Spectral characteristics of Dirichlet problems for nonlocal operators

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1 Overview

The thesis deals with the study of Dirichlet problems driven by nonlocal operators including those with small order. The content of the thesis is based on collection of research papers [39], [47], [45] and [46], that can be found in self-contained Chapters 2, 3, 4 and 5 and can be read independently. The main goal of the thesis is fourfold. First, we study a connection between qualitative properties of nodal solutions to a semilinear elliptic problem involving the fractional Laplacian $(-\Delta)^s$, $s \in (0,1)$ and their Morse index. Secondly, we study the small order asymptotics with respect to the parameter $s \to 0^+$ of the Dirichlet eigenvalues and corresponding eigenfunctions of the fractional Laplacian. Thirdly, we provide an alternative method to derive the singular integral corresponding to the operator with Fourier symbol $\log(1+|\xi|^2)$. In particular, we introduce tools to study variational problems involving this operator. Finally, we study a general class of nonlocal operators of small order. In particular, we present some auxiliary results corresponding to function spaces and study interior Sobolev type regularity of the associated Poisson problems. A short overview of each chapter is given below. In particular, let us explain the main idea in light of the following semilinear elliptic problem involving the fractional Laplacian,

$$(-\Delta)^s u = f(u) \quad \text{in} \quad \Omega \qquad u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega, \tag{1.1}$$

where the nonlinearity f is a real value function of class C^1 and $\Omega \subset \mathbb{R}^N$ an open bounded set.

Firstly, using the spectral theory of the related linearized problem, we present a relationship between a qualitative properties of nodal solutions of (1.1) and their Morse index, that is, the number of the negative Dirichlet eigenvalues of the linearized operator $L_u := (-\Delta)^s - f'(u)$ counted with their multiplicity. Our first result is motivated by the seminal work of Aftalion and Pacella [1], where the authors studied qualitative properties of radially sign changing solutions of the local semilinear elliptic problem $-\Delta u = f(u)$ in Ω, subject to Dirichlet boundary condition on $\partial\Omega$, where $\Omega \subset \mathbb{R}^N$ is a ball or an annulus centered at zero and $f \in C^1(\mathbb{R})$. They have proved that any radially sign changing solution of the Dirichlet problem $-\Delta u = f(u)$ in Ω , has a Morse index greater than or equal to $N+1$. In particular, they have deduced the nonradiality of least energy nodal solutions when f is superlinear with subcritical growth. The nonlocal version of this result was still unknown before our work, that is, for $s \in (0,1)$, *any bounded radially symmetric sign changing weak solution of problem* (1.1) *has a Morse index bigger or equal to* $N+1$. The main goal of Chapter 2 is therefore to show this result by giving a nonlocal counterpart in the particular case where Ω is the unit ball $\mathscr{B}$ of $\mathbb{R}^N$ centered at zero. The general idea of our proof is inspired by the one in [1] for the local problem with $s = 1$, where partial derivatives of weak solutions are used to construct suitable test functions which allow to estimate the Morse index of u. In the nonlocal case, several difficulties arise since local PDEs techniques do not apply. The most severe difficulty is related to the fact that weak solutions for nonlocal problems have much less boundary regularity than the classical one. Moreover, even though there exists a fractional version of the Hopf boundary lemma related to the fractional boundary derivative $\frac{u}{\delta^s}$ [40], it does not apply to sign changing solutions of (1.1) due to the nonlocality of the problem. Therefore, it is difficult to deal with possible oscillations of the radial derivative of u close to the boundary of $\mathscr{B}$. To overcome these difficulties, we distinguish two cases with regard to the parameter $s \in (0,1)$. In the case $s \in (\frac{1}{2},1)$, we use a regularity result of Grubb given in [53] to complete the argument when $\frac{u}{\delta^s}$ vanishes on $\partial\mathscr{B}$. Note that $\delta(x) := \text{dist}(x, \mathbb{R}^N \setminus \mathscr{B})$. Moreover, in the case $s \in (0,\frac{1}{2}]$, we use an extra assumption to ensure that $\frac{u}{\delta^s}$ does not vanish on the boundary of $\mathscr{B}$. This assumption arises from the Pohozaev identity for the fractional Laplacian derived in [83] and leads to the additional condition $f(0) = 0$. In the case $s \in (\frac{1}{2},1)$, no extra assumption is needed on $f(0)$. Since in particular, the above assumption is satisfied in the linear case, that is, when $f(u) = \lambda u$, our result applies to Dirichlet eigenvalue problem for the fractional Laplacian and provides information on the geometric structure of the eigenfunctions u corresponding to the second eigenvalue $\lambda_2(\mathscr{B})$. We indeed deduce that, u is antisymmetric, i.e. it satisfies $u(-x) = -u(x)$ for $x \in \mathscr{B}$. This was a conjecture due to Bañuelos and Kulczycki and partial results towards this conjecture have been proved in the particular cases $N \leq 3$, $s \in (0,1)$ and $4 \leq N \leq 9$, $s = \frac{1}{2}$ in [7, 36, 44, 68]. As consequence of our result, we derive the conjecture in full generality: Any eigenfunction u corresponding to the second eigenvalue $\lambda_2(\mathscr{B})$ of the fractional Dirichlet eigenvalue problem in a ball is antisymmetric for all $s \in (0,1)$ and $N \geq 1$.

Secondly, we are concerned with the study of spectral asymptotics with respect to the parameter $s \to 0^+$ of the Dirichlet eigenvalue problem for the fractional Laplacian in open bounded set $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary. That is, we consider (1.1) with $f(u) = \lambda u$. Using the logarithmic Laplacian L_Δ, which is the pseudo-differential operator with Fourier symbol $2\log|\cdot|$, and, belongs to family of operators with close to zero order, H. Chen and T. Weth [29] gave a description of the small order asymptotics $s \to 0^+$ of the principal Dirichlet eigenvalue $\lambda_{1,s}$ and the corresponding eigenfunction $u_{1,s}$ of the fractional Laplacian. In fact, they have shown that $\frac{\lambda_{1,s}-1}{s} \to \lambda_{1,L}$ and $u_{1,s} \to u_{1,L}$ in $L^2(\Omega)$ as $s \to 0^+$, where $\lambda_{1,L}$ denotes the principal eigenvalue of the eigenvalue problem $L_\Delta u = \lambda u$ in Ω, $u = 0$ in Ω^c, and $u_{1,L}$ the corresponding (unique) positive L^2-normalized eigenfunction. Motivated by the aforementioned convergence of the principal Dirichlet eigenvalue and its corresponding eigenfunction, the main goal of Chapter 3 is twofold. First, we improve the L^2-convergence $u_{1,s} \to u_{1,L}$ as $s \to 0^+$ by showing that the set $\{u_{1,s} \,:\, s \in (0,\frac{1}{4}]\}$ is relatively compact in $C(K)$ for any compact subset $K \subset \Omega$. Secondly, we extend the convergence result to higher eigenvalues $\lambda_{k,s}$ and corresponding eigenfunctions $u_{k,s}$ for all $k \in \mathbb{N}$. New tools are needed in order to overcome the lack of uniform regularity estimates for the fractional Laplacian $(-\Delta)^s$ for s close to zero. Moreover, due to the multiplicity of eigenvalues and eigenfunctions for $k \geq 2$, new approaches are also required, including, the use of Fourier transform in combination with the Courant-Fischer minimax characterization of eigenvalues. In fact, for $s \in (0,\frac{1}{4}]$, we prove that if $\lambda_{k,s}$ denote the k-th Dirichlet eigenvalue of the fractional Laplacian, then it satisfies the expansion $\lambda_{k,s} = 1 + s\lambda_{k,L} + o(s)$ as $s \to 0^+$ and, if $(s_n)_n \subset (0,\frac{1}{4}]$ is a sequence with $s_n \to 0$ as $n \to \infty$, then, after passing to a subsequence, we have $u_{k,s_n} \to u_{k,L}$ as $n \to \infty$ in $L^p(\Omega)$ for all $p < \infty$ and locally uniformly in Ω, where $u_{k,L}$ is a L^2-normalized eigenfunction of the logarithmic Laplacian corresponding to the eigenvalue $\lambda_{k,L}$. Moreover, if Ω satisfies an exterior sphere condition, then the above convergence is uniform in $\overline{\Omega}$ and the set $\{u_{k,s} \,:\, s \in (0,\frac{1}{4}]\}$ is relatively compact in the space $C_0(\Omega) := \{u \in C(\mathbb{R}^N) :\ u \equiv 0 \text{ in } \Omega^c\}$. Let us briefly comment on the idea of the proof. Indeed, to obtain local equicontinuity, the strategy is first to prove locally uniform estimate of the difference $[L_\Delta - \frac{(-\Delta)^s - \mathrm{id}}{s}]u_{k,s}$ close to the boundary $\partial\Omega$ as $s \to 0^+$ and then apply regularity estimate from [63] for weakly singular integral operators which applies, in particular, to the logarithmic Laplacian L_Δ. However, since no uniform regularity theory is available for the fractional Laplacian $(-\Delta)^s$ in the case where s is close to zero, we are not able to obtain uniform estimates for the difference. Therefore, we first prove uniform bounds related to an s-dependent auxiliary integral operator instead and then complete the proof by a direct contradiction argument. The proof of the relative compactness of the set $\{u_{k,s} \,:\, s \in (0, \frac{1}{4}]\}$ in the space $C_0(\Omega)$ follows from the application of the Arzelà–Ascoli theorem. Moreover, a crucial step in the proof is to obtain a uniform decay property of the set of eigenfunctions, which, also requires new uniform small volume maximum principle for $u_{s,k}$, a uniform radial barrier function for the difference quotient operator $\frac{(-\Delta)^s - \mathrm{id}}{s}$ and a uniform L^∞-bound of the set $\{u_{k,s} \,:\, s \in (0, \frac{1}{4}]\}$. To achive the L^∞-bound, we use a new technique based on the splitting of the integral over $\mathbb{R}^N$ on a small ball of radius δ (δ-decomposition) and apply known results and conditions associated to the newly obtained quadratic form as in [43, 61]. We emphasize that this technique is much simpler than the general De Giorgi iteration method in combination with Sobolev embedding to prove L^∞-bounds. We also point out that this δ-decomposition method is applicable for general nonlocal operators and allows to get explicit constants for the boundedness. Combining the uniform decay property and the equicontinuity of the set $\{u_{k,s} \,:\, s \in (0, \frac{1}{4}]\}$, the conclusion follows. As a byproduct, we also obtain corresponding regularity properties of eigenfunctions of the logarithmic Laplacian.

Thirdly, in Chapter 4 we are concerned with the logarithmic Schrödinger operator $(I-\Delta)^{\log}$, which is a pseudo-differential operator with Fourier symbol $\log(1+|\cdot|^2)$. It is known in the probabilistic literature as the generator of the symmetric variance gamma process in $\mathbb{R}^N$. It belongs to the family of more general operators arising as Lévy generators of geometric stable processes with associated Fourier symbols $\xi \mapsto \log(1+|\xi|^{2s})$, $s > 0$ [11, 64, 82, 88, 91]. These operators have many applications in mathematical finance and other fields of sciences. There has not been much attention from the point of view of functional analysis and PDEs in domains of these processes. The main purpose of Chapter 4 is to give an account from a PDE point of view and present some proofs not relying on probabilistic techniques but instead on purely analytic methods which are to some extend, simpler and more accessible to PDE oriented readers. We indeed show that the operator $(I-\Delta)^{\log}$, with symbol $\log(1+|\cdot|^2)$, arises as formal derivative $\frac{d}{ds}\Big|_{s=0}(I-\Delta)^s$ of the fractional relativistic Schrödinger operator at $s=0$. If $u \in C^\beta(\mathbb{R}^N)$ for some $\beta > 0$, it satisfies $\lim_{s\to 0^+} \dfrac{(I-\Delta)^s u - u}{s} = (I-\Delta)^{\log} u$ in $L^p(\mathbb{R}^N)$ for $1 \le p \le \infty$. Once the integral representation for $(I-\Delta)^{\log}$ is obtained, we introduce tools to study variational problems involving this operator. Particularly, we characterize, using minimization techniques and the Lagrange multiplier theorem, the eigenvalues and corresponding eigenfunctions of $(I-\Delta)^{\log}$ in an open bounded set $\Omega \subset \mathbb{R}^N$. We show that the Dirichlet eigenvalue problem $(I-\Delta)^{\log}\varphi = \mu\varphi$ in Ω, admits an ordered sequence of eigenvalues $\mu_{1,\log} < \mu_{2,\log} \le \mu_{3,\log} \le \dots$, with $\mu_{k,\log} \to \infty$ as $k \to \infty$ and a corresponding L^2-orthonormal basis of eigenfunctions $\varphi_{k,s}$, $k \in \mathbb{N}$. We establish the Faber-Krahn type inequality, and, using the δ-decomposition technique, we establish the boundedness of the eigenfunctions. While it is easy to see that all eigenvalues

$\mu_{k,s}$, $k \in \mathbb{N}$ of the problem $(I-\Delta)^s\varphi = \mu\varphi$ in Ω, $\varphi = 0$ in Ω^c converge to 1 as $s \to 0^+$, we prove that the rate of convergent is linear in s with speed determined by the eigenvalues $\mu_{k,\log}$ of the operator $(I-\Delta)^{\log}$. In fact, for $s \in (0,1)$, we prove that if $\mu_{k,s}$ denote the k-th Dirichlet eigenvalue of the fractional relativistic Schrödinger operator $(I-\Delta)^s$, then it satisfies the expansion $\mu_{k,s} = 1 + s\mu_{k,\log} + o(s)$ as $s \to 0^+$ and, if $(s_n)_n \subset (0,1)$ is a sequence with $s_n \to 0$ as $n \to \infty$, then, $\varphi_{1,s_n} \to \varphi_{1,\log}$ and, after passing to a subsequence, $\varphi_{k,s_n} \to \varphi_{k,\log}$ as $n \to \infty$ in $L^2(\Omega)$ for $k \geq 2$, where $\varphi_{k,\log}$ is a L^2-normalized eigenfunction of the logarithmic Schrödinger operator corresponding to the eigenvalue $\mu_{k,\log}$. In addition, using the asymptotics approximations of the modified Bessel function K_ν (see (4.23)), we derive asymptotics estimates of the kernel J associated to the operator $(I-\Delta)^{\log}$ at zero and at infinity. We then close the chapter with the proof of decay estimates at zero and at infinity of the solutions $u = G * f$ of the Poisson problem $(I-\Delta)^{\log} u = f$ in $\mathbb{R}^N$, where G is the associated fundamental solution.

Finally in Chapter 5, we deal with a general class of singular integral operators of order strictly below one. Motivated by some concrete examples of nonlocal operators of small order like the logarithmic Laplacian L_Δ and the logarithmic Schrödinger operator $(I-\Delta)^{\log}$, the aim of this chapter is the study of interior regularity result of Poisson problems involving nonlocal operators of small order. More precisely, we consider the linear equation $L_k u = f$ in Ω, where Ω is an open bounded set of $\mathbb{R}^N$, $f : \Omega \to \mathbb{R}$ is a given function and L_k is a singular integral operator with a weak integrability condition on the associated kernel k. Assuming suitable conditions on k, we first present some density results corresponding to the associated function spaces and prove maximum principles for weak solutions. Depending on the regularity of the function f on the right hand side of the equation, we investigate the regularity of the weak solutions u. In particular, assuming that the kernel is translation invariant, we provide a local H^1-regularity of weak solutions when the function f is of class C^2. The proof exploits the variational structure of the problem and a local L^∞- bound of weak solutions obtained by the δ-decomposition technique. The proof also uses an intermediate estimate in Nikol'skii spaces. From this, assuming furthermore that the kernels satisfy certain regularity properties away from its singularity, we deduce the interior C^∞-regularity of weak solutions u if f is of class C^∞. It is worthy to mention that using a probabilistic and potential theoretic approach, a local smoothness of bounded harmonic solutions solving in a certain very weak sense, $L_k u = 0$ in Ω, have been obtained in [56] for radial kernel functions. The regularity assumption on the kernel is similar to ours. We point out that our approach only exploits the variational structure of the problem. Furthermore, we obtain, using localization and a induction argument, a local H^m-regularity for any $m \geq 1$, of bounded weak solutions. We also establish interior regularity for the corresponding Dirichlet eigenvalue problem, by showing that, every eigenfunction of the problem $L_k u = \lambda u$ in Ω, belongs to $C^\infty(\Omega)$.

Contribution of the thesis

The thesis consists of four independent chapters. Each chapter presents one of the following research articles and has the same title. As previously mentioned, they can be read separately. The beginning of each chapter has a preface providing some information about the structure of the chapter and changes.

All the works were done under the co-supervision of Prof. Dr. Tobias Weth and Prof. Dr. Mouhamed Moustapha Fall and some in collaboration with Dr. Sven Jarohs and Remi Yvant Temgoua.

[P1] M. M. Fall, P. A. Feulefack, R. Y. Temgoua and T. Weth. *Morse index versus radial symmetry for fractional Dirichlet problems.* Advances in Mathematics 384 (2021): 107728. `doi.org/10.1016/j.aim.2021.107728`.

[P2] P. A. Feulefack, S. Jarohs and T. Weth. Small order asymptotics of the Dirichlet eigenvalue problem for the fractional Laplacian. Journal of Fourier Analysis and Applications 28, 18 (2022). `doi.org/10.1007/s00041-022-09908-8`.

[P3] P. A. Feulefack. *The logarithmic Schrödinger operator and associated Dirichlet problems.* (2021) `arxiv.org/abs/2112.08783`.

[P4] P. A. Feulefack and S. Jarohs. *Nonlocal operators of small order* (2021), `arxiv.org/abs/2112.09364`.

1.1 Introduction and presentation of the main results

The thesis presents new results on nonlocal Dirichlet problems established by means of suitable spectral theoretic and variational methods, taking care of the nonlocal feature of the operators.

All chapters of the thesis treat equations driven by nonlocal operators and mainly address the following:

(i) We estimate the Morse index of radially symmetric sign changing bounded weak solutions to a semilinear Dirichlet problem involving the fractional Laplacian $(-\Delta)^s$.

(ii) We study a small order asymptotics with respect to the parameter $s \to 0^+$ of the Dirichlet eigenvalues problem for the fractional Laplacian.

(iii) We deal with the logarithmic Schrödinger operator $(I-\Delta)^{\log}$. In particular, we provide an alternative to derive the singular integral representation corresponding to the symbol $\xi \mapsto \log(1+|\xi|^2)$ and introduce tools and functional analytic framework for variational studies.

(iv) We study nonlocal operators of order strictly below one. In particular, we investigate interior regularity properties of weak solutions to the associated Poisson problem depending on the regularity of the right-hand side.

Let us first explain the terms *local* and *nonlocal* that recurrently appear in the manuscript. Formally, an operator L acting on an admissible function $u : \mathbb{R}^N \to \mathbb{R}$ is called *local* if to evaluate the value $Lu(x)$ at a point $x \in \mathbb{R}^N$, it suffices to know the values of u in an arbitrary small neighborhood of x. An example of local operator is the Laplacian $\Delta u = \sum_{j=1}^N \partial_{jj} u$. A nonlocal operator is then an operator which is not local.
The prototype of nonlocal operators we consider in the thesis is given in an abstract form for smooth function $u : \mathbb{R}^N \to \mathbb{R}$, by the following singular integral

$$L_k u(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} (u(x)-u(y))k(x,y)\, dy. \tag{1.2}$$

Here the function $k : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a measurable kernel. Note that the value $L_k u(x)$ depends of the value of $u(x)$ and $u(y)$ for all $y \in \operatorname{supp}(k(x,\cdot))$, which may be $\mathbb{R}^N$. Hence, it depends in a nonlocal way on u.
These operators naturally arise in the study of Lévy processes, which are stochastic processes with stationary and independent increments [12, 63, 78]. They generalize the concept of Brownian motion and may contain discontinuities. Motivated by real-world situations and the ability to describe large scale behavior with better efficiency, nonlocal operators appear in Mathematical Finance [10, 78, 93], in Ecology [21], in Fluid Mechanics, in phase segregation [51], in Quantum Physics [73], in Image Processing [23] and in many other fields of sciences.

Let us also define the order of a nonlocal operator. Suppose that the kernel k in (1.2) satisfies

$$\sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} \min\{1, |x-y|^\sigma\} k(x,y)\, dy < \infty \qquad \text{for some } \sigma \in (0,2]. \tag{1.3}$$

Then the value $L_k u$ in $x \in \mathbb{R}^N$ is well-defined for compactly supported functions $u : \mathbb{R}^N \to \mathbb{R}$ of class C^σ. We now define the order of L_k as the infimum of the value $\sigma > 0$ for which (1.2) holds.
The primary well-known and most studied example of nonlocal operator is the fractional Laplacian $(-\Delta)^s$ with $s \in (0,1)$, whose the kernel in (1.2) is given by

$$k(x,y) := C_{N,s}|x-y|^{-N-2s}, \qquad \text{with} \quad C_{N,s} = s4^s \frac{\Gamma(\frac{N}{2}+s)}{\pi^{\frac{N}{2}}\Gamma(1-s)}.$$

The fractional Laplacian is a nonlocal operator of order $2s$ with $s \in (0,1)$ and satisfies (1.3) in particular with $\sigma = 2$. The constant $C_{N,s}$ is normalized such that for smooth function u, equivalently, $(-\Delta)^s$ is given

$$\mathscr{F}((-\Delta)^s u) = |\cdot|^{2s}\mathscr{F}(u).$$

Here and in the following, $\mathscr{F}$ denotes the usual Fourier transform. Formally, for smooth function $u : \mathbb{R}^N \to \mathbb{R}$, the fractional Laplacian satisfies the asymptotics $(-\Delta)^s u \to u$ as $s \to 0^+$ and $(-\Delta)^s u \to \Delta u$ as $s \to 1^-$, connecting nonlocal PDEs to classical (local) ones.

We call L_k a nonlocal operator of small order, if the order of L_k is less thtan one, i.e., (1.3) is satifies for $\sigma \in (0,1)$. We have as example the logarithmic Schrödinger operator $(I-\Delta)^{\log}$ [45] and the logarithmic Laplacian L_Δ [29] which will be defined further below.

In the past years, the interest in nonlocal Dirichlet problems have undergone rapid a growth attention. Well-known results and properties of solutions to classical Dirichlet problems have so far been successfully adapted and extended to their nonlocal counterpart. In that direction, there is an extensive literature devoted to the topic. Various regularity results can be found in [41,53,63,84,89], variational formulations and existence results in [15,43,87], the fractional Pohozaev type identity and nonexistence results in [42,83], radially symmetry and monotocinity results via moving plane method in [42, 62], radially sign-changing solutions in [76, 95, 97], weak and strong maximum principles in [20,60]. Of course, these references do not exhaust the rich literature on the subject.
Although remarkable advances have been made in the subject, many results still need to be established. In particular, as we shall present in the sequel, the result of Chapter 2 extends to its nonlocal counterpart, an estimate obtained by Aftalion and Pacella [1], of the Morse index of radially symmetric sign changing solutions of the following classical semilinear elliptic problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathscr{B} \\ \quad u = 0 & \text{on } \partial\mathscr{B}, \end{cases}$$

where $\mathscr{B}$ is the unit ball of $\mathbb{R}^N$ centred at zero and $f : \mathbb{R} \to \mathbb{R}$ is of class C^1.

While the result of Chapter 2 essentially deals with nonlocal operato of order $2s$, the rest of the thesis, namely, Chapter 3, 4 and 5 deals more or less with nonlocal operators of small order. These operators are getting nowadays, increasing interest in the study of linear and nonlinear nonlocal partial differential equations [13, 29, 30, 32, 86] and also, are motivated by some applications to nonlocal models where small order of the operator captures the optimal accuracy and the efficiency of the model [4, 81].

Apart from Chapter 5 where a general class of admissible kernels for nonlocal operators of small order is considered, many results of the thesis deal with symmetric and translation invariant kernels, that is, there exists a function $J : \mathbb{R}^N \setminus \{0\} \to [0,\infty]$ with $k(x,y) = J(x-y)$ for $x, y \in \mathbb{R}^N$ with $J(-z) = J(z)$ for all $z \in \mathbb{R}^N \setminus \{0\}$ and satisfying, for some $\sigma \in (0,2]$ the following properties

$$\int_{\mathbb{R}^N} J(z)\, dz = \infty \qquad \text{and} \qquad \int_{\mathbb{R}^N} \min\{1, |z|^{\sigma}\} J(z)\, dz < \infty.$$

Many techniques used in the thesis to prove our results are purely nonlocal and are applicable for quite general nonlocal of operators. This is because our arguments do not rely on the extension method of Caffarelli and Silvestre introduced in [25], which allows to reformulate nonlocal problems driven by the fractional Laplacian $(-\Delta)^s$, as local boundary value problems where the operator $(-\Delta)^s$ arises as a Dirichlet-to-Neumann type operator.

1.1.1 Presentation of the main results

In the following, we present the main results of the thesis. These results are from research papers [39], [47], [45] and [46] and will be indicated by their title.

1.1.1.1 Morse index versus radial symmetry for fractional Dirichlet problems

The first result the thesis is contained in Chapter 2 from article [39] and provides an estimate of the Morse index of radially symmetric sign changing solutions u to the semilinear fractional Dirichlet problem

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \mathscr{B} \\ u = 0 & \text{on } \mathbb{R}^N \setminus \mathscr{B}, \end{cases} \tag{1.4}$$

where $\mathscr{B} \subset \mathbb{R}^N$ is the unit ball centred at zero and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is of class C^1. As already mentioned, the theorem is stated in Chapter 2 and involves the fractional Laplacian $(-\Delta)^s$. Our approach applies for more general operators of Lévy type of order $2s$ like (1.2) in place of the fractional Laplacian.

The result of this chapter is motivated by the seminal work of Aftalion and Pacella [1], where the authors studied qualitative properties of radial sign changing solutions of the local semilinear elliptic problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N$ is a ball or an annulus centered at zero and $f \in C^1(\mathbb{R})$. They have proved the following classical result,

Theorem 1.1 ([1]). *If Ω is a ball or an annulus, any radial sign changing solution of* (1.5) *has Morse index greater than or equal to $N+1$.*

As consequence of their result, they have deduced in particular the nonradiality of least energy nodal solution when f is superlinear with subcritical growth.

The nonlocal version of this result was still unknown before our work. The main goal of the chapter is therefore to extend the result in Theorem 1.1 by giving a nonlocal counterpart in the particular case where Ω is the unit ball $\mathscr{B} \subset \mathbb{R}^N$ centered at zero. Before we state the main result of the chapter, we fix first some notation. Consider the function space

$$\mathscr{H}_0^s(\mathscr{B}) := \{u \in H^s(\mathbb{R}^N) : u \equiv 0 \text{ on } \mathbb{R}^N \setminus \mathscr{B}\} \subset H^s(\mathbb{R}^N).$$

By definition, a function $u \in \mathscr{H}_0^s(\mathscr{B}) \cap L^\infty(\mathscr{B})$ is a weak solution of (1.4) if

$$\mathscr{E}_s(u,v) = \int_{\mathscr{B}} f(u) v \, dx \qquad \text{for all } v \in \mathscr{H}_0^s(\mathscr{B}),$$

where $(v,w) \mapsto \mathscr{E}_s(v,w)$ is the bilinear form associated with $(-\Delta)^s$, with

$$\mathscr{E}_s(v,w) := \frac{c(N,s)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^{N+2s}} \, dxdy.$$

The *Morse index* $m(u)$ of a weak solution $u \in \mathscr{H}_0^s(\mathscr{B}) \cap L^\infty(\mathscr{B})$ of (1.4) is the maximal dimension of a subspace $X \subset \mathscr{H}_0^s(\mathscr{B})$ where the quadratic form

$$(v,w) \mapsto \mathscr{E}_{s,L}(v,w) := \mathscr{E}_s(v,w) - \int_{\mathscr{B}} f'(u) v w \, dx \tag{1.6}$$

associated to the linearized operator $L := (-\Delta)^s - f'(u)$ is negative definite. Equivalently, $m(u)$ can be defined as the number of the negative Dirichlet eigenvalues of L counted with their multiplicity.
We recall that the n-th Dirichlet eigenvalue $\lambda_{n,L}(\mathscr{B})$ of the linearized operator L admits the variational characterization

$$\lambda_{n,L}(\mathscr{B}) = \min_{V \in \mathscr{V}_n} \max_{v \in S_V} \mathscr{E}_{s,L}(v,v) \tag{1.7}$$

where $\mathscr{V}_n$ denotes the family of n-dimensional subspaces of $\mathscr{H}_0^s(\mathscr{B})$ and

$$S_V := \{v \in V \,:\, \|v\|_{L^2(\mathscr{B})} = 1\} \quad \text{for} \quad V \in \mathscr{V}_n.$$

Our first result reads as follows.

Theorem 1.2. *Let u be a radially symmetric sign changing solution of problem* (1.4)*, and suppose that* <u>*one*</u> *of the following additional conditions holds.*

(A1) $s \in (\frac{1}{2}, 1)$.

(A2) $s \in (0, \frac{1}{2}]$, *and*

$$\int_0^t f(\tau) d\tau > \frac{N-2s}{2N} t f(t) \qquad \textit{for } t \in \mathbb{R} \setminus \{0\}. \tag{1.8}$$

Then u has Morse index greater than or equal to $N+1$.

We note that assumption in (1.8) is satisfied for homogeneous nonlinearities with subcritical growth, i.e., if

$$f(t) = \lambda |t|^{p-2} t \qquad \text{with} \quad \lambda > 0 \quad \text{and} \quad 2 \leq p < \frac{2N}{N-2s}.$$

On the other hand, in the supercritical case where

$$\int_0^t f(\tau)d\tau < \frac{N-2s}{2N} t f(t) \quad \text{for} \quad t \in \mathbb{R} \setminus \{0\},$$

problem (1.4) does not admit any nontrivial weak solutions $u \in \mathscr{H}_0^s(\mathscr{B}) \cap L^\infty(\mathscr{B})$ by the fractional Pohozaev identity stated in [83, Theorem 1.1]. In particular, in the linear case $t \mapsto \lambda t$, our results apply to Dirichlet eigenvalue problem for the fractional Laplacian

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \mathscr{B} \\ u = 0 & \text{on } \mathbb{R}^N \setminus \mathscr{B}, \end{cases} \tag{1.9}$$

providing thereby a complete positive answer to a conjecture by Bañuelos and Kulczycki [36].

This is the content of the following theorem.

Theorem 1.3. *Let $N \geq 1$ and $0 < s < 1$, and let $\lambda_2 > 0$ be the second eigenvalue of problem (1.9). Then every eigenfunction u corresponding to λ_2 is antisymmetric, i.e. it satisfies*

$$u(-x) = -u(x) \quad \textit{for } x \in \mathscr{B}.$$

This was in fact a conjecture due to by Bañuelos and Kulczycki on the geometric structure of the eigenfunctions u corresponding to the second eigenvalue $\lambda_2(\mathscr{B})$. Partial results towards this conjecture have been obtained in recent years in [7,36,44,68], covering the special cases $N \leq 3$, $s \in (0,1)$ and $4 \leq N \leq 9$, $s = \frac{1}{2}$. As consequence of our results, we derive the conjecture in full generality $s \in (0,1)$ and $N \geq 1$.

Let us give some steps of the proof of Theorem 1.2. The strategy of the proof is to use partial derivatives of u to construct suitable test functions which allow to estimate the Morse index of u as in [1]. For $j \in \{1, \dots, N\}$, we consider the partial derivatives of u given by

$$v^j : \mathbb{R}^N \to \mathbb{R}, \qquad v^j(x) = \begin{cases} \partial_j u(x) = \dfrac{\partial u}{\partial x_j}(x), & x \in \mathscr{B}, \\ 0, & x \in \mathbb{R}^N \setminus \mathscr{B}, \end{cases} \qquad j = 1, \dots, N.$$

We have the following key lemma.

Lemma 1.4. *For any $j \in \{1,\dots,N\}$, we have $Lv^j = (-\Delta)^s v^j - f'(u)v^j = 0$ in distributional sense in $\mathscr{B}$, i.e.*

$$\int_{\mathscr{B}} v^j (-\Delta)^s \varphi \, dx = \mathscr{E}_s(v^j, \varphi) = \int_{\mathscr{B}} f'(u) v^j \varphi \, dx \qquad \text{for all } \varphi \in \mathscr{C}_c^\infty(\mathscr{B}).$$

Moreover, if $\varphi \in \mathscr{H}_0^s(\mathscr{B})$ has compact support in $\mathscr{B}$, then we have

$$\mathscr{E}_s(v^j, \varphi) = \int_{\mathscr{B}} f'(u) v^j \varphi \, dx. \tag{1.10}$$

Furthermore, if $v^j \in \mathscr{H}_0^s(\mathscr{B})$, then (1.10) is true for all $\varphi \in \mathscr{H}_0^s(\mathscr{B})$.

As already mentioned, several difficulties arise since local PDEs techniques do not apply. We note the following regularity properties for weak solutions of (1.4). This is related to the fact that weak solutions of (1.4) have much less boundary regularity. For this we consider the distance function to the boundary

$$\delta : \overline{\mathscr{B}} \to \mathbb{R}, \qquad \delta(x) = \operatorname{dist}(x, \partial\mathscr{B}) = 1 - |x|.$$

Proposition 1.5. *(cf. [41, 53, 84, 89])*
Let $u \in \mathscr{H}_0^s(\mathscr{B}) \cap L^\infty(\mathscr{B})$ be a weak solution of (1.4). Then $u \in C^{2,s}_{loc}(\mathscr{B}) \cap C_0^s(\overline{\mathscr{B}})$. Moreover,

$$\psi := \frac{u}{\delta^s} \in C^\alpha(\overline{\mathscr{B}}) \qquad \text{for some } \alpha \in (0,1), \tag{1.11}$$

and the following properties hold with some constant $c > 0$:

(i) $|\nabla u(x)| \le c\delta^{s-1}(x)$ *for all* $x \in \mathscr{B}$.

(ii) $|\nabla \psi(x)| \le c\delta^{\alpha-1}(x)$ *for all* $x \in \mathscr{B}$.

(iii) For every $x_0 \in \partial\mathscr{B}$, we have $\lim_{x \to x_0} \delta^{1-s}(x) \partial_r u(x) = -s\psi(x_0)$, where $\partial_r u(x) = \nabla u(x) \cdot \frac{x}{|x|}$ denotes the radial derivative of u at x.

(iv) If $s \in (\frac{1}{2}, 1)$, then $\psi \in C^1(\overline{\mathscr{B}})$.

We consider the function ψ defined in (1.11) which is also radial. We write

$$\psi(x) = \psi_0(r) \text{ for } r = |x| \text{ with a function } \psi_0 : [0,1] \to \mathbb{R} \tag{1.12}$$

which is of class C^α for some $\alpha > 0$ by Proposition 1.5. Moreover, by Proposition 1.5 we have

$$\psi_0(1) = \lim_{|x| \to 1} \frac{u(|x|)}{(1-|x|)^s} = -\frac{1}{s} \lim_{|x| \to 1} (1-|x|)^{1-s} \partial_r u(x).$$

By the Pohozaev type identity given in [83, Theorem 1.1], this value also satisfies

$$\psi_0^2(1) = \frac{1}{|S^{N-1}|\Gamma(1+s)^2} \int_{\mathscr{B}} \Big[(2s-N) u f(u) + 2NF(u) \Big] dx.$$

Here $F:\mathbb{R}\to\mathbb{R}$ is given by $F(t)=\int_0^t f(\tau)\,d\tau$.

We now build suitable test functions from partial derivatives which allow to estimate Dirichlet eigenvalues of the linearized operator $L:=(-\Delta)^s-f'(u)$ and therefore the Morse index of u.

Definition 1.6. Let ψ_0 be the function defined in (1.12). For $j=1,\dots,N$, we define the open half spaces

$$H_\pm^j:=\{x\in\mathbb{R}^N\,:\,\pm x_j>0\} \tag{1.13}$$

and the functions $d_j:\mathbb{R}^N\to\mathbb{R}$ by

$$d_j:=\begin{cases}(v^j)^+1_{H_+^j}-(v^j)^-1_{H_-^j} & \text{if } \psi_0(1)\ge 0;\\ (v^j)^+1_{H_-^j}-(v^j)^-1_{H_+^j} & \text{if } \psi_0(1)<0.\end{cases}$$

We note that, for $j=1,\dots,N$, the function d_j is odd with respect to the reflection

$$\sigma_j:\mathbb{R}^N\to\mathbb{R}^N,\qquad x=(x_1,\cdots,x_j,\cdots,x_N)\mapsto\sigma_j(x)=(x_1,\dots,-x_j,\dots,x_N)$$

at the hyperplane $\{x_j=0\}$ since the function v^j is odd. Moreover, in accordance with the assumptions of Theorem 1.2, u changes sign, which implies that

$$(v^j)^\pm 1_{H_+^j}\not\equiv 0\quad\text{and}\quad (v^j)^\pm 1_{H_-^j}\not\equiv 0\qquad\text{for } j=1,\dots,N, \tag{1.14}$$

where the half spaces $H_\pm^j$ are defined in (1.13). The function $\psi_0(1)$ used in the definition of d_j allows to control the possible oscillations of the radial derivative of the weak solutions u close to the boundary of $\mathscr{B}$. The next lemma is of key importance for the proof of Theorem 1.2.

Lemma 1.7. *Let $j=1,\dots,N$.*

(i) If $\psi_0(1)\neq 0$, we have $d_j\in\mathscr{H}_0^s(\mathscr{B})$, and d_j has compact support in $\mathscr{B}$.

(ii) If $s\in(\frac{1}{2},1)$ and $\psi_0(1)=0$, then we have $v^j\in\mathscr{H}_0^s(\mathscr{B})$ and $d_j\in\mathscr{H}_0^s(\mathscr{B})$.

Next, using the oddness of the functions v^j and d_j with respect to the reflection σ_j, we then show that

$$\mathscr{E}_{s,L}(d_j,d_j)<0,\quad \text{for } j\in\{1,\dots,N\}.$$

This allows us to deduce the following lemma

Lemma 1.8. *Let $\alpha=(\alpha_1,\dots,\alpha_N)\in\mathbb{R}^N$ and $d=\sum_{j=1}^N\alpha_j d_j$. Then we have*

$$\mathscr{E}_{s,L}(d,d)=\sum_{j=1}^N\alpha_j^2\mathscr{E}_L(d_j,d_j)\le 0.$$

Moreover,

$$\mathscr{E}_{s,L}(d,d)<0\qquad\text{if and only if}\qquad\alpha\neq 0,$$

and therefore the functions $d_1,\dots,d_N$ are linearly independent.

Lemma 1.9. *The first eigenvalue $\lambda_{1,L}$ of the operator $L=(-\Delta)^s - f'(u)$ is simple, and the corresponding eigenspace is spanned by radially symmetric eigenfunction $\varphi_{1,L}$. Furthermore,*

$$\mathscr{E}_{s,L}(d_j,\varphi_{1,L})=0 \quad \text{for } j=1,2,\cdots,N \qquad \text{and} \qquad \lambda_{1,L}=\mathscr{E}_{s,L}(\varphi_{1,L},\varphi_{1,L})<0.$$

From lemma 1.8 and 1.9, we consider the subspace

$$V = \operatorname{span}\{\varphi_{1,L},d_1,\dots,d_N\}.$$

For $\alpha\in\mathbb{R}^{N+1}\setminus\{0\}$ and $d=\alpha_0\varphi_{1,L}+\sum_{j=1}^{N}\alpha_j d_j\in V$, we then have, by Lemma 1.8 and 1.9,

$$\mathscr{E}_{s,L}(d,d)=\alpha_0^2\,\mathscr{E}_{s,L}(\varphi_{1,L},\varphi_{1,L})+\mathscr{E}_{s,L}\Big(\sum_{j=1}^{N}\alpha_j d_j,\sum_{j=1}^{N}\alpha_j d_j\Big)<0.$$

In particular, it follows that the functions $\varphi_{1,L},d_1,\dots,d_N$ are linearly independent and therefore V is $N+1$-dimensional. By (1.7) and the compactness of $S_V=\{v\in V\,:\,\|v\|_{L^2(\mathscr{B})}=1\}$, it then follows that $\lambda_{N+1,L}<0$, which means that u has Morse index greater than or equal to $N+1\geq 2$.

1.1.1.2 Small order asymptotics of the Dirichlet eigenvalue problem for the fractional Laplacian

We next present our main results of Chapter 3 from article [47], which relatively deal with the logarithmic Laplacian L_Δ, which, for compactly supported Dini continuous functions u, it is pointwisely given by

$$L_\Delta u(x)=C_N\int_{\mathbb{R}^N}\frac{u(x)1_{B_1(x)}(y)-u(y)}{|x-y|^N}\,dy+\rho_N u(x),$$

where $C_N=\pi^{-\frac{N}{2}}\Gamma(\frac{N}{2})$, and $\rho_N=2\log 2+\psi(\frac{N}{2})-\gamma$. Here, $\psi=\frac{\Gamma'}{\Gamma}$ denotes the Digamma function, and $\gamma=-\Gamma'(1)$ is the Euler-Mascheroni constant.
It further satisfies the following two key properties: If $u\in C_c^\beta(\mathbb{R}^N)$ for some $\beta>0$, then

$$\mathscr{F}(L_\Delta u)(\xi)=2\log|\xi|\mathscr{F}(u)(\xi) \qquad \text{for a.e. } \xi\in\mathbb{R}^N,$$

and

$$\frac{d}{ds}\Big|_{s=0}(-\Delta)^s u=\lim_{s\to 0^+}\frac{(-\Delta)^s u-u}{s}=L_\Delta u \qquad \text{in } L^p(\mathbb{R}^N) \text{ for } 1<p\leq\infty.$$

The results of this chapter concern the spectral asymptotics $s\to 0^+$ of the Dirichlet eigenvalues problem for the fractional Laplacian

$$\begin{cases}(-\Delta)^s\varphi_s=\lambda\varphi_s & \text{in } \Omega,\\ \varphi_s=0 & \text{in } \Omega^c,\end{cases} \tag{1.15}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded open set with Lipschitz boundary and $\Omega^c:=\mathbb{R}^N\setminus\Omega$.

In fact, using the logarithmic Laplacan L_Δ, H. Chen and T. Weth [29] gave the following description of the small order asymptotics $s\to 0^+$ of the principal Dirichlet eigenvalue $\lambda_{1,s}$ and the corresponding eigenfunction $u_{1,s}$ of (1.15).

Theorem 1.10 ([29]). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, and let $\lambda_{1,s}(\Omega)$ denote the first Dirichlet eigenvalue for $(-\Delta)^s$ on Ω for $s \in (0,1)$ and $u_{1,s}$ its unique positive L^2-normalized eigenfunction. Then we have*

$$\frac{\lambda_{1,s}(\Omega)-1}{s} \to \lambda_{1,L}(\Omega) \quad \text{and} \quad u_{1,s} \to u_{1,L} \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad s \to 0^+ \tag{1.16}$$

where $\lambda_{1,L}$ denotes the principal eigenvalue of the Dirichlet eigenvalue problem

$$\begin{cases} L_\Delta u = \lambda u & \text{in } \Omega \\ \quad u = 0 & \text{in } \mathbb{R}^N \setminus \Omega^c \end{cases} \tag{1.17}$$

and $u_{1,L}$ denotes the unique positive L^2-normalized eigenfunction for L_Δ corresponding to $\lambda_{1,L}$.

Note here that we consider both (1.15) and (1.17) in a suitable weak sense. Motivated by the above result, the main aim of Chapter 3 is twofold: First, to improve the L^2-convergence in (1.16) and secondly, to extend it to higher eigenvalues $\lambda_{k,s}(\Omega)$ and corresponding eigenfunctions $u_{k,s}$ for all $k \in \mathbb{N}$. For this, new tools are needed in order to overcome the lack of uniform regularity estimates for the fractional Laplacian $(-\Delta)^s$ for s close to zero. Also due to the multiplicity of eigenvalues and eigenfunctions for $k \geq 2$, new approaches are required based on the use of Fourier transform in combination with the Courant-Fischer characterization of eigenvalues,

$$\lambda_{k,s}(\Omega) = \inf_{\substack{V \subset \mathscr{H}_0^s(\Omega) \\ \dim V = k}} \max_{\substack{v \in V \setminus \{0\} \\ \|v\|_{L^2(\Omega)}=1}} \mathscr{E}_s(v,v) = \inf_{\substack{V \subset C_c^2(\Omega) \\ \dim V = k}} \max_{\substack{v \in V \setminus \{0\} \\ \|v\|_{L^2(\Omega)}=1}} \mathscr{E}_s(v,v).$$

Let us point out first that in order to complete the proof of the main result of this chapter, many new uniform results with respect to the parameter s are needed. As already mentioned some of them above, for sake of keeping this summary not too long, we gently refer the reader to Remark 3.5 for more clarifications and only state the main result of the chapter here.

Our result reads as follows.

Theorem 1.11. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary and let $k \in \mathbb{N}$. Moreover, for $s \in (0, \frac{1}{4})$, let $\lambda_{k,s}$ resp. $\lambda_{k,L}$ denote the k-th Dirichlet eigenvalue of the fractional and logarithmic Laplacian, respectively, and let $\varphi_{k,s}$ denote an L^2-normalized eigenfunction. Then we have:*

(i) The eigenvalue $\lambda_{k,s}$ satisfies the expansion

$$\lambda_{k,s} = 1 + s\lambda_{k,L} + o(s) \qquad \text{as } s \to 0^+. \tag{1.18}$$

(ii) The set $\{\varphi_{k,s} : s \in (0, \frac{1}{4}]\}$ is bounded in $L^\infty(\Omega)$ and relatively compact in $L^p(\Omega)$ for every $p < \infty$.

(iii) The set $\{\varphi_{k,s} : s \in (0, \frac{1}{4}]\}$ is equicontinuous in every point $x_0 \in \Omega$ and therefore relative compact in $C(K)$ for any compact subset $K \subset \Omega$.

(iv) If Ω satisfies an exterior sphere condition, then the set $\{\varphi_{k,s} \,:\, s \in (0, \frac{1}{4}]\}$ is relative compact in the space $C_0(\Omega) := \{u \in C(\mathbb{R}^N) \,:\, u \equiv 0 \quad in\ \Omega^c\}$.

(v) If $(s_n)_n \subset (0, \frac{1}{4}]$ is a sequence with $s_n \to 0$ as $n \to \infty$, then, after passing to a subsequence, we have

$$\varphi_{k,s_n} \to \varphi_{k,L} \quad as\ n \to \infty \tag{1.19}$$

in $L^p(\Omega)$ for $p < \infty$ and locally uniformly in Ω, where $\varphi_{k,L}$ is an L^2-normalized eigenfunction of the logarithmic Laplacian corresponding to the eigenvalue $\lambda_{k,L}$.

If, moreover, Ω satisfies an exterior sphere condition, then the convergence in (1.19) is uniform in $\overline{\Omega}$.

As direct consequence, Theorem 1.10 and Theorem 1.11 give rise to the following corollary.

Corollary 1.12. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary and let, for $s \in (0, \frac{1}{4}]$, $\varphi_{1,s}$ denote the unique positive L^2-normalized eigenfunction of $(-\Delta)^s$ corresponding to the principal Dirichlet eigenvalue $\lambda_{1,s}$. Then we have*

$$\varphi_{1,s} \to \varphi_{1,L} \quad as\ s \to 0^+ \tag{1.20}$$

in $L^p(\Omega)$ for $p < \infty$ and locally uniformly in Ω, where $\varphi_{1,L}$ is the unique positive L^2-normalized eigenfunction of L_Δ corresponding to the principal Dirichlet eigenvalue $\lambda_{1,L}$.

If, moreover, Ω satisfies an exterior sphere condition, then the convergence in (1.20) is uniform in $\overline{\Omega}$.

As a further corollary of Theorem 1.11, we also derive the following regularity properties of eigenfunctions of the logarithmic Laplacian.

Corollary 1.13. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary, and let $\varphi \in \mathscr{H}_0^0(\Omega)$ be an eigenfunction of (1.17). Then $\varphi \in L^\infty(\Omega) \cap C_{loc}(\Omega)$. Moreover, if Ω satisfies an exterior sphere condition, then $\varphi \in C_0(\Omega)$.*

1.1.1.3 The logarithmic Schrödinger operator and associated Dirichlet problems

This chapter is based on paper [45] and it is devoted to the study of the logarithmic Schrödinger operator $(I - \Delta)^{\log}$, which is the singular integral operator with Fourier symbol $\log(1 + |\cdot|^2)$.

This operator has been studied extensively in the literature from a probabilistic and potential theoretic point of view [11, 63, 64, 82, 88, 91] and belongs to the family of more general operators arising as Lévy generators of geometric stable processes with associated Fourier symbols $\log(1 + |\cdot|^{2s})$, $s > 0$.

We provide an alternative method to derive the singular integral corresponding to the Fourier symbol $\log(1 + |\cdot|^2)$ and introduce tools to study variational problems involving this operator.

The first result of the chapter is the following

Theorem 1.14. *Let $u \in C^{\alpha}(\mathbb{R}^N)$ for some $\alpha > 0$ and $1 < p \leq \infty$. Then*

$$\begin{aligned}(I-\Delta)^{\log} u(x) &= \frac{d}{ds}\Big|_{s=0}[(I-\Delta)^s u](x) \\ &= d_N \int_{\mathbb{R}^N} \frac{u(x)-u(x+y)}{|y|^N}\omega(|y|)\,dy = \int_{\mathbb{R}^N}(u(x)-u(x+y))J(y)\,dy,\end{aligned} \tag{1.21}$$

for $x \in \mathbb{R}^N$, where $d_N := \pi^{-\frac{N}{2}} = -\lim_{s\to 0^+}\frac{d_{N,s}}{s}$, $J(y) = d_N \frac{\omega(|y|)}{|y|^N}$, *and*

$$\omega(|y|) := 2^{1-\frac{N}{2}}|y|^{\frac{N}{2}}K_{\frac{N}{2}}(|y|) = \int_0^\infty t^{-1+\frac{N}{2}}e^{-t-\frac{|y|^2}{4t}}\,dt.$$

Moreover,

(i) If $u \in L^p(\mathbb{R}^N)$ for $1 \leq p \leq \infty$, then $(I-\Delta)^{\log} u \in L^p(\mathbb{R}^N)$ and

$$\frac{(I-\Delta)^s u - u}{s} \to (I-\Delta)^{\log} u \;\text{ in }\; L^p(\mathbb{R}^N) \;\text{ as }\; s \to 0^+.$$

(ii) $\mathscr{F}((I-\Delta)^{\log} u)(\xi) = \log(1+|\xi|^2)\mathscr{F}(u)(\xi)$, *for almost every* $\xi \in \mathbb{R}^N$.

Where we recall that for $s \in (0,1)$, the operator $(I-\Delta)^s$ is the fractional relativistic Schrödinger operator. For compactly supported functions $u : \mathbb{R}^N \to \mathbb{R}$ of class C^2, it is well-defined and represented via hypersinglar integral (see [85, page 548] and [38])

$$(I-\Delta)^s u(x) = u(x) + d_{N,s}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\varepsilon(0)} \frac{u(x+y)-u(x)}{|y|^{N+2s}}\omega_s(|y|)\,dy,$$

where $d_{N,s} = \frac{\pi^{-\frac{N}{2}}4^s}{\Gamma(-s)}$ is a normalization constant and the function ω_s is given by

$$\omega_s(|y|) = 2^{1-\frac{N+2s}{2}}|y|^{\frac{N+2s}{2}}K_{\frac{N+2s}{2}}(|y|) = \int_0^\infty t^{-1+\frac{N+2s}{2}}e^{-t-\frac{|y|^2}{4t}}\,dt.$$

In the particular case $N = 1$, the representation in (1.21) is given by

$$(I-\Delta)^{\log} u(x) = P.V.\int_{\mathbb{R}} \frac{u(x)-u(y)}{|x-y|}e^{-|x-y|}\,dy. \tag{1.22}$$

We note that (1.22) appears in [75] and is identified as symmetrized Gamma process (see also [66]).

The logarithmic Schrödinger operator $(I-\Delta)^{\log}$ shares the same kernel singularity with the logarithmic Laplacian L_Δ, but does not have an integrability problem at infinity. Indeed, using the asymptotics of the modified Bessel function K_ν, we have the following asymptotics approximations for the kernel J

$$J(z) \sim \begin{cases} \pi^{-\frac{N}{2}}\Gamma(\frac{N}{2})|z|^{-N} & \text{as} \quad |z| \to 0 \\ \pi^{-\frac{N-1}{2}}2^{-\frac{N-1}{2}}|z|^{-\frac{N+1}{2}}e^{-|z|} & \text{as} \quad |z| \to \infty. \end{cases}$$

The Green function G of the logarithmic Schrödinger operator $(I-\Delta)^{\log}$ is known ([55]) and it given by the following expression

$$G(x) = \frac{2^{1-N}}{\pi^{N/2}} \int_0^\infty \frac{1}{\Gamma(t)} \left(\frac{|x|}{2}\right)^{t-\frac{N}{2}} K_{t-\frac{N}{2}}(|x|)\, dt. \tag{1.23}$$

We present the following asymptotics results for G and decay result of the solution for Poisson problem $(I-\Delta)^{\log} u = f$ in $\mathbb{R}^N$.

Proposition 1.15. *The function G in (1.23) satisfies the asymptotics properties*

$$G(x) \sim \begin{cases} c_N|x|^{-N} & as \quad |x| \to 0 \\ c_N 2^{\frac{N-1}{2}}\pi^{1/2}|x|^{-\frac{N+1}{2}}e^{-|x|} & as \quad |x| \to \infty. \end{cases}$$

In addition, for $f \in L^1(\mathbb{R}^N)$, the solution $u = G * f$ of the equation $(I-\Delta)^{\log}u = f$ in $\mathbb{R}^N$ satisfies

$$u(x) = \begin{cases} O(|x|^{-N}) & as \quad |x| \to 0 \\ O(e^{-|x|}) & as \quad |x| \to \infty. \end{cases}$$

Next, let $\Omega \subset \mathbb{R}^N$ be an open set and $u, v \in C_c^2(\mathbb{R}^N)$. In order to settle the corresponding functional analysis framework and energy space related to operator $(I-\Delta)^{\log}$, we introduce the following bilinear form

$$\mathscr{E}_\omega(u,v) := \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(x)-u(y))(v(x)-v(y))J(x-y)\, dxdy,$$

and define the space $H^{\log}(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \quad \mathscr{E}_\omega(u,u) < \infty\}$. Then $H^{\log}(\mathbb{R}^N)$ is a Hilbert space endowed with the scalar product

$$(u,v) \to \langle u,v\rangle_{H^{\log}(\mathbb{R}^N)} = \langle u,v\rangle_{L^2(\mathbb{R}^N)} + \mathscr{E}_\omega(u,u)$$

and with the corresponding norm given $\|u\|_{H^{\log}(\mathbb{R}^N)} = (\|u\|^2_{L^2(\mathbb{R}^N)} + \mathscr{E}_\omega(u,u))^{\frac{1}{2}}$. We denote by $\mathscr{H}_0^{\log}(\Omega)$, the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^{\log}(\mathbb{R}^N)}$.
Next,we let $\Omega \subset \mathbb{R}^N$ be bounded, $f \in L^2(\Omega)$ and consider the Dirichlet elliptic problem

$$\begin{cases} (I-\Delta)^{\log}u = f & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.24}$$

We have by the Riesz representation theorem that problem (1.24) admits a unique weak solution $u \in \mathscr{H}_0^{\log}(\Omega)$ with the property

$$\mathscr{E}_\omega(u,v) = \int_\Omega f(x)v(x)\,dx \qquad \text{for all } v \in \mathscr{H}_0^{\log}(\Omega).$$

In addition, if $f \in L^\infty(\Omega)$ and Ω satisfies a uniform exterior sphere condition, it follows from the Green function representation and the regularity estimates in [63,64,79] that $u \in C_0(\Omega)$ with

$$C_0(\Omega) := \{u \in C(\mathbb{R}^N) \;:\; u = 0 \text{ on } \mathbb{R}^N \setminus \Omega\}.$$

To avoid a priori regularity assumptions, we consider (1.24) with $f = \lambda u$ in weak sense. We call a function $u \in \mathscr{H}_0^{\log}(\Omega)$ an eigenfunction of (1.24) corresponding to the eigenvalue λ if

$$\mathscr{E}_\omega(u,\varphi) = \lambda \int_\Omega u\varphi\,dx \qquad \text{for all } \varphi \in \mathscr{H}_0^{\log}(\Omega). \tag{1.25}$$

We then have the following

Theorem 1.16. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Then*

(i) Problem (1.24) *admits an eigenvalue $\lambda_1(\Omega)$ that is positive and characterized by*

$$\lambda_1(\Omega) = \inf_{u \in \mathscr{H}_0^{\log}(\Omega)} \frac{\mathscr{E}_\omega(u,u)}{\|u\|^2_{L^2(\Omega)}} = \inf_{u \in \mathscr{P}_1(\Omega)} \mathscr{E}_0(u,u), \tag{1.26}$$

with $\mathscr{P}_1(\Omega) := \{u \in \mathscr{H}_0^{\log}(\Omega) : \|u\|_{L^2(\Omega)} = 1\}$ and there exists a positive function $\varphi_1 \in \mathscr{H}_0^{\log}(\Omega)$, which is an eigenfunction corresponding to $\lambda_1(\Omega)$ and that attains the minimum in (1.26)*, i.e. $\|\varphi_1\|_{L^2(\Omega)} = 1$ and $\lambda_1(\Omega) = \mathscr{E}_\omega(\varphi_1,\varphi_1)$.*

(ii) The first eigenvalue $\lambda_1(\Omega)$ is simple, that is, if $u \in \mathscr{H}_0^{\log}(\Omega)$ satisfies (1.25) *in weak sense with $\lambda = \lambda_1(\Omega)$, then $u = \alpha\varphi_1$ for some $\alpha \in \mathbb{R}$.*

(iii) Problem (1.24) *admits a sequence of eigenvalues $\{\lambda_k(\Omega)\}_{k\in\mathbb{N}}$ with*

$$0 < \lambda_1(\Omega) < \lambda_2(\Omega) \le \cdots \le \lambda_k(\Omega) \le \lambda_{k+1}(\Omega)\cdots,$$

with corresponding eigenfunctions φ_k, $k \in \mathbb{N}$ and $\lim_{k\to\infty}\lambda_k(\Omega) = +\infty$. Moreover, for any $k \in \mathbb{N}$, the eigenvalue $\lambda_k(\Omega)$ can be characterized as $\lambda_k(\Omega) = \inf_{u\in\mathscr{P}_k(\Omega)} \mathscr{E}_\omega(u,u)$ where $\mathscr{P}_k(\Omega)$ is given by

$$\mathscr{P}_k(\Omega) := \{u \in \mathscr{H}_0^{\log}(\Omega) : \int_\Omega u\varphi_j\,dx = 0 \;\; for \;\; j = 1,2,\cdots k-1 \;\; and \;\; \|\varphi_k\|_{L^2(\Omega)} = 1\}.$$

(iv) The sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ of eigenfunctions corresponding to eigenvalues $\lambda_k(\Omega)$ form a complete orthogonal basis of $L^2(\Omega)$ and an orthogonal system of $\mathscr{H}_0^{\log}(\Omega)$.

Next, using the δ-decomposition technique, we provide the following boundedness result of the eigenfunctions.

Proposition 1.17. *Let $u \in \mathscr{H}_0^{\log}(\Omega)$ and $\lambda > 0$ satisfying* (1.25)*. Then $u \in L^\infty(\Omega)$ and there exists a constant $C := C(N,\Omega) > 0$ such that*

$$\|u\|_{L^\infty(\Omega)} \le C\|u\|_{L^2(\Omega)}.$$

Our next result concerns the Faber-Krahn inequality for the logarithmic Schrödinger operator. We denote by B^* the open ball in $\mathbb{R}^N$ centered at zero with radius determined such that $|\Omega| = |B^*|$. We have

Theorem 1.18 (Faber-Krahn type inequality)**.** *Let $\Omega \subset \mathbb{R}^N$ be open and bounded, and $\lambda_{1,\log}(\Omega)$ be the principal eigenvalue of problem* (1.24)*, then*

$$\lambda_{1,\log}(\Omega) \ge \lambda_{1,\log}(B^*).$$

Moreover, if equality occurs, Ω is a ball. Consequently, if Ω is a ball in $\mathbb{R}^N$, the first eigenfunction $\varphi_{1,\log}$ corresponding to $\lambda_{1,\log}(B)$ is radially symmetric.

Our last result of this chapter is devoted to the small order asymptotic $s \to 0^+$ of the fractional relativistic Schrödinger operator and provide a partial analogue result of Theorem 1.11

Theorem 1.19. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, and $\lambda_{k,s}(\Omega)$ resp. $\lambda_{k,\log}(\Omega)$ be the k-th Dirichlet eigenvalue of $(I-\Delta)^s$ resp. of $(I-\Delta)^{\log}$ on Ω. Then for $s \in (0,1)$, the eigenvalue $\lambda_{k,s}(\Omega)$ satisfies the expansion*

$$\lambda_{k,s}(\Omega) = 1 + s\lambda_{k,\log}(\Omega) + o(s) \quad as \quad s \to 0^+.$$

Moreover, if $(s_n)_n \subset (0,s_0)$, $s_0 > 0$ is a sequence with $s_n \to 0$ as $n \to \infty$, then if $\psi_{1,s}$ is the unique nonnegative L^2-normalized eigenfunction of $(I-\Delta)^s$ corresponding to the principal eigenvalue $\lambda_{1,s}(\Omega)$, we have that

$$\psi_s \to \psi_{1,\log} \quad in \;\; L^2(\Omega) \quad as \quad s \to 0^+,$$

and after passing to a subsequence, we have that

$$\psi_{k,s} \to \psi_{k,\log} \quad in \;\; L^2(\Omega) \quad as \quad s \to 0^+,$$

where $\psi_{1,\log}$, resp. $\psi_{k,\log}$, $k \ge 2$ is the unique nonnegative L^2-normalized eigenfunction resp. a L^2-normalized eigenfunction corresponding to $\lambda_{1,\log}(\Omega)$ resp. to $\lambda_{k,\log}(\Omega)$.

1.1.1.4 Nonlocal operators of small order

This chapter is based on the paper [46]. We are concerned with nonlocal operators of order strictly below one, that is, we consider

$$I_k u(x) = \int_{\mathbb{R}^N} (u(x) - u(y))k(x,y)\, dy,$$

with $k : \mathbb{R}^N \times \mathbb{R}^N \to [0,\infty]$ satisfying

$$k(x,y) = k(y,x) \quad \text{for all } x,y \in \mathbb{R}^N, \text{ and there exists } \sigma \in (0,1) \text{ such that}$$
$$\sup_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N} \min\{1,|x-y|^\sigma\} k(x,y)\,dy < \infty. \tag{1.27}$$

The main goal of this chapter is to investigate interior Sobolev type regularity of weak solutions to the associated Poisson problem depending on the regularity one puts on the source function on the right-hand side.
Let $\Omega \subset \mathbb{R}^N$ be an open, and $u,v \in C_c^{0,1}(\Omega)$ and consider the bilinear form

$$b_{k,\Omega}(u,v) := \frac{1}{2} \int_\Omega \int_\Omega (u(x)-u(y))(v(x)-v(y))k(x,y)\,dxdy, \tag{1.28}$$

where we also write $b_k(u,v) := b_{k,\mathbb{R}^N}(u,v)$, $b_{k,\Omega}(u) := b_{k,\Omega}(u,u)$ and $b_k(u) = b_k(u,u)$. We denote

$$D^k(\Omega) := \{u \in L^2(\Omega) \;:\; b_{k,\Omega}(u) < \infty\},$$

which is a Hilbert space with scalar product

$$\langle\cdot,\cdot\rangle_{D^k(\Omega)} = \langle\cdot,\cdot\rangle_{L^2(\Omega)} + b_{k,\Omega}(u,v).$$

We define also the space

$$\mathscr{D}^k(\Omega) = \{u \in D^k(\mathbb{R}^N) \;:\; 1_{\mathbb{R}^N\setminus\Omega} u \equiv 0\}.$$

Clearly, $\mathscr{D}^k(\Omega) = D^k(\mathbb{R}^N)$ and also the space $\mathscr{D}^k(\Omega)$ is a Hilbert space with scalar product

$$\langle\cdot,\cdot\rangle_{\mathscr{D}^k(\Omega)} = \langle\cdot,\cdot\rangle_{L^2(\Omega)} + b_k(u,v).$$

We first discuss some properties corresponding to function spaces, starting with the following density result

Theorem 1.20. *Let either $\Omega = \mathbb{R}^N$ or $\Omega \subset \mathbb{R}^N$ open and bounded with Lipschitz boundary. In the following, let $X(\Omega) := \mathscr{D}^k(\Omega)$ or $D^k(\Omega)$. Then $C_c^\infty(\Omega)$ is dense in $X(\Omega)$. Moreover, if $u \in X(\Omega)$ is nonnegative, then we have:*

1. *There exists a sequence $(u_n)_n \subset X(\Omega) \cap L^\infty(\Omega)$ with $\lim_{n\to\infty} u_n = u$ in $X(\Omega)$ satisfying that for every $n \in \mathbb{N}$ there is $\Omega'_n \subset\subset \Omega$ with $u_n = 0$ on $\Omega \setminus \Omega'_n$ and $0 \le u_n \le u_{n+1} \le u$.*
2. *There exists a sequence $(u_n)_n \subset C_c^\infty(\Omega)$ with $u_n \ge 0$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} u_n = u$ in $X(\Omega)$.*

We recall that the first Dirichlet eigenvalue of I_k is given by

$$\Lambda_1(\Omega) := \inf_{\substack{u\in\mathscr{D}^k(\Omega)\\ u\neq 0}} \frac{\mathscr{E}_k(u,u)}{\|u\|^2_{L^2(\Omega)}} \in [0,\infty).$$

If the symmetrization function

$$j(z) := \operatorname{essinf}\{k(x, x \pm z) \;:\; z \in \mathbb{R}^N\} \tag{1.29}$$

of k satisfies $|\{j > 0\}| > 0$ and Ω is bounded in one direction, then $\Lambda_1(\Omega) > 0$ by [29, 43]. In the following, we assume the stronger assumption

$$\text{The function } j \text{ given in (1.29) satisfies } \quad \int_{\mathbb{R}^N} j(z)\, dz = \infty. \tag{1.30}$$

Corollary 1.21. *Let k satisfy additionally* (1.29) *and* (1.30) *and let* $\Omega \subset \mathbb{R}^N$ *open and bounded. Then,*

$$\mathscr{D}^k(\Omega) \text{ is compactly embedded in } L^2(\Omega).$$

In particular, there is a sequence of eigenvalues $(\Lambda_n(\Omega))_n$ *of* I_k *with*

$$0 < \Lambda_1(\Omega) < \Lambda_2(\Omega) \leq \ldots \leq \Lambda_n(\Omega) \to \infty \quad \text{for } n \to \infty,$$

that is, $\Lambda_1(\Omega)$ *is simple and the first normalized eigenfunction* φ_1 *of* I_k *can be chosen to be positive in the sense that*

$$\operatorname*{essinf}_K \varphi_1 > 0 \quad \text{for all } K \subset\subset \Omega.$$

Moreover, any eigenfunction of I_k *is bounded. To be precise, given* $\lambda > 0$ *and* $u \in \mathscr{D}^k(\Omega)$ *such that* $I_k u = \lambda u$*, then there is* $C = C(N, \Omega, k, \lambda) > 0$ *such that*

$$\|u\|_{L^\infty(\Omega)} \leq C \|u\|_{L^2(\Omega)}.$$

Next, we consider the function space

$$\mathscr{V}^k(\Omega) := \Big\{ u : \mathbb{R}^N \to \mathbb{R} \;:\; u|_\Omega \in D^k(\Omega) \text{ and, for all } r > 0,\ \sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N \setminus B_r(x)} |u(y)| k(x,y)\, dy < \infty \Big\}.$$

$$\mathscr{V}^k_{loc}(\Omega) := \Big\{ u : \mathbb{R}^N \to \mathbb{R} \;:\; u|_{\Omega'} \in \mathscr{V}^k(\Omega') \text{ for all } \Omega' \subset\subset \Omega \Big\}.$$

Then it follows from the definitions (see also [60, Section 3]) that for $U \subset \Omega \subset \mathbb{R}^N$ open and $u : \mathbb{R}^N \to \mathbb{R}$, the following hold:

$$\mathscr{D}^k(U) \subset \mathscr{D}^k(\Omega) \subset \mathscr{V}^k(\Omega) \subset \mathscr{V}^k(U) \subset \mathscr{V}^k_{loc}(U). \tag{1.31}$$

Given $f \in L^2_{loc}(\Omega)$, we then call $v \in \mathscr{V}^k(\Omega)$ a *(weak) supersolution* of $I_k v = f$ in Ω, if

$$b_k(v, u) \geq \int_\Omega f(x) u(x)\, dx \qquad \text{for all } u \in C_c^\infty(\Omega). \tag{1.32}$$

We emphasize that this definition of supersolution is larger than the one considered in [60]. Using a density result we can then extend the weak and strong maximum principles of [60], which is of independent interest.

