Symmetry Breaking in Semilinear Boundary Value Problems

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Für meine Mutter Iris

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Contents

Abstract

This thesis is concerned with the study of symmetry breaking phenomena for several different semilinear partial differential equations. Roughly speaking, this encompasses equations whose symmetries are not necessarily inherited by their solutions, which is particularly interesting for ground state solutions. Chapter 1 provides an introduction to this topic and gives a self-contained overview of the main results and the related techniques. The following four chapters (Chapter 2 to Chapter 5) then correspond to different publications and preprints containing the full results and proofs.

Chapter 2 corresponds to the article

[P1] J. Kübler and T. Weth, *Spectral asymptotics of radial solutions and nonradial bifurcation for the Hénon equation*, Discrete and Continuous Dynamical Systems. Series A, **40** (2020), 3629–3656.

Here, we consider the Hénon equation and study the asymptotics of radial solutions which allows us to carry out a detailed analysis of the spectral properties of the associated linearized operator. This enables us to prove bifurcation of nonradial solutions with a fixed number of nodal domains from the branch of (nodal) radial solutions.

Chapter 3 is then given by the article

[P2] O. Agudelo, J. Kübler and T. Weth, *Spiraling solutions of nonlinear Schrödinger equations*, Proceedings of the Royal Society of Edinburgh Section A: Mathematics (2021).

In this article, we consider a class of solutions of a nonlinear Schrödinger equation which exhibit only partial decay. This leads to the study of least energy sign-changing solutions for an associated elliptic equation on $\mathbb{R}^2$, for which we show symmetry breaking as the rotational slope increases.

Chapter 4 corresponds to the article

[P3] J. Kübler and T. Weth, *Rotating waves in nonlinear media and critical degenerate Sobolev inequalities*, `https://arxiv.org/abs/2203.07991` (2022), submitted to Analysis & PDE.

Here, we study rotating wave solutions of a nonlinear wave equation via a (degenerate) elliptic Dirichlet problem which turns out to be related to a degenerate Sobolev inequality in the half space. We prove this inequality and the existence of associated extremal functions. This is then used to show the existence of ground state solutions and symmetry breaking results for the original problem, as well as several generalizations.

Chapter 5 then consists of the article

[P4] J. Kübler, *On the spectrum of a mixed-type operator with applications to rotating waves*,

`https://arxiv.org/abs/2204.05824` (2022), submitted to Calculus of Variations and Partial Differential Equations.

Here, we extend the studies from Chapter 4 to rotating wave solutions with large angular velocity, which leads to a semilinear problem involving a mixed-type operator. By proving new spectral estimates for this operator we are able to formulate a variational setting to find ground states and prove their symmetry breaking.

Finally, Chapter 6 contains a German introduction to the topic of symmetry breaking and a summary of the results above.

Notation

Throughout this thesis, we will always consider space dimensions $N \geq 2$.

Sets and Topology

For $x, y \in \mathbb{R}^N$ and $r > 0$ we set:

$\lvert x\rvert$	$= \left(\sum_{i=1}^N \lvert x_i\rvert^2\right)^{\frac{1}{2}}$
$x \cdot y$	$= \sum_{i=1}^N x_i y_i$
$B_r(x)$	$= \left\{y \in \mathbb{R}^N : \lvert y - x\rvert < r\right\}$
$\mathbf{B}$	$= B_1(0)$
$\mathbb{S}_r^{N-1}(x)$	$= \left\{y \in \mathbb{R}^N : \lvert y - x\rvert = r\right\}$
$\mathbb{S}^{N-1}$	$= \mathbb{S}_1^{N-1}(0)$
$\mathbb{R}_+^N$	$= \left\{x \in \mathbb{R}^N : x_1 > 0\right\}$
$d\sigma$	standard surface measure
$O(N)$	orthogonal group

Functions and Constants

For $\Omega \subset \mathbb{R}^N$, and a function $u : \Omega \to \mathbb{R}$ we set:

$\mathbb{1}_\Omega$	characteristic function of Ω
u^+, u^-	positive and negative part of u, i.e., $u^+(x) = \max\{u(x), 0\}$, $u^-(x) = \max\{-u(x), 0\}$
$\operatorname{supp} u$	$= \overline{\{x \in \Omega : u(x) \neq 0\}}$, where the closure is taken in Ω
J_ν	Bessel function of the first kind of order $\nu \geq 0$
$j_{\nu,k}$	k-th zero of J_ν
2^*	$= \begin{cases} \frac{2N}{N-2} & \text{for } N \geq 3 \\ \infty & \text{for } N = 2. \end{cases}$
$\lambda_1(\Omega)$	the first Dirichlet eigenvalue of $-\Delta$ on Ω

Function Spaces

Unless explicitly stated otherwise, all function spaces consist of real-valued functions. For a given open set $\Omega \subset \mathbb{R}^N$, $k \in \mathbb{N}$ and $\gamma > 0$, we consider the following spaces equipped with the usual norms:

$C(\overline{\Omega})$	continuous functions on $\overline{\Omega}$
$C^k(\overline{\Omega})$	k times continuously differentiable functions on Ω all of whose derivatives of order $\leq k$ have continuous extensions to $\overline{\Omega}$
$C^k_c(\overline{\Omega})$	functions in $C^k(\overline{\Omega})$ whose support is a compact subset of Ω
$C^{k,\gamma}(\overline{\Omega})$	functions in $C^k(\overline{\Omega})$ with γ-Hölder continuous derivatives of order k.

For $1 \leq p \leq \infty$, we consider the classical Lebesgue spaces $L^p(\Omega)$ with their standard norms. Moreover, we let $L^p_{\mathrm{loc}}(\Omega)$ denote the space of functions whose restrictions to compact subsets are contained in the $L^p(\Omega)$.

Similarly, we consider the classical Sobolev spaces of order $k \in \mathbb{N}$ by $H^k(\Omega)$ with their standard norms, as well as the following variants:

$H^k_0(\Omega)$	closure of $C^\infty_c(\Omega)$ in $H^k(\Omega)$
$H^k_{\mathrm{loc}}(\Omega)$	functions in $H^k(\Omega)$ such that the restriction to K is contained in $H^k(K)$ for any $K \subset \Omega$ with $\overline{K} \subset \Omega$
$H^k_{0,rad}(\Omega)$	subspace of radial functions in $H^k_0(\Omega)$

CHAPTER 1

Overview

1.1 Introduction

Symmetry, as well as the lack thereof, plays a crucial role in the understanding of many observable phenomena. The laws of nature typically contain symmetries which can be used to derive central information such as conservation laws. Furthermore, this intuitively suggests that the solution or state describing a given system possesses the same symmetries. In many cases, the assumption that such a symmetry property holds reduces the complexity of the problem tremendously, and even enables us to find exact solutions in some cases, see examples from quantum mechanics [**52**] or general relativity [**123**].

In general, however, this intuitive relation between the symmetries of a problem and its solutions may fail to hold. Such phenomena are commonly referred to as (spontaneous) *symmetry breaking* and have been noted as early as 1834: Jacobi [**74**] and Liouville [**87**] observed that the equations determining the shape of a body of fluid rotating around a fixed axis possess a solution which is not axially symmetric and even minimizes the kinetic energy in certain parameter regimes. Similar effects occur in other problems related to classical mechanics, but have received considerable attention for their role in condensed matter systems and quantum field theory, particularly in the form of chiral symmetry breaking. We refer to [**68, 98**] for an overview of the role of symmetries and symmetry breaking in modern physics.

In this thesis, we will study symmetry breaking for different types of partial differential equations (*PDEs*) with *radial symmetry*, i.e., which are invariant with respect to rotations. More specifically, we let $\Omega \subset \mathbb{R}^N$ be a rotationally invariant domain, i.e., an open and connected subset of $\mathbb{R}^N$ such that $R(\Omega) = \Omega$ holds for any $R \in O(N)$. Thus, Ω is a ball or annulus centered at the origin, the complement of a ball, or the whole space $\mathbb{R}^N$. Similarly, we let L be a linear second order differential operator which is rotationally invariant in the sense that $L(u \circ R) = (Lu) \circ R$ holds for any $R \in O(N)$ and $u \in C^2(\mathbb{R}^N)$. The model case for L is given by the (negative) Laplacian $L = -\Delta$, though we will also consider different types of operators.

We will then study symmetry breaking phenomena appearing in semilinear problems of the form

$$Lu = f(|x|, u) \quad \text{in } \Omega, \tag{1.1.1}$$

complemented by appropriate Dirichlet boundary or decay conditions for bounded and unbounded Ω, respectively. Here, f is a continuously differentiable function on $[0, \infty) \times \mathbb{R}$. In this setting, the classical intuition discussed above would suggest that the radial symmetry of Ω, L and f should enforce radial symmetry for certain solutions of (1.1.1).

In order to further specify the term symmetry breaking in the following, we also assume that the differential operator and the nonlinearity depend on a parameter $\alpha > 0$, i.e., we consider the family of problems

$$L_\alpha u = f_\alpha(|x|, u) \quad \text{in } \Omega. \tag{1.1.2}$$

For such equations, we then study selected classes of solutions which will be specified below. In this setting, we say that *symmetry breaking occurs if these solutions exhibit different symmetry properties depending on the parameter α.* In the problems considered later, we will typically observe radial symmetry for α close to zero, whereas the solutions become nonradial as α grows.

In order to gain a better understanding of these phenomena, we omit the α-dependence in (1.1.2) for now, and first discuss under which conditions symmetry can be enforced for certain solutions of (1.1.1). In general, we cannot expect *all* solutions of (1.1.1) to be radial, as we can already observe in the simple example of the eigenvalue problem for the Laplacian on the unit ball $\mathbf{B}$ in $\mathbb{R}^N$, i.e.,

$$\begin{cases} -\Delta u = \lambda u & \text{in } \mathbf{B}, \\ \quad u = 0 & \text{on } \partial\mathbf{B}. \end{cases} \tag{1.1.3}$$

Indeed, while the first eigenfunction u_1 is radial, there are infinitely many nonradial eigenfunctions. It is important to note, however, that u_1 is the only positive solution (up to sign and normalization), and can further be characterized as the (up to sign) unique minimizer of the Rayleigh quotient

$$R(u) = \frac{\int_{\mathbf{B}} |\nabla u|^2 \, dx}{\int_{\mathbf{B}} u^2 \, dx}$$

among all functions $u \in H_0^1(\mathbf{B}) \setminus \{0\}$.

This leads to a crucial observation: *Symmetry properties are intimately connected to variational structures.* In particular, this mirrors the fact that the state of a physical system is often characterized as the minimizer of a suitable action functional. In order to define such a functional for (1.1.1), we now assume that L is in divergence form, i.e., there exist C^1-functions $a_{ij} : \Omega \to \mathbb{R}$, $i, j = 1, \dots, N$ and a function $c : \Omega \to \mathbb{R}$ such that

$$Lu(x) = -\sum_{i,j=1}^{N} \partial_i \left[a_{ij}(x)\partial_j u(x)\right] + c(x)u(x)$$

for $u \in C^2(\Omega)$. We will also assume for the moment that the operator L is *elliptic*, which means that the $N \times N$ matrix $a_{ij}(x)$ is positive definite for every $x \in \Omega$ (degenerate elliptic and mixed-type operators will be considered later in Sections 1.4 and 1.5). Multiplying (1.1.1) by u and integrating over Ω, integration by parts then leads to the following *energy functional* associated to (1.1.1) given by

$$E(u) = \frac{1}{2} \int_\Omega \left(\sum_{i,j=1}^{N} a_{ij} \partial_i u \partial_j u + cu^2 \right) dx - \int_\Omega F(|x|, u) \, dx,$$

where $F(|x|, \cdot)$ denotes an antiderivative of $f(|x|, \cdot)$. Depending on the properties of the functions a_{ij}, c and f, the functional is well-defined on spaces of functions whose weak derivatives satisfy suitable integrability properties, and leads to the consideration of suitable

Hilbert spaces. For example, the aforementioned case $L = -\Delta$ corresponds to the energy functional

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(|x|, u)\,dx$$

and thus leads to the classical Sobolev space $H_0^1(\Omega)$ if f has subcritical growth.

The Euler-Lagrange equations associated to E correspond to (1.1.1), i.e., solutions of (1.1.1) are necessarily critical points of E. In light of the comments above, we are particularly interested in solutions which have minimal energy among all solutions. Throughout this thesis, we will call such solutions *ground state solutions* or simply *ground states.* Due to their physical significance, we may often expect more distinguished qualitative properties of ground states. Indeed, in many cases the ground states are positive (up to sign), and this property is closely related to rigid symmetry properties. To discuss this connection in detail, we again turn to the case where L is given by the Laplacian $L = -\Delta$, which is rotationally invariant as mentioned above. One of the most striking results ensuring symmetries of positive solutions to symmetric problems is due to Gidas, Ni and Nirenberg [**61**] which we state here in the following form.

Theorem 1.1.1. ([61])
Let $\Omega \subset \mathbb{R}^N$ be an open ball centered at the origin and let $f \in C^1([0,\infty)\times\mathbb{R})$ be a function such that $f(\cdot, t)$ is nonincreasing for every $t \in \mathbb{R}$. Moreover, let $u \in C^2(\overline{\Omega})$ be a positive solution of

$$\begin{cases} -\Delta u = f(|x|, u) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1.4}$$

Then u is radially symmetric and a decreasing function of the radial variable.

This result is based on the moving plane method which had previously been employed by Serrin [**120**] and goes back even further to the work of Alexandrov [**3**]. Roughly speaking, the general idea is to reflect the solution u with respect to a hyperplane and move this hyperplane to a critical position. Then variants of the maximum principle can be used to show that u must be symmetric with respect to this critical hyperplane. Importantly, these ideas allow for considerable generalizations to different settings such as unbounded domains [**62**], different types of semilinear elliptic problems [**15**,**29**,**33**], parabolic equations [**114**] and more recently also nonlocal operators, see e.g. [**18**, **34**, **75**, **76**].

We note that while in some instances, symmetry properties can also be shown by other methods such as symmetrization [**131**, **132**], we will restrict our attention to the moving plane method and its variants in the following brief discussion.

As outlined above, results such as Theorem 1.1.1 have far-reaching consequences for the symmetry properties of ground states for radially symmetric problems. But while the moving plane method and its variants can be used in a variety of contexts, it also leaves open many important cases where such techniques generally fail. In these cases, symmetry breaking phenomena may occur. More specifically, for equations of the form (1.1.1), this could happen in the following cases:

Domain geometry: Symmetry properties in the spirit of Theorem 1.1.1 will generally not hold if the ball is replaced by an annulus. For the model nonlinearity $f(|x|, u) = |u|^{p-2}u$ with $p \in 2^*$, this has already been observed by Brezis and Nirenberg [**24**] and, in fact, the ground states of the associated semilinear boundary value problem are nonradial if the exponent p is sufficiently large, see also [**40**]. Moreover, similar results also hold for suitable exterior domains [**53**].

