Omega = (-1, 1)$.

We mention that the existence and uniqueness of poitnwise and weak solutions to the Dirichlet problem (6.1.8) with general bounded right-hand side was obtained in [44]. We notice also that the Hölder regularity up to the boundary of any weak solution of (6.1.8) was recently proved in [73], while regularity up to the boundary of pointwise solution of (6.1.8) was obtained earlier in [44]. We also mention that the boundary regularity of the ratio $u_{tor}/\delta_\Omega^{2s-1}$ has been established in [73] in the case when Ω is of class $C^{1,\beta}$ for some $\beta > 0$. Thus, it makes sense to evaluate $u_{tor}/\delta_\Omega^{2s-1}$ pointwisely on $\overline{\Omega}$.

The proof of Theorem 6.1.6 follows the same line of thought as the one of Theorem 6.1.5, although with some more technical difficulties due to the weak character of super-solutions involved. For example, when $c \in L^q(\Omega)$ the strong maximum principle involved in our strategy takes the following form.

Proposition 6.1.7 (Strong maximum principle for distributional super-solutions)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $u \in L^\infty_{\mathrm{loc}}(\Omega)$ be a distributional super-solution (in the sense of Definition 6.1.3) of* (6.1.3) *with*

$$c \in L^q_{loc}(\Omega), \quad q > \frac{N}{2s}. \tag{6.1.10}$$

If $u \geq 0$ in Ω, then

$$\textit{either} \qquad u \equiv 0 \quad \textit{in } \Omega \qquad \textit{or} \qquad \operatorname{essinf}_K u > 0 \quad \textit{for any } K \subset\subset \Omega.$$

The paper is organized as follows. In Section 6.2, we present some notations and definitions. Section 6.3 is devoted to the proof of Theorem 6.1.5, whereas in Section 6.4 we prove Theorem 6.1.6. Finally, in Section 6.5 we prove Proposition 6.1.7.

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6.2 Preliminaries

We collect in this section some notations and useful tools. For $s \in (0,1)$, $H^s(\Omega)$ denotes the space of functions $u \in L^2(\Omega)$ such that

$$[u]^2_{H^s(\Omega)} := \frac{c_{N,s}}{2} \int_\Omega \int_\Omega \frac{\big(u(x) - u(y)\big)^2}{|x-y|^{N+2s}}\, dx\, dy < \infty.$$

It is a Hilbert space endowed with the norm

$$\|u\|_{H^s(\Omega)} := \big(\|u\|^2_{L^2(\Omega)} + [u]^2_{H^s(\Omega)}\big)^{1/2}.$$

We denote by $H^s_0(\Omega)$ the completion of $C^\infty_c(\Omega)$ with respect to the norm $\|\cdot\|_{H^s(\Omega)}$. It is known that for $s \in (1/2, 1)$, $H^s_0(\Omega)$ is a Hilbert space with the norm $\|\cdot\|_{H^s_0(\Omega)} = [\cdot]_{H^s(\Omega)}$ (which is equivalent to the usual one in $H^s(\Omega)$ thanks to a Poincaré-type inequality) and it can be characterized as follows

$$H^s_0(\Omega) := \big\{u \in H^s(\Omega) : u = 0 \text{ on } \partial\Omega\big\}.$$

Next, we define $H^s_0(\Omega)_+$ by

$$H^s_0(\Omega)_+ := \big\{u \in H^s_0(\Omega) : u \geq 0 \text{ in } \Omega\big\}.$$

For $u, v \in H_0^s(\Omega)$, we consider the symmetric, continuous, and coercive bilinear form

$$\mathcal{E}(u,v) := \frac{c_{N,s}}{2} \int_\Omega \int_\Omega \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{N+2s}}\,dx\,dy.$$

The first Dirichlet eigenvalue of $(-\Delta)^s_\Omega$ in Ω can be defined by

$$\lambda_1(\Omega) = \min_{\substack{u \in H_0^s(\Omega)\\ u\neq 0}} \frac{\mathcal{E}(u,u)}{\|u\|^2_{L^2(\Omega)}}. \tag{6.2.1}$$

It holds $\lambda_1(\Omega) > 0$, with the corresponding eigenfunction unique and strictly positive in Ω.

Given $x \in \Omega$ and $r > 0$, we denote by $B_r(x)$ the open ball centred at x with radius r. We denote by $u^+ := \max\{u, 0\}$ and $u^- := \max\{-u, 0\}$ the positive and negative part of u respectively. We also recall that, if $u \in H^s(\Omega)$, then $u^+, u^- \in H^s(\Omega)$ as well: this follows from a simple calculation, indeed $u = u^+ - u^-$ and

$$[u]^2_{H^s(\Omega)} = \mathcal{E}(u,u) = \mathcal{E}(u^+,u^+) - 2\mathcal{E}(u^+,u^-) + \mathcal{E}(u^-,u^-)$$

where

$$\begin{aligned}\mathcal{E}(u^+,u^-) = \frac{c_{N,s}}{2} \int_\Omega \int_\Omega \frac{\big(u^+(x)-u^+(y)\big)\big(u^-(x)-u^-(y)\big)}{|x-y|^{N+2s}}\,dx\,dy = \\ = -c_{N,s} \int_\Omega \int_\Omega \frac{u^+(x)\,u^-(y)}{|x-y|^{N+2s}}\,dx\,dy \leq 0.\end{aligned}$$

6.3 Proof of the Hopf lemma: the case of pointwise super-solutions

The aim of this section is to prove Theorem 6.1.5. Before doing this, we need one key result: we state and prove a strong maximum principle for pointwise super-solutions of (6.1.3).

Proposition 6.3.1 (Strong maximum principle for pointwise super-solutions)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Let $c \in L^\infty(\Omega)$ and $u : \overline{\Omega} \to \mathbb{R}$ be a lower semicontinuous function super-solution (in the sense of Definition 6.1.1) of* (6.1.3)*.*

(i) If $c \leq 0$ in Ω and $u \geq 0$ on $\partial\Omega$, then either u vanishes identically in Ω, or $u > 0$ in Ω.

(ii) If $u \geq 0$ in $\overline{\Omega}$, then either u vanishes identically in Ω, or $u > 0$ in Ω.

Proof. Before going into the proof, we start by proving that the function u is nonnegative in $\overline{\Omega}$ as long as the hypotheses of assertion (i) are satisfy.

Let us assume that $c \leq 0$ in Ω, $u \geq 0$ on $\partial\Omega$, and that u does not vanish identically on Ω. Then we claim that

$$u \geq 0 \quad \text{in} \quad \overline{\Omega}. \tag{6.3.1}$$

Assume to the contrary that (6.3.1) does not hold, that is, u is negative somewhere in Ω. Then, using that $\overline{\Omega}$ is compact together with the hypotheses of lower semicontinuity of u, a negative minimum of the function u must be achieved in Ω. In other words, there exists $x_0 \in \Omega$ such that

$$u(x_0) = \min_{x\in\Omega} u(x) < 0. \tag{6.3.2}$$

Combining (6.3.2) with $u \geq 0$ on $\partial\Omega$, it follows that

$$(-\Delta)^s_\Omega u(x_0) = c_{N,s} P.V. \int_\Omega \frac{u(x_0) - u(y)}{|x_0 - y|^{N+2s}}\, dy < 0.$$

But, since by assumption $c(x_0) \leq 0$, we have that $c(x_0)u(x_0) \geq 0$. Therefore

$$0 > (-\Delta)^s_\Omega u(x_0) \geq c(x_0)u(x_0) \geq 0.$$

which is a contradiction. Consequently, claim (6.3.1) follows.

So we can now suppose $u \geq 0$ in Ω. Suppose that $u \not\equiv 0$ in Ω and let us prove that

$$u > 0 \qquad \text{in } \Omega. \tag{6.3.3}$$

First of all, we recall that by the lower semicontinuity of u, there exist $x_1 \in \Omega$ and $\varepsilon_1, r > 0$ such that

$$u(y) \geq \varepsilon_1 \qquad \text{for all} \quad y \in B_r(x_1) \subset \Omega.$$

If the inequality (6.3.3) were not true, that is, if $u(\tilde{x}) = 0$ at some $\tilde{x} \in \Omega$, then it would hold

$$(-\Delta)^s_\Omega u(\tilde{x}) = c_{N,s} P.V. \int_\Omega \frac{-u(y)}{|\tilde{x} - y|^{N+2s}}\, dy \leq c_{N,s} P.V. \int_{B_r(x_1)} \frac{-u(y)}{|\tilde{x} - y|^{N+2s}}\, dy < 0$$

Therefore

$$0 > (-\Delta)^s_\Omega u(\tilde{x}) \geq c(\tilde{x})u(\tilde{x}) = 0,$$

a contradiction. Thus, the strict inequality $u > 0$ in Ω must hold true. $\square$

Having the above strong maximum principle, we can now give the proof of Theorem 6.1.5 by following some ideas in [93].

Proof of Theorem 6.1.5. From Proposition 6.3.1 it follows that

$$u(x) > 0 \qquad \text{for all } x \in \Omega \tag{6.3.4}$$

provided that u does not vanish identically in Ω. In other words, if u does not vanish identically in Ω, then for every compact subset $K \subset \Omega$ we have

$$\inf_{y \in K} u(y) > 0. \tag{6.3.5}$$

Now suppose that u does not vanish identically in Ω and let us prove (6.1.4). To this end, it suffices to construct a barrier for u in terms of the solution problem (6.1.8). Let u_{tor} denote the pointwise solution of (6.1.8).

Next, for $n \in \mathbb{N}$, we set

$$v_n(x) = \frac{1}{n} u_{tor}(x) \qquad \text{for } x \in \overline{\Omega}. \tag{6.3.6}$$

Then, by definition and (6.1.9), by the boundedness of Ω it follows that

$$v_n \to 0 \qquad \text{uniformly in } \overline{\Omega}. \tag{6.3.7}$$

We wish now to show that there exists some $n_0 \in \mathbb{N}$ such that

$$u \geq v_n \qquad \text{in } \overline{\Omega}, \text{ for any } n \geq n_0. \tag{6.3.8}$$

In order to prove (6.3.8), we argue by contradiction: suppose that for every $n \in \mathbb{N}$ the function w_n defined by

$$w_n := v_n - u \qquad \text{in } \overline{\Omega}$$

is positive somewhere in $\overline{\Omega}$. Then, using that $w_n = v_n - u = -u \leq 0$ on $\partial\Omega$ and the compactness of $\overline{\Omega}$, a positive maximum of the upper semicontinuous function w_n (since u is lower semicontinuous by assumption) must be achieved at some $x_n \in \Omega$, that is, there exists $x_n \in \Omega$ such that

$$w_n(x_n) = \max_{x \in \Omega} w_n(x) > 0. \tag{6.3.9}$$

This implies together with (6.3.4) that $0 < u(x_n) < v_n(x_n)$. From this and thanks to (6.3.7), we find that

$$\lim_{n \to \infty} u(x_n) = 0. \tag{6.3.10}$$

Recalling (6.3.5), we deduce from (6.3.10) that $x_n \to \partial\Omega$ as $n \to \infty$. Taking this into account, one deduces that for any compact set $K \subset \Omega$ there exists $h > 0$ such that $|x_n - y| \geq h > 0$ for any $y \in K$ and n sufficiently large. As a direct consequence, there exist two positive constants $\gamma_1, \gamma_2 > 0$, independent of n such that

$$\gamma_1 < \int_K \frac{dy}{|x_n - y|^{N+2s}} < \gamma_2 \qquad \text{for } n \text{ sufficiently large (depending on } K). \tag{6.3.11}$$

Thus we have

$$c(x_n)u(x_n) \leq (-\Delta)^s_\Omega u(x_n) \leq c_{N,s} \int_K \frac{u(x_n) - u(y)}{|x_n - y|^{N+2s}}\, dy + c_{N,s} P.V. \int_{\Omega \setminus K} \frac{u(x_n) - u(y)}{|x_n - y|^{N+2s}}\, dy. \tag{6.3.12}$$

We now aim at estimating the integrals on the right-hand side of the above inequality. Concerning the first integral, we notice that by (6.3.5), there exists a positive constant $\gamma_3 > 0$ such that $u(y) \geq \gamma_3$ for $y \in K$. As a consequence of this and by using (6.3.10) and (6.3.11), it follows that

$$\limsup_{n \to \infty} \int_K \frac{u(x_n) - u(y)}{|x_n - y|^{N+2s}}\, dy \leq -\gamma_1\gamma_3 < 0. \tag{6.3.13}$$

Regarding the second integral in (6.3.12), we first recall that since x_n is the maximum of w_n in $\overline{\Omega}$, then by (6.3.9)

$$u(x_n) - u(y) \leq v_n(x_n) - v_n(y).$$

Using this, the second integral in (6.3.12) can be estimated as follows:

$$P.V. \int_{\Omega \setminus K} \frac{u(x_n) - u(y)}{|x_n - y|^{N+2s}}\, dy \leq P.V. \int_{\Omega \setminus K} \frac{v_n(x_n) - v_n(y)}{|x_n - y|^{N+2s}}\, dy. \tag{6.3.14}$$

Moreover, a simple calculation yields

$$c_{N,s} P.V. \int_{\Omega \setminus K} \frac{v_n(x_n) - v_n(y)}{|x_n - y|^{N+2s}}\, dy = (-\Delta)^s_\Omega v_n(x_n) - c_{N,s} \int_K \frac{v_n(x_n) - v_n(y)}{|x_n - y|^{N+2s}}\, dy. \tag{6.3.15}$$

Now, from (6.3.6) and (6.1.8), it follows that

$$(-\Delta)^s_\Omega v_n(x_n) = \frac{1}{n}(-\Delta)^s_\Omega u_{tor}(x_n) = \frac{1}{n}.$$

This yields

$$(-\Delta)^s_\Omega v_n(x_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{6.3.16}$$

Combining (6.3.16), (6.3.7) and (6.3.11), we observe that the right-hand side in the equality (6.3.15) goes to zero as $n \to \infty$ and therefore

$$\lim_{n\to\infty} P.V. \int_{\Omega\setminus K} \frac{v_n(x_n) - v_n(y)}{|x_n - y|^{N+2s}}\, dy = 0.$$

Consequently, from (6.3.14), we get

$$\limsup_{n\to\infty} P.V. \int_{\Omega\setminus K} \frac{u(x_n) - u(y)}{|x_n - y|^{N+2s}}\, dy \le 0. \tag{6.3.17}$$

However, using that c is bounded, it follows from (6.3.10) that

$$\lim_{n\to\infty} c(x_n)u(x_n) = 0. \tag{6.3.18}$$

Finally, (6.3.17) and (6.3.13) into (6.3.12), lead to a contradiction with (6.3.18). Therefore, the inequality (6.3.8) follows for some $n \in \mathbb{N}$ large enough. $\square$

6.4 Proof of the Hopf lemma: the case of weak super-solutions

In this section, we aim at proving Theorem 6.1.6. Here, the function u_{tor} defined via (6.1.8) above is understood to be a weak solution. Recall the double-sided estimate (6.1.9). We first state and prove a technical lemma and a strong maximum principle for weak super-solutions of (6.1.3).

Lemma 6.4.1. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and $c \in L^{\frac{N}{2s}}(\Omega)$. Then u is a weak super-solution (in the sense of Definition 6.1.2) of* (6.1.3) *if and only if*

$$\mathcal{E}(u,v) \ge \int_\Omega cuv \qquad \textit{for any } v \in H^s_0(\Omega)_+.$$

Proof. Fixed $v \in H^s_0(\Omega)_+$, let $(\psi_n)_{n\in\mathbb{N}} \subset C^\infty_c(\Omega)$ a sequence of nonnegative functions converging to v in the $H^s(\Omega)$-norm. By Definition 6.1.2 we have

$$\mathcal{E}(u,\psi_n) \ge \int_\Omega cu\psi_n \qquad \text{for any } n \in \mathbb{N}.$$

On the left-hand side we have the convergence $\mathcal{E}(u,\psi_n) \to \mathcal{E}(u,v)$ as $n \to \infty$ by construction; so, let us deal with right-hand side. By the Sobolev embedding we have $\psi_n \to v$ as $n \to \infty$ in $L^{2^*_s}(\Omega)$, with $2^*_s = \frac{2N}{N-2s}$. So, we have the convergence

$$\int_\Omega cu\psi_n \longrightarrow \int_\Omega cuv \qquad \text{as } n \to \infty,$$

if $cu \in L^{\frac{2N}{N+2s}}(\Omega)$ where $\frac{2N}{N+2s} = (2_s^*)'$ is the conjugate exponent of 2_s^*, which is what we show next. This indeed follows from the Hölder inequality:

$$\int_\Omega |cu|^{\frac{2N}{N+2s}} = \left(\int_\Omega |c|^{\frac{N}{2s}}\right)^{\frac{4s}{N+2s}} \left(\int_\Omega |u|^{\frac{2N}{N-2s}}\right)^{\frac{N-2s}{N+2s}} < \infty.$$

Then

$$\mathcal{E}(u,v) = \lim_{n\to\infty} \mathcal{E}(u,\psi_n) \geq \lim_{n\to\infty} \int_\Omega cu\psi_n = \int_\Omega cuv.$$

□

Proposition 6.4.2 (Strong maximum principle for weak super-solutions). *Let $c \in L^q(\Omega)$, with $q > \frac{N}{2s}$, and $u \in H^s(\Omega) \cap L^\infty_{loc}(\Omega)$ be a weak super-solution of*

$$(-\Delta)^s_\Omega u = c(x)u \qquad \text{in } \Omega. \tag{6.4.1}$$

(i) If $c \leq 0$ in Ω and $u \geq 0$ on $\partial\Omega$, then either u vanishes identically in Ω or $u > 0$ in Ω.

(ii) If $u \geq 0$ in Ω, then either u vanishes identically in Ω or $u > 0$ in Ω.

Proof. We first recall the following elementary inequality:

$$(u(x)-u(y))(u^-(x)-u^-(y)) \leq -(u^-(x)-u^-(y))^2, \qquad \text{for any } x,y\in\Omega. \tag{6.4.2}$$

Assume then $c \leq 0$ in Ω and $u \geq 0$ on $\partial\Omega$. Then $u^- = 0$ on $\partial\Omega$. Moreover, by standard arguments, we also know $u^- \in H^s(\Omega)$. Therefore $u^- \in H^s_0(\Omega)_+$. Hence, by testing (6.4.1) on u^- (which is allowed by Lemma 6.4.1), we have from inequality (6.4.2) that

$$\int_\Omega c(x)u(x)u^-(x)\,dx \leq \mathcal{E}(u,u^-) \leq -\mathcal{E}(u^-,u^-).$$

Moreover, $u = u^+ - u^-$ with $u^+u^- \equiv 0$ in Ω, which yields

$$\int_\Omega c(x)u^-(x)^2\,dx \geq \mathcal{E}(u^-,u^-) \geq \lambda_1(\Omega)\|u^-\|^2_{L^2(\Omega)},$$

where $\lambda_1(\Omega)$ has been defined in (6.2.1). Since $\lambda_1(\Omega) > 0$, then from the nonpositivity of c it follows

$$\|u^-\|^2_{L^2(\Omega)} = 0$$

implying that $u^- = 0$ a.e. in Ω, that is, $u \geq 0$ a.e. in Ω.

So we can at this point assume that $u \geq 0$ in Ω. Note that the fact that u is a weak super-solution implies in particular that u is also a distributional super-solution. Indeed, for any $\psi \in C^\infty_c(\Omega)$, $\psi \geq 0$ in Ω,

$$\begin{aligned}\int_\Omega cu\psi \leq \mathcal{E}(u,\psi) &= \frac{c_{N,s}}{2}\int_\Omega\int_\Omega \frac{\big(u(x)-u(y)\big)\big(\psi(x)-\psi(y)\big)}{|x-y|^{N+2s}}\,dxdy \\ &= c_{N,s}\int_\Omega u(x)\,P.V.\int_\Omega \frac{\psi(x)-\psi(y)}{|x-y|^{N+2s}}\,dy\,dx = \int_\Omega u(-\Delta)^s_\Omega\psi.\end{aligned}$$

Using this remark, we can use Proposition 6.1.7. □

Remark 6.4.3. It is possible to drop the assumption $u \in L^\infty_{loc}(\Omega)$ in Proposition 6.4.2 by paying the price of assuming $c \in L^\infty(\Omega)$. In this case, the first part of the proof still holds while, instead of using Proposition 6.1.7, the second part simply follows from [108, Theorem 1.2].

We now prove Theorem 6.1.6. For the sake of clarity, we split its proof into two different arguments.

Proof of Theorem 6.1.6 under assumption (6.1.5). Suppose that u does not vanish identically in Ω and let us prove (4.4.9). In other words, we want to prove that there exists a positive constant $C > 0$ such that

$$u \geq C\delta_\Omega^{2s-1} \qquad \text{in } \Omega. \tag{6.4.3}$$

From Proposition 6.4.2 and Remark 6.4.3 it follows that $u > 0$ in Ω. This means that for any $K \subset\subset \Omega$ there exists $\varepsilon > 0$ such that it holds

$$u(x) \geq \varepsilon > 0 \qquad \text{for } x \in K. \tag{6.4.4}$$

Now, let $w_n := v_n - u$ where v_n is the function defined in (6.3.6). Then, thanks to (6.3.7) and (6.4.4), we can assume without any ambiguity that

$$w_n^+ \equiv 0 \qquad \text{in } K \quad \text{for } n \text{ sufficiently large.} \tag{6.4.5}$$

Now, since $w_n^+ \in H_0^s(\Omega)_+$ (because $w_n^+ \geq 0$ in Ω, $w_n^+ \in H^s(\Omega)$ since w_n does, and $w_n^+ = 0$ on $\partial\Omega$ since $v_n = 0$ on $\partial\Omega$ and $u \geq 0$ on $\partial\Omega$), one can use it as a test function in Definition 6.1.2 (by Lemma 6.4.1) in order to have

$$\mathcal{E}(u, w_n^+) \geq \int_\Omega c u w_n^+. \tag{6.4.6}$$

Since in $\{w_n^+ > 0\}$ it holds

$$u < v_n \leq \frac{1}{n}\|u_{tor}\|_{L^\infty(\Omega)},$$

we have

$$\int_\Omega c(x)u(x)w_n^+(x)\,dx \geq -\frac{1}{n}\|c\|_{L^\infty(\Omega)}\|u_{tor}\|_{L^\infty(\Omega)}\|w_n^+\|_{L^1(\Omega)}. \tag{6.4.7}$$

On the other hand,

$$\begin{aligned}\mathcal{E}(u, w_n^+) = \mathcal{E}(u - v_n, w_n^+) + \mathcal{E}(v_n, w_n^+) = \mathcal{E}(-w_n, w_n^+) + \frac{1}{n}\mathcal{E}(u_{tor}, w_n^+) = \\ = -\mathcal{E}(w_n^+, w_n^+) + \mathcal{E}(w_n^-, w_n^+) + \frac{1}{n}\mathcal{E}(u_{tor}, w_n^+).\end{aligned}$$

Since the first term on the right-hand side of the above equality is nonpositive and $\mathcal{E}(u_{tor}, w_n^+) = \int_\Omega w_n^+ = \|w_n^+\|_{L^1(\Omega)}$ thanks to (6.1.8), then

$$\mathcal{E}(u, w_n^+) \leq \mathcal{E}(w_n^-, w_n^+) + \frac{1}{n}\|w_n^+\|_{L^1(\Omega)}. \tag{6.4.8}$$

Now,

$$\mathcal{E}(w_n^-, w_n^+) = -c_{N,s} \int_\Omega \int_\Omega \frac{w_n^-(x) w_n^+(y)}{|x-y|^{N+2s}} \, dx \, dy. \tag{6.4.9}$$

Recall that, by definition (see also (6.4.5)), $K \subset \{w_n < 0\} = \{w_n^- > 0\}$ and

$$w_n^- \geq \varepsilon - \frac{1}{n} \|u_{tor}\|_{L^\infty(\Omega)} \qquad \text{in } \Omega,$$

so, upon plugging this into (6.4.9), we obtain

$$\begin{aligned} \mathcal{E}(w_n^-, w_n^+) &\leq -c_{N,s} \int_\Omega \int_K \frac{w_n^-(x) w_n^+(y)}{|x-y|^{N+2s}} \, dx \, dy \\ &\leq c_{N,s} \left(\frac{1}{n} \|u_{tor}\|_{L^\infty(\Omega)} - \varepsilon \right) \int_\Omega \int_K \frac{w_n^+(y)}{|x-y|^{N+2s}} \, dx \, dy \\ &\leq C_0 c_{N,s} \left(\frac{1}{n} \|u_{tor}\|_{L^\infty(\Omega)} - \varepsilon \right) \|w_n^+\|_{L^1(\Omega)} \end{aligned}$$

for some $C_0 > 0$ and n sufficiently large. Plugging (6.4.7) into (6.4.6), using (6.4.8) and this last obtained inequality, we get

$$C_0 c_{N,s} \left(\frac{1}{n} \|u_{tor}\|_{L^\infty(\Omega)} - \varepsilon \right) \|w_n^+\|_{L^1(\Omega)} + \frac{1}{n} \|w_n^+\|_{L^1(\Omega)} \geq -\frac{1}{n} \|c\|_{L^\infty(\Omega)} \|u_{tor}\|_{L^\infty(\Omega)} \|w_n^+\|_{L^1(\Omega)}. \tag{6.4.10}$$

For n sufficiently large, we deduce from (6.4.10) that

$$w_n^+ \equiv 0 \qquad \text{in } \Omega.$$

Therefore, (6.4.3) follows. □

Proof of Theorem 6.1.6 under assumption (6.1.6). The very first part of the proof follows the argument given above. We start here from (6.4.6). We know from Proposition 6.4.2 that $u > 0$ in Ω and so $w_n < v_n$, from which it follows

$$\int_\Omega c u w_n^+ \geq -\frac{1}{n} \|u_{tor}\|_{L^\infty(\Omega)} \int_\Omega |cu|.$$

By the fractional Sobolev inequality we have that $u \in L^p(\Omega)$ for any $1 \leq p \leq 2_s^* = 2N/(N-2s)$. As the conjugate exponent of $2N/(N-2s)$ is $2N/(N+2s)$ which is smaller than $N/(2s)$, we have by an application of the Hölder's inequality that

$$\int_\Omega c u w_n^+ \geq -\frac{1}{n} \|u_{tor}\|_{L^\infty(\Omega)} \|u\|_{L^{2_s^*}(\Omega)} \|c\|_{L^q(\Omega)}.$$

By repeating the calculations in the preceding argument we then get the analog of (6.4.10) which reads in this case

$$C_0 c_{N,s} \left(\frac{1}{n} \|u_{tor}\|_{L^\infty(\Omega)} - \varepsilon \right) \|w_n^+\|_{L^1(\Omega)} + \frac{1}{n} \|w_n^+\|_{L^1(\Omega)} \geq -\frac{1}{n} \|u_{tor}\|_{L^\infty(\Omega)} \|u\|_{L^{2_s^*}(\Omega)} \|c\|_{L^q(\Omega)}.$$

This last inequality, for n sufficiently large, gives

$$w_n^+ \equiv 0 \qquad \text{in } \Omega.$$

Therefore, (6.4.3) follows also in this case. □

6.5 Proof of the strong maximum principle for distributional super-solutions

This last section is devoted to the proof of Proposition 6.1.7. In the following, we assume that $u:\Omega\to\mathbb{R}$ is a distributional super-solution (in the sense of Definition 6.1.3) of (6.1.3) and that c satisfies the assumptions in (6.1.10).