Proposition 1.22 (Weak maximum principle)**.** *Define $j:\mathbb{R}^N\to[0,\infty]$ as in* (1.29) *and assume that*

$$j \text{ does not vanish identically on } B_r(0) \text{ for any } r>0. \tag{1.33}$$

Let $\Omega\subset\mathbb{R}^N$ open, $c\in L^\infty_{loc}(\Omega)$, and assume either

1. *$c\le 0$ or*
2. *Ω and c are such that $\|c^+\|_{L^\infty(\Omega)}<\inf_{x\in\Omega}\int_{\mathbb{R}^N\setminus\Omega}k(x,y)\,dy$.*

If $u\in\mathscr{V}^k(\Omega)$ satisfies in weak sense

$$I_k u\ge c(x)u \quad \text{in } \Omega,\ u\ge 0 \text{ almost everywhere in } \mathbb{R}^N\setminus\Omega, \text{ and } \quad \liminf_{|x|\to\infty}u(x)\ge 0,$$

then $u\ge 0$ almost everywhere in $\mathbb{R}^N$.

Proposition 1.23 (Strong maximum principle)**.** *Assume k satisfies additionally* (1.30)*. Let $\Omega\subset\mathbb{R}^N$ open and $c\in L^\infty_{loc}(\Omega)$ with $\|c^+\|_{L^\infty(\Omega)}<\infty$. Moreover, let $u\in\mathscr{V}^k(\Omega)$, $u\ge 0$ satisfy in weak sense $I_k u\ge c(x)u$ in Ω.*

1. *If Ω is connected, then either $u\equiv 0$ in Ω or $\operatorname{essinf}_K u>0$ for any $K\subset\subset\Omega$.*
2. *j given in* (1.29) *satisfies $\operatorname{essinf}_{B_r(0)} j>0$ for any $r>0$, then either $u\equiv 0$ in $\mathbb{R}^N$ or $\operatorname{essinf}_K u>0$ for any $K\subset\subset\Omega$.*

Clearly, if $\Lambda_1(\Omega)$ is positive, then b_k denotes an equivalent scalar product on $\mathscr{D}^k(\Omega)$ and thus for any $f\in L^2(\Omega)$ there is a unique solution $u\in\mathscr{D}^k(\Omega)$ with $I_k u=f$ in Ω.

Before we state the main result of this chapter, we begin with a boundedness result for weak solutions.

Theorem 1.24. *Assume k satisfies* (1.30) *and is such that*

$$\sup_{x\in\mathbb{R}^N}\int_{K\setminus B_\varepsilon(x)}k(x,y)^2\,dy<\infty \quad \text{for all } K\subset\subset\mathbb{R}^N \text{ and } \varepsilon>0. \tag{1.34}$$

Let $\Omega\subset\mathbb{R}^N$ be an open set. Let $f\in L^\infty(\Omega)$, $h\in L^1(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$, and let $u\in\mathscr{V}^k_{loc}(\Omega)$ satisfy in weak sense

$$I_k u\le\lambda u+h*u+f \quad \text{in } \Omega \text{ for some } \lambda>0.$$

If $u^+\in L^\infty(\mathbb{R}^N\setminus\Omega')$ for some $\Omega'\subset\subset\Omega$, then $u^+\in L^\infty(\mathbb{R}^N)$ and there is $C=C(\Omega,\Omega',k,h,\lambda)>0$ such that

$$\|u^+\|_{L^\infty(\Omega')}\le C\Big(\|f\|_{L^\infty(\Omega)}+\|u\|_{L^2(\Omega')}+\|u^+\|_{L^\infty(\mathbb{R}^N\setminus\Omega')}\Big).$$

As a consequence of Theorem 1.24 for $u\in\mathscr{D}^k(\Omega)$, we have the following theorem.

Theorem 1.25. *Assume k satisfies* (1.30) *and* (1.34). *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Let $f \in L^\infty(\Omega)$, $h \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, and let $u \in \mathscr{D}^k(\Omega)$ satisfy in weak sense $I_k u = \lambda u + h * u + f$ in Ω for some $\lambda > 0$. Then $u \in L^\infty(\mathbb{R}^N)$ and there is $C = C(\Omega, k, \lambda, h) > 0$ such that*

$$\|u\|_{L^\infty(\Omega)} \leq C\Big(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^2(\Omega)}\Big).$$

In the particular case, where the kernel is translation invariant, that is, there is a function $J : \mathbb{R}^N \to [0,\infty]$ with $k(x,y) = J(x-y)$ for $x, y \in \mathbb{R}^N$, we are also able to recover differentiability of a solution u to the problem $I_k u = f$, if f and J satisfy certain regularity properties.

Our main result of the chapter is the following.

Theorem 1.26. *Assume k satisfies* (1.30) *and let $\Omega \subset \mathbb{R}^N$ open and bounded with Lipschitz boundary. Then for any $f \in L^2(\Omega)$ there is a unique solution $u \in \mathscr{D}^k(\Omega)$ of $I_k u = f$. Moreover, if k satisfies additionally* (1.30) *and $f \in L^\infty(\Omega)$, then $u \in L^\infty(\Omega)$ and there is $C = C(N, \Omega, k) > 0$ such that*

$$\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}.$$

Furthermore, if k satisfies (1.27) *with $\sigma < \frac{1}{2}$ and*

$$\begin{aligned} &\text{there is } J : \mathbb{R}^N \to [0,\infty] \text{ such that } k(x,y) = J(x-y), \text{ where } J \text{ satisfies for some } m \in \mathbb{N} \cup \{\infty\}\text{:}\\ &(A)_m \quad \text{It holds } J \in W^{n,1}(\mathbb{R}^N \setminus B_\varepsilon(0)) \text{ for any } \varepsilon > 0,\ n \leq 2m \text{ and, for some } C_J > 0,\\ &\qquad\quad |\nabla J(z)| \leq C_J |z|^{-1-\sigma-N} \text{ for } 0 < |z| \leq 3 \text{ with } \sigma \text{ as in (1.27),} \end{aligned} \tag{1.35}$$

then, if $m \in \mathbb{N}$ and $f \in C^{2m}(\overline{\Omega})$, we have $\partial^\beta u \in L^2_{loc}(\Omega)$ for all $\beta \in \mathbb{N}_0^N$, $|\beta| \leq m$, $u \in H^m_{loc}(\Omega)$, and for every $\Omega' \subset\subset \Omega$ there is $C = C(N, \Omega, \Omega', k, \beta) > 0$ such that

$$\|\partial^\beta u\|_{L^2(\Omega')} \leq C\|f\|_{C^{2m}(\Omega)}.$$

In particular, for $m = \infty$, we have $u \in C^\infty(\Omega)$.

We note from (1.31) that Theorem 5.6 is a particular case of more general result (see Section 5.6 and 5.7), for functions which are in a certain sense locally in $\mathscr{V}^k(\Omega)$. This general result also includes the eigenvalue problem and yields the following theorem.

Theorem 1.27. *If in the situation of Corollary 1.21 the kernel k additionally satisfies* (1.35) *with $m = \infty$, then every function $u \in \mathscr{D}^k(\Omega)$ satisfying $I_k u = \lambda u$ in Ω for some $\lambda \in \mathbb{R}$ also belongs to $C^\infty(\Omega)$.*

The proof of Theorem 1.26 uses Theorem 1.25 and an intermediate estimate in Nikol'skii and in classical Sobolev spaces.

We point out that using a probabilistic and potential theoretic approach, a local smoothness of bounded harmonic solutions solving in a certain very weak sense $I_k u = 0$ in Ω, have been obtained in [56, Theorem 1.7] for radial kernel functions using the same regularity $(A)_m$ (1.35) (see also [58, 79]). See also [54] for related regularity properties of solutions. Our approaches only exploits the variational structure of the problem and uses purely analytic properties of the operator.

Let us recall the notation and properties of Sobolev and Nikol'skii spaces as introduced in [31, 96]. For $p \in [1,\infty)$, if $k \in \mathbb{N}_0$, we consider the Sobolev space as usual,

$$W^{k,p}(\Omega) := \Big\{ u \in L^p(\Omega) \;:\; \partial^\alpha u \text{ exists for all } \alpha \in \mathbb{N}_0^n,\ |\alpha| \le k \text{ and belongs to } L^p(\Omega) \Big\}$$

for the Banach space of k-times (weakly) differentialable functions in $L^p(\Omega)$ and in the particular case $p = 2$ the space $H^s(\Omega) := W^{s,2}(\Omega)$ is a Hilbert space. For $u : \Omega \to \mathbb{R}$ and $h \in \mathbb{R}$ let $\Omega_h := \{x \in \Omega \;:\; \operatorname{dist}(x, \partial\Omega) > h\}$ and, with $e \in \partial B_1(0)$, we let

$$\delta_h u(x) = \delta_{h,e} u(x) := u(x + he) - u(x).$$

Moreover, for $l \in \mathbb{N}$ with $l > 1$, let $\delta_h^l u(x) = \delta_h(\delta_h^{l-1} u)(x)$. and for $s = k + \sigma > 0$ with $k \in \mathbb{N}_0$ and $\sigma \in (0,1]$ define the Nikol'skii spaces

$$N^{s,p}(\Omega) := \Big\{ u \in W^{k,p}(\Omega) \;:\; [\partial^\alpha u]_{N^{\sigma,p}(\Omega)} < \infty \text{ for all } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| = k \Big\},$$

where

$$[u]_{N^{\sigma,p}(\Omega)} = \sup_{\substack{e \in \partial B_1(0) \\ h > 0}} h^{-\sigma} \|\delta_{h,e}^2 u\|_{L^p(\Omega_{2h})}.$$

It follows that $N^{s,p}(\Omega)$ is a Banach space with norm $\|u\|_{N^{s,p}(\Omega)} := \|u\|_{W^{k,p}(\Omega)} + \sum_{|\alpha|=k} [\partial^\alpha u]_{N^{\sigma,p}(\Omega)}$. We then have the following embedding.

Proposition 1.28 (see e.g. Propositions 3 and 4 in [31])**.** *Let $\Omega \subset \mathbb{R}^N$ be an open with C^∞ boundray. Moreover, let $t > s > 0$ and $1 \le p < \infty$. Then*

$$N^{t,p}(\Omega) \subset W^{s,p}(\Omega) \subset N^{s,p}(\Omega).$$

In the following, $\Omega \subset \mathbb{R}^N$ is an open bounded set and k is in particular such that, there is $J : \mathbb{R}^N \to [0,\infty]$ such that $k(x,y) = J(x - y)$ for $x, y \in \mathbb{R}^N$. Moreover, given σ from assumption (1.27) we assume that $\sigma < \frac{1}{2}$ and fix

$$\alpha := 1 - \sigma \in (\frac{1}{2}, 1).$$

The following Theorem is the first step towards Nikol'skii spaces, since, the idea is to use Proposition 1.28 later.

Theorem 1.29. *Let $f \in C^1(\overline{\Omega})$, $\lambda \in \mathbb{R}$ and $u \in \mathcal{V}_{loc}^J(\Omega) \cap L^\infty(\mathbb{R}^N)$ satisfy in weak sense $I_k u = f + \lambda u$ in Ω. Then for any $\Omega' \subset\subset \Omega$ there is $C = C(N, \Omega, \Omega', J, \lambda) > 0$ such that*

$$\|\delta_{h,e} u\|_{L^2(\Omega')} \le h^\alpha C \Big(\|f\|_{C^1(\Omega)}^2 + \|u\|_{L^\infty(\mathbb{R}^N)}^2 \Big)^{\frac{1}{2}} \quad \textit{for all } h > 0,\ e \in \partial B_1(0). \tag{1.36}$$

In other to show that solutions belong to Nikol'skii spaces and use Proposition 1.28, we need to iterate the result in Theorem 1.29. We have the following corollary.

Corollary 1.30. *Assume $m = 1$ in (1.35). Let $f \in C^2(\overline{\Omega})$, $\lambda \in \mathbb{R}$, and let $u \in \mathcal{V}^J_{loc}(\Omega) \cap L^\infty(\mathbb{R}^N)$ satisfy in weak sense $I_k u = \lambda u + f$ in Ω. Then $u \in H^1(\Omega')$ and $\partial_i u \in D^J(\Omega')$ for any $\Omega' \subset\subset \Omega$. More precisely, with α as above there is for any $\Omega' \subset\subset \Omega$ a constant $C = C(N, \Omega, \Omega', J, \lambda) > 0$ such that*

$$\sup_{\substack{e \in \partial B_1(0) \\ h>0}} h^{-2\alpha} \|\delta^2_{h,e} u\|_{L^2(\Omega')} \leq C\Big(\|f\|^2_{C^2(\Omega)} + \|u\|_{L^2(\Omega')} + \|u\|^2_{L^\infty(\mathbb{R}^N \setminus \Omega')}\Big)^{\frac{1}{2}},$$

so that $u \in N^{2\alpha,2}(\Omega') \subset H^1(\Omega')$, that is, there is also $C' = C'(n, J, \Omega, \Omega', \alpha, \lambda) > 0$ such that

$$\|\nabla u\|_{L^2(\Omega')} \leq C'\Big(\|f\|^2_{C^2(\Omega)} + \|u\|_{L^2(\Omega')} + \|u\|^2_{L^\infty(\mathbb{R}^N \setminus \Omega')}\Big)^{\frac{1}{2}}$$

and, moreover,

$$b_{J,\Omega'}(\partial_i u, \partial_i u) \leq C' \quad \text{for } i = 1, \dots, N.$$

From Corollary 1.30, one iterate further to get the following corollary.

Corollary 1.31. *Let $f \in C^{2m}(\overline{\Omega})$, $\lambda \in \mathbb{R}$, and let $u \in \mathcal{V}^J_{loc}(\Omega) \cap L^\infty(\mathbb{R}^N)$ satisfy in weak sense $I_k u = \lambda u + f$ in Ω. Then $u \in H^m(\Omega')$ for any $\Omega' \subset\subset \Omega$ and there is $C = C(n, J, \Omega, \Omega', m) > 0$ such that*

$$\|u\|_{H^m(\Omega')} \leq C\Big(\|f\|^2_{C^m(\Omega)} + \|u\|^2_{L^2(\Omega')} + \|u\|^2_{L^\infty(\mathbb{R}^N \setminus \Omega')}\Big)^{\frac{1}{2}}.$$

In particular, if $f \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$.

Outline of the thesis: The rest of the thesis is organised as follows.

Chapter 2 contains the results from paper [39] on Morse index versus radial symmetry for fractional Dirichlet problems. Chapter 3 presents the results on small order asymptotic of the Dirichlet eigenvalue problem for the fractional Laplacian from paper [47]. Chapter 4 is devoted to the results on the logarithmic Schrödinger operator and associated Dirichlet prolems from paper [45] and finally, Chapter 5 presents the results on nonlocal operators of small order from paper [46]. All chapters are structured and presented in the same structure as the original papers without major changes.

1.2 Zusammenfassung

Die vorliegende Dissertation ist der Untersuchung nichtlokaler Dirichletprobleme für singuläre Integralgleichungen mit Operatoren niedriger Ordnung gewidmet. Sie beinhaltet die folgenden Forschungsarbeiten:

[P1] M. M. Fall, P. A. Feulefack, R. Y. Temgoua and T. Weth. *Morse index versus radial symmetry for fractional Dirichlet problems.* Advances in Mathematics 384 (2021): 107728. doi.org/10.1016/j.aim.2021.107728.

[P2] P. A. Feulefack, S. Jarohs and T. Weth. *Small order asymptotics of the Dirichlet eigenvalue problem for the fractional Laplacian.* (2020) arxiv.org/abs/2010.10448. Journal of Fourier Analysis and Applications 28, 18 (2022). doi.org/10.1007/s00041-022-09908-8.

[P3] P. A. Feulefack. *The logarithmic Schrödinger operator and associated Dirichlet problems.* (2021) arxiv.org/abs/2112.08783.

[P4] P. A. Feulefack and S. Jarohs. *Nonlocal operators of small order* (2021), arxiv.org/abs/2112.09364.

Das Hauptresultat der Arbeit [P1] liefert eine Abschätzung an den Morse-Index vorzeichenwechselnder radialer beschränkter Lösungen u des semilinearen Dirichletproblems

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \mathscr{B}, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \mathscr{B}. \end{cases} \tag{1.37}$$

in der Einheitskugel $\mathscr{B} \subset \mathbb{R}^N$ mit $s \in (0,1)$. Hier und im Folgenden sei $(-\Delta)^s$ der fraktionale Laplace-Operator, welcher, für $s \in (0,1)$, als spezieller singulärer Integraloperator für hinreichend glatte Funktionen durch

$$(-\Delta)^s u(x) = c_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \quad x \in \mathbb{R}^N$$

gegeben ist. Die Normierungskonstante

$$c_{N,s} = s(1-s)\pi^{-N/2} 2^{2s} \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(2-s)} \tag{1.38}$$

sei hier wie üblich so gewählt, dass das Fouriersymbol von $(-\Delta)^s$ durch $\xi \mapsto |\xi|^{2s}$ gegeben ist. Ferner sei die Nichtlinearität f in (1.37) als stetig differenzierbar vorausgesetzt.

Wir zeigen im Fall $s \in (\frac{1}{2}, 1)$, dass jede solche Lösung einen Morse-Index größer gleich $N+1$ hat. Im Fall $s \in (0, \frac{1}{2}]$ ist die gleiche Abschätzung unter der zusätzlichen subkritischen Wachstumsbedingung

$$\int_0^t f(\tau)d\tau > \frac{N-2s}{2N} t f(t) \qquad \text{für } t \in \mathbb{R} \setminus \{0\}. \tag{1.39}$$

an f gültig. Dieses Resultat erweitert eine Abschätzung von A. Aftalion and F. Pacella für den Fall $s = 1$, d.h. für das zugehörige lokale Problem zweiter Ordnung

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathscr{B}, \\ u = 0 & \text{auf } \partial \mathscr{B}. \end{cases}$$

Dem Beweis der Morse-Index-Abschätzung liegt die gleiche Strategie wie bei Aftalion und Pacella zugrunde. Diese basiert auf der Konstruktion geeigneter Testfunktionen mittels partieller Ableitungen von u zur Abschätzung der zum linearisierten Operator $(-\Delta)^s - f'(u)$ gehörenden quadratischen Form. Im nichtlokalen Fall $s \in (0,1)$ müssen dabei aber erhebliche zusätzliche Schwierigkeiten überwunden werden, welche insbesondere mit der geringeren Randregularität von Lösungen von (1.37) und der Nichtverfügbarkeit eines lokalen Hopf-Lemmas für vorzeichenwechselnde Lösungen zusammenhängen. Die im Fall $s \in (0, \frac{1}{2}]$ geringere Randregularität der Lösung ist auch der Grund für die Zusatzbedingung 2.6, welche aber zumindest für homogene Funktionen der Form

$$u \mapsto f(u) = \lambda |u|^{p-2} u \qquad \text{mit } \lambda > 0 \text{ und } 2 \leq p < \frac{2N}{N-2s}$$

erfüllt ist.
Durch Anwendung der Morse-Index-Abschätzung auf den *linearen* Fall $f(u) = \lambda u$ beweisen wir, dass unabhängig von der Ordnung $s \in (0,1)$ jede Eigenfunktion des fraktionalen Laplace-Operators $(-\Delta)^s$ in $\mathscr{B}$ zum zweiten Dirichleteigenwert eine antisymmetrische Funktion ist, also $u(-x) = -u(x)$ für $x \in \mathscr{B}$ erfüllt. Dies bestätigt eine Vermutung von Bañuelos and Kulczycki, welche bisher nur in den Fällen

$$N \leq 3, \quad s \in (0,1) \qquad \text{sowie} \qquad 4 \leq N \leq 9, \quad s = \frac{1}{2}$$

nachgewiesen werden konnte.

Die Resultate der Arbeit [P2] beziehen sich auf die spektrale Asymptotik der Dirichleteigenwerte und zugehörige Eigenfunktionen zum Problem

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

im Limes verschwindender Ordnung $s \to 0^+$, wobei hier $\Omega \subset \mathbb{R}^N$ eine beschränkte offene Menge mit Lipschitzrand sei. Genauer zeigen wir für die zugehörigen Dirichleteigenwerte $\lambda_{k,s}(\Omega)$, $k \in \mathbb{N}$ die Asymptotik

$$\lambda_{k,s}(\Omega) = 1 + s\lambda_{k,L}(\Omega) + o(s) \qquad \text{für } s \to 0^+,$$

wobei der erste nichttrivale Term $\lambda_{k,L}(\Omega)$ in dieser Entwicklung als k-ter Eigenwert des logarithmischen Laplace-Operators L_Δ gegeben ist. Dieser Operator ist formal als Ableitung

$$L_\Delta = \frac{d}{ds}\Big|_{s=0} (-\Delta)^s$$

und somit als schwach singulärer Integraloperator mit Fouriersymbol $2\log|\xi|$ definiert. Die Integraldarstellung von L_Δ ist dabei durch

$$L_\Delta u(x) = C_N \int_{\mathbb{R}^N} \frac{u(x)1_{B_1(x)}(y) - u(y)}{|x-y|^N}\, dy + \rho_N u(x),$$

gegeben. Die hier auftauchenden Konstanten sind dabei durch die Asymptotik der Normierungskonstante $c_{N,s}$ in (1.38) festgelegt; genauer ist $C_N = \pi^{-\frac{N}{2}}\Gamma(\frac{N}{2})$ und $\rho_N = 2\log 2 + \psi(\frac{N}{2}) - \gamma$, wobei $\psi = \frac{\Gamma'}{\Gamma}$ die Digamma-Funktion und $\gamma = -\Gamma'(1)$ die Euler-Mascheroni-Konstante bezeichne.

Unser Resultat verallgemeinert ein früheres Ergebnis von H. Chen und T. Weth, welches auf den niedrigsten Eigenwert beschränkt war. Zudem verbessern wir das von Chen und Weth bewiesene L^2-Konvergenzresultat für die zugehörige, geeignet normierte Familie von Eigenfunktionen $u_{1,s}$, indem wir die relative Kompaktheit der Menge

$$\{u_{1,s} \,:\, s \in (0, \frac{1}{4}]\}$$

in $C(K)$ für jede kompakte Teilmenge $K \subset \Omega$ zeigen. Dies liefert die lokal gleichmäßige Konvergenz

$$u_{1,s} \to u_{1,L} \qquad \text{in } \Omega, \tag{1.40}$$

wobei $u_{1,L}$ die (bis auf das Vorzeichen und Normierung) eindeutige erste Dirichlet-Eigenfunktion von L_Δ bezeichne.

Darüber hinaus verallgemeinern wir die obige Kompaktheitsaussage auf Eigenfunktionen $u_{k,s}$ zu höheren Eigenwerten $\lambda_{k,s}$, $k \in \mathbb{N}$. Falls zudem Ω eine äußere Sphärenbedingung erfüllt, so ist die Konvergenz sogar uniform und die Menge $\{u_{k,s} \,:\, s \in (0, \frac{1}{4}]\}$ für jedes $k \in \mathbb{N}$ relativ kompakt im Raum

$$C_0(\Omega) := \{u \in C(\mathbb{R}^N) : \; u \equiv 0 \;\text{ in }\; \Omega^c\}.$$

Somit erhält man auch eine Variante der Konvergenzaussage (1.40) für höhere Eigenfunktionen, wobei man aufgrund der möglichen Vielfachheit der Eigenwerte zu Teilfolgen übergehen muss.

Für den Beweis dieser spektralen Asymptotik etablieren wir neue s-unabhängige uniforme Regularitätsabschätzungen und uniforme Schranken für das Wachstumsverhalten der Eigenfunktionen am Gebietsrand. Als Konsequenz dieser uniformen Abschätzungen erhalten wir zudem Regularitätseigenschaften für Eigenfunktionen des logarithmischen Laplace-Operators.

Die Arbeit [P3] ist der Untersuchung des logarithmischen Schrödingeroperators $(I-\Delta)^{\log}$ gewidmet, welcher formal über das Fouriersymbol $\xi \mapsto \log(1+|\xi|^2)$ definiert ist. Wir präsentieren eine alternative Methode zur Herleitung der Darstellung von $(I-\Delta)^{\log}$ als singulärer Integraloperator in der Form

$$(I-\Delta)^{\log} u(x) = d_N \int\limits_{\mathbb{R}^N} \frac{u(x)-u(x+y)}{|y|^N} \omega(|y|)\, dy,$$

mit $d_N = \pi^{-\frac{N}{2}}$ und $\omega(r) = 2^{1-\frac{N}{2}} r^{\frac{N}{2}} K_{\frac{N}{2}}(r)$, wobei K_ν die modifizierte Besselfunktion zweiter Art vom Index ν sei. Wir zeigen, dass dieser Operator als Ableitung in s des fraktionalen relativistischen Schrödingeroperators $(I-\Delta)^s$ bei $s=0$ auftaucht.
Wir untersuchen variationelle Probleme für diesen Operator mit Hilfe nichtprobabilistischer und aus Sicht der partiellen Differentialgleichungen leichter zugänglicher Methoden. Insbesondere charakterisieren wir die Dirichleteigenwerte und zugehörige Eigenfunktionen von $(I-\Delta)^{\log}$ in einer offenen beschränkten Teilmenge $\Omega \subset \mathbb{R}^N$ u.a. mittels der Asymptotik des Dirichlet-Eigenwertproblems für den fraktionalen relativistischen Operator $(I-\Delta)^s$. Des Weiteren beweisen wir eine Ungleichung vom Faber-Krahn-Typ für den ersten Eigenwert von $(I-\Delta)^{\log}$.

Die Arbeit [P4] beschäftigt sich speziell mit singulären Integraloperatoren der Form

$$I_k u(x) = \int\limits_{\mathbb{R}^N} (u(x)-u(y))k(x,y)\, dy$$

mit einer symmetrischen Kernfunktion $k : \mathbb{R}^N \times \mathbb{R}^N \to [0,\infty]$, welche die Bedingung

$$\sup_{x\in\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \min\{1, |x-y|^\sigma\} k(x,y)\, dy < \infty$$

für ein $\sigma \in (0,1)$ erfüllen möge. Die Operatoren dieser Klasse sind also von einer Ordnung kleiner als eins.
Unter geeigneten weiteren Voraussetzungen an k leiten wir zunächst Dichtheitsaussagen für assoziierte Funktionenräume und Maximumsprinzipien im Zusammenhang mit dem Operator I_k her. Darauf aufbauend untersuchen wir Regularitätseigenschaften schwacher Lösungen u des zugehörigen Poissonproblems

$$I_k u = f$$

in einer offenen Teilmenge $\Omega \subset \mathbb{R}^N$ in Abhängigkeit der Regularität der Funktion f. Für translationsinvariante Kernfunktionen k zeigen wir insbesondere lokale H^1-Regularität schwacher Lösungen, falls f von der Klasse C^2 in Ω ist. In einem weiteren Resultat setzen wir zusätzlich ausreichende Regularität der Kernfunktion jenseits des Singulärbereichs auf der Diagonale in $\mathbb{R}^N \times \mathbb{R}^N$ voraus und zeigen die C^∞-Regularität der Lösungen unter der Annahme, dass die Funktion f ebenfalls von der Klasse C^∞ ist. Als Folgerung erhalten wir, dass jede Eigenfunktion des Dirichlet-Eigenwertproblems zur Gleichung $L_k u = \lambda u$ in Ω eine C^∞-Funktion in Ω ist.

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1.3 Notation

The number $N \in \mathbb{N}$ will denote the dimension of the Euclidean space $\mathbb{R}^N$ which shall be our space of reference in the manuscript, and for any $x \in \mathbb{R}^N$, we put $|x|^2 = \sum_{j=1}^N |x_j|^2$ the Euclidean norm. We let $\omega_{N-1} = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}$ denotes the measure of the unit sphere in $\mathbb{R}^N$ and for sets $A, B \subset \mathbb{R}^N$, the notation $A \subset\subset B$ means $\bar{A}$ is compact and contained in the interior of B. For sets $A_1, A_2 \subset \mathbb{R}^N$ we set

$$\operatorname{dist}(A_1, A_2) := \inf\{|x-y|,\ x \in A_1, y \in A_2\}$$

If $A_1 = \{x\}$ for $x \in \mathbb{R}^N$, we simply write $\operatorname{dist}(x, A_2)$, in particular we define

$$\delta_A(x) := \operatorname{dist}(x, A^c) \ \text{ with } \ A^c = \mathbb{R}^N \setminus A, \ \text{ the complement of } \ A.$$

If A is measurable, then $|A|$ denotes its Lebesgue measure. Moreover, for a given $r > 0$, we let

$$B_r(A) := \{x \in \mathbb{R}^N \ : \ \operatorname{dist}(x, A) < r\}$$

and $B_r(x) := B_r(\{x\})$ simply denotes the ball of radius r with x as its center. If $x = 0$ we also write B_r instead of $B_r(0)$. For $a > 0$ and $b \geq 0$,

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t}\, dt \qquad \text{and} \quad \Gamma(a,b) = \int_b^\infty t^{a-1} e^{-t}\, dt$$

stands for the Gamma and incomplete Gamma function respectively on $(0, +\infty)$.
If A is open, we denote by $\mathscr{C}(A)$, the space (class) of function $u : \mathbb{R}^N \to \mathbb{R}$ which are continuous and the subclasses $\mathscr{C}_c^k(A)$, the space of function $u : \mathbb{R}^N \to \mathbb{R}$ which are k-times continuously differentiable and with support compactly contained in A, $\mathscr{C}_0(A)$ the space of function in $\mathscr{C}(\mathbb{R}^N)$ that vanish in $\mathbb{R}^N \setminus A$ i.e.

$$\mathscr{C}_0(A) = \{u \in \mathscr{C}(\mathbb{R}^N) : \ u = 0 \text{ in } \mathbb{R}^N \setminus A\}.$$

If f and g are two functions, then, $f \sim g$ as $x \to a$ if $\frac{f(x)}{g(x)}$ converges to a constant as x converges to a.
For a function $u : \mathbb{R}^N \to \mathbb{R}$, we denote by $u^+ := \max\{u, 0\}$ and $u^- := -\min\{u, 0\}$ the positive part of and the negative part of u respectively such that $u = u^+ - u^-$. Moreover, we let

$$\operatorname{osc}_A u := \sup_A u - \inf_A u \in [0, \infty),$$

denote the oscillation of u over A. The notation ∂A will be the boundary of A and the notation $1_A : A \to \mathbb{R}$ is the characteristic function of A given by

$$1_A(x) := \begin{cases} 1 & \text{for} \quad x \in A \\ 0 & \text{for} \quad x \in \mathbb{R}^N \setminus A. \end{cases}$$

2 Morse index versus radial symmetry for fractional Dirichlet problems

The results of this chapter is based on the article [39], joint work with Mouhamed Moustapha Fall, Rémi Yvant Temgoua and Tobias Weth. The chapter is self-contained and can be read independently. In fact, we provide an estimate of the Morse index of bounded radially sign changing weak solutions to problem (2.1). In particular, our result applies to Dirichlet eigenvalue problem for the fractional Laplacian in the unit ball, resolving thereby a conjecture by Bañuelos and Kulczycki on the geometry structure of the second Dirichlet eigenfunctions for the fractional Laplacian. The chapter is organized in the same structure as the published article and only acknowledgements is removed.

2.1 Introduction and main results

The purpose of this paper is to estimate the Morse index of radial sign changing solutions of the problem

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \mathscr{B} \\ u = 0 & \text{in } \mathbb{R}^N \setminus \mathscr{B}, \end{cases} \tag{2.1}$$

where $s \in (0,1)$, $\mathscr{B} \subset \mathbb{R}^N$ is the unit ball centred at zero and where the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is of class C^1. The fractional Laplacian operator $(-\Delta)^s$ is defined for all $u \in C^2_c(\mathbb{R}^N)$ by

$$(-\Delta)^s u(x) = c(N,s) \lim_{\varepsilon \to 0^+} \int\limits_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy,$$

where $c(N,s) = 2^{2s}\pi^{-\frac{N}{2}} s \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(1-s)}$ is a normalization constant. The operator $(-\Delta)^s$ can be seen as the infinitesimal generator of an isotropic stable Lévy processes (see [5]), and it arises in specific mathematical models within several areas of physics, biology, chemistry and finance (see [5, 6, 24]). For basic properties of $(-\Delta)^s$ and associated function spaces, we refer to [33].

In recent years, the study of linear and nonlinear Dirichlet boundary value problems involving fractional Laplacian has attracted extensive and steadily growing attention, whereas, in contrast to the local case $s = 1$, even basic questions still remain largely unsolved up to now. Even in the linear case where $f(t) := \lambda t$, the structure of Dirichlet eigenvalues and eigenfunctions of the fractional Laplacian on the unit ball $\mathscr{B}$ is not completely understood. In particular, we mention a conjecture of Bañuelos and Kulczycki which states that every Dirichlet eigenfunction u of $(-\Delta)^s$ on $\mathscr{B}$ corresponding to the second Dirichlet eigenvalue is antisymmetric, i.e., it satisfies $u(-x) = -u(x)$ for $x \in \mathscr{B}$. So far, by the results in [7, 36, 44, 68], this conjecture has been verified in the special cases $N \leq 3$, $s \in (0,1)$ and $4 \leq N \leq 9$, $s = \frac{1}{2}$. In the present paper, we will derive the full conjecture essentially as a corollary of our main result on the semilinear Dirichlet problem (2.1), see Theorem 2.2 below.

Our main result on sign changing radial solutions of (2.1) is heavily inspired by the seminal work of Aftalion and Pacella [1], where the authors studied qualitative properties of sign changing solutions of the local semilinear elliptic problem

$$-\Delta u = f(u) \quad \text{in } \Omega, \quad u = 0 \text{ on } \quad \partial\Omega, \tag{2.2}$$

where $\Omega \subset \mathbb{R}^N$ is a ball or an annulus centered at zero and $f \in C^1(\mathbb{R})$. It is proved in [1, Theorem 1.1] that any radial sign changing solution of (2.2) has Morse index greater than or equal to $N+1$.

In the following, we present a nonlocal version of this result in the case where Ω is the unit ball in $\mathbb{R}^N$. We need to fix some notation first. Consider the function space

$$\mathscr{H}_0^s(\mathscr{B}) := \{u \in H^s(\mathbb{R}^N) : u \equiv 0 \text{ on } \mathbb{R}^N \setminus \mathscr{B}\} \subset H^s(\mathbb{R}^N). \tag{2.3}$$

By definition, a function $u \in \mathscr{H}_0^s(\mathscr{B}) \cap L^\infty(\mathscr{B})$ is a weak solution of (2.1) if

$$\mathscr{E}_s(u,v) = \int_{\mathscr{B}} f(u)v\,dx \qquad \text{for all } v \in \mathscr{H}_0^s(\mathscr{B}),$$

where

$$(v,w) \mapsto \mathscr{E}_s(v,w) := \frac{c(N,s)}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^{N+2s}}\,dxdy. \tag{2.4}$$

is the bilinear form associated with $(-\Delta)^s$. By definition, the *Morse index* $m(u)$ of a weak solution $u \in \mathscr{H}_0^s(\mathscr{B}) \cap L^\infty(\mathscr{B})$ of (2.1) is the maximal dimension of a subspace $X \subset \mathscr{H}_0^s(\mathscr{B})$ where the quadratic form

$$(v,w) \mapsto \mathscr{E}_s(v,w) - \int_{\mathscr{B}} f'(u)vw\,dx \tag{2.5}$$

associated to the linearized operator $L := (-\Delta)^s - f'(u)$ is negative definite. Equivalently, $m(u)$ can be defined as the number of the negative Dirichlet eigenvalues of L counted with their multiplicity.

Our first main result reads as follows.

Theorem 2.1. *Let u be a radially symmetric sign changing solution of problem* (2.1)*, and suppose that* <u>*one*</u> *of the following additional conditions holds.*

(A1) $s \in (\frac{1}{2}, 1)$.

(A2) $s \in (0, \frac{1}{2}]$, *and*

$$\int_0^t f(\tau)d\tau > \frac{N-2s}{2N}\,tf(t) \qquad \text{for } t \in \mathbb{R}\setminus\{0\}. \tag{2.6}$$

Then u has Morse index greater than or equal to $N+1$.

We briefly comment on the inequality (2.6). In our proof of Theorem 2.1, this assumption arises when we use the Pohozaev identity for the fractional Laplacian, see [83, Theorem 1.1]. It is satisfied for homogeneous nonlinearities with subcritical growth, i.e., if

$$f(t) = \lambda |t|^{p-2} t \qquad \text{with} \quad \lambda > 0 \quad \text{and} \quad 2 \leq p < \frac{2N}{N-2s}.$$

We also note that, in the supercritical case where $\int_0^t f(\tau)d\tau < \frac{N-2s}{2N} t f(t)$ for $t \in \mathbb{R} \setminus \{0\}$, problem (2.1) does not admit any nontrivial weak solutions $u \in \mathcal{H}_0^s(\mathcal{B}) \cap L^\infty(\mathcal{B})$. This is a consequence of the Pohozaev identity stated in [83, Theorem 1.1].

In particular, assumption (2.6) is satisfied in the linear case $t \mapsto \lambda t$ with $\lambda > 0$. In fact, we can deduce the following result for the Dirichlet eigenvalue problem

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \mathcal{B} \\ \qquad u = 0 & \text{in } \mathbb{R}^N \setminus \mathcal{B}, \end{cases} \tag{2.7}$$

from Theorem 2.1, thereby providing a complete positive answer to a conjecture by Bañuelos and Kulczycki (see [36]).

Theorem 2.2. *Let $N \geq 1$ and $0 < s < 1$, and let $\lambda_2 > 0$ be the second eigenvalue of problem (2.7). Then every eigenfunction u corresponding to λ_2 is antisymmetric, i.e. it satisfies*

$$u(-x) = -u(x) \quad \text{for } x \in \mathcal{B}.$$

In recent years, partial results towards this conjecture have been obtained in [7, 36, 44, 68], covering the special cases $N \leq 3$, $s \in (0,1)$ and $4 \leq N \leq 9$, $s = \frac{1}{2}$. More precisely, in [7, Theorem 5.3], Bañuelos and Kulczycki proved antisymmetry of second eigenfunctions in the special case $N = 1$, $s = \frac{1}{2}$. In [68], this result was extended to $N = 1$, $s \in [\frac{1}{2}, 1)$. Recently in [36], the conjecture was proved in the cases $N \leq 2$, $s \in (0,1)$ and $3 \leq N \leq 9$, $s = \frac{1}{2}$. Moreover, in [44], the result has been proved for $N = 3$, $s \in (0,1)$.

While the proofs in these papers are based on fine eigenvalue estimates, our proof of Theorem 2.2 is completely different: In addition to Theorem 2.1, we shall only use the following important alternative which is implicitely stated in [36, p. 503]: *Either (2.7) admits a radially symmetric eigenfunction corresponding to the second eigenvalue λ_2, or every eigenfunction corresponding to λ_2 is a product of a linear and a radial function.* Since every such eigenfunction u is a sign changing solution of (2.1) with $t \mapsto f(t) = \lambda_2 t$ and has Morse index $1 < N+1$, it cannot be radially symmetric as a consequence of Theorem 2.1. Hence u must be a product of a linear and a radial function, and therefore u is antisymmetric. This completes the proof of Theorem 2.2. For a more detailed presentation of this argument and the underlying results from [36], see Section 2.5 below.

We briefly comment on the proof of Theorem 2.1. The general strategy, inspired by the paper [1] of Aftalion and Pacella for the local problem (2.2), is to use partial derivatives of u to construct suitable test functions which allow to estimate the Morse index of u. In the nonlocal case, several difficulties arise since local PDEs techniques do not apply. The most severe difficulty is related to the fact that weak solutions $u \in \mathcal{H}_0^s(\mathcal{B}) \cap L^\infty(\mathcal{B})$ of (2.1) have much

less boundary regularity than solutions of (2.2), see Proposition 2.7 for details. Moreover, even though there exists a fractional version of the Hopf boundary lemma related to the fractional boundary derivative $\frac{u}{\delta^s}$ (see [40, Proposition 3.3]), it does not apply to sign changing solutions of (2.1) due to the non-locality of the problem. We mention at this point that the classical Hopf boundary lemma is used in [1] together with an extra assumption on $f(0)$, but a slight change of the proof, exploiting the local character of the problem, allows to deal with solutions u having a vanishing derivative on the boundary; therefore [1, Theorem 1.1] extends to arbitrary nonlinearities $f \in C^1(\mathbb{R})$[1]. In the nonlocal case of radial solutions u of (2.1), it is more difficult to deal with possible oscillations of the radial derivative of u close to the boundary. In our proof of Theorem 2.1, we distinguish two cases. In the case $s \in (\frac{1}{2}, 1)$, we use a regularity result of Grubb given in [53, Theorem 2.2] to complete the argument in the case where $\frac{u}{\delta^s}$ vanishes on $\partial\mathscr{B}$. Moreover, in the case $s \in (0, \frac{1}{2}]$, we use the extra assumption (2.6) to ensure that $\frac{u}{\delta^s}$ does not vanish on the boundary. Here we point out that (2.6) implies $f(0) = 0$, while no extra assumption on $f(0)$ is needed in the case $s \in (\frac{1}{2}, 1)$.
We point out our proof of Theorem 2.1 does not use the extension method of Caffarelli and Silvestre [25], which allows to reformulate (2.1) as a boundary value problem where $(-\Delta)^s$ arises as a Dirichlet-to-Neumann type operator. We therefore expect that our approach applies to a more general class of nonlocal operators in place of $(-\Delta)^s$.
We wish to add some remarks on the role of Morse index estimates in the variational study of (2.1). In the case where $f \in C^1(\mathbb{R})$ has subcritical growth, weak solutions of (2.1) are precisely the critical points of the associated energy functional $J : \mathscr{H}_0^s(\mathscr{B}) \to \mathbb{R}$ defined by

$$J(u) = \frac{c(N,s)}{2} \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy - \int\limits_{\mathscr{B}} F(u)\, dx,$$

where $F(t) = \int_0^t f(s)\, ds$. Moreover, J is of class C^2, and thus the behaviour of J near a critical point u is closely related to the Morse index $m(u)$. Typically, critical points detected via minimax principles lead to bounds on the Morse index. In combination with Theorem 2.1, this allows to show the non-radiality of certain classes of sign changing critical points. In this spirit, it is proved in [1] that, under suitable additional assumptions on f, least energy sign changing solutions of the local problem (2.2) are non-radial functions.
With regard to the existence of least energy sign changing solutions of the nonlocal problem (2.1), we refer to the recent paper [95]. For existence results for sign changing solutions to related nonlocal problems, see e.g. [76, 97] and the references therein.

The paper is organized as follows. In Section 3.2 we introduce preliminary notions and collect preliminary results on function spaces. In Section 2.3, we investigate radial solutions of (2.1) and properties of their partial derivatives. In Section 2.4 we complete the proof of Theorem 2.1. Finally, in Section 2.5, we complete the proof of Theorem 2.2.

[1] We wish to thank the referee for pointing out this fact.

2.2 Preliminary definitions and results

In this section, we introduce some notation and state preliminary results to be used throughout this paper. We first introduce and recall some notation related to sets and functions. If $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ are open subsets, we write $\Omega_1 \subset\subset \Omega_2$ if $\overline{\Omega}_1$ is compact and contained in Ω_2. We denote by $1_U : \mathbb{R}^N \to \mathbb{R}$ the characteristic function of a subset $U \subset \mathbb{R}^N$. For a function $u : \mathbb{R}^N \to \mathbb{R}$, we use $u^+ := \max\{u, 0\}$ and $u^- := -\min\{u, 0\}$ to denote the positive and negative part of u, respectively.
Next we recall notation related to function spaces associated with the fractional power $s \in (0,1)$. We consider the space

$$\mathscr{L}_s^1 := \left\{ u \in L^1_{loc}(\mathbb{R}^N) : \|u\|_{\mathscr{L}_s^1} < \infty \right\}, \quad \text{where} \quad \|u\|_{\mathscr{L}_s^1} := \int_{\mathbb{R}^N} \frac{|u(x)|}{1+|x|^{N+2s}}\, dx. \tag{2.8}$$

If $w \in \mathscr{L}_s^1$, then $(-\Delta)^s w$ is well defined as a distribution on $\mathbb{R}^N$ by setting

$$[(-\Delta)^s w](\varphi) = \int_{\mathbb{R}^N} w(-\Delta)^s \varphi\, dx \qquad \text{for } \varphi \in \mathscr{C}_c^\infty(\mathbb{R}^N).$$

Here and in the following, for an open subset $\Omega \subset \mathbb{R}^N$, we denote by $\mathscr{C}_c^\infty(\Omega)$ the space of smooth functions on $\mathbb{R}^N$ with compact support in Ω. We recall a maximum principle for the fractional Laplacian in distributional sense due to Silvestre.

Proposition 2.3. *[89, Proposition 2.17] Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, and let $w \in \mathscr{L}_s^1$ be lower-semicontinuous function in $\overline{\Omega}$ such that $w \geq 0$ in $\mathbb{R}^N \setminus \Omega$ and $(-\Delta)^s w \geq 0$ in Ω in distributional sense, i.e.,*

$$\int_{\mathbb{R}^N} w(-\Delta)^s \varphi\, dx \geq 0 \qquad \textit{for all nonnegative functions } \varphi \in \mathscr{C}_c^\infty(\Omega).$$

Then $w \geq 0$ in $\mathbb{R}^N$.

For an open subset $\Omega \subset \mathbb{R}^N$, we now consider the fractional Sobolev space

$$H^s(\Omega) = \left\{ u \in L^2(\Omega) \ : \ \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy < \infty \right\}. \tag{2.9}$$

Setting

$$[u]_{s,\Omega} := \Big(\frac{1}{2} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy \Big)^{\frac{1}{2}} \qquad \text{for } u \in H^s(\Omega),$$

we note that $H^s(\Omega)$ is a Hilbert space whose norm can be written as

$$\|u\|_{H^s(\Omega)} = \Big(\|u\|^2_{L^2(\Omega)} + [u]^2_{s,\Omega} \Big)^{\frac{1}{2}} \tag{2.10}$$

We will also use the local fractional Sobolev space $H^s_{loc}(\Omega)$ defined as the space of functions $\psi \in L^2_{loc}(\Omega)$ with $\psi \in H^s(\Omega')$ for every $\Omega' \subset\subset \Omega$.
For a *bounded* open subset $\Omega \subset \mathbb{R}^N$, we let $\mathscr{H}^s_0(\Omega)$ denote the closure of $C^\infty_c(\Omega)$ in $H^s(\mathbb{R}^N)$. Then $\mathscr{H}^s_0(\Omega)$ is a Hilbert space with scalar product

$$(u,v) \mapsto \mathscr{E}_s(u,v) := \langle u,v\rangle_{\mathscr{H}^s_0(\Omega)} = \frac{c(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{N+2s}}\,dxdy$$

and corresponding norm

$$\|u\|_{\mathscr{H}^s_0(\Omega)} = \sqrt{\mathscr{E}_s(u,u)} = \sqrt{c(N,s)}\,[u]_{s,\mathbb{R}^N}.$$

This is a consequence of the fact that

$$\inf\{\mathscr{E}_s(u,u)\,:\,u \in \mathscr{H}^s_0(\Omega),\, \|u\|_{L^2(\Omega)} = 1\} \;>\; 0,$$

which in turn follows from the fractional Sobolev inequality (see e.g. [33, Theorem 6.5]) and the boundedness of Ω. In particular, $\mathscr{H}^s_0(\Omega)$ embeds into $L^2(\Omega)$. We also note that, by definition,

$$\mathscr{H}^s_0(\tilde{\Omega}) \subset \mathscr{H}^s_0(\Omega) \qquad \text{for bounded open sets } \Omega, \tilde{\Omega} \text{ with } \tilde{\Omega} \subset \Omega. \tag{2.11}$$

We also recall the following property, see e.g. [52, Theorem 1.4.2.2]:

$$\begin{aligned}&\textit{For any bounded domain } \Omega \textit{ with continuous boundary,}\\ &\textit{we have } \mathscr{H}^s_0(\Omega) := \{u \in H^s(\mathbb{R}^N) : u \equiv 0 \;\; on \;\; \mathbb{R}^N \setminus \Omega\}.\end{aligned} \tag{2.12}$$

Consequently, the definition of $\mathscr{H}^s_0(\Omega)$ is consistent with (2.3).
For the remainder of this section, we fix a bounded open subset $\Omega \subset \mathbb{R}^N$. The following lemma is known, but we include a short proof for the convenience of the reader.

Lemma 2.4. *Let $\varphi \in H^s_{loc}(\Omega)$ be compactly supported in Ω. Then $\varphi \in \mathscr{H}^s_0(\Omega)$.*

Here and in the following, we identify φ with its trivial extension to $\mathbb{R}^N$.

Proof. Without loss of generality, we may assume that Ω has a continuous boundary, since otherwise we may use (2.11) after replacing Ω by a bounded open subset $\tilde{\Omega}$ with continuous boundary containing the support of φ.
Let $\Omega' \subset\subset \Omega$ be an open subset of Ω which contains the support K of φ. Then we have

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{N+2s}}\,dxdy = [\varphi]^2_{s,\Omega'} + \int_{\Omega'}\int_{\mathbb{R}^N\setminus\Omega'} \frac{|\varphi(x)|^2}{|x-y|^{N+2s}}\,dydx, \tag{2.13}$$

where $[\varphi]^2_{s,\Omega'} < \infty$ since $\varphi \in H^s_{loc}(\Omega)$. Moreover,

$$\int_{\Omega'}\int_{\mathbb{R}^N\setminus\Omega'} \frac{|\varphi(x)|^2}{|x-y|^{N+2s}}\,dydx = \int_K |\varphi(x)|^2 \int_{\mathbb{R}^N\setminus\Omega'} \frac{dy}{|x-y|^{N+2s}}dx$$

$$\leq \|\varphi\|_{L^2(K)}^2 \sup_{x\in K} \int\limits_{\mathbb{R}^N\setminus\Omega'} \frac{dy}{|x-y|^{N+2s}} < \infty$$

since $\operatorname{dist}(K, \mathbb{R}^N \setminus \Omega') > 0$. Since Ω has a continuous boundary and $\varphi \equiv 0$ in $\mathbb{R}^N \setminus \Omega$, we conclude that $\varphi \in \mathscr{H}_0^s(\Omega)$ as a consequence of (2.12). □

We also need the following lemma.

Lemma 2.5. *Let $v \in \mathscr{L}_s^1 \cap H^s_{loc}(\Omega)$, and let $\varphi \in H^s_{loc}(\Omega)$ be a function with compact support. Then the integral*

$$\mathscr{E}_s(v,\varphi) = \frac{c(N,s)}{2} \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{\big(v(x)-v(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+2s}}\, dxdy$$

is well defined in Lebesgue sense. More precisely, for any choice of open subsets

$$\Omega' \subset\subset \Omega'' \subset\subset \Omega$$

with $\operatorname{supp}\varphi \subset \Omega'$, there exist constants c_1, c_2 – depending only on Ω', Ω'', N and s but not on v and φ —such that

$$\frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{\big|v(x)-v(y)\big|\big|\varphi(x)-\varphi(y)\big|}{|x-y|^{N+2s}}\, dxdy \tag{2.14}$$
$$\leq [v]_{s,\Omega''}[\varphi]_{s,\Omega''} + c_1\|v\|_{L^2(\Omega')}\|\varphi\|_{L^2(\Omega')} + c_2\|\varphi\|_{L^1(\Omega')}\|v\|_{\mathscr{L}_s^1}.$$

Proof. We put $\mathbf{k}(z) = |z|^{-N-2s}$. Since $\operatorname{supp}\varphi \subset \Omega'$, we see that

$$\frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} |v(x)-v(y)||\varphi(x)-\varphi(y)|\mathbf{k}(x-y)\, dxdy =$$
$$\frac{1}{2}\int\limits_{\Omega''}\int\limits_{\Omega''} \frac{\big|v(x)-v(y)\big|\big|\varphi(x)-\varphi(y)\big|}{|x-y|^{N+2s}}\, dxdy + \int\limits_{\Omega'}\int\limits_{\mathbb{R}^N\setminus\Omega''} \frac{\big|v(x)-v(y)\big|\big|\varphi(x)\big|}{|x-y|^{N+2s}}\, dydx$$
$$\leq [v]_{s,\Omega''}[\varphi]_{s,\Omega''} + \int\limits_{\Omega'} |\varphi(x)| \int\limits_{\mathbb{R}^N\setminus\Omega''} |v(x)-v(y)|\mathbf{k}(x-y)\, dydx,$$

where

$$\int\limits_{\Omega'} |\varphi(x)| \int\limits_{\mathbb{R}^N\setminus\Omega''} |v(x)-v(y)|\mathbf{k}(x-y)\, dydx$$
$$\leq \int\limits_{\Omega'} |\varphi(x)||v(x)|\kappa_{\Omega''}(x)\, dx + \int\limits_{\Omega'} |\varphi(x)| \int\limits_{\mathbb{R}^N\setminus\Omega''} |v(y)|\mathbf{k}(x-y) dydx$$
$$\leq c_1\|\varphi\|_{L^2(\Omega')}\|v\|_{L^2(\Omega')} + c_2\|\varphi\|_{L^1(\Omega')}\|v\|_{\mathscr{L}_s^1}$$

with

$$\kappa_{\Omega''}(x) = \int\limits_{\mathbb{R}^N \setminus \Omega''} \mathbf{k}(x-y)\,dy, \qquad x \in \Omega'$$

and

$$c_1 := \sup_{x\in\Omega'} \kappa_{\Omega''}(x), \qquad c_2 := \sup_{x\in\Omega', y\in\mathbb{R}^N\setminus\Omega''} \mathbf{k}(x-y)(1+|y|)^{N+2s}.$$

Note that the values c_1 and c_2 are finite since $\Omega' \subset\subset \Omega''$. It thus follows that $\mathscr{E}_s(u,v)$ is well-defined in Lebesgue sense and that (2.14) holds. □

Corollary 2.6. *Let $v \in \mathscr{L}_s^1 \cap H^s_{loc}(\Omega)$. If $\Omega' \subset\subset \Omega$ and $(\varphi_n)_n$ is a sequence in $H^s_{loc}(\Omega)$ with* $\operatorname{supp}\varphi$, $\operatorname{supp}\varphi_n \subset \Omega'$ *for all $n \in \mathbb{N}$ and $\varphi_n \to \varphi$ in $H^s_{loc}(\Omega)$, then we have*

$$\mathscr{E}_s(v,\varphi_n) \to \mathscr{E}_s(v,\varphi) \qquad \text{as } n \to \infty.$$

Proof. By Lemma 2.5,

$$|\mathscr{E}_s(v,\varphi_n-\varphi)| \le$$
$$c(N,s)[v]_{s,\Omega'}[\varphi_n-\varphi]_{s,\Omega'} + \mathbf{C}_1\|v\|_{L^2(\Omega')}\|\varphi_n-\varphi\|_{L^2(\Omega')} + \mathbf{C}_2\|\varphi_n-\varphi\|_{L^1(\Omega')}\|v\|_{\mathscr{L}_s^1},$$

where $\mathbf{C}_1$ and $\mathbf{C}_2$ are positive constants. Thanks to the embeddings

$$H^s_{loc}(\Omega) \hookrightarrow L^2_{loc}(\Omega) \hookrightarrow L^1_{loc}(\Omega),$$

we conclude that $\mathscr{E}_s(v,\varphi_n-\varphi) \to 0$ as $n \to \infty$. □

2.3 Properties of radial solutions and their partial derivatives

In the following, we restrict our attention to the case $\Omega = \mathscr{B}$ and to bounded weak solutions of equation (2.1). Here and in the following, we fix a nonlinearity $f : \mathbb{R} \to \mathbb{R}$ of class C^1, and we call a function $u \in \mathscr{H}^s_0(\mathscr{B}) \cap L^\infty(\mathscr{B})$ a weak solution of (2.1) if

$$\mathscr{E}_s(u,\varphi) = \int\limits_{\mathscr{B}} f(u)\varphi\,dx \qquad \text{for all } \varphi \in \mathscr{H}^s_0(\mathscr{B}).$$

We note the following regularity properties for weak solutions of (2.1). For this we consider the distance function to the boundary

$$\delta : \overline{\mathscr{B}} \to \mathbb{R}, \qquad \delta(x) = \operatorname{dist}(x,\partial\mathscr{B}) = 1-|x|.$$

Proposition 2.7. *(cf. [41, 53, 84, 89])*
Let $u \in \mathscr{H}^s_0(\mathscr{B}) \cap L^\infty(\mathscr{B})$ be a weak solution of (2.1). Then $u \in C^{2,s}_{loc}(\mathscr{B}) \cap C^s_0(\overline{\mathscr{B}})$. Moreover,

$$\psi := \frac{u}{\delta^s} \in C^\alpha(\overline{\mathscr{B}}) \qquad \text{for some } \alpha \in (0,1), \tag{2.15}$$

and the following properties hold with some constant $c > 0$:

(i) $|\nabla u(x)| \le c\delta^{s-1}(x)$ *for all* $x \in \mathscr{B}$.

(ii) $|\nabla \psi(x)| \le c\delta^{\alpha-1}(x)$ *for all* $x \in \mathscr{B}$.

(iii) *For every* $x_0 \in \partial\mathscr{B}$, *we have* $\lim\limits_{x\to x_0} \delta^{1-s}(x)\partial_r u(x) = -s\psi(x_0)$, *where* $\partial_r u(x) = \nabla u(x) \cdot \frac{x}{|x|}$ *denotes the radial derivative of* u *at* x.

(iv) *If* $s \in (\frac{1}{2}, 1)$, *then* $\psi \in C^1(\overline{\mathscr{B}})$.

Proof. Since $u \in L^\infty(\mathscr{B})$ and f is of class C^1, we have $f(u(\cdot)) \in L^\infty(\mathscr{B})$. Hence the regularity theory for the fractional Dirichlet-Possion problem developed in [84] shows that $u \in C_0^s(\mathscr{B})$, and that (i) holds. It is also shown in [84] that $\psi := \frac{u}{\delta^s} \in C^\alpha(\overline{\mathscr{B}})$ for some $\alpha \in (0,1)$. Moreover, (ii) and (iii) are proved in [41].
Finally, noting that $f(u(\cdot)) \in C^s(\mathscr{B})$ since $u \in C_0^s(\mathscr{B})$, it follows from interior regularity (see e.g. [89]) that $u \in C^{2,s}_{loc}(\mathscr{B})$. Moreover, if $s \in (\frac{1}{2}, 1)$ we have $\psi \in C^{2s}(\overline{\mathscr{B}}) \subset C^1(\overline{\mathscr{B}})$ by [53, Theorem 2.2]. □

The regularity estimates above allow to apply the following simple integration by parts formula to weak solutions of (2.1).

Lemma 2.8. *Let* $u \in C^0(\overline{\mathscr{B}}) \cap C^1_{loc}(\mathscr{B})$ *be a function satisfying* $u \equiv 0$ *on* $\partial\mathscr{B}$ *and* $|\nabla u| \in L^1(\mathscr{B})$. *Then*

$$\int_{\mathscr{B}} (\partial_j u)\varphi \, dx = -\int_{\mathscr{B}} u\partial_j\varphi \, dx \qquad \text{for } \varphi \in C^1(\overline{\mathscr{B}}),\ j = 1, \dots, N. \tag{2.16}$$

Proof. Let $\varphi \in C^1(\overline{\mathscr{B}})$, and let $\Omega_n := B_{1-\frac{1}{n}}(0) \subset \mathscr{B}$ for $n \in \mathbb{N}$. Then $u \in C^1(\overline{\Omega}_n)$ for $n \in \mathbb{N}$ since $u \in C^1_{loc}(\mathscr{B})$. Integrating by parts over Ω_n and using a change of variables, we find that

$$\int_{\Omega_n} \big((\partial_j u)\varphi + u\partial_j\varphi\big)\, dx = \int_{\partial\Omega_n} u\varphi\nu_j \, d\sigma = (1-\frac{1}{n})^{N-1} \int_{\partial\mathscr{B}} u((1-\frac{1}{n})\sigma)\varphi((1-\frac{1}{n})\sigma)\nu_j \, d\sigma,$$

where ν_j is the j-th component of the unit outward normal to $\partial\mathscr{B}$ at x. Since $u \in C^0(\overline{\mathscr{B}})$, $u = 0$ on $\partial\mathscr{B}$, $\Omega_n \uparrow \mathscr{B}$ and $\varphi \in C^1(\overline{\mathscr{B}})$, we can apply the Lebesgue dominated convergence theorem to both sides of the equation above to deduce (2.16). □

In the following, we fix a *radial* solution $u \in \mathscr{H}_0^s(\mathscr{B}) \cap L^\infty(\mathscr{B})$ of (2.1), and we consider the function ψ defined in (2.15) which is also radial. Hence we write

$$\psi(x) = \psi_0(r) \text{ for } r = |x| \text{ with a function } \psi_0 : [0,1] \to \mathbb{R} \tag{2.17}$$

which is of class C^α for some $\alpha > 0$ by Proposition 2.7. Moreover, by Proposition 2.7 we have

$$\psi_0(1) = \lim_{|x|\to 1} \frac{u(|x|)}{(1-|x|)^s} = -\frac{1}{s} \lim_{|x|\to 1} (1-|x|)^{1-s}\partial_r u(x). \tag{2.18}$$

By the Pohozaev type identity given in [83, Theorem 1.1], this value also satisfies

$$\psi_0^2(1) = \frac{1}{|S^{N-1}|\Gamma(1+s)^2} \int_{\mathscr{B}} \Big[(2s-N)uf(u) + 2NF(u)\Big] dx. \tag{2.19}$$

Here $F:\mathbb{R}\to\mathbb{R}$ is given by $F(t)=\int_0^t f(\tau)\,d\tau$.

The aim of this section is to construct test functions, related to partial derivatives of u, which allow to estimate Dirichlet eigenvalues of the linearized operator

$$L:=(-\Delta)^s-f'(u). \tag{2.20}$$

For $j\in\{1,\dots,N\}$, we consider the partial derivatives of u given by

$$v^j:\mathbb{R}^N\to\mathbb{R},\qquad v^j(x)=\begin{cases}\partial_j u(x)=\dfrac{\partial u}{\partial x_j}(x), & x\in\mathscr{B},\\ 0, & x\in\mathbb{R}^N\setminus\mathscr{B},\end{cases}\qquad j=1,\dots,N.$$

From Proposition 2.7, it then follows that

$$v^j\in\mathscr{L}^1_s\cap H^s_{loc}(\mathscr{B})\qquad\text{for } j\in\{1,\dots,N\}. \tag{2.21}$$

Hence $\mathscr{E}_s(v^j,\varphi)$ is well defined for every $\varphi\in\mathscr{H}^s_0(\mathscr{B})$ with compact support by Lemma 2.5. We have the following key lemma.

Lemma 2.9. *For any $j\in\{1,\dots,N\}$, we have $Lv^j=(-\Delta)^s v^j-f'(u)v^j=0$ in distributional sense in $\mathscr{B}$, i.e.*

$$\int_{\mathscr{B}} v^j(-\Delta)^s\varphi\,dx=\mathscr{E}_s(v^j,\varphi)=\int_{\mathscr{B}} f'(u)v^j\varphi\,dx\qquad\textit{for all }\varphi\in\mathscr{C}^\infty_c(\mathscr{B}). \tag{2.22}$$

Moreover, if $\varphi\in\mathscr{H}^s_0(\mathscr{B})$ has compact support in $\mathscr{B}$, then we have

$$\mathscr{E}_s(v^j,\varphi)=\int_{\mathscr{B}} f'(u)v^j\varphi\,dx. \tag{2.23}$$

Furthermore, if $v^j\in\mathscr{H}^s_0(\mathscr{B})$, then (2.23) is true for all $\varphi\in\mathscr{H}^s_0(\mathscr{B})$.

