Formal and non-formal approaches to the notion of "Risk":

A historical perspective

PD Dr. Frank Linhard

Goethe-University Frankfurt
Department of Physics
Max-von-Laue-Str. 1
D- 60438 Frankfurt am Main / Germany

Mail: F.Linhard@em.uni-frankfurt.de

Abstract: One of the problems modern theories of risk have to cope with is the variety of approaches to the subject. While the interest in solutions for applied fields like risk management and financial mathematics in general increased during the last years due to systems of regulations like Basel II and Solvency, particular problems were discussed employing mathematical approaches since the 16^{th} century. The variety of the particular problems investigated and the multitude of perspectives and interests in the related formulations of the questions lead to miscellaneous approaches during the centuries. These were the more or the less applicable to specific problems under consideration and had to be completed and supplemented and often additional aspects were added.

The multitude of isolated considerations and solutions of problems connected with the notion of risk led to a development of many single and equally isolated approaches and nowadays there is still no conclusive and integrated theory of risk in sight.

The paper will focus on the early texts of Galileo Galilei (1613~1623) and Daniel Bernoulli (1738) as examples of pure combinatorical analysis and perspectively considerations within the mathematical discipline of probability theory. It is argued that Bernoulli's approach needed to be developed further in order to achieve a successful and satisfactory theory of risk. In modern economy the need for a proper definition of a notion of risk is seen and currently discussed within the frame of ISO standards. But as already mentioned this interest is mainly owed to the governmental demands of the Basel II and Solvency standards and therefore an external demand. On the other hand an intrinsic understanding of the meaning of risk, as could be provided by a conclusive theory, could lead to a better success in modelling various risks and help to achieve better prognosis.

Introduction

In ancient times, few people could understand even the simplest arithmetic and geometry, and the confusion of mathematics with magic has a long history.
People who had knowledge of the regular movements of the heavens were able to predict the position of planets, and the particular the times when astronomical events appeared in certain sections of the sky.

In Europe, after the arrival of Christianity, the religious aspect of these practices was condemned as superstition. Because numbers were used in these processes, anyone who used numbers was regarded with considerable suspicion. A contemporary quote of St. Augustine of Hippo (354-430 CE), an early christian bishop, reads:

> *"The good Christian should beware of mathematicians and all those who make empty prophecies. The danger already exists that mathematicians have made a covenant with the devil to darken the spirit and confine man in the bonds of Hell."*

As the predictive power of astronomy and other practical uses of mathematics became apparent, mathematicians were able to dispel the idea that many events were not controlled by the goddess Fortuna, but could be explained in a rational way. It is a historical fact, that the beginnings of probability theory are connected with dice. Dice were used in gambling, in religious ceremonies and for divination. And I will start this considerations by scetching this tradition.

Probabilities and dice

The first document showing the possibilities with three dice was the Latin poem *De Vetula* , which shows all the combinations for the fall of three dice, and is believed to have been written in the early 13th century.
Later Dante refered to a game played with three dice in his *Divina Comedia*:

> "Quando si parte *il giuoco della zara*
> Colui che perde si riman dolente,
> Repetendo le volte, e tristo impara;
> Con l’altro se ne va tutta la gente;
> Qual va dinnanzi, e qual di dietro il prende,
> E qual da lato li si reca a mente.
> Ei non s’arresta, e questo e quello intende;
> A cui porge la man, più non fa pressa;
> E così dalla calca si difende."[1]

1 English translation: „WHENE'ER is broken up the game of Zara,
He who has lost remains behind despondent,
The throws repeating, and in sadness learns;
The people with the other all depart; One goes in front, and one behind doth pluck
And at his side one brings himself to mind;
He pauses not, and this and that one hears;
They crowd no more to whom his hand he stretches,
And from the throng he thus defends“ (Dante Divina Comedia [Pur, VI, 1-12])

Another twohundred years later Cardano (1501-1576) turned to the topic. He was writing with considerable personal knowledge of gambling. And it was him who recognised that if the die was honest, each face would have an equal chance of appearing. His *Liber De Ludo Aleae* was written about 1526 but only found after his death, and not published until 1663. He gave tables of the results for one, two and three dice, but these are not all correct.

