Fractional Hadamard formulas, Pohozaev type identities and Applications

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Contents

Abstract

The boundary expression of the one-sided shape derivative of nonlocal Sobolev best constants is derived. As a simple consequence, we obtain the fractional version of the so-called Hadamard formula for the torsional rigidity and the first Dirichlet eigenvalue. An application to the optimal obstacle placement problem for the torsional rigidity and the first eigenvalue of the fractional Laplacian is given. These results are the contains of the paper $[P1]$.

In the paper $[P2]$ we introduce and prove a new maximum principle for doubly antisymmetric functions. The latter can be seen as the first step towards studying the optimal obstacle placement problem for the second fractional eigenvalue. Using the new maximum principle we derive new symmetry results for odd solutions to semilinear Dirichlet boundary value problems with Lipschitz nonlinearity.

In the paper $[P3]$ we derive new integration by parts formula for $(-\Delta)^s$ with a general globally Lipschitz vector field X and in particular, we obtain a new Pohozaev type identity generalizing the one obtained by X. Ros-Oton and J. Serra in [95]. As an application we obtain nonexistence results for semilinear Dirichlet boundary problems in bounded domains that are not necessarly starshaped.

In the paper $[P4]$, we study symmetry properties of second eigenfunctions of annuli. Using results from the paper $[P1]$ and the maximum principle in $[P2]$ we extend the result on the optimal obstacle placement problem from the first eigenvalue to the second eigenvalue.

This manuscript presents the following results obtained during the course of my PhD studies. These have been published as follows

- [P1] Sidy. M. Djitte, M. M. Fall and T. Weth. "A fractional Hadamard formula and applications." Calculus of Variations and Partial Differential Equations 60.6 (2021): 1-31.
- [P2] Sidy. M. Djitte and S. Jarohs. "Symmetry of odd solutions to equations with fractional Laplacian." Journal of Elliptic and Parabolic equations.
- [P3] Sidy. M. Djitte, M. M. Fall and T. Weth, "A Generalized fractional Pohozaev identity and applications", submitted to Advances in Calculus of Variations.
- [P4] Sidy. M. Djitte and S. Jarohs, "Nonradiality of second fractional eigenfunctions of thin annuli", preprint.

0.1 General Introduction

Variational problems where the variables are shapes appear naturally in many areas of natural sciences. A classical example is the isoperimetric problem where one seeks to optimize the perimeter of a set subject to a volume constraint. Also in fluid dynamics, there is an interested in finding the form of a pipe such that a fluid moving inside it has the same tangential stress on all points of its wall. The corresponding mathematical model can be seen as a variational problem where one seeks the shape of a domain minimizing a Dirichlet energy subject to a volume constraint see e.g [107]. Understanding the dependence of the function on the shape variable plays an important role when studying these variational problems. In this thesis, the dependence, and in particular the first variation, of nonlocal shape functionals on the domain is studied. The shape functionals under consideration are related to the domain via a solution of a nonlocal equations posed in Ω. A particular instance of such shape functionals is the best constants in Sobolev embedding theorems. The latter has been studied very intensively and in different points of view due to their important role in studying, for instance, elliptic partial differential equations with critical growth. The dependence of these Sobolev constants on the domain Ω has attracted many attention and many works have been done in this regard see e.g [13, 14, 30] and the references therein.

Of particular interest is the variation, also called shape derivative, of Sobolev best constants under perturbation of the domain. One of the earliest result in this direction is due to J. Hadamard in [60]. Using a perturbation Φ_ε generated by a weighted unit normal $g\nu$, $g \in C^\infty(\partial\Omega)$, ν denotinh the outer unit normal, he computed the first variation of the fundamental frequency of a smooth vibrating object Ω and showed that the latter can be expressed as a boundary integral involving the normal derivative of the solution u of the underlying equation. In other words, he showed that

$$\frac{d}{d\varepsilon}\lambda_1(\Phi_\varepsilon(\Omega))\bigg|_{\varepsilon=0} = -\int_{\partial\Omega} |\nabla u|^2 g\, dx, \tag{0.1.1}$$

where u is the solution to $\Delta u + \lambda_1(\Omega)u = 0$ in Ω and $u = 0$ on $\partial\Omega$. Here $\lambda_1(\Omega)$ denotes the first Dirichlet eigenvalue of the Laplacian in Ω. This result can be seen as the starting point of a very active topic of research today known as shape calculus. Main contributions in the spirit of Hadamard are the works [51, 75]. See also [65, 108, 111] for more recent developements. We adopt the notation $dJ(\Omega)X$ for the first variation, whenever it exists, of a given functional $\Omega \mapsto J(\Omega)$ under a deformation field X. The structure theorem of "shape calculus", due to [29, 110], states that under certain conditions, the first variation or the shape derivative is a distribution acting on the normal part $X \cdot \nu$ of the deformations field X on the boundary of the domain. If the data of the problem is regular enough, the shape derivative of a given functional $\Omega \mapsto J(\Omega)$ is often written in the form

$$dJ(\Omega)X = \int_{\partial\Omega} G(x)X(x) \cdot \nu(x)\, dx, \tag{0.1.2}$$

where $G : \partial\Omega \to \mathbb{R}$ is some function depending on the solution of the underlying equation. We call the integral over the boundary in (1.2) the boundary expression of the shape derivative. In the literature, (0.1.2) is usually referred to as the Hadamard formula. Writing the shape derivative into this form has many advantages. Among others, it allows –under some regularity assumption– to characterised critical shapes of the shape optimization problem

$$\inf_{\Omega \in \mathscr{A}} J(\Omega),$$

where $\mathscr{A}$ is the set of admissible shapes, as an overdetermined boundary value problem which is sometimes easier to analyse to get certain geometric properties of optimal shapes see e.g [65, Chapter 6]. In general, it is a challenging problem to write the expression of the shape derivative into the form (0.1.2). In the classical case, this is done by using the divergence theorem or the integration by parts formula. The approach most commonly used, see e.g [65, 108], utilizes the notion of material derivative introduced in [108]. It consists of differentiating the state function, the solution of the underlying equation, with respect to the geometric variable. However, the latter being closely related to the uniqueness of solutions of the underlying PDE, makes this approach rather restrictive in the sense that it cannot be employed to compute one-sided shape derivative of a general shape functionals. Moreover, employing this approach successfully requires many intermediate steps, which in general are not easy to check. These include justifying the existence of the material derivative and finding the equation that it solves. It may also happen that the material derivative is not in the solution space of the equation rendering impossible to perform an integration by parts to obtain the boundary expression of the shape derivative. One example, where this situation occurs is the case of the p-Laplacian where it is known that the material derivative only belongs to some weighted Sobolev space and not to the solution space of the PDE. Because the boundary expression of shape derivative does not depend on the material derivative, several others approaches avoiding the use of the material derivative have been proposed, see e.g [15, 21, 69, 109] and the references therein.

Identity (0.1.1) plays a fundamental role in the proof of many results concerning Dirichlet eigenvalues see e.g [63, 64, 65]. For instance, it can be used –under certain regularity assumption– to recovery the celebrated Faber-Krahn inequality stating that balls have the lowest fundamental frequency among open set of $\mathbb{R}^N$ of fixed volume. It is also used in a crucial way in optimal obstacle placement problems. This amounts of finding the position of an obstacle (mostly spherical) within a bigger domain so as to optimize the corresponding functional. For instance in [77] it was proven, using (0.1.1), that among sets of the form $B_1(0)\setminus\overline{B_\tau(x)}$ with $x\in\mathbb{R}^N$ such that $B_\tau(x)\subset B_1(0)\}$, the concentric spheres must has the largest fundamental frequency. See also [3, 61, 91] for further extensions of this result. For extensions of (0.1.1) under other boundary conditions and for more general and nonelliptic PDEs we refer the reader to [7, 46, 65, 84, 112]. Apart from its own mathematical interest, shape calculus is also an essential tool for the numerical treatment of shape optimization problems see e.g [86, 88]. The applications of shape derivative go beyond the area of shape optimization: it can be used to prove symmetry breaking result as it was remarked in [3, 11]. See also [65, Chapter 6, Paragraph 6.1.5]. For an application to nonexistence result for the semilinear elliptic PDE $-\Delta u = f(u)$ in Ω, $u=0$ on $\partial\Omega$, we refer to [103].

0.1.1 Shape derivative of nonlocal domain dependent functionals

While in the classical case, the variation of shape functionals on domains has been studied widely since the pioneering work of J. Hadamard, it's only recently that these issues have been addressed for functionals involving nonlocal operators such as the fractional Laplacian $(-\Delta)^s$, see e.g [26]. The fractional Laplacian has attracted extensive attention due to its appearence in mathematical model describing phenomenon in quantum mechanics, biology, and finance [2, 68, 81, 80, 83, 105]. It can be pointwisely defined, when acted on smooth functions $u\in C_c^\infty(\mathbb{R}^N)$, by

$$(-\Delta)^s u(x) = \frac{b_{N,s}}{2}\int_{\mathbb{R}^N}\frac{2u(x)-u(x-y)-u(x+y)}{|y|^{N+2s}}dy, \quad x\in\mathbb{R}^N, \tag{0.1.3}$$

with a normalization constant $b_{N,s} = 4^s\pi^{-\frac{N}{2}}s\frac{\Gamma(\frac{N+2s}{2})}{\Gamma(1-s)}$ and $s\in(0,1)$. Here Γ is the usual Gamma function. For other equivalent definition, we refer the reader to [79, 17, 52]. Throughout this manuscript, by solution to the equation

$$(-\Delta)^s u = f \quad \text{in} \quad \Omega \quad \text{and} \quad u=0 \quad \mathbb{R}^N\setminus\Omega,$$

we mean

$$u\in\mathscr{H}_0^s(\Omega) \quad \text{and} \quad \mathscr{E}_s(u,v) = \int_\Omega f(x)\varphi(x)\,dx \qquad \forall\varphi\in\mathscr{H}_0^s(\Omega), \tag{0.1.4}$$

with

$$\mathscr{E}_s(u,v) := \frac{b_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy. \tag{0.1.5}$$

Here $\Omega\subset\mathbb{R}^N$ is a bounded open set and $\mathscr{H}_0^s(\Omega)$ is the fractional Sobolev space of order s defined as the completion of $C_c^\infty(\Omega)$ with respect to the norm $\mathscr{E}_s(u,u)$. We recall that when Ω has a continuous boundary, the space $\mathscr{H}_0^s(\Omega)$ coincide with those L^2-functions belonging to

$H^s(\mathbb{R}^N)$ and such that $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$, see e.g [56]. Here, and in the following, $H^s(\mathbb{R}^N)$ denote the set of measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ such that

$$[u]^2_{H^s(\mathbb{R}^N)} := \frac{b_{N,s}}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dxdy < \infty.$$

For more detailed introduction to the operator $(-\Delta)^s$ and the related Sobolev spaces, see [56].

In contrast to the classical case, computing the boundary expression of shape derivative of nonlocal shape functionals is a highly nontrivial task. This is mainly due to two things: firstly, lack of sufficient boundary regularity for solution to nonlocal equations (see e.g [104]) and secondly the absence of a divergence theorem or an integration by part formula with a general vector field. In fact, solutions to nonlocal equations involving the fractional Laplacian are known not to be better than $C^s(\overline{\Omega})$ this can be seen by looking at the solution to $(-\Delta)^s u = 1$ in B_1 and $u = 0$ in $\mathbb{R}^N \setminus B_1$ which is explicitly given by $u(x) = c_{N,s}(1-|x|^2)^s_+$ for some constant $c_{N,s} > 0$. Here, and in the following, $a^+ = \max(a,0)$ denotes the positive part of a. As for an integration by parts formula the best known results were [95, Proposition 1.6, Theorem 1.9] which only hold for the identity vector field $X = \mathrm{id}_{\mathbb{R}^N}$ and the constant vector field $X \equiv e_j$, $j \in \{1,\cdots,N\}$, with e_j being the j-coordinate unit vector.

To the best of our knowledge, apart from the present work, only the paper [26] addresses the question of computing the boundary expression of shape derivative of nonlocal domain dependent functionals. In there, the authors considered the energy functional $J_f(\Omega)$ associated to the solution of the problem

$$(-\Delta)^{1/2} u_\Omega = f \quad \text{in} \quad \Omega \quad \text{and} \quad u_\Omega = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded open set of class C^∞. Precisely, they considered the functional

$$\Omega \mapsto J_f(\Omega) = \inf_{u \in \mathscr{H}^{1/2}_0(\Omega)} \Big(\frac{1}{2} \Big\langle (-\Delta)^{1/2} u, u \Big\rangle_{H^{1/2},H^{-1/2}} - \int_{\mathbb{R}^2} f u\, dx \Big) = -\frac{1}{2} \int_\Omega u_\Omega(x) f(x)\, dx,$$

for some $f \in C^\infty(R^2, \mathbb{R})$ and proved the following

Theorem 0.1.1. *[26, Theorem 1] Let Φ_t be a flow generated by a smooth vector fields $X \in C^\infty_c(\mathbb{R}^2, \mathbb{R}^2)$. Then the maps $t \mapsto J_f(\Omega_t)$, with $\Omega_t := \Phi_t(\Omega)$, is derivable at 0 and*

$$\frac{d}{dt} J_f(\Omega_t)\Big|_{t=0} = C_0 \int_{\partial\Omega} \Big(\frac{u_\Omega}{d^{1/2}}\Big)^2 X \cdot \nu \, d\sigma \tag{0.1.6}$$

for some explicit constant C_0. Here $d = \mathrm{dist}(\cdot, \mathbb{R}^N \setminus \Omega)$ denotes the distance function to the boundary $\partial\Omega$ and ν the interior unit normal to the boundary.

We note that by the standard regularity theory for the operator $(-\Delta)^s$, the ratio u_Ω/d^s is well defined on the boundary $\partial\Omega$ see e.g [40, 58, 104] and therefore the integral in (0.1.6) makes

sense. The proof of the identity (0.1.6) given in [26] uses the material derivative approach and consist of differentiating the identity $J_f(\Omega_t) = -\frac{1}{2}\int_{\Omega_t} u_{\Omega_t}(x)f(x)\,dx$; and this requires, after changing variables, the differentiability of the functional $t \mapsto v_t := u_{\Omega_t} \circ \Phi_t^{-1}$. The process of checking the latter requires a long computations and it may not exists in some instances since it is related to the uniqueness of solutions to the underlying equations, here $(-\Delta)^{1/2}u_\Omega = f$ in Ω, $\quad u = 0$ in $\mathbb{R}^N \setminus \Omega$. For this reason, this approach cannot be employed to compute, for instance, the one-side shape derivative of more general shape functionals like the one we shall consider below. We also point out that in order to prove the differentiability with respect to t of the function v_t, they used the so called Caffarelli-Silvestre extension result which transforms a nonlocal equation driven by the fractional Laplacian to a local PDE with one extra dimension [19]. We do not pursue this approach here.
Related to (0.1.6), it was also proven [26, Theorem 3], by adapting the classical moving plane method, that discs are the only minimizers of the problem

$$\inf\{J_1(\Omega) : \Omega \text{ of class } C^\infty \text{ \& connected with } |\Omega| = c\}.$$

As far as we know, the formula (0.1.6) was the best known regarding first variation of nonlocal shape functionals and a general formula like (0.1.1) was missing in the literature. The paper $[P1]$ fills that gap. More generally, in the paper $[P1]$ we compute the boundary expression of the one sided shape derivative of the best constant $\lambda_{s,p}(\Omega)$ in the Sobolev embedding $\mathscr{H}_0^s(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$ and as a simple consequence we derive the fractional version of (0.1.1). We refer to Section 0.2 below for a precise statement of the result.

0.1.2 Maximum principles for the fractional Laplacian

A classical but powerful tool widely used in the analysis of elliptic PDEs is the so called maximum principle. For nonlocal operator such as the fractional Laplacian $(-\Delta)^s$, it can be stated, in its simplest form, as follows: Let Ω be a bounded open set of $\mathbb{R}^N$ and let $u : \mathbb{R}^N \to \mathbb{R}$ be a weak solution to

$$(-\Delta)^s u \geq 0 \quad \text{in} \quad \Omega \qquad \text{and} \qquad u \geq 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega, \tag{0.1.7}$$

in the sense that

$$u \in H^s(\mathbb{R}^N),\ u \geq 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega \quad \text{and} \quad \mathscr{E}_s(u, \varphi) \geq 0 \qquad \forall \varphi \in \mathscr{H}_0^s(\Omega),\ \text{with}\ \varphi \geq 0 \quad \text{in} \quad \Omega,$$

then

$$u \equiv 0 \quad \text{in} \quad \mathbb{R}^N \qquad \text{or} \qquad u > 0 \quad \text{in} \quad \Omega \quad \text{almost everywhere.}$$

We emphasize that such a version of the maximum principle doest not hold in the classical case. This simple result plays a fundamental role in studying many problems involving the operator $(-\Delta)^s$. One of its basics consequences are uniqueness results and regularity estimates for solutions to the equation $(-\Delta)^s u = f$ in Ω, $\quad u = 0$ in $\mathbb{R}^N \setminus \Omega$, see e.g [93].
Several variants and extensions of (0.1.7) have been considered in the literature, see e.g [73] and the references therein. Let us mention in particular the following types of maximum principle considered in [43], see also [24, 74].

Proposition 0.1.2. *[43, Proposition 3.1] Let H be a halfspace and let $\Omega \subset H$ be any open, bounded set, let $c \in L^\infty(\Omega)$ be such that $c \le c_\infty < \lambda_{1,s}(\Omega)$ in Ω for some $c_\infty \ge 0$ and let $g \in L^2(\Omega)$ be, such that $g \ge \kappa$ with*

$$0 \le \kappa < \frac{\lambda_{1,s}(\Omega) - c_\infty}{|\Omega|^{1/2}}.$$

If $u \in \mathscr{D}^s(\Omega)$ is antisymmetric with respect to the reflection across the hyperplane ∂H and solves the equation

$$(-\Delta)^s u \ge c(x)u + g(x) \quad in \quad \Omega \qquad and \qquad u \ge 0 \quad in \quad H \setminus \Omega \tag{0.1.8}$$

in the sense that

$$\mathscr{E}_s(u,\varphi) \ge \int_\Omega (c(x)u(x) + g(x))\varphi(x)\,dx \quad \forall \varphi \in \mathscr{H}_0^s(\Omega),\ \varphi \ge 0 \tag{0.1.9}$$

then $\|u_-\|_{L^2(\Omega)} \le \frac{\kappa|\Omega|^{1/2}}{\lambda_{1,s}(\Omega)-c_\infty}$. Here $\mathscr{D}^s(\Omega)$ is the set of measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ for which the LHS of (0.1.9) is finite and $u_- := -\min(u,0)$. In particular, if $\kappa = 0$, then $u \ge 0$ in Ω almost everywhere.

Problems of types (0.1.8) arise naturally when carrying out the moving plane method to prove symmetry results for equations of the types $(-\Delta)^s u = f(u)$ in Ω and $u = 0$ in $\mathbb{R}^N \setminus \Omega$. They also appear in shape optimization problems (see Chapter 1, Section 1.5). Proposition 0.1.2 was in particular used in [43] to obtained the fractional version of Serrin's result.

A natural and interesting question, which can be seen as an extension of (0.1.8), is the following: Let H and $\widetilde{H}$ be two half spaces such that $\partial H \perp \partial \widetilde{H}$ and let $\Omega \subset H \cap \widetilde{H}$ be a bounded open set and $c \in L^\infty(\Omega)$. Let $r, \widetilde{r}$ be the reflections with respect to the hyperplanes ∂H and $\partial \widetilde{H}$ respectively. Under which condition on c, a solution to the problem

$$(-\Delta)^s u \ge c(x)u \quad \text{in} \quad \Omega, \quad u \ge 0 \quad \text{in} \quad H \cap \widetilde{H} \setminus \Omega \quad \text{and} \quad u \circ r = -u = u \circ \widetilde{r} \quad \text{in} \quad \mathbb{R}^N, \tag{0.1.10}$$

satisfies the maximum principle. i.e, $u \ge 0$ in Ω?
The motivation to consider this type of questions comes from shape optimization as we shall see in Chapter 4 further below. The fundamental difference between (0.1.10) and (0.1.8) is that in (0.1.10) we set the boundary condition in the smaller set $H \cap \widetilde{H} \setminus \Omega \subset H \setminus \Omega$. To compensate the latter, we ask the function u to be also antisymmetric with respect to the reflection across the hyperplane $\partial \widetilde{H}$.
We shall see later in Chapter 2 that, under some reasonable assumption on c, a solution to (0.1.10) satisfies the maximum principle. This is one of our main findings in the paper [$P2$]. We refer to Section 0.2.2 for a precise statement of the result.

0.1.3 Symmetry of sign changing solutions to equations with the fractional Laplacian

A classical topic in the analysis of functions solving certain differential type equations is the study of symmetry properties of these functions. One of the earliest result in this direction is due to J. Serrin who in [100] proved that if u solves $-\Delta u = 1$ in Ω, $u = 0$ in $\partial\Omega$ and $\nabla u \cdot \nu = c$ on $\partial\Omega$ where Ω is connected, smooth and bounded, then Ω must necessary be a ball and u must be radial. After this pioneering work by J. Serrin, several other results have followed, one may see, for instance the work by Gidas, Ni and Nirenberg [54]. In the fractional setting, the problem has also been studied extensively see e.g [70, 74, 6, 47] and the references therein. A typical problem mostly studied is the following. Let Ω be convex and symmmetric and let u be a solution to the equation

$$(-\Delta)^s u = f(u) \quad \text{in} \quad \Omega, \quad u \geq 0 \quad \text{in} \quad \Omega \quad \text{and} \quad u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega. \tag{0.1.11}$$

Under which condition on f, does u inherit the symmetry properties of Ω? A typical answer is that if f is locally Lipschitz, then u inherits the symmetry property of Ω. In particular, if Ω is radially symmetric, so is u. These results are mostly derived by using the so called moving plane method which is based on forms of maximum principle see e.g [74]. If u is a sign changing solution, in general the moving plane method cannot be applied and therefore radial symmetry cannot be expected even if Ω is a radial set. For instance, in [41] it has been recently proved that a solution to the problem $(-\Delta)^s u = \lambda_2 u$ in B and $u = 0$ in $\mathbb{R}^N \setminus B$, where B is a ball and λ_2 is the second eigenvalue, cannot be radial. This result extends also to annuli with small width. This is one of our main results in the paper [$P4$].

Even though the moving plane method fails in general for sign changing solutions, in some instances it is possible to prove certains types of symmetry results for sign changing solutions to (0.1.11). This is the case if for instance Ω is a bounded radial set and $f(t)$ is locally Lipschitz. In this situation, it has been proven in [70] that any continuous bounded solution of $(-\Delta)^s u = f(u)$ in Ω and $u = 0$ in $\mathbb{R}^N \setminus \Omega$ is axial symmetric once it satisfies a certain reflection inequality with respect to a hyperplane. Another interesting case is when Ω has two perpendicular symmetry axis ∂H and $\partial\widetilde{H}$, and u is antisymmetric which respect to one of them, say, for instance $\partial\widetilde{H}$. In this case, if u has a sign in the half domain $\Omega \cap \widetilde{H}$, then it is possible to carry out the moving plane method to obtain symmetry results. An important example that fits into this framework is the minimizers of the functional

$$\lambda_{s,p}^{-}(\Omega) := \min_{\substack{u \in \mathscr{H}_0^s(\Omega) \\ u \neq 0 \\ u \circ \tilde{r} = -u}} \left\{ \frac{\mathscr{E}_s(u,u)}{\left(\int_\Omega |u(x)|^p \, dx \right)^{2/p}} \right\}.$$

Here $\tilde{r}$ denotes the reflection with respect to the hyperplane $\partial\widetilde{H}$ and $p \in [1, \frac{2N}{N-2s})$ is subcritical. This types of problem will be studied in Chapter 2 below.

0.1.4 Pohozaev identities

A celebrated identity due to Pohozaev [89] states that any –sufficiently regular– solution to the boundary value problem $-\Delta u = f(u)$ in Ω and $u = 0$ on $\partial\Omega$ where f is a continuous nonlinearity and Ω is a domain with C^2 boundary, satisfies

$$(2-N)\int_\Omega u f(u)\, dx + 2N \int_\Omega F(u)\, dx = \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2 x \cdot \nu \, dx. \tag{0.1.12}$$

Here $F(t) = \int_0^t f(s)ds$ and ν is the outward unit normal to the boundary of Ω. One of the immediate consequences of this identity is a nonexistence result in starshaped domains for the semilinear problem $-\Delta u = f(u)$ in Ω, $u = 0$ on $\partial\Omega$ when f has a critical or supercritical growth. Several identities generalizing (0.1.12) have appeared in the literature. We cite in particular the paper [90] by P. Pucci and J. Serrin where a Pohozaev type identity for elliptic PDEs in divergence form with a general vector field is obtained. The latter is very often used to obtained nonexistence result to variational problems in a larger class of domains that are not necessary starshaped [27, 85]. These identities turn out to be very useful in the analysis of many elliptic PDEs. Among others, they are used to prove uniqueness results, unique continuation properties and radial symmetry of solutions.
The fractional version of (0.1.12) is due to X. Ros-Oton and J. Serra. In their celebrated paper [95] it was proved that any bounded solution to the semilinear problem $(-\Delta)^s u = f(u)$ in Ω, $u = 0$ in $\mathbb{R}^N \setminus \Omega$, where Ω is a bounded open set of class $C^{1,1}$ and f is a locally Lipschitz nonlinearity, satisfies

$$(2s-N)\int_{\Omega} uf(u)\,dx + 2N\int_{\Omega} F(u)\,dx = \Gamma^2(1+s)\int_{\partial\Omega} (u/d^s)^2 x\cdot\nu\,dx. \qquad (0.1.13)$$

Here F is defined as above and ν is again the outer unit normal. As in the classical case, one immediate consequence is nonexistence results in starshaped domains Ω for the semilinear problem $(-\Delta)^s u = |u|^{p-1}u$ in Ω, $u = 0$ in $\mathbb{R}^N \setminus \Omega$, in the supercritical regime, that is, for $p > \frac{N+2s}{N-2s}$. Since its discovery, identity (0.1.13) has been used extensively in the study of elliptic equations involving $(-\Delta)^s$. We cite in particular the recent work [41] where this identity has been useful to answer a conjecture by Bañuelos and Kulczycki regarding the shape of second eigenfunctions of the fractional Laplacian in a ball. For other applications to uniqueness results we refer to [33, Section 5]. For an extension of (0.1.13) to a more general fractional types operators, we refer to the papers [59, 96]. The identity (0.1.13) was derived from the following more general identity (see [95, Proposition 1.6]) which, under a suitable assumption on u, states that

$$2\int_{\Omega} x\cdot\nabla u(-\Delta)^s u\,dx = (2s-N)\int_{\Omega} u(-\Delta)^s u\,dx - \Gamma^2(1+s)\int_{\partial\Omega} (u/d^s)^2 x\cdot\nu\, d\sigma, \qquad (0.1.14)$$

where ν is the outward unit normal to the boundary. Our main contributions in the paper $[P3]$ are the generalization of identities (0.1.13) and (0.1.14) in the sense that we allow the identity vector field x to be any globally Lipschitz vector field X. We refer to Section 0.2.4 below for a precise statement of the results.

0.2 Statement of the results of the thesis

In the rest of this introduction, we will present and discuss briefly the main achievements of this thesis.

0.2.1 $[P1]$: A fractional Hadamard formula and applications

One of the main achievement of this thesis is the computation of the boundary expression of the one-sided shape derivative of the best constant $\lambda_{s,p}(\Omega)$, with $p \in [1, \frac{2N}{(N-2s)^+})$, in the embedding $\mathscr{H}_0^s(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$. Here, and in the following, $a^+ = \max(a,0)$ and we use the convention $\frac{a}{0} = \infty$. We recall that this constant is characterised by

$$\lambda_{s,p}(\Omega) = \inf_{\substack{u \in \mathscr{H}_0^s(\Omega) \\ u \neq 0}} \left\{ \frac{\frac{b_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dxdy}{(\int_\Omega |u|^p dx)^{2/p}} \right\}. \tag{0.2.1}$$

Here $\Omega \subset \mathbb{R}^N$ is assumed to be a bounded open set of class $C^{1,1}$. It is a standard fact that the infimum in (0.2.1) is achieved by some $u \in \mathscr{H}_0^s(\Omega) \cap L^\infty(\Omega)$ which can be chosen positive and it solves (when normalized) the equation

$$(-\Delta)^s u = \lambda_{s,p}(\Omega) u^{p-1} \quad \text{in} \quad \Omega \tag{0.2.2}$$

in the sense of (0.1.4). By the standard regularity theory it is known that (0.2.2) holds also in the pointwise sense. Moreover $u \in C^s(\mathbb{R}^N)$ and $u/d^s \in C^\alpha(\overline{\Omega})$ for some $\alpha > 0$, where $d = \mathrm{dist}(\cdot, \mathbb{R}^N \setminus \Omega)$ is the distance function to the boundary $\partial\Omega$, see e.g [104].
Before the present work not much was known about the computation of the boundary expression of shape derivative of nonlocal shape functionals. As far as we know, only the paper [26] addressed this question. However, a general formula like (0.1.1) was missing in the litarature. The present work fills that gap. To state our results, we need to fix some notations. For $N \geq 1$, we fix a family of deformations $(\Phi_\varepsilon)_{\varepsilon \in (-1,1)}$ with the properties

$$\begin{aligned} &\Phi_\varepsilon \in C^{1,1}(\mathbb{R}^N; \mathbb{R}^N) \text{ for } \varepsilon \in (-1,1),\ \Phi_0 = \mathrm{id}_{\mathbb{R}^N}, \text{ and} \\ &\text{the map } (-1,1) \to C^{0,1}(\mathbb{R}^N, \mathbb{R}^N),\ \varepsilon \to \Phi_\varepsilon \text{ is of class } C^2. \end{aligned} \tag{0.2.3}$$

We note that under the assumption (0.2.3), there exists $\varepsilon_0 > 0$ so that Φ_ε is a global diffeomorphism for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. For simplicity we write $\lambda_{s,p}(\varepsilon) := \lambda_{s,p}(\Phi_\varepsilon(\Omega))$. The main achievement of $[P1]$ reads as follows.

Theorem 0.2.1. *For all $p \in [1, \frac{2N}{(N-2s)^+})$, the mapping $(-\varepsilon_0, \varepsilon_0) \ni \varepsilon \mapsto \lambda_{s,p}(\varepsilon)$ is right differentiable at* 0. *Moreover,*

$$\partial_\varepsilon^+ \lambda_{s,p}(0) := \lim_{\varepsilon \downarrow 0} \frac{\lambda_{s,p}(\varepsilon) - \lambda_{s,p}(0)}{\varepsilon} = \min \left\{ \Gamma^2(1+s) \int_{\partial\Omega} (u/d^s)^2 X \cdot \nu \, dx, \quad u \in \mathscr{H} \right\}, \tag{0.2.4}$$

where $\mathscr{H}$ is the set of positive minimizers of (0.2.1) *with the normalization condition $\int_\Omega u^p dx =$ 1, ν is the interior unit normal to the boundary, $X = \frac{d}{d\varepsilon}\Phi_\varepsilon \Big|_{\varepsilon=0}$ and Γ is the usual gamma function.*

Let us mention the following immediate consequence of Theorem 0.2.1. Since for $p \in [1,2]$, the normalized minimer u of $\lambda_{s,p}(\Omega)$ is unique (see Lemma 1.7.1 in the appendix for more details), Theorem 0.2.1 reduces in this case to.

Corollary 0.2.2. *Let $p \in [1,2]$ and let $\lambda_{s,p}(\varepsilon)$ be as above. Then the maps $\varepsilon \mapsto \lambda_{s,p}(\varepsilon)$ is differentiable at* 0*. Moreover,*

$$\frac{d}{d\varepsilon}\lambda_{s,p}(\varepsilon)\Big|_{\varepsilon=0} = \Gamma^2(1+s)\int_{\partial\Omega}(u/d^s)^2 X\cdot\nu\,dx. \tag{0.2.5}$$

where u is the unique solution to (0.2.2).

A further, less direct, consequence of the formula (0.2.4) is the characterization of local minima of the shape functional $\Omega \mapsto \lambda_{s,p}(\Omega)$ under a volume constraint.

Corollary 0.2.3. *Let $p \in \{1\}\cup[2,\infty)$. Suppose that Ω, an open set of class C^3, is a volume constrained local minimum for the functional $\Omega \mapsto \lambda_{s,p}(\Omega)$. Then Ω is a ball.*

Corollary 0.2.3 is a consequence of Theorem 0.2.1, from which we derive that if Ω is a constrained local minimum then, any element $u \in \mathscr{H}$ satisfies the overdetermined condition $u/d^s \equiv$ *const* on $\partial\Omega$. Therefore by the rigidity result in [43] we find that Ω must be a ball. We refer to Chapter 1, Section 1.5 for more details. Note that one can also prove the minimality of balls with other methods like rearrangement techniques see e.g [87]. The strength of our result is the uniqueness of the minimizing set under C^3 regularity assumption. Corollary 0.2.3 improved considerably [26, Theorem 1.3], where the authors considered the shape minimization problem for $\lambda_{s,p}(\Omega)$ in the case $p=1$, $s=\frac{1}{2}$, $N=2$ among domains Ω of class C^∞ of fixed volume and showed that such minimizers are discs.

Our argument of proving Theorem 0.2.1 takes advantage of the variational characterization (0.2.1) of the functional $\lambda_{s,p}(\Omega)$. Using this, we first derive the "interior expression" of the one-sided shape derivative of $\lambda_{s,p}(\Omega)$. This interior expression reads as

$$\partial_\varepsilon^+\lambda_{s,p}(0) = \min\left\{\mathscr{V}_u'(0) + \frac{2\lambda_{s,p}(\Omega)}{p}\int_\Omega u^p \operatorname{div}X\,dx, \quad u \in \mathscr{H}\right\},$$

with

$$\mathscr{V}_u'(0) = -\frac{1}{2}\int_{\mathbb{R}^{2N}}(u(x)-u(y))^2 K_X(x,y)\,dxdy,$$

where we denote

$$K_X(x,y) := \frac{b_{N,s}}{|x-y|^{N+2s}}\left\{(N+2s)\frac{(x-y)\cdot(X(x)-X(y))}{|x-y|^2} - (\operatorname{div}X(x)+\operatorname{div}X(y))\right\}.$$

Therefore identity (0.2.4) reduces to the following:

$$-\frac{1}{2}\int_{\mathbb{R}^{2N}}(u(x)-u(y))^2 K_X(x,y)\,dxdy + \frac{2\lambda_{s,p}(\Omega)}{p}\int_\Omega u^p\operatorname{div}X\,dx = -\Gamma^2(1+s)\int_{\partial\Omega}(\frac{u}{d^s})^2 X\cdot\nu\,dx.$$

The idea would be now to perform an integration by parts in the LHS of the identity above to arrive to the desired formula. This is where the difficulties arise. In the classical case, this is mostly solved by using the divergence theorem, which requires , among other things, a sufficient boundary regularity of the functions involved. However, this boundary regularity fails for solutions of nonlocal equations and also no divergence theorem with general vector field in this fractional setting had been available. To overcome the boundary regularity issue, we use an approximation argument. We fix $\rho \in C_c^\infty(-2,2)$ so that $\rho \equiv 1$ in $(-1,1)$ and define

$$\rho_k(x) = \rho(kd(x)) \quad \text{and} \quad \varsigma_k(x) = 1 - \rho_k(x) \quad \text{for all } x \in \mathbb{R}^N \quad \text{and} \quad \text{for all } k > 0. \qquad (0.2.6)$$

Here, $d \in C^{0,1}(\mathbb{R}^N, \mathbb{R})$ is any function that coincides with the signed distance function near the boundary and that is chosen to be positive in Ω. Using the cut-off function ς_k we approximate $\mathscr{V}_u'(0)$ by $\mathscr{V}_{\varsigma_k u}'(0)$. This is an easy consequence of the convergence $u\varsigma_k \to u$ in $\mathscr{H}_0^s(\Omega)$. Using the regularity of the approximating functions, and performing integrations by parts, we write $\mathscr{V}_{\varsigma_k u}'(0)$ in the following compact form

$$\mathscr{V}_{\varsigma_k u}'(0) = -2 \int_{\mathbb{R}^N} \nabla(\varsigma_k u) \cdot X (-\Delta)^s (\varsigma_k u) dx.$$

We expand the identity above by using the product rule for the fractional Laplacian and take the limit to arrive at

$$\mathscr{V}_u'(0) + \frac{2\lambda_{s,p}(\Omega)}{p} \int_\Omega u^p \mathrm{div} X dx = -2 \lim_{k\to\infty} \int_{\mathbb{R}^N} \nabla(u\zeta_k) \cdot X \big(u(-\Delta)^s \zeta_k - I_s(\zeta_k, u)\big) dx.$$

We observe that by the choice of the cut-off function ς_k the integral at the LSH of the above identity can be replaced by the integral in the ε-neighbourhood of the boundary $\Omega_\varepsilon^+ := \{x \in \Omega : d(x) < \varepsilon\}$ for any $\varepsilon > 0$. From this observation, the proof of Theorem 0.2.1 reduces finally to the following:

$$\lim_{k\to\infty} -2 \int_{\Omega_\varepsilon^+} \nabla(u\varsigma_k) \cdot X \big(u(-\Delta)^s \varsigma_k - I_s(\varsigma_k, u)\big) dx = -\Gamma^2(1+s) \int_{\partial\Omega} (u/d^s)^2 X \cdot \nu \, dx, \quad \forall \varepsilon > 0.$$

We achieved this by making the change of variables $\Psi : \partial\Omega \times (0,\varepsilon) \to \Omega_\varepsilon^+$, $(\sigma, r) \mapsto \sigma + r\nu(\sigma)$ and using the dominated convergence theorem. For this we used in a crucial way the fact that if $u \in \mathscr{H}_0^s(\Omega) \cap L^\infty(\Omega)$ is a solution to (0.2.2), then there exist $\alpha \in (0,1)$ so that $d^{1-\alpha}\nabla(u/d^s) \in L^\infty(\Omega)$. This gradient estimate has recently been obtained in [42].
Corollary 0.2.2 is used to established the following.

Theorem 0.2.4. *Let* $p \in \{1,2\}$, $B_1(0)$ *be the unit centered ball and* $\tau \in (0,1)$. *Define*

$$\mathscr{A} := \{a \in B_1(0) : B_\tau(a) \subset B_1(0)\}.$$

Then the map $\mathscr{A} \to \mathbb{R}$, $a \mapsto \lambda_{s,p}(B_1(0) \setminus \overline{B_\tau(a)})$ *takes its maximum at* $a = 0$.

The proof of Theorem 0.2.4 is inspired by the argument given in [77, 91] for the local case $s = 1$. We take advantage of the rotational invariance of the problem to reduce it into finding the maximum of the function

$$m : [0, 1-\tau) \to \mathbb{R}, \ a \mapsto m(a) := \lambda_{s,p}(B_1(0) \setminus \overline{B_\tau(ae_1)}), \tag{0.2.7}$$

where $e_1 = (1, 0, \cdots, 0)$ is the first unitary vector. We show that the map m is strictly decreasing. This is done by using the formula (0.2.5) and reflection techniques combined with maximum principle for anti-symmetric functions, see Section 1.5 for more details.

0.2.2 $[P2]$: Symmetry of odd solutions to equations with the fractional Laplacian

This paper is a joint work with S. Jarohs and it is motivated by the following question:
"Where to place a spherical obstacle B_τ within a ball $B_1(0)$ so as to maximize the second Dirichlet eigenvalue $\lambda_{2,s}(B_1(0) \setminus \overline{B_\tau})$ of the fractional Laplacian ?".
In fact, to study this question, we encounter mainly two difficulties. Firstly, due to non-simplicity, the functional $\Omega \mapsto \lambda_{2,s}(\Omega)$ is not differentiable with respect to perturbations of domain, that is, a formula like (0.2.5) does not holds for the second eigenvalue. Secondly, since the eigenfunctions corresponding to $\lambda_{2,s}$ change sign, the argument used in the proof of Theorem 1.1.4 cannot directly be applied. To be precise, the fractional Hopf lemma, which was among the tools used, does not apply to sign changing solutions. The corresponding problem in the classical case was treated in [39]. In their, the authors overcame these difficulties by noting that the second eigenvalue of the eccentric annulus $\Omega(t) := B_1(0) \setminus \overline{B_\tau(te_1)}$ is controlled by the its first antisymmetric eigenvalue defined by

$$\lambda_1^-(\Omega(t)) = \inf \left\{ \frac{\int_{\Omega(t)} |\nabla u(x)|^2 \, dx}{\int_{\Omega(t)} u^2(x) \, dx}, \ u \in \mathscr{H}_0^1(\Omega(t)) : u \neq 0 \ \text{and} \ u \circ r_N = -u \right\}, \tag{0.2.8}$$

and that the map $t \mapsto \lambda_1^-(\Omega(t))$ is a strictly decreasing function in $t \in [0, 1-\tau)$. The latter was obtained by exploiting the identity

$$\lambda_1^-(\Omega(t)) = \lambda_1\big(\{x \in \Omega(t) : x_N > 0\}\big). \tag{0.2.9}$$

Here r_N is the reflection with respect to the hyperplane $\partial H_N := \{x \in \mathbb{R}^N : x_N = 0\}$. The other important observation made is that for an annulus the two numbers coincide, i.e,

$$\lambda_2(B_1(0) \setminus \overline{B_\tau(0)}) = \lambda_1^-(B_1(0) \setminus \overline{B\tau(0)}). \tag{0.2.10}$$

An ODE techniques is used to derive (0.2.10), see [39, Lemma 2.1].

Our idea in $[P4]$ is to adapt this argument in the nonlocal setting. For this, one needs to study, among others, the variation of the nonlocal counterpart of (0.2.8), i.e, the variation of the real valued function

$$t \mapsto \lambda_{1,s}^-(t) = \inf \left\{ \frac{\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} \, dxdy}{\int_{\Omega(t)} u^2(x) \, dx}, \ u \in \mathscr{H}_0^s(\Omega(t)) : u \neq 0 \ \text{and} \ u \circ r_N = -u \right\} \tag{0.2.11}$$

where we denote $\lambda_{1,s}^{-}(t) := \lambda_{1,s}^{-}(\Omega(t))$. Obviously, an identity like (0.2.9) fails for $\lambda_{1,s}^{-}(\Omega(t))$ due to nonlocality and also because the minimizers corresponding to (0.2.11) change sign, the reflection techniques used to study the variation of (0.2.7) cannot directly be applied, since the latter relies heavily on the positivity of the solution of the underlying equation. To study the variation of $t \mapsto \lambda_{1,s}^{-}(t)$ in the spirit of Theorem 0.2.4, one needs an appropriate maximum principle. This is the main topic investigated in this paper $[P2]$. To state the results we fix some conventions. Let H_1 and H_N be two half spaces such that the hyperplanes ∂H_1 and ∂H_N are perpendicular. Without loss we take

$$H_1 = \{x \in \mathbb{R}^N : x_1 > 0\} \quad \text{and} \quad H_N := \{x \in \mathbb{R}^N : x_N > 0\}.$$

Let's denote by r_1 and r_N the reflection with respect to ∂H_1 and ∂H_N respectively. Fix $U \subset H_1 \cap H_N$ bounded and open and let $w \in H^s(\mathbb{R}^N)$. We call $w \in H^s(\mathbb{R}^N)$ a doubly antisymmetric supersolution to

$$(-\Delta)^s w = c(x)w \quad \text{in} \quad U, \qquad w \geq 0 \quad \text{in} \quad H_1 \cap H_N \setminus U, \tag{0.2.12}$$

if

$$w \circ r_1 = -w = w \circ r_N \quad \text{and} \quad \mathscr{E}_s(w,\varphi) \geq \int_U c(x)w(x)\varphi(x)dx \quad \forall\, \varphi \in \mathscr{H}_0^s(U),\, \varphi \geq 0 \text{ in } U,$$

with $\mathscr{E}_s(u,v)$ being defined as in (0.1.5). And, lastly, we define

$$\lambda_{1,s}^{-}(U) := \inf_{\substack{u \in \mathscr{H}_0^s\left(U \cup r_N(U)\right)\\ u \neq 0\\ u \circ r_N = -u}} \left\{ \frac{\mathscr{E}_s(u,u)}{\int_{U \cup r_N(U)} u^2(x)dx} \right\}. \tag{0.2.13}$$

One of the main result of this paper is the following maximum principle.

Proposition 0.2.5. *(Maximum principle for doubly antisymmetric functions) Let w be a doubly antisymmetric supersolution to* (5.0.5) *and assume* $\|c\|_{L^\infty(U)} \leq \lambda_{1,s}^{-}(U)$. *Then we have* $w \geq 0$ *in* $H_1 \cap H_N$.

Proposition 0.2.5 extends [43, Propsoition 3.1] to doubly antisymmetric functions. It will be crucial in the proof of Theorem 0.2.11 below. An important consequence of Proposition 0.2.5 is the following Hopf type lemma.

Proposition 0.2.6. *Let* $U \subset H_1 \cap H_N$ *be bounded and open. Furthermore, let* $c \in L^\infty(U)$ *and let* $w \in H^s(\mathbb{R}^N)$ *be a doubly antisymmetric supersolution of* (5.0.5). *Assume* $w \geq 0$ *in* $H_1 \cap H_N$. *Then either* $w \equiv 0$ *or* $w > 0$ *in U in the sense that*

$$\inf_K w > 0 \quad \textit{for all compact sets } K \subset U.$$

Moreover, if there is $x_0 \in \partial U \setminus [\partial H_1 \cup \partial H_N]$ *such that*

(*i*) *there exists a ball* $B \subset U$ *with* $\partial B \cap \partial U = \{x_0\}$ *and* $\lambda_{1,s}^{-}(B) \geq c$ *and*

(ii) $w(x_0) = 0$,

then there exists $C > 0$ *such that*

$$w \geq C d_B^s \qquad in \qquad B,$$

where d_B *denote the distance to boundary of* B*. In particular, if* $w \in C(B)$*, then*

$$\liminf_{t \downarrow 0} \frac{w(x_0 - t\nu(x_0))}{t^s} > 0.$$

We deduce Proposition 2.2.4 from Proposition 0.2.5, by considering in the case $w \neq 0$, the function $u = \alpha 1_K + \Psi_B$ where α is an arbitrary positive parameter and by noticing that for α large enough, its antisymmetric part with respect to r_1 and r_N, i.e, the function $-\overline{u} := -(w \circ r_1 \circ r_N - w \circ r_1 - w \circ r_N + w)$ is a supersolution to (5.0.5) with $U = B$. Here $\Psi_B \in \mathscr{H}_0^s(B)$ is the solution to the torsion problem $(-\Delta)^s v = 1$ in B, and $K \subset U$ is chosen in such a way that $\mathrm{dist}(B,K) > 0$ and $\varepsilon := \inf_K w > 0$. Once we have this, we conclude by applying Proposition 0.2.5 (with $U = B$) to the function $u_\varepsilon = w - \frac{\varepsilon}{\alpha}\overline{u}$ which solves (5.0.5) by the choice of ε.

Let us mention that Proposition 0.2.6 can be used to prove nonexistence result, in some special domains Ω, for sign changing solutions to the critical problem $(-\Delta)^s u = |u|^{2_s^*-2}u$ in Ω, $u = 0$ in $\mathbb{R}^N \setminus \Omega$, where $2_s^* = \frac{2N}{N-2s}$, see Remark 2.2.5 below.
A further, less direct, consequence of Proposition 0.2.5 is the following symmetry result regarding solutions to the semilinear boundary valued problem

$$(-\Delta)^s u = f(x,u) \quad \text{in} \quad \Omega, \qquad u = 0 \quad \mathbb{R}^N \setminus \Omega. \tag{0.2.14}$$

Let Ω and f satisfy the following assumptions:

(D) $\Omega \subset \mathbb{R}^N$ with $N \in \mathbb{N}$, $N \geq 2$ is open and bounded and, moreover, convex and symmetric in the directions x_1 and x_N. That is, for every $(x_1,\dots,x_N) \in \Omega$, $t,\tau \in [-1,1]$ we have $(tx_1, x_2 \dots, x_{N-1}, \tau x_N) \in \Omega$.

(F1) $f \in C(\Omega \times \mathbb{R})$ and for every bounded set $K \subset \mathbb{R}$ there is $L = L(K) > 0$ such that

$$\sup_{x \in \Omega} |f(x,u) - f(x,v)| \leq L|u - v| \quad \text{for all } u,v \in K.$$

(F2) f is symmetric in x_1 and monotone in $|x_1|$. That is, for every $u \in \mathbb{R}$, $x \in \Omega$, and $t \in [-1,1]$ we have $f(tx_1, x_2, \dots, x_N, u) \geq f(x,u)$.

Then we have

Theorem 0.2.7. *Let* $\Omega \subset \mathbb{R}^N$ *satisfy (D),* $f \in C(\Omega \times \mathbb{R})$ *satisfy (F1) and (F2), and let* $u \in \mathscr{H}_0^s(\Omega)$ *be a continuous bounded solution of* (2.1.1)*, which is antisymmetric with respect to* H_N *and* $u \geq 0$ *in* $H_N \cap \Omega$*. Then* u *is symmetric with respect to* ∂H_1 *and either* $u \equiv 0$ *in* Ω *or* $u|_{\Omega \cap H_1 \cap H_N}$ *is strictly decreasing in* x_1*, that is, for every* $x,y \in \Omega \cap H_1 \cap H_N$ *with* $x_1 < y_1$ *we have* $u(x) > u(y)$*.*

Theorem 0.2.7 represents a purely nonlocal feature. In fact in the classical case of the Laplacian, we may restrict u in its part of nonnegativity and apply the usual symmetry results due to [54]. However, in this fractional setting this is not possible due to the nonlocality. Theorem 0.2.7 can quite easily be derived from the classical moving plane method once we have the right ingredient. The latter being here the following variant of Proposition 0.2.5: let $c_\infty > 0$, then there is $\delta > 0$ such that for all $U \subset H_1 \cap H_2$ open with $|U| \leq \delta$, $c \in L^\infty(U)$ with $c \leq c_\infty$, and all doubly antisymmetric supersolutions w of (5.0.5), it holds that $w \geq 0$ in $H_1 \cap H_2$. To get this it is enough to observe that one can make $\delta > 0$ sufficiently small so that $c_\infty \leq \lambda^-_{1,s}(U)$, and then apply Proposition 0.2.5. We refer to [34] or Chapter 2 below for more details.
Theorem 0.2.7 applies in particular to minimers of

$$\lambda^-_{s,p}(\Omega) := \inf_{\substack{u\in\mathscr{H}^s_0(\Omega)\\ u\neq 0\\ u\circ r_N=-u}} \left\{ \frac{\mathscr{E}_s(u,u)}{\left(\int_\Omega |u(x)|^p\,dx\right)^{2/p}} \right\}, \tag{0.2.15}$$

where Ω is a bounded open set of class $C^{1,1}$ that satisfies (D) and $p \in [1, \frac{2N}{N-2s})$ with $N > 2s$. The reason for this is that a minimizer u of (0.2.15) could be taken to be nonnegative in $\Omega \cap H_N$ and therefore satisfies the hypothesis of Theorem 2.1.1. To see this, we argue by contradiction: Assuming that u changes sign in $\Omega \cap H_N$ we let $\Omega_1^+ = \{x \in \Omega \cap H_N : u(x) > 0\}$ and $\Omega_2^- = \{x \in \Omega \cap H_N : u(x) \leq 0\}$. To reach a contradiction, we polarize u with respect to the hyperplane ∂H_1 and use this polarization as a test function in (0.2.15), i.e, we use $\overline{u} = 1_{\Omega\cap H_N}|u| - 1_{\Omega\cap H_N^c}|u|$ as a test function. This gives, after reflecting several times,

$$0 \leq \int_{\Omega_1^+\times\Omega_2^-} u(x)u(y)\Big[\frac{1}{|x-y|^{N+2s}} - \frac{1}{|r_N(x)-y|^{N+2s}}\Big]dxdy, \tag{0.2.16}$$

from which we deduce $u \equiv 0$ in Ω_2^- since the integrand of (0.2.16) is nonpositive, and therefore $u \geq 0$ in $H_N \cap \Omega$ this gives a contradiction. We refer to Chapter 2, Section 2.4 for more details.

0.2.3 $[P4]$: Nonradiality of fractional second eigenfunctions of thin annuli

This paper is a joint work with Sven Jarohs. The first part of the work is concerned with the nonradiality of the solutions to the eigenvalue problem

$$u \in \mathscr{H}^s_0(A_{r,\rho}) \quad \text{and} \quad (-\Delta)^s u = \lambda_{2,s}(A_{r,\rho})u \quad \text{in} \quad A_{r,\rho}, \tag{0.2.17}$$

where $A_{r,\rho}$ is an annulus of given radii $0 < r < \rho$ and $\lambda_{2,s}(A_{r,\rho})$ is the second eigenvalue characterized by

$$\lambda_{2,s}(A_{r,\rho}) = \min_{\substack{u\in\mathscr{H}^s_0(A_{r,\rho})\\ u\neq 0}} \left\{ \frac{\mathscr{E}_s(u,u)}{\int_\Omega |u(x)|^2\,dx} : \int_{A_{r,\rho}} \varphi_1(x)u(x)\,dx = 0 \right\}. \tag{0.2.18}$$

Here φ_1 denote the first (normalised) fractional eigenfunction of $A_{r,\rho}$. This question is related to a conjecture attributed to Bañuelos and Kulczycki stating that a second eigenfunction of a ball cannot be radial. The conjecture has been solved recently in [41] by estimating the Morse index of a second radial eigenfunction(see also [9] for another argument proving the conjecture). The main motivation to investigate the equation above comes from the question we raised in the previous discussion concerning the optimal placement of an obstacle in order to maximize the second fractional eigenvalue. By the strategy described in Section 0.2.2, to answer the question one needs, among other things, to prove that the second fractional eigenvalue of an annulus coincides with its first antisymmetric eigenvalue as defined in (0.2.11). In other words, we need to established the nonlocal counterpart of (0.2.10). The latter happens to be closely related to nonradiality of solutions to the nonlocal equation (0.2.17), see below or Chapter 4, Lemma 4.6.2 for more details.

Our main result regarding (0.2.17) reads as follows.

Theorem 0.2.8. *There is $\tau_0 > 0$ such that for any $\tau \in [\tau_0, 1)$ any solution $u \in \mathscr{H}_0^s(A_\tau)$ of*

$$(-\Delta)^s u = \lambda_{2,s}(A_\tau)u \quad in \quad A_\tau \tag{0.2.19}$$

is nonradial.

One of the consequences of Theorem 0.2.8 is that for annuli A of small width, the second eigenvalue coincides with the first antisymmetric eigenvalue $\lambda_{1,s}^-(A)$. That is, one has

$$\lambda_{2,s}(A_\tau) = \lambda_{1,s}^-(A_\tau) := \inf_{\substack{u \in \mathscr{H}_0^s(A_\tau) \\ u \neq 0 \\ u \circ r_N = -u}} \left\{ \frac{\mathscr{E}_s(u,u)}{\int_{A_\tau} |u(x)|^2 \, dx} \right\} \quad \text{for } \tau \text{ sufficiently close to 1.} \tag{0.2.20}$$

The identity (0.2.20) plays an important role in the proof of Theorem 0.2.11 below. The proof of Theorem 0.2.8 follows the same pattern as that of [41, Theorem 1.2] and it consists of estimating the Morse index of a second radial eigenfunction u (if any) to get a contradiction. However, in our case, an additional difficulty arises due to the fact that the boundary of the annulus is disconnected. The idea is, starting from a *second radial eigenfunction* $u = \overline{u}(|\cdot|)$, one can construct, using the partial derivatives of u, a family $\{d_j\}_j$ of N-linearly independent test functions for the variational eigenvalues of the linearised operator $L := (-\Delta)^s - \lambda_{2,s}(A_{\tau,1})$ with the properties that $\mathscr{E}_{s,L}(d_j, d_j) < 0$ and $d_j \circ r_j = -d_j$ where $\mathscr{E}_{s,L}$ is the bilinear form associated to the operator L and r_j the reflection across the hyperplane $\partial H_+^j := \{x \in \mathbb{R}^N : x_j = 0\}$. Since the first eigenfunction φ_1 of L (which is the same as that of $(-\Delta)^s$) is radial and has negative energy with respect to the operator L, we thus obtain $(N+1)$-linearly independent test functions $\{\varphi_1, d_j\}_j$ each of which has a negative energy. This would implies that $u = \overline{u}(|\cdot|)$ has Morse index greater than or equal to $N+1$ which contradicts the fact of u is a second eigenfunction. The details to the construction of the d_j- functions is as in [41].
To run the argument of [41] successfully for the annulus $A_{\tau,1}$, one needs

$$\psi(1)\psi(\tau) < 0, \tag{0.2.21}$$

where $\psi(x) := \lim_{A_{\rho,\beta} \ni y \to x} \frac{u(y)}{d^s(y)}$ is the fractional normal derivative of a second radial eigenfunction $u(\cdot) = \overline{u}(|\cdot|)$ (if any). The property (0.2.21) cannot be proved by using the classical fractional Hopf lemma, since u is sign changing, and it also does not follow from the fractional Pohozaev identity either. To get (0.2.21), for τ sufficiently close to 1, we use a compactness argument. By the rescaling property of the fractional Laplacian, it is enough to prove

$$\overline{\psi}_R(0)\overline{\psi}_R(1) < 0 \quad \text{for } R > 0 \text{ sufficiently large,} \tag{0.2.22}$$

where $\overline{\psi}_R := \frac{\overline{u}_R(\cdot+R)}{\min(\cdot,1-\cdot)^s} : [0,1] \to \mathbb{R}$ is the fractional normal derivative of a second radial eigenfunction $\overline{u}_R(|\cdot|)$ of the annulus $A_R = \{x \in \mathbb{R}^N : R < |x| < R+1\}$. The get (0.2.22) the idea is to show that

$$\overline{\psi}_R \to \frac{\varphi_2}{d^s} \quad \text{uniformly in } [0,1] \quad \text{as} \quad R \to \infty, \tag{0.2.23}$$

with

$$\varphi_2 \in \mathscr{H}^s_0(0,1) \quad \text{and} \quad (-\Delta)^s \varphi_2 = \lambda_{2,s}(0,1)\varphi_2 \quad \text{in} \quad (0,1).$$

Once we have this, the claim follows since the limiting function φ_2/d^s has the desire property. To establish (0.2.23), we remark that the function $w_R(\cdot) = \overline{u}_R(\cdot+R) \in \mathscr{H}^s_{loc}(\mathbb{R})$ solves weakly the equation

$$\mathscr{L}_{K_R} w_R + w_R V_R = \lambda_{2,s}(A_R)\big(1+r/R\big)^{N-1} w_R \qquad \text{in} \qquad (0,1) \tag{0.2.24}$$

where

$$\langle \mathscr{L}_{K_R} u, w\rangle = \int_{\mathbb{R}^2} (u(r)-u(\widetilde{r}))(w(r)-w(\widetilde{r}))K_R(r,\widetilde{r})\,dr\,d\widetilde{r},$$

with V_R satisfying $0 < V_R \le C(N,s)$ for R large enough. Here

$$K_R(r,\widetilde{r}) = 1_{(-2,2)}(r)1_{(-2,2)}(\tilde{r})\overline{K}_R(r,\tilde{r}),$$

and

$$\overline{K}_R(r,\tilde{r}) = \frac{1}{|r-\tilde{r}|^{1+2s}} \frac{(r+R)^{\frac{N-1}{2}}(\tilde{r}+R)^{\frac{N-1}{2}}}{R^{N-1}} \int_{\frac{\sqrt{(r+R)(\tilde{r}+R)}}{|r-\tilde{r}|}(\mathbb{S}^{N-1}-e_1)} \frac{d\theta}{(1+|\theta|^2)^{\frac{N+2s}{2}}}.$$

We note that the kernel $K_R(\cdot,\cdot)$ satisfies

$$\overline{C}(N,s)|r-\widetilde{r}|^{-1-2s} \le K_R(r,\tilde{r}) \le C(N,s)|r-\widetilde{r}|^{-1-2s} \quad \text{for } R > 0 \text{ large enough.} \tag{0.2.25}$$

Now the point is to apply regularity estimates of types [40, Theorem 1.8] to solutions to (0.2.24) to deduce a uniform Hölder estimate of ψ_R and then use the Arzela-Ascoli compactness theorem. To do that the following two lemmas are of key importance.

Lemma 0.2.9. *There exists $C(N,s) > 0$ such that $\lambda_{2,s}(A_R) \le C(N,s)$ for all $R \ge 1$.*

Lemma 0.2.10. *Let $R > 1$ and define*

$$F_R : [-2,+2] \times [0,2] \to \mathbb{R}_+$$

by

$$F_R(t,r) = \begin{cases} \displaystyle\int_{\frac{\sqrt{(t+R)(t\pm r+R)}}{r}(\mathbb{S}^{N-1}-e_1)} \frac{d\theta}{(1+|\theta|^2)^{\frac{N+2s}{2}}}, & \forall r \neq 0; \\ \dfrac{\pi^{\frac{N-1}{2}}\Gamma(\frac{1+2s}{2})}{\Gamma(\frac{N+2s}{2})}, & \text{for } r = 0. \end{cases}$$

Then, there exists $C > 0$ (independent of R) so that

$$|F_R(t,r) - F_R(t',r')| \leq C(|r-r'|^{s+\delta} + |t-t'|^{s+\delta}), \quad \forall\, r,r' \in [0,2] \quad \textit{and} \quad \forall\, t,t' \in [-2,+2],$$

for some $\delta > 0$.

Thanks to Lemma 0.2.10 and [40, Theorem 1.8] we deduce the estimate

$$\|\psi_R\|_{C^{s-\beta}([0,1])} \leq C\big(\|w_R\|_{L^2(0,1)} + \|w_R\|_{\mathscr{L}^1_s} + \|\lambda_{2,s}(A_R)\big(1+r/R\big)^{N-1} w_R\|_{L^\infty(0,1)}\big) \quad \forall \beta \in (0,s),$$

with $C = C(N,s) > 0$. Lemma 0.2.9 is used to control the RHS of the estimate above uniformly in R. Applying the Arzela-Ascoli theorem and passing into a limit in (0.2.24) we arrive at (0.2.23). We refer to Chapter 4 for more details.

A combination of Proposition 0.2.5, Proposition 0.2.6 and the identity (0.2.20) gives the following extension of Theorem 0.2.4.

Theorem 0.2.11. *Assume a second eigenfunction of an annulus cannot be radial. Let $B_1(0)$ be the unit centered ball and $\tau \in (0,1)$. Define*

$$\mathscr{A} := \{a \in B_1(0) : B_\tau(a) \subset B_1(0)\}.$$

Then the map $\mathscr{A} \to \mathbb{R}_+,\ a \mapsto \lambda_{2,s}(B_1(0) \setminus \overline{B_\tau(a)})$ is maximized if and only if $a = 0$.