Proof. Since $u\in C^{2,s}_{loc}(\mathscr{B})$ by Proposition 2.7, we have $v^j\in C^{1,s}_{loc}(\mathscr{B})\subset H^s_{loc}(\mathscr{B})$. Let $\varphi\in\mathscr{C}^\infty_c(\mathscr{B})\subset\mathscr{C}^\infty_c(\mathbb{R}^N)$. Then

$$\partial_j\varphi\in\mathscr{C}^\infty_c(\mathscr{B}),\quad(-\Delta)^s\varphi\in C^\infty(\mathbb{R}^N),\qquad\text{and}\qquad\partial_j(-\Delta)^s\varphi=(-\Delta)^s\partial_j\varphi\quad\text{on }\mathbb{R}^N.$$

Consequently, since u satisfies the assumptions of Lemma 2.8, (2.16) implies that

$$\begin{aligned}\int_{\mathscr{B}} v^j(-\Delta)^s\varphi\,dx&=-\int_{\mathscr{B}} u\partial_j(-\Delta)^s\varphi\,dx=-\int_{\mathscr{B}} u(-\Delta)^s\partial_j\varphi\,dx\\&=-\mathscr{E}_s(u,\partial_j\varphi)=-\int_{\mathscr{B}} f(u)\partial_j\varphi\,dx=\int_{\mathscr{B}}\partial_j f(u)\varphi\,dx=\int_{\mathscr{B}} f'(u)v^j\varphi\,dx.\end{aligned}$$

Hence v^j solves $Lv^j = (-\Delta)^s v^j - f'(u)v^j = 0$ in distributional sense. Next we show that

$$\mathscr{E}_s(v^j, \varphi) = \int_{\mathscr{B}} f'(u)v^j \varphi \, dx \qquad \text{for all } \varphi \in \mathscr{C}_c^\infty(\mathscr{B}). \tag{2.24}$$

Since $v^j \in \mathscr{L}_s^1 \cap H_{loc}^s(\mathscr{B})$, the integral

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v^j(x) - v^j(y)||\varphi(x) - \varphi(y)|}{|x-y|^{N+2s}} \, dxdy$$

exists by Lemma 2.5, and therefore we have, by Lebesgue's Theorem,

$$\begin{aligned}
\mathscr{E}_s(v^j, \varphi) &= \frac{c(N,s)}{2} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \int_{|x-y| \geq \varepsilon} \frac{\big(v^j(x) - v^j(y)\big)\big(\varphi(x) - \varphi(y)\big)}{|x-y|^{N+2s}} \, dxdy \\
&= c(N,s) \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} v^j(x) \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}} \, dydx \\
&= c(N,s) \int_{\mathbb{R}^N} v^j(x) \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}} \, dydx \\
&= \int_{\mathbb{R}^N} v^j(-\Delta)^s \varphi \, dx = \int_{\mathscr{B}} v^j(-\Delta)^s \varphi \, dx = \int_{\mathscr{B}} f'(u)v^j \varphi \, dx.
\end{aligned}$$

Next, let $\varphi \in \mathscr{H}_0^s(\mathscr{B})$ with compact support in $\mathscr{B}$, and choose an open subset $\Omega' \subset\subset \mathscr{B}$ such that $\operatorname{supp} \varphi \subset \Omega'$. By definition of $\mathscr{H}_0^s(\Omega')$, there exists a sequence $(\varphi_n)_n$ in $\mathscr{C}_c^\infty(\Omega') \subset \mathscr{C}_c^\infty(\mathscr{B})$ with $\varphi_n \to \varphi$ in $\mathscr{H}_0^s(\Omega')$, hence also $\varphi_n \to \varphi$ in $\mathscr{H}_0^s(\mathscr{B})$. Then Corollary 2.6 and (2.24) imply that

$$\mathscr{E}_s(v^j, \varphi) = \lim_{n \to \infty} \mathscr{E}_s(v^j, \varphi_n) = \lim_{n \to \infty} \int_{\mathscr{B}} f'(u)v^j \varphi_n \, dx = \int_{\mathscr{B}} f'(u)v^j \varphi \, dx, \tag{2.25}$$

and thus (2.23) holds.
Finally, assume that $v^j \in \mathscr{H}_0^s(\mathscr{B})$, let $\varphi \in \mathscr{H}_0^s(\mathscr{B})$, and let $(\varphi_n)_n$ be a sequence in $\mathscr{C}_c^\infty(\mathscr{B})$ with $\varphi_n \to \varphi$ in $\mathscr{H}_0^s(\mathscr{B})$. Then (2.25) holds again by the continuity of the quadratic form $\mathscr{E}_s$ on $\mathscr{H}_0^s(\mathscr{B})$, as claimed. $\square$

We now have all the tools to build suitable test functions from partial derivatives in order to estimate the Morse index of u as a solution of (2.1). As remarked before, the construction is inspired by [1].

Definition 2.10. Let ψ_0 be the function defined in (2.17). For $j = 1, \dots, N$, we define the open half spaces

$$H_\pm^j := \{x \in \mathbb{R}^N \,:\, \pm x_j > 0\} \tag{2.26}$$

and the functions $d_j : \mathbb{R}^N \to \mathbb{R}$ by

$$d_j := \begin{cases} (v^j)^+ 1_{H_+^j} - (v^j)^- 1_{H_-^j} & \text{if } \psi_0(1) \geq 0; \\ (v^j)^+ 1_{H_-^j} - (v^j)^- 1_{H_+^j} & \text{if } \psi_0(1) < 0. \end{cases}$$

We note that, for $j = 1,\dots,N$, the function d_j is odd with respect to the reflection

$$\sigma_j : \mathbb{R}^N \to \mathbb{R}^N, \qquad x = (x_1,\cdots,x_j,\cdots,x_N) \mapsto \sigma_j(x) = (x_1,\dots,-x_j,\dots,x_N)$$

at the hyperplane $\{x_j = 0\}$ since the function v^j is odd.

Lemma 2.11. *$d_j \in H^s_{loc}(\mathscr{B})$ for $j = 1,\dots,N$.*

Proof. By definition of d_j, it suffices to show that

$$(v^j)^{\pm}1_{H^j_\pm} \in H^s_{loc}(\mathscr{B}) \tag{2.27}$$

We only consider the function $(v^j)^+1_{H^j_+}$, the proof for the other functions is essentially the same. As noted in (2.21), we have $v^j \in H^s_{loc}(\mathscr{B})$, and therefore also $(v^j)^+ \in H^s_{loc}(\mathscr{B})$ by a standard estimate. To abbreviate, we now put $\chi = 1_{H^j_+}$, $v := (v^j)^+$, and we let $\Omega' \subset\subset \mathscr{B}$ be an open subset of $\mathscr{B}$. Making Ω' larger if necessary, we may assume that Ω' is symmetric with respect to the reflection σ_j. To show that $v\chi \in H^s_{loc}(\Omega')$, we write

$$\begin{aligned}
[v\chi]^2_{s,\Omega'} &= [v]^2_{s,\Omega'\cap H^j_+} + \int_{\Omega'\cap H^j_+} |v(x)|^2 \int_{\Omega'\cap H^j_-} |x-y|^{-N-2s}\,dydx \\
&\le [v]^2_{s,\Omega'} + \int_{\Omega'\cap H^j_+} |v(x)|^2 \int_{\{y\in\mathbb{R}^N, |y-x|\ge|x_j|\}} |x-y|^{-N-2s}dydx \\
&= [v]^2_{s,\Omega'} + \int_{\Omega'\cap H^j_+} |v(x)|^2 \int_{\{z\in\mathbb{R}^N, |z|\ge|x_j|\}} |z|^{-N-2s}dzdx \\
&= [v]^2_{s,\Omega'} + \frac{|S^{N-1}|}{2s} \int_{\Omega'\cap H^j_+} |v(x)|^2|x_j|^{-2s}dx
\end{aligned}$$

Since $v = (v^j)^+ \in C^s_{loc}(\mathscr{B})$ by Proposition 2.7 and $v \equiv 0$ on $\{x_j = 0\}$, we have $|v(x)| \le C|x_j|^s$ for $x \in \Omega' \cap H^j_+$. Therefore, the latter integral is finite, and $(v^j)^+1_{H^j_+} = v\chi \in H^s_{loc}(\mathscr{B})$. $\square$

The next lemma is of key importance for the proof of Theorem 2.1.

Lemma 2.12. *Let $j = 1,\dots,N$.*

(i) If $\psi_0(1) \ne 0$, we have $d_j \in \mathscr{H}^s_0(\mathscr{B})$, and d_j has compact support in $\mathscr{B}$.

(ii) If $s \in (\frac{1}{2},1)$ and $\psi_0(1) = 0$, then we have $v^j \in \mathscr{H}^s_0(\mathscr{B})$ and $d_j \in \mathscr{H}^s_0(\mathscr{B})$

Proof. (i) By Lemma 2.4 and Lemma 2.11, it suffices to show that d_j has compact support in $\mathscr{B}$. We now distinguish the cases $\psi_0(1) > 0$ and $\psi_0(1) < 0$.
If $\psi_0(1) > 0$, we have $\partial_r u(x) \le 0$ in $\mathscr{B} \setminus B_{r_*}(0)$ for some $r_* \in (0,1)$ by (2.18), and therefore

$$v^j(x) = \partial_j u(x) = \frac{x_j}{|x|}\partial_r u(x) \le 0 \qquad \text{for } x \in \mathscr{B} \setminus B_{r_*}(0) \text{ with } x_j \ge 0.$$

Consequently, $d_j(x) = (v^j)^+(x) = 0$ for $x \in \mathscr{B} \setminus B_{r_*}(0)$ with $x_j \geq 0$. Since d_j is odd with respect to the reflection σ_j it follows that $\operatorname{supp} d_j \subset \overline{B_{r_*}(0)}$, so d_j is compactly supported in $\mathscr{B}$.
If $\psi_0(1) < 0$, we have $\partial_r u(x) \geq 0$ in $\mathscr{B} \setminus B_{r_*}(0)$ for some $r_* \in (0,1)$ by (2.18), which in this case, similarly as above, implies that $d_j(x) = -(v^j)^-(x) = 0$ for $x \in \mathscr{B} \setminus B_{r_*}(0)$ with $x_j \geq 0$. Again we conclude that d_j is compactly supported in $\mathscr{B}$ since it is odd with respect to the reflection σ_j.
(ii) Since $s \in (\frac{1}{2}, 1)$, it follows from Proposition 2.7(iv) that $\psi \in C^1(\overline{\mathscr{B}})$ and therefore $\psi_0 \in C^1([0,1])$, whereas $\psi_0(1) = 0$ by assumption. Consequently, $\psi(x)\delta^{s-1}(x) \to 0$ as $|x| \to 1$, and therefore

$$\nabla u(x) = \delta^s(x)\nabla\psi(x) + s\psi(x)\delta^{s-1}(x)\nabla\delta(x) \to 0 \qquad \text{as } |x| \to 1.$$

It thus follows that $u \in C^1(\mathbb{R}^N)$ with $u \equiv 0$ on $\mathbb{R}^N \setminus \mathscr{B}$, and therefore $v^j \in C^0(\mathbb{R}^N)$ with $v^j \equiv 0$ in $\mathbb{R}^N \setminus \mathscr{B}$. To see that $v^j \in \mathscr{H}_0^s(\mathscr{B})$, we shall use Proposition 2.3 as follows: Since the function $f'(u)v^j$ is continuous and therefore bounded in $\overline{\mathscr{B}}$, there exists a unique weak solution $w \in \mathscr{H}_0^s(\mathscr{B})$ to the Poisson problem

$$(-\Delta)^s w = f'(u)v^j \quad \text{in } \mathscr{B}, \qquad\qquad w = 0 \quad \text{in } \mathbb{R}^N \setminus \mathscr{B} \tag{2.28}$$

which satisfies $w \in C_0^s(\mathscr{B})$ by [84, Proposition 1.1]. By setting $V := w - v^j$, it follows that $V \in C^0(\mathbb{R}^N)$ with $V \equiv 0$ in $\mathbb{R}^N \setminus \mathscr{B}$. Moreover, by Lemma 2.9 the function V satisfies the equation $(-\Delta)^s V = 0$ in $\mathscr{B}$ in the sense of distributions. Since V is continuous, Proposition 2.3 – applied to $\pm V$ – implies that $V \equiv 0$ in $\mathbb{R}^N$, i.e.,

$$v^j = w \in \mathscr{H}_0^s(\mathscr{B}) \cap C_0^s(\mathscr{B}). \tag{2.29}$$

By a similar argument as in the proof of Lemma 2.11, we will now see that $d_j \in \mathscr{H}_0^s(\mathscr{B})$. For the convenience of the reader, we give the details. It is clearly sufficient to show that

$$(v^j)^{\pm} 1_{H^j_{\pm}} \in \mathscr{H}_0^s(\mathscr{B}), \tag{2.30}$$

We only consider the function $(v^j)^+ 1_{H^j_+}$, the proof for the other functions is the same. Since $v^j \in \mathscr{H}_0^s(\mathscr{B})$, we also have $(v^j)^{\pm} \in \mathscr{H}_0^s(\mathscr{B})$ by a standard estimate. To abbreviate, we now put $\chi = 1_{H^j_+}$ and $v := (v^j)^+$. To show that $v\chi \in \mathscr{H}_0^s(\mathscr{B})$, we note that $v\chi \equiv 0$ in $\mathbb{R}^N \setminus \mathscr{B}$, and we estimate

$$\begin{aligned}
[v\chi]^2_{s,\mathbb{R}^N} &= [v]^2_{s,H^j_+} + \int\limits_{H^j_+ \cap \mathscr{B}} |v(x)|^2 \int\limits_{H^j_-} |x-y|^{-N-2s}\, dydx \\
&\leq [v]^2_{s,\mathbb{R}^N} + \int\limits_{H^j_+ \cap \mathscr{B}} |v(x)|^2 \int\limits_{\{z \in \mathbb{R}^N, |z| \geq |x_j|\}} |z|^{-N-2s} dzdx \\
&= [v]^2_{s,\mathbb{R}^N} + \frac{|S^{N-1}|}{2s} \int\limits_{H^j_+ \cap \mathscr{B}} |v(x)|^2 |x_j|^{-2s} dx.
\end{aligned}$$

Since $v = (v^j)^+ \in C^s(\overline{\mathscr{B}})$ by (2.29) and $v \equiv 0$ on $\{x_j = 0\}$, we have $|v(x)| \leq C|x_j|^s$ for $x \in H^j_+ \cap \mathscr{B}$. Therefore, the latter integral is finite, and $(v^j)^+ 1_{H^j_+} = v\chi \in \mathscr{H}_0^s(\mathscr{B})$. □

Corollary 2.13. *If $\psi_0(1) \neq 0$ or $s \in (\frac{1}{2}, 1)$, then the values $\mathscr{E}_s(d_j, d_k)$ and $\mathscr{E}_s(v^j, d_k)$ are well-defined and satisfy*

$$\mathscr{E}_s(v^j, d_k) = \int_{\mathscr{B}} f'(u) v^j d_k \, dx \qquad \textit{for } j,k = 1, \ldots, N.$$

Proof. This follows from Lemma 2.5, Lemma 2.9 and Lemma 2.12. □

2.4 Proof of Theorem 2.1

In this section we complete the proof of Theorem 2.1. As before, we consider a fixed radial weak solution $u \in \mathscr{H}_0^s(\mathscr{B}) \cap L^\infty(\mathscr{B})$ of (2.1), and we will continue using the notation related to u as introduced in Section 2.3. Moreover, in accordance with the assumptions of Theorem 2.1, we assume that u changes sign, which implies that

$$(v^j)^\pm 1_{H_+^j} \not\equiv 0 \quad \text{and} \quad (v^j)^\pm 1_{H_-^j} \not\equiv 0 \qquad \text{for } j = 1, \ldots, N, \tag{2.31}$$

where the half spaces $H_\pm^j$ are defined in (2.26). We first note that, under the assumptions of Theorem 2.1, we have

$$\psi_0(1) \neq 0 \qquad \text{or} \qquad s \in (\frac{1}{2}, 1). \tag{2.32}$$

Indeed, if $s \in (0, \frac{1}{2}]$, then $\psi_0^2(1) > 0$ by (2.6) and (2.19).
Next we recall that the n-th Dirichlet eigenvalue $\lambda_{n,L}$ of the linearized operator L defined in (2.20) admits the variational characterization

$$\lambda_{n,L} = \min_{V \in \mathscr{V}_n} \max_{v \in S_V} \mathscr{E}_{s,L}(v, v) \tag{2.33}$$

where

$$(v, w) \mapsto \mathscr{E}_{s,L}(v, w) := \mathscr{E}_s(v, w) - \int_{\mathscr{B}} f'(u) v w \, dx \tag{2.34}$$

is the bilinear form associated to L, $\mathscr{V}_n$ denotes the family of n-dimensional subspaces of $\mathscr{H}_0^s(\mathscr{B})$ and $S_V := \{v \in V : \|v\|_{L^2(\mathscr{B})} = 1\}$ for $V \in \mathscr{V}_n$.
To estimate $\lambda_{n,L}$ from above, we wish to build test function spaces V by using the functions d_j introduced in Definition 2.10. By Lemma 2.12 and (2.32), we have $d_j \in \mathscr{H}_0^s(\Omega)$ for $j = 1, \ldots, N$. Moreover, as a consequence of Corollary 2.13, the values $\mathscr{E}_s(v^j, d_k)$ are well-defined and satisfy

$$\mathscr{E}_{s,L}(v^j, d_k) = 0 \qquad \text{for } j,k = 1, \ldots, N. \tag{2.35}$$

We need the following key inequality.

Lemma 2.14. *For $j \in \{1, \ldots, N\}$ we have $\mathscr{E}_{s,L}(d_j, d_j) < 0$.*

Proof. To simplify notation, we put $k(z) = c(N,s)|z|^{-N-2s}$ for $z \in \mathbb{R}^N \setminus \{0\}$. Since $v^j d_j = d_j^2$ in $\mathbb{R}^N$ by definition of d_j and therefore

$$\int_{\mathscr{B}} f'(u) v^j d_j \, dx = \int_{\mathscr{B}} f'(u) d_j^2 \, dx,$$

we have, by (2.35),

$$\begin{aligned}
\mathcal{E}_{s,L}(d_j,d_j) &= \mathcal{E}_{s,L}(d_j - v^j, d_j)\\
&= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Big(\big(d_j(x) - v^j(x) - (d_j(y) - v^j(y))\big)\big(d_j(x) - d_j(y)\big)\Big)k(x-y)dxdy\\
&= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Big(v^j(x)d_j(y) + v^j(y)d_j(x) - 2d_j(x)d_j(y)\Big)k(x-y)\;dxdy
\end{aligned}$$

In the following, we put

$$\ell_j(x,y) := k(x-y) - k(\sigma_j(x) - y) \qquad \text{for } x,y \in \mathbb{R}^N,\; x \neq y.$$

Using the oddness of the functions v^j and d_j with respect to the reflection σ_j, we deduce that

$$\begin{aligned}
\mathcal{E}_{s,L}(d_j,d_j) &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{H_+^j} \Big(v^j(x)d_j(y) + v^j(y)d_j(x) - 2d_j(x)d_j(y)\Big)\ell_j(x,y)\;dxdy\\
&= \frac{1}{2}\int_{H_+^j}\int_{H_+^j} \Big(v^j(x)d_j(y) + v^j(y)d_j(x) - 2d_j(x)d_j(y)\Big)\big(\ell_j(x,y) - \ell_j(x,\sigma_j(y))\big)\;dxdy\\
&= \int_{H_+^j}\int_{H_+^j} \Big(v^j(x)d_j(y) + v^j(y)d_j(x) - 2d_j(x)d_j(y)\Big)\ell_j(x,y)dxdy. \qquad (2.36)
\end{aligned}$$

Here we used in the last step that

$$k(\sigma_j(x) - \sigma_j(y)) = k(x-y) \qquad \text{and} \qquad k(\sigma_j(x) - y) = k(x - \sigma_j(y))$$

for $x,y \in \mathbb{R}^N$, $x \neq y$ and therefore

$$\begin{aligned}
\ell_j(x,y) - \ell_j(x,\sigma_j(y)) &= k(x-y) - k(\sigma_j(x) - y) - \big(k(x - \sigma_j(y)) - k(\sigma_j(x) - \sigma_j(y))\big)\\
&= 2\ell_j(x,y).
\end{aligned}$$

Next, we note that

$$\ell_j(x,y) = k(x-y) - k(\sigma_j(x) - y) > 0 \qquad \text{for } x,y \in H_+^j. \qquad (2.37)$$

Moreover, we claim that the function

$$\begin{aligned}
(x,y) \mapsto h_j(x,y) &= v^j(x)d_j(y) + v^j(y)d_j(x) - 2d_j(x)d_j(y)\\
&= (v^j(x) - d_j(x))d_j(y) + (v^j(y) - d^j(y))d_j(x)
\end{aligned}$$

satisfies

$$h_j \leq 0 \quad \text{and} \quad h_j \not\equiv 0 \qquad \text{on } H_+^j \times H_+^j. \qquad (2.38)$$

Indeed, if $\psi_0(1) \geq 0$, we have $d_j = (v^j)^+$ and therefore $v^j - d_j = -(v^j)^-$ on H_+^j. Hence (2.38) follows from (2.31). Moreover, if $\psi_0(1) < 0$, we have $d_j = -(v^j)^-$ and therefore $v^j - d_j = (v^j)^+$ on H_+^j. Again (2.38) follows from (2.31). The claim now follows by combining (2.36), (2.37) and (2.38). □

Lemma 2.15. *Let* $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$ *and* $d = \sum_{j=1}^{N} \alpha_j d_j$. *Then we have*

$$\mathscr{E}_{s,L}(d,d) = \sum_{j=1}^{N} \alpha_j^2 \mathscr{E}_L(d_j,d_j) \leq 0.$$

Moreover,

$$\mathscr{E}_{s,L}(d,d) < 0 \qquad \textit{if and only if} \qquad \alpha \neq 0, \tag{2.39}$$

and therefore the functions $d_1, \dots, d_N$ *are linearly independent.*

Proof. We first note that

$$\mathscr{E}_{s,L}(d_j,d_k) = 0 \qquad \text{for } j,k \in \{1,\dots,N\},\ j \neq k. \tag{2.40}$$

Indeed, since u is radially symmetric, the function d_j is odd with respect to the reflection σ_j and even with respect to the reflection σ_k for $k \neq j$. Hence, by a change of variable,

$$\begin{aligned}
\mathscr{E}_{s,L}(d_j,d_k) = \frac{c(N,s)}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} & \frac{\Big(d_j(\sigma_j(x)) - d_j(\sigma_j(y))\Big)\Big(d_k(\sigma_j(x)) - d_k(\sigma_j(y))\Big)}{|\sigma_j(x) - \sigma_j(y)|^{N+2s}}\, dxdy \\
& - \int_{\mathscr{B}} f'(u(\sigma_j(x)))d_j(\sigma_j(x))d_k(\sigma_j(x))\, dx \\
= \frac{c(N,s)}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} & \frac{\Big(d_j(y) - d_j(x)\Big)\Big(d_k(x) - d_k(y)\Big)}{|x-y|^{N+2s}}\, dxdy + \int_{\mathscr{B}} f'(u(x))d_j(x)d_k(x)\, dx \\
= -\mathscr{E}_{s,L}(d_j,d_k). &
\end{aligned}$$

Hence (2.40) is true. Now, for $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$ and $d = \sum_{j=1}^{N} \alpha_j d_j$, we have

$$\mathscr{E}_{s,L}(d,d) = \sum_{j=1}^{N} \alpha_j^2 \mathscr{E}_{s,L}(d_j,d_j) + \sum_{\substack{j,k=1 \\ j\neq k}}^{N} \alpha_j\alpha_k \mathscr{E}_{s,L}(d_j,d_k) = \sum_{j=1}^{N} \alpha_j^2 \mathscr{E}_{s,L}(d_j,d_j) \leq 0$$

by 2.40 and Lemma 2.14. Moreover, if $\alpha \neq 0$, it follows from Lemma 2.14 that $\mathscr{E}_{s,L}(d,d) < 0$, which in particular implies that $d \neq 0$. Consequently, the functions $d_1, \dots, d_N$ are linearly independent, as claimed. □

Lemma 2.16. *The first eigenvalue* $\lambda_{1,L}$ *of the operator* $L = (-\Delta)^s - f'(u)$ *is simple, and the corresponding eigenspace is spanned by radially symmetric eigenfunction* $\varphi_{1,L}$. *Furthermore,*

$$\mathscr{E}_{s,L}(d_j, \varphi_{1,L}) = 0 \quad \textit{for } j = 1,2,\cdots,N \qquad \textit{and} \qquad \lambda_{1,L} = \mathscr{E}_{s,L}(\varphi_{1,L}, \varphi_{1,L}) < 0.$$

Proof. The simplicity of $\lambda_{1,L}$ and the radial symmetry of $\varphi_{1,L}$ are well known, but we recall the proof for the convenience of the reader. The variational characterization of $\lambda_{1,L}$ is given by

$$\lambda_{1,L} = \inf_{v \in \mathscr{H}_0^s(\mathscr{B}) \setminus \{0\}} \frac{\mathscr{E}_{s,L}(v,v)}{\|v\|^2_{L^2(\mathscr{B})}} = \inf_M \mathscr{E}_{s,L}(v,v) \quad \text{with} \quad M = \{v \in \mathscr{H}_0^s(\mathscr{B}) : \|v\|_{L^2(\mathscr{B})} = 1\},$$

and the associated minimizers $\varphi \in M$ are precisely the L^2-normalized eigenfunctions of L corresponding ot $\lambda_{1,L}$, i.e., the L^2-normalized (weak) solutions of

$$L\varphi = \lambda_{1,L}\varphi \quad \text{in } \mathscr{B}, \qquad \varphi \equiv 0 \quad \text{in } \mathbb{R}^N \setminus \mathscr{B}. \tag{2.41}$$

Moreover, if $\varphi \in M$ is such a minimizer, then also $|\varphi| \in M$ and

$$\lambda_{1,L} = \mathscr{E}_{s,L}(\varphi,\varphi) \geq \mathscr{E}_{s,L}(|\varphi|,|\varphi|) \geq \inf_M \mathscr{E}_{s,L}(v,v) = \lambda_{1,L},$$

which implies that $|\varphi|$ is also a minimizer and therefore a weak solution of (2.41). By the strong maximum principle for nonlocal operators (see e.g. [20, p.312–313] or [60]), $|\varphi|$ is strictly positive in $\mathscr{B}$. Consequently, every eigenfunction φ of L is either strictly positive or strictly negative in $\mathscr{B}$. Consequently, $\lambda_{1,L}$ does not admit two L^2-orthogonal eigenfunctions, and therefore $\lambda_{1,L}$ is simple.

Next we note that, by a simple change of variable, if φ is an eigenfunction of L corresponding to $\lambda_{1,L}$, then also $\varphi \circ \mathscr{R}$ is an eigenfunction for every rotation $\mathscr{R} \in O(N)$. Consequently, the simplicity of $\lambda_{1,L}$ implies that the associated eigenspace is spanned by a radially symmetric eigenfunction $\varphi_{1,L}$.

Next, using the radially symmetry of u and $\varphi_{1,L}$ and the oddness of d_j with respect to the reflection σ_j, we find, by a change of variable, that

$$\begin{aligned}
\mathscr{E}_{s,L}(d_j,\varphi_{1,L}) &= \frac{c(N,s)}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(d_j(\sigma_j(x)) - d_j(\sigma_j(y)))(\varphi_{1,L}(\sigma_j(x)) - \varphi_{1,L}(\sigma_j(x)))}{|x-y|^{N+2s}} dxdy \\
&\qquad - \int_{\mathscr{B}} f'(u(\sigma_j(x)))d_j(\sigma_j(x))\varphi_{1,L}(\sigma_j(x))\, dx \\
&= \frac{c(N,s)}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(d_j(y) - d_j(x))(\varphi_{1,L}(x) - \varphi_{1,L}(y))}{|x-y|^{N+2s}}\, dxdy + \int_{\mathscr{B}} f'(u(x))d_j(x)\varphi_{1,L}(x)dx \\
&= -\mathscr{E}_{s,L}(d_j,\varphi_{1,L})
\end{aligned}$$

and therefore $\mathscr{E}_{s,L}(d_j,\varphi_{1,L}) = 0$ for $j = 1,\ldots,N$. Finally, by Lemma 2.14 and the variational characterization of $\lambda_{1,L}$, we have $\lambda_{1,L} = \mathscr{E}_{s,L}(\varphi_{1,L},\varphi_{1,L}) < 0$, as claimed. □

Proof of Theorem 2.1(completed). Let $\varphi_{1,L} \in \mathscr{H}_0^s(\mathscr{B})$ be an eigenfunction of L corresponding to the first eigenvalue $\lambda_{1,L}$ as given in Lemma 2.16. We consider the subspace

$$V = \operatorname{span}\{\varphi_{1,L}, d_1, \ldots, d_N\}.$$

For $\alpha \in \mathbb{R}^{N+1} \setminus \{0\}$ and $d = \alpha_0 \varphi_{1,L} + \sum_{j=1}^{N} \alpha_j d_j \in V$, we then have, by Lemma 2.15 and Lemma 2.16,

$$\mathscr{E}_{s,L}(d,d) = \alpha_0^2 \mathscr{E}_{s,L}(\varphi_{1,L}, \varphi_{1,L}) + \mathscr{E}_{s,L}(\sum_{j=1}^{N} \alpha_j d_j, \sum_{j=1}^{N} \alpha_j d_j) < 0.$$

In particular, it follows that the functions $\varphi_{1,L}, d_1, \dots, d_N$ are linearly independent and therefore V is $N+1$-dimensional. By (2.33) and the compactness of $S_V = \{v \in V : \|v\|_{L^2(\mathscr{B})} = 1\}$, it then follows that $\lambda_{N+1,L} < 0$, which means that u has Morse index greater than or equal to $N+1 \geq 2$, as claimed. □

2.5 The linear case

In this section we discuss the linear eigenvalue problem (2.7) and complete the proof of Theorem 2.2. In particular, we wish to recall a useful characterization of eigenvalues and eigenfunctions of (2.7) derived in [36]. For this we need to consider the following radially symmetric version of (2.7) in general dimensions $d \in \mathbb{N}$:

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \mathscr{B} \subset \mathbb{R}^d \\ u \in \mathscr{H}_0^s(\mathscr{B}), & u \text{ radially symmetric.} \end{cases} \tag{2.42}$$

In the following, we let $\lambda_{d,0} < \lambda_{d,1} \leq \dots$ denote the increasing sequence of eigenvalues of this problem (counted with multiplicity).
The following characterization is essentially a reformulation of [36, Proposition 1.1].

Proposition 2.17. *The eigenvalues of (2.7) in $\mathscr{B} \subset \mathbb{R}^N$ are of the form $\lambda = \lambda_{N+2\ell,n}$ with integers $\ell, n \geq 0$. Moreover, if*

$$Z_\lambda := \{(\ell,n) : \lambda_{N+2\ell,n} = \lambda\},$$

then the eigenspace corresponding to λ is spanned by functions of the form

$$u(x) = V_\ell(x) \varphi_{N+2\ell,n}(|x|),$$

where $(\ell,n) \in Z_\lambda$, V_ℓ is a solid harmonic polynomial of degree ℓ and $x \mapsto \varphi_{N+2\ell,n}(|x|)$ is a (radial) eigenfunction of the problem (2.42) in dimension $d = N+2\ell$ corresponding to the eigenvalue $\lambda_{N+2\ell,n}$.

Here and in the following, a solid harmonic polynomial V of degree ℓ is a function of the form $V(x) = |x|^\ell Y(\frac{x}{|x|})$, where Y is a spherical harmonic of degree ℓ. Hence $V : \mathbb{R}^N \to \mathbb{R}$ is a homogenous polynomial of degree ℓ satisfying $\Delta V = 0$.
Regarding the eigenvalues $\lambda_{d,n}$ of (2.42), it is also proved in [36, Section 3] that

$$\text{the sequence } (\lambda_{d,0})_d \text{ is strictly increasing in } d \geq 1. \tag{2.43}$$

Moreover,

$$\lambda_{d,n} > \lambda_{d,0} \qquad \text{for every } d,n \geq 1 \tag{2.44}$$

by the simplicity of the first eigenvalue of (2.42). Consequently, the first eigenvalue λ_1 of (2.7) equals $\lambda_{N,0}$, whereas the second eigenvalue λ_2 of (2.7) is given as the minimum of $\lambda_{N+2,0}$ and $\lambda_{N,1}$.
Theorem 2.2 is now a direct consequence of the following result, which we will derive from Theorem 2.1 and from the observations above.

Theorem 2.18. *We have $\lambda_{N+2,0} < \lambda_{N,1}$. Consequently, the second eigenvalue λ_2 of (2.7) is given by $\lambda_{N+2,0}$, and every corresponding eigenfunction u is antisymmetric, i.e., it satisfies $u(-x) = -u(x)$ for every $x \in \mathscr{B}$.*

Proof. Suppose by contradiction that $\lambda_2 = \lambda_{N,1} \leq \lambda_{N+2,0}$. Then, noting that the only solid harmonic polynomials of degree zero are the constants, it follows from Proposition 2.17 that (2.7) admits a radially symmetric eigenfunction corresponding to λ_2. But then u is a radially symmetric sign changing solution of (2.1) with $t \mapsto f(t) = \lambda_2 t$, so it must have Morse index greater than or equal to $N+1$. This contradicts the fact that λ_2 is the second eigenvalue.
We thus conclude that $\lambda_2 = \lambda_{N+2,0} < \lambda_{N,1}$. Combining this inequality with (2.43) and (2.44), we then deduce that $Z_{\lambda_2} = \{(1,0)\}$, and therefore the eigenspace corresponding to λ_2 is spanned by functions of the form $x \mapsto V_1(x)\varphi_{N+2,0}(|x|)$, where V_1 is a solid harmonic polynomial of degree one, hence a linear function, and $x \mapsto \varphi_{N+2,0}(|x|)$ is an eigenfunction of the problem (2.42) in dimension $d = N+2$ corresponding to the eigenvalue $\lambda_{N+2,0}$. Since every such function is antisymmetric, the claim follows. $\square$

3 Small order asymptotics of the Dirichlet eigenvalue problem for the fractional Laplacian

This chapter is devoted to spectral asymptotics with respect to parameter s. We are concerned with the study of small order limit $s \to 0^+$ of the eigenvalue problem (3.3) in a bounded open set with Lipschitz boundary. while it is easy to see that all eigenvalues of $(-\Delta)^s$ converges to 1, we prove that the rate of convergence is linear in s, with speed determined by the eigenvalues of the logarithmic Laplacian L_Δ. Moreover, the set of L^2-normalized Dirichlet eigenfunctions of $(-\Delta)^s$ corresponding to the k-th eigenvalue are uniformly bounded and converge to the set of L^2-normalized eigenfunctions for L_Δ. The chapter is self-contained and can be read independently. The content of the chapter has the same structure as the article [47] except the missing of acknowledgements. It is based on joint work done with Sven Jarohs and Tobias Weth.

3.1 Introduction

Fueled by various applications and important links to stochastic processes and partial differential equations, the interest in nonlocal operators and associated Dirichlet problems has been growing rapidly in recent years. In this context, the fractional Laplacian has received by far the most attention, see e.g. [7, 8, 15, 18, 19, 24, 25, 68, 84] and the references therein. We recall that, for compactly supported functions $u : \mathbb{R}^N \to \mathbb{R}$ of class C^2 and $s \in (0,1)$, the fractional Laplacian $(-\Delta)^s$ is well-defined by

$$(-\Delta)^s u(x) = C_{N,s} \lim_{\varepsilon \to 0^+} \int\limits_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \quad \text{where} \quad C_{N,s} = s4^s \frac{\Gamma(\frac{N}{2}+s)}{\pi^{\frac{N}{2}}\Gamma(1-s)}. \tag{3.1}$$

The normalization constant $C_{N,s}$ is chosen such that $(-\Delta)^s$ is equivalently given by

$$\mathscr{F}\big((-\Delta)^s u\big) = |\cdot|^{2s}\mathscr{F}u, \tag{3.2}$$

where, here and in the following, $\mathscr{F}$ denotes the usual Fourier transform. We emphasize that the fractional Laplacian is an operator of order $2s$ and many related regularity properties – in particular of associated eigenfunctions – rely on this fact.
The present paper is concerned with the small order asymptotics $s \to 0^+$ of the Dirichlet eigenvalue problem

$$\begin{cases} (-\Delta)^s \varphi_s = \lambda \varphi_s & \text{in } \Omega, \\ \qquad\quad \varphi_s = 0 & \text{in } \Omega^c, \end{cases} \tag{3.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set with Lipschitz boundary and $\Omega^c := \mathbb{R}^N \setminus \Omega$. It is well known (see [87, Proposition 9] or [15, Proposition 3.1]) that, for every $s \in (0,1)$, (3.3) admits an ordered sequence of eigenvalues

$$\lambda_{1,s} < \lambda_{2,s} \le \lambda_{3,s} \le \dots \tag{3.4}$$

with $\lambda_{k,s} \to \infty$ as $k \to \infty$ and a corresponding L^2-orthonormal basis of eigenfunctions $\varphi_{k,s}$, $k \in \mathbb{N}$. Moreover, $\varphi_{1,s}$ is unique up to sign and can be chosen as a positive function.
The starting point of the present work is the basic observation that

$$(-\Delta)^s u \to u \qquad \text{as } s \to 0^+ \text{ for every } u \in C_c^2(\mathbb{R}^N), \tag{3.5}$$

which readily follows from (3.2) and standard properties of the Fourier transform (see also [33, Proposition 4.4]. Similarly, we have

$$\mathscr{E}_s(u,u) \to \|u\|^2_{L^2(\mathbb{R}^N)} \qquad \text{as } s \to 0^+ \text{ for every } u \in C_c^1(\mathbb{R}^N), \tag{3.6}$$

where $\mathscr{E}_s$ denotes the quadratic form associated with $(-\Delta)^s$ given by

$$(u,v) \mapsto \mathscr{E}_s(u,v) = \frac{C_{N,s}}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{N+2s}}\, dxdy.$$

We remark that these convergence properties in the limit $s \to 0^+$ extend to a non-Hilbertian setting of quasilinear operators where the Fourier transform cannot be employed, see e.g. [2] and the references therein. It is not difficult to deduce from (3.5) that

$$\lambda_{k,s} \to 1 \qquad \text{as } s \to 0^+ \text{ for all } k \in \mathbb{N}, \tag{3.7}$$

see Section 3.2 below for details. However, there is no straightforward approach to obtain the asymptotics of associated eigenfunctions as $s \to 0^+$ since, as a consequence of (3.5) and (3.6), no uniform regularity theory is available for the fractional Laplacian $(-\Delta)^s$ in the case where s is close to zero. For general bounded open sets with Lipschitz boundary, the only available result regarding these asymptotics is contained in [29], where Chen and the third author introduced the Dirichlet problem for the logarithmic Laplacian operator L_Δ to give a more detailed description of the first eigenvalue $\lambda_{1,s}$ and the corresponding eigenfunction $\varphi_{1,s}$ as $s \to 0^+$. On compactly supported Dini continuous functions, the operator L_Δ is pointwisely given by

$$L_\Delta u(x) = C_N \int_{\mathbb{R}^N} \frac{u(x)1_{B_1(x)}(y) - u(y)}{|x-y|^N}\, dy + \rho_N u(x), \tag{3.8}$$

where $C_N = \pi^{-\frac{N}{2}}\Gamma(\frac{N}{2})$, and $\rho_N = 2\log 2 + \psi(\frac{N}{2}) - \gamma$. Here, $\psi = \frac{\Gamma'}{\Gamma}$ denotes the Digamma function, and $\gamma = -\Gamma'(1)$ is the Euler-Mascheroni constant.
We note two key properties of the operator L_Δ shown in [29]. If $u \in C_c^\beta(\mathbb{R}^N)$ for some $\beta > 0$, then

$$\mathscr{F}(L_\Delta u) = 2\log|\xi|\mathscr{F}(u)(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R}^N, \tag{3.9}$$

so the operator L_Δ has the Fourier symbol $\xi \mapsto 2\log|\xi|$. Moreover,

$$\frac{d}{ds}\Big|_{s=0}(-\Delta)^s u = \lim_{s\to 0^+} \frac{(-\Delta)^s u - u}{s} = L_\Delta u \qquad \text{in } L^p(\mathbb{R}^N) \text{ for } 1 < p \le \infty. \tag{3.10}$$

Hence, L_Δ arises as a formal derivative of fractional Laplacians at $s = 0$. As a consequence of (3.9), L_Δ is an operator of *logarithmic order*, and it belongs to a class of weakly singular

integral operators having an intrinsic scaling property. Operators of this type have also been studied e.g. in [30, 56, 57, 60, 61, 63–65, 79], while the most attention has been given to Lévy generators of geometric stable processes. These operators have a Fourier symbol of the form $\xi \mapsto \log(1+|\xi|^{2\beta})$ with some $\beta > 0$. The particular case $\beta = 1$ corresponds to the variance gamma process, and the kernel of the associated Lévy generator has the same weakly singular behavior as the one of L_Δ. The operator L_Δ also arises in a geometric context of the 0-fractional perimeter studied recently in [32].
Using (3.10) and related functional analytic properties, it has been shown in [29, Theorem 1.5] that

$$\frac{\lambda_{1,s}-1}{s} \to \lambda_{1,L} \quad \text{and} \quad \varphi_{1,s} \to \varphi_{1,L} \quad \text{in } L^2(\Omega) \qquad \text{as } s \to 0^+, \tag{3.11}$$

where $\lambda_{1,L}$ denotes the principal eigenvalue of the Dirichlet eigenvalue problem

$$\begin{cases} L_\Delta u = \lambda u & \text{in } \Omega, \\ \quad u = 0 & \text{in } \Omega^c, \end{cases} \tag{3.12}$$

and $\varphi_{1,L}$ denotes the corresponding (unique) positive L^2-normalized eigenfunction. Here we note that we consider both (3.3) and (3.12) in a suitable weak sense which we will make more precise below.
The main aim of the present paper is twofold. First, we wish to improve the L^2-convergence $\varphi_{1,s} \to \varphi_{1,L}$ in (3.11). For this, new tools are needed in order to overcome the lack of uniform regularity estimates for the fractional Laplacian $(-\Delta)^s$ for s close to zero. Secondly, we wish to extend the convergence result from [29] to higher eigenvalues and eigenfunctions. Due to the multiplicity of eigenvalues and eigenfunctions for $k \geq 2$, this also requires a new approach based on the use of Fourier transform in combination with the Courant-Fischer characterization of eigenvalues.
In order to state our main results, we need to introduce some notation regarding the weak formulations of (3.3) and (3.12). For the weak formulation of (3.3), we consider the standard Sobolev space

$$\mathcal{H}_0^s(\Omega) := \left\{ u \in H^s(\mathbb{R}^N) \ : \ u \equiv 0 \text{ on } \Omega^c \right\} \tag{3.13}$$

and we call $\varphi \in \mathcal{H}_0^s(\Omega)$ an eigenfunction of (3.3) corresponding to the eigenvalue λ if

$$\mathcal{E}_s(\varphi, v) = \lambda \int_\Omega \varphi v \, dx \qquad \text{for all } v \in \mathcal{H}_0^s(\Omega).$$

For the weak formulation of (3.12), we follow [29] and define the space

$$\mathcal{H}_0^0(\Omega) := \Big\{ u \in L^2(\mathbb{R}^N) \ : \ u \equiv 0 \text{ on } \Omega^c, \quad \langle u, u \rangle_{\mathcal{H}_0^0(\Omega)} < +\infty \Big\}, \tag{3.14}$$

where the quadratic form $\langle \cdot, \cdot \rangle_{\mathcal{H}_0^0(\Omega)}$ is given by

$$(u, v) \mapsto \langle u, v \rangle_{\mathcal{H}_0^0(\Omega)} := \frac{C_N}{2} \iint\limits_{\substack{x,y \in \mathbb{R}^N \\ |x-y|<1}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^N} \, dxdy. \tag{3.15}$$

A function $\varphi \in \mathscr{H}_0^0(\Omega)$ is called an eigenfunction of (3.12) corresponding to the eigenvalue λ if

$$\mathscr{E}_L(\varphi, v) = \lambda \int_\Omega \varphi v \, dx \qquad \text{for all } v \in \mathscr{H}_0^0(\Omega),$$

where

$$(u,v) \mapsto \mathscr{E}_L(u,v) = \langle u, v\rangle_{\mathscr{H}_0^0(\Omega)} - C_N \iint_{\substack{x,y\in\mathbb{R}^N \\ |x-y|\geq 1}} \frac{u(x)v(y)}{|x-y|^N}\, dxdy + \rho_N \int_{\mathbb{R}^N} uv \, dx \tag{3.16}$$

is the quadratic form associated with L_Δ. For more details, see Section 3.2 below and [29].

The first main result of this paper now reads as follows.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary and let $k \in \mathbb{N}$. Moreover, for $s \in (0, \frac{1}{4})$, let $\lambda_{k,s}$ resp. $\lambda_{k,L}$ denote the k-th Dirichlet eigenvalue of the fractional and logarithmic Laplacian, respectively, and let $\varphi_{k,s}$ denote an L^2-normalized eigenfunction. Then we have:*

(i) The eigenvalue $\lambda_{k,s}$ satisfies the expansion

$$\lambda_{k,s} = 1 + s\lambda_{k,L} + o(s) \qquad \text{as } s \to 0^+. \tag{3.17}$$

(ii) The set $\{\varphi_{k,s} : s \in (0, \frac{1}{4}]\}$ is bounded in $L^\infty(\Omega)$ and relatively compact in $L^p(\Omega)$ for every $p < \infty$.

(iii) The set $\{\varphi_{k,s} : s \in (0, \frac{1}{4}]\}$ is equicontinuous in every point $x_0 \in \Omega$ and therefore relative compact in $C(K)$ for any compact subset $K \subset \Omega$.

(iv) If Ω satisfies an exterior sphere condition, then the set $\{\varphi_{k,s} : s \in (0, \frac{1}{4}]\}$ is relative compact in the space $C_0(\Omega) := \{u \in C(\mathbb{R}^N) : u \equiv 0 \quad \text{in } \Omega^c\}$.

(v) If $(s_n)_n \subset (0, \frac{1}{4}]$ is a sequence with $s_n \to 0$ as $n \to \infty$, then, after passing to a subsequence, we have

$$\varphi_{k,s_n} \to \varphi_{k,L} \quad \text{as } n \to \infty \tag{3.18}$$

in $L^p(\Omega)$ for $p < \infty$ and locally uniformly in Ω, where $\varphi_{k,L}$ is an L^2-normalized eigenfunction of the logarithmic Laplacian corresponding to the eigenvalue $\lambda_{k,L}$.

If, moreover, Ω satisfies an exterior sphere condition, then the convergence in (3.18) *is uniform in $\overline{\Omega}$.*

Here and in the following, we identify the space $L^p(\Omega)$ with the space of functions $u \in L^p(\mathbb{R}^N)$ with $u \equiv 0$ on Ω^c.

Remark 3.2. (i) Theorem 3.1 complements [29, Theorem 1.5] by emphasizing the relevance of higher Dirichlet eigenvalues and eigenfunctions of L_Δ for the spectral asymptotics of the fractional Laplacian as $s \to 0^+$. We note that upper and lower bounds for the Dirichlet eigenvalues $\lambda_{k,L}$ of the logarithmic Laplacian and corresponding Weyl type asymptotics in the limit $k \to +\infty$ have been derived in [72] and more recently in [28].

(ii) The number $\frac{1}{4}$ in the above theorem is chosen for technical reasons, as it allows to reduce the number of case distinctions in the arguments. In the case $N \geq 2$, it can be replaced by any fixed number smaller than 1, and in the case $N = 1$ it can be replaced by any fixed number smaller than $\frac{1}{2}$. Since we are only interested in parameters s close to zero in this paper, we omit the details of such an extension.

As noted already, the principal eigenvalue $\lambda_{1,s}(\Omega)$ admits, up to sign, a unique L^2-normalized eigenfunction which can be chosen to be positive. Hence Theorem 3.1 and [29, Theorem 1.5] give rise to the following corollary.

Corollary 3.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary and let, for $s \in (0, \frac{1}{4}]$, $\varphi_{1,s}$ denote the unique positive L^2-normalized eigenfunction of $(-\Delta)^s$ corresponding to the principal Dirichlet eigenvalue $\lambda_{1,s}$. Then we have*

$$\varphi_{1,s} \to \varphi_{1,L} \quad \text{as } s \to 0^+ \tag{3.19}$$

in $L^p(\Omega)$ for $p < \infty$ and locally uniformly in Ω, where $\varphi_{1,L}$ is the unique positive L^2-normalized eigenfunction of L_Δ corresponding to the principal Dirichlet eigenvalue $\lambda_{1,L}$.

If, moreover, Ω satisfies an exterior sphere condition, then the convergence in (3.19) *is uniform in $\overline{\Omega}$.*

As a further corollary of Theorem 3.1, we shall derive the following regularity properties of eigenfunctions of the logarithmic Laplacian.

Corollary 3.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary, and let $\varphi \in \mathscr{H}_0^0(\Omega)$ be an eigenfunction of* (3.12). *Then $\varphi \in L^\infty(\Omega) \cap C_{loc}(\Omega)$. Moreover, if Ω satisfies an exterior sphere condition, then $\varphi \in C_0(\Omega)$.*

Remark 3.5. We briefly comment on the main steps and difficulties in the proof of Theorem 3.1. The first step is to prove the asymptotic expansion (3.17) and the L^2-convergence property asserted in Theorem 3.1(v). Then, we prove the uniform L^∞-bound on eigenfunctions as stated in Theorem 3.1(ii). For this, we use a new technique based on the splitting of the integral over $\mathbb{R}^N$ on a small ball of radius δ (δ-decomposition) and apply known results and conditions associated to the newly obtained quadratic form as in [43, 61]. We emphasize that this technique strongly simplifies the general De Giorgi iteration method in combination with Sobolev embedding to prove L^∞-bounds. We also point out that this δ-decomposition method is applicable for general nonlocal operators and allows to get explicit constants for the boundedness.
As a third step, we prove the local equicontinuity result stated in Theorem 3.1(iii). A natural strategy of proving this result is to first obtain a locally uniform estimate for the difference

$$\left[L_\Delta - \frac{(-\Delta)^s - \mathrm{id}}{s}\right] \varphi_{k,s} \tag{3.20}$$

and then to use the local regularity estimates available for the class of weakly singular operators containing L_Δ, see e.g. [63] and the references therein. However, we are not able to obtain uniform estimates for the difference in (3.20). Therefore we first prove uniform bounds related to an s-dependent auxiliary integral operator family instead (see Lemma 3.20 below), and then we complete the proof by a direct contradiction argument. We recall here that regularity estimates for $(-\Delta)^s$ alone, even those with explicit constants, cannot yield sufficient uniform control on continuity modules of the functions $\varphi_{k,s}$ since $(-\Delta)^s$ converges to the identity operator, as noted in (3.5). Once local equicontinuity is established, we then prove, assuming a uniform exterior sphere condition for Ω, a uniform decay property in the sense that there exists, for every fixed $k \in \mathbb{N}$, a function $h_k \in C_0(\Omega)$ with the property that $|\varphi_{k,s}| \leq h_k$ in Ω for all $s \in (0, \frac{1}{4}]$. This will be done with the help of a uniform small volume maximum principle and uniform radial barrier function for the difference quotient operator $\frac{(-\Delta)^s - \mathrm{id}}{s}$, see Section 3.5 below. We point out that the lack of uniform estimates for the difference in (3.20) prevents us from using directly the boundary decay estimates in [64] and [29, Section 5]. On the other hand, the estimates in [64] allow to deduce, together with Corollary 3.4, that every eigenfunction $\varphi \in \mathscr{H}_0^0(\Omega)$ of L_Δ satisfies

$$|\varphi(x)| = O\Big(\big(-\ln \operatorname{dist}(x, \Omega^c)\big)^{-1/2}\Big) \qquad \text{as } x \to \partial\Omega$$

at least in the case when the underlying domain Ω is of class $C^{1,1}$. As a consequence, we conjecture that also the majorizing functions h_k above can be chosen with the property that $h_k(x) \sim \big(-\ln \operatorname{dist}(x, \Omega^c)\big)^{-1/2}$ as $x \to \partial\Omega$.

The paper is organized as follows. In Section 3.2, we collect preliminary results on the functional analytic setting. Moreover, we prove the asymptotic expansion (3.17) and the L^2- convergence property asserted in Theorem 3.1(v). In Section 3.3, we prove the uniform L^∞-bound on eigenfunctions as stated in Theorem 3.1(ii). In Section 3.4, we then prove the local equicontinuity result stated in Theorem 3.1(iii). In Section 3.5, we prove, assuming a uniform exterior sphere condition for Ω, a uniform decay property for the set of eigenfunctions $\{\varphi_{k,s} : s \in (0, \frac{1}{4}]\}$. Combining this uniform decay property with the local equicontinuity proved in Section 3.4, the relative compactness in $C_0(\Omega)$ then follows, as claimed in Theorem 3.1(iv). In Section 3.6, we finally complete the proof of the main results stated here in the introduction.

Notation. We let $\omega_{N-1} = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} = \frac{2}{C_N}$ denote the measure of the unit sphere in $\mathbb{R}^N$. For a set $A \subset \mathbb{R}^N$ and $x \in \mathbb{R}^N$, we define $\delta_A(x) := \operatorname{dist}(x, A^c)$ with $A^c = \mathbb{R}^N \setminus A$ and, if A is measurable, then $|A|$ denotes its Lebesgue measure. Moreover, for given $r > 0$, let $B_r(A) := \{x \in \mathbb{R}^N : \operatorname{dist}(x, A) < r\}$, and let $B_r(x) := B_r(\{x\})$ denote the ball of radius r with x as its center. If $x = 0$ we also write B_r instead of $B_r(0)$.
For $A \subset \mathbb{R}^N$ and $u : A \to \mathbb{R}$ we denote $u^+ := \max\{u, 0\}$ as the positive and $u^- = -\min\{u, 0\}$ as the negative part of u, so that $u = u^+ - u^-$. Moreover, we let

$$\operatorname*{osc}_A u := \sup_A u - \inf_A u \quad \in [0, \infty]$$

denote the oscillation of u over A. If A is open, we denote by $C_c^k(A)$ the space of function $u : \mathbb{R}^N \to \mathbb{R}$ which are k-times continuously differentiable and with support compactly contained in A.

3.2 First order expansion of eigenvalues and L^2-convergence of eigenfunctions

In this section, we first collect some preliminary notions and observations. After this, we complete the proof Theorem 3.1(i), see Theorem 3.15 below.
For $s \in (0,1)$, we use the fractional Sobolev space $H^s(\mathbb{R}^N)$ defined as

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) \, : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy < \infty \right\}, \tag{3.21}$$

with corresponding norm given by

$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\|u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy \right)^{\frac{1}{2}}. \tag{3.22}$$

We recall that this norm is induced by the scalar product

$$(u,v) \mapsto \langle u,v\rangle_{H^s(\mathbb{R}^N)} = \langle u,v\rangle_{L^2(\mathbb{R}^N)} + \mathscr{E}_s(u,v),$$

where

$$\mathscr{E}_s(u,v) = \frac{C_{N,s}}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{N+2s}}\, dxdy = \int_{\mathbb{R}^N} |\xi|^{2s}\hat{u}(\xi)\hat{v}(\xi)\, d\xi \tag{3.23}$$

for $u,v \in H^s(\mathbb{R}^N)$ and the constant $C_{N,s}$ is given in (3.1). The following elementary observations involving the asymptotics of $C_{N,s}$ are used frequently in the paper.

Lemma 3.6. *With $C_N = \pi^{-\frac{N}{2}}\Gamma(\frac{N}{2}) = \frac{2}{\omega_{N-1}}$ and $\rho_N = 2\log 2 + \psi(\frac{N}{2}) - \gamma$ as defined in the introduction, we have*

$$\frac{C_{N,s}}{sC_N} = \frac{\omega_{N-1}C_{N,s}}{2s} = 1 + s\rho_N + o(s) \qquad \text{as } s \to 0^+. \tag{3.24}$$

Consequently, there exists a constant $D_N > 0$ with

$$\left|1 - \frac{C_{N,s}}{C_N s}\right| \le sD_N \quad \text{and therefore} \quad \left|C_N - \frac{C_{N,s}}{s}\right| \le sC_ND_N \qquad \text{for } s \in (0,\frac{1}{4}]. \tag{3.25}$$

Proof. The function

$$s \mapsto \tau(s) := \frac{C_{N,s}}{sC_N} = 4^s \frac{\Gamma(\frac{N}{2}+s)}{\Gamma(\frac{N}{2})\Gamma(1-s)}$$

is of class C^1 on $[0,1)$ and satisfies $\tau(0) = 1$ and $\tau'(0) = \rho_N$. Hence (3.24) follows, and (3.25) is an immediate consequence of (3.24) and the fact that the function $s \mapsto C_{N,s}$ is continuous. $\square$

In the remainder of this paper, we assume that $\Omega \subset \mathbb{R}^N$ is an open bounded subset with Lipschitz boundary. As noted already in the introduction, we identify, for $p \in [1,\infty]$, the space $L^p(\Omega)$ with the space of functions $u \in L^p(\mathbb{R}^N)$ satisfying $u \equiv 0$ on Ω^c.
For $s \in (0,1)$, we then consider the subspace $\mathscr{H}_0^s(\Omega) \subset H^s(\mathbb{R}^N)$ as defined in (3.13). Due to the boundedness of Ω, we have

$$\lambda_{1,s}(\Omega) := \inf_{\substack{u \in \mathscr{H}_0^s(\Omega) \\ u \neq 0}} \frac{\mathscr{E}_s(u,u)}{\|u\|^2_{L^2(\mathbb{R}^N)}} > 0 \tag{3.26}$$

so we can equip the Hilbert space $\mathscr{H}_0^s(\Omega)$ with the scalar product $\mathscr{E}_s$ and induced norm

$$u \mapsto \|u\|_{\mathscr{H}_0^s(\Omega)} := \mathscr{E}_s(u,u)^{\frac{1}{2}}.$$

Moreover, $\mathscr{H}_0^s(\Omega)$ is compactly embedded in $L^2(\Omega)$, $C_c^2(\Omega)$ is dense in $\mathscr{H}_0^s(\Omega)$, and we have

$$\mathscr{E}_s(u,v) = \int_{\mathbb{R}^N} u(x)(-\Delta)^s v(x)\,dx \qquad \text{for all } u \in H^s(\mathbb{R}^N) \text{ and } v \in C_c^2(\mathbb{R}^N),$$

see [33]. We now set up the corresponding framework of problem (3.12) for the logarithmic Laplacian. We let as in the introduction, see (3.15), (3.14),

$$\mathscr{H}_0^0(\Omega) := \left\{ u \in L^2(\mathbb{R}^N) \;:\; u \equiv 0 \text{ on } \Omega^c \text{ and } \iint_{\substack{x,y \in \mathbb{R}^N \\ |x-y|<1}} \frac{(u(x)-u(y))^2}{|x-y|^N} dxdy < +\infty \right\}. \tag{3.27}$$

Then the map

$$(u,v) \mapsto \langle u,v \rangle_{\mathscr{H}_0^0(\Omega)} := \frac{C_N}{2} \iint_{\substack{x,y \in \mathbb{R}^N \\ |x-y|<1}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^N}\,dxdy, \tag{3.28}$$

is a scalar product on $\mathscr{H}_0^0(\Omega)$ by [43, Lemma 2.7], and the space $\mathscr{H}_0^0(\Omega)$ is a Hilbert space. Here, $C_N = \pi^{-N/2}\Gamma(\frac{N}{2}) = \frac{2}{\omega_{N-1}}$ is as in the introduction. We denote the induced norm by $\|\cdot\|_{\mathscr{H}_0^0(\Omega)}$. Moreover, by [30, Theorem 2.1]),

$$\text{the embedding } \mathscr{H}_0^0(\Omega) \hookrightarrow L^2(\Omega) \text{ is compact,} \tag{3.29}$$

and the space $C_c^2(\Omega)$ is dense in $\mathscr{H}_0^0(\Omega)$ by [29, Theorem 3.1].

Remark 3.7. We stress that, despite the similarities noted above, $\mathscr{H}_0^0(\Omega)$ should *not* be considered as a limit of the Hilbert spaces $\mathscr{H}_0^s(\Omega)$ as $s \to 0^+$. In particular, it is not the limit in the sense of [71]. Instead, the space $\mathscr{H}_0^0(\Omega)$ arises naturally when considering a first oder expansion of $\langle \cdot,\cdot \rangle_{H^s(\mathbb{R}^N)}$, cf. Lemma 3.11 below.

Next we note that, setting

$$\mathscr{E}_0(u,v) = \langle u,v\rangle_{\mathscr{H}_0^0(\Omega)} - C_N \iint\limits_{\substack{x,y\in\mathbb{R}^N\\|x-y|\geq 1}} \frac{u(x)v(y)}{|x-y|^N}\,dxdy + \rho_N \int\limits_{\mathbb{R}^N} uv\,dx \tag{3.30}$$

with $\rho_N = 2\log 2 + \psi(\frac{N}{2}) - \gamma$ as in the introduction, we have

$$\mathscr{E}_0(u,v) = \int\limits_{\Omega} u(x)L_\Delta v(x)\,dx \quad \text{for } u \in \mathscr{H}_0^0(\Omega) \text{ and } v \in C_c^1(\Omega),$$

see [29]. In order to get a convenient parameter-dependent notation for the remainder of this section, we now put

$$L^s = (-\Delta)^s \quad \text{for } s \in (0,1) \qquad \text{and} \qquad L^0 = L_\Delta.$$

Then, for $s \in [0,1)$, we call $\lambda \in \mathbb{R}$ a Dirichlet-eigenvalue of L^s in Ω with corresponding eigenfunction $u \in \mathscr{H}_0^s(\Omega)\setminus\{0\}$ if

$$\begin{cases} L^s u = \lambda u & \text{in } \Omega \\ \quad u = 0 & \text{in } \Omega^c, \end{cases} \tag{3.31}$$

holds in weak sense, i.e., if

$$\mathscr{E}_s(u,\psi) = \lambda \int\limits_{\Omega} u\psi\,dx \qquad \text{for all } \psi \in \mathscr{H}_0^s(\Omega).$$

In the following Proposition we collect the known properties on the eigenvalues and eigenfunctions of the fractional Laplacian and the logarithmic Laplacian, see e.g. [15, Prosition 3.1] and the references in there for the fractional Laplacian and [29, Theorem 3.4] for the logarithmic Laplacian.

Proposition 3.8. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary, and let $s \in [0,1)$. Then the following holds:*

(a) The eigenvalues of problem (3.31) *consist of a sequence $\{\lambda_{k,s}(\Omega)\}_{k\in\mathbb{N}}$ with $0 < \lambda_{1,s}(\Omega) < \lambda_{2,s}(\Omega) \leq \cdots \leq \lambda_{k,s}(\Omega) \leq \lambda_{k+1,s}(\Omega) \leq \cdots$ and $\lim\limits_{k\to\infty} \lambda_{k,s}(\Omega) = +\infty$.*

(b) The sequence $\{\varphi_{k,s}\}_{k\in\mathbb{N}}$ of eigenfunctions corresponding to eigenvalues $\lambda_{k,s}(\Omega)$ forms a complete orthonormal basis of $L^2(\Omega)$ and an orthogonal system of $\mathscr{H}_0^s(\Omega)$.

(c) For any $k \in \mathbb{N}$, the eigenvalue $\lambda_{k,s}(\Omega)$ is characterized as

$$\lambda_{k,s}(\Omega) = \min\Big\{\mathscr{E}_s(u,u) : u \in \mathbb{P}_{k,s}(\Omega) \quad \textit{and} \quad \|u\|_{L^2(\Omega)} = 1\Big\},$$

where $\mathbb{P}_{1,s}(\Omega) = \mathscr{H}_0^s(\Omega)$ and

$$\mathbb{P}_{k,s}(\Omega) = \big\{u \in \mathscr{H}_0^s(\Omega) : \mathscr{E}_s(u,\varphi_{j,s}) = 0 \textit{ for } j = 1,\cdots,k-1\big\} \qquad \textit{for } k > 1.$$

(d) *The first eigenvalue $\lambda_{1,s}(\Omega)$ is simple and the corresponding eigenfunction $\varphi_{1,s}$ does not change its sign in Ω and can be chosen to be strictly positive in Ω.*

Remark 3.9. (i) The characterization in Proposition 3.8(c) implies that $\lambda_{1,s}(\Omega)$, as defined in (3.26), is indeed the first Dirichlet eigenvalue of $(-\Delta)^s$ on Ω, so the notation is consistent.

(ii) We emphasize that in the case $s=0$ the eigenvalues $\lambda_{k,0}$ and corresponding eigenfunctions $\varphi_{k,0}$ for $k \in \mathbb{N}$ are also denoted by $\lambda_{k,L}$ and $\varphi_{k,L}$ resp. as in the introduction for consistency.

(iii) By the Courant-Fischer minimax principle and due to the density of $C_c^2(\Omega)$ in $\mathscr{H}_0^s(\Omega)$, the eigenvalues $\lambda_{k,s}$, $s \in [0,1)$, $k \in \mathbb{N}$ can be characterized equivalently as

$$\lambda_{k,s}(\Omega) = \inf_{\substack{V \subset \mathscr{H}_0^s(\Omega)\\ \dim V = k}} \max_{\substack{v \in V\setminus\{0\}\\ \|v\|_{L^2(\Omega)}=1}} \mathscr{E}_s(v,v) = \inf_{\substack{V \subset C_c^2(\Omega)\\ \dim V = k}} \max_{\substack{v \in V\setminus\{0\}\\ \|v\|_{L^2(\Omega)}=1}} \mathscr{E}_s(v,v). \tag{3.32}$$

This fact will be used in the sequel.

Next, we need the following elementary estimates.

Lemma 3.10. *For $s \in (0,1)$ and $r > 0$ we have*

$$\Big|\frac{r^{2s}-1}{s}\Big| \leq 2\Big(|\ln r| 1_{(0,1]}(r) + 1_{(1,\infty)}(r) r^4\Big) \tag{3.33}$$

and

$$\Big|\frac{r^{2s}-1}{s} - 2\log r\Big| \leq 4s\Big(\ln^2(r) 1_{(0,1]}(r) + 1_{(1,\infty)}(r) r^4\Big) \tag{3.34}$$

Proof. Fix $r>0$ and let $h_r(s) = r^{2s}$, $r>0$. Then we have $h_r'(\tau) = 2r^{2\tau}\ln r$ and $h_r''(\tau) = 4r^{2\tau}\ln^2(r)$ for $\tau > 0$. Consequently,

$$\Big|\frac{r^{2s}-1}{s}\Big| = \frac{2|\ln r|}{s}\int_0^s r^{2\tau}\,d\tau \leq 2|\ln r| \max\{1, r^{2s}\} \leq 2\Big(|\ln r| 1_{(0,1]}(r) + 1_{(1,\infty)}(r) r^4\Big),$$

where in the last step we used that $r^{2s} \leq 1$ for $r \leq 1$ and, since $s<1$,

$$r^{2s}\ln r \leq r^{2s+1} \leq r^4 \quad \text{for } r > 1.$$

Hence (3.33) is true. Moreover, by Taylor expansion,

$$h_r(s) = 1 + s h_r'(0) + \int_0^s h_r''(\tau)(s-\tau)\,d\tau = 1 + 2s\ln r + 4\ln^2 r \int_0^s r^{2\tau}(s-\tau)\,d\tau$$

and therefore

$$\Big|\frac{r^{2s}-1}{s} - 2\log r\Big| \leq \frac{4\ln^2(r)}{s}\Big|\int_0^s r^{2\tau}(s-\tau)\,d\tau\Big| \leq 4s\ln^2(r)\max\{r^{2s},1\}.$$

Hence (3.34) follows since for $r \in (0,1]$ we have $r^{2s} \le 1$ and, since $s < 1$,

$$r^{2s} \ln^2 r \le r^{2s+2} \le r^4 \quad \text{for } r > 1.$$

$\square$

Lemma 3.11. *For every $u \in C_c^2(\Omega)$ and $s \in (0,1)$ we have*

$$\Big|\mathscr{E}_s(u,u) - \|u\|^2_{L^2(\mathbb{R}^N)}\Big| \le 2s\Big(\kappa_N \|u\|^2_{L^1(\mathbb{R}^N)} + \|\Delta u\|^2_{L^2(\mathbb{R}^N)}\Big) \tag{3.35}$$

and

$$\Big|\mathscr{E}_s(u,u) - \|u\|^2_{L^2(\mathbb{R}^N)} - s\mathscr{E}_0(u,u)\Big| \le 4s^2\Big(\kappa_N \|u\|^2_{L^1(\mathbb{R}^N)} + \|\Delta u\|^2_{L^2(\mathbb{R}^N)}\Big) \tag{3.36}$$

with $\kappa_N = (2\pi)^{-N} \int_{B_1(0)} \ln^2|\xi|\, d\xi$.