Galilei's consideration of dice

Galileo (1564 - 1642) wrote on probability but his work was not published until 1718. He stated that with three dice there can only be one way of obtaining a 3 (1,1,1) and an 18 (6,6,6) but there are three ways of obtaining a 6, and four ways for a 7.

The manuscript was found without a title. The titles known today were given by the editors. One can find "Considerazione sopra il gioco dei dadi" (1718) as well as "Sopra le scoperte dei dadi".[2]

The problem Galilei solved was stated by a nobleman, as can be deduced from the text. But it seems clear, that it was his employer Cosimo II de' Medici, Duke of Tuscany. Therefore the paper can be dated as of 1610-1621. Cosimo II. de’ Medici (1590-1621) was the Grandduke of Tuscany since 1608. In autumn 1610 Galilei was appointed mathematician of the court.

The fact that in a dice-game certain numbers are more advantageous than others has a very obvious reason, i.e. that some are more easily and more frequently made than others, which depends on their being able to be made up with more variety of numbers. Thus a 3 and an 18, which are throws which can only be made in one way with 3 numbers (that is, the latter with 6.6.6 and the former with 1.1.1, and in no other way), are more difficult to make than, e.g. 6 or 7, which can be made up in several ways:

6 can be achieved by 3 different combinations
7 can be achieved by 4 different combinations
Therefore the output 7 should occur more often than 6.

One can easily convince oneself by simply counting the possible outcomes:

6	**7**
1.2.3	1.1.5
2.2.2	1.2.4
1.1.4	1.3.3
	2.2.3

"Nevertheless, although 9 and 12 can be made up in as many ways as 10 and 11, and therefore they should be considered as being of equal utility to these, yet it is known that long observation has made dice-players consider 10 and 11 to be more advantageous than 9 and 12. And it is clear that 9 and 10 can be made up by an equal diversity of numbers, [...] which are six triple numbers, [...], and in no other ways, and these also are six combinations."[3]

2 Title given by Favaro in the Edizione Nazionale, vol. 8
3 Galilei: Considerazione sopra il giuoco dei dadi.

9	**10**
1.2.6	1.3.6
1.3.5	1.4.5
1.4.4	2.2.6
2.2.5	2.3.5
2.3.4	2.4.4
3.3.3	3.3.4

Galilei hopes to open the way to a precise understanding of the reasons for which all the details of the game have been with great care and judgement arranged and adjusted. His analysis reads:

> „Three special points must be noted for a clear understanding of what follows. The first is that that sum of the points of 3 dice, which is composed of 3 equal numbers, can only be produced by *one single throw* of the dice: and thus a 3 can only be produced by the three ace-faces, and a 6, if it is to be made up of 3 twos, can only be made by a single throw."

Therefore Galilei's conclusion is: here exists only 1 partition. Accordingly 4 and 8 can be composed by 3 partitions, namely

4 = (2.1.1), (1.2.1), (1.1.2)

and

8 = (2.2.3), (3.2.3), (3.3.2).[4]

In general each partition of 3 distinct numbers can be made of 6 permutations. The combinations for 8 are still considered as an example, using the partition 1.3.4:

8 = (1.3.4), (1.4.3), (3.1.4), (4.1.3), (3.4.1), (4.3.1).[5]

Galilei sums up his findings:

4 In Galilei's words this reads: "Secondly: the sum which is made up of 3 numbers, of which two are the same and the third different, can be produced by *three throws*: as e.g., a 4 which is made up of a 2 and of two aces, can be produced by three different throws; that is, when the first die shows 2 and the second and third show the ace, or the second die a 2 and the first and third the ace; or the third a 2 and the first and second the ace. And so e.g., an 8, when it is made up of 3.3.2, can be produced also in three ways: i.e. when the first die shows 2 and the others 3 each, or when the second die shows 2 and the first and third 3, or finally when the third shows 2 and the first and second 3."

5 Galilei writes: "Thirdly the sum of points which is made up of three different numbers can be produced in six ways. As for example, an 8 which is made up of 1.3.4. can be made with six different throws: first, when the first die shows 1, the second 3 and the third 4; second, when the first die still shows 1, but the second 4 and the third 3; third, when the second die shows 1, and the first 3 and the third 4; fourth, when the second still shows 1, and the first 4 and the third 3; fifth, when the third die shows 1, the first 3, and the second 4; sixth, when the third shows 1, the first 4 and the second 3."