The proof uses the same strategy as in the proof of Theorem 0.2.4. However, due to the non-simplicity of $\lambda_{2,s}$, the map $m : (-1+\tau, 1-\tau) \ni a \mapsto \lambda_{2,s}(B_1(0) \setminus \overline{B_\tau(ae_1)})$ is not differentiable. To overcome this difficulty, we observe that the second eigenvalue of the eccentric annulus $B_1(0) \setminus \overline{B_\tau(ae_1)}$ is always controlled by its first antisymmetric eigenvalue $\lambda^-_{1,2}(a) := \lambda^-_{1,s}(B_1(0) \setminus \overline{B_\tau(ae_1)})$ and that for the annulus the two numbers coincide, i.e, $\lambda^-_{1,s}B_1(0) \setminus \overline{B_\tau(0)} = \lambda_{2,s}(B_1(0) \setminus \overline{B_\tau(0)})$. These two properties allow to reduce the proof of Theorem 0.2.11 into proving that the map $m : [0, 1-\tau) \ni a \mapsto \lambda^-_{1,2}(a)$ is strictly decreasing. To get the latter, we use a shape derivative argument combined with the maximum principle for doubly antisymmetric functions Proposition 0.2.5 and Proposition 0.2.6. We refer to Chapter 4, Section 4.6 for more details.

0.2.4 [$P3$]: A generalized fractional Pohozaev identity and applications

This paper is concerned with integral identities for solutions to the semilinear boundary valued problem

$$u \in \mathscr{H}^s_0(\Omega) \quad \text{and} \quad (-\Delta)^s u = f(u) \quad \text{in} \quad \Omega, \tag{0.2.26}$$

where $f:\mathbb{R}\to\mathbb{R}$ is a locally Lipschitz nonlinearity. These types of identities, known as Pohozaev's identities, were first obtained by S. I. Pohozaev in [89] for the classical case of the Laplacian. The corresponding identity in this fractional setting was discovered by X. Ros-Oton and J. Serra in [95]. In there, it was shown that any bounded weak solution to (0.2.26) satisfies

$$\Gamma(1+s)^2\int_{\partial\Omega}(u/d^s)^2 x\cdot\nu\,dx = 2N\int_{\Omega}F(u)\,dx-(N-2s)\int_{\Omega}f(u)u\,dx. \tag{0.2.27}$$

where $F(t)=\int_0^t f(s)ds$ and ν is the *outer unit normal* to the boundary $\partial\Omega$. This identity happens to be very useful in studying the semilinear problem (0.2.26). Among others, it allows to obtain unique continuation properties, nonexistence results for f critical or supercritical and also simplicity results in the case $f(u)=\lambda u$.

One of our main contributions in here is an extension of (0.1.13) with a general globally Lipchitz vector field X. To state the result we need to fix some notations. Let $X\in C^{0,1}(\mathbb{R}^N,\mathbb{R}^N)$ be a globally Lipschitz vector field and define the *fractional deformation kernel* by

$$K_X(x,y):=\frac{b_{N,s}}{2}\left\{\big(\operatorname{div}X(x)+\operatorname{div}X(y)\big)-(N+2s)\frac{\big(X(x)-X(y)\big)\cdot(x-y)}{|x-y|^2}\right\}|x-y|^{-N-2s}.$$

Moreover, for all $u,v\in H^s(\mathbb{R}^N)$, we let $\mathscr{E}_{K_X}(u,v)$ be the bilinear form associated to K_X, i.e,

$$\mathscr{E}_{K_X}(u,v):=\int_{\mathbb{R}^{2N}}K_X(x,y)(u(x)-u(y))(v(x)-v(y))\,dxdy. \tag{0.2.28}$$

Our main result for the problem (0.2.26) is the following generalization of (0.1.13).

Theorem 0.2.12. *Let $u\in\mathscr{H}_0^s(\Omega)\cap L^\infty(\Omega)$ be a (weak) solution of the problem (0.2.26) with $f:\mathbb{R}\to\mathbb{R}$ locally Lipschitz. Then we have*

$$\Gamma(1+s)^2\int_{\partial\Omega}\big(u/d^s\big)^2 X\cdot\nu\,dx = 2\int_{\Omega}F(u)\operatorname{div}X\,dx-\mathscr{E}_{K_X}(u,u), \tag{0.2.29}$$

where $F(t)=\int_0^t f(s)ds$ and ν is the outer unit normal to the boundary $\partial\Omega$.

We remark that when $X=\mathrm{id}_{\mathbb{R}^N}$, the kernel $K_X(x,y)$ reduces to $(N-2s)\frac{b_{N,s}}{2}|x-y|^{-N-2s}$ and hence the remainder term $\mathscr{E}_{k_X}(u,u)$ becomes $(2s-N)\int_\Omega f(u)u\,dx$. In this case, identity (0.2.29) reduces to identity (0.1.13).

Let us mention some consequences of the Theorem 0.2.12.

(i) Let Ω be a bounded open set of class $C^{1,1}$ given together with a globally Lipschitz vector field X satisfying

$$(X(x)-X(y))\cdot(x-y)=c|x-y|^2\quad\forall x,y\in\mathbb{R}^N \text{ with } c>0 \text{ and } X\cdot\nu\geq 0 \text{ on } \partial\Omega. \tag{0.2.30}$$

Then, for $p > \frac{2N}{(N-2s)}$, the problem

$$(-\Delta)^s u = |u|^{p-2}u \quad \text{in} \quad \Omega \subset \mathbb{R}^N,\ N \geq 2, \qquad u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega,$$

admits no nontrivial bounded solution. By the first condition of (0.2.30), the RHS of (0.2.29) reduces to $c\big(\frac{2N}{p} - (N-2s)\big)\int_\Omega |u|^p\,dx$. From this and (0.2.29), one immediately sees that, $p > \frac{2N}{N-2s}$ and $X\cdot\nu \geq 0$ on $\partial\Omega$, imply $\int_\Omega |u|^p dx = 0$. We note that for the critical case $p = \frac{2N}{N-2s}$, the same conclusion holds under the additional assumption that u is nonnegative.

(ii) Let Ω be a bounded open set of class $C^{1,1}$ given together with a globally Lipschitz vector field X satisfying

$$\operatorname{div} X(x) \geq c_1, \quad \big(X(x) - X(y)\big)\cdot(x-y) \leq c_2|x-y|^2 \quad \text{for all } x,y \in \mathbb{R}^N, \tag{0.2.31}$$

and

$$X\cdot\nu \geq 0 \quad \text{on} \quad \partial\Omega \tag{0.2.32}$$

with some constants $c_2 > 0$ and $c_1 \in (\frac{c_2 N}{2}, c_2 N]$. Moreover, suppose that

$$0 < s < \Big(\frac{c_1}{c_2} - \frac{N}{2}\Big), \qquad p > \frac{2N}{\frac{2c_1}{c_2} - (N+2s)}.$$

Then, the problem

$$(-\Delta)^s u = |u|^{p-2}u \quad \text{in}\ \Omega, \qquad u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega \tag{0.2.33}$$

admits no nontrivial bounded solution.

(iii) Identity (3.1.8) could also be applied to compute the shape derivative of simple eigenvalues of the Dirichlet problem $u \in \mathscr{H}^s_0(\Omega)$ and $(-\Delta)^s u = \lambda u$ in Ω. If λ is a simple eigenvalue and Φ_ε is a family of deformations satisfying (0.2.3), then one shows, by the implicit function theorem, that there exists a C^1-curve $(-\varepsilon_0, \varepsilon_0) \ni \varepsilon \mapsto (u_\varepsilon, \lambda_\varepsilon) \in \mathscr{H}^s_0(\Omega) \times (0,\infty)$ with $(u_0, \lambda_0) = (u, \lambda)$ so that

$$\int_{\mathbb{R}^{2N}} (u_\varepsilon(x) - u_\varepsilon(y))^2 K_\varepsilon(x,y)dxdy = \lambda_\varepsilon \int_\Omega u_\varepsilon^2 Jac_{\varphi_\varepsilon}(x)dx$$

with $K_\varepsilon(x,y) := \frac{b_{N,s}}{2}\frac{\text{Jac}_{\Phi_\varepsilon}(x)\text{Jac}_{\Phi_\varepsilon}(y)}{|\Phi_\varepsilon(x)-\Phi_\varepsilon(y)|^{N+2s}}$. Here Jac_{Φ_ε} is the Jacobian of Φ_ε. Differentiating the identity above we obtain

$$\mathscr{E}_{K_X}(u,u) = \frac{d}{d\varepsilon}\lambda_\varepsilon\Big|_{\varepsilon=0} \int_\Omega u^2(x)\,dx + \lambda_0 \int_\Omega u^2 \operatorname{div} X\,dx. \tag{0.2.34}$$

Applying the identity (0.2.29) with $f(u) = \lambda_0 u$, we deduce from (0.2.34) that

$$\frac{d}{d\varepsilon}\lambda_\varepsilon\Big|_{\varepsilon=0} = \frac{-\Gamma^2(1+s)\int_{\partial\Omega}(u/d^s)^2 X\cdot\nu\,dx}{\int_\Omega u^2(x)\,dx} \qquad \text{with} \qquad X = \frac{d}{d\varepsilon}\Phi_\varepsilon\Big|_{\varepsilon=0}. \tag{0.2.35}$$

Identity (0.2.35) applies in particular to radial Dirichlet eigenvalues of balls or annuli. In fact, for balls and annuli, the simplicity of radial eigenvalues happens to be a directy consequence of the Pohozaev identity (0.1.13), see Chapter 3, Section 3.4 for more details.

We obtain Theorem 0.2.12 as a particular case of the following more general identity which extends the identity (0.1.14) obtained in [95, Proposition 1.6].

Theorem 0.2.13. *Let Ω be a bounded open set of class $C^{1,1}$ and let $u \in H^s(\mathbb{R}^N)$ such that $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$. Moreover, assume $(-\Delta)^s u \in L^\infty(\Omega)$ if $2s > 1$ and $(-\Delta)^s u \in C^\alpha_{loc}(\Omega) \cap L^\infty(\Omega)$ with $\alpha > 1-2s$ if $2s \leq 1$. Then, the following identity holds*

$$2\int_\Omega \nabla u \cdot X(-\Delta)^s u\,dx = -\Gamma^2(1+s)\int_{\partial\Omega} \left(u/d^s\right)^2 X\cdot\nu\,dx - \mathscr{E}_{K_X}(u,u) \tag{0.2.36}$$

where $\mathscr{E}_{K_X}(v,w)$ is defined as in (0.2.28).

We note that under the assumptions of the theorem, the integrals appearing in (0.2.36) are all well defined by the standard regularity theory see e.g [42, 104]. To deduce the formula (0.2.29) from (0.2.36) it simply suffices to use the pointwise identity $(-\Delta)^s u = f(u)$, $\nabla F(u) = f(u)\nabla u$ and integrate by parts and noting that $F(0) = 0$. Note that since $f : \mathbb{R} \to \mathbb{R}$ is assumed to be locally Lipschitz, the assumptions of Theorem 0.2.13 are satisfied.

One immediate consequence of Theorem 0.2.13 is the following identity: Let u, w satisfy the assumption in the theorem above. Then, we have

$$\int_\Omega \nabla u \cdot X(-\Delta)^s w\,dx = -\int_\Omega \nabla w \cdot X(-\Delta)^s u\,dx - \Gamma^2(1+s)\int_{\partial\Omega} \frac{u}{d^s}\frac{w}{d^s} X\cdot\nu\,dx - \mathscr{E}_{K_X}(u,w). \tag{0.2.37}$$

The identity (0.2.37) follows by simply applying identity (0.2.36) to $u+tw$ and differentiate with respect to t. The identity (0.2.37) with $X = e_j$, the j-unit coordinate vector was stated in [95, Theorem 1.9].
Our proof of Theorem 0.2.13 is based on approximation arguments. It uses the same strategy used in the proof of Theorem 0.2.1. We refer to Chapter 3 for more details.

Chapter 1

A fractional Hadamard formula and applications

This Chapter is devoted to the paper $[P1]$ a joint work with M. M. Fall and T. Weth. The exposition is as in the original paper, except in here we added in the appendix an alternative computation of the constant κ_s apprearing in (1.5.1) below.

1.1 Introduction

Let $s \in (0,1)$ and $\Omega \subset \mathbb{R}^N$ be a bounded open set. The present paper is devoted to the study of best constants $\lambda_{s,p}(\Omega)$ in the family of subcritical Sobolev inequalities

$$\lambda_{s,p}(\Omega)\|u\|^2_{L^p(\Omega)} \leq [u]^2_s \qquad \text{for all } u \in \mathscr{H}^s_0(\Omega), \tag{1.1.1}$$

where $p \in [1, \frac{2N}{N-2s})$ if $2s < N$ and $p \in [1,\infty)$ if $2s \geq N = 1$. Here, the Sobolev space $\mathscr{H}^s_0(\Omega)$ is given as completion of $C^\infty_c(\Omega)$ with respect to the norm $[\cdot]_s$ defined by

$$[u]^2_s = \frac{b_{N,s}}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dxdy \qquad \text{with} \quad b_{N,s} = \pi^{-\frac{N}{2}} s 4^s \frac{\Gamma(\frac{N}{2}+s)}{\Gamma(1-s)}. \tag{1.1.2}$$

The normalization constant $b_{N,s}$ is chosen such that $[u]^2_s = \int_{\mathbb{R}^N} |\xi|^{2s}|\hat{u}(\xi)|^2 d\xi$ for $u \in \mathscr{H}^s_0(\Omega)$, where $\hat{u}$ denotes the Fourier transform of u. The best (i.e., largest possible) constant in (1.1.1) is given by

$$\lambda_{s,p}(\Omega) := \inf\left\{[u]^2_s \,:\, u \in \mathscr{H}^s_0(\Omega), \quad \|u\|_{L^p(\Omega)} = 1\right\}. \tag{1.1.3}$$

As a consequence of the subcriticality assumption on p and the boundedness of Ω, the space $\mathscr{H}^s_0(\Omega)$ compactly embeds into $L^p(\Omega)$. Therefore a direct minimization argument shows that $\lambda_{s,p}(\Omega)$ admits a nonnegative minimizer $u \in \mathscr{H}^s_0(\Omega)$ with $\|u\|_{L^p(\Omega)} = 1$. Moreover, every such minimizer solves, in the weak sense, the semilinear problem

$$(-\Delta)^s u = \lambda_{s,p}(\Omega) u^{p-1} \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \tag{1.1.4}$$

where $(-\Delta)^s$ stands for the fractional Laplacian. It therefore follows from regularity theory and the strong maximum principle for $(-\Delta)^s$ that u is strictly positive in Ω, see Lemma 1.2.3 below. We recall that, for functions $\varphi \in C_c^{1,1}(\mathbb{R}^N)$, the fractional Laplacian is given by

$$(-\Delta)^s \varphi(x) = b_{N,s} PV \int_{\mathbb{R}^N} \frac{\varphi(x)-\varphi(y)}{|x-y|^{N+2s}}\, dy = \frac{b_{N,s}}{2} \int_{\mathbb{R}^N} \frac{2\varphi(x)-\varphi(x+y)-\varphi(x-y)}{|y|^{N+2s}}\, dy.$$

Of particular interest are the cases $p=1$ and $p=2$ which correspond to the fractional torsion problem

$$(-\Delta)^s u = \lambda_{s,1}(\Omega) \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \tag{1.1.5}$$

and the eigenvalue problem

$$(-\Delta)^s u = \lambda_{s,2}(\Omega) u \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \tag{1.1.6}$$

associated with the first Dirichlet eigenvalue of the fractional Laplacian, respectively. In these cases, the minimization problem for $\lambda_{s,p}(\Omega)$ in (1.1.3) possesses a unique positive minimizer. Indeed, it is a well-known consequence of the fractional maximum principle that (1.1.5) admits a unique solution, and that (1.1.6) has a unique positive eigenfunction with $\|u\|_{L^2(\Omega)} = 1$. Incidentally, the uniqueness of positive minimizers extends to the full range $1 \le p \le 2$, as we shall show in Lemma 1.7.1 in the appendix of this paper.

Our first goal in this paper is to analyze the dependence of the best constants on the underlying domain Ω. For this we shall derive a formula for a one-sided shape derivative of the map $\Omega \mapsto \lambda_{s,p}(\Omega)$. We assume from now on that $\Omega \subset \mathbb{R}^N$ is a bounded open set of class $C^{1,1}$, and we consider a family of deformations $\{\Phi_\varepsilon\}_{\varepsilon\in(-1,1)}$ with the following properties:

$$\begin{aligned} &\Phi_\varepsilon \in C^{1,1}(\mathbb{R}^N;\mathbb{R}^N) \text{ for } \varepsilon \in (-1,1),\ \Phi_0 = \mathrm{id}_{\mathbb{R}^N}, \text{ and}\\ &\text{the map } (-1,1) \to C^{0,1}(\mathbb{R}^N,\mathbb{R}^N),\ \varepsilon \to \Phi_\varepsilon \text{ is of class } C^2. \end{aligned} \tag{1.1.7}$$

We note that (3.3.1) implies that $\Phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ is a global diffeomorphism if $|\varepsilon|$ is small enough, see e.g. [28, Chapter 4.1]. To clarify, we stress that we only need the C^2-dependence of Φ_ε on ε with respect to Lipschitz-norms, while Φ_ε is assumed to be a $C^{1,1}$-function for $\varepsilon \in (-1,1)$ to guarantee $C^{1,1}$-regularity of the perturbed domains $\Phi_\varepsilon(\Omega)$.

From the variational characterization of $\lambda_{s,p}(\Omega)$ it is not difficult to see that the map $\varepsilon \mapsto \lambda_{s,p}(\Phi_\varepsilon(\Omega))$ is continuous. However, since $\lambda_{s,p}(\Omega)$ may not have a unique positive minimizer, we cannot expect this map to be differentiable. We therefore rely on determining the right derivative of $\varepsilon \mapsto \lambda_{s,p}(\Phi_\varepsilon(\Omega))$ from which we derive differentiability whenever $\lambda_{s,p}(\Omega)$ admits a unique positive minimizer, thereby extending the classical Hadamard shape derivative formula for the first Dirichlet eigenvalue of the Laplacian $-\Delta$.

Throughout this paper, we consider a fixed function $d \in C^{1,1}(\mathbb{R}^N)$ which coincides with the *signed* distance function $\mathrm{dist}(\cdot,\mathbb{R}^N\setminus\Omega) - \mathrm{dist}(\cdot,\Omega)$ in a neighborhood of the boundary $\partial\Omega$. We note here that, since we assume that Ω is of class $C^{1,1}$, the signed distance function is also of class $C^{1,1}$ in a neighborhood of $\partial\Omega$ but not globally on $\mathbb{R}^N$. We also suppose that d is chosen with the property that d is positive in Ω and negative in $\mathbb{R}^N \setminus \overline{\Omega}$, as it is the case for the signed distance function.

Our first main result is the following.

Theorem 1.1.1. *Let* $\lambda_{s,p}(\Omega)$ *be given by* (1.1.3) *and consider a family of deformations* Φ_ε *satisfying* (3.3.1). *Then the map* $\varepsilon \mapsto \theta(\varepsilon) := \lambda_{s,p}(\Phi_\varepsilon(\Omega))$ *is right differentiable at* $\varepsilon = 0$. *Moreover,*

$$\partial_+ \theta(0) = \min \left\{ \Gamma(1+s)^2 \int_{\partial\Omega} (u/d^s)^2 X \cdot \nu \, dx \, : \, u \in \mathscr{H} \right\}, \tag{1.1.8}$$

where ν *denotes the interior unit normal on* $\partial\Omega$, $\mathscr{H}$ *is the set of positive minimizers for* $\lambda_{s,p}(\Omega)$ *and* $X := \partial_\varepsilon\big|_{\varepsilon=0}\Phi_\varepsilon$.

Here the function u/d^s is defined on $\partial\Omega$ as a limit. Namely, for $x_0 \in \partial\Omega$, the limit

$$\frac{u}{d^s}(x_0) = \lim_{\substack{x \to x_0 \\ x \in \Omega}} \frac{u}{d^s}(x) \tag{1.1.9}$$

exists, as the function u/d^s extends to a function in $C^\alpha(\overline{\Omega})$ for some $\alpha > 0$, see [94]. In addition, the function $d^{1-s}\nabla u$ also admits a Hölder continuous extension on $\overline{\Omega}$ satisfying $d^{1-s}\nabla u \cdot \nu = su/d^s$ on $\partial\Omega$, see [42]. As a consequence, the expression u/d^s, restricted on $\partial\Omega$, plays the role of an inner fractional normal derivative. Note that, for $s = 1$, the limit on the RHS of (1.1.9) coincides with the classical inner normal derivative of u at x_0.
We observe that the constant $\Gamma(1+s)^2$ appears also in the fractional Pohozaev identity, see e.g. [95]. This is, to some extend, not surprising at least in the classical case since Pohozaev's identity can be obtained using techniques of domain variation, see e.g. [103].
We also remark that one-sided derivatives naturally arise in the analysis of parameter-dependent minimization problems, see e.g. [28, Section 10.2.3] for an abstract result in this direction. Related to this, they also appear in the analysis of the domain dependence of eigenvalue problems with possible degeneracy, see e.g. [44] and the references therein.
A natural consequence of Theorem 1.1.1 is that the map $\varepsilon \mapsto \theta(\varepsilon) = \lambda_{s,p}(\Phi_\varepsilon(\Omega))$ is differentiable at $\varepsilon = 0$ whenever $\lambda_{s,p}(\Omega)$ admits a unique positive minimizer. Indeed, applying Theorem 1.1.1 to the map $\varepsilon \mapsto \widetilde{\theta}(\varepsilon) := \lambda_{s,p}(\Phi_{-\varepsilon}(\Omega))$ yields

$$\partial_- \theta(0) = -\partial_+ \widetilde{\theta}(0) = \max \left\{ \Gamma(1+s)^2 \int_{\partial\Omega} (u/d^s)^2 X \cdot \nu \, dx \, : \, u \in \mathscr{H} \right\},$$

where $\mathscr{H}$ is given as in Theorem 1.1.1. As a consequence, we obtain the following result.

Corollary 1.1.2. *Let* $\lambda_{s,p}(\Omega)$ *be given by* (1.1.3) *and consider a family of deformations* Φ_ε *satisfying* (3.3.1). *Suppose that* $\lambda_{s,p}(\Omega)$ *admits a unique positive minimizer* $u \in \mathscr{H}_0^s(\Omega)$. *Then the map* $\varepsilon \mapsto \theta(\varepsilon) = \lambda_{s,p}(\Phi_\varepsilon(\Omega))$ *is differentiable at* $\varepsilon = 0$. *Moreover*

$$\theta'(0) = \Gamma(1+s)^2 \int_{\partial\Omega} (u/d^s)^2 X \cdot \nu \, dx, \tag{1.1.10}$$

where $X := \partial_\varepsilon\big|_{\varepsilon=0}\Phi_\varepsilon$.

As mentioned earlier, $\lambda_{s,p}(\Omega)$ admits a unique positive minimizer $u \in \mathscr{H}_0^s(\Omega)$ for $1 \le p \le 2$, see Lemma 1.7.1 in the appendix. Therefore Corollary 1.1.2 extends, in particular, the classical Hadamard formula, for the first Dirichlet eigenvalue $\lambda_{1,2}(\Omega)$ of $-\Delta$, to the fractional setting. We recall, see e.g. [65], that the classical Hadamard formula is given by

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\lambda_{1,2}(\Phi_\varepsilon(\Omega)) = \int_{\partial\Omega} |\nabla u|^2 X \cdot \nu \, dx. \tag{1.1.11}$$

An analogue of Corollary 1.1.2 for the case of the local r-Laplace operator was obtained in [84, 20]. We also point out that, prior to this paper, a Hadamard formula in the fractional setting of the type (1.1.10) was obtained in [26] for the special case $p = 1$, $s = \frac{1}{2}$, $N = 2$ and Ω of class C^∞. We are not aware of any other previous work related to Theorem 1.1.1 or 1.1.2 in the fractional setting.
Our next result provides a characterization of constrained local minima of $\lambda_{s,p}$. Here and in the following, we call a bounded open subset Ω of class $C^{1,1}$ a constrained local minimum for $\lambda_{s,p}$ if for all families of deformations Φ_ε satisfying (3.3.1) and the volume invariance condition $|\Phi_\varepsilon(\Omega)| = |\Omega|$ for $\varepsilon \in (-1,1)$, there exists $\varepsilon_0 \in (0,1)$ with $\lambda_{s,p}(\Phi_\varepsilon(\Omega)) \ge \lambda_{s,p}(\Omega)$ for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Our classification result reads as follows.

Corollary 1.1.3. *Let $p \in \{1\} \cup [2,\infty)$. If an open subset Ω of $\mathbb{R}^N$ of class C^3 is a volume constrained local minimum for $\Omega \mapsto \lambda_{s,p}(\Omega)$, then Ω is a ball.*

Corollary 1.1.3 is a consequence of Theorem 1.1.1, from which we derive that if Ω is a constrained local minimum then any element $u \in \mathscr{H}$ satisfies the overdetermined condition $u/d^s \equiv constant$ on $\partial\Omega$. Therefore by the rigidity result in [43] we find that Ω must be a ball. We point out that we are not able to include the case $p \in (1,2)$ in Corollary 1.1.3, since the rigidity result in [43] is based on the moving plane method and therefore requires the nonlinearity in (1.1.4) to be Lipschitz. The case $p \in (1,2)$ therefore remains an open problem in Corollary 1.1.3.
We note that the authors in [26] considered the shape minimization problem for $\lambda_{s,p}(\Omega)$ in the case $p = 1$, $s = \frac{1}{2}$, $N = 2$ among domains Ω of class C^∞ of fixed volume. They showed in [26] that such minimizers are discs.
Next we consider the optimization problem of $\Omega \mapsto \lambda_{s,p}(\Omega)$ for $p \in \{1,2\}$ and Ω a punctured ball, with the hole having the shape of ball. We show that, as the hole moves in Ω then $\lambda_{s,p}(\Omega)$ is maximal when the two balls are concentric. In the local case $s = 1$ and $N = 2$, this is a classical result by Hersch [67]. For subsequent generalizations in the case of the local problem, see [61, 25, 77].

Theorem 1.1.4. *Let $p \in \{1,2\}$, $B_1(0)$ be the unit centered ball and $\tau \in (0,1)$. Define*

$$\mathscr{A} := \{a \in B_1(0) : B_\tau(a) \subset B_1(0)\}.$$

Then the map $\mathscr{A} \to \mathbb{R}$, $a \mapsto \lambda_{s,p}(B_1(0) \setminus \overline{B_\tau(a)})$ takes its maximum at $a = 0$.

The proof of Theorem 1.1.4 is inspired by the argument given in [61, 77] for the local case $s = 1$. It uses the fractional Hadamard formula in Corollary 1.1.2 and maximum principles for anti-symmetric functions. Our proof also shows that the map $a \mapsto \lambda_{s,p}(B_1(0) \setminus \overline{B_\tau(a)})$ takes its minimum when the boundary of the ball $B_\tau(a)$ touches the one of $B_1(0)$, see Section 1.5 below. The proof of Theorem 1.1.1 is based on the use of test functions in the variational characterization of $\lambda_{s,p}(\Omega)$ and $\lambda_{s,p}(\Phi_\varepsilon(\Omega))$. The general strategy is inspired by the direct approach in [44], which is related to a Neumann eigenvalue problem on manifolds. In the case of $\lambda_{s,p}(\Phi_\varepsilon(\Omega))$, it is important to make a change of variables so that $\lambda_{s,p}(\Phi_\varepsilon(\Omega))$ is determined by minimizing an ε-dependent family of seminorms among functions $u \in \mathscr{H}^s_0(\Omega)$, see Section 1.2 below. An obvious choice of test functions are minimizers u and v_ε for $\lambda_{s,p}(\Omega)$ and $\lambda_{s,p}(\Phi_\varepsilon(\Omega))$, respectively. However, due to the fact that u is only of class C^s up to the boundary, we cannot obtain a boundary integral term directly from the divergence theorem. In particular, the integration by parts formula given in [95, Theorem 1.9] does not apply to general vector fields X which appear in (1.1.8). Hence, we need to replace u with $\varsigma_k u$, where ς_k is a cut-off function vanishing in a $\frac{1}{k}$-neighborhood of $\partial\Omega$. This leads to upper and lower estimates of $\lambda_{s,p}(\Phi_\varepsilon(\Omega))$ up to order $o(\varepsilon)$, where the first order term is given by an integral involving $(-\Delta)^s(\varsigma_k u)$ and $\nabla(\varsigma_k u)$. We refer the reader to Section 1.4 below for more precise information. A highly nontrivial task is now to pass to the limit as $k \to \infty$ in order to get boundary integrals involving $\psi := u/d^s$. This is the most difficult part of the paper. We refer to Proposition 1.2.4 and Section 1.6 below for more details.

The Chapter is organized as follows. In Section 1.2, we provide preliminary results on convergence properties of integral functional, inner approximations of functions in $\mathscr{H}^s_0(\Omega)$ and on properties of minimizers of (1.1.3). In Section 1.3, we introduce notation related to domain deformations and related quantities. In Section 1.4 we establish a preliminary variant of Theorem 1.1.1, which is given in Proposition 1.4.1. In this variant, the constant $\Gamma(1+s)^2$ in (1.1.8) is replaced by an implicitly given value which still depends on cut-off data. The proofs of the main results, as stated in this introduction, are then completed in Section 1.5. Section 1.6 is devoted to the proof of the main technical ingredient of the paper, which is given by Proposition 1.2.4. In the last Section, we provided an alternative computation of the constant κ_s appearing in (1.5.1).

1.2 Notations and preliminary results

Throughout this section, we fix a bounded open set $\Omega \subset \mathbb{R}^N$. As noted in the introduction, we define the space $\mathscr{H}^s_0(\Omega)$ as completion of $C^\infty_c(\Omega)$ with respect to the norm $[\cdot]_s$ given in (1.1.2). Then $\mathscr{H}^s_0(\Omega)$ is a Hilbert space with scalar product

$$(u,v) \mapsto [u,v]_s = \frac{b_{N,s}}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dxdy,$$

where $c_{N,s}$ is given in (1.1.2). It is well known and easy to see that $\mathscr{H}^s_0(\Omega)$ coincides with the closure of $C^\infty_c(\Omega)$ in the standard fractional Sobolev space $H^s(\mathbb{R}^N)$. Moreover, if Ω has a continuous boundary, then $\mathscr{H}^s_0(\Omega)$ admits the highly useful characterization

$$\mathscr{H}^s_0(\Omega) = \left\{ w \in L^1_{loc}(\mathbb{R}^N) \,:\, [w]^2_s < \infty, \quad w \equiv 0 \text{ on } \mathbb{R}^N \setminus \Omega \right\}, \tag{1.2.1}$$

see e.g. [56, Theorem 1.4.2.2]. We start with an elementary but useful observation.

Lemma 1.2.1. *Let $\mu \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$, and let $(v_k)_k$ be a sequence in $\mathscr{H}_0^s(\Omega)$ with $v_k \to v$ in $\mathscr{H}_0^s(\Omega)$ as $k \to \infty$. Then we have*

$$\lim_{k\to\infty} \int\limits_{\mathbb{R}^{2N}} \frac{(v_k(x)-v_k(y))^2\mu(x,y)}{|x-y|^{N+2s}}dxdy = \int\limits_{\mathbb{R}^{2N}} \frac{(v(x)-v(y))^2\mu(x,y)}{|x-y|^{N+2s}}dxdy.$$

Proof. We have

$$\begin{aligned}&\Big|\int\limits_{\mathbb{R}^{2N}} \frac{(v_k(x)-v_k(y))^2-(v(x)-v(y))^2\mu(x,y)}{|x-y|^{N+2s}}dxdy\Big|\\ &\le \|\mu\|_{L^\infty}\int\limits_{\mathbb{R}^{2N}} \frac{|(v_k(x)-v_k(y))^2-(v(x)-v(y))^2|}{|x-y|^{N+2s}}dxdy,\end{aligned}$$

where

$$\begin{aligned}&\int\limits_{\mathbb{R}^{2N}} \frac{|(v_k(x)-v_k(y))^2-(v(x)-v(y))^2|}{|x-y|^{N+2s}}dxdy\\ &= \int\limits_{\mathbb{R}^{2N}} \frac{|[(v_k(x)-v(x))-(v_k(y)-v(y))][(v_k(x)+v(x))-(v_k(y)+v(y))]|}{|x-y|^{N+2s}}dxdy\\ &\le \frac{2}{c_{N,s}}[v_k-v]_s[v_k+v]_s \to 0 \qquad \text{as } k\to\infty.\end{aligned}$$

□

Throughout the remainder of this paper, we fix $\rho \in C_c^\infty(-2,2)$ with $0 \le \rho \le 1$, $\rho \equiv 1$ on $(-1,1)$, and we define

$$\zeta \in C^\infty(\mathbb{R}), \qquad \zeta(t) = 1-\rho(t). \tag{1.2.2}$$

Moreover, for $k \in \mathbb{N}$, we define the functions

$$\rho_k,\, \varsigma_k \in C^{1,1}(\mathbb{R}^N), \qquad \rho_k(x) = \rho(k\delta(x)), \quad \varsigma_k(x) = \zeta(k\delta(x)). \tag{1.2.3}$$

We note that the function ρ_k is supported in the $\frac{2}{k}$-neighborhood of the boundary, while the function ς_k vanishes in the $\frac{1}{k}$-neighborhood of the boundary.

Lemma 1.2.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and let $u \in \mathscr{H}_0^s(\Omega)$. Moreover, for $k \in \mathbb{N}$, let $u_k := u\varsigma_k \in \mathscr{H}_0^s(\Omega)$ denote inner approximations of u. Then we have*

$$u_k \to u \qquad \text{in } \mathscr{H}_0^s(\Omega).$$

Proof. In the following, the letter $C > 0$ stands for various constants independent of k. Since $\rho_k = 1-\varsigma_k$, it suffices to show that

$$u\rho_k \in \mathscr{H}_0^s(\Omega) \text{ for } k \text{ sufficiently large} \qquad \text{and} \qquad [u\rho_k]_s \to 0 \quad \text{as } k\to\infty. \tag{1.2.4}$$

For $\varepsilon > 0$, we put $A_\varepsilon = \{x \in \Omega : \delta(x) < \varepsilon\}$. Since $u\rho_k$ vanishes in $\mathbb{R}^N \setminus A_{\frac{2}{k}}$, $0 \le \rho_k \le 1$ on $\mathbb{R}^N$ and $|\rho_k(x) - \rho_k(y)| \le C\min\{k|x-y|, 1\}$ for $x, y \in \mathbb{R}^N$, we observe that

$$
\begin{aligned}
&\frac{1}{b_{N,s}}[\rho_k u]_s^2 = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u(x)\rho_k(x) - u(y)\rho_k(y)]^2}{|x-y|^{N+2s}}dydx \\
&= \frac{1}{2}\int_{A_{\frac{4}{k}}}\int_{A_{\frac{4}{k}}} \frac{[u(x)\rho_k(x) - u(y)\rho_k(y)]^2}{|x-y|^{N+2s}}\, dydx + \int_{A_{\frac{2}{k}}} u(x)^2\rho_k(x)^2 \int_{\mathbb{R}^N\setminus A_{\frac{4}{k}}} |x-y|^{-N-2s}\, dydx \\
&\le \frac{1}{2}\int_{A_{\frac{4}{k}}}\int_{A_{\frac{4}{k}}} \frac{\left[u(x)\big(\rho_k(x) - \rho_k(y)\big) + \rho_k(y)\big(u(x) - u(y)\big)\right]^2}{|x-y|^{N+2s}}\, dydx \\
&\quad + C\int_{A_{\frac{2}{k}}} u(x)^2 \mathrm{dist}(x, \mathbb{R}^N \setminus A_{\frac{4}{k}})^{-2s}dx \\
&\le \int_{A_{\frac{4}{k}}} u^2(x)\int_{A_{\frac{4}{k}}} \frac{(\rho_k(x) - \rho_k(y))^2}{|x-y|^{N+2s}}\, dydx + \int_{A_{\frac{4}{k}}}\int_{A_{\frac{4}{k}}} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}}dydx \\
&\qquad + C\int_{A_{\frac{2}{k}}} u(x)^2\delta^{-2s}(x)dx \\
&\quad \le Ck^2\int_{A_{\frac{4}{k}}} u^2(x)\int_{B_{\frac{1}{k}}(x)} |x-y|^{2-2s-N}dydx + C\int_{A_{\frac{4}{k}}} u^2(x)\int_{\mathbb{R}^N\setminus B_{\frac{1}{k}}(x)} |x-y|^{-N-2s}dydx \\
&\qquad + \int_{A_{\frac{4}{k}}}\int_{A_{\frac{4}{k}}} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}}dydx + C\int_{A_{\frac{2}{k}}} u(x)^2\delta^{-2s}(x)dx \\
&\quad \le Ck^{2s}\int_{A_{\frac{4}{k}}} u^2(x)dx + \int_{A_{\frac{4}{k}}}\int_{A_{\frac{4}{k}}} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}}dydx + C\int_{A_{\frac{2}{k}}} u(x)^2\delta^{-2s}(x)dx \\
&\quad \le C\int_{A_{\frac{4}{k}}} u^2(x)d^{-2s}(x)dx + \int_{A_{\frac{4}{k}}}\int_{A_{\frac{4}{k}}} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}}dydx. \qquad (1.2.5)
\end{aligned}
$$

Now, since Ω has a Lipschitz boundary, using $\int_{\mathbb{R}^N\setminus\Omega} |x-y|^{-N-2s}\, dy \sim d^{-2s}(x)$ see e.g [22], we get

$$
\int_{\Omega} u^2(x)d^{-2s}(x)dx \le C\int_{\Omega} u^2(x)\int_{\mathbb{R}^N\setminus\Omega} |x-y|^{-N-2s}\, dydx \le C[u]_s^2,
$$

and therefore

$$
\int_{A_{\frac{4}{k}}} u^2(x)d^{-2s}(x)dx \to 0 \qquad \text{as } k \to \infty. \qquad (1.2.6)
$$

Moreover, since also

$$\int_\Omega \int_\Omega \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dydx \le \frac{2}{b_{N,s}} [u]_s^2,$$

we have

$$\int_{A_{\frac{4}{k}}} \int_{A_{\frac{4}{k}}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dydx \to 0 \qquad \text{as } k \to \infty. \tag{1.2.7}$$

Combining (1.2.5), (1.2.6) and (1.2.7), we obtain (1.2.4), as required. □

From now on, we fix a bounded $C^{1,1}$-domain $\Omega \subset \mathbb{R}^N$. We also let

$$C_0^s(\overline{\Omega}) = \left\{ w \in C^s(\overline{\Omega}) : w = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega \right\},$$

and we recall the following regularity and positivity properties of nonnegative minimizers for $\lambda_{s,p}(\Omega)$ as defined in (1.1.3).

Lemma 1.2.3. *Let $u \in \mathcal{H}_0^s(\Omega)$ be a nonnegative minimizer for $\lambda_{s,p}(\Omega)$. Then $u \in C_0^s(\overline{\Omega}) \cap C_{loc}^\infty(\Omega)$. Moreover, $\psi := \frac{u}{d^s} \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$, and there exists a constant $c = c(N,s,\Omega,\alpha,p) > 0$ with the property that*

$$\|\psi\|_{C^\alpha(\overline{\Omega})} \le c \tag{1.2.8}$$

and

$$|\nabla \psi(x)| \le c d^{\alpha-1}(x) \qquad \text{for all } x \in \Omega. \tag{1.2.9}$$

Moreover, $\psi > 0$ on $\overline{\Omega}$, so in particular $u > 0$ in Ω.

Proof. By standard arguments in the calculus of variations, u is a weak solution of (1.1.4). By [96, Proposition 1.3] we have that $u \in L^\infty(\Omega)$, and therefore the RHS of (1.1.4) is a function in $L^\infty(\Omega)$. Thus the regularity up to the boundary $u \in C_0^s(\overline{\Omega})$ is proved in [94], where also the C^α-bound (3.2.8) for the function $\psi = \frac{u}{d^s}$ is established for some $\alpha > 0$. Moreover, (1.2.9) is proved in [42]. It also follows from (1.1.4), the strong maximum principle and the Hopf lemma for the fractional Laplacian that ψ is a strictly positive function on $\overline{\Omega}$. In particular, $u > 0$ in Ω, Therefore $u \in C_{loc}^\infty(\Omega)$ follows by interior regularity theory (see e.g. [97]) and the fact that the function $t \mapsto t^{p-1}$ is of class C^∞ on $(0,\infty)$. □

The computation of one-sided shape derivatives as given in Theorem 1.1.1 will be carried out in Section 1.4, and it requires the following key technical proposition. Since its proof is long and quite involved, we postpone the proof to Section 1.6 below.

Proposition 1.2.4. *Let $X \in C^0(\overline{\Omega}, \mathbb{R}^N)$, let $u \in C_0^s(\overline{\Omega}) \cap C^1(\Omega)$, and assume that $\psi := \frac{u}{d^s}$ extends to a function on $\overline{\Omega}$ satisfying* (3.2.8) *and* (1.2.9). *Moreover, put $U_k := u\varsigma_k \in C_c^{1,1}(\Omega)$, where ς_k is defined in (1.2.3). Then*

$$\lim_{k\to\infty} \int_\Omega \nabla U_k \cdot X \Big(u(-\Delta)^s \varsigma_k - I(u,\varsigma_k) \Big)\, dx = -\kappa_s \int_{\partial\Omega} \psi^2 X \cdot \nu \, dx,$$

where

$$\kappa_s := -\int_{\mathbb{R}} h'(r)(-\Delta)^s h(r)\,dr \qquad \text{with} \quad h(r) := r_+^s \zeta(r) = \max(r,0)^s \zeta(r) \tag{1.2.10}$$

and ζ given in (1.2.2), and where we use the notation

$$I(u,v)(x) := b_{N,s} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dy \tag{1.2.11}$$

for $u \in C_c^s(\mathbb{R}^N)$, $v \in C^{0,1}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$.

Remark 1.2.5. *The minus sign in the definition of the constant κ_s in* (1.2.10) *might appear a bit strange at first glance. We shall see later that, defined in this way, κ_s has a positive value. A priori it is not clear that the value of κ_s does not depend on the particular choice of the function ζ. This follows a posteriori once we have established in Proposition 1.4.1 below that this constant appears in Theorem 1.1.1. This will then allow us to show that $\kappa_s = \frac{\Gamma(1+s)^2}{2}$ by applying the resulting shape derivative formula to a one-parameter family of concentric balls, see Section 1.5 below. A more direct, but somewhat lengthy computation of κ_s is possible via the logarithmic Laplacian, which has been introduced in [23].*

1.3 Domain perturbation and the associated variational problem

Here and in the following, we define $\Omega_\varepsilon := \Phi_\varepsilon(\Omega)$. In order to study the dependence of $\lambda_{s,p}(\Omega_\varepsilon)$ on ε, it is convenient to pull back the problem on the fixed domain Ω via a change of variables. For this we let $\mathrm{Jac}_{\Phi_\varepsilon}$ denote the Jacobian determinant of the map $\Phi_\varepsilon \in C^{1,1}(\mathbb{R}^N)$, and we define the kernels

$$K_\varepsilon(x,y) := b_{N,s} \frac{\mathrm{Jac}_{\Phi_\varepsilon}(x)\mathrm{Jac}_{\Phi_\varepsilon}(y)}{|\Phi_\varepsilon(x)-\Phi_\varepsilon(y)|^{N+2s}} \qquad \text{and} \qquad K_0(x,y) = c_{N,s}\frac{1}{|x-y|^{N+2s}}. \tag{1.3.1}$$

Then (3.3.1) gives rise to the well known expansions

$$\mathrm{Jac}_{\Phi_\varepsilon}(x) = 1 + \varepsilon \mathrm{div} X(x) + O(\varepsilon^2), \qquad \partial_\varepsilon \mathrm{Jac}_{\Phi_\varepsilon}(x) = \mathrm{div} X(x) + O(\varepsilon) \tag{1.3.2}$$

uniformly in $x \in \mathbb{R}^N$, where $X := \partial_\varepsilon\big|_{\varepsilon=0}\Phi_\varepsilon \in C^{0,1}(\mathbb{R}^N;\mathbb{R}^N)$ and therefore $\mathrm{div}X$ is a.e. defined on $\mathbb{R}^N$. From (3.3.1), we also get

$$|\Phi_\varepsilon(x)-\Phi_\varepsilon(y)|^{-N-2s} = |x-y|^{-N-2s}\left(1 + 2\varepsilon \frac{x-y}{|x-y|}\cdot P_X(x,y) + O(\varepsilon^2)\right)^{-\frac{N+2s}{2}},$$

and

$$\partial_\varepsilon |\Phi_\varepsilon(x)-\Phi_\varepsilon(y)|^{-N-2s} = |x-y|^{-N-2s}\left(-(N+2s)\frac{x-y}{|x-y|}\cdot P_X(x,y) + O(\varepsilon)\right),$$

uniformly in $x,y \in \mathbb{R}^N$, $x \neq y$ with

$$P_X \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N), \qquad P_X(x,y) = \frac{X(x) - X(y)}{|x-y|}.$$

Moreover by (1.3.2) and the fact that $\partial_\varepsilon \Phi_\varepsilon, X \in C^{0,1}(\mathbb{R}^N)$, we have that

$$K_\varepsilon(x,y) = K_0(x,y) + \varepsilon \partial_\varepsilon \Big|_{\varepsilon=0} K_\varepsilon(x,y) + O(\varepsilon^2) K_0(x,y), \tag{1.3.3}$$

and

$$\partial_\varepsilon K_\varepsilon(x,y) = \partial_\varepsilon \Big|_{\varepsilon=0} K_\varepsilon(x,y) + O(\varepsilon) K_0(x,y), \tag{1.3.4}$$

uniformly in $x,y \in \mathbb{R}^N$, $x \neq y$, where

$$\partial_\varepsilon \Big|_{\varepsilon=0} K_\varepsilon(x,y) = -\Big[(N+2s)\frac{x-y}{|x-y|} \cdot P_X(x,y) - (\operatorname{div}X(x) + \operatorname{div}X(y))\Big] K_0(x,y). \tag{1.3.5}$$

In particular, it follows from (1.3.3) and (3.3.9) that there exist $\varepsilon_0, C > 0$ with the property that

$$\frac{1}{C} K_0(x,y) \leq K_\varepsilon(x,y) \leq C K_0(x,y) \qquad \text{for all } x,y \in \mathbb{R}^N,\ x \neq y \text{ and } \varepsilon \in (-\varepsilon_0, \varepsilon_0). \tag{1.3.6}$$

For $v \in \mathscr{H}_0^s(\Omega)$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we now define

$$\mathscr{V}_v(\varepsilon) := \frac{1}{2} \int_{\mathbb{R}^{2N}} (v(x) - v(y))^2 K_\varepsilon(x,y)\, dx dy. \tag{1.3.7}$$

Then, by (1.1.3), (3.3.1) and a change of variables, we have the following variational characterization for $\lambda_{s,p}(\Omega_\varepsilon)$:

$$\begin{aligned}
\lambda_{s,p}^\varepsilon := \lambda_{s,p}(\Omega_\varepsilon) &= \inf \left\{ [u]_s^2 \,:\, u \in \mathscr{H}_0^s(\Omega_\varepsilon), \quad \int_{\Omega_\varepsilon} |u|^p \, dx = 1 \right\} \\
&= \inf \left\{ \mathscr{V}_v(\varepsilon) \,:\, v \in \mathscr{H}_0^s(\Omega), \quad \int_{\Omega} |v|^p \operatorname{Jac}_{\Phi_\varepsilon}(x) \, dx = 1 \right\} \quad \text{for } \varepsilon \in (-\varepsilon_0, \varepsilon_0).
\end{aligned} \tag{1.3.8}$$

As mentioned earlier, we prefer to use (1.3.8) from now on where the underlying domain is fixed and the integral terms depend on ε instead. It follows from (1.3.3) and (1.3.4) that, for given $v \in \mathscr{H}_0^s(\Omega)$, the function $\mathscr{V}_v : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ is of class C^1 with

$$\mathscr{V}_v'(0) = \frac{1}{2} \int_{\mathbb{R}^{2N}} (v(x) - v(y))^2 \partial_\varepsilon \big|_{\varepsilon=0} K_\varepsilon(x,y)\, dx dy, \tag{1.3.9}$$

where $\partial_\varepsilon \big|_{\varepsilon=0} K_\varepsilon(x,y)$ is given in (3.3.9),

$$|\mathscr{V}_v'(0)| \leq C[v]_s^2 \qquad \text{with a constant } C > 0 \tag{1.3.10}$$

and we have the expansions

$$\mathcal{V}_v(\varepsilon) = \mathcal{V}_v(0) + \varepsilon \mathcal{V}_v'(0) + O(\varepsilon^2)[v]_s^2, \qquad \mathcal{V}_v'(\varepsilon) = \mathcal{V}_v'(0) + O(\varepsilon)[v]_s^2 \tag{1.3.11}$$

with $O(\varepsilon)$, $O(\varepsilon^2)$ independent of v. From (1.3.2), (1.3.6) and the variational characterization (1.3.8), it is easy to see that

$$\frac{1}{C} \le \lambda_{s,p}^{\varepsilon} \le C \qquad \text{for all } \varepsilon \in (-\varepsilon_0, \varepsilon_0) \text{ with some constant } C > 0.$$

Using this and (1.3.2), (1.3.6) once more, we can show that

$$\frac{1}{C} \le \|v_\varepsilon\|_{L^p(\Omega)} \le C \qquad \text{and} \qquad \frac{1}{C} \le [v_\varepsilon]_s \le C. \tag{1.3.12}$$

for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and every minimizer $v_\varepsilon \in \mathcal{H}_0^s(\Omega)$ for (1.3.8) with a constant $C > 0$. The following lemma is essentially a corollary of Lemma 1.2.1.

Lemma 1.3.1. *Let $(v_k)_k$ be a sequence in $\mathcal{H}_0^s(\Omega)$ with $v_k \to v$ in $\mathcal{H}_0^s(\Omega)$. Then we have*

$$\lim_{k\to\infty} \mathcal{V}_{v_k}(0) = \mathcal{V}_v(0) \qquad \textit{and} \qquad \lim_{k\to\infty} \mathcal{V}_{v_k}'(0) = \mathcal{V}_v'(0).$$

Proof. The first limit is trivial since $\mathcal{V}_v(0) = [v]_s^2$ for $v \in \mathcal{H}_0^s(\Omega)$. The second limit follows from Lemma 1.2.1, (3.3.9) and (3.3.8) by noting that $\mu \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ for the function

$$\mu(x,y) = -(N+2s)\frac{x-y}{|x-y|} \cdot P_X(x,y) + (\operatorname{div} X(x) + \operatorname{div} X(y)).$$

□

1.4 One-sided shape derivative computations

We keep using the notation of the previous sections, and we recall in particular the variational characterization of $\lambda_{s,p}^{\varepsilon} = \lambda_{s,p}(\Omega_\varepsilon)$ given in (1.3.8). The aim of this section is to prove the following result.

Proposition 1.4.1. *We have*

$$\partial_\varepsilon^+\Big|_{\varepsilon=0} \lambda_{s,p}^{\varepsilon} = \min\left\{ 2\kappa_s \int_{\partial\Omega} (u/d^s)^2 X \cdot \nu \, dx \, : \, u \in \mathcal{H} \right\},$$

where $\mathcal{H}$ is the set of positive minimizers for $\lambda_{s,p}^0 := \lambda_{s,p}(\Omega)$, $X := \partial_\varepsilon\big|_{\varepsilon=0}\Phi_\varepsilon$ and κ_s is given by (1.2.10).

The proof of Proposition 1.4.1 requires several preliminary results. We start with a formula for the derivative of the function given by (3.3.6).

Lemma 1.4.2. *Let $U \in C_c^{1,1}(\Omega)$. Then*

$$\mathscr{V}_U'(0) = -2 \int_{\mathbb{R}^N} \nabla U \cdot X(-\Delta)^s U dx. \tag{1.4.1}$$

Proof. By (3.3.9), (1.3.11) and Fubini's theorem, we have

$$\begin{aligned}
\mathscr{V}_U'(0) =& \frac{-(N+2s)b_{N,s}}{2} \int_{\mathbb{R}^{2N}} (U(x)-U(y))^2 \frac{(x-y)\cdot(X(x)-X(y))}{|x-y|^{N+2s+2}} dxdy \\
&+ \frac{1}{2} \int_{\mathbb{R}^{2N}} (U(x)-U(y))^2 K_0(x,y)(\mathrm{div}X(x)+\mathrm{div}X(y))dxdy \\
=& \frac{-(N+2s)b_{N,s}}{2} \lim_{\mu\to 0} \int_{|x-y|>\mu} (U(x)-U(y))^2 \frac{(x-y)\cdot(X(x)-X(y))}{|x-y|^{N+2s+2}} dxdy \\
&+ \int_{\mathbb{R}^{2N}} (U(x)-U(y))^2 K_0(x,y)\mathrm{div}X(x)dxdy \\
=& -(N+2s)b_{N,s} \lim_{\mu\to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N\setminus \overline{B_\mu(y)}} (U(x)-U(y))^2 \frac{(x-y)\cdot X(x)}{|x-y|^{N+2s+2}} dxdy \\
&+ \int_{\mathbb{R}^{2N}} (U(x)-U(y))^2 K_0(x,y)\mathrm{div}X(x)dxdy
\end{aligned}$$

Applying, for fixed $y \in \mathbb{R}^N$ and $\mu > 0$, the divergence theorem in the domain $\{x \in \mathbb{R}^N : |x-y| > \mu\}$ and using that $\nabla_x |x-y|^{-N-2s} = -(N+2s)\frac{x-y}{|x-y|^{N+2s+2}}$, we obtain

$$\begin{aligned}
\mathscr{V}_U'(0) =& b_{N,s} \lim_{\mu\to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N\setminus \overline{B_\mu(y)}} (U(x)-U(y))^2 \nabla_x |x-y|^{-N-2s} \cdot X(x) dxdy \\
&+ \int_{\mathbb{R}^{2N}} (U(x)-U(y))^2 K_0(x,y)\mathrm{div}X(x)dxdy \\
=& -\lim_{\mu\to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N\setminus \overline{B_\mu(y)}} (U(x)-U(y))^2 K_0(x,y)\mathrm{div}X(x)dxdy \\
&- \lim_{\mu\to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N\setminus \overline{B_\mu(y)}} (U(x)-U(y))\nabla U(x)\cdot X(x)K_0(x,y)dxdy \\
&+ \lim_{\mu\to 0} \int_{\mathbb{R}^N} \int_{\partial B_\mu(y)} (U(x)-U(y))^2 \frac{y-x}{|x-y|} \cdot X(x)K_0(x,y)\, d\sigma(y)\, dx \\
&+ \int_{\mathbb{R}^{2N}} (U(x)-U(y))^2 K_0(x,y)\mathrm{div}X(x)dxdy \\
=& -\lim_{\mu\to 0} \int_{|x-y|>\mu} (U(x)-U(y))\nabla U(x)\cdot X(x)K_0(x,y)d(x,y)
\end{aligned}$$

$$+\lim_{\mu\to 0}\mu^{-N-1-2s}\int\limits_{|x-y|=\mu}(U(x)-U(y))^2(y-x)\cdot X(x)\,d\sigma(x,y)$$

$$=-\frac{b_{N,s}}{2}\lim_{\mu\to 0}\int\limits_{\mathbb{R}^N}\nabla U(x)\cdot X(x)\int\limits_{\mathbb{R}^N\setminus\overline{B_\mu(0)}}\frac{2U(x)-U(x+z)-U(x-z)}{|z|^{N+2s}}dzdx$$

$$+\frac{1}{2}\lim_{\mu\to 0}\mu^{-N-1-2s}\int\limits_{|x-y|=\mu}(U(x)-U(y))^2(y-x)\cdot(X(x)-X(y))\,d\sigma(x,y) \qquad (1.4.2)$$

Since $U\in C^{1,1}_c(\Omega)$, we have that

$$\frac{b_{N,s}}{2}\lim_{\mu\to 0}\int\limits_{\mathbb{R}^N}\nabla U(x)\cdot X(x)\int\limits_{\mathbb{R}^N\setminus\overline{B_\mu(0)}}\frac{2U(x)-U(x+z)-U(x-z)}{|z|^{N+2s}}dzdx$$

$$=\frac{b_{N,s}}{2}\int\limits_{\mathbb{R}^N}\nabla U(x)\cdot X(x)\int\limits_{\mathbb{R}^N}\frac{2U(x)-U(x+z)-U(x-z)}{|z|^{N+2s}}dzdx$$

$$=\int\limits_{\mathbb{R}^N}(-\Delta)^sU(x)\nabla U(x)\cdot X(x)dx. \qquad (1.4.3)$$

Moreover, since U is compactly supported, we may fix $R>0$ large enough such that $(U(x)-U(y))^2=0$ for all $x,y\in B_R(0)$ with $|x-y|<1$. Setting $N_\mu:=\{(x,y)\in B_R(0)\times B_R(0):|x-y|=\mu\}$ for $0<\mu<1$ and using that $U,X\in C^{0,1}(\mathbb{R}^N)$, we thus deduce that

$$\mu^{-N-1-2s}\int\limits_{|x-y|=\mu}(U(x)-U(y))^2(y-x)\cdot(X(x)-X(y))\,d\sigma(x,y)$$

$$=\mu^{-N-1-2s}\int\limits_{N_\mu}(U(x)-U(y))^2(y-x)\cdot(X(x)-X(y))\,d\sigma(x,y)=O(\mu^{3-1-2s})\to 0, \qquad (1.4.4)$$

as $\mu\to 0$, since the $2N-1$-dimensional measure of the set N_μ is of order $O(N-1)$ as $\mu\to 0$. The claim now follows by combining (1.4.2),(1.4.3) and (1.4.4). □

We cannot apply Lemma 3.2.1 directly to minimizers $u\in\mathscr{H}^s_0(\Omega)$ of $\lambda_{s,p}(\Omega)$ since these are not contained in $C^{1,1}_c(\Omega)$. The aim is therefore to apply Lemma 3.2.1 to $U_k:=u\varsigma_k\in C^{1,1}_c(\Omega)$ with ς_k given in (1.2.3), and to use Proposition 1.2.4. This leads to the following derivative formula which plays a key role in the proof of Proposition 1.4.1.

Lemma 1.4.3. *Let $u\in\mathscr{H}^s_0(\Omega)$ be a solution to* (1.1.4). *Then we have*

$$\mathscr{V}_u'(0)=\frac{2\lambda_{s,p}(\Omega)}{p}\int\limits_{\Omega}u^p\operatorname{div}X\,dx+2\kappa_s\int\limits_{\partial\Omega}(u/d^s)^2X\cdot\nu\,dx.$$

Proof. By Lemma 1.2.3 and since Ω is of class $C^{1,1}$, we have $U_k:=u\zeta_k\in C^{1,1}_c(\Omega)\subset\mathscr{H}^s_0(\Omega)$ for $k\in\mathbb{N}$, and $U_k\to u$ in $\mathscr{H}^s_0(\Omega)$ by Lemma 1.2.2. Consequently, $\mathscr{V}_u'(0)=\lim\limits_{k\to\infty}\mathscr{V}_{U_k}'(0)$ by

Corollary 1.3.1, so it remains to show that

$$\lim_{k\to\infty} \mathscr{V}'_{U_k}(0) = \frac{2\lambda_{s,p}(\Omega)}{p}\int_\Omega u^p \operatorname{div} X\, dx + 2\kappa_s \int_{\partial\Omega} (u/d^s)^2 X\cdot\nu\, dx. \tag{1.4.5}$$

Applying Lemma 3.2.1 to U_k, we find that

$$\mathscr{V}'_{U_k}(0) = -2\int_{\mathbb{R}^N} \nabla U_k \cdot X(-\Delta)^s U_k dx \qquad \text{for } k\in\mathbb{N}.$$

By the standard product rule for the fractional Laplacian, we have $(-\Delta)^s U_k = u(-\Delta)^s \varsigma_k + \varsigma_k(-\Delta)^s u - I(u,\varsigma_k)$ with $I(u,\varsigma_k)$ given by (1.2.11). We thus obtain

$$\begin{aligned}\mathscr{V}'_{U_k}(0) &= -2\int_{\mathbb{R}^N} \nabla U_k\cdot X\varsigma_k(-\Delta)^s u\, dx - 2\int_{\mathbb{R}^N}[\nabla U_k\cdot X]u(-\Delta)^s\varsigma_k\, dx \\ &\quad + 2\int_{\mathbb{R}^N}\nabla U_k\cdot XI(u,\varsigma_k)\, dx \\ &= -2\lambda_{s,p}(\Omega)\int_\Omega \nabla U_k\cdot X\varsigma_k u^{p-1}\, dx - 2\int_{\mathbb{R}^N}\nabla U_k\cdot X\Big(u(-\Delta)^s\varsigma_k - I(u,\varsigma_k)\Big)\, dx,\end{aligned} \tag{1.4.6}$$

where we used that $(-\Delta)^s u = \lambda_{s,p}(\Omega)u^{p-1}$ in Ω. Consequently, Proposition 1.2.4 yields that

$$\lim_{k\to\infty}\mathscr{V}'_{U_k}(0) = -2\lambda_{s,p}(\Omega)\lim_{k\to\infty}\int_\Omega \nabla U_k\cdot X\varsigma_k u^{p-1}\, dx + 2\kappa_s\int_{\partial\Omega}\psi^2 X\cdot\nu\, dx. \tag{1.4.7}$$

Moreover, integrating by parts, we obtain, for $k\in\mathbb{N}$,

$$\begin{aligned}\int_\Omega [\nabla U_k\cdot X]\varsigma_k u^{p-1}\, dx &= \frac{1}{p}\int_\Omega [\nabla u^p\cdot X]\varsigma_k^2\, dx + \int_\Omega [\nabla\varsigma_k\cdot X]\varsigma_k u^p\, dx \\ &= -\frac{1}{p}\int_\Omega u^p \operatorname{div} X\varsigma_k^2\, dx - \frac{2}{p}\int_\Omega u^p\varsigma_k[X\cdot\nabla\varsigma_k]\, dx + \int_\Omega u^p\varsigma_k[X\cdot\nabla\varsigma_k]\, dx.\end{aligned} \tag{1.4.8}$$

Since $u^p\in C_0^s(\overline{\Omega})$ by Lemma 1.2.3, it is easy to see from the definition of ς_k that the last two terms in (1.4.8) tend to zero as $k\to\infty$, whereas

$$\lim_{k\to\infty}\int_\Omega u^p\operatorname{div} X\varsigma_k^2\, dx = \int_\Omega u^p \operatorname{div} X\, dx.$$

Hence

$$\lim_{k\to\infty}\int_\Omega \nabla U_k\cdot X\varsigma_k u^{p-1}\, dx = -\frac{1}{p}\int_\Omega u^p\operatorname{div} X\, dx.$$

Plugging this into (1.4.7), we obtain (1.4.5), as required. □

Our next lemma provides an upper estimate for $\partial_\varepsilon^+\Big|_{\varepsilon=0}\lambda_{s,p}^\varepsilon$.