Proof. Let $u \in C_c^2(\Omega)$ and $s \in (0,1)$. By (3.23) and (3.33), we have

$$\begin{aligned}
\Big|\mathscr{E}_s(u,u) - \|u\|^2_{L^2(\mathbb{R}^N)}\Big| &\le \int_{\mathbb{R}^N} \big||\xi|^{2s} - 1\big|\, |\hat{u}(\xi)|^2\, d\xi \\
&\le 2s\Big(\int_{B_1(0)} |\ln|\xi|||\hat{u}(\xi)|^2\, d\xi + \int_{\mathbb{R}^N\setminus B_1} |\xi|^4 |\hat{u}(\xi)|^2\, d\xi\Big) \\
&\le 2s\Big(\|\hat{u}\|^2_{L^\infty(\mathbb{R}^N)} \int_{B_1(0)} |\ln|\xi||\, d\xi + \|\Delta u\|^2_{L^2(\mathbb{R}^N)}\Big) \\
&\le 2s\Big((2\pi)^{-N}\|u\|^2_{L^1(\mathbb{R}^N)} \int_{B_1(0)} \ln^2|\xi|\, d\xi + \|\Delta u\|^2_{L^2(\mathbb{R}^N)}\Big).
\end{aligned}$$

Thus (3.35) follows. Moreover, by (3.34) we have

$$\begin{aligned}
\Big|\mathscr{E}_s(u,u) - \|u\|^2_{L^2(\mathbb{R}^N)} - s\mathscr{E}_0(u,u)\Big| &\le \int_{\mathbb{R}^N} \big||\xi|^{2s} - 1 - 2s\log|\xi|\big|\, |\hat{u}(\xi)|^2\, d\xi \\
&\le 4s^2\Big(\int_{B_1(0)} \ln^2|\xi||\hat{u}(\xi)|^2\, d\xi + \int_{\mathbb{R}^N\setminus B_1} |\xi|^4 |\hat{u}(\xi)|^2\, d\xi\Big) \\
&\le 4s^2\Big(\|\hat{u}\|^2_{L^\infty} \int_{B_1(0)} \ln^2|\xi|\, d\xi + \|\Delta u\|^2_{L^2(\mathbb{R}^N)}\Big) \\
&\le 4s^2\Big((2\pi)^{-N}\|u\|^2_{L^1(\mathbb{R}^N)} \int_{B_1(0)} \ln^2|\xi|\, d\xi + \|\Delta u\|^2_{L^2(\mathbb{R}^N)}\Big)
\end{aligned}$$

Hence (3.36) follows. $\square$

Lemma 3.12. *For all $k \in \mathbb{N}$ we have*

$$\lambda_{1,0}(\Omega) \le \liminf_{s\to 0^+} \frac{\lambda_{k,s}(\Omega) - 1}{s} \le \limsup_{s\to 0^+} \frac{\lambda_{k,s}(\Omega) - 1}{s} \le \lambda_{k,0}(\Omega) \tag{3.37}$$

and

$$\lambda_{k,s}(\Omega) \le 1 + sC \qquad \textit{for all } s \in (0,1) \tag{3.38}$$

with a constant $C = C(N,\Omega,k) > 0$.

Proof. We fix a subspace $V \subset C_c^2(\Omega)$ of dimension k and let $S_V := \{u \in V \ : \ \|u\|_{L^2(\Omega)} = 1\}$. Using (3.32) and (3.35), we find that, for $s \in (0,1)$,

$$\frac{\lambda_{k,s}(\Omega) - 1}{s} \le \max_{u \in S_V} \frac{\mathscr{E}_s(u,u) - 1}{s} \le C \tag{3.39}$$

with

$$C = C(N,\Omega,k) = 2 \max_{u \in S_V} \Big(\kappa_N \|u\|^2_{L^1(\mathbb{R}^N)} + \|\Delta u\|^2_{L^2(\mathbb{R}^N)} \Big).$$

Hence (3.38) holds. Technically speaking, the constant C depends on the choice of V, but V can be chosen merely in dependence of Ω. Moreover, setting $\mathscr{R}_s(u) = \frac{\mathscr{E}_s(u,u)-1}{s} - \mathscr{E}_0(u,u)$ for $u \in C_c^2(\Omega)$, we deduce from (3.39) that

$$\frac{\lambda_{k,s}(\Omega) - 1}{s} \le \max_{u \in S_V} \mathscr{E}_0(u,u) + \max_{u \in S_V} |\mathscr{R}_s(u)|$$

while, by Lemma 3.11,

$$|\mathscr{R}_s(u)| \le 4s \Big(\kappa_N \|u\|^2_{L^1(\mathbb{R}^N)} + \|\Delta u\|^2_{L^2(\mathbb{R}^N)} \Big) \to 0 \quad \text{as } s \to 0^+ \text{ uniformly in } u \in S_V.$$

Consequently,

$$\limsup_{s \to 0^+} \frac{\lambda_{k,s}(\Omega) - 1}{s} \le \max_{u \in S_V} \mathscr{E}_0(u,u).$$

Since V was chosen arbitrarily, the characterization of the Dirichlet eigenvalues of the logarithmic Laplacian given in (3.32) with $s = 0$ implies that

$$\limsup_{s \to 0^+} \frac{\lambda_{k,s}(\Omega) - 1}{s} \le \inf_{\substack{V \subset C_c^2(\Omega) \\ \dim(V) = k}} \max_{\substack{u \in V \\ \|u\|_{L^2(\Omega)} = 1}} \mathscr{E}_L(u,u) = \lambda_{k,0}(\Omega), \tag{3.40}$$

In particular, the last inequality in (3.37) holds. Moreover, since $\lambda_{k,s}(\Omega) \ge \lambda_{1,s}(\Omega)$ for every $k \in \mathbb{N}$ and

$$\lim_{s \to 0^+} \frac{\lambda_{1,s}(\Omega) - 1}{s} = \lambda_{1,0}(\Omega)$$

by [29, Theorem 1.5], the first inequality in (3.37) also follows. □

Corollary 3.13. *For all* $k \in \mathbb{N}$ *we have* $\lim_{s \to 0^+} \lambda_{k,s}(\Omega) = 1$.

Proof. This immediately follows from (3.37). □

Lemma 3.14. *Let* $k \in \mathbb{N}$, $s_0 \in (0,1)$, *and let, for* $s \in (0,s_0)$, $\varphi_{k,s} \in \mathscr{H}_0^s(\Omega)$ *denote an* L^2-*normalized eigenfunction of* $(-\Delta)^s$ *in* Ω. *Then the set*

$$\{\varphi_{k,s} \ : \ s \in (0,s_0)\}$$

is uniformly bounded in $\mathscr{H}_0^0(\Omega)$ *and therefore relatively compact in* $L^2(\Omega)$.

Proof. By (3.38), there exists a constant $C = C(N,\Omega,k) > 0$ with the property that

$$C \geq \frac{\lambda_{k,s}(\Omega)-1}{s} = \frac{\mathscr{E}_s(\varphi_{k,s},\varphi_{k,s})-1}{s} = \frac{C_{N,s}}{2s}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{|\varphi_{k,s}(x)-\varphi_{k,s}(y)|^2}{|x-y|^{N+2s}}dxdy - \frac{1}{s}$$
$$= \frac{C_{N,s}}{2s}\iint\limits_{|x-y|<1}\frac{|\varphi_{k,s}(x)-\varphi_{k,s}(y)|^2}{|x-y|^{N+2s}}dxdy - \frac{C_{N,s}}{s}\iint\limits_{|x-y|\geq 1}\frac{\varphi_{k,s}(x)\varphi_{k,s}(y)}{|x-y|^{N+2s}}dxdy + f_N(s), \tag{3.41}$$

where, due to the L^2-normalization of $\varphi_{k,s}$,

$$f_N(s) := \frac{1}{s}\Big(C_{N,s}\int\limits_{\Omega}|\varphi_{k,s}(x)|^2\int\limits_{\mathbb{R}^N\setminus B_1(x)}\frac{1}{|x-y|^{N+2s}}dy\,dx - 1\Big) = \frac{1}{s}\left(\frac{C_{N,s}\omega_{N-1}}{2s}-1\right). \tag{3.42}$$

Therefore, using the definition of $\|\cdot\|_{\mathscr{H}_0^0(\Omega)}$, we deduce that

$$C \geq \frac{C_{N,s}}{sC_N}\|\varphi_{k,s}\|^2_{\mathscr{H}_0^0(\Omega)} - \frac{C_{N,s}}{s}\iint\limits_{|x-y|\geq 1}\frac{|\varphi_{k,s}(x)\varphi_{k,s}(y)|}{|x-y|^{N+2s}}dxdy + f_N(s), \tag{3.43}$$

where, by Hölder's inequality,

$$\iint\limits_{|x-y|\geq 1}\frac{|\varphi_{k,s}(x)\varphi_{k,s}(y)|}{|x-y|^{N+2s}}dxdy \leq \int\limits_{\Omega}\int\limits_{\Omega\cap\{|x-y|\geq 1\}}\frac{|\varphi_{k,s}(x)|^2}{|x-y|^N}dydx \leq |\Omega|\|\varphi_{k,s}\|_{L^2(\Omega)} = |\Omega|,$$

using again the L^2-normalization. Combining this with (3.43), we find that

$$\|\varphi_{k,s}\|^2_{\mathscr{H}_0^0(\Omega)} \leq \frac{sC_N}{C_{N,s}}\Big(C+|\Omega|-f_N(s)\Big).$$

Since moreover $\frac{sC_N}{C_{N,s}} \to 1$ and $f_N(s) \to \rho_N$ as $s \to 0^+$ by Lemma 3.6, we conclude that there exists a constant $K = K(N,k,\Omega) > 0$ and $s_1 \in (0,1)$ such that

$$\|\varphi_{k,s}\|_{\mathscr{H}_0^0(\Omega)} \leq K \qquad \text{for all } s \in (0,s_1).$$

Consequently, the set $\{\varphi_{k,s} : s \in (0,s_1)\}$ is uniformly bounded in $\mathscr{H}_0^0(\Omega)$ and thus relatively compact in $L^2(\Omega)$ by (3.29). Hence the claim follows for $s_0 \leq s_1$.
If $s_0 \in (s_1,1)$, we can use the fact that by (3.38) we have, for $s \in [s_1,s_0]$,

$$1+C \geq \lambda_{k,s}(\Omega) = \mathscr{E}_s(\varphi_{k,s},\varphi_{k,s}) = \frac{C_{N,s}}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{|\varphi_{k,s}(x)-\varphi_{k,s}(y)|^2}{|x-y|^{N+2s}}dxdy$$
$$\geq \frac{C_{N,s}}{2}\iint\limits_{|x-y|\leq 1}\frac{|\varphi_{k,s}(x)-\varphi_{k,s}(y)|^2}{|x-y|^N}dxdy = \frac{C_{N,s}}{C_N}\|\varphi_{k,s}\|^2_{\mathscr{H}_0^0(\Omega)}$$

with a constant $C = C(N,\Omega,k) > 0$ and hence

$$\sup_{s\in[s_1,s_0]} \|\varphi_{k,s}\|^2_{\mathscr{H}_0^0(\Omega)} \leq C_N(1+C) \sup_{s\in[s_1,s_0]} \frac{1}{C_{N,s}} < \infty.$$

We thus conclude that the set $\{\varphi_{k,s} \,:\, s \in (0,s_0)\}$ is uniformly bounded in $\mathscr{H}_0^0(\Omega)$ and thus relatively compact in $L^2(\Omega)$ by (3.29), as claimed. □

We finish this section with the the following theorem which, in particular, completes the proof of Theorem 3.2(i).

Theorem 3.15. *For every $k \in \mathbb{N}$ we have*

$$\lim_{s\to 0^+} \frac{\lambda_{k,s}(\Omega)-1}{s} = \lambda_{k,0}(\Omega). \tag{3.44}$$

Moreover, if $(s_n)_n \subset (0,1)$ is a sequence such that $\lim\limits_{n\to\infty} s_n = 0$ and φ_{k,s_n} is an L^2-normalized Dirichlet eigenfunction of $(-\Delta)^s$ corresponding to the eigenvalue $\lambda_{k,s}(\Omega)$, then, after passing to a subsequence,

$$\varphi_{k,s} \to \varphi_{k,0} \qquad \text{in } L^2(\Omega) \text{ as } n\to\infty,$$

where $\varphi_{k,0}$ is an L^2-normalized Dirichlet eigenfunction of the logarithmic Laplacian corresponding to $\lambda_{k,0}(\Omega)$.

Proof. To establish (3.44), it suffices, in view of (3.37), to consider an arbitrary sequence $(s_n)_n \subset (0,1)$ with $\lim\limits_{n\to\infty} s_n = 0$, and to show that, after passing to a subsequence,

$$\lim_{n\to\infty} \frac{\lambda_{k,s_n}(\Omega)-1}{s_n} = \lambda_{k,0}(\Omega) \qquad \text{for } k\in\mathbb{N}. \tag{3.45}$$

Let $\{\varphi_{k,s_n} \,:\, k\in\mathbb{N}\}$ be an orthonormal system of eigenfunctions corresponding to the Dirichlet eigenvalue $\lambda_{k,s_n}(\Omega)$ of $(-\Delta)^{s_n}$. By Lemma 3.14, it follows that, for every $k\in\mathbb{N}$, the sequence of functions φ_{k,s_n}, $n\in\mathbb{N}$ is bounded in $\mathscr{H}_0^0(\Omega)$ and relatively compact in $L^2(\Omega)$. Consequently, we may pass to a subsequence such that, for every $k\in\mathbb{N}$,

$$\varphi_{k,s_n} \rightharpoonup \varphi_{k,0} \text{ weakly in } \mathscr{H}_0^0(\Omega) \text{ and } \varphi_{k,s_n} \to \varphi_{k,0} \text{ strongly in } L^2(\Omega) \quad \text{as } n\to\infty. \tag{3.46}$$

Here a diagonal argument is used to have convergence for all $k\in\mathbb{N}$. Moreover, by (3.37) we may, after passing again to a subsequence if necessary, assume that, for every $k\in\mathbb{N}$,

$$\frac{\lambda_{k,s_n}(\Omega)-1}{s_n} \to \lambda_k^\star \in \Big[\lambda_{1,0}(\Omega), \lambda_{k,0}(\Omega)\Big] \qquad \text{as } n\to\infty. \tag{3.47}$$

To prove (3.45), it now suffices to show that

$$\lambda_{k,0}(\Omega) = \lambda_k^\star \qquad \text{for every } k\in\mathbb{N}. \tag{3.48}$$

It follows from (3.46) that

$$\|\varphi_{k,0}\|_{L^2(\Omega)} = 1 \quad \text{and} \quad \langle \varphi_{k,0}, \varphi_{\ell,0}\rangle_{L^2(\Omega)} = 0 \qquad \text{for } k,\ell \in \mathbb{N},\, \ell \neq k. \tag{3.49}$$

Moreover, for $w \in C_c^2(\Omega)$ and $n \in \mathbb{N}$ we have

$$\mathscr{E}_{s_n}(\varphi_{k,s_n}, w) = \lambda_{k,s_n}(\Omega)\langle \varphi_{k,s_n}, w\rangle_{L^2(\Omega)} \tag{3.50}$$

and therefore, by [29, Theorem 1.1 (i)],

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\lambda_{k,s_n}(\Omega)-1}{s_n}\langle \varphi_{k,s_n}, w\rangle_{L^2(\Omega)} = \lim_{n\to\infty}\frac{1}{s_n}\Big(\mathscr{E}_s(\varphi_{k,s_n}, w) - \langle \varphi_{k,s_n}, w\rangle_{L^2(\Omega)}\Big)\\
&= \lim_{n\to\infty}\Big\langle \varphi_{k,s_n}, \frac{(-\Delta)^{s_n}w - w}{s_n}\Big\rangle_{L^2(\Omega)} = \langle \varphi_{k,0}, L_\Delta w\rangle_{L^2(\Omega)} = \mathscr{E}_L(\varphi_{k,0}, w).
\end{aligned}$$

Since moreover $\langle \varphi_{k,s_n}, w\rangle_{L^2(\Omega)} \to \langle \varphi_{k,0}, w\rangle_{L^2(\Omega)}$ for $n \to \infty$, it follows from (3.47) that

$$\mathscr{E}_L(\varphi_{k,0}, w) = \lambda_k^\star \langle \varphi_{k,0}, w\rangle_{L^2(\Omega)} \qquad \text{for all } w \in C_c^2(\Omega). \tag{3.51}$$

Thus $\varphi_{k,0}$ is a Dirichlet eigenfunction of the logarithmic Laplacian L_Δ corresponding to $\lambda_k^\star$. Next, for fixed $k \in \mathbb{N}$, we consider $E_{k,0} := \operatorname{span}\{\varphi_{1,0}, \varphi_{2,0}, \cdots, \varphi_{k,0}\}$, which is a k-dimensional subspace of $\mathscr{H}_0^0(\Omega)$ by (3.49). Since

$$\lambda_1^\star \leq \lambda_2^\star \leq \ldots \leq \lambda_k^\star$$

as a consequence of (3.47) and since $\lambda_{i,s_n} \leq \lambda_{j,s_n}$ for $1 \leq i \leq j \leq k$, $n \in \mathbb{N}$, we have the following estimate for every $w = \sum\limits_{i=1}^{k} \alpha_i \varphi_{i,0} \in E_{k,0}$ with $\alpha_1, \cdots, \alpha_k \in \mathbb{R}$:

$$\mathscr{E}_0(w,w) = \sum_{i,j=1}^{k} \alpha_i\alpha_j \mathscr{E}_0(\varphi_{i,0}, \varphi_{j,0}) = \sum_{i,j=1}^{k} \alpha_i\alpha_j\lambda_i^\star \langle \varphi_{i,0}, \varphi_{j,0}\rangle_{L^2(\Omega)} \tag{3.52}$$

$$= \sum_{i=1}^{k} \alpha_i^2 \lambda_i^\star \|\varphi_{i,0}\|_{L^2(\Omega)}^2 \leq \lambda_k^\star \sum_{i=1}^{k}\alpha_i^2 = \lambda_k^\star \|w\|_{L^2(\Omega)}^2. \tag{3.53}$$

The characterization in (3.32) now yields that

$$\lambda_{k,0}(\Omega) \leq \max_{\substack{w\in E_{k,0}\\ \|w\|_{L^2(\Omega)}=1}} \mathscr{E}_0(w,w) \leq \lambda_k^\star.$$

Since also $\lambda_k^\star \leq \lambda_{k,0}(\Omega)$ by (3.47), (3.48) follows. We thus conclude that (3.45) holds. Moreover, the second statement of the theorem also follows a posteriori from the equality $\lambda_k^\star = \lambda_{k,0}(\Omega)$, since we have already seen that $\varphi_{k,s_n} \to \varphi_{k,0}$ in $L^2(\Omega)$, where $\varphi_{k,0}$ is a Dirichlet eigenfunction of the logarithmic Laplacian L_Δ corresponding to the eigenvalue $\lambda_k^\star$. The proof is thus finished. $\square$

3.3 Uniform L^∞-bounds on eigenfunctions

Through the remainder of this paper, we fix $k \in \mathbb{N}$, and we consider, for $s \in (0, \frac{1}{4}]$, eigenfunctions $\varphi_s := \varphi_{k,s}$ of $(-\Delta)^s$ in Ω corresponding to $\lambda_s := \lambda_{k,s}$. Furthermore, we assume that φ_s is L^2-normalized, that is $\|\varphi_s\|_{L^2(\Omega)} = 1$ for all $s \in (0, \frac{1}{4}]$. The main result of this section is the following.

Theorem 3.16. *There exists a constant $C = C(N, \Omega, k)$ with the property that $\|\varphi_s\|_{L^\infty(\Omega)} \le C$ for all $s \in (0, \frac{1}{4}]$.*

To prove this result, we use a new approach based on a so-called δ-decomposition of nonlocal quadratic forms.

For $\delta > 0$ and $u, v \in H^s(\mathbb{R}^N)$, we can write

$$\begin{aligned}\mathscr{E}_s(u,v) &= \mathscr{E}_s^\delta(u,v) + \frac{C_{N,s}}{2} \iint\limits_{|x-y|>\delta} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy \\ &= \mathscr{E}_s^\delta(u,v) + \kappa_{\delta,s} \langle u, v\rangle_{L^2(\mathbb{R}^N)} - \langle k_{\delta,s} * u, v\rangle_{L^2(\mathbb{R}^N)}\end{aligned}$$

with the δ-dependent quadratic form

$$(u,v) \mapsto \mathscr{E}_s^\delta(u,v) = \frac{C_{N,s}}{2} \iint\limits_{|x-y|<\delta} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy,$$

the function $k_{\delta,s} = C_{N,s} 1_{\mathbb{R}^N \setminus B_\delta(0)} |\cdot|^{-N-2s} \in L^1(\mathbb{R}^N)$ and the constant

$$\kappa_{\delta,s} = \frac{C_{N,s}\, \omega_{N-1}\, \delta^{-2s}}{2s}.$$

In particular, this decomposition is valid if $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain and $u, v \in \mathscr{H}_0^s(\Omega)$.

Proof of Theorem 3.16. Let $\delta \in (0,1)$, $c > 0$, and consider the function $w_c = (\varphi_s - c)^+ : \Omega \to \mathbb{R}$ for $s \in (0,1)$. Then $w_c \in \mathscr{H}_0^0(\Omega)$ by [60, Lemma 3.2]. Moreover, for $x, y \in \mathbb{R}^N$ we have

$$\begin{aligned}&(\varphi_s(x) - \varphi_s(y))(w_c(x) - w_c(y)) = ([\varphi_s(x) - c] - [\varphi_s(y) - c])(w_c(x) - w_c(y)) \\ &= [\varphi_s(x) - c] w_c(x) + [\varphi_s(y) - c] w_c(y) - [\varphi_s(x) - c] w_c(y) - w_c(x)[\varphi_s(y) - c] \\ &= w_c^2(x) + w_c^2(y) - 2 w_c(x) w_c(y) + [\varphi_s(x) - c]^- w_c(y) + w_c(x)[\varphi_s(y) - c]^- \\ &\ge w_c^2(x) + w_c^2(y) - 2 w_c(x) w_c(y) = (w_c(x) - w_c(y))^2,\end{aligned}$$

which implies that

$$\mathscr{E}_s^\delta(w_c, w_c) = \frac{C_{N,s}}{2} \iint\limits_{|x-y|<\delta} \frac{(w_c(x) - w_c(y))^2}{|x-y|^{N+2s}}\, dxdy \tag{3.54}$$

$$
\begin{aligned}
&\leq \frac{C_{N,s}}{2} \iint\limits_{|x-y|<\delta} \frac{(\varphi_s(x)-\varphi_s(y))(w_c(x)-w_c(y))}{|x-y|^{N+2s}}\,dxdy \\
&= \mathscr{E}_s^{\delta}(\varphi_s,w_c) = \mathscr{E}_s(\varphi_s,w_c) - \kappa_{\delta,s}\langle \varphi_s, w_c\rangle_{L^2(\Omega)} + \langle k_{\delta,s} * \varphi_s\,, w_c\rangle_{L^2(\Omega)} \\
&= \big(\lambda_s - \kappa_{\delta,s}\big)\langle \varphi_s, w_c\rangle_{L^2(\Omega)} + \langle k_{\delta,s} * \varphi_s\,, w_c\rangle_{L^2(\Omega)} = g_\delta(s)\langle \varphi_s, w_c\rangle_{L^2(\Omega)} + \langle k_{\delta,s} * \varphi_s\,, w_c\rangle_{L^2(\Omega)}
\end{aligned}
$$

with the function

$$
g_\delta : (0,1) \to \mathbb{R}, \qquad g_\delta(s) = \lambda_s - \kappa_{\delta,s} = \lambda_s - \frac{C_{N,s}\omega_{N-1}\delta^{-2s}}{2s}. \tag{3.55}
$$

Since $\lambda_s = 1 + \lambda_L s + o(s)$ by Theorem 3.15, where $\lambda_L = \lambda_{k,0}$ denotes the k-the eigenvalue of the logarithmic Laplacian, and

$$
\frac{C_{N,s}\omega_{N-1}\delta^{-2s}}{2s} = 1 + \big(\rho_N + 2\ln\delta\big)s + o(s) \qquad \text{as } s \to 0^+
$$

by Lemma 3.6, we have

$$
g_\delta(s) = \big(\lambda_L - \rho_N + 2\ln\delta\big)s + o(s) \qquad \text{as } s \to 0^+.
$$

Here the remainder term $o(s)$ depends on $\delta > 0$. Nevertheless, we may first fix $\delta \in (0,1)$ sufficiently small such that $\lambda_L - \rho_N + 2\ln\delta < -1$, and then we may fix $s_0 \in (0,\frac{1}{4}]$ with the property that

$$
g_\delta(s) \leq -s \leq 0 \quad \text{ for all } s \in (0,s_0]. \tag{3.56}
$$

Since also $\varphi_s(x)w_c(x) \geq c w_c(x) \geq 0$ for $x \in \Omega$, $s \in (0,s_0]$, we deduce from (3.54) that

$$
\mathscr{E}_s^{\delta}(w_c,w_c) \leq \int\limits_{\Omega} [k_{\delta,s} * \varphi_s - sc] w_c\,dx \leq \big(\|k_{\delta,s} * \varphi_s\|_{L^\infty(\Omega)} - sc\big)\int\limits_{\Omega} w_c\,dx. \tag{3.57}
$$

Here we note that, by Hölder's (or Young's) inequality,

$$
\|k_{\delta,s} * \varphi_s\|_{L^\infty(\Omega)} \leq \|k_{\delta,s}\|_{L^2(\mathbb{R}^N)}\|\varphi_s\|_{L^2(\Omega)} = \|k_{\delta,s}\|_{L^2(\mathbb{R}^N)}
$$

with

$$
\|k_{\delta,s}\|_{L^2(\mathbb{R}^N)} = C_{N,s}\Bigg(\int\limits_{\mathbb{R}^N\setminus B_\delta} |y|^{-2N-4s}\,dy\Bigg)^{1/2} = \frac{C_{N,s}\omega_{N-1}^{\frac{1}{2}}\delta^{-\frac{N}{2}-2s}}{\sqrt{N+4s}}.
$$

Since

$$
\tilde{d} := \sup_{s\in(0,s_0]} \frac{\|k_{\delta,s}\|_{L^2(\mathbb{R}^N)}}{s} = \sup_{s\in(0,s_0]} \frac{C_{N,s}\omega_{N-1}^{\frac{1}{2}}\delta^{-\frac{N}{2}-2s}}{s\sqrt{N+4s}} < \infty,
$$

we deduce from (3.57) that for $c > \tilde{d}$ and $s \in (0,s_0]$ we have

$$
0 \leq \mathscr{E}_s^{\delta}(w_c,w_c) \leq s\big(\tilde{d} - c\big)\int\limits_{\Omega} w_c\,dx \leq 0
$$

and therefore $\mathscr{E}_s^{\delta}(w_c, w_c) = 0$. Consequently, $w_c = 0$ in Ω for $s \in (0, s_0]$ by the Poincaré type inequality given in [43, Lemma 2.7] . But then $\varphi_s(x) \le c$ a.e. in Ω, and therefore

$$\sup_{s \in (0,s_0]} \|\varphi_s^+\|_{L^\infty(\Omega)} \le c.$$

Repeating the above argument for $-\varphi_s$ in place of φ_s, we also find that $\sup\limits_{s \in (0,s_0]} \|\varphi_s^-\|_{L^\infty(\Omega)} \le c$ and therefore

$$\sup_{s \in (0,s_0]} \|\varphi_s\|_{L^\infty(\Omega)} \le c. \tag{3.58}$$

It remains to prove that

$$\sup_{s \in [s_0,\frac{1}{4}]} \|\varphi_s\|_{L^\infty(\Omega)} < \infty. \tag{3.59}$$

To see this, we argue as above, but with different values of $\delta \in (0,1)$ and $c > 0$. For this we first note that, by (3.55), we may choose $\delta \in (0,1)$ sufficiently small so that (3.56) holds for $s \in [s_0, \frac{1}{4}]$. With this new value of δ and $\tilde{d}$ redefined as

$$\tilde{d} := \sup_{s \in [s_0,\frac{1}{4}]} \frac{\|k_{\delta,s}\|_{L^2(\mathbb{R}^N)}}{s} = \sup_{s \in [s_0,\frac{1}{4}]} \frac{C_{N,s}\,\omega_{N-1}^{\frac{1}{2}}\,\delta^{-\frac{N}{2}-2s}}{s\sqrt{N+4s}} < \infty,$$

we may now fix $c > \tilde{d}$ and complete the argument as above to see that also

$$\sup_{s \in [s_0,\frac{1}{4}]} \|\varphi_s\|_{L^\infty(\Omega)} \le c.$$

Hence (3.59) holds. The proof is now finished by combining (3.58) and (3.59). $\square$

3.4 Local equicontinuity

This section is devoted to prove local equicontinuity of the set $\{\varphi_s \,:\, s \in (0, \frac{1}{4}]\}$ in Ω. The first step of the proof consists in deriving s-dependent Hölder estimates for the functions *with uniform (i.e., s-independent) constants as* $s \to 0^+$. As a preliminary tool, we need to consider the *Riesz kernel*

$$F_s : \mathbb{R}^N \setminus \{0\} \to [0,\infty), \qquad F_s(z) = \kappa_{N,s}|z|^{2s-N} \quad \text{with} \quad \kappa_{N,s} = \frac{s\Gamma(\frac{N}{2}-s)}{4^s \pi^{N/2}\Gamma(1+s)}. \tag{3.60}$$

The following two lemmas contain estimates which are essentially standard but hard to find in the literature in this form with s-independent constants. For closely related estimates, see e.g. [90, Section 2] and [69, Section 7].

Lemma 3.17. *Let $s \in (0, \frac{1}{4}]$, $r \in (0,1)$ and $f \in L^\infty(\overline{B_r})$. Moreover, let*

$$u_f : \mathbb{R}^N \to \mathbb{R}, \qquad u_f(x) := \int_{B_r} F_s(x-y) f(y)\, dy. \tag{3.61}$$

Then $u_f \in C^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and there is a constant $C = C(N) > 0$ such that

$$|u_f(x) - u_f(y)| \le C r^s \|f\|_{L^\infty(B_r)} |x-y|^s \quad \textit{for all } x, y \in \mathbb{R}^N. \tag{3.62}$$

If, moreover, $f \in C^\alpha(\overline{B_r})$ for some $\alpha \in (0, 1-s)$, then we also have

$$|u_f(x) - u_f(y)| \le C r^{s-\alpha} \|f\|_{C^\alpha(\overline{B_r})} |x-y|^{s+\alpha} \quad \textit{for } x, y \in B_{3r/4} \tag{3.63}$$

after making $C = C(N)$ larger if necessary.

Proof. For $x \in B_1$ we have

$$u_f(rx) = \int_{B_r} F_s(rx-y) f(y)\, dy = r^{2s} \int_{B_1} F_s(x-z) f(rz)\, dz,$$

so that we may assume $r = 1$ in the following. Next, we recall the following standard estimate:

$$\int_{B_t} |x-z|^{\tau-N}\, dz \le \int_{B_t} |z|^{\tau-N}\, dz = \frac{\omega_{N-1} t^\tau}{\tau} \qquad \text{for every } t > 0,\ \tau \in (0,N) \text{ and } x \in \mathbb{R}^N. \tag{3.64}$$

From this we deduce that $u_f \in L^\infty(\mathbb{R}^N)$ with

$$\begin{aligned} \|u_f\|_{L^\infty(\mathbb{R}^N)} &\le \|f\|_{L^\infty(B_1)} \kappa_{N,s} \sup_{x \in \mathbb{R}^N} \int_{B_1} |x-y|^{2s-N}\, dy \le \|f\|_{L^\infty(B_1)} \frac{\kappa_{N,s}\omega_{N-1}}{2s} \\ &= \frac{\Gamma(\frac{N}{2}-s)\omega_{N-1}}{2^{2s+1}\pi^{N/2}\Gamma(1+s)} \|f\|_{L^\infty(B_1)} \le C_1 \|f\|_{L^\infty(B_1)} \end{aligned} \tag{3.65}$$

with a constant $C_1 = C(N)$ independent of $s \in (0, \frac{1}{4}]$. Next, by e.g. [42, Eq. (A.3)] we use

$$|a^{2s-N} - b^{2s-N}| \le \frac{N-2s}{N-s} |a-b|^s (a^{s-N} + b^{s-N}) \le |a-b|^s (a^{s-N} + b^{s-N}) \quad \text{for } a, b > 0. \tag{3.66}$$

With this estimate and (3.64), we deduce that

$$\begin{aligned} |u(x+h) - u(x)| &= \left| \int_{B_1} \big(F_s(x-z+h) - F_s(x-z)\big) f(z)\, dz \right| \\ &\le |h|^s \|f\|_{L^\infty(B_1)} \kappa_{N,s} \int_{B_1} (|x-z-h|^{s-N} + |x-z|^{s-N})\, dz \\ &\le \frac{2\omega_{N-1}\kappa_{N,s}}{s} \|f\|_{L^\infty(B_1)} |h|^s = \frac{2\omega_{N-1}\Gamma(\frac{N}{2}-s)}{4^s \pi^{N/2}\Gamma(1+s)} \|f\|_{L^\infty(B_1)} |h|^s \qquad \text{for } x, h \in \mathbb{R}^N. \end{aligned}$$

Hence there is $C_2 = C_2(N)$ independent of $s \in (0, \frac{1}{4}]$ such that

$$|u(x+h) - u(x)| \le C_2 \|f\|_{L^\infty(B_1)} |h|^s \qquad \text{for all } x, h \in \mathbb{R}^N. \tag{3.67}$$

We thus deduce (3.62).
Next we assume that $f \in C^{\alpha}(\overline{B_1})$ for some $\alpha \in (0,1-s)$, and we establish (3.63) in the case $r = 1$.
We choose a cut-off function $\eta \in C_c^{\infty}(\mathbb{R}^N)$ with $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{7/8}$ and $\eta = 0$ on $\mathbb{R}^N \setminus B_1$.
We then define $w \in C_c^{\alpha}(\mathbb{R}^N)$ by $w(x) = \eta(x)f(x)$ for $x \in B_1$ and $w(x) = 0$ for $x \in \mathbb{R}^N \setminus B_1$. Then $u_f(x) = u_1(x) + u_2(x)$ for $x \in B_1$ with

$$u_1(x) = \int_{B_1} F_s(x-z)(1-\eta(z))f(z)\, dz = \int_{B_1 \setminus B_{7/8}} F_s(x-z)(1-\eta(z))f(z)\, dz$$

and

$$u_2(x) = \int_{\mathbb{R}^N} F_s(x-z)w(z)\, dz \qquad \text{for } x \in \mathbb{R}^N.$$

Since $|x-z| \ge \frac{1}{8}$ for $x \in B_{3/4}$ and $z \in B_1 \setminus B_{7/8}$, for all $\beta \in \mathbb{N}_0^d$, $|\beta| \le 1$ we have

$$\begin{aligned}
|\partial^{\beta} u_1(x)| &= \Big| \int_{B_1} \partial_x^{\beta} F_s(x-z)(1-\eta(z))f(z)\, dz \Big| \le \|f\|_{L^{\infty}(B_1)} \|\partial^{\beta} F_s\|_{L^1(B_2 \setminus B_{\frac{1}{8}})} \\
&\le \|f\|_{L^{\infty}(B_1)} \kappa_{N,s}\, \omega_{N-1} \Big((N-2s) \int_{1/8}^{2} t^{2s-2}\, dt + \int_{1/8}^{2} t^{2s-1}\, dt \Big) \\
&\le \|f\|_{L^{\infty}(B_1)} \kappa_{N,s}\, \omega_{N-1} (N+2) \int_{1/8}^{2} t^{2s-2}\, dt \le C_3 \|f\|_{L^{\infty}(B_1)}
\end{aligned}$$

for $x \in B_{3/4}$, $s \in (0,1)$ with a constant $C_3 = C_3(N) > 0$. Hence $u_1 \in C^1(\overline{B_{3/4}})$, and

$$|u_1(x) - u_1(y)| \le C_3 \|f\|_{L^{\infty}(B_1)} |x-y| \quad \text{ for all } x, y \in B_{3/4}. \tag{3.68}$$

To estimate u_2, we first note that, by the same estimate as in (3.65), we find that

$$\|u_2\|_{L^{\infty}(B_1)} \le C \|w\|_{L^{\infty}(B_1)} \le C \|f\|_{L^{\infty}(B_1)}. \tag{3.69}$$

Moreover, we write $\delta_h w(x) = w(x+h) - w(x)$ for $x, h \in \mathbb{R}^N$. Since w has a compact support contained in B_1 and η is smooth, there is $C_4 = C_4(N)$ such that

$$|\delta_h w(x)| \le C_4 \|f\|_{C^{\alpha}(\overline{B_1})} |h|^{\alpha} \quad \text{ for all } x, h \in \mathbb{R}^N.$$

For $x, h \in \mathbb{R}^N$, $|h| \le 1$ we now have, by (3.66) and since $\delta_h w$ is supported in B_2,

$$\begin{aligned}
&|u_2(x+2h) - 2u_2(x+h) + u_2(x)| \\
&= \big| \delta_h^2 u_2(x) \big| = \Big| \int_{\mathbb{R}^N} \delta_h F_s(x-z) \delta_h w(z)\, dz \Big| = \Big| \int_{B_2} \delta_h F_s(x-z) \delta_h w(z)\, dz \Big|
\end{aligned}$$

$$\leq |h|^{\alpha+s} C_4 \|f\|_{C^\alpha(\overline{B_1})} \kappa_{N,s} \int_{B_2} (|x-z-h|^{s-N} + |x-z|^{s-N})\, dz.$$

Using now (3.64) again, we deduce that

$$\begin{aligned} &|u_2(x+2h) - 2u_2(x+h) + u_2(x)| \\ &\leq \frac{\kappa_{N,s}}{s} C_4 \omega_{N-1} 2^{s+1} \|f\|_{C^\alpha(\overline{B_1})} |h|^{\alpha+s} = \frac{C_4 \omega_{N-1} 2^{s+1} \Gamma(\frac{N}{2}-s)}{4^s \pi^{N/2} \Gamma(1+s)} \|f\|_{C^\alpha(\overline{B_1})} |h|^{\alpha+s}. \end{aligned}$$

Hence there is $C_5 = C_5(N)$ such that

$$|u_2(x+2h) - 2u_2(x+h) + u_2(x)| \leq C_5 \|f\|_{C^\alpha(\overline{B_1})} |h|^{\alpha+s} \qquad \text{for all } x \in \mathbb{R}^N,\ |h| \leq 1. \tag{3.70}$$

By (3.69), we may make $C_5 > 0$ larger if necessary so that (3.70) holds for all $x, h \in \mathbb{R}^N$. Since $\alpha + s < 1$ by assumption, it now follows, by a well known argument, that

$$|u_2(x+h) - u_2(x)| \leq C_6 \|f\|_{C^\alpha(\overline{B_1})} |h|^{\alpha+s} \qquad \text{for all } x, h \in \mathbb{R}^N \tag{3.71}$$

with a constant $C_6 = C_6(N) > 0$. For the convenience of the reader, we recall this argument in the appendix. The estimate (3.63) now follows by combining (3.68) and (3.71). □

Lemma 3.18. *Let $r > 0$, $f \in L^\infty(B_r)$, and suppose that $u \in L^\infty(\mathbb{R}^N)$ is a distributional solution of the equation $(-\Delta)^s u = f$ in B_r for some $s \in (0, \frac{1}{4}]$. Moreover, let $u_f : \mathbb{R}^N \to \mathbb{R}$ be defined as in* (3.61)*, and let $u_* := u - u_f$.*
Then we have the estimate

$$|u_*(x) - u_*(y)| \leq C|x-y|^{3s} \Big(r^{-3s} \|u\|_{L^\infty(\mathbb{R}^N \setminus B_r)} + r^{-s} \|f\|_{L^\infty(B_r)} \Big) \qquad \text{for } x, y \in B_{\frac{r}{2}} \tag{3.72}$$

with a constant $C = C(N) > 0$.

Proof. By scaling invariance, it suffices to consider the case $r = 1$. In this case, we may follow the proof of [59, Lemma A.1], using the fact that u_* solves the problem

$$(-\Delta)^s u_* = 0 \quad \text{in } B_r \qquad u_* = u - u_f \quad \text{in } \mathbb{R}^N \setminus B_r.$$

Using the corresponding Poisson representation of u_*, it was shown in [59, Proof of Lemma A.1] that

$$|u_*(x) - u_*(y)| \leq c_1 |x-y| \Big(\tau_{N,s} \int_{\mathbb{R}^N \setminus B_1} \frac{|u(z)|}{|z|^N (|z|^2 - 1)^s} dz + \|f\|_{L^\infty(B_1)} \Big) \qquad \text{for } x, y \in B_{\frac{1}{2}} \tag{3.73}$$

with a constant $c_1 = c_1(N)$ and $\tau_{N,s} = \frac{2}{\Gamma(s)\Gamma(1-s)|S^{N-1}|}$, see [59, P. 48]. From this, we deduce (3.72) in the case $r = 1$ since $s \in (0, \frac{1}{4}]$. □

Corollary 3.19. *Let $s \in (0, \frac{1}{4}]$. Then $\varphi_s \in C^{3s}(\overline{B_{r/8}(x_0)})$ for all $x_0 \in \Omega$ and $0 < r \leq \min\{1, \delta_\Omega(x_0)\}$. Moreover, there is $C = C(N, \Omega, k) > 0$ such that*

$$\sup_{x,y \in B_{r/8}(x_0)} \frac{|\varphi_s(x) - \varphi_s(y)|}{|x-y|^{3s}} \leq C r^{-3s} \qquad \text{for } s \in (0, \frac{1}{4}].$$

Proof. By translation invariance we may assume $x_0 = 0 \in \Omega$. Let $r \in (0, \min\{1, \delta_\Omega(0)\})$. We write $\varphi_s = u_{s,1} + u_{s,2}$ with

$$u_{s,1}(x) = \int_{B_r} F_s(x-z)\lambda_s \varphi_s(z)\, dz, \quad \text{for } x \in \mathbb{R}^N, \qquad u_{s,2} = \varphi_s - u_{s,1},$$

where F_s is the Riesz kernel defined in Lemma 3.17. Moreover, in the following, the letter $C > 0$ denotes different constants depending only on N, Ω and k. By Theorem 3.16 and Lemma 3.17, we have

$$|u_{s,1}(x) - u_{s,1}(y)| \le Cr^s|x-y|^s \quad \text{for all } x, y \in \mathbb{R}^N.$$

Moreover, by Lemma 3.18 we have

$$|u_{s,2}(x) - u_{s,2}(y)| \le Cr^{-3s}|x-y|^{3s} \le Cr^{-2s}|x-y|^{2s} \le Cr^{-s}|x-y|^s \quad \text{for all } x, y \in B_{r/2}. \tag{3.74}$$

Hence

$$|\varphi_s(x) - \varphi_s(y)| \le Cr^{-s}|x-y|^s \quad \text{for all } x, y \in B_{r/2}.$$

Applying now the second claim in Lemma 3.17 with $\alpha = s$, we deduce that

$$|u_{s,1}(x) - u_{s,1}(y)| \le Cr^{-s}|x-y|^{2s} \quad \text{for all } x, y \in B_{r/4}.$$

Combining this estimate with (3.74), we deduce that

$$|\varphi_s(x) - \varphi_s(y)| \le Cr^{-2s}|x-y|^{2s} \quad \text{for all } x, y \in B_{r/4}.$$

Finally, applying the second claim in Lemma 3.17 with $\alpha = 2s$, we deduce that

$$|u_{s,1}(x) - u_{s,1}(y)| \le Cr^{-2s}|x-y|^{3s} \quad \text{for all } x, y \in B_{r/8}.$$

Combining this estimate with (3.74), we deduce that

$$|\varphi_s(x) - \varphi_s(y)| \le Cr^{-3s}|x-y|^{3s} \quad \text{for all } x, y \in B_{r/8},$$

as claimed. □

We now state a key local bound related to an auxiliary integral operator.

Lemma 3.20. *Let $t_0, r > 0$. Then there exists a constant $C = C(N, \Omega, k, r, t_0) > 0$ with the property that*

$$\Big| \int_{B_{t_0}} \frac{\varphi_s(x) - \varphi_s(x+y)}{|y|^{N+2s}}\, dy \Big| \le C \qquad \text{for all } s \in (0, \tfrac{1}{4}] \text{ and all } x \in \Omega \text{ with } \delta_\Omega(x) > r.$$

Proof. Without loss of generality, we may assume that $r < 1$. Moreover, we fix $x \in \Omega$ with $\delta_\Omega(x) > r$. In the following, we fix $t = \min\{\frac{t_0}{2}, \frac{r}{8}\} < 1$, and we write

$$\int_{B_{t_0}} \frac{\varphi_s(x) - \varphi_s(x+y)}{|y|^{N+2s}}\, dy = \int_{B_t} \frac{\varphi_s(x) - \varphi_s(x+y)}{|y|^{N+2s}} dy - \int_{B_{t_0} \setminus B_t} \frac{\varphi_s(x+y)}{|y|^{N+2s}} dy + \omega_{N-1} \frac{t^{-2s} - {t_0}^{-2s}}{2s} \varphi_s(x)$$

and

$$(-\Delta)^s\varphi_s(x) = C_{N,s}\int\limits_{B_t}\frac{\varphi_s(x)-\varphi_s(x+y)}{|y|^{N+2s}}dy - C_{N,s}\int\limits_{\mathbb{R}^N\setminus B_t}\frac{\varphi_s(x+y)}{|y|^{N+2s}}dy + \frac{\omega_{N-1}C_{N,s}}{2s}t^{-2s}\varphi_s(x).$$

Since $C_N\omega_{N-1} = 2$, we can thus write

$$C_N\int\limits_{B_{t_0}}\frac{\varphi_s(x)-\varphi_s(x+y)}{|y|^{N+2s}}\,dy - \Big(\frac{(-\Delta)^s-1}{s}\Big)\varphi_s(x) = I_1^s(x)+I_2^s(x)+I_3^s(x) \tag{3.75}$$

with

$$I_1^s(x) := \Big(C_N - \frac{C_{N,s}}{s}\Big)\int\limits_{B_t}\frac{\varphi_s(x)-\varphi_s(x+y)}{|y|^{N+2s}}\,dy$$

$$I_2^s(x) := \Big(\frac{C_{N,s}}{s} - C_N\Big)\int\limits_{B_{t_0}\setminus B_t}\frac{\varphi_s(x+y)}{|y|^{N+2s}}\,dy + \frac{C_{N,s}}{s}\int\limits_{\mathbb{R}^N\setminus B_{t_0}}\frac{\varphi_s(x+y)}{|y|^{N+2s}}\,dy \quad \text{and}$$

$$I_3^s(x) := \frac{\varphi_s(x)}{s}\Big(C_N\omega_{N-1}\frac{t^{-2s}-t_0{}^{-2s}}{2} + 1 - \frac{\omega_{N-1}C_{N,s}}{2s}t^{-2s}\Big) = \frac{\varphi_s(x)}{s}\Big[\Big(1-\frac{C_{N,s}}{C_N s}\Big)t^{-2s} + 1 - t_0^{-2s}\Big].$$

By (3.25) and since

$$t^{-2s} \le t^{-\frac{1}{2}} \quad \text{and} \quad \Big|\frac{1-t_0^{-2s}}{s}\Big| \le \frac{|\ln t_0|}{2}\max\{1,t_0^{-2s}\} \le \frac{|\ln t_0|}{2}\max\{1,t_0^{-\frac{1}{2}}\} \qquad \text{for } s\in(0,\frac{1}{4}],$$

it follows that

$$|I_3^s(x)| \le \Big[D_N t^{-\frac{1}{2}} + \frac{|\ln t_0|}{2}\max\{1,t_0^{-\frac{1}{2}}\}\Big]\sup_{s\in(0,\frac{1}{4}]}\|\varphi_s\|_{L^\infty(\Omega)}, \tag{3.76}$$

where the RHS is a finite constant by Theorem 3.16. To estimate I_2^s, we let $R := 1+\operatorname{diam}(\Omega)$ and note that, by (3.25), Theorem 3.16, and since $\varphi_s \equiv 0$ on Ω^c,

$$\begin{aligned}
|I_2^s(x)| &\le \Big(\Big|\frac{C_{N,s}}{s} - C_N\Big| + \frac{C_{N,s}}{s}\Big)\int\limits_{B_R\setminus B_t}\frac{|\varphi_s(x+y)|}{|y|^{N+2s}}\,dy \\
&\le \Big(\Big|\frac{C_{N,s}}{s} - C_N\Big| + \frac{C_{N,s}}{s}\Big)\omega_{N-1}\frac{t^{-2s}-R^{-2s}}{2s}\|\varphi_s\|_{L^\infty(\Omega)} \\
&= \Big(\Big|\frac{C_{N,s}}{sC_N} - 1\Big| + \frac{C_{N,s}}{sC_N}\Big)\frac{t^{-2s}-R^{-2s}}{s}\|\varphi_s\|_{L^\infty(\Omega)} \le (2sD_N+1)\frac{t^{-2s}-R^{-2s}}{s}\|\varphi_s\|_{L^\infty(\Omega)} \\
&\le \Big(\frac{D_N}{2}+1\Big)\,\frac{t^{-2s}-R^{-2s}}{s}\|\varphi_s\|_{L^\infty(\Omega)} \qquad \text{for } s\in(0,\frac{1}{4}].
\end{aligned}$$

Since $\big(t^{-2s}-R^{-2s}\big) = 2(\ln R - \ln t)s + o(s)$ as $s\to 0^+$, it follows that

$$|I_2^s(x)| \le \Big(\frac{D_N}{2}+1\Big)\sup_{s\in(0,\frac{1}{4}]}\frac{t^{-2s}-R^{-2s}}{s}\sup_{s\in(0,\frac{1}{4}]}\|\varphi_s\|_{L^\infty(\Omega)}, \tag{3.77}$$

where the RHS is a finite constant depending on t but not on s.
Finally, to estimate $I_1^s(x)$, we note that our choice of $t = \min\{\frac{t_0}{2}, \frac{r}{8}\}$ allows us to apply Corollary 3.19, which gives that

$$|\varphi_s(x+h) - \varphi_s(x)| \le \tilde{C}|y|^{3s} \qquad \text{for } s \in (0, \frac{1}{4}], \, y \in B_t$$

with a constant $\tilde{C} = \tilde{C}(N, \Omega, k, r, t_0) > 0$. Using this together with (3.25) we may estimate

$$|I_1^s(x)| \le \Big|C_N - \frac{C_{N,s}}{s}\Big|\tilde{C}\int_{B_t} |y|^{s-N} dy \le \omega_{N-1}\tilde{C}\big(sC_N D_N\big)\frac{t^s}{s} = 2\tilde{C}D_N t^s \le 2\tilde{C}D_N \quad \text{for } s \in (0, \frac{1}{4}].$$

Going back to (3.75), we now find that

$$\sup_{s\in(0,\frac{1}{4}]}\Big|C_N\int_{B_{t_0}} \frac{w(x)-w(x+y)}{|y|^{N+2s}}\,dy - \Big(\frac{(-\Delta)^s - 1}{s}\Big)\varphi_s(x)\Big| < \infty.$$

Since also

$$\sup_{s\in(0,\frac{1}{4}]}\Big\|\Big(\frac{(-\Delta)^s - 1}{s}\Big)\varphi_s(x)\Big\|_{L^\infty(\Omega)} = \sup_{s\in(0,\frac{1}{4}]}\Big(\Big|\frac{\lambda_s - 1}{s}\Big|\|\varphi_s(x)\|_{L^\infty(\Omega)}\Big) < \infty$$

by Theorems 3.15 and 3.16, the claim now follows. □

We now have all tools to complete the proof of Theorem 3.1(iii) which we restate here for the reader's convenience.

Theorem 3.21. *The set $\{\varphi_s : s \in (0, \frac{1}{4}]\}$ is equicontinuous in every point $x_0 \in \Omega$ and therefore relative compact in $C(K)$ for every compact subset $K \subset \Omega$.*

Proof. We only have to prove the equicontinuity of the set $M := \{\varphi_s : s \in (0, \frac{1}{4}]\}$ in every point $x_0 \in \Omega$. Once this is shown, it follows from Theorem 3.16 and the Arzela-Ascoli Theorem that, for every compact subset $K \subset \Omega$, the set M is relative compact when regarded as a subset of $C(K)$.
Arguing by contradiction, we now assume that there exists a point $x_0 \in \Omega$ such that M is not equicontinuous at x_0, which means that

$$\lim_{t\to 0^+}\sup_{s\in(0,\frac{1}{4}]}\operatorname*{osc}_{B_t(x_0)}\varphi_s = \varepsilon > 0. \tag{3.78}$$

Here, we note that this limit exists since the function

$$(0,\infty) \to [0,\infty), \qquad t \mapsto \sup_{s\in(0,\frac{1}{4}]}\operatorname*{osc}_{B_t(x_0)}\varphi_s$$

is nondecreasing. Without loss of generality, to simplify the notation, we may assume that $x_0 = 0 \in \Omega$. We first choose $\delta > 0$ sufficiently small so that

$$\frac{\varepsilon - \delta}{2^{N+2}} - 2\cdot 3^N\delta > 0 \tag{3.79}$$

The relevance of this condition will become clear later. Moreover, we choose $t_0 > 0$ sufficiently small so that

$$B_{3t_0} \subset \Omega \tag{3.80}$$

and

$$\varepsilon \leq \sup_{s\in(0,\frac{1}{4}]} \operatorname*{osc}_{B_t} \varphi_s \leq \varepsilon + \delta \qquad \text{for } 0 < t \leq 2t_0. \tag{3.81}$$

By Lemma 3.20 and (3.80), there exists a constant $C_1 > 0$ with the property that

$$\Big| \int_{B_{t_0}} \frac{\varphi_s(x) - \varphi_s(x+y)}{|y|^{N+2s}} dy \Big| \leq C_1 \qquad \text{for all } x \in B_{t_0},\ s \in (0, \frac{1}{4}]. \tag{3.82}$$

Next, we choose a sequence of numbers $t_n \in (0, \frac{t_0}{5})$ with $t_n \to 0^+$ as $n \to \infty$. By (3.81), there exists a sequence $(s_n)_n \subset (0, \frac{1}{4}]$ such that

$$\operatorname*{osc}_{B_{t_n}} \varphi_{s_n} \geq \varepsilon - \delta \qquad \text{for all } n \in \mathbb{N}, \tag{3.83}$$

whereas, by Lemma 3.19, we have

$$\operatorname*{osc}_{B_{t_n}} \varphi_{s_n} \leq C_2 (2t_n)^{3s_n} \qquad \text{for all } n \in \mathbb{N} \text{ with a constant } C_2 > 0.$$

Hence,

$$t_n^{s_n} \geq 2^{-s_n} \Big(\frac{1}{C_2} \operatorname*{osc}_{B_{t_n}} \varphi_{s_n} \Big)^{\frac{1}{3}} \geq 2^{-\frac{1}{4}} \Big(\frac{\varepsilon - \delta}{C_2} \Big)^{\frac{1}{3}} \qquad \text{for all } n \in \mathbb{N} \tag{3.84}$$

which implies, in particular, that

$$s_n \to 0 \qquad \text{as } n \to \infty. \tag{3.85}$$

To simplify the notation, we now set $\varphi_n := \varphi_{s_n}$. By (3.83), we may write

$$\varphi_n(\overline{B_{t_n}}) = [d_n - r_n, d_n + r_n] \qquad \text{for } n \in \mathbb{N} \text{ with some } d_n \in \mathbb{R} \text{ und } r_n \geq \frac{\varepsilon - \delta}{2}. \tag{3.86}$$

Together with (3.81) and the fact that $\overline{B_{t_n}} \subset B_{2t_0}$, we deduce that

$$\varphi_n(B_{2t_0}) \subset [d_n - \frac{\varepsilon + 3\delta}{2}\, , d_n + \frac{\varepsilon + 3\delta}{2}]. \tag{3.87}$$

Indeed,

$$\sup_{B_{2t_0}} \varphi_n \leq \inf_{\overline{B_{t_n}}} + \operatorname*{osc}_{B_{2t_0}} \varphi_n \leq d_n - r_n + \varepsilon + \delta \leq d_n + \frac{\varepsilon + 3\delta}{2}$$

and, similarly, $\inf_{B_{2t_0}} \varphi_n \geq d_n - \frac{\varepsilon + 3\delta}{2}$. Next, we let

$$c_n := \int_{B_{t_0} \setminus B_{3t_n}} |y|^{-N-2s_n} dy = \omega_{N-1} \frac{(3t_n)^{-2s_n} - t_0^{-2s_n}}{2s_n} \qquad \text{for } n \in \mathbb{N},$$

and we note that

$$c_n \to \infty \qquad \text{as } n \to \infty \tag{3.88}$$

since $c_n \geq \omega_{N-1}\big(\log t_0 - \log(3t_n)\big)$ for $n \in \mathbb{N}$ and $t_n \to 0$ for $n \to \infty$. We also put

$$A^n_+ := \{y \in B_{t_0} \setminus B_{3t_n} \,:\, \varphi_n(y) \geq d_n\} \quad \text{and} \quad A^n_- := \{y \in B_{t_0} \setminus B_{3t_n} \,:\, \varphi_n(y) \leq d_n\}.$$

Since

$$c_n \leq \int_{A^n_+} |y|^{-N-2s_n}\,dy + \int_{A^n_-} |y|^{-N-2s_n}\,dy \qquad \text{for all } n \in \mathbb{N},$$

we may pass to a subsequence such that

$$\int_{A^n_+} |y|^{-N-2s_n}\,dy \geq \frac{c_n}{2} \quad \text{for all } n \in \mathbb{N} \qquad \text{or} \qquad \int_{A^n_-} |y|^{-N-2s_n}\,dy \geq \frac{c_n}{2} \quad \text{for all } n \in \mathbb{N}.$$

Without loss of generality, we may assume that the second case holds (otherwise we may replace φ_n by $-\varphi_n$ and d_n by $-d_n$). We then define the Lipschitz function $\psi_n \in C_c(\mathbb{R}^N)$ by

$$\psi_n(x) = \begin{cases} 2\delta, & |x| \leq t_n \\ 0, & |x| \geq 2t_n \\ \dfrac{2\delta}{t_n}(2t_n - |x|), & t_n \leq |x| \leq 2t_n. \end{cases}$$

We also let $\tau_n := \varphi_n + \psi_n$ for all $n \in \mathbb{N}$. By (3.87), we have

$$\tau_n = \varphi_n \leq d_n + \frac{\varepsilon + 3\delta}{2} \leq d_n + r_n + 2\delta \qquad \text{in } B_{2t_0} \setminus B_{2t_n}.$$

Moreover, since $d_n + r_n \in \varphi_n(\overline{B_{t_n}})$ by (3.86), we have

$$d_n + r_n + 2\delta \in \tau_n(\overline{B_{t_n}}) \subset \tau_n(B_{2t_n}).$$

Consequently, $\max\limits_{B_{2t_0}} \tau_n$ is attained at a point $x_n \in B_{2t_n}$ with

$$\tau_n(x_n) \geq d_n + r_n + 2\delta$$

which implies that

$$\varphi_n(x_n) \geq d_n + r_n \geq d_n + \frac{\varepsilon - \delta}{2}. \tag{3.89}$$

By (3.82) and since $B_{3t_n} \subset B_{t_0}(x_n)$ for $n \in \mathbb{N}$ by construction, we have that

$$\begin{aligned} C_1 &\geq \int_{B_{t_0}} \frac{\varphi_n(x_n) - \varphi_n(x_n + y)}{|y|^{N+2s_n}}\,dy = \int_{B_{t_0}(x_n)} \frac{\varphi_n(x_n) - \varphi_n(y)}{|x_n - y|^{N+2s_n}}\,dy \\ &= \int_{B_{3t_n}} \frac{\varphi_n(x_n) - \varphi_n(y)}{|x_n - y|^{N+2s_n}}\,dy + \int_{B_{t_0}(x_n)\setminus B_{3t_n}} \frac{\varphi_n(x_n) - \varphi_n(y)}{|x_n - y|^{N+2s_n}}\,dy. \end{aligned} \tag{3.90}$$

To estimate the first integral, we note that, by definition of the function ψ_n,

$$|\psi_n(x)-\psi_n(y)| \le \frac{2\delta}{t_n}|x-y| \qquad \text{for all } x,z\in\mathbb{R}^N.$$

Moreover, by the choice of x_n we have $\tau_n(x_n)\ge\tau_n(y)$ for all $y\in B_{3t_n}$. Consequently,

$$\begin{aligned}
&\int\limits_{B_{3t_n}} \frac{\varphi_n(x_n)-\varphi_n(y)}{|x_n-y|^{N+2s_n}}dy = \int\limits_{B_{3t_n}} \frac{\tau_n(x_n)-\tau_n(y)}{|x_n-y|^{N+2s_n}}dy - \int\limits_{B_{3t_n}} \frac{\psi_n(x_n)-\psi_n(y)}{|x_n-y|^{N+2s_n}}dy\\
&\ge -\int\limits_{B_{3t_n}} \frac{\psi(x_n)-\psi(y)}{|x_n-y|^{N+2s_n}}dy \ge -\frac{2\delta}{t_n}\int\limits_{B_{3t_n}} |x_n-y|^{1-N-2s_n}dy \ge -\frac{2\delta}{t_n}\int\limits_{B_{3t_n}} |y|^{1-N-2s_n}dy\\
&= -\frac{3^{1-2s_n}\omega_{N-1}2\delta t_n^{-2s_n}}{1-2s_n} \ge -12\omega_{N-1}\delta t_n^{-2s_n} \ge -C_3
\end{aligned} \tag{3.91}$$

with a constant $C_3>0$ independent of n. Here we used (3.64) and (3.84).
To estimate the second integral in (3.90) we first note, since $x_n\in B_{2t_n}$, we have that

$$2|y| \ge |y-x_n| \ge \frac{|y|}{3} \qquad \text{for every } n\in\mathbb{N} \text{ and } y\in\mathbb{R}^N\setminus B_{3t_n}.$$

Moreover, by (3.81), (3.87), and (3.89) we have

$$\varepsilon+\delta \ge \varphi_n(x_n)-\varphi_n(y) \ge d_n+\frac{\varepsilon-\delta}{2}-\varphi_n(y) \ge -2\delta \qquad \text{for } y\in B_{t_0}(x_n)\subset B_{2t_0}.$$

Consequently, combining (3.90) and (3.91), using again (3.89), we may estimate as follows:

$$\begin{aligned}
C_1+C_3 &\ge \int\limits_{B_{t_0}(x_n)\setminus B_{3t_n}} \frac{\varphi_n(x_n)-\varphi_n(y)}{|y-x_n|^{N+2s_n}}dy\\
&\ge \int\limits_{B_{t_0}(x_n)\setminus B_{3t_n}} \frac{[\varphi_n(x_n)-\varphi_n]_+(y)}{|y-x_n|^{N+2s_n}}dy - 2\delta \int\limits_{B_{t_0}(x_n)\setminus B_{3t_n}} |y-x_n|^{-N-2s_n}dy\\
&\ge \frac{1}{2^{N+2s_n}} \int\limits_{B_{t_0}(x_n)\setminus B_{3t_n}} \frac{[\varphi_n(x_n)-\varphi_n]_+(y)}{|y|^{N+2s_n}}dy - 2\cdot 3^{N+2s_n}\delta \int\limits_{B_{t_0}(x_n)\setminus B_{3t_n}} |y|^{-N-2s_n}dy\\
&\ge \frac{1}{2^{N+2s_n}}\Big(\int\limits_{B_{t_0}\setminus B_{3t_n}} \frac{[\varphi_n(x_n)-\varphi_n]_+(y)}{|y|^{N+2s_n}}dy - \int\limits_{B_{t_0}\setminus B_{t_0}(x_n)} \frac{[\varphi_n(x_n)-\varphi_n]_+(y)}{|y|^{N+2s_n}}dy\Big)\\
&\quad - 2\cdot 3^{N+2s_n}\delta\Big(\int\limits_{B_{t_0}\setminus B_{3t_n}} |y|^{-N-2s_n}dy + \int\limits_{B_{t_0}(x_n)\setminus B_{t_0}} |y|^{-N-2s_n}dy\Big)\\
&\ge \frac{1}{2^{N+2s_n}}\Big(r_n\int\limits_{A_n^-} |y|^{-N-2s_n}dy - (\varepsilon+\delta)\int\limits_{B_{t_0}\setminus B_{t_0}(x_n)} |y|^{-N-2s_n}dy\Big)\\
&\quad - 2\cdot 3^{N+2s_n}\delta\Big(c_n + \int\limits_{B_{t_0}(x_n)\setminus B_{t_0}} |y|^{-N-2s_n}dy\Big)
\end{aligned}$$

$$\begin{aligned}
&\geq \Big(\frac{r_n}{2\cdot 2^{N+2s_n}} - 2\cdot 3^{N+2s_n}\delta\Big)c_n \\
&\quad - \frac{(\varepsilon+\delta)}{2^{N+2s_n}} \int\limits_{B_{t_0}\setminus B_{t_0-2t_n}} |y|^{-N-2s_n}dy \;-\; 2\cdot 3^{N+2s_n}\delta \int\limits_{B_{t_0+2t_n}\setminus B_{t_0}} |y|^{-N-2s_n}dy \\
&\geq \Big(\frac{\varepsilon-\delta}{2^{N+2+2s_n}} - 2\cdot 3^{N+2s_n}\delta\Big)c_n - o(1) = \Big(\frac{\varepsilon-\delta}{2^{N+2}} - 2\cdot 3^{N}\delta + o(1)\Big)c_n - o(1) \qquad \text{as } n\to\infty,
\end{aligned}$$

where we used (3.86). By our choice of $\delta > 0$ satisfying (3.79), we arrive at a contradiction to (3.88). The proof is thus finished. □

3.5 Uniform boundary decay

Throughout this section, we assume that Ω is a bounded Lipschitz domain satisfying a uniform exterior sphere condition. By definition, this means that there exists a radius $R_0 > 0$ such that for every point $x_* \in \partial\Omega$ there exists a ball B^{x_*} of radius R_0 contained in $\mathbb{R}^N \setminus \overline{\Omega}$ and with $\overline{B^{x_*}} \cap \overline{\Omega} = \{x^*\}$.

We first note the following boundary decay estimate.