Spatial properties of the nonlinearity: If the nonlinearity f does not satisfy the monotonicity condition stated in Theorem 1.1.1, i.e., if $f(\cdot, t)$ is not decreasing, we cannot expect symmetry. A prominent example is given by the nonlinearity $f(|x|, t) = |x|^\alpha |t|^{p-2} t$ associated to the Hénon equation, where $\alpha > 0$ and $p > 2$. In this case, nonradial ground states exist for certain values of α and p, while this example also shows, however, that an increasing nonlinearity does not guarantee nonradial ground states, since those of the Hénon equation can be radial for suitable parameters, see [**122**].

Sign-changing solutions: In general, the moving plane method and related techniques cannot be extended to sign-changing solutions, as we already observed for the linear eigenvalue problem (1.1.3). Nonetheless, suitable variational properties may still enforce certain symmetries for sign-changing solutions, such as assumptions on the Morse index of a solution [**66**], or by considering least energy sign-changing solutions [**11, 133**].

Differential operator properties: If the operator L in the general problem (1.1.1) is not given by the Laplacian, its specific properties may prohibit the use of the moving plane method altogether. Even if L is rotationally invariant, we will observe that a lack of translation invariance may still allow for nonradial solutions. Moreover, the situation may be completely different if L is not elliptic or even of higher order, see e.g. recent results on biharmonic nonlinear Schrödinger equations in [**82**].

In each of these cases, nonradial solutions may exist but their existence is not guaranteed. Moreover, even if we can prove their existence, these nonradial solutions may not necessarily be ground states. As outlined earlier, instead of working with a fixed problem of the form (1.1.1), we will consider a family of equations depending on a parameter $\alpha > 0$, i.e.,

$$L_\alpha u = f_\alpha(|x|, u) \quad \text{in } \Omega. \tag{1.1.5}$$

In particular, this parameter-dependent problem allows for different approaches regarding the existence and structural properties of nonradial solutions.

The first technique we discuss is the use of bifurcation theory. Typically, for each α problem (1.1.5) possesses a unique positive radial solution u_α. The existence of nonradial solutions can then be shown by proving that the curve $\alpha \mapsto u_\alpha$ bifurcates at a suitable point $\alpha_0 \in \mathbb{R}$, i.e., there exists a sequence $\alpha_n \to \alpha_0$ and distinct solutions u^n of (1.1.5) for $\alpha = \alpha_n$ such that $u^n \to u_{\alpha_0}$ in $C^2(\overline{\Omega})$. It can then be shown that this bifurcating branch must consist of nonradial solutions, giving symmetry breaking. Even in cases where the existence of nonradial solutions is already known, such results are of interest as they give more insight into the structural relation between radial and nonradial solutions. A major difficulty arising in this approach is the need for more detailed information on the associated linearized operator and its degeneracy properties. For the cases we will discuss below, this is most suitable if the parameter-dependence only appears in the nonlinearity f_α.

Note, however, that the use of bifurcation theory generally does not yield information on the variational properties of nonradial solutions and, in particular, generally does not allow us to deduce whether the bifurcating branch consists of ground states. To this end, it is instead necessary to study properties of the ground state energy, which will be one of our main tools to derive symmetry breaking for problems of the form (1.1.5). The strategy will be to first characterize the minimal energy among radial solutions, and compare it with suitable estimates for the ground state energy. If the latter turns out to be smaller for certain parameter regimes, this implies symmetry breaking.

Clearly, this energy approach displays a major difficulty since both the ground state energy and the minimal radial energy need to be controlled sufficiently well. Unfortunately, the values of these energies are usually not known explicitly, so the estimates can at best be of an asymptotic nature. We therefore often wish to identify an appropriate limit problem for (1.1.5) as the parameter α goes to infinity, for example. Properties of the limit problem may then allow us to estimate the energy in some cases. Of course, both the limiting process as well as finding sufficient information on the limit problem can pose many difficulties. Moreover, we will later encounter interesting problems which do not admit a limit problem so that different methods are required altogether.

In the following, we discuss our results obtained in the articles **[P1]**, **[P2]**, **[P3]** and **[P4]**. While each of these papers is concerned with symmetry breaking for equations of the form (1.1.1), both the problems themselves and the methods used to treat them differ substantially in each case. We therefore give brief introductions to each of these problems and discuss the main results as well as the general ideas and methods used in each case. In our presentation, we largely follow the notation used in the respective papers, which may vary from the notation used in the preceding introduction.

1.2 Symmetry Breaking for the Hénon Equation

The article **[P1]** is concerned with the Hénon equation

$$\begin{cases} -\Delta u = |x|^{\alpha}|u|^{p-2}u & \text{in } \mathbf{B}, \\ \quad u = 0 & \text{on } \partial\mathbf{B}, \end{cases} \tag{1.2.1}$$

where $\mathbf{B} \subset \mathbb{R}^N$ denotes the unit ball, $p > 2$ and $\alpha > 0$. This equation was introduced by Hénon [**71**] in 1973 in order to study stellar clusters. More specifically, u models the mass density of such a cluster with a black hole at the center, whose mass is characterized by the parameter $\alpha > 0$.

It was first noted by Ni [**108**] that this equation has nontrivial solutions for $2 < p < 2^*_\alpha := \frac{2N+2\alpha}{N-2}$. Interestingly, 2^*_α is strictly larger than the critical Sobolev exponent for $H^1_0(\mathbf{B})$ if $N \geq 3$, giving a larger existence range than is typically expected in semilinear problems with power-type nonlinearities. In fact, this range is optimal in the sense that the equation only has a trivial solution for $p \geq 2^*_\alpha$, which can be deduced from a suitable Pohozaev identity.

As noted above, the nonlinearity $f(r,u) = r^\alpha|u|^{p-2}u$ appearing in (1.2.1) is strictly increasing in r and therefore does not satisfy the conditions of the symmetry result of Gidas, Ni and Nirenberg stated in Theorem 1.1.1. This suggests that nonradial positive solutions may exist. This question has attracted a lot of interest, as we will discuss in the following. We first consider ground states of (1.2.1) which can be characterized as minimizers of the Rayleigh functional

$$R_\alpha : H^1_0(\mathbf{B}) \setminus \{0\} \to \mathbb{R}, \quad R_\alpha(u) := \frac{\int_{\mathbf{B}} |\nabla u|^2\,dx}{\left(\int_{\mathbf{B}} |x|^\alpha |u|^p\,dx\right)^{\frac{2}{p}}},$$

and are positive (up to sign reflection). It is thus natural to ask not only whether nonradial positive solutions exist, but whether even the ground states are nonradial for suitable $\alpha > 0$. In fact, the latter had been suggested by numerical computations by Chen, Ni and Zhou [**35**] and was later proven by Smets, Willem and Su [**122**]. More specifically, they show that for any $2 < p < 2^*$ there exists $\alpha^* > 0$ such that the ground states of (1.2.1) are nonradial for $\alpha > \alpha^*$. As remarked above, they also show that the ground states are radial if α is sufficiently small. These remarkable observations have sparked extensive further research and led to a plethora of results, see e.g. [**4–6**, **27**, **28**, **31**, **89**, **113**, **119**, **121**].

In [**4**] it has been shown that for fixed $\alpha \in (0, 1]$, nonradial solutions bifurcate from the branch of positive solutions with respect to the exponent p. On the other hand, the question had remained open whether nonradial nodal (i.e., sign-changing) solutions may bifurcate from branches of radial nodal solutions as $\alpha \to \infty$ for fixed p. Our main contribution in **[P1]** answers this question and states that each branch of radial solutions possesses infinitely many bifurcation points with respect to α.

In order to state our results more precisely, we fix $K \in \mathbb{N}$, $p > 2$ and consider

$$\alpha > \alpha_p := \max\left\{\frac{(N-2)p - 2N}{2}, 0\right\},$$

which ensures that the exponent satisfies $p < 2^*_\alpha$. We then first study radial solutions with precisely K nodal domains where, here and in the following, the nodal domains of a function $u : \Omega \to \mathbb{R}$ are defined as the connected components of $\{x \in \Omega : u(x) \neq 0\}$. Consequently, we study radial solutions with $K-1$ zeros in the radial variable $r = |x| \in (0, 1)$. Nagasaki [**106**] has shown that (1.2.1) admits a unique classical radial solution $u_\alpha \in C^2(\overline{\mathbf{B}})$ with $u_\alpha(0) > 0$ and exactly K nodal domains. Our main result then states that nonradial bifurcation occurs in the following sense:

Theorem 1.2.1. *Let $2 < p < \frac{2N}{N-2}$ and $K \in \mathbb{N}$ be fixed. Then there exists a sequence $(\alpha_\ell)_\ell$ with $\alpha_\ell \to \infty$ such that each point α_ℓ is a bifurcation point for nonradial solutions of (1.2.1). More precisely, for every ℓ, there exists a sequence $(\alpha_n^\ell, u_n^\ell)_n$ in $(0, \infty) \times C^2(\overline{\mathbf{B}})$ with the following properties:*

(i) $\alpha_n^\ell \to \alpha_\ell$ and $u_n^\ell \to u_{\alpha_\ell}$ in $C^2(\overline{\mathbf{B}})$ as $n \to \infty$.

(ii) For every $n \in \mathbb{N}$, u_n^ℓ is a nonradial solution of (1.2.1) with $\alpha = \alpha_n^\ell$ having precisely K nodal domains $\Omega_1, \dots, \Omega_K$ such that $0 \in \Omega_1$, Ω_1 is homeomorphic to a ball and $\Omega_2, \dots, \Omega_K$ are homeomorphic to annuli.

In fact, we will prove a somewhat more general result below.
The proof is based on several intermediate results, most importantly a detailed characterization of the asymptotics of the negative eigenvalues of a related weighted eigenvalue problem, which is interesting on its own and has already found applications, as we will discuss below. In order to state these results, we require some ideas from bifurcation theory, starting with the following classical observations:

Letting $E := L^2(\mathbf{B})$, $D := H^2(\mathbf{B}) \cap H_0^1(\mathbf{B})$ and fixing $p > 2$, solutions of (1.2.1) can be characterized as zeros of the map $G : (-\alpha_p, \infty) \times D \to E$ given by

$$[G(\alpha, u)] = -\Delta u - |x|^\alpha |u|^{p-2} u.$$

In order for nonradial solutions to bifurcate from the branch $\alpha \mapsto u_\alpha$, the implicit function theorem then necessitates that the derivative of G with respect to u is not invertible, which is equivalent to the property that the linearized operator

$$\begin{aligned} &L^\alpha : H^2(\mathbf{B}) \cap H_0^1(\mathbf{B}) \to L^2(\mathbf{B}) \\ &L^\alpha \varphi := -\Delta\varphi - (p-1)|x|^\alpha |u_\alpha|^{p-2}\varphi, \qquad \alpha > \alpha_p, \end{aligned}$$

is not invertible.

Since bifurcation can only happen if L^α is not an isomorphism, we call u_α nondegenerate if the equation $L^\alpha\varphi = 0$ only has the trivial solution $\varphi = 0$, otherwise u_α is called degenerate. Thus values α such that u_α is degenerate are candidates for bifurcation points. In order to identify such points, we study the asymptotics of the eigenvalues of L^α as $\alpha \to \infty$. It is important to note that nondegeneracy is generally not a sufficient condition for bifurcation. Hence we also study the Morse index of u_α, which is defined as the number of negative eigenvalues of the operator L^α. We will then be able to deduce occurrences of bifurcation from suitable changes of the Morse index in degenerate points.

In order to study the Morse index as $\alpha \to \infty$, we use the crucial observation that the Morse index of u_α equals the number of negative eigenvalues (counted with multiplicity) of the weighted eigenvalue problem

$$L^\alpha\varphi = \frac{\lambda}{|x|^2}\varphi, \qquad \varphi \in H_0^1(\mathbf{B}). \tag{1.2.2}$$

For the case of the Hénon equation, this was proven by Amadori and Gladiali [5]. Note that (1.2.2) needs to be formulated in a suitable weak setting due to the lack of regularity of the weight $\frac{1}{|x|^2}$, in particular in the case $N = 2$.

The main advantage of the weighted problem (1.2.2) is the fact that its eigenfunctions can be found via a product ansatz, i.e., they are given as sums of functions of the form

$$x \mapsto \varphi(x) = \psi(x)Y_\ell\left(\frac{x}{|x|}\right), \tag{1.2.3}$$

where $\psi \in H_{0,rad}^1(\mathbf{B})$ and Y_ℓ is a spherical harmonic of degree ℓ. We recall that the functions Y_ℓ are eigenfunctions of the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^{N-1}$ corresponding to the eigenvalue $\lambda_\ell := \ell(\ell + N - 2)$. Consequently, functions φ of the form (1.2.3) reduce the eigenvalue problem (1.2.2) to an eigenvalue problem for radial functions given by

$$L^\alpha\psi = \frac{\mu}{|x|^2}\psi, \qquad \psi \in H_{0,rad}^1(\mathbf{B}), \tag{1.2.4}$$

where $\mu = \lambda - \lambda_\ell$. As observed by Amadori and Gladiali [5], the problem (1.2.4) admits precisely K negative eigenvalues

$$\mu_1(\alpha) < \mu_2(\alpha) < \cdots < \mu_K(\alpha) < 0 \qquad \text{for } \alpha > \alpha_p.$$

Overall, we find that the Morse index of u_α is given by

$$m(u_\alpha) = \sum_{(i,\ell)\in E^-} d_\ell,$$

where E^- denotes the set of pairs (i, ℓ) with $i \in \mathbb{N}$, $\ell \in \mathbb{N} \cup \{0\}$ and $\mu_i(\alpha) + \lambda_\ell < 0$. Moreover, u_α is nondegenerate if and only if

$$\mu_i(\alpha) + \lambda_\ell \neq 0 \qquad \text{for every } i \in \{1, \dots, K\},\ \ell \in \mathbb{N} \cup \{0\}.$$

Consequently, information on the asymptotics of the curves $\alpha \mapsto \mu_i(\alpha)$, $i = 1, \dots, K$ allows us to study the Morse index of u_α.