1. triples, that is the sum of three-dice throws, which are made up of three equal numbers, can only be produced in one way
2. that the triples which are made up of two equal numbers and the third different, are produced in three ways
3. that those triples which are made up of three different numbers are produced in six ways

His final result is given by the table:

1 3 6	10		9		8		7		6		5		4		3	
10	631	6	621	6	611	3	511	3	411	3	311	3	211	3	111	1
15	622	3	531	6	521	6	421	6	321	6	221	3				
21	541	6	522	3	431	6	331	3	222	1						
25	532	6	441	3	422	3	322	3								
27	442	3	432	6	332	3										
108	433	3	333	1												
108		27		25		21		15		10		6		3		1
216																

This table gives indeed the correct results and it makes Cosimo's problem comprehensible. The comprehension is not only achieved by counting the possible outcomes, but Galilei shows how one can thoroughly analyse complex systems which may be not overlooked at a first glance. His text was a large step towards theoretical analysis of complex systems, as it showed how to analyse risks and chances in formerly unknown or less understood systems. Therefore Galilei's paper marks the dawn of a theory of probability and decision theory.

He shows that it is most important to *discern* between questions of different types. In the example one has to deal with three levels. These are:

• Level 1: event of interest: outcome if you throw 3 dice

• Level 2: event (at medium level): partition

• Level 3: elementary event (sub-structure): permutation

Ian Hacking in his book on probability (1975/2006)[6] makes two important observations on this:

> "In case the probabilities of compound outcomes with dice should now be so well established as to seem a priori, it may be useful to update the example. Consider elementary particles of microphysics. The natural generalization of dice takes r dice with n faces, thus giving n' equiprobable alternatives."

Therefore one has the problem of the distribution of r objects into n cells, but this is related to one of the particle statistics in physics:

> "This is called the Maxwell-Boltzmann system. So far as is known, no particles obey the Maxwell-Boltzmann laws, which apply to indistinguishable particles. If however we consider that arrangements of particles are indistinguishable, we get the Bose-Einstein

6 Ian Hacking 1975/2006:

> statistics. If we add the further condition that it is impossible for two or more particles to be in the same cell, and assign equal probability to all arrangements satisfying that condition we get the Fermi-Dirac statistics."[7]

Hacking comments further on the relation to modern particle statistics and connects this to Leibniz[8]:

> "It is an empirical fact that photons obey the Bose-Einstein statistics, electrons obey the Fermi-Dirac statistics, and dice obey the Maxwell-Boltzmann statistics. Leibniz once made the mistake of supposing that with dice partitions (rather than permutations) form the Fundamental Probability Set of equal alternatives. That is, he accepted the Bose-Einstein statistics for dice!"[9]

Galilei's conclusion and the solution of Cosimo's problem was: If permutations are equally probable, the 11 is more advantageous than 12 in the ratio 27:25. It is also agreed by Hacking, that this is one of the first examples in history where a statistical proof is given by long term observations.

It should be observed that Leibniz had metaphysical problems with the concept of indiscernibility and therefore supposed that all kinds of particles should stay within the classical domain. He uses properties like those of the Fermi-Dirac statistics to disprove the existence of 'particles' obeying this in several passages where he is dealing with the concept of indiscernibility.[10]

Further Galilei almost 'invented' the concept of 'long term observation' and refers to it in several places, e.g. in his "Discorsi" where he claims that he performed his experiment proofing the independence of mass in free fall "more than one hundred times" at the tower of Pisa.
[11]

Finally the difference between 9 and 10 (3 dice experiment) is in terms of probability (25/216) to (27/216), which is even in *long terms* difficult to observe. Although this may be possible, indeed.