Lemma 1.4.4. *Let $u \in \mathscr{H}$ be a positive minimizer for $\lambda_{s,p}^0 = \lambda_{s,p}(\Omega)$. Then*

$$\limsup_{\varepsilon\to 0^+} \frac{\lambda_{s,p}^\varepsilon - \lambda_{s,p}^0}{\varepsilon} \leq 2\kappa_s \int_{\partial\Omega} (u/d^s)^2 X\cdot\nu\, dx. \tag{1.4.9}$$

Proof. For $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we define

$$j(\varepsilon) := \frac{\mathscr{V}_u(\varepsilon)}{\tau(\varepsilon)} \quad \text{for } k \in \mathbb{N} \text{ with} \quad \tau(\varepsilon) := \left(\int_\Omega |u|^p \mathrm{Jac}_{\Phi_\varepsilon}(x)\, dx\right)^{2/p}.$$

By (1.3.8), we then have $\lambda_{s,p}^\varepsilon \leq j(\varepsilon)$ for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Moreover,

$$\tau(0) = \|u\|_{L^p(\Omega)}^{2/p} = 1, \quad \mathscr{V}_u(0) = [u]_s^2 = \lambda_{s,p}(\Omega) \quad \text{and} \quad j(0) = \frac{\mathscr{V}_u(0)}{\tau(0)} = \lambda_{s,p}^0,$$

which implies that

$$\partial_\varepsilon^+\Big|_{\varepsilon=0}\lambda_{s,p}^\varepsilon \leq j'(0) = 2\kappa_s \int_{\partial\Omega} (u/d^s)^2 X\cdot\nu\, dx,$$

by Lemma 1.4.3 and (1.3.2), as claimed. □

Next, we shall prove a lower estimate for $\partial_\varepsilon^+\Big|_{\varepsilon=0}\lambda_{s,p}^\varepsilon$.

Lemma 1.4.5. *We have*

$$\liminf_{\varepsilon\searrow 0^+} \frac{\lambda_{s,p}^\varepsilon - \lambda_{s,p}^0}{\varepsilon} \geq \inf\left\{2\kappa_s \int_{\partial\Omega} (u/d^s)^2 X\cdot\nu\, dx \,:\, u \in \mathscr{H}\right\}.$$

Proof. Let $(\varepsilon_n)_n$ be a sequence of positive numbers converging to zero and with the property that

$$\lim_{n\to\infty} \frac{\lambda_{s,p}^{\varepsilon_n} - \lambda_{s,p}^0}{\varepsilon_n} = \liminf_{\varepsilon\searrow 0^+} \frac{\lambda_{s,p}^\varepsilon - \lambda_{s,p}^0}{\varepsilon}. \tag{1.4.10}$$

For $n \in \mathbb{N}$, we let v_{ε_n} be a positive minimizer corresponding to the variational characterization of $\lambda_{s,p}^{\varepsilon_n}$ given in (1.3.8), i.e. we have

$$\lambda_{s,p}^{\varepsilon_n} = \mathscr{V}_{v_{\varepsilon_n}}(\varepsilon_n) \qquad \text{and} \qquad \int_\Omega v_{\varepsilon_n}^p \mathrm{Jac}_{\Phi_{\varepsilon_n}} dx = 1. \tag{1.4.11}$$

Since v_{ε_n} remains bounded in $\mathscr{H}_0^s(\Omega)$ by (1.3.12), we may pass to a sub-sequence with the property that $v_{\varepsilon_n} \rightharpoonup u$ in $\mathscr{H}_0^s(\Omega)$ for some $u \in \mathscr{H}_0^s(\Omega)$. Moreover, $v_{\varepsilon_n} \to u$ in $L^p(\Omega)$ as $n \to \infty$

since the embedding $\mathscr{H}_0^s(\Omega) \to L^p(\Omega)$ is compact. In the following, to keep the notation simple, we write ε in place of ε_n. By (1.3.10), (1.3.11) and (1.4.11), we have

$$\mathscr{V}_{v_\varepsilon}(0) = \mathscr{V}_{v_\varepsilon}(\varepsilon) - \varepsilon \mathscr{V}'_{v_\varepsilon}(0) + O(\varepsilon^2)[v_\varepsilon]_s^2 = \lambda_{s,p}^\varepsilon - \varepsilon \mathscr{V}'_{v_\varepsilon}(0) + O(\varepsilon^2) = \lambda_{s,p}^\varepsilon + O(\varepsilon) \quad (1.4.12)$$

and therefore

$$\mathscr{V}_u(0) = [u]_s^2 \le \liminf_{\varepsilon\to 0}[v_\varepsilon]_s^2 = \liminf_{\varepsilon\to 0} \mathscr{V}_{v_\varepsilon}(0) \le \limsup_{\varepsilon\to 0} \lambda_{s,p}^\varepsilon \le \lambda_{s,p}^0, \quad (1.4.13)$$

where the last inequality follows from Lemma 1.4.4. In view of (1.3.2) and the strong convergence $v_\varepsilon \to u$ in $L^p(\Omega)$, we see that

$$1 = \int_\Omega v_\varepsilon^p \mathrm{Jac}_{\Phi_\varepsilon} dx = \int_\Omega v_\varepsilon^p (1 + \varepsilon \mathrm{div} X) dx + O(\varepsilon^2) = \int_\Omega u^p \, dx + o(1) \quad (1.4.14)$$

as $\varepsilon \to 0$, and hence $\|u\|_{L^p(\Omega)} = 1$. Combining this with (1.4.13), we see that $u \in \mathscr{H}$ is a minimizer for $\lambda_{s,p}^0$, and that equality must hold in all inequalities of (1.4.13). From this we deduce that

$$v_\varepsilon \to u \text{ strongly in } \mathscr{H}_0^s(\Omega). \quad (1.4.15)$$

Now (1.4.12) and the variational characterization of $\lambda_{s,p}^0$ imply that

$$\lambda_{s,p}^0 \left(\int_\Omega v_\varepsilon^p dx \right)^{2/p} \le \mathscr{V}_{v_\varepsilon}(0) = \lambda_{s,p}(\Omega_\varepsilon) - \varepsilon \mathscr{V}'_{v_\varepsilon}(0) + O(\varepsilon^2) \quad (1.4.16)$$

whereas by (1.4.14) we have

$$\int_\Omega v_\varepsilon^p \, dx = 1 - \varepsilon \int_\Omega v_\varepsilon^p \mathrm{div} X dx + O(\varepsilon^2) = 1 - \varepsilon \int_\Omega u^p \mathrm{div} X dx + o(\varepsilon)$$

and therefore

$$\left(\int_\Omega v_\varepsilon^p dx \right)^{2/p} = 1 - \frac{2\varepsilon}{p} \int_\Omega u^p \mathrm{div} X dx + o(\varepsilon). \quad (1.4.17)$$

Plugging this into (1.4.16), we get the inequality

$$\lambda_{s,p}^\varepsilon \ge \Big(1 - \frac{2\varepsilon}{p} \int_\Omega u^p \mathrm{div} X dx\Big) \lambda_{s,p}^0 + \varepsilon \mathscr{V}'_{v_\varepsilon}(0) + o(\varepsilon).$$

Since, moreover, $\mathscr{V}'_{v_\varepsilon}(0) \to \mathscr{V}'_u(0)$ as $\varepsilon \to 0$ by Lemma 1.3.1 and (1.4.15), it follows that

$$\lambda_{s,p}^\varepsilon - \lambda_{s,p}^0 \ge \varepsilon \Big(\mathscr{V}'_u(0) - \frac{2\lambda_{s,p}^0}{p} \int_\Omega u^p \mathrm{div} X dx \Big) + o(\varepsilon)$$

and therefore

$$\lambda_{s,p}^{\varepsilon} - \lambda_{s,p}^{0} \geq 2\varepsilon\kappa_s \int_{\partial\Omega} (u/d^s)^2 X \cdot \nu \, dx + o(\varepsilon)$$

by Lemma 1.4.3. We thus conclude that

$$\lim_{\varepsilon\to 0^+} \frac{\lambda_{s,p}^{\varepsilon} - \lambda_{s,p}^{0}}{\varepsilon} \geq 2\kappa_s \int_{\partial\Omega} (u/d^s)^2 X \cdot \nu \, dx.$$

Taking the infinimum over $u \in \mathscr{H}$ in the RHS of this inequality and using (1.4.10), we get the result. □

Proof of Proposition 1.4.1 (completed). Proposition 1.4.1 is a consequence of Lemma 1.4.4 and Lemma 1.4.5. Indeed, let

$$A_{s,p}(\Omega) := \inf \left\{ 2\kappa_s \int_{\partial\Omega} (u/d^s)^2 X \cdot \nu \, dx \, : \, u \in \mathscr{H} \right\}.$$

Thanks to (3.2.8) the infinimum $A_{s,p}(\Omega)$ is attained. Finally by Lemma 1.4.4 and Lemma 1.4.5 we get

$$A_{s,p}(\Omega) \geq \partial_{\varepsilon}^{+}\Big|_{\varepsilon=0} \lambda_{s,p}^{\varepsilon} \geq \liminf_{\varepsilon\searrow 0} \frac{\lambda_{s,p}^{\varepsilon} - \lambda_{s,p}^{0}}{\varepsilon} \geq A_{s,p}(\Omega).$$

□

1.5 Proof of the main results

In this section we complete the proofs of the main results stated in the introduction.

Proof of Theorem 1.1.1 (completed). In view of Proposition 1.4.1, the proof of Theorem 1.1.1 is complete once we show that

$$2\kappa_s = \Gamma(1+s)^2, \tag{1.5.1}$$

where Γ is the usual Gamma function. In view of (1.2.10), the constant κ_s does not depend on N, p and Ω, we consider the case $N = p = 1$ and the family of diffeomorphisms Φ_ε on $\mathbb{R}^N$ given by $\Phi_\varepsilon(x) = (1+\varepsilon)x$, $\varepsilon \in (-1,1)$, so that $X := \partial_\varepsilon\big|_{\varepsilon=0}\Phi_\varepsilon$ is simply given by $X(x) = x$. Letting $\Omega_0 := (-1,1)$, we define $\Omega_\varepsilon = \Phi_\varepsilon(\Omega_0) = (-1-\varepsilon, 1+\varepsilon)$. Moreover, we consider $w_\varepsilon \in \mathscr{H}_0^s(\Omega_\varepsilon) \cap C_0^s([-1-\varepsilon, 1+\varepsilon])$ given by

$$w_\varepsilon(x) = \ell_s((1+\varepsilon)^2 - |x|^2)_+^s \qquad \text{with} \qquad \ell_s := \frac{2^{-2s}\Gamma(1/2)}{\Gamma(s+1/2)\Gamma(1+s)}. \tag{1.5.2}$$

It is well known that w_ε is the unique solution of the problem

$$(-\Delta)^s w_\varepsilon = 1 \quad \text{in } \Omega_\varepsilon, \qquad w_\varepsilon \equiv 0 \quad \text{on } \mathbb{R}^N \setminus \Omega_\varepsilon,$$

see e.g. [95] or [43]. Recalling (1.1.4), we thus deduce that $u_\varepsilon = \lambda_{s,1}(\Omega_\varepsilon)w_\varepsilon$ is the unique positive minimizer corresponding to (1.1.3) in the case $N = p = 1$, which implies that $\|u_\varepsilon\|_{L^1(\mathbb{R})} = 1$ and therefore

$$\lambda_{s,1}(\Omega_\varepsilon) = \|w_\varepsilon\|^{-1}_{L^1(\mathbb{R})} = (1+\varepsilon)^{-(2s+1)}\|w_0\|^{-1}_{L^1(\mathbb{R})}. \tag{1.5.3}$$

Moreover, by standard properties of the Gamma function,

$$\begin{aligned}\|w_0\|_{L^1(\mathbb{R})} &= \ell_s \int_{-1}^{1} (1-|x|^2)^s\,dx = 2\ell_s \int_0^1 (1-r^2)^s\,dr = \ell_s \int_0^1 t^{-1/2}(1-t)^s\,dt \\ &= \ell_s \frac{\Gamma(1/2)\Gamma(s+1)}{\Gamma(s+3/2)} = \ell_s \frac{\Gamma(1/2)\Gamma(s+1)}{(s+1/2)\Gamma(s+1/2)} = \frac{2^{2s}\ell_s^2\Gamma(s+1)^2}{s+1/2}.\end{aligned}$$

By differentiating (1.5.3), we get

$$\partial_\varepsilon\Big|_{\varepsilon=0}\lambda_{s,1}(\Omega_\varepsilon) = -\frac{2s+1}{\|w_0\|_{L^1(\mathbb{R})}}. \tag{1.5.4}$$

On the other hand, by Proposition 1.4.1 and the fact that u_0 is the unique positive minimizer for $\lambda_{s,1}$, we deduce that

$$\partial_\varepsilon^+\Big|_{\varepsilon=0}\lambda_{s,1}(\Omega_\varepsilon) = -2\kappa_s[(u_0/d^s)^2(1) + (u_0/d^s)^2(-1)] = -2^{2+2s}\kappa_s\ell_s^2\lambda_{s,1}(\Omega_0)^2 = -\frac{2^{2+2s}\kappa_s\ell_s^2}{\|w_0\|^2_{L^1(\mathbb{R})}}.$$

We thus conclude that

$$2\kappa_s = \frac{(2s+1)\|w_0\|_{L^1(\mathbb{R})}}{2^{1+2s}\ell_s^2} = \Gamma(s+1)^2.$$

Thus, by Proposition 1.4.1, we get the result as stated in the theorem. □

Proof of Corollary 1.1.3. Let $h \in C^3(\partial\Omega)$, with $\int_{\partial\Omega} h\,dx = 0$. Then it is well known (see e.g. [44, Lemma 2.2]) that there exists a family of diffeomorphisms $\Phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$, $\varepsilon \in (-1,1)$ satisfying (3.3.1) and having the following properties:

$$|\Phi_\varepsilon(\Omega)| = |\Omega| \text{ for } \varepsilon \in (-1,1), \text{ and } X := \partial_\varepsilon\big|_{\varepsilon=0}\Phi_\varepsilon \text{ equals } h\nu \text{ on } \partial\Omega. \tag{1.5.5}$$

By assumption, there exists $\varepsilon_0 \in (0,1)$ with $\lambda_{s,p}(\Phi_\varepsilon(\Omega)) \geq \lambda_{s,p}(\Omega)$ for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Applying Theorem 1.1.1 and noting that $X \cdot \nu \equiv h$ on $\partial\Omega$ by (1.5.5), we get

$$\min\left\{\Gamma(1+s)^2\int_{\partial\Omega}(u/d^s)^2 h\,dx \,:\, u \in \mathscr{H}\right\} \geq 0.$$

By the same argument applied to $-h$, we get

$$\max\left\{\Gamma(1+s)^2\int_{\partial\Omega}(u/d^s)^2 h\,dx \,:\, u \in \mathscr{H}\right\} \leq 0. \tag{1.5.6}$$

We thus conclude that

$$\int_{\partial\Omega} (u/d^s)^2 h\,dx = 0 \qquad \text{for every } u \in \mathscr{H} \text{ and for all } h \in C^3(\partial\Omega), \text{ with } \int_{\partial\Omega} h\,dx = 0.$$

By a standard argument, this implies that u/d^s is constant on $\partial\Omega$. Now, since u solves (1.1.4) and $p \in \{1\} \cup [2,\infty)$, we deduce from [43, Theorem 1.2] that Ω is a ball. □

Proof of Theorem 1.1.4. Consider the unit centered ball $B_1 = B_1(0)$. For $\tau \in (0,1)$ and $t \in (\tau-1, 1-\tau)$, we define $B^t := B_\tau(te_1)$, where e_1 is the first coordinate direction. To prove Theorem 1.1.4, we can take advantage of the invariance under rotations of the problem and may restrict our attention to domains of the form $\Omega(t) = B_1 \setminus \overline{B^t}$. We define

$$\theta : (\tau-1, 1-\tau) \to \mathbb{R}, \qquad \theta(t) := \lambda_{s,p}(\Omega(t)). \tag{1.5.7}$$

We claim that θ is differentiable and satisfies

$$\theta'(t) < 0 \qquad \text{for } t \in (0, 1-\tau). \tag{1.5.8}$$

For this we fix $t \in (\tau-1, 1-\tau)$ and a vector field $X : \mathbb{R}^N \to \mathbb{R}^N$ given by $X(x) = \rho(x)e_1$, where $\rho \in C_c^\infty(B_1)$ satisfies $\rho \equiv 1$ in a neighborhood of B^t. For $\varepsilon \in (-1,1)$, we then define $\Phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ by $\Phi_\varepsilon(x) = x + \beta\varepsilon X(x)$, where $\beta > 0$ is chosen sufficiently small to guarantee that Φ_ε, $\varepsilon \in (-1,1)$ is a family of diffeomorphisms satisfying (3.3.1) and satisfying $\Phi_\varepsilon(B_1) = B_1$ for $\varepsilon \in (-1,1)$. Then, by construction, we have

$$\Phi_\varepsilon(\Omega(t)) = \Phi_\varepsilon\left(B_1 \setminus \overline{B^t}\right) = B_1 \setminus \overline{\Phi_\varepsilon(B^t)} = B_1 \setminus \overline{B^{t+\beta\varepsilon}} = \Omega(t+\beta\varepsilon). \tag{1.5.9}$$

Next we recall that, since $p \in \{1,2\}$, there exists a unique positive minimizer $u \in \mathscr{H}_0^s(\Omega(t))$ corresponding to the variational characterization (1.1.3) of $\lambda_{s,p}(\Omega(t))$. Hence, by Corollary 1.1.2, the map $\varepsilon \mapsto \lambda_{s,p}(\Phi_\varepsilon(\Omega(t)))$ is differentiable at $\varepsilon = 0$. In view of (1.5.9), we thus find that the map θ in (1.5.7) is differentiable at t, and

$$\theta'(t) = \frac{1}{\beta}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \lambda_{s,p}(\Phi_\varepsilon(\Omega(t))) = \Gamma(1+s)^2 \int_{\partial\Omega(t)} \left(\frac{u}{d^s}\right)^2 X \cdot \nu\,dx = \Gamma(1+s)^2 \int_{\partial B^t} \left(\frac{u}{d^s}\right)^2 \nu_1\,dx \tag{1.5.10}$$

by (1.1.10). Here ν denotes the interior unit normal on $\partial\Omega(t)$ which coincides with the exterior unit normal to B^t on ∂B^t, and we used that

$$X \equiv e_1 \quad \text{on } \partial B^t, \qquad X \equiv 0 \quad \text{on } \partial B_1 = \partial\Omega(t) \setminus \partial B^t$$

to get the last equality in (1.5.10). Next, for fixed $t \in (0, 1-\tau)$, let H be the half space defined by $H = \{x \in \mathbb{R}^N : x \cdot e_1 > t\}$ and let $\Theta = H \cap \Omega(t)$. We also let $r_H : \mathbb{R}^N \to \mathbb{R}^N$ be the reflection map with respect to he hyperplane $\partial H := \{x \in \mathbb{R}^N : x \cdot e_1 = t\}$. For $x \in \mathbb{R}^N$, we denote $\bar{x} := r_H(x)$, $\overline{u}(x) := u(\bar{x})$. Using these notations, we have

$$\theta'(t) = \Gamma(1+s)^2 \int_{\partial B^t} \left(\frac{u}{d^s}\right)^2 \nu_1\,dx$$

$$= \Gamma(1+s)^2 \int_{\partial B^t \cap \Theta} \left(\left(\frac{u}{d^s}\right)^2 (x) - \left(\frac{\overline{u}}{d^s}\right)^2 (x) \right) \nu_1 \, dx. \tag{1.5.11}$$

Let $w = \overline{u} - u \in H^s(\mathbb{R}^N)$. Then w is a (weak) solution of the problem

$$(-\Delta)^s w = \lambda_{s,p}(\Omega(t))\overline{u}^{p-1} - \lambda_{s,p}(\Omega(t))u^{p-1} = c_p w \qquad \text{in} \quad \Theta, \tag{1.5.12}$$

where

$$\begin{cases} c_p := \lambda_{s,p}(\Omega(t)) & \text{for } p = 2, \\ c_p = 0 & \text{for } p = 1. \end{cases}$$

Moreover, by definition, $w \equiv \overline{u} \geq 0$ in $H \setminus \overline{\Theta}$, and $w \equiv \overline{u} > 0$ in the subset $[r_H(B_1) \cap H] \setminus \overline{\Theta}$ which has positive measure since $t > 0$. Using that w is anti-symmetric with respect to H and the fact that $\lambda_{s,p}(\Theta) > c_p$ (which follows since Θ is a proper subdomain of $\Omega(t)$), we can apply the weak maximum principle for antisymmetric functions (see [43, Proposition 3.1] or [43, Proposition 3.5]) to deduce that $w \geq 0$ in Θ. Moreover, since $w \not\equiv 0$ in $\mathbb{R}^N$, it follows from the strong maximum principle for antisymmetric functions given in [43, Proposition 3.6] that $w > 0$ in Θ. Now by the fractional Hopf lemma for antisymmetric functions (see [43, Proposition 3.3]) we conclude that

$$0 < \frac{w}{d^s} = \frac{\overline{u}}{d^s} - \frac{u}{d^s} \quad \text{and therefore} \quad \frac{\overline{u}}{d^s} > \frac{u}{d^s} \geq 0 \qquad \text{on } \partial B^t \cap \Theta.$$

From this and (1.5.11) we get (1.5.8), since $\nu_1 > 0$ on $\partial B^t \cap \Theta$.
To conclude, we observe that the function $t \mapsto \lambda_{s,p}(t) = \lambda_{s,p}(\Omega(t))$ is even, thanks to the invariance of the problem under rotations. Therefore the function θ attains its maximum uniquely at $t = 0$. □

1.6 Proof of Proposition 1.2.4

The aim of this section is to prove Proposition 1.2.4. For the readers convenience, we repeat the statement here.

Proposition 1.6.1. *Let $X \in C^0(\overline{\Omega}, \mathbb{R}^N)$, let $u \in C_0^s(\overline{\Omega}) \cap C^1(\Omega)$, and assume that $\psi := \frac{u}{d^s}$ extends to a function on $\overline{\Omega}$ satisfying* (3.2.8) *and* (1.2.9). *Moreover, put $U_k := u\varsigma_k \in C_c^{1,1}(\Omega)$, where ς_k is defined in (1.2.3). Then*

$$\lim_{k\to\infty} \int_{\Omega} \nabla U_k \cdot X \Big(u(-\Delta)^s \varsigma_k - I(u, \varsigma_k) \Big) \, dx = -\kappa_s \int_{\partial\Omega} \psi^2 X \cdot \nu \, dx, \tag{1.6.1}$$

where

$$\kappa_s := -\int_{\mathbb{R}} h'(r)(-\Delta)^s h(r) \, dr \qquad \text{with} \quad h(r) := r_+^s \zeta(r) \tag{1.6.2}$$

and ζ given in (1.2.2), and where we use the notation

$$I(u,v)(x) := \int_{\mathbb{R}^N} (u(x) - u(y))(v(x) - v(y)) K_0(x,y) \, dy \tag{1.6.3}$$

for $u \in C^s_c(\mathbb{R}^N)$, $v \in C^{0,1}(\mathbb{R}^N)$ *and* $x \in \mathbb{R}^N$.

The remainder of this section is devoted to the proof of this proposition. For $k \in \mathbb{N}$, we define

$$g_k := \nabla U_k \cdot X\Big(u(-\Delta)^s \varsigma_k - I(u,\varsigma_k)\Big) \quad : \quad \Omega \to \mathbb{R}. \tag{1.6.4}$$

For $\varepsilon > 0$, we put

$$\Omega^\varepsilon = \{x \in \mathbb{R}^N : |d(x)| < \varepsilon\} \qquad \text{and} \qquad \Omega^\varepsilon_+ = \{x \in \mathbb{R}^N : 0 < d(x) < \varepsilon\} = \{x \in \Omega : d(x) < \varepsilon\}.$$

For every $\varepsilon > 0$, we then have

$$\lim_{k\to\infty} \int_{\Omega\setminus\Omega^\varepsilon} g_k\, dx = 0. \tag{1.6.5}$$

To see this, we first note that $\varsigma_k \to 1$ pointwise on $\mathbb{R}^N \setminus \partial\Omega$, and therefore a.e. on $\mathbb{R}^N$. Moreover, choosing a compact neighborhood $K \subset \Omega$ of $\Omega \setminus \Omega^\varepsilon$, we have

$$(-\Delta)^s \varsigma_k(x) = c_{N,s} \int_{\mathbb{R}^N\setminus K} \frac{1-\varsigma_k(y)}{|x-y|^{N+2s}} dy \qquad \text{for } x \in \Omega\setminus\Omega^\varepsilon \text{ and } k \text{ sufficiently large,}$$

where $\frac{|1-\varsigma_k(y)|}{|x-y|^{N+2s}} \le \frac{C}{1+|y|^{N+2s}}$ for $x \in \Omega \setminus \Omega^\varepsilon$, $y \in \mathbb{R}^N \setminus K$ and $C > 0$ independent of x and y. Consequently, $\|(-\Delta)^s \varsigma_k\|_{L^\infty(\Omega\setminus\Omega^\varepsilon)}$ remains bounded independently of k and $(-\Delta)^s \varsigma_k \to 0$ pointwise on $\Omega \setminus \Omega^\varepsilon$ by the dominated convergence theorem. Similarly, we see that $\|I(u,\varsigma_k)\|_{L^\infty(\Omega\setminus\Omega^\varepsilon)}$ remains bounded independently of k and $I(u,\varsigma_k) \to 0$ pointwise on $\Omega\setminus\Omega^\varepsilon$. Consequently, we find that

$$\|g_k\|_{L^\infty(\Omega\setminus\Omega^\varepsilon)} \text{ is bounded independently of } k \text{ and } g_k \to 0 \text{ pointwise on } \Omega\setminus\Omega^\varepsilon.$$

Hence (1.6.5) follows again by the dominated convergence theorem. As a consequence,

$$\lim_{k\to\infty} \int_\Omega g_k(x)\, dx = \lim_{k\to\infty} \int_{\Omega^\varepsilon_+} g_k(x)\, dx \qquad \text{for every } \varepsilon > 0. \tag{1.6.6}$$

Let, as before, $\nu : \partial\Omega \to \mathbb{R}^N$ denotes the unit interior normal vector field on Ω. Since we assume that $\partial\Omega$ is of class $C^{1,1}$, the map ν is Lipschitz, which means that the derivative $d\nu : T\partial\Omega \to \mathbb{R}^N$ is a.e. well defined and bounded. Moreover, we may fix $\varepsilon > 0$ from now on with the property that the map

$$\Psi : \partial\Omega \times (-\varepsilon,\varepsilon) \to \Omega^\varepsilon, \qquad (\sigma, r) \mapsto \Psi(\sigma,r) = \sigma + r\nu(\sigma) \tag{1.6.7}$$

is a bi-Lipschitz map with $\Psi(\partial\Omega \times (0,\varepsilon)) = \Omega^\varepsilon_+$. In particular, Ψ is a.e. differentiable, and the variable r is precisely the signed distance of the point $\Psi(\sigma,r)$ to the boundary $\partial\Omega$, i.e.,

$$d(\Psi(\sigma,r)) = r \qquad \text{for } \sigma \in \partial\Omega,\ 0 \le r < \varepsilon. \tag{1.6.8}$$

Moreover, for $0 < \varepsilon' \leq \varepsilon$, it follows from (1.6.6) that

$$\lim_{k\to\infty}\int\limits_{\Omega} g_k\,dx = \lim_{k\to\infty}\int\limits_{\Omega_+^{\varepsilon'}} g_k\,dx = \lim_{k\to\infty}\int\limits_{\partial\Omega}\int\limits_{0}^{\varepsilon'} \mathrm{Jac}_\Psi(\sigma,r)g_k(\Psi(\sigma,r))\,drd\sigma$$

$$= \lim_{k\to\infty}\frac{1}{k}\int\limits_{\partial\Omega}\int\limits_{0}^{k\varepsilon'} j_k(\sigma,r)G_k(\sigma,r)\,drd\sigma, \tag{1.6.9}$$

where we define

$$j_k(\sigma,r) = \mathrm{Jac}_\Psi(\sigma,\frac{r}{k}) \quad \text{and} \quad G_k(\sigma,r) = g_k(\Psi(\sigma,\frac{r}{k})) \qquad \text{for a.e. } \sigma\in\partial\Omega,\ 0\leq r<k\varepsilon. \tag{1.6.10}$$

We note that

$$\begin{aligned} &\|j_k\|_{L^\infty(\partial\Omega\times[0,k\varepsilon))} \leq \|\mathrm{Jac}_\Psi\|_{L^\infty(\Omega_\varepsilon)} < \infty \quad \text{for all } k, \text{ and}\\ &\lim_{k\to\infty} j_k(\sigma,r) = \mathrm{Jac}_\Psi(\sigma,0) = 1 \quad \text{for a.e. } \sigma\in\partial\Omega,\ r>0. \end{aligned} \tag{1.6.11}$$

By definition of the functions g_k in (1.6.4), we may write

$$G_k(\sigma,r) = G_k^0(\sigma,r)[G_k^1(\sigma,r)-G_k^2(\sigma,r)] \qquad \text{for } \sigma\in\partial\Omega,\ 0\leq r<k\varepsilon \tag{1.6.12}$$

with

$$\begin{aligned} &G_k^0(\sigma,r) = [\nabla U_k\cdot X](\Psi(\sigma,\frac{r}{k}))\\ &G_k^1(\sigma,r) = [u(-\Delta)^s\varsigma_k](\Psi(\sigma,\frac{r}{k})) \qquad \text{and}\\ &G_k^2(\sigma,r) = I(u,\varsigma_k)(\Psi(\sigma,\frac{r}{k})). \end{aligned} \tag{1.6.13}$$

In order to analyze the limit in (1.6.9) for suitable $\varepsilon'\in(0,\varepsilon]$, we provide estimates for the functions G_k^0, G_k^1, G_k^2 separately in the following. We start with an estimate for G_k^0 given by the following lemma.

Lemma 1.6.2. *Let $\alpha\in(0,1)$ be given by Lemma 1.2.3. Then we have*

$$k^{s-1}|G_k^0(\sigma,r)| \leq C(r^{s-1}+r^{s-1+\alpha}) \qquad \text{for } k\in\mathbb{N},\ 0\leq r<k\varepsilon \tag{1.6.14}$$

with a constant $C>0$, and

$$\lim_{k\to\infty} k^{s-1}G_k^0(\sigma,r) = h'(r)\psi(\sigma)[X(\sigma)\cdot\nu(\sigma)] \quad \text{for } \sigma\in\partial\Omega,\ r>0 \tag{1.6.15}$$

with the function $r\mapsto h(r) = r_+^s\zeta(r)$ given in (1.6.2).

Proof. Since $u=\psi\delta^s$, we have

$$\nabla u = sd^{s-1}\psi\nabla d + d^s\nabla\psi = sd^{s-1}\psi\nabla d + O(d^{s-1+\alpha}) \qquad \text{in } \Omega$$

by Lemma 1.2.3, and therefore, since $\varsigma_k = \zeta \circ (kd)$ by (1.2.3),

$$\nabla U_k = \nabla\Big(u\varsigma_k\Big) = \Big(s\zeta \circ (kd) + kd\zeta' \circ (kd)\Big)\psi d^{s-1}\nabla d + O(d^{s-1+\alpha}) \qquad \text{in } \Omega.$$

Consequently, by (1.6.8) we have

$$\big[(\nabla U_k)\circ\Psi\big](\sigma,\frac{r}{k}) = \Big(s\zeta(r) + r\zeta'(r)\Big)\psi(\sigma+\frac{r}{k}\nu(\sigma))\big(\frac{r}{k}\big)^{s-1}\nabla d(\sigma+\frac{r}{k}\nu(\sigma)) + O\Big(\big(\frac{r}{k}\big)^{s-1+\alpha}\Big)$$

for $\sigma \in \partial\Omega$, $0 \le r < \varepsilon$ with $O(r^{s-1+\alpha})$ independent of k, and therefore

$$G_k^0(\sigma,r) = \Big(s\zeta(r) + r\zeta'(r)\Big)\psi(\sigma+\frac{r}{k}\nu(\sigma))\nabla d(\sigma+\frac{r}{k}\nu(\sigma))\cdot X(\sigma+\frac{r}{k}\nu(\sigma))k^{1-s}r^{s-1}$$
$$+ k^{1-s-\alpha}O(r^{s-1+\alpha}) \qquad \text{for } \sigma \in \partial\Omega,\ 0 \le r < k\varepsilon.$$

Since $\alpha > 0$, we deduce that

$$k^{s-1}G_k^0(\sigma,r) \to \Big(s\zeta(r) + r\zeta'(r)\Big)\psi(\sigma)\nabla d(\sigma)\cdot X(\sigma)r^{s-1} = h'(r)\psi(\sigma)X(\sigma)\cdot\nu(\sigma) \qquad \text{as } k \to \infty$$

for $\sigma \in \partial\Omega$, $r > 0$, while

$$k^{s-1}|G_k^0(\sigma,r)| \le C(r^{s-1} + r^{s-1+\alpha}) \qquad \text{for } k \in \mathbb{N},\ 0 \le r < k\varepsilon$$

with a constant $C > 0$ independent of k and r, as claimed. □

Next we consider the functions G_k^1 defined in (1.6.13), and we first state the following estimate.

Proposition 1.6.3. *There exists $\varepsilon' > 0$ with the property that*

$$|k^{-2s}(-\Delta)^s\varsigma_k(\Psi(\sigma,\frac{r}{k}))| \le \frac{C}{1+r^{1+2s}} \qquad \text{for } k \in \mathbb{N},\ 0 \le r < k\varepsilon' \tag{1.6.16}$$

with a constant $C > 0$. Moreover,

$$\lim_{k\to\infty} k^{-2s}(-\Delta)^s\varsigma_k(\Psi(\sigma,\frac{r}{k})) = (-\Delta)^s\zeta(r) \quad \text{for } \sigma \in \partial\Omega,\ r > 0. \tag{1.6.17}$$

Before giving the somewhat lengthy proof of this proposition, we infer the following corollary related to the functions G_k^1.

Corollary 1.6.4. *There exists $\varepsilon' > 0$ with the property that*

$$|k^{-s}G_k^1(\sigma,r)| \le \frac{Cr^s}{1+r^{1+2s}} \qquad \text{for } k \in \mathbb{N},\ 0 \le r < k\varepsilon' \tag{1.6.18}$$

with a constant $C > 0$. Moreover,

$$\lim_{k\to\infty} k^{-s}G_k^1(\sigma,r) = \psi(\sigma)r^s(-\Delta)^s\zeta(r) \quad \text{for } \sigma \in \partial\Omega,\ r > 0. \tag{1.6.19}$$

Proof. Since $u = \psi\delta^s$ we have $u(\Psi(\sigma, \frac{r}{k})) = k^{-s}\psi(\sigma + \frac{r}{k}\nu(\sigma))r^s$ for $k \in \mathbb{N}$, $0 \le r < k\varepsilon$, and

$$\lim_{k\to\infty} k^s u(\Psi(\sigma, \frac{r}{k})) = \psi(\sigma)r^s \quad \text{for } \sigma \in \partial\Omega,\ r > 0.$$

Since moreover $\|\psi\|_{L^\infty(\Omega_\varepsilon)} < \infty$, the claim now follows from Proposition 1.6.3 by recalling the definition in G_k^1 in (1.6.13). □

We now turn to the proof of Proposition 1.6.3, and we need some preliminary considerations. Since $\partial\Omega$ is of class $C^{1,1}$ by assumption, there exists an open ball $B \subset \mathbb{R}^{N-1}$ centered at the origin and, for every $\sigma \in \partial\Omega$, a parametrization $f_\sigma : B \to \partial\Omega$ of class $C^{1,1}$ with the property that $f_\sigma(0) = \sigma$ and $df_\sigma(0) : \mathbb{R}^{N-1} \to \mathbb{R}^N$ is a linear isometry. For $z \in B$ we then have

$$f_\sigma(z) - f_\sigma(0) = df_\sigma(0)z + O(|z|^2)$$

and therefore

$$|f_\sigma(0) - f_\sigma(z)|^2 = |df_\sigma(0)z|^2 + O(|z|^3) = |z|^2 + O(|z|^3), \tag{1.6.20}$$

$$(f_\sigma(0) - f_\sigma(z)) \cdot \nu(\sigma) = -df_\sigma(0)z \cdot \nu(\sigma) + O(|z|^2) = O(|z|^2), \tag{1.6.21}$$

where we used in (1.6.21) that $df_\sigma(0)z$ belongs to the tangent space $T_\sigma\partial\Omega = \{\nu(\sigma)\}^\perp$. Here and in the following, the term $\mathscr{O}(\tau)$ stands for a function depending on τ and possibly other quantities but satisfying $|\mathscr{O}(\tau)| \le C\tau$ with a constant $C > 0$.
Recalling the definition of the map Ψ in (1.6.7) and writing $\nu_\sigma(z) := \nu(f_\sigma(z))$ for $z \in B$, we now define

$$\Psi_\sigma : (-\varepsilon, \varepsilon) \times B \to \Omega^\varepsilon, \qquad \Psi_\sigma(r, z) = \Psi(f_\sigma(z), r) = f_\sigma(z) + r\nu_\sigma(z). \tag{1.6.22}$$

Then Ψ_σ is a bi-Lipschitz map which maps $(-\varepsilon, \varepsilon) \times B$ onto a neighborhood of σ. Consequently, there exists $\varepsilon' \in (0, \frac{\varepsilon}{2})$ with the property that

$$|\sigma - y| \ge 3\varepsilon' \qquad \text{for all } y \in \mathbb{R}^N \setminus \Psi_\sigma((-\varepsilon, \varepsilon) \times B). \tag{1.6.23}$$

Moreover, ε' can be chosen independently of $\sigma \in \partial\Omega$.
Coming back to the proof of Proposition 1.6.3, we now write, for $\sigma \in \partial\Omega$ and $r \in [0, k\varepsilon')$,

$$(-\Delta)^s \varsigma_k(\Psi(\sigma, \frac{r}{k})) = c_{N,s}\Big(A_k(\sigma, r) + B_k(\sigma, r)\Big) \tag{1.6.24}$$

with

$$A_k(\sigma, r) := \int\limits_{\Psi_\sigma((-\varepsilon,\varepsilon)\times B)} \frac{\zeta(r) - \varsigma_k(y)}{|\Psi(\sigma, \frac{r}{k}) - y|^{N+2s}}\, dy$$

and

$$B_k(\sigma, r) := \int\limits_{\mathbb{R}^N \setminus \Psi_\sigma((-\varepsilon,\varepsilon)\times B)} \frac{\zeta(r) - \varsigma_k(y)}{|\Psi(\sigma, \frac{r}{k}) - y|^{N+2s}}\, dy.$$

Here we used that $\varsigma_k(\Psi(\sigma, \frac{r}{k})) = \zeta(r)$ for $\sigma \in \partial\Omega$, $r \in [0, k\varepsilon')$ by (1.6.8) and the definition of ς_k. We first provide a rather straightforward estimate for the functions B_k.

Lemma 1.6.5. *We have*

$$k^{-2s}|B_k(\sigma,r)| \leq \frac{C}{1+r^{1+2s}} \qquad \text{for } k \in \mathbb{N},\ 0 \leq r < k\varepsilon',\ \sigma \in \partial\Omega \tag{1.6.25}$$

with a constant $C > 0$ and

$$\lim_{k\to\infty} k^{-2s}|B_k(\sigma,r)| = 0 \qquad \text{for every } \sigma \in \Omega,\ r \geq 0. \tag{1.6.26}$$

Proof. By (1.6.23) and since $r < k\varepsilon'$, we have

$$|\Psi(\sigma,\frac{r}{k}) - y| = |\sigma - y + \frac{r}{k}\nu(\sigma)| \geq |\sigma - y| - \frac{r}{k} \geq \frac{|\sigma - y|}{3} + \varepsilon' \quad \text{for } y \in \mathbb{R}^N \setminus \Psi_\sigma((-\varepsilon,\varepsilon)\times B).$$

Recalling that $\zeta = 1-\rho$, $\varsigma_k = 1-\rho_k$ and that ρ_k is supported in $\Omega^{\frac{2}{k}}$, we thus estimate

$$\begin{aligned} |B_k(\sigma,r)| &\leq \int_{\mathbb{R}^N\setminus\Psi_\sigma((-\varepsilon,\varepsilon)\times B)} \frac{|\rho(r)-\rho_k(y)|}{|\Psi(\sigma,\frac{r}{k})-y|^{N+2s}}\,dy \\ &\leq 3^{N+2s}|\rho(r)| \int_{\mathbb{R}^N} \Big(|\sigma - y| + 3\varepsilon'\Big)^{-N-2s} dy + \big(\varepsilon'\big)^{-N-2s} \int_{\mathbb{R}^N} |\rho_k(y)|\,dy \\ &\leq C\Big(|\rho(r)| + |\Omega^{\frac{2}{k}}|\Big) \leq C\Big(|\rho(r)| + k^{-1}\Big). \end{aligned}$$

Here and in the following, the letter C stands for various positive constants. This estimate readily yields (1.6.26). Moreover,

$$k^{-2s}|B_k(\sigma,r)| \leq Ck^{-2s}\Big(|\rho(r)| + k^{-1}\Big) \leq \frac{C}{1+r^{1+2s}} + k^{-1-2s} \leq \frac{C}{1+r^{1+2s}}$$

for $k \in \mathbb{N}$, $0 \leq r < k\varepsilon'$, $\sigma \in \partial\Omega$, as claimed in (1.6.25). □

To complete the proof of Proposition 1.6.3, it thus remains to consider the functions A_k in the following. For this, we need the following additional estimates for the maps Ψ_σ, $\sigma \in \partial\Omega$. We note here that Ψ_σ is a.e. differentiable since it is Lipschitz, so the Jacobian determinant $\mathrm{Jac}_{\Psi_\sigma}$ is a.e. well-defined on $(-\varepsilon,\varepsilon)\times B$.

Lemma 1.6.6. *There exists a constant C_0 with the property that for every $\sigma \in \partial\Omega$ we have the following estimates:*

(i) $|\mathrm{Jac}_{\Psi_\sigma}(r,z)| \leq C_0$ *for a.e.* $r \in (-\varepsilon,\varepsilon)$, $z \in B$*;*

(ii) $|\mathrm{Jac}_{\Psi_\sigma}(r,z) - 1| \leq C_0(|r|+|z|)$ *for a.e.* $r \in (-\varepsilon,\varepsilon)$, $z \in B$*;*

(iii) $|\mathrm{Jac}_{\Psi_\sigma}(r+t,z) - \mathrm{Jac}_{\Psi_\sigma}(r-t,z)| \leq C_0|t|$ *for a.e.* $r \in (-\varepsilon,\varepsilon)$, $z \in B$, $t \in (-\varepsilon - r, \varepsilon - r)$*;*

Moreover, for $\sigma \in \partial\Omega$, $r \in (-\varepsilon,\varepsilon)$, $z \in B$, $t \in (-\varepsilon - r,\varepsilon - r)$ we have

(iv) $\frac{1}{C_0}\big(t^2+|z|^2\big)^{\frac{1}{2}} \leq |\Psi_\sigma(r,0) - \Psi_\sigma(r+t,z)| \leq C_0\big(t^2+|z|^2\big)^{\frac{1}{2}}$,

and for $\sigma \in \partial\Omega$, $r \in (-\varepsilon, \varepsilon)$, $t \in (-\varepsilon - r, \varepsilon - r) \setminus \{0\}$ *and* $z \in \frac{1}{|t|}B$ *we have*

(v) $\left| \frac{|\Psi_\sigma(r,0)-\Psi_\sigma(r+t,|t|z)|^2}{t^2} - (1+|z|^2) \right| \le C_0(|t|+|r|+|tz|)|z|^2$;

(vi) $$\left| \left|\Psi_\sigma(r,0) - \Psi_\sigma(r+t,|t|z)\right|^{-N-2s} - \left|\Psi_\sigma(r,0) - \Psi_\sigma(r-t,|t|z)\right|^{-N-2s} \right| \le C_0|t|^{1-N-2s}(1+|z|^2)^{-\frac{N+2s}{2}}.$$

Proof. The inequalities (i) and (iv) are direct consequences of the fact that Ψ_σ is bi-Lipschitz. In particular, if C_0 is a Lipschitz constant for Ψ_σ^{-1}, we have

$$\left(t^2+|z|^2\right)^{\frac{1}{2}} = |(-t,z)| = |(r,0)-(r+t,z)| \le C_0|\Psi_\sigma(r,0) - \Psi_\sigma(r+t,z)|$$

for $\sigma \in \partial\Omega$, $r \in (-\varepsilon, \varepsilon)$, $z \in B$ and $t \in (-\varepsilon - r, \varepsilon - r)$, so the first inequality in (iv) follows. By making C_0 larger if necessary so that it is also a Lipschitz constant for Ψ_σ, we then deduce the second inequality in (iv).
To see (ii) and (iii), we note that $d\Psi_\sigma$ is a.e. given by

$$d\Psi_\sigma(r,z)(r',z') = [df_\sigma(z) + r d\nu_\sigma(z)]z' + r'\nu_\sigma(z)$$

for $(r,z) \in (-\varepsilon, \varepsilon) \times B$, $(r',z') \in \mathbb{R} \times \mathbb{R}^{N-1}$, which implies that

$$[d\Psi_\sigma(r,z) - d\Psi_\sigma(0,0)](r',z') = [df_\sigma(z) - df_\sigma(0)]z' + rd\nu_\sigma(z) + r'\big(\nu_\sigma(z) - \nu_\sigma(0)\big)$$

and

$$[d\Psi_\sigma(r+t,z) - d\Psi_\sigma(r-t,z)](r',z') = 2td\nu_\sigma(z)z'.$$

Since df_σ, ν_σ are Lipschitz functions on B, $d\nu_\sigma$ is a bounded function on B and the determinant is a locally Lipschitz continuous function on the space of linear endomorphisms of $\mathbb{R}^N$, it follows that

$$|\mathrm{Jac}_{\Psi_\sigma}(r,z) - \mathrm{Jac}_{\Psi_\sigma}(0,0)| \le C_0(|r|+|z|) \quad \text{and} \quad |\mathrm{Jac}_{\Psi_\sigma}(r+t,z) - \mathrm{Jac}_{\Psi_\sigma}(r-t,z)| \le C_0|t|$$

for a.e. $r \in (-\varepsilon, \varepsilon)$, $z \in B$, $t \in (-\varepsilon - r, \varepsilon - r)$. Moreover, $\mathrm{Jac}_{\Psi_\sigma}(0,0) = 1$ since the map

$$\mathbb{R} \times \mathbb{R}^{N-1} \to \mathbb{R}^N, \qquad (r',z') \mapsto d\Psi_\sigma(0,0)(r',z') = df_\sigma(0)z' + r'\nu_\sigma(0)$$

is an isometry. Hence (ii) and (iii) follow.
To see (v) and (vi), we note that by definition of Ψ_σ we have

$$\Psi_\sigma(r,0) - \Psi_\sigma(r+t,z) = f_\sigma(0) - f_\sigma(z) - t\nu_\sigma(0) + (r+t)(\nu_\sigma(0) - \nu_\sigma(z))$$

for $z \in B$, $r \in (0, \varepsilon')$ and $t \in (-\varepsilon - r, \varepsilon - r)$. Using moreover that $(\nu_\sigma(0) - \nu_\sigma(z)) \cdot \nu_\sigma(0) = \frac{1}{2}|\nu_\sigma(0) - \nu_\sigma(z)|^2$, we get

$$|\Psi_\sigma(r,0) - \Psi_\sigma(r+t,z)|^2 = t^2 + |f_\sigma(0) - f_\sigma(z)|^2 + (r+t)^2|\nu_\sigma(0) - \nu_\sigma(z)|^2$$
$$-2t(f_\sigma(0) - f_\sigma(z)) \cdot \nu_\sigma(0) - t(r+t)|\nu_\sigma(0) - \nu_\sigma(z)|^2 + 2(r+t)(f_\sigma(0) - f_\sigma(z)) \cdot (\nu_\sigma(0) - \nu_\sigma(z))$$

$$
\begin{aligned}
&= t^2 + |f_\sigma(0) - f_\sigma(z)|^2 + r(r+t)|\nu_\sigma(0) - \nu_\sigma(z)|^2 \\
&\qquad - 2t(f_\sigma(0) - f_\sigma(z)) \cdot \nu_\sigma(0) + 2(r+t)(f_\sigma(0) - f_\sigma(z)) \cdot (\nu_\sigma(0) - \nu_\sigma(z)) \\
&= t^2 + |z|^2 + \big[|z| m_\sigma(z) + r(r+t) n_\sigma(z) - 2t p_\sigma(z) + 2(r+t) q_\sigma(z)\big] |z|^2 \qquad (1.6.27)
\end{aligned}
$$

for $z \in B$, $r \in (-\varepsilon, \varepsilon)$ and $t \in (-\varepsilon - r, \varepsilon - r)$ with the functions

$$
m_\sigma(z) = \frac{|f_\sigma(0) - f_\sigma(z)|^2 - |z|^2}{|z|^3}, \quad n_\sigma(z) = \frac{|\nu_\sigma(0) - \nu_\sigma(z)|^2}{|z|^2}, \quad p_\sigma(z) = \frac{(f_\sigma(0) - f_\sigma(z)) \cdot \nu_\sigma(0)}{|z|^2}
$$

and

$$
q_\sigma(z) = \frac{(f_\sigma(0) - f_\sigma(z)) \cdot (\nu_\sigma(0) - \nu_\sigma(z))}{|z|^2}, \qquad z \in B \setminus \{0\},
$$

which are all bounded as a consequence of the Lipschitz continuity of f_σ and ν_σ and of (1.6.20) and (1.6.21). We deduce that

$$
\begin{aligned}
&\Big| \frac{|\Psi_\sigma(r,0) - \Psi_\sigma(r+t, |t|z)|^2}{t^2} - (1 + |z|^2) \Big| \\
&= \Big| |tz| m_\sigma(|t|z) + r(r+t) n_\sigma(|t|z) - 2t p_\sigma(|t|z) + 2(r+t) q_\sigma(|t|z) \Big| |z|^2 \le C_0 (|tz| + |r| + |t|) |z|^2
\end{aligned}
$$

for $\sigma \in \partial\Omega$, $r \in (-\varepsilon, \varepsilon)$, $t \in (-\varepsilon - r, \varepsilon - r) \setminus \{0\}$ and $z \in \frac{1}{|t|} B$ if C_0 is chosen sufficiently large, as claimed in (v).

For the proof of (vi), we now set $w_\sigma(r,t,z) := \frac{1}{t^2} |\Psi_\sigma(r,0) - \Psi_\sigma(r+t, |t|z)|^2$, and we note that

$$
w_\sigma(r,t,z) \ge \frac{1 + |z|^2}{C_0^2} \quad \text{for } \sigma \in \partial\Omega,\ r \in (-\varepsilon, \varepsilon),\ t \in (-\varepsilon - r, \varepsilon - r) \setminus \{0\},\ z \in \frac{1}{|t|} B
$$

by (iv). Moreover, from (1.6.27) we infer that

$$
\Big| w_\sigma(r,t,z) - w_\sigma(r,-t,z) \Big| = \Big| 2rt n_\sigma(|t|z) + 4t \big(q_\sigma(|t|z) - p_\sigma(|t|z) \big) \Big| |z|^2 \le C_0 |t| |z|^2
$$

for $\sigma \in \partial\Omega$, $r \in (-\varepsilon, \varepsilon)$, $t \in (-\varepsilon - r, \varepsilon - r) \setminus \{0\}$ and $z \in \frac{1}{|t|} B$ if C_0 is made larger if necessary. Using these estimates together with the mean value theorem, we get that, for some $\tau = \tau(\sigma, r, t, z)$ with $-t < \tau < t$,

$$
\begin{aligned}
&\Big| |\Psi_\sigma(r,0) - \Psi_\sigma(r+t, |t|z)|^{-N-2s} - |\Psi_\sigma(r,0) - \Psi_\sigma(r-t, |t|z)|^{-N-2s} \Big| \\
&= |t|^{-N-2s} \Big| w_\sigma(r,t,z)^{-\frac{N+2s}{2}} - w_\sigma(r,-t,z)^{-\frac{N+2s}{2}} \Big| \\
&= \frac{(N+2s)|t|^{-N-2s}}{2} w_\sigma(r,\tau,z)^{-\frac{N+2s+2}{2}} \Big| w_\sigma(r,t,z) - w_\sigma(r,-t,z) \Big| \\
&\le C_0 |t|^{1-N-2s} (1 + |z|^2)^{-\frac{N+2s+2}{2}} |z|^2 \le C_0 |t|^{1-N-2s} (1 + |z|^2)^{-\frac{N+2s}{2}}
\end{aligned}
$$

for $z \in B$, $r \in (0, \varepsilon')$ and $t \in (-\varepsilon + r, \varepsilon - r)$ after making C_0 larger if necessary, as claimed in (vi). □

We now have all the tools to study the quantity $A_k(\sigma,r)$ in (1.6.24).

Lemma 1.6.7. *We have*

$$k^{-2s}|A_k(\sigma,r)| \leq \frac{C}{1+r^{1+2s}} \qquad \text{for } k\in\mathbb{N},\ 0\leq r<k\varepsilon',\ \sigma\in\partial\Omega \tag{1.6.28}$$

with a constant $C>0$ and

$$\lim_{k\to\infty} k^{-2s}A_k(\sigma,r) = \frac{(-\Delta)^s\zeta(r)}{b_{N,s}} \qquad \text{for every } \sigma\in\Omega,\ r\geq 0. \tag{1.6.29}$$

Proof. For $\sigma\in\partial\Omega$ and $0<r<k\varepsilon'$, we write, with a change of variables,

$$\begin{aligned} &A_k(\sigma,r) \qquad\qquad (1.6.30)\\ &= \int_{\Psi_\sigma((-\varepsilon,\varepsilon)\times B)} \frac{\zeta(r)-\varsigma_k(y)}{|\Psi(\sigma,\frac{r}{k})-y|^{N+2s}}\,dy = \int_{-\varepsilon}^{\varepsilon}\int_B \mathrm{Jac}_{\Psi_\sigma}(\tilde r,z)\,\frac{\zeta(r)-\zeta(k\tilde r)}{|\Psi_\sigma(\frac{r}{k},0)-\Psi_\sigma(\tilde r,z)|^{N+2s}}\,dzd\tilde r\\ &= \frac{1}{k}\int_{-k\varepsilon-r}^{k\varepsilon-r}\int_B \mathrm{Jac}_{\Psi_\sigma}(\frac{r+t}{k},z)\,\frac{\zeta(r)-\zeta(r+t)}{|\Psi_\sigma(\frac{r}{k},0)-\Psi_\sigma(\frac{r+t}{k},z)|^{N+2s}}\,dzdt\\ &= \int_{-k\varepsilon-r}^{k\varepsilon-r}\frac{|t|^{N-1}}{k^N}\int_{\frac{k}{|t|}B}\mathrm{Jac}_{\Psi_\sigma}(\frac{r+t}{k},\frac{|t|z}{k})\,\frac{\zeta(r)-\zeta(r+t)}{|\Psi_\sigma(\frac{r}{k},0)-\Psi_\sigma(\frac{r+t}{k},\frac{|t|z}{k})|^{N+2s}}\,dzdt\\ &= k^{2s}\int_{\mathbb{R}}\frac{\zeta(r)-\zeta(r+t)}{|t|^{1+2s}}\mathscr{K}_k(r,t)dt \end{aligned}$$

with the kernels $K_k:(0,k\varepsilon')\times\mathbb{R}\to\mathbb{R}$ defined by

$$K_k(r,t)=\begin{cases}\Big(\frac{|t|}{k}\Big)^{N+2s}\displaystyle\int_{\frac{k}{|t|}B}\frac{\mathrm{Jac}_{\Psi_\sigma}(\frac{r+t}{k},\frac{|t|z}{k})}{\big|\Psi_\sigma(\frac{r}{k},0)-\Psi_\sigma(\frac{r+t}{k},\frac{|t|z}{k})\big|^{N+2s}}dz, & t\in(-k\varepsilon-r,k\varepsilon-r),\\ 0, & t\notin(-k\varepsilon-r,k\varepsilon-r).\end{cases}$$

Consequently,

$$A_k(\sigma,r)=k^{2s}\Big(J_k^1(\sigma,r)+J_k^2(\sigma,r)\Big) \tag{1.6.31}$$

with

$$J_k^1(\sigma,r):=\frac{1}{4}\int_{\mathbb{R}}\frac{2\zeta(r)-\zeta(r+t)-\zeta(r-t)}{|t|^{1+2s}}\big(\mathscr{K}_k(r,t)+\mathscr{K}_k(r,-t)\big)dt$$

and

$$J_k^2(\sigma,r):=-\frac{1}{4}\int_{\mathbb{R}}\frac{\zeta(r+t)-\zeta(r-t)}{|t|^{2s}}\,\frac{\mathscr{K}_k(r,t)-\mathscr{K}_k(r,-t)}{|t|}dt.$$

By Lemma 1.6.6(i),(iv) and the definition of $\mathscr{K}_k$, we have

$$|K_k(r,t)| \le C_0^{N+2s+1} \int\limits_{\frac{k}{|t|}B} \Big(1+\big|z|^2\Big)^{-\frac{N+2s}{2}} dz \le C_0^{N+2s+1} a_{N,s} \tag{1.6.32}$$

for $r \in (-k\varepsilon', k\varepsilon')$ and $t \in \mathbb{R}\setminus\{0\}$ with

$$a_{N,s} := \int\limits_{\mathbb{R}^{N-1}} (1+|z|^2)^{-\frac{N+2s}{2}} dz < \infty. \tag{1.6.33}$$

Moreover, by Lemma 1.6.6(i)(ii),(iv),(v) and the dominated convergence theorem, we have

$$\lim_{k\to\infty} K_k(r,t) = \int\limits_{\mathbb{R}^{N-1}} (1+|z|^2)^{-\frac{N+2s}{2}} dz = a_{N,s} \qquad \text{for every } r \ge 0,\, t \in \mathbb{R}\setminus\{0\}. \tag{1.6.34}$$

Using (1.6.32) and the fact that $\rho = 1-\zeta \in C_c^\infty(\mathbb{R})$, we obtain the estimate

$$\begin{aligned} |J_k^1(\sigma,r)| &\le C \int\limits_{\mathbb{R}} \frac{|2\zeta(r)-\zeta(r+t)-\zeta(r-t)|}{|t|^{1+2s}} dt \\ &= C \int\limits_{\mathbb{R}} \frac{|2\rho(r)-\rho(r+t)-\rho(r-t)|}{|t|^{1+2s}} dt \le \frac{C}{1+r^{1+2s}} \end{aligned} \tag{1.6.35}$$

for $k \in \mathbb{N}$, $r \in (0,k\varepsilon')$ and $\sigma \in \partial\Omega$. Here and in the following, the letter $C>0$ stands for different positive constants. Moreover, by (1.6.32), (1.6.34) and the dominated convergence theorem, we find that

$$\lim_{k\to\infty} J_k^1(\sigma,r) = \frac{a_{N,s}}{2} \int\limits_{\mathbb{R}} \frac{2\zeta(r)-\zeta(r+t)-\zeta(r-t)}{|t|^{1+2s}} dt = \frac{a_{N,s}}{b_{1,s}}(-\Delta)^s\zeta(r) = \frac{(-\Delta)^s\zeta(r)}{b_{N,s}}. \tag{1.6.36}$$

Here we have used the fact that

$$b_{N,s}a_{N,s} = b_{1,s}, \tag{1.6.37}$$

see e.g. [45].

Next we deal with $J_k^2(\sigma,r)$, and for this we have to estimate the kernel differences $|\mathscr{K}_k(r,t) - \mathscr{K}_k(r,-t)|$. By Lemma 1.6.6 (i), (iii), (iv) and (vi), we have

$$\Bigg|\frac{\mathrm{Jac}_{\Psi_\sigma}(\frac{r+t}{k},\frac{|t|z}{k})}{\big|\Psi_\sigma(\frac{r}{k},0)-\Psi_\sigma(\frac{r+t}{k},\frac{|t|z}{k})\big|^{N+2s}} - \frac{\mathrm{Jac}_{\Psi_\sigma}(\frac{r-t}{k},\frac{|t|z}{k})}{\big|\Psi_\sigma(\frac{r}{k},0)-\Psi_\sigma(\frac{r-t}{k},\frac{|t|z}{k})\big|^{N+2s}}\Bigg| \le C\Big(\frac{|t|}{k}\Big)^{1-N-2s}(1+|z|^2)^{-\frac{N+2s}{2}}$$

for $z \in \frac{k}{|t|}B$, $r \in (0,k\varepsilon')$ and $t \in (-k\varepsilon+r, k\varepsilon-r)$ and therefore

$$\frac{|K_k(r,t)-K_k(r,-t)|}{|t|} \le \frac{C}{k}\int\limits_{\frac{k}{|t|}B}(1+|z|^2)^{-\frac{N+2s}{2}} dz \le \frac{C}{k}\int\limits_{\mathbb{R}^{N-1}}(1+|z|^2)^{-\frac{N+2s}{2}} dz \le \frac{C}{k} \tag{1.6.38}$$

for $r \in (0, k\varepsilon')$ and $t \in (-k\varepsilon + r, k\varepsilon - r)$. Moreover, by definition we have

$$|K_k(r,t) - K_k(r,-t)| = 0 \qquad \text{for } t \in \mathbb{R} \setminus (-k\varepsilon - r, k\varepsilon + r), \tag{1.6.39}$$

while for $t \in (-k\varepsilon - r, -k\varepsilon + r) \cup (k\varepsilon - r, k\varepsilon + r)$ we have $|t| \geq k\varepsilon - \varepsilon' \geq \frac{k\varepsilon}{2}$ and therefore, similarly as in (1.6.32),

$$\frac{|K_k(r,t)|}{|t|} \leq \frac{C}{|t|} \int_{\frac{k}{|t|}B} \left(1 + |z|^2\right)^{-\frac{N+2s}{2}} dz \leq \frac{C}{k} \int_{\frac{2}{\varepsilon}B} \left(1 + |z|^2\right)^{-\frac{N+2s}{2}} dz \leq \frac{C}{k}. \tag{1.6.40}$$

Note here that the constant $C > 0$ on the RHS depends on ε, but this is not a problem. Combining (1.6.38), (1.6.39), (1.6.40) and using that $\rho = 1 - \zeta \in C_c^\infty(\mathbb{R})$, we get

$$\begin{aligned} |J_k^2(\sigma, r)| &\leq \frac{1}{4} \int_{\mathbb{R}} \frac{|\zeta(r+t) - \zeta(r-t)|}{|t|^{2s}} \frac{|K_k(r,t) - K_k(r,-t)|}{|t|} dt \\ &\leq \frac{C}{k} \int_{\mathbb{R}} \frac{|\zeta(r+t) - \zeta(r-t)|}{t^{2s}} dt = \frac{C}{k} \int_{\mathbb{R}} \frac{|\rho(r+t) - \rho(r-t)|}{t^{2s}} dt \leq \frac{C(1+r)^{-2s}}{k} \end{aligned}$$

for $k \in \mathbb{N}$, $\sigma \in \partial\Omega$ and $0 \leq r < k\varepsilon'$. Hence

$$|J_k^2(\sigma, r)| \leq \frac{C}{1 + r^{1+2s}} \qquad \text{for } k \in \mathbb{N},\ 0 \leq r < k\varepsilon' \tag{1.6.41}$$

and

$$\lim_{k \to \infty} |J_k^2(\sigma, r)| = 0 \qquad \text{for all } r \geq 0. \tag{1.6.42}$$

Now (1.6.28) follows by combining (1.6.31), (1.6.35) and (1.6.41). Moreover, (1.6.29) follows by combining (1.6.31), (1.6.36) and (1.6.42). □

Proof of Proposition 1.6.3. The proof is completed by combining (1.6.24) with Lemmas 1.6.5 and 1.6.7. □

It finally remains to estimate the function G_k^2 in (1.6.12).