The proof of Theorem 1.2.1 thus relies on two major steps: Firstly, we need suitable information on the behavior of the radial solutions u_α as $\alpha \to \infty$. In the second step, this information can then be used to study the asymptotics of the eigenvalues $\mu_i(\alpha)$ and hence the Morse index.

To this end, it is necessary to characterize the limit shape of the solutions u_α after suitable transformations. Recalling that u_α is a radial function, we may interpret it as a function of the radial variable $r = |x| \in [0, 1]$ and give the following asymptotic characterization of radial solutions:

Proposition 1.2.2. *Let $p > 2$, $K \in \mathbb{N}$. Moreover, for $\alpha > \alpha_p$, let u_α denote the unique radial solution of (1.2.1) with K nodal domains and $u_\alpha(0) > 0$, and define*

$$U_\alpha : [0,\infty) \to \mathbb{R}, \qquad U_\alpha(t) = (N+\alpha)^{-\frac{2}{p-2}}\, u_\alpha(e^{-\frac{t}{N+\alpha}}). \tag{1.2.5}$$

Then $U_\alpha \to (-1)^{K-1}U_\infty$ uniformly on $[0,\infty)$ as $\alpha \to \infty$, where $U_\infty \in C^2([0,\infty))$ is characterized as the unique bounded solution of the limit problem

$$-U'' = e^{-t}|U|^{p-2}U \quad \text{in } [0,\infty), \qquad U(0) = 0 \tag{1.2.6}$$

with $U'(0) > 0$ and with precisely $K-1$ zeros in $(0,\infty)$.

The transformation (1.2.5) had first been used by Byeon and Wang [**27**] to study the asymptotics of ground states. The proof of our result is based on the observation that this transformation turns the Hénon equation (1.2.1) into a one-dimensional problem on $[0,\infty)$ given by

$$-(e^{-\gamma t}U')' = e^{-t}|U|^{p-2}U \quad \text{in } I := [0,\infty), \qquad U(0) = 0, \tag{1.2.7}$$

depending on the new parameter $\gamma = \frac{N-2}{N+\alpha}$. This transformed problem then admits a well-defined limit problem as $\gamma \to 0^+$ (and hence $\alpha \to \infty$), which is precisely given by (1.2.6). The convergence result then follows from an application of the implicit function theorem at $\gamma = 0$ with respect to appropriate function spaces, which are given by C^1-functions with suitable exponential decay. Moreover, we also need to ensure sufficient uniqueness and degeneracy properties for (1.2.7). Finally, we note that the proof of Proposition 1.2.2 is simpler in the case $N = 2$ since we then have $U_\alpha = U_\infty$ for all $\alpha > 0$, as suggested by the fact that $\gamma = \frac{N-2}{N+\alpha} = 0$ in this case.

Importantly, we can apply an almost identical transformation to the weighted eigenvalue problem (1.2.4). This yields an eigenvalue problem on the interval $[0,\infty)$, which depends on γ. This constitutes the second part of our aforementioned strategy and leads to our other main result, the following detailed asymptotic characterization of the curves $\alpha \mapsto \mu_i(\alpha)$.

Theorem 1.2.3. *Let $p > 2$ and $\alpha > \alpha_p$. Then the negative eigenvalues of (1.2.4) are given as C^1-functions $(\alpha_p,\infty) \to \mathbb{R}$, $\alpha \mapsto \mu_i(\alpha)$, $i = 1,\dots,K$ satisfying the asymptotic expansions*

$$\mu_i(\alpha) = \nu_i^*\alpha^2 + c_i^*\alpha + o(\alpha) \quad \text{and} \quad \mu_i'(\alpha) = 2\nu_i^*\alpha + c_i^* + o(1) \qquad \text{as } \alpha \to \infty, \tag{1.2.8}$$

where c_i^*, $i = 1,\dots,K$ are constants and the values $\nu_1^* < \nu_2^* < \dots < \nu_K^* < 0$ are precisely the negative eigenvalues of the eigenvalue problem

$$\begin{cases} -\Psi'' - (p-1)e^{-t}|U_\infty(t)|^{p-2}\Psi = \nu\Psi & \text{in } [0,\infty), \\ \quad \Psi(0) = 0, \quad \Psi \in L^\infty(0,\infty), \end{cases} \tag{1.2.9}$$

with U_∞ given in Proposition 1.2.2. In particular, there exists $\alpha^* > 0$ such that the curves μ_i, $i = 1,\dots,K$ are strictly decreasing on $[\alpha^*,\infty)$.

This result in particular has found applications in the study of the monotonicity of the Morse index of radial solutions in [**81**].

As mentioned above, the proof of Theorem 1.2.8 is based on a transformation of the weighted eigenvalue problem (1.2.4) into an γ-dependent eigenvalue problem on the interval $[0,\infty)$ given by

$$\begin{cases} -(e^{-\gamma t}\Psi')' - (p-1)e^{-t}|U_\gamma(t)|^{p-2}\Psi = \nu e^{-\gamma t}\Psi & \text{in } (0,\infty), \\ \Psi(0) = 0, \quad \Psi \in L^\infty(0,\infty). \end{cases} \tag{1.2.10}$$

Letting $\gamma \to 0^+$ we then find that (1.2.9) serves as a limit problem for (1.2.10). Similar to Proposition 1.2.2, the C^1-expansions of eigenvalue curves rely on an application of the implicit function theorem at $\gamma = 0$ for each $i \in \{1,\dots,K\}$. In this case, however, the arguments involve much more technical difficulties for several reasons. Firstly, the choice of function spaces and of the associated map relies on the variational characterization of the eigenvalues $\nu_1,\dots,\nu_K$ given by

$$\nu_j(\gamma) = \inf_{\substack{W \subset H_0^1(I) \\ \dim W = j}} \max_{\Psi \in W\setminus\{0\}} \frac{\int_0^\infty e^{-\gamma t}\Psi'^2 - (p-1)e^{-t}|U_\gamma|^{p-2}\Psi^2\,dt}{\int_0^\infty e^{-\gamma t}\Psi^2\,dt} \tag{1.2.11}$$

for $j = 1,\dots,K$. Secondly, for $p \in (2,3]$ the map $U \mapsto |U|^{p-2}$ is no longer differentiable between standard function spaces. Instead, we restrict this map to the subset of C^1-functions on $[0,\infty)$ having only a finite number of simple zeros, endowed with a suitable weighted uniform L^1-norm. A substantial part of **[P1]** therefore consists of overcoming these difficulties.

We note that constants c_i^* can be characterized explicitly in terms of U_∞, normalized eigenfunctions of (1.2.9) associated with the eigenvalue ν_i^* and the solution of a related ODE.

As mentioned above, the most important consequence of this result is the fact that the curves μ_i are strictly decreasing for large α. Crucially, this yields suitable candidates for bifurcation points, i.e., values of α for which u_α is degenerate, as well as more detailed information on the changes of the Morse index of u_α as $\alpha \to \infty$.

Corollary 1.2.4. *Let $p > 2$. For every $i \in \{1,\dots,K\}$, there exist $\ell_i \in \mathbb{N}\cup\{0\}$ and sequences of numbers $\alpha_{i,\ell} \in (\alpha_p,\infty)$, $\varepsilon_{i,\ell} > 0$, $\ell \geq \ell_i$ with the following properties:*

(i) $\alpha_{i,\ell} \to \infty$ *as* $\ell \to \infty$.

(ii) $\mu_i(\alpha_{i,\ell}) + \lambda_\ell = 0$. *In particular,* $u_{\alpha_{i,\ell}}$ *is degenerate.*

(iii) u_α *is nondegenerate for* $\alpha \in (\alpha_{i,\ell} - \varepsilon_{i,\ell}, \alpha_{i,\ell} + \varepsilon_{i,\ell})$, $\alpha \neq \alpha_{i,\ell}$.

(iv) For $\varepsilon \in (0,\varepsilon_{i,\ell})$ *the Morse index of* $u_{\alpha_{i,l}+\varepsilon}$ *is strictly larger than the Morse index of* $u_{\alpha_{i,l}-\varepsilon}$.

The previous information on the eigenvalue curves can then be used to prove the bifurcation result stated in Theorem 1.2.1. More specifically, we recall that for $E = L^2(\mathbf{B})$, $D := H^2(\mathbf{B}) \cap H_0^1(\mathbf{B})$ and fixed $\alpha := \alpha_{i,\ell}$, solutions of (1.2.1) correspond to zeros of the map $G : (-\alpha,\infty)\times D \to E$ given by

$$[G(\lambda,u)] = -\Delta(u + u_{\alpha+\lambda}) - |x|^{\alpha+\lambda}|u + u_{\alpha+\lambda}|^{p-2}(u + u_{\alpha+\lambda}).$$

Corollary 1.2.4 then implies that the crossing number of an associated operator family is nonzero at the points $\alpha_{i,\ell}$. This allows us to use a bifurcation result by Kielhöfer [**77**] which implies that the points $\alpha_{i,\ell}$, $\ell \geq \ell_i$ are bifurcation points for solutions of (1.2.1) along the branch $\alpha \mapsto u_\alpha$.

The second part of Theorem 1.2.1, i.e., the nonradiality of the solutions in the sequence $(u^n)_n$, is based on results by Amadori and Gladiali [**5**], who showed that the kernel of L^α does not contain radial functions, i.e., the solutions u_α are radially nondegenerate for $\alpha > 0$. Alternatively, this can also be deduced from earlier results by Yanagida [**137**].

1.3 Spiraling Solutions of Nonlinear Schrödinger Equations

In **[P2]**, we consider a nonlinear Schrödinger equation. Due to its key role in the description of quantum mechanical systems, the Schrödinger equation has become one of the most studied PDEs since its introduction in 1926 [**117**]. In the following, we will consider the special case of a stationary nonlinear Schrödinger equation of the form

$$-\Delta v + qv = |v|^{p-2}v \qquad \text{in } \mathbb{R}^N, \tag{1.3.1}$$

where $p > 2$ and $q > 0$ is a constant. For subcritical exponents $p < 2^*$, there are countless results for solutions of (1.3.1) in $H^1(\mathbb{R}^N)$ with exponential decay, for which we refer to the more comprehensive treatments in [**7**, **80**, **125**, **126**, **136**] and the references therein.

In contrast, solutions of (1.3.1) with only partial decay, i.e., with decay in some but not all variables, are significantly less well understood. Such solutions in particular appear as the limits of similar equations in bounded domains. In the following, we will focus on solutions of (1.3.1) which can be characterized as follows: Setting $\bar{x} = (x, t) \in \mathbb{R}^N$ with $x \in \mathbb{R}^{N-1}$ and $t \in \mathbb{R}$, we consider solutions $v : \mathbb{R}^N \to \mathbb{R}$ satisfying

$$\lim_{|x| \to \infty} v(x, t) = 0 \quad \text{uniformly in } t. \tag{1.3.2}$$

Note that if $\tilde{v}$ is a radial solution of (1.3.1) in $\mathbb{R}^{N-1}$ satisfying $\tilde{v}(x) \to 0$ as $|x| \to \infty$, the function $v(x,t) = \tilde{v}(x)$ is a solution of (1.3.1) satisfying (1.3.2). Such a solution is therefore necessarily t-invariant and axially symmetric. Here and in the following, axial symmetry always refers to the axis $\{(0_{\mathbb{R}^{N-1}}, t) : t \in \mathbb{R}\} \subset \mathbb{R}^N$. Naturally, this begs the question of whether different solutions with partial decay exist.

One of the first major results concerning this question is due to Dancer [**43**] and sparked several subsequent results, see e.g. [**26**,**45**,**91**] and the references therein. The solutions found in [**43**] bifurcate from the unique family of t-invariant axially symmetric positive solutions of (1.3.1) and are shown to be t-periodic, axially symmetric and positive. Importantly, a result due to Farina, Malchiodi and Rizzi [**56**] gives an analogue to the symmetry result by Gidas, Ni and Nirenberg stated in Theorem 1.1.1 for solutions with partial decay and, in particular, implies that all t-periodic, positive solutions of (1.3.1) satisfying (1.3.2) are necessarily axially symmetric up to translations.

In the following, we will consider solutions of (1.3.1) satisfying the partial decay condition (1.3.2) which are periodic in t, but do not exhibit axial symmetry. As a consequence of the previous remarks, such solutions must be sign-changing and with the exception of solutions which are in fact independent of t, such solutions appear to be new. Similar to the positive case, there are several results for nonradial sign-changing solutions of (1.3.1) in $\mathbb{R}^N$ with exponential decay in all directions, see e.g. [**8**, **10**, **88**, **105**]. Since the case $q > 0$ is equivalent to $q = 1$ by rescaling, it suffices to consider the cases $q = 1$ in the following. Moreover, we will focus on the case $N = 3$ and thus consider solutions of

$$-\Delta v + v = |v|^{p-2}v \qquad \text{in } \mathbb{R}^3, \tag{1.3.3}$$

which are invariant under the action of a screw motion. More specifically, we let $\lambda > 0$ and call a function $v : \mathbb{R}^3 \to \mathbb{R}$ λ-spiraling if for any $\theta \in \mathbb{R}$,

$$v(R_\theta x, t + \lambda\theta) = v(x,t) \qquad \text{for } x \in \mathbb{R}^2,\, t \in \mathbb{R},$$

where $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the counter-clockwise rotation with angle θ in $\mathbb{R}^2$. In particular, λ-spiraling functions are $2\lambda\pi$-periodic in t. Similar spiraling solutions have also been considered for the classical and fractional Allen-Cahn equation in [**46**] and [**39**], respectively, though the variational structure is vastly different in these cases.

In order to treat spiraling solutions of (1.3.3) we note that in cylindrical coordinates $(x,t) = (r\cos\varphi, r\sin\varphi, t)$ with $(r,\varphi,t) \in [0,\infty) \times \mathbb{R} \times \mathbb{R}$, a λ-spiraling function is given by

$$v(r,\varphi,t) = u\left(r, \varphi - \frac{t}{\lambda}\right)$$

with a function $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ which is 2π-periodic in the second variable. Here, the function u can be interpreted as the profile at $t = 0$, i.e., $v(\cdot, 0)$. Using this as an ansatz and going back to Cartesian coordinates, we find that u must satisfy

$$\begin{cases} -\Delta u - \dfrac{1}{\lambda^2}\partial_\theta^2 u + u = |u|^{p-2}u & \text{on } \mathbb{R}^2, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.3.4}$$

where, by construction, the operator

$$\partial_\theta := x_1 \partial_{x_2} - x_2 \partial_{x_1}$$

corresponds to the angular derivative in polar coordinates. Subsequently, we use variational methods to study (1.3.4) and consider the space

$$H := \left\{ u \in H^1(\mathbb{R}^2) \;:\; \int_{\mathbb{R}^2} |\partial_\theta u|^2 dx < \infty \right\}.$$

For $\lambda > 0$, we endow H with the λ-dependent scalar product

$$\langle u, v\rangle_\lambda := \int_{\mathbb{R}^2} \left(\nabla u \cdot \nabla v + \frac{1}{\lambda^2}(\partial_\theta u)(\partial_\theta v) + uv \right) dx$$

and find that $(H, \langle\cdot,\cdot\rangle_\lambda)$ is a Hilbert space. Importantly, for any $\lambda > 0$ the space $(H, \langle\cdot,\cdot\rangle_\lambda)$ compactly embeds into $L^p(\mathbb{R}^2)$ for any $p > 2$.