Bernoulli's Specimen Theoriae Novae

The other important step towards a theory of risks, I want to mention, was taken 1738 by Daniel Bernoulli (1700-1782) in his analysis of a problem posed by his cousin Nicolaus Bernoulli in 1713.[12]

At the beginning of the paper Bernoulli states, that all mathematicians who had studied the measurement of risk agree, that:

7 Ian Hacking 1975/2006:
8 The problem is connected to Leibniz's philosophy through the notion of indiscernibility. For details see Linhard 2000, pp.155-201.
9 Ian Hacking 1975/2006 pp.51-52.
10 Frank Linhard 2000 and Frank Linhard 2008.
11 Everybody who climbed the Pisa tower (when this was still possible) will agree that this claim of Galilei is in doubt. He most probably did not try his experiment with free fall for more than 100 times.
12 Daniel Bernoulli: "Specimen Theoriae Novae de Mensura Sortis", 1738, *Commentarii Academiae Scientiarum Imperialis Petropolitanae*., (trans. in 1954, *Econometrica, Vol. 22, No.1, p.23-36*).

> "Expected values are computed by multiplying each possible gain by the number of ways in which it can occur, and then dividing the sum of these products by the total number of cases where, in this theory, the consideration of cases which are all of the same probability is insisted upon."

and that:

> "If this rule be accepted, what remains to be done within the framework of this theory amounts to the enumeration of all alternatives, their breakdown into equi-probable cases and, finally, their insertion into corresponding classifications."[13]

Expected values are computed by multiplying each possible gain by the number of ways in which it can occur, and then dividing the sum of these products by the total number of cases. It is supposed that the cases are all of the same probability.

If X is a discrete random variable with probability mass function p(x), the expected value becomes E(X). Be f(x) the probability mass function of *X,* the outcome of a single coin toss, assigning 0 to tails and 1 to heads. The probability that $X = x$ is 0.5 on the state space {0, 1}.

Before Bernoulli turns to the famous Petersburg Paradox, he starts with an observation from the multiple publications on the topic, all claiming that: "since there is no reason to assume that of two persons encountering identical risks, either should expect to have his desires more closely fulfilled, the risks anticipated by each must be deemed equal in value."

This is a formulation of the principle of sufficient reason.

He further tries to formulate the problem in general terms by stripping individual characteristics of the persons themselves and banning them from the consideration; only those matters should be weighed carefully that pertain to the terms of the risk. Using this method he prepares the topic for a theoretical approach.

In §3 he introduces his kind of problem by an example:

> "Somehow a very poor fellow obtains a lottery ticket that will yield with equal probability either nothing or twenty thousand ducats. Will this man evaluate his chance of winning at ten thousand ducats? Would he not be ill-advised to sell this lottery ticket for nine thousand ducats? To me it seems that the answer is in the negative. On the other hand I am inclined to believe that a rich man would be ill-advised to refuse to buy the lottery ticket for nine thousand ducats. If I am not wrong then it seems clear that all men cannot use the same rule to evaluate the gamble."[14]

The rule established in §1 must, therefore, be discarded. But anyone who considers the problem with perspicacity and interest will ascertain that the concepts of *value* which we have used in this rule may be defined in a way which renders the entire procedure universally acceptable without reservation. To do this the determination of the *value* of an item must not be based on its *price*, but rather on the *utility* it yields.

13 Bernoulli §1.
14 Bernoulli §3.

Bernoulli deduces from his concept of utility a fundamental rule which he gives in §4: *If the utility of each possible profit expectation is multiplied by the number of ways in which it can occur, and we then divide the sum of these products by the total number of possible cases, a mean utility [moral expectation] will be obtained, and the profit which corresponds to this utility will equal the value of the risk in question.*

For Bernoulli it becomes evident that no valid measurement of the value of a risk can be obtained without consideration being given to its *utility*, that is to say, the utility of whatever gain accrues to the individual or, conversely, how much profit is required to yield a given utility. However, it hardly seems plausibly to make any precise generalizations since the utility of an item may change with circumstances.[15]

The question if this kind of problem is accessible to mathematics was answered by Bernoulli with: yes. He claims "Now it is highly probably that *any increase in wealth, no matter how insignificant, will always result in an increase in utility which is inversely proportional to the quantity of goods already possessed.*"[16]

This is the first published exposition of the *Principle of Decreasing Marginal utility*. This principle, later widely accepted in the theory of economic behavior, states that marginal utility (the extra utility obtained from consuming a good) decreases as the quantity consumed increases; in other words, that each additional good consumed is less satisfying than the previous one.