6.5.1 Regional v. restricted fractional Laplacian

Note that

$$(-\Delta)^s_\Omega\psi=(-\Delta)^s\psi-\kappa_\Omega\psi\quad\text{in }\Omega,\qquad \kappa_\Omega(x)=c_{N,s}\int_{\mathbb{R}^N\setminus\Omega}\frac{dy}{|x-y|^{N+2s}}\quad\text{for }x\in\Omega,$$

where we recall (6.1.2), so that Definition 6.1.3 is equivalent to (if we extend $u=0$ in $\mathbb{R}^N\setminus\overline{\Omega}$)

$$(-\Delta)^s u\geq\big(c+\kappa_\Omega\big)u\quad\text{in }\mathcal{D}'(\Omega),\qquad c\in L^q_{\text{loc}}(\Omega),\ q>\frac{N}{2s}. \tag{6.5.1}$$

6.5.2 Approximation and representation of distributional solutions

Consider a solution $u:\mathbb{R}^N\to[0,+\infty)$ to (6.5.1) with

$$u\in L^\alpha_{\text{loc}}(\Omega),\qquad \alpha>\frac{Nq}{2sq-N},\ \text{and}\quad u=0\text{ in }\mathbb{R}^N\setminus\overline{\Omega}. \tag{6.5.2}$$

Take $\eta_\varepsilon\in C^\infty_c(B_\varepsilon)$ a mollifier. If we take an open $\Omega'\subset\subset\Omega$ then for any $\psi\in C^\infty_c(\Omega')$, $\psi\geq 0$, it holds $\psi*\eta_\varepsilon\in C^\infty_c(\Omega)$ for ε small independently of ψ and we can say

$$\begin{aligned}\int_{\mathbb{R}^N}\big(u*\eta_\varepsilon\big)(-\Delta)^s\psi=\int_{\mathbb{R}^N}u\big(\eta_\varepsilon*(-\Delta)^s\psi\big)=\int_\Omega u(-\Delta)^s\big(\psi*\eta_\varepsilon\big)\geq\int_\Omega\big(c+\kappa_\Omega\big)u\big(\psi*\eta_\varepsilon\big)=\\ =\int_\Omega\Big[\Big(\big(c+\kappa_\Omega\big)u\Big)*\eta_\varepsilon\Big]\psi\end{aligned}$$

which implies that

$$(-\Delta)^s\big(u*\eta_\varepsilon\big)\geq\Big(\big(c+\kappa_\Omega\big)u\Big)*\eta_\varepsilon\qquad\text{in }\Omega'.$$

As $u*\eta_\varepsilon\in C^\infty(\overline{\Omega'})$, the above inequality also holds in a pointwise sense. We can then exploit a Green representation on $u*\eta_\varepsilon$ (see [33]) to deduce that for any $x\in\Omega''\subset\subset\Omega'$ and $0<r<\text{dist}(\Omega'',\mathbb{R}^N\setminus\Omega')$

$$\begin{aligned}\big(u*\eta_\varepsilon\big)(x)\geq r^{2s}\int_{B_1}G(0,y)\Big[\Big(\big(c+\kappa_\Omega\big)u\Big)*\eta_\varepsilon\Big](x+ry)\,dy\,+\\ +\int_{\mathbb{R}^N\setminus B_1}P(0,y)\big(u*\eta_\varepsilon\big)(x+ry)\,dy.\end{aligned} \tag{6.5.3}$$

Here we have used the kernels G and P which are respectively the Green function and the Poisson kernel of the fractional Laplacian $(-\Delta)^s$ on the unitary ball B_1, which are explicitly known, see [33]:

$$G(x,y)=\frac{k_{N,s}}{|x-y|^{N-2s}}\int_0^{\frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2}}\frac{t^{s-1}}{(t+1)^{N/2}}\,dt\qquad\qquad x,y\in B_1,$$

$$P(x,y) = \frac{\gamma_{N,s}}{|x-y|^N}\left(\frac{1-|x|^2}{|y|^2-1}\right)^s \qquad x \in B_1,\ y \in \mathbb{R}^N \setminus B_1.$$

From now on, we assume that $u \geq 0$ in Ω. We want to send $\varepsilon \to 0$ in (6.5.3) and deduce a representation for u. For the Poisson integral we use the nonnegativity of u and the Fatou's Lemma to say

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^N\setminus B_1} P(0,y)\big(u * \eta_\varepsilon\big)(x+ry)\,dy \geq \int_{\mathbb{R}^N\setminus B_1} P(0,y)\,u(x+ry)\,dy.$$

For the Green integral we use that

$$G(0,\cdot) \in L^p(B_1) \qquad \text{for any } p \in \Big[1, \frac{N}{N-2s}\Big)$$

and

$$\Big\|\Big((c+\kappa_\Omega)u\Big) * \eta_\varepsilon\Big\|_{L^\beta(\Omega')} \leq C\Big\|(c+\kappa_\Omega)u\Big\|_{L^\beta(\Omega')} \leq C\|cu\|_{L^\beta(\Omega')} + C\big\|\kappa_\Omega\big\|_{L^\infty(\Omega')}\|u\|_{L^\alpha(\Omega')} \qquad \text{for any } \beta \in \Big(\frac{N}{2s}, \alpha\Big)$$

where, moreover, by the Hölder inequality

$$\int_{\Omega'} |cu|^\beta \leq \|c\|_{L^q(\Omega')}^{\frac{1}{\beta}}\big\|u^\beta\big\|_{L^{q/(q-\beta)}(\Omega')} \qquad \text{for } \frac{N}{2s} < \beta < q,$$

$$\int_{\Omega'} u^{\beta q/(q-\beta)} < \infty \qquad \text{for } \frac{\beta q}{q-\beta} < \alpha,$$

where the second inequality holds for β close to $\frac{N}{2s}$ in view of (6.5.2). Therefore, using the weak topology in Lebesgue spaces,

$$\lim_{\varepsilon\to 0}\int_{B_1} G(0,y)\Big[\Big((c+\kappa_\Omega)u\Big) * \eta_\varepsilon\Big](x+ry)\,dy = \\ = \int_{B_1} G(0,y)\,\big(c+\kappa_\Omega\big)(x+ry)\,u(x+ry)\,dy.$$

Thus

$$u(x) \geq r^{2s}\int_{B_1} G(0,y)\,\big(c+\kappa_\Omega\big)(x+ry)\,u(x+ry)\,dy + \int_{\mathbb{R}^N\setminus B_1} P(0,y)\,u(x+ry)\,dy \qquad \text{for a.e. } x \in \Omega''. \tag{6.5.4}$$

6.5.3 The Hardy-Littlewood maximal function

Recall that, given $f \in L^p_{\mathrm{loc}}(\mathbb{R}^N)$, $p > 1$, the Hardy-Littlewood maximal function is defined as

$$\mathbf{M}[f](x) = \sup_{r>0}\frac{1}{r^N}\int_{B_r(x)} |f| \qquad x \in \mathbb{R}^N. \tag{6.5.5}$$

In the following we are going to need the following fact

$$\big\|\mathbf{M}[f]\big\|_{L^p(K)} \leq C\|f\|_{L^p(K)} \qquad \text{for } p > 1 \text{ and } K \subset\subset \mathbb{R}^N \text{ measurable.} \tag{6.5.6}$$

6.5.4 The strong maximum principle

Having the above ingredients, in this subsection, we are ready to give the proof of Proposition 6.1.7.

Proof of Proposition 6.1.7. We argue by contradiction. Assume that $|\{u > \delta\} \cap \Omega'| > 0$ for some $\delta > 0$.

In the notations of the previous subsection, and without loss of generality, we assume that

$$\begin{aligned} &\text{there exist } \ (x_j)_{j\in\mathbb{N}} \subset \Omega'' \ \text{ and } \ (r_j)_{j\in\mathbb{N}} \subset (0,\infty), \ r_j \to 0 \text{ as } j \to \infty, \\ &\text{such that } \ \lim_{j\to\infty} \frac{1}{(2r_j)^N} \int_{B_{2r_j}(x_j)} u = 0. \end{aligned}$$

Without loss of generality, we can assume that $(r_j)_{j\in\mathbb{N}}$ is decreasing. Extract a subsequence $(\rho_j)_{j\in\mathbb{N}} \subset (r_j)_{j\in\mathbb{N}}$ in such a way that[2]

$$\frac{1}{\rho_j^N} \int_{B_{\rho_j}(x_j)} u \le \frac{r_j^{2s}}{j} \qquad \text{and} \qquad \rho_j \le r_j \qquad \text{for any } j \in \mathbb{N}. \tag{6.5.7}$$

In order to ease notation, relabel $c_\Omega = c + \kappa_\Omega$. We apply representation (6.5.4) with $r = r_j$ and we then integrate it over $B_{\rho_j}(x_j)$, obtaining

$$\begin{aligned} \frac{1}{\rho_j^N} \int_{B_{\rho_j}(x_j)} u \ge \frac{r_j^{2s}}{\rho_j^N} \int_{B_1} G(0,y) \int_{B_{\rho_j}(x_j)} c_\Omega(x + r_j y)\, u(x + r_j y)\, dx\, dy \,+ \\ + \frac{1}{\rho_j^N} \int_{\mathbb{R}^N \setminus B_1} P(0,y) \int_{B_{\rho_j}(x_j)} u(x + r_j y)\, dx\, dy. \end{aligned} \tag{6.5.8}$$

The Poisson integral can be estimated as follows:

$$\begin{aligned} \int_{\mathbb{R}^N\setminus B_1} P(0,y)\, u(x + r_j y)\, dy &= \gamma_{N,s} \int_{\mathbb{R}^N \setminus B_1} \frac{u(x + r_j y)}{|y|^N \big(|y|^2 - 1\big)^s}\, dy \\ &\ge \gamma_{N,s} r_j^{2s} \int_{\Omega' \setminus B_{r_j}(x)} \frac{u(y)}{|y - x|^N \big(|y - x|^2 - r^2\big)^s}\, dy \\ &\ge C r_j^{2s} \int_{\Omega' \setminus B_{r_j}(x)} u \end{aligned}$$

which entails

$$\frac{1}{\rho_j^N} \int_{\mathbb{R}^N \setminus B_1} P(0,y) \int_{B_{\rho_j}(x_j)} u(x + r_j y)\, dx\, dy \ge C r_j^{2s} \tag{6.5.9}$$

[2] Here we briefly comment on inequality (6.5.7). As we know by assumption that $\frac{1}{(2r_j)^N} \int_{B_{2r_j}(x_j)} u \to 0$ as $j \to \infty$, one has also $\frac{2^N}{(2r_j)^N} \int_{B_{2r_j}(x_j)} u \to 0$ as $j \to \infty$. Now, using that $B_{r_j}(x_j) \subset B_{2r_j}(x_j)$ and that u is nonnegative, one can write

$$0 \le \frac{1}{r_j^N} \int_{B_{r_j}(x_j)} u \le \frac{2^N}{(2r_j)^N} \int_{B_{2r_j}(x_j)} u \longrightarrow 0 \qquad \text{as } j \to \infty.$$

One can then extract a subsequence $(\rho_j)_{j\in\mathbb{N}} \subset (r_j)_{j\in\mathbb{N}}$ with $\rho_j \le r_j$ such that (6.5.7) holds.

for some $C > 0$. Mind that here we have used the assumption that $|\{u > \delta\} \cap \Omega'| > 0$ for some $\delta > 0$.

We now deal with the Green integral in (6.5.8). Fix $p \in (1, \min\{q, N/(N-2s)\})$. We estimate

$$
\begin{aligned}
&\frac{1}{\rho_j^N} \int_{B_1} G(0,y) \int_{B_{\rho_j}(x_j)} c_\Omega(x + r_j y)\, u(x + r_j y)\, dx\, dy \geq \\
&\geq -\frac{C}{\rho_j^N} \int_{B_1} |y|^{2s-N} \int_{B_{\rho_j}(x_j)} |c_\Omega(x + r_j y)|\, u(x + r_j y)\, dx\, dy \\
&\geq -C \int_{B_1} |y|^{2s-N} \bigg(\frac{1}{\rho_j^N} \int_{B_{\rho_j}(x_j)} |c_\Omega(x + r_j y)|^{\overline{q}}\, dx \bigg)^{\frac{1}{\overline{q}}} \times \\
&\qquad \times \bigg(\frac{1}{\rho_j^N} \int_{B_{\rho_j}(x_j)} u(x + r_j y)^{\frac{\overline{q}}{\overline{q}-1}}\, dx \bigg)^{\frac{\overline{q}-1}{\overline{q}}} dy \\
&\geq -C \bigg[\int_{B_1} |y|^{(2s-N)p} \bigg(\frac{1}{\rho_j^N} \int_{B_{\rho_j}(x_j)} |c_\Omega(x + r_j y)|^{\overline{q}}\, dx \bigg)^{\frac{p}{\overline{q}}} dy \bigg]^{\frac{1}{p}} \times \qquad (6.5.10) \\
&\qquad \times \bigg[\int_{B_1} \bigg(\frac{1}{\rho_j^N} \int_{B_{\rho_j}(x_j)} |u(x + r_j y)|^{\frac{\overline{q}}{\overline{q}-1}}\, dx \bigg)^{\frac{p}{p-1} \frac{\overline{q}-1}{\overline{q}}} dy \bigg]^{\frac{p-1}{p}}. \qquad (6.5.11)
\end{aligned}
$$

Using that

$$
\frac{1}{\rho_j^N} \int_{B_{\rho_j}(x_j)} |c_\Omega(x + r_j y)|^{\overline{q}}\, dx \leq \mathbf{M}\big[|c_\Omega|^{\overline{q}}\big](x_j + r_j y)
$$

by definition (6.5.5), we obtain for (6.5.10) the following estimates by means of a Hölder inequality

$$
\begin{aligned}
&\int_{B_1} |y|^{(2s-N)p} \bigg(\frac{1}{\rho_j^N} \int_{B_{\rho_j}(x_j)} |c_\Omega(x + r_j y)|^{\overline{q}}\, dx \bigg)^{\frac{p}{\overline{q}}} dy \leq \\
&\leq \int_{B_1} |y|^{(2s-N)p} \mathbf{M}\big[|c_\Omega|^{\overline{q}}\big](x_j + r_j y)^{\frac{p}{\overline{q}}}\, dy \\
&\leq \bigg(\int_{B_1} |y|^{\frac{(2s-N)pq}{q-p}}\, dy \bigg)^{\frac{q-p}{pq}} \bigg(\int_{B_1} \mathbf{M}\big[|c_\Omega|^{\overline{q}}\big](x_j + r_j y)^{\frac{q}{\overline{q}}}\, dy \bigg)^{\frac{p}{q}} \\
&\leq \bigg(\int_{B_1} |y|^{\frac{(2s-N)pq}{q-p}}\, dy \bigg)^{\frac{q-p}{pq}} \big\| \mathbf{M}\big[|c_\Omega|^{\overline{q}}\big] \big\|_{L^{q/\overline{q}}(\Omega')}^{p/\overline{q}} \\
&\leq \bigg(\int_{B_1} |y|^{\frac{(2s-N)pq}{q-p}}\, dy \bigg)^{\frac{q-p}{pq}} \big\| |c_\Omega|^{\overline{q}} \big\|_{L^{q/\overline{q}}(\Omega')}^{p/\overline{q}} \\
&\leq \bigg(\int_{B_1} |y|^{\frac{(2s-N)pq}{q-p}}\, dy \bigg)^{\frac{q-p}{pq}} \big\| c_\Omega \big\|_{L^{q}(\Omega')}^{pq/\overline{q}} \qquad (6.5.12)
\end{aligned}
$$

by (6.5.6). Remark that the assumption $1 < p < N/(N-2s)$ ensures that

$$
\frac{(2s-N)pq}{q-p} > -\frac{Nq}{q-p} > -N,
$$

which guarantees the finiteness of the first factor in (6.5.12).

Fix now $\overline{q} \in (p,q)$ and notice how this implies

$$\frac{p}{p-1}\frac{\overline{q}-1}{\overline{q}} > 1.$$

Using this, we estimate (6.5.11) as follows:

$$\begin{aligned}
&\int_{B_1}\left(\frac{1}{\rho_j^N}\int_{B_{\rho_j}(x_j)}|u(x+r_jy)|^{\frac{\overline{q}}{\overline{q}-1}}\,dx\right)^{\frac{p}{p-1}\frac{\overline{q}-1}{\overline{q}}}dy \leq\\
&\leq C\|u\|_{L^\infty(\Omega')}^{\frac{\overline{q}}{\overline{q}-1}(\frac{p}{p-1}\frac{\overline{q}-1}{\overline{q}}-1)}\int_{B_1}\left(\frac{1}{\rho_j^N}\int_{B_{\rho_j}(x_j)}|u(x+r_jy)|^{\frac{\overline{q}}{\overline{q}-1}}\,dx\right)dy\\
&\leq C\|u\|_{L^\infty(\Omega')}^{(\frac{p}{p-1}-\frac{\overline{q}}{\overline{q}-1})(\frac{\overline{q}}{\overline{q}-1}-1)}\int_{B_1}\left(\frac{1}{\rho_j^N}\int_{B_{\rho_j}(x_j)}u(x+r_jy)\,dx\right)dy.
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_{B_1}\frac{1}{\rho_j^N}\int_{B_{\rho_j}(x_j)}u(x+r_jy)\,dx\,dy = \frac{1}{r_j^N}\int_{B_{r_j}}\frac{1}{\rho_j^N}\int_{B_{\rho_j}}u(x_j+x+y)\,dx\,dy\\
&= \frac{1}{\rho_j^N}\int_{B_{\rho_j}}\frac{1}{r_j^N}\int_{B_{r_j}}u(x_j+x+y)\,dy\,dx \leq \frac{1}{\rho_j^N}\int_{B_{\rho_j}}\frac{1}{r_j^N}\int_{B_{r_j+\rho_j}}u(x_j+z)\,dz\,dx\\
&= \omega_N\Big(\frac{r_j+\rho_j}{r_j}\Big)^N\frac{1}{\big(r_j+\rho_j\big)^N}\int_{B_{r_j+\rho_j}}u(x_j+z)\,dz\\
&\leq \frac{C}{\big(r_j+\rho_j\big)^N}\int_{B_{r_j+\rho_j}}u(x_j+z)\,dz\\
&\leq C\Big(\frac{2r_j}{r_j+\rho_j}\Big)^N\frac{1}{(2r_j)^N}\int_{B_{2r_j}}u(x_j+z)\,dz \;\longrightarrow\; 0 \qquad \text{as } j\to\infty. \qquad (6.5.13)
\end{aligned}$$

We therefore deduce, by plugging in (6.5.8) the estimates contained in (6.5.7), (6.5.9), (6.5.12), and (6.5.13),

$$\frac{r_j^{2s}}{j} \geq -C_1 r_j^{2s}\varepsilon_j + C_2 r_j^{2s}, \qquad \text{for some } (\varepsilon_j)_{j\in\mathbb{N}}\subset(0,\infty),\ \varepsilon_j\to 0 \text{ as } j\to\infty.$$

But this gives a contradiction for j large enough. $\square$

Chapter 7

Qualitative properties of positive solutions for elliptic problem driven by the regional fractional Laplacian in the half-space

In this chapter, we use the moving plane method to derive the symmetry and monotonicity for the regional fractional Laplacian in the half-space. The presentation of this chapter has the same form as the original article [R6]. The notation may slightly differ from those in the previous chapters.