Lemma 1.6.8. *There exists $\varepsilon' > 0$ with the property that the function G_k^2 defined in (1.6.13) satisfies*

$$|k^{-s} G_k^2(\sigma, r)| \leq \frac{C}{1 + r^{1+s}} \qquad \text{for } k \in \mathbb{N},\ 0 \leq r < k\varepsilon',\ \sigma \in \partial\Omega \tag{1.6.43}$$

with a constant $C > 0$. Moreover,

$$\lim_{k \to \infty} k^{-s} G_k^2(\sigma, r) = \psi(\sigma) \tilde{I}(r) \tag{1.6.44}$$

with

$$\tilde{I}(r) = b_{1,s} \int_{\mathbb{R}} \frac{\left(r_+^s - (r+t)_+^s\right)\left(\zeta(r) - \zeta(r+t)\right)}{|t|^{1+2s}} dt.$$

Proof. The proof is similar to the one of Proposition 1.6.3, but there are some differences we need to deal with. First, as in the proof of Proposition 1.6.3, we choose $\varepsilon' \in (0, \frac{\varepsilon}{2})$ small enough, so that (1.6.23) holds. Similarly as in (1.6.24) we can then write

$$G_k^2(\sigma, r) = b_{N,s}\Big(\widetilde{A}_k(\sigma, r) + \widetilde{B}_k(\sigma, r)\Big) \tag{1.6.45}$$

with

$$\widetilde{A}_k(\sigma, r) := \int\limits_{\Psi_\sigma((-\varepsilon,\varepsilon)\times B)} \frac{(u(\Psi(\sigma, \frac{r}{k})) - u(y))(\zeta(r) - \varsigma_k(y))}{|\Psi(\sigma, \frac{r}{k}) - y|^{N+2s}}\, dy$$

and

$$\widetilde{B}_k(\sigma, r) = \int\limits_{\mathbb{R}^N \setminus \Psi_\sigma((-\varepsilon,\varepsilon)\times B)} \frac{(u(\Psi(\sigma, \frac{r}{k})) - u(y))(\zeta(r) - \varsigma_k(y))}{|\Psi(\sigma, \frac{r}{k}) - y|^{N+2s}}\, dy.$$

As noted in the proof of Lemma 1.6.5, we have

$$|\Psi(\sigma, \frac{r}{k}) - y| \geq \frac{|\sigma - y|}{3} + \varepsilon' \quad \text{for } y \in \mathbb{R}^N \setminus \Psi_\sigma((-\varepsilon, \varepsilon) \times B),\ 0 < r < k\varepsilon'.$$

Therefore, since $u \in L^\infty(\mathbb{R}^N)$, we may estimate as in the proof of Lemma 1.6.5 to get

$$|\widetilde{B}_k(\sigma, r)| \leq 2\|u\|_{L^\infty} \int\limits_{\mathbb{R}^N \setminus \Psi_\sigma((-\varepsilon,\varepsilon)\times B)} \frac{|\rho(r) - \rho_k(y)|}{|\Psi(\sigma, \frac{r}{k}) - y|^{N+2s}} dy \leq C\Big(|\rho(r)| + k^{-1}\Big).$$

Here, as before, the letter C stands for various positive constants. Consequently,

$$\lim_{k\to\infty} k^{-s} |\widetilde{B}_k(\sigma, r)| = 0 \qquad \text{for every } \sigma \in \Omega,\ r \geq 0, \tag{1.6.46}$$

since ρ has compact support in $\mathbb{R}$, and

$$k^{-s}|\widetilde{B}_k(\sigma, r)| \leq Ck^{-s}\Big(|\rho(r)| + k^{-1}\Big) \leq \frac{C}{1 + r^{1+s}} \qquad \text{for } k \in \mathbb{N},\ 0 \leq r < k\varepsilon',\ \sigma \in \partial\Omega. \tag{1.6.47}$$

Hence it remains to estimate $\widetilde{A}_k(\sigma, r)$. For this we note that, by the same change of variables as in (1.6.30), we have

$$\begin{aligned} \widetilde{A}_k(\sigma, r) &= \int\limits_{-\varepsilon}^{\varepsilon} \int\limits_{B} \operatorname{Jac}_{\Psi_\sigma}(z, \tilde{r})\, \frac{(u(\Psi(\frac{r}{k}, 0)) - u(\Psi_\sigma(\tilde{r}, z)))(\zeta(r) - \zeta(k\tilde{r}))}{|\Psi_\sigma(\frac{r}{k}, 0) - \Psi_\sigma(\tilde{r}, z)|^{N+2s}}\, dz d\tilde{r} \\ &= k^s \int\limits_{\mathbb{R}} \frac{\zeta(r) - \zeta(r+t)}{|t|^{1+s}} \widetilde{\mathscr{K}_k}(r, t) dt \end{aligned} \tag{1.6.48}$$

with the kernel

$\widetilde{K}_k(r, t)$

$$= \begin{cases} \Big(\frac{|t|}{k}\Big)^{N+s} \int\limits_{\frac{k}{|t|}B} \frac{\big(u(\Psi_\sigma(\frac{r}{k},0)) - u(\Psi_\sigma(\frac{r+t}{k},\frac{|t|}{k}z))\big)\mathrm{Jac}_{\Psi_\sigma}(\frac{r+t}{k},\frac{|t|z}{k})}{\big|\Psi_\sigma(\frac{r}{k},0)-\Psi_\sigma(\frac{r+t}{k},\frac{|t|z}{k})\big|^{N+2s}} dz, & t \in (-k\varepsilon - r, k\varepsilon - r), \\ 0, & t \notin (-k\varepsilon - r, k\varepsilon - r). \end{cases}$$

Since $u \in C^s(\mathbb{R}^N)$ and Ψ_σ is Lipschitz, we have

$$\big|u(\Psi_\sigma(\frac{r}{k},0)) - u(\Psi_\sigma(\frac{r+t}{k},\frac{|t|}{k}z)\big| \le C\Big(\big(\frac{|t|}{k}\big)^2 + \big(\frac{|tz|}{k}\big)^2\Big)^{\frac{s}{2}} \le C\Big(\frac{|t|}{k}\Big)^s(1+|z|^s),$$

for $\sigma \in \partial\Omega$, $r \in (-k\varepsilon, k\varepsilon)$, $t \in (-k\varepsilon - r, k\varepsilon - r) \setminus \{0\}$ and $z \in \frac{k}{|t|}B$. Therefore, by using Lemma 1.6.6(i),(iv) as in (1.6.32),

$$|\widetilde{K}_k(r,t)| \le C \int\limits_{\mathbb{R}^{N-1}} (1+|z|^s)(1+|z|^2)^{-\frac{N+2s}{2}} dz \le C \int\limits_{\mathbb{R}^{N-1}} (1+|z|)^{-N-s} dz < \infty. \tag{1.6.49}$$

Inserting this estimate in (1.6.48), we conclude that

$$k^{-s}|\widetilde{A}_k(\sigma,r)| \le C \int\limits_{\mathbb{R}} \frac{|\zeta(r)-\zeta(r+t)|}{|t|^{1+s}} dt = C \int\limits_{\mathbb{R}} \frac{|\rho(r)-\rho(r+t)|}{|t|^{1+s}} dt \le \frac{C}{1+r^{1+s}}.$$

for $k \in \mathbb{N}$, $0 \le r < k\varepsilon'$, $\sigma \in \partial\Omega$. Combining this inequality with (1.6.47), we obtain (1.6.43). Moreover, since $u \in C_0^s(\overline{\Omega})$ and $\psi = \frac{u}{d^s} \in C^0(\overline{\Omega})$, we have

$$\lim_{k\to\infty} k^s \left[u(\Psi_\sigma(\frac{r}{k},0)) - u(\Psi_\sigma(\frac{r+t}{k},\frac{|t|}{k}z))\right] = \psi(\sigma)(r_+^s - (r+t)_+^s) \tag{1.6.50}$$

for $\sigma \in \partial\Omega$, $r > 0$ and $t \in \mathbb{R}$ and $z \in \mathbb{R}^{N-1}$. Consequently, arguing as for (1.6.34) with Lemma 1.6.6(i)(ii),(iv),(v) and the dominated convergence theorem, we find that

$$\lim_{k\to\infty} \widetilde{K}_k(r,t) = \psi(\sigma)\frac{(r_+^s - (r+t)_+^s)}{|t|^s} \int\limits_{\mathbb{R}^{N-1}} (1+|z|^2)^{-\frac{N+2s}{2}} dz = a_{N,s}\psi(\sigma)\frac{(r_+^s - (r+t)_+^s)}{|t|^s} \tag{1.6.51}$$

for $\sigma \in \partial\Omega$, $r > 0$ and $t \in \mathbb{R}$ with $a_{N,s}$ given in (1.6.33). Hence, by (1.6.48), (1.6.49), (1.6.51) and the dominated convergence theorem,

$$\lim_{k\to\infty} k^{-s}\widetilde{A}_k(\sigma,r) = a_{N,s}\psi(\sigma)\int\limits_{\mathbb{R}} \frac{(r_+^s - (r+t)_+^s)(\zeta(r)-\zeta(r+t))}{|t|^{1+2s}} dt = \frac{a_{N,s}}{b_{1,s}}\psi(\sigma)\tilde{I}(r) = \frac{\psi(\sigma)\tilde{I}(r)}{b_{N,s}},$$

where we used again (1.6.37) for the last equality. Combining this with (1.6.45) and (1.6.46), we obtain (1.6.44). □

We are now ready to complete the

Proof of Proposition 1.6.1. Combining (1.6.14), (1.6.18) and (1.6.43), we see that there exists $\varepsilon' > 0$ with the property that the functions G_k defined in (1.6.4) satisfy

$$\frac{G_k(\sigma,r)}{k} \le C\frac{r^{s-1}+r^{s-1+\alpha}}{1+r^{1+s}} \qquad \text{for } k\in\mathbb{N},\, 0\le r< k\varepsilon' \tag{1.6.52}$$

with a constant $C>0$ independent of k and r. Since $s,\alpha\in(0,1)$, the RHS of this inequality is integrable over $[0,\infty)$. Moreover, by (1.6.15), (1.6.19) and (1.6.44),

$$\frac{1}{k}G_k(\sigma,r)\to[X(\sigma)\cdot\nu(\sigma)]\psi^2(\sigma)h'(r)\big(r^s(-\Delta)^s\zeta(r)-\tilde{I}(r)\big) \tag{1.6.53}$$

for every $r>0$, $\sigma\in\partial\Omega$ as $k\to\infty$. Next we note that, by a standard computation,

$$(-\Delta)^s h(r)=(-\Delta)^s[r_+^s\zeta(r)]=\zeta(r)(-\Delta)^s r_+^s+r_+^s(-\Delta)^s\zeta(r)-\tilde{I}(r)=r_+^s(-\Delta)^s\zeta(r)-\tilde{I}(r) \tag{1.6.54}$$

for $r>0$ since r_+^s is an s-harmonic function on $(0,\infty)$ see e.g [16]. Hence, by (1.6.9), (1.6.9), (1.6.52), (1.6.53), (1.6.54) and the dominated convergence theorem, we conclude that

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega g_k dx &= \int_0^\infty h'(r)(-\Delta)^s h(r)dr\int_{\partial\Omega}[X(\sigma)\cdot\nu(\sigma)]\psi^2(\sigma)d\sigma\\ &=\int_{\mathbb{R}} h'(r)(-\Delta)^s h(r)dr\int_{\partial\Omega}[X(\sigma)\cdot\nu(\sigma)]\psi^2(\sigma)d\sigma,\end{aligned}$$

as claimed in (1.6.1). □

1.7 Appendix

Here we give a short proof of the uniqueness of positive minimizers of the problem (1.1.3) for $1\le p\le 2$.

Lemma 1.7.1. *Let $\Omega\subset\mathbb{R}^N$ be a bounded open set of class $C^{1,1}$, let $p\in[1,2]$, and let u_1 and u_2 be two positive minimizers of* (1.1.3). *Then $u_1=u_2$.*

Proof. Suppose by contradiction that there are two different positive minimizers u_1,u_2 for the minimization problem. Then, since $\|u_1\|_{L^p(\Omega)}=\|u_2\|_{L^p(\Omega)}=1$, the difference u_1-u_2 changes sign. Since moreover $\frac{u_1}{d^s}$ and $\frac{u_2}{d^s}$ are continuous positive functions on $\overline{\Omega}$ by Lemma 1.2.3, there exists a maximal $\tau\in(0,1)$ with

$$\tau u_1\le u_2 \qquad \text{on } \overline{\Omega}.$$

Moreover, $\tau u_1\not\equiv u_2$ since u_1-u_2 changes sign. Consequently, $v:=u_2-\tau u_1$ satisfies $v\ge 0$ on $\overline{\Omega}$ and $v\not\equiv 0$. Moreover, using that $p-1\in[0,1]$ and $\tau\in(0,1)$, we find that

$$(-\Delta)^s v=\lambda\big(u_2^{p-1}-\tau u_1^{p-1}\big)\ge\lambda\big(u_2^{p-1}-(\tau u_1)^{p-1}\big)\ge 0 \quad \text{in } \Omega, \qquad v=0 \qquad \text{in } \mathbb{R}^N\setminus\Omega$$

with $\lambda:=\lambda_{s,p}(\Omega)>0$. Now the strong maximum principle for the fractional Laplacian and the fractional Hopf lemma implies that $v=u_2-\tau u_1$ is strictly positive in Ω and $\frac{v}{d^s}>0$ on $\partial\Omega$. This contradicts the maximality of τ. Hence uniqueness holds. □

1.7.1 Alternative computation of the constant κ_s

In this appendix, we provide an alternative way to derive the explicit value of the constant κ_s given by (1.2.10). Since this part was omitted in the Paper [32] we give here the full argument that uses the Logarithmic Laplacian introduced in [23]. For clarity of the exposition, we sketch the proof into several steps. Recall the expression of κ_s given by (1.6.2). Recall also that $\zeta = 1-\rho$ where $\rho \in C_c^\infty(-2,2)$ such that $\rho \equiv 1$ in $(-1,1)$.
We start the journey with the following key observation

Lemma 1.7.2. *Let*

$$\kappa_s = \int_{\mathbb{R}} h'(r)(-\Delta)^s h(r)\,dr,$$

where $h(r) := r_+^s \zeta(r) = \max(r,0)^s \zeta(r)$. *Then*

$$2\kappa_s := -s\int_{\mathbb{R}} r_+^{s-1}(-\Delta)^s(r_+^s\zeta)\,dr = s\int_{\mathbb{R}} r_+^{s-1}(-\Delta)^s(r_+^s\rho)\,dr.$$

Proof. To obtain the identity above the idea is to regularized h' by mean of the new cut-off $\widetilde{\varsigma}_k(x) = 1-\varsigma(x/k) = \rho(x/k)$, for $x\in\mathbb{R}$. Note that $\widetilde{\varsigma}_k \in C_c^\infty(-2k,2k)$ and $\widetilde{\varsigma}_k \equiv 1$ on $(-k,k)$. Using this we write

$$\begin{aligned}
\kappa_s &= \int_{\mathbb{R}} \widetilde{\zeta}_k(r)h'(r)(-\Delta)^s h(r)\,dr + \int_{\mathbb{R}} \zeta(r/k)h'(r)(-\Delta)^s h(r)\,dr\\
&= -\int_{\mathbb{R}} \widetilde{\zeta}_k(r)h(r)(-\Delta)^s h'(r)\,dr - \int_{\mathbb{R}} \widetilde{\zeta}_k'(r)h(r)(-\Delta)^s h(r)\,dr + \int_{\mathbb{R}} \zeta(r/k)h'(r)(-\Delta)^s h(r)\,dr\\
&= -\int_{\mathbb{R}} h'(r)(-\Delta)^s(\widetilde{\zeta}_k h)(r)\,dr - \int_{\mathbb{R}} \widetilde{\zeta}_k'(r)h(r)(-\Delta)^s h(r)\,dr + \int_{\mathbb{R}} \zeta(r/k)h'(r)(-\Delta)^s h(r)\,dr
\end{aligned} \tag{1.7.1}$$

$$\begin{aligned}
&= -\int_{\mathbb{R}} \widetilde{\zeta}_k(r)h'(r)(-\Delta)^s h(r)\,dr - \int_{\mathbb{R}} h'(r)h(r)(-\Delta)^s\widetilde{\zeta}_k(r)\,dr + \int_{\mathbb{R}} h'(r)I(\widetilde{\zeta}_k,h)(r)\,dr\\
&\quad -\int_{\mathbb{R}} \widetilde{\zeta}_k'(r)h(r)(-\Delta)^s h(r)\,dr + \int_{\mathbb{R}} \zeta(r/k)h'(r)(-\Delta)^s h(r)\,dr.
\end{aligned}$$

Here we used that

$$\int_{\mathbb{R}} \widetilde{\varsigma}_k(r)h'(r)(-\Delta)^s h(r)\,dr = -\int_{\mathbb{R}} \widetilde{\varsigma}_k(r)h(r)(-\Delta)^s h'(r)\,dr - \int_{\mathbb{R}} \widetilde{\varsigma}_k'(r)h(r)(-\Delta)^s h(r)\,dr.$$

This can easily be seen by integration by parts and using that $\partial_r\circ(-\Delta)^s = (-\Delta)^s\circ\partial_r$ when acted on $C_c^\infty(\mathbb{R})$ functions. We observe that $\lim_{k\to\infty}\int_{\mathbb{R}}\widetilde{\zeta}_k(r)h'(r)(-\Delta)^s h(r)\,dr = \kappa_s$. On the other hand since $\widetilde{\zeta}_k(r) = \zeta(r/k) = 0$ for all $r\in(-k,k)$ we can show that

$$\lim_{k\to\infty}\int_k^\infty \widetilde{\zeta}_k'(r)h(r)(-\Delta)^s h(r)\,dr = -\lim_{k\to\infty}\int_k^\infty \widetilde{\zeta}_k'(r)h(r)(-\Delta)^s(r_+^s\rho)(r)\,dr = 0$$

and

$$\lim_{k\to\infty}\int_k^\infty \zeta(r/k)h'(r)(-\Delta)^s h(r)\,dr = -\lim_{k\to\infty}\int_k^\infty \zeta(r/k)h'(r)(-\Delta)^s(r_+^s\rho)(r)\,dr = 0,$$

where we used that $(-\Delta)^s r_+^s = 0$ on $(0,\infty)$. From these and (1.7.1), we get

$$2\kappa_s = -\lim_{k\to\infty}\int_{\mathbb{R}} h(r)h'(r)(-\Delta)^s\widetilde{\zeta}_k(r)\,dr + \lim_{k\to\infty}\int_{\mathbb{R}} h'(r)I(\widetilde{\zeta}_k,h)(r)\,dr =: I_1 + I_2.$$

By rescaling, we get

$$\int_{\mathbb{R}} h(r)h'(r)(-\Delta)^s\widetilde{\zeta}_k(r)\,dr = \int_{\mathbb{R}} k^{1-2s}h(kr)h'(kr)(-\Delta)^s\rho(r)\,dr$$

Since for k large enough, $\rho'(kr)=0$, By using the expression of h and that $\rho \in C_c^\infty(\mathbb{R})$, we bound

$$\left|k^{1-2s}h(kr)h'(kr)(-\Delta)^s\rho(r)\right| \le C(\rho)\frac{s r_+^{2s-1}}{1+|r|^{2s+1}}, \quad \text{for some } C(\rho)>0.$$

and by dominated convergence theorem, we get

$$I_1 = -\int_{\mathbb{R}} s\, r_+^{2s-1}(-\Delta)^s\rho(r)\,dr$$

Similarly, we have

$$\int_{\mathbb{R}} h'(r)I(\widetilde{\zeta}_k,h)(r)\,dr = \int_{\mathbb{R}^2} k^{1-2s}h'(kr)\frac{(\rho(r)-\rho(\widetilde{r}))(h(kr)-h(k\widetilde{r}))}{|r-\widetilde{r}|^{1+2s}}$$

By a similar argument as above, we bound

$$\left|k^{1-2s}h'(kr)\frac{(\rho(r)-\rho(\widetilde{r}))(h(kr)-h(k\widetilde{r}))}{|r-\widetilde{r}|^{1+2s}}\right| \le C(\rho)r_+^{s-1}\frac{|\rho(r)-\rho(\widetilde{r})||r_+^s-\widetilde{r}_+^s|}{|r-\widetilde{r}|^{1+2s}},$$

for some $C(\rho)>0$. Applying dominated convergence theorem, we get

$$I_2 = \int_{\mathbb{R}^2} r_+^{s-1}\frac{(\rho(r)-\rho(\widetilde{r}))(r_+^s-\widetilde{r}_+^s)}{|r-\widetilde{r}|^{1+2s}}\,dr\,d\widetilde{r}$$

Adding I_1 and I_2, we get

$$2\kappa_s = -s\int_{\mathbb{R}} r_+^{s-1}(-\Delta)^s(r_+^s\zeta)\,dr.$$

□

In the second step we show that the constant κ_s does not depend on the cut-off function ρ. This is the contain of the following lemma

Lemma 1.7.3. *We have*

$$2\kappa_s = s\int_{\mathbb{R}} r_+^{s-1}(-\Delta)^s(r_+^s\rho)\,dr = sb_{1,s}p.v\int_0^\infty \frac{u^s\log u}{|1-u|^{1+2s}}\,du.$$

Proof. The identity above is obtained by using the principal value definition of $(-\Delta)^s$ combined with some elementary tools such as fundamental theorem of calculus and the Fubini theorem. By expanding, and using that, for fix $r>0$,

$$\lim_{\varepsilon\to 0^+}\int_{\{|r-\tilde r|>\varepsilon\}} \tilde r_+^s\frac{\rho(r)-\rho(\tilde r)}{|r-\tilde r|^{1+2s}}d\tilde r = \lim_{\varepsilon\to 0^+}\int_{\{|r-\tilde r|>r\varepsilon\}} \tilde r_+^s\frac{\rho(r)-\rho(\tilde r)}{|r-\tilde r|^{1+2s}}d\tilde r,$$

we obtain

$$\begin{aligned}
&-\int_{\mathbb{R}} r_+^{s-1}(-\Delta)^s(r_+^s\zeta)\,dr = b_{1,s}\int_{-\infty}^{\infty} r_+^{s-1}p.v\int_{-\infty}^{\infty}\tilde r_+^s\frac{\rho(r)-\rho(\tilde r)}{|r-\tilde r|^{1+2s}}d\tilde r\,dr\\
&= b_{1,s}\int_{-\infty}^{\infty} r_+^{s-1}\lim_{\varepsilon\to 0^+}\int_{\{|r-\tilde r|>r\varepsilon\}}\tilde r_+^s\frac{\rho(r)-\rho(\tilde r)}{|r-\tilde r|^{1+2s}}d\tilde r\\
&= b_{1,s}\int_0^\infty r_+^{s-1}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}}\chi_{|r-\tilde r|>r\varepsilon}\frac{\tilde r_+^s}{|r-\tilde r|^{1+2s}}\int_0^1\rho'(\tilde r t+(1-t)r)(r-\tilde r)\,dt\\
&= b_{1,s}\int_0^\infty dr\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}}\frac{d\tilde r}{r}\left(\frac{\tilde r}{r}\right)_+^s\chi_{|1-\tilde r/r|>\varepsilon}\frac{1}{|1-\frac{\tilde r}{r}|^{1+2s}}\int_0^1\rho'\left(r(t\frac{\tilde r}{r}+1-t)\right)(1-\frac{\tilde r}{r})\,dt\\
&= b_{1,s}\int_0^\infty dr\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}}\chi_{|1-u|>\varepsilon}\frac{u_+^s(1-u)}{|1-u|^{1+2s}}\int_0^1\rho'\left(r(tu+1-t)\right)\,dt\,du.
\end{aligned}$$

Now the idea is to take out the limit in the identity above, and use Funibi theorem to arrive at the result. To do so one needs to regularize the integrand. For that we fix δ small so that $1-2\delta>0$ and we consider the function $\varphi\in C_c^\infty(1-2\delta,1+2\delta)$ such that $\varphi\equiv 1$ in $(1-\delta,1+\delta)$. Replacing u_+^s by $u_+^s=(1-\varphi)u_+^s+\varphi u_+^s$ in (**??**) we get

$$\begin{aligned}
-\int_{\mathbb{R}} r_+^{s-1}(-\Delta)^s(r_+^s\zeta)\,dr &= b_{1,s}\int_0^\infty dr\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}}\chi_{|1-u|>\varepsilon}\frac{u_+^s\varphi(u)(1-u)}{|1-u|^{1+2s}}\int_0^1\rho'\left(r(tu+1-t)\right)\,dt\,du\\
&+b_{1,s}\int_0^\infty dr\int_{\mathbb{R}}\frac{(1-\varphi(u))u_+^s(1-u)}{|1-u|^{1+2s}}\int_0^1\rho'\left(r(tu+1-t)\right)\,dt\,du := J_1+J_2. \qquad (1.7.2)
\end{aligned}$$

Define $F_r \in C_c^\infty(1-2\delta, 1+2\delta)$ by

$$F_r(u) = \varphi(u)u_+^s(1-u)\int_0^1 \rho'\left(r(tu+1-t)\right)dt,$$

We then have that

$$\int_{\mathbb{R}} \chi_{|1-u|>\varepsilon}\frac{\varphi u_+^s(1-u)}{|1-u|^{1+2s}}\int_0^1 \rho'\left(r(tu+1-t)\right)dt = -\int_{\{|1-u|>\varepsilon\}} \frac{F_r(1)-F_r(u)}{|1-u|^{N+2s}}\,du$$

Since F_r has compact support, we have (see e.g [46, Lemma 2.1],)

$$\left|\int_{\{|x-y|>\varepsilon\}} \frac{F_r(x)-F_r(y)}{|x-y|^{N+2s}}\,dy\right| \le \frac{C\|F_r\|_{C^2(\mathbb{R}^N)}}{1+|x|^{N+2s}}, \quad \text{whenever } F \in C_c^\infty(\Omega).$$

By the dominated convergence theorem

$$\begin{aligned}
&\int_0^\infty dr \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}} \chi_{|1-u|>\varepsilon}\frac{\varphi u_+^s(1-u)}{|1-u|^{1+2s}}\int_0^1 \rho'\left(r(tu+1-t)\right)dt\,du\\
&= \lim_{\varepsilon\to 0^+}\int_0^\infty dr\int_{\mathbb{R}} \chi_{|1-u|>\varepsilon}\frac{\varphi u_+^s(1-u)}{|1-u|^{1+2s}}\int_0^1 \rho'\left(r(tu+1-t)\right)dt\,du.
\end{aligned}$$

Consequently, by using Fubuni's theorem, we obtain

$$\begin{aligned}
2\kappa_s &= sb_{1,s}\lim_{\varepsilon\to 0^+}\int_0^\infty dr\int_{\mathbb{R}} \chi_{|1-u|>\varepsilon}\frac{\varphi u_+^s(1-u)}{|1-u|^{1+2s}}\int_0^1 \rho'\left(r(tu+1-t)\right)dt\,du\\
&+ sb_{1,s}\int_0^\infty dr\int_{\mathbb{R}} \frac{(1-\varphi)u_+^s(1-u)}{|1-u|^{1+2s}}\int_0^1 \rho'\left(r(tu+1-t)\right)dt\,du\\
&= sb_{1,s}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}} \chi_{|1-u|>\varepsilon}\frac{\varphi u_+^s(1-u)}{|1-u|^{1+2s}}\int_0^1\int_0^\infty dr\,\rho'\left(r(tu+1-t)\right)dt\,du\\
&+ sb_{1,s}\int_{\mathbb{R}} \frac{(1-\varphi)u_+^s(1-u)}{|1-u|^{1+2s}}\int_0^1\int_0^\infty dr\,\rho'\left(r(tu+1-t)\right)dt\,du\\
&= -sb_{1,s}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}} \chi_{|1-u|>\varepsilon}\frac{\varphi u_+^s(1-u)}{|1-u|^{1+2s}}\int_0^1 \frac{dt}{tu+1-t} - sb_{1,s}\int_{\mathbb{R}} \frac{(1-\varphi)u_+^s(1-u)}{|1-u|^{1+2s}}\int_0^1 \frac{dt}{tu+1-t}\,du\\
&= -sb_{1,s}\lim_{\varepsilon\to 0^+}\int_0^\infty \chi_{|1-u|>\varepsilon}\frac{\varphi u^s(1-u)}{|1-u|^{1+2s}}\frac{\log u}{u-1}\,du - sb_{1,s}\int_0^\infty \frac{(1-\varphi)u^s(1-u)}{|1-u|^{1+2s}}\frac{\log u}{u-1}du
\end{aligned}$$

$$= sb_{1,s} \lim_{\varepsilon \to 0^+} \int_0^\infty \chi_{|1-u|>\varepsilon} \frac{u^s \log u}{|1-u|^{1+2s}}\, du = sb_{1,s} p.v \int_0^\infty \frac{u^s \log u}{|1-u|^{1+2s}}\, du.$$

□

Before we continue, let us make the following observation. Recall that the function $f : \mathbb{R} \to (0,\infty)$, $r \mapsto r_+^s$ is s-harmonic in $(0,\infty)$, i.e,

$$(-\Delta)^s r_+^s = 0 \quad \text{in} \quad (0,+\infty) \tag{1.7.3}$$

Differentiating, at least formally, (1.7.3) in s gives

$$[\partial_s(-\Delta)^s] r_+^s + (-\Delta)^s [\partial_s(\cdot)_+^s](r) = 0 \quad \text{in} \quad (0,\infty) \tag{1.7.4}$$

Evaluating (1.7.4) at 1 we obtain

$$b_{1,s} p.v \int_0^\infty \frac{u^s \log u}{|1-u|^{1+2s}}\, du = \Big([\partial_s(-\Delta)^s] r_+^s \Big)(1).$$

Thus, if one can give sense to the differentiation formula in (1.7.4), one can deduce the value of k_s by evaluating the identity at 1. In fact we only need to give sense of (1.7.4) in the neighbourhood of 1. This will be the aim of the following lines. Let us firs recall the following zero order nonlocal operator introduced in [23]. Let $u \in C_c^\alpha(\mathbb{R}^N)$ with $\alpha > 0$. It has been proved in [23] that the map $s \mapsto (-\Delta)^s u \in L^p(\mathbb{R}^N)$ is differentiable at 0 for all $p \in (1,\infty]$ and that the derivative called the Logarithmic Laplacian and denoted by L_Δ is given, for all $x \in \mathbb{R}^N$, by

$$L_\Delta u(x) = b_N \int_{B_1(x)} \frac{u(x)-u(y)}{|x-y|^N}\, dy - b_N \int_{\mathbb{R}^N \setminus B_1(x)} \frac{u(y)}{|x-y|^N}\, dy + \rho_N u(x), \tag{1.7.5}$$

with

$$b_N = \pi^{-N/2}\Gamma(N/2) \quad \text{and} \quad \rho_N = 2\log 2 + \frac{\Gamma'(N/2)}{\Gamma(N/2)} + \Gamma'(1).$$

To make sense of the differentiation formula (1.7.4), we regularised the function $f(r) = r_+^s$ by mean of the cut-off $\Psi_k \in C_c^\infty\big((-\frac{1}{k}, -2k) \cup (\frac{1}{k}, 2k)\big)$ defined by

$$\Psi_k(r) = \varsigma_k(r)\widetilde{\rho}_k(r), \tag{1.7.6}$$

where $\widetilde{\rho}_k(r) = \rho(r/k)$. The starting point is to established the following

$$(-\Delta)^s \big(\partial_s[(\cdot)_+^s]\big) + \big[\big(-\Delta\big)^s \circ L_\Delta\big] (\cdot)_+^s = 0 \qquad \text{in} \qquad \mathscr{D}'(1/2,2). \tag{1.7.7}$$

To see (1.7.7) one notices that since $(\cdot)_+^s \Psi_k \in C_c^\infty(\mathbb{R}_+)$ and $\partial_s(-\Delta)^s = L_\Delta \circ (-\Delta)^s$, when acted on smooth enough functions, then

$$(-\Delta)^s(\partial_s[\Psi_k(\cdot)_+^s])(x) = \partial_s[(-\Delta)^s \Psi_k(\cdot)_+^s](x) - L_\Delta \circ (-\Delta)^s[\Psi_k(\cdot)_+^s](x) \quad \forall (s,x) \in (0,1)\times\mathbb{R} \tag{1.7.8}$$

Here L_Δ is given by (1.7.5). Next, let $\chi \in C_c^\infty(0,1)$ and $\varphi \in C_c^\infty(1/2,2)$. Define $g = \chi\varphi \in C_c^\infty((0,1)\times\mathbb{R}_+)$. We multiply (1.7.8) by g and integrate by parts over $(0,1)\times\mathbb{R}_+$ to get

$$\int_{(0,1)\times\mathbb{R}} (\partial_s[\Psi_k(\cdot)_+^s])(x)\chi(s)(-\Delta)^s\varphi(x),dxds$$

$$= -\int_{(0,1)\times\mathbb{R}} [\Psi_k(\cdot)_+^s](x)\chi'(s)(-\Delta)^s\varphi(x)\,dxds - \int_{(0,1)\times\mathbb{R}} \left(\Psi_k(\cdot)_+^s\right)(x)\chi(s)(-\Delta)^s\circ L_\Delta\varphi(x)\,dxds.$$

We have the following estimates

$$|(-\Delta)^s\varphi(x)| \le \frac{b_{1,s}}{s(1-s)}\frac{C(\varphi)}{1+|x|^{1+2s}} \qquad \text{and} \qquad |L_\Delta\varphi(x)| \le \frac{C(\varphi)}{1+|x|}. \tag{1.7.9}$$

Since $(-\Delta)^s(\cdot)_+^s = 0$ in $\mathbb{R}_+$, we have

$$\lim_{k\to\infty}\int_{(0,1)\times\mathbb{R}_+} \Psi_k(x)x_+^s(x)\chi'(s)(-\Delta)^s\varphi(x)\,dxds = \int_0^1 \chi'(s)\int_{\mathbb{R}} x_+^s(-\Delta)^s\varphi(x)dxds = 0. \tag{1.7.10}$$

Using the definition of Ψ_k and the dominated convergence theorem, we easily find that

$$\int_0^1 \chi(s)\left[\int_{\mathbb{R}} (\partial_s[(\cdot)_+^s])(x)(-\Delta)^s\varphi(x)dx\right]ds = -\int_0^1 \chi(s)\left[\int_{\mathbb{R}} x_+^s(-\Delta)^s\circ L_\Delta\varphi(x)\,dx\right]ds. \tag{1.7.11}$$

Since the maps $s\mapsto \int_{\mathbb{R}}(\partial_s[(\cdot)_+^s])(x)(-\Delta)^s\varphi(x)dx$ and $s\mapsto \int_{\mathbb{R}} x_+^s(-\Delta)^s\circ L_\Delta\varphi(x)\,dx$ are locally integrable on $(0,1)$ (even continuous!), we have

$$\int_{\mathbb{R}} (\partial_s[(\cdot)_+^s])(x)(-\Delta)^s\varphi(x)dx = -\int_{\mathbb{R}} x_+^s(-\Delta)^s\circ L_\Delta\varphi(x)\,dx, \tag{1.7.12}$$

as wanted. In the next lemma we improve the identity (1.7.7) to

$$(-\Delta)^s\left(\partial_s[(\cdot)_+^s]\right) + d_sL_\Delta\left(1_{(-\infty,0)}(\cdot)|\cdot|^{-s}\right) = 0 \quad \text{in} \quad \mathscr{D}'(1/2,2), \tag{1.7.13}$$

with $d_s := b_{1,s}\int_0^\infty \frac{y^s}{(1+y)^{1+2s}}\,dy$. Equivalently, we shall proved that

$$[(-\Delta)^s\circ L_\Delta](\cdot)_+^s = d_sL_\Delta\left(1_{(-\infty,0)}(\cdot)|\cdot|^{-s}\right) \quad \text{in} \quad \mathscr{D}'(1/2,2)$$

This is done by an integration by parts and using the identity (see e.g [16, Theorem 2.1.10])

$$(-\Delta)^s(\cdot)_+^s = d_s 1_{(-\infty,0)}(\cdot)|\cdot|^{-s} \quad \text{in} \quad \mathbb{R} \tag{1.7.14}$$

We now prove the

Lemma 1.7.4. *Let* Ψ_k *be given by* (1.7.6). *Then we have*

$$\int_{\mathbb{R}} x_+^s (-\Delta)^s \circ L_\Delta \varphi(x)\, dx = d_s \int_{\mathbb{R}} 1_{(-\infty,0)}(x)|x|^{-s} L_\Delta \varphi(x)\, dx, \tag{1.7.15}$$

where $d_s := b_{1,s} \int_0^\infty \frac{y^s}{(1+y)^{1+2s}}\, dy$.

Proof. We note that

$$\begin{aligned}\int_{\mathbb{R}} x_+^s (-\Delta)^s \circ L_\Delta \varphi(x)\, dx &= \lim_{k\to\infty} \int_{\mathbb{R}} \Psi_k(x) x_+^s (-\Delta)^s \circ L_\Delta \varphi(x)\, dx \\ &= \lim_{k\to\infty} \int_{\mathbb{R}} (-\Delta)^s (\Psi_k(\cdot)_+^s)(x) L_\Delta \varphi(x)\, dx. \end{aligned} \tag{1.7.16}$$

We start by proving the following two estimates, for all $k \geq 2$,

$$|(-\Delta)^s \Psi_k(x)| \leq C\left(\frac{k^{-2s}}{1+|x/k|^{1+2s}} + \frac{k^{2s}}{1+|kx|^{1+2s}} + k^{-2s} 1_{\{|x|<1\}}(x) + |x|^{-1-2s} 1_{\{|x|>k\}}(x)\right) \tag{1.7.17}$$

and

$$|I(\Psi_k, (\cdot)_+^s)(x)| \leq C\left(\frac{k^s}{1+|kx|^{1+s}} + \frac{k^{-s}}{1+|x/k|^{1+s}}\right) \tag{1.7.18}$$

We start with (1.7.17). We have $\Psi_k = \varsigma_k(x)\widetilde{\rho}_k(x)$, where $\widetilde{\rho}_k(x) = \rho(x/k)$. Then

$$(-\Delta)^s \Psi_k(x) = \varsigma_k(x)(-\Delta)^s \widetilde{\rho}_k(x) - \widetilde{\rho}_k(-\Delta)^s \rho_k(x) + I(\rho_k, \widetilde{\rho}_k)(x)$$

By the scaling property of the fractional Laplacian,

$$(-\Delta)^s \widetilde{\rho}_k(x) = k^{-2s}(-\Delta)^s \rho(x/k), \qquad (-\Delta)^s \widetilde{\rho}_k(x) = k^{2s}(-\Delta)^s \rho(kx).$$

Now for $|x| < 1$, we have

$$|I(\rho_k, \widetilde{\rho}_k)(x)| = b_{1,s}\left| \int_{|y|\geq k} \frac{\rho_k(x)(1-\widetilde{\rho}_k(y))}{|x-y|^{1+2s}}\, dy \right| \leq Ck^{-2s}.$$

If $k > |x| > 1$, then

$$I(\rho_k, \widetilde{\rho}_k)(x) = 0.$$

It $|x| \geq k$ then

$$|I(\rho_k, \widetilde{\rho}_k)(x)| = b_{1,s}\left| \int_{|y|\leq 2/k} \frac{\rho_k(y)(\widetilde{\rho}_k(x) - \widetilde{\rho}_k(y))}{|x-y|^{1+2s}}\, dy \right| \leq Ck^{-1}|x|^{-1-2s}.$$

We thus get (1.7.17) from the above estimates and (1.7.9).

Next, we estimate

$$I(\Psi_k,(\cdot)_+^s)(x) = -b_{1,s}\widetilde{\rho}_k(x)\int_{\mathbb{R}} \frac{(\rho_k(x)-\rho_k(x+y)))(x_+^s-(x+y)_+^s)}{|y|^{1+2s}}dy$$
$$+b_{1,s}\int_{\mathbb{R}}\zeta_k(y)\frac{(\widetilde{\rho}_k(x)-\widetilde{\rho}_k(x+y)))(x_+^s-(x+y)_+^s)}{|y|^{1+2s}}dy$$

Hence by change of variable

$$|I(\Psi_k,(\cdot)_+^s)(x)| \le Ck^s\int_{\mathbb{R}} \frac{|\rho(kx)-\rho(kx+z)|}{|z|^{1+s}}\,dz + Ck^{-s}\int_{\mathbb{R}}\frac{|\rho(x/k)-\rho(x/k+z)|}{|z|^{1+s}}\,dz$$
$$\le Ck^s\frac{1}{1+|kx|^{1+s}}+Ck^{-s}\frac{1}{1+|x/k|^{1+s}}.$$

That is (1.7.18).

Next, we put $v = L_\Delta\varphi \in L_0^1(\mathbb{R})$. Then by (1.7.17) we have

$$|\int_{\mathbb{R}} x_+^s(-\Delta)^s\Psi_k(x)v(x)\,dx| \le Ck^{-2s}\int_{\mathbb{R}_+}\frac{x^{s-1}}{1+|x/k|^{1+2s}}\,dx + Ck^{2s}\int_{\mathbb{R}_+}\frac{x^s}{1+|kx|^{1+2s}}\,dx$$
$$+Ck^{-2s}\int_0^1 x^s\,dx + C\int_k^\infty x^{s-2-2s}\,dx$$
$$\le Ck^{-s}\int_{\mathbb{R}_+}\frac{y^{s-1}}{1+|y|^{1+2s}}\,dy + Ck^{s-1}\int_{\mathbb{R}_+}\frac{y^s}{1+|y|^{1+2s}}\,dy$$
$$+Ck^{-2s}+Ck^{-1-s}.$$

Therefore

$$\lim_{k\to\infty}\int_{\mathbb{R}} x_s^+(-\Delta)^s\Psi_k(x)v(x)\,dx = 0. \tag{1.7.19}$$

Next, we estimate, using (1.7.18),

$$\int_{\mathbb{R}} I(\Psi_k,(\cdot)_+^s)(x)v(x)\,dx| \le C\left(\int_{\mathbb{R}}\frac{k^s}{1+|kx|^{1+s}}dx + \int_{\mathbb{R}}\frac{k^{-s}\min(1,|x|^{-1})}{1+|x/k|^{1+s}}dx\right)$$
$$\le C\int_{\mathbb{R}}\frac{k^{s-1}}{1+|y|^{1+s}}\,dy + C\int_{|x|<1}\frac{k^{-s}\min(1,|x|^{-1})}{1+|x/k|^{1+s}}dx + C\int_{1<|x|<k}\frac{k^{-s}\min(1,|x|^{-1})}{1+|x/k|^{1+s}}dx$$
$$+C\int_{k<|x|}\frac{k^{-s}\min(1,|x|^{-1})}{1+|x/k|^{1+s}}dx$$

$$
\begin{aligned}
&\leq C k^{s-1}+C k^{-s}+k^{-s} \int_{1<|x|<k}|x|^{-1} d x+C k^{-s} \int_{1<|y|} \frac{|y|^{-1}}{1+|y|^{1+s}} d y \\
&\leq C k^{s-1}+C k^{-s} \log k .
\end{aligned}
$$

It follows that, for $v=L_\Delta \varphi$,

$$
\lim_{k \to \infty} \int_{\mathbb{R}} I(\Psi_k,(\cdot)_+^s)(x) v(x)\, dx=0. \tag{1.7.20}
$$

Recalling (1.7.14) we obtain

$$
(-\Delta)^s(\Psi_k(\cdot)_+^s)=(\cdot)_+^s(-\Delta)^s \Psi_k+d_s 1_{(-\infty, 0)}|\cdot|^{-s} \Psi_k-I(\Psi_k,(\cdot)_+^s). \quad \text{on} \quad \mathbb{R} \setminus 0 \tag{1.7.21}
$$

In view of this, (1.7.19) and (1.7.20), we can use the dominated convergence theorem to to get

$$
\lim_{k \to \infty} \int_{\mathbb{R}}(-\Delta)^s(\Psi_k(\cdot)_+^s)(x) L_\Delta \varphi(x)\, d x=d_s \int_{\mathbb{R}} 1_{(-\infty, 0)}(x)|x|^{-s} L_\Delta \varphi(x) d x .
$$

Using this in (1.7.16) we get the result. □

We finally prove the following

Lemma 1.7.5.

$$
2 \kappa_s=\Gamma^2(1+s)
$$

Proof. Now we are going to compute the integral above. Thanks to Lemma 1.7.4 and 1.7.12

$$
\int_{\mathbb{R}}(\partial_s[(\cdot)_+^s])(-\Delta)^s \varphi(x) d x=d_s \int_{\mathbb{R}} 1_{(-\infty, 0)}(x)|x|^{-s} L_\Delta \varphi(x) d x \qquad \forall \varphi \in C_c^\infty(1/2,2).
$$

Since $\partial_s(\cdot)_+^s \in C^2(1/2,2) \cap L_s^1(\mathbb{R})$ and $1_{(-\infty, 0)}|\cdot|^{-s} \in C^2(1/2,2) \cap L_s^1(\mathbb{R})$, we have that

$$
(-\Delta)^s(\partial_s(\cdot)_+^s)(x)=d_s L_\Delta\left(1_{(-\infty, 0)}|\cdot|^{-s}\right)(x) \qquad \text{for all } x \in(1/2,2). \tag{1.7.22}
$$

In particular, we can evaluate at 1 and use (1.7.5) to obtain

$$
\begin{aligned}
(-\Delta)^s(\partial_s(\cdot)_+^s)(1) &=-p.v s b_{1, s} \int_0^{\infty} \frac{y^s \log y}{|1-y|^{1+2 s}} d y \\
&=-d_s \int_0^{\infty} \frac{y^{-s}}{1+y} d y \\
&=-b_{1, s} \int_0^{\infty} \frac{y^s}{(1+y)^{1+2 s}} d y \int_0^{\infty} \frac{y^{-s}}{1+y} d y .
\end{aligned}
$$

After rescaling and using the formula for beta function, see e.g [101], we get

$$\begin{aligned}
-2\kappa_s &= -sb_{1,s}p.v\int_0^\infty \frac{y^s \log y}{|1-y|^{1+2t}}\,dy = -sb_{1,s}\int_0^1 (1-y)^s y^{s-1}\,dy \int_0^1 (1-y)^s y^{-1-s}\,dy \\
&= -s^2(1-s)\pi^{-1/2}4^s \frac{\Gamma(\frac{1}{2}+s)}{\Gamma(2-s)}\frac{\Gamma(1+s)\Gamma(s)}{\Gamma(1+2s)}\Gamma(1+s)\Gamma(-s) \\
&= -\Gamma^2(1+s).
\end{aligned}$$

□

Chapter 2

Symmetry of odd solutions to equations with the fractional Laplacian

This chapter is devoted to the paper [$P2$], a joint work with S. Jarohs. The exposition is as in the original paper. The main finding of the paper is a new maximum priciple for doubly antisymmetric functions and a corresponding Hopf type lemma. The result is used to obtain new symmetry results for odd solutions to equations with the fractional Laplacian. It is also used in Chapter 4 to study optimal obstacle placement problem for the second fractional eigenvalue.

2.1 Introduction

In the following, we study symmetries of odd solutions to the nonlinear problem

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{2.1.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open set, $f \in C(\Omega \times \mathbb{R})$, and $(-\Delta)^s$, $s \in (0,1)$ is the fractional Laplacian given for $\varphi \in C_c^\infty(\mathbb{R}^N)$ by

$$(-\Delta)^s \varphi(x) = \frac{b_{N,s}}{2} \int_{\mathbb{R}^N} \frac{2\varphi(x) - \varphi(x+y) - \varphi(x-y)}{|y|^{N+2s}}\, dy,$$

with a normalization constant $b_{N,s} > 0$.

Symmetries of nonnegative solutions to problem (2.1.1) have been studied in detail by various authors (see [10, 71, 74]), where f satisfies some monotonicity and symmetry in x_1 and Ω is symmetric in x_1. Here, we aim at investigating (2.1.1), where Ω has two perpendicular symmetries and the solution u is odd in one of these directions. For the variational framework, see also [98, 99, 4, 5] and the references in there. To give a precise framework of our statements, we assume the following:

(D) $\Omega \subset \mathbb{R}^N$ with $N \in \mathbb{N}$, $N \geq 2$ is open and bounded and, moreover, convex and symmetric in the directions x_1 and x_N. That is, for every $(x_1, \dots, x_N) \in \Omega$, $t, \tau \in [-1,1]$ we have $(tx_1, x_2 \dots, x_{N-1}, \tau x_N) \in \Omega$.

(F1) $f \in C(\Omega \times \mathbb{R})$ and for every bounded set $K \subset \mathbb{R}$ there is $L = L(K) > 0$ such that

$$\sup_{x\in\Omega} |f(x,u) - f(x,v)| \le L|u-v| \quad \text{for all } u,v \in K.$$

(F2) f is symmetric in x_1 and monotone in $|x_1|$. That is, for every $u \in \mathbb{R}$, $x \in \Omega$, and $t \in [-1,1]$ we have $f(tx_1, x_2, \dots, x_N, u) \ge f(x,u)$.

In this work, we consider *weak* solutions of (2.1.1), i.e., $u \in \mathscr{H}_0^s(\Omega) := \{v \in H^s(\mathbb{R}^N) \,:\, u = 0 \text{ on } \mathbb{R}^N \setminus \Omega\}$ is called a (weak) solution of (2.1.1), if

$$\mathscr{E}_s(u,v) = \int_\Omega f(x,u(x))v(x)\,dx \quad \text{for all } v \in \mathscr{H}_0^s(\Omega),$$

whenever the right-hand side exists, where

$$\mathscr{E}_s(u,v) = \frac{b_{N,s}}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy \tag{2.1.2}$$

is the bilinearform associated to $(-\Delta)^s$. Here, $H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) \,:\, \mathscr{E}_s(u,u) < \infty\}$ is the usual fractional Hilbert space of order s (see e.g. [4, 5, 17]).
Denote $e_i := (\delta_{ij})_{1\le j\le N} \in \mathbb{R}^N$, where $\delta_{ij} = 1$ if $j = i$ and 0 otherwise is the usual Kronecker Delta. Moreover, for $\lambda \in \mathbb{R}$, consider the halfspace

$$H_{i,\lambda} := \{x \in \mathbb{R}^N \,:\, x \cdot e_i > \lambda\} = \{x \in \mathbb{R}^N \,:\, x_i > \lambda\} \tag{2.1.3}$$

and denote by

$$r_{i,\lambda} : \mathbb{R}^N \to \mathbb{R}^N, \quad r_{i,\lambda}(x) = 2(\lambda - x\cdot e_i)e_i + x \tag{2.1.4}$$

the reflection of x at $\partial H_{i,\lambda}(\lambda)$. Note that $r_{1,0}(\Omega) = r_{N,0}(\Omega) = \Omega$, if assumption (D) is satisfied. We call $u : \mathbb{R}^N \to \mathbb{R}$ *symmetric with respect to* $H_{i,\lambda}$, if $u \circ r_{i,\lambda} = u$ and we call u *antisymmetric with respect to* $H_{i,\lambda}$, if $u \circ r_{i,\lambda} = -u$.

Theorem 2.1.1. *Let $\Omega \subset \mathbb{R}^N$ satisfy (D), $f \in C(\Omega \times \mathbb{R})$ satisfy (F1) and (F2), and let $u \in \mathscr{H}_0^s(\Omega)$ be a continuous bounded solution of* (2.1.1)*, which is antisymmetric with respect to $H_{N,0}$ and $u \ge 0$ in $H_{N,0} \cap \Omega$. Then u is symmetric with respect to $H_{1,0}$ and either $u \equiv 0$ in Ω or $u|_{\Omega\cap H_{1,0}\cap H_{N,0}}$ is strictly decreasing in x_1, that is, for every $x,y \in \Omega \cap H_{1,0} \cap H_{N,0}$ with $x_1 < y_1$ we have $u(x) > u(y)$.*

We note that Theorem 2.1.1 is not surprising in the local case, where $(-\Delta)^s$ is considered with $s = 1$, if we have $u > 0$ in $H_{N,0} \cap \Omega$. In this case, the conclusion follows by simply considering the solution restricted to its part of nonnegativity and apply the usual symmetry result due to [54]. We emphasize however, that if this positivity assumption is reduced to a nonnegativity assumption, then in general the claimed monotonicity is not true in the local case and presents a purely nonlocal feature. Moreover, in the nonlocal setting, we are not able to simply restrict the solution to its set of nonnegativity. Due to this, we present in Section 2.2 below new maximum principles for *doubly antisymmetric* functions to certain linear problems, which we believe are of independent interest.

Let us emphasize that if $u \in L^\infty(\mathbb{R}^N)$ is antisymmetric with respect to $H_{N,0}$, it follows that for any $x \in H_{N,0}$, such that u is regular enough at x, we have with a change of variables

$$(-\Delta)^s u(x) = b_{N,s} \lim_{\varepsilon \to 0^+} \int_{H_{N,0} \setminus B_\varepsilon(x)} (u(x)-u(y)) \Big(\frac{1}{|x-y|^{N+2s}} - \frac{1}{|x-r_{N,0}(y)|^{N+2s}} \Big)\, dy = (-\Delta|_{H_{N,0}})^s u(x),$$

where $(-\Delta|_{H_{N,0}})^s$ denotes the so-called spectral fractional Laplacian (c.f. [18] for $s = 1/2$). In particular, this *difference of the kernel function* does not meet the assumptions needed to conclude the symmetry result by a restriction to $H_{N,0}$ and applying statements of [74].
In the particular case, where $\Omega = B_1(0)$ is the unitary ball, it was shown in [41] that the second eigenfunction of the fractional Laplacian in $B_1(0)$, denoted by φ_2, is odd and can be chosen to be positive in $\{x_N > 0\}$. Due to the regularity of φ_2, Theorem 2.1.1 yields that for $i = 1, \dots, N-1$ we have

i. φ_2 is symmetric with respect to $H_{i,0}$ (see also [70]) and

ii. $\varphi_2|_{\{x_1>0\}}$ is decreasing in $x_i > 0$.

We emphasize that such a statement already follows due to [41] combined with [38], since thus the second eigenfunction can be written as a product of the first eigenfunction with a homogeneous function.

To give a more generalized application of our results to a class of nonlinear problems, we consider for $1 < p < \frac{2N}{N-2s}$ the minimization problem

$$\lambda_{1,p}(\Omega) := \min_{\substack{u \in \mathscr{H}_0^s(\Omega) \\ u \neq 0}} \frac{\mathscr{E}_s(u,u)}{\Big(\int_\Omega |u(x)|^p \, dx\Big)^{2/p}}. \tag{2.1.5}$$

Clearly, the minimizer exists and is a solution of (2.1.1) with $f(x,u) = |u|^{p-2}u$ (see e.g. [98, 99]) and, since $\mathscr{E}_s(|u|,|u|) \leq \mathscr{E}_s(u,u)$ it can be chosen to be positive. For more information about the minimization problem (2.1.5) we refer to [82]. In the local case $s = 1$, this is a well known problem, see e.g. [49, 76]. If Ω satisfies (D), then it follows that this minimizer is also symmetric with respect to the symmetries of Ω (see [71]). In this case, we can also consider the minimizer in the set of $\mathscr{H}_0^s(\Omega)$-functions, which satisfy $u = -u \circ r_{N,0}$, that is

$$\lambda_{1,p}^-(\Omega) := \min_{\substack{u \in \mathscr{H}_0^s(\Omega) \\ u \neq 0 \\ u = -u \circ r_{N,0}}} \frac{\mathscr{E}_s(u,u)}{\Big(\int_\Omega |u(x)|^p \, dx\Big)^{2/p}}. \tag{2.1.6}$$

In the next theorem, we prove that minimizers of (2.1.6) have constant sign in $\Omega \cap H_{N,0}$ and in the particular case $p = 2$ we also prove a simplicity result for $\lambda_{1,p}^-(\Omega)$.

Theorem 2.1.2. *Let $1 < p < \frac{2N}{N-2s}$ with $N \geq 2$ and let $\Omega \subset \mathbb{R}^N$ satisfy (D) with $\partial\Omega$ of class $C^{1,1}$. Then there is a nontrivial solution $u \in \mathscr{H}_0^s(\Omega)$ of*

$$\begin{cases} (-\Delta)^s u = \lambda_{1,p}^-(\Omega)|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{2.1.7}$$

which is continuous, bounded, and antisymmetric with respect to $H_{N,0}$. Moreover, u is of one sign in $\Omega \cap H_{N,0}$ and hence it is symmetric with respect to $H_{1,0}$ and $u|_{\Omega\cap H_{1,0}\cap H_{N,0}}$ is strictly decreasing in x_1. In particular, u can be chosen to be positive in $H_N \cap \Omega$. Furthermore, if $p = 2$, then the minimizer is unique up to a sign.

The existence, as mentioned above, follows immediately from a minimization problem. Moreover, by the known regularity theory it follows that indeed we have $u \in C^\infty(\Omega) \cap C^s(\mathbb{R}^N)$, see e.g. [94, 57]. We show here that this minimizer can actually be chosen to be nonnegative in $\Omega \cap H_N$ and thus the conclusion follows from Theorem 2.1.1.

The Chapter is organized as follows. In Section 2.2 we give the framework for supersolutions and maximum principles used later on. Section 2.3 is devoted to prove Theorem 2.1.1 and in Section 2.4 we show Theorem 2.1.2.

Notation The following notation is used. For subsets $D, U \subset \mathbb{R}^N$ we write $\text{dist}(D,U) := \inf\{|x-y| \,:\, x \in D,\, y \in U\}$. If $D = \{x\}$ is a singleton, we write $\text{dist}(x,U)$ in place of $\text{dist}(x,U)$. For $U \subset \mathbb{R}^N$ and $r > 0$ we consider $B_r(U) := \{x \in \mathbb{R}^N \,:\, \text{dist}(x,U) < r\}$, and we let, as usual $B_r(x) = B_r(\{x\})$ be the open ball in $\mathbb{R}^N$ centered at $x \in \mathbb{R}^N$ with radius $r > 0$. For any subset $M \subset \mathbb{R}^N$, we denote by $1_M : \mathbb{R}^N \to \mathbb{R}$ the characteristic function of M and by $\text{diam}(M)$ the diameter of M. If M is measurable, $|M|$ denotes the Lebesgue measure of M. Moreover, if $w : M \to \mathbb{R}$ is a function, we let $w^+ = \max\{w, 0\}$ resp. $w^- = -\min\{w, 0\}$ denote the positive and negative part of w, respectively, so that $w = w^+ - w^-$. Finally, $H_{i,\lambda}$ and $r_{i,\lambda}$ are as defined in (2.1.3) and resp. (2.1.4) for $i \in \{1, \ldots, N\}$ and $\lambda \in \mathbb{R}$. Finally, $\Omega \subset \mathbb{R}^N$ is always an open set satisfying (D).

2.2 A linear problem

For this section, we fix $\lambda, \mu \in \mathbb{R}$ and denote $H_1 := H_{1,\mu}$ and $H_2 := H_{N,\lambda}$. Similarly, $r_1 := r_{1,\mu}$ and $r_2 := r_{N,\lambda}$. We call $w : \mathbb{R}^N \to \mathbb{R}$ *doubly antisymmetric (with respect to H_1 and H_2)*, if

$$w \circ r_i = -w \quad \text{in } \mathbb{R}^N \text{ for } i = 1, 2.$$

Moreover, if $U \subset \mathbb{R}^N$ is open, we let

$$\mathscr{V}^s(U) = \Big\{ u \in L^2_{loc}(\mathbb{R}^N) \,:\, \rho_s(w,U) := \int_U \int_{\mathbb{R}^N} \frac{(w(x)-w(y))^2}{|x-y|^{N+2s}}\, dxdy < \infty \Big\}.$$

Note that clearly $\mathscr{H}_0^s(U) \subset H^s(\mathbb{R}^N) \subset \mathscr{V}^s(\mathbb{R}^N) \subset \mathscr{V}^s(U)$. In the following Lemma we collect some statements corresponding to the space $\mathscr{V}^s(U)$. The proofs can be found e.g. in [74, 73, 72].

Lemma 2.2.1. *Let $U \subset \mathbb{R}^N$ open and bounded. Then the following hold.*

i. *$\mathscr{E}_s$ is well defined on $\mathscr{V}^s(U) \times \mathscr{H}_0^s(U)$.*

ii. *If $w \in \mathscr{V}^s(U)$, then also $w^{\pm}, |w| \in \mathscr{V}^s(U)$. Moreover, if $w \geq 0$ in $\mathbb{R}^N \setminus U$, then $w^- \in \mathscr{H}_0^s(U)$ and we have*

$$\mathscr{E}_s(w^-, w^-) \leq -\mathscr{E}_s(w, w^-).$$

iii. *Let $i = 1$ or $i = 2$ and $U \subset H_i$. If $w \in \mathscr{V}^s(U)$ is antisymmetric in x_i, then $w1_{H_i} \in \mathscr{V}^s(U)$. Moreover, if $w \geq 0$ in $H_i \setminus U$, then $w^- 1_{H_i} \in \mathscr{H}_0^s(U)$ and $\mathscr{E}_s(w^- 1_{H_i}, w^- 1_{H_i}) \leq -\mathscr{E}_s(w, w^- 1_{H_i})$.*

The following Lemma gives an extension of Lemma 2.2.1.iii to the case of doubly antisymmetric functions.

Lemma 2.2.2. *Let $U \subset H_1 \cap H_2$ and $w \in \mathscr{V}^s(U_{1,2})$ be doubly antisymmetric, where $U_{1,2} = U \cup r_1(U) \cup r_2(U) \cup r_1(r_2(U))$. Then $v = w^- 1_{H_1} 1_{H_2} - w^+ 1_{H_1^c} 1_{H_2} \in \mathscr{H}_0^s(U \cup r_1(U))$ and we have*

$$\mathscr{E}_s(w, v) + \mathscr{E}_s(v, v) \leq 0,$$

where equality can only hold if $v \equiv 0$, that is, if $w \geq 0$ in $H_1 \cap H_2$.