The St. Petersburg paradox

The reason why Bernoulli's paper became famous is the St. Petersburg Paradox. It is introduced in the following example: A fair coin will be tossed until a head appears; if the first head appears on the nth toss, then the payoff is 2 nducats. How much should one pay to play this game? The paradox is that the expected return is infinite, namely:

E(w) = (1/2)·2 + (1/4)22 + (1/8)23 + ... = 1 + 1 + 1 + ... = infinite

(calculated by the Definition of §1)

Yet while the expected payoff is infinite, one would not suppose, at least intuitively, that real-world people would be willing to pay an infinite amount of money to play this.

The problem was posed by Nicolaus Bernoulli in 1713 and already discussed by the mathematician Gabriel Cramer, who gave Bernoulli's solution in a letter to Nicolaus in 1728. Daniel Bernoulli's solution involved two ideas that have since revolutionized economics: firstly, that people's utility from wealth u(w), is not linearly related to wealth w but rather increases at a decreasing rate - the famous idea of diminishing marginal utility;
secondly that a person's valuation of a risky venture is not the expected return of that venture, but rather the expected utility from that venture.

15 Bernoulli §5.
16 Bernoulli §5.

In the St. Petersburg case, the *value* of the game to an agent (assuming initial wealth is zero) is:

$$E(u) = (1/2)\cdot u(2) + (1/4)\cdot u(22) + (1/8)\cdot u(23) + ... < \text{infinite}$$

which Bernoulli conjectured is finite because of the principle of diminishing marginal utility. (Bernoulli originally used a logarithmic function of the type u(x) = a log x). Consequently, people would only be willing to pay a *finite* amount of money to play this, even though its expected return is infinite.
In general, by Bernoulli's logic, the valuation of any risky venture takes the expected utility form:

$$E(u \mid p, X) = \sum_x p(x)u(x)$$

where X is the set of possible outcomes, p(x) is the probability of a particular outcome $x \in X$ and $u: X \rightarrow R$ is a utility function over outcomes.

The St. Petersburg game offers the possibility of huge prizes. A run of forty would, for example, pay a whopping €1.1 trillion. Of course, this prize happens rarely: only once in about 1.1 trillion times. Half the time, the game pays only €2, and you're 75% likely to wind up with a payment of €4 or less. Your chances of getting more than €20 (the entry price which had suggested is a reasonable maximum) are less than one in 20. Very low payments are very probable, and very high ones very rare. It's a foolish risk to invest more than €20 to play.

It was argued that this sort of consideration in fact solves the St. Petersburg paradox. Complicated ways were offered of including a risk-aversion factor in calculations of expected utility, with the result that there is a finite upper limit to the rational entrance fee for the game.

The inclusion of a risk-aversion factor is as artificial as Bernoulli's original logarithmic function, to get an upper bound.
On the other hand Bernoulli's principle of decreasing marginal utility was widely accepted within economic theory.

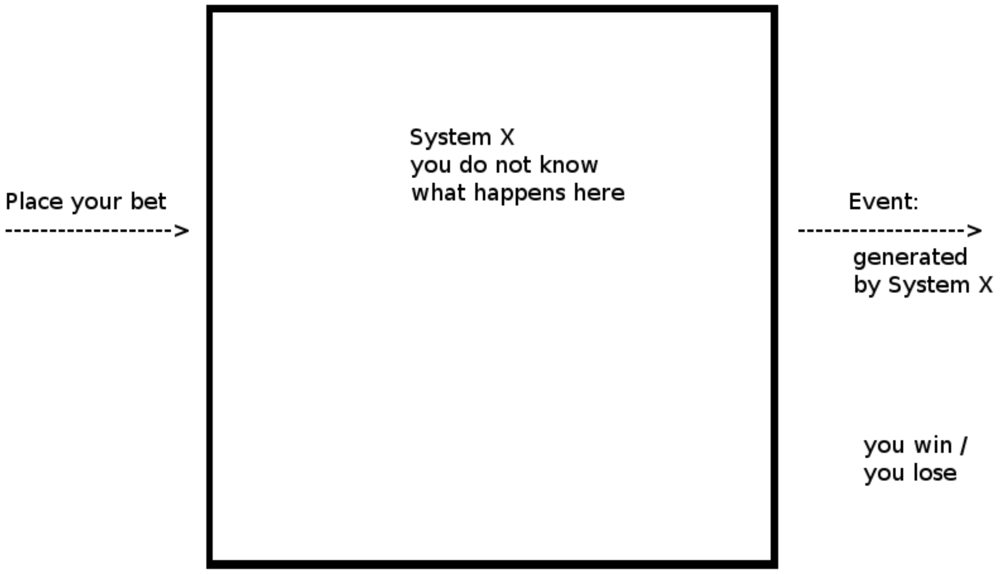
The problem already in Bernoulli's approach seems that the way to deal with the paradox in mathematical terms is entirely ad hoc. Bernoulli introduced a logarithmic function to gain some kind of artificial cut-off. But why is the function logarithmic? It simply serves the purpose of the fast cut-off he wants to have.