7.1 Introduction and main result

The aim of the present paper is to prove symmetry and monotonicity result of positive solutions to the Dirichlet problem

$$\begin{cases} (-\Delta)^s_{\mathbb{R}^N_+} u = u^{2^*_s - 1}, \quad u > 0 \text{ in } \mathbb{R}^N_+ \\ \qquad u = 0 \quad \text{on} \quad \partial\mathbb{R}^N_+ = \mathbb{R}^{N-1}, \end{cases} \tag{7.1.1}$$

where $s \in (0, 1/2) \cup (1/2, 1)$, $N \geq 2$, $\mathbb{R}^N_+ = \{x = (x', x_N) \in \mathbb{R}^N : x' \in \mathbb{R}^{N-1},\ x_N > 0\}$ the upper half-space and $2^*_s := 2N/(N-2s)$ is the so-called fractional critical Sobolev exponent. The regional fractional Laplacian $(-\Delta)^s_{\mathbb{R}^N_+}$ is a nonlocal operator defined for all $u \in C^2_c(\mathbb{R}^N_+)$ by

$$(-\Delta)^s_{\mathbb{R}^N_+} u(x) = c_{N,s} P.V. \int_{\mathbb{R}^N_+} \frac{u(x) - u(y)}{|x - y|^{N+2s}}\, dxdy, \quad x \in \mathbb{R}^N_+, \tag{7.1.2}$$

where $c_{N,s}$ and $P.V.$ are respectively a normalization constant and the principal value of the integral. For functions u belonging to the class $C^{2s+\varepsilon}_{loc}(\mathbb{R}^N_+) \cap L^\infty(\mathbb{R}^N_+)$ for some $\varepsilon > 0$, the integral in the above definition is finite. In this case, the definition (7.1.2) is understood in pointwise sense in $\mathbb{R}^N_+$. Next, we denote by $\mathcal{L}^1_s(\mathbb{R}^N_+)$ the space of functions $u \in L^1_{loc}(\mathbb{R}^N_+)$ such that $\|u\|_{\mathcal{L}^1_s(\mathbb{R}^N_+)} := \int_{\mathbb{R}^N_+} \frac{|u(x)|}{1+|x|^{N+2s}}\, dx$ is finite. Additionnaly to the pointwise definition, there are other fashion of defining the operator $(-\Delta)^s_{\mathbb{R}^N_+}$: the weak and distributional definitions, that is,

$$\langle (-\Delta)^s_{\mathbb{R}^N_+} u, \varphi \rangle = \int_{\mathbb{R}^N_+} (-\Delta)^{\frac{s}{2}}_{\mathbb{R}^N_+} u (-\Delta)^{\frac{s}{2}}_{\mathbb{R}^N_+} \varphi\, dx$$

$$= \frac{c_{N,s}}{2} \int_{\mathbb{R}^N_+} \int_{\mathbb{R}^N_+} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} \, dxdy$$

for $u, \varphi \in H^s(\mathbb{R}^N_+)$, corresponding to the weak sense definition, and

$$\begin{aligned} \langle (-\Delta)^s_{\mathbb{R}^N_+} u, \varphi \rangle &= \int_{\mathbb{R}^N_+} u (-\Delta)^s_{\mathbb{R}^N_+} \varphi \, dx \\ &= \frac{c_{N,s}}{2} \int_{\mathbb{R}^N_+} \int_{\mathbb{R}^N_+} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} \, dxdy \end{aligned}$$

for $u \in \mathcal{L}^1_s(\mathbb{R}^N_+)$ and $\varphi \in C^\infty_c(\mathbb{R}^N_+)$, for the distributional definition. These definitions are completely different (but they coincide for sufficiently regular functions) and throughout this paper, we will focus on the definition of $(-\Delta)^s_{\mathbb{R}^N_+}$ in *weak* sense.

The study of qualitative properties for solutions to a given problem is one of the relevant tasks in partial differential equation theory. This allows a better understanding as regards the classifications of solutions. Among other things, Monotonicity and symmetry are one of the most studied qualitative properties of positive solutions. In general, this is done by exploiting the celebrated method of *moving planes* that goes back to the work of Alexandrov [4], Serrin [135] and Gidas, Ni, and Nirenberg [89].

Several works have been devoted to the moving plane method for nonlocal operators, especially, the fractional Laplacian. Just to cite a few references on this topic, we refer to [17, 18, 22, 45, 49–52, 68, 78, 79, 100, 107, 109, 119].

The main result of this paper reads as follows.

Theorem 7.1.1. *Let $s \in (0, 1/2) \cup (1/2, 1)$ and $N \geq 2$. Any solution to (7.1.1) is radially symmetric in x' and monotonic in the radial variable. In other words, there exists a monotonic function $(0,\infty) \times (0,\infty) \ni (r, x_N) \mapsto v(r, x_N)$ with respect to r such that*

$$u(x', x_N) = v(r, x_N) \quad \textit{with} \quad r = |x'|. \tag{7.1.3}$$

As a first observation, when dealing with the regional fractional Laplacian in bounded domains, e.g., $(-\Delta)^s_B$, the counterpart of Theorem 7.1.1 seems to fail. The main difficulty relies on the fact that the operator depends on the domain, and therefore, it is not invariant under scaling. A break of symmetry is strongly expected.

The paper is organized as follows. In Section 7.2 we give some useful notations and definitions that are needed in other to make the paper as self-contained as possible. Section 7.3 deals with the proof of Theorem 7.1.1. We also provide in this section an L^∞-bounds of weak solutions via the Moser's iteration method.

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7.2 Notation and preliminary setting

In this section, we introduce some preliminaries that will be useful throughout this article. First of all, we start with some notations. Given $x \in \mathbb{R}^N_+$ and $r > 0$, we denote by $B_r(x)$ the open ball centered at x with radius r. We also denote by $\mathbb{1}_A$ the characteristic function of any subset $A \subset \mathbb{R}^N$. Next, for all function $u : \mathbb{R}^N_+ \to \mathbb{R}$, we define respectively by $u^+ = \max\{u, 0\}$ and $u^- = -\min\{u, 0\}$ the positive and negative part of u.

Now, for $s \in (0, 1)$, the fractional Sobolev space $H^s(\mathbb{R}^N_+)$ is defined as the space of measurable functions $u : \mathbb{R}^N_+ \to \mathbb{R}$ such that

$$[u]^2_{H^s(\mathbb{R}^N_+)} := \int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}}\, dxdy$$

is finite. It is a Hilbert space with the associated norm $\|\cdot\|_{H^s(\mathbb{R}^N_+)}$ define as

$$\|u\|^2_{H^s(\mathbb{R}^N_+)} := \|u\|^2_{L^2(\mathbb{R}^N_+)} + [u]^2_{H^s(\mathbb{R}^N_+)}.$$

On the other hand, $H^s_0(\mathbb{R}^N_+)$ is defined as the completion of $C^\infty_c(\mathbb{R}^N_+)$ under the norm $\|\cdot\|_{H^s(\mathbb{R}^N_+)}$. It is also a Hilbert space equipped with the norm

$$\|u\|^2_{H^s_0(\mathbb{R}^N_+)} := \int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}}\, dxdy$$

which is equivalent to the usual one in $H^s(\mathbb{R}^N_+)$. Customarily, $C^\infty_c(\mathbb{R}^N_+)$ denotes the space of smooth functions on $\mathbb{R}^N$ with compact support in $\mathbb{R}^N_+$. We notice that $H^s_0(\mathbb{R}^N_+)$ can be equivalently define as

$$H^s_0(\mathbb{R}^N_+) := \{u \in H^s(\mathbb{R}^N_+) : u \equiv 0 \ \text{ on } \ \partial\mathbb{R}^N_+\}.$$

Finaly, in the same spirit, we define the Hilbert space $\mathcal{H}^s_0(\mathbb{R}^N_+)$ as

$$\mathcal{H}^s_0(\mathbb{R}^N_+) := \{u \in H^s(\mathbb{R}^N) : u \equiv 0 \ \text{ in } \ \mathbb{R}^N \setminus \mathbb{R}^N_+\},$$

which is the completion of $C^\infty_c(\mathbb{R}^N_+)$ with respect to the norm $\|\cdot\|_{H^s(\mathbb{R}^N)}$.

We have the following

Definition 7.2.1. Let $s \in (0, 1/2) \cup (1/2, 1)$ and $N \geq 2$. We say that $u \in H^s_0(\mathbb{R}^N_+)$ is a weak solution to problem (7.1.1) if $u > 0$ in $\mathbb{R}^N_+$ and

$$\frac{c_{N,s}}{2}\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}}\, dxdy = \int_{\mathbb{R}^N_+} u^{2^*_s - 1}\varphi\, dx \tag{7.2.1}$$

for every $\varphi \in H^s_0(\mathbb{R}^N_+)$.

We notice that the existence of weak solution to the Dirichlet problem has been established in [83]. We end this section by recalling the following definition for weak superharmonic functions with respect to $(-\Delta)^s_{\mathbb{R}^N_+}$.

Definition 7.2.2. Let $s \in (0,1)$ and $\Omega \subset \mathbb{R}^N_+$. We say that $v \in H^s_0(\Omega)$ satisfies

$$(-\Delta)^s_{\mathbb{R}^N_+} v \geq 0 \quad \text{in} \quad \Omega, \tag{7.2.2}$$

if

$$\frac{c_{N,s}}{2} \int_{\mathbb{R}^N_+} \int_{\mathbb{R}^N_+} \frac{(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} \, dxdy \geq 0, \tag{7.2.3}$$

for all nonnegative function $\varphi \in C^\infty_c(\Omega)$.

7.3 Proof of Theorem 7.1.1

The purpose of this section is to prove the main result on symmetry and monotonicity of positive solutions for problem (7.1.1), that is, Theorem 7.1.1. For this and as is stated earlier in the introduction, we will make use of the celebrated method of *moving planes*. This requires, first of all, a key result on the strong maximum principle for odd functions, that we state and prove in the next proposition.

Before doing so, we first introduce some notation. Let $\lambda \in \mathbb{R}$ be real number and let $T_\lambda = \{x \in \mathbb{R}^N_+ : x_1 = \lambda\}$ be a hyperplane perpendicular to the x_1-direction. Define $\Sigma_\lambda = \{x \in \mathbb{R}^N_+ : x_1 < \lambda\}$ as the region on the left of the plane and $x_\lambda = (2\lambda - x_1, x_2, \ldots, x_N)$ the reflection of x with respect to T_λ. Finally, we put $u_\lambda(x) = u(x_\lambda)$.

We have the following.

Proposition 7.3.1 (Strong maximum principle)**.** *Let $s \in (0,1/2) \cup (1/2,1)$, $\lambda \in \mathbb{R}$ and $U \subset\subset \Sigma_\lambda$ be a bounded set. Let $v \in H^s_0(\mathbb{R}^N_+)$ be a continuous function on $\overline{U}$, satisfying*

$$(-\Delta)^s_{\mathbb{R}^N_+} v \geq 0 \quad \text{in} \quad U \tag{7.3.1}$$

in the sense of Definition 7.2.2. If v is nonnegative in Σ_λ and odd with respect to the hyperplane T_λ, then either $v \equiv 0$ in $\mathbb{R}^N_+$ or $v > 0$ in U.

Proof. We recall first of all that as a consequence of Hardy inequality, the space $H^s_0(\mathbb{R}^N_+)$ can be identified with $\mathcal{H}^s_0(\mathbb{R}^N_+)$ for $s \in (0,1/2) \cup (1/2,1)$, see [94]. From this, identifying v with its trivial extension on $\mathbb{R}^N$ one gets that $v \in \mathcal{H}^s_0(\mathbb{R}^N_+)$.

Now, let $\varphi \in C^\infty_c(U)$, $\varphi \geq 0$. We have

$$\begin{aligned}
&\frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} \, dxdy \\
&= \frac{c_{N,s}}{2} \int_{\mathbb{R}^N_+} \int_{\mathbb{R}^N_+} \frac{(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} \, dxdy + c_{N,s} \int_{\mathbb{R}^N_+} \int_{\mathbb{R}^N \setminus \mathbb{R}^N_+} \frac{v(x)\varphi(x)}{|x-y|^{N+2s}} \, dydx \\
&= \frac{c_{N,s}}{2} \int_{\mathbb{R}^N_+} \int_{\mathbb{R}^N_+} \frac{(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} \, dxdy + c_{N,s} \int_{\Sigma_\lambda} \int_{\mathbb{R}^N \setminus \mathbb{R}^N_+} \frac{v(x)\varphi(x)}{|x-y|^{N+2s}} \, dydx.
\end{aligned}$$

In the latter, we used that

$$\int_{\mathbb{R}^N_+ \setminus \Sigma_\lambda} \int_{\mathbb{R}^N \setminus \mathbb{R}^N_+} \frac{v(x)\varphi(x)}{|x-y|^{N+2s}} \, dydx = 0 \tag{7.3.2}$$

since φ has compact support in Σ_λ. However, from (7.3.1), it follows that

$$\frac{c_{N,s}}{2}\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dxdy \geq 0. \tag{7.3.3}$$

Moreover, using that v is nonnegative in Σ_λ and that $\varphi \geq 0$, we get

$$c_{N,s}\int_{\Sigma_\lambda}\int_{\mathbb{R}^N\setminus\mathbb{R}^N_+}\frac{v(x)\varphi(x)}{|x-y|^{N+2s}}\,dydx \geq 0. \tag{7.3.4}$$

Combining (7.3.3) and (7.3.4), we deduce that

$$\frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dxdy \geq 0 \tag{7.3.5}$$

for all nonnegative function $\varphi \in C_c^\infty(U)$.
In other words, $v \in \mathcal{H}_0^s(\mathbb{R}^N_+)$ is a continuous function on $\overline{U}$ satisfying

$$(-\Delta)^s v \geq 0 \quad \text{in} \quad U. \tag{7.3.6}$$

The proof now follows from [68, Proposition 3.1] (see also [18, Proposition 3.2]). □

Having the above strong maximum principle, we are now ready to prove our main result by using the *moving plane* method. We mention further that some techniques from [68] will be borrowed.

Proof of Theorem 7.1.1. For every $\varphi \in H_0^s(\mathbb{R}^N_+)$ there holds that $\varphi_\lambda \in H_0^s(\mathbb{R}^N_+)$. Therefore, as far as φ can be used as a test function in (7.2.1), so is φ_λ. Now, given a weak solution u in the sense of Definition 7.2.1, we determine the equation satisfying by u_λ. We claim that u_λ weakly solves the problem

$$\begin{cases} (-\Delta)^s_{\mathbb{R}^N_+} u_\lambda = u_\lambda^{2_s^*-1}, \quad u_\lambda > 0 \quad \text{in} \quad \mathbb{R}^N_+ \\ \qquad u_\lambda = 0 \quad \text{in} \quad \partial\mathbb{R}^N_+ = \mathbb{R}^{N-1}. \end{cases} \tag{7.3.7}$$

Indeed, for every $\varphi \in H_0^s(\mathbb{R}^N_+)$, we have,

$$\begin{aligned} &\frac{c_{N,s}}{2}\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{(u_\lambda(x)-u_\lambda(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dxdy \\ &= \frac{c_{N,s}}{2}\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{(u(x_\lambda)-u(y_\lambda))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dxdy \\ &= \frac{c_{N,s}}{2}\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{(u(x)-u(y))(\varphi(x_\lambda)-\varphi(y_\lambda))}{|x_\lambda-y_\lambda|^{N+2s}}\,dxdy \\ &= \frac{c_{N,s}}{2}\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{(u(x)-u(y))(\varphi_\lambda(x)-\varphi_\lambda(y))}{|x-y|^{N+2s}}\,dxdy \\ &= \int_{\mathbb{R}^N_+} u^{2_s^*-1}(x)\varphi_\lambda(x)\,dx = \int_{\mathbb{R}^N_+} u_\lambda^{2_s^*-1}(x)\varphi(x)\,dx. \end{aligned}$$

From this, we conclude that u_λ is a weak solution to the problem (7.3.7).
Now, let $\lambda \in \mathbb{R}$ and let us define

$$v_\lambda(x) := \begin{cases} (u - u_\lambda)^+(x) & \text{if} \quad x \in \Sigma_\lambda \\ -(u - u_\lambda)^-(x) & \text{if} \quad x \in \mathbb{R}^N_+ \setminus \Sigma_\lambda. \end{cases} \tag{7.3.8}$$

We notice that $v_\lambda \in L^\infty(\mathbb{R}^N_+) \subset \mathcal{L}^1_s(\mathbb{R}^N_+)$ since $u \in L^\infty(\mathbb{R}^N_+)$, thanks to Proposition 7.3.3.
Next, we aim to prove that

$$v_\lambda \equiv 0 \quad \text{for} \quad \lambda \quad \text{sufficiently negative.} \tag{7.3.9}$$

First of all, it is easily seen that $v_\lambda \in H^s_0(\mathbb{R}^N_+)$. This follows by a standard argument and we omit the proof. Therefore, using v_λ as an test function in Definition 7.2.1 and in the weak formulation of problem (7.3.7), we get that

$$\frac{c_{N,s}}{2}\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{(u(x)-u(y))(v_\lambda(x)-v_\lambda(y))}{|x-y|^{N+2s}}\,dxdy = \int_{\mathbb{R}^N_+} u^{2^*_s-1}(x)v_\lambda(x)\,dx$$
$$\frac{c_{N,s}}{2}\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{(u_\lambda(x)-u_\lambda(y))(v_\lambda(x)-v_\lambda(y))}{|x-y|^{N+2s}}\,dxdy = \int_{\mathbb{R}^N_+} u_\lambda^{2^*_s-1}(x)v_\lambda(x)\,dx$$

and substracting the above two equations, we find that

$$\begin{aligned}&\frac{c_{N,s}}{2}\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{(u(x)-u_\lambda(x)-(u(y)-u_\lambda(y)))(v_\lambda(x)-v_\lambda(y))}{|x-y|^{N+2s}}\,dxdy\\ &\quad= \int_{\mathbb{R}^N_+}(u^{2^*_s-1}(x)-u_\lambda^{2^*_s-1}(x))v_\lambda(x)\,dx.\end{aligned} \tag{7.3.10}$$

On the other hand, using that

$$\begin{aligned}&(u(x)-u_\lambda(x)-(u(y)-u_\lambda(y)))(v_\lambda(x)-v_\lambda(y))\\ &= (v_\lambda(x)-v_\lambda(y))^2 + \Big((u(x)-u_\lambda(x))-(u(y)-u_\lambda(y))-(v_\lambda(x)-v_\lambda(y))\Big)(v_\lambda(x)-v_\lambda(y))\end{aligned}$$

then the double integral in the left-hand side of (7.3.10) can be equivalently writen as

$$\begin{aligned}&\frac{c_{N,s}}{2}\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{(u(x)-u_\lambda(x)-(u(y)-u_\lambda(y)))(v_\lambda(x)-v_\lambda(y))}{|x-y|^{N+2s}}\,dxdy\\ &\quad= \frac{c_{N,s}}{2}\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{(v_\lambda(x)-v_\lambda(y))^2}{|x-y|^{N+2s}}\,dxdy + \frac{c_{N,s}}{2}\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{\mathcal{C}(x,y)}{|x-y|^{N+2s}}\,dxdy\end{aligned} \tag{7.3.11}$$

where

$$\mathcal{C}(x,y) := \Big((u(x)-u_\lambda(x))-(u(y)-u_\lambda(y))-(v_\lambda(x)-v_\lambda(y))\Big)(v_\lambda(x)-v_\lambda(y)).$$

We wish now to show that the last double integral in (7.3.11) is nonnegative. To achieve this goal, we argue as follows.
We put $K_\lambda := \text{supp } v_\lambda$ and we define the sets below

$$G_\lambda := K_\lambda \cap \Sigma_\lambda, \qquad\qquad G^c_\lambda := \Sigma_\lambda \setminus G_\lambda \tag{7.3.12}$$

$$\Omega_\lambda := K_\lambda \cap \Big(\mathbb{R}^N_+ \setminus \Sigma_\lambda\Big), \qquad \Omega^c_\lambda := \Big(\mathbb{R}^N_+ \setminus \Sigma_\lambda\Big) \setminus \Omega_\lambda. \tag{7.3.13}$$

Then, by the definition of v_λ it follows that Ω_λ is the reflection of G_λ about the hyperplane T_λ. We will exploit this fact in the sequel. Moreover, the above sets cover the whole upper half-space $\mathbb{R}^N_+$. Therefore, we can decompose the product $\mathbb{R}^N_+ \times \mathbb{R}^N_+$ as follows

$$\mathbb{R}^N_+ \times \mathbb{R}^N_+ = (G_\lambda \cup G^c_\lambda \cup \Omega_\lambda \cup \Omega^c_\lambda) \times (G_\lambda \cup G^c_\lambda \cup \Omega_\lambda \cup \Omega^c_\lambda). \tag{7.3.14}$$

By using the definition of v_λ, we find that

$$\begin{array}{lll}
\mathcal{C}(x,y) = -(u(x) - u_\lambda(x))v_\lambda(y) & \text{in} & (G^c_\lambda \times G_\lambda) \\
\mathcal{C}(x,y) = -(u(x) - u_\lambda(x))v_\lambda(y) & \text{in} & (G^c_\lambda \times \Omega_\lambda) \\
\mathcal{C}(x,y) = -(u(y) - u_\lambda(y))v_\lambda(x) & \text{in} & (G_\lambda \times G^c_\lambda) \\
\mathcal{C}(x,y) = -(u(y) - u_\lambda(y))v_\lambda(x) & \text{in} & (G_\lambda \times \Omega^c_\lambda) \\
\mathcal{C}(x,y) = -(u(x) - u_\lambda(x))v_\lambda(y) & \text{in} & (\Omega^c_\lambda \times G_\lambda) \\
\mathcal{C}(x,y) = -(u(x) - u_\lambda(x))v_\lambda(y) & \text{in} & (\Omega^c_\lambda \times \Omega_\lambda) \\
\mathcal{C}(x,y) = -(u(y) - u_\lambda(y))v_\lambda(x) & \text{in} & (\Omega_\lambda \times G^c_\lambda) \\
\mathcal{C}(x,y) = -(u(y) - u_\lambda(y))v_\lambda(x) & \text{in} & (\Omega_\lambda \times \Omega^c_\lambda) \\
\mathcal{C}(x,y) = 0 & & \text{elsewhere.}
\end{array}$$

By the definition of v_λ, we see that $v_\lambda \geq 0$ in Σ_λ and $v_\lambda \leq 0$ in $\mathbb{R}^N_+ \setminus \Sigma_\lambda$. Therefore, for $x \in G^c_\lambda$ and $y \in G_\lambda$, we have that

$$\begin{aligned}
0 \leq \mathcal{C}(x,y) &= -(u(x) - u_\lambda(x))v_\lambda(y) \\
&= -(u(x) - u_\lambda(x))(u(y) - u_\lambda(y)) \\
&= -(u(x) - u_\lambda(x))(u_\lambda(y_\lambda) - u(y_\lambda)) \\
&= (u(x) - u_\lambda(x))v_\lambda(y_\lambda) \\
&= -\mathcal{C}(x, y_\lambda).
\end{aligned}$$

Moreover, since $|x-y| \leq |x - y_\lambda|$ for $x \in G^c_\lambda$ and $y \in G_\lambda$, and by recalling that Ω_λ is the reflexion of G_λ with respect to the hyperplane T_λ, it follows that

$$\begin{aligned}
&\int_{G^c_\lambda}\int_{G_\lambda} \frac{\mathcal{C}(x,y)}{|x-y|^{N+2s}}\,dxdy + \int_{G^c_\lambda}\int_{\Omega_\lambda} \frac{\mathcal{C}(x,y)}{|x-y|^{N+2s}}\,dxdy \\
&= \int_{G^c_\lambda}\int_{G_\lambda} \frac{\mathcal{C}(x,y)}{|x-y|^{N+2s}}\,dxdy + \int_{G^c_\lambda}\int_{G_\lambda} \frac{\mathcal{C}(x,y_\lambda)}{|x-y_\lambda|^{N+2s}}\,dxdy \\
&= \int_{G^c_\lambda}\int_{G_\lambda} \mathcal{C}(x,y)\Big(\frac{1}{|x-y|^{N+2s}} - \frac{1}{|x-y_\lambda|^{N+2s}}\Big)\,dxdy \geq 0.
\end{aligned}$$

This yields

$$\int_{G^c_\lambda}\int_{G_\lambda} \frac{\mathcal{C}(x,y)}{|x-y|^{N+2s}}\,dxdy + \int_{G^c_\lambda}\int_{\Omega_\lambda} \frac{\mathcal{C}(x,y)}{|x-y|^{N+2s}}\,dxdy \geq 0. \tag{7.3.15}$$

By the same manner, we show that

$$\int_{G_\lambda}\int_{G^c_\lambda} \frac{\mathcal{C}(x,y)}{|x-y|^{N+2s}}\,dxdy + \int_{G_\lambda}\int_{\Omega^c_\lambda} \frac{\mathcal{C}(x,y)}{|x-y|^{N+2s}}\,dxdy \geq 0, \tag{7.3.16}$$

$$\int_{\Omega_\lambda^c}\int_{G_\lambda}\frac{\mathcal{C}(x,y)}{|x-y|^{N+2s}}\,dxdy+\int_{\Omega_\lambda^c}\int_{\Omega_\lambda}\frac{\mathcal{C}(x,y)}{|x-y|^{N+2s}}\,dxdy\ge 0, \tag{7.3.17}$$

and

$$\int_{\Omega_\lambda}\int_{G_\lambda^c}\frac{\mathcal{C}(x,y)}{|x-y|^{N+2s}}\,dxdy+\int_{\Omega_\lambda}\int_{\Omega_\lambda^c}\frac{\mathcal{C}(x,y)}{|x-y|^{N+2s}}\,dxdy\ge 0. \tag{7.3.18}$$

Combining (7.3.15), (7.3.16), (7.3.17) and (7.3.18), we deduce that

$$\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{\mathcal{C}(x,y)}{|x-y|^{N+2s}}\,dxdy\ge 0$$

and from this,

$$\begin{aligned}
&\frac{c_{N,s}}{2}\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{(v_\lambda(x)-v_\lambda(y))^2}{|x-y|^{N+2s}}\,dxdy\\
&\le \frac{c_{N,s}}{2}\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{(v_\lambda(x)-v_\lambda(y))^2}{|x-y|^{N+2s}}\,dxdy+\frac{c_{N,s}}{2}\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{\mathcal{C}(x,y)}{|x-y|^{N+2s}}\,dxdy\\
&=\int_{\mathbb{R}^N_+}(u^{2_s^*-1}(x)-u_\lambda^{2_s^*-1}(x))v_\lambda(x)\,dx
\end{aligned} \tag{7.3.19}$$

thanks to (7.3.10) and (7.3.11).
On the other hand, using the well-known inequality [1]

$$\frac{1}{p}(|a|^p-|b|^p)\ge |b|^{p-2}b(a-b)\quad\text{for } p\in(1,\infty)\text{ and } a,b\in\mathbb{R} \tag{7.3.20}$$

with $p=2_s^*-1$ (since $2_s^*>2$) and Hölder inequality with exponents $2_s^*/(2_s^*-2)$ and $2_s^*/2$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N_+}(u^{2_s^*-1}(x)-u_\lambda^{2_s^*-1}(x))v_\lambda(x)\,dx\\
&=\int_{G_\lambda}(u^{2_s^*-1}(x)-u_\lambda^{2_s^*-1}(x))v_\lambda(x)\,dx+\int_{\Omega_\lambda}(u^{2_s^*-1}(x)-u_\lambda^{2_s^*-1}(x))v_\lambda(x)\,dx\\
&\le\gamma_1\int_{G_\lambda}u^{2_s^*-2}(x)v_\lambda^2(x)\,dx+\gamma_1\int_{\Omega_\lambda}u_\lambda^{2_s^*-2}(x)v_\lambda^2\,dx\\
&\le\gamma_1\left(\int_{G_\lambda}u^{2_s^*}(x)\,dx\right)^{(2_s^*-2)/2_s^*}\left(\int_{G_\lambda}v_\lambda^{2_s^*}(x)\,dx\right)^{2/2_s^*}\\
&\qquad+\gamma_1\left(\int_{\Omega_\lambda}u_\lambda^{2_s^*}(x)\,dx\right)^{(2_s^*-2)/2_s^*}\left(\int_{\Omega_\lambda}v_\lambda^{2_s^*}(x)\,dx\right)^{2/2_s^*}.
\end{aligned} \tag{7.3.21}$$

In the above equality, we used that $v_\lambda=0$ in $\mathbb{R}^N_+\setminus K_\lambda$. Moreover, since Ω_λ is the reflection of G_λ about the hyperplane T_λ, if follows that

$$\int_{\Omega_\lambda}u_\lambda^{2_s^*}(x)\,dx=\int_{G_\lambda}u^{2_s^*}(x)\,dx. \tag{7.3.22}$$

[1] It suffices to use the convexity of the function $t\mapsto|t|^p$ for $p>1$.