Proof. First note that since w is antisymmetric with respect to H_i, $i = 1, 2$, Lemma 2.2.1 and its proof imply $w_i := w1_{H_i} \in \mathscr{V}^s(U \cup r_j(U))$, $i, j = 1, 2$, $i \neq j$ and

$$\rho_s(w_i, U \cup r_j(U)) \leq \rho_s(w, U_{1,2}) \quad \text{and} \quad \mathscr{E}_s(w_i^-, w_i^-) \leq -\mathscr{E}_s(w, w_i^-) \quad \text{for } i, j = 1, 2,\ i \neq j.$$

Similarly, also w_2 is antisymmetric with respect to H_1 (resp. w_1 with respect to H_2) and thus also $w_{1,2} := w_1 1_{H_2} = w_2 1_{H_1} \in \mathscr{V}^s(U)$ with

$$\rho_s(w_{1,2}, U) \leq \min\Big\{\rho_s(w_1, U \cup r_2(U)), \rho_s(w_2, U \cup r_1(U))\Big\}$$

and it holds

$$\mathscr{E}_s(w_{1,2}^-, w_{1,2}^-) \leq -\max\Big\{\mathscr{E}_s(w_1, w_{1,2}^-), \mathscr{E}_s(w_2, w_{1,2}^-)\Big\}.$$

Similarly, we also have $w_{r_1,2} = w_2 1_{H_1^c} \in \mathscr{V}^s(r_1(U))$ with

$$\rho_s(w_{r_1,2}, r_1(U)) \leq \rho_s(w_2, U \cup r_1(U)).$$

It thus follows that $v = w^- 1_{H_1} 1_{H_2} - w^+ 1_{H_1^c} 1_{H_2} = w_{1,2}^- - w_{r_1,2}^+ \in \mathscr{H}_0^s(U \cup r_1(U))$. Using the monotonicity of $|\cdot|$ and the antisymmetry of w and denoting $r_{1,2} := r_1 \circ r_2 = r_2 \circ r_1$ we have by several rearrangements and substitutions

$$\frac{2}{c_{N,s}}\Big(\mathscr{E}_s(w, v) + \mathscr{E}_s(v, v)\Big) = \int_{H_2}\int_{H_2} \frac{[(w+v)(x) - (w+v)(y)][v(x) - v(y)]}{|x-y|^{N+2s}}\, dxdy + \int_{H_2^c}\int_{H_2^c} \ldots + 2\int_{H_2}\int_{H_2^c} \ldots$$

$$= \int_{H_2}\int_{H_2} \frac{[(w+v)(x) - (w+v)(y)][v(x) - v(y)]}{|x-y|^{N+2s}}\, dxdy - 2\int_{H_2}\int_{H_2} \frac{[w(r_2(x)) - (w+v)(y)]v(y)}{|r_2(x) - y|^{N+2s}}\, dxdy$$

$$= \int_{H_2}\int_{H_2} \frac{[(w+v)(x)-(w+v)(y)][v(x)-v(y)]}{|x-y|^{N+2s}}\,dxdy - 2\int_{H_2}\int_{H_2} \frac{[-w(x)-(w+v)(y)]v(y)}{|r_2(x)-y|^{N+2s}}\,dxdy$$

$$= \int_{H_1\cap H_2}\int_{H_1\cap H_2} \frac{[(w+v)(x)-(w+v)(y)][v(x)-v(y)]}{|x-y|^{N+2s}}\,dxdy + \int_{H_2\setminus H_1}\int_{H_2\setminus H_1} \dots + 2\int_{H_2\cap H_1}\int_{H_2\setminus H_1} \dots$$

$$-2\int_{H_2\cap H_1}\int_{H_2\cap H_1} \frac{[-w(x)-(w+v)(y)]v(y)}{|r_2(x)-y|^{N+2s}}\,dxdy - 2\int_{H_2\setminus H_1}\int_{H_2\setminus H_1} \dots - 2\int_{H_2\cap H_1}\int_{H_2\setminus H_1} \dots - 2\int_{H_2\setminus H_1}\int_{H_2\cap H_1} \dots$$

$$= \int_{H_1\cap H_2}\int_{H_1\cap H_2} \frac{[(w+w^-)(x)-(w+w^-)(y)][w^-(x)-w^-(y)]}{|x-y|^{N+2s}}\,dxdy$$

$$- \int_{H_2\setminus H_1}\int_{H_2\setminus H_1} \frac{[(w-w^+)(x)-(w-w^+)(y)][w^+(x)-w^+(y)]}{|x-y|^{N+2s}}\,dxdy$$

$$-2\int_{H_2\cap H_1}\int_{H_2\setminus H_1} \frac{[(w-w^+)(x)-(w+w^-)(y)][w^+(x)+w^-(y)]}{|x-y|^{N+2s}}\,dxdy$$

$$-2\int_{H_2\cap H_1}\int_{H_2\cap H_1} \frac{[-w(x)-(w+w^-)(y)]w^-(y)}{|r_2(x)-y|^{N+2s}}\,dxdy + 2\int_{H_2\setminus H_1}\int_{H_2\setminus H_1} \frac{[-w(x)-(w-w^+)(y)]w^+(y)}{|r_2(x)-y|^{N+2s}}\,dxdy$$

$$+2\int_{H_2\setminus H_1}\int_{H_2\cap H_1} \frac{[-w(x)-(w-w^+)(y)]w^+(y)}{|r_2(x)-y|^{N+2s}}\,dxdy - 2\int_{H_2\cap H_1}\int_{H_2\setminus H_1} \frac{[-w(x)-(w+w^-)(y)]w^-(y)}{|r_2(x)-y|^{N+2s}}\,dxdy$$

$$= \int_{H_1\cap H_2}\int_{H_1\cap H_2} \frac{[w^+(x)-w^+(y)][w^-(x)-w^-(y)]}{|x-y|^{N+2s}}\,dxdy$$

$$- \int_{H_2\setminus H_1}\int_{H_2\setminus H_1} \frac{[-w^-(x)+w^-(y)][w^+(x)-w^+(y)]}{|x-y|^{N+2s}}\,dxdy$$

$$-2\int_{H_2\cap H_1}\int_{H_2\setminus H_1} \frac{[-w^-(x)-w^+(y)][w^+(x)+w^-(y)]}{|x-y|^{N+2s}}\,dxdy$$

$$-2\int_{H_2\cap H_1}\int_{H_2\cap H_1} \frac{[-w(x)-w^+(y)]w^-(y)}{|r_2(x)-y|^{N+2s}}\,dxdy + 2\int_{H_2\setminus H_1}\int_{H_2\setminus H_1} \frac{[-w(x)+w^-(y)]w^+(y)}{|r_2(x)-y|^{N+2s}}\,dxdy$$

$$+2\int_{H_2\setminus H_1}\int_{H_2\cap H_1} \frac{[-w(x)+w^-(y)]w^+(y)}{|r_2(x)-y|^{N+2s}}\,dxdy - 2\int_{H_2\cap H_1}\int_{H_2\setminus H_1} \frac{[-w(x)-w^+(y)]w^-(y)}{|r_2(x)-y|^{N+2s}}\,dxdy$$

$$= -\int_{H_1\cap H_2}\int_{H_1\cap H_2} \frac{w^+(x)w^-(y)+w^+(y)w^-(x)}{|x-y|^{N+2s}}\,dxdy - \int_{H_2\setminus H_1}\int_{H_2\setminus H_1} \frac{w^-(x)w^+(y)+w^-(y)w^+(x)}{|x-y|^{N+2s}}\,dxdy$$

$$+2\int_{H_2\cap H_1}\int_{H_2\setminus H_1} \frac{w^-(x)w^-(y)+w^+(y)w^+(x)}{|x-y|^{N+2s}}\,dxdy$$

$$+2\int_{H_2\cap H_1}\int_{H_2\cap H_1}\frac{w(x)w^-(y)}{|r_2(x)-y|^{N+2s}}\,dxdy-2\int_{H_2\setminus H_1}\int_{H_2\setminus H_1}\frac{w(x)w^+(y)}{|r_2(x)-y|^{N+2s}}\,dxdy$$
$$-2\int_{H_2\setminus H_1}\int_{H_2\cap H_1}\frac{w(x)w^+(y)}{|r_2(x)-y|^{N+2s}}\,dxdy+2\int_{H_2\cap H_1}\int_{H_2\setminus H_1}\frac{w(x)w^-(y)}{|r_2(x)-y|^{N+2s}}\,dxdy$$
$$=-2\int_{H_1\cap H_2}\int_{H_1\cap H_2}\frac{w^+(x)w^-(y)}{|x-y|^{N+2s}}\,dxdy-2\int_{H_2\setminus H_1}\int_{H_2\setminus H_1}\frac{w^-(x)w^+(y)}{|x-y|^{N+2s}}\,dxdy$$
$$+2\int_{H_2\cap H_1}\int_{H_2\setminus H_1}\frac{w^-(x)w^-(y)+w^+(y)w^+(x)}{|x-y|^{N+2s}}\,dxdy$$
$$+2\int_{H_2\cap H_1}\int_{H_2\cap H_1}\frac{w^+(x)w^-(y)-w^-(x)w^-(y)}{|r_2(x)-y|^{N+2s}}\,dxdy-2\int_{H_2\setminus H_1}\int_{H_2\setminus H_1}\frac{w^+(x)w^+(y)-w^-(x)w^+(y)}{|r_2(x)-y|^{N+2s}}\,dxdy$$
$$-2\int_{H_2\setminus H_1}\int_{H_2\cap H_1}\frac{w^+(x)w^+(y)-w^-(x)w^+(y)}{|r_2(x)-y|^{N+2s}}\,dxdy+2\int_{H_2\cap H_1}\int_{H_2\setminus H_1}\frac{w^+(x)w^-(y)-w^-(x)w^-(y)}{|r_2(x)-y|^{N+2s}}\,dxdy$$
$$=-2\int_{H_2\cap H_1}\int_{H_2\cap H_1}\frac{w^-(x)w^-(y)}{|r_2(x)-y|^{N+2s}}\,dxdy-2\int_{H_2\setminus H_1}\int_{H_2\setminus H_1}\frac{w^+(x)w^+(y)}{|r_2(x)-y|^{N+2s}}\,dxdy$$
$$-2\int_{H_2\cap H_1}\int_{H_2\cap H_1}w^+(x)w^-(y)\Big(\frac{1}{|x-y|^{N+2s}}-\frac{1}{|r_2(x)-y|^{N+2s}}\Big)\,dxdy$$
$$-2\int_{H_2\setminus H_1}\int_{H_2\setminus H_1}w^-(x)w^+(y)\Big(\frac{1}{|x-y|^{N+2s}}-\frac{1}{|r_2(x)-y|^{N+2s}}\Big)\,dxdy$$
$$+2\int_{H_2\cap H_1}\int_{H_2\cap H_1}\frac{w^-(r_1(x))w^-(y)+w^+(y)w^+(r_1(x))}{|r_1(x)-y|^{N+2s}}\,dxdy$$
$$-2\int_{H_2\cap H_1}\int_{H_2\cap H_1}\frac{w^+(x)w^+(r_1(y))-w^-(x)w^+(r_1(y))}{|r_2(x)-r_1(y)|^{N+2s}}\,dxdy$$
$$+2\int_{H_2\cap H_1}\int_{H_2\cap H_1}\frac{w^+(r_1(x))w^-(y)-w^-(r_1(x))w^-(y)}{|r_{1,2}(x)-y|^{N+2s}}\,dxdy$$
$$=-2\int_{H_2\cap H_1}\int_{H_2\cap H_1}\frac{w^-(x)w^-(y)}{|r_2(x)-y|^{N+2s}}\,dxdy-2\int_{H_2\setminus H_1}\int_{H_2\setminus H_1}\frac{w^+(x)w^+(y)}{|r_2(x)-y|^{N+2s}}\,dxdy$$
$$-2\int_{H_2\cap H_1}\int_{H_2\cap H_1}w^+(x)w^-(y)\Big(\frac{1}{|x-y|^{N+2s}}-\frac{1}{|r_2(x)-y|^{N+2s}}\Big)\,dxdy$$
$$-2\int_{H_2\setminus H_1}\int_{H_2\setminus H_1}w^-(x)w^+(y)\Big(\frac{1}{|x-y|^{N+2s}}-\frac{1}{|r_2(x)-y|^{N+2s}}\Big)\,dxdy$$

$$
\begin{aligned}
&+2\int\limits_{H_2\cap H_1}\int\limits_{H_2\cap H_1}\frac{w^+(x)w^-(y)+w^+(y)w^-(x)}{|r_1(x)-y|^{N+2s}}\,dxdy\\
&-2\int\limits_{H_2\cap H_1}\int\limits_{H_2\cap H_1}\frac{w^+(x)w^-(y)-w^-(x)w^-(y)}{|r_2(x)-r_1(y)|^{N+2s}}\,dxdy+2\int\limits_{H_2\cap H_1}\int\limits_{H_2\cap H_1}\frac{w^-(x)w^-(y)-w^+(x)w^-(y)}{|r_{1,2}(x)-y|^{N+2s}}\,dxdy\\
=&-2\int\limits_{H_2\cap H_1}\int\limits_{H_2\cap H_1}\frac{w^-(x)w^-(y)}{|r_2(x)-y|^{N+2s}}\,dxdy-2\int\limits_{H_2\cap H_1}\int\limits_{H_2\cap H_1}\frac{w^-(x)w^-(y)}{|r_{1,2}(x)-r_1(y)|^{N+2s}}\,dxdy\\
&-2\int\limits_{H_2\cap H_1}\int\limits_{H_2\cap H_1}w^+(x)w^-(y)\Big(\frac{1}{|x-y|^{N+2s}}-\frac{1}{|r_2(x)-y|^{N+2s}}\Big)\,dxdy\\
&-2\int\limits_{H_2\cap H_1}\int\limits_{H_2\cap H_1}w^+(x)w^-(y)\Big(\frac{1}{|x-y|^{N+2s}}-\frac{1}{|r_{1,2}(x)-r_1(y)|^{N+2s}}\Big)\,dxdy\\
&+2\int\limits_{H_2\cap H_1}\int\limits_{H_2\cap H_1}\frac{w^+(x)w^-(y)+w^+(y)w^-(x)}{|r_1(x)-y|^{N+2s}}\,dxdy\\
&-2\int\limits_{H_2\cap H_1}\int\limits_{H_2\cap H_1}\frac{w^+(x)w^-(y)-w^-(x)w^-(y)}{|r_2(x)-r_1(y)|^{N+2s}}\,dxdy\\
&+2\int\limits_{H_2\cap H_1}\int\limits_{H_2\cap H_1}\frac{w^-(x)w^-(y)-w^+(x)w^-(y)}{|r_{1,2}(x)-y|^{N+2s}}\,dxdy\\
=&-4\int\limits_{H_2\cap H_1}\int\limits_{H_2\cap H_1}\frac{w^-(x)w^-(y)}{|r_2(x)-y|^{N+2s}}\,dxdy-4\int\limits_{H_2\cap H_1}\int\limits_{H_2\cap H_1}w^+(x)w^-(y)\Big(\frac{1}{|x-y|^{N+2s}}-\frac{1}{|r_2(x)-y|^{N+2s}}\Big)\,dxdy\\
&+2\int\limits_{H_2\cap H_1}\int\limits_{H_2\cap H_1}\frac{w^+(x)w^-(y)+w^+(y)w^-(x)}{|r_1(x)-y|^{N+2s}}\,dxdy\\
&-4\int\limits_{H_2\cap H_1}\int\limits_{H_2\cap H_1}\frac{w^+(x)w^-(y)}{|r_{1,2}(x)-y|^{N+2s}}\,dxdy+4\int\limits_{H_2\cap H_1}\int\limits_{H_2\cap H_1}\frac{w^-(x)w^-(y)}{|r_{1,2}(x)-y|^{N+2s}}\,dxdy\\
=&-4\int\limits_{H_2\cap H_1}\int\limits_{H_2\cap H_1}w^-(x)w^-(y)\Big(\frac{1}{|r_2(x)-y|^{N+2s}}-\frac{1}{|r_{1,2}(x)-y|^{N+2s}}\Big)\,dxdy\\
&-4\int\limits_{H_2\cap H_1}\int\limits_{H_2\cap H_1}w^+(x)w^-(y)\Big(\frac{1}{|x-y|^{N+2s}}-\frac{1}{|r_2(x)-y|^{N+2s}}\Big)\,dxdy\\
&+4\int\limits_{H_2\cap H_1}\int\limits_{H_2\cap H_1}\frac{w^+(x)w^-(y)}{|r_1(x)-y|^{N+2s}}\,dxdy-4\int\limits_{H_2\cap H_1}\int\limits_{H_2\cap H_1}\frac{w^+(x)w^-(y)}{|r_{1,2}(x)-y|^{N+2s}}\,dxdy\\
=&-4\int\limits_{H_2\cap H_1}\int\limits_{H_2\cap H_1}w^-(x)w^-(y)\Big(\frac{1}{|r_2(x)-y|^{N+2s}}-\frac{1}{|r_{1,2}(x)-y|^{N+2s}}\Big)\,dxdy\\
&-4\int\limits_{H_2\cap H_1}\int\limits_{H_2\cap H_1}w^+(x)w^-(y)\Big(\frac{1}{|x-y|^{N+2s}}-\frac{1}{|r_2(x)-y|^{N+2s}}-\frac{1}{|r_1(x)-y|^{N+2s}}+\frac{1}{|r_{1,2}(x)-y|^{N+2s}}\Big)\,dxdy.
\end{aligned}
$$

From here the statement of the Lemma follows, once we show the following claim:

$$\textit{Claim:}\quad \frac{1}{|x-y|^{N+2s}}-\frac{1}{|r_2(x)-y|^{N+2s}}-\frac{1}{|r_1(x)-y|^{N+2s}}+\frac{1}{|r_{1,2}(x)-y|^{N+2s}}\geq 0, \qquad (2.2.1)$$

for all $x,y\in H_1\cap H_2$ We write

$$\frac{1}{|x-y|^{N+2s}}-\frac{1}{|r_2(x)-y|^{N+2s}}=\frac{1}{|x-y|^{N+2s}}\left(1-\Big(\frac{|x-y|^2}{|r_2(x)-y|^2}\Big)^{\frac{N}{2}+s}\right)$$

and

$$\frac{1}{|r_1(x)-y|^{N+2s}}-\frac{1}{|r_{1,2}(x)-y|^{N+2s}}=\frac{1}{|r_1(x)-y|^{N+2s}}\left(1-\Big(\frac{|r_1(x)-y|^2}{|r_{1,2}(x)-y|^2}\Big)^{\frac{N}{2}+s}\right).$$

In the following, fix $x,y\in H_1\cap H_2$ and without loss we may assume $e_1=(1,0,\dots,0)$ and $e_2=(0,1,0,\dots,0)$. Indeed, otherwise we may rotate the half spaces and since $\mathscr{E}_s$ is invariant under rotations the situation remains the same. Then with $D:=\sum_{k=2}^{N}(x_k-y_k)^2$

$$\begin{aligned}
|r_1(x)-y|^2&=(x_1+y_1)^2+(x_2-y_2)^2+D=4x_1y_1+(x_1-y_1)^2+(x_2-y_2)^2+D=4x_1y_1+|x-y|^2\\
|r_2(x)-y|^2&=(x_1-y_1)^2+(x_2+y_2)^2+D=4x_2y_2+(x_1-y_1)^2+(x_2-y_2)^2+D=4x_2y_2+|x-y|^2\\
|r_{1,2}(x)-y|^2&=(x_1+y_1)^2+(x_2+y_2)^2+D=4x_1y_1+4x_2y_2+(x_1-y_1)^2+(x_2-y_2)^2+D\\
&=4x_1y_1+4x_2y_2+|x-y|^2
\end{aligned}$$

Thus with $M:=|x-y|^2$ we have

$$\begin{aligned}
&\frac{1}{M^{\frac{N}{2}+s}}-\frac{1}{|r_2(x)-y|^{N+2s}}-\frac{1}{|r_1(x)-y|^{N+2s}}+\frac{1}{|r_{1,2}(x)-y|^{N+2s}}\\
&=\frac{1}{M^{\frac{N}{2}+s}}\left(1-\Big(\frac{M}{|r_2(x)-y|^2}\Big)^{\frac{N}{2}+s}-\Big(\frac{M}{|r_1(x)-y|^2}\Big)^{\frac{N}{2}+s}\left(1-\Big(\frac{|r_1(x)-y|^2}{|r_1(x)-r_2(y)|^2}\Big)^{\frac{N}{2}+s}\right)\right)\\
&=\frac{1}{M^{\frac{N}{2}+s}}\left(1-\Big(\frac{M}{|r_2(x)-y|^2}\Big)^{\frac{N}{2}+s}-\Big(\frac{M}{|r_1(x)-y|^2}\Big)^{\frac{N}{2}+s}+\Big(\frac{M}{|r_{1,2}(x)-y|^2}\Big)^{\frac{N}{2}+s}\right)\\
&=\frac{1}{M^{\frac{N}{2}+s}}\left(1+\Big(\frac{M}{4x_1y_1+4x_2y_2+M}\Big)^{\frac{N}{2}+s}-\Big(\frac{M}{4x_1y_1+M}\Big)^{\frac{N}{2}+s}-\Big(\frac{M}{4x_2y_2+M}\Big)^{\frac{N}{2}+s}\right).
\end{aligned}$$

Using the notation $a=4x_1y_1>0$ and $b=4x_2y_2>0$, we may consider for fixed $M>0$ the map

$$f:[0,\infty)^2\to\mathbb{R},\quad (a,b)\mapsto 1+\Big(\frac{M}{a+b+M}\Big)^{\frac{N}{2}+s}-\Big(\frac{M}{a+M}\Big)^{\frac{N}{2}+s}-\Big(\frac{M}{b+M}\Big)^{\frac{N}{2}+s}.$$

Then (2.2.1) follows once $f\geq 0$. Note that

$$\nabla f(a,b)=-(\frac{N}{2}+s)M^{\frac{N}{2}+s}\begin{pmatrix}\Big(\frac{1}{a+b+M}\Big)^{\frac{N}{2}+1+s}-\Big(\frac{1}{a+M}\Big)^{\frac{N}{2}+1+s}\\ \Big(\frac{1}{a+b+M}\Big)^{\frac{N}{2}+1+s}-\Big(\frac{1}{b+M}\Big)^{\frac{N}{2}+1+s}\end{pmatrix}.$$

Clearly, f has a saddle node at $(0,0)$, but note that for any $(c,d) \in [0,\infty)^2$ we have

$$\nabla f(a,b)\begin{pmatrix} c \\ d \end{pmatrix} > 0,$$

so that f is increasing in any direction (c,d). In particular, since $f(0,0) = 0$, it follows that $f(a,b) \geq 0$ for $a,b \geq 0$. Hence (2.2.1) follows, which implies the assertion of the lemma. □

In view of Lemma 2.2.1 we may define *doubly antisymmetric supersolutions* as follows. Let $U \subset H_1 \cap H_2$ and $c \in L^\infty(U)$. Then $w \in \mathscr{V}^s(U)$ is called a doubly antisymmetric supersolution of

$$\begin{cases} (-\Delta)^s w \geq c(x)w & \text{in } U, \\ \qquad w \geq 0 & \text{in } H_1 \cap H_2 \setminus U, \end{cases} \tag{2.2.2}$$

if w is doubly antisymmetric and satisfies

$$\mathscr{E}_s(w,\varphi) \geq \int_U c(x)w(x)\varphi(x)\,dx \quad \text{for all nonnegative } \varphi \in \mathscr{H}_0^s(U).$$

In the following, for an open set $U \subset H_1 \cap H_2$ let

$$\lambda_1^-(U) := \min_{\substack{u \in \mathscr{H}_0^s(U \cup r_1(U)) \\ u \neq 0 \\ u \circ r_1 \equiv -u}} \frac{\mathscr{E}_s(u,u)}{\|u\|^2_{L^2(U \cup r_1(U))}}.$$

We emphasize that $\lambda_1^-(U) > \lambda_1(U \cup r_1(U))$, where $\lambda_1(D)$ denotes the first Dirichlet eigenvalue of $(-\Delta)^s$ in D. Since (see e.g. [74, Lemma 2.1])

$$\sup_{\substack{D \subset \mathbb{R}^N \text{ open} \\ |D| \leq \delta}} \lambda_1(D) \to \infty \quad \text{as } \delta \to 0,$$

it follows also that

$$\sup_{\substack{U \subset \mathbb{R}^N \text{ open} \\ |U| \leq \delta}} \lambda_1^-(U) \to \infty \quad \text{as } |U| \to 0. \tag{2.2.3}$$

We thus can show the following version of a small volume maximum principle for doubly antisymmetric supersolutions.

Proposition 2.2.3. *Let $c_\infty > 0$. Then there is $\delta > 0$ such that the following is true. For all $U \subset H_1 \cap H_2$ open with $|U| \leq \delta$, $c \in L^\infty(U)$ with $c \leq c_\infty$, and all doubly antisymmetric supersolutions w of* (2.2.2) *it follows that $w \geq 0$ in $H_1 \cap H_2$.*

Proof. Let $c_\infty > 0$. By (2.2.3), we may fix $\delta > 0$ such that $c_\infty \leq \lambda_1^-(U)$ for all open sets $U \subset H_1 \cap H_2$ with $|U| \leq \delta$. Fix such an open set U and let $c \in L^\infty(U)$. Then note that we may reflect c evenly across ∂H_1. Then we have for any with respect to ∂H_1 antisymmetric function $\varphi \in \mathscr{H}_0^s(V)$, $V = U \cup r_1(U)$ with $\varphi \geq 0$ in U:

$$\mathscr{E}_s(w,\varphi) = \mathscr{E}_s(w, 1_U\varphi) + \mathscr{E}_s(w, 1_{r_1(U)}\varphi) \geq \int_U c(x)w(x)\varphi(x) + \int_{r_1(U)} c(x)w(x)\varphi(x)\,dx$$

$$= \int_V c(x)w(x)\varphi(x)\, dx.$$

Here, we have used the antisymmetry of w and φ with respect to ∂H_1 and Lemma 2.2.1 to have $1_U\varphi \in \mathscr{H}_0^s(U)$, $1_{r_1(U)}\varphi \in \mathscr{H}_0^s(r_1(U))$, and

$$\mathscr{E}_s(w, 1_{r_1(U)}\varphi) = \mathscr{E}_s(w\circ r_1, 1_U\varphi\circ r_1) = \mathscr{E}_s(w, 1_U\varphi) \geq \int_U c(x)w(x)\varphi(x)\, dx = \int_{r_1(U)} c(x)w(x)\varphi(x)\, dx,$$

since we extended c evenly across ∂H_1. Then $v = w^- 1_{H_1}1_{H_2} - w^+ 1_{H_1^c}1_{H_2} \in \mathscr{H}_0^s(U\cup r_1(U))$ by Lemma 2.2.2 and we have by symmetry

$$\begin{aligned}
\mathscr{E}_s(w,v) &= \int_U c(x)w(x)w^-(x)\, dx - \int_{r_1(U)} c(x)w(x)w^+(x)\, dx \\
&= -\int_U c(x)(w^-(x))^2\, dx - \int_{r_1(U)} c(x)(w^+(x))^2\, dx \\
&\geq -\lambda_{1,s}^-\left(\int_U (w^-(x))^2\, dx + \int_{r_1(U)} (w^+(x))^2\, dx\right) = -\lambda_{1,s}^-\|v\|_{L^2(V)}^2 \geq -\mathscr{E}_s(v,v).
\end{aligned}$$

Hence with Lemma 2.2.2 we have $0 \leq \mathscr{E}_s(w,v) + \mathscr{E}_s(v,v) \leq 0$ and this can only be true if $v \equiv 0$. □

In the next statement, we give a Hopf type lemma for equation (2.2.2) similar to [43, Proposition 3.3].

Proposition 2.2.4. *Let $U \subset H_1 \cap H_2$ open. Furthermore, let $c \in L^\infty(U)$ and let $u \in \mathscr{V}^s(U)$ be a doubly antisymmetric supersolution of* (2.2.2). *Assume $u \geq 0$ in $H_1 \cap H_2$. Then either $u \equiv 0$ or $u > 0$ in U in the sense that*

$$\operatorname{essinf}_K u > 0 \quad \textit{for all compact sets } K \subset U.$$

Moreover, if there is $x_0 \in \partial U \setminus [\partial H_1 \cup \partial H_2]$ such that

i. there exists a ball $B \subset U$ with $\partial B \cap \partial U = \{x_0\}$ and $\lambda_{1,s}^-(B) \geq c$ and

ii. $u(x_0) = 0$,

then there exists $C > 0$ such that

$$u \geq C d_B^s \qquad \textit{in} \qquad B,$$

where δ_B denote the distance to boundary of B, and, in particular, if $u \in C(B)$, then

$$\liminf_{t\downarrow 0} \frac{u(x_0 - t\nu(x_0))}{t^s} > 0.$$

Proof. Assume $u \not\equiv 0$. Then there exists a set $K \subset H_1 \cap H_2$ such that $|K| > 0$ and such that

$$\varepsilon := \text{essinf}_K u > 0. \tag{2.2.4}$$

Let $B \subset U$ be an open ball such that $\text{dist}(B,K) > 0$ and $\partial B \cap \partial H_i = \emptyset$ for $i = 1,2$. By making B smaller if necessary, we may assume

$$\lambda_{1,s}^{-}(B) \geq c \tag{2.2.5}$$

Let $\psi_B \in \mathscr{H}_0^s(B)$ be the solution to

$$(-\Delta)^s \psi_B = 1 \qquad \text{in} \qquad B$$

Recall that there exists $c_i = c_i(N,s,B) > 0$, $i = 1,2$ such that $c_1 d_B^s \leq \psi_B \leq c_2 d_B^s$. For any $\alpha > 0$, we define

$$\overline{u} := \psi_B + \alpha 1_K - \psi_{r_1(B)} - \alpha 1_{r_1(K)} \qquad \text{and} \qquad w := \overline{u} - \overline{u} \circ r_2$$

It is clear that $w \circ r_1 = -w = w \circ r_2$, that is, w is doubly antisymmetric. Let $\varphi \in \mathscr{H}_0^s(B)$ with $\varphi \geq 0$. Then, we have

$$\begin{aligned}
&\mathscr{E}_s(w,\varphi) = \mathscr{E}_s(\overline{u},\varphi) - \mathscr{E}_s(\overline{u} \circ r_2, \varphi) \\
&= \int_B \varphi(x)\,dx - \alpha b_{N,s} \int_B \varphi(x) \int_K \frac{dy}{|x-y|^{N+2s}} dx + \alpha b_{N,s} \int_B \varphi(x) \int_{r_1(K)} \frac{dy}{|x-y|^{N+2s}} dx \\
&+ b_{N,s} \int_B \varphi(x) \int_{r_1(B)} \frac{\psi_B(y)}{|x-y|^{N+2s}} dy\,dx + b_{N,s} \int_{B\times B} \frac{\psi_B(x)\varphi(y)}{|x-r_2(y)|^{N+2s}} dx\,dy + \alpha b_{N,s} \int_K \int_B \frac{\varphi(y)}{|x-r_2(y)|^{N+2s}} \\
&- \alpha b_{N,s} \int_{r_1(K)\times r_2(B)} \frac{\varphi(r_2(y))}{|x-y|^{N+2s}} dx\,dy - b_{N,s} \int_{r_1(B)\times r_2(B)} \frac{\psi_{r_1(B)}(x)\varphi(r_2(y))}{|x-y|^{N+2s}} dx\,dy \\
&= \int_B \varphi(x) \Big(1 - \alpha b_{N,s} \int_K \Big[\frac{1}{|x-y|^{N+2s}} - \frac{1}{|x-r_1(y)|^{N+2s}} - \frac{1}{|x-r_2(y)|^{N+2s}} + \frac{1}{|x-r_{1,2}(y)|^{N+2s}}\Big] dy \\
&+ b_{N,s} \int_B \frac{\psi_B(r_1(y))}{|x-r_1(y)|^{N+2s}} dy + b_{N,s} \int_B \frac{\psi_B(y)}{|x-r_2(y)|^{N+2s}} dy - b_{N,s} \int_B \frac{\psi_{r_1(B)}(r_1(y))}{|x-r_{1,2}(y)|^{N+2s}} dy\Big) \\
&\leq \int_B \varphi(x) \Big(1 - \alpha b_{N,s} \int_K \Big[\frac{1}{|x-y|^{N+2s}} - \frac{1}{|x-r_1(y)|^{N+2s}} - \frac{1}{|x-r_2(y)|^{N+2s}} + \frac{1}{|x-r_{1,2}(y)|^{N+2s}}\Big] dy \\
&+ b_{N,s} \|\psi_B\|_{L^\infty(\mathbb{R}^N)} \int_B \Big[\frac{1}{|x-r_1(y)|^{N+2s}} + \frac{1}{|x-r_2(y)|^{N+2s}} + \frac{1}{|x-r_{1,2}(y)|^{N+2s}}\Big] dy\Big) \\
&\leq \int_B \varphi(x) \Big(\kappa - \alpha b_{N,s} \int_K \Big[\frac{1}{|x-y|^{N+2s}} - \frac{1}{|x-r_1(y)|^{N+2s}} - \frac{1}{|x-r_2(y)|^{N+2s}} + \frac{1}{|x-r_{1,2}(y)|^{N+2s}}\Big] dy\Big),
\end{aligned} \tag{2.2.6}$$

with

$$\kappa := 1 + b_{N,s} \|\psi_B\|_{L^\infty(\mathbb{R}^N)} \int_B \Big[\frac{1}{|x-r_1(y)|^{N+2s}} + \frac{1}{|x-r_2(y)|^{N+2s}} + \frac{1}{|x-r_{1,2}(y)|^{N+2s}}\Big] dy < \infty,$$

where we have used that the boundary of B does not touch $\partial H_1 \cup \partial H_2$. Since $\overline{B}$ and K are compactly contained in $H_1 \cap H_2$, it follows that

$$C := \inf_{x\in B,\, y\in K} \left(\frac{1}{|x-y|^{N+2s}} - \frac{1}{|x-r_1(y)|^{N+2s}} - \frac{1}{|x-r_2(y)|^{N+2s}} + \frac{1}{|x-r_{1,2}(y)|^{N+2s}} \right) > 0.$$

Since $c, \psi_B \in L^\infty(U)$, we may hence choose α large enough so that

$$\kappa - \alpha b_{N,s} \int_K \Big[\frac{1}{|x-y|^{N+2s}} - \frac{1}{|x-r_1(y)|^{N+2s}} - \frac{1}{|x-r_2(y)|^{N+2s}} + \frac{1}{|x-r_{1,2}(y)|N+2s} \Big] dy \le c(x)\psi_B(x)$$

for all $x \in B$. Consequently, equation (2.2.6) gives

$$\mathscr{E}_s(w,\varphi) \le \int_B c(x)\varphi(x)\psi_B(x)\,dx \qquad \text{for all nonnegative } \varphi \in \mathscr{H}^s_0(B).$$

Therefore $-w$ satisfies in weak sense

$$\begin{cases} (-\Delta)^s(-w) \ge c(x)(-w) & \text{in } B, \\ (-w) \ge 0 & \text{in } H_1 \cap H_2 \setminus B, \\ -w \circ r_i = w & \text{in } \mathbb{R}^N \text{ for } i = 1,2. \end{cases} \tag{2.2.7}$$

Next, consider $u_\varepsilon := u - \frac{\varepsilon}{\alpha} w$ with ε given in (2.2.4). Then u_ε also satisfies in weak sense (2.2.7) where the *nonlocal boundary condition* is satisfied by the choice of ε. By (2.2.5) and Lemma 2.2.3 we conclude that $u \ge \frac{\varepsilon}{\alpha}\psi_B \ge \frac{\varepsilon}{\alpha} c_1 d_B^s$ in B. Since B is chosen arbitrary, the above implies that $u > 0$ in U as stated. If in addition there is $x_0 \in \partial U \setminus [\partial H_1 \cup \partial H_2]$ with the given properties, the above argument yields in particular

$$\liminf_{t\downarrow 0} \frac{u(x_0 - t\nu(x_0))}{t^s} \ge \varepsilon \lim_{t\downarrow 0} \frac{\psi_B(x_0 - t\nu(x_0))}{t^s} > 0.$$

This finishes the proof. □

Remark 2.2.5. *To put the Hopf type statement in Proposition 2.2.4 into perspective, consider in Problem* (2.1.1) *the nonlinearity* $f(x,u) = |u|^{2^*_s-2}u$ *with* $2^*_s := \frac{2N}{N-2s}$*, the critical fractional exponent. It was shown in [95] that there is no positive bounded solution if* Ω *is starshaped. Up to our knowledge, it remains an open question, if there is a sign-changing solution to this problem. Assuming that* Ω *is bounded and starshaped with* $C^{1,1}$ *boundary and there exists a bounded solution of* (2.1.1) *with* $f(x,u) = |u|^{2^*_s-2}u$*, it first follows that* $u \in C^s(\mathbb{R}^N) \cap C^\infty(\Omega)$ *(see e.g. [94]) and the fractional Pohozaev identity from [95] implies*

$$\int_{\partial\Omega} \Big(\frac{u}{\mathrm{dist}(\cdot,\partial\Omega)^s} \Big)^2 (x\cdot\nu)\, d\sigma = 0.$$

However, by [43, Proposition 3.3] it then follows that if Ω *has additionally a symmetry hyperplane* T *and* u *is odd with respect to reflections across this hyperplane and of one sign on one side of the hyperplane, then* $\left(\frac{u}{\mathrm{dist}(\cdot,\partial\Omega)^s}\right)^2 > 0$ *on* $\partial\Omega \setminus T$*. Whence, there cannot be such an odd solution of the problem. Similarly, using instead Proposition 2.2.4, it follows that there can also be no doubly antisymmetric solution of this problem if* Ω *satisfies (D).*

2.3 Symmetry of solutions

In the following, we use the notation from Section 2.2 and assume $\Omega \subset \mathbb{R}^N$ satisfies (D). Moreover, $f \in C(\Omega \times \mathbb{R})$ satisfies (F1) and (F2) and let $u \in L^\infty(U) \cap \mathscr{H}_0^s(\Omega)$ be a solution of problem (2.1.1) which satisfies $u \circ r_{N,0} = -u$. Note that by (F1) and [94] it follows that $u \in C^s(\mathbb{R}^N)$. For $\lambda \in \mathbb{R}$ we may than define

$$v_\lambda(x) = u(r_{\lambda,1}(x)) - u(x).$$

Then it follows that v_λ is antisymmetric with respect to $H_{N,0}$ and $H_{1,\lambda}$, hence doubly antisymmetric, and it satisfies due to (F2)

$$\begin{cases} (-\Delta)^s v_\lambda \geq c_\lambda(x) v_\lambda & \text{in } \Omega_\lambda := \Omega \cap H_{N,0} \cap H_{1,\lambda}, \\ v_\lambda \geq 0 & \text{in } H_{N,0} \cap H_{1,\lambda} \setminus \Omega_\lambda, \end{cases} \tag{2.3.1}$$

where

$$c_\lambda(x) = \begin{cases} \dfrac{f(x,u(r_{\lambda,1}(x))) - f(x,u(x))}{u(r_{\lambda,1}(x)) - u(x)} & u(r_{\lambda,1}(x)) \neq u(x) \\ 0 & u(r_{\lambda,1}(x)) = u(x) \end{cases}$$

Note that by assumption (F1) we have

$$\sup_{\lambda \in \mathbb{R}} \sup_{x \in \Omega_\lambda} |c_\lambda(x)| =: c_\infty < \infty.$$

Finally, let $\lambda_1 := \sup_{x \in \Omega} x_1$.

Proof of Theorem 2.1.1. Assume that u is nontrivial. We apply the moving plane method to then prove that u is symmetric with respect to $H_{1,0}$ and decreasing in x_1. For this let

$$\lambda_0 := \inf\{\lambda \in (0,\lambda_1) \,:\, v_\mu > 0 \text{ in } \Omega_\mu \text{ for all } \mu \in (\lambda,\lambda_1)\}$$

Next note that by (D) and Proposition 2.2.3 it follows that there is $\varepsilon > 0$ such that $v_\mu \geq 0$ for all $\lambda \in (\lambda_1 - \varepsilon, \lambda_1)$ and thus by Proposition 2.2.4 we have $\lambda_0 \leq \lambda_1 - \varepsilon$. Assume next by contradiction that $\lambda_0 > 0$. Then by continuity $v_{\lambda_0} \geq 0$ in $H_{N,0} \cap H_{1,\lambda_0}$. By Proposition 2.2.4 it follows that either $v_{\lambda_0} \equiv 0$ or $v_{\lambda_0} > 0$.
If $v_{\lambda_0} \equiv 0$, this implies that we have $u \equiv 0$ in $\Omega \setminus H_{1,\lambda_0 - \lambda_1}$. But then, we can also start moving the hyperplane from the left (working instead with $\mathbb{R}^N \setminus H_{1,\lambda}$), up to the same λ_0. It then follows that u has two different parallel symmetry hyperplanes, but since u vanishes outside of Ω, this implies $u \equiv 0$, which cannot be the case.
If $v_{\lambda_0} > 0$, let $\delta > 0$ be given by Proposition 2.2.3 according to c_∞. Then by continuity there is $\mu > 0$ such that and a compact set $K \subset \Omega_{\lambda_0}$ such that $|\Omega_{\lambda_0} \setminus K| \leq \frac{\delta}{2}$ and $v_{\lambda_0} \geq 2\mu$ in K. Again, by continuity, we can find $\tau \in (0, \lambda_1 - \lambda_0)$ such that $v_\lambda \geq \mu$ for all $\lambda \in [\lambda_0 - \tau, \lambda_0]$. Let $U_\lambda := \{x \in \Omega_\lambda \,:\, v_\lambda < 0\}$. Then, by making τ smaller if necessary, we may also assume $|U_\lambda| \leq \delta$ for all $\lambda \in [\lambda_0 - \tau, \lambda_0]$. A combination of Proposition 2.2.3 and 2.2.4 gives a contradiction to the definition of λ_0.
Whence, $\lambda_0 > 0$ is not possible. Thus $\lambda_0 = 0$ and this finishes the proof. □

2.4 A symmetric sign-changing solution

Let $\Omega \subset \mathbb{R}^N$ open and bounded and consider the functional

$$J : \mathscr{H}_0^s(\Omega) \to \mathbb{R}, \qquad J(u) = \mathscr{E}_s(u,u).$$

Let $M := \{u \in \mathscr{H}_0^s(\Omega) \ : \ u = -u \circ r_N, \ \int_\Omega |u(x)|^p \ dx = 1\}$ with $1 < p < \frac{2N}{N-2s}$. Then by a constraint minimization argument using the framework as explained e.g. in [98, 99], see also [12], it follows that there exists such a minimizer u of $J|_M$. That is, the minimum

$$\lambda_{1,p}^- = \min_{u \in M} \mathscr{E}_s(u,u) \tag{2.4.1}$$

is attained. Similar to [12, Theorem 3.1], it can be shown that this minimizer is bounded and then, by an iteration of the results of [94, 57], we have $u \in C^\infty(\Omega)$. If in addition $\partial\Omega$ is of class $C^{1,1}$, then [94, 57] also imply that $u \in C^s(\mathbb{R}^N)$.

Proof of Theorem 2.1.2. Let $\lambda_{1,p}^-$ be as in (2.4.1) and let u be the minimizer as explained in the above remarks. In view of Theorem 2.1.1 it remains to show that u can be chosen of one sign in $\Omega^+ := \Omega \cap H_{N,0}$. In the following $\Omega^- = \Omega \setminus \Omega^+$. Assume by contradiction that u changes sign in Ω^+ and let $\Omega_1^+ := \{x \in \Omega^+ : u(x) > 0\}$ and $\Omega_2^+ := \{x \in \Omega^+ : u(x) \leq 0\}$. We also let $\Omega_1^- = r_{N,0}(\Omega_1^+)$, and $\Omega_2^- = r_{N,0}(\Omega_2^+)$. By the property of u, it is clear that $u < 0$ in Ω_1^- and $u \geq 0$ in Ω_2^-. Now let $\bar{u}$ be defined by

$$\bar{u} = 1_{\Omega^+}|u| - 1_{\Omega^-}|u|. \tag{2.4.2}$$

Then $\bar{u} \in M$, that is $\bar{u} \in \mathscr{H}_0^s(\Omega)$ satisfies $\bar{u} \circ r_{N,0} = -\bar{u}$ and

$$\int_\Omega |\bar{u}|^p dx = \int_\Omega (|\bar{u}|^2)^{p/2} dx = \int_\Omega (1_{\Omega^+}|u|^2 + 1_{\Omega^-}|u|^2)^{p/2} dx = \int_\Omega |u|^p dx = 1. \tag{2.4.3}$$

Moreover, we have

$$\begin{aligned}
&\frac{2}{b_{N,s}} \mathscr{E}_s(\bar{u},\bar{u}) \\
&= \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\big(\bar{u}(x) - \bar{u}(y)\big)^2}{|x-y|^{N+2s}} dxdy = \int_{\Omega \times \Omega} \frac{\big(\bar{u}(x) - \bar{u}(y)\big)^2}{|x-y|^{N+2s}} dxdy + 2\int_\Omega \bar{u}^2(x) \int_{\mathbb{R}^N \setminus \Omega} \frac{dy}{|x-y|^{N+2s}} dx \\
&= \int_{\Omega^+ \times \Omega} \frac{\big(\bar{u}(x) - \bar{u}(y)\big)^2}{|x-y|^{N+2s}} dxdy + \int_{\Omega^- \times \Omega} \frac{\big(\bar{u}(x) - \bar{u}(y)\big)^2}{|x-y|^{N+2s}} dxdy + 4\int_{\Omega^+} \int_{\mathbb{R}^N \setminus \Omega} \frac{u^2(x)dy}{|x-y|^{N+2s}} dx \\
&= \int_{\Omega^+ \times \Omega^+} \frac{\big(|u(x)| - |u(y)|\big)^2}{|x-y|^{N+2s}} dxdy + \int_{\Omega^- \times \Omega^-} \frac{\big(|u(x)| - |u(y)|\big)^2}{|x-y|^{N+2s}} dxdy \\
&+ 2\int_{\Omega^- \times \Omega^+} \frac{\big(|u|(x) + |u|(y)\big)^2}{|x-y|^{N+2s}} dxdy + 4\int_{\Omega^+} u^2(x) \int_{\mathbb{R}^N \setminus \Omega} \frac{dy}{|x-y|^{N+2s}} dx.
\end{aligned} \tag{2.4.4}$$

Using the notation above, we rewrite

$$\begin{aligned}
&\int\limits_{\Omega^+\times\Omega^+} \frac{\big(|u(x)|-|u(y)|\big)^2}{|x-y|^{N+2s}}dxdy \\
&= \int\limits_{\Omega^+\times\Omega^+} \frac{\big(u(x)-u(y)\big)^2}{|x-y|^{N+2s}}dxdy + 2\int\limits_{\Omega_1^+\times\Omega_2^+} \frac{(u(x)+u(y))^2-(u(x)-u(y))^2}{|x-y|^{N+2s}}dxdy \\
&= \int\limits_{\Omega^+\times\Omega^+} \frac{\big(u(x)-u(y)\big)^2}{|x-y|^{N+2s}}dxdy + 4\int\limits_{\Omega_1^+\times\Omega_2^+} \frac{u(x)u(y)}{|x-y|^{N+2s}}. \qquad (2.4.5)
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\int\limits_{\Omega^-\times\Omega^-} \frac{\big(|u(x)|-|u(y)|\big)^2}{|x-y|^{N+2s}}dxdy &= \int\limits_{\Omega^-\times\Omega^-} \frac{\big(u(x)-u(y)\big)^2}{|x-y|^{N+2s}}dxdy + 4\int\limits_{\Omega_1^-\times\Omega_2^-} \frac{u(x)u(y)}{|x-y|^{N+2s}}dxdy \\
&= \int\limits_{\Omega^-\times\Omega^-} \frac{\big(u(x)-u(y)\big)^2}{|x-y|^{N+2s}}dxdy + 4\int\limits_{\Omega_1^+\times\Omega_2^+} \frac{u(x)u(y)}{|x-y|^{N+2s}}dxdy. \qquad (2.4.6)
\end{aligned}$$

Now using that $\Omega_j^- = r_N(\Omega_j^+)$, $j = 1,2$ we get

$$\begin{aligned}
&\int\limits_{\Omega^-\times\Omega^+} \frac{\big(|u(x)|-|u(y)|\big)^2}{|x-y|^{N+2s}}dxdy \\
&= \int\limits_{\Omega^-\times\Omega^+} \frac{\big(u(x)-u(y)\big)^2}{|x-y|^{N+2s}}dxdy + \int\limits_{\Omega_2^-\times\Omega_1^+} \frac{(u(x)+u(y))^2-(u(x)-u(y))^2}{|x-y|^{N+2s}}dxdy \\
&+ \int\limits_{\Omega_1^-\times\Omega_2^+} \frac{(u(x)+u(y))^2-(u(x)-u(y))^2}{|x-y|^{N+2s}}dxdy \\
&= \int\limits_{\Omega^-\times\Omega^+} \frac{\big(u(x)-u(y)\big)^2}{|x-y|^{N+2s}}dxdy + 2\int\limits_{\Omega_2^-\times\Omega_1^+} \frac{u(x)u(y)}{|x-y|^{N+2s}}dxdy + 2\int\limits_{\Omega_1^-\times\Omega_2^+} \frac{u(x)u(y)}{|x-y|^{N+2s}}dxdy \\
&= \int\limits_{\Omega^-\times\Omega^+} \frac{\big(u(x)-u(y)\big)^2}{|x-y|^{N+2s}}dxdy - 4\int\limits_{\Omega_1^+\times\Omega_2^+} \frac{u(x)u(y)}{|r_{N,0}(x)-y|^{N+2s}}dxdy. \qquad (2.4.7)
\end{aligned}$$

Summing up (2.4.5), (2.4.6) and (2.4.7), and taking into account (2.4.4), we obtain

$$\frac{2}{C_{N,s}}\mathscr{E}_s(\overline{u},\overline{u}) - 4\int\limits_{\Omega^+} u^2(x)\int\limits_{\mathbb{R}^N\setminus\Omega} \frac{dy}{|x-y|^{N+2s}}\,dx$$

$$= \int\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}dxdy - 2\int\limits_{\Omega} u^2(x) \int\limits_{\mathbb{R}^N\setminus\Omega} \frac{dy}{|x-y|^{N+2s}}$$
$$+8 \int\limits_{\Omega_1^+\times\Omega_2^+} \frac{u(x)u(y)}{|x-y|^{N+2s}}dxdy - 8\int\limits_{\Omega_2^+\times\Omega_1^+} \frac{u(x)u(y)}{|r_{N,0}(x)-y|^{N+2s}}dxdy. \tag{2.4.8}$$

By a change of variable it is clear that

$$2\int\limits_{\Omega} u^2(x) \int\limits_{\mathbb{R}^N\setminus\Omega} \frac{dy}{|x-y|^{N+2s}}dx = 4\int\limits_{\Omega^+} u^2(x) \int\limits_{\mathbb{R}^N\setminus\Omega} \frac{dy}{|x-y|^{N+2s}}\,dx.$$

Putting that into (2.4.8) gives

$$\frac{2}{C_{N,s}}\mathcal{E}_s(\overline{u},\overline{u})$$
$$= \int\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}dxdy + 8\int\limits_{\Omega_1^+\times\Omega_2^+} u(x)u(y)\Big[\frac{1}{|x-y|^{N+2s}} - \frac{1}{|r_{N,0}(x)-y|^{N+2s}}\Big]dxdy. \tag{2.4.9}$$

Now since $\overline{u}\circ r_N = -\overline{u}$, it follows from the variational characterization of $\lambda_{1,p}^-(\Omega)$ in (2.4.1) with (2.4.9) and (2.4.3) that

$$\lambda_{1,p}^-(\Omega) \le \mathcal{E}_s(\overline{u},\overline{u}) = \mathcal{E}_s(u,u) + 4C_{N,s}\int\limits_{\Omega_1^+\times\Omega_2^+} u(x)u(y)\Big[\frac{1}{|x-y|^{N+2s}} - \frac{1}{|r_{N,0}(x)-y|^{N+2s}}\Big]dxdy$$
$$= \lambda_{1,p}^-(\Omega) + 4C_{N,s}\int\limits_{\Omega_1^+\times\Omega_2^+} u(x)u(y)\Big[\frac{1}{|x-y|^{N+2s}} - \frac{1}{|r_{N,0}(x)-y|^{N+2s}}\Big]dxdy.$$

That is

$$0 \le \int\limits_{\Omega_1^+\times\Omega_2^+} u(x)u(y)\Big[\frac{1}{|x-y|^{N+2s}} - \frac{1}{|r_{N,0}(x)-y|^{N+2s}}\Big]dxdy \le 0.$$

Whence $u\equiv 0$ in Ω_2^+ and therefore $u\ge 0$ in Ω^+. This is in contradiction with the hypothesis. It follows that u does not change sign in Ω^+ and, without loss of generality, we may assume $u\ge 0$ in Ω^+. By the strong maximum principle [43, Corollary 3.4] we have $u>0$ in Ω^+.
For the additional statement let $p=2$ and let u,v be two normalized minimizers for $\lambda_{1,2}^-(\Omega)$. Assume further they satisfy the sign property in Theorem 2.1.2, i.e. they are of one sign in $\Omega\cap H_{N,0}$. Then, if $u-v$ is not identically zero, it must change sign in $\Omega\cap H_{N,0}$. Indeed, if not, we may assume $u-v>0$ in $\Omega\cap H_{N,0}$ by [43, Corollary 3.4]. Therefore $1=\int_\Omega u^2dx = 2\int_{\Omega\cap H_{N,0}} u^2dx > 2\int_{\Omega\cap H_{N,0}} v^2dx = 1$ a contradiction. Note that if $u\not\equiv v$, then also $(u-v)/\|u-v\|_{L^2(\Omega)}$ is a minimizer. But by the above argument, $u-v$ cannot change sign in $\Omega\cap H_{N,0}$. Whence $u\equiv v$ as claimed. □

Chapter 3

A generalized fractional Pohozaev identity and applications

This Chapter is based on the paper $[P4]$, a joint work with M. M. Fall and T. Weth. The exposition is as in the original paper. The main achievement of the paper is a generalization of the fractional Pohozaev identity obtained by X. Ros-Oton and J. Serra in [95]. The result can be seen as a fractional version of [90, Identity (4)] by P. Puccin and J. Serrin. We apply the result to derive nonnexistence results for fractional semilinear Dirichlet boundary value problem with Lipschitz nonlinearity. We also apply the identity to compute the shape derivative of the fractional Dirichlet simple eigenvalues.

3.1 Introduction

Let Ω be a bounded open set of class $C^{1,1}$ and $s \in (0,1)$. We consider the semilinear fractional Dirichlet problem

$$(-\Delta)^s u = f(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \mathbb{R}^N \setminus \Omega. \tag{3.1.1}$$

Here $(-\Delta)^s$ denotes the fractional Laplacian, which, for sufficiently regular functions φ, is pointwisely given by

$$(-\Delta)^s \varphi(x) = b_{N,s}\, PV \int_{\mathbb{R}^N} \frac{\varphi(x)-\varphi(y)}{|x-y|^{N+2s}}\, dy = \frac{b_{N,s}}{2} \int_{\mathbb{R}^N} \frac{2\varphi(x)-\varphi(x+y)-\varphi(x-y)}{|y|^{N+2s}}\, dy.$$

with $b_{N,s} = \pi^{-\frac{N}{2}} s 4^s \frac{\Gamma(\frac{N}{2}+s)}{\Gamma(1-s)}$. Moreover, we assume that

$$f : \mathbb{R} \to \mathbb{R} \text{ in (3.1.1) is locally Lipschitz,} \tag{3.1.2}$$

and we let $F \in C^1(\mathbb{R})$ be defined by $F(t) = \int_0^t f(s)\, ds$. We consider (3.1.1) in weak sense. For this we define

$$\mathcal{H}_0^s(\Omega) := \{u \in H^s(\mathbb{R}^N) : u \equiv 0 \text{ on } \mathbb{R}^N \setminus \Omega\} \subset H^s(\mathbb{R}^N). \tag{3.1.3}$$

Here $H^s(\mathbb{R}^N)$ is the set of those functions u for which $\mathscr{E}(u,u)$, with $\mathscr{E}$ define as in (3.1.4), is finite. By definition, a function $u \in \mathscr{H}_0^s(\Omega) \cap L^\infty(\Omega)$ is a weak solution of (3.1.1) if

$$\mathscr{E}(u,v) = \int_\Omega f(u)v\,dx \qquad \text{for all } v \in \mathscr{H}_0^s(\Omega),$$

where

$$(v,w) \mapsto \mathscr{E}(v,w) := \frac{b_{N,s}}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^{N+2s}}\,dxdy. \tag{3.1.4}$$

From (3.1.2) and the elliptic regularity theory for weak solutions developed in recent years (see [94, 102]), it follows that every weak solution $u \in \mathscr{H}_0^s(\Omega) \cap L^\infty(\Omega)$ is contained in the space $C_0^s(\overline{\Omega}) \cap C_{loc}^{2s+1-\varepsilon}(\Omega)$ for every $\varepsilon \in (0, 2s+1)$. Here $C_0^s(\overline{\Omega}) = \{u \in C^s(\overline{\Omega}) : u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega\}$. Moreover, it has been proved in [94] that

$$\text{the function } \psi_u := \frac{u}{d^s} \text{ extends uniquely to a function in } C^\alpha(\overline{\Omega}) \text{ for some } \alpha > 0,$$

where, here and in the following, we let $d(x) = \operatorname{dist}(x, \mathbb{R}^N \setminus \Omega)$ for $x \in \mathbb{R}^N$.
In the seminal paper [95], Ros-Oton and Serra introduced and proved a fractional Pohozaev identity which states that every (weak) solution of (3.1.1) satisfies

$$\Gamma(1+s)^2 \int_{\partial\Omega} \psi_u^2\, x \cdot \nu\, d\sigma = 2N \int_\Omega F(u)\,dx - (N-2s) \int_\Omega f(u)u\,dx, \tag{3.1.5}$$

see [95, Theorem 1.1]. Here ν in (3.1.5) is the unit outer normal vector field. This identity has proved to be highly relevant in the study of (3.1.1). In particular, it yields a nonexistence result for (3.1.1) in the case where Ω is starshaped and f satisfies a supercritical growth condition, see [95, Corollary 1.3]. Somewhat surprisingly, (3.1.5) is already useful in the linear case $f(u) = \lambda u$, as it gives valuable information on the fractional boundary derivative $\psi_u := \frac{u}{d^s}$ of Dirichlet eigenfunctions of the fractional Laplacian $(-\Delta)^s$. In particular, as we shall see in Section 4.1.1 below, it allows to show the simplicity of radial Dirichlet eigenvalues of $(-\Delta)^s$ in the case where Ω is a ball or an annulus. Moreover, (3.1.5) has been used recently in [41] to prove the nonradiality of second Dirichlet eigenfunctions of $(-\Delta)^s$ in the case $\Omega = B_1(0)$. Note that these properties are standard in the local case $s = 1$, where tools like separation of variables and ODE techniques are available.
The main purpose of this paper is to present a generalization of the identity (3.1.5) depending on a given Lipschitz vector field $X \in C^{0,1}(\mathbb{R}^N, \mathbb{R}^N)$. We recall that every such vector field is a.e. differentiable on $\mathbb{R}^N$, so its derivative dX and also $\operatorname{div} X$ are a.e. well defined on $\mathbb{R}^N$. For every such vector field, we let

$$K_X(x,y) := \frac{b_{N,s}}{2}\Big[\big(\operatorname{div} X(x) + \operatorname{div} X(y)\big) - (N+2s)\frac{\big(X(x)-X(y)\big)\cdot(x-y)}{|x-y|^2}\Big]\frac{1}{|x-y|^{N+2s}} \tag{3.1.6}$$

for $x, y \in \mathbb{R}^N$, $x \neq y$, and we call K_X the *fractional deformation kernel associated with the vector field X*. We will justify this name further below. Moreover, we denote by $\mathscr{E}_X$ the bilinear form associated to the Kernel K_X, i.e,

$$\mathscr{E}_{K_X}(v,w) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (v(x)-v(y))(w(x)-w(y))K_X(x,y)\,dxdy \quad \text{for all } v,w \in H^s(\mathbb{R}^N). \quad (3.1.7)$$

Our first main result for problem (3.1.1) is the following.

Theorem 3.1.1. *Let $u \in \mathscr{H}_0^s(\Omega) \cap L^\infty(\Omega)$ be a (weak) solution of the problem (3.1.1). Then we have*

$$\Gamma(1+s)^2 \int_{\partial\Omega} (\frac{u}{d^s})^2 X\cdot\nu\,dx = 2\int_{\Omega} F(u)\operatorname{div}X\,dx - \mathscr{E}_{K_X}(u,u) \quad \textit{for all } X \in C^{0,1}(\mathbb{R}^N,\mathbb{R}^N) \quad (3.1.8)$$

with $F(t) = \int_0^t f(s)\,ds$. Here ν is the outer unit normal to the boundary and $\mathscr{E}_{K_X}(u,w)$ is defined as in (3.1.7)*.*

Theorem 3.1.1 is a particular case of the following more general identity.

Theorem 3.1.2. *Let $u \in H^s(\mathbb{R}^N)$ such that $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$. Moreover, assume $(-\Delta)^s u \in L^\infty(\Omega)$ if $2s > 1$ and $(-\Delta)^s u \in C^\alpha_{loc}(\Omega) \cap L^\infty(\Omega)$ with $\alpha > 1-2s$ if $2s \leq 1$. Then we have*

$$2\int_{\Omega} \nabla u \cdot X(-\Delta)^s u\,dx = -\Gamma^2(1+s)\int_{\partial\Omega} (\frac{u}{d^s})^2 X\cdot\nu\,dx - \mathscr{E}_X(u,u), \quad (3.1.9)$$

for any vector field $X \in C^{0,1}(\mathbb{R}^N,\mathbb{R}^N)$.

To deduce formula (3.1.8) from (3.1.9) it simply suffices to use the pointwise identities $(-\Delta)^s u = f(u)$, $\nabla F(u) = f(u)\nabla u$ and to integrate by parts, noting that $F(0) = 0$. As noted already above, the regularity assumptions of Theorem 3.1.2 are satisfied in this case as a consequence of assumption (3.1.2) and the elliptic regularity theory for weak solutions developed in [102, 104]. We note that Theorem 3.1.2 generalizes [95, Proposition 1.6] where the particular vector field $X \equiv \text{id} : \mathbb{R}^N \to \mathbb{R}^N$ is considered. Indeed, in the case $X \equiv \text{id}$, we have

$$\operatorname{div}X \equiv N \qquad \text{and} \qquad K_X(x,y) = \frac{b_{N,s}}{2}(N-2s)|x-y|^{-N-2s} \quad \text{for } x,y \in \mathbb{R}^N,$$

so (3.1.9) reduces to

$$2\int_{\Omega} (x\cdot\nabla u)(-\Delta)^s u\,dx = -\Gamma^2(1+s)\int_{\partial\Omega} (\frac{u}{d^s})^2 x\cdot\nu\,dx - (N-2s)\int_{\Omega} u(-\Delta)^s u\,dx.$$

This is the identity stated in [95, Proposition 1.6]. Moreover, for every weak solution of (3.1.1) we have

$$\mathscr{E}_{K_X}(u,u) = (N-2s)\mathscr{E}(u,u) = (N-2s)\int_{\Omega} f(u)u\,dx$$

in this case, and therefore (3.1.8) reduces to (3.1.5).

We also note the following integration-by-parts formula, which is an immediate consequence of Theorem 3.1.2.

Theorem 3.1.3. *Let $u, w \in H^s(\mathbb{R}^N)$ be functions with $u \equiv 0 \equiv w$ in $\mathbb{R}^N \setminus \Omega$. Moreover, assume $(-\Delta)^s u, (-\Delta)^s w \in L^\infty(\Omega)$ if $2s > 1$ and $(-\Delta)^s u$, $(-\Delta)^s w \in C^\alpha_{loc}(\Omega) \cap L^\infty(\Omega)$ with $\alpha > 1-2s$ if $2s \leq 1$. Then, for any vector field $X \in C^{0,1}(\mathbb{R}^N, \mathbb{R}^N)$, it holds that*

$$\int_\Omega \nabla u \cdot X(-\Delta)^s w\,dx = -\int_\Omega \nabla w \cdot X(-\Delta)^s u\,dx - \Gamma^2(1+s)\int_{\partial\Omega} \frac{u}{d^s}\frac{w}{d^s} X\cdot \nu\,dx - \mathscr{E}_{K_X}(u,w). \quad (3.1.10)$$

To deduce this theorem from Theorem 3.1.2, it suffices to apply (3.1.9) to u, w and $u+w$ and to evaluate the difference $\mathscr{E}_{K_X}(u+w, u+w) - \mathscr{E}_{K_X}(w,w) - \mathscr{E}_{K_X}(u,u)$. We note that Theorem 3.1.3 is stated in [95, Theorem 1.9] in the particular case of constant coordinate vector fields $X \equiv e_i$, $i = 1, \ldots, N$, in which $K_X \equiv 0$ and therefore (3.1.10) reduces to

$$\int_\Omega u_{x_i}(-\Delta)^s w\,dx = -\int_\Omega w_{x_i}(-\Delta)^s u\,dx - \Gamma^2(1+s)\int_{\partial\Omega} \frac{u}{d^s}\frac{w}{d^s}\nu_i\,dx.$$

The following corollary of Theorem 3.1.1 is devoted again to problem (3.1.1) and deals with a class of vector fields leading to the same RHS as in (3.1.5) (up to a constant).