And here is also the main difference between Galilei's solution of a former not entirely understood problem and Bernoulli's observation that a mathematical description may not always picture the aspects of a system that are of special interest to man. While Galilei had to complete an uncomplete analysis to give the proper mathematics for three dice – which may even be used to model any problem, Bernoulli has to introduce additional aspects to take care of human interest in his model of a system. Therefore he is not improving an existing mathematical model, but he shows that a model does not deal with important aspects currently at stage.

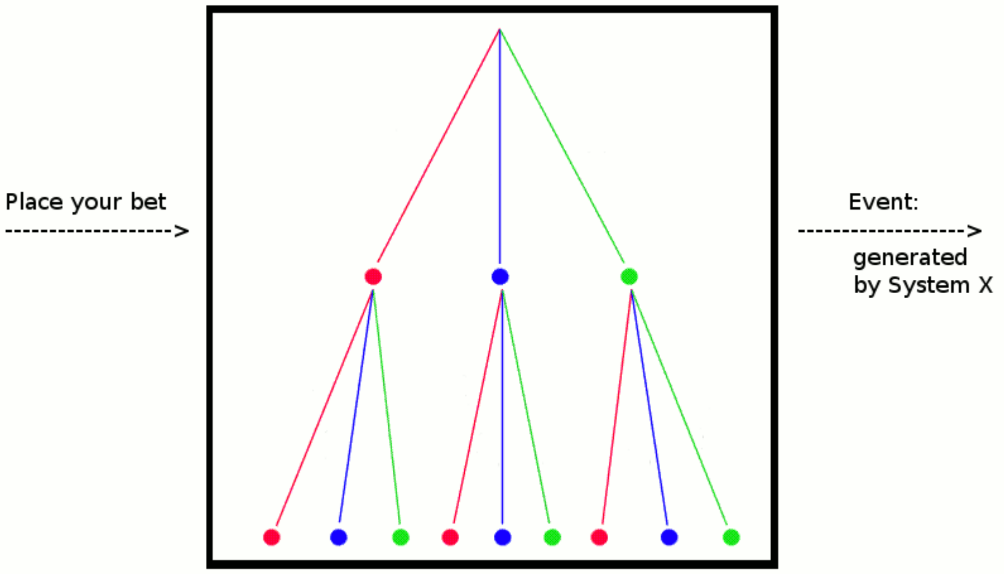
Both approaches are pivotal for the development of an integrated theory of risk. First, one needs a thorough analysis of the system under consideration, second, one needs mathematical descriptions of aspects that are important or interesting to humans.