Plugging (7.3.22) into (7.3.21), we get that

$$\begin{aligned}
&\int_{\mathbb{R}^N_+}(u^{2^*_s-1}(x)-u_\lambda^{2^*_s-1}(x))v_\lambda(x)\ dx\\
&\leq \gamma_2\bigg(\int_{G_\lambda} u^{2^*_s}(x)\ dx\bigg)^{(2^*_s-2)/2^*_s}\bigg(\int_{\mathbb{R}^N_+} v_\lambda^{2^*_s}(x)\ dx\bigg)^{2/2^*_s}\\
&\leq \gamma_3\bigg(\int_{G_\lambda} u^{2^*_s}(x)\ dx\bigg)^{(2^*_s-2)/2^*_s}\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{(v_\lambda(x)-v_\lambda(y))^2}{|x-y|^{N+2s}}\ dxdy
\end{aligned} \tag{7.3.23}$$

thanks to Sobolev inequality (see [65]). Combining (7.3.19) and (7.3.23), we get

$$\begin{aligned}
&\frac{c_{N,s}}{2}\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{(v_\lambda(x)-v_\lambda(y))^2}{|x-y|^{N+2s}}\ dxdy\\
&\leq \gamma_3\bigg(\int_{G_\lambda} u^{2^*_s}(x)\ dx\bigg)^{(2^*_s-2)/2^*_s}\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{(v_\lambda(x)-v_\lambda(y))^2}{|x-y|^{N+2s}}\ dxdy
\end{aligned} \tag{7.3.24}$$

Here, $\gamma_i,\ i\in\{1,2,3\}$ is a positive constant independent of λ and that may change from line to line. However, using that $u\in L^{2^*_s}(\mathbb{R}^N_+)$, there exists $l>0$ such that for $\lambda<-l$, we get

$$\bigg(\int_{G_\lambda} u^{2^*_s}(x)\ dx\bigg)^{(2^*_s-2)/2^*_s} < \frac{c_{N,s}}{2\gamma_3}, \tag{7.3.25}$$

we deduce from (7.3.24) that

$$\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+}\frac{(v_\lambda(x)-v_\lambda(y))^2}{|x-y|^{N+2s}}\ dxdy=0 \tag{7.3.26}$$

which implies that $v_\lambda = const$ in $\mathbb{R}^N_+$ and recalling that $v_\lambda\equiv 0$ on $\{x\in\mathbb{R}^N_+ : x_1=\lambda\}$, we conclude that $v_\lambda\equiv 0$ in $\mathbb{R}^N_+$ and (7.3.9) follows.
We now move the plane to the right as long as (7.3.9) holds true to its limiting position. Next, we define

$$\Lambda=\{\lambda\in\mathbb{R} : u\leq u_\mu \ \text{ in } \ \Sigma_\lambda,\ \ \forall\mu\leq\lambda\}. \tag{7.3.27}$$

From (7.3.9), it follows that Λ is nonnempty, that is $\Lambda\neq\emptyset$. Since also Λ is bounded, the following is well-defined

$$\lambda_* := \sup\Lambda. \tag{7.3.28}$$

We claim that $\lambda_*<\infty$. Indeed, if the claim does not holds true, that is, if $\lambda_*=\infty$, then we can choose λ_j with $\lambda_j\to\infty$ and $u\leq u_{\lambda_j}$ a.e. in Σ_{λ_j}. Now, by integrating over Σ_{λ_j} we get that

$$\int_{\Sigma_{\lambda_j}} u^{2^*_s}\ dx \leq \int_{\Sigma_{\lambda_j}} u_{\lambda_j}^{2^*_s}\ dx = \int_{\mathbb{R}^N_+\setminus\Sigma_{\lambda_j}} u^{2^*_s}\ dx, \tag{7.3.29}$$

which implies that

$$\int_{\mathbb{R}^N_+\setminus\Sigma_{\lambda_j}} u^{2^*_s}\ dx \geq \frac{1}{2}\int_{\mathbb{R}^N_+} u^{2^*_s}\ dx, \tag{7.3.30}$$

that is,

$$\int_{\mathbb{R}^N_+} u^{2^*_s} \mathbb{1}_{\mathbb{R}^N_+ \setminus \Sigma_{\lambda_j}} \, dx \geq \frac{1}{2} \int_{\mathbb{R}^N_+} u^{2^*_s} \, dx. \tag{7.3.31}$$

Since $\mathbb{1}_{\mathbb{R}^N_+ \setminus \Sigma_{\lambda_j}} \to 0$ a.e. on $\mathbb{R}^N_+$ and $u \in L^{2^*_s}(\mathbb{R}^N_+)$ then, by applying Lebesgue's Dominated Convergence Theorem, the left-hand side of the above inequality converges to zero as $j \to \infty$. This implies that $u \equiv 0$ a.e. in $\mathbb{R}^N_+$, contradicting the fact that u is nonconstant. Therefore, the claim follows, that is, $\lambda_* < \infty$.
By setting $w_{\lambda_*} := u_{\lambda_*} - u$, we get by continuity that

$$w_{\lambda_*} \geq 0 \quad \text{in} \ \ \Sigma_{\lambda_*}. \tag{7.3.32}$$

This means that

$$\text{either} \ \ w_{\lambda_*} \equiv 0 \ \ \text{in} \ \ \Sigma_{\lambda_*} \qquad \text{or} \quad w_{\lambda_*} \gneqq 0 \ \ \text{in} \ \ \Sigma_{\lambda_*}. \tag{7.3.33}$$

We now show that the second option in (7.3.33) cannot occurs. By contradiction, assume that the second option in (7.3.33) is true, that is, there exist two positive constants $c > 0$ and $\rho = \rho(\lambda_*)$ such that

$$w_{\lambda_*} \geq c > 0 \quad \text{in} \quad B_\rho(\tilde{x}) \subset \Sigma_{\lambda_*} \tag{7.3.34}$$

for some $\tilde{x} \in \Sigma_{\lambda_*}$. Now, in order to get a contradiction, one prove that the plane can be move further to the right. To this end, we show the following strict inequality

$$w_{\lambda_*} > 0 \quad \text{in all of} \ \ \Sigma_{\lambda_*}. \tag{7.3.35}$$

Consider now an arbitrary point $z \in \Sigma_{\lambda_*} \setminus \{\tilde{x}\}$ and we fix ρ_* such that $B_{\rho_*}(z) \subset \Sigma_{\lambda_*}$. Then, it easy to see that w_{λ_*} weakly solves the equation

$$(-\Delta)^s_{\mathbb{R}^N_+} w_{\lambda_*} \geq 0 \quad \text{in} \quad B_{\rho_*}(z). \tag{7.3.36}$$

Indeed,

$$\begin{aligned}
&\frac{c_{N,s}}{2} \int_{\mathbb{R}^N_+} \int_{\mathbb{R}^N_+} \frac{(w_{\lambda_*}(x) - w_{\lambda_*}(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dxdy \\
&= \frac{c_{N,s}}{2} \int_{\mathbb{R}^N_+} \int_{\mathbb{R}^N_+} \frac{(u_{\lambda_*}(x) - u_{\lambda_*}(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dxdy \\
&\quad - \frac{c_{N,s}}{2} \int_{\mathbb{R}^N_+} \int_{\mathbb{R}^N_+} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dxdy \\
&= \int_{\mathbb{R}^N_+} (u^{2^*_s - 1}_{\lambda_*}(x) - u^{2^*_s - 1}(x))\varphi(x) \, dx \\
&= \int_{\Sigma_{\lambda_*}} (u^{2^*_s - 1}_{\lambda_*}(x) - u^{2^*_s - 1}(x))\varphi(x) \, dx \geq 0,
\end{aligned}$$

for all nonnegative function $\varphi \in C^\infty_c(B_{\rho_*}(z))$. Notice that to obtain the latter inequality, we used (7.3.32). Therefore, we conclude that w_{λ_*} satisfies (7.3.36), as claimed. Moreover, w_{λ_*} is continuous on $\overline{B_{\rho_*}(z)}$, thanks to Proposition 7.3.4. Consequently, from the strong maximum principle, see Proposition 7.3.1, either $w_{\lambda_*} \equiv 0$ in $\mathbb{R}^N_+$ or $w_{\lambda_*} > 0$ in $B_{\rho_*}(z)$. However, if $w_{\lambda_*} \equiv 0$ in $\mathbb{R}^N_+$, then this contradicts the inequality (7.3.34). From this, we deduce that $w_{\lambda_*} > 0$ in $B_{\rho_*}(z)$, that is,

$u < u_{\lambda_*}$ in $B_{\rho_*}(z)$. Using the fact that z is arbitrarily takeng in $\Sigma_{\lambda_*} \setminus \{\tilde{x}\}$, then the strict inequality (7.3.35) follows. Therefore, the plane T_{λ_*} can be moved further to the right, say, to $\lambda_* + \varepsilon$, with $\varepsilon \in (0, \varepsilon_*)$, for some $\varepsilon_* > 0$. With regards to $\lambda_* + \varepsilon$, we associate the function $v_{\lambda_*+\varepsilon}$ and we set $K_{\lambda_*+\varepsilon} := \text{supp } v_{\lambda_*+\varepsilon}$ so that

$$K_{\lambda_*+\varepsilon} \equiv G_{\lambda_*+\varepsilon} \cup \Omega_{\lambda_*+\varepsilon} \tag{7.3.37}$$

with $G_{\lambda_*+\varepsilon}$ and $\Omega_{\lambda_*+\varepsilon}$ define as above, see (7.3.12) and (7.3.13). Arguing as above with $v_{\lambda_*+\varepsilon}$, we have similar to (7.3.24) that

$$\begin{aligned} &\frac{c_{N,s}}{2} \int_{\mathbb{R}^N_+} \int_{\mathbb{R}^N_+} \frac{(v_{\lambda_*+\varepsilon}(x) - v_{\lambda_*+\varepsilon}(y))^2}{|x-y|^{N+2s}} \, dxdy \\ &\leq C \left(\int_{G_{\lambda_*+\varepsilon}} u^{2^*_s}(x) \, dx \right)^{(2^*_s-2)/2^*_s} \int_{\mathbb{R}^N_+} \int_{\mathbb{R}^N_+} \frac{(v_{\lambda_*+\varepsilon}(x) - v_{\lambda_*+\varepsilon}(y))^2}{|x-y|^{N+2s}} \, dxdy, \end{aligned} \tag{7.3.38}$$

where $C = C(N, s)$ is a positive constant independent on ε.
Now, we choose a sufficiently large compact set K of $\mathbb{R}^N_+$ such that

$$\int_{\mathbb{R}^N_+ \setminus K} u^{2^*_s} \, dx < \frac{\delta}{2} \tag{7.3.39}$$

for some δ to be specified later. Notice further that K is choosing so that $K \cap T_{\lambda_*} \neq \emptyset$. Next, we also choose ε_0 such that

$$\int_{B_{2\varepsilon_0}(T_{\lambda_*})} u^{2^*_s} \, dx < \frac{\delta}{2}. \tag{7.3.40}$$

Here, $B_{2\varepsilon_0}(T_{\lambda_*}) := \{x \in \mathbb{R}^N : \text{dist}(x, T_{\lambda_*}) < 2\varepsilon_0\}$, which is an ε_0-neighborhood of T_{λ_*}. By continuity, there is $c > 0$ such that

$$u_{\lambda_*} - u \geq c > 0 \quad \text{in} \quad (K \cap \Sigma_{\lambda_*}) \setminus B_{2\varepsilon_0}(T_{\lambda_*}), \tag{7.3.41}$$

thanks to (7.3.35). Consequently, there exists $\varepsilon \in (0, \varepsilon_0)$ such that

$$u_{\lambda_*+\varepsilon} - u \geq \frac{c}{2} > 0 \quad \text{in} \quad (K \cap \Sigma_{\lambda_*}) \setminus B_{2\varepsilon_0}(T_{\lambda_*}) = (K \cap \Sigma_{\lambda_*+\varepsilon}) \setminus B_{2\varepsilon_0}(T_{\lambda_*}). \tag{7.3.42}$$

To simplify the notation, we put $\widetilde{K} := (K \cap \Sigma_{\lambda_*+\varepsilon}) \setminus B_{2\varepsilon_0}(T_{\lambda_*})$. Then from (7.3.42), we have that $v_{\lambda_*+\varepsilon} = 0$ in $\widetilde{K}$. This implies that $G_{\lambda_*+\varepsilon} \subset K_{\lambda_*+\varepsilon} \subset \Sigma_{\lambda_*+\varepsilon} \setminus \widetilde{K}$.
Using (7.3.39), (7.3.40) and the fact that $\Sigma_{\lambda_*+\varepsilon} \setminus \widetilde{K} \subset (\mathbb{R}^N_+ \setminus K) \cup B_{2\varepsilon_0}(T_{\lambda_*})$, we find that

$$\int_{G_{\lambda_*+\varepsilon}} u^{2^*_s} \, dx \leq \int_{\Sigma_{\lambda_*+\varepsilon} \setminus \widetilde{K}} u^{2^*_s} \, dx \leq \int_{\mathbb{R}^N_+ \setminus K} u^{2^*_s} \, dx + \int_{B_{2\varepsilon_0}(T_{\lambda_*})} u^{2^*_s} \, dx < \delta. \tag{7.3.43}$$

By choosing now $\delta > 0$ sufficiently small such that

$$\delta \leq \left(\frac{c_{N,s}}{2C} \right)^{\frac{2^*_s}{2^*_s - 2}}, \quad \text{where } C \text{ is the constant appearing in (7.3.38)}, \tag{7.3.44}$$

we find from (7.3.43) that

$$\left(\int_{G_{\lambda_*+\varepsilon}} u^{2^*_s}(x) \, dx \right)^{(2^*_s-2)/2^*_s} < \frac{c_{N,s}}{2C}, \tag{7.3.45}$$

and therefore, by using (7.3.45) in (7.3.38), we obtain

$$\int_{\mathbb{R}^N_+}\int_{\mathbb{R}^N_+} \frac{(v_{\lambda_*+\varepsilon}(x) - v_{\lambda_*+\varepsilon}(y))^2}{|x-y|^{N+2s}}\, dxdy = 0. \tag{7.3.46}$$

This implies that $v_{\lambda_*+\varepsilon} = const$ in $\mathbb{R}^N_+$ and recalling that $v_{\lambda_*+\varepsilon} \equiv 0$ in $\{x \in \mathbb{R}^N_+ : x_1 = \lambda_* + \varepsilon\}$, we deduce that

$$v_{\lambda_*+\varepsilon} \equiv 0 \quad \text{in} \quad \mathbb{R}^N_+. \tag{7.3.47}$$

This contradicts (7.3.28). Finally,

$$w_{\lambda_*} \equiv 0 \quad \text{in} \quad \Sigma_{\lambda_*} \tag{7.3.48}$$

that is

$$u_{\lambda_*} = u \quad \text{in} \quad \Sigma_{\lambda_*} \tag{7.3.49}$$

as wanted. To conclude the reflexion with respect to the x_1-direction, we argue as follows. For any $y \in \mathbb{R}^N_+ \setminus \Sigma_{\lambda_*}$, there exists $x \in \Sigma_{\lambda_*}$ such that $y = x_{\lambda_*}$. Using this property and (7.3.49), we find that

$$u_{\lambda_*}(y) = u_{\lambda_*}(x_{\lambda_*}) = u(x) = u(x_{\lambda_*}) = u(y). \tag{7.3.50}$$

This implies that

$$u_{\lambda_*} = u \quad \text{in} \quad \mathbb{R}^N_+ \setminus \Sigma_{\lambda_*}. \tag{7.3.51}$$

Combining (7.3.49) and (7.3.51), we deduce that

$$u_{\lambda_*} = u \quad \text{in} \quad \mathbb{R}^N_+. \tag{7.3.52}$$

Using the same argument in the opposite direction, that is, $(-x_1)$-direction, equality (7.3.52) also holds true. Consequently, the symmetry with respect to the x_1-direction follows. Let put $\lambda_* := \lambda^1_*$. Now we prove the monotonicity of u with respect to the x_1 variable. Let $(x_1, \dots, x_N)$ and $(\overline{x}_1, \dots, x_N)$ be two points in $\Sigma_{\lambda_*} \equiv \Sigma_{\lambda^1_*}$ with $x_1 < \overline{x}_1$. Then, by setting $\lambda := \frac{x_1+\overline{x}_1}{2}$ it follows from the arguments developed above that

$$w_\lambda > 0 \quad \text{in} \quad \Sigma_\lambda. \tag{7.3.53}$$

Consequently,

$$\begin{aligned} 0 < w_\lambda(x_1, \dots, x_N) &= u_\lambda(x_1, \dots, x_N) - u(x_1, \dots, x_N) \\ &= u(\overline{x}_1, \dots, x_N) - u(x_1, \dots, x_N), \end{aligned}$$

that is

$$u(x_1, \dots, x_N) < u(\overline{x}_1, \dots, x_N). \tag{7.3.54}$$

Therefore, u is strictly increasing in $x_1 < \lambda^1_*$. By the same manner, we prove that u is strictly decreasing in $x_1 > \lambda^1_*$. Repeating this process in the remaining directions, we find a sequence $\lambda^2_*, \dots, \lambda^{N-1}_*$ for which u is symmetric in $x_2, \dots, x_{N-1}$ across the hyperplanes $T_{\lambda^2_*}, \dots, T_{\lambda^{N-1}_*}$, and monotonic as well (strictly increasing and decreasing on the left and on the right of each plane respectively). Since the problem (7.1.1) is invariant under dilations and translations parallel to the boundary, we have by mean of the change of variables $x_i \mapsto x_i - \lambda^i_*,\ i = 1, \dots, N-1$ that u is symmetric in $x_1, \dots, x_{N-1}$ with respect to the plane T_0 and monotonic as well, up to translation. Therefore, we can assume without loss of generality that $\lambda^1_* = \lambda^2_* = \cdots = \lambda^{N-1}_* = 0$. From this,

we deduce in particular that u is even in x', that is, $u(-x', x_N) = u(x', x_N)$ for all $(x', x_N) \in \mathbb{R}^{N-1} \times (0, \infty)$.
To complete the proof, it remains to show that the above symmetry and monotonicity are valid for any direction in $\mathbb{R}^{N-1}$. Let then e be any direction in $\mathbb{R}^{N-1}$, that is, $e \in \mathbb{R}^{N-1}$ and $|e| = 1$. Then as above, there is λ_*^e for which u is symmetric in the e-direction about the hyperplane $T_{\lambda_*^e}$. The proof will be done if we show that, in fact,

$$\lambda_*^e \equiv 0. \tag{7.3.55}$$

Assume to the contrary that (7.3.55) does not hold, that is, $\lambda_*^e \neq 0$. Then from the argument above, the function $t \mapsto \psi(t) = u(te, 1)$ is strictly increasing in $(-\infty, \lambda_*^e]$ and strictly decreasing in $[\lambda_*^e, \infty)$. Moreover, it also satisfies $\psi(-t) = \psi(t)$, that is, ψ is even. From this, we get in particular that

$$\psi(-\lambda_*^e) = \psi(\lambda_*^e) \tag{7.3.56}$$

and since $-\lambda_*^e$ belongs in one of the set $(-\infty, \lambda_*^e)$ and (λ_*^e, ∞), then by taking into account the monotonicity of ψ, we find that the equality (7.3.56) holds true if and only if $\psi \equiv const$, that is, $u(te, 1) \equiv const$. Since e can be chosen arbitrarily in $\mathbb{R}^{N-1}$, the latter equality gives that $u \equiv const$ in $\mathbb{R}^N_+$. This contradicts the fact that $u \in L^{2_s^*}(\mathbb{R}^N_+)$. Finally, equality (7.3.55) must be true, and therefore, we easily complete the proof. In fact, we have shown that, up to a translation parallel to the boundary, every hyperplane T containing the upper half-space $\mathbb{R}^N_+$ is a symmetry hyperplane for u. This yields that, up to a translation parallel to the boundary, u is radially symmetric in x' and monotonic in the radial variable as well. $\square$

7.3.1 L^∞-bounds of weak solution

In this subsection, we provide L^∞-bounds of weak solutions to the problem (7.1.1) via Moser's iteration method. Before doing so, let us recall in the next proposition, an elementary result regarding the effect of convex functions on $(-\Delta)^s_{\mathbb{R}^N_+}$.

Proposition 7.3.2. *Assume that $\varphi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz convex function such that $\varphi(0) = 0$. Then if $u \in H^s_0(\mathbb{R}^N_+)$ we have*

$$(-\Delta)^s_{\mathbb{R}^N_+} \varphi(u) \le \varphi'(u) (-\Delta)^s_{\mathbb{R}^N_+} u \quad \textit{weakly in} \quad \mathbb{R}^N_+. \tag{7.3.57}$$

Proof. The proof of the above lemma is standard. In fact, using that every convex φ satisfies $\varphi(a) - \varphi(b) \le \varphi'(a)(a - b)$ for all $a, b \in \mathbb{R}$, the proof follows. $\square$

Our main result on the boundedness of weak solutions to problem (7.1.1) reads as follows.