The energy functional $E_\lambda : H \to \mathbb{R}$ associated to (1.3.4) is then given by

$$E_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla u|^2 + \frac{1}{\lambda^2}|\partial_\theta u|^2 + u^2 \right) dx - \frac{1}{p}\int_{\mathbb{R}^2} |u|^p \, dx,$$

and it can be shown that E_λ is a C^1-functional and that critical points of E_λ are weak solutions of (1.3.4). Consequently, we study spiraling solutions by finding critical points of E_λ. Since the functional is neither bounded from above nor below and only possesses a local minimum at the trivial solution $u \equiv 0$, this amounts to the study of saddle points. In order to find such critical points, we then consider the Nehari manifold

$$\mathcal{N}_\lambda := \left\{ u \in H \setminus \{0\} : E_\lambda'(u)u = 0 \right\}.$$

It can be shown that $E_\lambda\big|_{\mathcal{N}_\lambda}$ attains a positive minimum and any minimizer is a critical point of E_λ and thus a weak solution of (1.3.4). Moreover, since $\mathcal{N}_\lambda$ contains all nontrivial critical points of E_λ by construction, any minimizer of $E_\lambda\big|_{\mathcal{N}_\lambda}$ is in fact a ground state.

It is important to note, however, that the characterization of ground states as minimizers of $E_\lambda\big|_{\mathcal{N}_\lambda}$ can be used to show that ground states cannot change sign, so the symmetry of E_λ implies that any ground state u is either positive or $-u$ is a positive ground state. Consequently, the resulting spiraling solution v of (1.3.1) is also positive (or $-v$ is positive), so that the result of Farina, Malchiodi and Rizzi [**56**] mentioned above then implies that v must be radial in x and therefore constant in t.

We subsequently focus on nonradial solutions of (1.3.4) which correspond to solutions of (1.3.3) that are $2\lambda\pi$-periodic in t but neither axially symmetric nor t-invariant. We therefore restrict our attention to nodal (i.e., sign-changing) solutions of (1.3.4). More specifically, we study *least energy nodal solutions* of (1.3.4), i.e., minimizers of E_λ within the class of sign-changing solutions of (1.3.4). Variationally, these solutions can be characterized as minimizers of E_λ over

$$\begin{aligned}\mathcal{M}_\lambda &:= \left\{u \in H : u^+ \not\equiv 0, u^- \not\equiv 0,\ E'_\lambda(u)u^+ = E'_\lambda(u)u^- = 0\right\}\\ &= \left\{u \in H \setminus \{0\} : u^+, u^- \in \mathcal{N}_\lambda\right\}\end{aligned}$$

and set

$$\beta_\lambda := \inf_{u \in \mathcal{M}_\lambda} E_\lambda(u).$$

It can be shown that $E_\lambda\big|_{\mathcal{M}_\lambda}$ attains a positive minimum and that any minimizer is a critical point of E_λ. By construction, such a minimizer is in fact sign-changing and thus a least energy nodal solution.

Our main result states that least energy nodal solutions exist and characterizes their symmetry for small and large $\lambda > 0$, respectively.

Theorem 1.3.1. *Let $p > 2$. For every $\lambda > 0$ there exists a least energy nodal solution of* (1.3.4). *Furthermore, there exist $0 < \lambda_0 \le \Lambda_0 < \infty$ with the following properties:*

(i) For $\lambda < \lambda_0$, every least energy nodal solution of (1.3.4) *is radial.*

(ii) For $\lambda > \Lambda_0$, every least energy nodal solution of (1.3.4) *is nonradial.*

Notably, this implies that least energy nodal solutions exhibit symmetry breaking within the interval $[\lambda_0, \Lambda_0]$. To our knowledge, such a phenomenon has not been observed for least energy nodal solutions before.

We first discuss the proof of Theorem 1.3.1(i), which turns out to be a consequence of the following more general radiality result for solutions of (1.3.4) with small $\lambda > 0$.

Theorem 1.3.2. *Let $p > 2$.*

(i) If $u \in H$ is a nontrivial weak solution of (1.3.4) for some $\lambda > 0$ satisfying $\lambda < \left(\frac{1}{(p-1)\|u\|_{L^\infty}^{p-2}}\right)^{\frac{1}{2}}$, then u is a radial function.

(ii) For every $c > 0$, there exists $\lambda_c > 0$ with the property that every weak solution $u \in H$ of (1.3.4) for some $\lambda \in (0, \lambda_c)$ with $E_\lambda(u) \le c$ is radial.

The proof is based on several ingredients. Firstly, we use the radial averaging operator

$$H \to H, \quad u \mapsto u^\#$$
$$u^\#(x) = \frac{1}{2\pi}\int_{\mathbb{S}^1} u(|x|\omega)\, d\omega$$

to give a Poincaré type inequality in the angular variable given by

$$\|u\|_{L^2}^2 \leq \|\partial_\theta u\|_{L^2}^2 + \|u^\#\|_{L^2}^2 \tag{1.3.5}$$

for $u \in H$. This allows us to study the difference $u - u^\#$ and leads to the estimate

$$\frac{1}{\lambda^2}\|\partial_\theta u\|_{L^2}^2 \leq (p-1)\|u\|_{L^\infty}^{p-2}\|\partial_\theta u\|_{L^2}^2,$$

which readily implies (i). Secondly, we can derive uniform elliptic L^∞-estimates for solutions of (1.3.4) in terms of their H-norms. Combined with estimates for the energy, this is used to prove (ii).

Before we discuss the nonradiality of least energy nodal solutions as stated in Theorem 1.3.1(i), we first note that Theorem 1.3.2(ii) implies that nonradial sign-changing solutions of (1.3.4) for small values $\lambda > 0$ only exist if their energy is sufficiently large. An important tool in our study is a suitable class of solutions whose energy lies above this threshold, which are given as nonradial nodal solutions of (1.3.4) which are odd with respect to the reflection at the hyperplane $\{x_1 = 0\}$. Any such solution satisfies the boundary value problem

$$\begin{cases} -\Delta u - \dfrac{1}{\lambda^2}\partial_\theta^2 u + u = |u|^{p-2}u & \text{on } \mathbb{R}^2_+, \\ u = 0 & \text{on } \partial\mathbb{R}^2_+ \end{cases} \tag{1.3.6}$$

in the half space $\mathbb{R}^2_+ := \{x \in \mathbb{R}^2 \,:\, x_1 > 0\}$.

Similarly to the previous considerations, we wish to find solutions of (1.3.6) via variational methods and consider the space

$$H^+ := \left\{ u \in H^1_0(\mathbb{R}^2_+) \,:\, \int_{\mathbb{R}^2_+} |\partial_\theta u|^2 dx < \infty \right\},$$

which can be interpreted as a closed subspace of H by trivially extending the elements of H^+ to $\mathbb{R}^2$. The energy functional $E_\lambda^+ : H^+ \to \mathbb{R}$ associated to (1.3.6) is then given by

$$E_\lambda^+(u) := \frac{1}{2}\int_{\mathbb{R}^2_+} \left(|\nabla u|^2 + \frac{1}{\lambda^2}|\partial_\theta u|^2 + u^2\right)dx - \frac{1}{p}\int_{\mathbb{R}^2_+} |u|^p \, dx,$$

and weak solutions of (1.3.6) correspond to critical points of E_λ^+. Similar to the comments above, nontrivial critical points can be found by minimizing E_λ^+ over the Nehari manifold

$$\mathcal{N}_\lambda^+ := \left\{ u \in H^+ \setminus \{0\} : (E_\lambda^+)'(u)u = 0 \right\}.$$

Using the compact embedding $H^+ \hookrightarrow L^p(\mathbb{R}^2)$, it can be shown that a minimizer exists and is a critical point of E_λ^+. Moreover, such a minimizer has minimal energy among all critical points of E_λ^+ and is thus referred to as a least energy solution or ground state of (1.3.6). Importantly, the associated minimal energy admits the minimax characterization

$$c_\lambda = \inf_{u \in H^+ \setminus \{0\}} \sup_{t \geq 0} E_\lambda^+(tu).$$

Our main result for (1.3.6) then reads as follows.

Theorem 1.3.3. *Let $p > 2$ and $\lambda > 0$.*

(i) ***(Existence)*** *Problem* (1.3.6) *admits a positive least energy solution.*

(ii) ***(Symmetry)*** *Any positive solution u of (1.3.6) is symmetric with respect to reflection at the x_1-axis and decreasing in the angle $|\theta|$ from the x_1-axis. In particular, u takes its maximum on the x_1-axis.*

(iii) ***(Asymptotics)*** *If $\lambda_k \geq 1$ are given with $\lambda_k \to +\infty$ as $k \to \infty$ and u_k is a positive least energy solution of (1.3.6) with $\lambda = \lambda_k$, then, after passing to a subsequence, there exists a sequence of numbers $\tau_k > 0$ with*

$$\tau_k \to +\infty, \qquad \frac{\tau_k}{\lambda_k} \to 0 \qquad \text{as } k \to \infty$$

such that the translated functions $w_k \in H^1(\mathbb{R}^2)$, $w_k(x) = u_k(x_1 + \tau_k, x_2)$ satisfy

$$w_k \to w_\infty \qquad \text{strongly in } H^1(\mathbb{R}^2),$$

where w_∞ is the unique positive radial solution of

$$-\Delta w_\infty + w_\infty = |w_\infty|^{p-2} w_\infty, \qquad w_\infty \in H^1(\mathbb{R}^2). \tag{1.3.7}$$

Here we note that the uniqueness of the positive radial solution to (1.3.7) is due to Kwong [**78**]. While the existence result stated in Theorem 1.3.3(i) essentially follows from a minimization over the Nehari manifold as mentioned above, the symmetry property (ii) is based upon the method of rotating planes.

The proof of Theorem 1.3.3(iii) relies on the observation that $\|u_k\|_{L^p}$ remains bounded away from zero, so that Lions' Lemma [**86**, Lemma I.1] and (ii) imply the existence of numbers τ_k such that the translated functions w_k satisfy $w_k \rightharpoonup w \not\equiv 0$ in $H^1(\mathbb{R}^2)$. A further analysis then shows that the numbers τ_k tend to infinity and satisfy $\tau_k/\lambda_k \to 0$ as $k \to \infty$, which allows us to show that w is a weak solution of (1.3.7).

In fact, the results stated in Theorem 1.3.3(i) and (ii) can be extended to more general equations of the form

$$\begin{cases} -\Delta u - \dfrac{1}{\lambda^2}\partial_\theta^2 u = f(u) & \text{on } \mathbb{R}^2_+, \\ u(x) \to 0 & \text{for } |x| \to \infty, \end{cases}$$

where the nonlinearity f satisfies suitable superlinear growth conditions. In particular, this holds for $f(u) = -qu + |u|^{p-2}u$, where $q \geq 0$ and $p > 2$. Moreover, we may also replace the underlying half space $\mathbb{R}^2_+$ by suitable cones and use successive reflection to find solutions with precisely $2j$ nodal domains.

In view of Theorem 1.3.1, it is crucial to note that Theorem 1.3.3(iii) allows us to show that c_λ converges to c_∞ as $\lambda \to \infty$, where c_∞ is the least energy of nontrivial solutions of the limit problem (1.3.7). In particular, this implies that the energy of the least energy nodal solution of (1.3.4), as considered in Theorem 1.3.1, tends to $2c_\infty$ as $\lambda \to \infty$. On the other hand, it follows from [**135**] that there exists a fixed constant $\varepsilon_* > 0$ such that every radial nodal solution $u \in H$ of (1.3.4) satisfies

$$E_\lambda(u) > 2c_\infty + \varepsilon_*.$$

Combining these two observations consequently implies Theorem 1.3.1(ii).

Additionally, we can characterize the asymptotics of positive least energy solutions of (1.3.6) as $\lambda \to 0$. More precisely, we observe the following concentration behavior at the origin.

Theorem 1.3.4. *Let $(\lambda_k)_k$ be a sequence of numbers $\lambda_k \le 1$ such that $\lambda_k \to 0$ as $k \to \infty$. Moreover, let $u_k \in H^+$ be a positive least energy solution of (1.3.6), and let $v_k \in H^+$ be defined by $v_k(x) = \lambda_k^{\frac{2}{p-2}} u_k(\lambda_k x)$.*

Then, after passing to a subsequence, we have $v_k \to v^*$ in H^+, where v is a positive least energy solution of the problem

$$\begin{cases} -\Delta v^* - \partial_\theta^2 v^* = |v^*|^{p-2} v^* & \text{on } \mathbb{R}^2_+, \\ v = 0 & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{1.3.8}$$

Note that the existence of ground states for (1.3.8) relies on a stronger version of the Poincaré type inequality (1.3.5) for functions in H^+. The convergence result then uses the fact that v_k is a least energy solution of

$$\begin{cases} -\Delta v - \partial_\theta^2 v + \lambda_k^2 v = |v|^{p-2} v & \text{in } \mathbb{R}^2_+, \\ v = 0 & \text{on } \partial\mathbb{R}^2_+, \end{cases}$$

and can be characterized variationally via Rayleigh quotients, which allows us to show Γ-convergence of v_k to v^* in a suitable topology. Combined with the compact embedding properties of H^+, this ultimately yields convergence in H^+.

1.4 Rotating Waves in Nonlinear Media

The article **[P3]** is devoted to symmetry breaking phenomena related to a class of solutions of a nonlinear wave equation. In general, equations of the form

$$\partial_t^2 v - \Delta v + mv = f(v) \qquad \text{in } \mathbb{R} \times \Omega, \tag{1.4.1}$$

model the propagation of waves in an ambient domain $\Omega \subset \mathbb{R}^N$ with mass parameter $m \ge 0$ and nonlinear response function f. In the case $m > 0$, (1.4.1) is also known as a nonlinear Klein-Gordon equation. For suitable nonlinearities f, equations of this type can possess many different types of solutions with completely different behaviors, such as travelling wave solutions and scattering solutions, while a particularly important class of time-periodic solutions is given by standing wave solutions. These can be found via the ansatz

$$v(t, x) = e^{-ikt} u(x), \qquad k > 0 \tag{1.4.2}$$

and lead to the study of stationary nonlinear Schrödinger equations or Helmholtz equations depending on the frequency k. We refer to [**7**, **55**] for more details. We stress that standing wave solutions given by the ansatz above have a stationary amplitude $|v|$. In particular, the ansatz (1.4.2) cannot be used to find a real-valued non-stationary solution.