Lemma 3.22. *There is a constant $C = C(N,\Omega,k) > 0$ such that*

$$|\varphi_s(x)| \leq C\delta_\Omega^s(x) \quad \textit{for } x \in \mathbb{R}^N,\ s \in (0,\frac{1}{4}]. \tag{3.92}$$

Proof. We note that φ_s is a weak solution of

$$(-\Delta)^s\varphi_s = f_s \quad \text{in } \Omega, \qquad \varphi_s \equiv 0 \quad \text{in } \Omega^c,$$

where the functions $f_s := \lambda_s\varphi_s$, $s \in (0,\frac{1}{4}]$ are uniformly bounded in $L^\infty(\Omega)$ by Theorem 3.16. Therefore, the decay estimate in (3.92) essentially follows from [84, Lemma 2.7], although it is not stated there that the constant C can be chosen independently of s. For an alternative proof of the latter fact, see [59, Appendix]. We stress here that the use of radial barrier functions as in [84] and [59, Appendix] only requires a uniform exterior sphere condition and no further regularity assumptions on Ω. □

For $\delta > 0$, we now consider the one-sided neighborhood of the boundary

$$\Omega^\delta := \{x \in \Omega \,:\, \delta_\Omega(x) < \delta\}$$

The main result of the present section is the following.

Theorem 3.23. *We have*

$$\lim_{\delta\to 0^+} \sup_{s\in(0,\frac{1}{4}]} \|\varphi_s\|_{L^\infty(\Omega^\delta)} = 0.$$

In other words, for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ with the property that

$$|\varphi_s(x)| \leq \varepsilon \qquad \textit{for all } s \in (0,\frac{1}{4}],\ x \in \Omega^{\delta_\varepsilon}.$$

The remainder of this section is devoted to the proof of this theorem. We need some preliminaries. In the following, for $s \geq 0$, we let $L^1_s(\mathbb{R}^N)$ denotes the space of locally integrable functions $u : \mathbb{R}^N \to \mathbb{R}$ such that

$$\|u\|_{L^1_s} := \int_{\mathbb{R}^N} \frac{|u(x)|}{(1+|x|)^{N+2s}}\, dx < +\infty.$$

We note that $L^1_s(\mathbb{R}^N) \subset L^1_t(\mathbb{R}^N)$ for $0 \leq s < t$. Next, we need the following generalization of [29, Theorem 1.1].

Lemma 3.24. *Let $A \subset \mathbb{R}^N$ be a compact set, let $U \subset \mathbb{R}^N$ be an open neighborhood of A, and let $u \in L^1_0(\mathbb{R}^N)$ be a function with $u \in C^\alpha_{loc}(U)$ for some $\alpha > 0$. Then*

$$\lim_{s\to 0^+} \sup_{x\in A} \Big| \frac{(-\Delta)^s u(x) - u(x)}{s} - L_\Delta u(x) \Big| = 0.$$

Proof. In the following, we assume $\alpha < 1$. Moreover, without loss of generality, we may assume that $u \in C^\alpha(U)$, otherwise we replace U by a compact neighborhood $U' \subset U$ of A. Next, since A is compact, we may fix $r \in (0,1)$ such that for all $x \in A$ we have $\operatorname{dist}(x, \mathbb{R}^N \setminus U) > r$. For $x \in A$ we split the expression of the logarithmic Laplacian as

$$L_\Delta u(x) = C_N \int_{B_r} \frac{u(x)-u(x+y)}{|y|^N}\, dy - C_N \int_{\mathbb{R}^N\setminus B_r} \frac{u(x+y)}{|y|^N}\, dy + u(x)\Big(\int_{B_1\setminus B_r} \frac{C_N}{|y|^N}\, dy + \rho_N \Big).$$

With $D_{r,N}(s) := \frac{C_{N,s}\omega_{N-1}}{2s} r^{-2s}$ and since $C_N \omega_{N-1} = 2$, this splitting gives rise to the inequality

$$\begin{aligned}
\sup_{x\in A}\Big| \frac{(-\Delta)^s - 1}{s} u(x) - L_\Delta u(x) \Big| &\leq \sup_{x\in A} \int_{B_r} \frac{|u(x)-u(x+y)|}{|y|^N} \Big| \frac{C_{N,s}}{s}|y|^{-2s} - C_N \Big|\, dy \\
&\quad + \sup_{x\in A} \int_{\mathbb{R}^N\setminus B_r} \frac{|u(x+y)|}{|y|^N} \Big| \frac{C_{N,s}}{s}|y|^{-2s} - C_N \Big|\, dy + \|u\|_{L^\infty(A)} \Big| \frac{D_{r,N}(s)-1}{s} - \rho_N + 2\log r \Big| \\
&\leq \|u\|_{C^\alpha(U)} I_1(s) + \sup_{x\in A} I_2(s,x) + \|u\|_{L^\infty(A)} I_3(s),
\end{aligned} \tag{3.93}$$

where

$$I_1(s) = \int_{B_r} |y|^{\alpha-N} \Big| \frac{C_{N,s}}{s}|y|^{-2s} - C_N \Big|\, dy, \quad I_2(s,x) = \int_{\mathbb{R}^N\setminus B_r} \frac{|u(x+y)|}{|y|^N} \Big| \frac{C_{N,s}}{s}|y|^{-2s} - C_N \Big|, \quad \text{and}$$

$$I_3(s) = \Big| \frac{D_{r,N}(s)-1}{s} - \rho_N + 2\log r \Big|.$$

By Lemma 3.6, we have $\lim\limits_{s\to 0^+} \frac{D_{r,N}(s)-1}{s} = \rho_N - 2\log r$ and therefore

$$\lim_{s\to 0^+} I_3(s) = \lim_{s\to 0^+} \Big| \frac{D_{r,N}(s)-1}{s} - \rho_N + 2\log r \Big| = 0. \tag{3.94}$$

Moreover, by (3.25), we have the inequality

$$\Big|\frac{C_{N,s}}{s}|y|^{-2s}-C_N\Big| \le \Big|\frac{C_{N,s}}{s}-C_N\Big||y|^{-2s}+C_N\Big||y|^{-2s}-1\Big| \le C_N\Big(sD_N|y|^{-2s}+\Big||y|^{-2s}-1\Big|\Big). \tag{3.95}$$

for $y\in\mathbb{R}^N\setminus\{0\}$. Using that $\big||y|^{-2s}-1\big| \le \frac{4s}{\alpha}\big(|y|^{-2s-\frac{\alpha}{2}}+|y|^{\frac{\alpha}{2}}\big)$ by [59, Lemma 2.1] it follows that

$$\Big|\frac{C_{N,s}}{s}|y|^{-2s}-C_N\Big| \le sC_N\Big(D_N|y|^{-2s}+\frac{4}{\alpha}\big(|y|^{-2s-\frac{\alpha}{2}}+|y|^{\frac{\alpha}{2}}\big)\Big) \qquad \text{for } y\in\mathbb{R}^N\setminus\{0\}. \tag{3.96}$$

In particular,

$$\Big|\frac{C_{N,s}}{s}|y|^{-2s}-C_N\Big| \le sC_N\Big(D_N+\frac{8}{\alpha}\Big)|y|^{-2s-\frac{\alpha}{2}} \qquad \text{for } 0<|y|\le r \tag{3.97}$$

and

$$\Big|\frac{C_{N,s}}{s}|y|^{-2s}-C_N\Big| \le sC_Nr^{-2s+\alpha}\big(D_N+\frac{8}{\alpha}\big)|y|^{\frac{\alpha}{2}} \qquad \text{for } |y|>r. \tag{3.98}$$

Therefore, (3.97) gives

$$\lim_{s\to0^+} I_1(s) \le \lim_{s\to0^+} sC_N\Big(D_N+\frac{8}{\alpha}\Big)\int_{B_r}|y|^{\frac{\alpha}{2}-N-2s}dy = \lim_{s\to0^+} 2s\Big(D_N+\frac{8}{\alpha}\Big)\frac{r^{\frac{\alpha}{2}-2s}}{\frac{\alpha}{2}-2s}=0 \tag{3.99}$$

It remains to consider $I_2(s,x)$ for $x\in A$. For this, let $\varepsilon>0$ and note that there is $R_0>0$ such that for any $R\ge R_0$ we have

$$\int_{\mathbb{R}^N\setminus B_R}\frac{|u(y)|}{|y|^N}\,dy \le \frac{\varepsilon}{C_N2^N}. \tag{3.100}$$

Indeed, this is possible since $u\in L^1_0$ and thus $\lim\limits_{R\to0}\int_{\mathbb{R}^N\setminus B_R}\frac{|u(y)|}{|y|^N}\,dy=0$. In the following, we fix $R>\max\{2,R_0\}$ such that $B_{\frac{R}{2}}(A)\subset B_R$. Note that by this choice we have in particular $\sup\limits_{z\in A}|z|\le\frac{R}{2}$. Using (3.98) we then split for $x\in A$

$$\begin{aligned} I_2(s,x) &= \int_{\mathbb{R}^N\setminus B_r(x)}\frac{|u(y)|}{|x-y|^N}\Big|\frac{C_{N,s}}{s}|x-y|^{-2s}-C_N\Big|\,dy \\ &\le sC_Nr^{-2s-\alpha}(D_N+\frac{8}{\alpha})\int_{B_R\setminus B_r(x)}\frac{|u(y)|}{|x-y|^{N-\frac{\alpha}{2}}}\,dy + C_N\int_{\mathbb{R}^N\setminus B_R}\frac{|u(y)|}{|x-y|^N}\Big|\frac{C_{N,s}}{sC_N}|x-y|^{-2s}-1\Big|\,dy. \end{aligned} \tag{3.101}$$

To estimate the first integral in this decomposition, we use the fact that $|x-y|\ge r\ge\frac{r}{R+1}(1+|y|)$ for $y\in B_R\setminus B_r(x)$ and therefore

$$\int_{B_R\setminus B_r}\frac{|u(y)|}{|x-y|^{N-\frac{\alpha}{2}}}\,dy \le \Big(\frac{r}{R+1}\Big)^{\frac{\alpha}{2}-N}\int_{B_R\setminus B_r(x)}|u(y)|(1+|y|)^{\frac{\alpha}{2}-N}dy$$

$$\leq \Big(\frac{r}{R+1}\Big)^{\frac{\alpha}{2}-N}(1+R)^{\frac{\alpha}{2}}\|u\|_{L^1_0} \leq (1+R)^N r^{\frac{\alpha}{2}-N}\|u\|_{L^1_0}. \tag{3.102}$$

For the second integral in this decomposition, we note that, since $|x-y| \geq \max\{1, \frac{|y|}{2}\}$ for $y \in \mathbb{R}^N \setminus B_R$, we have for $y \in \mathbb{R}^N \setminus B_R$ by (3.24)

$$\Big|\frac{C_{N,s}}{sC_N}|x-y|^{-2s}-1\Big| \leq 1-4^s|y|^{-2s}(1+s\rho_N+o(s)) \leq 1+O(s) \quad \text{for } s\to 0^+ \text{ (uniform in } x \text{ and } y).$$

Combining this with (3.102) in (3.101) we find

$$\lim_{s\to 0^+}\sup_{x\in A} I_2(s,x) \leq C_N \sup_{x\in A}\int\limits_{\mathbb{R}^N\setminus B_R}\frac{|u(y)|}{|x-y|^N}\,dy \leq C_N 2^N \int\limits_{\mathbb{R}^N\setminus B_R}\frac{|u(y)|}{|y|^N}\,dy \leq \varepsilon. \tag{3.103}$$

Combining (3.94), (3.99), and (3.103), we get from (3.93)

$$\lim_{s\to 0^+}\sup_{x\in A}\Big|\frac{(-\Delta)^s u(x)-u(x)}{s} - L_\Delta u(x)\Big| \leq \varepsilon.$$

Here, $\varepsilon > 0$ is chosen arbitrary and this completes the proof of the lemma. □

Next we state a uniform small volume maximum principle. For this we define, for $s \in (0,1)$ and any open set $U \subset \mathbb{R}^N$, the function space

$$\mathscr{V}^s(U) := \{u \in L^2_{loc}(\mathbb{R}^N) \,:\, \int\limits_U\int\limits_{\mathbb{R}^N}\frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dxdy < \infty\}$$

It is easy to see that the quadratic form

$$\mathscr{E}_s(u,v) = C_{N,s}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy$$

is well-defined in Lebesgue sense for $u \in \mathscr{V}^s(U)$, $v \in \mathscr{H}^s_0(U)$, see e.g. [60] and the references therein. If functions $u \in \mathscr{V}^s(U)$ and $g \in L^2(U)$ are given, we say that $(-\Delta)^s u \geq g$ *in U weak sense* if

$$\mathscr{E}_s(u,v) - \int\limits_U gv\,dx \geq 0 \qquad \text{for all } v \in \mathscr{H}^s_0(U),\, v \geq 0.$$

Remark 3.25. Let $U \subset \mathbb{R}^N$ be an open bounded set. Moreover, let $g \in L^2(U)$, and let $u \in L^1_s(\mathbb{R}^N)\cap L^2_{loc}(\mathbb{R}^N)$ be a function satisfying $u \in C^\alpha(K)$ for a compact neighborhood K of $\overline{U}$ and, for some $s \in (0, \frac{\alpha}{2})$,

$$(-\Delta)^s u \geq g \qquad \text{in } U \text{ in pointwise sense.}$$

Then $u \in \mathscr{V}^s(U)$, and u satisfies $(-\Delta)^s u \geq g$ also in weak sense. This follows since, under the stated assumptions, we have

$$\int\limits_U [(-\Delta)^s u]v\,dx = \mathscr{E}_s(u,v) \qquad \text{for all } v \in \mathscr{H}^s_0(U).$$

The latter property follows easily by Fubini's theorem.

Our uniform small volume weak maximum principle now reads as follows.

Proposition 3.26. *There exists $\mu_0 = \mu_0(N) > 0$ with the property that the operators*

$$(-\Delta)^s - \mathrm{id}, \qquad s \in (0,1)$$

satisfy the following weak maximum principle on every open set $U \subset \mathbb{R}^N$ with $|U| \le \mu_0$:
For every $s \in (0,1)$ and every function $u \in \mathcal{V}^s(U)$ satisfying

$$(-\Delta)^s u \ge u \quad \textit{in } U, \qquad u \ge 0 \quad \textit{in } \mathbb{R}^N \setminus U$$

we have $u \ge 0$ on $\mathbb{R}^N$.

Proof. By [60, Prop. 2.3], it suffices to find $\mu_0 > 0$ with the property that

$$\lambda_{1,s}(U) > 1 \text{ for every open set } U \subset \mathbb{R}^N \text{ with } |U| \le \mu_0 \text{ and every } s \in (0,1), \tag{3.104}$$

where $\lambda_{1,s}(U)$ denotes the first Dirichlet eigenvalue of $(-\Delta)^s$ on U.
Let $r_0 = r_0(N) := 2e^{\frac{1}{2}(\psi(\frac{N}{2})-\gamma)}$. It then follows from [29, Section 4] that $\lambda_{1,L}(B_{r_0}) > 0$, where $\lambda_{1,L}(B_{r_0})$ denotes the first Dirichlet eigenvalue of L_Δ on $B_{r_0} := B_{r_0}(0)$.
Since

$$\frac{\lambda_{1,s}(B_{r_0}) - 1}{s} \to \lambda_{1,L}(B_{r_0}) \qquad \text{as } s \to 0^+,$$

there exists $s_0 \in (0,1)$ with the property that

$$\lambda_{1,s}(B_{r_0}) > 1 \qquad \text{for } s \in (0,s_0).$$

By the scaling properties of the fractional Laplacian, this also implies that

$$\lambda_{1,s}(B_r) = \left(\frac{r_0}{r}\right)^{2s} \lambda_{1,s}(B_{r_0}) \ge \lambda_{1,s}(B_{r_0}) > 1 \qquad \text{for } s \in (0,s_0),\ r \in (0,r_0]. \tag{3.105}$$

To obtain a similar estimate for $s \in [s_0,1)$, we use a lower eigenvalue bound given by Bañuelos and Kulczycki. In [7, Corollary 2.2], they proved that

$$\lambda_{1,s}(B_1) \ge 2^{2s} \frac{\Gamma(1+s)\Gamma(\frac{N}{2}+s)}{\Gamma(\frac{N}{2})} \qquad \text{for } s \in (0,1).$$

From this we deduce that

$$\lambda_{1,s}(B_r) \ge \left(\frac{2}{r}\right)^{2s} \frac{\Gamma(1+s)\Gamma(\frac{N}{2}+s)}{\Gamma(\frac{N}{2})} \ge \left(\frac{2}{r}\right)^{2s_0} \frac{\Gamma_{min}}{\Gamma(\frac{N}{2})} > 1 \quad \text{for } s \in [s_0,1) \text{ and } 0 < r \le r_1, \tag{3.106}$$

where $r_1 := 2\left(\frac{\Gamma_{min}}{\Gamma(\frac{N}{2})}\right)^{\frac{1}{2s_0}}$ and $\Gamma_{min} > 0$ denotes the minimum of the Gamma function on $(0,\infty)$.
Setting $r_* := \min\{r_0, r_1\}$, we thus find, by combining (3.105) and (3.106), that

$$\lambda_{1,s}(B_r) > 1 \qquad \text{for } s \in (0,1),\ r \in (0,r_*]. \tag{3.107}$$

Next, let $\mu_0 := |B_{r_*}|$, and let $U \subset \mathbb{R}^N$ be a nonempty open set with $|U| \le \mu_0$. Moreover, let $r \in (0, r_*]$ with $|B_r| = |U|$. Combining (3.107) and the Faber-Krahn type principle given in [8, Theorem 5], we deduce that

$$\lambda_{1,s}(U) \ge \lambda_{1,s}(B_r) > 1 \qquad \text{for } s \in (0,1),$$

as required. □

We recall a result from [29] regarding a radial barrier type function for the logarithmic Laplacian, see [29, Lemma 5.3, Case $\tau = \frac{1}{4}$].

Lemma 3.27. *Let $R \in (0, \frac{1}{2})$. Then there exists $\delta_0 = \delta_0(R) > 0$ and a continuous function $V \in L_0^1(\mathbb{R}^N)$ with the following properties:*

(i) $V \equiv 0$ in B_R and $V > 0$ in $\mathbb{R}^N \setminus \overline{B_R}$;

(ii) $V \in C^1_{loc}(\mathbb{R}^N \setminus \overline{B_R})$;

(iii) $L_\Delta V(x) \to \infty$ as $|x| \to R$, $|x| > R$.

In fact, in [29, Lemma 5.3] it was only stated that V is locally uniformly Dini continuous on $\mathbb{R}^N \setminus \overline{B_R}$ since this was sufficent for the considerations in this paper. However, the construction in the proof of this lemma immediately yields that $V \in C^1_{loc}(\mathbb{R}^N \setminus \overline{B_R})$.

Proof of Theorem 3.23 (completed). We need some more notation. For $R > 0$ and $R_1 > R$, we consider the open annulus

$$A_{R,R_1} := \{x \in \mathbb{R}^N : R < |x| < R_1\} \subset \mathbb{R}^N$$

and its translations

$$A_{R,R_1}(y) := \{x \in \mathbb{R}^N : R < |x-y| < R_1\}, \qquad y \in \mathbb{R}^N.$$

In the following, we let $\partial^i\Omega \subset \partial\Omega$ denote the subset of boundary points $x_* \in \partial\Omega$ for which there exists an (inner) open ball $B_{x_*} \subset \Omega$ with $x_* \in \partial B_{x_*}$.
Since Ω satisfies a uniform exterior sphere condition, there exists a radius $0 < R_0 < \frac{1}{2}$ such that for every point $x_* \in \partial^i\Omega$ there exists a (unique) ball B^{x_*} of radius R_0 contained in $\mathbb{R}^N \setminus \overline{\Omega}$ and tangent to ∂B_{x_*} at x_*. Let $c(x_*)$ denote the center of B^{x_*}.

Applying Lemma 3.27 with the value $R := \frac{R_0}{2}$ now yields a function $V \in L_0^1(\mathbb{R}^N)$ such that the properties (i)-(iii) of Lemma 3.27 are satisfied.

We now choose $\delta_0 \in (0, R)$ sufficiently small such that

$$|A_{R,R+\delta_0}| < \mu_0,$$

where $\mu_0 > 0$ is given by Proposition 3.26.

Next we consider the finite values

$$m_1 := \sup_{s\in(0,\frac{1}{4}]} \|\varphi_s\|_{L^\infty(\Omega)} \qquad \text{and} \qquad m_2 := \sup_{s\in(0,\frac{1}{4}]} \Big\|\frac{\lambda_s-1}{s}\varphi_s\Big\|_{L^\infty(\Omega)}$$

By Lemma 3.27(iii), we can make $\delta_0 > 0$ smaller if necessary to guarantee that

$$L_\Delta V(x) \geq 2m_2 \qquad \text{in } A_{R,R+\delta_0}. \tag{3.108}$$

Next, for $x_* \in \partial^i\Omega$ and $t \in [0,R]$, we consider the point

$$z(t,x_*) := x_* + (t+R)\frac{c(x_*)-x_*}{|c(x_*)-x_*|} \qquad \text{in } \mathbb{R}^N \setminus \overline{\Omega}.$$

which lies on the extension of the line segment spanned by the points x_* and $c(x_*)$ beyond $c(x_*)$. By construction, $\overline{B_R(z(t,x_*))} \cap \overline{\Omega} = \varnothing$ for $t \in (0,R]$, while, for $t \in (0,\delta_0)$, the intersection

$$\Omega_{t,x_*} := \Omega \cap A_{R,R+\delta_0}(z(t,x_*)) = \Omega \cap A_{R+t,R+\delta_0}(z(t,x_*))$$

is nonempty. Since Ω is bounded, there exists $R_1 > R$ such that

$$\Omega \subset A_{R,R_1}(z(t,x_*)) \qquad \text{for all } x_* \in \partial^i\Omega,\ t \in (0,\delta_0),$$

which implies that

$$\Omega \setminus \Omega_{t,x_*} \subset \overline{A_{R+\delta_0,R_1}(z(t,x_*))} \qquad \text{for all } x_* \in \partial^i\Omega,\ t \in (0,\delta_0). \tag{3.109}$$

Next, we define the translated functions

$$V_{t,x_*} \in L^1_0(\mathbb{R}^N), \qquad V_{t,x_*}(x) = V(x - z(t,x^*)), \qquad x_* \in \partial\Omega,\ t \in [0,R].$$

Since V is positive on the compact set $\overline{A_{R+\delta_0,R_1}}$ by Lemma 3.27(i), we may choose $c > 1$ sufficiently large such that $V \geq \frac{m_1}{c}$ in $\overline{A_{R+\delta_0,R_1}}$ and thus, by (3.109), also

$$V_{t,x_*} \geq \frac{m_1}{c} \qquad \text{in } \Omega \setminus \Omega_{t,x_*} \text{ for all } x_* \in \partial^i\Omega,\ t \in (0,\delta_0). \tag{3.110}$$

To finish the proof of the theorem, we now let $\varepsilon > 0$ be given. Since V is continous and $V \equiv 0$ on B_R by Lemma 3.27(i), we may fix $\delta \in (0,\frac{\delta_0}{2})$ such that

$$0 \leq V \leq \frac{\varepsilon}{c} \qquad \text{in } B_{R+2\delta}. \tag{3.111}$$

Since $A_{R+\delta,R+\delta_0} \subset\subset \mathbb{R}^N \setminus \overline{B_R}$, we find, as a consequence of Lemma 3.24 and Lemma 3.27, that

$$\frac{(-\Delta)^s V - V}{s} \to L_\Delta V \qquad \text{uniformly on } A_{R+\delta,R+\delta_0} \text{ as } s \to 0^+.$$

Hence, by (3.108), we may fix $s_1 \in (0,\frac{1}{4}]$ with the property that

$$\frac{(-\Delta)^s V - V}{s} \geq m_2 \geq \frac{m_2}{c} \qquad \text{on } A_{R+\delta,R+\delta_0} \text{ for } s \in (0,s_1). \tag{3.112}$$

We now claim that
$$|\varphi_s(x)| \le \varepsilon \qquad \text{for } s \in (0,s_1),\ x \in \Omega^\delta. \tag{3.113}$$
To show (3.113), we let $x \in \Omega^\delta$, and we let $x_* \in \partial\Omega$ with $\delta_\Omega(x) = |x - x_*|$. By definition, we then have $x_* \in \partial^i\Omega$. Moreover, by construction we have

$$x \in \Omega \cap A_{R+\delta,R+2\delta}(z(\delta,x_*)) \subset B_{R+2\delta}(z(\delta,x_*)). \tag{3.114}$$

We now define $W := cV_{\delta,x_*} \in L^1_0(\mathbb{R}^N)$. By (3.112), we then have that

$$(-\Delta)^s W \ge W + sm_2 \qquad \text{in } A_{R+\delta,R+\delta_0}(z(\delta,x^*)) \text{ for } s \in (0,s_1). \tag{3.115}$$

Consequently, in weak sense,

$$\begin{aligned}(-\Delta)^s\big(W \pm \varphi_s\big) = (-\Delta)^s W \pm \lambda_s\varphi_s \ge \big(W \pm \varphi_s\big) + s\big(m_2 \pm \frac{\lambda_s - 1}{s}\varphi_s\big)&\\ \ge W \pm \varphi_s \qquad \text{in } \Omega_{\delta,x_*} = \Omega \cap A_{R+\delta,R+\delta_0}(z(\delta,x^*))&\end{aligned} \tag{3.116}$$

by the definition of m_2. Moreover, it follows from (3.110) and the definition of m_1 that

$$W \pm \varphi_s \ge 0 \qquad \text{in } \mathbb{R}^N \setminus \Omega_{\delta,x_*} \text{ for } s \in (0,s_1). \tag{3.117}$$

Using Proposition 3.26, (3.116), and (3.117) together with the fact that $|\Omega_{\delta,x_*}| \le |A_{R,R+\delta_0}| \le \mu_0$, we deduce that
$$W \pm \varphi_s \ge 0 \qquad \text{in } \mathbb{R}^N,$$

and thus, in particular,

$$|\varphi_s| \le W \le \varepsilon \qquad \text{in } B_{R+2\delta}(z(\delta,x_*)) \text{ for } s \in (0,s_1)$$

by (3.111). Consequently, $|\varphi_s(x)| \le \varepsilon$ for $s \in (0,s_1)$ by (3.114), and this yields (3.113). Making $\delta > 0$ smaller if necessary, we may, by Lemma 3.22, also assume that

$$|\varphi_s(x)| \le \varepsilon \qquad \text{for } s \in [s_1, \frac{1}{4}],\ x \in \Omega^\delta. \tag{3.118}$$

Combining (3.113) and (3.118), we conclude that

$$|\varphi_s(x)| \le \varepsilon \qquad \text{for } s \in (0, \frac{1}{4}],\ x \in \Omega^\delta.$$

The proof of Theorem 3.23 is thus finished. □

3.6 Completion of the proofs

In this section, we complete the proofs of Theorem 3.1, Corollary 3.3 and Corollary 3.4.
We start with the

Proof of Theorem 3.1. Part (i) is proved in Theorem 3.15. Part (iii) is proved in Theorem 3.21. Moreover, the first claim in Part (ii), the boundedness of the set $M := \{\varphi_{k,s} : s \in (0, \frac{1}{4}]\}$ in $L^\infty(\Omega)$, has been proved in Theorem 3.16. Combining this fact with the relative compactness of the set M in $C(K)$ for every compact subset $K \subset \Omega$, it follows from Theorem 3.23 together with the Kolmogorov–Riesz compactness theorem that M is relative compact in $L^p(\Omega)$ for every $p \in [1, \infty)$, this completes the claim in Part (ii).
To prove Part (iv) of Theorem 3.1, we first observe that, since Ω satisfies an exterior sphere condition by assumption, it follows from Lemma 3.22 that $\varphi_{k,s} \in C_0(\Omega)$ for any $k \in \mathbb{N}$ and $s \in (0, \frac{1}{4}]$. Furthermore, M is equicontinuous in all points in Ω by Theorem 3.21 and in all points in $\partial\Omega$ by Theorem 3.23. Since moreover M is uniformly bounded with respect to $\|\cdot\|_{L^\infty(\Omega)}$ by Part (ii), the Arzelà-Ascoli Theorem implies that M is relative compact in $C_0(\Omega)$.
To prove Part (v), let $(s_n)_n \subset (0, \frac{1}{4}]$ be a sequence of numbers with $s_n \to 0^+$. By Theorem 3.15, we may pass to a subsequence with the property that

$$\varphi_{k,s_n} \to \varphi_{k,L} \qquad \text{in } L^2(\Omega) \text{ as } n \to \infty. \tag{3.119}$$

Due to the relative compactness of the set M in $L^p(\Omega)$ already proved in Part (ii), we also have L^p-convergence in (3.119) for $1 \le p < \infty$, and the locally uniform convergence follows from Part (iii). Moreover, in the case where Ω satisfies an exterior sphere condition, the convergence in $C_0(\Omega)$ follows from the relative compactness in the space $C_0(\Omega)$ stated in Part (iv). □

Next we complete the

Proof of Corollary 3.3. For the particular case $1 \le p \le 2$, the convergent in (3.19) follows directly from [29, Theorem 1.5] combined with the Hölder inequality. But using the relative compactness of the set M in $L^p(\Omega)$ proved in Part (ii) of Theorem 3.1 and the uniqueness of $\varphi_{1,s}$, the L^p-convergence in (3.19) for $1 \le p < \infty$ and the locally uniform convergence in Ω also follows by Part (iv) of Theorem 3.1. The additional assertion follows from the additional assertion in Theorem 3.1(v). □

Proof of Corollary 3.4. Let $(s_n)_n \subset (0, \frac{1}{4}]$ be a sequence of numbers with $s_n \to 0^+$. Moreover, for every $n \in \mathbb{N}$, let φ_{k,s_n}, $k \in \mathbb{N}$ denote L^2-orthonormal Dirichlet eigenfunctions of $(-\Delta)^{s_n}$ on Ω corresponding to the eigenvalues λ_{k,s_n}. Passing to a subsequence, we may assume, by Theorem 3.1, that

$$\frac{\lambda_{k,s_n} - 1}{s_n} \to \lambda_{k,L} \qquad \text{and} \qquad \varphi_{k,s_n} \to \varphi_{k,L} \quad \text{in } L^2(\Omega) \tag{3.120}$$

as $n \to \infty$, where, for every $k \in \mathbb{N}$, $\varphi_{k,L}$ is a Dirichlet eigenfunction of L_Δ on Ω corresponding to the eigenvalue $\lambda_{k,L}$. Parts (iii) and (v) of Theorem 3.1 then imply that

$$\varphi_{k,L} \in L^\infty(\Omega) \cap C_{loc}(\Omega) \qquad \text{for every } k \in \mathbb{N}.$$

Moreover, it follows that $\varphi_{k,L} \in C_0(\Omega)$ in the case where Ω satisfies an exterior sphere condition. Finally, the L^2-convergence in (3.120) implies that the sequence of functions $\varphi_{k,L}$, $k \in \mathbb{N}$ is L^2-orthonormal. It then follows that every Dirichlet eigenfunction of L_Δ on Ω can be written as a finite linear combination of the functions $\varphi_{k,L}$, and therefore it has the same regularity properties as the functions $\varphi_{k,L}$, $k \in \mathbb{N}$. □

4 The logarithmic Schrödinger operator and associated Dirichlet problems

This chapter is devoted to the study of the operator corresponding to the logarithmic symbol $\log(1+|\cdot|^2)$ and assiociated Dirichlet problems. We present an alternative method to derive the corresponding singular inetgral $(I-\Delta)^{\log}$ and settle the functional analytic properties that allow to study equations involving this operator and related variational characterizations. The structure of the chapter has the same form as paper [45] only acknowledgements is removed.

4.1 Introduction and main results

The present paper is devoted to the study of the integrodifferential operator corresponding to the logarithmic symbol $\log(1+|\cdot|^2)$ and associated Dirichlet problems in domains. This symbol is known in the probability literature as the characteristic exponent of the symmetric variance gamma process in $\mathbb{R}^N$ [9]. As particular case of geometric stable processes $\log(1+|\cdot|^{2s})$ for $s \in (0,1)$, it plays an important role in the study of Markov process [12] and finds applications to many different fields such as engineering reliability, credit risk theory in structure models, option pricing in mathematical finance [10] and it is used to study the heavy-tailed financial models [67, 77, 88]. It was recently used in wave equation to model damping mechanism in $\mathbb{R}^N$ (see [26]).

Let us emphasize that the associated operator $(I-\Delta)^{\log}$, which we call the logarithmic Schrödinger operator in the following, has been studied extensively in the literature from a probabilistic and potential theoretic point of view, see e.g. [11, 63, 64, 82, 88, 91] and the references therein. The main purpose of the present paper is to give an account on functional analytic properties of this operator from a PDE point of view. So some of the results we present here are not new but are stated under somewhat different assumptions related to the concept of weak solutions. Moreover, we present proofs not relying on probabilistic techniques but instead on purely analytic methods which are to some extend simpler and more accessible to PDE oriented readers.

Integrodifferential operators of order close to zero are getting increasing interest in the study of linear and nonlinear integrodifferential equations, see for e.g. [29, 30, 63, 72, 79, 86] with references therein. In particular, the logarithmic Schrödinger operator $(I-\Delta)^{\log}$ has the same singular local behavior as that of the logarithmic Laplacian L_Δ studied in [29], while it eliminates the integrability problem of L_Δ at infinity. We recall that for compactly supported Dini continuous functions $\varphi : \mathbb{R}^N \to \mathbb{R}$, the logarithmic Laplacian L_Δ is defined by

$$L_\Delta \varphi(x) = c_N \lim_{\varepsilon \to 0} \int\limits_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{\varphi(x) 1_{B_1(x)}(y) - \varphi(y)}{|x-y|^N} \, dy + \rho_N \varphi(x), \tag{4.1}$$

with the constants $c_N := \frac{\Gamma(N/2)}{\pi^{N/2}}$ and $\rho_N := 2\ln 2 + \psi(\frac{N}{2}) - \gamma$, see [29] for more details. Similarly as in [29], the starting point of the present paper is the observation

$$\lim_{s\to 0^+}(I-\Delta)^s u = u \quad \text{for } u\in C^2(\mathbb{R}^N), \tag{4.2}$$

where for $s\in(0,1)$, the operator $(I-\Delta)^s$ stands for the relativistic Schrödinger operator which, for sufficiently regular function $u:\mathbb{R}^N\to\mathbb{R}$, is represented via hypersinglar integral (see [85, page 548] and [38])

$$(I-\Delta)^s u(x) = u(x) + d_{N,s}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\varepsilon(0)} \frac{u(x+y)-u(x)}{|y|^{N+2s}}\omega_s(|y|)\,dy, \tag{4.3}$$

where $d_{N,s}=\frac{\pi^{-\frac{N}{2}}4^s}{\Gamma(-s)}$ is a normalization constant and the function ω_s is given by

$$\omega_s(|y|) = 2^{1-\frac{N+2s}{2}}|y|^{\frac{N+2s}{2}}K_{\frac{N+2s}{2}}(|y|) = \int_0^\infty t^{-1+\frac{N+2s}{2}}e^{-t-\frac{|y|^2}{4t}}\,dt. \tag{4.4}$$

In particular, if $u\in C^2(\mathbb{R}^N)$, then $(I-\Delta)^s u(x)$ is well defined by (4.3) for every $x\in\mathbb{R}^N$. Here the function K_ν is the modified Bessel function of the second kind with index $\nu>0$ and it is given by the expression

$$K_\nu(r) = \frac{(\pi/2)^{\frac{1}{2}}r^\nu e^{-r}}{\Gamma(\frac{2\nu+1}{2})}\int_0^\infty e^{-rt}t^{\nu-\frac{1}{2}}(1+t/2)^{\nu-\frac{1}{2}}dt.$$

The normalization constant $d_{N,s}$ in (4.3) is chosen such that the operator $(I-\Delta)^s$ is equivalently defined via its Fourier representation given by

$$\mathscr{F}((I-\Delta)^s u)(\xi) = (1+|\xi|^2)^s\mathscr{F}(u)(\xi), \quad \text{for a.e } \xi\in\mathbb{R}^N, \tag{4.5}$$

where $\mathscr{F}$ denotes the usual Fourier transform. It therefore follows from (4.2) that one may expect a Taylor expansion with respect to parameter s of the operator $(I-\Delta)^s$ near zero for $u\in C^2(\mathbb{R}^N)$ and $x\in\mathbb{R}^N$ as

$$(I-\Delta)^s u(x) = u(x) + s(I-\Delta)^{\log}u(x) + o(s) \quad \text{as } s\to 0^+,$$

where, the logarithmic Schrödinger operator $(I-\Delta)^{\log}$ appears as the first order term in the above expansion. Indeed, we have the following.

Theorem 4.1. *Let $u\in C^\alpha(\mathbb{R}^N)$ for some $\alpha>0$ and $1<p\le\infty$. Then*

$$\begin{aligned}(I-\Delta)^{\log}u(x) &= \frac{d}{ds}\Big|_{s=0}[(I-\Delta)^s u](x)\\ &= d_N\int_{\mathbb{R}^N}\frac{u(x)-u(x+y)}{|y|^N}\omega(|y|)\,dy = \int_{\mathbb{R}^N}(u(x)-u(x+y))J(y)\,dy,\end{aligned} \tag{4.6}$$

for $x \in \mathbb{R}^N$, *where* $d_N := \pi^{-\frac{N}{2}} = -\lim_{s\to 0^+} \frac{d_{N,s}}{s}$, $J(y) = d_N \frac{\omega(|y|)}{|y|^N}$, *and*

$$\omega(|y|) := 2^{1-\frac{N}{2}} |y|^{\frac{N}{2}} K_{\frac{N}{2}}(|y|) = \int_0^\infty t^{-1+\frac{N}{2}} e^{-t-\frac{|y|^2}{4t}} \, dt. \tag{4.7}$$

Moreover,

(i) If $u \in L^p(\mathbb{R}^N)$ *for* $1 \le p \le \infty$, *then* $(I-\Delta)^{\log} u \in L^p(\mathbb{R}^N)$ *and*

$$\frac{(I-\Delta)^s u - u}{s} \to (I-\Delta)^{\log} u \ \text{ in } \ L^p(\mathbb{R}^N) \ \text{ as } \ s \to 0^+$$

.

(ii) $\mathscr{F}((I-\Delta)^{\log} u)(\xi) = \log(1+|\xi|^2)\mathscr{F}(u)(\xi)$, *for almost every* $\xi \in \mathbb{R}^N$.

We note that in the particular case $N = 1$, it follows from the definition of ω in (4.7) (see also [55, (2.4)] and [88, Remark 4.5]) that $\omega(r) = \pi^{N/2} e^{-r}$ and

$$(I-\Delta)^{\log} u(x) = P.V. \int_{\mathbb{R}} \frac{u(x)-u(y)}{|x-y|} e^{-|x-y|} \, dy. \tag{4.8}$$

We note here that the operator in (4.8) appears in [75] and is identified as symmetrized Gamma process (see also [66, Example 1]). We stress however that the symbol of this operator is $\log(1+|\xi|^2)$ and not $\log(1+|\xi|)$ as claimed in [75, Page 183]. The representation of J in (4.6) provides an explicit expression for the kernel of the variance Gamma process in $\mathbb{R}^N$ and gives the following asymptotics expansions

$$J(z) \sim \begin{cases} \pi^{-\frac{N}{2}} \Gamma(\frac{N}{2}) |z|^{-N} & \text{as} \quad |z| \to 0 \\ \pi^{-\frac{N-1}{2}} 2^{-\frac{N-1}{2}} |z|^{-\frac{N+1}{2}} e^{-|z|} & \text{as} \quad |z| \to \infty. \end{cases} \tag{4.9}$$

Indeed, these expansions follow directly from (4.7) and the asymptotics expansions of the modified Bessel function K_ν (see Section 4.2), (see also [88, Theorem 3.4 and 3.6] for other proof). The Green function of the operator $(I-\Delta)^{\log}$ is given (see [55, 64]) by

$$G(x) = \int_0^\infty q_t(x) \, dt \qquad x \in \mathbb{R}^N, \tag{4.10}$$

where for $t > 0$, $q_t : \mathbb{R}^N \to \mathbb{R}$ is the density of the symmetry variance Gamma process i.e., for all $t > 0$ and $x \in \mathbb{R}^N$,

$$q_t(x) \ge 0, \qquad \int_{\mathbb{R}^N} q_t(x) \, dx = 1 \qquad \text{and} \qquad \mathscr{F}(q_t)(\xi) = e^{-t\log(1+|\xi|^2)}.$$

It follows from (4.7) that for any $t > 0$,

$$q_t(x) = \frac{2^{1-N}}{\pi^{N/2}\Gamma(t)} \left(\frac{|x|}{2}\right)^{t-\frac{N}{2}} K_{t-\frac{N}{2}}(|x|), \tag{4.11}$$

and the Green function for $(I-\Delta)^{\log}$ then writes

$$G(x) = \frac{2^{1-N}}{\pi^{N/2}} \int_0^\infty \frac{1}{\Gamma(t)} \left(\frac{|x|}{2}\right)^{t-\frac{N}{2}} K_{t-\frac{N}{2}}(|x|)\, dt. \tag{4.12}$$

Using the asymptotics expansions for the modified Bessel function (see (4.23) Section 4.2), we have the following proposition.

Proposition 4.2. *The function G in (4.12) satisfies the asymptotics properties*

$$G(x) \sim \begin{cases} c_N|x|^{-N} & \text{as} \quad |x| \to 0 \\ c_N 2^{\frac{N-1}{2}} \pi^{1/2} |x|^{-\frac{N+1}{2}} e^{-|x|} & \text{as} \quad |x| \to \infty. \end{cases} \tag{4.13}$$

Moreover, for $f \in L^1(\mathbb{R}^N)$, the solution $u = G * f$ of the equation $(I-\Delta)^{\log} u = f$ in $\mathbb{R}^N$ satisfies

$$u(x) = \begin{cases} O(|x|^{-N}) & \text{as} \quad |x| \to 0 \\ O(e^{-|x|}) & \text{as} \quad |x| \to \infty. \end{cases} \tag{4.14}$$

The next task is the study in weak sense with the source function $f \in L^2(\Omega)$, the following related Dirichlet elliptic problem in open bounded set $\Omega \subset \mathbb{R}^N$

$$\begin{cases} (I-\Delta)^{\log} u = f & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.15}$$

In order to settle the corresponding functional analytic framework and energy space related to integro-differential operator $(I-\Delta)^{\log}$, we introduce the following space

$$H^{\log}(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \quad \mathscr{E}_\omega(u,u) < \infty \right\}$$

where with J as in (4.6), the bilinear form considered here is given by

$$\mathscr{E}_\omega(u,v) := \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (u(x)-u(y))(v(x)-v(y)) J(x-y)\, dxdy.$$

We shall see in Section 4.2 that $H^{\log}(\mathbb{R}^N)$ is a Hilbert space endowed with the scalar product

$$(u,v) \to \langle u,v \rangle_{H^{\log}(\mathbb{R}^N)} = \langle u,v \rangle_{L^2(\mathbb{R}^N)} + \mathscr{E}_\omega(u,u),$$

where $\langle u,v \rangle_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} u(x)v(x)\, dx$ with corresponding norm

$$\|u\|_{H^{\log}(\mathbb{R}^N)} = \left(\|u\|^2_{L^2(\mathbb{R}^N)} + \mathscr{E}_\omega(u,u) \right)^{\frac{1}{2}}.$$

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set of $\mathbb{R}^N$. Here and the following we identify the space $L^2(\Omega)$ with the space of functions $u \in L^2(\mathbb{R}^N)$ with $u \equiv 0$ on $\mathbb{R}^N \setminus \Omega$. We denote by $\mathscr{H}_0^{\log}(\Omega)$ the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^{\log}(\mathbb{R}^N)}$. We have, by the Riesz representation theorem that problem (4.15) admits a unique weak solution $u \in \mathscr{H}_0^{\log}(\Omega)$ with

$$\mathscr{E}_\omega(u,v) = \int_\Omega f(x)v(x)\,dx \quad \text{ for all } v \in \mathscr{H}_0^{\log}(\Omega).$$

Moreover, if $f \in L^\infty(\Omega)$ and Ω satisfies a uniform exterior sphere condition, it follows from the Green function representation and the regularity estimates in [63,64,79] that $u \in C_0(\Omega) := \{u \in C(\mathbb{R}^N) \;:\; u = 0 \text{ on } \mathbb{R}^N \setminus \Omega\}$.
We aim next to study the eigenvalue problem in bounded domain $\Omega \subset \mathbb{R}^N$ involving the logarithmic Schrödinger operator $(I-\Delta)^{\log}$, that is, we consider (4.15) with $f = \lambda u$. To avoid a priori regularity assumption, we consider the eigenvalue problem (4.15) in weak sense. We call a function $u \in \mathscr{H}_0^{\log}(\Omega)$ an eigenfunction of (4.15) corresponding to the eigenvalue λ if

$$\mathscr{E}_\omega(u,\varphi) = \lambda \int_\Omega u\varphi\,dx \qquad \text{for all } \varphi \in \mathscr{C}_c^\infty(\Omega). \tag{4.16}$$

We then have the following characterisation of the eigenvalues and eigenfunctions for the operator $(I-\Delta)^{\log}$ in an open bounded set Ω of $\mathbb{R}^N$.

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Then*

(i) Problem (4.15) admits an eigenvalue $\lambda_1(\Omega) > 0$ characterized by

$$\lambda_1(\Omega) = \inf_{\substack{u \in \mathscr{H}_0^{\log}(\Omega) \\ u \neq 0}} \frac{\mathscr{E}_\omega(u,u)}{\|u\|^2_{L^2(\Omega)}} = \inf_{u \in \mathscr{P}_1(\Omega)} \mathscr{E}_\omega(u,u), \tag{4.17}$$

with $\mathscr{P}_1(\Omega) := \{u \in \mathscr{H}_0^{\log}(\Omega) : \|u\|_{L^2(\Omega)} = 1\}$ and there exists a positive function $\varphi_1 \in \mathscr{H}_0^{\log}(\Omega)$, which is an eigenfunction corresponding to $\lambda_1(\Omega)$ and that attains the minimum in (4.17), i.e. $\|\varphi_1\|_{L^2(\Omega)} = 1$ and $\lambda_1(\Omega) = \mathscr{E}_\omega(\varphi_1,\varphi_1)$.

(ii) The first eigenvalue $\lambda_1(\Omega)$ is simple, that is, if $u \in \mathscr{H}_0^{\log}(\Omega)$ satisfies (4.16) with $\lambda = \lambda_1(\Omega)$, then $u = \alpha\varphi_1$ for some $\alpha \in \mathbb{R}$.

(iii) Problem (4.15) admits a sequence of eigenvalues $\{\lambda_k(\Omega)\}_{k\in\mathbb{N}}$ with

$$0 < \lambda_1(\Omega) < \lambda_2(\Omega) \le \cdots \le \lambda_k(\Omega) \le \lambda_{k+1}(\Omega)\cdots,$$

*with corresponding eigenfunctions φ_k, $k \in \mathbb{N}$ and $\lim_{k\to\infty} \lambda_k(\Omega) = +\infty$.
Moreover, for any $k \in \mathbb{N}$, the eigenvalue $\lambda_k(\Omega)$ can be characterized as*

$$\lambda_k(\Omega) = \inf_{u \in \mathscr{P}_k(\Omega)} \mathscr{E}_\omega(u,u) \tag{4.18}$$

where $\mathscr{P}_k(\Omega)$ is given by

$$\mathscr{P}_k(\Omega) := \{u \in \mathscr{H}_0^{\log}(\Omega) : \int_\Omega u\varphi_j\,dx = 0 \;\; for \;\; j = 1,2,\cdots k-1 \;\; and \;\; \|\varphi_k\|_{L^2(\Omega)} = 1\}.$$

(iv) The sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ of eigenfunctions corresponding to eigenvalues $\lambda_k(\Omega)$ form a complete orthonormal basis of $L^2(\Omega)$ and an orthogonal system of $\mathscr{H}_0^{\log}(\Omega)$.

Using the δ-decomposition technique introduced in [47], we provide a boundedness result of the eigenfunctions introduced in Theorem 4.3.

Proposition 4.4. *Let $u \in \mathscr{H}_0^{\log}(\Omega)$ and $\lambda > 0$ satisfying* (4.16). *Then $u \in L^\infty(\Omega)$ and there exists a constant $C := C(N,\Omega) > 0$ such that*

$$\|u\|_{L^\infty(\Omega)} \le C\|u\|_{L^2(\Omega)}.$$

Our next result concerns the Faber-Krahn inequality for the logarithmic Schrödinger operator, which says: Among all open sets in $\mathbb{R}^N$ with given measure, ball uniquely gives the smallest first Dirichlet eigenvalue of the logarithmic Schrödinger operator $(I-\Delta)^{\log}$. Here and in the following, we denote by B^* the open ball in $\mathbb{R}^N$ centered at zero with radius determined such that $|\Omega| = |B^*|$

Theorem 4.5 (Faber-Krahn inequality)**.** *Let $\Omega \subset \mathbb{R}^N$ be open and bounded, and $\lambda_{1,\log}(\Omega)$ be the principal eigenvalue of $(I-\Delta)^{\log}$ in Ω. Then*

$$\lambda_{1,\log}(\Omega) \ge \lambda_{1,\log}(B^*). \tag{4.19}$$

Moreover, if equality occurs, Ω is a ball. Consequently, if Ω is a ball in $\mathbb{R}^N$, the first eigenfunction $\varphi_{1,\log}$ corresponding to $\lambda_{1,\log}(B)$ is radially symmetric.

Our last result concerns small order asymptotics $s \to 0^+$ of eigenvalues and corresponding eigenfunctions of the relativistic Schrödinger operator $(I-\Delta)^s$ on bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$, which is an analogue, but a part of the result of the small order asymptotics $s \to 0^+$ proved in [47] for the fractional Laplacian.

Theorem 4.6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, and $\lambda_{k,s}(\Omega)$ resp. $\lambda_{k,\log}(\Omega)$ be the k-th Dirichlet eigenvalue of $(I-\Delta)^s$ resp. of $(I-\Delta)^{\log}$ on Ω. Then for $s \in (0,1)$, the eigenvalue $\lambda_{k,s}(\Omega)$ satisfies the expansion*

$$\lambda_{k,s}(\Omega) = 1 + s\lambda_{k,\log}(\Omega) + o(s) \quad \textit{as} \quad s \to 0^+. \tag{4.20}$$

Moreover, if $(s_n)_n \subset (0,s_0)$, $s_0 > 0$ is a sequence with $s_n \to 0$ as $n \to \infty$, then if $\psi_{1,s}$ is the unique nonnegative L^2-normalized eigenfunction of $(I-\Delta)^s$ corresponding to the principal eigenvalue $\lambda_{1,s}(\Omega)$, we have that

$$\psi_s \to \psi_{1,\log} \quad \textit{in } L^2(\Omega) \quad \textit{as} \quad s \to 0^+, \tag{4.21}$$

and after passing to a subsequence, we have that

$$\psi_{k,s} \to \psi_{k,\log} \quad \textit{in } L^2(\Omega) \quad \textit{as} \quad s \to 0^+, \tag{4.22}$$

where $\psi_{1,\log}$, resp. $\psi_{k,\log}$, $k \ge 2$ is the unique nonnegative L^2-normalized eigenfunction resp. a L^2-normalized eigenfunction corresponding to $\lambda_{1,\log}(\Omega)$ resp. to $\lambda_{k,\log}(\Omega)$.

The paper is organized as follows. In Section 4.2, we provide the proof of Theorem 4.1 and establish some properties of $(I-\Delta)^{\log}$ and functional spaces. In Section 4.3, we prove Theorem 4.3 and, using the δ-decomposition tecnique introduced in [47], we give the proof of Proposition 4.4 on the L^{∞}-bound of eigenfunctions and close the section with the proof of Theorem 4.5 on Faber-Krahn inequality. Section 4.4 is dedicated to the proof of Theorem 4.6 on small order asymptotics $s \to 0^+$ of the eigenvalues and corresponding eigenfunctions of $(I-\Delta)^s$. In section 4.5, we establish the proof of Proposition 4.2 concerning the decay of the solution of Poisson problem in $\mathbb{R}^N$. Finally, Section 4.6 collects some theorems that can be directly deduced from known results in the literature.

Notation: We let $\omega_{N-1} = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}$ denote the measure of the unit sphere in $\mathbb{R}^N$ and, for a set $A \subset \mathbb{R}^N$ and $x \in \mathbb{R}^N$, we define $\delta_A(x) := \mathrm{dist}(x, A^c)$ with $A^c = \mathbb{R}^N \setminus A$ and, if A is measurable, then $|A|$ denotes its Lebesgue measure. Moreover, for given $r > 0$, let $B_r(A) := \{x \in \mathbb{R}^N \,:\, \mathrm{dist}(x, A) < r\}$, and let $B_r(x) := B_r(\{x\})$ denote the ball of radius r with x as its center. If $x = 0$ we also write B_r instead of $B_r(0)$. If A is open, we denote by $C_c^k(A)$ the space of function $u : \mathbb{R}^N \to \mathbb{R}$ which are k-times continuously differentiable and with support compactly contained in A. If f and g are two functions, then, $f \sim g$ as $x \to a$ if $\frac{f(x)}{g(x)}$ converges to a constant as x converges to a.

4.2 Properties of the operator and Functional spaces

We commence this section with the establishment of the integral representation of the operator $(I-\Delta)^{\log}$ for a function $u \in C^{\alpha}(\mathbb{R}^N)$, that is, we provide the proof of Theorem 4.1. After that, we also provide some properties of the functional spaces related to $(I-\Delta)^{\log}$. We first introduce the following asymptotics approximations (see [80]) for the modified Bessel function K_ν. It well-known that

$$K_\nu(r) \sim \begin{cases} 2^{|\nu|-1}\Gamma(|\nu|)r^{-|\nu|}, & r \to 0, \quad \nu \neq 0, \\ \log \frac{1}{r}, & r \to 0, \quad \nu = 0, \\ \sqrt{\pi/2}\, r^{-\frac{1}{2}} e^{-r}, & r \to +\infty, \end{cases} \tag{4.23}$$

and the monotonicity (see [80, 10.37.1])

$$|K_\nu(r)| < |K_\mu(r)| \quad \text{for} \quad 0 \le \nu < \mu. \tag{4.24}$$

Consequently,

$$\omega_s(r) \sim \begin{cases} \Gamma(\frac{N+2s}{2}), & r \to 0, \\ 2^{-\frac{N+2s-1}{2}} r^{\frac{N+2s-1}{2}} e^{-r}, & r \to +\infty. \end{cases} \tag{4.25}$$

Note also that the functions $s \mapsto \omega_s$ and $s \mapsto d_{N,s}$ defined in (4.3) are continuous function of s and we have that $\lim\limits_{s \to 0^+} d_{N,s} = 0$ and, by dominated convergent theorem,

$$\omega(|y|) := \lim_{s \to 0^+} \omega_s(|y|) = 2^{1-\frac{N}{2}} |y|^{\frac{N}{2}} K_{\frac{N}{2}}(|y|) = \int_0^\infty t^{-1+\frac{N}{2}} e^{-t-\frac{|y|^2}{4t}}\, dt. \tag{4.26}$$

We now give the

Proof of Theorem 4.1. Let $u \in C^{\alpha}(\mathbb{R}^N)$ with $0 < s < \min\{\frac{\alpha}{2}, \frac{1}{2}\}$. Then, from the definition of $(I-\Delta)^s$ in (4.3), the principal value can be dropped out and we have the different quotient

$$\frac{(I-\Delta)^s u - u}{s} = \frac{d_{N,s}}{s} \int_{\mathbb{R}^N} \frac{u(x+y)-u(x)}{|y|^{N+2s}} \omega_s(|y|)\, dy = A_\varepsilon(s,x) + D_\varepsilon(s,x),$$

where $\varepsilon > 0$, with $A_\varepsilon(s,x)$ and $D_\varepsilon(s,x)$ given respectively by

$$A_\varepsilon(s,x) := \frac{d_{N,s}}{s} \int_{|y|<\varepsilon} \frac{u(x+y)-u(x)}{|y|^{N+2s}} \omega_s(|y|)\, dy,$$

$$D_\varepsilon(s,x) := \frac{d_{N,s}}{s} \int_{|y|\geq\varepsilon} \frac{u(x+y)-u(y)}{|y|^{N+2s}} \omega_s(|y|)\, dy.$$

First, from (4.4) and (4.7) and the fact that $|y|^{-2s} \leq \varepsilon^{-2}$ for $|y| \geq \varepsilon$ and $s \in (0,1)$, we have by dominated convergent theorem that

$$D_\varepsilon(s,x) = \frac{d_{N,s}}{s} \int_{|y|\geq\varepsilon} \frac{u(x+y)-u(x)}{|y|^{N+2s}} \omega_s(|y|)\, dy \to D_\varepsilon(0,x) \quad \text{as } s \to 0^+,$$

with

$$D_\varepsilon(0,x) := d_N \int_{|x-y|\geq\varepsilon} \frac{u(x)-u(y)}{|x-y|^N} \omega(|x-y|)\, dy = \int_{|x-y|\geq\varepsilon} (u(x)-u(y))J(x-y)\, dy.$$

Since next $u \in C^{\alpha}(\mathbb{R}^N)$, it also follows that

$$A_\varepsilon(s,x) = \frac{d_{N,s}}{s} \int_{|y|<\varepsilon} \frac{u(x+y)-u(x)}{|y|^{N+2s}} \omega_s(|y|)\, dy \to A_\varepsilon(0,x) \quad \text{as} \quad s \to 0^+,$$

with

$$A_\varepsilon(0,x) = d_N \int_{|y|<\varepsilon} \frac{u(x)-u(x+y)}{|y|^N} \omega(|y|)\, dy = \int_{|x-y|<\varepsilon} (u(x)-u(y))J(x-y)\, dy.$$

We recall that $\lim_{s\to 0} d_{N,s}/s = -d_N$. It is easy to see that $A_\varepsilon(0,x) \to 0$ as $\varepsilon \to 0^+$, and from the the fact that $u \in C^{\alpha}(\mathbb{R}^N)$, we also infer that

$$\left|(I-\Delta)^{\log} u(x) - D_\varepsilon(0,x)\right| \leq C \int_{|y|<\varepsilon} \min\{1, |y|^{\alpha}\}\, dy \to 0 \quad \text{as } \varepsilon \to 0^+.$$

Since $u \in C^{\alpha}(\mathbb{R}^N)$, setting $\kappa_{N,s,u} = \left|\frac{d_{N,s}}{s}\right| \Gamma((N+2s)/2)\|u\|_{C^{\alpha}(\mathbb{R}^N)}\omega_{N-1}$ it follows from (4.25) that

$$|A_\varepsilon(s,x)| \le \left|\frac{d_{N,s}}{s}\right| \int\limits_{|y|<\varepsilon} \frac{\|u\|_{C^{\alpha}(\mathbb{R}^N)}}{|y|^{N+2s-\alpha}}\omega_s(|y|)\, dy \le \kappa_{N,s,u}\frac{\varepsilon^{\alpha-2s}}{\alpha-2s}.$$

Consequently,

$$\|A_\varepsilon(s,\cdot)\|_{L^p(B_\varepsilon)} \le \kappa_{N,s,u}\frac{\varepsilon^{\frac{N}{p}+\alpha-2s}}{\alpha-2s} \quad \text{for } 1 \le p \le \infty.$$

On the other hand, using again (4.25) with $s=0$, we infer that

$$\begin{aligned}
|D_\varepsilon(0,x)| &\le \int\limits_{|x-y|\ge\varepsilon} |u(x)-u(x+y)|J(y)\, dy \\
&\le 2\|u\|_{C^{\alpha}(\mathbb{R}^N)}\big(\int\limits_{B_1\setminus B_\varepsilon} |y|^{\alpha-N}dy + \int\limits_{|y|\ge 1} e^{-|y|}\, dy\big) \\
&\le 2\|u\|_{C^{\alpha}(\mathbb{R}^N)}\Big(2\frac{1-\varepsilon^{\alpha}}{\alpha} + \omega_{N-1}\Gamma(N,1)\Big) = \ C_{N,\varepsilon}\|u\|_{C^{\alpha}(\mathbb{R}^N)}.
\end{aligned}$$

Therefore,

$$\|D_\varepsilon(0,\cdot)\|_{L^\infty(\mathbb{R}^N\setminus B_\varepsilon)} < \infty.$$

Next, by the Minkowski's integral inequality, we have

$$\begin{aligned}
\|D_\varepsilon(0,\cdot)\|_{L^p(\mathbb{R}^N\setminus B_\varepsilon)} &\le \Big(\int\limits_{\mathbb{R}^N\setminus B_\varepsilon} \Big|\int\limits_{|y|\ge\varepsilon} (u(x)-u(x+y))J(y)\, dy\Big|^p dx\Big)^{\frac{1}{p}} \\
&\le \int\limits_{\mathbb{R}^N\setminus B_\varepsilon} \Big(\int\limits_{\mathbb{R}^N\setminus B_\varepsilon} |u(x)-u(x+y)|^p\, dx\Big)^{\frac{1}{p}} J(y)\, dy \\
&\le 2^{\frac{p-1}{p}}\|u\|_{L^p(\mathbb{R}^N\setminus B_\varepsilon)} \int\limits_{\mathbb{R}^N\setminus B_\varepsilon} J(y)\, dy < \infty.
\end{aligned}$$

Therefore, we conclude that $D_\varepsilon(0,\cdot) \in L^p(\mathbb{R}^N\setminus B_\varepsilon)$ for all $1 \le p \le \infty$ and thus

$$\|D_\varepsilon(s,\cdot) - D_\varepsilon(0,\cdot)\|_{L^p(\mathbb{R}^N\setminus B_\varepsilon)} \to 0 \quad \text{uniformly in } \varepsilon \text{ as } s \to 0^+. \tag{4.27}$$

This allows to conclude for $x \in \mathbb{R}^N$ that

$$\lim_{\varepsilon\to 0^+} D_\varepsilon(0,x) = \lim_{\varepsilon\to 0^+} \int\limits_{|y|\ge\varepsilon} (u(x)-u(x+y))J(y)\, dy = (I-\Delta)^{\log}u(x). \tag{4.28}$$

Taking into account the above facts, we find with $1 \le p < \infty$ that

$$\left\|\frac{(I-\Delta)^s u - u}{s} - (I-\Delta)^{\log}u\right\|_{L^p(\mathbb{R}^N)} = \left\|A_\varepsilon(s,\cdot) + D_\varepsilon(s,\cdot) - (I-\Delta)^{\log}u\right\|_{L^p(\mathbb{R}^N)}$$

$$\leq \|A_\varepsilon(s,\cdot)\|_{L^p(\mathbb{R}^N)} + \left\|D_\varepsilon(s,\cdot) - (I-\Delta)^{\log}u\right\|_{L^p(\mathbb{R}^N)}$$
$$\leq \kappa_{N,s,u}\frac{\varepsilon^{\frac{N}{p}+\alpha-2s}}{\alpha-2s} + \left\|D_\varepsilon(s,\cdot) - (I-\Delta)^{\log}u\right\|_{L^p(\mathbb{R}^N)}.$$

Therefore, using (4.27) and (4.28) , we have for every $1 \leq p < \infty$ that

$$\limsup_{s\to 0^+}\left\|\frac{(I-\Delta)^s u - u}{s} - (I-\Delta)^{\log}u\right\|_{L^p(\mathbb{R}^N)} \leq \kappa_{N,u}\frac{\varepsilon^{\frac{N}{p}+\alpha}}{\alpha} \quad \text{for every } \varepsilon > 0,$$

where $\kappa_{N,u}$ is independent of ε. The case $p = \infty$ follows by the same computation and

$$\limsup_{s\to 0^+}\left\|\frac{(I-\Delta)^s u - u}{s} - (I-\Delta)^{\log}u\right\|_{L^\infty(\mathbb{R}^N)} \leq \kappa_{N,u}\frac{\varepsilon^{\alpha}}{\alpha} \quad \text{for every } \varepsilon > 0.$$

Moreover, it follows from the arbitrary of ε that

$$\lim_{s\to 0^+}\left\|\frac{(I-\Delta)^s u - u}{s} - (I-\Delta)^{\log}u\right\|_{L^p(\mathbb{R}^N)} = 0 \quad \text{for every } 1 \leq p \leq \infty.$$

This completes the of item (i). The proof of item (ii) is a particular case with $p = 2$. Moreover, using the continuity of the Fourier transform in $L^2(\mathbb{R}^N)$, we have that

$$\mathscr{F}((I-\Delta)^{\log}u) = \lim_{s\to 0^+}\frac{\mathscr{F}((I-\Delta)^s u) - \mathscr{F}(u)}{s} = \lim_{s\to 0^+}\left(\frac{(1+|\cdot|^2)^s - 1}{s}\right)\mathscr{F}(u)$$
$$= \log(1+|\cdot|^2)\mathscr{F}(u) \quad \text{in} \quad L^2(\mathbb{R}^N).$$

We therefore infer that

$$\mathscr{F}((I-\Delta)^{\log}u)(\xi) = \log(1+|\cdot|^2)\mathscr{F}(u)(\xi), \qquad \text{for almost every} \quad \xi \in \mathbb{R}^N.$$

The proof of Theorem 4.1 is henceforth completed. □

In the following, we let $L_0(\mathbb{R}^N)$ denotes the space

$$L_0(\mathbb{R}^N) := \left\{u:\ \mathbb{R}^N \to \mathbb{R}:\ \|u\|_{L_0(\mathbb{R}^N)} < \infty\right\} \ \text{with} \ \|u\|_{L_0(\mathbb{R}^N)} = \int_{\mathbb{R}^N}\frac{|u(x)|e^{-|x|}}{(1+|x|)^{\frac{N+1}{2}}}\,dx.$$

Let U be a measurable subset and $u : U \to \mathbb{R}$ be a measurable function. The modulus of continuity of u at a point $x \in U$ is defined by

$$\omega_{u,x,U} : (0,+\infty) \to [0,+\infty), \qquad \omega_{u,x,U}(r) = \sup_{y\in U,\ |x-y|\leq r}|u(x) - u(y)|.$$

The function u is called Dini continuous at x if

$$\int_0^1 \frac{\omega_{u,x,U}(r)}{r}\,dr < \infty.$$

Moreover, we call u uniformly Dini continuous in U for the uniform modulus of continuity

$$\omega_{u,U}(r) := \sup_{x\in U} \omega_{u,x,U}(r) \quad \text{if} \quad \int_0^1 \frac{\omega_{u,U}(r)}{r}\, dr < \infty.$$

In the following proposition, we list some properties the operator $(I-\Delta)^{\log}$.