Proposition 7.3.3. *Let $s \in (0, 1/2) \cup (1/2, 1)$, $N \ge 2$ and $u \in H^s_0(\mathbb{R}^N_+)$ be a positive solution to problem (7.1.1). Then $u \in L^\infty(\mathbb{R}^N_+)$.*

Proof. Let $\alpha \ge 1$ and $T > 0$ large. Let us we define the following convex function

$$\varphi_{T,\alpha}(t) = \begin{cases} 0, & \text{if} \quad t \le 0 \\ t^\alpha, & \text{if} \quad 0 < t < T \\ \alpha T^{\alpha-1}(t - T) + T^\alpha, & \text{if} \quad t \ge T. \end{cases}$$

For simplicity, we put $\varphi_{T,\alpha} =: \varphi$. Here and in the rest of the proof, we will use φ instead of $\varphi_{T,\alpha}$. Using that φ is Lipschitz, with constant $\Lambda_\varphi = \alpha T^{\alpha-1}$, and $\varphi(0) = 0$, we have $\varphi(u) \in H_0^s(\mathbb{R}^N_+)$. Now the convexity of φ yields

$$(-\Delta)^s_{\mathbb{R}^N_+}\varphi(u) \leq \varphi'(u)(-\Delta)^s_{\mathbb{R}^N_+}u, \tag{7.3.58}$$

thanks to Proposition 7.3.2. Moreover, by Sobolev inequality (see for instance [65]) and the inequality (7.3.58), we have that

$$\begin{aligned}\|\varphi(u)\|^2_{L^{2^*_s}(\mathbb{R}^N_+)} \leq C\|\varphi(u)\|^2_{H^s_0(\mathbb{R}^N_+)} &= C\int_{\mathbb{R}^N_+}\varphi(u)(-\Delta)^s_{\mathbb{R}^N_+}\varphi(u)\ dx\\ &\leq C\int_{\mathbb{R}^N_+}\varphi(u)\varphi'(u)(-\Delta)^s_{\mathbb{R}^N_+}u\ dx\\ &= C\int_{\mathbb{R}^N_+}\varphi(u)\varphi'(u)u^{2^*_s-1}\ dx.\end{aligned}$$

Exploiting that $u\varphi'(u) \leq \alpha\varphi(u)$, we get from the above inequality that

$$\|\varphi(u)\|^2_{L^{2^*_s}(\mathbb{R}^N_+)} \leq C\alpha\int_{\mathbb{R}^N_+}(\varphi(u))^2u^{2^*_s-2}\ dx. \tag{7.3.59}$$

We notice that the integral on the right-hand side of the above inequality is finite. This follows from a simple argument. Indeed, using that $\alpha \geq 1$ and $\varphi(u)$ is linear when $u \geq T$, it follows that

$$\begin{aligned}\int_{\mathbb{R}^N_+}(\varphi(u))^2u^{2^*_s-2}\ dx &= \int_{\{u\leq T\}}(\varphi(u))^2u^{2^*_s-2}\ dx + \int_{\{u>T\}}(\varphi(u))^2u^{2^*_s-2}\ dx\\ &\leq T^{2\alpha-2}\int_{\mathbb{R}^N_+}u^{2^*_s}\ dx + C\int_{\mathbb{R}^N_+}u^{2^*_s}\ dx < \infty.\end{aligned}$$

We now choose α in (7.3.59) so that $2\alpha - 1 = 2^*_s$. Let α_1 be this value. Then it holds that

$$\alpha_1 := \frac{2^*_s+1}{2}. \tag{7.3.60}$$

Let $M > 0$ be a positive number whose value will be fixed later. Then exploiting Hölder's inequality with exponents $q := 2^*_s/2$ and $q' := 2^*_s/(2^*_s - 2)$, the integral on the right-hand side of (7.3.59) can be estimate as

$$\begin{aligned}&\int_{\mathbb{R}^N_+}(\varphi(u))^2u^{2^*_s-2}\ dx = \int_{\{u\leq M\}}(\varphi(u))^2u^{2^*_s-2}\ dx + \int_{\{u>M\}}(\varphi(u))^2u^{2^*_s-2}\ dx\\ &\leq \int_{\{u\leq M\}}\frac{(\varphi(u))^2}{u}M^{2^*_s-1}\ dx + \left(\int_{\mathbb{R}^N_+}(\varphi(u))^{2^*_s}\ dx\right)^{2/2^*_s}\left(\int_{\{u>M\}}u^{2^*_s}\ dx\right)^{\frac{2^*_s-2}{2^*_s}}.\end{aligned} \tag{7.3.61}$$

From the Monotone Convergence Theorem, we can choose M large so that

$$\left(\int_{\{u>M\}}u^{2^*_s}\ dx\right)^{\frac{2^*_s-2}{2^*_s}} \leq \frac{1}{2C\alpha_1}, \tag{7.3.62}$$

where C is the positive constant appearing in (7.3.59). Consequently, taking into account (7.3.62) in (7.3.61) and by using (7.3.60), we deduce from (7.3.59) that

$$\|\varphi(u)\|^2_{L^{2_s^*}(\mathbb{R}^N_+)} \leq 2C\alpha_1 \left(M^{2_s^*-1} \int_{\mathbb{R}^N_+} \frac{(\varphi(u))^2}{u} \, dx \right),$$

Since $\varphi(u) \leq u^{\alpha_1}$ and recalling (7.3.60), we get by letting $T \to \infty$ that

$$\left(\int_{\mathbb{R}^N_+} u^{2_s^*\alpha_1} \, dx \right)^{2/2_s^*} \leq 2C\alpha_1 \left(M^{2_s^*-1} \int_{\mathbb{R}^N_+} u^{2_s^*} \, dx \right) < \infty,$$

and therefore

$$u \in L^{2_s^*\alpha_1}(\mathbb{R}^N_+). \tag{7.3.63}$$

Suppose now that $\alpha > \alpha_1$. Thus, using that $\varphi(u) \leq u^\alpha$ in the right hand side of (7.3.59) and letting $T \to \infty$, we obtain

$$\left(\int_{\mathbb{R}^N_+} u^{2_s^*\alpha} \, dx \right)^{2/2_s^*} \leq C\alpha \left(\int_{\mathbb{R}^N_+} u^{2\alpha+2_s^*-2} \, dx \right). \tag{7.3.64}$$

Therefore,

$$\left(\int_{\mathbb{R}^N_+} u^{2_s^*\alpha} \, dx \right)^{\frac{1}{2_s^*(\alpha-1)}} \leq (C\alpha)^{\frac{1}{2(\alpha-1)}} \left(\int_{\mathbb{R}^N_+} u^{2\alpha+2_s^*-2} \, dx \right)^{\frac{1}{2(\alpha-1)}}. \tag{7.3.65}$$

We are now ready to use an iterative argument as in [16, Proposition 2.2]. To this end, we define inductively the sequence $\alpha_{m+1}, \ m \geq 1$ by

$$2\alpha_{m+1} + 2_s^* - 2 = 2_s^*\alpha_m,$$

from which we deduce that,

$$\alpha_{m+1} - 1 = \left(\frac{2_s^*}{2}\right)^m (\alpha_1 - 1).$$

Now by using α_{m+1} in place of α, in (7.3.65), it follows that

$$\left(\int_{\mathbb{R}^N_+} u^{2_s^*\alpha_{m+1}} \, dx \right)^{\frac{1}{2_s^*(\alpha_{m+1}-1)}} \leq (C\alpha_{m+1})^{\frac{1}{2(\alpha_{m+1}-1)}} \left(\int_{\mathbb{R}^N_+} u^{2_s^*\alpha_m} \, dx \right)^{\frac{1}{2_s^*(\alpha_m-1)}}.$$

Next, we set

$$C_{m+1} := (C\alpha_{m+1})^{\frac{1}{2(\alpha_{m+1}-1)}} \quad \text{and} \quad A_m := \left(\int_{\mathbb{R}^N_+} u^{2_s^*\alpha_m} \, dx \right)^{\frac{1}{2_s^*(\alpha_m-1)}}$$

so that

$$A_{m+1} \leq C_{m+1} A_m, \quad m \geq 1. \tag{7.3.66}$$

We now iterate the above inequality to see that

$$A_{m+1} \leq \prod_{i=2}^{m+1} C_i A_1,$$

from which it follows that

$$\log A_{m+1} \leq \sum_{i=2}^{m+1} \log C_i + \log A_1 \leq \sum_{i=2}^{\infty} \log C_i + \log A_1.$$

Notice that the serie $\sum_{i=2}^{\infty} \log C_i$ converges since $\alpha_{m+1} = (\alpha_1 - 1/2)^m(\alpha_1 - 1) + 1$ and also, since $u \in L^{2_s^*\alpha_1}(\mathbb{R}^N_+)$ (see for instance (7.3.63)), then $A_1 \leq C$. From this, we deduce that

$$\log A_{m+1} \leq C_0 \tag{7.3.67}$$

with being $C_0 > 0$ a positive constant independent of m. By letting $m \to \infty$, it follows that

$$\|u\|_{L^\infty(\mathbb{R}^N_+)} \leq C_0' < \infty,$$

as wanted. □

As a direct consequence of the above result, we deduce from [122, Theorem D] the following.

Proposition 7.3.4. *Let $s \in (0, 1/2) \cup (1/2, 1)$, $N \geq 2$ and $u \in H^s_0(\mathbb{R}^N_+)$ be a nonnegative solution to problem* (7.1.1)*. Then, $u \in C(\mathbb{R}^N_+)$.*

Chapter 8

Mountain pass solutions for the regional fractional Laplacian

In this last chapter, we apply the mountain pass Theorem to the regional fractional Laplacian. The presentation of this chapter follows verbatim the one of the original paper [R7]. The notation may slightly differ from those in the previous chapters.

8.1 Introduction and main result

The aim of this note is to study the existence of nontrivial mountain pass solutions of the following nonlinear Dirichlet problem

$$\begin{cases} (-\Delta)^s_\Omega u = f(u) & \text{in} \quad \Omega \\ \qquad u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{8.1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded Lipschitz domain, $s \in (1/2, 1)$ and $(-\Delta)^s_\Omega$ is the *regional fractional Laplacian* defined for all $u \in C^2(\overline{\Omega})$ by

$$(-\Delta)^s_\Omega u(x) = C_{N,s} P.V. \int_\Omega \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy, \quad x \in \Omega.$$

Here, $C_{N,s}$ is a suitable positive normalization constant and *P.V.* stands for the principal value of the integral.

In recent years, the study of nonlinear problem involving the regional fractional Laplacian has received a great attention. In [83] the authors established the existence of nonnegative solutions for problem of the type (8.1.1) with critical nonlinearity. More precisely, they showed that the minimization problem

$$S_{N,s}(\Omega) = \inf_{u \in H^s_0(\Omega)} \frac{Q_{N,s,\Omega}[u]}{\|u\|^2_{L^{2^*_s}(\Omega)}}, \quad \text{with} \quad Q_{N,s,\Omega}[u] := \int_\Omega \int_\Omega \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\, dxdy \tag{8.1.2}$$

is attained. Here, $2^*_s := 2N/(N-2s)$ is the so-called fractional critical Sobolev exponent. Notice that at the critical power 2^*_s there is a lack of compactness. Therefore, classical technique such as mountain pass argument cannot be applied to (8.1.2). To bypass this difficulty, the authors in [83] used the missing mass method. Interior regularity and boundary regularity of positive weak dual solutions of problem of type (8.1.1) were obtain in [26]. The existence an nonexistence of blowing-up solution for nonlinear equation driven by the regional fractional Laplacian has been addressed in [46].

However, to the best of our knowledge, much less is known regarding the existence of mountain pass solutions. In [115], the author used the mountain pass theorem to prove the existence of solutions for semilinear problem driven by a variational version of the regional fractional Laplacian $(-\Delta)^s_\rho$, with a range of scope determined by a positive function $\rho \in C(\overline{\Omega})$ (with additional condition that $\lambda\delta_\Omega \leq \rho \leq \delta_\Omega$ for all $\lambda \in [0, 1]$), which agrees with the following definition (see e.g. Eq. (1.2) in [115])

$$\int_\Omega (-\Delta)^s_\rho u(x)v(x)\ dx = \int_\Omega \int_{B_{\rho(x)}(0)} \frac{(u(x+z) - u(x))(v(x+z) - v(x))}{|z|^{N+2s}}\ dz. \tag{8.1.3}$$

Apart from the reference [115], nothing is known in the literature about the existence of mountain pass solutions for the *full* regional fractional Laplacian. Therefore, in this note, our aim is to study the existence of nontrivial solutions of (8.1.1) by using the mountain pass theorem. We stress that the operator in (8.1.1) and the one treated in [115] are completely different and also the arguments developed here are different from those in [115]. It therefore makes sense to study problem (8.1.1).

Before stating our main theorem, we make the following assumptions on the nonlinearity $f : \mathbb{R} \to \mathbb{R}$.

(F_1) There is $C > 0$ and $p \in (2, 2^*_s)$ such that

$$|f(t)| \leq C(1 + |t|^{p-1});$$

(F_2) $\lim\limits_{t\to 0} \frac{f(t)}{t} \leq 0$;

(F_3) $\lim\limits_{|t|\to\infty} \frac{F(t)}{t^2} = +\infty$;

(F_4) Denote by $H(t) = tf(t) - 2F(t)$. Then there is $c_0 > 0$ such that

$$H(t_1) \leq H(t_2) + c_0$$

for all $0 < t_1 < t_2$ or $t_2 < t_1 < 0$.

Here $F(t) = \int_0^t f(\tau)\ d\tau$ is the primitive of f. We notice that condition (F_4) is a weaker form of the following assumption

$(F_4)^*$ There exists $T_0 > 0$ such that $\frac{f(t)}{|t|}$ is increasing for $|t| > T_0$.

With the model case $F(t) = t^2 \log(1+|t|)$, we have in particular that f satisfies the assumptions $(F_1) - (F_4)$. This function can be found in [148].

With the above hypotheses on f, we are now ready to state our main result. It reads as follows:

Theorem 8.1.1. *Let f be a function satisfying conditions (F_1)-(F_4). Then, there exists nontrivial mountain pass solution to the problem* (8.1.1).

We mention that Theorem 8.1.1 remains valid if $f(u)$ is replaced by $f(x,u)$ provided that assumptions (F_2) and (F_3) hold uniformly in the first variable, that is, in x. In the case of fractional Laplacian with homogeneous exterior Dirichlet data, such type of existence result has been obtained in [136, 148].

The strategy behind the proof of Theorem 8.1.1 reads as follows. Let introduce J as the corresponding energy functional to (8.1.1). Then, the first step of the proof is to show that the functional J has the geometric features needed in order to apply the mountain pass theorem. Next, in the last step, we prove that the functional J satisfies the Palais-Smale condition. We believe our result produces a preliminary step in studying nonlinear problems involving regional fractional Laplacian.

The rest of the note is organized as follows. in Section 8.2 we give some preliminaries that will be useful throughout this paper whereas in Section 8.3 we prove Theorem 8.1.1.

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8.2 Preliminary results and notation

In this section, we introduce some preliminary properties which will be useful in this work. For all $s \in (0,1)$, the fractional Sobolev space $H^s(\Omega)$ is defined as the set of all measurable functions u such that

$$[u]^2_{H^s(\Omega)} := \int_\Omega \int_\Omega \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\, dxdy$$

is finite. It is a Hilbert space endowed with the norm

$$\|u\|^2_{H^s(\Omega)} = \|u\|^2_{L^2(\Omega)} + [u]^2_{H^s(\Omega)}.$$

We refer to [65] for more details on this fractional Sobolev spaces. Next, we denote by $H^s_0(\Omega)$ the completion of $C^\infty_c(\Omega)$ under the norm $\|\cdot\|_{H^s(\Omega)}$. Moreover, for $s \in (1/2,1)$, $H^s_0(\Omega)$ is a Hilbert space equipped with the norm

$$\|u\|^2_{H^s_0(\Omega)} = \int_\Omega \int_\Omega \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\, dxdy$$

which is equivalent to the usual one in $H^s(\Omega)$ thanks to Poincaré inequality.

For every set $A \subset \mathbb{R}^N$, we denote by $|A|$ its N-dimensional Lebesgue measure. As usual, for a given function u, we denote respectively by $u^+ = \max\{u,0\}$ and $u^- = \max\{-u,0\}$ its positive and negative part. Finally, if X is a Banach space, then by X^*, we refer to its dual.

We have the following

Definition 8.2.1. Let $c \in \mathbb{R}$, X be a Banach space and $J \in C^1(X,\mathbb{R})$.

(i) $\{u_i\}$ is a Palais-Smale sequence in X for J if $J(u_i) = c + o(1)$ and $J'(u_i) = o(1)$ strongly in X^* as $i \to \infty$.

(ii) We say that J satisfies the Palais-Smale condition if any Palais-Smale sequence $\{u_i\}$ for J in X has a convergent subsequence.

In the next proposition, we recall Sobolev embeddings.

Proposition 8.2.2 (see [63,65])**.** *The embedding $H_0^s(\Omega) \hookrightarrow L^p(\Omega)$ is continuous for any $p \in [2, 2_s^*]$, and compact for any $p \in [2, 2_s^*)$.*

The following proposition will be of key importance.

Proposition 8.2.3. *(i) For every $w \in H_0^s(\Omega)$ then both w^- and w^+ belongs to $H_0^s(\Omega)$.*

(ii) For any measurable function w, the following inequalities hold true

(a) $(w^+(x) - w^+(y))^2 \le (w^+(x) - w^+(y))(w(x) - w(y))$,

(b) $(w^-(x) - w^-(y))^2 \le (w^-(x) - w^-(y))(w(x) - w(y))$.

Proof. (i) From the definition of w^+ and w^-, we immediately see that $w^+ = w^- = 0$ on $\partial\Omega$. Now,

$$\begin{cases} |w^-(x) - w^-(y)| = |0| \le |w(x) - w(y)|, & \text{if} \quad w(x) \ge 0, w(y) \ge 0 \\ |w^-(x) - w^-(y)| = |-w(y)| \le |w(x) - w(y)| & \text{if} \quad w(x) \ge 0, w(y) < 0 \\ |w^-(x) - w^-(y)| = |-w(x)| \le |w(x) - w(y)| & \text{if} \quad w(x) < 0, w(y) \ge 0 \\ |w^-(x) - w^-(y)| = |-w(x) + w(y)| = |w(x) - w(y)| & \text{if} \quad w(x) < 0, w(y) < 0. \end{cases}$$

Therefore,

$$\|w^-\|^2_{H_0^s(\Omega)} = \int_\Omega \int_\Omega \frac{|w^-(x) - w^-(y)|^2}{|x-y|^{N+2s}}\, dxdy \le \int_\Omega \int_\Omega \frac{|w(x) - w(y)|^2}{|x-y|^{N+2s}}\, dxdy = \|w\|^2_{H_0^s(\Omega)}$$

which implies that $w^- \in H_0^s(\Omega)$. Furthernore, using that $w^+ = w + w^-$ then it is easy to see that $\|w^+\|^2_{H_0^s(\Omega)} \le \|w\|^2_{H_0^s(\Omega)} + \|w^-\|^2_{H_0^s(\Omega)}$. Thus $w^+ \in H_0^s(\Omega)$ too.
(ii) It is enough to prove (a). Using that $w = w^+ - w^-$, we get

$$\begin{aligned} (w(x) - w(y))(w^+(x) - w^+(y)) &= (w^+(x) - w^+(y))^2 - (w^+(x) - w^+(y))(w^-(x) - w^-(y)) \\ &= (w^+(x) - w^+(y))^2 + w^+(x)w^-(y) + w^+(y)w^-(x) \\ &\ge (w^+(x) - w^+(y))^2 \end{aligned}$$

since $w^+(x)w^-(y) + w^+(y)w^-(x) \ge 0$. $\square$

We conclude this section with the following elementary result regarding convex functions applied to $(-\Delta)^s_\Omega$.

Proposition 8.2.4. *Assume that $\varphi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz convex function such that $\varphi(0) = 0$. Then if $u \in H_0^s(\Omega)$ we have*

$$(-\Delta)^s_\Omega \varphi(u) \le \varphi'(u)(-\Delta)^s_\Omega u \quad \textit{weakly in} \quad \Omega. \tag{8.2.1}$$

Proof. The proof of the above lemma is standard. In fact, using that every convex φ satisfies $\varphi(a) - \varphi(b) \le \varphi'(a)(a - b)$ for all $a, b \in \mathbb{R}$, the proof follows. $\square$

8.3 Mountain pass solutions: proof of Theorem 8.1.1

The purpose of this section is to prove Theorem 8.1.1. Before doing this, we mention that solutions of (8.1.1) correspond to critical points of the associated Euler-Lagrange functional $J : H_0^s(\Omega) \to \mathbb{R}$ defined by

$$J(u) := \frac{C_{N,s}}{4} \int_\Omega \int_\Omega \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} \, dxdy - \int_\Omega F(u) \, dx, \tag{8.3.1}$$

where $F(\xi) = \int_0^\xi f(t) \, dt$ is the primitive of f. By the property of F, it is easy to check that $J \in C^1(H_0^s(\Omega), \mathbb{R})$ in the sense of Fréchet and

$$\langle J'(u), \varphi \rangle = \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dxdy - \int_\Omega f(u)\varphi \, dx \tag{8.3.2}$$

for any $\varphi \in H_0^s(\Omega)$. In order to find the critical points of J, we wish to apply the mountain pass theorem which goes back to Ambrosetti and Rabinowitz [7]. For this ends, we have to check that J has an appropriated geometrical structure and that it satisfies the Palais-Smale compactness condition as well.

In the sequel, we collect some prelimany results. We note that the analysis here has similarities to that of [148], where the authors treated the case of fractional Laplacian.

Lemma 8.3.1. *Let f be a function satisfying condition (F_3). Then, the functional J is unbounded from below.*

Proof. From condition (F_3) we deduce that for all $A > 0$, there exists $C_A > 0$ such that

$$F(t) \geq At^2 - C_A \quad \text{for all} \quad t > 0. \tag{8.3.3}$$

Now, we fix $\varphi \in H_0^s(\Omega)$. Without any loss of generality, we may assume that $\varphi \geq 0$. This is possible since if $\varphi \in H_0^s(\Omega)$, then $\varphi^+ \in H_0^s(\Omega)$ too (see part (i) of Proposition 8.2.3).
Hence from (8.3.3), we have that

$$\begin{aligned} J(t\varphi) &= \frac{C_{N,s}}{4} t^2 \|\varphi\|^2_{H_0^s(\Omega)} - \int_\Omega F(t\varphi) \, dx \\ &\leq \frac{C_{N,s}}{4} t^2 \|\varphi\|^2_{H_0^s(\Omega)} - At^2 \|\varphi\|^2_{L^2(\Omega)} + C_A|\Omega| \\ &= t^2 \Big(\frac{C_{N,s}}{4} \|\varphi\|^2_{H_0^s(\Omega)} - A\|\varphi\|^2_{L^2(\Omega)} \Big) + C_A|\Omega|. \end{aligned}$$

By choosing in particular $A = \dfrac{\frac{C_{N,s}}{4}\|\varphi\|^2_{H_0^s(\Omega)}}{\|\varphi\|^2_{L^2(\Omega)}} + 1$, then

$$\lim_{t \to \infty} J(t\varphi) = -\infty,$$

as needed. $\square$

Lemma 8.3.2. *Let f be a function satisfying conditions (F_1) and (F_2). Then, there exist $\rho, \beta > 0$ such that for any $u \in H_0^s(\Omega)$ with $\|u\|_{H_0^s(\Omega)} = \rho$, it follows that $J(u) \geq \beta$.*

Proof. From (F_1) and (F_2), we have that for every $\varepsilon > 0$, there exists C_ε such that

$$F(t) \leq \varepsilon t^2 + C_\varepsilon t^p \quad \text{for all} \quad t > 0.$$

Therefore, taking this into account and by using Hölder inequality, we get that

$$\begin{aligned} J(u) &\geq \frac{C_{N,s}}{4}\|u\|^2_{H^s_0(\Omega)} - \varepsilon \int_\Omega |u|^2 \ dx - C_\varepsilon \int_\Omega |u|^p \ dx \\ &\geq \frac{C_{N,s}}{4}\|u\|^2_{H^s_0(\Omega)} - \varepsilon|\Omega|^{\frac{2^*_s-2}{2^*_s}}\|u\|^2_{L^{2^*_s}(\Omega)} - C_\varepsilon|\Omega|^{\frac{2^*_s-p}{2^*_s}}\|u\|^p_{L^{2^*_s}(\Omega)} \\ &\geq \left(\frac{C_{N,s}}{4} - \varepsilon\gamma_0|\Omega|^{\frac{2^*_s-2}{2^*_s}}\right)\|u\|^2_{H^s_0(\Omega)} - \gamma_0^{\frac{p}{2}} C_\varepsilon|\Omega|^{\frac{2^*_s-p}{2^*_s}}\|u\|^p_{H^s_0(\Omega)}. \end{aligned}$$

Choosing $\varepsilon > 0$ such that $\varepsilon\gamma_0|\Omega|^{\frac{2^*_s-2}{2^*_s}} \leq \frac{C_{N,s}}{8}$, it easily follows from the above inequality that

$$J(u) \geq \theta\|u\|^2_{H^s_0(\Omega)}\left(1 - \eta\|u\|^{p-2}_{H^s_0(\Omega)}\right),$$

where θ and η are positive real numbers. Now, let $u \in H^s_0(\Omega)$ satisfying $\|u\|_{H^s_0(\Omega)} = \rho > 0$. Then, recalling that $p > 2$, we can choose ρ sufficiently small (i.e. we choose ρ such that $1 - \eta\rho^{p-2} > 0$), so that

$$\inf_{\substack{u \in H^s_0(\Omega) \\ \|u\|_{H^s_0(\Omega)} = \rho}} J(u) \geq \theta\rho^2(1 - \eta\rho^{p-2}) =: \beta > 0,$$

concluding the proof of Lemma 8.3.2. □

In our next lemma, we analyze the compactness property of every Palais-Smale sequence of J in $H^s_0(\Omega)$.