Corollary 3.1.4. *Let $X \in C^{0,1}(\mathbb{R}^N, \mathbb{R}^N)$, and suppose that*

$$\big(X(x) - X(y)\big)\cdot(x-y) = c|x-y|^2 \qquad \text{for all } x, y \in \mathbb{R}^N \quad (3.1.11)$$

with some constant $c \in \mathbb{R}$. Moreover, let $u \in \mathscr{H}^s_0(\Omega) \cap L^\infty(\Omega)$ be a (weak) solution of the problem (3.1.1). Then we have

$$\Gamma(1+s)^2 \int_{\partial\Omega} \Big(\frac{u}{d^s}\Big)^2 X\cdot\nu\,dx = c\Big(2N\int_\Omega F(u)\,dx - (N-2s)\int_\Omega f(u)u\,dx\Big). \quad (3.1.12)$$

Remark 3.1.5. It is easy to see that condition (3.1.11) is equivalent to

$$\big(dX(y)h\big)\cdot h = c|h|^2 \qquad \text{for a.e. } y \in \mathbb{R}^N \text{ and every } h \in \mathbb{R}^N. \quad (3.1.13)$$

Applying (3.1.13) to the coordinate vectors $e_1, \ldots, e_N \in \mathbb{R}^N$, we deduce that $\operatorname{div} X \equiv cN$ a.e. on $\mathbb{R}^N$. We note that condition (3.1.11) is satisfied if

$$x \mapsto X(x) = cx + Y(x) + v, \quad (3.1.14)$$

where $v \in \mathbb{R}^N$ is a constant vector and Y is any linear combination of the vector fields

$$x \mapsto Y^{ij}(x) = x_i e_j - x_j e_i, \qquad 1 \leq i < j \leq N.$$

In Section 3.2 we also deduce the following corollary from Theorem 3.1.1.

Corollary 3.1.6. *Let $X \in C^{0,1}(\mathbb{R}^N, \mathbb{R}^N)$, and suppose that*

$$\operatorname{div} X(x) \geq c_1 \qquad \text{and} \qquad \big(X(x) - X(y)\big)\cdot(x-y) \leq c_2|x-y|^2 \qquad \text{for all } x, y \in \mathbb{R}^N \quad (3.1.15)$$

with constants $c_1, c_2 \in \mathbb{R}$. Moreover, let $u \in \mathcal{H}_0^s(\Omega) \cap L^\infty(\Omega)$ be a (weak) solution of the problem (3.1.1) with a nonlinearity f satisfying (3.1.2) and $F(t) = \int_0^t f(s)\,ds \geq 0$ for $t \in \mathbb{R}$. Then we have

$$\Gamma(1+s)^2 \int_{\partial\Omega} \Big(\frac{u}{d^s}\Big)^2 X \cdot \nu \, dx \leq \int_{\Omega} \Big(2c_2 N F(u) - \big[2c_1 - (N+2s)c_2\big] f(u)u\Big) dx. \qquad (3.1.16)$$

In particular, if $f(u) = |u|^{p-2}u$ for some $p > 2$, then

$$\Gamma(1+s)^2 \int_{\partial\Omega} \Big(\frac{u}{d^s}\Big)^2 X \cdot \nu \, dx \leq \Big(\frac{2c_2 N}{p} - \big[2c_1 - (N+2s)c_2\big]\Big) \int_{\Omega} |u|^p dx. \qquad (3.1.17)$$

Corollary 3.1.6 gives rise to the nonexistence of nontrivial solutions of (3.1.1) in the case where $u \mapsto f(u) = |u|^{p-2}u$ is a homogeneous nonlinearity with supercritical growth. In particular, the following non-existence result is an immediate consequence.

Corollary 3.1.7. *Let $X \in C^{0,1}(\mathbb{R}^N, \mathbb{R}^N)$ be a vector field satisfying* (3.1.15) *with some constants $c_2 > 0$ and $c_1 \in (\frac{c_2 N}{2}, c_2 N]$. Moreover, suppose that*

$$0 < s < \Big(\frac{c_1}{c_2} - \frac{N}{2}\Big), \qquad p > \frac{2N}{\frac{2c_1}{c_2} - (N+2s)},$$

and let $u \in \mathcal{H}_0^s(\Omega) \cap L^\infty(\Omega)$ be a (weak) solution of the problem

$$(-\Delta)^s u = |u|^{p-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \mathbb{R}^N \setminus \Omega. \qquad (3.1.18)$$

If $X \cdot \nu \geq 0$ on $\partial\Omega$, then $u \equiv 0$.

If (3.1.11) holds for a vector field $X \in C^{0,1}(\mathbb{R}^N, \mathbb{R}^N)$ with some $c > 0$, then, by Remark 3.1.5, condition (3.1.15) holds with $c = c_2$ and $c_1 = Nc_2$. In this case, Corollary 3.1.7 reduces to the following statement.

Corollary 3.1.8. *Let $X \in C^{0,1}(\mathbb{R}^N, \mathbb{R}^N)$ be a vector field satisfying (3.1.11) for some $c > 0$. Moreover, suppose that*

$$N \geq 2s \qquad \text{and} \qquad p > \frac{2N}{N-2s},$$

and let $u \in \mathcal{H}_0^s(\Omega) \cap L^\infty(\Omega)$ be a (weak) solution of problem (3.1.18). If $X \cdot \nu \geq 0$ on $\partial\Omega$, then $u \equiv 0$.

Example 3.1.9. We briefly discuss applications of Corollaries 3.1.7 and 3.1.8 to some specific domains.

(i) In the special case $X = \mathrm{id}$, Corollary 3.1.8 yields the nonexistence of nontrivial solutions of (3.1.18) for starshaped domains, as stated in [95, Corollary 1.3].

(ii) A specific example of a non-sharshaped domain $\Omega \subset \mathbb{R}^2$ to which Corollary 3.1.8 applies is given by

$$\Omega = \{x \in \mathbb{R}^2 \,:\, x_1^2 + 10(x_2^3 + x_1)^2 < 1\}$$

Here, we choose the vector field

$$X : \mathbb{R}^2 \to \mathbb{R}^2, \qquad X(x_1, x_2) = (5x_1 - 4x_2, 5x_2 + 4x_1),$$

so $X \equiv 5\,\mathrm{id} - 4Y^{12}$ with the notation of Remark 3.1.5. Hence (3.1.11) is satisfied with $c = 5$. Moreover, a careful estimate shows that $X \cdot \nu \geq 0$ on $\partial\Omega$ (see Figure 1). In [92, p. 92],

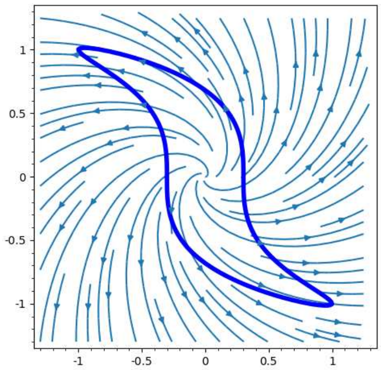

Figure 3.1: Domain Ω and flow lines of the vector field X.

further (non-explicit) examples of non-sharshaped domains $\Omega \subset \mathbb{R}^N$ and vector fields X of the form (3.1.14) for some $c > 0$ satisfying $X \cdot \nu \geq 0$ on $\partial\Omega$ are given in the context of the Dirichlet problem for the classical local equation $-\Delta u = |u|^{p-2}u$.

(iii) We consider $N \geq 2$ and, for $\varepsilon \in [0,1)$, the vector field $X_\varepsilon \in C^{0,1}(\mathbb{R}^N, \mathbb{R}^N)$ given by $X(x) = (\varepsilon x_1, x_2, \dots, x_N)$, which satisfies $\operatorname{div} X_\varepsilon \equiv N - 1 + \varepsilon$ and $\big(X(x) - X(y)\big) \cdot (x - y) \leq |x - y|^2$ for $x, y \in \mathbb{R}^N$. Hence (3.1.15) is satisfied with $c_2 = 1$ and $c_1 := N - 1 + \varepsilon \in [\frac{c_2 N}{2}, c_2 N]$. Consequently, for any bounded domain $\Omega \subset \mathbb{R}^N$ satisfying

$$X_\varepsilon \cdot \nu \geq 0 \qquad \text{on } \partial\Omega, \tag{3.1.19}$$

Corollary 3.1.7 yields nonexistence of nontrivial solutions to (3.1.18) if $0 < s < \min\{1, \frac{N}{2} - 1 + \varepsilon\}$ and $p > \frac{2N}{N - 2(1 + s - \varepsilon)}$. To give specific examples, we restrict our attention to rotationally symmetric domains of the form

$$\Omega = \{x \in \mathbb{R}^N \,:\, g(x_1^2) + \kappa \sum_{\ell=2}^{N} x_\ell^2 < 0\}$$

with $\kappa > 0$ and a C^2-function $g : [0, \infty) \to \mathbb{R}$ having a simple zero at some point $r > 0$ with the property that $g < 0$ on $[0, r)$ and $g > 0$ on (r, ∞). Then Ω is a bounded domain of class C^2, and it can easily be shown that (3.1.19) holds for $\varepsilon \geq 0$ sufficiently small, so in particular for $\varepsilon = 0$. As an explicit example in dimension $N = 2$, we consider the non-starshaped domain

$$\Omega = \{x \in \mathbb{R}^2 \,:\, 3x_1^2 - 5x_1^4 + x_1^6 - 1 + 4y^4 < 0\}.$$

In this case, a careful estimate shows that (3.1.19) holds with $\varepsilon = \frac{1}{2}$ (see Figure 2 below). Hence Corollary 3.1.7 yields nonexistence of nontrivial solutions to (3.1.18) for $s \in (0, \frac{1}{2})$ and

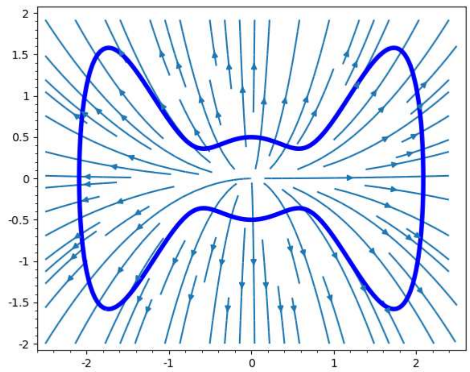

Figure 3.2: Domain Ω and flow lines of the vector field $X_{1/2}$.

$p > \frac{4}{1-2s}$. A related study of non-starshaped rotationally symmetric domains in the context of the second order semilinear elliptic PDEs is contained in [85, Section 2].

Next, we briefly comment on the proof of Theorems 3.1.1 and 3.1.2, which relies on some integral identities and boundary estimates obtained recently by the authors in [32] to obtain a Hadamard formula for the rate of change of best constants in subcritical Sobolev embeddings with respect to domain deformations. In particular, this Hadamard formula applies to the first Dirichlet eigenvalue of $(-\Delta)^s$. It is one aim of the paper to indicate close connections between Hadamard formulas and Pohozaev identies in the fractional setting. In fact, both in the Hadamard formula given in [32, Theorem 1.1] and in fractional Pohozaev type identies, the boundary term on the LHS of (3.1.8) appears. Moreover, the bilinear form $\mathscr{E}_{K_X}(u,v)$ related to the fractional deformation kernel K_X defined in (3.1.6) arises as a derivative $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\mathscr{E}(u\circ\Phi_\varepsilon, v\circ\Phi_\varepsilon)$, where $\mathscr{E}$ is the unperturbed bilinear form given in (3.1.4) and $\varepsilon \to \Phi_\varepsilon$ is a family of diffeomorphisms $\mathbb{R}^N \to \mathbb{R}^N$ with $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\Phi_\varepsilon = X$, see [32] and Section 3.3 below for more details. The connections will be further stressed in Theorem 3.3.1 below, which is a variant of the Hadamard formula given in [32, Corollary 1.2] dealing with *arbitrary* Dirichlet eigenvalues of $(-\Delta)^s$.

As a further application of the fractional Pohozaev identity in the form (3.1.5), we derive, in Theorem 3.4.1, the simplicity of radial eigenvalues of $(-\Delta)^s$ in a ball or an annulus, and in Theorem 3.4.3 we provide a multiplicity estimate in general (disconnected) radial open bounded sets. The proofs of these facts are extremely simple but have not been noticed in the literature up to our knowledge. As an application of Theorem 3.3.1, we also derive, in Theorem 3.4.1, a rate of change formula for radial eigenvalues with respect to radial deformations of balls or annuli.

The Chapter is organized as follows: Section 3.2 is devoted to the proof of Theorem 3.1.2, Corollary 3.1.4 and Corollary 3.1.6. In the last section, we use the identity (3.1.8) to derive Hadamard formula for simple eigenvalues of the Dirichlet fractional Laplacian and we apply the latter to radial eigenvalues of bounded radial domains.

3.2 Proof of the generalized integration by parts formula Theorem 3.1.2

This section is mainly devoted to the proof of Theorem 3.1.2. Throughout this section, let X be a vector field of class $C^{0,1}$ which satisfies a global Lipschitz bound. Recall the definition in (3.1.7) of the bilinear form $\mathscr{E}_{K_X}$ associated to X. We first need the following result.

Lemma 3.2.1. *Let $U \in C_c^\alpha(\Omega)$ for some $\alpha > \max\{1, 2s\}$. Then we have*

$$\mathscr{E}_{K_X}(U,U) = -2 \int_{\mathbb{R}^N} \nabla U \cdot X(-\Delta)^s U dx. \tag{3.2.1}$$

A similar statement has been proved under slightly stronger regularity assumptions on U in [32, Lemma 4.2]. Here we give a somewhat simpler proof which is also consistent with the present notation.

Proof. By symmetry of the kernel and Fubini's theorem, we have

$$\begin{aligned}
&\mathscr{E}_{K_X}(U,U)\\
&= \frac{b_{N,s}}{2} \int_{\mathbb{R}^{2N}} (U(x)-U(y))^2 \Big[\frac{\operatorname{div} X(x) + \operatorname{div} X(y)}{|x-y|^{N+2s}} - (N+2s)\frac{\big(X(x)-X(y)\big)\cdot(x-y)}{|x-y|^{N+2s+2}}\Big] dxdy\\
&= b_{N,s} \lim_{\mu\to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N \setminus \overline{B_\mu(y)}} (U(x)-U(y))^2 \Big[\frac{\operatorname{div} X(x)}{|x-y|^{N+2s}} - (N+2s)\frac{(x-y)\cdot X(x)}{|x-y|^{N+2s+2}} dxdy\\
&= b_{N,s} \lim_{\mu\to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N \setminus \overline{B_\mu(y)}} (U(x)-U(y))^2 \nabla_x \Big(|x-y|^{-N-2s} X(x)\Big) dxdy.
\end{aligned}$$

Applying, for fixed $y \in \mathbb{R}^N$ and $\mu > 0$, the divergence theorem in the domain $\mathbb{R}^N \setminus \overline{B_\mu(y)}$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} K_X(x,y)(U(x)-U(y))^2 dxdy \qquad (3.2.2)\\
&= b_{N,s} \lim_{\mu\to 0} \int_{\mathbb{R}^N} \int_{\partial B_\mu(y)} (U(x)-U(y))^2 \frac{y-x}{|x-y|^{N+2s+1}} \cdot X(x)\, d\sigma(y)\, dx\\
&- 2b_{N,s} \lim_{\mu\to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N \setminus \overline{B_\mu(y)}} \frac{(U(x)-U(y))\nabla U(x)\cdot X(x)}{|x-y|^{N+2s}} dxdy\Big] =: I_1 - 2I_2.
\end{aligned}$$

Since $U \in C_c^\alpha(\Omega)$ for some $\alpha > 2s$, we may use Fubini's theorem and the change of variable $(x,y) \mapsto (y, y-x)$ to see that

$$I_2 = b_{N,s} \lim_{\mu\to 0} \int_{\mathbb{R}^N \setminus \overline{B_\mu(0)}} \int_{\mathbb{R}^N} \frac{U(y)-U(y-x)}{|x|^{N+2s}} \nabla U(y)\cdot X(y) dydx$$

$$
\begin{aligned}
&= \frac{b_{N,s}}{2} \int\limits_{\mathbb{R}^{2N}} \frac{2U(y)-U(y-x)-U(y+x)}{|x|^{N+2s}} \nabla U(y)\cdot X(y)\,dy dx \\
&= \frac{b_{N,s}}{2} \int\limits_{\mathbb{R}^{N}} \nabla U(y)\cdot X(y) \int\limits_{\mathbb{R}^{N}} \frac{2U(y)-U(y-x)-U(y+x)}{|x|^{N+2s}} dx dy \\
&= \int\limits_{\mathbb{R}^{N}} \nabla U(y)\cdot X(y)(-\Delta)^s U(y) dy.
\end{aligned}
\tag{3.2.3}
$$

Moreover, also by Fubini's theorem, we have

$$
I_1 = \frac{1}{2}\lim_{\mu\to 0} \mu^{-N-1-2s} \int\limits_{|x-y|=\mu} (U(x)-U(y))^2 (y-x)\cdot (X(x)-X(y))\, d\sigma(x,y) \tag{3.2.4}
$$

Moreover, since U is compactly supported, we may fix $R>0$ large enough such that $(U(x)-U(y))^2 = 0$ for all $x,y\in B_R(0)$ with $|x-y|<1$. Setting $N_\mu := \{(x,y)\in B_R(0)\times B_R(0) : |x-y| = \mu\}$ for $0<\mu<1$ and using that $U,X\in C^{0,1}(\mathbb{R}^N)$, we thus deduce that

$$
\begin{aligned}
&\mu^{-N-1-2s} \int\limits_{|x-y|=\mu} (U(x)-U(y))^2 (y-x)\cdot (X(x)-X(y))\, d\sigma(x,y) \\
&= \mu^{-N-1-2s} \int\limits_{N_\mu} (U(x)-U(y))^2 (y-x)\cdot (X(x)-X(y))\, d\sigma(x,y) = O(\mu^{3-1-2s}) \to 0,
\end{aligned}
$$

as $\mu\to 0$, since the $2N-1$-dimensional measure of the set N_μ is of order $O(N-1)$ as $\mu\to 0$. Thus (3.2.4) yields $I_1 = 0$, and together with (3.2.2) and (3.2.3) the claim follows. □

Next we consider $u\in \mathscr{H}_0^s(\Omega)$ satisfying the assumptions of Theorem 3.1.2, and we recall that $u\in C_0^s(\overline{\Omega})\cap C^{\alpha}_{loc}(\Omega)$ with $\alpha > \max(1,2s)$ by the standard regularity theory. We cannot apply Lemma 3.2.1 directly to u since u does not have compact support. We therefore consider inner approximations $U_k := u\varsigma_k$ for $k\in\mathbb{N}$ for suitable functions $\varsigma_k \in C^{1,1}(\mathbb{R}^N)$. To define ς_k, we note that, since Ω is of class $C^{1,1}$ by assumption, the signed distance function to $\partial\Omega$ is also of class $C^{1,1}$ in a neighborhood of $\partial\Omega$. We therefore may consider a function $\tilde{d}\in C^{1,1}(\mathbb{R}^N)$ which is is positive in Ω, negative in $\mathbb{R}^N\setminus\overline{\Omega}$ and coincides with the signed distance to $\partial\Omega$ in a neighborhood of $\partial\Omega$. We then define $\varsigma_k\in C^{1,1}(\mathbb{R}^N)$ by

$$
\varsigma_k(x) = 1-\rho(k\tilde{d}(x)).
$$

where $\rho\in C_c^\infty(-2,2)$ is fixed with $0\le\rho\le 1$ and $\rho\equiv 1$ on $(-1,1)$. By [32, Lemma 2.1 and 2.2], we have, for arbitrary $u\in\mathscr{H}_0^s(\Omega)$,

$$
u\varsigma_k \to u \qquad \text{in } \mathscr{H}_0^s(\Omega) \quad \text{and} \quad \mathscr{E}_{K_X}(u\varsigma_k, u\varsigma_k) \to \mathscr{E}_{K_X}(u,u) \quad \text{as} \quad k\to\infty. \tag{3.2.5}
$$

Indeed, the latter is true for more general kernel functions $(x,y)\mapsto K(x,y)$ in place of K_X as long as $(x,y)\mapsto K(x,y)|x-y|^{N+2s}$ defines a function in $L^\infty(\mathbb{R}^N\times\mathbb{R}^N)$. To complete the proof of Theorem 3.1.2, we need, as a final tool taken from[32], the following limit identity.

Proposition 3.2.2. *We have*[1]

$$\lim_{k\to\infty}\int_{\Omega}\nabla(u\zeta_k)\cdot X\Big(u(-\Delta)^s\zeta_k - I(u,\zeta_k)\Big)\,dx = \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\big(\frac{u}{d^s}\big)^2 X\cdot\nu\,dx, \tag{3.2.6}$$

where

$$I(u,\varsigma_k)(x) := \frac{b_{N,s}}{2}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(\varsigma_k(x)-\varsigma_k(y))}{|x-y|^{N+2s}}\,dy. \tag{3.2.7}$$

Proof. As stated in [32, Prop. 2.4 and Remark 2.5], the identity (3.2.6) holds for functions $u\in C^s_0(\overline{\Omega})\cap C^1_{loc}(\Omega)$ satisfying, for some $\alpha>0$, the following regularity properties:

$$\frac{u}{d^s}\in C^{\alpha}(\overline{\Omega}) \qquad \text{and} \qquad d^{1-\alpha}\nabla\big(\frac{u}{d^s}\big) \quad \text{is bounded in a neighborhood of } \partial\Omega. \tag{3.2.8}$$

The first property is satisfied under the assumption of Theorem 3.1.2 by the regularity theory in [42]. Moreover, the gradient estimate for $\frac{u}{d^s}$ holds as well in this case, as proved in [42].
Hence the identity (3.2.6) holds if u satisfies the assumptions of Theorem 3.1.2[2] □

We may now complete the

Proof of Theorem 3.1.2 . Applying Lemma 3.2.1 to $U_k = u\varsigma_k$, we find that

$$\mathscr{E}_{K_X}(U_k,U_k) = -2\int_{\mathbb{R}^N}\nabla U_k\cdot X(-\Delta)^s U_k dx = -2\int_{\Omega}\nabla U_k\cdot X(-\Delta)^s U_k dx \qquad \text{for } k\in\mathbb{N}.$$

By the standard product rule for the fractional Laplacian, we have

$$(-\Delta)^s U_k = \varsigma_k(-\Delta)^s u + u(-\Delta)^s\varsigma_k - I(u,\varsigma_k) \qquad \text{in } \Omega$$

with $I(u,\varsigma_k)$ given in 3.2.7. We thus obtain

$$\begin{aligned}\mathscr{E}_{K_X}(U_k,U_k) &= -2\int_{\Omega}\varsigma_k\nabla U_k\cdot X(-\Delta)^s u\,dx - 2\int_{\mathbb{R}^N}\nabla U_k\cdot X\Big(u(-\Delta)^s\varsigma_k - I(u,\varsigma_k)\Big)\,dx,\\ &= -2\int_{\Omega}\varsigma_k^2\nabla u\cdot X(-\Delta)^s u\,dx - 2\int_{\mathbb{R}^N}\nabla U_k\cdot X\Big(u(-\Delta)^s\varsigma_k - I(u,\varsigma_k)\Big)\,dx\\ &\quad -2\int_{\Omega}u\varsigma_k\nabla\varsigma_k\cdot X(-\Delta)^s u\,dx.\end{aligned}$$

Since $(-\Delta)^s u\in L^\infty(\Omega)$ and $u\in C^s_0(\overline{\Omega})$, we easily find, by definition of ζ_k, that

$$\int_{\Omega}u\varsigma_k\nabla\varsigma_k\cdot X(-\Delta)^s u\,dx \to 0 \qquad \text{as} \qquad k\to\infty. \tag{3.2.9}$$

[1] Note that sign of the RHS of (3.2.6) differs from [32] since the inner unit normal is used in [32]

[2] In [32, Prop. 2.4], the property $u\varsigma_k\in C^{1,1}_c(\Omega)$ for all k is unnecessarily stated as an assumption. Only the bounds 3.2.8 and the $C^{1,1}$-regularity of the functions ζ_k are used in the proof.

Taking the limit into the identity above and using (3.2.5), (3.2.6) and (3.2.9) we deduce

$$\mathscr{E}_{K_X}(u,u) = \lim_{k\to\infty} c_k(u) = -2\int_{\Omega} \nabla u \cdot X(-\Delta)^s u\,dx - \Gamma^2(1+s)\int_{\partial\Omega} \left(\frac{u}{d^s}\right)^2 X\cdot\nu\,dx. \tag{3.2.10}$$

The proof is finished. □

Next we give the

Proof of Corollary 3.1.6. As noted in Remark 3.1.5, it follows from assumption (3.1.11) that

$$\operatorname{div} X \equiv cN \qquad \text{and therefore} \qquad K_X(x,y) = \frac{b_{N,s}c}{2}(N-2s)|x-y|^{-N-2s} \quad \text{for } x,y\in\mathbb{R}^N.$$

Hence for every weak solution $u\in\mathscr{H}_0^s(\Omega)\cap L^\infty(\Omega)$ of (3.1.1) we have

$$\mathscr{E}_{K_X}(u,u) = c(N-2s)\mathscr{E}(u,u) = c(N-2s)\int_{\Omega} f(u)u\,dx.$$

Therefore (3.1.8) reduces to (3.1.12) in this case, as claimed. □

We close this section with the

Proof of Corollary 3.1.6. We first note that the second condition in (3.1.15) implies that

$$\big(dX(y)h\big)\cdot h \le c_2|h|^2 \qquad \text{for a.e. } y\in\mathbb{R}^N \text{ and every } h\in\mathbb{R}^N. \tag{3.2.11}$$

Applying (3.2.11) to the coordinate vectors $e_1,\dots,e_N\in\mathbb{R}^N$ and combining the result with the first condition in (3.1.15), we deduce that

$$c_1 \le \operatorname{div} X \le c_2 N \qquad \text{a.e. on } \mathbb{R}^N. \tag{3.2.12}$$

In particular, this implies that

$$K_X(x,y) \ge \frac{b_{N,s}}{2}\,\frac{2c_1 - c_2(N+2s)}{|x-y|^{N+2s}} \qquad \text{for } x,y\in\mathbb{R}^N.$$

Hence for every weak solution $u\in\mathscr{H}_0^s(\Omega)\cap L^\infty(\Omega)$ of (3.1.1) we have

$$\mathscr{E}_{K_X}(u,u) \ge \big[2c_1 - c_2(N+2s)\big]\mathscr{E}(u,u) = \big[2c_1 - c_2(N+2s)\big]\int_{\Omega} f(u)u\,dx.$$

Since $F(u)$ is nonnegative in Ω by assumption, we thus conclude from (3.1.8) and (3.2.12) that

$$\begin{aligned}\Gamma(1+s)^2\int_{\partial\Omega}\left(\frac{u}{d^s}\right)^2 X\cdot\nu\,dx &= 2\int_{\Omega} F(u)\operatorname{div} X\,dx - \mathscr{E}_X(u,u)\\ &\le \int_{\Omega}\Big(2c_2NF(u) - \big[2c_1 - c_2(N+2s)\big]f(u)u\Big)dx,\end{aligned}$$

as claimed in (3.1.16). Moreover, (3.1.17) is a direct consequence of (3.1.16). □

3.3 A Hadamard formula for the fractional Dirichlet eigenvalue problem

Let $\Omega \subset \mathbb{R}^N$ denote a bounded open set with $C^{1,1}$-boundary. From now on we fix $\varepsilon_0 > 0$ and a family of deformations $\{\Phi_\varepsilon\}_{\varepsilon \in (-\varepsilon_0, \varepsilon_0)}$ with the following properties:

$$\begin{aligned}&\Phi_\varepsilon \in C^{1,1}(\mathbb{R}^N;\mathbb{R}^N) \text{ for } \varepsilon \in (-1,1),\ \Phi_0 = \mathrm{id}_{\mathbb{R}^N}, \text{ and}\\ &\text{the map } (-1,1) \to C^{1,1}(\mathbb{R}^N,\mathbb{R}^N),\ \varepsilon \to \Phi_\varepsilon \text{ is of class } C^2.\end{aligned} \tag{3.3.1}$$

By making $\varepsilon_0 > 0$ smaller if necessary, we may then assume that $\Phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ is a global diffeomorphism for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, see e.g. [28, Chapter 4.1]. For $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we write $\Omega_\varepsilon = \Phi_\varepsilon(\Omega)$.

The aim of this section is to establish the following rate of change formula for Dirichlet eigenvalues of the fractional Laplacian with respect to the domain deformation given by Φ_ε.

Theorem 3.3.1. *Consider a C^1-curve*

$$(-\varepsilon_0, \varepsilon_0) \to \mathcal{H}_0^s(\Omega), \qquad \varepsilon \mapsto u_\varepsilon$$

with the property that, for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, the function

$$v_\varepsilon := u_\varepsilon \circ \Phi_\varepsilon^{-1} \quad \in \mathcal{H}_0^s(\Omega_\varepsilon)$$

is a nontrivial weak solution of the Dirichlet eigenvalue problem

$$(-\Delta)^s v_\varepsilon = \lambda(\varepsilon) v_\varepsilon \quad \text{in } \Omega_\varepsilon, \qquad v_\varepsilon \equiv 0 \quad \text{in } \mathbb{R}^N \setminus \Omega_\varepsilon, \tag{3.3.2}$$

for some $\lambda(\varepsilon) \in \mathbb{R}$. Then the function $\varepsilon \mapsto \lambda_\varepsilon$ is of class C^1 on $(-\varepsilon_0, \varepsilon_0)$, and we have the identity

$$\partial_\varepsilon\Big|_{\varepsilon=0} \lambda(\varepsilon) = -\frac{\Gamma(1+s)^2 \int_{\partial\Omega} \left(u/d^s\right)^2 X \cdot \nu \, dx}{\int_\Omega u^2 \, dx}$$

with $u := u_0$ and the vector field

$$X := \partial_\varepsilon\Big|_{\varepsilon=0} \Phi_\varepsilon \in C^{1,1}(\mathbb{R}^N, \mathbb{R}^N). \tag{3.3.3}$$

For the proof of this theorem, we introduce some notation. In weak sense, the eigenvalue equation for v_ε reads

$$\mathcal{E}(v_\varepsilon, \varphi) = \lambda_\varepsilon \int_{\Omega_\varepsilon} v_\varepsilon \varphi \, dx \qquad \text{for all } \varphi \in \mathcal{H}_0^s(\Omega_\varepsilon), \tag{3.3.4}$$

Using the fact that the map

$$\mathcal{H}_0^s(\Omega) \to \mathcal{H}_0^s(\Omega_\varepsilon), \qquad \psi \mapsto \psi \circ \Phi_\varepsilon^{-1}$$

is a topological isomorphism, we may rewrite this property, by means of integral transformations, in the form

$$\mathscr{E}^{\varepsilon}(u_\varepsilon,\varphi)=\lambda(\varepsilon)\int_\Omega u_\varepsilon\varphi \mathrm{Jac}_{\Phi_\varepsilon}\,dx \qquad \text{for all } \varphi\in\mathscr{H}_0^s(\Omega). \tag{3.3.5}$$

Here $\mathrm{Jac}_{\Phi_\varepsilon}$ denotes the Jacobian determinant of the map $\Phi_\varepsilon\in C^{1,1}(\mathbb{R}^N,\mathbb{R}^N)$, and

$$\mathscr{E}^{\varepsilon}(v,\varphi):=\frac{1}{2}\int_{\mathbb{R}^{2N}}(v(x)-v(y))(\varphi(x)-\varphi(y))K_\varepsilon(x,y)dxdy \qquad \text{for } v,\varphi\in\mathscr{H}_0^s(\Omega) \tag{3.3.6}$$

with the kernel

$$K_\varepsilon(x,y):=b_{N,s}\frac{\mathrm{Jac}_{\Phi_\varepsilon}(x)\mathrm{Jac}_{\Phi_\varepsilon}(y)}{|\Phi_\varepsilon(x)-\Phi_\varepsilon(y)|^{N+2s}} \qquad \text{for } \varepsilon\in(-\varepsilon_0,\varepsilon_0). \tag{3.3.7}$$

The first step in the proof of Theorem 3.3.1 is the following lemma.

Lemma 3.3.2. *(i) The map*

$$(-\varepsilon_0,\varepsilon_0)\times\mathscr{H}_0^s(\Omega)\times\mathscr{H}_0^s(\Omega)\to\mathbb{R}, \qquad (\varepsilon,v,w)\mapsto\mathscr{E}^{\varepsilon}(v,w)$$

is of class C^1 with

$$\widetilde{\mathscr{E}}(v,w):=\partial_\varepsilon\Big|_{\varepsilon=0}\mathscr{E}^{\varepsilon}(v,w)=\mathscr{E}_{\mathscr{X}}(u,v) \tag{3.3.8}$$

for $v,w\in\mathscr{H}_0^s(\Omega)$, where X is given in (3.3.3) and the kernel $\mathscr{E}_X(v,w)$ is defined in (3.1.7).

(ii) The map

$$(-\varepsilon_0,\varepsilon_0)\times L^2(\Omega)\times L^2(\Omega)\to\mathbb{R}, \qquad (\varepsilon,v,w)\mapsto\int_\Omega vwJac_{\Phi_\varepsilon}\,dx$$

is of class C^1 with

$$\partial_\varepsilon\Big|_{\varepsilon=0}\int_\Omega vwJac_{\Phi_\varepsilon}\,dx=\int_\Omega vw\,div X\,dx \qquad \text{for } v,w\in L^2(\Omega).$$

Proof. We only give the proof of (i), the proof of (ii) is similar but easier. We first note that, by direct computation, we have

$$\partial_\varepsilon\Big|_{\varepsilon=0}\Big(|x-y|^{N+2s}K_\varepsilon(x,y)\Big) \tag{3.3.9}$$
$$=b_{N,s}\left\{(\mathrm{div}X(x)+\mathrm{div}X(y))-(N+2s)\frac{(X(x)-X(y))(x-y)}{|x-y|^2}-\right\}=2|x-y|^{N+2s}K_X(x,y)$$

uniformly in $x, y \in \mathbb{R}^N$. Next we consider the space $\mathscr{L}_S^2(\mathscr{H}_0^s(\Omega))$ of continuous symmetric bilinear forms on $\mathscr{H}_0^s(\Omega)$, which is endowed with the norm

$$\|b\|_{\mathscr{L}_s^2} := \sup\Big\{ \frac{|b(v,w)|}{\|v\|_{H^s}\|w\|_{H^s}} \,:\, v, w \in \mathscr{H}_0^s(\Omega) \setminus \{0\} \Big\}$$

It then suffices to show that

$$\text{the map } (-\varepsilon_0, \varepsilon_0) \to \mathscr{L}_S^2(\mathscr{H}_0^s(\Omega)),\ \varepsilon \mapsto \mathscr{E}^\varepsilon \text{ is of class } C^1 \text{ with } \partial_\varepsilon\big|_{\varepsilon=0} \mathscr{E}^\varepsilon = \widetilde{\mathscr{E}}, \tag{3.3.10}$$

where $\widetilde{\mathscr{E}}$ is defined in (3.3.8). To see the differentiability at $\varepsilon = 0$, we note that, since

$$K_\varepsilon(x,y) - b_{N,s}|x-y|^{-N-2s} - 2K_X(x,y) = o(\varepsilon)|x-y|^{-N-2s}$$

by (3.3.9), we have, for $v, w \in \mathscr{H}_0^1(\Omega)$,

$$\begin{aligned}
&\mathscr{E}^\varepsilon(v,w) - \mathscr{E}(v,w) - \widetilde{\mathscr{E}}(v,w) \\
&\quad = \frac{1}{2}\int_{\mathbb{R}^{2N}} (v(x)-v(y))(w(x)-w(y))\big(K_\varepsilon(x,y) - c_{N,s}|x-y|^{-N-2s} - 2K_X(x,y)\big)dxdy \\
&\quad = o(\varepsilon)\int_{\mathbb{R}^{2N}} (v(x)-v(y))(w(x)-w(y))K_0(x,y)dxdy \le o(\varepsilon)\|v\|_{H_0^1}\|w\|_{H_0^1}
\end{aligned}$$

as $\varepsilon \to 0$, where $o(\varepsilon)$ is independent of v and w. Consequently,

$$\|\mathscr{E}^\varepsilon - \mathscr{E} - \varepsilon\widetilde{\mathscr{E}}\|_{\mathscr{L}_S^2} = \sup\Big\{ \frac{|\mathscr{E}^\varepsilon(v,w) - \mathscr{E}(v,w) - \widetilde{\mathscr{E}}(v,w)|}{\|v\|_{H_0^1}\|w\|_{H_0^1}} \,:\, v, w \in \mathscr{H}_0^1(\Omega) \setminus \{0\} \Big\} = o(\varepsilon),$$

and this shows that the map $\varepsilon \mapsto \mathscr{E}^\varepsilon$ is differentiable at $\varepsilon = 0$ with $\partial_\varepsilon\big|_{\varepsilon=0}\mathscr{E}^\varepsilon = \widetilde{\mathscr{E}}$. For $\varepsilon_* \in (-\varepsilon_0, \varepsilon_0)$ different from 0, the same argument shows that $\varepsilon \mapsto \mathscr{E}^\varepsilon$ is differentiabe at ε_*, where $\partial_\varepsilon\big|_{\varepsilon=\varepsilon_*}\mathscr{E}^\varepsilon$ has the same form as $\partial_\varepsilon\big|_{\varepsilon=0}\mathscr{E}^\varepsilon$ in (3.3.8) with X replaced by $X_{\varepsilon_*} := \partial_\varepsilon\big|_{\varepsilon=\varepsilon_*}\Phi_\varepsilon$. Finally, the continuity of the map

$$(-\varepsilon_0, \varepsilon_0) \to \mathscr{L}_S^2(\mathscr{H}_0^s(\Omega)), \qquad \varepsilon_* \mapsto \partial_\varepsilon\big|_{\varepsilon=\varepsilon_*}\mathscr{E}^\varepsilon,$$

follows in a straightforward way from the fact that the map

$$(-\varepsilon_0, \varepsilon_0) \to L^\infty(\mathbb{R}^N \times \mathbb{R}^N), \qquad \varepsilon \mapsto |x-y|^{N+2s}K_{X_\varepsilon}(x,y)$$

is continuous. The proof of (3.3.10) is thus finished. □

We may then complete the

Proof of Theorem 3.3.1. Since $\lambda(\varepsilon) = \frac{\mathscr{E}^\varepsilon(u_\varepsilon, u_\varepsilon)}{\int_\Omega u_\varepsilon^2 \mathrm{Jac}_{\Phi_\varepsilon}\,dx}$ for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, it follows from Lemma 3.3.2 that the map $\varepsilon \mapsto \lambda(\varepsilon)$ is of class C^1. So we may differentiate the equation

$$\mathscr{E}^\varepsilon(u_\varepsilon, u_\varepsilon) = \lambda(\varepsilon)\int_\Omega u_\varepsilon^2 \mathrm{Jac}_{\Phi_\varepsilon}\,dx,$$

at $\varepsilon = 0$, noting that

$$\partial_\varepsilon\Big|_{\varepsilon=0}\mathscr{E}^\varepsilon(u_\varepsilon,u_\varepsilon) = \widetilde{\mathscr{E}}(u,u) + 2\mathscr{E}(\partial_\varepsilon\big|_{\varepsilon=0}u_\varepsilon,u)$$

and

$$\partial_\varepsilon\Big|_{\varepsilon=0}\Big(\lambda(\varepsilon)\int_\Omega u_\varepsilon^2 \mathrm{Jac}_{\Phi_\varepsilon}\,dx\Big) = \Big(\partial_\varepsilon\big|_{\varepsilon=0}\lambda(\varepsilon)\Big)\int_\Omega u^2\,dx + 2\lambda(0)\int_\Omega\Big(\partial_\varepsilon\big|_{\varepsilon=0}v_\varepsilon\Big)\varphi\,dx + \lambda(0)\int_\Omega u^2 \mathrm{div}X\,dx.$$

Since

$$\mathscr{E}(\partial_\varepsilon\big|_{\varepsilon=0}u_\varepsilon,u) = \lambda(0)\int_\Omega\Big(\partial_\varepsilon\big|_{\varepsilon=0}u_\varepsilon\Big)u\,dx$$

as a consequence of (3.3.5), we deduce that

$$\widetilde{\mathscr{E}}(u,u) = \Big(\partial_\varepsilon\big|_{\varepsilon=0}\lambda(\varepsilon)\Big)\int_\Omega u^2\,dx + \lambda(0)\int_\Omega u^2 \mathrm{div}X\,dx.$$

On the other hand, the generalized Pohozaev identity (3.1.8) for u gives, with $F(u) = \frac{\lambda_0}{2}u^2$,

$$\int_{\mathbb{R}^{2N}} K_X(x,y)(u(x)-u(y))^2\,dxdy = \lambda_0\int_\Omega u^2 \mathrm{div}X\,dx - \Gamma(1+s)^2\int_{\partial\Omega}(\frac{u}{d^s})^2 X\cdot\nu\,dx$$

Recalling (3.3.8), we conclude that

$$\partial_\varepsilon\big|_{\varepsilon=0}\lambda(\varepsilon) = -\frac{\Gamma(1+s)^2\int_{\partial\Omega}\left(u/d^s\right)^2 X\cdot\nu\,dx}{\int_\Omega u^2\,dx},$$

as claimed. □

3.4 Application to the radial eigenvalue problem

In this section we study the eigenvalue problem

$$(-\Delta)^s w = \mu w \quad \text{in } \Omega, \qquad u \equiv 0 \quad \text{in } \Omega^c, \tag{3.4.1}$$

among radial functions. For this, we let H^s_{rad} denote the subspace of radially symmetric functions in the space $\mathscr{H}^s_0(\Omega)$. By definition, a function $w \in H^s_{rad}$ is an eigenfunction of (4.1.1) corresponding to the eigenvalue μ if

$$\mathscr{E}_s(w,\psi) = \mu(\Omega)\int_\Omega w(x)\psi(x)dx \qquad \text{for all } \psi \in H^s_{rad}. \tag{3.4.2}$$

In the following, we will call μ a *radial eigenvalue* for μ if there exists an eigenfunction $w \in H^s_{rad}$ for μ. It is a well-known fact that the radial eigenvalues of (4.1.1) form an increasing sequence of numbers $0 < \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \nearrow +\infty$, counted with possible multiplicity.

While the simplicity of μ_1 is a classical fact (see e.g [55]), the same property seems unavailable in the literature for higher eigenvalues. In this section, we shall show, by means of the fractional Pohazaev identity (3.1.5), that all radial eigenvalues are simple in the case where Ω is a ball or an annulus in $\mathbb{R}^N$.
For a related question, we refer to [50] where for $\Omega = \mathbb{R}^N$ simplicity result has been obtained for Schrödinger operator with a increasing radially symmetric potential. The second aim of this section is to derive, from Theorem 3.3.1, a Hadamard formula for the dependence of the k-th eigenvalue μ_k on the inner and outer radius of Ω.
The following is the main result of this section. Here and in the following, we identify a radial function $u = u(x)$ with the associated function $u = u(r)$ of the radial variable.

Theorem 3.4.1. *Let* $0 < r_{inn} < r_{out} < \infty$ *and suppose that either*

$$\Omega = B_{r_{out}}(0) \qquad or \qquad \Omega = A(r_{inn}, r_{out}) := \{x \in \mathbb{R}^N \,:\, r_{inn} < |x| < r_{out}\}.$$

Let $k \geq 1$ *and let* λ_k *be the k-th radial eigenvalue of* (4.1.1). *Then we have:*

(i) $\lambda_k(\Omega)$ *is simple.*

(ii) λ_k *depends in a differentiable way on* r_{out} *with*

$$\frac{\partial_{\lambda_k}}{\partial r_{out}} = -\Gamma(1+s)^2|S^{N-1}|r_{out}^{N-1}\psi_u^2(r_{out}). \tag{3.4.3}$$

Moreover, in the case where $\Omega = A(r_{inn}, r_{out})$, λ_k *depends in a differentiable way on* r_{inn} *with*

$$\frac{\partial_{\lambda_k}}{\partial r_{inn}} = \Gamma(1+s)^2|S^{N-1}|r_{inn}^{N-1}\psi_u^2(r_{inn}) \tag{3.4.4}$$

Here $u \in H^s_{rad}$ *is the (up to sign unique)* L^2*-normalized eigenfunction associated with* λ_k*, and* ψ_u *is the continuous extension of* $\frac{u}{d^s}$ *to* $\overline{\Omega}$*, as before.*

Notice that the statement of the theorem is new for $k > 1$ but is already known for $k = 1$. In fact, as we already mentioned above, the simplicity of the first (radial) eigenvalue is a classical fact, while the identities (3.4.3) and (3.4.4) follow from [32, Corollary 1.2] in this special case.
In the case $N = 1$ the annulus $A(r_{inn}, r_{out})$ is a disconnected set. It is therefore natural to ask whether at least a weaker variant of Theorem 3.4.1 still holds on other disconnected radial open sets. The following result gives a partial answer to this question.

Theorem 3.4.2. *Consider, for some* $n \in \mathbb{N}$, *real positive numbers*

$$0 < r_{inn}^1 < r_{out}^1 < r_{inn}^2 < r_{out}^2 < \cdots < r_{inn}^n < r_{out}^n < \infty$$

and suppose that either

$$\Omega = B_{r_{out}^1}(0) \cup \bigcup_{i=2}^{n} A(r_{inn}^i, r_{out}^i) \quad or \quad \Omega = \bigcup_{i=1}^{n} A(r_{inn}^i, r_{out}^i).$$

Then every radial eigenvalue of 4.1.1 on Ω *has multiplicity at most n.*

Note that Theorem 3.4.1(i) is a special case of Theorem 3.4.2. In the following, we therefore give the proof of Theorem 3.4.2 first and then add the proof of Theorem 3.4.1(ii).
We start by noting the following direct consequence of the fractional Pohozaev type identity (3.1.5).

Proposition 3.4.3. *Consider, for some $n \in \mathbb{N}$, real positive numbers*

$$0 < r^1_{inn} < r^1_{out} < r^2_{inn} < r^2_{out} < \cdots < r^n_{inn} < r^n_{out} < \infty$$

and suppose that either

$$\Omega = B_{r^1_{out}}(0) \cup \bigcup_{i=2}^{n} A(r^i_{inn}, r^i_{out}) \quad or \quad \Omega = \bigcup_{i=1}^{n} A(r^i_{inn}, r^i_{out}).$$

Moreover, let $u \in H^s_{rad}$ be a solution of (4.1.1).

(i) We have

$$u = 0 \qquad \text{if and only if} \qquad \psi_u(r^1_{out}) = \psi_u(r^2_{out}) = \cdots = \psi_u(r^n_{out}) = 0. \tag{3.4.5}$$

(ii) If $\Omega = \bigcup_{i=1}^{n} A(r^i_{inn}, r^i_{out})$, then

$$\int_\Omega u^2\,dx = \frac{|S^{N-1}|\Gamma(1+s)^2}{2s\mu} \sum_{i=i}^{n} \Big(\big(r^i_{out}\big)^N \psi_u^2(r^i_{out}) - \big(r^i_{inn}\big)^N \psi_u^2(r^i_{inn}) \Big). \tag{3.4.6}$$

(iii) If $\Omega = B_{r^1_{out}}(0) \cup \bigcup_{i=2}^{n} A(r^i_{inn}, r^i_{out})$, then

$$\int_\Omega u^2\,dx = \tag{3.4.7}$$

$$\frac{|S^{N-1}|\Gamma(1+s)^2}{2s\mu} \Big(\big(r^1_{out}\big)^N \psi_u^2(r^1_{out}) + \sum_{i=2}^{n} \big(r^i_{out}\big)^N \psi_u^2(r^i_{out}) - \big(r^i_{inn}\big)^N \psi_u^2(r^i_{inn}) \Big).$$

Here, as noted before, we write u and ψ_u as a function of the radial variable.

Proof. Applying (3.1.5) with $f(u) = \mu u$ and $F(u) = \frac{\mu}{2}u^2$, we obtain

$$\int_\Omega u^2\,dx = \frac{\Gamma(1+s)^2}{2s\mu} \int_{\partial\Omega} \Big(\frac{u}{d^s}\Big)^2 x \cdot \nu \, d\sigma(x), \tag{3.4.8}$$

where, as before, ν denotes the outer unit normal on $\partial\Omega$. The formulas (3.4.7) and (3.4.6) follow directly from (3.4.8) and the radiality of u in view of the fact that the outward unit normal ν on $\partial\Omega$ is given by

$$\nu(x) = \begin{cases} \dfrac{x}{|x|} & \text{if } |x| = r^i_{out} \text{ for some } i \in \{1,\dots,N\}; \\ -\dfrac{x}{|x|} & \text{if } |x| = r^i_{inn} \text{ for some } i \in \{1,\dots,N\}. \end{cases}$$

To see (3.4.5), we note that $u = 0$ trivially implies $\psi_u(r^i_{out}) = 0$ for $i = 1,\dots,n$. On the other hand, if $\psi_u(r^i_{out}) = 0$ for $i = 1,\dots,n$, it follows from (3.4.7) and (3.4.6) that $\int_\Omega u^2\,dx \le 0$ and therefore $u = 0$. Hence (3.4.5) follows. □

We may now complete the

Proof of Theorem 3.4.2. Let, for $\mu > 0$, $V_\mu \subset H^s_{rad}$ denote the space of radial solutions of the eigenvalue problem (4.1.1). From (3.4.5), it follows that the linear map

$$\ell : V_\mu \to \mathbb{R}^n, \qquad \ell(u) = \Big(\psi_u(r^1_{out}),\dots,\psi_u(r^n_{out})\Big)$$

is injective. Hence the space V_μ is at most n-dimensional. It thus follows that every positive eigenvalue μ of (4.1.1) has multiplicity at most n. □

In the remainder of this section, we give the

Proof of Theorem 3.4.1(ii). We only prove the differentiability of λ_k as a function of r_{out} and the formula (3.4.3), the differentiability as a function of r_{inn} and the formula (3.4.4) follow in a similar way.

We fix $\delta > 0$ with $\delta < r_{out}$ in case $\Omega = B_{r_{out}}(0)$ and $\delta < r_{out} - r_{inn}$ in case $\Omega = \Omega = A(r_{inn}, r_{out})$. Moreover, we let $\mathscr{X} \in C^{1,1}(\mathbb{R}^N,\mathbb{R}^N)$ be a vector field with

$$X(x) = \frac{x}{|x|} \qquad \text{for } x \in A(r_{out} - \frac{\delta}{2}, r_{out} + \frac{\delta}{2})$$

and

$$X \equiv 0 \qquad \text{in } \mathbb{R}^N \setminus A(r_{out} - \delta, r_{out} + \delta).$$

For $\varepsilon \in \mathbb{R}$, we now define $\Phi_\varepsilon \in C^{1,1}(\mathbb{R}^N,\mathbb{R}^N)$, $\Phi_\varepsilon(x) = x + \varepsilon X(x)$. Then Φ_ε satisfies the assumptions (3.3.1), so we may fix $\varepsilon_0 \in (0, \frac{\delta}{2})$ sufficiently small so that $\Phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ is a global diffeomorphism for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Moreover, for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we write $\Omega_\varepsilon = \Phi_\varepsilon(\Omega)$. By our choice of δ we have

$$\Omega_\varepsilon = B_{r_{out}+\varepsilon}(0) \qquad \text{if} \quad \Omega = B_{r_{out}}(0) \tag{3.4.9}$$

and

$$\Omega_\varepsilon = A(r_{inn}, r_{out} + \varepsilon) \qquad \text{if} \quad \Omega = A(r_{inn}, r_{out}). \tag{3.4.10}$$

Next, for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we let $\lambda(\varepsilon)$ denote the k-th eigenvalue of (4.1.1) on $\Omega = \Omega_\varepsilon$. By Proposition 3.4.3(i), there exists a unique eigenfunction $v_\varepsilon \in \mathscr{H}^s_0(\Omega_\varepsilon)$ corresponding to $\lambda(\varepsilon)$ with $\psi_{v_\varepsilon}(r_{out} + \varepsilon) > 0$ and the normalization $\|u_\varepsilon\|_{L^2(\Omega)} = 1$, where we define

$$u_\varepsilon := v_\varepsilon \circ \Phi_\varepsilon \qquad \text{for } \varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

We claim that

$$\text{the curve } (-\varepsilon, \varepsilon) \to \mathscr{H}^s_0(\Omega), \quad \varepsilon \mapsto u_\varepsilon \text{ is of class } C^1. \tag{3.4.11}$$

Once this is proved, it follows from Theorem 3.3.1 and the definition of $\mathscr{X}$ that $\varepsilon \mapsto \lambda(\varepsilon)$ is a differentiable function with

$$\partial_\varepsilon\big|_{\varepsilon=0}\lambda(\varepsilon) = -\Gamma(1+s)^2 \int_{\partial\Omega} \big(\frac{u}{d^s}\big)^2 X\cdot\nu\, dx = -\Gamma(1+s)^2|S^{N-1}|r_{out}^{N-1}\psi_u^2(r_{out}).$$

Thus (3.4.3) follows by (3.4.9) and (3.4.10). As mentioned before, (3.4.4) follows by a similar argument.

It thus remains to prove (3.4.11). More precisely, it suffices to prove, using the simplicity of the eigenvalue $\lambda(\varepsilon)$ and the implicit function theorem, the differentiability of the map $\varepsilon \to u_\varepsilon$ in a neighborhood of $\varepsilon = 0$. For $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we define the linear maps

$$L_\varepsilon \in \mathscr{L}\big(H^s_{rad}, (H^s_{rad})'\big), \qquad [L_\varepsilon v]w = \mathscr{E}^\varepsilon(v,w)$$

and

$$J_\varepsilon \in \mathscr{L}\big(H^s_{rad}, (H^s_{rad})'\big), \qquad [J_\varepsilon v]w = \int_\Omega vw\mathrm{Jac}_{\Phi_\varepsilon}\, dx.$$

Here, as usual, $(H^s_{rad})'$ denotes the topological dual of H^s_{rad}. With this notation, we can write the property (3.3.5) in the form

$$L_\varepsilon u = \lambda J_\varepsilon u \qquad \text{in } (H^s_{rad})'. \tag{3.4.12}$$

Moreover, as a consequence of Lemma 3.3.2, we see that the maps

$$(-\varepsilon_0,\varepsilon_0) \to \mathscr{L}\big(H^s_{rad}, \big(H^s_{rad}\big)'\big), \qquad \varepsilon \mapsto L_\varepsilon, \qquad \varepsilon \mapsto J_\varepsilon$$

are of class C^1. Consequently, the map

$$\Sigma : (-\varepsilon_0,\varepsilon_0)\times(0,\infty)\times H^s_{rad} \to \mathbb{R}\times(H^1_{rad})', \qquad \Sigma(\varepsilon,\lambda,u) = (\|u\|^2_{L^2(\Omega)} - 1, L_\varepsilon u - \lambda J_\varepsilon u) \tag{3.4.13}$$

is also of class C^1, and by definition we have

$$\Sigma(\varepsilon,\lambda(\varepsilon),u_\varepsilon) = 0 \qquad \text{for } \varepsilon \in (-\varepsilon_0,\varepsilon_0). \tag{3.4.14}$$

Moreover, we have

$$\frac{\partial\Sigma}{\partial(\lambda,u)}(0,\lambda,u)(\mu,v) = \Big(2\langle u,v\rangle_{L^2(\Omega)}, L_0 v - \lambda v - \mu J_0(u\Big)$$

for $(\lambda,u) \in (0,\infty)\times H^s_{rad}$ and $(\mu,v) \in \mathbb{R}\times H^s_{rad}$. We claim that

$$\frac{\partial\Sigma}{\partial(\lambda,u)}(0,\lambda(0),u_0) \in \mathscr{L}\Big(\mathbb{R}\times H^1_{rad}, \mathbb{R}\times\big(H^1_{rad}\big)'\Big) \qquad \text{is a topological isomorphism.} \tag{3.4.15}$$

Indeed, since the radial eigenvalue $\lambda(0)$ is simple by Theorem 3.4.1(i), the linear map

$$v \mapsto L_0 v - \lambda(0)J_0 v$$

defines a topological isomorphism between the spaces $\{v \in H^1_{rad} : \langle v,u_0\rangle_{L^2(\Omega)} = 0\}$ and $Y := \{\varphi \in (H^1_{rad})' : \varphi(u_0) = 0\}$. From this we readily deduce (3.4.15).

From (3.4.14), (3.4.15) and the simplicity of the eigenvalue $\lambda(\varepsilon)$, it follows by the implicit function theorem that the map $\varepsilon \mapsto u_\varepsilon$ is of class C^1 in a neighborhood of $\varepsilon = 0$, as claimed. □

Chapter 4

Nonradiality of second fractional eigenfunctions of thin annuli

This Chapter is based on the paper $[P4]$, a joint work with S. Jarohs. The exposition is as in the orignial paper. The paper deals with symmetry properties of second fractional eigenfunctions of annuli. It is proven that for annuli with small width, a second fractional eigenfunction cannot be radial. The latter is used to maximized the second fractional eigenvalue in annular-shaped domains of the type $B \setminus B'$ where B is a fixed ball and B' is ball whose position is varied within B.

4.1 Introduction

Let Ω be a radial open bounded subset of $\mathbb{R}^N$, $N \geq 1$. In the first part of this note, we are interested to symmetry properties of weak solutions to the eigenvalue problem

$$(-\Delta)^s \varphi = \mu \varphi \quad \text{in} \quad \Omega \qquad \text{and} \qquad \varphi = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega. \tag{4.1.1}$$

where $(-\Delta)^s$ is the fraction Laplacian operator which is defined, when acted on smooth function $\varphi \in C_c^\infty(\Omega)$ by

$$(-\Delta)^s \varphi(x) = \frac{b_{N,s}}{2} \int_{\mathbb{R}^N} \frac{2\varphi(x) - \varphi(x+y) - \varphi(x-y)}{|y|^{N+2s}} dy, \qquad x \in \mathbb{R}^N$$

with $b_{N,s} = \frac{s4^s\Gamma(\frac{N}{2}+s)}{\pi^{\frac{N}{2}}\Gamma(1-s)}$. Here, a function φ is called a weak solution of (4.1.1), if $\varphi \in \mathscr{H}_0^s(\Omega)$ and for all $\psi \in \mathscr{H}_0^s(\Omega)$ it holds that

$$\mathscr{E}_s(\varphi, \psi) =: \frac{b_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{(\varphi(x) - \varphi(y))(\psi(x) - \psi(y))}{|x-y|^{N+2s}} dxdy = \mu \int_{\Omega} \varphi(x)\psi(x)dx.$$

As usual, $\mathscr{H}_0^s(\Omega)$ is defined as the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|\psi\|^2_{H^s(\mathbb{R}^N)} := \mathscr{E}_s(\psi, \psi)$. Recall here, that since Ω is bounded, it follows that $\mathscr{E}_s$ is a scalar product on $\mathscr{H}_0^s(\Omega)$.

Recall moreover that when Ω has a continuous boundary then the space $\mathscr{H}_0^s(\Omega)$ coincides also with the space $\{v \in L^2_{loc}(\mathbb{R}^N) : \mathscr{E}_s(v,v) < \infty \text{ and } v = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}$. We refer to [31, 56] for more details and for more information about fractional Sobolev spaces. By standard theory it follows that there is an increasing sequence of real numbers $0 < \lambda_{1,s}(\Omega) < \lambda_{2,s}(\Omega) \leq \cdots \lambda_{j,s}(\Omega) \leq \cdots \nearrow +\infty$ such that for each $\lambda_{j,s}(\Omega)$ the equation (4.1.1) has a nontrivial solution φ and that the first eigenvalue $\lambda_{1,s}(\Omega)$ is simple, i.e, the corresponding solution φ_1 is unique up to a multiplicative constant and, moreover, can be chosen to be positive. As a consequence of the latter, it is known that φ_1 always inherits the symmetry properties of the underlying domain Ω. In particular when Ω is radial, one obtains that φ_1 has to be radial. However for $j \geq 2$, since simplicity fails in general, it is a nontrivial task to decide whether the corresponding solutions would inherit the symmetry properties of the domain Ω or not.
A conjecture by Bañuelos and Kulczycki (see [38]) states that when Ω is a ball and $j = 2$, then a solution corresponding to (4.1.1) with $\mu = \lambda_{2,s}(\Omega)$ cannot be radial. Several partial answers were obtained in [9, 8, 38, 48, 78] and it is only recently that the conjecture is fully solved in [41] by estimating the Morse index of a radial eigenfunction (see also [9], with a different approach to prove this conjecture). The study of the Morse index of radial functions was in particular studied in [1] for the Laplacian, that is the case $s = 1$, in balls and annuli. In [41] this approach has been extended to the nonlocal framework in balls. Our first result concerns the extension of the Bañuelos-Kulczycki conjecture to annuli and is in the spirit of [1, 41] in annuli. We show the following for $A_R := \{x \in \mathbb{R}^N : 0 < R < |x| < R+1\}$.

Theorem 4.1.1. *There is $R_0 > 0$ such that for any $R \geq R_0$ any second eigenfunction corresponding to $\lambda_{2,s}(A_R)$ is nonradial.*

Note that due to the scaling properties of the fractional Laplacian we immediately deduce the following corollary to this result.

Corollary 4.1.2. *There is $\tau_0 > 0$ such that for any $\tau \in [\tau_0, 1)$ any second eigenfunction corresponding to $\lambda_{2,s}(B_1(0) \setminus B_\tau(0))$ is nonradial.*

To prove Theorem 4.1.1, the main point is to establish that any second radial eigenfunction u_R of A_R satisfies $\frac{u_R}{d^s}(R)\frac{u_R}{d^s}(R+1) < 0$ for R sufficiently large. Once we have this, one can argue exactly as in [41] to conclude the proof of the Theorem. As already mentioned in [41], such property cannot be proved by using the classical Hopf lemma since u_R is sign changing; and it also does not follow from the fractional Pohozaev identity (see e.g [104]) either. To obtain the latter, we use compactness argument to show that, along some subsequence, $\psi_R := \frac{u_R(\cdot+R)}{d^s} \to \frac{\varphi_2}{d^s}$ in $C^0([0,1])$ where φ_2 is a second eigenfunction of the unit interval $(0,1)$. From there we deduce the claim since the limiting problem has already the desire property by the result of [41].
We note that indeed we expect the conclusion of Corollary 4.1.2 to hold for any $\tau \in (0,1)$, and thus we only give here a partial answer.

Our next results concerns the maximization of $\lambda_{2,s}$ of certain *shifted* annuli. To be precise, given $\tau \in (0,1)$ we aim at finding

$$\sup_{a \in B_{1-\tau}(0)} \lambda_{2,s}(B_1(0) \setminus B_\tau(a)).$$

For the classical case of the Laplacian the corresponding problem was studied in [39], where it was proven that concentric spheres maximize the second eigenvalue, i.e. $\sup_{a\in B_{1-\tau}(0)} \lambda_{2,1}(B_1(0)\setminus B_\tau(a)) = \lambda_{2,1}(B_1(0)\setminus B_\tau(0))$. We show the following.