Despite the vast literature on standing wave solutions, significantly less is known about the existence of non-stationary real-valued time-periodic solutions. If $\Omega = \mathbb{R}^N$ and the nonlinearity f is multiplied with a compactly supported function, the class of breather solutions has recently received much attention, see e.g. [**73**, **94**]. For bounded domains Ω, however, fewer results are available. Notably, the one dimensional case was first studied in the works of Rabinowitz [**116**] and Brézis, Coron and Nirenberg [**22**], and a radial setting in higher dimensions with sublinear nonlinearities has been considered by Ben-Naoum and Mawhin [**13**].

In the following, we will consider the case where Ω is a ball centered at the origin. For such radially symmetric Ω, another interesting class of real-valued time-periodic solutions is given by *rotating wave solutions*. More specifically, we consider the problem

$$\begin{cases} \partial_t^2 v - \Delta v + mv = |v|^{p-2}v & \text{in } \mathbb{R}\times\mathbf{B}, \\ v = 0 & \text{on } \mathbb{R}\times\partial\mathbf{B} \end{cases} \tag{1.4.3}$$

for $N \geq 2$, where $\mathbf{B} \subset \mathbb{R}^N$ denotes the unit ball, $2 < p < 2^*$ and $m > -\lambda_1(\mathbf{B})$. Here, $\lambda_1(\mathbf{B})$ denotes the first Dirichlet eigenvalue of $-\Delta$ on $\mathbf{B}$.

Rotating wave solutions of (1.4.3) are then characterized by the ansatz

$$v(t,x) = u(R_{\alpha t}(x)), \tag{1.4.4}$$

where, for $\theta \in \mathbb{R}$, we let $R_\theta \in O(N)$ denote a planar rotation in $\mathbb{R}^N$ with angle θ, so the constant $\alpha > 0$ in (1.4.4) is the angular velocity of the rotation. Without loss of generality, we may assume that

$$R_\theta(x) = (x_1\cos\theta + x_2\sin\theta, -x_1\sin\theta + x_2\cos\theta, x_3, \dots, x_N) \qquad \text{for } x \in \mathbb{R}^N,$$

so R_θ is the rotation in the x_1-x_2-plane with fixed point set $\{0_{\mathbb{R}^2}\}\times\mathbb{R}^{N-2}$. Clearly, the ansatz (1.4.4) could also lead to a stationary solution, in particular when the profile function u is radial. This motivates the following definition.

Definition 1.4.1. *We say that a function $u : \mathbf{B} \to \mathbb{R}$ is **x_1-x_2-nonradial** if there exists at least one angle $\theta \in \mathbb{R}$ such that u is not R_θ-invariant.*

Consequently, profile functions u which are x_1-x_2-nonradial yield non-stationary rotating waves v via the ansatz (1.4.4). The ansatz (1.4.4) reduces (1.4.3) to

$$\begin{cases} -\Delta u + \alpha^2\partial_\theta^2 u + mu = |u|^{p-2}u & \text{in } \mathbf{B}, \\ u = 0 & \text{on } \partial\mathbf{B}, \end{cases} \tag{1.4.5}$$

where $\partial_\theta = x_1\partial_{x_2} - x_2\partial_{x_1}$ denotes the associated angular derivative operator.

Note that the operator $\alpha^2\partial_\theta^2 u$ also appeared in a similar fashion in our study of spiraling solutions of nonlinear Schrödinger equations discussed in Section 1.3. However, the different sign leads to a completely different problem in this case. This can be observed in polar coordinates as

$$-\Delta + \alpha^2\partial_\theta^2 = -\Delta_r u - \frac{1}{r^2}\Delta_{\mathbb{S}^{N-1}} u + \alpha^2\partial_\theta^2 u, \tag{1.4.6}$$

so, in particular, the operator loses uniform ellipticity for $\alpha \geq 1$. This observation will play a key role later on.

As outlined above, it is important to ensure that a solution u of (1.4.5) is x_1-x_2-nonradial in order to yield a genuine rotating wave solution of (1.4.3). Clearly, x_1-x_2-nonradiality can be characterized by the derivative ∂_θ, i.e., a function $u \in H_0^1(\mathbf{B})$ is x_1-x_2-nonradial if and only if $\partial_\theta u \not\equiv 0$. If, on the other hand, a solution u of (1.4.5) satisfies $\partial_\theta u \equiv 0$ in $\mathbf{B}$, then u solves the classical stationary nonlinear Schrödinger equation $-\Delta u + mu = |u|^{p-2}u$ in $\mathbf{B}$ with Dirichlet boundary conditions on $\partial\mathbf{B}$, so it satisfies (1.4.5) with $\alpha = 0$. If u is positive as well, the symmetry result of Gidas, Ni and Nirenberg stated in Theorem 1.1.1 implies that u is a radial function. With respect to the previous discussion, our main goal is thus the existence of positive solutions of (1.4.5) which do not satisfy $\partial_\theta u \equiv 0$.

More specifically, we study ground state solutions which may be characterized as minimizers of the Rayleigh quotient $R_{\alpha,m,p} : H_0^1(\mathbf{B}) \setminus \{0\} \to \mathbb{R}$ given by

$$R_{\alpha,m,p}(u) = \frac{\int_{\mathbf{B}} \left(|\nabla u|^2 - \alpha^2|\partial_\theta u|^2 + mu^2\right)\, dx}{\left(\int_{\mathbf{B}} |u|^p\, dx\right)^{\frac{2}{p}}},$$

for $\alpha, m \in \mathbb{R}$ and $p \in [2, 2^*)$, i.e., we study functions attaining

$$\mathscr{C}_{\alpha,m,p}(\mathbf{B}) := \inf_{u \in H_0^1(\mathbf{B}) \setminus \{0\}} R_{\alpha,m,p}(u). \tag{1.4.7}$$

We will further restrict our study to the case $\alpha \in [0, 1]$ in this section, as

$$\mathscr{C}_{\alpha,m,p}(\mathbf{B}) = -\infty \tag{1.4.8}$$

holds for every $p \in [2, 2^*)$, $m \in \mathbb{R}$ and $\alpha > 1$. This is essentially due to the fact that the operator given in (1.4.6) is hyperbolic in $\mathbf{B} \setminus \overline{B_{1/\alpha}(0)}$ for $\alpha > 1$. The case $\alpha > 1$ thus requires different methods which will be presented in Section 1.5 below.

For $0 \leq \alpha < 1$, the operator is uniformly elliptic and the existence of minimizers of $R_{\alpha,m,p}$ on $H_0^1(\mathbf{B}) \setminus \{0\}$ follows from the compactness of the embedding $H_0^1(\mathbf{B}) \hookrightarrow L^p(\mathbf{B})$ and the weak lower semicontinuity of the enumerator of $R_{\alpha,m,p}$. Initially, however, it is completely unclear whether minimizers are radial or nonradial functions. While the term $-\alpha^2 \|\partial_\theta u\|^2_{L^2(\mathbf{B})}$ favors x_1-x_2-nonradial functions as energy minimizers, the Pólya-Szegö inequality yields $\int_{\mathbf{B}} |\nabla u^*|^2 \, dx \leq \int_{\mathbf{B}} |\nabla u|^2 \, dx$, where u^* denotes the (radial) Schwarz symmetrization of a function $u \in H_0^1(\mathbf{B})$, and therefore favors radiality. The first effect is weak if α is close to zero, therefore one might expect radiality of ground states in this case. Indeed, in our first result, we find that the ground states are radial if α is sufficiently small.

Theorem 1.4.2. *Let $m \geq 0$ and $2 < p < 2^*$. Then there exists $\alpha_0 > 0$ such that*

$$\mathscr{C}_{\alpha,m,p}(\mathbf{B}) = \mathscr{C}_{0,m,p}(\mathbf{B}) \qquad \text{for } \alpha \in [0, \alpha_0). \tag{1.4.9}$$

Moreover, for $\alpha \in [0, \alpha_0)$, there is, up to sign, a unique ground state solution of (1.4.5) which is a radial function.

The proof is based on the implicit function theorem, using known nondegeneracy results for the unique positive radial solutions of the classical problem

$$\begin{cases} -\Delta u + mu = |u|^{p-2}u & \text{in } \mathbf{B}, \\ u = 0 & \text{on } \partial\mathbf{B}, \end{cases}$$

corresponding to the case $\alpha = 0$ in (1.4.5).

In order to further shed light on the symmetry properties of ground states for larger α, we note that for every $p \in [2, 2^*)$ and $m \in \mathbb{R}$, the map

$$\alpha \mapsto \mathscr{C}_{\alpha,m,p}(\mathbf{B})$$

is continuous and nonincreasing on $[0, 1]$. Combined with the fact that $R_{\alpha,m,p}(u) = R_{0,m,p}(u)$ for every radial function $u \in H_0^1(\mathbf{B}) \setminus \{0\}$ and every $\alpha \in [0, 1]$, a sufficient condition for the x_1-x_2-nonradiality of all ground state solutions is the inequality

$$\mathscr{C}_{\alpha,m,p}(\mathbf{B}) < \mathscr{C}_{0,m,p}(\mathbf{B}). \tag{1.4.10}$$

In light of the competing effects in $R_{\alpha,m,p}$ and the monotonicity stated above, we expect the validity of (1.4.10) to be particularly clear in the case α close to 1.

Notably, the limit case $\alpha = 1$ needs to be treated separately from $0 \leq \alpha < 1$, as the differential operator $-\Delta + \partial_\theta^2$ is not uniformly elliptic on $\mathbf{B}$. Indeed, (1.4.6) suggests that the operator $-\Delta + \partial_\theta^2$ is no longer uniformly elliptic in a neighborhood of the great circle

$\{x \in \partial\mathbf{B} : x_3 = \cdots = x_N = 0\}$. Instead, the minimization problem in the case $\alpha = 1$ is closely related to a degenerate anisotropic critical Sobolev inequality in the half space, where the corresponding critical exponent is given by

$$2_1^* := \frac{4N+2}{2N-3}. \tag{1.4.11}$$

Before we discuss this connection in detail, we state our next main result which highlights the relevance of the exponent 2_1^*.

Theorem 1.4.3. *Let $m > -\lambda_1(\mathbf{B})$ and $p \in (2, 2^*)$.*

(i) If $\alpha \in (0,1)$, then there exists a ground state solution of (1.4.5).

(ii) We have

$$\mathscr{C}_{1,m,p}(\mathbf{B}) = 0 \quad \text{for } p > 2_1^*, \qquad \text{and} \quad \mathscr{C}_{1,m,p}(\mathbf{B}) > 0 \quad \text{for } p \le 2_1^*. \tag{1.4.12}$$

Moreover, for any $p \in (2_1^*, 2^*)$, there exists $\alpha_p \in (0,1)$ with the property that

$$\mathscr{C}_{\alpha,m,p}(\mathbf{B}) < \mathscr{C}_{0,m,p}(\mathbf{B}) \qquad \text{for } \alpha \in (\alpha_p, 1]$$

and therefore every ground state solution of (1.4.5) is x_1-x_2-nonradial for $\alpha \in (\alpha_p, 1)$.

In particular, Theorems 1.4.3 and 1.4.2 imply that for fixed $p > 2_1^*$, symmetry breaking of ground state solutions occurs as the parameter α varies from 0 to 1 (we will discuss further symmetry breaking for the case $p \le 2_1^*$ later). Notably, Theorem 1.4.3 also yields the following new degenerate Sobolev inequality as a consequence of the case $m = 0$, $\alpha = 1$.

Corollary 1.4.4. *For $u \in H_0^1(\mathbf{B})$ we have*

$$\Big(\int_{\mathbf{B}} |u|^{2_1^*}\,dx\Big)^{\frac{2}{2_1^*}} \le \frac{1}{\mathscr{C}_{1,0,p}(\mathbf{B})} \int_{\mathbf{B}} \big(|\nabla u|^2 - |\partial_\theta u|^2\big)dx \tag{1.4.13}$$

and the exponent 2_1^* is optimal in the sense that no such inequality holds for $p > 2_1^*$.