Lemma 8.3.3 (Palais-Smale condition)**.** *Let f be a function satisfying conditions (F_1), (F_3) and (F_4). Then every Palais-Smale sequence for J strongly converges in $H^s_0(\Omega)$, up to a subsequence.*

Proof. Let $\{u_i\}_{i\in\mathbb{N}} \subset H^s_0(\Omega)$ be a Palais-Smale sequence for J i.e., $\{u_i\}_{i\in\mathbb{N}}$ is a sequence such that (see part (i) in Definition 8.2.1)

$$J(u_i) \to c \quad \text{and} \quad \langle J'(u_i), \varphi\rangle \to 0 \quad \text{for all} \quad \varphi \in H^s_0(\Omega). \tag{8.3.4}$$

The aim is to prove that $\{u_i\}_{i\in\mathbb{N}}$ is bounded in $H^s_0(\Omega)$. Seeking contradiction,

$$\|u_i\|_{H^s_0(\Omega)} \to \infty \quad \text{as} \quad i \to \infty.$$

Define $v_i := \frac{u_i}{\|u_i\|_{H^s_0(\Omega)}}$. Then $\|v_i\|_{H^s_0(\Omega)} = 1$, that is, v_i is bounded in $H^s_0(\Omega)$. Since $H^s_0(\Omega)$ is Hilbert, then after passing to a subsequence, there is $v \in H^s_0(\Omega)$ such that

$$\begin{aligned} &v_i \rightharpoonup v \quad \text{weakly in} \quad H^s_0(\Omega) \\ &v_i \to v \quad \text{strongly in} \quad L^2(\Omega) \\ &v_i \to v \quad \text{a.e. in} \quad \Omega, \end{aligned}$$

thanks to Proposition 8.2.2. Now, we claim that $v \equiv 0$ a.e. in Ω. To prove this claim, we set $\Omega_0 := \{x \in \Omega : v(x) \neq 0\}$ and we aim to prove that $\Omega_0 = \emptyset$. If Ω_0 where nonempty i.e., $\Omega_0 \neq \emptyset$ then for $x \in \Omega_0$, $|u_i(x)| \to \infty$ as $i \to \infty$. Therefore from assumption (F_3), we have that

$$\lim_{i\to\infty} \frac{F(u_i(x))}{(u_i(x))^2}(v_i(x))^2 = \infty. \tag{8.3.5}$$

Using (8.3.4) along with Fatou's lemma, we find that

$$\begin{aligned}
\int_\Omega \lim_{i\to\infty} \frac{F(u_i(x))}{(u_i(x))^2}(v_i(x))^2 \; dx &= \int_\Omega \lim_{i\to\infty} \frac{F(u_i(x))}{(u_i(x))^2} \frac{(u_i(x))^2}{\|u_i\|^2_{H^s_0(\Omega)}} \; dx \\
&\leq \liminf_{i\to\infty} \frac{1}{\|u_i\|^2_{H^s_0(\Omega)}} \int_\Omega F(u_i(x)) \; dx \\
&\leq \liminf_{i\to\infty} \frac{1}{\|u_i\|^2_{H^s_0(\Omega)}} \int_\Omega \Big(\frac{C_{N,s}}{4} \|u_i\|^2_{H^s_0(\Omega)} - J(u_i(x)) \Big) \; dx \\
&= \frac{C_{N,s}}{4}.
\end{aligned}$$

This with (8.3.5) imply that Ω_0 has zero measure and therefore $v(x) \equiv 0$ a.e. in Ω.
Next, following the strategy developed in [110], we take $t_i \in [0,1]$ suh that

$$J(t_i u_i) = \max_{t\in[0,1]} J(tu_i).$$

This yields that

$$\frac{d}{dt}\Big|_{t=t_i} J(tu_i) = \frac{C_{N,s}}{2} t_i \|u_i\|^2_{H^s_0(\Omega)} - \int_\Omega f(t_i u_i) \cdot u_i \; dx = 0. \tag{8.3.6}$$

Moreover, since

$$\langle J'(t_i u_i), t_i u_i \rangle = \frac{C_{N,s}}{2} t_i^2 \|u_i\|^2_{H^s_0(\Omega)} - \int_\Omega f(t_i u_i) \cdot t_i u_i \; dx,$$

we have with (8.3.6) that

$$\langle J'(t_i u_i), t_i u_i \rangle = t_i \cdot \frac{d}{dt}\Big|_{t=t_i} J(tu_i) = 0.$$

Consequently, from (F_4) and thanks to (8.3.4), we find that

$$\begin{aligned}
2J(tu_i) &\leq 2J(t_i u_i) - \langle J'(t_i u_i), t_i u_i \rangle \\
&= \int_\Omega (t_i u_i \cdot f(t_i u_i) - 2F(t_i u_i)) \; dx \\
&\leq \int_\Omega (u_i \cdot f(u_i) - 2F(u_i) + c_0) \; dx \\
&= 2J(u_i) - \langle J'(u_i), u_i \rangle + c_0|\Omega| \\
&\to 2c + c_0|\Omega| \quad \text{as} \quad i \to \infty.
\end{aligned} \tag{8.3.7}$$

But for every $l > 0$, there holds

$$2J(lv_i) = \frac{C_{N,s}}{2} l^2 - 2\int_\Omega F(lv_i) \; dx = \frac{C_{N,s}}{2} l^2 + o(1),$$

yielding a contradiction with (8.3.7), for l and i large enough. From this, we conclude that $\{u_i\}$ is boubed in $H_0^s(\Omega)$ and since $H_0^s(\Omega)$ is Hilbert space, there exists $u \in H_0^s(\Omega)$ such that after passing to a subsequence

$$\begin{aligned} u_i &\rightharpoonup u \quad \text{weakly in } H_0^s(\Omega), \\ u_i &\to u \quad \text{strongly in } L^p(\Omega), \quad 2 < p < 2_s^* \\ u_i &\to u \quad \text{a.e. in } \Omega, \end{aligned} \tag{8.3.8}$$

and there exists $h \in L^p(\Omega)$ such that (see [27, Theorem 4.9])

$$|u_i(x)| \le h(x) \quad \text{a.e. in} \quad \Omega \quad \text{for all } i \in \mathbb{N}. \tag{8.3.9}$$

Combining assumption (F_1) with (8.3.8) and (8.3.9), and using Dominated Convergence Theorem, it follows that

$$\int_\Omega f(u_i)u_i\ dx \to \int_\Omega f(u)u\ dx \tag{8.3.10}$$

and

$$\int_\Omega f(u_i)u\ dx \to \int_\Omega f(u)u\ dx. \tag{8.3.11}$$

On the other hand, we have by (8.3.4) that

$$0 \leftarrow \langle J'(u_i), u_i\rangle = \frac{C_{N,s}}{2}\|u_i\|^2_{H_0^s(\Omega)} - \int_\Omega f(u_i)u_i\ dx. \tag{8.3.12}$$

From this and by using (8.3.10), we get that

$$\frac{C_{N,s}}{2}\|u_i\|_{H_0^s(\Omega)} \to \int_\Omega f(u)u\ dx \quad \text{as} \quad i \to \infty. \tag{8.3.13}$$

Moreover, since

$$0 \leftarrow \langle J'(u_i), u\rangle = \frac{C_{N,s}}{2}\int_\Omega\int_\Omega \frac{(u_i(x)-u_i(y))(u(x)-u(y))}{|x-y|^{N+2s}}\ dxdy - \int_\Omega f(u_i)u\ dx, \tag{8.3.14}$$

it follows from (8.3.8) and (8.3.11) that

$$\frac{C_{N,s}}{2}\|u\|^2_{H_0^s(\Omega)} = \int_\Omega f(u)u\ dx. \tag{8.3.15}$$

Consequently, putting together (8.3.13) and (8.3.15) we find that

$$\|u_i\|^2_{H_0^s(\Omega)} \to \|u\|^2_{H_0^s(\Omega)} \quad \text{as} \quad i \to \infty. \tag{8.3.16}$$

Finally from straightforward calculations, we get, thanks to (8.3.8) and (8.3.16) that

$$\begin{aligned} \|u_i - u\|^2_{H_0^s(\Omega)} &= \|u_i\|^2_{H_0^s(\Omega)} + \|u\|^2_{H_0^s(\Omega)} - 2\int_\Omega\int_\Omega \frac{(u_i(x)-u_i(y))(u(x)-u(y))}{|x-y|^{N+2s}}\ dxdy \\ &\to 2\|u\|^2_{H_0^s(\Omega)} - 2\|u\|^2_{H_0^s(\Omega)} = 0 \end{aligned}$$

as $i \to \infty$. This completes the proof. $\square$

We now complete the proof of Theorem 8.1.1.

Proof of Theorem 8.1.1 (completed). From Lemmas 8.3.1 and 8.3.2, we see that the functional J possesses the mountain pass geometry and satisfies moreover the Palais-Smale condition, thanks to Lemma 8.3.3. Then by mountain pass Theorem (see [7]), there exists a critical point $u \in H_0^s(\Omega)$ for J. Furthermore, since by Lemma 8.3.2, $J(u) \geq \beta > 0 = J(0)$, it follows that $u \not\equiv 0$, that is, u is a nontrivial mountain pass solution. This concludes the proof. $\square$

In the next proposition, we analyze the sign of the mountain pass solution.

Proposition 8.3.4. *Assume that the assumptions of Theorem 8.1.1 are satisfied. Then, problem* (8.1.1) *admits at least a nonnegative and nonpositive mountain pass solution* $u_+ \in H_0^s(\Omega)$ *and* $u_- \in H_0^s(\Omega)$ *with* $u_+, u_- \not\equiv 0$.

Proof. Consider the following nonlinearities

$$f_+(t) := \begin{cases} f(t) & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0 \end{cases} \qquad \text{and} \qquad f_-(t) := \begin{cases} 0 & \text{if} \quad t > 0 \\ f(t) & \text{if} \quad t \leq 0 \end{cases}$$

and define also $F_+(\xi) := \int_0^\xi f_+(t)\, dt$ and $F_-(\xi) := \int_0^\xi f_-(t)\, dt$. We first prove the existence of nontrivial and nonnegative solution of problem (8.1.1). For that, we consider the following problem

$$\begin{cases} (-\Delta)^s_\Omega u = f_+(u) & \text{in} \quad \Omega \\ \qquad u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{8.3.17}$$

where the corresponding functional $J_+ : H_0^s(\Omega) \to \mathbb{R}$ is defined by

$$J_+(u) = \frac{C_{N,s}}{4} \int_\Omega \int_\Omega \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}}\, dxdy - \int_\Omega F_+(u)\, dx.$$

Clearly, $F_+ \in C^1(H_0^s(\Omega), \mathbb{R})$ and f_+ satisfies all the assumptions of Theorem 8.1.1. Therefore, there is a nontrivial critical point $u_+ \in H_0^s(\Omega)$ for J_+, which is a mountain pass (weak) solution of (8.3.17). On the other hand, $(u_+)^- \in H_0^s(\Omega)$ (see part (i) of Proposition 8.2.3). Using now $(u_+)^-$ as a test function in (8.3.17) and recalling (ii) of Proposition 8.2.3, we find that

$$\begin{aligned} 0 \leq \frac{C_{N,s}}{2} \|(u_+)^-\|^2_{H_0^s(\Omega)} &\leq \int_\Omega f_+(u_+)(u_+)^-\, dx \\ &= \int_{\{u_+ \geq 0\}} f_+(u_+)(u_+)^-\, dx + \int_{\{u_+ < 0\}} f_+(u_+)(u_+)^-\, dx = 0. \end{aligned}$$

In the latter, we used the definition of f_+. Therefore, $\|(u_+)^-\|^2_{H_0^s(\Omega)} = 0$ which implies that $u_+ \geq 0$ a.e. in Ω. From this, we finally get that $u_+ \geq 0$, $u_+ \not\equiv 0$ a.e. in Ω. Moreover, since $J(u_+) = J_+(u_+)$, then u_+ is also a nontrivial and nonnegative weak solution of problem (8.1.1). This completes the proof of the existence of nonegative solution of (8.1.1).

By the same manner, by considering now the following problem

$$\begin{cases} (-\Delta)^s_\Omega u = f_-(u) & \text{in} \quad \Omega \\ \qquad u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{8.3.18}$$

we also obtain a nontrivial and nonpositive solution $u_- \in H_0^s(\Omega)$ of problem (8.1.1), completing the proof $\square$

By using the classical De Giorgi iteration method, we obtain in next proposition, a priori L^∞-bounds of weak solution to (8.1.1). Before, let us recall the following.

Definition 8.3.5. We say that $u \in H_0^s(\Omega)$ is a weak solution of (8.1.1) if

$$\frac{C_{N,s}}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dxdy = \int_\Omega f\varphi\,dx \quad \text{for all} \quad \varphi \in H_0^s(\Omega). \tag{8.3.19}$$

Now, we have the following.

Proposition 8.3.6. *Let $u \in H_0^s(\Omega)$ be a weak solution of* (8.1.1). *Then $u \in L^\infty(\Omega)$.*

Proof. As mention above, we use the classical De Giorgi iteration method to prove the boundedness of solution of (8.1.1). More precisely, we will show that

$$\|u\|_{L^\infty(\Omega)} \le 1 \quad \text{whenever} \quad \|u^+\|_{L^p(\Omega)} \le \delta \quad \text{for some} \quad \delta > 0. \tag{8.3.20}$$

For all $k \in \mathbb{N}$, we consider the following monotone truncation $v_k := (u-(1-2^{-k}))^+$ in Ω. Then $v_k \in H_0^s(\Omega)$ satisfies the following properties

$$v_{k+1} \le v_k \text{ in } \Omega,\ u < (2^{k+1}-1)v_k \text{ where } \{v_{k+1}>0\} \text{ and } \{v_{k+1}>0\} \subseteq \{v_k > 2^{-k-1}\}. \tag{8.3.21}$$

Using the inequality (ii)-(a) of Proposition 8.2.3 with $w = u-(1-2^{-k-1})$ (so that $w^+ = v_{k+1}$), we get thanks to (8.3.19) that

$$\begin{aligned}
\int_\Omega\int_\Omega \frac{(v_{k+1}(x)-v_{k+1}(y))^2}{|x-y|^{N+2s}}\,dxdy &\le \int_\Omega\int_\Omega \frac{(u(x)-u(y))(v_{k+1}(x)-v_{k+1}(y))}{|x-y|^{N+2s}}\,dxdy\\
&= \frac{2}{C_{N,s}}\int_\Omega f(u)v_{k+1}\,dx \le \frac{2}{C_{N,s}}\int_\Omega |f(u)|v_{k+1}\,dx\\
&\le \frac{2C}{C_{N,s}}\int_\Omega (1+|u|^{p-1})v_{k+1}\,dx\\
&= \frac{2C}{C_{N,s}}\int_{\{v_{k+1}>0\}} (1+|u|^{p-1})v_{k+1}\,dx\\
&= \frac{2C}{C_{N,s}}\Big(\int_{\{v_{k+1}>0\}} v_{k+1}\,dx + \int_{\{v_{k+1}>0\}} |u|^{p-1}v_{k+1}\,dx\Big)\\
&\le \frac{2C}{C_{N,s}}|\{x\in\Omega : v_{k+1}(x)>0\}|^{1-\frac{1}{p}}\|v_k\|_{L^p(\Omega)}\\
&\quad + \frac{2C}{C_{N,s}}\int_{\{v_{k+1}>0\}} (2^{k+1}+1)^{p-1}v_k^p\,dx\\
&\le C|\{x\in\Omega : v_{k+1}(x)>0\}|^{1-\frac{1}{p}}U_k^{\frac{1}{p}} + C(2^{k+1}+1)^{p-1}U_k.
\end{aligned}$$

Here, we have set $U_k := \|v_k\|^p_{L^p(\Omega)}$.
On the other hand,

$$U_k = \int_\Omega v_k^p\,dx \ge \int_{\{v_{k+1}>0\}} v_k^p\,dx \ge 2^{-(k+1)p}|\{x\in\Omega : v_{k+1}(x)>0\}|.$$

Therefore,

$$\|v_{k+1}\|^2_{H^s_0(\Omega)} \leq (C2^{(k+1)(p-1)} + C(2^{k+1}+1)^{p-1})U_k \leq C'(2^{k+1}+1)^{p-1}U_k. \tag{8.3.22}$$

Moreover, applying Hölder inequality and using Sobolev embedding (see Proposition 8.2.2), we get that

$$\begin{aligned} U_{k+1} = \int_{\{v_{k+1}>0\}} v^p_{k+1}\, dx &\leq \|v_{k+1}\|^p_{L^{2^*_s}(\Omega)} |\{x \in \Omega : v_{k+1}(x) > 0\}|^{\frac{2^*_s-p}{2^*_s}} \\ &\leq \left(C_0\|v_{k+1}\|^2_{H^s_0(\Omega)}\right)^{\frac{p}{2}} (2^{(k+1)p}U_k)^{\frac{2^*_s-p}{2^*_s}} \\ &\leq \left(C_0C'(2^{k+1}+1)^{p-1}U_k\right)^{\frac{p}{2}} (2^{(k+1)p}U_k)^{\frac{2^*_s-p}{2^*_s}}. \end{aligned}$$

Since $p > 2$ then we can write $\frac{p}{2} = 1 + \varepsilon$ for some $\varepsilon > 0$ depending on p. From this, we have with the above inequality that

$$\begin{aligned} U_{k+1} &\leq C_1(2^{k+1}+1)^{\frac{p(p-1)}{2} + \frac{p(2^*_s-p)}{2^*_s}} U_k^{1+(\varepsilon + \frac{2^*_s-p}{2^*_s})} \\ &= \kappa_0 b^k U_k^{1+\beta}, \end{aligned}$$

where we have set $\kappa_0 > 0$, $b > 1$ and $\beta = \varepsilon + \frac{2^*_s-p}{2^*_s}$.
Now by [91, Lemma 7.1],

$$U_k \to 0 \quad \text{provided that} \quad \|u^+\|_{L^p(\Omega)} = U_0 \leq \kappa_0^{-\frac{1}{\beta}} b^{-\frac{1}{\beta^2}}.$$

Hence, by Fatou's lemma,

$$\|(u-1)^+\|^p_{L^p(\Omega)} \leq \liminf_{k\to\infty} U_k = 0 \quad \text{provided that} \quad \|u^+\|_{L^p(\Omega)} \leq \delta.$$

i.e.,

$$(u-1)^+ = 0 \ \text{ a.e. in } \ \Omega \quad \text{provided that} \quad \|u^+\|_{L^p(\Omega)} \leq \delta.$$

i.e.,

$$\operatorname*{ess\,sup}_{\Omega} u \leq 1 \quad \text{provided that} \quad \|u^+\|_{L^p(\Omega)} \leq \delta.$$

Replacing u by $-u$ in arguments above, we also find that

$$\operatorname*{ess\,sup}_{\Omega}(-u) \leq 1 \quad \text{provided that} \quad \|u^+\|_{L^p(\Omega)} \leq \delta.$$

Combining the above result, we conclude that

$$\|u\|_{L^\infty(\Omega)} \leq 1 \quad \text{provided that} \quad \|u^+\|_{L^p(\Omega)} \leq \delta,$$

as wanted. □

Remark 8.3.7. Owing to Proposition 8.3.6 and assumption (F_1), $f(u(\cdot)) \in L^\infty(\Omega)$. Therefore, all the regularities established in [73] can be carried out to solutions of (8.1.1).

Appendix

In this last part of the thesis, we obtain some asymptotic results for $(-\Delta)^s_\Omega u$ as $s \to 0^+$. We recall that for all $s \in (0,1)$ and $u \in C^2(\overline{\Omega})$,

$$(-\Delta)^s_\Omega u(x) = c_{N,s} P.V. \int_\Omega \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \quad x \in \Omega, \tag{8.3.23}$$

where $c_{N,s}$ is the normalized constant define in (1.1.6). Our first result is the following.

Proposition 8.3.8. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be an unbounded domain and $u \in C^\beta_c(\Omega)$ for some $\beta > 0$. Then*

$$(-\Delta)^s_\Omega u \to 0 \quad as \quad s \to 0^+$$

provided that

$$\int_{\Omega \setminus B_R} |z|^{-N}\, dz \to 0 \quad as \quad R \to \infty. \tag{8.3.24}$$

Proof. Let $R > 4$ be such that supp$u \subseteq B_{\frac{R}{4}}$. Since we are interested in the asymptotic behavior as $s \to 0$, then we can assume without losing generality that $0 < s < \min\{\frac{\beta}{3}, \frac{1}{2}\}$. From this, the integral in (8.3.23) can be understood in Lebesgue sense. We can therefore drop the "P.V." to obtain

$$\begin{aligned}
|(-\Delta)^s_\Omega u(x)| &\leq c_{N,s} \int_\Omega \frac{|u(x)-u(x+z)|}{|z|^{N+2s}}\, dz \\
&= c_{N,s}\Big(\int_{B_R} \frac{|u(x)-u(x+z)|}{|z|^{N+2s}}\, dz + \int_{\Omega\setminus B_R} \frac{|u(x)-u(x+z)|}{|z|^{N+2s}}\, dz \Big) \\
&= J^1_R(s,x) + J^2_R(s,x),
\end{aligned}$$

where

$$J^1_R(s,x) = c_{N,s} \int_{B_R} \frac{|u(x)-u(x+z)|}{|z|^{N+2s}}\, dz \quad \text{and} \quad J^2_R(s,x) = c_{N,s} \int_{\Omega\setminus B_R} \frac{|u(x)-u(x+z)|}{|z|^{N+2s}}\, dz.$$

Estimate of $J^1_R(s,x)$.

$$J^1_R(s,x) = c_{N,s} \int_{B_R} \frac{|u(x)-u(x+z)|}{|z|^{N+2s}}\, dz \leq \frac{c_{N,s}\|u\|_{C^\beta(\Omega)}|\mathbb{S}^{N-1}|}{\beta - 2s} R^{\beta-2s}. \tag{8.3.25}$$

Estimate of $J_R^2(s,x)$.
• If $|x| \le \frac{R}{2}$ then $|x+z| \ge |z| - |x| \ge \frac{R}{2}$ whenever $z \in \Omega \setminus B_R$. This shows that $x+z \notin \operatorname{supp} u$ and therefore $u(x+z) = 0$. Thus

$$J_R^2(s,x) = c_{N,s} \int_{\Omega \setminus B_R} \frac{|u(x)|}{|z|^{N+2s}}\, dz \le c_{N,s} \|u\|_{L^\infty(\Omega)} \int_{\Omega \setminus B_R} |z|^{-N}\, dz. \tag{8.3.26}$$

• If $|x| \ge \frac{R}{2}$ then $u(x) = 0$ since in this case $x \notin \operatorname{supp} u$. Consequently,

$$J_R^2(s,x) = c_{N,s} \int_{\Omega \setminus B_R} \frac{|u(x+z)|}{|z|^{N+2s}}\, dz \le c_{N,s} \sup_{x \in \Omega} \|u(x+\cdot)\|_{L^\infty(\Omega \setminus B_R)} \int_{\Omega \setminus B_R} |z|^{-N}\, dz. \tag{8.3.27}$$

Combining (8.3.26) and (8.3.27), we find that

$$J_R^2(s,x) \le c_{N,s} \max\Big\{ \|u\|_{L^\infty(\Omega)}, \sup_{x\in\Omega} \|u(x+\cdot)\|_{L^\infty(\Omega \setminus B_R)} \Big\} \int_{\Omega \setminus B_R} |z|^{-N}\, dz. \tag{8.3.28}$$

Now recalling (1.1.7) we get from (8.3.25) and (8.3.28) that

$$(-\Delta)^s_\Omega u \to 0 \quad \text{in} \quad \Omega \quad \text{as} \quad s \to 0^+$$

provided that (8.3.24) holds true. We notice also that this limit holds uniformly. □

Remark 8.3.9. In the case when Ω is bounded, we also find that for $u \in C^\beta(\overline{\Omega})$ for some $\beta \in (0,1)$,

$$(-\Delta)^s_\Omega u \to 0 \qquad \text{uniformly as} \ \ s \to 0^+. \tag{8.3.29}$$

Indeed, assuming without loss of generality that $s \in (0, \frac{\beta}{2})$, a simple calculation yields

$$\|(-\Delta)^s_\Omega u\|_{L^\infty(\Omega)} \le \frac{|\mathbb{S}^{N-1}| \|u\|_{C^\beta(\overline{\Omega})} \operatorname{diam}(\Omega)^{\beta - 2s}}{\beta - 2s} c_{N,s} \tag{8.3.30}$$

where $\operatorname{diam}(\Omega) = \sup\{|x-y| : x, y \in \Omega\}$ is the diameter of Ω. Now, recalling (1.1.7), we obtain (8.3.29) from (8.3.30) by letting $s \to 0^+$.