Proposition 4.7. *(i) Let $u \in L_0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. If u is locally Dini continuous at some point $x \in \mathbb{R}^N$, then the operator $(I-\Delta)^{\log}u$ is well defined by*

$$(I-\Delta)^{\log}u(x) = \int_{\mathbb{R}^N} (u(x)-u(y))J(x-y)\, dy.$$

(ii) Let $\varphi \in C_c^\alpha(\mathbb{R}^N)$ for some $\alpha > 0$, there is $C = C(N,\varphi)$ such that

$$|(I-\Delta)^{\log}\varphi(x)| \leq C\|\varphi\|_{C^\alpha(\mathbb{R}^N)} \frac{e^{-|x|}}{(1+|x|)^{\frac{N+1}{2}}}.$$

In particular, for $u \in L_0(\mathbb{R}^N)$, $(I-\Delta)^{\log} u$ defines a distribution via the map

$$\varphi \mapsto \langle (I-\Delta)^{\log} u, \varphi\rangle = \int_{\mathbb{R}^N} u(I-\Delta)^{\log}\varphi\, dx.$$

(iii) Let $u \in L_0(\mathbb{R}^N)$ and $r > 0$ such that $u \in C^\alpha(B_r(0))$ for some $\alpha > 0$. Then there exists a constant $C := C(N,\alpha) > 0$ such that

$$|(I-\Delta)^{\log}u(x)| \leq C(\|u\|_{C^\alpha(B_{r(0)})} + \|u\|_{\mathscr{L}_0(\mathbb{R}^N)})$$

(iv) If $u \in C^\beta(\mathbb{R}^N)$ for some $\beta > 0$, then $(I-\Delta)^{\log}u \in C^{\beta-\varepsilon}(\mathbb{R}^N)$ for every ε such that $0 < \varepsilon < \beta$ and there exists a constant $C := C(N,\beta,\varepsilon) > 0$ such that

$$[(I-\Delta)^{\log}u]_{\beta-\varepsilon} \leq C\|u\|_{C^\beta(\mathbb{R}^N)}.$$

(v) Let $\varphi, \psi \in \mathscr{C}_c^\infty(\Omega)$. Then we have the product rule

$$(I-\Delta)^{\log}(\varphi\psi)(x) = \varphi(x)(I-\Delta)^{\log}\psi(x) + \psi(x)(I-\Delta)^{\log}\varphi(x) - \Lambda(\varphi,\psi).$$

with

$$\Lambda(\varphi,\psi) := \int_{\mathbb{R}^N} (\varphi(x)-\varphi(y))(\psi(x)-\psi(y))J(x-y)\, dy$$

If ρ_ε, $\varepsilon > 0$ is a family of mollified, then

$$[(I-\Delta)^{\log}(\rho_\varepsilon * \varphi)](x) = \rho_\varepsilon * [(I-\Delta)^{\log}\varphi](x).$$

Proof. Let $x \in \mathbb{R}^N$. By splitting the integral and using the asymptotic of J in (4.9), we have the following,

$$\begin{aligned}
|(I-\Delta)^{\log}u(x)| &\le \int_{B_1(x)} |u(x)-u(y)|J(x-y)\,dy + \int_{\mathbb{R}^N\setminus B_1(x)} (|u(x)|+|u(y)|)J(x-y)\,dy \\
&\le \Gamma(N/2)\omega_{N-1}\int_0^1 \frac{\omega_{u,x}(r)}{r}\,dr + C\|u\|_{L^\infty}\int_{\mathbb{R}^N\setminus B_1} e^{-|y|}\,dy + C\int_{\mathbb{R}^N\setminus B_1(x)} \frac{|u(y)|e^{-|x-y|}}{|x-y|^{\frac{N+1}{2}}}\,dy \\
&\le C(1+\|u\|_{L^\infty(\mathbb{R}^N)}) + C\Big(\int_{B_{1+2|x|}(0)\setminus B_1(x)} + \int_{\mathbb{R}^N\setminus B_{1+2|x|}(0)}\Big)\frac{|u(y)|e^{-|x-y|}}{|x-y|^{\frac{N+1}{2}}}\,dy \\
&\le C(1+\|u\|_{L^\infty(\mathbb{R}^N)}) + C\int_{B_{1+2|x|}(0)} |u(y)|\,dy + C\int_{\mathbb{R}^N\setminus B_{1+2|x|}(0)} \frac{|u(y)|e^{-|x-y|}}{|x-y|^{\frac{N+1}{2}}}\,dy.
\end{aligned}$$

Now, since $|x-y| \ge \frac{1}{2}(1+|y|)$ for $|y| \ge 1+2|x|$, it follows that

$$|(I-\Delta)^{\log}u(x)| \le C(1+\|u\|_{L^\infty(\mathbb{R}^N)} + \|u\|_{L_0(\mathbb{R}^N)}) < \infty.$$

This shows that $(I-\Delta)^{\log}u(x)$ is well-defined.
To prove (ii), for $x \in \mathbb{R}^N$, we use again 4.9 and the representation

$$(I-\Delta)^{\log}\varphi(x) = \frac{d_N}{2}\int_{\mathbb{R}^N} \frac{2\varphi(x)-\varphi(x+y)-\varphi(x-y)}{|y|^N}\omega(|y|)\,dy.$$

Put $A := \|\varphi\|_{C^\alpha(\mathbb{R}^N)}$. Note first that, since $\varphi \in C_c^\alpha(\mathbb{R}^N)$, we have

$$|2\varphi(x)-\varphi(x+y)-\varphi(x-y)| \le A\min\{1,|y|^\alpha\}.$$

Therefore, for any $x \in \mathbb{R}^N$, we have with $0<r<1$ that

$$\begin{aligned}
|(I-\Delta)^{\log}\varphi(x)| &\le \frac{d_N}{2}\int_{\mathbb{R}^N} \frac{|2\varphi(x)-\varphi(x+y)-\varphi(x-y)|}{|y|^N}\omega(|y|)\,dy \\
&\le A\int_{\mathbb{R}^N} \frac{\min\{1,|y|^\alpha\}}{|y|^N}\omega(|y|)\,dy \\
&\le C_N A\Big(\int_{B_r} |y|^{\alpha-N}\,dy + \int_{B_1\setminus B_r} \frac{1}{|y|^N}\,dy + \int_{\mathbb{R}^N\setminus B_1} e^{-|y|}\,dr\Big) \\
&\le C(N,r,\alpha)A.
\end{aligned}$$

Next, Let $R>0$ be such that $B_1(\text{supp }\varphi) \subset B_R(0)$. Let $x \in \mathbb{R}^N$ satisfying $\frac{|x|}{2} > R$, then $1+|y| \le \frac{|x|}{2}$ for $y \in B_1(\text{supp }\varphi)$ and $|x-y| \ge |x|-|y| \ge \frac{|x|}{2}+1 \ge \frac{1}{2}(|x|+1)$. Moreover, since $\varphi(x) \equiv 0$ for $x \in \mathbb{R}^N \setminus B_R(0)$, it follows that

$$|(I-\Delta)^{\log}\varphi(x)| \le 2d_N A\int_{\text{supp }\varphi} \frac{\omega(|x-y|)}{|x-y|^N}\,dy \le C_N A\int_{\text{supp }\varphi} \frac{e^{-|x-y|}}{|x-y|^{\frac{N+1}{2}}}\,dy$$

$$\leq C_N A \int\limits_{\text{supp } \varphi} \frac{e^{-\frac{|x|}{2}}}{(1+|x|)^{\frac{N+1}{2}}}\, dy \leq C_N |\text{supp } \varphi| A \frac{e^{-\frac{|x|}{2}}}{(1+|x|)^{\frac{N+1}{2}}}.$$

Therefore, combining the above computations, we find that

$$|(I-\Delta)^{\log}\varphi(x)| \leq C_{N,\varphi} A \frac{e^{-|x|}}{(1+|x|)^{\frac{N+1}{2}}} \qquad \text{for all } x \in \mathbb{R}^N.$$

From the above computations, we have that $|\langle (I-\Delta)^{\log} u, \varphi\rangle| \leq C_{N,\varphi}\|\varphi\|_{C^\alpha(\mathbb{R}^N)}\|u\|_{L_0(\mathbb{R}^N)}$ and if the sequence $\{u_n\}_n$ converges to u in $L_0(\mathbb{R}^N)$ as $n \to \infty$ then

$$|\langle (I-\Delta)^{\log} u_n - (I-\Delta)^{\log} u, \varphi\rangle| \leq C_{N,\varphi} A \|u_n - u\|_{L_0(\mathbb{R}^N)} \to 0 \quad \text{ as } n \to \infty.$$

Proof of (iii). This follows from (i) and the inequality

$$|2u(x) - u(x+y) - u(x-y)| \leq \|u\|_{C^\alpha(B_r(0))}|y|^\alpha \qquad \text{for } \; y \in B_{r/2}(0).$$

Proof of (iv). Let $0 < r < 1$ be small. We have the following estimate of the difference,

$$|(I-\Delta)^{\log}u(x_1) - (I-\Delta)^{\log}u(x_2)| \leq d_N(I_1 + I_2)$$

where I_1 and I_2 are given by

$$I_1 := \int\limits_{B_r} \frac{|u(x_1) - u(x_1+y)| + |u(x_2) - u(x_2+y)|}{|y|^N}\omega(|y|)\, dy$$

$$I_2 := \int\limits_{\mathbb{R}^N\setminus B_r} \frac{|u(x_1) - u(x_2)| + |u(x_1+y) - u(x_2+y)|}{|y|^N}\omega(|y|)\, dy$$

For I_1, we use the inequality $|u(x_1) - u(x_1+y)| \leq \|u\|_{C^\beta(\mathbb{R}^N)}|y|^\beta$ to get

$$I_1 \leq 2\|u\|_{C^\beta(\mathbb{R}^N)} \int\limits_{B_r} |y|^{\beta-N}\omega(|y|)\, dx \leq \frac{2\omega_{N-1}\Gamma(N/2)}{\beta}\|u\|_{C^k(\mathbb{R}^N)} r^\beta$$

For I_2, we use $|u(x_1) - u(x_2)| + |u(x_1+y) - u(x_2+y)| \leq 2\|u\|_{C^\beta(\mathbb{R}^N)}|x_1 - x_2|^\beta$ and,

$$\begin{aligned}
I_2 &\leq 2|x_1 - x_2|^\beta \|u\|_{C^\beta(\mathbb{R}^N)}\Big(\int\limits_{B_1\setminus B_r} \frac{\omega(|y|)}{|y|^N}\, dy + \int\limits_{\mathbb{R}^N\setminus B_1} \frac{\omega(|y|)}{|y|^N}\, dy\Big) \\
&\leq 2|x_1 - x_2|^\beta \|u\|_{C^\beta(\mathbb{R}^N)}\Big(\Gamma(N/2)\int\limits_{B_1\setminus B_r} \frac{1}{|y|^N}\, dy + \int\limits_{\mathbb{R}^N\setminus B_1} \frac{e^{-|y|}}{|y|^{\frac{N+1}{2}}}\, dy\Big) \\
&\leq 2|x_1 - x_2|^\beta \|u\|_{C^\beta(\mathbb{R}^N)}\omega_{N-1}\Big(\Gamma(N/2)\log\frac{1}{r} + \Gamma(N,1)\Big) \\
&\leq 2|x_1 - x_2|^\beta \omega_{N-1}\|u\|_{C^\beta(\mathbb{R}^N)}\Big(\frac{\Gamma(N/2)r^{-\varepsilon}}{\varepsilon} + \Gamma(N,1)\Big),
\end{aligned}$$

where we have used the inequality $\log(\rho) \leq \frac{\rho^{\varepsilon}}{\varepsilon}$ for $\varepsilon > 0$ and $\rho \geq 1$ (see [59]). Therefore, taking $r = |x_1 - x_2|$, we ends with

$$|(I-\Delta)^{\log}u(x_1) - (I-\Delta)^{\log}u(x_2)| \leq C(N,\beta,\varepsilon)\|u\|_{C^{\beta}(\mathbb{R}^N)}|x_1 - x_2|^{\beta-\varepsilon}.$$

Proof of (v). This easily follows by integrating the following equality

$$(\varphi(x)\psi(x) - \varphi(y)\psi(y)) = (\varphi(x)-\varphi(y))\psi(x) + (\psi(x)-\psi(y))\varphi(x) - (\varphi(x)-\varphi(y))(\psi(x)-\psi(y)),$$

while the second statement is an application of Fubini's theorem. This completes the proof of Proposition 5.13. □

We next list some properties for functions belonging to the space $H^{\log}(\mathbb{R}^N)$.

Lemma 4.8. *The following assertions hold true*

1. *If $u \in H^{\log}(\mathbb{R}^N)$, then $|u|, u^{\pm} \in H^{\log}(\mathbb{R}^N)$ with $\||u|\|_{H^{\log}(\mathbb{R}^N)}, \|u^{\pm}\|_{H^{\log}(\mathbb{R}^N)} \leq \|u\|_{H^{\log}(\mathbb{R}^N)}$.*
2. *The space $\mathscr{C}_c^{0,\alpha}(\mathbb{R}^N) \subset H^{\log}(\mathbb{R}^N)$ for any $\alpha > 0$.*
3. *If $\varphi \in \mathscr{C}_c^{0,\alpha}(\mathbb{R}^N)$ and $u \in H^{\log}(\mathbb{R}^N)$, then $\varphi u \in H^{\log}(\mathbb{R}^N)$ and there a constant $C := C(N,\varphi) > 0$ such that*
$$\|\varphi u\|^2_{H^{\log}(\mathbb{R}^N)} \leq C\|u\|^2_{H^{\log}(\mathbb{R}^N)}$$

Proof. It straightforward to see by integrating the inequality

$$||u(x)| - |u(y)|| \leq |u(x) - u(y)|$$

that $\mathscr{E}_{\omega}(|u|,|u|) \leq \mathscr{E}_{\omega}(u,u,)$ and $\||u|\|_{H^{\log}(\mathbb{R}^N)} \leq \|u\|_{H^{\log}(\mathbb{R}^N)}$. Using also the inequality

$$2(u^+(x) - u^+(y))(u^-(x) - u^-(y)) = -2(u^-(x)u^+(y) + u^-(y)u^+(x)) \leq 0 \quad \text{for } x,y \in \mathbb{R}^N,$$

it follows that

$$\mathscr{E}_{\omega}(u,u) = \mathscr{E}_{\omega}(u^+,u^+) + \mathscr{E}_{\omega}(u^-,u^-) - 2\mathscr{E}_{\omega}(u^+,u^-) \geq \mathscr{E}_{\omega}(u^+,u^+) + \mathscr{E}_{\omega}(u^-,u^-),$$

proving clearly that the first item holds. Now, for the second item, we let $u \in \mathscr{C}_c^{0,\alpha}(\mathbb{R}^N)$ be such that supp $u \subset B_r$, $r > 0$. without loss of generality we may assume that $0 < r < 1$ such that we can directly apply the asymptotics in (4.9). We therefore have

$$\begin{aligned}
\mathscr{E}_{\omega}(u,u) &= \frac{1}{2}\int_{B_r}\int_{B_r} |u(x)-u(y)|^2 J(x,y)\,dxdy + \int_{B_r} u^2(x) \int_{\mathbb{R}^N\setminus B_r} J(x-y)\,dydx \\
&\leq C_1 \int_{B_r}\int_{B_r} |x-y|^{2\alpha-N} dxdy + C_2 \int_{B_r} u^2(x)\Big(\int_{B_1\setminus B_r} |x-y|^{-N}\,dy \\
&\qquad + \int_{\mathbb{R}^N\setminus B_1} e^{-|x-y|} dy\Big) dx \leq C\frac{|B_r(0)|}{2\alpha} r^{-2\alpha} + C_3,
\end{aligned}$$

where the constants $C := C(N) > 0$, $C_2 := C_2(r,N) > 0$ and $C_3 := C_3(r,N) > 0$. The second item is proved. We next prove item 3. Let $u \in H^{\log}(\mathbb{R}^N)$ and $\varphi \in \mathscr{C}_c^{0,\alpha}(\mathbb{R}^N)$ with supp $\varphi \subset B_r$, for $0 < r < 1$. Then using the inequality

$$|\varphi(x)u(x) - \varphi(y)u(y)|^2 \leq 2(|u(x) - u(y)|^2|\varphi(x)|^2 + |u(y)|^2|\varphi(x) - \varphi(y)|^2),$$

we get

$$\begin{aligned}
\mathscr{E}_\omega(u,u) \leq & \int_{B_r}\int_{B_r} |\varphi(x)|^2|u(x) - u(y)|^2 J(x,y)dxdy \\
& + 2\int_{B_r} u^2(x) \int_{B_r} |\varphi(x) - \varphi(y)|^2 J(x-y)dydx \\
& + C\int_{B_r} |\varphi(x)u(x)|^2 \Big(\int_{B_1\setminus B_r} |x-y|^{-N}\, dy + \int_{\mathbb{R}^N\setminus B_1} e^{-|x-y|}dy\Big)dx \\
\leq & \; 2\|\varphi\|^2_{L^\infty(\mathbb{R}^N)}\mathscr{E}_\omega(u,u) + C_2\int_{B_r} u^2(x)\int_{B_r} |x-y|^{2\alpha-N}dydx + C_3 < \infty.
\end{aligned}$$

Since $\|\varphi u\|_{L^2(\mathbb{R}^N)} \leq C_\varphi\|u\|_{L^2(\mathbb{R}^N)}$, we have that $\varphi u \in H^{\log}(\mathbb{R}^N)$ and item 3 is proved. $\square$

We recall the space $\mathscr{H}_0^0(\Omega)$, corresponding to the analytical framework for the logarithmic Laplacian L_Δ introduced in [29], see also [47], given by

$$\mathscr{H}_0^0(\Omega) = \Big\{ u \in L^2(\mathbb{R}^N) : u \equiv 0 \text{ on } \Omega^c \text{ and } \iint_{\substack{x,y\in\mathbb{R}^N \\ |x-y|<1}} \frac{|u(x)-u(y)|^2}{|x-y|^N}dxdy < \infty \Big\}. \tag{4.29}$$

Here $\Omega^c = \mathbb{R}^N \setminus \Omega$, and the map

$$(u,v) \mapsto \langle u,v\rangle_{\mathscr{H}_0^0(\Omega)} := \frac{C_N}{2}\iint_{\substack{x,y\in\mathbb{R}^N \\ |x-y|<1}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^N}\, dxdy,$$

is a scalar product on $\mathscr{H}_0^0(\Omega)$. The space $\mathscr{H}_0^0(\Omega)$ is a Hilbert space with induced norm $\|\cdot\|_{\mathscr{H}_0^0(\Omega)} = \langle\cdot,\cdot\rangle^{\frac{1}{2}}_{\mathscr{H}_0^0(\Omega)}$. Moreover, The space $C_c^2(\Omega)$ is dense in $\mathscr{H}_0^0(\Omega)$ and

$$\text{the embedding } \; \mathscr{H}_0^0(\Omega) \hookrightarrow L^2(\Omega) \; \text{ is compact.}$$

We have the following Lemma

Lemma 4.9. *(i) the space $H^{\log}(\mathbb{R}^N)$ is a Hilbert space and, $H^m(\mathbb{R}^N) \subset H^{\log}(\mathbb{R}^N)$ for all $m > 0$.*

(ii) If $\Omega \subset \mathbb{R}^N$ is an open set with finite measure then we have the following Poincaré inequality with $C := C(N,\Omega)$

$$\|u\|^2_{L^2(\Omega)} \le C \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^2 J(x-y)dxdy, \qquad u \in \mathscr{H}_0^{\log}(\Omega) \tag{4.30}$$

(iii) If $\Omega \subset \mathbb{R}^N$ is bounded, then there a constant $C_j := C(N,\Omega)$, $j = 1,2$ such that

$$C_1\mathscr{E}_\omega(u,u) \le \|u\|^2_{\mathscr{H}_0^0(\Omega)} \le C_2\mathscr{E}_\omega(u,u)$$

(iv) The space $\mathscr{C}_c^\infty(\Omega)$ is dense in $\mathscr{H}_0^{\log}(\Omega)$ and

$$\text{the embedding } \mathscr{H}_0^{\log}(\Omega) \hookrightarrow L^2(\Omega) \text{ is compact.} \tag{4.31}$$

Proof. Let $\{u_n\}_n \subset H^{\log}(\mathbb{R}^N)$ be a Cauchy sequence. Then $\{u_n\}_n$ is in particular a Cauchy sequence in $L^2(\mathbb{R}^N)$ and hence there exists a $u \in L^2(\mathbb{R}^N)$ such that $u_n \to u$ as $n \to \infty$. Passing to a subsequence we get that $u_n \to u$ a.e in $\mathbb{R}^N$ as $n \to \infty$ and by Fatou Lemma we have

$$\mathscr{E}_\omega(u,u) \le \liminf_{n\to\infty} \mathscr{E}_\omega(u_n,u_n) \le \sup_{n\in\mathbb{N}} \mathscr{E}_\omega(u_n,u_n) < \infty,$$

showing that $u \in H^{\log}(\mathbb{R}^N)$. Apply once more Fatou Lemma it follows that

$$\|u_n-u\|^2_{H^{\log}(\mathbb{R}^N)} = \|u_n-u\|^2_{L^2(\mathbb{R}^N)} + \mathscr{E}_\omega(u_n-u,u_n-u) \le \liminf_{n\to\infty} \|u_n-u_m\|^2_{H^{\log}(\mathbb{R}^N)},$$

for $n,m \in \mathbb{N}$. The claim follows since $\{u_n\}_n$ is a Cauchy sequence in $H^{\log}(\mathbb{R}^N)$.
By Plancherel thereon the norm in $H^{\log}(\mathbb{R}^N)$ is also given via Fourier representation

$$\|u\|_{H^{\log}(\mathbb{R}^N)} = \Big(\|u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \log(1+|\xi|^2)|\mathscr{F}(u)(\xi)|^2\, d\xi\Big)^{\frac{1}{2}}.$$

Threfore, using the standard inequality $\log\rho \le \frac{\rho^m}{m}$ for $\rho \ge 1$ for $m > 0$ (see e.g. [59]) one see that the space $H^{\log}(\mathbb{R}^N)$ is larger than any Sobolev space $H^m(\mathbb{R}^N) := W^{m,2}(\mathbb{R}^N)$. In fact if $u \in H^m(\mathbb{R}^N)$ then the proof of (i) is completed by the following inequality,

$$\begin{aligned}\|u\|^2_{H^{\log}(\mathbb{R}^N)} &= \|u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \log(1+|\xi|^2)|\mathscr{F}(u)(\xi)|^2 d\xi \\ &\le \|u\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{m}\int_{\mathbb{R}^N} (1+|\xi|^2)^m|\mathscr{F}(u)(\xi)|^2 d\xi \le C_m\|u\|^2_{H^m(\mathbb{R}^N)}.\end{aligned} \tag{4.32}$$

The Poincaré inequality in (ii) follows from [43, Lemma 2.9] and [61] if Ω is bounded or bounded in one direction. We provide the proof here for $\Omega \subset \mathbb{R}^N$ with $|\Omega| < \infty$. Since $u = 0$ in $\mathbb{R}^N \setminus \Omega$, we first have by Hölder inequality that

$$|\hat{u}(\xi)|^2 \le (2\pi)^{-N}|\Omega|\|u\|^2_{L^2(\Omega)} \qquad \text{for every} \quad \xi \in \mathbb{R}^N.$$

Next, by Plancherel theorem and every $R > 0$, we get

$$\|u\|^2_{L^2(\Omega)} = \int_{\mathbb{R}^N} |\hat{u}(\xi)|^2\, d\xi = \int_{|\xi|<R} |\hat{u}(\xi)|^2\, d\xi + \int_{|\xi|\geq R} \log(1+|\xi|^2)|\hat{u}(\xi)|^2 \log(1+|\xi|^2)^{-1}\, d\xi$$
$$\leq (2\pi)^{-N} R^N |\Omega||B_1(0)|\|u\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2\log(1+R^2)} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(x)-u(y))^2 J(x-y)\, dxdy.$$

Therefore, choosing $R < 2\pi(|\Omega||B_1(0)|)^{-\frac{1}{N}} = 2\pi\Big(\frac{N}{\omega_{N-1}|\Omega|}\Big)^{\frac{1}{N}}$ we find that

$$\|u\|^2_{L^2(\Omega)} \leq \frac{2}{\log(1+R^2)\Big(1-(2\pi)^{-N}R^N|\Omega||B_1(0)|\Big)} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(x)-u(y))^2 J(x-y)\, dxdy.$$

The proof of (ii) follows here by minimizing in R the coefficient in the right hand side.
For item (iii), we use the asymptotics in (4.9) to get

$$\|u\|^2_{\mathscr{H}_0^0(\Omega)} = \frac{1}{2} \iint_{\substack{x,y\in\mathbb{R}^N \\ |x-y|<1}} \frac{|u(x)-u(y)|^2}{|x-y|^N}\, dxdy \leq C_1 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^N} \omega(|x-y|)\, dxdy.$$

Next, using Poincaré inequality for $\mathscr{H}_0^0(\Omega)$ again with (4.9) we get that

$$\mathscr{E}_\omega(u,u) = \frac{d_N}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^N} \omega(|x-y|)\, dxdy$$
$$\leq \Gamma(\frac{N}{2}) \iint_{\substack{x,y\in\Omega \\ |x-y|<1}} \frac{|u(x)-u(y)|^2}{|x-y|^N}\, dxdy + 2\int_\Omega |u(x)|^2 \int_{\Omega\cap\{|x-y|\geq 1\}} \omega(|x-y|)\, dydx$$
$$+ \int_\Omega |u(x)|^2 \int_{\mathbb{R}^N\setminus\Omega} \frac{\omega(|x-y|)}{|x-y|^N}\, dydx \ \leq C_2 \|u\|_{\mathscr{H}_0^0(\Omega)}$$

with

$$C_2 := C\Big(1 + \sup_{x\in\Omega}\Big(\int_{\mathbb{R}^N\setminus\Omega} \frac{\omega(|x-y|)}{|x-y|^N}\, dy + \int_{\Omega\cap\{|x-y|\geq 1\}} \omega(|x-y|)\, dy\Big)\Big) < \infty.$$

The proof of (iv) follows from [29, Theorem 3.1] and (iii) since The space $C_c^\infty(\Omega)$ is dense in $\mathscr{H}_0^0(\Omega)$ and the embedding $\mathscr{H}_0^0(\Omega) \hookrightarrow L^2(\Omega)$ is compact. The proof ends here. □

As consequence of the Poincaré inequality, we have for bounded Ω with continuous boundary that the space $\mathscr{H}_0^{\log}(\Omega)$ can be identified by

$$\mathscr{H}_0^{\log}(\Omega) = \Big\{u \in H^{\log}(\mathbb{R}^N): \quad u \equiv 0 \ \text{ on } \ \mathbb{R}^N \setminus \Omega\Big\}.$$

and it is a Hilbert space endowed with the scalar product $(v,w) \mapsto \mathscr{E}_\omega(v,w)$ and the corresponding norm $\|u\|_{\mathscr{H}_0^{\log}(\Omega)} = \sqrt{\mathscr{E}_\omega(u,u)}$.

4.3 Eigenvalue problem

In this section, we provide the proof of Theorem 4.3, proposition 4.4 and Theorem 4.5 concerning the study of the Dirichlet eigenvalue problem in bounded open set Ω,

$$\begin{cases} (I-\Delta)^{\log} u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.33}$$

We start with the

Proof of Theorem 4.3. Let $\Psi : \mathscr{H}_0^{\log}(\Omega) \to \mathbb{R}$ be the functional defined by

$$\Psi(u) := \mathscr{E}_\omega(u,u) = \|u\|^2_{\mathscr{H}_0^{\log}(\Omega)}.$$

We use the direct method of minimization. Let $\{u_n\}_{n\in\mathbb{N}}$ be a minimizing sequence for Ψ in $\mathscr{P}_1(\Omega) := \{u \in \mathscr{H}_0^{\log}(\Omega) : \|u\|_{L^2(\Omega)} = 1\}$, that is

$$\lim_{n\to\infty} \Psi(u_n) = \inf_{u\in\mathscr{P}_1(\Omega)} \Psi(u) \geq 0 > -\infty.$$

Then by the definition of Ψ, the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathscr{H}_0^{\log}(\Omega)$ and up to subsequence, there exists $u_0 \in \mathscr{H}_0^{\log}(\Omega)$ such that thanks to (4.31),

$$u_n \rightharpoonup u_0 \quad \text{weakly} \quad \text{in } \mathscr{H}_0^{\log}(\Omega) \tag{4.34}$$

$$u_n \to u_0 \quad \text{strongly} \quad \text{in } L^2(\Omega). \tag{4.35}$$

It follows from (4.35) that $\|u_0\|_{L^2(\Omega)} = 1$ and that $u_0 \in \mathscr{P}_1(\Omega)$. Using the lower-semi-continuity of the norm in $\mathscr{H}_0^{\log}(\Omega)$, we deduce that

$$\inf_{u\in\mathscr{P}_1(\Omega)} \Psi(u) = \lim_{n\to\infty} \Psi(u_n) \geq \Psi(u_0) \geq \inf_{u\in\mathscr{P}_1(\Omega)} \Psi(u).$$

This yields that $\Psi(u_0) = \inf_{u\in\mathscr{P}_1(\Omega)} \Psi(u)$ and, the first eigenvalue is $\lambda_1(\Omega) = \Psi(u_0)$, with the corresponding eigenfunction $\varphi_1 = u_0 \in \mathscr{P}_1(\Omega)$. By the Lagrange multipliers theorem, there exists $\lambda \in \mathbb{R}$ such that

$$\mathscr{E}_\omega(\varphi_1, v) = \langle \Psi'(\varphi_1), v\rangle = \lambda \int_\Omega \varphi_1 v\, dx \quad \text{for all} \quad v \in \mathscr{H}_0^{\log}(\Omega). \tag{4.36}$$

Taking in particular $v = \varphi_1$, we find that $\lambda = \lambda_1(\Omega) = \mathscr{E}_\omega(\varphi_1,\varphi_1)$. We next show that φ_1 does not change sign in Ω. Indeed, since $\mathscr{E}_\omega(|v|,|v|) \leq \mathscr{E}_\omega(v,v)$ for $v \in \mathscr{H}_0^{\log}(\Omega)$, it follows that $|\varphi_1| \in \mathscr{P}_1(\Omega)$ and by the definition of $\lambda_1(\Omega)$ we have that

$$\lambda_1(\Omega) = \mathscr{E}_\omega(|\varphi_1|, |\varphi_1|),$$

showing that φ_1 does not change sign in Ω. We may assume that φ_1 is nonnegative. Suppose then that $\varphi_1(x_0) = 0$ for some $x_0 \in \Omega$. Then

$$0 = \lambda_1(\Omega)\varphi_1(x_0) = -d_N \int_{\mathbb{R}^N} \frac{\varphi_1(x_0)}{|x-y|^N} \omega(|x-y|)\, dy < 0$$

which yields a contradiction. Therefore $\varphi_1 > 0$ in Ω and (i) is proved.
We prove (ii) via contradiction. Suppose that there exists a function $v \in \mathscr{P}_1(\Omega)$ satisfying $(I-\Delta)^{\log} v = \lambda_1 v$ with $v \neq \alpha\varphi_1$ for every $\alpha \in \mathbb{R}$. Then $w := v - \alpha\varphi_1$ satisfies also $(I-\Delta)^{\log} w = \lambda_1 w$. But since $\varphi_1 > 0$ in Ω, by choosing $\alpha = \frac{v(x_0)}{\varphi_1(x_0)}$, $x_0 \in \Omega$, it follows that w vanishes at $x_0 \in \Omega$ and therefore must change sign. This contradicts (i) and thus the eigenvalue $\lambda_1(\Omega)$ is simple.
We prove (iii) by induction. We first note that, if follows from the simplicity of $\lambda_1(\Omega)$ in (ii) that $\lambda_1(\Omega) < \lambda_2(\Omega)$. By the same construction as in the case $k = 1$, we get a sequence of eigenfunctions $\varphi_2, \cdots, \varphi_k \in \mathscr{H}_0^{\log}(\Omega)$ and eigenvalues $\lambda_2(\Omega) \leq \cdots \leq \lambda_k(\Omega)$, $k \in \mathbb{N}$ with the properties that

$$\lambda_j(\Omega) = \inf_{u \in \mathscr{P}_j(\Omega)} \mathscr{E}_\omega(u,u) = \mathscr{E}_\omega(\varphi_j, \varphi_j), \qquad j = 1, \cdots, k \quad \text{and}$$

$$\mathscr{E}_\omega(\varphi_j, v) = \lambda_j(\Omega) \int_\Omega \varphi_j v\, dx \quad \text{for all} \quad v \in \mathscr{H}_0^{\log}(\Omega).$$

Next, we define $\lambda_{k+1}(\Omega)$ as in (4.18), that is

$$\lambda_{k+1}(\Omega) = \inf_{u \in \mathscr{P}_{k+1}(\Omega)} \mathscr{E}_\omega(u,u).$$

By the same argument as above, the value $\lambda_{k+1}(\Omega)$ is attained by a function $\varphi_{k+1} \in \mathscr{P}_{k+1}(\Omega)$ and by the Lagrange multipliers theorem, there exists $\lambda \in \mathbb{R}$ such that

$$\mathscr{E}_\omega(\varphi_{k+1}, v) = \lambda \int_\Omega \varphi_{k+1} v\, dx \quad \text{for all} \quad v \in \mathscr{P}_{k+1}(\Omega). \tag{4.37}$$

Taking in particular $v = \varphi_{k+1}$ in (4.37), we get that $\lambda = \lambda_{k+1}(\Omega)$. Moreover, for $j = 1, \cdots k$, it follows from the definition of $\mathscr{P}_{k+1}(\Omega)$ and taking $v = \varphi_j$ in (4.37), we find that

$$\mathscr{E}_\omega(\varphi_{k+1}, \varphi_j) = 0 = \lambda_j(\Omega) \int_\Omega \varphi_{k+1}\varphi_j\, dx. \tag{4.38}$$

In other to conclude that φ_{k+1} is an eigenfunction corresponding to eigenvalue $\lambda_{k+1}(\Omega)$, we need to show that (4.37) holds for all $v \in \mathscr{H}_0^{\log}(\Omega)$. To see this we write

$$\mathscr{H}_0^{\log}(\Omega) = span\{\varphi_1, \cdots, \varphi_k\} \oplus \mathscr{P}_{k+1}(\Omega)$$

such that any $v \in \mathscr{H}_0^{\log}(\Omega)$ can be written as $v = v_1 + v_2$ with $v_1 \in span\{\varphi_1, \cdots, \varphi_k\}$ and $v_2 \in \mathscr{P}_{k+1}(\Omega)$. It follows from (4.37) with v replaced by $v_2 = v - v_1 \in \mathscr{P}_{k+1}(\Omega)$ that

$$0 = \mathscr{E}_\omega(\varphi_{k+1}, v_2) - \lambda_{k+1}(\Omega) \int_\Omega \varphi_{k+1} v_2\, dx$$

$$= \mathscr{E}_\omega(\varphi_{k+1}, v) - \mathscr{E}_\omega(\varphi_{k+1}, v_1) - \lambda_{k+1}(\Omega) \int_\Omega \varphi_{k+1}(v - v_1)\, dx$$
$$= \mathscr{E}_\omega(\varphi_{k+1}, v) - \lambda_{k+1}(\Omega) \int_\Omega \varphi_{k+1} v\, dx,$$

where we used equality in (4.38). This shows that (4.37) holds for all $v \in \mathscr{H}_0^{\log}(\Omega)$. We have just constructed inductively an L^2-normalized sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ in $\mathscr{H}_0^{\log}(\Omega)$ and a nondecreasing sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}$ such that (4.18) holds and such that φ_k is an eigenfunction of (4.15) corresponding to $\lambda = \lambda_k(\Omega)$ for every $k \in \mathbb{N}$. Moreover, we have by construction that $\{\varphi_k\}_{k\in\mathbb{N}}$ form an orthogonal system in $L^2(\Omega)$. To complete the proof of (iii), it remains to show that $\lim_{k\to+\infty} \lambda_k(\Omega) = +\infty$. Suppose by contradiction that

$$\mathscr{E}_\omega(\varphi_k, \varphi_k) = \lambda_k(\Omega) \to c_0 \in \mathbb{R} \quad \text{as} \quad k \to +\infty \qquad \text{for every } k \in \mathbb{N}.$$

Then the sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ is bounded in $\mathscr{H}_0^{\log}(\Omega)$ and, up to subsequence, there is $\varphi_0 \in \mathscr{H}_0^{\log}(\Omega)$ such that

$$\varphi_k \to \varphi_0 \quad \text{in} \quad L^2(\Omega) \quad \text{as } k \to +\infty.$$

It follows in particular that $\{\varphi_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega)$. But orthogonality in $L^2(\Omega)$ implies that $\|\varphi_k - \varphi_j\|_{L^2(\Omega)} = 2$ for every k and j, which leads to a contradiction.
For the proof of assertion (iv), the orthogonality follows from from (iii). we then need to show that the sequence of eigenfunctions $\{\varphi_k\}_{k\in\mathbb{N}}$ is a basis for both $L^2(\Omega)$ and $\mathscr{H}_0^{\log}(\Omega)$. Let suppose by contradiction that there exists a nontrivial $u \in \mathscr{H}_0^{\log}(\Omega)$ with

$$\|u\|_{L^2(\Omega)} = 1 \;\text{ and } \int_\Omega \varphi_k u\, dx = 0 \;\text{ for any } k \in \mathbb{N}\,. \tag{4.39}$$

Since we have that $\lim\limits_{k\to+\infty} \lambda_k(\Omega) = +\infty$, there exists an integer $k_0 > 0$ such that

$$\Psi(u) < \lambda_{k_0}(\Omega) = \inf_{v\in\mathscr{P}_{k_0}(\Omega)} \Psi(v).$$

This implies that $u \notin \mathscr{P}_{k_0}(\Omega)$ and, by the definition of $\mathscr{P}_{k_0}(\Omega)$, we have that $\int_\Omega \varphi_j u\, dx \neq 0$ for some $j \in \{1, \cdots, k_0 - 1\}$. This contradicts (4.39). We conclude that $\mathscr{H}_0^{\log}(\Omega)$ is contained in the L^2-closure of the span of $\{\varphi_k : k \in \mathbb{N}\}$. Since $\mathscr{H}_0^{\log}(\Omega)$ is dense in $L^2(\Omega)$, we conclude that the span of $\{\varphi_k : k \in \mathbb{N}\}$ is dense in $L^2(\Omega)$, and hence, the sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$. This complete the proof of Theorem 4.3. □

We next give the

Proof of Proposition 4.4. We work here with the δ-decomposition of the nonlocal operators as described in [47, Theorem 3.1]. For this, let $\Omega \subset \mathbb{R}^N$ be open and bounded set of $\mathbb{R}^N$. For $\delta > 0$, we let $J_\delta := 1_{B_\delta} J$ and $K_\delta := J - J_\delta$. Note that for $u, v \in \mathscr{H}_0^{\log}(\Omega)$,

$$\mathscr{E}_\omega(u, v) = \mathscr{E}_\omega^\delta(u, v) + \frac{d_N}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(x) - u(y))(v(x) - v(y)) K_\delta(x - y)\, dxdy$$
$$= \mathscr{E}_\omega^\delta(u, v) + \kappa_\delta \langle u, v\rangle_{L^2(\mathbb{R}^N)} - \langle K_\delta * u, v\rangle_{L^2(\mathbb{R}^N)}$$

where the δ-dependent quadratic form $\mathscr{E}_\omega^\delta$ is given by

$$(u,v) \mapsto \mathscr{E}_\omega^\delta(u,v) = \frac{d_N}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (u(x)-u(y))(v(x)-v(y)) J_\delta(x-y)\, dxdy,$$

the function $K_\delta \in L^1(\mathbb{R}^N)$ and the constant κ_δ is

$$\kappa_\delta = \int_{\mathbb{R}^N} K_\delta(z)\, dz > \int_{B_1 \setminus B_\delta} \frac{1}{|z|^N}\, dz = -c_N \ln \delta \to +\infty \quad \text{as} \quad \delta \to 0.$$

Next, let $c > 0$ be a constant to be chosen later. Consider the function $w_c = (u-c)^+ : \Omega \to \mathbb{R}$. Then $w_c \in \mathscr{H}_0^{\log}(\Omega)$ by Lemma 4.8 see also [60, Lemma 3.2]. Moreover, for $x, y \in \mathbb{R}^N$ we have that $(u(x)-u(y))(w_c(x)-w_c(y)) \geq (w_c(x)-w_c(y))^2$. Indeed,

$$\begin{aligned}
&(u(x)-u(y))(w_c(x)-w_c(y)) = ([u(x)-c]-[u(y)-c])(w_c(x)-w_c(y))\\
&= [u(x)-c]w_c(x) + [u(y)-c]w_c(y) - [u(x)-c]w_c(y) - w_c(x)[u(y)-c]\\
&= w_c^2(x) + w_c^2(y) - 2w_c(x)w_c(y) + [u(x)-c]^- w_c(y) + w_c(x)[u(y)-c]^-\\
&\geq w_c^2(x) + w_c^2(y) - 2w_c(x)w_c(y) = (w_c(x)-w_c(y))^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
\mathscr{E}_\omega^\delta(w_c, w_c) &= \frac{d_N}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (w_c(x)-w_c(y))^2 J_\delta(x-y)\, dxdy\\
&\leq \frac{d_N}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (u(x)-u(y))(w_c(x)-w_c(y)) J_\delta(x-y)\, dxdy\\
&= \mathscr{E}_\omega(u, w_c) - \kappa_\delta \langle u, w_c \rangle_{L^2(\Omega)} + \langle K_\delta * u\, , w_c \rangle_{L^2(\Omega)} \qquad (4.40)\\
&\leq \big(\lambda - \kappa_\delta\big) \langle u, w_c \rangle_{L^2(\Omega)} + \|K_\delta * u\|_{L^\infty(\mathbb{R}^N)} \langle 1\, , w_c \rangle_{L^2(\Omega)}.
\end{aligned}$$

Note that $\kappa_\delta \to +\infty$ as $\delta \to 0$. Hence, we may fix $\delta > 0$ such that $\lambda + \kappa_\delta < -1$. Moreover, with this choice of δ, together with the trivial inequality $u(x)w_c(x) \geq c w_c(x)$ for $x \in \Omega$, we infer that

$$\begin{aligned}
\mathscr{E}_\omega^\delta(w_c, w_c) &\leq \int_\Omega (\|K_\delta * u\|_{L^\infty(\mathbb{R}^N)} - c) w_c\, dx\\
&\leq \int_\Omega (c_{N,\delta} \|u\|_{L^2(\mathbb{R}^N)} - c) w_c\, dx.
\end{aligned} \qquad (4.41)$$

The quantity $c_{N,\delta}\|u\|_{L^2(\mathbb{R}^N)}$ is obtained in the following computation using Höder's (or Young's) inequality combined with the asymptotics in (4.9),

$$\|k_\delta * u\|_{L^\infty(\mathbb{R}^N)} \leq c_{N,\delta} \|u\|_{L^2(\mathbb{R}^N)}.$$

We then deduce from (4.41) with $c > c_{N,\delta}\|u\|_{L^2(\mathbb{R}^N)}$ that

$$0 \leq \mathscr{E}_\omega^\delta(w_c, w_c) \leq 0, \qquad (4.42)$$

which implies that $\mathscr{E}^{\delta}_{\omega}(w_c, w_c) = 0$. Consequently, $w_c = 0$ in Ω by the Poincaré type inequality. But then $u(x) \le c \quad a.e.$ in Ω, and therefore

$$u(x) \le c_N \|u\|_{L^2(\mathbb{R}^N)}.$$

Repeating the above argument for $-u$ in place of u , we conclude that

$$\|u\|_{L^\infty(\Omega)} \le c\|u\|_{L^2(\mathbb{R}^N)}.$$

This complete the proof of Proposition 4.4. □

For the proof of Theorem 4.5, we first state a Polya-Szegö type inequality for $(I-\Delta)^{\log}$.

Lemma 4.10. *Let u^* be the symmetric radial decreasing rearrangement of u. Then,*

$$\mathscr{E}_{\omega}(u^*, u^*) \le \mathscr{E}_{\omega}(u, u). \tag{4.43}$$

Moreover, the equality occurs for radial decreasing functions. Here,

Proof. By a changes of variable, we write the kernel J as

$$J(z) = d_N |z|^{-N} \omega(|z|) = 4(\frac{\pi}{2})^{-\frac{N}{2}} \int_0^\infty e^{-t|z|^2} t^{\frac{N}{2}-1} e^{-\frac{1}{4t}}\, dt.$$

Then by Fubuni's theorem, we write the quadratic form as

$$\mathscr{E}_{\omega}(u,u) = \frac{1}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^2 J(x,y)\, dxdy = 2(\frac{\pi}{2})^{-\frac{N}{2}} \int_0^\infty G(t,u)\, t^{\frac{N}{2}-1} e^{-\frac{1}{4t}}\, dt,$$

where,

$$G(t,u) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^2 e^{-t|x-y|^2}\, dxdy.$$

Noticing that

$$\left(e^{-t|z|^2}\right)^* = e^{-t|z|^2}, \quad \text{for all} \quad t \ge 0,$$

It follows from [3, corollary 2.3 and Theorem 9.2] see also [49, Theorem A_1] that

$$G(t,u^*) \le G(t,u) \quad \text{for all} \quad t \ge 0.$$

This gives that

$$\mathscr{E}_{\omega}(u^*, u^*) \le \mathscr{E}_{\omega}(u,u) \quad \text{for} \quad u \in H^{\log}(\mathbb{R}^N). \tag{4.44}$$

The proof of Lemma 4.10 is completed. □

Proof of Theorem 4.5. This is a direct consequence of lemma 4.10 and the characterization of the first eigenvalue $\lambda_{1,\log}(\Omega)$ of $(I-\Delta)^{\log}$ in Ω. Since we know by Theorem 4.3 that the first eigenfunction $\varphi_{1,\log}$ corresponding to $\lambda_{1,\log}(\Omega)$ is unique and strictly positive in Ω, we have thanks to Lemma 4.10 that

$$\lambda_{1,\log}(\Omega) = \frac{\mathscr{E}_\omega(\varphi_{1,\log},\varphi_{1,\log})}{\|\varphi_{1,\log}\|^2_{L^2(\Omega)}} \geq \frac{\mathscr{E}_\omega(\varphi^*_{1,\log},\varphi^*_{1,\log})}{\|\varphi^*_{1,\log}\|^2_{L^2(B^*)}} \geq \inf_{u\in\mathscr{H}_0^{\log}(B^*)} \frac{\mathscr{E}_\omega(u,u)}{\|u\|^2_{L^2(B^*)}} = \lambda_{1,\log}(B^*),$$

where we have used (see [22, Lemma 3.3]) the fact that

$$\int_\Omega |u|^2\,dx = \int_{B^*} |u^*|^2\,dx.$$

This gives the proof of (4.19). For the equality, if we suppose that $\lambda_{1,\log}(\Omega) = \lambda_{1,\log}(B^*)$ with $|\Omega| = |B^*|$, then we must have the following equality

$$\mathscr{E}_L(\varphi_{1,\log},\varphi_{1,\log}) = \mathscr{E}_L(\varphi^*_{1,\log},\varphi^*_{1,\log})$$

and by [49, Lemma A_2] we deduce that the first eigenfunction $\varphi_{1,\log}$ has to be proportional to a translate of a radially symmetric decreasing function such that the level set

$$\Omega_0 := \{x\in\mathbb{R}^N: \quad \varphi_{1,\log} > 0\}$$

is a ball. Since $\varphi_{1,\log} > 0$ in Ω by definition and it is unique, it follows that Ω must coincide with Ω_0 and has to be a ball. The proof of Theorem 4.5 is then completed. □

4.4 Small order Asymptotics

This section is dedicated to the proof of Theorem 4.6. We first introduce some notions and preliminary lemmas that shall be used. For $0 < s < 1$, we introduce the Sobolev space (see [85, 92])

$$H^s(\mathbb{R}^N) = \Big\{u\in L^2(\mathbb{R}^N): \quad \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\omega_s(|x-y|)dxdy < \infty\Big\}$$

with corresponding norm given by

$$\|u\|_{H^s(\mathbb{R}^N)} = \Big(\|u\|_{L^2}(\mathbb{R}^N) + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\omega_s(|x-y|)dxdy\Big)^{\frac{1}{2}}.$$

Let $\Omega\subset\mathbb{R}^N$ be an open bounded set. We will use the fact that (see [92])

$$\text{the space } C_c^2(\Omega) \text{ is dense in } \mathscr{H}_0^s(\Omega),$$

where the space $\mathscr{H}_0^s(\Omega)$ is the completion of $\mathscr{C}_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^s(\mathbb{R}^N)}$. We start with the following Dirichlet eigenvalue problem

$$\begin{cases} (I-\Delta)^s u = \lambda u & \text{in } \Omega \\ \quad u = 0 & \text{on } \mathbb{R}^N\setminus\Omega, \end{cases} \tag{4.45}$$

where Ω is a bounded Lipschitz open set of $\mathbb{R}^N$. We define the first Dirichlet eigenvalue of $(I-\Delta)^s$ in Ω by

$$\lambda_{1,s}(\Omega) = \inf_{u \in C_c^2(\Omega)} \frac{\mathscr{E}_{\omega,s}(u,u)}{\|u\|_{L^2(\Omega)}} = \inf_{\substack{u \in C_c^2(\Omega) \\ \|u\|_{L^2(\Omega)}=1}} \mathscr{E}_{\omega,s}(u,u), \tag{4.46}$$

where the quadratic form $(u,v) \mapsto \mathscr{E}_{\omega,s}(u,v)$ is defined by

$$\begin{aligned}\mathscr{E}_{\omega,s}(u,v) &= \int_\Omega u(x)v(x)\,dx - \frac{d_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\omega_s(|x-y|)dxdy \\ &= \int_{\mathbb{R}^N} (1+|\xi|^2)^s \mathscr{F}(u)(\xi)\mathscr{F}(u)(\xi)\,d\xi.\end{aligned}$$

By the Courant-Fischer minimax principle, the eigenvalues $\lambda_{k,s}(\Omega)$, $k \in \mathbb{N}$ can be characterized equivalently as

$$\lambda_{k,s}(\Omega) = \inf_{\substack{V \subset \mathscr{H}_0^s(\Omega) \\ \dim V = k}} \max_{\substack{v \in V\setminus\{0\} \\ \|v\|_{L^2(\Omega)}=1}} \mathscr{E}_{\omega,s}(v,v) = \inf_{\substack{V \subset C_c^2(\Omega) \\ \dim V = k}} \max_{\substack{v \in V\setminus\{0\} \\ \|v\|_{L^2(\Omega)}=1}} \mathscr{E}_{\omega,s}(v,v). \tag{4.47}$$

Remark 4.11. Noticing that $(1+|\xi|^2)^s \geq |\xi|^{2s}$ for $s \in (0,1)$ and $\xi \in \mathbb{R}^N$, we have via the Fourier transform of the functional $\mathscr{E}_{\omega,s}(\cdot,\cdot)$ for $(I-\Delta)^s$ and $\mathscr{E}_s(\cdot,\cdot)$ for the fractional Laplacian $(-\Delta)^s$ that

$$\lambda_{k,s}(\Omega) = \mathscr{E}_{\omega,s}(\psi_{k,s},\psi_{k,s}) \geq \mathscr{E}_s(\psi_{k,s},\psi_{k,s}) \geq \inf_{\substack{v \in C_c^2(\Omega) \\ \|v\|_{L^2(\Omega)}=1}} \mathscr{E}_s(v,v) = \lambda_{1,s}^F(\Omega),$$

where $\psi_{k,s}$ is a L^2-normalized eigenfunction of $(I-\Delta)^s$ corresponding to $\lambda_{k,s}(\Omega)$ and $\lambda_{1,s}^F(\Omega)$ is the first Dirichlet eigenvalue of the fractional Laplacian $(-\Delta)^s$ in Ω with

$$\mathscr{E}_s(u,v) := \frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy.$$

We need the following elementary estimates and inequalities.

Lemma 4.12. *For $s \in (0,1)$ and $r > 0$ we have*

$$\Big|\frac{(1+r^2)^s-1}{s}\Big| \leq 2\Big(1+r^4\Big) \tag{4.48}$$

and

$$\Big|\frac{(1+r^2)^s-1}{s} - \log(1+r^2)\Big| \leq 2s\Big(1+r^4\Big). \tag{4.49}$$

Consequently, for every $u \in C_c^2(\Omega)$ and $s \in (0,1)$ we have

$$\Big|\mathscr{E}_{\omega,s}(u,u) - \|u\|^2_{L^2(\mathbb{R}^N)}\Big| \leq 2s\Big(\|u\|^2_{L^2(\mathbb{R}^N)} + \|\Delta u\|^2_{L^2(\mathbb{R}^N)}\Big) \tag{4.50}$$

and

$$\Big|\mathscr{E}_{\omega,s}(u,u) - \|u\|^2_{L^2(\mathbb{R}^N)} - s\mathscr{E}_\omega(u,u)\Big| \leq 2s^2\Big(\|u\|^2_{L^2(\mathbb{R}^N)} + \|\Delta u\|^2_{L^2(\mathbb{R}^N)}\Big) \tag{4.51}$$

Proof. For fix $r > 0$, let $h_r(s) = (1+r^2)^s$. Then we have

$$h_r'(\tau) = (1+r^2)^\tau \ln(1+r^2) \quad \text{and} \quad h_r''(\tau) = (1+r^2)^\tau \ln^2(1+r^2).$$

Consequently, since $(1+r^2)^s \leq (1+r^2)$ for $s \in (0,1)$ and $\ln(1+r^2) \leq (1+r^2)$,

$$\Big|\frac{(1+r^2)^s - 1}{s}\Big| = \frac{\ln(1+r^2)}{s} \int_0^s (1+r^2)^\tau \, d\tau \leq \ln(1+r^2)(1+r^2)^s \leq 2\Big(1+r^4\Big)$$

where in the last step we used that $(1+r^2)^2 \leq 2(1+r^4)$ for $r > 0$. Hence (4.48) holds. Moreover, by Taylor expansion,

$$h_r(s) = 1 + s\ln(1+r^2) + \ln^2(1+r^2) \int_0^s (1+r^2)^\tau (s-\tau) d\tau$$

and therefore

$$\Big|\frac{(1+r^2)^s - 1}{s} - \log(1+r^2)\Big| \leq \frac{\ln^2(1+r^2)}{s} \Big|\int_0^s (1+r^2)^\tau (s-\tau) d\tau\Big| \leq s(1+r^2)^s \ln^2(1+r^2).$$

But since $\ln^2(1+r^2) \leq (1+r^2)$ and $(1+r^2)^s \leq (1+r^2)$ for $s \in (0,1)$, (4.49) follows. Next, let $u \in C_c^2(\Omega)$ and $s \in (0,1)$. By (4.48) and Fourier transform for $\mathscr{E}_{\omega,s}$, we have

$$\begin{aligned}
\Big|\mathscr{E}_{\omega,s}(u,u) - \|u\|^2_{L^2(\mathbb{R}^N)}\Big| &\leq \int_{\mathbb{R}^N} \big|(1+|\xi|^2)^s - 1\big| \, |\hat{u}(\xi)|^2 \, d\xi \\
&\leq 2s \int_{\mathbb{R}^N} \left(1+|\xi|^4\right) |\hat{u}(\xi)|^2 \, d\xi \leq 2s\Big(\|u\|^2_{L^2(\mathbb{R}^N)} + \|\Delta u\|^2_{L^2(\mathbb{R}^N)}\Big).
\end{aligned}$$

Thus (4.50) follows. Moreover, by (4.49) we have

$$\begin{aligned}
\Big|\mathscr{E}_{\omega,s}(u,u) - \|u\|^2_{L^2(\mathbb{R}^N)} - s\mathscr{E}_\omega(u,u)\Big| &\leq \int_{\mathbb{R}^N} \big|(1+|\xi|^2)^s - 1 - s\log(1+|\xi|^2)\big| \, |\hat{u}(\xi)|^2 \, d\xi \\
&= s\int_{\mathbb{R}^N} \Big|\frac{(1+|\xi|^2)^s - 1}{s} - \log(1+|\xi|^2)\Big| \, |\hat{u}(\xi)|^2 \, d\xi \\
&\leq 2s^2\Big(\|u\|^2_{L^2(\mathbb{R}^N)} + \|\Delta u\|^2_{L^2(\mathbb{R}^N)}\Big).
\end{aligned}$$

Hence (4.51) follows. This completes the proof of Lemma 4.12. □

Lemma 4.13. *For all $k \in \mathbb{N}$ we have*

$$\lambda_{k,s}(\Omega) \leq 1 + sC \qquad \textit{for all } s \in (0,1) \tag{4.52}$$

with a constant $C = C(N,\Omega,k) > 0$*, and*

$$\limsup_{s\to 0^+} \frac{\lambda_{k,s}(\Omega)-1}{s} \le \lambda_{k,\log}(\Omega). \tag{4.53}$$

Consequently,

$$\lim_{s\to 0^+} \lambda_{k,s}(\Omega) = 1 \qquad \text{for all } k \in \mathbb{N}. \tag{4.54}$$

Proof. We fix a subspace $V \subset C_c^2(\Omega)$ of dimension k and let $S_V := \{u \in V \ : \ \|u\|_{L^2(\Omega)} = 1\}$. Using (4.47) and (4.50), we find that, for $s \in (0,1)$,

$$\frac{\lambda_{k,s}(\Omega)-1}{s} \le \max_{u\in S_V} \frac{\mathscr{E}_{\omega,s}(u,u)-1}{s} \le C \tag{4.55}$$

with

$$C = C(N,\Omega,k) = 2\max_{u\in S_V}\Big(\|u\|^2_{L^2(\mathbb{R}^N)} + \|\Delta u\|^2_{L^2(\mathbb{R}^N)}\Big).$$

Hence (4.52) holds. Moreover, setting $\mathscr{R}_s(u) = \frac{\mathscr{E}_{\omega,s}(u,u)-1}{s} - \mathscr{E}_\omega(u,u)$ for $u \in C_c^2(\Omega)$, we deduce from (4.55) that

$$\frac{\lambda_{k,s}(\Omega)-1}{s} \le \max_{u\in S_V} \mathscr{E}_\omega(u,u) + \max_{u\in S_V} |\mathscr{R}_s(u)|$$

while, by (4.51),

$$|\mathscr{R}_s(u)| \le 2s\Big(\|u\|^2_{L^2(\mathbb{R}^N)} + \|\Delta u\|^2_{L^2(\mathbb{R}^N)}\Big) \to 0 \quad \text{as } s \to 0^+ \text{ uniformly in } u \in S_V.$$

Consequently,

$$\limsup_{s\to 0^+} \frac{\lambda_{k,s}(\Omega)-1}{s} \le \max_{u\in S_V} \mathscr{E}_\omega(u,u).$$

Since V was chosen arbitrarily, the characterization of the Dirichlet eigenvalues of $(I-\Delta)^{\log}$ given in (4.47) implies that

$$\limsup_{s\to 0^+} \frac{\lambda_{k,s}(\Omega)-1}{s} \le \inf_{\substack{V\subset C_c^2(\Omega)\\ \dim(V)=k}} \max_{\substack{u\in V\\ \|u\|_{L^2(\Omega)}=1}} \mathscr{E}_\omega(u,u) = \lambda_{k,\log}(\Omega). \tag{4.56}$$

This shows that the inequality in (4.53) holds. It follows directly from (4.52) that

$$\limsup_{s\to 0^+} \lambda_{k,s}(\Omega) \le 1 \qquad \text{for all } k \in \mathbb{N}.$$

From Remark 4.11 we have that $\lambda_{k,s}(\Omega) \ge \lambda^F_{1,s}(\Omega)$. It therefore follows from [47, Lemma 2.8] that

$$\liminf_{s\to 0^+} \lambda_{k,s}(\Omega) \ge 1 \qquad \text{for all } k \in \mathbb{N}.$$

This proves (4.54) and the proof of Lemma 4.13 is completed. □

Lemma 4.14. *Let $k \in \mathbb{N}$. If $\psi_{k,s} \in \mathscr{H}_0^s(\Omega)$ denote an L^2-normalized eigenfunction of $(I-\Delta)^s$, then the set*

$$\{\psi_{k,s} : s \in (0,1)\}$$

is uniformly bounded in $\mathscr{H}_0^{\log}(\Omega)$ and therefore relatively compact in $L^2(\Omega)$.