Theorem 4.1.3. *Let $N \geq 2$ and assume any second eigenfunction of the annulus $B_1(0)\setminus\overline{B_\tau(0)}$ cannot be radial. Then*

$$\lambda_{2,s}\big(B_1(0)\setminus\overline{B_\tau(a)}\big) \leq \lambda_{2,s}\big(B_1(0)\setminus\overline{B_\tau(0)}\big) \quad \text{for all } a\in B_{1-\tau}(0) \tag{4.1.2}$$

and equality holds in (4.1.2) *if and only if* $a = 0$.

Note that by Corollary 4.1.2, the assumption of Theorem 4.1.3 is satisfied when τ is sufficiently close to 1. To prove the Theorem 4.1.3, the key observation is that the second eigenvalue of an eccentric annulus is always controlled by its first antisymmetric eigenvalue and that for an annulus the two numbers coincide. These two properties allow to reduce the proof of (4.1.2) into proving that the first antisymmetric eigenvalue of an eccentric annulus decreases when the obstacle (the inner ball) moves from the center to the boundary of the unitary ball. To get the latter, we use a shape derivative argument combined with the maximum principle for doubly antisymmetric functions that we established in [34]. We refer to Section 4.6 for more details.

The Chapter is organized as follows: Section 4.2 and 4.3 contains preliminaries results that will be used later in Section 4.4 to obtain uniform Hölder estimates of the fractional normal derivative. Section 4.5 is devoted to the proof Theorem 4.1.1 and in the last Section we prove Theorem 4.1.3.

4.2 Preliminary results

Let $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$, $z>0$ be the Gamma-function. In the following, we use strongly the identity

$$\int_0^\infty \frac{h^{a-1}}{(1+h)^{a+b}}\,dh = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad \text{for } a,b>0.$$

Lemma 4.2.1. *Let $N\in\mathbb{N}$, $N\geq 2$, $s\in(0,1)$, and consider the function*

$$\varphi:[0,1]\to\mathbb{R},\quad \varphi(t) = \begin{cases} t^{1+2s}\displaystyle\int_0^1 \frac{\big(h(1-h)\big)^{\frac{N-3}{2}}}{\big(t^2+h\big)^{\frac{N+2s}{2}}}\,dh, & t>0;\\[2ex] \dfrac{\Gamma(\frac12+s)\Gamma(\frac{N-1}{2})}{\Gamma(\frac{N+2s}{2})}, & t=0.\end{cases}$$

Then the following holds.

(*i*) *If $s>\frac12$, then $\varphi\in C^{0,1}([0,1])$.*

(*ii*) *If $s\in(0,\frac12)$, then $\varphi\in C^{0,2s}([0,1])$.*

(*iii*) *If* $s=\frac{1}{2}$, *then* $\varphi \in C^{0,\sigma}([0,1])$ *for all* $\sigma \in (0,1)$.

In particular, $\varphi \in C^{s+\delta}([0,1])$ *for some* $\delta = \delta(s) > 0$.

Proof. First note that if $N=3$, then it directly follows that $\varphi(0) = \frac{\Gamma(\frac{1+2s}{2})}{\Gamma(\frac{3+2s}{2})} = \frac{\Gamma(\frac{1+2s}{2})}{\Gamma(1+\frac{1+2s}{2})} = \frac{2}{1+2s}$ and for $t>0$ we have

$$\varphi(t) = t^{1+2s}\int_0^1 \frac{1}{(t^2+h)^{\frac{3+2s}{2}}}\,dh = \frac{2t^{1+2s}}{1+2s}\Big(\frac{1}{t^{1+2s}} - \frac{1}{(1+t^2)^{\frac{1+2s}{2}}}\Big) = \frac{2}{1+2s}\Big(1-\frac{t^{1+2s}}{(1+t^2)^{\frac{1+2s}{2}}}\Big),$$

so that it follows easily that $\varphi \in C^{0,1}([0,1])$.
In the following let $N \neq 3$. We begin by transforming the integral slightly. Note that for $t>0$ and with the substitution $h = f(\tau) := \frac{\tau}{1+\tau}$ we find

$$\begin{aligned}
\varphi(t) &= t^{1+2s}\int_0^1 \frac{(h(1-h))^{\frac{N-3}{2}}}{(t^2+h)^{\frac{N+2s}{2}}}\,dh = t^{1+2s}\int_0^\infty \frac{f'(\tau)(f(\tau)(1-f(\tau)))^{\frac{N-3}{2}}}{(t^2+f(\tau))^{\frac{N+2s}{2}}}\,d\tau\\
&= t^{1+2s}\int_0^\infty \frac{\tau^{\frac{N-3}{2}}}{(1+\tau)^{N-1}(t^2+\frac{\tau}{1+\tau})^{\frac{N+2s}{2}}}\,d\tau\\
&= t^{1+2s}\int_0^\infty \frac{\tau^{\frac{N-3}{2}}}{(1+\tau)^{\frac{N}{2}-1-s}(t^2+(1+t^2)\tau)^{\frac{N+2s}{2}}}\,d\tau\\
&= \frac{t^{1+2s}}{(1+t^2)^{\frac{N+2s}{2}}}\int_0^\infty \frac{\tau^{\frac{N-3}{2}}}{(1+\tau)^{\frac{N}{2}-1-s}(\frac{t^2}{1+t^2}+\tau)^{\frac{N+2s}{2}}}\,d\tau\\
&= (f^{-1}(f(t^2)))^{\frac{1-N}{2}} f(t^2)^{\frac{N+2s}{2}}\int_0^\infty \frac{\tau^{\frac{N-3}{2}}}{(1+\tau)^{\frac{N}{2}-1-s}(f(t^2)+\tau)^{\frac{N+2s}{2}}}\,d\tau\\
&= \Big(\frac{f(t^2)}{1-f(t^2)}\Big)^{\frac{1-N}{2}} f(t^2)^{\frac{N+2s}{2}}\int_0^\infty \frac{\tau^{\frac{N-3}{2}}}{(1+\tau)^{\frac{N}{2}-1-s}(f(t^2)+\tau)^{\frac{N+2s}{2}}}\,d\tau\\
&= (1-T)^{\frac{N-1}{2}} T^{\frac{1+2s}{2}}\int_0^\infty \frac{\tau^{\frac{N-3}{2}}}{(1+\tau)^{\frac{N}{2}-1-s}(T+\tau)^{\frac{N+2s}{2}}}\,d\tau,
\end{aligned}$$

where we put $T = f(t^2)$ and used that $f^{-1}(a) = \frac{a}{1-a}$ for $a \in [0,1)$. Whence

$$\begin{aligned}
\varphi(t) &= (1-T)^{\frac{N-1}{2}} T^{\frac{1+2s}{2}}\int_0^\infty \frac{\tau^{\frac{N-3}{2}}}{(1+\tau)^{\frac{N}{2}-1-s}(T+\tau)^{\frac{N+2s}{2}}}\,d\tau\\
&= (1-T)^{\frac{N-1}{2}} T^{\frac{1+2s}{2}}\int_0^\infty \frac{T^{\frac{N-1}{2}}h^{\frac{N-3}{2}}}{(1+Th)^{\frac{N}{2}-1-s}(T+Th)^{\frac{N+2s}{2}}}\,dh
\end{aligned}$$

$$= (1-T)^{\frac{N-1}{2}} \int_0^\infty \frac{h^{\frac{N-3}{2}}}{(1+Th)^{\frac{N}{2}-1-s}(1+h)^{\frac{N+2s}{2}}}\, dh =: \Phi(T),$$

that is, $\Phi(T) = \varphi(\sqrt{f^{-1}(T)})$ for $T \in (0, \frac{1}{2}]$. Moreover, note that

$$\begin{aligned}(1-T)^{\frac{N-1}{2}} \int_0^\infty \frac{h^{\frac{N-3}{2}}}{(1+Th)^{\frac{N}{2}-1-s}(1+h)^{\frac{N+2s}{2}}}\, dh &\le \int_0^\infty \frac{h^{\frac{N-3}{2}}}{(1+Th)^{\frac{N}{2}-1}(1+h)^{\frac{N}{2}}}\Big(\frac{1+Th}{1+h}\Big)^s dh \\ &\le \int_0^\infty \frac{h^{\frac{N-3}{2}}}{(1+h)^{\frac{N}{2}}}\, dh = \frac{\sqrt{\pi}\,\Gamma(\frac{N-1}{2})}{\Gamma(\frac{N}{2})} =: C_1\end{aligned} \tag{4.2.1}$$

Thus by dominated convergence and the above we find

$$\Phi(0) := \lim_{t\to 0} \Phi(T) = \int_0^\infty \frac{h^{\frac{N-3}{2}}}{(1+h)^{\frac{N+2s}{2}}}\, dh = \frac{\Gamma(\frac{1}{2}+s)\Gamma(\frac{N-1}{2})}{\Gamma(\frac{N+2s}{2})} = \varphi(0).$$

Whence $\Phi : [0, \frac{1}{2}] \to \mathbb{R}$ is continuous and thus it follows that also φ is continuous. To show the Lipschitz continuity of φ, we use the representation via Φ and consider different cases.

The case $s > \frac{1}{2}$: Let $0 \le A < B \le \frac{1}{2}$. Then

$$\begin{aligned}|\Phi(B) - \Phi(A)| \le{}& (1-B)^{\frac{N-1}{2}} \int_0^\infty \frac{h^{\frac{N-3}{2}}}{(1+h)^{\frac{N+2s}{2}}} \left| \frac{1}{(1+Bh)^{\frac{N}{2}-1-s}} - \frac{1}{(1+Ah)^{\frac{N}{2}-1-s}} \right| dh \\ &+ \int_0^\infty \frac{h^{\frac{N-3}{2}}}{(1+ah)^{\frac{N}{2}-1-s}(1+h)^{\frac{N+2s}{2}}}\, dh \left| (1-B)^{\frac{N-1}{2}} - (1-A)^{\frac{N-1}{2}} \right| \\ \le{}& |\frac{N}{2} - 1 - s| \int_0^\infty \frac{h^{\frac{N-1}{2}}}{(1+h)^{\frac{N+2s}{2}}} \sup_{x\in[A,B]} \frac{1}{(1+xh)^{\frac{N}{2}-s}}\, dh |A-B| \\ &+ \frac{N-1}{2} \int_0^\infty \frac{h^{\frac{N-3}{2}}}{(1+ah)^{\frac{N}{2}-1-s}(1+h)^{\frac{N+2s}{2}}}\, dh \sup_{x\in[A,B]} (1-x)^{\frac{N-3}{2}} |A-B|.\end{aligned}$$

Clearly, using (4.2.1),

$$\frac{N-1}{2} \int_0^\infty \frac{h^{\frac{N-3}{2}}}{(1+Ah)^{\frac{N}{2}-1-s}(1+h)^{\frac{N+2s}{2}}}\, dh \sup_{x\in[A,B]} (1-x)^{\frac{N-3}{2}} \le \frac{\sqrt{\pi}\,\Gamma(\frac{N+1}{2})}{\Gamma(\frac{N}{2})} = C_1.$$

Moreover, if $\frac{N}{2} \ge s$ we have $\sup_{x\in[A,B]} \frac{1}{(1+xh)^{\frac{N}{2}-s}} = 1$ and thus also

$$|\frac{N}{2} - 1 - s| \int_0^\infty \frac{h^{\frac{N-1}{2}}}{(1+h)^{\frac{N+2s}{2}}} \sup_{x\in[A,B]} \frac{1}{(1+xh)^{\frac{N}{2}-s}}\, dh$$

$$\leq |\frac{N}{2}-1-s|\int_0^\infty \frac{h^{\frac{N-1}{2}}}{(1+h)^{\frac{N}{2}+s}}\,dh = |\frac{N}{2}-1-s|\frac{\Gamma(s-\frac{1}{2})\Gamma(\frac{N+1}{2})}{\Gamma(\frac{N+2s}{2})} =: C_2.$$

Whence Φ is Lipschitz continuous. Thus, for $0 \leq a < b \leq 1$ we find with $C_3 = \max\{C_1, C_2\}$

$$\begin{aligned}|\varphi(a)-\varphi(b)| &= \Big|\Phi(\frac{a^2}{1+a^2})-\Phi(\frac{b^2}{1+b^2})\Big| \leq C_3\Big|\frac{a^2}{1+a^2}-\frac{b^2}{1+b^2}\Big| \\ &\leq C_3 \sup_{x\in[a^2,b^2]}|\frac{2x}{(1+x^2)^2}||a^2-b^2| \leq C_3|a-b|,\end{aligned}$$

using that $a,b \leq 1$. Whence, φ is Lipschitz continuous in this case.

The case $s < \frac{1}{2}$: We use the inequality

$$|x^{2s-N}-y^{2s-N}| \leq c|x-y|^{2s}\Big(x^{-2-N}+y^{-2-N}\Big) \quad \text{for } x,y>0$$

for a constant $c = c_{s,N} > 0$. Let $0 \leq A < B \leq \frac{1}{2}$. Proceeding as in the previous case, we find with the above inequality

$$\begin{aligned}|\Phi(B)-\Phi(A)| &\leq \int_0^\infty \frac{h^{\frac{N-3}{2}}}{(1+h)^{\frac{N+2s}{2}}}\Big|(1+Bh)^{s+1-\frac{N}{2}}-(1+Ah)^{s+1-\frac{N}{2}}\Big|\,dh + C_1|A-B| \\ &\leq C_1|A-B| + c|A-B|^{2s}\int_0^\infty \frac{h^{\frac{N-3}{2}+2s}}{(1+h)^{\frac{N+2s}{2}}}\max\{(1+Ah)^{1-s-\frac{N}{2}},(1+Bh)^{1-s-\frac{N}{2}}\}\,dh \\ &\leq C_1|A-B| + c|A-B|^{2s}\int_0^\infty \frac{h^{\frac{N-3}{2}+2s}}{(1+h)^{\frac{N+2s}{2}}}\,dh \\ &= \underbrace{\Big(C_1 + c\frac{\Gamma(\frac{1}{2}-s)\Gamma(\frac{N-1}{2}+2s)}{\Gamma(\frac{N+2s}{2})}\Big)}_{=:C_4}|A-B|^{2s}.\end{aligned}$$

Similarly as in the previous case we find

$$|\varphi(a)-\varphi(b)| = \Big|\Phi(\frac{a^2}{1+a^2})-\Phi(\frac{b^2}{1+b^2})|\Big| \leq C_4|a^2-b^2|^{2s} \leq C_4|a-b|^{2s}.$$

This shows the case for $s < \frac{1}{2}$.

The case $s = \frac{1}{2}$: Let $\sigma \in (0,1)$. Then we have for $0 < x < y$ by Hölder's inequality with $\frac{1}{p} = \sigma$, $\frac{1}{q} = 1-\sigma$

$$\Big|x^{\frac{3-N}{2}}-y^{\frac{3-N}{2}}\Big| = \Big|\frac{3-N}{2}\int_x^y t^{\frac{1-N}{2}}\,dt\Big| \leq \Big|\frac{3-N}{2}\Big||x-y|^{\sigma}\Big(\int_x^y t^{q(\frac{1-N}{2})}\,dt\Big)^{1-\sigma}$$

$$\leq \frac{|N-3|}{2(q(\frac{N-1}{2})+1)^{1-\sigma}}|x-y|^{\sigma}\max\left\{x^{\frac{3-N}{2}-\sigma},y^{\frac{3-N}{2}-\sigma}\right\}$$
$$=\underbrace{\frac{|N-3|}{2(\frac{N+1}{2}-\sigma)^{1-\sigma}}(1-\sigma)^{1-\sigma}}_{=:C_5}|x-y|^{\sigma}\max\left\{x^{\frac{3-N}{2}-\sigma},y^{\frac{3-N}{2}-\sigma}\right\}.$$

Then we find similar to the previous case for $0\leq A<B\leq \frac{1}{2}$

$$\begin{aligned}|\Phi(B)-\Phi(A)| &\leq \int_0^\infty \frac{h^{\frac{N-3}{2}}}{(1+h)^{\frac{N+1}{2}}}\left|(1+Bh)^{\frac{3-N}{2}}-(1+Ah)^{\frac{3-N}{2}}\right| dh+C_1|A-B|\\ &\leq C_1|A-B|+C_5|A-B|^{\sigma}\int_0^\infty \frac{h^{\frac{N-3}{2}+\sigma}}{(1+h)^{\frac{N+1}{2}}}\max\{(1+Ah)^{\frac{3-N}{2}-\sigma},(1+Bh)^{\frac{3-N}{2}-\sigma}\}\,dh\\ &\leq C_1|A-B|+C_5|A-B|^{\sigma}\int_0^\infty \frac{h^{\frac{N-3}{2}+\sigma}}{(1+h)^{\frac{N+1}{2}}}\,dh\\ &=\Big(C_1+C_5\frac{\Gamma(1-\sigma)\Gamma(\frac{N-1}{2}+\sigma)}{\Gamma(\frac{N+1}{2})}\Big)|A-B|^{\sigma}.\end{aligned}$$

As before we conclude that $\varphi\in C^{0,\sigma}([0,1])$. This finishes the proof. □

A a corollary we have the following

Corollary 4.2.2. *Let $R>1$ and define*

$$F_R:[-2,+2]\times[0,2]\to\mathbb{R}_+,\quad (t,r)\mapsto F_R(t,r)=\begin{cases}\displaystyle\int_{\frac{\sqrt{(t+R)(t\pm r+R)}}{r}(\mathbb{S}^{N-1}-e_1)}\frac{d\theta}{(1+|\theta|^2)^{\frac{N+2s}{2}}}, & \forall r\neq 0;\\ \dfrac{\pi^{\frac{N-1}{2}}\Gamma(\frac{1+2s}{2})}{\Gamma(\frac{N+2s}{2})}, & for\quad r=0.\end{cases}$$

Then, there exists $C,\delta>0$ (independent of R) so that

$$|F_R(t,r)-F_R(t',r')|\leq C(|r-r'|^{s+\delta}+|t-t'|^{s+\delta}),\quad \forall\, r,r'\in[0,2]\quad and\quad \forall\, t,t'\in[-2,+2].$$

Proof. For simplicity we let

$$\tau_R(t,r):=\frac{\sqrt{(t+R)(t\pm r+R)}}{r}.$$

Since the integrand is invariant under rotation, by changing variables $\overline{y}=y+\tau_R(t,r)e_N$ we get

$$F_R(t,r)=\int_{\tau_R(t,r)(\mathbb{S}^{N-1}-e_N)}\frac{dy}{(1+|y|^2)^{\frac{N+2s}{2}}}=\int_{\tau_R(t,r)\mathbb{S}^{N-1}}\frac{dy}{(1+|y-\tau_R(t,r)e_N|^2)^{\frac{N+2s}{2}}}$$

$$= \int_{\tau_R(t,r)\mathbb{S}^{N-1}} \frac{dy}{(1+|y|^2 - 2\tau_R(t,r) y\cdot e_N + \tau_R^2(t,r))^{\frac{N+2s}{2}}}$$
$$= \int_{\tau_R(t,r)\mathbb{S}^{N-1}} \frac{dy}{(1+2\tau_R^2(t,r) - 2\tau_R(t,r) y\cdot e_N)^{\frac{N+2s}{2}}}.$$

We first start with the case $N \geq 3$. Passing into spherical coordinates we get

$$F_R(t,r) = \int_{(0,\pi)^{N-2}} d\theta_1 \cdots d\theta_{N-2} \tau_R^{N-1}(t,r) \sin^{N-2}(\theta_1)\sin^{N-3}(\theta_2)\cdots\sin(\theta_{N-2})$$
$$\times \int_0^{2\pi} \frac{d\varphi}{(1+2\tau_R^2(t,r) - 2\tau_R^2(t,r)\cos\theta_1)^{\frac{N+2s}{2}}}$$
$$= 2\pi c_N \int_0^{\pi} \frac{\tau_R^{N-1}(t,r)\sin^{N-2}(\theta_1)}{(1+2\tau_R^2(t,r) - 2\tau_R^2(t,r)\cos\theta_1)^{\frac{N+2s}{2}}},$$

where $c_N = 1$ for $N = 3$ and $c_N = \int_{(0,\pi)^{N-3}} \sin^{N-3}(\theta_2)\cdots\sin(\theta_{N-2}) d\theta_2 \cdots d\theta_{N-2} = \frac{\pi^{\frac{N-1}{2}}}{\pi\Gamma(\frac{N-1}{2})}$ for $N > 3$. Using the change of variables $h = \cos\theta_1$, we obtain

$$F_R(t,r) = 2\pi c_N \int_{-1}^{+1} \frac{\tau_R^{N-1}(r,t)(1-h^2)^{\frac{N-2}{2}}}{\Big(1+2\tau_R^2(t,r) - 2\tau_R^2(t,r)h\Big)^{\frac{N+2s}{2}}} \frac{dh}{(1-h^2)^{1/2}}$$
$$= 2\pi c_N \tau_R^{N-1}(t,r) \int_{-1}^{+1} \frac{\Big(1-h^2\Big)^{\frac{N-3}{2}}}{(1+2\tau_R^2(t,r) - 2\tau_R^2(t,r)h\Big)^{\frac{N+2s}{2}}} dh$$
$$= 2\pi c_N \tau_R^{N-1}(t,r) \int_0^2 \frac{h^{\frac{N-3}{2}}(2-h)^{\frac{N-3}{2}}}{(1+2\tau_R^2(t,r)h)^{\frac{N+2s}{2}}} dh$$
$$= 2\pi c_N \tau_R^{-1-2s}(t,r) \int_0^2 \frac{h^{\frac{N-3}{2}}(2-h)^{\frac{N-3}{2}}}{\big(\frac{1}{\tau_R^2(t,r)} + 2h\big)^{\frac{N+2s}{2}}} dh$$
$$= \frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})}\Big(\frac{1}{2\tau_R(t,r)}\Big)^{1+2s} \int_0^1 \frac{h^{\frac{N-3}{2}}(1-h)^{\frac{N-3}{2}}}{\Big(\big(\frac{1}{2\tau_R(t,r)}\big)^2 + h\Big)^{\frac{N+2s}{2}}} dh$$
$$= \frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})}\varphi\Big(\frac{1}{2\tau_R(t,r)}\Big) \qquad \forall\, t \in [-2,+2], \forall\, r \in (0,2],$$

with φ defined as in Lemma 4.2.1. By Lemma 4.2.1 we deduce that for $N \geq 3$ and $s \in (0,1)$, for all $r, r' \in [0,1]$ and for all $t, t' \in [-2,2]$,

$$|F_R(t,r) - F_R(t',r')| \leq C\Big|\varphi\Big(\frac{1}{2\tau_R(t,r)}\Big) - \varphi\Big(\frac{1}{2\tau_R(t,r)}\Big)\Big| \leq C\Big|\frac{1}{\tau_R(t,r)} - \frac{1}{\tau_R(t',r')}\Big|^{s+\delta}$$

$$\leq C\Big|\frac{r}{\sqrt{(r+R)(r+t+R)}}-\frac{r'}{\sqrt{(r'+R)(r'+t'+R)}}\Big|^{s+\delta}$$
$$\leq C(|r-r'|^{s+\delta}+|t-t'|^{s+\delta}),$$

for some $\delta>0$ and $C>0$ independent of R. The case $N=2$ is treated similarly. □

4.3 Uniform estimates of the first and second eigenvalues of the annulus A_R.

The aim of this section is to obtain a uniform control of the second fractional eigenvalue of the annulus $A_R=\{x\in\mathbb{R}^N : R<|x|<R+1\}$. For that, we let $\lambda_{1,s}(A_R)$ and $\lambda_{2,s}(A_R)$ be respectively the first and the second fractional eigenvalue of A_R. We start with the following

Lemma 4.3.1. *There exists a positive constant $C(N,s)>0$ so that*

$$\lambda_{1,s}(A_R)\leq C(N,s)\quad \textit{for all } R\geq 1. \tag{4.3.1}$$

Proof. Let $\lambda_{1,s}(0,1)$ be the first eigenvalue of the interval $(0,1)$ and φ_1 be the corresponding (normalized) eigenfunction. Consider $\Phi_R : x\mapsto \varphi_1(|x|-R)$. It is clear that $\Phi_R=0$ in $\mathbb{R}^N\setminus A_R$. Moreover

$$\begin{aligned}
&R^{-(N-1)}\int\limits_{\mathbb{R}^{2N}}\frac{(\Phi_R(x)-\Phi_R(y))^2}{|x-y|^{N+2s}}dxdy\\
&=\int_0^\infty\int_0^\infty\int\limits_{\mathbb{S}^{N-1}\times\mathbb{S}^{N-1}}\frac{r^{N-1}\tilde r^{N-1}}{R^{N-1}}\frac{\big(\varphi_1(r-R)-\varphi_1(r-R)\big)^2}{|r\theta-\tilde r\tilde\theta|^{N+2s}}d\theta d\tilde\theta drd\tilde r\\
&=\int_{-R}^\infty\int_{-R}^\infty\frac{\big[(r+R)(\tilde r+R)\big]^{N-1}}{R^{N-1}}\big(\varphi_1(r)-\varphi_1(\tilde r)\big)^2\int\limits_{\mathbb{S}^{N-1}}d\theta\int\limits_{\mathbb{S}^{N-1}}\frac{d\tilde\theta}{|(r+R)\theta-(\tilde r+R)\tilde\theta|^{N+2s}}drd\tilde r\\
&=\omega_N\int_{-R}^\infty\int_{-R}^\infty\frac{\big[(r+R)(\tilde r+R)\big]^{N-1}}{R^{N-1}}\big(\varphi_1(r)-\varphi_1(\tilde r)\big)^2\int\limits_{\mathbb{S}^{N-1}}\frac{d\tilde\theta}{|(r+R)e_1-(\tilde r+R)\tilde\theta|^{N+2s}}drd\tilde r.
\end{aligned}\tag{4.3.2}$$

Here ω_N denotes the volume of the $(N-1)$-dimensional unit sphere $\mathbb{S}^{N-1}$. On the other hand, we have

$$\begin{aligned}
&\int\limits_{\mathbb{S}^{N-1}}\frac{d\theta}{|(r+R)e_1-(\tilde r+R)\theta|^{N+2s}}=\int\limits_{\mathbb{S}^{N-1}}\frac{d\theta}{\Big((r+R)^2+(\tilde r+R)^2-2(r+R)(\tilde r+R)e_1\cdot\theta\Big)^{\frac{N+2s}{2}}}\\
&=\int\limits_{\mathbb{S}^{N-1}}\frac{d\theta}{\Big((r-\tilde r)^2+2(r+R)(\tilde r+R)(1-e_1\cdot\theta)\Big)^{(N+2s)/2}}
\end{aligned}$$

$$
\begin{aligned}
&= \int\limits_{\mathbb{S}^{N-1}} \frac{d\theta}{\Big((r-\tilde r)^2+(r+R)(\tilde r+R)|e_1-\theta|^2\Big)^{\frac{N+2s}{2}}} \\
&= \frac{1}{|r-\tilde r|^{N+2s}} \int\limits_{\mathbb{S}^{N-1}-e_1} \frac{d\theta}{\Big(1+\frac{(r+R)(\tilde r+R)}{|r-\tilde r|^2}|\theta|^2\Big)^{\frac{N+2s}{2}}} \\
&= \frac{|r-\tilde r|^{-1-2s}}{(r+R)^{\frac{N-1}{2}}(\tilde r+R)^{\frac{N-1}{2}}} \int\limits_{\frac{\sqrt{(r+R)(\tilde r+R)}}{|r-\tilde r|}(\mathbb{S}^{N-1}-e_1)} \frac{d\theta}{(1+|\theta|^2)^{\frac{N+2s}{2}}}. \qquad (4.3.3)
\end{aligned}
$$

In the following define

$$
K_R(r,\tilde r) = \frac{(r+R)^{\frac{N-1}{2}}(\tilde r+R)^{\frac{N-1}{2}}}{R^{N-1}} \int\limits_{\frac{\sqrt{(r+R)(\tilde r+R)}}{|r-\tilde r|}(\mathbb{S}^{N-1}-e_1)} \frac{d\theta}{(1+|\theta|^2)^{\frac{N+2s}{2}}}. \qquad (4.3.4)
$$

Plugging (4.3.3) into (4.3.2) gives

$$
\begin{aligned}
&R^{-(N-1)} \int\limits_{\mathbb{R}^{2N}} \frac{(\Phi_R(x)-\Phi_R(y))^2}{|x-y|^{N+2s}}dxdy \\
&= \omega_N \int\limits_{-R}^{\infty}\int\limits_{-R}^{\infty} \frac{\big(\varphi_1(r)-\varphi_1(\tilde r)\big)^2}{|r-\tilde r|^{1+2s}} \frac{(r+R)^{\frac{N-1}{2}}(\tilde r+R)^{\frac{N-1}{2}}}{R^{N-1}} \int\limits_{\frac{\sqrt{(r+R)(\tilde r+R)}}{|r-\tilde r|}(\mathbb{S}^{N-1}-e_1)} \frac{d\theta}{(1+|\theta|^2)^{\frac{N+2s}{2}}} \\
&= \omega_N \int\limits_{-2}^{2}\int\limits_{-2}^{2} \frac{\big(\varphi_1(r)-\varphi_1(\tilde r)\big)^2}{|r-\tilde r|^{1+2s}} K_R(r,\tilde r)drd\tilde r + 2\omega_N \int\limits_0^1 \varphi_1^2(r) \int\limits_{(-R,\infty)\setminus(-2,2)} \frac{K_R(r,\tilde r)}{|r-\tilde r|^{1+2s}}d\tilde r. \qquad (4.3.5)
\end{aligned}
$$

We know, see Lemma 4.2.1 and the change of variables in the proof of Corollary 4.2.2, that

$$
K_R(r,\tilde r) = \frac{(r+R)^{\frac{N-1}{2}}(\tilde r+R)^{\frac{N-1}{2}}}{R^{N-1}} \frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})} \varphi\Big(\frac{|r-\tilde r|}{\sqrt{(r+R)(\tilde r+R)}}\Big) \to \frac{\pi^{\frac{N-1}{2}}\Gamma(\frac{1+2s}{2})}{\Gamma(\frac{N+2s}{2})} > 1 \quad \text{as} \quad R\to\infty \qquad (4.3.6)
$$

for all $r,\tilde r \in (-2,2)$, where φ is defined as in Lemma 4.2.1 (see also [23, Lemma 5.1] for a different proof). Hence, the first integral in (4.3.5) is comparable to $\int_{-2}^{2}\int_{-2}^{2} \frac{\big(\varphi_1(r)-\varphi_1(\tilde r)\big)^2}{|r-\tilde r|^{1+2s}}drd\tilde r$ for R sufficiently large. In other words

$$
C_1(N,s)[\varphi_1]^2_{H^s(-2,2)} \le \int\limits_{-2}^{2}\int\limits_{-2}^{2} \frac{\big(\varphi_1(r)-\varphi_1(\tilde r)\big)^2}{|r-\tilde r|^{1+2s}} K_R(r,\tilde r)drd\tilde r \le C_2(N,s)[\varphi_1]^2_{H^s(-2,2)} \qquad (4.3.7)
$$

for R sufficiently large. To estimate the second integral in (4.3.5), we write

$$
\int\limits_{(-R,\infty)\setminus(-2,2)} \frac{K_R(r,\tilde r)}{|r-\tilde r|^{1+2s}}d\tilde r = \int\limits_{-R}^{-2}\cdots d\tilde r + \int\limits_{2}^{\infty}\cdots d\tilde r
$$

Using the new variable $\overline{r}=\frac{(r+R)(R-\tilde{r})}{(r+\tilde{r})}$, we may write

$$\int_{\frac{\sqrt{(r+R)(R-\tilde{r})}}{(r+\tilde{r})}(\mathbb{S}^{N-1}-e_1)}\frac{d\theta}{(1+|\theta|^2)^{\frac{N+2s}{2}}}=\frac{(r+R)^{\frac{N-1}{2}}(R-\tilde{r})^{\frac{N-1}{2}}}{(r+\tilde{r})^{N-1}}\int_{\mathbb{S}^{N-1}-e_1}\frac{d\theta}{(1+\frac{(r+R)(R-\tilde{r})}{(r+\tilde{r})^2}|\theta|^2)^{\frac{N+2s}{2}}}$$

$$=(r+R)^{-(N-1)}\overline{r}^{\frac{N-1}{2}}(\overline{r}+r+R)^{\frac{N-1}{2}}\int_{\mathbb{S}^{N-1}-e_1}\frac{d\theta}{\left(1+\left|\frac{\sqrt{\overline{r}(\overline{r}+r+R)}}{R+r}\theta\right|^2\right)^{\frac{N+2s}{2}}}=\overline{f}\Big(\frac{\overline{r}}{r+R}\Big),$$

where we set

$$\overline{f}(\rho)=\int_{\sqrt{\rho(\rho+1)}(\mathbb{S}^{N-1}-e_1)}\frac{d\theta}{(1+|\theta|^2)^{\frac{N+2s}{2}}}. \tag{4.3.8}$$

We also have

$$\int_{-R}^{-2}\cdots d\tilde{r}=\frac{(r+R)^{\frac{N-1}{2}}}{R^{N-1}}\int_{-R}^{-2}(\tilde{r}+R)^{\frac{N-1}{2}}\frac{1}{|r-\tilde{r}|^{1+2s}}\int_{\frac{\sqrt{(r+R)(\tilde{r}+R)}}{|r-\tilde{r}|}(\mathbb{S}^{N-1}-e_1)}\frac{d\theta}{(1+|\theta|^2)^{\frac{N+2s}{2}}}$$

$$=\frac{(r+R)^{\frac{N-1}{2}}}{R^{N-1}}\int_{2}^{R}(R-\tilde{r})^{\frac{N-1}{2}}\frac{1}{(r+\tilde{r})^{1+2s}}\int_{\frac{\sqrt{(r+R)(R-\tilde{r})}}{(r+\tilde{r})}(\mathbb{S}^{N-1}-e_1)}\frac{d\theta}{(1+|\theta|^2)^{\frac{N+2s}{2}}}$$

$$=\frac{(r+R)^{\frac{N-1}{2}}}{R^{N-1}}\int_0^{\frac{(R+r)(R-2)}{r+2}}(R+r)^{-2s}(1+\frac{\overline{r}}{r+R})^{2s-1}\overline{r}^{\frac{N-1}{2}}(1+\frac{\overline{r}}{r+R})^{-\frac{N-1}{2}}\overline{f}(\frac{\overline{r}}{r+R})\frac{d\overline{r}}{r+R}$$

$$=\frac{(r+R)^{N-1}}{R^{N-1}}(R+r)^{-2s}\int_0^{\frac{R-2}{r+2}}(1+\rho)^{2s-1}\Big(\frac{\rho}{1+\rho}\Big)^{\frac{N-1}{2}}\overline{f}(\rho)d\rho$$

$$\leq\frac{(r+R)^{N-1}}{R^{N-1}}(R+r)^{-2s}\int_0^{\frac{R}{2}}(1+\rho)^{2s-1}\int_{\sqrt{\rho(\rho+1)}(\mathbb{S}^{N-1}-e_1)}\frac{d\theta}{(1+|\theta|^2)^{\frac{N+2s}{2}}}d\rho.$$

Now since

$$\overline{f}(\rho)=\int_{\sqrt{\rho(\rho+1)}(\mathbb{S}^{N-1}-e_1)}\frac{d\theta}{(1+|\theta|^2)^{\frac{N+2s}{2}}}\to\frac{\pi^{\frac{N-1}{2}}\Gamma(\frac{1+2s}{2})}{\Gamma(\frac{N+2s}{2})}\quad\text{as}\quad\rho\to\infty,$$

there exists $m>0$ large enough so that $f(\rho)\leq C(N,s,m)$ for $\rho\geq m$. From this, the continuity of $\rho\mapsto\overline{f}(\rho)$ and the estimate above, we get

$$\int_{-R}^{-2}\cdots d\tilde{r}\leq\frac{(r+R)^{N-1}}{R^{N-1}}(R+r)^{-2s}\int_0^{\frac{R}{2}}(1+\rho)^{2s-1}\int_{\sqrt{\rho(\rho+1)}(\mathbb{S}^{N-1}-e_1)}\frac{d\theta}{(1+|\theta|^2)^{\frac{N+2s}{2}}}d\rho$$

$$\leq 2^{N-1}R^{-2s}\int_0^m (1+\rho)^{2s-1}\overline{f}(\rho)d\rho + C(N,s,m)\int_m^{\frac{R}{2}}(1+\rho)^{2s-1}d\rho$$
$$\leq C(N,s,m)R^{-2s}(1+R^{2s}) \leq C(N,s). \tag{4.3.9}$$

Similarly, by the change of variable $\overline{r} = \frac{(r+R)(\tilde{r}+R)}{\tilde{r}-r}$ we get

$$\int_2^\infty \cdots d\tilde{r} = \int_2^\infty \frac{K_R(r,\tilde{r})}{(\tilde{r}-r)^{1+2s}}d\tilde{r}$$
$$= \frac{(r+R)^{\frac{N-1}{2}}}{R^{N-1}}\int_2^\infty (\tilde{r}+R)^{\frac{N-1}{2}}\frac{1}{(\tilde{r}-r)^{1+2s}}\int_{\frac{\sqrt{(r+R)(\tilde{r}+R)}}{(\tilde{r}-r)}(\mathbb{S}^{N-1}-e_1)}\frac{d\theta}{(1+|\theta|^2)^{\frac{N+2s}{2}}}$$
$$= \frac{(r+R)^{\frac{N-1}{2}}}{R^{N-1}}\int_{r+R}^{\frac{(r+R)(R+2)}{2-r}}(R+r)^{-2s}\Big(\frac{\overline{r}}{r+R}-1\Big)^{2s-1}\overline{r}^{\frac{N-1}{2}}\Big(\frac{\overline{r}}{r+R}-1\Big)^{-\frac{N-1}{2}}\overline{g}\Big(\frac{\overline{r}}{r+R}\Big)\frac{d\overline{r}}{r+R}$$
$$= \frac{(r+R)^{N-1}}{R^{N-1}}(R+r)^{-2s}\int_1^{\frac{R+2}{2-r}}(\rho-1)^{2s-1}\rho^{\frac{N-1}{2}}(\rho-1)^{-\frac{N-1}{2}}\overline{g}(\rho)d\rho$$
$$\leq 2^{N-1}R^{-2s}\int_1^{R+2}(\rho-1)^{2s-1}\Big(\frac{\rho}{\rho-1}\Big)^{\frac{N-1}{2}}\overline{g}(\rho)d\rho$$
$$\leq 2^{N-1}R^{-2s}\int_0^{2R}\rho^{2s-1}\Big(\frac{\rho+1}{\rho}\Big)^{\frac{N-1}{2}}\overline{g}(\rho+1)d\rho = 2^{N-1}\int_0^2\rho^{2s-1}\Big[\frac{\rho+1/R}{\rho}\Big]^{\frac{N-1}{2}}g(R\rho+1)d\rho,$$

with

$$\overline{g}(\rho) = \int_{\sqrt{\rho(\rho-1)}(\mathbb{S}^{N-1}-e_1)}\frac{d\theta}{(1+|\theta|^2)^{\frac{N+2s}{2}}} \quad \text{for all} \quad \rho > 1. \tag{4.3.10}$$

Since

$$g(R\rho+1) = \int_{\sqrt{R\rho(R\rho+1)}(\mathbb{S}^{N-1}-e_1)}\frac{d\theta}{(1+|\theta|^2)^{\frac{N+2s}{2}}} \to \frac{\pi^{\frac{N-1}{2}}\Gamma(\frac{1+2s}{2})}{\Gamma(\frac{N+2s}{2})} \quad \text{uniformly in } \rho,$$

and that $\frac{\rho+1/R}{\rho}$ remain bounded for R large enough, we deduce from above that

$$\int_2^\infty \frac{K_R(r,\tilde{r})}{(\tilde{r}-r)^{1+2s}}d\tilde{r} \leq C(N,s)\int_0^2 \rho^{2s-1}d\rho \leq C(N,s) \qquad \text{as } R\to\infty. \tag{4.3.11}$$

Estimates (4.3.9) and (4.3.11) yield

$$\int_{(-R,\infty)\setminus(-2,2)} \frac{K_R(r,\tilde r)}{|r-\tilde r|^{1+2s}} d\tilde r \le C(N,s). \tag{4.3.12}$$

Hence,

$$\int_0^1 \varphi_1^2(r) \int_{(-R,\infty)\setminus(-2,2)} \frac{K_R(r,\tilde r)}{|r-\tilde r|^{1+2s}} d\tilde r \le C(N,s) \int_0^1 \varphi_1^2(r)(1+r^{-2s})dr \le C(N,s)[\varphi_1]^2_{H^s(\mathbb{R})}. \tag{4.3.13}$$

Putting together (4.3.5), (4.3.7), and (4.3.13) we obtain that

$$R^{-(N-1)} \int_{\mathbb{R}^{2N}} \frac{(\Phi_R(x)-\Phi_R(y))^2}{|x-y|^{N+2s}} dxdy \le C(N,s) \qquad \text{as } R\to\infty. \tag{4.3.14}$$

Next, since

$$R^{-(N-1)} \int_{A_R} \Phi_R^2(x)dx = \omega_N \int_0^1 (1+r/R)^{N-1}\varphi_1^2(r)dr \to \omega_N \int_0^1 \varphi_1^2 dr = \omega_N \tag{4.3.15}$$

as $R\to\infty$, it follows that

$$\lambda_{1,s}(A_R) = \inf_{u\in\mathscr{H}_0^s(A_R)} \left\{ \frac{\frac{b_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}dxdy}{\int_{A_R} u^2(x)dx} \right\}$$
$$\le \frac{\frac{b_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{(\Phi_R(x)-\Phi_R(y))^2}{|x-y|^{N+2s}}dxdy}{\int_{A_R} \Phi_R^2(x)dx} \le C(N,s)$$

as $R\to\infty$ by (4.3.14) and (4.3.15). This shows the claim holds for $R \ge R_0$ for some fixed $R_0 > 1$. The assertion of the Lemma then easily follows. □

From Lemma 4.3.1, we deduce the following

Lemma 4.3.2. *There exists $C(N,s) > 0$ such that*

$$\lambda_{2,s}(A_R) \le C(N,s) \quad \textit{for all } R \le 1 \quad R\to\infty. \tag{4.3.16}$$

Proof. As in the proof of Lemma 4.3.1, it is enough to show that the claim holds for $R \ge R_0$ for some $R_0 \ge 1$. Before we start the proof of this, we explain very briefly the idea of the argument.. Let φ_2 be a (normalised) second eigenfunction of the interval $(0,1)$. We define

$$\Psi_R(x) = \varphi_2(|x|-R) \quad \text{for all } x\in\mathbb{R}^N.$$

Next we let Υ_R to be the projection of Ψ_R into the $\mathbb{R}\varphi_{1,A_R}$ where φ_{1,A_R} is the first eigenfunction of A_R. That is,

$$\Upsilon_R := \Psi_R - \Big[\int_{A_R} \varphi_{1,A_R}(x)\Psi_R(x)\,dx\Big]\varphi_{1,A_R} := \Psi_R - \alpha_R\varphi_{1,A_R}.$$

It is clear that $\Upsilon_R = 0$ in $\mathbb{R}^N \setminus A_R$ and that $\int_{A_R}\Upsilon_R(x)\varphi_{1,A_R}(x)dx = 0$. Moreover, $\Upsilon_R \in H^s(\mathbb{R}^N)$ by construction. By the variational characterization of $\lambda_{2,s}(A_R)$, we get

$$\begin{aligned}\frac{2}{c_{N,s}}\lambda_{2,s}(A_R) &\le \frac{\int_{\mathbb{R}^{2N}} \frac{\left(\Upsilon_R(x)-\Upsilon_R(y)\right)^2}{|x-y|^{N+2s}}dxdy}{\int_{A_R}\Upsilon_R^2(x)dx}\\ &= \frac{\int_{\mathbb{R}^{2N}} \frac{\left(\Psi_R(x)-\Psi_R(y)\right)^2}{|x-y|^{N+2s}}dxdy + \alpha_R^2\int_{\mathbb{R}^{2N}}\frac{(\varphi_{1,A_R}(x)-\varphi_{1,A_R}(y))^2}{|x-y|^{N+2s}}dxdy - 2\alpha_R^2\lambda_{1,s}(A_R)}{\int_{A_R}\Psi_R^2(x)dx-\alpha_R^2}\\ &= \frac{\int_{\mathbb{R}^{2N}} \frac{\left(\Psi_R(x)-\Psi_R(y)\right)^2}{|x-y|^{N+2s}}dxdy - \alpha_R^2\lambda_{1,s}(A_R)}{\int_{A_R}\Psi_R^2(x)\,dx-\alpha_R^2} \qquad (4.3.17)\end{aligned}$$

To prove the lemma, we need a uniform upper estimate of the RHS of (4.3.17). First of all, following the argument of the proof of Lemma 4.3.1, we get

$$R^{-(N-1)}\int_{\mathbb{R}^{2N}} \frac{\left(\Psi_R(x)-\Psi_R(y)\right)^2}{|x-y|^{N+2s}}dxdy \le C(N,s) \quad \text{as } R\to\infty. \qquad (4.3.18)$$

Therefore we need to show that $R^{-(N-1)}\left(\int_{A_R}\Psi_R^2(x)\,dx-\alpha_R^2\right) > c > 0$ as $R\to\infty$. Since $R^{-(N-1)}\int_{A_R}\Psi_R^2(x) = \omega_N\int_0^1(1+\frac{r}{R})^{N-1}\varphi_2^2(r)dr \to \omega_N$ as $R\to\infty$, we need to show that $R^{-\frac{N-1}{2}}\alpha_R \to c' < \omega_N$ as $R\to\infty$. But this follows, once we show

$$\frac{\alpha_R}{R^{\frac{N-1}{2}}\omega_N} = \int_0^1 (1+\frac{r}{R})^{N-1}\varphi_{1,A_R}(\cdot+R)\varphi_2(r)dr \to \int_0^1\varphi_1(r)\varphi_2(r)dr = 0, \qquad (4.3.19)$$

as $R\to\infty$ where φ_1 is the first eigenfunction of $\lambda_{1,s}(0,1)$. The rest of this section is devoted to the proof of (4.3.19). To that aim, we let φ_{1,A_R} to be the normalized first eigenfunction so that

$$\begin{aligned}1 = \int_{A_R}\varphi_{1,A_R}^2(x)dx &= \omega_N R^{N-1}\int_0^1(1+\frac{r}{R})^{N-1}\varphi_{1,A_R}^2(\cdot+R)dr \qquad (4.3.20)\\ &\ge \omega_N R^{N-1}\int_0^1\varphi_{1,A_R}^2(\cdot+R)dr.\end{aligned}$$

By a similar computation as above, passing into polar coordinates in the identity

$$\frac{2}{b_{N,s}}\lambda_{1,s}(A_R)R^{-(N-1)}\int_{A_R}\varphi_{1,A_R}^2(x)dx = R^{-(N-1)}\int_{\mathbb{R}^{2N}}\frac{\left(\varphi_{1,A_R}(x)-\varphi_{1,A_R}(y)\right)^2}{|x-y|^{N+2s}}dxdy,$$

we obtain for R sufficiently large

$$\begin{aligned}
\overline{C}(N,s) &\geq \frac{2}{b_{N,s}}\lambda_{1,s}(A_R)\int_0^1 (1+r/R)^{N-1}\varphi_{1,A_R}^2(\cdot+R)dr \\
&= \int_{-R}^{\infty}\int_{-R}^{\infty} \frac{\Big(\varphi_{1,A_R}(r+R)-\varphi_{1,A_R}(\tilde r+R)\Big)^2}{|r-\tilde r|^{1+2s}} K_R(\tilde r,r)drd\tilde r \\
&= \int_{-a}^{a}\int_{-a}^{a} \frac{\big(\varphi_{1,A_R}(r+R)-\varphi_{1,A_R}(\tilde r+R)\big)^2}{|r-\tilde r|^{1+2s}} K_R(r,\tilde r)drd\tilde r + 2\int_0^1 \varphi_1^2(r)\int_{(-R,\infty)\setminus(-a,a)} \frac{K_R(r,\tilde r)}{|r-\tilde r|^{1+2s}}d\tilde r \\
&\geq \int_{-a}^{a}\int_{-a}^{a} \frac{\big(\varphi_{1,A_R}(r+R)-\varphi_{1,A_R}(\tilde r+R)\big)^2}{|r-\tilde r|^{1+2s}} K_R(r,\tilde r)drd\tilde r \\
&\geq C(N,s)\Big[\varphi_{1,A_R}(\cdot+R)\Big]^2_{H^s(-a,a)} \qquad \text{for all } a: 1<a<\infty, \qquad (4.3.21)
\end{aligned}$$

with $K_R(\cdot,\cdot)$ given by (4.3.4) . In the last line we used that

$$c(N,s) \geq K_R(r,\tilde r) \geq C(N,s) \qquad \forall\, r,\tilde r \in [-a,a],$$

for R sufficiently large. Hence, we may assume, up to passing to a subsequence, that

$$\varphi_{1,A_R}(\cdot+R) \to u_\infty \quad \text{weakly in } H^s_{loc}(\mathbb{R}) \quad \text{and} \quad \varphi_{1,A_R}(\cdot+R) \to u_\infty \quad \text{in } L^2(0,1) \qquad (4.3.22)$$

and

$$\varphi_{1,A_R}(\cdot+R) \to u_\infty \qquad \text{pointwise in } (0,1). \qquad (4.3.23)$$

To see the equation that u_∞ solves, we let $\psi \in C_c^\infty(0,1)$ and define $\psi_R(x) = \psi(|x|-R)$ so that $\psi_R \in C_c^\infty(A_R)$. Then, we get as above

$$\begin{aligned}
&\frac{2}{b_{N,s}}\lambda_{1,s}(A_R)\int_0^1 R^{N-1}(1+\frac{r}{R})^{N-1}\varphi_{1,A_R}(r+R)\psi(r)dr = \lambda_{1,s}(A_R)\frac{1}{\omega_N}\int_{A_R}\varphi_{1,A_R}(x)\psi_R(x)dx \\
&= \int_{-2}^{2}\int_{-2}^{2} \frac{\big(\varphi_{1,A_R}(r+r)-\varphi_{1,A_R}(\tilde r+R)\big)\big(\psi(r)-\psi(\tilde r)\big)}{|r-\tilde r|^{1+2s}} K_R(r,\tilde r)drd\tilde r \\
&+2\int_0^1 \varphi_{1,A_R}(r+R)\psi(r)\int_{(-R,\infty)\setminus(-2,2)} \frac{K_R(r,\tilde r)}{|r-\tilde r|^{1+2s}}d\tilde r dr. \qquad (4.3.24)
\end{aligned}$$

By the strong convergence there exists a subsequence, still denoted by R, so that

$$|\varphi_{1,A_R}(\cdot+R)| \leq h \quad \text{with} \quad h \in L^2(0,1). \qquad (4.3.25)$$

Moreover, for any $r \in (0,1)$, we know from above that

$$\begin{aligned}
&\int\limits_{(-R,\infty)\setminus(-2,2)} \frac{K_R(r,\tilde r)}{|r-\tilde r|^{1+2s}}d\tilde r = \int\limits_{-R}^{-2} \frac{K_R(r,\tilde r)}{|r-\tilde r|^{1+2s}}d\tilde r + \int\limits_{2}^{\infty} \frac{K_R(r,\tilde r)}{|r-\tilde r|^{1+2s}}d\tilde r \\
&= \frac{(r+R)^{N-1}}{R^{N-1}}(R+r)^{-2s}\int\limits_0^{\frac{R-2}{r+2}}(1+\rho)^{2s-1}\Big(\frac{\rho}{1+\rho}\Big)^{\frac{N-1}{2}}\overline{f}(\rho)d\rho \\
&+ \frac{(r+R)^{N-1}}{R^{N-1}}(R+r)^{-2s}\int\limits_1^{\frac{R+2}{2-r}}(\rho-1)^{2s-1}\rho^{\frac{N-1}{2}}(\rho-1)^{-\frac{N-1}{2}}\overline{g}(\rho)d\rho \\
&\le C(N,s)(1+r^{-2s}),
\end{aligned} \tag{4.3.26}$$

where $\overline{f}$ and $\overline{g}$ are defined in (4.3.8) and (4.3.10) respectively. Putting together (4.3.25) and (4.3.26) gives

$$\Big|\varphi_{1,A_R}(r+R)\psi(r)\int\limits_{\mathbb{R}\setminus(0,1)}\frac{K_R(r,\tilde r)}{|r-\tilde r|^{1+2s}}d\tilde r\Big| \le C(N,s)h(r)\psi(r)(1+r^{-2s}). \tag{4.3.27}$$

Moreover

$$\int\limits_0^1 h(r)\psi(r)(1+r^{-2s})dr = \int\limits_{\sup(\psi)} h(r)\psi(r)(1+r^{-2s})dr < \infty. \tag{4.3.28}$$

Consequently, the dominated convergence theorem yields

$$\begin{aligned}
&\lim_{R\to\infty}\int\limits_0^1 \varphi_{1,A_R}(r+R)\psi(r)\int\limits_{\mathbb{R}\setminus(-2,2)}\frac{K_R(r,\tilde r)}{|r-\tilde r|^{1+2s}}d\tilde r dr \\
&= \int\limits_0^1 u_\infty(r)\psi(r)\lim_{R\to\infty}\int\limits_{\mathbb{R}\setminus(0,1)}\frac{K_R(r,\tilde r)}{|r-\tilde r|^{1+2s}}d\tilde r dr \\
&= \int\limits_0^1 u_\infty(r)\psi(r)\Big[\lim_{R\to\infty}\big(\frac{r+R}{R}\big)^{N-1}\big(\frac{R-2}{R+r}\big)^{2s}\int\limits_0^{\frac{1}{r+2}}\big(\frac{1}{R-2}+\rho\big)^{2s-1}\Big(\frac{(R-2)\rho}{1+(R-2)\rho}\Big)^{\frac{N-1}{2}}\overline{f}((R-2)\rho)d\rho\Big]dr \\
&+ \int\limits_0^1 u_\infty(r)\psi(r)\Big[\lim_{R\to\infty}\big(\frac{r+R}{R}\big)^{N-1}\big(\frac{R+2}{R+r}\big)^{2s}\int\limits_{\frac{1}{R+2}}^{\frac{1}{2-r}}\big(\rho-\frac{1}{R+2}\big)^{2s-1}\Big(\frac{(R+2)\rho}{(R+2)\rho-1}\Big)^{\frac{N-1}{2}}\overline{g}((R+2)\rho)d\rho\Big]dr \\
&= \kappa_{N,s}\int\limits_0^1 u_\infty(r)\psi(r)\Big(\int\limits_0^{\frac{1}{r+2}}\rho^{2s-1}d\rho + \int\limits_0^{\frac{1}{2-r}}\rho^{2s-1}d\rho\Big)dr
\end{aligned}$$

$$= \kappa_{N,s} \int_0^1 u_\infty(r)\psi(r)\frac{1}{2s}\Big[(r+2)^{-2s} + (2-r)^{-2s}\Big]dr$$
$$= \kappa_{N,s} \int_0^1 u_\infty(r)\psi(r)\Big(\int_{-\infty}^{-2} \frac{d\tilde r}{|r-\tilde r|^{1+2s}} + \int_2^{\infty} \frac{d\tilde r}{|r-\tilde r|^{1+2s}}\Big)$$
$$= \kappa_{N,s} \int_0^1 u_\infty(r)\psi(r) \int_{\mathbb{R}\setminus(-2,2)} \frac{d\tilde r}{|r-\tilde r|^{1+2s}} \tag{4.3.29}$$

with

$$\kappa(N,s) = \lim_{R\to\infty} \overline{f}((R-2)\rho) = \lim_{R\to\infty} \overline{g}((R+2)\rho) = \frac{\pi^{\frac{N-1}{2}}\Gamma(\frac{1+2s}{2})}{\Gamma(\frac{N+2s}{2})}.$$

As for the first integral in (4.3.24), we write

$$\int_{-2}^{2}\int_{-2}^{2} \frac{\big(\varphi_{1,A_R}(r+R) - \varphi_{1,A_R}(\tilde r+R)\big)\big(\psi(r)-\psi(\tilde r)\big)}{|r-\tilde r|^{1+2s}} K_R(r,\tilde r)drd\tilde r$$
$$= \kappa(N,s)\int_{-2}^{2}\int_{-2}^{2} \frac{\big(\varphi_{1,A_R}(r+R) - \varphi_{1,A_R}(\tilde r+R)\big)\big(\psi(r)-\psi(\tilde r)\big)}{|r-\tilde r|^{1+2s}} drd\tilde r$$
$$+ \int_{-2}^{2}\int_{-2}^{2} \frac{\big(\varphi_{1,A_R}(r+R) - \varphi_{1,A_R}(\tilde r+R)\big)\big(\psi(r)-\psi(\tilde r)\big)}{|r-\tilde r|^{1+2s}} \Big(K_R(r,\tilde r) - \kappa(N,s)\Big)drd\tilde r$$
$$=: \mathscr{E}_{(-2,2)}\big(\varphi_{1,A_R}, \psi(\cdot+R)\big) + \mathscr{E}_{(-2,2),R}\big(\varphi_{1,A_R}(\cdot+R), \psi\big). \tag{4.3.30}$$

On one hand, by the weak convergence (4.3.22) we have

$$\lim_{R\to\infty} \mathscr{E}_{(-2,2)}\big(\varphi_{1,A_R}(\cdot+R),\psi\big) = \kappa(N,s)\int_{-2}^{2}\int_{-2}^{2} \frac{\big(u_\infty(r) - u_\infty(\tilde r)\big)\big(\psi(r)-\psi(\tilde r)\big)}{|r-\tilde r|^{1+2s}} drd\tilde r. \tag{4.3.31}$$

On the other hand,

$$\Big|\mathscr{E}_{(-2,2),R}\big(\varphi_{1,A_R}(\cdot+R),\psi\big)\Big|$$
$$\le \sup_{r,\tilde r\in(-2,2)} \big|K_R(\cdot,\cdot) - \kappa(N,s)\big| \int_{-2}^{2}\int_{-2}^{2} \frac{\big|\big(\varphi_{1,A_R}(r+R) - \varphi_{1,A_R}(\tilde r+R)\big)\big(\psi(r)-\psi(\tilde r)\big)\big|}{|r-\tilde r|^{1+2s}}$$
$$\le \sup_{r,\tilde r\in(-2,2)} \big|K_R(\cdot,\cdot) - \kappa(N,s)\big| [\varphi_{1,A_R}(\cdot+R)]_{H^s(-2,+2)}[\psi]_{H^s(-2,+2)}$$
$$\le C(N,s,\psi) \sup_{r,\tilde r\in(-2,2)} \big|K_R(\cdot,\cdot) - \kappa(N,s)\big| \to 0 \quad \text{as} \quad R\to\infty, \tag{4.3.32}$$

where we used (4.3.6) and (4.3.21). Plugging (4.3.31) and (4.3.32) into (4.3.30) yields

$$\int_{-2}^{2}\int_{-2}^{2} \frac{\big(\varphi_{1,A_R}(r+R)-\varphi_{1,A_R}(\tilde r+R)\big)\big(\psi(r)-\psi(\tilde r)\big)}{|r-\tilde r|^{1+2s}} K_R(r,\tilde r)drd\tilde r$$

$$= \kappa(N,s)\int_{-2}^{2}\int_{-2}^{2} \frac{\big(u_\infty(r)-u_\infty(\tilde r)\big)\big(\psi(r)-\psi(\tilde r)\big)}{|r-\tilde r|^{1+2s}}drd\tilde r. \tag{4.3.33}$$

Now with (4.3.29) and (4.3.33) we can pass to a limit in (4.3.24) to obtain (along a subsequence)

$$\lambda_\infty \int_0^1 u_\infty(r)\psi(r)dr = \kappa(N,s)\frac{b_{N,s}}{2}\int_{\mathbb{R}^2} \frac{(u_\infty(r)-u_\infty(\tilde r))(\psi(r)-\psi(\tilde r))}{|r-\tilde r|^{1+2s}}drd\tilde r$$

$$= \frac{b_{1,s}}{2}\int_{\mathbb{R}^2} \frac{(u_\infty(r)-u_\infty(\tilde r))(\psi(r)-\psi(\tilde r))}{|r-\tilde r|^{1+2s}}drd\tilde r \qquad \text{for all } \psi \in C_c^\infty(0,1),$$

where we used that $b_{N,s}\kappa(N,s) = \frac{s4^s\Gamma(\frac{N}{2}+s)}{\pi^{\frac{N}{2}}\Gamma(1-s)}\frac{\pi^{\frac{N-1}{2}}\Gamma(s+\frac{1}{2})}{\Gamma(\frac{N}{2}+s)} = \frac{s4^s\Gamma(s+\frac{1}{2})}{\sqrt{\pi}\Gamma(1-s)} = b_{1,s}$. Since $u_\infty \geq 0$ in $(0,1)$ by the pointwise convergence (4.3.23), it follows that $\lambda_\infty = \lambda_{1,s}(0,1)$ and $u_\infty = \varphi_1 > 0$ by the normalization condition. In conclusion, we have for $R \to \infty$

$$\frac{\alpha_R}{R^{\frac{N-1}{2}}\omega_N} = \int_0^1 (1+\frac{r}{R})^{N-1}\varphi_{1,A_R}(\cdot+R)\varphi_2(r)dr$$

$$\to \int_0^1 u_\infty(r)\varphi_2(r)dr = \int_0^1 \varphi_1(r)\varphi_2(r)dr = 0. \tag{4.3.34}$$

Whence, (4.3.19) holds and Lemma 4.3.1 follows. □

4.4 Uniform estimate of fractional normal derivative of second radial eigenfunctions of A_R

For simplicity we denote by u_R a fix radial eigenfunction (if any) corresponding to $\lambda_{2,s}(A_R)$ and define $w_R := \overline{u}_R(\cdot+R)$ where $u_R(x) = \overline{u}_R(|x|)$. From the calculation above, we know $w_R \in H^s_{loc}(\mathbb{R})$. Moreover, it solves weakly the equation

$$\mathscr{L}_{K_R}w_R + w_R V_R = \lambda_{2,s}(A_R)\big(1+r/R\big)^{N-1}w_R \qquad \text{in} \qquad (0,1) \tag{4.4.1}$$

where

$$\langle \mathscr{L}_{K_R}u, w\rangle = \int_{\mathbb{R}^2} (u(r)-u(\tilde r))(w(r)-w(\tilde r))K_R(r,\tilde r)drd\tilde r,$$

with $K_R(r,\widetilde{r}) = 1_{(-2,2)}(r)1_{(-2,2)}(\tilde{r})\overline{K}_R(r,\tilde{r})$, where

$$\overline{K}_R(r,\tilde{r}) = \frac{1}{|r-\tilde{r}|^{1+2s}} \frac{(r+R)^{\frac{N-1}{2}}(\tilde{r}+R)^{\frac{N-1}{2}}}{R^{N-1}} \int_{\frac{\sqrt{(r+R)(\tilde{r}+R)}}{|r-\tilde{r}|}(\mathbb{S}^{N-1}-e_1)} \frac{d\theta}{(1+|\theta|^2)^{\frac{N+2s}{2}}},$$

and

$$V_R(r) = \int_{(-R,\infty)\setminus(-2,2)} \frac{\overline{K}_R(r,\tilde{r})}{|r-\tilde{r}|^{1+2s}} d\tilde{r}.$$

Note that for R large enough we have

$$\overline{C}(N,s)|r-\widetilde{r}|^{-1-2s} \leq K_R(r,\tilde{r}) \leq C(N,s)|r-\widetilde{r}|^{-1-2s} \quad \text{and} \tag{4.4.2}$$

$$0 < V_R(r) \leq C(N,s) \tag{4.4.3}$$

Let $w_R \in H^s_{loc}(\mathbb{R})$ be a solution to (4.4.1) and $d(\cdot) = \min(\cdot, 1-\cdot)$ be the distance function to the boundary of $(0,1)$. Then we have

Proposition 4.4.1. *There exist a constant $C(N,s) > 0$ so that for all $\beta \in (0,s)$ we have*

$$\|\frac{w_R}{d^s}\|_{C^{s-\beta}([0,1])} \leq C(N,s) \quad \textit{for all } R \geq 2. \tag{4.4.4}$$

Proof. Let

$$\tau_R(t,r) := \frac{\sqrt{(t+R)(t\pm r+R)}}{r},$$

and consider the function $\overline{\lambda}_{K_R} : \mathbb{R} \times [0,\infty) \times \{\pm 1\} \to \mathbb{R}$ defined by

$$\begin{aligned}\overline{\lambda}_{K_R}(t,r,\pm 1) &= r^{1+2s}\overline{K}_R(t,t\pm r) \\ &= \frac{(t+R)^{\frac{N-1}{2}}(t\pm r+R)^{\frac{N-1}{2}}}{R^{N-1}} \int_{\tau_R(t,r)(\mathbb{S}^{N-1}-e_1)} \frac{d\theta}{(1+|\theta|^2)^{\frac{N+2s}{2}}},\end{aligned}$$

for all $r \neq 0$ and

$$\overline{\lambda}_{K_R}(t,0,\pm 1) := \lim_{r\to 0^+} \overline{\lambda}_{K_R}(t,r,\pm 1) = (1+t/R)^{N-1} \frac{\pi^{\frac{N-1}{2}}\Gamma(\frac{1+2s}{2})}{\Gamma(\frac{N+2s}{2})}.$$

By Corollary, 4.2.2 we have

$$\|\overline{\lambda}_{K_R}\|_{C^{0,s+\delta}\left([-2,2]\times[0,2]\times\{\pm 1\}\right)} \leq C_0 \quad \text{uniformly in } R >> 1. \tag{4.4.5}$$

Quoting [40, Theorem 1.8] we get

$$\|\frac{w_R}{d^s}\|_{C^{s-\beta}([0,1])} \leq C\big(\|w_R\|_{L^2(0,1)} + \|w_R\|_{\mathscr{L}^1_s} + \|\lambda_{2,s}(A_R)\big(1+r/R\big)^{N-1} w_R\|_{L^\infty(0,1)}\big), \tag{4.4.6}$$

for all $\beta \in (0,s)$ with $C>0$ independent of R by (4.4.5). Here $\|w_R\|_{\mathscr{L}^1_s} := \int_{\mathbb{R}} \frac{|w_R(r)|}{1+r^{1+2s}}dr$. By [12, Remark 3.2] we get

$$\sup_{r\in(0,1)} w_R(r) = \sup_{r\in(0,1)} u_R(r+R) = \|u_R\|_{L^\infty(A_R)} \le C(N,s)\Big[\lambda_{2,s}(A_R)\Big]^{N/4s}\|u_R\|_{L^2(A_R)}$$
$$\le C(N,s) \qquad \text{as} \qquad R\to\infty, \tag{4.4.7}$$

where we used Lemma 4.3.2. Combining (4.4.2), (4.4.3), (4.4.5) and (4.4.7), we see that the RHS of (4.4.6) is bounded independently of R, as claimed. □

Corollary 4.4.2. *There exists* $\varphi_2 \in \mathscr{H}^s_0(0,1)$ *solving*

$$(-\Delta)^s\varphi_2 = \lambda_{s,2}(0,1)\varphi_2 \quad \textit{in} \quad (0,1) \tag{4.4.8}$$

so that, along some subsequence still denoted by R, it holds

$$\frac{w_R}{d^s} \to \frac{\varphi_2}{d^s} \quad \textit{in} \quad C^0([0,1]). \tag{4.4.9}$$

Proof. By Proposition 4.4.1 and the Arzela-Ascoli theorem, there exist $q_\infty \in C^{s-\beta}([0,1])$ with

$$\frac{w_R}{d^s} \to q_\infty \qquad \text{uniformly in} \quad [0,1].$$

It remains to show that $q_\infty = \frac{\varphi}{d^s}$ for some φ solving (4.4.8). By a similar argument as for the case of the first eigenvalue, one obtains (along some subsequence):

$$w_R \to w_\infty \quad \text{weakly in} \quad H^s_{loc}(\mathbb{R}) \qquad \text{and} \qquad w_R \to w_\infty \quad \text{in} \quad L^2(0,1) \tag{4.4.10}$$

and

$$w_R \to w_\infty \quad \text{pointwise in} \quad (0,1) \tag{4.4.11}$$

Passing to a limit in (4.4.1) as we did in the proof of Lemma 4.3.2, we get

$$\lambda_\infty \int_0^1 \psi(r)w_\infty(r)dr = \frac{\kappa(N,s)c_{N,s}}{2}\int_{\mathbb{R}^2}\frac{(w_\infty(r)-w_\infty(\tilde r))(\psi(r)-\psi(\tilde r))}{|r-\tilde r|^{1+2s}}drd\tilde r$$
$$= \frac{c_{1,s}}{2}\int_{\mathbb{R}^2}\frac{(w_\infty(r)-w_\infty(\tilde r))(\psi(r)-\psi(\tilde r))}{|r-\tilde r|^{1+2s}}drd\tilde r,$$

for all $\psi \in C^\infty_c(0,1)$. Since w_∞ changes sign (by (4.4.11)), we have that $\lambda_\infty \ge \lambda_{2,s}(0,1)$. Moreover, since $R^{-(N-1)}\alpha_R^2 \to 0$ as $R\to\infty$ by (4.3.34), passing to the limit (along a subsequence) in (4.3.17) we get

$$\lambda_\infty = \lim_{R\to\infty}\lambda_{2,s}(A_R) \le \frac{\kappa(N,s)c_{N,s}}{2}\int_{\mathbb{R}^2}\frac{\varphi_2(r)-\varphi_2(\tilde r))^2}{|r-\tilde r|^{1+2s}}drd\tilde r = \lambda_{2,s}(0,1),$$

where φ_2 is a normalized eigenfunction of $\lambda_{2,s}(0,1)$. We deduce that $\lambda_\infty = \lambda_{2,s}(0,1)$ and therefore w_∞ is a second eigenfunction. By the pointwise convergence (4.4.11) and regularity we see that $\frac{w_R}{d^s} \to \frac{w_\infty}{d^s}$ in $C^0_{loc}(0,1)$ and by uniqueness of the limit we deduce that $q_\infty = \frac{w_\infty}{d^s}$ with w_∞ solving (4.4.8). □

4.5 Proof of the nonradiality of second eigenfunctions

We now have the necessary ingredient needed to proceed to the proof of Theorem 4.1.1.