In what follows, we turn our attention to the special case $\Omega \equiv \mathbb{R}^N_+ = \{x = (x', x_N) \in \mathbb{R}^N : x_N > 0\}$. First, from the definition of $c_{N,s}$ (see (1.1.6)), we have

$$t_N(s) := \frac{c_{N,s}}{s} \to t_N(0) := c_N = \pi^{-\frac{N}{2}} \Gamma(\frac{N}{2}) = \frac{2}{|\mathbb{S}^{N-1}|} \quad \text{as} \quad s \to 0^+. \tag{8.3.31}$$

Also,

$$\begin{aligned} t'_N(0) = \partial_s|_{s=0} t_N(s) &= \pi^{-\frac{N}{2}} \Big((2\log 2 - \gamma)\Gamma(\frac{N}{2}) + \Gamma'(\frac{N}{2}) \Big) \\ &= c_N \Big(2\log 2 - \gamma + \frac{\Gamma'(\frac{N}{2})}{\Gamma(\frac{N}{2})} \Big) = c_N \rho_N, \end{aligned}$$

where $\rho_N := 2\log 2 - \gamma + \Psi(\frac{N}{2})$ and $\gamma = -\Gamma'(1)$ is the Euler Mascheroni constant. Here, $\Psi = \frac{\Gamma'}{\Gamma}$ is the Digamma function.

Our next result analyzes the asymptotic behavior of $(-\Delta)^s_{\mathbb{R}^N_+}$ as $s \to 0$ among functions in $C^\beta_c(\mathbb{R}^N_+)$. It reads as follows.

Proposition 8.3.10. *Let $\beta > 0$. For any $u \in C_c^{\beta}(\mathbb{R}^N_+)$, the following holds*

$$\lim_{s\to 0^+} (-\Delta)^s_{\mathbb{R}^N_+} u = \frac{1}{2}u \quad in \quad \mathbb{R}^N_+.$$

Equivalently,

$$(-\Delta)^s_{\mathbb{R}^N_+} \to \frac{id}{2} \quad in \quad \mathbb{R}^N_+ \quad as \quad s \to 0^+$$

among functions in $C_c^{\beta}(\mathbb{R}^N_+)$.

Proof. Fix $x \in \mathbb{R}^N_+$ and let $R_0 > 0$ such that $\operatorname{supp} u \subseteq B^+_{R_0}$. We set $R = R_0 + |x| + 1$. For $0 < s < \min\{\frac{\beta}{2}, \frac{1}{2}\}$, the integral (8.3.23) is well defined in Lebesgue sense. Now, for all $x \in \mathbb{R}^N_+$,

$$\int_{\mathbb{R}^N_+} \frac{u(x) - u(x+z)}{|z|^{N+2s}}\, dz = \int_{B^+_R} \frac{u(x) - u(x+z)}{|z|^{N+2s}}\, dz + \int_{\mathbb{R}^N_+ \setminus B^+_R} \frac{u(x) - u(x+z)}{|z|^{N+2s}}\, dz.$$

Next, a simple calculation shows that

$$\begin{aligned} \Big| \int_{B^+_R} \frac{u(x) - u(x+z)}{|z|^{N+2s}}\, dz \Big| &\le \|u\|_{C^{\beta}(\mathbb{R}^N_+)} \int_{B^+_R} |z|^{\beta - N - 2s}\, dz \\ &= \frac{1}{2}\|u\|_{C^{\beta}(\mathbb{R}^N_+)} \int_{B_R} |z|^{\beta - N - 2s}\, dz \\ &= \frac{|\mathbb{S}^{N-1}| \|u\|_{C^{\beta}(\mathbb{R}^N_+)}}{2(\beta - 2s)} R^{\beta - 2s}. \end{aligned} \tag{8.3.32}$$

Hence, from (1.1.7) and (8.3.32) we get that

$$c_{N,s} \int_{B^+_R} \frac{u(x) - u(x+z)}{|z|^{N+2s}}\, dz \to 0 \quad \text{as} \quad s \to 0^+. \tag{8.3.33}$$

Moreover, for $|z| \ge R$, we have $|x+z| \ge |z| - |x| \ge R - |x| = R_0 + 1$. This implies that $x + z \notin \operatorname{supp} u$ and therefore $u(x+z) = 0$. Thus,

$$\begin{aligned} \int_{\mathbb{R}^N_+ \setminus B^+_R} \frac{u(x) - u(x+z)}{|z|^{N+2s}}\, dz &= u(x) \int_{\mathbb{R}^N_+ \setminus B^+_R} |z|^{-N-2s}\, dz \\ &= \frac{1}{2} u(x) \int_{\mathbb{R}^N \setminus B_R} |z|^{-N-2s}\, dz = \frac{|\mathbb{S}^{N-1}| R^{-2s}}{4s} u(x). \end{aligned}$$

From the above equality, we get thanks to (8.3.31), that

$$c_{N,s} \int_{\mathbb{R}^N_+ \setminus B^+_R} \frac{u(x) - u(x+z)}{|z|^{N+2s}}\, dz \to \frac{1}{2} u(x) \quad \text{as} \quad s \to 0^+. \tag{8.3.34}$$

It follows from (8.3.33) and (8.3.34) that

$$(-\Delta)^s_{\mathbb{R}^N_+} u(x) \to \frac{1}{2} u(x) \quad \text{as} \quad s \to 0^+.$$

i.e.,

$$\lim_{s\to 0^+} (-\Delta)^s_{\mathbb{R}^N_+} = \frac{id}{2}$$

among functions in $C_c^{\beta}(\mathbb{R}^N_+)$, for $\beta > 0$. This concludes the proof. $\square$

As a consequence of Proposition 8.3.10, we derive an operator denoted by $L_\Delta^{\mathbb{R}^N_+}$ which appears as the derivative $\partial_s|_{s=0}(-\Delta)^s_{\mathbb{R}^N_+}$ at $s=0$. This operator is called *regional logarithmic Laplacian* in the half-space.

Proposition 8.3.11. *Let $\beta > 0$ and $1 < p \le \infty$. For any $u \in C_c^\beta(\mathbb{R}^N_+)$ we have*

$$\frac{(-\Delta)^s_{\mathbb{R}^N_+}u - \frac{1}{2}u}{s} \to L_\Delta^{\mathbb{R}^N_+}u \quad in \quad L^p(\mathbb{R}^N_+) \quad as \quad s \to 0^+,$$

where

$$[L_\Delta^{\mathbb{R}^N_+}u](x) := c_N \int_{\mathbb{R}^N_+} \frac{u(x)\mathbb{1}_{B_1^+(x)}(y) - u(y)}{|x-y|^N}\,dy + \frac{\rho_N}{2}u(x), \quad x \in \mathbb{R}^N_+.$$

Proof. As above, for $0 < s < \min\{\frac{\beta}{2}, \frac{1}{2}\}$ the integral in definition (8.3.23) is not singular near the origin. Therefore the "P.V." can be dropped. Now, let $R > 4$ be such that $\mathrm{supp} u \subseteq B^+_{\frac{R}{4}}$. For $x \in \mathbb{R}^N_+$, we have

$$\begin{aligned}
(-\Delta)^s_{\mathbb{R}^N_+}u(x) &= c_{N,s}\int_{\mathbb{R}^N_+} \frac{u(x)-u(x+z)}{|z|^{N+2s}}\,dz \\
&= c_{N,s}\int_{B_R^+} \frac{u(x)-u(x+z)}{|z|^{N+2s}}\,dz + c_{N,s}\int_{\mathbb{R}^N_+\setminus B_R^+} \frac{u(x)-u(x+z)}{|z|^{N+2s}}\,dz \\
&= c_{N,s}\Big(\int_{B_R^+} \frac{u(x)-u(x+z)}{|z|^{N+2s}}\,dz - \int_{\mathbb{R}^N_+\setminus B_R^+} \frac{u(x+z)}{|z|^{N+2s}}\,dz\Big) \\
&\qquad + c_{N,s}u(x)\int_{\mathbb{R}^N_+\setminus B_R^+} |z|^{-N-2s}\,dz.
\end{aligned}$$

i.e.,

$$(-\Delta)^s_{\mathbb{R}^N_+}u(x) = U_R(s,x) + u(x)V_R(s), \tag{8.3.35}$$

where

$$U_R(s,x) = c_{N,s}\Big(\int_{B_R^+} \frac{u(x)-u(x+z)}{|z|^{N+2s}}\,dz - \int_{\mathbb{R}^N_+\setminus B_R^+} \frac{u(x+z)}{|z|^{N+2s}}\,dz\Big) \tag{8.3.36}$$

and

$$V_R(s) = c_{N,s}\int_{\mathbb{R}^N_+\setminus B_R^+} |z|^{-N-2s}\,dz = \frac{c_{N,s}}{2}\int_{\mathbb{R}^N\setminus B_R} |z|^{-N-2s}\,dz = \frac{c_{N,s}|\mathbb{S}^{N-1}|}{4s}R^{-2s}.$$

We wish now to estimate $U_R(s,x)$ $\forall\, x \in \mathbb{R}^N_+$. For that we distinguish two cases: $|x| \ge \frac{R}{2}$ and $|x| \le \frac{R}{2}$.

• If $|x| \ge \frac{R}{2}$ then $u(x) = 0$ (since $x \notin \mathrm{supp} u$) and $|z| = |x+z-x| \ge |x| - |x+z| \ge \frac{R}{4} > 1$ whenever $x+z \in \mathrm{supp} u$. Therefore,

$$\begin{aligned}
|U_R(s,x)| &= c_{N,s}\Big|\int_{B_R^+} \frac{u(x+z)}{|z|^{N+2s}}\,dz + \int_{\mathbb{R}^N_+\setminus B_R^+} \frac{u(x+z)}{|z|^{N+2s}}\,dz\Big| \\
&= c_{N,s}\Big|\int_{\mathbb{R}^N_+} \frac{u(x+z)}{|z|^{N+2s}}\,dz\Big| \le c_{N,s}\int_{\mathbb{R}^N_+} \frac{|u(x+z)|}{|z|^{N+2s}}\,dz
\end{aligned}$$

$$\leq c_{N,s}\int_{\mathbb{R}^N_+}\frac{|u(x+z)|}{|z|^N}\,dz.$$

Using the change of variables $y=x+z$ so that $|y|\leq\frac{R}{4}$ and $|z|\geq|x|-|y|\geq|x|-\frac{R}{4}\geq\frac{|x|}{2}$ whenever $x+z\in\mathrm{supp}u$, we obtain

$$|U_R(s,x)|\leq c_{N,s}\int_{\mathbb{R}^N_+}\frac{|u(y)|}{(\frac{|x|}{2})^N}\,dy=2^Nc_{N,s}|x|^{-N}\|u\|_{L^1(\mathbb{R}^N_+)}.$$

Consequently,

$$\begin{aligned}\int_{\mathbb{R}^N_+\setminus B^+_{\frac{R}{2}}}|U_R(s,x)|^p\,dx&\leq 2^{pN}c^p_{N,s}\|u\|^p_{L^1(\mathbb{R}^N_+)}\int_{\mathbb{R}^N_+\setminus B^+_{\frac{R}{2}}}|x|^{-pN}\,dx\\&=2^{pN-1}c^p_{N,s}\|u\|^p_{L^1(\mathbb{R}^N_+)}\int_{\mathbb{R}^N\setminus B_{\frac{R}{2}}}|x|^{-pN}\,dx\\&=\frac{2^{2pN-N-1}c^p_{N,s}|\mathbb{S}^{N-1}|\|u\|^p_{L^1(\mathbb{R}^N_+)}}{N(p-1)}R^{N(1-p)}.\end{aligned}$$

i.e.,

$$\|U_R(s,\cdot)\|_{L^p(\mathbb{R}^N_+\setminus B^+_{\frac{R}{2}})}\leq 2^{2N-\frac{N+1}{p}}c_{N,s}\|u\|_{L^1(\mathbb{R}^N_+)}\Big(\frac{|\mathbb{S}^{N-1}|}{N(p-1)}\Big)^{\frac{1}{p}}R^{\frac{N}{p}-N}\tag{8.3.37}$$

for every $1<p<\infty$ and

$$\|U_R(s,\cdot)\|_{L^\infty(\mathbb{R}^N_+\setminus B^+_{\frac{R}{2}})}\leq 4^Nc_{N,s}\|u\|_{L^1(\mathbb{R}^N_+)}R^{-N}.$$

From (8.3.37) we see that

$$\|U_R(s,\cdot)\|_{L^p(\mathbb{R}^N_+\setminus B^+_{\frac{R}{2}})}\leq s\vartheta_N(s)R^{\frac{N}{p}-N},\tag{8.3.38}$$

where $\vartheta_N(s):=2^{2N-\frac{N+1}{p}}t_N(s)\|u\|^p_{L^1(\mathbb{R}^N_+)}\Big(\frac{|\mathbb{S}^{N-1}|}{N(p-1)}\Big)^{\frac{1}{p}}$. We observe that $\vartheta_N(s)$ depends uniformly on s.

• If $|x|\leq\frac{R}{2}$ then $|x+z|\geq|z|-|x|\geq\frac{R}{2}$ whenever $z\in\mathbb{R}^N_+\setminus B^+_R$. This implies that $x+z\notin\mathrm{supp}u$ and therefore, the second integral in (8.3.36) vanishes. Moreover, since $u\in C^\beta_c(\mathbb{R}^N_+)$, we obtain via Dominated Convergenge Theorem that

$$\begin{aligned}\frac{U_R(s,x)}{s}&=\frac{c_{N,s}}{s}\int_{B^+_R}\frac{u(x)-u(x+z)}{|z|^{N+2s}}\,dz\\&\to\widetilde{U}_R(x):=c_N\int_{B^+_R}\frac{u(x)-u(x+z)}{|z|^N}\,dz\quad\text{as}\quad s\to 0^+.\end{aligned}\tag{8.3.39}$$

The above convergence holds uniformly in $B^+_{\frac{R}{2}}$.

Now, since $V_R(s)=\frac{|\mathbb{S}^{N-1}|}{4}t_N(s)R^{-2s}$ with $V_R(0)=\frac{|\mathbb{S}^{N-1}|}{4}t_N(0)=\frac{1}{2}$, we have

$$\lim_{s\to 0^+}\frac{V_R(s)-\frac{1}{2}}{s}=\partial_s|_{s=0}V_R(s)=\frac{|\mathbb{S}^{N-1}|}{4}\Big(t'_N(0)-2t_N(0)\log R\Big)$$

$$= \frac{1}{2}(\rho_N - 2\log R) =: \tau_R.$$

Therefore, for every $u \in C_c^\beta(\mathbb{R}^N_+)$ we find that

$$\Big\| \frac{V_R(s)u - \frac{1}{2}u}{s} - \tau_R u \Big\|_{L^p(\mathbb{R}^N_+)} \to 0 \quad \text{as} \quad s \to 0^+ \tag{8.3.40}$$

for $1 < p \leq \infty$.

On the other hand,

$$\begin{aligned}
\widetilde{U}_R(x) + \tau_R u(x) &= c_N \int_{B_R^+} \frac{u(x) - u(x+z)}{|z|^N}\ dz + \frac{1}{2}(\rho_N - 2\log R)u(x) \\
&= c_N \Big(\int_{B_R^+} \frac{u(x) - u(x+z)}{|z|^N}\ dz - \frac{1}{c_N} u(x)\log R \Big) + \frac{\rho_N}{2} u(x) \\
&= c_N \Big(\int_{B_R^+} \frac{u(x) - u(x+z)}{|z|^N}\ dz - \frac{|\mathbb{S}^{N-1}|}{2} u(x)\log R \Big) + \frac{\rho_N}{2} u(x) \\
&= c_N \Big(\int_{B_R^+} \frac{u(x) - u(x+z)}{|z|^N}\ dz - \frac{1}{2} u(x) \int_{B_R \setminus B_1} \frac{1}{|z|^N}\ dz \Big) + \frac{\rho_N}{2} u(x) \\
&= c_N \Big(\int_{B_R^+} \frac{u(x) - u(x+z)}{|z|^N}\ dz - u(x) \int_{B_R^+ \setminus B_1^+} \frac{1}{|z|^N}\ dz \Big) + \frac{\rho_N}{2} u(x).
\end{aligned}$$

Since also

$$\begin{aligned}
&\int_{B_R^+} \frac{u(x) - u(x+z)}{|z|^N}\ dz \\
&= \int_{B_1^+} \frac{u(x) - u(x+z)}{|z|^N}\ dz + u(x) \int_{B_R^+ \setminus B_1^+} \frac{1}{|z|^N}\ dz - \int_{\mathcal{B}_R^+ \setminus B_1^+} \frac{u(x+z)}{|z|^N}\ dz,
\end{aligned}$$

then

$$\begin{aligned}
&\widetilde{U}_R(x) + \tau_R u(x) \\
&= c_N \Big(\int_{B_1^+} \frac{u(x) - u(x+z)}{|z|^N}\ dz - \int_{\mathcal{B}_R^+ \setminus B_1^+} \frac{u(x+z)}{|z|^N}\ dz \Big) + \frac{\rho_N}{2} u(x) \\
&= c_N \int_{B_R^+} \frac{u(x)\mathbb{1}_{B_1^+(x)}(z) - u(x+z)}{|z|^N}\ dz + \frac{\rho_N}{2} u(x) \\
&= c_N \Big(\int_{\mathbb{R}^N_+} \frac{u(x)\mathbb{1}_{B_1^+(x)}(z) - u(x+z)}{|z|^N}\ dz - \int_{\mathbb{R}^N_+ \setminus B_R^+} \frac{u(x)\mathbb{1}_{B_1^+(x)}(z) - u(x+z)}{|z|^N}\ dz \Big) + \frac{\rho_N}{2} u(x) \\
&= c_N \Big(\int_{\mathbb{R}^N_+} \frac{u(x)\mathbb{1}_{B_1^+(x)}(z) - u(x+z)}{|z|^N}\ dz + \int_{\mathbb{R}^N_+ \setminus B_R^+} \frac{u(x+z)}{|z|^N}\ dz \Big) + \frac{\rho_N}{2} u(x).
\end{aligned}$$

This implies that

$$\widetilde{U}_R(x) + \tau_R u(x) = [L_\Delta^{\mathbb{R}^N_+} u](x) + f_R(x), \tag{8.3.41}$$

where

$$[L_\Delta^{\mathbb{R}^N_+} u](x) = c_N \int_{\mathbb{R}^N_+} \frac{u(x)\mathbb{1}_{B_1^+(x)}(z) - u(x+z)}{|z|^N}\, dz + \frac{\rho_N}{2} u(x)$$

and

$$f_R(x) = c_N \int_{\mathbb{R}^N_+ \setminus B_R^+} \frac{u(x+z)}{|z|^N}\, dz, \qquad x \in \mathbb{R}^N_+.$$

For $|x| \le \frac{R}{2}$ we have $u(x+z) = 0$ since $x+z \notin \operatorname{supp} u$ whenever $z \in \mathbb{R}^N_+ \setminus B_R^+$. Therefore, $f_R(x) = 0$. Likewise for $|x| \ge \frac{R}{2}$, $|z| = |x+z-x| \ge |x| - |x+z| \ge \frac{|x|}{2}$ whenever $x+z \in \operatorname{supp} u$. Thus,

$$|f_R(x)| \le 2^N c_N \|u\|_{L^1(\mathbb{R}^N_+)} |x|^{-N},$$

which implies that

$$\|f_R\|_{L^p(\mathbb{R}^N_+ \setminus B_R^+)} \le \nu_N R^{\frac{N}{p} - N}, \tag{8.3.42}$$

where

$$\nu_N = 2^{2N - \frac{N+1}{p}} c_N \|u\|_{L^1(\mathbb{R}^N_+)} \Big(\frac{|\mathbb{S}^{N-1}|}{N(p-1)} \Big)^{\frac{1}{p}}.$$

Finally, for all $x \in \mathbb{R}^N_+$, we have with (8.3.35) and (8.3.41) that

$$\begin{aligned}
&\frac{(-\Delta)^s_{\mathbb{R}^N_+} u(x) - \frac{1}{2}u(x)}{s} - [L_\Delta^{\mathbb{R}^N_+} u](x) \\
&= \frac{U_R(s,x) + u(x)V_R(s) - \frac{1}{2}u(x)}{s} - \Big(\widetilde{U}_R(x) + \tau_R u(x) - f_R(x) \Big) \\
&= \frac{U_R(s,x)}{s} - \widetilde{U}_R(x) + \frac{V_R(s)u(x) - \frac{1}{2}u(x)}{s} - \tau_R u(x) + f_R(x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\Big\| \frac{(-\Delta)^s_{\mathbb{R}^N_+} u - \frac{1}{2}u}{s} - L_\Delta^{\mathbb{R}^N_+} u \Big\|_{L^p(\mathbb{R}^N_+)} \\
&\le \Big\| \frac{U_R(s,\cdot)}{s} - \widetilde{U}_R \Big\|_{L^p(\mathbb{R}^N_+)} + \Big\| \frac{V_R(s)u - \frac{1}{2}u}{s} - \tau_R u \Big\|_{L^p(\mathbb{R}^N_+)} + \|f_R\|_{L^p(\mathbb{R}^N_+)}
\end{aligned}$$

From (8.3.38), (8.3.39), (8.3.40) and (8.3.42) we find that

$$\limsup_{s \to 0^+} \Big\| \frac{(-\Delta)^s_{\mathbb{R}^N_+} u - \frac{1}{2}u}{s} - L_\Delta^{\mathbb{R}^N_+} u \Big\|_{L^p(\mathbb{R}^N_+)} \le (\vartheta_N + \nu_N) R^{\frac{N}{p} - N} \quad \forall R > 0,\ 1 < p \le \infty.$$

Hence

$$\Big\| \frac{(-\Delta)^s_{\mathbb{R}^N_+} u - \frac{1}{2}u}{s} - L_\Delta^{\mathbb{R}^N_+} u \Big\|_{L^p(\mathbb{R}^N_+)} \to 0 \quad \text{as} \quad s \to 0^+.$$

i.e.,

$$\frac{(-\Delta)^s_{\mathbb{R}^N_+} u - \frac{1}{2}u}{s} \to L_\Delta^{\mathbb{R}^N_+} u \quad \text{in} \quad L^p(\mathbb{R}^N_+) \quad \text{as} \quad s \to 0^+,$$

for all $1 < p \le \infty$. The proof is therefore finished. $\square$

Summary

In this thesis, we study elliptic problems driven by two particular nonlocal operators: the fractional Laplacian $(-\Delta)^s$ and the regional fractional Laplacian $(-\Delta)^s_\Omega$, where Ω is an open set in $\mathbb{R}^N$. The thesis contains results of the following papers.