In order to illustrate the proof of Theorem 1.4.3, we consider the case $N = 2$ and use polar coordinates $(r, \theta) \in (0,1) \times (-\pi, \pi)$, giving

$$\int_{\mathbf{B}} \big(|\nabla u|^2 - |\partial_\theta u|^2\big)\,dx = \int_0^1 r \int_{-\pi}^{\pi} \left(|\partial_r u|^2 + \left(\frac{1}{r^2} - 1\right)|\partial_\theta u|^2\right) d\theta\,dr.$$

For any $s \in (0,1)$, the right hand side induces an equivalent norm on the subspace $H_0^1(B_s(0)) \subset H_0^1(\mathbf{B})$ and classical Sobolev embeddings can thus be used to show the inequality (1.4.13) for functions in $H_0^1(B_s(0))$. For functions $u \in H_0^1(\mathbf{B})$ whose support is contained in a neighborhood of the boundary $\partial\mathbf{B}$, however, a more careful analysis is required. For such functions, the fact that

$$\frac{1}{r^2} - 1 = \frac{(1+r)(1-r)}{r^2} \approx 2(1-r) \qquad \text{for } r \text{ close to } 1$$

suggests

$$\int_{\mathbf{B}} \big(|\nabla u|^2 - |\partial_\theta u|^2\big)\,dx \approx \int_0^1 \int_{-\pi}^{\pi} \big(|\partial_r u|^2 + 2(1-r)|\partial_\theta u|^2\big)\,d\theta\,dr.$$

This can be made more rigorous for $\delta > 0$, $x_0 \in \partial\mathbf{B}$ by setting $\Omega_{x_0,\delta} = \{x \in \mathbf{B} : |x - x_0| < \delta\}$ and considering a function $u \in C_c^1(\Omega_{x_0,\delta})$. We may then set $v(x_1, x_2) := u(1 - x_1, x_2)$ and trivially extend v to the half space $\mathbb{R}^2_+ := \{x \in \mathbb{R}^2 : x_1 > 0\}$. For $\varepsilon > 0$ we can then show that there exists $\delta > 0$ such that

$$\int_{\mathbf{B}} \left(|\nabla u|^2 - |\partial_\theta u|^2\right) dx \geq (1 - \varepsilon) \int_{\mathbb{R}^2_+} \left(|\partial_1 v|^2 + 2x_1|\partial_2 v|^2\right) dx$$

holds for $u \in C_c^1(\Omega_{x_0,\delta})$. For $p > 2$, similar arguments allow us to estimate $\int_{\mathbf{B}} |u|^p \, dx \leq \int_{\mathbb{R}^2_+} |v|^p \, dx$ and hence

$$\begin{aligned} R_{\alpha,0,p}(u) &= \frac{\int_{\mathbf{B}} \left(|\nabla u|^2 - \alpha^2|\partial_\theta u|^2\right) dx}{\left(\int_{\mathbf{B}} |u|^p \, dx\right)^{\frac{2}{p}}} \\ &\geq (1 - \varepsilon) \inf_{v \in C_c^1(\mathbb{R}^2_+)} \frac{\int_{\mathbb{R}^2_+} \left(|\partial_1 v|^2 + 2x_1|\partial_2 v|^2\right) dx}{\left(\int_{\mathbb{R}^2_+} |v|^p \, dx\right)^{\frac{2}{p}}} \end{aligned} \tag{1.4.14}$$

for $u \in C_c^1(\Omega_{x_0,\delta}) \setminus \{0\}$. By a scaling argument, we find that the right hand side can only be positive if $p = 10$, which corresponds precisely to 2_1^* in the case $N = 2$. In fact, it can be shown that the infimum on the right hand side is indeed positive for $p = 10$, yielding the inequality

$$\left(\int_{\mathbb{R}^2_+} |v|^{10} \, dx\right)^{\frac{1}{5}} \leq C \int_{\mathbb{R}^2_+} \left(|\partial_1 v|^2 + 2x_1|\partial_2 v|^2\right) dx \qquad \text{for } u \in C_c^1(\mathbb{R}^2_+),$$

which is also known as a Grushin inequality, as we will further discuss below. Ultimately, (1.4.14) allows us to show the characterization (1.4.12).

For $N \geq 3$, we are similarly led to an inequality on the half space

$$\mathbb{R}^N_+ := \left\{x \in \mathbb{R}^N : x_1 > 0\right\},$$

which is given by

$$\left(\int_{\mathbb{R}^N_+} |u|^{2_1^*} \, dx\right)^{\frac{2}{2_1^*}} \leq C \int_{\mathbb{R}^N_+} \left(\sum_{i=1}^{N-1} |\partial_i u|^2 + 2x_1|\partial_N u|^2\right) dx \qquad \text{for } u \in C_c^1(\mathbb{R}^N_+). \tag{1.4.15}$$

This inequality is no longer related to a Grushin inequality and appears to be new. Indeed, writing $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^k$, where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^k$ and $s > 0$, classical Grushin inequalities usually take the form

$$\|u\|_{L^{\frac{2m+2k(s+1)}{m+k(s+1)-2}}(\mathbb{R}^N)} \leq C\left(\int_{\mathbb{R}^N} |\nabla_x u|^2 + c|x|^{2s}|\nabla_y u|^2 \, d(x, y)\right)^{1/2}, \qquad \text{for } u \in C_c^1(\mathbb{R}^N),$$

which does not cover (1.4.15) for $N \geq 3$. We note, however, that the critical exponents coincide for $m = N - 1$, $k = 1$ and $s = \frac{1}{2}$. A more extensive exposition on this topic can be found in [**70**], though we particularly mention symmetry results for positive entire solutions to related semilinear problems [**100**], as well as the existence of extremal functions for Grushin inequalities on $\mathbb{R}^N$ shown in [**12**] and [**99**]. We point out that a more general family of Grushin type operators and their associated inequalities is studied in [**58**, Theorem 1.7], which bears more similarity to (but does not cover) the inequality (1.4.15).

The validity of (1.4.15) as well as the existence of minimizers is summarized in the following much more general result.

Theorem 1.4.5. *Let $s > 0$ and set $2_s^* := \frac{4N+2s}{2N-4+s}$. Then we have*

$$\mathcal{S}_s(\mathbb{R}^N_+) := \inf_{u \in C^1_c(\mathbb{R}^N_+)} \frac{\displaystyle\int_{\mathbb{R}^N_+} \sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2 \, dx}{\left(\int_{\mathbb{R}^N_+} |u|^{2_s^*} \, dx\right)^{\frac{2}{2_s^*}}} > 0. \tag{1.4.16}$$

Moreover, the value $\mathcal{S}_s(\mathbb{R}^N_+)$ is attained in $H_s \setminus \{0\}$, where H_s denotes the closure of $C^1_c(\mathbb{R}^N_+)$ in the space

$$\left\{ u \in L^{2_s^*}(\mathbb{R}^N_+) : \|u\|^2_{H_s} := \int_{\mathbb{R}^N_+} \sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2 \, dx < \infty \right\} \tag{1.4.17}$$

with respect to the norm $\|\cdot\|_{H_s}$.

Here, distributional derivatives are considered in (1.4.17). The scaling properties of the quotient mentioned above lead to a lack of compactness, necessitating the use of concentration-compactness methods in order to deduce the existence of minimizers. Moreover, we also point out that the case $s > 0$ greatly differs from the case $s = 0$, since it is known that the classical Sobolev inequality only admits extremal functions in the entire space $\mathbb{R}^N$. For $s > 0$, however, minimizers also exist for the half space problem, as stated above. This is related to the fact that the weight x_1^s causes the quotient to lose its invariance with respect to translations in the x_1-direction.

As outlined above, the case $s = 1$ in Theorem 1.4.5 is intimately connected to the characterization of $\mathscr{C}_{1,m,p}(\mathbf{B})$. The same arguments can also be used, however, to give an analogue of Theorem 1.4.3 on annuli with outer radius equal to 1. The more general case $s \in (0, 2]$ can be used to treat variants of (1.4.5) in the context of Riemannian models with boundary instead of $\mathbf{B}$. In particular, this includes hypersurfaces of revolution with boundary in $\mathbb{R}^{N+1}$ such as hemispheres. In fact, the study of (1.4.5) on hemispheres allows us to relate our results to works of Taylor [**130**] and Mukherjee [**103**, **104**] on rotating solutions on the unit sphere. Their works rely on Fourier analytic and pseudodifferential arguments in order to prove different degenerate Sobolev embeddings, for which we can give new proofs.

While Theorem 1.4.3 gives detailed information on symmetry breaking for fixed m and large α and p, it is not clear if nonradial ground states may exist for p close to 2. To this end, we have the following result for large m.

Theorem 1.4.6. *Let $\alpha \in (0, 1)$ and $2 < p < 2^*$. Then there exists $m_0 > 0$ with the property that (1.4.10) holds for $m \geq m_0$ and therefore every ground state solution of (1.4.5) is x_1-x_2-nonradial for $m \geq m_0$.*

The proof is based on a rescaling of functions $u \in H^1_0(\mathbf{B})$ by setting $u_\varepsilon(x) := u(\varepsilon x)$. This gives a function $u_\varepsilon \in H^1_0(B_{1/\varepsilon})$ where $B_{1/\varepsilon} := B_{1/\varepsilon}(0)$ which can be used to prove the identity

$$\begin{aligned}\mathscr{C}_{\alpha\varepsilon,1,p}(B_{1/\varepsilon}) &:= \inf_{v \in H^1_0(B_{1/\varepsilon}) \setminus \{0\}} \frac{\int_{B_{1/\varepsilon}} \left(|\nabla v|^2 - \alpha^2 \varepsilon^2 |\partial_\theta v|^2 + v^2\right) dx}{\left(\int_{B_{1/\varepsilon}} |v|^p \, dx\right)^{\frac{2}{p}}} \\ &= \varepsilon^{2-N+\frac{2N}{p}} \mathscr{C}_{\alpha,\frac{1}{\varepsilon^2},p}(\mathbf{B}).\end{aligned}$$

Consequently, it suffices to show that functions $v_\varepsilon \in H_0^1(B_{1/\varepsilon}) \setminus \{0\}$ attaining $\mathscr{C}_{\alpha\varepsilon,1,p}(B_{1/\varepsilon})$ are nonradial for sufficiently small ε. If this is not the case, the inclusion $H_0^1(B_{1/\varepsilon}) \subset H^1(\mathbb{R}^N)$ allows us to deduce

$$\begin{aligned}\mathscr{C}_{\alpha\varepsilon,1,p}(B_{1/\varepsilon}) &= \frac{\int_{B_{1/\varepsilon}} \left(|\nabla v_\varepsilon|^2 + v^2\right) dx}{\left(\int_{B_{1/\varepsilon}} |v|^p \, dx\right)^{\frac{2}{p}}} \\ &\geq \inf_{v \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} \left(|\nabla v|^2 + v^2\right) dx}{\left(\int_{\mathbb{R}^N} |v|^p \, dx\right)^{\frac{2}{p}}} =: \mathscr{C}_{0,1,p}(\mathbb{R}^N).\end{aligned}$$

This inequality can then be brought to a contradiction by constructing suitable nonradial functions in $H_0^1(B_{1/\varepsilon} \setminus \{0\})$ to estimate $\mathscr{C}_{\alpha\varepsilon,1,p}(B_{1/\varepsilon})$. Recalling that $\mathscr{C}_{0,1,p}(\mathbb{R}^N)$ is attained by any translation of the unique positive radial solution $\tilde{u}_0 \in H^1(\mathbb{R}^N)$ of the nonlinear Schrödinger equation

$$-\Delta u + u = |u|^{p-2}u \qquad \text{in } \mathbb{R}^N,$$

we consider a translation of $\tilde{u}_0$ in the x_1 direction and multiply with a suitable cutoff function. This yields a nonradial function $w_\varepsilon \in H_0^1(B_{1/\varepsilon})$. Using the fact that $\tilde{u}_0$ has exponential decay, it can be shown that

$$\frac{\int_{B_{1/\varepsilon}} \left(|\nabla w_\varepsilon|^2 + w^2\right) dx}{\left(\int_{B_{1/\varepsilon}} |w|^p \, dx\right)^{\frac{2}{p}}} \leq \mathscr{C}_{0,1,p}(\mathbb{R}^N) + C_1 e^{-\frac{\delta}{\varepsilon}}$$

with constants $C_1, \delta > 0$ which are independent of ε. On the other hand, there exists $C_2 > 0$ such that

$$\frac{\int_{B_{1/\varepsilon}} |\partial_\theta w_\varepsilon|^2 \, dx}{\left(\int_{B_{1/\varepsilon}} |w_\varepsilon|^p \, dx\right)^{\frac{2}{p}}} \geq C_2 \varepsilon^2$$

holds for $\varepsilon > 0$, which implies

$$\mathscr{C}_{0,1,p}(\mathbb{R}^N) \leq \mathscr{C}_{\alpha\varepsilon,1,p}(B_{1/\varepsilon}) \leq \mathscr{C}_{0,1,p}(\mathbb{R}^N) - C_2\varepsilon^2 + C_1 e^{-\frac{\delta_1}{\varepsilon}}$$

and the right hand side of this inequality is strictly smaller than $\mathscr{C}_{0,1,p}(\mathbb{R}^N)$ if $\varepsilon > 0$ is sufficiently small. This yields a contradiction and thus completes the proof.

Finally, we discuss the limit case $\alpha = 1$ in the minimization problem (1.4.7). Since

$$u \mapsto \left(\int_{\mathbf{B}} \left(|\nabla u|^2 - |\partial_\theta u|^2\right) \, dx\right)^{\frac{1}{2}}$$

does not define an equivalent norm on $H_0^1(\mathbf{B})$, we need to consider the larger space $\mathcal{H}(\mathbf{B})$ which is given as closure of $C_c^1(\mathbf{B})$ in

$$\left\{u \in L^{2_1^*}(\mathbf{B}) : \|u\|_{\mathcal{H}(\mathbf{B})}^2 := \int_{\mathbf{B}} \left(|\nabla u|^2 - |\partial_\theta u|^2\right) \, dx < \infty\right\}$$

with respect to the norm $\|\cdot\|_{\mathcal{H}(\mathbf{B})}$. We then have the following result, which complements Theorems 1.4.3 and 1.4.6 by extending the arguments used there to the case $\alpha = 1$.

Theorem 1.4.7. *Let $2 < p < 2_1^*$ and $\alpha = 1$.*

(i) For every $m > -\lambda_1(\mathbf{B})$, there exists a ground state solution of (1.4.5).

(ii) There exists $m_0 > 0$ with the property that (1.4.10) holds for $m \geq m_0$ and therefore every ground state solution $u \in \mathcal{H}(\mathbf{B})$ of (1.4.5) is x_1-x_2-nonradial for $m \geq m_0$.

Since the embedding $\mathcal{H} \hookrightarrow L^{2_1^*}(\mathbf{B})$ is not compact, the existence of ground states in the critical case $\alpha = 1$, $p = 2_1^*$ is not clear. We have a partial result on the existence of ground state solutions which relates problem (1.4.5) to the optimal constant for (1.4.15) given by (1.4.16).

Theorem 1.4.8. *If*

$$\mathscr{C}_{1,m,2_1^*}(\mathbf{B}) < 2^{\frac{1}{2}-\frac{1}{2_1^*}} \mathcal{S}_1(\mathbb{R}^N_+) \tag{1.4.18}$$

for some $m > -\lambda_1(\mathbf{B})$, then the value $\mathscr{C}_{1,m,2_1^*}(\mathbf{B})$ is attained in $\mathcal{H}(\mathbf{B}) \setminus \{0\}$ by a ground state solution of (1.4.5). Moreover, there exists $\varepsilon > 0$ with the property that (1.4.18) holds for every $m \in (-\lambda_1(\mathbf{B}), -\lambda_1(\mathbf{B}) + \varepsilon)$.

This result crucially relies on the fact that the ground state energy and $\mathcal{S}_1(\mathbb{R}^N_+)$ are related via (1.4.14), which allows us to estimate the energies of minimizing sequences. The additional factor $2^{1/2-1/2_1^*}$ in (1.4.18) is then due to the factor 2 appearing in the right hand side of (1.4.14), which is not present in the definition of $\mathcal{S}_1(\mathbb{R}^N_+)$.

1.5 A Mixed-Type Operator with Applications to Rotating Waves

Since the previous results give extensive information on rotating wave solutions of (1.4.3) with angular velocity $\alpha \leq 1$, it is natural to ask what happens in the case $\alpha > 1$. The article **[P4]** discusses this question in detail for the case $N = 2$.