Proof. Assume there is a second radial eigenfunction u_R corresponding to $\lambda_{2,s}(A_R)$. To reach a contradiction, we prove that such u_R must have Morse index greater than or equal to $N+1$ and this would contradict the fact of u_R being a second eigenfunction. Let $w_R := \overline{u}_R(\cdot + R)$, we know by Corollary 4.4.2 that there exists some subsequence still denoted by R along which we have $\frac{w_R}{d^s} \to \frac{\varphi_2}{d^s}$ in $C^0([0,1])$ where $d = \min(r, 1-r)$ and φ_2 is a second eigenfunction. We also know that φ_2 is antisymmetric and it vanishes only at $r = 1/2$ see e.g [41, Theorem 5.2]. Moreover, $\frac{\varphi_2}{d^s}(1)\frac{\varphi_2}{d^s}(0) < 0$ (the latter can be seen, for instance, by applying the Hopf lemma for entire antisymmetric supersolutions in [43, Proposition 3.3]). Consequently, we have

$$\left(\frac{u_R}{d^s}\bigg|_{\partial_{in}A_R}\right) \cdot \left(\frac{u_R}{d^s}\bigg|_{\partial_{out}A_R}\right) < 0 \qquad \text{for} \quad R > 0 \quad \text{sufficiently large.}$$

Here, $d(x) = \min\big(|x| - R, R + 1 - |x|\big)$. Without loss of generality we may assume

$$\frac{u_R}{d^s}\bigg|_{\partial_{out}A_R} > 0 \qquad \text{and} \qquad \frac{u_R}{d^s}\bigg|_{\partial_{in}A_R} < 0. \tag{4.5.1}$$

For simplicity we let $\overline{\psi}_R := \frac{\overline{u}_R(\cdot + R)}{\min(\cdot, 1-\cdot)^s} : [0,1] \to \mathbb{R}$ so that $\overline{\psi}_R(1) > 0$ and $\overline{\psi}_R(0) < 0$. In the spirit of [41], for any fix direction $j \in \{1, \cdots, N\}$, we define

$$d_R^j = (v_R^j)^+ 1_{H_+^j} - (v_R^j)^- 1_{H_-^j}, \tag{4.5.2}$$

where

$$v_R^j : \mathbb{R}^N \to \mathbb{R}^N, \quad x \mapsto v_R^j(x) = \begin{cases} \dfrac{\partial u_R}{\partial x_j}(x), & \text{if } x \in A_R; \\ 0 & \text{if } x \in \mathbb{R}^N \setminus A_R. \end{cases}$$

Now the point is because of (4.5.1) we find that $d_R^j \in \mathscr{H}_0^s(A_R)$. Indeed, By [42] we know that

$$d^{1-s}(x)\nabla u_R \cdot \nu(x) = -s\psi_R(x) \qquad \forall\, x \in \partial A_R, \quad \text{with} \quad \psi_R(x) := \lim_{A_R \ni y \to x} \frac{u_R(y)}{d^s(y)}. \tag{4.5.3}$$

Applying this to the inner and outer boundary of A_R gives respectively

$$s\overline{\psi}_R(0) = \lim_{t \to 0^+} t^{1-s}\partial_r \overline{w}_R(t) \qquad \text{and} \qquad -s\overline{\psi}_R(1) = \lim_{t \to 1^-} (1-t)^{1-s}\partial_r \overline{w}_R(t), \tag{4.5.4}$$

from which we deduce that $v_R^j(x) = \partial_r \overline{u}_R(|x|)\frac{x_j}{|x|} < 0$ near $\partial_{out}A_R$ whenever $x_j > 0$ since $\overline{\psi}_R(1) > 0$ by (4.5.1), that is, $(v_R^j)^+ 1_{H_+^j} = 0$ near $\partial_{out}A_R$. Hence, $d_R^j = 0$ near $\partial_{out}A_R \cap H_+^j$. By the same reasoning we show that $d_R^j = 0$ near $\partial_{out}A_R \cap H_-^j$. Consequently $d_R^j = 0$ near $\partial_{out}A_R$. Similarly, since $\overline{\psi}_R(0) < 0$ by (4.5.1), using the first identity in (4.5.4) we show that $d_R^j = 0$

near $\partial_{in} A_R$. In conclusion we have $\text{supp}(d_R^j) \subset\subset A_R$. By [41, lemma 2.2], to conclude that $d_R^j \in \mathscr{H}_0^s(A_R)$, one simply needs to check that $d_R^j \in \mathscr{H}_{loc}^s(A_R)$. For this one argue as in [41, Lemma 3.5]. In fact, since for all $\Omega' \subset\subset A_R$, we have that $\Omega' \cup \sigma_j(\Omega') \subset\subset A_R$ and $\left[(v_R^j)^+ 1_{H_+^j}\right]^2_{H^s(\Omega')} \leq \left[(v_R^j)^+ 1_{H_+^j}\right]^2_{H^s\left(\Omega' \cup \sigma_j(\Omega')\right)}$, we may assume without loss that Ω' is symmetric with respect to the reflection σ_j across the hyperplane ∂H_+^j and then argue as in [41, Lemma 3.5]. Having $d_R^j \in \mathscr{H}_0^s(A_R)$ for all $j \in \{1, \cdots, N\}$, we can argue as in [41] to show that u_R has Morse index greater than or equal $N+1$ and this contradicts the fact of u_R is a second eigenfunction. □

4.6 Maximization of the second eigenvalue

We first start with some preliminaries. In the following, let $\tau \in (0,1)$ be fixed. For simplicity, we let $\Omega_a := B_1(0) \setminus \overline{B_\tau(ae_1)}$ for all $a \in (-1+\tau, 1-\tau)$ and defined $\lambda_{1,s}^-(a) := \lambda_{1,s}^-(\Omega_a)$ by

$$\lambda_{1,s}^-(a) = \inf\left\{ \frac{\frac{b_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dxdy}{\int_{\Omega_a} u^2(x)dx} : u \in \mathscr{H}_0^s(\Omega_a), u \circ \sigma_N = -u \right\}, \tag{4.6.1}$$

where σ_N is the reflection with respect to the hyperplane $\partial H_+^N := \{x \in \mathbb{R}^N : x_N = 0\}$. It is a standard fact that the infimum in (4.6.1) is achieved by some u which solves the equation $(-\Delta)^s u = \lambda_{1,s}^-(a) u$ in Ω_a in the weak sense. We recall the following properties of minimizers of (4.6.1).

Lemma 4.6.1 (Theorem 1.2, [34])**.** *The function u achieving the infimum in* (4.6.1) *is unique (up to a multiplicative constant) and it is of one sign in* $\Omega_a^+ := \Omega_a \cap \{x \in \mathbb{R}^N : x_N > 0\}$. *Without lose we may assume* $u > 0$ *in* Ω_a^+.

Using the variational characterization

$$\lambda_{2,s}(a) = \inf\left\{ \frac{\frac{b_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dxdy}{\int_{\Omega_a} u^2(x)dx} : u \in \mathscr{H}_0^s(\Omega_a), \int_{\Omega_a} u(x)\varphi_1(x)\,dx = 0 \right\},$$

and recalling that the first eigenfunction φ_1 of Ω_a is symmetric with respect to σ_N, one easily checks that

$$\lambda_{2,s}(\Omega_a) \leq \lambda_{1,s}^-(a) \qquad \forall a \in (-1+\tau, 1-\tau) \tag{4.6.2}$$

The following observation is of key importance for the proof of Theorem 4.1.3.

Lemma 4.6.2. *Assume a second eigenfunction of the annulus* $B_1(0) \setminus \overline{B_\tau(0)}$ *cannot be radial. Then we have*

$$\lambda_{2,s}(0) = \lambda_{1,s}^-(0). \tag{4.6.3}$$

Proof. By (4.6.2), it remains to prove $\lambda_{2,s}(0) \geq \lambda_{1,s}^-(0)$. For that we let $\lambda_j^{(N+2l)}$, $l \geq 0$ and $j \geq 1$ be the set of radial eigenvalues of the problem $(-\Delta)^s u = \lambda u$ in $\Omega_0 \subset \mathbb{R}^{N+2l}$ and $u = 0$ in $\mathbb{R}^{N+2l} \setminus \Omega_0$. By [37, Proposition 3.5, (ii)], we know that the first eigenfunction $\varphi_1^{(N+2)}(|\cdot|)$ gives rise to N-linearly independent eigenfunctions $x_1\varphi_1^{(N+2)}(|x|),\cdots,x_N\varphi_1^{(N+2)}(|x|)$ to the problem $(-\Delta)^s u = \lambda u$ in $\Omega_0 \subset \mathbb{R}^N$ and $u = 0$ in $\mathbb{R}^N \setminus \Omega_0$ with the same eigenvalue $\lambda_1^{(N+2)}$. Consequently, the second eigenvalue $\lambda_{2,s}$ of Ω_0 is given by $\lambda_{2,s}(0) = \min(\lambda_1^{(N+2)}, \lambda_2^{(N)})$. Since by assumption a second eigenfunction cannot be radial, we deduce that $\lambda_{2,s} = \lambda_1^{(N+2)}$. Therefore the eigenspace corresponding to $\lambda_{2,s}$ is spanned by the functions $x \mapsto x_j\varphi_1^{(N+2)}(|x|)$ with $j = 1,\cdots,N$. Using $x_N\varphi_1^{(N+2)}(|\cdot|)$ as a test function in (4.6.1) we get $\lambda_{2,s}(0) \geq \lambda_{1,s}^-(0)$ as wanted. □

Next we prove

Proposition 4.6.3. *The mapping* $(-1+\tau, 1-\tau) \to \mathbb{R}_+$, $a \mapsto \lambda_{1,s}^-(a)$ *is differentiable. Moreover,* $(\lambda_{1,s}^-)'(0) = 0$ *and* $(\lambda_{1,s}^-)'(a) < 0$ *for all* $a \in (0, 1-\tau)$.

Proof. Fix $a \in (-1+\tau, 1-\tau)$ and let $\rho \in C_c^\infty(B_1)$ so that $\rho \equiv 1$ near $B_\tau(ae_1)$ and $\rho \circ \sigma_N = \rho$. For any $\varepsilon \in (-\varepsilon_0, +\varepsilon_0)$, we consider the maps $\Phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N, x \mapsto \Phi_\varepsilon(x) = x + \varepsilon\rho(x)e_1$. By making ε_0 sufficiently small if necessary, one may assume that $\Phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ is a global diffeomorphism for all $\varepsilon \in (-\varepsilon_0, +\varepsilon_0)$. Moreover, it is easily verifiable that $\Phi_\varepsilon(\Omega_a) = \Omega_{a+\varepsilon}$ and hence $\lambda_{1,s}^-(a+\varepsilon) = \lambda_{1,s}^-\big(\Phi_\varepsilon(\Omega_a)\big)$. Now it suffices to prove that the function $\varepsilon \mapsto \lambda_{1,s}^-\big(\Phi_\varepsilon(\Omega_a)\big)$ is differentiable at 0 and this is a consequence of the simplicity of $\lambda_{1,s}^-(a)$ stated in Lemma 4.6.1. Indeed, let u be the unique normalized minimizer corresponding to $\lambda_{1,s}^-(a)$, then the function

$$w_\varepsilon := \frac{u \circ \Phi_\varepsilon^{-1} - [u \circ \Phi_\varepsilon^{-1}] \circ \sigma_N}{2}$$

is admissible in the variational characterization of $\lambda_{1,s}^-(a+\varepsilon)$ and therefore

$$\lambda_{1,s}^-(a+\varepsilon) \leq \frac{\frac{b_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{(w_\varepsilon(x)-w_\varepsilon(y))^2}{|x-y|^{N+2s}}dxdy}{\int_{\Omega_{a+\varepsilon}} w_\varepsilon^2(x)dx} =: m(\varepsilon).$$

Since $m(0) = \lambda_{1,s}^-(a)$, we deduce that

$$\limsup_{\varepsilon \to 0^+} \frac{\lambda_{1,s}^-(a+\varepsilon) - \lambda_{1,s}^-(a)}{\varepsilon} \leq \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} m(\varepsilon). \tag{4.6.4}$$

By a simple change of variables and using that $\varphi_\varepsilon \circ \sigma_N = \sigma_N \circ \varphi_\varepsilon$, one obtains

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} m(\varepsilon) = \int_{\mathbb{R}^{2N}} (u(x)-u(y))^2 K_X(x,y)dxdy + \lambda_{1,s}^-(a)\int_{\Omega_a} u^2(x)\operatorname{div}X(x)dx, \tag{4.6.5}$$

where

$$K_X(x,y) = \frac{b_{N,s}}{2}\left\{\big(\operatorname{div}X(x) + \operatorname{div}X(y)\big) - (N+2s)\frac{\big(X(x)-X(y)\big)\cdot(x-y)}{|x-y|^2}\right\}|x-y|^{-N-2s}$$

and $X = \rho e_1 \in C_c^\infty(B_1, \mathbb{R}^N)$. Since u solves weakly the equation

$$\begin{cases} (-\Delta)^s u = \lambda_{1,s}^-(a) u \quad \text{in} \quad \Omega_a; \\ u \in \mathscr{H}_0^s(\Omega_a), \end{cases}$$

the standard regularity theory see e.g [94] gives $u \in L^\infty(\Omega_a) \cap C^1_{loc}(\Omega_a)$ and therefore u satisfies the assumptions of [33, Theorem 1.2]. Consequently

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} (u(x)-u(y))^2 K_X(x,y)\,dxdy \\
&= \Gamma^2(1+s) \int_{\partial\Omega_a} (u/^s)^2 X\cdot\nu\, dx + 2\int_{\Omega_a} \nabla u \cdot X (-\Delta)^s u\, dx \\
&= \Gamma^2(1+s) \int_{\partial\Omega_a} (u/d^s)^2 X\cdot\nu\, dx + 2\lambda_{1,s}^- \int_{\Omega_a} u\nabla u\cdot X\, dx \\
&= \Gamma^2(1+s) \int_{\partial\Omega_a} (u/d^s)^2 X\cdot\nu\, dx - \lambda_{1,s}^-(a) \int_{\Omega_a} u^2(x) \mathrm{div} X(x)\, dx
\end{aligned}$$

Here ν is the outer unit normal to the boundary. Plugging this into (4.6.5) and recalling (4.6.4), we conclude that

$$\begin{aligned}
\limsup_{\varepsilon\to 0^+} \frac{\lambda_{1,s}^-(a+\varepsilon)-\lambda_{1,s}^-(a)}{\varepsilon} &\le \Gamma^2(1+s)\int_{\partial\Omega_a} (u/d^s)^2 X\cdot\nu\, dx \\
&= \Gamma^2(1+s)\int_{\partial B_\tau(ae_1)} (u/d^s)^2 e_1\cdot\nu\, dx
\end{aligned} \tag{4.6.6}$$

Furthermore, arguing as in [32, Lemma 4.5], we also obtain

$$\liminf_{\varepsilon\to 0^+} \frac{\lambda_{1,s}^-(a+\varepsilon)-\lambda_{1,s}^-(a)}{\varepsilon} \ge \Gamma^2(1+s)\int_{\partial B_\tau(ae_1)} (u/d^s)^2 e_1\cdot\nu\, dx \tag{4.6.7}$$

Combining (4.6.6) and (4.6.7) yields

$$\partial_\varepsilon^+ \lambda_{1,s}^-(a+\varepsilon) := \lim_{\varepsilon\to 0^+} \frac{\lambda_{1,s}^-(a+\varepsilon)-\lambda_{1,s}^-(a)}{\varepsilon} = \Gamma^2(1+s)\int_{\partial B_\tau(ae_1)} (u/d^s)^2 e_1\cdot\nu\, dx. \tag{4.6.8}$$

Applying (4.6.8) to $\varepsilon \mapsto \lambda_{1,s}^-(a-\varepsilon)$ gives

$$\lim_{\varepsilon\to 0^-} \frac{\lambda_{1,s}^-(a+\varepsilon)-\lambda_{1,s}^-(a)}{\varepsilon} = -\partial_\varepsilon^+ \lambda_{1,s}^-\big(a-\varepsilon\big) = \Gamma^2(1+s)\int_{\partial B_\tau(ae_1)} (u/d^s)^2 e_1\cdot\nu\, dx \tag{4.6.9}$$

Consequently

$$(\lambda_{1,s}^{-})'(a) = \lim_{\varepsilon \to 0} \frac{\lambda_{1,s}^{-}(a+\varepsilon) - \lambda_{1,s}^{-}(a)}{\varepsilon} = \Gamma^2(1+s) \int_{\partial B_\tau(ae_1)} (u/d^s)^2 e_1 \cdot \nu \, dx, \tag{4.6.10}$$

for all $a \in (-1+\tau, 1-\tau)$. Since $\lambda_{1,s}^{-}(a) = \lambda_{1,s}^{-}(-a)$, we have $(\lambda_{1,s}^{-})'(0) = 0$. Next fix $a \in (0, 1-\tau)$ and let $H_a := \{x \in \mathbb{R}^N : x_1 > a\}$. Moreover, let us denote by σ_a the reflection with respect to the hyperplane ∂H_a. For simplicity, we let $\overline{u} := u \circ \sigma_a$, $w := \overline{u} - u$ and $B^a := B_\tau(ae_1)$. It is not difficult to check that w solves

$$(-\Delta)^s w = \lambda_{1,s}^{-}(a) w \qquad \text{in} \qquad \Theta := H_a \cap \Omega_a^{+}.$$

Moreover, by Lemma 4.6.1 we have

$$w \geq 0 \quad \text{in} \quad H_a \cap H_+^N \setminus \Theta.$$

By [34, Proposition 2.3], the maximum principle for doubly antisymmetric functions, we get that $w \geq 0$. Consequently $w > 0$ in Θ by the strong maximum principle [34, Proposition 2.4] since $w > 0$ in $\sigma_a(B_1) \cap H_a \cap H_+^N \setminus \Theta$. Finally, by [34, Proposition 2.4], we deduce that

$$0 < \frac{w}{d^s} = \frac{\overline{u}}{d^s} - \frac{u}{d^s} \qquad \text{on} \qquad \partial B^a \cap H_+^N \cap H_a = \partial B^a \cap H_a \cap \Omega_a^{+}. \tag{4.6.11}$$

It is also clear that

$$\frac{u}{d^s} \geq 0 \qquad \text{on} \qquad \partial B^a \cap H_a \cap \Omega_a^{+}. \tag{4.6.12}$$

Now using the fact that $\sigma_N\Big(\partial B^a \cap H_a \cap \Omega_a^{+}\Big) = \partial B^a \cap H_a \cap \Omega_a^{-}$, $\sigma_N(\nu \cdot e_1) = \nu \cdot e_1$ and $\sigma_N \circ \sigma_a = \sigma_a \circ \sigma_N$, we get

$$\begin{aligned}
\int_{\partial B^a} \big(\frac{u}{d^s}\big)^2 \nu \cdot e_1 dx &= \int_{\partial B^a \cap H_a} \Big[\big(\frac{u}{d^s}\big)^2 - \big(\frac{u \circ \sigma_a}{d^s}\big)^2\Big] \nu \cdot e_1 dx \\
&= \int_{\partial B^a \cap H_a \cap \Omega_a^{+}} \Big[\big(\frac{u}{d^s}\big)^2 - \big(\frac{\overline{u}}{d^s}\big)^2\Big] \nu \cdot e_1 dx + \int_{\partial B^a \cap H_a \cap \Omega_a^{-}} \Big[\big(\frac{u}{d^s}\big)^2 - \big(\frac{\overline{u}}{d^s}\big)^2\Big] \nu \cdot e_1 dx \\
&= \int_{\partial B^a \cap H_a \cap \Omega_a^{+}} \Big[\big(\frac{u}{d^s}\big)^2 - \big(\frac{\overline{u}}{d^s}\big)^2\Big] \nu \cdot e_1 dx + \int_{\partial B^a \cap H_a \cap \Omega_a^{+}} \Big[\big(\frac{u \circ \sigma_a}{d^s}\big)^2 - \big(\frac{\overline{u} \circ \sigma_a}{d^s}\big)^2\Big] \nu \cdot e_1 dx \\
&= 2 \int_{\partial B^a \cap H_a \cap \Omega_a^{+}} \Big[\big(\frac{u}{d^s}\big)^2 - \big(\frac{\overline{u}}{d^s}\big)^2\Big] \nu \cdot e_1 dx \\
&= -2 \int_{\partial B^a \cap H_a \cap \Omega_a^{+}} \frac{w}{d^s}\Big(\frac{u}{d^s} + \frac{\overline{u}}{d^s}\Big) \nu \cdot e_1 dx.
\end{aligned} \tag{4.6.13}$$

Combining (4.6.10), (4.6.11), (4.6.12) and (4.6.13) we conclude that

$$(\lambda_{1,s}^{-})'(a) = \Gamma^2(1+s) \int_{\partial B^a} \big(u/d^s\big)^2 \nu \cdot e_1 dx < 0 \qquad \forall a \in (0, 1-\tau),$$

as wanted. □

We are now ready to complete the proof of Theorem 4.1.3.

Proof of Theorem 4.1.3. Since the problem is invariant under rotation, we may consider without loss domains of the form

$$\Omega_a := B_1 \setminus \overline{B_\tau(ae_1)}, \qquad \text{for} \quad a \in [0, 1-\tau).$$

By (4.6.2), Lemma 4.6.2, and Proposition 4.6.3, it immediately follows that

$$\lambda_{2,s}(a) \leq \lambda_{1,s}^{-}(a) < \lambda_{1,s}^{-}(0) = \lambda_{2,s}(0),$$

for all $a \in (0, 1-\tau)$. The proof is finished. □

Chapter 5

Summary

The main topic investigated in this thesis is the computation of the boundary expression of the shape derivative of the best constants in the Sobolev embedding $\mathcal{H}_0^s(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$ in the subcritical regime $p \in [1, \frac{2N}{(N-2s)^+})$. Here $a^+ = \max(a,0)$ and by convention we set $\frac{a}{0} = \infty$. Before the present work, very little was known about the shape derivative in this fractional setting. Up to our knowledge, the only work addressing this topic is the paper [26]. In there, the authors compute the shape derivative of the functional $\Omega \mapsto J_f(\Omega) = -\frac{1}{2}\int_\Omega f(x)u_f(x)dx$, associated to the solution of the Dirichlet boundary value problem $(-\Delta)^{1/2}u = f$ in Ω, $u = 0$ in $\mathbb{R}^N \setminus \Omega$, where $f \in C^\infty(\mathbb{R}^2)$ and Ω is a bounded open set of $\mathbb{R}^2$ of class C^∞. They showed that the latter can be expressed as a boundary integral involving the fractional normal derivative $u_f/d^{1/2}$. Precisely, they showed that

$$\frac{d}{dt}J_f(\Omega_t)\bigg|_{t=0} = C_0 \int_{\partial\Omega} \big(\frac{u_f}{d^{1/2}}\big)^2 X\cdot\nu\, dx \tag{5.0.1}$$

with some explicit constant C_0, where $\Omega_t := \Phi_t(\Omega)$ with Φ_t being the flow associated to the deformation field $X \in C_c^\infty(\mathbb{R}^2,\mathbb{R}^2)$. Here $d = \mathrm{dist}(\cdot,\mathbb{R}^N \setminus \Omega)$ denotes the distance function to the boundary and ν is the interior unit normal vector field.

The formula (5.0.1) was the best known regarding the boundary expression of shape derivative of nonlocal shape functionals and a more general formula was missing in the literature. The paper [$P1$] fills that gap. More generally, in [$P1$] we compute the one-sided shape derivative of the best constants $\lambda_{s,p}(\Omega)$ in the Sobolev embedding $\mathcal{H}_0^s(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$ and as a consequence we obtain the fractional version of the so called Hadamard formula established in [60] for the classical Laplacian. Before we state the result explicitly, we first fix the following notations. Let $N \geq 1$ and fix $\Phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ a family of deformations with the property

$$\begin{aligned}&\Phi_\varepsilon \in C^{1,1}(\mathbb{R}^N;\mathbb{R}^N) \text{ for } \varepsilon \in (-1,1),\ \Phi_0 = \mathrm{id}_{\mathbb{R}^N}, \text{ and}\\ &\text{the map } (-1,1) \to C^{0,1}(\mathbb{R}^N,\mathbb{R}^N),\ \varepsilon \to \Phi_\varepsilon \text{ is of class } C^2.\end{aligned} \tag{5.0.2}$$

Then we have the following.

Theorem 5.0.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set of class $C^{1,1}$, and let $\lambda_{s,p}(\Omega)$ be given by*

$$\lambda_{s,p}(\Omega) = \inf_{\substack{u \in \mathscr{H}_0^s(\Omega) \\ u \neq 0}} \left\{ \frac{\frac{b_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dxdy}{(\int_\Omega |u|^p dx)^{2/p}} \right\},$$

with $p \in [1, \frac{2N}{(N-2s)^+})$. Consider a family of deformations Φ_ε satisfying (5.0.2)*, then for $\varepsilon_0 > 0$ sufficiently small, the map $(-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$, $\quad \varepsilon \mapsto \lambda_{s,p}(\Phi_\varepsilon(\Omega))$ is right differentiable at $\varepsilon = 0$. Moreover,*

$$\partial_\varepsilon^+ \big|_{\varepsilon=0} \lambda_{s,p}(\Phi_\varepsilon(\Omega)) = \min \left\{ \Gamma(1+s)^2 \int_{\partial\Omega} (u/d^s)^2 X \cdot \nu \, dx \, : \, u \in \mathscr{H} \right\}, \tag{5.0.3}$$

where ν denotes the interior unit normal on $\partial\Omega$, $\mathscr{H}$ the set of positive normalised minimizers for $\lambda_{s,p}(\Omega)$, $X := \frac{d}{d\varepsilon}\Phi_\varepsilon \Big|_{\varepsilon=0}$ and Γ the usual Gamma function.

We mention that for p in the range $[1,2]$, $\lambda_{s,p}(\Omega)$ has a unique normalized minimizer u. In this case, Theorem 5.0.1 reduces to the following.

Corollary 5.0.2. *Let $p \in [1,2]$ and let $\lambda_{s,p}(\varepsilon)$ be as above. Then the maps $\varepsilon \mapsto \lambda_{s,p}(\varepsilon)$ is differentiable at* 0*. Moreover,*

$$\frac{d}{d\varepsilon}\lambda_{s,p}(\varepsilon)\Bigg|_{\varepsilon=0} = -\Gamma^2(1+s) \int_{\partial\Omega} (u/d^s)^2 X \cdot \nu \, dx, \tag{5.0.4}$$

where u is the unique normalized minimizer corresponding to $\lambda_{s,p}(\Omega)$.

Identity (5.0.4) extends in this fractional setting the so called Hadamard formula obtained in [60] for the classical Laplacian. The identity (5.0.4) is used to study the optimal obstacle placement problem for the torsional rigidity and the first eigenvalue of the fractional Laplacian in the spirit of [77, 91]. This amounts of finding the position of a spherical obstacle B_τ within a bigger ball, for instance within $B_1(0)$ which maximizes or minimizers the corresponding functional. We prove that in order to maximize the eigenvalue or the torsional rigidity, the obstacle must be located at the center of the ball. The precise statement of the result is as follows.

Theorem 5.0.3. *Let $p \in \{1,2\}$, $B_1(0)$ be the unit centered ball and $\tau \in (0,1)$. Define*

$$\mathscr{A} := \{a \in B_1(0) \, : \, B_\tau(a) \subset B_1(0)\}.$$

Then the map $\mathscr{A} \to \mathbb{R}$, $a \mapsto \lambda_{s,p}(B_1(0) \setminus \overline{B_\tau(a)})$ takes its maximum at $a = 0$.

The idea of the argument is inspired from [61, 77] and it consists of analysing the derivative of the eigenvalue with respect to the position of the obstacle. Thanks to (5.0.4), the latter is expressed as a boundary integral involving the fractional normal derivative u/d^s of the solution

of the underlying equation $(-\Delta)^s u = \lambda_{s,p}(B_1(0)\setminus\overline{B_\tau(a)})u^{p-1}$ in $B_1(0)\setminus\overline{B_\tau(a)}$, $u=0$ in $\mathbb{R}^N\setminus(B_1(0)\setminus\overline{B_\tau(a)})$ with $p\in\{1,2\}$. To analyse its sign we use reflection techniques based on maximum principles for antisymmetric functions established in [43].
In the paper [$P4$] we extend the result of Theorem 5.0.3 to the fractional second eigenvalue $\lambda_{2,s}$. In order to do that we introduce and prove in [$P2$] a new maximum principle for doubly antisymmetric functions. This is stated as follows.

Proposition 5.0.4. *Let H and $\widetilde{H}$ be two half spaces such that the hyperplanes ∂H and $\partial\widetilde{H}$ are perpendicular, and let r, $\tilde{r}$ be the reflections with respect to ∂H and $\partial\widetilde{H}$ respectively. Let $w\in H^s(\mathbb{R}^N)$ be a weak doubly antisymmetric supersolution to the problem*

$$(-\Delta)^s w = c(x)w \quad in \quad U\subset H\cap\widetilde{H}, \qquad w\geq 0 \quad in \quad H\cap\widetilde{H}\setminus U, \tag{5.0.5}$$

in the sense that

$$w\circ r = -w = w\circ\tilde{r} \quad and \quad \mathcal{E}_s(w,\varphi)\geq\int_U c(x)w(x)\varphi(x)dx \quad \forall\,\varphi\in\mathcal{H}_0^s(U),\,\varphi\geq 0 \ in\ U.$$

$$Assume \quad \|c\|_{L^\infty(U)}\leq\lambda_{1,s}^-(U) \quad with \quad \lambda_{1,s}^-(U):=\inf_{\substack{u\in\mathcal{H}_0^s\left(U\cup\tilde{r}(U)\right)\\ u\neq 0\\ u\circ\tilde{r}=-u}}\left\{\frac{\mathcal{E}_s(u,u)}{\int_{U\cup\tilde{r}(U)}u^2(x)dx}\right\}.$$

Then $w\geq 0$ in U.

As a consequence of Proposition 5.0.4, we deduce the following symmetry result regarding solutions to the equation $(-\Delta)^s u = f(x,u)$ in Ω, $u=0$ in $\mathbb{R}^N\setminus\Omega$.

Theorem 5.0.5. *Let $\Omega\subset\mathbb{R}^N$ with $N\in\mathbb{N}$, $N\geq 2$ be open and bounded and, moreover, convex and symmetric in the directions x_1 and x_N. Let $f\in C(\Omega\times\mathbb{R})$ be locally Lipschitz in its second variable, that is, for every bounded set $K\subset\mathbb{R}$ we have*

$$\sup_{x\in\Omega}|f(x,u)-f(x,v)|\leq L(K)|u-v| \quad for\ all\ u,v\in K.$$

Assume further that f is symmetric in x_1 and monotone in $|x_1|$. Then, every continuous bounded weak solution of $(-\Delta)^s u = f(x,u)$ in Ω, $u=0$ in $\mathbb{R}^N\setminus\Omega$, which is antisymmetric with respect to $\partial H_{N,0}$ and $u\geq 0$ in $H_{N,0}\cap\Omega$ is symmetric with respect to $\partial H_{1,0}$. Moreover, either $u\equiv 0$ in Ω or $u|_{\Omega\cap H_{1,0}\cap H_{N,0}}$ is strictly decreasing in x_1, that is, for every $x,y\in\Omega\cap H_{1,0}\cap H_{N,0}$ with $x_1<y_1$ we have $u(x)>u(y)$. Here, we denote $H_{j,0}=\{x\in\mathbb{R}^N : x_j>0\}$.

The other main achievement of this thesis concerns the Dirichlet boundary value problem $(-\Delta)^s u = f(u)$ in Ω and $u=0$ in $\mathbb{R}^N\setminus\Omega$ where f is a locally Lipschitz nonlinearity and Ω is a bounded open set of class $C^{1,1}$. In the seminal work [95], X. Ros-Oton and J. Serra found and established the so called Pohozaev identity for the fractional Laplacian, which states that any bounded weak solution u of the problem $(-\Delta)^s u = f(u)$ in Ω and $u=0$ in $\mathbb{R}^N\setminus\Omega$ satisfies

$$\Gamma(1+s)^2\int_{\partial\Omega}(u/d^s)^2 x\cdot\nu\,dx = 2N\int_\Omega F(u)\,dx-(N-2s)\int_\Omega f(u)u\,dx. \tag{5.0.6}$$

Here, ν is the unit outer normal vector field, $F(t) = \int_0^t f(s)ds$ and Γ the usual Gamma function. Since its discovery, identity (5.0.6) has been used widely in the analysis of the equation $(-\Delta)^s u = f(u)$ in Ω, $u = 0$ in $\mathbb{R}^N \setminus \Omega$. In particular, it has been used in the recent work [41] to answer a conjecture by Bañuelos and Kulczycki regarding the shape of fractional second eigenfunctions of balls. Our main achievement in the paper [$P3$] is a generalization of (5.0.6). The result is stated as follows.

Theorem 5.0.6. *Let $X \in C^{0,1}(\mathbb{R}^N, \mathbb{R}^N)$ be a globally Lipschitz vector field and let $K_X(x,y)$, $x \neq y$ be the fractional deformation kernel defined by*

$$K_X(x,y) := \frac{b_{N,s}}{2}\left\{ \big(\operatorname{div} X(x) + \operatorname{div} X(y)\big) - (N+2s)\frac{\big(X(x)-X(y)\big)\cdot(x-y)}{|x-y|^2} \right\} |x-y|^{-N-2s}.$$

Let

$$\mathscr{E}_{K_X}(u,w) := \int_{\mathbb{R}^{2N}} (u(x)-u(y))(w(x)-w(y))K_X(x,y)\,dxdy \quad \text{for all } u,w \in H^s(\mathbb{R}^N),$$

be the bilinear form associated to the kernel K_X. Then, any bounded weak solution of the problem $(-\Delta)^s u = f(u)$ in Ω, $u = 0$ in $\mathbb{R}^N \setminus \Omega$ satisfies

$$\Gamma(1+s)^2 \int_{\partial\Omega} \big(u/d^s\big)^2 X\cdot\nu\,dx = 2\int_{\Omega} F(u)\operatorname{div} X\,dx - \mathscr{E}_{K_X}(u,u), \tag{5.0.7}$$

where $F(t) = \int_0^t f(s)ds$, ν the outer unit normal to the boundary $\partial\Omega$ and Γ the usual Gamma function.

Theorem 5.0.6 is used to derive nonexistence results for the Dirichlet problem $(-\Delta)^s u = |u|^{p-2}u$ in Ω and $u = 0$ in $\mathbb{R}^N \setminus \Omega$ in the supercritical regime $p \geq \frac{2N}{N-2s}$. It is also used to compute the boundary expression of the shape derivative of simple eigenvalues of the Dirichlet problem $(-\Delta)^s u = \lambda u$ in Ω and $u = 0$ in $\mathbb{R}^N \setminus \Omega$.
We obtained Theorem 5.0.6 as a particular case of the following integration by parts formula, which can be seen as a generalization of [95, Proposition 1.6] by X. Ros-Oton and J. Serra.

Theorem 5.0.7. *Let $u \in H^s(\mathbb{R}^N)$ such that $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$. Moreover, assume $(-\Delta)^s u \in L^\infty(\Omega)$ if $2s > 1$ and $(-\Delta)^s u \in C^\alpha_{loc}(\Omega) \cap L^\infty(\Omega)$ with $\alpha > 1-2s$ if $2s \leq 1$. Then for any vector field $X \in C^{0,1}(\mathbb{R}^N, \mathbb{R}^N)$, it holds that*

$$2\int_{\Omega} \nabla u \cdot X(-\Delta)^s u\,dx = -\Gamma^2(1+s)\int_{\partial\Omega} (u/d^s)^2 X\cdot\nu\,dx - \mathscr{E}_{K_X}(u,u). \tag{5.0.8}$$

The proof of Theorem 5.0.7, similar to that of Theorem 5.0.1, is based on an approximation argument. It mainly consists of two steps: We introduce a cut-off function ς_k that vanishes in the $\frac{1}{k}$-neighbourhood of the boundary $\partial\Omega$ and approximate the quantity $\mathscr{E}_{K_X}(u,u)$ by $\mathscr{E}_{K_X}(\varsigma_k u, \varsigma_k u)$. By an integration by parts, we write the quantity $\mathscr{E}_{K_X}(\varsigma_k u, \varsigma_k u)$ into the compact form $\mathscr{E}_{K_X}(\varsigma_k u, \varsigma_k u) = -2\int_{\mathbb{R}^N} \nabla(\varsigma_k u)\cdot X(-\Delta)^s(\varsigma_k u)dx$. The problem then reduces into finding the limit $\lim_{k\to\infty} c_k(u) = -2\lim_{k\to\infty}\int_{\mathbb{R}^N} \nabla(\varsigma_k u)\cdot X(-\Delta)^s(\varsigma_k u)dx$. By the choice of the cut-off function, the domain of integration in the latter equality is replaced by $\Omega_\varepsilon^+ := \{x \in \Omega : d(x) < \varepsilon\}$ for all $\varepsilon > 0$ and therefore we are lead to establishing the identity

$$-2\lim_{k\to\infty}\int_{\Omega_\varepsilon^+} \nabla(\varsigma_k u)\cdot X(-\Delta)^s(\varsigma_k u)dx = -\Gamma^2(1+s)\int_{\partial\Omega}(u/d^s)^2 X\cdot\nu\, dx - 2\int_{\Omega}\nabla u\cdot X(-\Delta)^s u\, dx,$$

for all $\varepsilon > 0$. This is done by flattening the boundary and using the dominated convergence theorem.

Bibliography

[1] A. Aftalion and F. Pacella, *Qualitative properties of nodal solutions of semilinear elliptic equations in radially symmetric domains*, C. R. Math. 339.5 (2004): 339-344.

[2] V. Akgiray and G. G. Booth, *The siable-law model of stock returns*, Journal of Business & Economic Statistics 6.1 (1988): 51-57.

[3] T. V. Anoop, V. Bobkov, and S. Sasi, *On the strict monotonicity of the first eigenvalue of the p-Laplacian on annuli*, Trans. Am. Math. Soc 370.10 (2018): 7181-7199.

[4] V. Ambrosio, *Mountain pass solutions for the fractional Berestycki-Lions problem*, Adv. Differential Equations 23.5-6 (2018), 455–488.

[5] V. Ambrosio, *Nonlinear fractional Schrödinger equations in* $\mathbb{R}^N$, Frontiers in Elliptic and Parabolic Problems. Birkhäuser/Springer, [Cham]; 2021.

[6] B. Barrios, L. Montoro, and B. Sciunzi, *On the moving plane method for nonlocal problems in bounded domains*, Journal d'Analyse Mathématique 135.1 (2018): 37-57.

[7] C. Bandle and A. Wagner, *Second domain variation for problems with Robin boundary conditions*, Journal of Optimization Theory and Applications 167.2 (2015): 430-463.

[8] R. Bañuelos and T. Kulczycki, *The Cauchy process and the Steklov problem*, Journal of Functional Analysis 211.2 (2004): 355-423.

[9] J. Benedikt et al, *Nonradiality of second eigenfunctions of the fractional Laplacian in a ball*, arXiv preprint arXiv:2102.08298 (2021).

[10] M. Birkner, J. A. López-Mimbela, and A. Wakolbinger, *Comparison results and steady states for the Fujita equation with fractional Laplacian*, Ann. Inst. H. Poincaré Anal. Non Linéaire 22 (2005), 83–97.

[11] V. Bobkov and S. Kolonitskii, *On qualitative properties of solutions for elliptic problems with the p-Laplacian through domain perturbations*, Communications in Partial Differential Equations 45.3 (2020): 230-252.

[12] L. Brasco and E. Parini, *The second eigenvalue of the fractional p-Laplacian*, Advances in Calculus of Variations 9.4 (2016): 323-355.

[13] J. F. Bonder, S. Martinez, and J. D. Rossi, *The behavior of the best Sobolev trace constant and extremals in thin domains*, Journal of Differential Equations 198.1 (2004): 129-148.

[14] J. F. Bonder and J. D. Rossi, *Asymptotic behavior of the best Sobolev trace constant in expanding and contracting domains*, Communications on Pure & Applied Analysis 1.3 (2002): 359.

[15] G. Bouchitté, I. Fragala, and I. Lucardesi, *Shape derivatives for minima of integral functionals*, Mathematical Programming 148.1 (2014): 111-142.

[16] C. Bucur, *Some nonlocal operators and effects due to nonlocality*, arXiv preprint arXiv:1705.00953 (2017).

[17] C. Bucur and E. Valdinoci, *Nonlocal diffusion and applications*, volume 20 of *Lecture Notes of the Unione Matematica Italiana* Springer, [Cham]; Unione Matematica Italiana, Bologna, 2016.

[18] X. Cabré and J. Tan, *Positive solutions of nonlinear problems involving the square root of the Laplacian*, Adv. Math. 224.5 (2010), 2052–2093.

[19] L. A. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Communications in partial differential equations 32.8 (2007): 1245-1260.

[20] T. Caroll, M. M. Fall and J. Ratzkin, *On the rate of change of the best constant in the Sobolev inequality*. Math. Nachr. 290 (2017), no. 14-15, 2185-2197.

[21] J. Céa, *Conception optimale ou identification de formes, calcul rapide de la dérivée directionnelle de la fonction coût*, ESAIM: Mathematical Modelling and Numerical Analysis 20.3 (1986): 371-402.

[22] H. Chen, *The Dirichlet elliptic problem involving regional fractional Laplacian*, Journal of Mathematical Physics 59.7 (2018): 071504.

[23] H. Chen and T. Weth, *The Dirichlet problem for the logarithmic Laplacian*, Communications in Partial Differential Equations 44.11 (2019): 1100-1139.

[24] W. Chen, C. Li, and Y. Li, *A direct method of moving planes for the fractional Laplacian*, Advances in Mathematics 308 (2017): 404-437.

[25] A. M. H. Chorwadwala and R. Mahadevan, *A shape optimization problem for the p-Laplacian*. Proc. Roy. Soc. Edinburg Sect. A145. (2015), no. 6, 1145–1151.

[26] A. L. Dalibard and D. Gérard-Varet, *On shape optimization problems involving the fractional Laplacian*, ESAIM: Control, Optimisation and Calculus of Variations 19.4 (2013): 976-1013.

[27] E. N. Dancer and K. Zhang, *Uniqueness of solutions for some elliptic equations and systems in nearly star-shaped domains*, Nonlinear Analysis: Theory, Methods & Applications 41.5-6 (2000): 745-761.

[28] M. Delfour and J. Zolesio, *Shapes and Geometries*. Analysis, Differential Calculus, and Optimization, Advances in Design and Control, Vol. 4, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001.

[29] M. C. Delfour and J. P. Zolésio, *Structure of shape derivatives for nonsmooth domains*, Journal of Functional Analysis 104.1 (1992): 1-33.

[30] M. del Pino and C. Flores, *Asymptotic behavior of best constants and extremals for trace embeddings in expanding domains*, Communications in Partial Differential Equations 26.11-12 (2001): 2189-2210.

[31] E. Di Nezza, Giampiero Palatucci and Enrico Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bulletin des sciences mathématiques 136.5 (2012): 521-573.

[32] S. M. Djitte, M. M. Fall and T. Weth, *A fractional Hadamard formula and applications*, Calculus of Variations and Partial Differential Equations 60.6 (2021): 1-31.

[33] S. M. Djitte, M. M. Fall, and T. Weth, *A generalized fractional Pohozaev identity and applications*, preprint available at `https://arxiv.org/abs/2112.10653` (2021).

[34] S. M. Djitte and S. Jarohs, *Symmetry of odd solutions to equations with fractional Laplacian*, to appear in Journal of Elliptic and Parabolic equations, preprint available at `https://arxiv.org/abs/2109.14024` (2021).

[35] S. M. Djitte and S. Jarhos, *Nonradility of second fractional eigenfunctions of thin annuli*, preprint available at `https://arxiv.org/abs/2201.04907` (2022).

[36] B. Dyda, *Fractional calculus for power functions and eigenvalues of the fractional Laplacian*, Fractional calculus and applied analysis 15.4 (2012): 536-555.

[37] B. Dyda, A. Kuznetsov, and M. Kwaśnicki, *Fractional Laplace operator and Meijer G-function*, Constr. Approx. 45.3 (2017): 427-448.

[38] B. Dyda, A. Kuznetsov and M. Kwaśnicki, *Eigenvalues of the fractional Laplace operator in the unit ball*, Journal of the London Mathematical Society 95.2 (2017): 500-518.

[39] A. El Soufi and R. Kiwan, *Where to place a spherical obstacle so as to maximize the second Dirichlet eigenvalue*, arXiv preprint arXiv:0712.2033 (2007).

[40] M. M. Fall, *Regularity estimates for nonlocal Schrödinger equations*, Discrete & Continuous Dynamical Systems - A, 2019, 39 (3) : 1405-1456. doi: 10.3934/dcds.2019061.

[41] M. M. Fall, P. A. Feulefack, R. Y. Temgoua, and T. Weth, *Morse index versus radial symmetry for fractional Dirichlet problems*, Advances in Mathematics 384 (2021)

[42] M. M. Fall and S. Jarohs, *Gradient estimates in fractional Dirichlet problems*, Potential Anal. 54 (2021), no. 4, 627–636. 35 (47).

[43] M. M. Fall and S. Jarohs, *Overdetermined problems with fractional Laplacian*, ESAIM Control Optim. Calc. Var. 21 (2015), no.4, 924–938.

[44] M. M. Fall and T. Weth, *Critical domains for the first nonzero Neumann eigenvalue in Riemannian manifolds*, The Journal of Geometric Analysis (2018): 1-27.

[45] M. M. Fall and T. Weth, *Monotonicity and nonexistence results for some fractional elliptic problems in the half-space*. Commun. Contemp. Math. 18 (2016), no. 1, 1550012, 25 pp.

[46] M. M. Fall and T. Weth, *Nonexistence results for a class of fractional elliptic boundary value problems*, Journal of Functional Analysis 263.8 (2012): 2205-2227.

[47] P. Felmer and Y. Wang, *Radial symmetry of positive solutions to equations involving the fractional Laplacian*, Communications in Contemporary Mathematics 16.01 (2014): 1350023.

[48] R. AC. Ferreira, *Anti-symmetry of the second eigenfunction of the fractional Laplace operator in a 3-D ball*, arXiv preprint arXiv:1706.04960 (2017).

[49] G. Franzina and D. Lamberti. *Existence and uniqueness for a p-Laplacian nonlinear eigenvalue problem*, Electronic Journal of Differential Equations (EJDE) 26 (2010), available online at `http://eudml.org/doc/230413`.

[50] R. L. Frank, E. Lenzmann, and L. Silvestre, *Uniqueness of radial solutions for the fractional Laplacian*, Communications on Pure and Applied Mathematics 69.9 (2016): 1671-1726.

[51] P. R. Garabedian and M. Schiffer, *Convexity of domain functionals*, Stanford Univ Ca Applied Mathematics And Statistic LABS, 1953.

[52] N. Garofalo, *Fractional thoughts*, arXiv preprint arXiv:1712.03347 (2017).

[53] R. K. Getoor, *First passage times for symmetric stable processes in space*, Transactions of the American Mathematical Society 101.1 (1961): 75-90.

[54] B. Gidas, W. N. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. 68.3 (1979), 209–243.

[55] F. Giovanni, and G. Palatucci, *Fractional p-eigenvalues*, arXiv preprint arXiv:1307.1789 (2013).

[56] P. Grisvard, *Elliptic problems in nonsmooth domains*, volume 69 of Classics in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011. Reprint of the 1985 original.

[57] G. Grubb, *Fractional Laplacians on domains, a development of Hörmander's theory of μ-transmission pseudodifferential operators*, Advances in Mathematics 268 (2015), 478–528.

[58] G. Grubb, *Fractional Laplacians on domains, a development of Hörmander's theory of μ-transmission pseudodifferential operators*, Advances in Mathematics 268 (2015): 478-528.

[59] G. Grubb, *Integration by parts and Pohozaev identities for space-dependent fractional-order operators*, Journal of Differential Equations 261.3 (2016): 1835-1879.

[60] J. Hadamard, *Mémoire sur le probleme d'analyse relatif a l'équilibre des plaques élastiques*, In Mémoire des savants étrangers, 33, 1907, Oeuvres de Jacques Hadamard, pages 515 641. Editions du C.N.R.S., Paris, 1968.

[61] E. M. Harrell and P. Kröger, K. Kurata, *On the placement of an obstacle or a well so as to optimize the fundamental eigenvalue*, SIAM journal on mathematical analysis 33.1 (2001): 240-259.

[62] A. Henrot, *Extremum problems for eigenvalues of elliptic operators*, Springer Science & Business Media, 2006.

[63] A. Henrot, *Minimization problems for eigenvalues of the Laplacian*, Nonlinear Evolution Equations and Related Topics. Birkhäuser, Basel, 2003. 443-461.

[64] A. Henrot and E. Oudet, *Minimizing the second eigenvalue of the Laplace operator with Dirichlet boundary conditions*, Archive for rational mechanics and analysis 169.1 (2003): 73-87.

[65] A. Henrot and M. Pierre, *Variation et optimisation de formes: une analyse géométrique*, Vol. 48. Springer Science & Business Media, 2006.

[66] D. Henry, *Perturbation of the boundary in boundary-value problems of partial differential equations*, No. 318. Cambridge University Press, 2005.

[67] J. Hersch, *The method of interior parallels applied to polygonal or multiply connected membranes*, Pacific J. Math., 13 (1963) 1229-1238.

[68] N. E. Humphries et al, *Environmental context explains Lévy and Brownian movement patterns of marine predators*, Nature 465.7301 (2010): 1066-1069.

[69] K. Ito, K. Kunisch, and G. H. Peichl, *Variational approach to shape derivatives*, ESAIM: Control, Optimisation and Calculus of Variations 14.3 (2008): 517-539.

[70] S. Jarohs, *Symmetry of solutions to nonlocal nonlinear boundary value problems in radial sets*, Nonlinear Differential Equations and Applications NoDEA 23.3 (2016): 32.

[71] S. Jarohs and T. Weth, *Asymptotic symmetry for a class of nonlinear fractional reaction-diffusion equations*, Discrete Contin. Dyn. Syst. 34.6 (2014), 2581–2615, `http://dx.doi.org/10.3934/dcds.2014.34.2581`.

[72] S. Jarohs and T. Weth, *Local compactness and nonvanishing for weakly singular nonlocal quadratic forms*, Nonlinear Anal. 193 (2020), 111431.

[73] S. Jarohs and T. Weth, *On the strong maximum principle for nonlocal operators*, Math. Z. 293.1-2 (2019), 81–111.

[74] S. Jarohs and T. Weth, *Symmetry via antisymmetric maximum principles in nonlocal problems of variable order*, Ann. Mat. Pura Appl. (4) 195.1 (2016), 273–291.

[75] D. D. Joseph, *Parameter and domain dependence of eigenvalues of elliptic partial differential equations*, Archive for Rational Mechanics and Analysis 24.5 (1967): 325-351.

[76] B. Kawohl and P. Lindqvist. *Positive eigenfunctions for the p-Laplace operator revisited*, Analysis 26.4 (2006), 539–544.

[77] S. Kesavan, *On two functionals connected to the Laplacian in a class of doubly connected domains*, Proc. Roy. Soc. Edinburgh Sect. A, 133(2003) 617-624.

[78] M. Kwaśnicki, *Eigenvalues of the fractional Laplace operator in the interval*, Journal of Functional Analysis 262.5 (2012): 2379-2402.

[79] M. Kwaśnicki, *Ten equivalent definitions of the fractional Laplace operator*, Fractional Calculus and Applied Analysis 20.1 (2017): 7-51.

[80] N. Laskin, *Fractional quantum mechanics and Lévy path integrals*, Physics Letters A 268.4-6 (2000): 298-305.

[81] S. Z. Levendorskii, *Pricing of the American put under Lévy processes*, International Journal of Theoretical and Applied Finance 7.03 (2004): 303-335.

[82] E. Lindgren and P. Lindqvist, *Fractional eigenvalues*, Calc. Var. PDE 49 (2014), 795–826.

[83] A. Massaccesi and E. Valdinoci, *Is a nonlocal diffusion strategy convenient for biological populations in competition?*, Journal of mathematical biology 74.1-2 (2017): 113-147.

[84] J. Garcia Melián and J. J. Sabina de Lis, *On the perturbation of eigenvalues for the p-Laplacian*, C. R. Acad. Sci. Paris Sér. I Math. 332 (2001), 893-898.

[85] J. McGough and J. Mortensen, *Pohozaev obstructions on non-starlike domains*, Calc. Var. Partial Differential Equations 18 (2003), no. 2, 189–205.

[86] E. Oudet, *Numerical minimization of eigenmodes of a membrane with respect to the domain*, ESAIM: Control, Optimisation and Calculus of Variations 10.3 (2004): 315-330.

[87] Y. J. Park, *Fractional polya-Szegö inequality*, Journal of the Chungcheong Mathematical Society 24.2 (2011): 267-271.

[88] M. Pierre and J. R. Roche, *Numerical simulation of tridimensional electromagnetic shaping of liquid metals*, Numerische Mathematik 65.1 (1993): 203-217.

[89] S. I. Pohozaev, *On the eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Dokl.

[90] P. Pucci and J. Serrin, *A general variational identity*, Indiana University mathematics journal 35.3 (1986): 681-703. Akad. Nauk SSSR 165 (1965), 1408-1411.

[91] A. G. Ramm and P. N. Shivakumar, *Inequalities for the minimal eigenvalue of the Laplacian in an annulus*, arXiv preprint math-ph/9911040 (1999).

[92] W. Reichel, *Uniqueness theorems for variational problems by the method of transformation groups*, Lecture Notes in Mathematics, 1841. Springer-Verlag, Berlin, 2004. xiv+152 pp.

[93] X. Ros-Oton, *Nonlocal elliptic equations in bounded domains: a survey*, Publicacions matematiques (2016): 3-26.

[94] X. Ros-Oton and J. Serra, *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary*, J. Math. Pures Appl. (9) 101. (2014), no.3, 275–302.

[95] X. Ros-Oton and J. Serra, *The Pohozaev identity for the fractional Laplacian*, Arch. Ration. Mech. Anal. 213.2 (2014), 587–628.

[96] X. Ros-Oton, J. Serra, and E. Valdinoci, *Pohozaev identities for anisotropic integrodifferential operators*, Communications in Partial Differential Equations 42.8 (2017): 1290-1321.

[97] X. Ros-Oton and J. Serra, *Regularity theory for general stable operators*, J. Differential Equations 260 (2016), 8675-8715.

[98] R. Servadei and E. Valdinoci, *Mountain Pass solutions for non-local elliptic operators*, J. Math. Anal. Appl. 389 (2012), 887–898.

[99] R. Servadei and E. Valdinoci, *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst. 33.5 (2013), 2105–2137.

[100] J. Serrin, *A symmetry problem in potential theory*, Arch. Rational Mech. Anal. 43 (1971), 304-318.

[101] R. A. Silverman et al., *Special functions and their applications*, Courier Corporation, 1972.

[102] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*. Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences 60, no. 1 (2007): 67–112.

[103] A. Wagner, *Pohozaev's Identity from a Variational Viewpoint*, Journal of Mathematical Analysis and Applications 266, 149-159 (2002).

[104] X. Ros-Oton and J. Serra, *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary*, Journal de Mathématiques Pures et Appliquées 101.3 (2014): 275-302.

[105] W. Schoutens, *Lévy processes in finance: pricing financial derivatives*, J. Wiley, 2003.

[106] Y. Sire, J. L. Vázquez and B. Volzone, *Symmetrization for fractional elliptic and parabolic equations and an isoperimetric application*, Chinese Annals of Mathematics, series B 38.2 (2017): 661-686.

[107] B. Sirakov, *Overdetermined elliptic problems in physics*, Nonlinear PDEs? in Condensed Matter and Reactive Flows. Springer, Dordrecht, 2002. 273-295.

[108] J. Sokolowski and J. P. Zolésio, *Introduction to shape optimization*, Springer, Berlin, Heidelberg, 1992. 5-12.

[109] K. Sturm, *Minimax Lagrangian approach to the differentiability of nonlinear PDE constrained shape functions without saddle point assumption*, SIAM Journal on Control and Optimization 53.4 (2015): 2017-2039.

[110] J. P. Zolésio (1979), *Identification de domaines par déformations (Doctoral dissertation)*

[111] J. P. Zolésio and M. C. Delfour, *Shapes and Geometries: Analysis, Differential Calculus and Optimization*, SIAM, 2001.

[112] D. Z. Zanger, *Eigenvalue variations for the Neumann problem*, Applied Mathematics Letters 14.1 (2001): 39-44.