Historical Summary

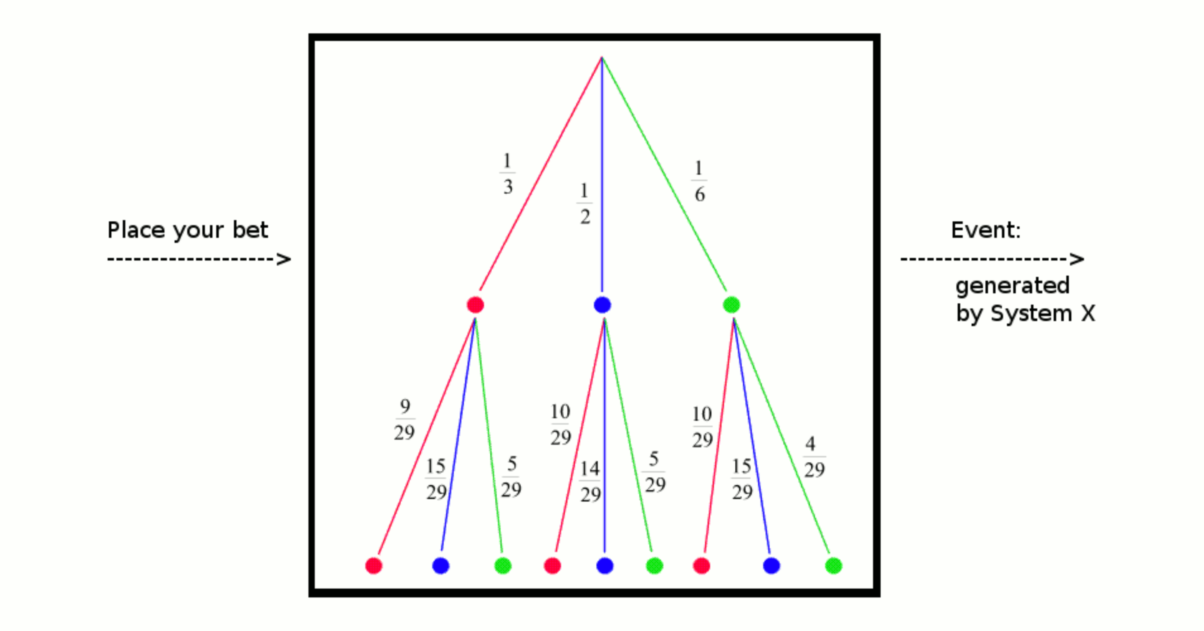
Here I will shortly sketch the historical steps the analysis of risks took during the last centuries. This is illustrated by some drawings.
Situation before Cosimo: The system you bet on is a black box:



Situation at the time of Cosimo: The system you bet on is only rudimentary analysed:



Situation at the time of Cosimo with Galilei's help: The system you bet on is thoroughly analysed:



Situation now: The system you bet on is *not* thoroughly analysed.

This means that we are somehow close to the situation at the time of Galilei. Although there is considerable interest in the problem, because of regulations given by the state (government) like BASEL II or Solvency, there are only "island and solitary solutions" and no conclusive theory. In the final section I will try to sketch a possible solution.

Conclusion and solutions

The notion of *risk* has always a personal component, as was already observed by Bernoulli. Furthermore the notion of *risk* has a temporal component. Something is always at risk for a certain time and sooner or later the risk is realised or not. Therefore there is no risk in the past. Things that have already happened in a certain way cannot be considered in terms of risk. On the other hand the temporary component is of great importance in any description of a system which is actually at risk. The observer wants to know every risk at every instant of time.

A theory of risk has always to deal with complex systems and to take care of a multitude of constraints and boundary conditions. One of the main problems is that not all the risks in a system or all the risks thereatening a system can be identified properly. Modern risk management deals with this problems by so called risk assessments, where experts are expected to identify certain types of risks. After the risks are identified, experts try to model them with mathematical tools.

The problems of this approach are evident: Not all the risks can be identified and not all the identified risks are modelled properly. The reasons for this are multifarious. One problem is a lack of time and the improper analysis of already identified risks. So Galilei's approach is still worthwile. Another point is the mathematical modelling of man's needs and this corresponds to the historical example of Bernoulli. It is important to find a regulator for mathematical ad hoc implementations into theories.

To deal with this I propose a modelling method already successful in many-particle physics, where one takes care of unknown interactions by an effective potential and where it is formally supposed that these interactions are known. This approach was elaborated already in the 1960s in what is now called Density-Functional-Theory, and the formal fundamentals were laid by Hohenberg and Kohn.[17] The method was very successful in atomic physics and theoretical chemistry, as it allows comparably easily for numerical modelling. Especially the Kohn-Sham theory has to be mentioned in this context, because it provides differential equations that can be solved in a self-consistent manner.[18]

As a conclusion of this paper I will give a short sketch how to deal with the modelling of systems by effective potentials, using a symbolic notation.