Recall that the rotating wave ansatz (1.4.4) reduced the nonlinear wave equation (1.4.3) to

$$\begin{cases} -\Delta u + \alpha^2 \partial_\theta^2 u + mu = |u|^{p-2}u & \text{in } \mathbf{B}, \\ u = 0 & \text{on } \partial\mathbf{B}, \end{cases} \tag{1.5.1}$$

where $\partial_\theta = x_1\partial_{x_2} - x_2\partial_{x_1}$ denotes the angular derivative, $2 < p < \infty$, $m \in \mathbb{R}$ and $\mathbf{B} \subset \mathbb{R}^2$ now denotes the unit disk. Again, we are interested in the existence of nonradial solutions of (1.5.1).

Compared to the previous section, however, the variational structure of the problem is vastly different for $\alpha > 1$, as the quotient $R_{\alpha,m,p}$ is no longer bounded from below on $C_c^1(\mathbf{B})$. Indeed, taking a sequence of functions of the form $u_k(x) = \varphi(|x|)Y_k(\theta)$, where $\varphi \in C_c^1((\frac{1}{\alpha}, 1))$ and $Y_k(\theta) = \sin(k\theta)$, we find that $R_{\alpha,m,p}(u_k) \to -\infty$ as $k \to \infty$. Consequently, we can no longer define ground state solutions as minimizers of $R_{\alpha,m,p}$ and it is in fact unclear if ground states even exist in a suitable sense or how they could be characterized.

These difficulties are essentially rooted in the fact that the operator

$$L_\alpha := -\Delta + \alpha^2 \partial_\theta^2$$

is no longer (degenerate) elliptic for $\alpha > 1$. Indeed, note that in polar coordinates $(r, \theta) \in (0, 1) \times (-\pi, \pi)$ we have

$$L_\alpha u = -\partial_r^2 u - \frac{1}{r}\partial_r u - \left(\frac{1}{r^2} - \alpha^2\right) \partial_\theta^2 u$$

and the term $1/r^2 - \alpha^2$ is clearly sign-changing. More specifically, L_α is elliptic in $B_{1/\alpha}(0)$, parabolic on $\mathbb{S}^1_{1/\alpha}(0)$ and hyperbolic in $\mathbf{B} \setminus \overline{B_{1/\alpha}(0)}$. Consequently, L_α is of *mixed-type* for

$\alpha > 1$. This complicates the situation tremendously, since the general theory for such operators is much less comprehensive and even the associated linear problems are significantly less well understood. Consequently, very few results on variational approaches are available, and solutions are often found by treating the different regions separately. In particular, the use of direct variational methods will be met with several obstacles, due to the structure of the associated energy functional, as we will discuss below.

In order to overcome these difficulties, we first study the spectrum of L_α in detail. It turns out that the Dirichlet eigenvalues of L_α are given by

$$j_{\ell,k}^2 - \alpha^2 \ell^2,$$

where $\ell \in \mathbb{N}_0$, $k \in \mathbb{N}$ and $j_{\ell,k}$ denotes the k-th zero of the Bessel function J_ℓ. In order to gain a better understanding of the spectrum, we thus need to analyze the asymptotic behavior of these zeros more carefully. In general, it is unclear whether the spectrum of L_α only consists of isolated points. Initially, we cannot exclude the existence of finite accumulation points or even density in $\mathbb{R}$. These cases could pose a serious obstruction for the use of variational methods.

Based upon a detailed analysis of the asymptotic behavior of different sequences of Bessel function zeros, our first main result characterizes the spectrum of L_α for certain values of α as follows.

Theorem 1.5.1. *For any $\alpha > 1$ the spectrum of L_α is unbounded from above and below. Moreover, there exists an unbounded sequence $(\alpha_n)_n \subset (1, \infty)$ such that the following properties hold for $n \in \mathbb{N}$:*

(i) The spectrum of L_{α_n} consists of eigenvalues with finite multiplicity.

(ii) There exists $c_n > 0$ such that for each $\ell \in \mathbb{N}_0$, $k \in \mathbb{N}$ we either have $j_{\ell,k}^2 - \alpha_n^2 \ell^2 = 0$ or

$$|j_{\ell,k}^2 - \alpha_n^2 \ell^2| \geq c_n j_{\ell,k}. \tag{1.5.2}$$

(iii) The spectrum of L_{α_n} has no finite accumulation points.

The proof is based on several results for the asymptotics of $j_{\ell,k}$ as $\ell, k \to \infty$. More specifically, we first observe that the formula

$$j_{\ell,k}^2 - \alpha^2 \ell^2 = \ell (j_{\ell,k} + \alpha \ell) \left(\frac{j_{\ell,k}}{\ell} - \alpha \right)$$

suggests that a sequence of points $j_{\ell_i,k_i}^2 - \alpha^2 \ell_i^2$ in the spectrum of L_α can remain bounded if and only if $j_{\ell_i,k_i}/\ell_i \to \alpha$. Using known estimates for $j_{\ell,k}$, we then find that this may only occur if $\ell_i/k_i \to \sigma$ for some $\sigma > 0$. Consequently, we need to study such sequences in detail. Firstly, we note that results by Elbert and Laforgia [**50**] imply that for each $\alpha > 1$ there exists a unique $\sigma = \sigma(\alpha) > 0$ such that

$$\frac{j_{\sigma k,k}}{\sigma k} \to \alpha \qquad \text{as } k \to \infty. \tag{1.5.3}$$

If we choose $\alpha > 1$ such that the associated $\sigma > 0$ is a rational number, sequences of the form $\ell_i = \sigma i$, $k_i = i$, $i \in \mathbb{N}$ could therefore be problematic. An important technical step is thus a precise characterization of the order of convergence in (1.5.3) using the Watson formula.

For certain values of α, these arguments give good control over sequences satisfying $\ell_i/k_i = \sigma$ and allow us to exclude accumulation points of the spectrum. Sequences which only satisfy $\ell_i/k_i = \sigma + o(1)$ as $i \to \infty$ on the other hand, could still remain bounded. If such a sequence also satisfies $\ell_i/k_i \leq \sigma$, we can use known estimates for the zeros and the characterization of the order of convergence in (1.5.3) to show that such sequences cannot lead to accumulation points either. Notably, these arguments work for arbitrary $\alpha > 1$.

Overall, this reduces the problem to the case where $\ell_i = \sigma k_i - \delta_i$, with $\delta_i > 0$, $\sigma_i = o(k_i)$. Further estimates based on our characterization of (1.5.3) subsequently yield

$$\liminf_{i\to\infty} j_{\ell_i,k_i} - \alpha\ell_i \geq -c_\sigma + \left(\alpha - \frac{\pi}{2}\right) \inf_{i\in\mathbb{N}} \delta_i$$

with a constant $c_\sigma > 0$, provided $\alpha > \frac{\pi}{2}$. In particular, sufficient control over the term $\inf_{i\in\mathbb{N}} \delta_i$ is required to give a lower estimate. This leads to the following crucial observation: If $\sigma = 1/n$ for some $n \in \mathbb{N}$, the fact that $\ell_i = \sigma k_i - \delta_i$ must be a natural number implies that we must have $\delta_i = n_i'/n$ for some $n_i' \in \mathbb{N}$ since $k_i \in \mathbb{N}$. Consequently, we then have $\inf_{i\in\mathbb{N}} \delta_i \geq 1/n$ for $\sigma = 1/n$ and thus

$$\liminf_{i\to\infty} j_{\ell_i,k_i} - \alpha\ell_i \geq -c_{1/n} + \left(\alpha_n - \frac{\pi}{2}\right) \frac{1}{n}, \tag{1.5.4}$$

where $\alpha_n > 1$ is then uniquely determined by $\sigma = 1/n$, as mentioned above. Using an explicit characterization of α in terms of a transcendental equation, we ultimately find that the right hand side in (1.5.4) is positive for large n, which yields the claim.

We note that this argument can be extended to some other rational values of σ, while the behavior of the spectrum for irrational σ remains open in general. In earlier treatments of the wave operator similar phenomena can be found, where the solvability of the radially symmetric periodic Dirichlet-problem in balls is intimately connected to the arithmetical properties of the period length (see e.g. [**17**, **95**]), though similar observations for the one dimensional problem go back to the work of Borel [**20**] in 1895.

Importantly, Theorem 1.5.1 provides a characterization of the spectrum of L_α which allows us to set up a variational framework in order to find solutions. We consider the space

$$E_{\alpha,m} := \left\{ u \in L^2(\mathbf{B}) : \sum_{\ell=0}^{\infty}\sum_{k=1}^{\infty} \left| j_{\ell,k}^2 - \alpha^2\ell^2 + m \right| \left(|\langle u, \varphi_{\ell,k}\rangle|^2 + |\langle u, \psi_{\ell,k}\rangle|^2 \right) < \infty \right\},$$

so that the quadratic form

$$u \mapsto \int_{\mathbf{B}} \left(|\nabla u|^2 - \alpha^2 |\partial_\theta u|^2 + mu^2 \right) \, dx \tag{1.5.5}$$

is well-defined on $E_{\alpha,m}$. Crucially, the estimate (1.5.2) allows us to deduce that $E_{\alpha,m}$ compactly embeds into $L^p(\mathbf{B})$ for $p \in (2,4)$. More specifically, this is a consequence of fractional Sobolev embeddings since (1.5.2) implies $E_{\alpha,m} \hookrightarrow H_0^{1/2}(\mathbf{B})$.

For $p \in (2,4)$, weak solutions of (1.5.1) can therefore be found as critical points of the energy functional $\Phi_{\alpha,m} : E_{\alpha,m} \to \mathbb{R}$ given by

$$\Phi_{\alpha,m}(u) := \frac{1}{2} \int_{\mathbf{B}} \left(|\nabla u|^2 - \alpha^2 |\partial_\theta u|^2 + mu^2 \right) \, dx - \frac{1}{p} \int_{\mathbf{B}} |u|^p \, dx.$$

In the case $\alpha \in [0,1]$, such critical points could be found via the mountain pass lemma or by minimizing over the Nehari manifold, which is equivalent to minimizing the Rayleigh functional as we had done for the arguments in Section 1.4. In the case $\alpha > 1$, however, $\Phi_{\alpha,m}$ no longer possesses a mountain pass structure. This is due to the fact that the associated quadratic form (1.5.5) is *strongly indefinite*, i.e., it is negative on an infinite dimensional subspace.

We note that similar functionals appear in the classical study of periodic solutions of nonlinear wave equations in one dimension (see e.g. [**125**, Chapter I.6] and the references therein), as well as nonlinear stationary Schrödinger equations on $\mathbb{R}^N$. The existence of nontrivial solutions for the latter has been treated extensively via dual variational methods [**2**] or more abstract critical point theory [**9**, **25**]. In our case, however, the existence of nonzero solutions to (1.5.1) is already known since the equation is satisfied by radial solutions. Hence we require a setting which allows for the distinction of such solutions.

In order to overcome this difficulty, we first consider the subspaces spanned by positive, zero and negative eigenvalues denoted by $E^+_{\alpha,m}$, $E^0_{\alpha,m}$, $E^-_{\alpha,m}$, respectively. Note that Theorem 1.5.1 implies that $E^0_{\alpha,m}$ is finite dimensional for $\alpha = \alpha_n$. In fact, the subsequent results hold for any $\alpha > 1$ such that an estimate of the form (1.5.2) holds and $E^0_{\alpha,m}$ is finite dimensional.

In the following, we set $F_{\alpha,m} := E^0_{\alpha,m} \oplus E^-_{\alpha,m}$ and employ the methods of Szulkin and Weth [**128**], who studied the following generalized Nehari manifold which had been introduced by Pankov [**112**]:

$$\mathcal{N}_{\alpha,m} := \left\{ u \in E_{\alpha,m} \setminus F_{\alpha,m} : \ \Phi'_{\alpha,m}(u)u = 0 \text{ and } \Phi'_{\alpha,m}(u)v = 0 \text{ for all } v \in F_{\alpha,m} \right\}.$$

In particular, $\mathcal{N}_{\alpha,m}$ contains every critical point of $\Phi_{\alpha,m}$. It can then be shown that the infimum

$$c_{\alpha,m} = \inf_{u \in \mathcal{N}_{\alpha,m}} \Phi_{\alpha,m}(u)$$

is positive and attained by a critical point of $\Phi_{\alpha,m}$ if $\alpha = \alpha_n$, where α_n was given in Theorem 1.5.1 and $m \in \mathbb{R}$. In particular, this allows us to define *ground states* of (1.5.1) as such minimizers. Our second main result states that (1.5.1) has nonradial ground state solutions for certain choices of parameters.

Theorem 1.5.2. *Let $p \in (2,4)$ and let the sequence $(\alpha_n)_n \subset (1,\infty)$ be given by Theorem 1.5.1. Then the following properties hold:*

(i) For any $n \in \mathbb{N}$ and $m \in \mathbb{R}$ there exists a ground state solution of (1.5.1) *for $\alpha = \alpha_n$.*

(ii) For any $n \in \mathbb{N}$ there exists $m_n > 0$ such that the ground state solutions of (1.5.1) *are nonradial for $\alpha = \alpha_n$ and $m > m_n$.*

The proof relies on two ingredients. Firstly, the value $c_{\alpha,m}$ admits a minimax characterization given by

$$c_{\alpha,m} = \inf_{w \in E_{\alpha,m} \setminus F_{\alpha,m}} \max_{w \in \widehat{E}_{\alpha,m}(u)} \Phi_{\alpha,m}(w), \tag{1.5.6}$$

where we set

$$\widehat{E}_{\alpha,m}(u) := \{tu + w : \ t \geq 0, \ w \in F_{\alpha,m}\} = \mathbb{R}^+ u \oplus F_{\alpha,m}$$

for $u \in E_{\alpha,m} \setminus F_{\alpha,m}$. This allows us to compare the asymptotics of the ground state energy to the radial energy more directly. To this end, we let β^{rad}_m denote the energy of the unique positive radial solution $u_m \in H^1_{0,rad}(\mathbf{B})$ of (1.5.1) and show that there exists $c > 0$ such that

$$\beta^{rad}_m \geq c m^{\frac{2}{p-2}} \tag{1.5.7}$$

holds for all $\alpha > 1$ and $m \geq 0$, based on a rescaling argument similar to the proof of Theorem 1.4.6.