Proof. To ease notation, we set $\psi_s := \psi_{k,s}$, the k-th L^2-normalized eigenfunction corresponding to $\lambda_{k,s}(\Omega)$, $k \in \mathbb{N}$. By (4.54), there exits a constant $C = C(N,\Omega,k) > 0$ such that

$$\begin{aligned} C \geq \frac{\lambda_{k,s}(\Omega)-1}{s} = \frac{\mathscr{E}_{\omega,s}(\psi_s,\psi_s)-1}{s} &= \int_{\mathbb{R}^N} \frac{(1+|\xi|^2)^s-1}{s}|\psi_s(\xi)|^2\,d\xi \\ &= \int_0^1\int_{\mathbb{R}^N} \log(1+|\xi|^2)|\psi_s(\xi)|^2(1+|\xi|^2)^{st}\,d\xi dt \\ &\geq \frac{1}{2}\int_0^1\int_{\mathbb{R}^N} \log(1+|\xi|^2)|\psi_s(\xi)|^2\,d\xi dt = \frac{1}{2}\mathscr{E}_{\omega}(\psi_s,\psi_s). \end{aligned}$$

Therefore, there exist a constant $M := M(\Omega,k,N) > 0$ such that

$$\sup_{s\in(0,1)} \|\psi_s\|_{\mathscr{H}^{\log}(\Omega)} \leq M \tag{4.57}$$

We conclude from (4.57) that ψ_s remains uniformly bounded in $\mathscr{H}_0^{\log}(\Omega)$ for $s \in (0,1)$. Consequently $\{\psi_{k,s} : s \in (0,1)\}$ is uniformly bounded in $\mathscr{H}_0^{\log}(\Omega)$ and relatively compact in $L^2(\Omega)$ since we have from (4.31) that $\mathscr{H}_0^{\log}(\Omega) \hookrightarrow L^2(\Omega)$ is compact. □

We now give the

Proof of Theorem 4.6. The proof follows the idea in article [47, Theorem 2.10] by the author combined with [29, Theorem 3.5]. It then suffices, in view of Lemma 4.13, to consider an arbitrary sequence $(s_n)_n \subset (0,1)$ with $\lim_{n\to\infty} s_n = 0$, and to show that, after passing to a subsequence,

$$\lim_{n\to\infty} \frac{\lambda_{k,s_n}(\Omega)-1}{s} = \lambda_{k,\log}(\Omega) \qquad \text{for } k \in \mathbb{N}. \tag{4.58}$$

Let $\{\psi_{k,s_n} : k \in \mathbb{N}\}$ be an orthonormal system of eigenfunctions corresponding to the Dirichlet eigenvalue $\lambda_{k,s_n}(\Omega)$ of $(I-\Delta)^{s_n}$. By Lemma 4.14, it follows that, for every $k \in \mathbb{N}$, the sequence of functions ψ_{k,s_n}, $n \in \mathbb{N}$ is bounded in $\mathscr{H}_0^{\log}(\Omega)$ and relatively compact in $L^2(\Omega)$. Consequently, we may pass to a subsequence such that, for every $k \in \mathbb{N}$,

$$\psi_{k,s_n} \rightharpoonup \psi^\star_{k,\log} \text{ weakly in } \mathscr{H}_0^{\log}(\Omega) \text{ and } \psi_{k,s_n} \to \psi^\star_{k,\log} \text{ strongly in } L^2(\Omega) \quad \text{as } n \to \infty. \tag{4.59}$$

Moreover, by Lemma 4.13, we may, after passing again to a subsequence if necessary, assume that, for every $k \in \mathbb{N}$,

$$\frac{\lambda_{k,s_n}(\Omega)-1}{s_n} \to \lambda_k^\star \in \Big[-\infty, \lambda_{k,\log}(\Omega)\Big] \qquad \text{as } n \to \infty. \tag{4.60}$$

To prove then (4.58), it now suffices to show that

$$\lambda_{k,\log}(\Omega) = \lambda_k^\star \qquad \text{for every } k \in \mathbb{N}. \tag{4.61}$$

It follows from (4.59) that

$$\|\psi_{k,\log}^\star\|_{L^2(\Omega)} = 1 \quad \text{and} \quad \langle \psi_{k,\log}^\star, \psi_{\ell,\log}^\star\rangle_{L^2(\Omega)} = 0 \qquad \text{for } k,\ell \in \mathbb{N},\, \ell \neq k. \tag{4.62}$$

Moreover, for $v \in C_c^2(\Omega)$ and $n \in \mathbb{N}$, we have from Theorem 4.3 that

$$\mathscr{E}_{\omega,s_n}(\psi_{k,s_n}, v) = \lambda_{k,s_n}(\Omega)\langle \psi_{k,s_n}, v\rangle_{L^2(\Omega)} \tag{4.63}$$

and therefore, rearranging (4.63), it follows from (i) in Theorem 4.1 with $p = 2$ that

$$\begin{aligned}
\lim_{n\to\infty} \frac{\lambda_{k,s_n}(\Omega) - 1}{s_n} \langle \psi_{k,s_n}, v\rangle_{L^2(\Omega)} &= \lim_{n\to\infty} \frac{1}{s_n}\Big(\mathscr{E}_{\omega,s_n}(\psi_{k,s_n}, v) - \langle \psi_{k,s_n}, v\rangle_{L^2(\Omega)}\Big) \\
&= \lim_{n\to\infty} \Big\langle \psi_{k,s_n}, \frac{(I-\Delta)^{s_n} v - v}{s_n}\Big\rangle_{L^2(\Omega)} \\
&= \langle \psi_{k,\log}^\star, (I-\Delta)^{\log} v\rangle_{L^2(\Omega)} = \mathscr{E}_\omega(\psi_{k,\log}^\star, v).
\end{aligned} \tag{4.64}$$

Since moreover $\langle \psi_{k,s_n}, v\rangle_{L^2(\Omega)} \to \langle \psi_{k,\log}^\star, v\rangle_{L^2(\Omega)}$ as $n \to \infty$ for any $k \in \mathbb{N}$ and $v \in C_c^2(\Omega)$, in particular, for $k = 1$, we may choose $v \in C_c^2(\Omega)$ such that $\langle \psi_{1,\log}^\star, v\rangle_{L^2(\Omega)} > 0$. It follows from (4.60) and (4.64) that $\lambda_1^\star$ satisfies $-\infty < \lambda_1^\star \le \lambda_{1,\log}(\Omega)$ and

$$\mathscr{E}_\omega(\psi_{1,\log}^\star, v) = \lambda_1^\star \langle \psi_{1,\log}^\star, v\rangle_{L^2(\Omega)} \qquad \text{for all } v \in \mathscr{H}_0^{\log}(\Omega). \tag{4.65}$$

Thus $\psi_{1,\log}^\star$ is an eigenfunction of $(I-\Delta)^{\log}$ corresponding to the eigenvalue $\lambda_1^\star$. Since $\lambda_1^\star \le \lambda_{1,\log}(\Omega)$, it follows from the definition of the principal eigenvalue (4.17) that $\lambda_1^\star = \lambda_{1,\log}(\Omega)$ and then $\lambda_{1,\log}(\Omega) = \lambda_1^\star \le \liminf\limits_{s\to 0^+} \dfrac{\lambda_{1,s}(\Omega) - 1}{s}$. From the uniqueness of the first eigenfunction, we get that $\psi_{1,\log}^\star = \psi_{1,\log}$ is the nonnegative L^2-normalized eigenfunction of $(I-\Delta)^{\log}$ corresponding to $\lambda_{1,\log}(\Omega)$. In short, we have just shown that as $s \to 0^+$,

$$\frac{\lambda_{1,s}(\Omega) - 1}{s} \to \lambda_{1,\log}(\Omega) \quad \text{and} \quad \psi_{1,s} \to \psi_{1,\log} \quad \text{in} \quad L^2(\Omega).$$

This completes the proof for $k = 1$. Now for $k \ge 2$, it still follows from (4.60) and (4.64) that

$$\mathscr{E}_\omega(\psi_{k,\log}^\star, v) = \lambda_k^\star \langle \psi_{k,\log}^\star, v\rangle_{L^2(\Omega)} \qquad \text{for all } v \in C_c^2(\Omega), \tag{4.66}$$

where $\psi_{k,\log}^\star$ is a Dirichlet eigenfunction of $(I-\Delta)^{\log}$ corresponding to $\lambda_k^\star$, now with

$$\lambda_k^\star \in [\lambda_{1,\log}(\Omega), \lambda_{k,\log}(\Omega)]. \tag{4.67}$$

Next, for fixed $k \in \mathbb{N}$ we consider $E_k^\star := \operatorname{span}\{\psi_{1,\log}^\star, \psi_{2,\log}^\star, \cdots, \psi_{k,\log}^\star\}$, which is a k-dimensional subspace of $\mathscr{H}_0^{\log}(\Omega)$ by (4.62). Since

$$\lambda_1^\star \le \lambda_2^\star \le \ldots \le \lambda_k^\star$$

as a consequence of (4.67) and since $\lambda_{i,s_n}(\Omega) \leq \lambda_{j,s_n}(\Omega)$ for $1 \leq i \leq j \leq k$, $n \in \mathbb{N}$, we have the following estimate for every $v = \sum_{i=1}^{k} \alpha_i \psi^{\star}_{i,\log} \in E^{\star}_k$ with $\alpha_1, \cdots, \alpha_k \in \mathbb{R}$:

$$\mathscr{E}_\omega(v,v) = \sum_{i,j=1}^{k} \alpha_i \alpha_j \mathscr{E}_\omega(\psi^{\star}_{i,\log}, \psi^{\star}_{j,\log}) = \sum_{i,j=1}^{k} \alpha_i \alpha_j \lambda^{\star}_i \langle \psi^{\star}_{i,\log}, \psi^{\star}_{j,\log} \rangle_{L^2(\Omega)} \tag{4.68}$$

$$= \sum_{i=1}^{k} \alpha_i^2 \lambda^{\star}_i \|\psi^{\star}_{i,\log}\|^2_{L^2(\Omega)} \leq \lambda^{\star}_k \sum_{i=1}^{k} \alpha_i^2 = \lambda^{\star}_k \|v\|^2_{L^2(\Omega)}. \tag{4.69}$$

The characterization in (4.47) now yields that

$$\lambda_{k,\log}(\Omega) \leq \max_{\substack{v \in E^{\star}_k \\ \|v\|_{L^2(\Omega)}=1}} \mathscr{E}_\omega(v,v) \leq \lambda^{\star}_k.$$

Since also $\lambda^{\star}_k \leq \lambda_{k,\log}(\Omega)$ by (4.60), the equality in (4.61) follows. We thus conclude that (4.58) holds and also (4.20) follows. Moreover, the statement (4.22) of the theorem follows a posteriori from the equality $\lambda^{\star}_k = \lambda_{k,\log}(\Omega)$, since we have already seen that $\psi_{k,s_n} \to \psi^{\star}_{k,\log}$ in $L^2(\Omega)$, the proof is thus finished here. □

4.5 Decay Estimates

This section deals with the proof of Proposition 4.2 concerning the decay estimates at infinity and at zero of the solution u corresponding to Poisson problem,

$$(I-\Delta)^{\log} u = f \qquad \text{in} \quad \mathbb{R}^N. \tag{4.70}$$

The fundamental solution of equation (4.70) can be is given in term of the Green function $G : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ (see(4.10)) defined by

$$G(x) = \int_0^\infty \frac{1}{\Gamma(t)} \int_0^\infty p_s(x) s^{t-1} e^{-s} \, ds dt,$$

We have in the sense of distributional that $\mathscr{F}(G)(\xi) = \frac{1}{\log(1+|\xi|^2)}, \quad \xi \in \mathbb{R}^N \setminus \{0\}$. Indeed, for $\varphi \in S$, we have by Fubini's theorem that

$$\begin{aligned}\int_{\mathbb{R}^N} G(\xi)\mathscr{F}(\varphi)(\xi)\, d\xi &= \int_0^\infty \frac{1}{\Gamma(t)} \int_0^\infty \int_{\mathbb{R}^N} p_s(\xi)\mathscr{F}(\varphi)(\xi)\, d\xi s^{t-1} e^{-s} \, ds dt \\ &= \int_{\mathbb{R}^N} \int_0^\infty \frac{1}{\Gamma(t)} \int_0^\infty e^{-s(1+|\xi|^2)} s^{t-1} \, ds dt \; \varphi(\xi)\, d\xi \\ &= \int_{\mathbb{R}^N} \int_0^\infty (1+|\xi|^2)^{-t} dt \; \varphi(\xi)\, d\xi = \int_{\mathbb{R}^N} \frac{1}{\log(1+|\xi|^2)} \varphi(\xi)\, d\xi,\end{aligned}$$

and then
$$\mathscr{F}^{-1}\Big(\frac{1}{\log(1+|\xi|^2)}\Big)(x) = G(x) \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}.$$
We then define the solution u of equation (4.70) for a $f \in \mathscr{C}_c^\infty(\mathbb{R}^N)$ by
$$u(x) = [G * f](x) = \int_{\mathbb{R}^N} G(x-y)f(y)\,dy \quad \text{for } x \in \mathbb{R}^N. \tag{4.71}$$
This follows from the property of Fourier transform and convolution since
$$\mathscr{F}(u) = \mathscr{F}(G)\mathscr{F}(f) \quad \text{and} \quad \log(1+|\xi|^2)\mathscr{F}(u) = \log(1+|\xi|^2)\mathscr{F}(G)\mathscr{F}(f) = \mathscr{F}(f).$$
We now give the

Proof of Proposition 4.2. For $|x|$ small, We split the integral representation of G in two pieces as follows
$$G_1(x) = \frac{2^{1-N}}{\pi^{N/2}} \int_0^{\frac{N}{2}} \frac{1}{\Gamma(t)} \left(\frac{|x|}{2}\right)^{t-\frac{N}{2}} K_{t-\frac{N}{2}}(|x|)\,dt$$
and
$$G_2(x) = \frac{2^{1-N}}{\pi^{N/2}} \int_{\frac{N}{2}}^{\infty} \frac{1}{\Gamma(t)} \left(\frac{|x|}{2}\right)^{t-\frac{N}{2}} K_{t-\frac{N}{2}}(|x|)\,dt.$$
Since $t \le \frac{N}{2}$, it follows from the asymptotics property (4.23) for K_ν (see [50]) that as $|x| \to 0$,
$$K_{t-\frac{N}{2}}(|x|) \sim 2^{|t-\frac{N}{2}|-1}\Gamma(|t-\frac{N}{2}|)|x|^{-|t-\frac{N}{2}|} \sim \begin{cases} 2^{\frac{N}{2}-t-1}\Gamma(N/2-t)|x|^{-\frac{N}{2}+t} & \text{if} \quad t < \frac{N}{2}, \\ \log\frac{1}{|x|} \quad \text{if} \quad t = \frac{N}{2}. \end{cases}$$
Plugging the above approximations in G_1, we end up with
$$G_1(x) \sim \begin{cases} \dfrac{2^{1-N}}{\pi^{N/2}} \log\dfrac{1}{|x|} & \text{as} \quad |x| \to 0 \quad \text{if} \quad t = \dfrac{N}{2} \\ \dfrac{2^N}{\pi^{N/2}}|x|^{-N} \displaystyle\int_0^{\frac{N}{2}} \frac{\Gamma(N/2-t)}{4^t\Gamma(t)}\,dt & \text{as} \quad |x| \to 0 \quad \text{if} \quad t < \dfrac{N}{2}, \end{cases} \tag{4.72}$$
where we have used that since $N > 2t$, $|x|^{-N+2t} \sim |x|^{-N}$ as $|x| \to 0$. Since also $t < \frac{N}{2}$, we have $\int_0^{\frac{N}{2}} \frac{\Gamma(N/2-t)}{4^t\Gamma(t)}\,dt < \infty$. Now, for $t > \frac{N}{2}$, again by using (4.23), we have
$$K_{t-\frac{N}{2}}(|x|) \sim 2^{t-\frac{N}{2}-1}\Gamma(t-\frac{N}{2})|x|^{-t+\frac{N}{2}} \quad \text{as} \quad |x| \to 0.$$
Taken the above approximations into account, we get the approximation for G_2,
$$G_2(x) \sim \frac{2^{-N}}{\pi^{N/2}} \int_{\frac{N}{2}}^{\infty} \frac{\Gamma(t-N/2)}{\Gamma(t)}\,dt \quad \text{as} \quad |x| \to 0 \tag{4.73}$$

Since $\lim\limits_{t\to+\infty} \dfrac{\Gamma(t-\frac{N}{2})}{4^t\Gamma(t)} = 0$ and $t > \frac{N}{2}$, we infer that $\int_{2N}^{\infty} \frac{\Gamma(t-\frac{N}{2})}{\Gamma(t)}\, dt < \infty$. Therefore, combining the approximations of G_1 in (4.72) and G_2 in (4.73) we get

$$|x|^N G(x) \sim \frac{2^N}{\pi^{N/2}} \int\limits_0^{\frac{N}{2}+1} \frac{\Gamma(N/2-t)}{4^t\Gamma(t)}\, dt \quad \text{as} \quad |x| \to 0.$$

We next investigate the case with the modulus of $|x|$ large. From the asymptotics property (4.23) we have for all $t \geq 0$ that

$$\begin{aligned} |x|^{t-\frac{N}{2}} K_{t-\frac{N}{2}}(|x|) &\sim \frac{\pi^{\frac{1}{2}}}{\sqrt{2}} |x|^{-\frac{N+1}{2}+t} e^{-|x|} \quad \text{as} \quad |x| \to \infty \\ &\sim \frac{\pi^{\frac{1}{2}}}{\sqrt{2}} |x|^{-\frac{N+1}{2}} e^{-|x|} \quad \text{as} \quad |x| \to \infty. \end{aligned}$$

From this, we infer that

$$G(x) \sim\ 2^{-\frac{N+1}{2}} \pi^{-\frac{N-1}{2}} |x|^{-\frac{N+1}{2}} e^{-|x|} \int\limits_0^{\infty} \frac{1}{2^t\Gamma(t)}\, dt \quad \text{as} \quad |x| \to \infty.$$

Noticing that $\lim\limits_{t\to 0} 1/(2^t\Gamma(t)) = 0 = \lim\limits_{t\to+\infty} 1/(2^t\Gamma(t))$, the above integral is finite and

$$\int\limits_0^{\infty} \frac{1}{2^t\Gamma(t)}\, dt \sim 1.$$

We therefore infer that

$$G(x) \sim\ 2^{-\frac{N+1}{2}} \pi^{-\frac{N-1}{2}} |x|^{-\frac{N+1}{2}} e^{-|x|} \quad \text{as } |x| \to \infty.$$

For $f \in L^1(\mathbb{R}^N)$, we write

$$u(x) = \int\limits_{\mathbb{R}^N} G(x-y) f(y)\, dy = \int\limits_{\mathbb{R}^N} G(y) f(x-y)\, dy$$

First observe that if $f \geq 0$, we have that

$$u(x) \geq \int\limits_{B(x,|x|)} G(x-y) f(y)\, dy \geq Ce^{-|x|} \int\limits_{B(x,|x|)} f(y)\, dy.$$

Since $B(x,|x|) \to \mathbb{R}^N$ as $|x| \to \infty$ and $f \in L^1(\mathbb{R}^N)$, we see that $u(x) = O(e^{-|x|})$ as $|x| \to \infty$. Moreover, Since $G(x)$ decays as $e^{-|x|}$ at infinity, there exists a constant $M > 0$ such that

$$\|e^{|\cdot|} G\|_{L^\infty(\mathbb{R}^N)} < C \quad \text{for} \quad |x| \geq M,$$

where $C > 0$ is a positive constant. We then write

$$e^{|x|}u(x) = [e^{|\cdot|}G * f](x) = \int_{\mathbb{R}^N} e^{|y|}G(y)f(x-y)\,dy.$$

Thus,

$$\begin{aligned}|e^{|x|}u(x)| \le \Big|\int_{\mathbb{R}^N} e^{|y|}G(y)f(x-y)\,dy\Big| &\le \|e^{|\cdot|}G\|_{L^\infty(\mathbb{R}^N)}\int_{\mathbb{R}^N}|f(x-y)|\,dy\\ &\le C\|f\|_{L^1(\mathbb{R}^N)}.\end{aligned}$$

This allows to conclude that $u(x)$ decays as $e^{-|x|}$ at infinity, that is

$$u(x) = O(e^{-|x|}) \qquad \text{as} \quad |x| \to \infty.$$

As before, there exists $\delta > 0$ such that

$$\||\cdot|^N u\|_{L^\infty(\mathbb{R}^N)} < C \quad \text{for } |x| < \delta.$$

Therefore,

$$||x|^N u(x)| \le C\int_{\mathbb{R}^N}|f(x-y)|\,dy \le C\|f\|_{L^1(\mathbb{R}^N)}.$$

This allows to conclude that

$$u(x) = O(|x|^{-N}) \quad \text{as} \quad |x| \to 0.$$

This completes the proof of Theorem 4.2. □

4.6 Additional remarks

We present in this section some results concerning the logarithmic Schrödinger operator $(I-\Delta)^{\log}$ that can be directly deduced from known results in the literature. For this fact, we introduce the following space $\mathscr{V}_\omega(\Omega)$, being the space of all functions $u \in L^2_{loc}(\mathbb{R}^N)$ such that

$$\rho(u,\Omega) := \int_\Omega\int_{\mathbb{R}^N}\frac{(u(x)-u(y))^2}{|x-y|^N}\omega(|x-y|)\,dxdy < \infty.$$

Then the quantity $\mathscr{E}_\omega(u,v)$ is well defined for $u \in \mathscr{H}_0^{\log}(\Omega)$ and $v \in \mathscr{V}_\omega(\Omega)$ (see [60, Lemma 3.1]). The proof of the following results on the maximum principle for the operator $(I-\Delta)^{\log}$ on an open set Ω of $\mathbb{R}^N$ can be deduced from [60].

Theorem 4.15. *(i) (Strong maximum principle) Let $\Omega \subset \mathbb{R}^N$ be a bounded subset and $u \in \mathscr{L}_0(\mathbb{R}^N)$ be a continuous function on $\overline{\Omega}$ satisfying*

$$(I-\Delta)^{\log}u \ge 0 \quad in \quad \Omega, \qquad u \ge 0 \quad in \quad \mathbb{R}^N\setminus\Omega.$$

Then $u > 0$ in Ω or $u \equiv 0$ a.e. in $\mathbb{R}^N$.

(ii) (Weak maximum principle) Let $u \in \mathscr{V}_\omega(\Omega)$ *with* $(I-\Delta)^{\log} u \geq 0$ *in* Ω *weakly and* $u \geq 0$ *in* $\mathbb{R}^N \setminus \Omega$. *Then* $u \geq 0$ *in* $\mathbb{R}^N$.

(iii) (Small volume maximum principle)There exists $\delta > 0$ *such that for every open bounded set* Ω *of* $\mathbb{R}^N$ *with* $|\Omega| \leq \delta$ *and every function* $u \in \mathscr{V}_\omega(\Omega)$ *satisfying*

$$(I-\Delta)^{\log} u \geq c(x)u \quad \text{in } \Omega \quad \text{and} \quad u \geq 0 \ \text{ in } \quad \mathbb{R}^N \setminus \Omega,$$

with $c \in L^\infty(\mathbb{R}^N)$, *then* $u \geq 0$ *in* $\mathbb{R}^N$.

We recall that, $u \in \mathscr{V}_\omega(\Omega)$ satisfies $(I-\Delta)^{\log} u \geq 0$ in Ω weakly means,

$$\mathscr{E}_\omega(u, \varphi) \geq 0 \text{ for all nonnegative } \varphi \in \mathscr{C}_c^\infty(\Omega).$$

Next, consider the following semilinear elliptic problem involving the operator $(I-\Delta)^{\log}$ in a bounded set Ω of $\mathbb{R}^N$,

$$(I-\Delta)^{\log} u = f(x,u) \quad \text{in} \quad \Omega \qquad u = 0 \text{ on } \mathbb{R}^N \setminus \Omega, \tag{4.74}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous. The following result on the radially symmetry of the solution can deduced from [62]

Theorem 4.16. *Assume that* f *is locally Lipschitz with respect to the second variable and radially symmetry and strictly decreasing in* $r = |x|$. *Then every positive solution of* (4.74) *is radially symmetry and strictly decreasing in* $|x|$.

5 Nonlocal operators of small order

This chapter focus particularly on singular integral operators with order strictly less than one. Exploiting the variational structure of the associated Poisson problem, we study corresponding spaces and investigate regularity properties of weak solutions depending on the regularity of the right-hand side. The content below is in the same form as in the paper [46] except the missing of acknowledgements. It is based on results from work done in collaboration with Sven Jarohs.

5.1 Introduction and Main results

In the following we let $k:\mathbb{R}^N\times\mathbb{R}^N\to[0,\infty]$ be such that

$$\begin{aligned} k(x,y)=k(y,x) \quad &\text{for all } x,y\in\mathbb{R}^N\text{, and there exists } \sigma\in(0,1) \text{ such that}\\ &\sup_{x\in\mathbb{R}^N}\int_{\mathbb{R}^N}\min\{1,|x-y|^{\sigma}\}k(x,y)\,dy<\infty \end{aligned} \tag{5.1}$$

and we refer to k as a *nonlocal kernel (function of small order)*. Note here, that the integrability assumption on k is usually done with $\sigma=2$. Here, this assumption ensures that k is associated to an operator of order strictly below one. In particular, we are interested in the study of opertators with order near zero. Motivated by some applications to nonlocal models, where a small order of the operator captures the optimal efficiency of the model [4, 81], nonlocal operators with possibly differential order close to zero have been studied in linear and nonlinear integro-differential equations, see [29, 30, 43, 45, 47, 86, 94] and references in there. From a stochastic point of view, general classes of nonlocal operators appear as the generator of jump processes, where the jump behavior is modelled through types of *Lévy measures* and properties of associated *harmonic* functions have been studied, see [56, 58, 63, 79] and there references in there. In particular, operators of the form $\varphi(-\Delta)$ for certain classes of functions φ are of interest from a stochastic and analytic point of view, see e.g. [14, 16] and the references in there.
In the following, we aim at investigating properties of bilinear forms and operators associated to a kernel k as in (5.1) from a variational point of view. Suitably, we give additional assumptions on k focusing, however, on minimizing these and we present certain explicit examples at the end of this introduction.
To present our results, let $\Omega\subset\mathbb{R}^N$ be an open, and $u,v\in C_c^{0,1}(\Omega)$ and consider the bilinear form

$$b_{k,\Omega}(u,v):=\frac{1}{2}\int_{\Omega}\int_{\Omega}(u(x)-u(y))(v(x)-v(y))k(x,y)\,dxdy, \tag{5.2}$$

where we also write $b_k(u,v):=b_{k,\mathbb{R}^N}(u,v)$ and $b_{k,\Omega}(u):=b_{k,\Omega}(u,u)$, $b_k(u)=b_k(u,u)$ resp. We denote

$$D^k(\Omega):=\{u\in L^2(\Omega)\;:\;b_{k,\Omega}(u)<\infty\}, \tag{5.3}$$

which is a Hilbert space with scalar product

$$\langle\cdot,\cdot\rangle_{D^k(\Omega)}=\langle\cdot,\cdot\rangle_{L^2(\Omega)}+b_{k,\Omega}(u,v).$$

Associated to b_k there is a *nonlocal* self-adjoint operator I_k which for $u,v \in C_c^{0,1}(\Omega)$ satisfies

$$b_k(u,v) = \int_{\mathbb{R}^N} I_k u(x)v(x)\,dx \quad \text{and is represented by} \quad I_k u(x) = \int_{\mathbb{R}^N} (u(x)-u(y))k(x,y)\,dy, \quad x \in \Omega. \tag{5.4}$$

Here, the first equality can be extended, see Section 5.3, to functions $v \in \mathscr{V}^k(\Omega)$, the space of those functions $v : \mathbb{R}^N \to \mathbb{R}$ such that $v|_\Omega \in D^k(\Omega)$ and

$$\sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N \setminus B_r(x)} |v(y)|k(x,y)\,dy < \infty \quad \text{for all } r > 0. \tag{5.5}$$

Given $f \in L^2_{loc}(\Omega)$, we then call $v \in \mathscr{V}^k(\Omega)$ a *(weak) supersolution* of $I_k v = f$ in Ω, if

$$b_k(v,u) \geq \int_\Omega f(x)u(x)\,dx \qquad \text{for all } u \in C_c^\infty(\Omega). \tag{5.6}$$

In this situation, we also say that v satisfies in weak sense $I_k v \geq f$ in Ω. Similarly, we define *subsolutions* and *solutions*.
We emphasize that this definition of supersolution is larger than the one considered in [60]. Using a density result we then can extend the weak maximum principles presented in [60] as follows.

Proposition 5.1 (Weak maximum principle)**.** *Define $j : \mathbb{R}^N \to [0,\infty]$ as the symmetric rearrangement of k, that is*

$$j(z) := \operatorname{essinf}\{k(x, x \pm z) \ : \ x \in \mathbb{R}^N\} \quad \textit{for } z \in \mathbb{R}^N, \tag{5.7}$$

and assume that

$$j \textit{ does not vanish identically on } B_r(0) \textit{ for any } r > 0. \tag{5.8}$$

Let $\Omega \subset \mathbb{R}^N$ open, $c \in L^\infty_{loc}(\Omega)$, and assume either

1. $c \leq 0$ *or*
2. Ω *and* c *are such that* $\|c^+\|_{L^\infty(\Omega)} < \inf_{x\in\Omega} \int_{\mathbb{R}^N\setminus\Omega} k(x,y)\,dy$.

If $u \in \mathscr{V}^k(\Omega)$ satisfies in weak sense

$$I_k u \geq c(x)u \quad \textit{in } \Omega,\ u \geq 0 \textit{ almost everywhere in } \mathbb{R}^N \setminus \Omega, \textit{ and } \quad \liminf_{|x|\to\infty} u(x) \geq 0,$$

then $u \geq 0$ almost everywhere in $\mathbb{R}^N$.

We note that assumption 5.8 on the function j defined in (5.7) implies the positivity of the first Dirichlet eigenvalue of I_k in bounded (or thin) sets $\Omega \subset \mathbb{R}^N$: The operator I_k on an open set $\Omega \subset \mathbb{R}^N$ can be associated to a form domain given by the space

$$\mathscr{D}^k(\Omega) = \{u \in D^k(\mathbb{R}^N) \ : \ 1_{\mathbb{R}^N\setminus\Omega} u \equiv 0\}. \tag{5.9}$$

Clearly, $\mathscr{D}^k(\Omega) = D^k(\mathbb{R}^N)$ and also the space $\mathscr{D}^k(\Omega)$ is a Hilbert space with scalar product

$$\langle \cdot, \cdot \rangle_{\mathscr{D}^k(\Omega)} = \langle \cdot, \cdot \rangle_{L^2(\Omega)} + b_k(u,v).$$

Then the first (variational) Dirichlet eigenvalue of I_k is given for $\Omega \subset \mathbb{R}^N$ open by

$$\Lambda_1(\Omega) := \inf_{\substack{u \in \mathscr{D}^k(\Omega)\\ u \neq 0}} \frac{b_k(u)}{\|u\|^2_{L^2(\Omega)}} \in [0,\infty). \tag{5.10}$$

Then, if (5.8) is satisfied and Ω is contained (after a rotation) in a strip $(-a,a)\times\mathbb{R}^{N-1}$ for some $a>0$, then $\Lambda_1(\Omega)>0$ (see [43, 61]).
In the following, we assume the stronger assumption

$$\text{The function } j \text{ given in (5.7) satisfies } \int_{\mathbb{R}^N} j(z)\,dz = \infty \tag{5.11}$$

and conclude the

Proposition 5.2 (Strong maximum principle)**.** *Assume k satisfies additionally* (5.11)*. Let $\Omega \subset \mathbb{R}^N$ open and $c \in L^\infty_{loc}(\Omega)$ with $\|c^+\|_{L^\infty(\Omega)} < \infty$. Moreover, let $u \in \mathscr{V}^k(\Omega)$, $u \geq 0$ satisfy in weak sense $I_k u \geq c(x)u$ in Ω. If*

1. *If Ω is connected, then either $u \equiv 0$ in Ω or $\operatorname{essinf}_K u > 0$ for any $K \subset\subset \Omega$.*
2. *j given in* (5.7) *satisfies $\operatorname{essinf}_{B_r(0)} j > 0$ for any $r > 0$, then either $u \equiv 0$ in $\mathbb{R}^N$ or $\operatorname{essinf}_K u > 0$ for any $K \subset\subset \Omega$.*

Clearly, if $\Lambda_1(\Omega)$ is positive, then b_k denotes an equivalent scalar product on $\mathscr{D}^k(\Omega)$ and thus for any $f \in L^2(\Omega)$ there is a unique solution $u \in \mathscr{D}^k(\Omega)$ with $I_k u = f$ in Ω. The main results of this article then are concerned with the regularity of u, if f has a certain regularity.
We begin with a boundedness result for solutions.

Theorem 5.3. *Assume k satisfies* (5.11) *and is such that*

$$\sup_{x\in\mathbb{R}^N} \int_{K\setminus B_\varepsilon(x)} k(x,y)^2\,dy < \infty \quad \textit{for all } K \subset\subset \mathbb{R}^N \textit{ and } \varepsilon > 0. \tag{5.12}$$

Let $\Omega \subset \mathbb{R}^N$ be an open set and $f \in L^\infty(\Omega)$, $h \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, and let $u \in \mathscr{V}^k(\Omega)$ satisfy in weak sense $I_k u = \lambda u + h * u + f$ in Ω for some $\lambda > 0$. If there is $\Omega' \subset\subset \Omega$ such that $u \in L^\infty(\mathbb{R}^N \setminus \Omega')$, then $u \in L^\infty(\mathbb{R}^N)$ and there is $C = C(\Omega,\Omega',k,\lambda,h) > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \leq C\Big(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^2(\Omega')} + \|u\|_{L^\infty(\mathbb{R}^N\setminus\Omega')}\Big).$$

Remark 5.4.

1. *Indeed, Theorem 5.3 is a consequence of a slightly more general result stated in Theorem 5.22 in Section 5.6 below, which concerns functions $u : \mathbb{R}^N \to \mathbb{R}$, which are in a certain sense locally in $\mathscr{V}^k(\Omega)$.*

2. *If $u \in \mathscr{D}^k(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is open and bounded with Lipschitz boundary, then in the above, $\Omega' = \Omega$ can be chosen. Indeed, here the regularity assumption on $\partial\Omega$ is only needed since our definition of weak solutions uses test-function in $C_c^\infty(\Omega)$. Replacing this directly with $\mathscr{D}^k(\Omega)$, the regularity assumption is not needed.*

As shown in [61], under assumption 5.11, it follows also that the embedding $\mathscr{D}^k(\Omega) \hookrightarrow L^2(\Omega)$ is indeed compact, if $\Omega \subset \mathbb{R}^N$ is bounded. Whence, from Theorem 5.3 we have the following Corollary.

Theorem 5.5. *Let k satisfy additionally* (5.11) *and* (5.12) *and let $\Omega \subset \mathbb{R}^N$ open and bounded. Then there is a sequence of Dirichlet eigenvalues $(\Lambda_n(\Omega))_n$ of I_k with*

$$0 < \Lambda_1(\Omega) < \Lambda_2(\Omega) \leq \ldots \leq \Lambda_n(\Omega) \to \infty \quad \textit{for } n \to \infty,$$

that is, $\Lambda_1(\Omega)$ is simple and the first normalized eigenfunction φ_1 of I_k can be chosen to be positive in the sense that

$$\operatorname*{essinf}_K \varphi_1 > 0 \quad \textit{for all } K \subset\subset \Omega.$$

Moreover, any eigenfunction of I_k is bounded. To be precise, given $\lambda > 0$ and $u \in \mathscr{D}^k(\Omega)$ such that $I_k u = \lambda u$, then there is $C = C(N,\Omega,k,\lambda) > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{L^2(\Omega)}.$$

In the particular case, where the kernel is translation invariant, that is, there is a function $J : \mathbb{R}^N \to [0,\infty]$ with $k(x,y) = J(x-y)$ for $x,y \in \mathbb{R}^N$, we are also able to recover differentiability of a solution u to the problem $I_k u = f$, if f and J satisfy certain regularity properties. Our result is as follows.

Theorem 5.6. *Assume k satisfies* (5.11) *and let $\Omega \subset \mathbb{R}^N$ open and bounded with Lipschitz boundary. Then for any $f \in L^2(\Omega)$ there is a unique solution $u \in \mathscr{D}^k(\Omega)$ of $I_k u = f$. Moreover, if k satisfies additionally* (5.12) *and $f \in L^\infty(\Omega)$, then $u \in L^\infty(\Omega)$ and there is $C = C(N,\Omega,k) > 0$ such that*

$$\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}.$$

Furthermore, if k satisfies (5.1) *with $\sigma < \frac{1}{2}$ and*

there is $J : \mathbb{R}^N \to [0,\infty]$ such that $k(x,y) = J(x-y)$, where J satisfies for some $m \in \mathbb{N} \cup \{\infty\}$:

$(A)_m$ *It holds $J \in W^{n,1}(\mathbb{R}^N \setminus B_\varepsilon(0))$ for any $\varepsilon > 0$, $n \leq 2m$ and, for some $C_J > 0$, it holds $|\nabla J(z)| \leq C_J |z|^{-1-\sigma-N}$ for $0 < |z| \leq 3$ with σ as in* (5.1), (5.13)

then, if $m \in \mathbb{N}$ and $f \in C^{2m}(\overline{\Omega})$, we have $\partial^\beta u \in L^2_{loc}(\Omega)$ for all $\beta \in \mathbb{N}_0^N$, $|\beta| \leq m$, $u \in H^m_{loc}(\Omega)$, and for every $\Omega' \subset\subset \Omega$ there is $C = C(N,\Omega,\Omega',k,\beta) > 0$ such that

$$\|\partial^\beta u\|_{L^2(\Omega')} \leq C\|f\|_{C^{2m}(\Omega)}.$$

In particular, for $m = \infty$, we have $u \in C^\infty(\Omega)$.

Remark 5.7. *We note that Theorem 5.6 is a particular case of more general result proved in Section 5.6 and 5.7. This general result also includes the eigenvalue problem and yields the following theorem.*

Theorem 5.8. *If in the situation of Theorem 5.5 the kernel k additionally satisfies* (5.13) *with $m=\infty$, then every function $u \in \mathscr{D}^k(\Omega)$ satisfying $I_k u = \lambda u$ in Ω for some $\lambda \in \mathbb{R}$ also belongs to $C^\infty(\Omega)$.*

It is worthy to mention our approaches only exploits the variational structure of the problem and uses purely analytic properties of the operator. Using a probabilistic and potential theoretic approach, a local smoothness of bounded harmonic solutions solving in a certain very weak sense $I_k u = 0$ in Ω, have been obtained in [56, Theorem 1.7] for radial kernel functions using the same regularity $(A)_m$ (see also [58, 79]). See also [54] for related regularity properties of solutions.

5.2 Examples

We close this introduction with a class of operators covered in the above discussion:

1. As introduced in [29, 47, 59] the *logarithmic Laplacian*

$$L_\Delta \varphi(x) = c_N P.V. \int\limits_{B_1(0)} \frac{\varphi(x)-\varphi(x+y)}{|y|^N}\, dy - c_N \int\limits_{\mathbb{R}^N \setminus B_1(0)} \frac{\varphi(x+y)}{|y|^N}\, dy + \rho_N \varphi(x), \quad (5.14)$$

appears as the operator with Fourier-symbol $-2\ln(|\cdot|)$ and can be seen as the formal derivative in s of $(-\Delta)^s$ at $s=0$. Here

$$c_N = \frac{\Gamma(\frac{N}{2})}{\pi^{N/2}} = \frac{2}{|S^{N-1}|} \qquad \text{and} \qquad \rho_N := 2\ln(2) + \psi(\frac{N}{2}) - \gamma \quad (5.15)$$

where $\psi := \frac{\Gamma'}{\Gamma}$ denotes the digamma function and $\gamma := -\psi(1) = -\Gamma'(1)$ is the Euler-Mascheroni constant. With $k(x,y) = c_N 1_{B_1(0)}(x-y)|x-y|^{-N}$, $h = -c_N 1_{\mathbb{R}^N \setminus B_1(0)}|y|^{-N}$ and $\lambda = \rho_N$, can be studied using Theorem 5.3. Moreover, also the generalizations of Theorem 5.6 considered in Section 5.6 and the regularity statements in Section 5.7 cover this operator —where in the latter situation a localization procedure is needed.

2. The logarithmic Schrödinger operator $(I-\Delta)^{\log}$ as in [45] is an integro-differential operator with Fourier-symbol $\log(1+|\cdot|^2)$ and also appears as the formal derivative in s of the relativistic Schrödinger operator $(I-\Delta)^s$ at $s=0$,

$$(I-\Delta)^{\log} u(x) = d_N P.V. \int\limits_{\mathbb{R}^N} \frac{u(x)-u(x+y)}{|y|^N} \omega(|y|)\, dy,$$

where $d_N = \pi^{-\frac{N}{2}}$, $\omega(r) = 2^{1-\frac{N}{2}} r^{\frac{N}{2}} K_{\frac{N}{2}}(r)$ and K_ν is the modified Bessel function of the second kind with index ν. More generally, operators with symbol $\log(1+|\cdot|^\beta)$ for some $\beta \in (0,2]$ are studied in [64].

3. Finally, also nonradial kernels of the type considered in [60] satisfy in particular the assumptions (5.1) and (5.11). See also also [58, 64, 79] and references in there.

The paper is organized as follows. In Section 5.3 we collect some general results concerning the spaces used in this paper and resulting definitions of weak sub- and supersolutions. Section 5.5 is devoted to show several density results of $C_c^\infty(\Omega)$ in $D^k(\Omega)$ and in $\mathscr{D}^k(\Omega)$, which then is used to show the Propositions 5.1 and 5.2. In Section 5.6 we present a general approach to show boundedness of solutions and in Section 5.7 we give the proof of an interior H^1-regularity of solutions and from this deduce the interior C^∞-regularity of solutions to conclude the proof of Theorem 5.6.

Notation In the remainder of the paper, we use the following notation. Let $U,V \subset \mathbb{R}^N$ be nonempty measurable sets, $x \in \mathbb{R}^N$ and $r > 0$. We denote by $1_U : \mathbb{R}^N \to \mathbb{R}$ the characteristic function, $|U|$ the Lebesgue measure, and $\operatorname{diam}(U)$ the diameter of U. The notation $V \subset\subset U$ means that $\overline{V}$ is compact and contained in the interior of U. The distance between V and U is given by $\operatorname{dist}(V,U) := \inf\{|x-y| \,:\, x \in V, y \in U\}$. Note that this notation does *not* stand for the usual Hausdorff distance. If $V = \{x\}$ we simply write $\operatorname{dist}(x,U)$. We let $B_r(U) := \{x \in \mathbb{R}^N \,:\, \operatorname{dist}(x,U) < r\}$, so that $B_r(x) := B_r(\{x\})$ is the open ball centered at x with radius r. We also put $B := B_1(0)$ and $\omega_N := |B|$. Finally, given a function $u : U \to \mathbb{R}$, $U \subset \mathbb{R}^N$, we let $u^+ := \max\{u,0\}$ and $u^- := -\min\{u,0\}$ denote the positive and negative part of u, and we write $\operatorname{supp} u$ for the support of u given as the closure in $\mathbb{R}^N$ of the set $\{x \in U \,:\, u(x) \neq 0\}$.

5.3 Preliminaries

Recall for $\Omega \subset \mathbb{R}^N$ open, the definitions

$$
\begin{aligned}
b_{k,\Omega}(u,v) &:= \frac{1}{2}\int_\Omega\int_\Omega (u(x)-u(y))(v(x)-v(y))k(x,y)\,dxdy,\\
\kappa_{k,\Omega}(x) &:= \int_{\mathbb{R}^N\setminus\Omega} k(x,y)\,dy \in [0,\infty] \quad \text{for } x \in \mathbb{R}^N, \text{ and}\\
K_{k,\Omega}(u,v) &:= \int_\Omega u(x)v(x)\kappa_{k,\Omega}(x)\,dx,
\end{aligned}
$$

where if $u = v$ we put

$$
b_{k,\Omega}(u) := b_{k,\Omega}(u,u) \quad \text{and} \quad K_{k,\Omega}(u) := K_{k,\Omega}(u,u).
$$

Note that we have for any fixed $x \in \Omega$ that $\kappa_{k,\Omega}(x) < \infty$ by (5.1). Moreover, we consider the function spaces

$$
D^k(\Omega) := \Big\{u \in L^2(\Omega) \,:\, b_{k,\Omega}(u) < \infty\Big\},
$$

$$D^k_{loc}(\Omega) := \Big\{ u : \Omega \to \mathbb{R} \; : \; u|_{\Omega'} \in D^k(\Omega') \text{ for all } \Omega' \subset\subset \Omega \Big\},$$
$$\mathscr{D}^k(\Omega) := \Big\{ u \in D^k(\mathbb{R}^N) \; : \; u = 0 \text{ on } \mathbb{R}^N \setminus \Omega \Big\},$$
$$\mathscr{V}^k(\Omega) := \Big\{ u : \mathbb{R}^N \to \mathbb{R} \; : \; u|_\Omega \in D^k(\Omega) \text{ and, for all } r > 0, \sup_{x \in \mathbb{R}^N} \int\limits_{\mathbb{R}^N \setminus B_r(x)} |u(y)| k(x,y)\, dy < \infty \Big\}, \quad \text{and}$$
$$\mathscr{V}^k_{loc}(\Omega) := \Big\{ u : \mathbb{R}^N \to \mathbb{R} \; : \; u|_{\Omega'} \in \mathscr{V}^k(\Omega') \text{ for all } \Omega' \subset\subset \Omega \Big\}.$$

Lemma 5.9. *Let $U \subset \Omega \subset \mathbb{R}^N$ open and $u : \mathbb{R}^N \to \mathbb{R}$. Then the following hold:*

1. $u \in \mathscr{D}^k(\Omega) \;\Rightarrow\; u|_\Omega \in D^k(\Omega) \;\Rightarrow\; u|_\Omega \in D^k_{loc}(\Omega)$.
2. $\mathscr{D}^k(U) \subset \mathscr{D}^k(\Omega) \subset \mathscr{V}^k(\Omega) \subset \mathscr{V}^k(U) \subset \mathscr{V}^k_{loc}(U)$.

Proof. This follows immediately from the definitions (see also [60, Section 3]). □

Lemma 5.10 (see Proposition 3.3 in [60] or Proposition 1.7 in [61])**.** *For $\Omega \subset \mathbb{R}^N$ open let $\Lambda_1(\Omega)$ be given as in* (5.10) *and let*

$$\lambda(r) = \inf\{\Lambda_1(\Omega) \; : \; \Omega \subset \mathbb{R}^N \textit{ open with } |\Omega| = r\}.$$

Then $\lim\limits_{r\to\infty} \lambda(r) \geq \int_{\mathbb{R}^N} j(z)\, dz$ with $j(z) := \operatorname{essinf}\{k(x, x \pm z) \; : \; z \in \mathbb{R}^N\}$ as in (5.7).

Lemma 5.11. *Let $\Omega \subset \mathbb{R}^N$ open and let X be any of the above function spaces. Then the following holds:*

1. *$b_{k,\Omega}$ is a bilinear form and in particular we have $b_{k,\Omega}(u,v) \leq b^{1/2}_{k,\Omega}(u) b^{1/2}_{k,\Omega}(v)$. Moreover, $D^k(\Omega)$ and $\mathscr{D}^k(\Omega)$ are Hilbert spaces with scalar products*
$$\langle u,v\rangle_{D^k(\Omega)} = \langle u,v\rangle_{L^2(\Omega)} + b_{k,\Omega}(u,v),$$
$$\langle u,v\rangle_{\mathscr{D}^k(\Omega)} = \langle u,v\rangle_{L^2(\Omega)} + b_{k,\mathbb{R}^N}(u,v).$$
2. *If $u \in X$, then $u^\pm, |u| \in X$ and we have $b_{k,\Omega'}(u^+,u^-) \leq 0$ for all $\Omega' \subset \Omega$ with $b_{k,\Omega'}(u) < \infty$.*
3. *If $g \in C^{0,1}(\mathbb{R}^N)$, $u \in X$, then $g \circ u \in X$.*
4. *$C^{0,1}_c(\Omega) \subset X$.*
5. *$\varphi \in C^{0,1}_c(\Omega)$, $u \in X$, then $\varphi u \in \mathscr{D}^k(\Omega)$, where if necessary we extend u trivially to a function on $\mathbb{R}^N$. Moreover, there is $C = C(N,k,\|\varphi\|_{C^{0,1}(\Omega)}) > 0$ such that*
$$b_{k,\mathbb{R}^N}(\varphi u) \leq C\Big(\|u\|^2_{L^2(\Omega')} + b_{k,\Omega'}(u) \Big)$$
for any $\Omega' \subset \Omega$ with $\operatorname{supp}\varphi \subset\subset \Omega'$.

Proof. Theses statements follow directly from the definition (c.f. [60, Section 3]). To be precise in the last part, let $\varphi \in C_c^{0,1}(\Omega)$ and fix $L := \|\varphi\|_{C^{0,1}(\Omega)}$. That is, we have

$$|\varphi(x)| \leq L \quad \text{and} \quad |\varphi(x) - \varphi(y)| \leq L|x-y|.$$

Then using the inequality for $x, y \in \mathbb{R}^N$

$$|\varphi(x)u(x) - \varphi(y)u(y)|^2 \leq 2|\varphi(x) - \varphi(y)|^2|u(x)|^2 + 2|\varphi(y)|^2|u(x) - u(y)|^2$$

we find by the assumptions (5.1)

$$\begin{aligned} b_{k,\mathbb{R}^N}(\varphi u) &\leq b_{k,\Omega'}(\varphi u) + L^2 \int\limits_{\operatorname{supp}\varphi} |u(x)|^2 \kappa_{k,\Omega'}(x)\, dx \\ &\leq 2L^2 \int\limits_{\Omega'}\int\limits_{\Omega'} |u(x)|^2|x-y|^2 k(x,y)\, dy\, dx + 2L^2 b_{k,\Omega'}(u) + L \sup_{x \in \operatorname{supp}\varphi} \kappa_{k,\Omega'}(x) \|u\|^2_{L^2(\Omega')} \\ &\leq 2L^2 \Big(\sup_{x\in\Omega'} \int\limits_{\Omega'} |x-y|^2 k(x,y)\, dy + \sup_{x\in\operatorname{supp}\varphi} \kappa_{k,\Omega'}(x) \Big) |\Omega'| \|u\|^2_{L^2(\Omega')} + 2L^2 b_{k,\Omega'}(u) < \infty. \end{aligned}$$

□

Remark 5.12. *1. Note that for $u, v \in \mathscr{D}^k(\Omega)$ we have*

$$b_k(u,v) = b_{k,\mathbb{R}^N}(u,v) = b_{k,\Omega}(u,v) + K_{k,\Omega}(u,v).$$

2. It follows in particular that there is a nonnegative self-adjoint operator I_k associated to $b_{k,\mathbb{R}^N} = b_k$ as mentioned in the introduction.

Lemma 5.13. *Let $\Omega \subset \mathbb{R}^N$ open, $u \in \mathscr{V}^k_{loc}(\Omega)$. Then $b_{k,\mathbb{R}^N}(u,\varphi)$ is well-defined for any $\varphi \in C_c^\infty(\Omega)$.*

Proof. Let $\varphi \in C_c^\infty(\Omega)$ and fix $U \subset\subset \Omega$ such that $\operatorname{supp}\varphi \subset\subset U$. Then with the symmetry of k

$$\begin{aligned} |b_k(u,\varphi)| &\leq |b_{k,U}(u,\varphi)| + \int\limits_U |\varphi(x)| \int\limits_{\mathbb{R}^N\setminus U} |u(x)-u(y)| k(x,y)\, dydx \\ &\leq b_{k,U}^{1/2}(u) b_{k,U}^{1/2}(\varphi) + \int\limits_{\operatorname{supp}\varphi} |\varphi(x)|\, dx \sup_{x\in\mathbb{R}^N} \int\limits_{\mathbb{R}^N\setminus B_\varepsilon(x)} |u(y)| k(x,y)\, dy \\ &\quad + \int\limits_{\operatorname{supp}\varphi} |\varphi(x)u(x)|\, dx \sup_{x\in\mathbb{R}^N} \int\limits_{\mathbb{R}^N\setminus B_\varepsilon(x)} k(x,y)\, dy < \infty, \end{aligned}$$

where $\varepsilon = \operatorname{dist}(\operatorname{supp}\varphi, \mathbb{R}^N \setminus U) > 0$. □

Definition 5.14. Let $\Omega \subset \mathbb{R}^N$ open and $f \in L^1_{loc}(\Omega)$. Then $u \in \mathscr{V}^k_{loc}(\Omega)$ is called a *weak supersolution* of $I_k u = f$ in Ω, if

$$b_{k,\mathbb{R}^N}(u,\varphi) \geq \int\limits_\Omega f(x)\varphi(x)\, dx \quad \text{for all nonnegative } \varphi \in C_c^\infty(\Omega).$$

We also say that *u satisfies $I_k u \geq f$ weakly in Ω*.
Similarly, we define *weak subsolutions* and *solutions*.

Remark 5.15. *1. We note that by Assumption 5.1 it follows that for any function $u \in \mathscr{V}^k_{loc}(\Omega)$ with $u|_\Omega \in C^{0,1}(\Omega)$ for $\Omega \subset \mathbb{R}^N$ open we have $I_k u|_U \in L^\infty(U)$ for any $U \subset\subset \Omega$ and*

$$I_k u(x) = \int_{\mathbb{R}^N} (u(x)-u(y))k(x,y)\, dy \quad \text{for } x \in \Omega.$$

This follows easily similarly to the proof of the statements in Lemma 5.11.

2. If $u \in \mathscr{V}^k(\Omega)$, then indeed also $b_k(u,\varphi)$ is well-defined for all $\varphi \in \mathscr{D}^k(\Omega)$. Whence also b_k is well defined on $\mathscr{V}^k_{loc}(\Omega) \times \mathscr{D}^k(U)$ for all $U \subset\subset \Omega$. In some of our results the statements need a Lipschitz-boundary of Ω, which comes into play due to approximation with $C_c^\infty(\Omega)$-functions (see Section 5.5 below). However, this can be weakened, if $u \in \mathscr{V}^k(\Omega)$ and the space of test-functions is adjusted.

Lemma 5.16. *Let $\Omega \subset \mathbb{R}^N$ open. Let $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega$. Let $\eta \in C_c^{0,1}(\mathbb{R}^N)$ such that $0 \le \eta \le 1$ in $\mathbb{R}^N$ and we have*

$$\eta = 1 \quad \text{in } \Omega_2 \quad \text{and} \quad \eta = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega_3.$$

Let $f \in L^1_{loc}(\Omega)$ and let $u \in \mathscr{V}^k_{loc}(\Omega)$ satisfy in weak sense $Iu \ge f$ in Ω. Then the function $v = \eta u \in \mathscr{D}^k(\Omega_3)$ satisfies in weak sense $I_k v \ge f + g_{\eta,u}(x)$ in Ω_1, where

$$g_{\eta,u}(x) = \int_{\mathbb{R}^N \setminus \Omega_2} (1-\eta(y))u(y)k(x,y)\, dy \quad \text{for } x \in \Omega_1.$$

Proof. The fact, that $v \in \mathscr{D}^k(\Omega_3)$ follows from Lemma 5.11. Let $\varphi \in C_c^\infty(\Omega_1)$, then

$$\int_{\mathbb{R}^N} v(x) I_k \varphi(x)\, dx \ge \int_{\mathbb{R}^N} f(x)\varphi(x)\, dx - \int_{\mathbb{R}^N} (1-\eta(x))u(x) I_k\varphi(x)\, dx.$$

Here, since $(1-\eta)u \equiv 0$ on Ω_2, we have

$$\begin{aligned}\int_{\mathbb{R}^N} (1-\eta(x))u(x) I_k\varphi(x)\, dx &= \int_{\mathbb{R}^N} \varphi(x)[I_k(1-\eta)u](x)\, dx \\ &= -\int_{\Omega_1} \varphi(x) \int_{\mathbb{R}^N \setminus \Omega_2} (1-\eta(y))u(y)k(x,y)\, dy\, dx.\end{aligned}$$

Thus the claim follows. □

Remark 5.17. *The same result as in Lemma 5.16 also holds if "$\ge$" in the solution type is replaced by "$\le$" or "$=$".*

In the following, it is useful to understand functions $u \in D^k(\Omega)$ satisfying $b_{k,\Omega}(u) = 0$.

Proposition 5.18. *Assume additionally 5.11 and let $\Omega \subset \mathbb{R}^N$ open and bounded and let $u \in D^k(\Omega)$ such that $b_{k,\Omega}(u) = 0$. Then u is constant.*

Proof. Let $x_0 \in \Omega$ and fix $r > 0$ such that $B_{2r}(x_0) \subset \Omega$. Denote $q(z) := \min\{c, j(z)\}1_{B_r(0)}(z)$, where we may fix $c > 0$ such that $|\{q > 0\}| > 0$ due to Assumption (5.11). Then by Lemma 6.1 we have

$$0 = 2b_{k,\Omega}(u) \geq 2b_{q,\Omega}(u) \geq \frac{1}{2\|q\|_{L^1(\mathbb{R}^N)}} b_{q*q,B_r(x_0)}(u),$$

where $a*b = \int_{\mathbb{R}^N} a(\cdot - y)b(y)\,dy$ denotes as usual the convolution. Note that since $q \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $q = 0$ on $\mathbb{R}^N \setminus B_r(0)$, it follows that $q*q \in C(\mathbb{R}^N)$ with support in $B_{2r}(0)$ and we have

$$q*q(0) = \int\limits_{\mathbb{R}^N} q(z)^2\,dz > 0$$

by Assumption (5.11). Hence there is $R > 0$ with $q*q \geq \varepsilon$ for some $\varepsilon > 0$ and thus we have

$$0 = b_{q*q,B_r(x_0)}(u) \geq b_{q*q,B_\rho(x_0)}(u) \geq \frac{\varepsilon}{2} \int\limits_{B_\rho(x_0)} \int\limits_{B_\rho(x_0)} (u(x) - u(y))^2\,dxdy.$$

for any $\rho \in (0, \frac{R}{2}]$. But then $u(x) = u(y)$ for almost every $x, y \in B_{R/2}(x_0)$ so that u is constant a.e. in $B_\rho(x_0)$. Since $b_{k,\Omega}(u) = b_{k,\Omega}(u - m)$ for any $m \in \mathbb{R}$, we may next assume that $u = 0$ in $B_{R/2}(x_0)$ and show that indeed we have $u = 0$ a.e. in Ω. Denote by W the set of points $x \in \Omega$ such that there is $r > 0$ with $u = 0$ a.e. in $B_r(x)$. By definition W is open and the above shows that W is nonempty. Next, let $(x_n)_n \subset W$ be a sequence with $x_n \to x \in \Omega$ for $n \to \infty$. Then there is $r_x > 0$ such that $B_{4r_x}(x) \subset \Omega$ and we can find $n_0 \in \mathbb{N}$ such that $x \in B_{r_x}(x_n) \subset B_{2r_x}(x_n) \subset \Omega$ for $n \geq n_0$. Repeating the above argument, it follows that u must be zero in $B_{r_x}(x_n)$ and thus $x \in W$. Hence, W is relatively open and closed in Ω and since W is nonempty, we have $W = \Omega$. That is $u = 0$ in Ω. □

5.4 On Sobolev and Nikol'skii spaces

We recall here the notations and properties of Sobolev and Nikol'skii spaces as introduced in [31, 96]. In the following, let $p \in [1, \infty)$ and $\Omega \subset \mathbb{R}^N$ open.

5.4.1 Sobolev spaces

If $k \in \mathbb{N}_0$, we set as usual

$$W^{k,p}(\Omega) := \Big\{u \in L^p(\Omega) \;:\; \partial^\alpha u \text{ exists for all } \alpha \in \mathbb{N}_0^n,\, |\alpha| \leq k \text{ and belongs to } L^p(\Omega)\Big\}$$

for the Banach space of k-times (weakly) differentialable functions in $L^p(\Omega)$. Moreover, as usual, for $\sigma \in (0,1)$, $p \in [1, \infty)$ we set

$$W^{\sigma,p}(\Omega) := \Big\{u \in L^p(\Omega) \;:\; \frac{u(x) - u(y)}{|x-y|^{\frac{n}{p}+\sigma}} \in L^p(\Omega \times \Omega)\Big\}.$$

With the norm

$$\|u\|_{W^{\sigma,p}(\Omega)} = \|u\|^p_{L^p(\Omega)} + [u]_{W^{\sigma,p}(\Omega)}, \quad \text{where} \quad [u]_{W^{\sigma,p}(\Omega)} = \left(\iint\limits_{\Omega\times\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+\sigma p}}\,dxdy\right)^{1/p}$$

the space $W^{\sigma,p}(\Omega)$ is a Banach space. For general $s=k+\sigma$, $k\in\mathbb{N}_0$, $\sigma\in[0,1)$ the Sobolev space is defined as

$$W^{s,p}(\Omega):=\Big\{u\in W^{k,p}(\Omega)\ :\ \partial^{\alpha}u\in W^{\sigma,p}(\Omega)\text{ for all }\alpha\in\mathbb{N}_0^n\text{ with }|\alpha|=k\Big\}.$$

Finally, in the particular case $p=2$ the space $H^s(\Omega):=W^{s,2}(\Omega)$ is a Hilbert space.

5.4.2 Nikol'skii spaces

For $u:\Omega\to\mathbb{R}$ and $h\in\mathbb{R}$, let $\Omega_h:=\{x\in\Omega\ :\ \mathrm{dist}(x,\partial\Omega)>h\}$ and, with $e\in\partial B_1(0)$, we let

$$\delta_h u(x)=\delta_{h,e}u(x):=u(x+he)-u(x).$$

Moreover, for $l\in\mathbb{N}$, $l>1$ let

$$\delta_h^l u(x)=\delta_h(\delta_h^{l-1}u)(x).$$

For $s=k+\sigma>0$ with $k\in\mathbb{N}_0$ and $\sigma\in(0,1]$ define

$$N^{s,p}(\Omega):=\Big\{u\in W^{k,p}(\Omega)\ :\ [\partial^{\alpha}u]_{N^{\sigma,p}(\Omega)}<\infty\text{ for all }\alpha\in\mathbb{N}_0^n\text{ with }|\alpha|=k\Big\},$$

where

$$[u]_{N^{\sigma,p}(\Omega)}=\sup_{\substack{e\in\partial B_1(0)\\ h>0}} h^{-\sigma}\|\delta_{h,e}^2 u\|_{L^p(\Omega_{2h})}.$$

It follows that $N^{s,p}(\Omega)$ is a Banach space with norm $\|u\|_{N^{s,p}(\Omega)}:=\|u\|_{W^{k,p}(\Omega)}+\sum_{|\alpha|=k}[\partial^{\alpha}u]_{N^{\sigma,p}(\Omega)}$. It can be shown that this norm is equivalent to

$$\|u\|_{L^p(\Omega)}+\sum_{|\alpha|=k}\sup_{\substack{e\in\partial B_1(0)\\ h>0}} h^{m-\sigma}\|\delta_{h,e}^l u\|_{L^p(\Omega_{lh})}$$

for any fixed $m,l\in\mathbb{N}_0$ with $m<\sigma$ and $l>\sigma-m$ (see [96, Theorem 4.4.2.1]).

Proposition 5.19 (see e.g. Propositions 3 and 4 in [31]). *Let $\Omega\subset\mathbb{R}^N$ open and with C^{∞} boundray. Moreover, let $t>s>0$ and $1\le p<\infty$. Then*

$$N^{t,p}(\Omega)\subset W^{s,p}(\Omega)\subset N^{s,p}(\Omega).$$

5.5 Density results and Maximum principles

The main goal of this section is to show the following.

Theorem 5.20. *Let either $\Omega=\mathbb{R}^N$ or $\Omega\subset\mathbb{R}^N$ open and bounded with Lipschitz boundary. In the following, let $X(\Omega):=\mathscr{D}^k(\Omega)$ or $D^k(\Omega)$. Then $C_c^{\infty}(\Omega)$ is dense in $X(\Omega)$. Moreover, if $u\in X(\Omega)$ is nonnegative, then we have*

1. *There exists a sequence $(u_n)_n\subset X(\Omega)\cap L^{\infty}(\Omega)$ with $\lim\limits_{n\to\infty}u_n=u$ in $X(\Omega)$ satisfying that for every $n\in\mathbb{N}$ there is $\Omega_n'\subset\subset\Omega$ with $u_n=0$ on $\Omega\setminus\Omega_n'$ and $0\le u_n\le u_{n+1}\le u$.*

2. *There exists a sequence* $(u_n)_n \subset C_c^\infty(\Omega)$ *with* $u_n \geq 0$ *for every* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} u_n = u$ *in* $X(\Omega)$.

Remark 5.21. *To put Theorem 5.20 into perspective, we consider the following examples.*

1. *In the case* $k(x,y) = |x-y|^{-2s-N}$ *for some* $s \in (0, \frac{1}{2})$, *the above Theorem is well-known and leads to the interesting property that for any open, bounded Lipschitz set* $\Omega \subset \mathbb{R}^N$ *we have*
$$D^k(\Omega) = H^s(\Omega) = H_0^s(\Omega).$$
We emphasize that the above equality also holds for $s = \frac{1}{2}$. *Moreover, if* $s < \frac{1}{2}$, *it also holds* $H^s(\Omega) = \{u \in H^s(\mathbb{R}^N) \ : \ 1_{\mathbb{R}^N \setminus \Omega} u \equiv 0\}$.

2. *If* $k(x,y) = 1_{B_1(0)}(x-y)|x-y|^{-N}$, $D^k(\Omega)$ *is associated to the function space of the localized logarithmic Laplacian (see [29]).*

The proof is split into several smaller steps. Recall that $\mathscr{D}^k(\mathbb{R}^N) = D^k(\mathbb{R}^N)$ by definition.

Lemma 5.22. *Let* $u \in D^k(\mathbb{R}^N)$. *Then there is a sequence* $(u_n)_n \subset D^k(\mathbb{R}^N)$ *with* $\lim_{n\to\infty} u_n = u$ *in* $D^k(\mathbb{R}^N)$ *satisfying that for every* $n \in \mathbb{N}$ *there is* $\Omega_n \subset\subset \mathbb{R}^N$ *with* $u_n = 0$ *on* $\mathbb{R}^N \setminus \Omega_n$. *Moreover, if* $u \geq 0$, *then* $(u_n)_n$ *can be chosen to satisfy in addition* $0 \leq u_n \leq u_{n+1} \leq u$.

Proof. For $n \in \mathbb{N}$ let $\varphi_n \in C_c^{0,1}(\mathbb{R}^N)$ be radially symmetric and such that $\varphi \equiv 1$ on $B_n(0)$, $\varphi_n \equiv 0$ on $B_{n+1}(0)^c$. Clearly, we may assume that $[\varphi_n]_{C^{0,1}(\mathbb{R}^N)} = 1$. By Lemma 5.11 there is hence some $C = C(N,k) > 0$ with $b_{k,\mathbb{R}^N}(\varphi_n u) \leq C\|u\|_{D^k(\mathbb{R}^N)}$ for all $n \in \mathbb{N}$. In the following, let $u_n := \varphi_n u$ and without loss of generality we may assume $u \geq 0$. Since then $0 \leq u - u_n \leq u$ on $\mathbb{R}^N$ and $u - u_n = 0$ on B_n, by dominated convergence we have $\lim_{n\to\infty} \|u-u_n\|_2 = 0$. Moreover, by choice of φ_n we have for $x, y \in \mathbb{R}^N$
$$\begin{aligned}|u(x)(1-\varphi_n(x)) - u(y)(1-\varphi_n(y))| &\leq |u(x)-u(y)|(1-\varphi_n(x)) + |u(y)||\varphi_n(x)-\varphi_n(y)| \\ &\leq |u(x)-u(y)| + |u(y)|\min\{1,|x-y|\} =: U(x,y).\end{aligned}$$
Here, $U(x,y) \in L^2(\mathbb{R}^N \times \mathbb{R}^N, k(x,y)\ d(x,y))$, since
$$\begin{aligned}\iint_{\mathbb{R}^N\times\mathbb{R}^N} U(x,y)k(x,y)\ dxdy &= b_{k,\mathbb{R}^N}(u) + \int_{\mathbb{R}^N} |u(y)|^2 \int_{\mathbb{R}^N} \min\{1,|x-y|^2\}k(x,y)\ dxdy \\ &\leq b_{k,\mathbb{R}^N}(u) + \int_{\mathbb{R}^N} |u(y)|^2\ dy\, \mathrm{supp}_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N} \min\{1,|x-y|^2\}k(x,y)dy < \infty.\end{aligned}$$
Thus $\lim_{n\to\infty} b_{k,\mathbb{R}^N}(u-u_n) = 0$ by the dominated convergence Theorem. $\square$

Proposition 5.23. *We have that* $C_c^\infty(\mathbb{R}^N)$ *is dense in* $D^k(\mathbb{R}^N)$. *Moreover, if* $u \in D^k(\mathbb{R}^N)$ *is nonnegative, then there exists* $(\varphi_n)_n \subset C_c^\infty(\mathbb{R}^N)$ *with* $\varphi_n \geq 0$ *for every* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} \varphi_n = u$ *in* $D^k(\mathbb{R}^N)$.

Proof. Let $u \in D^k(\mathbb{R}^N)$. Moreover, let $\varphi_n \in C_c^{0,1}(\mathbb{R}^N)$ for $n \in \mathbb{N}$ be given by Lemma 5.22 such that $\|u - \varphi u\|_{s,p} < \frac{1}{n}$. Then $v_n := \varphi_n u \in D^k(\mathbb{R}^N)$ and there is $R_n > 0$ with $v_n \equiv 0$ on $\mathbb{R}^N \setminus B_{R_n}(0)$. Next, let $(\rho_\varepsilon)_{\varepsilon \in (0,1]}$ by a Dirac sequence and denote $v_{n,\varepsilon} := \rho_\varepsilon * v_n$. Then $v_{n\varepsilon} \in C_c^\infty(\mathbb{R}^N)$ for all $n \in \mathbb{N}$, $\varepsilon \in (0,1]$ and

$$b_{k,\mathbb{R}^N}(u - v_{n,\varepsilon}) \le b_{k,\mathbb{R}^N}(u - v_n) + b_{k,\mathbb{R}^N}(v_n - v_{n,\varepsilon}) \le \frac{1}{n} + b_{k,\mathbb{R}^N}(v_n - v_{n,\varepsilon}).$$

It is hence enough to show that $v_{n,\varepsilon} \to v_n$ in $D^k(\mathbb{R}^N)$ for $\varepsilon \to 0$. In the following, we write v in place of v_n and $v_\varepsilon = \rho_\varepsilon * v$ in place of $v_{n,\varepsilon}$ for $\varepsilon \in (0,1]$. Moreover, let $R = R_n > 0$ with $v = v_n = 0$ on $\mathbb{R}^N \setminus B_R(0)$. Clearly, $v_\varepsilon \to v$ in $L^2(\mathbb{R}^N)$ for $\varepsilon \to 0$ and this convergence is also pointwise almost everywhere. Hence it is enough to analyze the convergence of $b_{k,\mathbb{R}^N}(v - v_\varepsilon)$ as $\varepsilon \to 0$. From here, the proof follows along the lines of [60, Proposition 4.1] noting that there it is not used that k only depends on the difference of x and y. Note here, that if u is nonnegative then the above constructed sequence is also nonnegative. □

Lemma 5.24. *Let $\Omega \subset \mathbb{R}^N$ open and such that $\partial\Omega$ is bounded. Denote $\delta(x) := \mathrm{dist}(x, \mathbb{R}^N \setminus \Omega)$. Then the following is true.*

1. *There is $C = C(N, \Omega, k) > 0$ such that $\kappa_{k,\Omega}(x) \le C\delta^{-\sigma}(x)$ for $x \in \Omega$.*
2. *If Ω is bounded, then $1_\Omega \in D^k(\mathbb{R}^N)$.*

Proof. Let $C = C(N, \Omega, k) > 0$ be constants varying from line to line and denote $U := \{x \in \mathbb{R}^N : \mathrm{dist}(x, \Omega) \le 1\}$. To see item 1., let $x \in \Omega$ and fix $p \in \partial\Omega$ such that $\delta(x) = |x - p|$. Then

$$\kappa_{k,\Omega}(x) \le C + \int_{U \setminus \Omega} \frac{|x-p|^\sigma}{|x-p|^\sigma} k(x,y)\, dy \le C + \delta(x)^{-\sigma} \int_{U \setminus \Omega} |x-y|^\sigma k(x,y)\, dy \le C\delta^{-\sigma}(x),$$

where we have used that $|x - p| \le |x - y|$ for $y \in \mathbb{R}^N \setminus \Omega$. Now 2. follows immediately from 1., since we have

$$b_{k,\mathbb{R}^N}(1_\Omega) = \int_\Omega \int_{\mathbb{R}^N \setminus \Omega} k(x,y)\, dydx \le C \int_\Omega \delta^{-\sigma}(x)\, dx < \infty.$$

□

Theorem 5.25 (See Theorem 5.20)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Then $C_c^\infty(\Omega)$ is dense in $\mathscr{D}^k(\Omega)$. Moreover, if $u \in \mathscr{D}^k(\Omega)$ is nonnegative, then we have*

1. *There exists a sequence $(u_n)_n \subset \mathscr{D}^k(\Omega)$ with $\lim_{n\to\infty} u_n = u$ in $\mathscr{D}^k(\Omega)$ satisfying that for every $n \in \mathbb{N}$ there is $\Omega_n' \subset\subset \Omega$ with $u_n = 0$ on $\mathbb{R}^N \setminus \Omega_n'$ and $0 \le u_n \le u_{n+1} \le u$.*
2. *There exists a sequence $(u_n)_n \subset C_c^\infty(\Omega)$ with $u_n \ge 0$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} u_n = u$ in $\mathscr{D}^k(\Omega)$.*

Proof. Note that the second claim follows immediately from the first one using [60, Proposition 4.1] as in the proof of Proposition 5.23. Then also the main claim follows by considering $u^{\pm}$ separately. Hence it is enough to show 1. We proceed similar to [29, Theorem 3.1]. Denote $\delta(x) := \operatorname{dist}(x, \mathbb{R}^N \setminus \Omega)$. For $r > 0$, define the Lipschitz map

$$\varphi_r : \mathbb{R}^N \to \mathbb{R}, \quad \varphi_r(x) = \begin{cases} 0 & \delta(x) \geq 2r, \\ 2 - \dfrac{\delta(x)}{r} & r \leq \delta(x) \leq 2r, \\ 1 & \delta(x) \leq r. \end{cases}$$

Note that we have $\varphi_s \leq \varphi_r$ for $0 < s \leq r$. We show

$$u\varphi_r \in \mathscr{D}^k(\Omega) \;\text{ for } r > 0 \text{ sufficiently small and } b_{k,\mathbb{R}^N}(u\varphi_r) \to 0 \;\text{ for } r \to 0. \tag{5.16}$$

Note that once this is shown, we have $u(1-\varphi_r) \in \mathscr{D}^k(\Omega)$ for $r > 0$ sufficiently small and $u(1-\varphi_r) \to u$ for $r \to 0$. Since also $0 \leq u(1-\varphi_r) \leq u(1-\varphi_s)$ for $0 < s \leq r$ and $u(1-\varphi_r) = 0$ for $x \in \mathbb{R}^N$ with $\delta(x) \leq r$, it follows that (5.16) implies 1.