Ordinary risk theory deals with systems, their developments and risks by applying a matrix M on a state vector S. Here the components of the matrix $M=m_{ij}$ picture development factors in general and risk factors in particular. As already mentioned the matrix M and the vector S have to be timedependent. If the matrix M is applied to the state vector S it results in the new vector S', giving the current state:

$$M\,S = S'$$

The first step is to show the equivalence of the matrix formulation with a wavefunction operator calculus. This equivalence should be expected in analogy to the formulations of quantum mechanics.[19] Call the “wavefunction” Φ and the potential which includes the system relevant operators V. This potential V should be set up as an effective potential as in many-body theory, where the effective potential represents the complete surrounding and all other components relating to a single component. These single components are represented by their “single-component-wavefunction”. From V_{eff} it is expected to separate some kind of kinetic operator taking care of the individual development of the components, which may be called T. Local properties of the environment of the system may be included in some potential V_{loc}, while a kind of component-component interaction may be given by the exchange potential V_{cc}. The exchange components not included in these spinoffs are included in some exchange-correlation term V_{xc}. Therefore the formally complete effective potential is:

$$V_{eff} = T + V_{cc} + V_{loc} + V_{xc}$$

The thus gained equations could be solved self-consistently, improving the value of the unknown exchange-correlation term V_{xc}. Of course the scheme is set up in analogy to Kohn-Sham theory, but it may solve the problem if the following steps are taken:

1. The formal equivalence of the matrix calculus and the effective potential calculus has to be proven. In quantum mechanics it has not been a simple task to show that Heisenberg's matrix formalism was formally equivalent to Schrödinger's wave

17 P. Hohenberg and W. Kohn: *Inhomogeneous Electron Gas*. Phys. Rev. 136 (1964) B864-B871
18 W. Kohn and L. J. Sham, Phys.Rev. 140 (1965) A1133
19 Details on the historical development in quantum mechanics can be found in Carlos Madrid Casado: *A brief history of the mathematical equivalence between the two quantum mechanics,* Lat. Am. J. Phys. Educ. Vol. 2, No. 2, May 2008.

formalism. It was John von Neumann who finally could show that the Heisenberg and Schrödinger formalism are operator calculi on isomorphic (isometric) realizations of the same Hilbert space and hence equivalent formulations of one and the same conceptual substratum.[20] For the conclusive theory of risk something similar has to be shown.

2. Proper formulations of the component functions and of the potential parts for the kinetic term, the component-component interaction and the local potential have to be constructed and tested.
3. It is necessary to find an approximation for the exchange correlation term V_{xc}. To do the self-consistent calculations successfully it is important to have a realistic approximation for the exchange part. In many-particle physics so called *local* approximations proved to be successful. Here the exchange and correlation effects are modelled in a short range of the component and the close surrounding is generalised to form some global background. A formulation in terms of density seems to be promising for this task.

To set up a consistent integrated theory of risk, there is obviously still a long way to go – but there is a way.

20 For more details see Madrid Casado 2008.

References

Daniel Bernoulli: *Specimen theoriae novae de mensura sortis*, Comentarii academiae scientiarium imperialis Petropolitanae, 1738, pp.175-192, english by Louise Sommer, in Econometrica, vol. 22, No. 1, January 1954, pp.23-36.

Peter L. Bernstein: *Against the Gods*, The Remarkable Story of Risk, Wiley, 1996

Galileo Galilei: *Sopra le Scoperte dei Dadi*, Galileo Opere, Edizione Nazionale, A. Favaro (ed.), Firenze 1898, vol.8, pp.591-594, english by E.H.Thorne (published in Games, Gods and Gambling by F N David).

Ian Hacking: *The Emergence of Probability*, Cambridge UP, 2nd ed., 2006

P. Hohenberg and W. Kohn: *Inhomogeneous Electron Gas*. Phys. Rev. 136 (1964) B864-871

Vassilios Hombas: *Generalizing Galileo's Passedix Game*, Interstat, ISSN 1941-689X, Statistics on the Internet, January 2004

W. Kohn and L. J. Sham, *Self-Consistent Equations Including Exchange and Correlation Effects*, Phys. Rev. 140 (1965) A1133

Frank Linhard: *Historische Elemente einer Prinzipienphysik*, Hildesheim, Zurich, New York, 2000.

Carlos Madrid Casado: *A brief history of the mathematical equivalence between the two quantum mechanics,* Lat. Am. J. Phys. Educ. Vol. 2, No. 2, May 2008

Robert Martin: *The St.Petersburg Paradox*, Stanford Encyclopedia of Philosophy, http://www.seop.leeds.ac.uk/entries/paradox-stpetersburg/