Paper 1 gives an estimate of the Morse index of any radially sign changing bounded weak solution to the Dirichlet problem

$$(-\Delta)^s u = f(u) \quad \text{in} \ \ \mathcal{B}, \quad u = 0 \quad \text{on} \ \ \mathbb{R}^N \setminus \mathcal{B} \tag{8.3.43}$$

where $\mathcal{B}$ is the unit ball in $\mathbb{R}^N$ and $f \in C^1(\mathbb{R})$. More precisely, when $s \in (\frac{1}{2}, 1)$, we show that any bounded weak radial sign-changing solution of (8.3.43) has Morse index greater than or equal to $N+1$. Moreover, in the case when $s \in (0, \frac{1}{2}]$ the above conclusion holds whenever the nonlinearity f satisfies some additional assumptions. Our proof is based on constructing test functions by means of partial derivatives $\partial_j u$ in order to estimate the Morse index. The nonlocality of the problem and the lack of boundary regularity are the main difficulties in the construction. Main tools for proving the result use a gradient estimate due to Fall and Jarohs, and boundary regularity result due to Grubb.

As a byproduct of our aforementioned main result, we deduce that every second Dirichlet eigenfunction of $(-\Delta)^s$ in $\mathcal{B}$ is antisymmetric. We notice that in the literature, this result was partially known only in the cases $N \leq 3$, $s \in (0,1)$ and $4 \leq N \leq 9$, $s = \frac{1}{2}$, and was fully conjectured by Bañuelos and Kulczycki. Our result therefore resolves this conjecture.

Paper 2 analyzes small order asymptotics for the regional fractional Laplacian $(-\Delta)^s_\Omega$ with being $\Omega \subset \mathbb{R}^N$ an open bounded Lipschitz set. Precisely, we study asymptotic behavior for eigenvalues and eigenfunctions of $(-\Delta)^s_\Omega$ as $s \to 0^+$. In our analysis, we first introduce the so-called *regional logarithmic Laplacian* L^Ω_Δ which arises as a formal derivative of $(-\Delta)^s_\Omega$ at $s = 0$. Moreover, it naturally appears in the description of asymptotic behavior of eigenvalues and eigenfunctions of $(-\Delta)^s_\Omega$ for s close to 0. Specifically, we prove that $\partial_s|_{s=0}\mu^\Omega_{n,s} = \mu^\Omega_{n,0}$, where $\mu^\Omega_{n,s}$ resp. $\mu^\Omega_{n,0}$ are the n-th eigenvalues of $(-\Delta)^s_\Omega$, L^Ω_Δ, respectively. Moreover, if for some sequence $s_k \to 0^+$, $\{\xi_{n,s_k}\}_k$ is a sequence of L^2-normalized eigenfunctions of $(-\Delta)^{s_k}_\Omega$ corresponding to μ^Ω_{n,s_k}, we also show that after passing to a subsequence, $\xi_{n,s_k} \to \xi_n$ uniformly in $\overline{\Omega}$ as $k \to \infty$, where ξ_n is an L^2-normalized eigenfunction of L^Ω_Δ corresponding to $\mu^\Omega_{n,0}$. To prove this convergence result, uniform boundedness and regularity estimates in s are required.

Paper 3 is devoted to analyzing the s-regularity of the unique weak solution of the Poisson problem

$$(-\Delta)^s_\Omega u_s = f \quad \text{in} \ \ \Omega \tag{8.3.44}$$

where $f \in L^{\infty}(\Omega)$ with being $\Omega \subset \mathbb{R}^N$ a $C^{1,1}$ bounded domain. It is worth recalling that a necessary condition for the unique existence of a solution of problem (8.3.44) is $\int_\Omega f\ dx = 0$. In our main result of this paper, we show that the solution map $(0,1) \to L^2(\Omega),\ s \mapsto u_s$ is continously differentiable. Our proof is based on estimating the difference quotient $v_h := \frac{u_{s+h}-u_s}{h}$ uniformly in h with respect to the $H^s(\Omega)$-norm. For this purposes, higher Sobolev regularity of u_s of the order $s+\varepsilon$ is needed. In (8.3.44), when f is replaced by the s-dependent function $f \equiv \mu_s u_s$ (with $\mu_s := \mu_{1,s}^{\Omega}$ the first nontrivial eigenvalue of $(-\Delta)^s_\Omega$), we also obtain a one-sided differentiability of the map $(0,1) \to (0,\infty),\ s \mapsto \mu_s$. The reason for which one-sided differentiability is considered in this case is that, in general, the first nontrivial eigenvalue of $(-\Delta)^s_\Omega$ is not simple.

Paper 4 is concerned with the existence of positive minimizers to the best Sobolev constant

$$S_{N,s}(\Omega) = \inf_{\substack{u\in H^s_0(\Omega)\\ u\not\equiv 0}} \frac{\frac{c_{N,s}}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\ dxdy}{\left(\int_\Omega |u|^{2^*_s}\ dx\right)^{2/2^*_s}}, \tag{8.3.45}$$

where $s \in (\frac{1}{2},1)$ and $2^*_s := \frac{2N}{N-2s}$ is the so-called fractional critical Sobolev exponent and Ω a C^1 domain of $\mathbb{R}^N$. Due to the lack of compactness of the Sobolev embedding $H^s_0(\Omega) \hookrightarrow L^{2^*_s}(\Omega)$, standard variational methods do not apply to this minimization problem, which has to be analysed with the so-called *missing mass method*. As noted in recent work of Frank, Jin and Xiong, the strict inequality $S_{N,s}(\Omega) < S_{N,s}(\mathbb{R}^N_+)$ plays a crucial role in this context. The goal of this paper is twofold: First, assuming $N \geq 2$, we show that, for any underlying C^1-domain Ω, the critical Sobolev constant $S_{N,s}(\Omega)$ is attained if $s \in (\frac{1}{2},1)$ is close to $\frac{1}{2}$. Second, when $N \geq 4s$ and the underlying domain is the unit ball $\mathcal{B}$ of $\mathbb{R}^N$, we prove that the infimum

$$S_{N,s,rad}(\mathcal{B},h) = \inf_{\substack{u\in H^s_{0,rad}(\mathcal{B})\\ u\not\equiv 0}} \frac{\frac{c_{N,s}}{2}\int_{\mathcal{B}}\int_{\mathcal{B}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\ dxdy + \int_{\mathcal{B}} hu^2\ dx}{\left(\int_{\mathcal{B}} |u|^{2^*_s}\ dx\right)^{2/2^*_s}} \tag{8.3.46}$$

is attained for every $s \in (\frac{1}{2},1)$. Here, $h \in L^{\infty}(\Omega)$ is a function with the property that $S_{N,s,rad}(\mathcal{B},h) > 0$. On the other hand, in low dimensions $2s < N < 4s$, we prove that the infimum $S_{N,s,rad}(\mathcal{B},h)$ is attained under the additional assumption that the *mass* associated to the Schrödinger operator $(-\Delta)^s + h$ on $\mathcal{B}$ at 0 is *strictly positive*. The sign of the mass is crucial for the validity the strict inequality $S_{N,s,rad}(\mathcal{B},h) < S_{N,s}(\mathbb{R}^N)$.

Paper 5 provides a Hopf lemma for the regional fractional Laplacian by analyzing the super-solutions to the equation

$$(-\Delta)^s_\Omega u = c(x)u \quad \text{in} \ \ \Omega, \tag{8.3.47}$$

where Ω is a bounded $C^{1,1}$ domain of $\mathbb{R}^N$ and $s \in (\frac{1}{2},1)$. Under some mild assumptions on the function c, we prove that if u is a pointwise or weak super-solution of (8.3.47), then the ratio $\frac{u(x)}{\delta_\Omega(x)^{2s-1}}$ is bounded below by a positive constant near the boundary $\partial\Omega$ of Ω. Here δ_Ω is the boundary distance function. The proof of this property is based on the construction of a suitable barrier for u. The major ingredient in the construction is the solution u_{tor} of the torsion problem for the regional fractional Laplacian

$$(-\Delta)^s_\Omega u_{tor} = 1 \quad \text{in} \ \ \Omega, \quad u_{tor} = 0 \quad \text{on} \ \ \partial\Omega. \tag{8.3.48}$$

By a result of Bonforte, Figalli and Vásquez, the function u_{tor} behaves like δ_Ω^{2s-1} close to the boundary.

Paper 6 establishes qualitative properties of positive solutions to the semi-linear Dirichlet problem

$$(-\Delta)^s_{\mathbb{R}^N_+} u = u^{2^*_s - 1}, \quad u > 0 \;\text{ in }\; \mathbb{R}^N_+, \quad u = 0 \;\text{ on }\; \partial\mathbb{R}^N_+ = \mathbb{R}^{N-1} \tag{8.3.49}$$

where $s \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, $N \geq 2$ and $\mathbb{R}^N_+ = \{x = (x', x_N) \in \mathbb{R}^N : x' \in \mathbb{R}^{N-1},\ x_N > 0\}$ is the upper half-space. Precisely, we prove that solutions of (8.3.49) are radially symmetric up to translation with respect to the horizontal variable x' and monotonic in the radial variable. For this, we adapt the celebrated *method of moving planes* to this setting.

Paper 7 studies the homogeneous Dirichlet problem

$$(-\Delta)^s_\Omega u = f(u) \quad \text{in } \;\Omega, \qquad u = 0 \quad \text{on } \;\partial\Omega \tag{8.3.50}$$

where $s \in (\frac{1}{2}, 1)$ and Ω is a bounded set of $\mathbb{R}^N$ with $N \geq 2$. Under some mild assumptions on the nonlinearity f including a subcritical growth assumption, we prove the existence of mountain pass solutions. Moreover, using the De Giorgi iteration method in a fractional setting, we obtain a priori L^∞-bounds of mountain pass solutions.

Zusammenfassung

In der vorliegenden Dissertation betrachten wir nichtlokale elliptische Probleme im Zusammenhang mit dem fraktionalen Laplace-Operator $(-\Delta)^s$ und dem regionalen fraktionalen Laplace-Operator $(-\Delta)^s_\Omega$, wobei Ω eine zugrunde liegende offene Teilmenge des $\mathbb{R}^N$ sei. Für hinreichend reguläre Funktionen u auf $\mathbb{R}^N$ bzw. auf Ω mit geeigneten Integrabilitätseigenschaften sind diese Operatoren punktweise durch die Hauptwertintegrale

$$(-\Delta)^s u(x) = c_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy, \quad x \in \mathbb{R}^N$$

bzw.

$$(-\Delta)^s_\Omega u(x) = c_{N,s} P.V. \int_{\Omega} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy, \quad x \in \Omega$$

gegeben, wobei die Normierungskonstante $c_{N,s} = s(1-s)\pi^{-N/2} 2^{2s} \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(2-s)}$ wie üblich so gewählt sei, dass das Fouriersymbol von $(-\Delta)^s$ durch $\xi \mapsto |\xi|^{2s}$ gegeben ist.

Die Dissertation beinhaltet Resultate folgender Forschungsarbeiten, welche die Kapitel 2-8 der Arbeit darstellen:

- **Artikel 1** (Chapter 2):
  M. M. Fall, P. A. Feulefack, R. Y. Temgoua, and T. Weth, *Morse index versus radial symmetry for fractional Dirichlet problems,* Advances in Mathematics 384 (2021): 107728, https://doi.org/10.1016/j.aim.2021.107728.

- **Artikel 2** (Chapter 3):
  R. Y. Temgoua and T. Weth, *The eigenvalue problem for the regional fractional Laplacian in the small order limit,* submitted to Potential Analysis,
  https://arxiv.org/abs/2112.08856v1 (2021).

- **Artikel 3** (Chapter 4):
  R. Y. Temgoua, *On the s-derivative of weak solutions of the Poisson problem for the regional fractional Laplacian,* arXiv preprint https://arxiv.org/abs/2112.09547v2 (2021).

- **Artikel 4** (Chapter 5):

  M. M. Fall and R. Y. Temgoua, *Existence results for nonlocal problems governed by the regional fractional Laplacian,* submitted to Nonlinear Differential Equations and Applications NoDEA, arXiv preprint https://arxiv.org/abs/2112.06272v3 (2021).

- **Artikel 5** (Chapter 6):

  N. Abatangelo, M. M. Fall, and R. Y. Temgoua, *A Hopf lemma for the regional fractional Laplacian,* submitted to Annali di Matematica Pura ed Applicata, arXiv preprint https://arxiv.org/abs/2112.09522v1 (2021).

- **Artikel 6** (Chapter 7):

  R. Y. Temgoua, *Qualitative properties of positive solutions for elliptic problem driven by the regional fractional Laplacian in the half-space,* preprint.

- **Artikel 7** (Chapter 8):
  R. Y. Temgoua, *Mountain pass solutions for the regional fractional Laplacian,* preprint.

Artikel 1 präsentiert eine Abschätzung des Morse-Index von radialen vorzeichenwechselnden schwachen Lösungen des Dirichletproblems

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \mathcal{B}, \\ u = 0 & \text{auf } \mathbb{R}^N \setminus \mathcal{B}, \end{cases} \tag{8.3.51}$$

wobei $\mathcal{B}$ die Einheitskugel im $\mathbb{R}^N$ and f eine C^1-Funktion auf $\mathbb{R}$ sei. Genauer zeigen wir im Fall $s \in (\frac{1}{2}, 1)$, dass der Morse-Index jeder radialen vorzeichenwechselnden schwachen Lösung von (8.3.51) durch $N+1$ nach unten beschränkt ist. Im Fall $s \in (0, \frac{1}{2}]$ ist diese Abschätzung zudem unter Zusatzvoraussetzungen an f erfüllt. Unser Beweis basiert auf einer Konstruktion von Testfunktionen mittels partieller Ableitungen. Die Nichtlokalität des Problems und die verminderte Randregularität sind die Hauptschwierigkeiten der Konstruktion. Wesentliche Werkzeuge im Beweis sind eine Gradientenabschätzung von Fall und Jarohs sowie ein Randregularitätsresultat von Grubb.

Als Folgerung der neuen Abschätzung für den Morse-Index radialsymmetrischer Lösungen von (8.3.51) erhalten wir durch Betrachtung des Spezialfalls $f(u) = \lambda u$, dass jede zweite Dirichleteigenfunktion von $(-\Delta)^s$ in $\mathcal{B}$ antisymmetrisch ist. Dieses Resultat bestätigt eine Vermutung von Bañuelos and Kulczycki und war bisher nur in den Spezialfällen $N \leq 3,\ s \in (0,1)$ sowie $4 \leq N \leq 9,\ s = \frac{1}{2}$ bekannt.

Artikel 2 analysiert die Asymptotik des regionalen Laplace-Operators $(-\Delta)^s_\Omega$ zu einer beschränkten, offenen Lipschitzmenge im Limes verschwindender Ordnung $s \to 0^+$. Genauer studieren wir asymptotische Eigenschaften der Eigenwerte und zugehöriger Eigenfunktionen. Unsere Analyse basiert auf der Einführung des sogenannten *regionalen logarithmischen Laplace-Operators* L^Ω_Δ, welcher als formale Ableitung von $(-\Delta)^s_\Omega$ in $s = 0$ gegeben ist. Der Operator taucht in natürlicher Weise in der Beschreibung des asymptotischen Verhaltens von Eigenwerten und Eigenfunktionen von $(-\Delta)^s_\Omega$ für s nahe bei 0 auf. Genauer beweisen wir in diesem Artikel die Identität

$$\partial_s\big|_{s=0}\mu_{n,s}^\Omega = \mu_{n,0}^\Omega,$$

wobei $\mu_{n,s}^\Omega$ und $\mu_{n,0}^\Omega$ die jeweils n-ten Eigenwerte von $(-\Delta)^s_\Omega$ bzw. L_Δ^Ω bezeichnen.

Ist zudem $s_k \to 0^+$ eine Folge und $\{\xi_{n,s_k}\}_k$ eine Folge L^2-normierter Eigenfunktionen von $(-\Delta)^{s_k}_\Omega$ zu den Eigenwerten μ_{n,s_k}^Ω, so zeigen wir gleichmäßige Konvergenz $\xi_{n,s_k} \to \xi_n$ in $\overline{\Omega}$ nach Übergang zu einer Teilfolge, wobei hier ξ_n eine L^2-normierte Eigenfunktion von L_Δ^Ω zum Eigenwert $\mu_{n,0}^\Omega$ sei. Der Beweis basiert auf uniformen Regularitätsabschätzungen in s.

Artikel 3 ist der Analyse der s-Regularität der eindeutigen schwachen Lösung des Poissonproblems

$$(-\Delta)^s_\Omega u_s = f \quad \text{in } \Omega \tag{8.3.52}$$

gewidmet, wobei hier $\Omega \subset \mathbb{R}^N$ ein $C^{1,1}$-Gebiet und $f \in L^\infty(\Omega)$ gegeben sei. Bekannterweise ist $\int_\Omega f\, dx = 0$ eine notwendige Bedingung für die eindeutige Lösbarkeit von (8.3.52). Im Hauptresultat unserer Arbeit zeigen wir, dass die Lösungsabbildung

$$(0,1) \to L^2(\Omega), \qquad s \mapsto u_s$$

stetig differenzierbar ist. Unser Beweis basiert auf uniformen Abschätzungen des Differenzenquotienten $v_h := \frac{u_{s+h}-u_s}{h}$ in der $H^s(\Omega)$-Norm. Dies erfordert insbesondere eine höhere Sobolevregularität der Ordnung $s+\varepsilon$ von u_s.

Für den Fall, dass f in (8.3.52) durch die s-abhängige Funktion $f \equiv \mu_s u_s$ mit dem ersten nichttrivialen Eigenwert $\mu_s := \mu_{1,s}^\Omega$ von $(-\Delta)^s_\Omega$ ersetzt wird, erhalten wir zumindest eine einseitige Differenzierbarkeit der Abbildung $(0,1) \to (0,\infty)$, $s \mapsto \mu_s$. Der Grund für die Betrachtung einseitiger Differenzierbarkeit an dieser Stelle ist die mögliche Vielfachheit des ersten nichttrivialen Eigenwerts von $(-\Delta)^s_\Omega$.

Artikel 4 befasst sich mit der Existenz positiver Minimierer für die beste Sobolev-Konstante

$$S_{N,s}(\Omega) = \inf_{\substack{u\in H_0^s(\Omega)\\ u\not\equiv 0}} \frac{\frac{c_{N,s}}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dxdy}{\left(\int_\Omega |u|^{2_s^*}\,dx\right)^{2/2_s^*}}, \tag{8.3.53}$$

wobei hier $\Omega \subset \mathbb{R}^N$ ein C^1-Gebiet, $s \in (\frac{1}{2},1)$ und $2_s^* := \frac{2N}{N-2s}$ der sogenannte fraktionale kritische Sobolevexponent seien. Aufgrund der fehlenden Kompaktheit der Sobolev-Einbettung $H_0^s(\Omega) \hookrightarrow L^{2_s^*}(\Omega)$ kann das Minimierungsproblem nicht mit Standardargumenten der Variationsrechnung untersucht werden und erfordert eine Analyse auf der Basis der sogenannten Missing-mass-Methode. Wie vor einigen Jahren von Frank, Jin und Xiong gezeigt wurde, ist hierfür die strikte Ungleichung $S_{N,s}(\Omega) < S_{N,s}(\mathbb{R}^N_+)$ entscheidend.

Die Hauptresultate des Artikels greifen offen gebliebene Fragen aus der Arbeit von Frank, Jin und Xiong auf. Zunächst zeigen wir im Fall $N \geq 2$ unabhängig vom zugrunde liegenden Gebiet Ω, dass für $s > \frac{1}{2}$ nahe bei $\frac{1}{2}$ die kritische Sobolev-Konstante $S_{N,s}(\Omega)$ angenommen wird.

Zweitens zeigen wir unter der Voraussetzung $N \geq 4s$ im Fall des Einheitsballs $\Omega = \mathcal{B} \subset \mathbb{R}^N$, dass das Infimum

$$S_{N,s,rad}(\mathcal{B}, h) = \inf_{\substack{u \in H^s_{0,rad}(\mathcal{B}) \\ u \not\equiv 0}} \frac{\frac{c_{N,s}}{2} \int_{\mathcal{B}} \int_{\mathcal{B}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} \ dxdy + \int_{\mathcal{B}} hu^2 \ dx}{\left(\int_{\mathcal{B}} |u|^{2^*_s} \ dx \right)^{2/2^*_s}} \tag{8.3.54}$$

für jedes $s \in (\frac{1}{2}, 1)$ angenommen wird. Hierbei ist $h \in L^\infty(\Omega)$ eine Funktion mit $S_{N,s,rad}(\mathcal{B}, h) > 0$. In niedrigen Dimensionen $2s < N < 4s$ beweisen wir die Existenz eines Minimierers zu $S_{N,s,rad}(\mathcal{B}, h)$ unter der Zusatzvoraussetzung, dass die *Masse* $\mathbf{k}(0)$ zum fraktionalen Schrödinger operator $(-\Delta)^s + h$ bei 0 *strikt positiv* ist. Diese Bedingung ist entscheidend für den Beweis der strikten Ungleichung $S_{N,s,rad}(\mathcal{B}, h) < S_{N,s}(\mathbb{R}^N)$.

Artikel 5 etabliert ein Hopf-Lemma für den regionalen fraktionalen Laplace-Operator auf der Basis einer Analyse von Superlösungen der Gleichung

$$(-\Delta)^s_\Omega u = c(x)u \quad \text{in} \ \ \Omega. \tag{8.3.55}$$

Hier sei $s \in (\frac{1}{2}, 1)$ und $\Omega \subset \mathbb{R}^N$ ein beschränktes $C^{1,1}$-Gebiet. Unter allgemeinen Voraussetzungen an die Funktion c zeigen wir für jede punktweise oder schwache Superlösung von (8.3.55) eine gleichmäßige untere Abschätzung für den Ausdruck $\frac{u(x)}{\delta_\Omega(x)^{2s-1}}$ in der Nähe des Randes $\partial\Omega$ von Ω, wobei δ_Ω die Randabstandsfunktion von Ω bezeichne. Der Beweis dieser Eigenschaft basiert auf der Konstruktion einer geeigneten Barrierefunktion für u. In dieser Konstruktion spielt die Lösung u_{tor} des Torsionsproblems

$$\begin{cases} (-\Delta)^s_\Omega u_{tor} = 1 & \text{in} \ \ \Omega, \\ \qquad u_{tor} = 0 & \text{auf} \ \ \partial\Omega. \end{cases} \tag{8.3.56}$$

eine zentrale Rolle. Gemäß eines Resultats von Bonforte, Figalli und Vásquez hat die Funktion u_{tor} dasselbe Randverhalten wie δ_Ω^{2s-1}.

Artikel 6 untersucht qualitative Eigenschaften positiver Lösungen des semilinearen Dirichletproblems

$$\begin{cases} (-\Delta)^s_{\mathbb{R}^N_+} u = u^{2^*_s - 1}, \ \ u > 0 & \text{in} \ \mathbb{R}^N_+, \\ \qquad u = 0 & \text{auf} \ \partial\mathbb{R}^N_+ = \mathbb{R}^{N-1} \end{cases} \tag{8.3.57}$$

wobei $s \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1), \ N \geq 2$ und

$$\mathbb{R}^N_+ = \{x = (x', x_N) \in \mathbb{R}^N : x' \in \mathbb{R}^{N-1}, \ x_N > 0\}$$

der obere Halbraum seien. Genauer zeigen wir, das Lösungen von (8.3.57) bzgl. der horizontalen Variable x' bis auf Translation radialsymmetrisch und monoton in der radialen Variable sind. Der Beweis basiert auf einer Adaptierung der klassischen Moving-Plane-Methode.

Artikel 7 ist der Analyse des homogenen Dirichletproblems

$$\begin{cases} (-\Delta)^s_\Omega u = f(u) & \text{in } \Omega, \\ u = 0 & \text{auf } \partial\Omega \end{cases} \tag{8.3.58}$$

für $s \in (\frac{1}{2}, 1)$ in einer beschränkten offenen Menge $\Omega \subset \mathbb{R}^N$, $N \geq 2$ gewidmet. Unter allgemeinen Voraussetzungen an f, welche u.a. eine subkritische Wachstumsbedingung beinhalten, beweisen wir die Existenz von Mountain-Pass-Lösungen. Ferner etablieren wir a priori L^∞-Schranken für Mountain-Pass-Lösungen unter Verwendung einer fraktionalen Variante der De Giorgi-Iterationsmethode.

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