The remainder of the proof is to show (5.16). For this, let $C = C(N, \Omega, k) > 0$ be a constant which may vary from line to line. Let $A_t := \{x \in \Omega \,:\, \delta(x) \leq t\}$. Note that $u\varphi_r$ vanishes on $\mathbb{R}^N \setminus A_{2r}$, we have $0 \leq \varphi_r \leq 1$ and, moreover,

$$|\varphi_r(x) - \varphi_r(y)| \leq \min\left\{C\frac{|x-y|}{r}, 1\right\} \quad \text{for } x, y \in \mathbb{R}^N.$$

Then proceeding similarly to the proof of Lemma 5.11 we find for r small enough

$$\begin{aligned}
b_{k,\mathbb{R}^N}(u\varphi_r) &= \frac{1}{2}\int_{A_{4r}}\int_{A_{4r}} (u(x)\varphi_r(x) - u(y)\varphi_r(y))^2 k(x,y)\,dxdy + \int_{A_{2r}} u^2(x)\varphi_r^2(x)\kappa_{k,A_{4r}}(x)\,dx \\
&\leq \int_{A_{4r}}\int_{A_{4r}} \Big(u(x)^2(\varphi_r(x)-\varphi_r(y))^2 + (u(x)-u(y))^2\varphi_r^2(y)^2\Big)k(x,y)\,dxdy \\
&\qquad + \int_{A_{2r}} u^2(x)\kappa_{k,A_{4r}}(x)\,dx \\
&\leq \frac{C}{r^2}\int_{A_{4r}} u(x)^2 \int_{B_r(x)} |x-y|^2 k(x,y)\,dydx + \int_{A_{4r}} u(x)^2 \int_{\mathbb{R}^N\setminus B_r(x)} k(x,y)\,dydx \\
&\qquad + \int_{A_{4r}}\int_{A_{4r}} (u(x)-u(y))^2 k(x,y)\,dxdy + \int_{A_{2r}} u^2(x)\kappa_{k,A_{4r}}(x)\,dx \\
&\leq \frac{C}{r^{\varepsilon}}\int_{A_{4r}} u(x)^2 \int_{B_1(x)} |x-y|^{\varepsilon} k(x,y)\,dydx + C\int_{A_{4r}} u(x)^2 dx + b_{k,A_{4r}}(u) + \int_{A_{2r}} u^2(x)\kappa_{k,A_{4r}}(x)\,dx \\
&\leq \frac{C}{r^{\varepsilon}}\int_{A_{4r}} u(x)^2\,dx + C\int_{A_{4r}} u(x)^2 dx + b_{k,A_{4r}}(u) + \int_{A_{2r}} u^2(x)\kappa_{k,A_{4r}}(x)\,dx.
\end{aligned}$$

Note here, since $u \in D^k(\mathbb{R}^N)$, we have $\int_{A_{4r}} u(x)^2 dx + b_{k,A_{4r}}(u) \to 0$ for $r \to 0$. Moreover, we have by Lebesgue's differentiation theorem

$$\frac{C}{r^\varepsilon} \int\limits_{A_{4r}} u(x)^2 dx \le \frac{C|B_{4r}|}{r^\varepsilon} \int\limits_{\partial\Omega} \frac{1}{|B_{4r}|} \int\limits_{B_{4r}(\theta)} u(x)^2 \, dx \sigma(d\theta)$$
$$\le C r^{1-\varepsilon} \int\limits_{\partial\Omega} \frac{1}{|B_{4r}|} \int\limits_{B_{4r}(\theta)} u(x)^2 \, dx \sigma(d\theta) \to 0 \quad \text{for } r \to 0^+.$$

Finally, since

$$K_{k,\Omega}(u) = \int\limits_{\Omega} u^2(x) \kappa_{k,\Omega}(x) \, dx < \infty \tag{5.17}$$

and, by Lemma 5.24, we have

$$\kappa_{k,A_{4r}}(x) \le \int\limits_{\mathbb{R}^N \setminus \Omega} k(x,y) \, dy + \int\limits_{\Omega \setminus A_{4r}} k(x,y) \, dy \le C \kappa_{k,\Omega}(x) + C r^{-\varepsilon}$$

for $x \in A_{2r}$, so that also $\int_{A_{2r}} u^2(x) \kappa_{k,A_{4r}}(x) \, dx \to 0$ for $r \to 0$ with a similar argument. □

Proof of Theorem 5.20 for $X(\Omega) = \mathscr{D}^k(\Omega)$. This statement now follows from Theorem 5.25, Lemma 5.22, and Proposition 5.23. □

Theorem 5.26 (See Theorem 5.20)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Then $C_c^\infty(\Omega)$ is dense in $D^k(\Omega)$. Moreover, if $u \in D^k(\Omega)$ is nonnegative, then we have*

1. *There exists a sequence $(u_n)_n \subset D^k(\Omega) \cap L^\infty(\Omega)$ with $\lim\limits_{n\to\infty} u_n = u$ in $D^k(\Omega)$ satisfying that for every $n \in \mathbb{N}$ there is $\Omega'_n \subset\subset \Omega$ with $u_n = 0$ on $\Omega \setminus \Omega'_n$ and $0 \le u_n \le u_{n+1} \le u$.*
2. *There exists a sequence $(u_n)_n \subset C_c^\infty(\Omega)$ with $u_n \ge 0$ for every $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} u_n = u$ in $D^k(\Omega)$.*

Proof. Consider the Lipschitz map

$$g_n : \mathbb{R} \to \mathbb{R}, \qquad g_n(t) = \begin{cases} 0 & t \le 0 \\ t & 0 < t < n \\ n & t \ge n. \end{cases}$$

Then $v_n := g_n(u) \in D^k(\Omega) \cap L^\infty(\Omega)$ and we have with φ_r as in the proof of Proposition 5.25

$$b_{k,\Omega}(u - (1-\varphi_r)v_n) \le b_{k,\Omega}(u - v_n) + b_{k,\Omega}(\varphi_r v_n).$$

Clearly, $b_{k,\Omega}(u - v_n) \to 0$ for $n \to \infty$ by dominated convergence and $b_{k,\Omega}(\varphi_r v_n) \to 0$ for $r \to 0$ analogously to the proof of Proposition 5.25, noting that the term in (5.17) reads in this case

$$K_{k,\Omega}(v_n) \le n^2 \int\limits_{\Omega} \kappa_{k,\Omega}(x) \, dx < \infty \quad \text{for every } n \in \mathbb{N}.$$

In particular, 1. follows. Now 2. and the density statement follow analogously, again, to the proof of Proposition 5.25. □

Proof of Theorem 5.20 for $X(\Omega) = D^k(\Omega)$. This statement now follows from Theorem 5.26, Lemma 5.22, and Proposition 5.23. □

Remark 5.27. *It is tempting to conjecture the following type of Hardy inequality: There is $C > 0$ such that*

$$K_{k,\Omega}(\varphi) \le C\Big(\|\varphi\|^2_{L^2(\Omega)} + b_{k,\Omega}(\varphi)\Big) \quad \textit{for all } \varphi \in C_c^\infty(\Omega)$$

if Ω is a bounded Lipschitz set and k satisfies addtionally (5.11)*. Let us mention that for $k(x,y) = |x-y|^{-2s-N}$ this holds for $s \in (0,1)$, $s \neq \frac{1}{2}$, see [27,34]. Moreover, for $k(x,y) = 1_{B_1(0)}(x-y)|x-y|^{-N}$, this has been shown in [29]. In the general framework presented here, however, it is not clear if this is true.*

Remark 5.28. *With the above density results, we can now note that our definition of weak supersolutions (and similarly of weak subsolutions and solutions), see Definition 5.14 can be extended slightly:*
Let $u \in \mathscr{V}^k_{loc}(\Omega)$ satisfy weakly $I_k u \ge f$ in Ω for some $f \in L^1_{loc}(\Omega)$ and $\Omega \subset \mathbb{R}^N$ open and bounded with Lipschitz boundary.

1. *If $f \in L^2_{loc}(\Omega)$, then by density it also holds*

$$b_k(u,v) \ge \int_U f(x)v(x)\, dx \quad \textit{for all nonnegative } v \in \mathscr{D}^k(U),\ U \subset\subset \Omega. \tag{5.18}$$

2. *If $u \in \mathscr{V}^k(\Omega) \cap L^\infty(\mathbb{R}^N)$ and $f \in L^2(\Omega)$, then by density it also holds*

$$b_k(u,v) \ge \int_\Omega f(x)v(x)\, dx \quad \textit{for all nonnegative } v \in \mathscr{D}^k(\Omega). \tag{5.19}$$

Finally note that if $u : \mathbb{R}^N \to \mathbb{R}$ satisfies $u1_U \in D^k(U)$ for some $U \subset\subset \mathbb{R}^N$ and $u \in L^\infty(\mathbb{R}^N \setminus U)$, then $u \in \mathscr{V}^k_{loc}(U)$.

Proof of Proposition 5.1. Note that also $u^- \in \mathscr{V}^k(\Omega)$ and in particular $u^- \in D^k(\Omega)$. Hence, we can find $(v_n)_n \subset C_c^\infty(\Omega)$ with $v_n \to u^-$ in $D^k(\Omega)$ for $n \to \infty$ with $0 \le v_n \le v_{n+1} \le u^-$ by Proposition 5.26. Then

$$b_{k,\mathbb{R}^N}(u,v_n) \ge \int_\Omega c(x)u(x)v_n(x)\, dx \ge -\|c^+\|_{L^\infty(\Omega)} \int_\Omega u^-(x)v_n(x)\, dx.$$

On the other hand, since $u^+v_n = 0$ for all $n \in \mathbb{N}$ and $u \ge 0$ almost everywhere in $\mathbb{R}^N \setminus \Omega$, we find

$$\begin{aligned} b_{k,\mathbb{R}^N}(u,v_n) &= b_{k,\Omega}(u,v_n) + \int_\Omega v_n(x) \int_{\mathbb{R}^N\setminus\Omega} (u(x)-u(y))k(x,y)\, dydx \\ &\le b_{k,\Omega}(u^+,v_n) - b_{k,\Omega}(u^-,v_n) + \int_\Omega v_n(x)u(x) \int_{\mathbb{R}^N\setminus\Omega} k(x,y)\, dydx \end{aligned}$$

$$\leq -b_{k,\Omega}(u^-,v_n) - K_{k,\Omega}(u^-,v_n).$$

Hence

$$0 \leq \int_\Omega u^-(x)v_n(x)\Big(\|c^+\|_{L^\infty(\Omega)} - \kappa_{k,\Omega}(x)\Big)\, dx - b_{k,\Omega}(u^-,v_n) \leq -b_{k,\Omega}(u^-,v_n).$$

Since $v_n \to u^-$ in $D^k(\Omega)$, it follows that $b_{k,\Omega}(u^-,u^-) = 0$, but then u^- is constant by Proposition 5.18 in Ω. Assume by contradiction that $u^- = m > 0$. Then the above calculation gives

$$0 \leq m \int_\Omega v_n(x)\Big(\|c^+\|_{L^\infty(\Omega)} - \kappa_{k,\Omega}(x)\Big)\, dx, \tag{5.20}$$

which is in both cases a contradiction: If in case 1. $c \leq 0$, then by (5.11) we have $\kappa_{k,\Omega}(x) \not\equiv 0$ and since $v_n \to m$ in $D^k(\Omega)$ the right-hand side of (5.20) is negative.
In case 2. this contradiction is immediate in a similar way. □

Remark 5.29. *Usually, the weak maximum principle is stated with an assumption on the first eigenvalue $\Lambda_1(\Omega)$ in place of $\inf_{x\in\Omega} \kappa_{k,\Omega}(x)$. This can be done once the Hardy inequality in Remark 5.27 is shown.*

Proof of Proposition 5.2. This statement follows by approximation from [60, Theorem 2.5 and 2.6]. Here, the statement $j \notin L^1(\mathbb{R}^N)$ comes into play since we need

$$\inf_{x\in B_r(x_0)} \kappa_{k,B_r(x_0)}(x) \to \infty \quad \text{for } r \to 0$$

to conclude the statement for arbitrary c as stated. □

5.6 On Boundedness

In the following, let $h * u(x) = \int_{\mathbb{R}^N} h(x-y)u(y)\, dy$ as usual denote the convolution of two functions.

Theorem 5.30. *Assume k satisfies* (5.11) *and is such that*

$$\sup_{x\in\mathbb{R}^N} \int_{K\setminus B_\varepsilon(x)} k(x,y)^2\, dy < \infty \quad \textit{for all } K \subset\subset \mathbb{R}^N \textit{ and } \varepsilon > 0. \tag{5.21}$$

Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $f \in L^\infty(\Omega)$, $h \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, and let $u \in \mathcal{V}^k_{loc}(\Omega)$ satisfy in weak sense

$$I_k u \leq \lambda u + h * u + f \quad \textit{in } \Omega \textit{ for some } \lambda > 0.$$

If $u^+ \in L^\infty(\mathbb{R}^N \setminus \Omega')$ for some $\Omega' \subset\subset \Omega$, then $u^+ \in L^\infty(\mathbb{R}^N)$ and there is $C = C(\Omega, \Omega', k, h, \lambda) > 0$ such that

$$\|u^+\|_{L^\infty(\Omega')} \leq C\Big(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^2(\Omega')} + \|u^+\|_{L^\infty(\mathbb{R}^N\setminus\Omega')}\Big).$$

Proof. Let $\Omega_1, \Omega_2, \Omega_3 \subset \mathbb{R}^N$ be with Lipschitz boundary and such that

$$\Omega' \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega.$$

Let $\eta \in C_c^{0,1}(\Omega_3)$ such that $0 \le \eta \le 1$ and $\eta = 1$ on Ω_2. Put $v = \eta u$ and, for $\delta > 0$, denote $J_\delta(x,y) := 1_{B_\delta(0)}(x-y)k(x,y)$ and $k_\delta(x,y) = k(x,y) - J_\delta(x,y)$. Note that by Assumption (5.1) it follows that $y \mapsto k_\delta(x,y) \in L^1(\mathbb{R}^N)$ for all $x \in \mathbb{R}^N$. Moreover, by Assumption (5.11)

$$c_\delta := \inf_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} k_\delta(x,y)\, dy \ge \int_{\mathbb{R}^N \setminus B_\delta(0)} j(z)\, dz \to \infty \quad \text{for } \delta \to 0.$$

Hence, we may fix $\delta > 0$ such that

$$c_\delta > \lambda.$$

In the following, $C_i > 0$, $i = 1, \dots$ denote constants depending on Ω', Ω_i, for $i = 1,2,3$, λ, δ, Ω, η, k, and h but may vary from line to line —clearly, by the choices these dependencies are actually only through λ, Ω, Ω', η, k, and h. First note that by Lemma 5.16 we have in weak sense

$$I_k v \le \lambda u + h * u + \tilde{f} \quad \text{in } \Omega_1 \quad \text{with} \quad \tilde{f}(x) = f(x) + \int_{\mathbb{R}^N \setminus \Omega_2} (1 - \eta(y)) u(y) k(x,y)\, dy.$$

In the following, put

$$A := \|f\|_{L^\infty(\Omega)} + \|u\|_{L^2(\Omega')} + \|u^+\|_{L^\infty(\mathbb{R}^N \setminus \Omega')}.$$

Then note that for $x \in \mathbb{R}^N$ we have

$$|h * u(x)| \le \|h\|_{L^2(\mathbb{R}^N)} \|u\|_{L^2(\Omega')} + \|u^+\|_{L^\infty(\mathbb{R}^N \setminus \Omega')} \|h\|_{L^1(\mathbb{R}^N)} \le C_1 A$$

and, since $\sup\limits_{x \in \Omega_1} \int_{\mathbb{R}^N \setminus \Omega_2} (1 - \eta(y)) k(x,y)\, dy \le C_2 \sup\limits_{x \in \Omega_1} \int_{\mathbb{R}^N \setminus \Omega_2} \min\{1, |x-y|^\sigma\} k(x,y)\, dy < \infty$, it also holds that

$$\|\tilde{f}\|_{L^\infty(\Omega_1)} \le \|f\|_{L^\infty(\Omega)} + \|u^+\|_{L^\infty(\mathbb{R}^N \setminus \Omega_1)} C_2 \le C_3 A.$$

Whence, since $u = v$ in Ω_1, we have in weak sense

$$I_k v \le \lambda v + C_4 A \quad \text{in } \Omega_1.$$

Next, let $\mu \in C_c^\infty(\Omega'')$ for some $\Omega' \subset\subset \Omega'' \subset\subset \Omega_1$ such that $0 \le \mu \le 1$, $\mu = 1$ on Ω', and $\mu = 0$ on $\mathbb{R}^N \setminus \Omega''$. Let $\varphi_t = \mu^2 (v - t)^+ \in \mathscr{D}^k(\Omega'')$ for $t > 0$ and note that

$$b_k(v, \varphi_t) \le \int_{\Omega''} (\lambda v(x) + C_4 A) \varphi_t(x)\, dx. \tag{5.22}$$

Fix $t > 0$ such that

$$t \ge \|u^+\|_{L^\infty(\mathbb{R}^N \setminus \Omega')} \quad \text{and} \quad C_6 A + (\lambda - c_\delta) t \le 0, \quad \text{where}$$

$$C_6 = C_4 + C_5 \quad \text{with} \quad C_5 = \sup_{x\in\Omega'} \int_{\Omega'} k_\delta^2(x,y)\, dy + \sup_{x\in\Omega'} \int_{\mathbb{R}^N\setminus\Omega'} k_\delta(x,y)\, dy.$$

That is, we fix

$$t = A\Big(1 + \frac{C_6}{c_\delta - \lambda}\Big).$$

Then with (5.22)

$$\begin{aligned} &b_{J_\delta}(v,\varphi_t) = b_k(v,\varphi_t) - b_{k_\delta}(v,\varphi_t) \\ &\le \int_{\Omega''} \lambda v(x)\varphi_t(x) + C_4 A\varphi_t(x)\, dx - \int_{\mathbb{R}^N} v(x)\varphi_t(x) \int_{\mathbb{R}^N} k_\delta(x,y)\, dydx + \int_{\mathbb{R}^N} \varphi_t(x) \int_{\mathbb{R}^N} v(y)k_\delta(x,y)\, dy\, dx. \end{aligned}$$

Note here, that for $x\in\mathbb{R}^N$ we have by the integrability assumptions on k_δ and k

$$\int_{\mathbb{R}^N} v(y)k_\delta(x,y)\, dy \le \int_{\mathbb{R}^N} u(y)k_\delta(x,y)\, dy \le C_5(\|u\|_{L^2(\Omega')} + \|u^+\|_{L^\infty(\mathbb{R}^N\setminus\Omega')}) \le C_5 A$$

so that using that $v \ge t$ in $\operatorname{supp}\varphi_t$ we have

$$b_{J_\delta}(v,\varphi_t) \le \int_{\Omega''} (C_6 A + (\lambda - c_\delta)v(x))\varphi_t(x)\, dx \le (C_6 A + (\lambda - c_\delta)t) \int_{\Omega''} \varphi_t(x)\, dx. \tag{5.23}$$

On the other hand, with $v_t(x) = v(x) - t$, we have

$$\begin{aligned} &(v(x)-v(y))(\varphi_t(x)-\varphi_t(y)) - (\mu(x)v_t^+(x) - \mu(y)v_t^+(y))^2 \\ &\qquad = 2\mu(x)\mu(y)v_t^+(x)v_t^+(y) - v_t(y)\mu^2(x)v_t^+(x) - \mu^2(y)v_t^+(y)v_t(x) \\ &\qquad = -v_t^+(x)v_t^+(y)(\mu(x)-\mu(y))^2 + v_t^-(y)\mu^2(x)v_t^+(x) + \mu^2(y)v_t^+(y)v_t^-(x) \\ &\qquad \ge -v_t^+(x)v_t^+(y)(\mu(x)-\mu(y))^2. \end{aligned}$$

Whence with Poincaré's inequality, using that by Assumption 5.11 there is for any $K\subset\mathbb{R}^N$ open and bounded some $C>0$ such that $b_{J_\delta}(u) \ge C\|u\|^2_{L^2(\mathbb{R}^N)}$ for $u\in\mathscr{D}^{J_\delta}(K)$, we find for some constant C_7

$$\begin{aligned} b_{J_\delta}(v,\varphi_t) &\ge b_{J_\delta}(\mu v_t^+) - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} v_t^+(x)v_t^+(y)(\mu(x)-\mu(y))^2 J_\delta(x,y)\, dxdy \\ &\ge C_7 \int_{\mathbb{R}^N} \mu^2(x)(v_t^+(x))^2\, dx - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} v_t^+(x)v_t^+(y)(\mu(x)-\mu(y))^2 J_\delta(x,y)\, dxdy \qquad (5.24) \\ &= C_7 \int_{\mathbb{R}^N} \mu^2(x)(v_t^+(x))^2\, dx - \frac{1}{2}\int_{\Omega'}\int_{\Omega'} v_t^+(x)v_t^+(y)(\mu(x)-\mu(y))^2 J_\delta(x,y)\, dxdy \qquad (5.25) \\ &= C_7 \int_{\mathbb{R}^N} \mu^2(x)(v_t^+(x))^2\, dx \ge C_7 \int_{\Omega'} (v_t^+(x))^2\, dx. \qquad (5.26) \end{aligned}$$

Combining (5.26) and (5.23) we have

$$C_7 \int_{\Omega'} (v_t^+(x))^2 \, dx \le \Big(C_6 A + (\lambda - c_\delta)t\Big) \int_{\Omega''} \varphi_t(x) \, dx \le 0.$$

Whence $v_t^+ = 0$ in Ω' and thus $u = v \le t = AC_{10}$ in Ω' as claimed. □

Corollary 5.31. *If in the situation of Theorem 5.30 we have in weak sense $I_k u = \lambda u + h * u + f$ in Ω, then we have $u \in L^\infty(\Omega')$ and there is $C = C(\Omega, \Omega', k, \lambda, h) > 0$ such that*

$$\|u\|_{L^\infty(\Omega')} \le C\Big(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^2(\Omega')} + \|u\|_{L^\infty(\mathbb{R}^N \setminus \Omega')}\Big).$$

Proof. This follows by replacing u with $-u$ (and f with $-f$) in the statement of Theorem 5.30. □

Proof of Theorem 5.3. This follows directly from Corollary 5.31. □

Theorem 5.32. *If in the situation of Theorem 5.30 we have in weak sense $I_k u = \lambda u + h * u + f$ in Ω and $u \in \mathscr{D}^k(\Omega)$, then we have $u \in L^\infty(\Omega)$ and there is $C = C(\Omega, k, \lambda, h) > 0$ such that*

$$\|u\|_{L^\infty(\Omega)} \le C\Big(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^2(\Omega)}\Big).$$

Proof. Using in the proof of Theorem 5.30 the test-function u_t^+ instead of φ_t (and similarly for Corollary 5.31), we find

$$\|u\|_{L^\infty(\Omega)} \le C\Big(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^2(\Omega)}\Big).$$

as claimed. □

Proof of Corollary 5.5. The compact embedding has been shown in [61], the fact that the first eigenfunction can be chosen to be positive follows from [60] and the final statement of the boundedness follows from Theorem 5.3 (with $h = f = 0$) if Ω has a Lipschitz boundary. If this is not the case, it can be easily seen that eigenfunctions u corresponding to an eigenvalue λ actually satisfy $b_k(u,v) = \lambda \int_\Omega uv \, dx$ for all $v \in \mathscr{D}^k(\Omega)$ (see Remark 5.4). Whence, as in the proof of Theorem 5.1 the test-function $u_t^+ \in \mathscr{D}^k(\Omega)$ can be used. □

5.7 On differentiability of solutions

In the following, $\Omega \subset \mathbb{R}^N$ is an open bounded set and k satisfies through out the assumptions (5.11), (5.12), and (5.13) for some $m \in \mathbb{N} \cup \{\infty\}$ —in particular, there is $J : \mathbb{R}^N \to [0, \infty]$ such that $k(x,y) = J(x-y)$ for $x, y \in \mathbb{R}^N$. We hence also write J in place of k. Moreover, given σ from assumption (5.1) we assume that $\sigma < \frac{1}{2}$ and fix

$$\alpha := 1 - \sigma \in (\frac{1}{2}, 1).$$

Theorem 5.33. *Let $f \in C^1(\overline{\Omega})$, $\lambda \in \mathbb{R}$ and $u \in \mathscr{V}^J_{loc}(\Omega) \cap L^\infty(\mathbb{R}^N)$ satisfy in weak sense $I_k u = f + \lambda u$ in Ω. Then for any $\Omega' \subset\subset \Omega$ there is $C = C(N, \Omega, \Omega', J, \lambda) > 0$ such that*

$$\|\delta_{h,e} u\|_{L^2(\Omega')} \leq h^\alpha C\Big(\|f\|^2_{C^1(\Omega)} + \|u\|^2_{L^\infty(\mathbb{R}^N)}\Big)^{\frac{1}{2}} \quad \text{for all } h > 0,\ e \in \partial B_1(0). \tag{5.27}$$

Proof. Let $\Omega' \subset\subset \Omega$ and fix $r \in (0, \frac{1}{8})$ small such that $8r \leq \operatorname{dist}(\Omega', \mathbb{R}^N \setminus \Omega)$. Moreover, fix $x_0 \in \Omega'$ and denote $B_n := B_{nr}(x_0)$. Note that by using assumption (5.11) with Lemma 5.10 we achieve, by making $r > 0$ small enough,

$$\lambda < \lambda_1 = \min_{\substack{w \in \mathscr{D}^J(B_4) \\ w \neq 0}} \frac{\rho_J(w)}{\|w\|_{L^2(\mathbb{R}^N)^2}}.$$

Let $\eta \in C^{0,1}_c(B_4)$ with $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_2. Note that it holds

$$|\eta(x) - \eta(y)| \leq 2\|\eta\|_{C^{0,1}(\mathbb{R}^N)} \min\{1, |x-y|\},$$

where we put as usual

$$\|\eta\|_{C^{0,1}(\mathbb{R}^N)} := \sup_{x \in \mathbb{R}^N} |\eta(x)| + \sup_{\substack{x,y \in \mathbb{R}^N \\ x \neq y}} \frac{|\eta(x) - \eta(y)|}{|x-y|}.$$

Note that by choice we have $\|\eta\|_{C^{0,1}(\mathbb{R}^N)} \leq 1 + \frac{1}{r} \leq \frac{2}{r}$, so that for all $x, y \in \mathbb{R}^N$

$$|\eta(x) - \eta(y)| \leq \frac{4}{r} \min\{1, |x-y|\}. \tag{5.28}$$

Fix $e \in \partial B_1(0)$ and $h \in (0, r)$. Let

$$A := \|u\|_{L^\infty(\mathbb{R}^N)}.$$

Let $\psi = \eta^2 \delta_h u \in \mathscr{D}^J(B_4)$, where in the following $\delta_h u := \delta_{h,e} u$. Note that

$$\begin{aligned}(\delta_h u(x) - \delta_h u(y))(\psi(x) - \psi(y)) = (\eta(x)\delta_h u(x) - \eta(y)\delta_h u(y))^2 \\ - \delta_h u(x)\partial_h u(y)(\eta(x) - \eta(y))^2.\end{aligned}$$

Hence, we have

$$b_J(\delta_h u, \psi) = b_J(\eta \delta_h u) - \frac{1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \delta_h u(x)\delta_h u(y)(\eta(x) - \eta(y))^2 J(x-y)\, dxdy.$$

and using the translation invariance, we also have

$$b_J(\delta_h u, \psi) = \int_\Omega [\delta_h f(x) + \lambda \delta_h u]\psi(x)\, dx.$$

In the following, for simplicity, we put $v(x) = \eta(x)\delta_h u(x)$, $x \in \mathbb{R}^N$. Note that by Definition, $v \in \mathscr{D}^J(B_4)$. Then with the help of Young's inequality for some $\mu \in (0,1)$ such that

$$2\mu < \lambda_1 - \lambda \tag{5.29}$$

we find

$$\begin{aligned}
\lambda_1 \|v\|^2_{L^2(\Omega'')} &\le b_J(v) = b_J(\delta_h u, \psi) + \frac{1}{2} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^N} \delta_h u(x)\delta_h u(y)(\eta(x)-\eta(y))^2 J(x-y)\, dxdy \\
&= \int\limits_{\Omega''} [\delta_h f(x) + \lambda\,\delta_h u]\eta^2(x)\delta_h u(x)\, dx + \frac{1}{2} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^N} \delta_h u(x)\delta_h u(y)(\eta(x)-\eta(y))^2 J(x-y)\, dxdy \\
&\le (\mu+\lambda)\|v\|^2_{L^2(\Omega'')} + \mu^{-1}h^2 \|\frac{\delta_h f}{h}\|^2_{L^2(\Omega'')} + \frac{1}{2} \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \delta_h u(x)\delta_h u(y)(\eta(x)-\eta(y))^2 J(x-y)\, dxdy.
\end{aligned} \tag{5.30}$$

By a rearrangement of the double integral with Young's inequality for the same $\mu \in (0,1)$ as above we have

$$\begin{aligned}
&\frac{1}{2} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^N} \delta_h u(x)\delta_h u(y)(\eta(x)-\eta(y))^2 J(x-y)\, dxdy \\
&\quad = \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \delta_h u(x)\delta_h u(y)\eta(x)(\eta(x)-\eta(y))J(x-y)\, dxdy \\
&\quad = \int\limits_{\mathbb{R}^N} \eta(x)\delta_h u(x) \int\limits_{\mathbb{R}^N} u(y)\delta_{-h,y}\Big((\eta(x)-\eta(y))J(x-y)\Big)\, dydx \\
&\quad \le \mu\|v\|^2_{L^2(B_4)} + \mu^{-1} \int\limits_{B_4} \left(\int\limits_{\mathbb{R}^N} |u(y)| \Big|\delta_{-h,y}\Big((\eta(x)-\eta(y))J(x-y)\Big)\Big|\, dy \right)^2 dx \\
&\quad \le \mu\|v\|^2_{L^2(B_4)} + \mu^{-1}A^2 \int\limits_{B_4} \left(\int\limits_{\mathbb{R}^N} \Big|\delta_{-h,y}\Big((\eta(x)-\eta(y))J(y-x)\Big)\Big|\, dy \right)^2 dx \\
&\quad \le \mu\|v\|^2_{L^2(B_4)} + \mu^{-1}A^2 \int\limits_{B_4} \left(\int\limits_{\mathbb{R}^N} \Big|\delta_{-h,z}\Big((\eta(x)-\eta(z+x))J(z)\Big)\Big|\, dz \right)^2 dx.
\end{aligned} \tag{5.31}$$

Here, we indicate with $\delta_{-h,y}$ (resp. $\delta_{-h,z}$) that δ_{-h} acts on the y (resp. z) variable. Note that

$$\delta_{-h,z}\Big((\eta(x)-\eta(z+x))J(z)\Big)$$
$$= \delta_{-h,z}(\eta(x)-\eta(z+x))J(z) + (\eta(x)-\eta(z+x-he))\delta_{-h}J(z)$$
$$= \Big(\eta(z+x)-\eta(z+x-he)\Big)J(z) + (\eta(x)-\eta(z+x-he))\Big(J(z-he)-J(z)\Big) \tag{5.32}$$
$$= \Big(\eta(z+x)-\eta(z)\Big)J(z) + \Big(\eta(x)-\eta(z+x-he)\Big)J(z-he). \tag{5.33}$$

Note here, that (5.32) satisfies

$$\begin{aligned}
&\left|\Big(\eta(z+x)-\eta(z+x-he)\Big)J(z)+(\eta(x)-\eta(z+x-he))\Big(J(z-he)-J(z)\Big)\right| \\
&\qquad\qquad \leq \frac{4h}{r}J(z)+\frac{4h}{r}\min\{1,|z-he|\}\int_0^1 |\nabla J(z-\tau he)|\,d\tau
\end{aligned} \tag{5.34}$$

and (5.33) can be written as

$$\begin{aligned}
&\left|\Big(\eta(z+x)-\eta(z)\Big)J(z)+\Big(\eta(x)-\eta(z+x-he)\Big)J(z-he)\right| \\
&\qquad\qquad \leq \frac{4}{r}\min\{1,|z|\}J(z)+\frac{4}{r}\min\{1,|z-he|\}J(z-he).
\end{aligned} \tag{5.35}$$

For $h\in(0,r)$, $z\in\mathbb{R}^N\setminus\{0\}$ put

$$\begin{aligned}
k_h(z)=\min\Bigg\{&h\Big(J(z)+\min\{1,|z-he|\}\int_0^1 |\nabla J(z-\tau he)|\,d\tau\Big), \\
&\min\{1,|z|\}J(z)+\min\{1,|z-he|\}J(z-he)\Bigg\}.
\end{aligned}$$

Then, by combining (5.30) and (5.31), we find

$$\begin{aligned}
\|\delta_h u\|^2_{L^2(B_2)} &\leq \|v\|^2_{L^2(B_4)} \\
&\leq \frac{\mu^{-1}|B_4|}{\lambda_1-\lambda-2\mu}\left(h^2\|\frac{\delta_h f}{h}\|^2_{L^2(B_4)}+\frac{16}{r^2}\|u\|^2_{L^\infty(\mathbb{R}^N)}\Big(\int_{\mathbb{R}^N} k_h(z)\,dz\Big)^2\right).
\end{aligned} \tag{5.36}$$

Next we show that we have $\int_{\mathbb{R}^N} k_h(z)\,dz\leq Ch^\alpha$ for some $C>0$. Clearly, we can bound

$$\int_{\mathbb{R}^N\setminus B_2(0)} k_h(z)\,dz\leq C_1 h \tag{5.37}$$

for some $C_1=C_1(n,J)>0$, using that $B_1(0)\cup B_1(he)\subset B_2(0)$ and the properties of J. In the following, by making C_J larger if necessary, we may also assume that assumption (5.1) reads

$$\sup_{x\in\mathbb{R}^N}\int_{\mathbb{R}^N}\min\{1,|x-y|^\sigma\}J(x-y)\,dy=\int_{\mathbb{R}^N}\min\{1,|z|^\sigma\}J(z)\,dz\leq C_J.$$

Then note that $B_{2h}(he)\subset B_{3h}(0)$ and we have

$$\int_{B_{2h}(0)}\min\{1,|z|\}J(z)+\min\{1,|z-he|\}J(z-he)\,dz$$

$$\leq C_J \int\limits_{B_{3h}(0)} |z|^{1-\sigma-n}\, dz + C_J \int\limits_{B_{3h}(he)} |z-he|^{1-\sigma-n}\, dz$$

$$= \frac{2|B_1(0)|C_J}{n} \int\limits_0^{3h} \rho^{-\sigma}\, d\rho = \frac{2|B_1(0)|C_J}{n(1-\sigma)}(3h)^{1-\sigma}. \tag{5.38}$$

While with $b_\sigma(t) = \frac{1}{\sigma}t^{-\sigma}$ we have

$$h \int\limits_{B_2(0)\setminus B_{2h}(0)} J(z) + \min\{1, |z-he|\} \int\limits_0^1 |\nabla J(z-\tau he)|\, d\tau\, dz$$

$$\leq \frac{h|B_1(0)|C_J}{n} \int\limits_{2h}^{2} \rho^{-\sigma-1}\, d\rho + hC_J \int\limits_0^1 \int\limits_{B_3(\tau he)\setminus B_h(\tau he)} |z||z-\tau he|^{-1-\sigma-n}\, dz\, d\tau$$

$$\leq \frac{h|B_1(0)|C_J}{n} b_\sigma(2h) + hC_J \int\limits_0^1 \int\limits_{B_3(0)\setminus B_h(0)} |z+\tau he||z|^{-1-\sigma-n}\, dz\, d\tau$$

$$\leq \frac{h|B_1(0)|C_J}{n} b_\sigma(2h) + hC_J \int\limits_{B_3(0)\setminus B_h(0)} |z|^{-\sigma-n}\, dz + h^2 C_J \int\limits_{B_3(0)\setminus B_h(0)} |z|^{-1-\sigma-n}\, dz$$

$$\leq \frac{2h|B_1(0)|C_J}{n} b_\sigma(h) + \frac{h^2|B_1(0)|C_J}{n} \int\limits_h^3 \rho^{-2-\sigma}\, d\rho$$

$$\leq \frac{2|B_1(0)|C_J}{n} h b_\sigma(h) + \frac{|B_1(0)|C_J}{n(1+\sigma)} h^{1-\sigma}. \tag{5.39}$$

Combining (5.37) with (5.38) and (5.39) and the choice $\alpha = 1-\sigma \in (0,1)$ we find $C_2 = C_2(n, J, \alpha) > 0$ such that

$$\int\limits_{\mathbb{R}^N} k_h(z)\, dz \leq C_2 h^\alpha. \tag{5.40}$$

Whence, from (5.36) with (5.40) we have

$$\|\delta_h u\|^2_{L^2(B_2)} \leq \|v\|^2_{L^2(B_4)} \leq h^{2\alpha} C_4 \Big(\|\frac{\delta_h f}{h}\|^2_{L^2(B_4)} + \|u\|^2_{L^\infty(\mathbb{R}^N)} \Big), \tag{5.41}$$

for a constant $C_4 = C_4(N, J, r, \alpha, \lambda) > 0$. By a standard covering argument, we then also find with a constant $C_5 = C_5(N, J, \Omega, \Omega', \alpha, \lambda) > 0$ and $\Omega'' = \{x \in \Omega \,:\, \mathrm{dist}(x, \mathbb{R}^N \setminus \Omega) > 4r\}$

$$\|\delta_h u\|^2_{L^2(\Omega')} \leq h^{2\alpha} C_4 \Big(\|\frac{\delta_h f}{h}\|^2_{L^2(\Omega'')} + \|u\|^2_{L^\infty(\mathbb{R}^N)} \Big), \tag{5.42}$$

The claim (5.27) then follows since $f \in C^1(\Omega)$. $\square$

Remark 5.34. *Combining Theorem 5.33 with Corollary 5.31 it follows that we have in the situation of Theorem 5.33 for every $\Omega' \subset\subset \Omega$*

$$\|\delta_{h,e}u\|_{L^2(\Omega')} \le h^{\alpha} C\Big(\|f\|^2_{C^1(\Omega)} + \|u\|^2_{L^2(\Omega')} + \|u\|^2_{L^\infty(\mathbb{R}^N\setminus\Omega')}\Big)^{\frac{1}{2}} \quad \text{for all } h>0,\ e \in \partial B_1(0). \tag{5.43}$$

Corollary 5.35. *Assume $m=1$. Let $f \in C^2(\overline{\Omega})$, $\lambda \in \mathbb{R}$, and let $u \in \mathcal{V}^J_{loc}(\Omega) \cap L^\infty(\mathbb{R}^N)$ satisfy in weak sense $I_k u = \lambda u + f$ in Ω. Then $u \in H^1(\Omega')$ and $\partial_i u \in D^J(\Omega')$ for any $\Omega' \subset\subset \Omega$. More precisely, with α as above there is for any $\Omega' \subset\subset \Omega$ a constant $C = C(N,\Omega,\Omega',J,\lambda) > 0$ such that*

$$\sup_{\substack{e\in\partial B_1(0)\\ h>0}} h^{-2\alpha}\|\delta^2_{h,e}u\|_{L^2(\Omega')} \le C\Big(\|f\|^2_{C^2(\Omega)} + \|u\|_{L^2(\Omega')} + \|u\|^2_{L^\infty(\mathbb{R}^N\setminus\Omega')}\Big)^{\frac{1}{2}}, \tag{5.44}$$

so that $u \in N^{2\alpha,2}(\Omega') \subset H^1(\Omega')$, that is, there is also $C' = C'(n,J,\Omega,\Omega',\alpha,\lambda) > 0$ such that

$$\|\nabla u\|_{L^2(\Omega')} \le C'\Big(\|f\|^2_{C^2(\Omega)} + \|u\|_{L^2(\Omega')} + \|u\|^2_{L^\infty(\mathbb{R}^N\setminus\Omega')}\Big)^{\frac{1}{2}} \tag{5.45}$$

and, moreover,

$$b_{J,\Omega'}(\partial_i u) \le C' \quad \text{for } i = 1,\dots,N.$$

Proof. Let $\Omega_i \subset\subset \Omega$, $i = 1,\dots,7$ such that

$$\Omega' \subset\subset \Omega_i \subset\subset \Omega_j \quad \text{for } 1 \le i < j \le 7.$$

Let $\eta \in C_c^\infty(\Omega_7)$ with $\eta = 1$ on Ω_6 and $0 \le \eta \le 1$. Fix $e \in \partial B_1(0)$ and $h \in (0, \frac{1}{2}r)$, where $r = \min\{\operatorname{dist}(\Omega_i, \Omega\setminus\Omega_{i+1}) \,:\, i = 1,\dots,6\}$. Then by Lemma 5.16 the function $v = \eta\delta_h u$, where we write δ_h instead of $\delta_{h,e}$, satisfies $I_k v = \lambda v + \tilde{f}$ in Ω_5, where $\tilde{f} = \delta_h f + g_{\eta,\delta_h u}$. Following the proof of Theorem 5.33 to (5.42) it follows with Theorem 5.30 that there is $C = C(n,J,r,\alpha,\lambda) > 0$ (changing from line to line) such that

$$\begin{aligned}
\|\delta_h^2 u\|^2_{L^2(\Omega')} = \|\delta_h v\|^2_{L^2(\Omega')} &\le h^{2\alpha} C\Big(\|\frac{\delta_h\tilde{f}}{h}\|^2_{L^2(\Omega_1)} + \|v\|^2_{L^\infty(\mathbb{R}^N)}\Big)\\
&\le h^{2\alpha} C\Big(\|\frac{\delta_h\tilde{f}}{h}\|^2_{L^2(\Omega_1)} + \|\tilde{f}\|^2_{L^\infty(\Omega_4)} + \|v\|^2_{L^2(\Omega_3)} + \|v\|^2_{L^\infty(\mathbb{R}^N\setminus\Omega_3)}\Big)\\
&\le h^{2\alpha} C\Big(\|\frac{\delta_h\tilde{f}}{h}\|^2_{L^2(\Omega_1)} + \|\tilde{f}\|^2_{L^\infty(\Omega_4)} + \|\delta_h u\|^2_{L^2(\Omega_3)}\Big)\\
&\le h^{2\alpha} C\Big(\|\frac{\delta_h\tilde{f}}{h}\|^2_{L^2(\Omega_1)} + \|\tilde{f}\|^2_{L^\infty(\Omega_4)} + h^{2\alpha}\Big(\|f\|^2_{C^1(\Omega)} + \|u\|^2_{L^\infty(\mathbb{R}^N)}\Big)\Big),
\end{aligned}$$

where we applied once more Theorem 5.33. Here, for $x \in \Omega_4$ using assumption (5.13) it follows that there is $C = C(J) > 0$ such that

$$|\tilde{f}(x)| \le |\delta_h f(x)| + \Big|\int_{\mathbb{R}^N\setminus\Omega_6} (1-\eta(y))\delta_h u(y) J(x-y)\, dy\Big|$$

$$= |\delta_h f(x)| + \left| \int_{\mathbb{R}^N\setminus\Omega_5} |u(y)|\delta_h[(1-\eta(y))J(x-y)]\,dy \right.$$
$$\leq hC\Big(\|\nabla f\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\mathbb{R}^N\setminus\Omega')}\Big).$$

Moreover, for $x \in \Omega_1$ in a similar way there is $C = C(J) > 0$ such that

$$|\delta_h \tilde{f}(x)| \leq |\delta_h^2 f(x)| + \left| \int_{\mathbb{R}^N\setminus\Omega_6} (1-\eta(y))\delta_h u(y)\delta_h J(x-y)\,dy \right|$$
$$\leq h^2\|f\|_{C^2(\Omega)} + \|u\|_{L^\infty(\mathbb{R}^N\setminus\Omega')} \left| \int_{\mathbb{R}^N\setminus\Omega_5} \delta_h[(1-\eta(y))\delta_h J(x-y)]\,dy \right|$$
$$\leq h^2 C\left(\|f\|_{C^2(\Omega)} + \|u\|_{L^\infty(\mathbb{R}^N\setminus\Omega')}\right).$$

Thus we have

$$\|\delta_h^2 u\|^2_{L^2(\Omega')} \leq Ch^{4\alpha}\left(\|f\|^2_{C^2(\Omega)} + \|u\|^2_{L^2(\Omega')} + \|u\|^2_{L^\infty(\mathbb{R}^N\setminus\Omega')}\right).$$

The proof of the first part then is finished with Proposition 5.19 since $2\alpha > 1$. Next, write $D_h p(x) = \frac{p(x+he)-p(x)}{h}$ for any function $p : \mathbb{R}^N \to \mathbb{R}$, with $e \in \partial B_1(0)$ fixed and $h \in \mathbb{R}\setminus\{0\}$. Then with Lemma 5.16 for some $\eta \in C_c^\infty(\Omega)$ such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on $\Omega_2 \subset\subset \Omega$ with $\Omega' \subset\subset \Omega_1 \subset\subset \Omega_2$ we have with $v = \eta u$,

$$Iv = f + \lambda v + g_{\eta,u} \quad \text{in } \Omega_1, \text{ where} \quad g_{\eta,u} = \int_{\mathbb{R}^N\setminus\Omega_2} (1-\eta(y))u(y)J(x-y)\,dy.$$

Next, let $\mu \in C_c^\infty(\Omega_1)$ with $0 \leq \mu \leq 1$ and $\mu \equiv 1$ on Ω'. Then with $\varphi = D_{-h}[\mu^2 D_h v] \in \mathscr{D}^J(\Omega_1)$ for h small enough we have for some $C > 0$ (which may change from line to line independently of h)

$$|b_J(v,\varphi)| = \left| \int_{\Omega_1} D_h f \mu^2 D_h v + \lambda(\mu D_h v)^2 + D_h g_{\eta,u}\mu^2 D_h v\,dx \right| \leq C, \tag{5.46}$$

since

$$\int_{\Omega_1} |D_h f \mu^2 D_h v|\,dx \leq C\|f\|_{C^1(\Omega)}\|\nabla u\|_{L^2(\Omega_2)} < \infty, \qquad \int_{\Omega_1} |\lambda(\mu D_h v)^2|\,dx \leq 2|\lambda|\|\nabla u\|^2_{L^2(\Omega_2)} < \infty,$$

and

$$\int_{\Omega_1} |D_h g_{\eta,u}\mu^2 D_h v|\,dx \leq C\Big(\int_{\Omega_1}\int_{\mathbb{R}^N\setminus\Omega_2} |(1-\eta(y))u(y)|[D_h J](x-y)|\,dy\,dx\Big)^{1/2}\|\nabla u\|_{L^2(\Omega_2)} < \infty$$

due to assumption (5.13). Moreover, with a similar calculation as in the proof of Theorem 5.33 we have

$$b_J(v,\varphi) = b_J(\mu D_h v, \mu D_h v) - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} D_h v(x) D_h v(y)(\mu(x)-\mu(y))^2 J(x-y)\,dxdy,$$

where for some $\Omega_2 \subset\subset \Omega_3 \subset\subset \Omega_4 \subset\subset \Omega$ with h small enough

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |D_h v(x) D_h v(y)(\mu(x)-\mu(y))^2 J(x-y)|\,dxdy \\
&\qquad \le C\int_{\Omega_3}\int_{\Omega_3} |D_h(\eta u)(x) D_h(\eta u)(y)||x-y|^2 J(x-y)\,dxdy \\
&\qquad \le C\int_{\Omega_3} |D_h(\eta u)(x)|^2 \int_{\Omega_3} |x-y|^2 J(x-y)\,dydx \\
&\qquad \le C\|\nabla u\|_{L^2(\Omega_4)} \int_{\mathbb{R}^N} \min\{1,|z|^2\} J(z)\,dz < \infty.
\end{aligned}$$

Combining this with (5.46) we find

$$b_J(\mu D_h v, \mu D_h v) \le C \quad \text{for all } h>0 \text{ small enough.}$$

Since also $\mu D_h v \in \mathscr{D}^J(\Omega_2)$ for all $h>0$ small enough (see Lemma 5.11) and since $D^J(\Omega_2)$ is a Hilbert space, we conclude that $\mu\partial_e v \in \mathscr{D}^J(\Omega_2)$ with

$$b_J(\mu\partial_e v) \le C$$

for $h \to 0$. This finishes the proof. □

Corollary 5.36. *Let $f \in C^{2m}(\overline{\Omega})$, $\lambda \in \mathbb{R}$, and let $u \in \mathscr{V}^J_{loc}(\Omega) \cap L^\infty(\mathbb{R}^N)$ satisfy in weak sense $I_k u = \lambda u + f$ in Ω. Then $u \in H^m(\Omega')$ for any $\Omega' \subset\subset \Omega$ and there is $C = C(n,J,\Omega,\Omega',m) > 0$ such that*

$$\|u\|_{H^m(\Omega')} \le C\Big(\|f\|^2_{C^m(\Omega)} + \|u\|^2_{L^2(\Omega')} + \|u\|^2_{L^\infty(\mathbb{R}^N\setminus\Omega')}\Big)^{\frac{1}{2}}. \tag{5.47}$$

In particular, if $m=\infty$, then $u \in C^\infty(\Omega)$.

Proof. By Corollary 5.35 the claim holds for $m=1$ in particular with $u|_{\Omega'} \in D^J(\Omega')$ for all $\Omega' \subset\subset \Omega$. Assume next, the claim holds for $m-1$ with $m \in \mathbb{N}$, $m \ge 2$ in the following way: We have $u \in H^{m-1}(\Omega')$ and $\partial^\beta u|_{\Omega'} \in D^J(\Omega')$ for any $\Omega' \subset\subset \Omega$ and $\beta \in \mathbb{N}_0^N$ with $|\beta| \le m-1$, and there is $C = C(n,J,\Omega,\Omega',m) > 0$ such that

$$\|u\|_{H^{m-1}(\Omega')} \le C\Big(\|f\|^2_{C^{m-1}(\Omega)} + \|u\|_{L^2(\Omega')} + \|u\|^2_{L^\infty(\mathbb{R}^N\setminus\Omega')}\Big)^{\frac{1}{2}}. \tag{5.48}$$

Fix $\Omega' \subset\subset \Omega$ and let $\Omega_i \subset\subset \Omega$, $i = 1,\dots,7$ and $\eta \in C_c^\infty(\Omega_7)$ as in the proof of Corollary 5.35. Put $v = \partial^\beta(\eta u)$ for some $\beta \in \mathbb{N}_0^N$, $|\beta| = m-1$. Then $I_k v = \partial^\beta f + \lambda v + \partial^\beta g_{\eta,u}$ in Ω_5 by Lemma 5.16 and direct computation using the assumptions on J. From here, proceeding as in the proof of Corollary 5.35 by applying Theorem 5.33 the claim follows. □

Proof of Theorem 5.6. The first part follows from the Poincaré inequality Lemma 5.10 and Theorem 5.3 with $h = 0 = \lambda$. The last assertion follows from Corollary 5.36. $\square$

Proof of Theorem 5.8. This statement follows directly from Corollary 5.36. $\square$

6 Appendix

6.1 An inequality

The following is a variant of [35, Lemma 10] (see also [59, Lemma 5.1]).

Lemma 6.1. *Let $q \in L^1(\mathbb{R}^N)$ be a nonnegative even function with $q = 0$ on $\mathbb{R}^N \setminus B_r(0)$ for some $r > 0$. Let $\Omega \subset \mathbb{R}^N$ open and $x_0 \in \Omega$ such that $B_{2r}(x_0) \subset \Omega$. Then for all measurable functions $u : \Omega \to \mathbb{R}$ we have*

$$b_{q*q,B_r(x_0)}(u) \leq 4\|q\|_{L^1(\mathbb{R}^N)} b_{q,\Omega}(u),$$

where the bilinear form $b_{k,A}$ for an open $A \subset \mathbb{R}^N$ is defined in (5.2) *by*

$$b_{k,A}(u,v) := \frac{1}{2}\int_A \int_A (u(x)-u(y))(v(x)-v(y))k(x,y)\, dxdy \tag{6.1}$$

with $b_{k,A}(u,u) = b_{k,A}(u)$.

Proof. Let u be as stated and we extend u trivially to a function on $\mathbb{R}^N$. Denote $g(x,y) = (u(x)-u(y))^2$ for $x, y \in \mathbb{R}^N$. Note that we have

$$0 \leq g(x,y) = g(y,x) \leq 2g(x,z) + 2g(y,z) \qquad \text{for all } x,y,z \in \mathbb{R}^N.$$

By Fubini's theorem we have

$$\begin{aligned}
\int_{B_r(x_0)} \int_{B_r(x_0)} g(x,y)(q*q)(x-y)\, dxdy &= \int_{B_r(x_0)} \int_{B_r(x_0)} \int_{\mathbb{R}^N} g(x,y)q(x-z)q(y-z)\, dzdxdy \\
&\leq 2 \int_{B_r(x_0)} \int_{B_r(x_0)} \int_{\mathbb{R}^N} [g(x,z)+g(y,z)]q(x-z)q(y-z)\, dzdxdy \\
&\leq 4 \int_{B_r(x_0)} \int_{\mathbb{R}^N} g(x,z)q(x-z) \int_{\mathbb{R}^N} q(y-z)\, dydzdx = 4\|q\|_{L^1(\mathbb{R}^N)} \int_{B_r(x_0)} \int_{\mathbb{R}^N} g(x,z)q(x-z)\, dzdx.
\end{aligned}$$

Note that since $q = 0$ on $\mathbb{R}^N \setminus B_r(x_0)$, q is even, and $B_r(x) \subset B_{2r}(x_0) \subset \Omega$ for any $x \in B_r(x_0)$, we have

$$\int_{B_r(x_0)} \int_{\mathbb{R}^N} g(x,z)q(x-z)\, dzdx = \int_{B_r(x_0)} \int_{B_r(x)} (u(x)-u(z))^2 q(x-z)\, dzdx \leq 2b_{q,\Omega}(u).$$

□

6.2 On equivalent Hölder estimates

Here we recall that by the notion of Hölder-Zygmund spaces we have for $\tau \in (0,1)$ and $r > 0$ that $v \in C^\tau(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ if and only if

$$\|v\|_{L^\infty(\mathbb{R}^N)} + \sup_{\substack{x,h \in \mathbb{R}^N \\ h \neq 0}} \frac{|2v(x+h) - v(x+2h) - v(x)|}{|h|^\tau} =: v_\tau < \infty. \tag{6.2}$$

Indeed, if $v \in C^\tau(B_r(0)) \cap L^\infty(\mathbb{R}^N)$, then clearly (6.2) holds. To see the reverse implication, first note that we have $\|v\|_{L^\infty(\mathbb{R}^N)} \le v_\tau < \infty$ by (6.2). Next, let $x \in \mathbb{R}^N$ and we claim that there is C_2 independent of x such that

$$\sup_{\substack{y\in\mathbb{R}^N\\ h\neq 0}} \frac{|v(x+h)-v(x)|}{|h|^\tau} \le C_2.$$

Since $v(x+h)-v(x) = (v-c)(x+h)-(v-c)(x)$ for all constants $c \in \mathbb{R}$, we may assume $v(x)=0$. Next, let $h \in \mathbb{R}^N$, then

$$|2v(x+2^k h) - v(x+2^{k+1}h)| = |2v(x+2^k h) - v(x+2^{k+1}h) - v(x)| \le v_\tau 2^{k\tau}|h|^\tau \quad \text{for } k \in \mathbb{N}_0.$$

But then, for $n \in \mathbb{N}$ and since $\tau < 1$,

$$\begin{aligned}|2^n v(x+h) - v(x+2^n h)| &\le \sum_{k=0}^{n-1} 2^{n-1-k}|2v(x+2^k h) - v(x+2^{k+1}h)| \\ &\le C|h|^\tau \sum_{k=0}^{n-1} 2^{n-1-k+k\tau} \le v_\tau 2^n |h|^\tau \sum_{k=0}^{\infty} 2^{-(1-\tau)k} = \frac{v_\tau 2^n}{1-2^{\tau-1}}|h|^\tau.\end{aligned}$$

Hence, for all $n \in \mathbb{N}$,

$$\begin{aligned}|v(x+h)-v(x)| = |v(x+h)| &\le 2^{-n}|2^n v(x+h) - v(x+2^n h)| + 2^{-n}|v(x+2^n h)| \\ &\le \frac{v_\tau}{1-2^{\tau-1}}|h|^\tau + 2^{-n}v_\tau\end{aligned}$$

and, for $n \to \infty$, we have $|v(x+h)-v(x)| \le \frac{v_\tau}{1-2^{\tau-1}}|h|^\tau$ so that $v \in C^\tau(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

6.3 The Arzelà-Ascoli and The Riesz-Fréchet-Kolmogorov theorems

We start with some definitions. The space $\mathscr{C}(K)$ is equipped with the norm $\|u\|_{\mathscr{C}(K)} : \sup_{x\in K}|u(x)|$ where K is a compact subset of $\mathbb{R}^N$. Let $\mathscr{F} \subset \mathscr{C}(K)$ be a collection of functions defined on K.

Definition 6.2. i) $\mathscr{F}$ is said to be bounded (uniformly bounded) on $\Omega \subset \mathbb{R}^N$ if there exists a constant $C > 0$ such that

$$|u(x)| \le C \quad \text{for all } x \in \Omega \quad \text{and for all } u \in \mathscr{F}$$

ii) $\mathscr{F}$ is said to be equicontinuous in $\mathscr{C}(\Omega)$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x-y| < \delta \quad \text{implies that} \qquad |f(x)-f(y)| < \varepsilon \text{ for } x,y \in \Omega \text{ and for all } u \in \mathscr{F}$$

We now state the Arzelà-Ascoli theorem in two version.

Theorem 6.3 (Arzelà-Ascoli). *[17] Let K be a compact subset of $\mathbb{R}^N$ and let $\{u_n\}$ be a sequence of continuous functions from K to $\mathbb{R}^N$. If the sequence u_n is uniformly bounded and equicontinuous, then the sequance $\{u_n\}$ has a subsequence that converges uniformly on K*

Theorem 6.4 (Arzelà-Ascoli theorem). *Let K be a compact subset of $\mathbb{R}^N$ a subset $M \subset \mathscr{C}(K)$ is relatively compact if and only if it is bounded and equicontinuous in $\mathscr{C}(K)$.*

We will also need a L^p-version of the the Arzelà-Ascoli theorem

Theorem 6.5 (Riesz-Fréchet-Kolmogorov). *Let $\mathscr{F}$ be a bounded subset in $L^p(\Omega)$ with $1 \leq p < \infty$. Assume further that*

$$\lim_{|h| \to 0} \|u(x+h) - u(x)\|_{L^p(\mathbb{R}^N)} = 0 \quad \textit{uniformly in } f \in \mathscr{F} \tag{6.3}$$

Then, $\mathscr{F}_{|\Omega}$ the restriction to Ω of the function in $\mathscr{F}$, is relatively compact in $L^p(\Omega)$ for any measurable set $\Omega \subset \mathbb{R}^N$ with finite measure.

7 Summary

The thesis deals with the study of Dirichlet problems driven by nonlocal operators including those with small order.
The result of paper [P1] provides an estimate of the Morse index of radially symmetric sign changing bounded weak solutions u to the semilinear fractional Dirichlet problem

$$(-\Delta)^s u = f(u) \qquad \text{in } \mathscr{B}, \qquad\qquad u = 0 \qquad \text{in} \quad \mathbb{R}^N \setminus \mathscr{B},$$

where $s \in (0,1)$, $\mathscr{B} \subset \mathbb{R}^N$ is the unit ball centred at zero and the nonlinearity f is of class C^1. We prove that for $s \in (1/2, 1)$, any radially symmetric sign changing solutions of the above problem has a Morse index greater than or equal to $N+1$. If $s \in (0, 1/2]$, the same conclusion holds under additional assumption on f. This extends the estimate proved by A. Aftalion and F. Pacella for the local problem with $s = 1$. In particular, our results apply to the Dirichlet eigenvalue problem for the fractional Laplacian $(-\Delta)^s$ in $\mathscr{B}$ for all $s \in (0,1)$, and implies that eigenfunctions corresponding to the second Dirichlet eigenvalue in $\mathscr{B}$ are antisymmetric i.e., it satisfies $u(-x) = -u(x)$ for $x \in \mathscr{B}$. This resolves a conjecture by Bañuelos and Kulczycki.

The result of paper [P2] deals with spectral asymptotics in the small order limit $s \to 0^+$ of the Dirichlet eigenvalue problem

$$(-\Delta)^s u = \lambda u \quad \text{in} \quad \Omega, \qquad\qquad u = 0 \qquad \text{in} \quad \mathbb{R}^N \setminus \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set with Lipschitz boundary. More precisely, we study the asymptotics of Dirichlet eigenvalues $\lambda_{k,s}(\Omega)$, $k \in \mathbb{N}$ and corresponding eigenfunctions $u_{k,s}$ of the fractional Laplacian $(-\Delta)^s$. We show that

$$\lambda_{k,s}(\Omega) = 1 + s\lambda_{k,L}(\Omega) + o(s) \qquad \text{as } s \to 0^+,$$

where the first order correction in these asymptotics is given by the eigenvalues $\lambda_{k,L}(\Omega)$ of the logarithmic Laplacian operator L_Δ, i.e., the singular integral operator with Fourier symbol $2\log|\xi|$. By this we extend a result of H. Chen and T. Weth which was restricted to the principal eigenvalue. Moreover, we improve their L^2-convergence result of the corresponding first eigenfunction by showing that the set $\{u_{1,s} : s \in (0, \frac{1}{4}]\}$ is relatively compact in $C(K)$ for any compact subset $K \subset \Omega$, and we extend the convergence result to higher eigenfunctions $u_{k,s}$ corresponding to eigenvalues $\lambda_{k,s}$ for all $k \in \mathbb{N}$. In addition, if Ω satisfies an exterior sphere condition, then the above convergence is uniform in $\overline{\Omega}$ and the set $\{u_{k,s} \,:\, s \in (0, \frac{1}{4}]\}$ is relatively compact in the space $C_0(\Omega) := \{u \in C(\mathbb{R}^N) : \ u \equiv 0 \ \text{ in } \ \Omega^c\}$. In order to derive these spectral asymptotics, we establish new uniform regularity and boundary decay estimates for Dirichlet eigenfunctions for the fractional Laplacian. As a byproduct, we also obtain corresponding regularity properties of eigenfunctions of the logarithmic Laplacian.

The result of paper [P3] is devoted to the study of the logarithmic Schrödinger operator $(I - \Delta)^{\log}$, which is the singular integral operator corresponding to the logarithmic symbol $\xi \mapsto$

$\log(1+|\xi|^2)$. We provide an alternative method to derive the singular integral representation corresponding to $(I-\Delta)^{\log}$. It is given by

$$(I-\Delta)^{\log}u(x) = d_N \int_{\mathbb{R}^N} \frac{u(x)-u(x+y)}{|y|^N}\omega(|y|)\, dy,$$

where $d_N = \pi^{-\frac{N}{2}}$, $\omega(r) = 2^{1-\frac{N}{2}} r^{\frac{N}{2}} K_{\frac{N}{2}}(r)$ and K_ν is the modified Bessel function of second kind with index ν. We show that $(I-\Delta)^{\log}$ arises as derivative in s of fractional relativistic Schrödinger operators $(I-\Delta)^s$ at $s=0$. If $u \in C^\beta(\mathbb{R}^N)$ for some $\beta > 0$, we have

$$\lim_{s\to 0^+} \frac{(I-\Delta)^s u - u}{s} = (I-\Delta)^{\log} u \quad \text{in } L^p(\mathbb{R}^N) \quad \text{for } 1 \le p \le \infty.$$

We introduce tools to study variational problems involving this operator and present some proofs not relying on probabilistic techniques but instead on purely analytic methods which are to some extend, simpler and more accessible to PDE oriented readers. In particular, we characterize the eigenvalues and corresponding eigenfunctions of $(I-\Delta)^{\log}$ in an open bounded set $\Omega \subset \mathbb{R}^N$ and prove the Faber-Krahn type inequality. We also derive a decay estimate in $\mathbb{R}^N$ of the Poisson problem and investigate small order asymptotics $s \to 0^+$ of the Dirichlet eigenvalue problem for the fractional relativistic operator $(I-\Delta)^s$ in a bounded open set with Lipschitz boundary.

The result of paper [P4] focuses on nonlocal operators of order strictly below one, that is, we consider singular integral operators

$$I_k u(x) = \int_{\mathbb{R}^N} (u(x)-u(y))k(x,y)\, dy$$

with the kernel $k : \mathbb{R}^N \times \mathbb{R}^N \to [0,\infty]$ satisfying $\quad k(x,y) = k(y,x) \quad$ for all $x,y \in \mathbb{R}^N$ and

$$\sup_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N} \min\{1, |x-y|^\sigma\} k(x,y)\, dy < \infty \qquad \text{for some } \sigma \in (0,1).$$

Assuming suitable conditions on the kernel k, we first present some density results corresponding to the associated function spaces and prove maximum principles for weak solutions. We investigate regularity properties of weak solutions u to the associated Poisson problem $I_k u = f$ in an open bounded set $\Omega \subset \mathbb{R}^N$, depending on the regularity of the function f. In particular, assuming that the kernel is translation invariant, we prove local H^1-regularity of weak solutions when the function f is of class C^2. Assuming furthermore that the kernels satisfy certain regularity properties away from its singularity, we deduce the interior C^∞-regularity of weak solutions u if f is of class C^∞. We also establish interior regularity for the corresponding Dirichlet eigenvalue problem, by showing that, every eigenfunction of the problem $I_k u = \lambda u$ in Ω, belongs to $C^\infty(\Omega)$.

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