On the other hand, (1.5.6) implies

$$c_{\alpha,m} \le \max_{w \in \widehat{E}_{\alpha,m}(u)} \Phi_{\alpha,m}(w)$$

for any $w \in E_{\alpha,m} \setminus F_{\alpha,m}$. In particular, we can choose w to be an eigenfunction of $-\Delta$ such that the associated eigenvalue of $-\Delta + \alpha^2\partial_\theta^2 + m$ given by $j_{\ell,k}^2 - \alpha^2\ell^2 + m$ is positive. A further analysis of the zeros $j_{\ell,1}$ then shows that we can always find $\ell \in \mathbb{N}$ such that

$$0 < j_{\ell,1}^2 - \alpha^2\ell^2 + m \le Cm^{\frac{1}{2}}$$

holds, and consequently

$$c_{\alpha,m} \le \left(\frac{1}{2} - \frac{1}{p}\right) |\mathbf{B}| C m^{\frac{p}{2(p-2)}}.$$

Since $p < 4$, the estimate (1.5.7) finally yields

$$c_{\alpha,m} < \beta_m^{rad}$$

for sufficiently large m as claimed.

CHAPTER 2

Spectral Asymptotics of Radial Solutions and Nonradial Bifurcation for the Hénon Equation

In this chapter, we present our results concerning nonradial bifurcation for the Hénon equation as outlined in Section 1.2. Up to minor changes, the subsequent content has appeared in **[P1]**.

2.1 Introduction

We consider the Dirichlet problem for the generalized Hénon equation

$$\begin{cases} -\Delta u = |x|^{\alpha}|u|^{p-2}u & \text{in } \mathbf{B}, \\ \quad u = 0 & \text{on } \partial\mathbf{B}, \end{cases} \tag{2.1.1}$$

where $\mathbf{B} \subset \mathbb{R}^N$ is the unit ball and $p > 2$, $\alpha > 0$. Here and in the following, we assume that $N \geq 3$, whereas the planar case $N = 2$ will be discussed separately in Section 2.6 below. The equation in (2.1.1) originally arose through the study of stellar clusters in [**71**]. One of the first results on (2.1.1) is due to Ni [**108**], who proved the existence of a positive radial solution in the subcritical range of exponents $2 < p < 2^*_\alpha$, where $2^*_\alpha := \frac{2N+2\alpha}{N-2}$. In another seminal paper, Smets, Willem and Su [**122**] observed that symmetry breaking occurs for fixed p and large α, i.e., there exists $\alpha^* > 0$ depending on p such that ground state solutions of (2.1.1) are nonradial for $\alpha > \alpha^*$. In the sequel, the existence and shape of radial and nonradial solutions of the Hénon equation has received extensive attention, see e.g. [**4–6**, **27**, **28**, **31**, **89**, **113**, **119**, **121**]. In particular, bifurcation of nonradial positive solutions in the parameter p is studied in [**4**] for fixed $\alpha > 0$. Moreover, a related critical parameter-dependent equation on $\mathbb{R}^N$ is considered in [**65**]. The main motivation for the present paper is the investigation of bifurcation of nonradial nodal (i.e., sign changing) solutions – in the parameter $\alpha > 0$ – from the set of radial nodal solutions. To explain this in more detail, let us fix $K \in \mathbb{N}$, an exponent $p > 2$ and consider

$$\alpha > \alpha_p := \max\left\{\frac{(N-2)p - 2N}{2}, 0\right\},$$

which amounts to the subcriticality condition $p < 2^*_\alpha$. Under these assumptions, it has been proved by Nagasaki [**106**] that (2.1.1) admits a unique classical radial solution $u_\alpha \in C^2(\overline{\mathbf{B}})$ with $u_\alpha(0) > 0$ and with precisely K nodal domains (i.e., $K - 1$ zeros in the radial variable $r = |x| \in (0, 1)$). In order to decide whether the branch $\alpha \to u_\alpha$ admits bifurcation of nonradial solutions for large α, we need to analyze its spectral asymptotics as $\alpha \to \infty$. More precisely, we wish to derive asymptotic expansions of the eigenvalues of the linearizations of (2.1.1) at u_α as $\alpha \to \infty$. For this we consider the linearized operators

$$\varphi \mapsto L^\alpha \varphi := -\Delta\varphi - (p-1)|x|^\alpha |u_\alpha|^{p-2}\varphi, \qquad \alpha > \alpha_p, \tag{2.1.2}$$

which are self-adjoint operators in $L^2(\mathbf{B})$ with compact resolvent, domain $H^2(\mathbf{B}) \cap H^1_0(\mathbf{B})$ and form domain $H^1_0(\mathbf{B})$. In particular, they are Fredholm operators of index zero.

As usual, u_α is called nondegenerate if $L^\alpha : H^2(\mathbf{B}) \cap H^1_0(\mathbf{B}) \to L^2(\mathbf{B})$ is an isomorphism, which amounts to the property that the equation $L^\alpha\varphi = 0$ only has the trivial solution $\varphi = 0$ in $H^2(\mathbf{B}) \cap H^1_0(\mathbf{B})$. Otherwise, u_α is called degenerate. By a classical observation, only values α such that u_α is degenerate can give rise to bifurcation from the branch $\alpha \mapsto u_\alpha$. Moreover, properties of the kernel of L^α and the change of the Morse index are of key importance to establish bifurcation. Here we recall that the Morse index of u_α is defined as the number of negative eigenvalues of the operator L^α.

The first step in deriving asymptotic spectral information of the operator family L^α, $\alpha > \alpha_p$ is to characterize the limit shape of the solutions u_α after suitable transformations. Inspired by Byeon and Wang [**27**], we transform the radial variable and derive a corresponding limit problem. Here, for simplicity, we also regard $u_\alpha = u_\alpha(r)$ as a function of the radial variable $r = |x| \in [0, 1]$. Our first preliminary result is the following.

Proposition 2.1.1. *Let $p > 2$, $K \in \mathbb{N}$. Moreover, for $\alpha > \alpha_p$, let u_α denote the unique radial solution of (2.1.1) with K nodal domains and $u_\alpha(0) > 0$, and define*

$$U_\alpha : [0, \infty) \to \mathbb{R}, \qquad U_\alpha(t) = (N+\alpha)^{-\frac{2}{p-2}}\, u_\alpha(e^{-\frac{t}{N+\alpha}}). \tag{2.1.3}$$

Then $U_\alpha \to (-1)^{K-1} U_\infty$ uniformly on $[0,\infty)$ as $\alpha \to \infty$, where $U_\infty \in C^2([0,\infty))$ is characterized as the unique bounded solution of the limit problem

$$-U'' = e^{-t}|U|^{p-2}U \quad \text{in } [0,\infty), \qquad U(0) = 0 \tag{2.1.4}$$

with $U'(0) > 0$ and with precisely $K-1$ zeros in $(0,\infty)$.

The asymptotic description derived in Proposition 2.1.1 implies that the solutions u_α blow up everywhere in $\mathbf{B}$ as $\alpha \to \infty$, in contrast to the nonradial ground states considered in [**122**]. It is therefore reasonable to expect that the Morse index of u_α tends to infinity as $\alpha \to \infty$. This fact has been proved recently and independently for more general classes of problems in [**5**, **89**], extending a result for the case $N = 2$ given in [**102**]. To obtain a more precise description of the distribution of eigenvalues of L^α as $\alpha \to \infty$, we rely on complementary approaches of [**5**, **89**] and implement new tools. We note here that [**89**] uses the transformation (2.1.3) in a more general context together with Liouville type theorems for limiting problems on the half line. In the present paper, we build on very useful results obtained recently by Amadori and Gladiali in [**5**]. In particular, we use the fact that the Morse index of u_α equals the number of negative eigenvalues (counted with multiplicity) of the weighted eigenvalue problem

$$L^\alpha\varphi = \frac{\lambda}{|x|^2}\varphi, \qquad \varphi \in H^1_0(\mathbf{B}), \tag{2.1.5}$$

see [**5**, Prop. 5.1]. In various special cases, this observation had already been used before, see e.g. [**44**, Section 5]. In order to avoid regularity issues related to the singularity of the weight $\frac{1}{|x|^2}$, it is convenient to consider (2.1.5) in weak sense via the quadratic form q_α associated with L^α, see Section 2.3 below. The problem (2.1.5) is easier to analyze than the standard eigenvalue problem $L^\alpha \varphi = \lambda\varphi$ without weight. Indeed, every eigenfunction of (2.1.5) is a sum of functions of the form

$$x \mapsto \varphi(x) = \psi(x) Y_\ell \left(\frac{x}{|x|}\right), \tag{2.1.6}$$

where $\psi \in H^1_{0,rad}(\mathbf{B})$ and Y_ℓ is a spherical harmonic of degree ℓ, see [**5**, Prop. 4.1]. Here $H^1_{0,rad}(\mathbf{B})$ denotes the space of radial functions in $H^1_0(\mathbf{B})$. We recall that the space of spherical harmonics of degree $\ell \in \mathbb{N} \cup \{0\}$ has dimension $d_\ell := \binom{N+\ell-1}{N-1} - \binom{N+\ell-3}{N-1}$, and that every such spherical harmonic is an eigenfunction of the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^{N-1}$ corresponding to the eigenvalue $\lambda_\ell := \ell(\ell + N - 2)$. For functions φ of the form (2.1.6), the eigenvalue problem (2.1.5) reduces to an eigenvalue problem for radial functions given by

$$L^\alpha \psi = \frac{\mu}{|x|^2}\psi, \qquad \psi \in H^1_{0,rad}(\mathbf{B}), \tag{2.1.7}$$

where $\mu = \lambda - \lambda_\ell$. In [**5**, p.19 and Prop. 3.7], it has been proved that (2.1.7) admits precisely K negative eigenvalues

$$\mu_1(\alpha) < \mu_2(\alpha) < \cdots < \mu_K(\alpha) < 0 \qquad\qquad \text{for } \alpha > \alpha_p. \tag{2.1.8}$$

Combining this fact with the observations summarized above, one may then derive the following facts which we cite here in a slightly modified form from [**5**].

Proposition 2.1.2. *(see [**5**, Prop. 1.3 and 1.4]) Let $p > 2$ and $\alpha > \alpha_p$. Then the Morse index of u_α is given by*

$$m(u_\alpha) = \sum_{(i,\ell)\in E^-} d_\ell,$$

where E^- denotes the set of pairs (i, ℓ) with $i \in \mathbb{N}$, $\ell \in \mathbb{N} \cup \{0\}$ and $\mu_i(\alpha) + \lambda_\ell < 0$. Moreover, u_α is nondegenerate if and only if

$$\mu_i(\alpha) + \lambda_\ell \neq 0 \qquad \text{for every } i \in \{1, \dots, K\},\ \ell \in \mathbb{N} \cup \{0\}.$$

In order to describe the asymptotic distribution of negative eigenvalues of L^α, it is essential to study the asymptotics of the eigenvalues $\alpha \mapsto \mu_i(\alpha)$, $i = 1, \dots, K$. With regard to this aspect, we mention the estimate

$$\mu_i(\alpha) < -\frac{(\alpha+2)\big(\alpha + 2(N-1)\big)}{4} \qquad \text{for } \alpha > \alpha_p,\ i = 1, \dots, K-1, \tag{2.1.9}$$

which has been derived in [**5**, Lemma 5.11 and Remark 5.12]. In particular, it follows that $\mu_i(\alpha) \to -\infty$ as $\alpha \to \infty$ for $i = 1, \dots, K-1$. In our first main result, we complement this estimate by deriving asymptotics for $\mu_i(\alpha)$.

Theorem 2.1.3. *Let $p > 2$ and $\alpha > \alpha_p$. Then the negative eigenvalues of (2.1.7) are given as C^1-functions $(\alpha_p, \infty) \to \mathbb{R}$, $\alpha \mapsto \mu_i(\alpha)$, $i = 1, \dots, K$ satisfying the asymptotic expansions*

$$\mu_i(\alpha) = \nu_i^* \alpha^2 + c_i^* \alpha + o(\alpha) \quad \text{and} \quad \mu_i'(\alpha) = 2\nu_i^*\alpha + c_i^* + o(1) \qquad \text{as } \alpha \to \infty, \tag{2.1.10}$$

where c_i^*, $i = 1, \dots, K$ are constants and the values $\nu_1^* < \nu_2^* < \cdots < \nu_K^* < 0$ are precisely the negative eigenvalues of the eigenvalue problem

$$\begin{cases} -\Psi'' - (p-1)e^{-t}|U_\infty(t)|^{p-2}\Psi = \nu\Psi & \text{in } [0,\infty), \\ \quad \Psi(0) = 0, \quad \Psi \in L^\infty(0,\infty), \end{cases} \tag{2.1.11}$$

with U_∞ given in Proposition 2.1.1. In particular, there exists $\alpha^* > 0$ such that the curves μ_i, $i = 1, \dots, K$ are strictly decreasing on $[\alpha^*, \infty)$.

Remark 2.1.4. *The strict monotonicity of the curves μ_i on $[\alpha^*, \infty)$ will be of key importance for the derivation of bifurcation of nonradial solutions via variational bifurcation theory. For this we require the derivative expansion in (2.1.10), but we do not need additional information on the constants c_i^* since $\nu_i^* < 0$ for $i = 1, \dots, K$. Our proof of (2.1.10) gives rise to the following characterization of the constants c_i^*: For fixed $i \in \{1, \dots, K\}$, we have*

$$c_i^* = -(2N\nu_i^* + N - 2)(p-1)\int_0^\infty \left(te^{-t}|U_\infty|^{p-2}\Psi^2 + (p-2)e^{-t}|U_\infty|^{p-4}U_\infty V\Psi^2\right)\,dt,$$

where U_∞ is given in Proposition 2.1.1, V is the unique bounded solution of the problem

$$-V'' - (p-1)e^{-t}|U_\infty|^{p-2}V = U_\infty' - te^{-t}|U_\infty|^{p-2}U_\infty \quad \text{in } [0,\infty), \qquad V(0) = 0$$

and Ψ is the (up to sign unique) eigenfunction of (2.1.11) associated with the eigenvalue ν_i^* with $\int_0^\infty \Psi^2\,dt = 1$.

The strict monotonicity of the curves μ_i for large α asserted in Theorem 2.1.3 allows us to deduce the following useful properties related to nondegeneracy and a change of the Morse index of the functions u_α.

Corollary 2.1.5. *Let $p > 2$. For every $i \in \{1, \dots, K\}$, there exist $\ell_i \in \mathbb{N} \cup \{0\}$ and sequences of numbers $\alpha_{i,\ell} \in (\alpha_p, \infty)$, $\varepsilon_{i,\ell} > 0$, $\ell \geq \ell_i$ with the following properties:*

(i) $\alpha_{i,\ell} \to \infty$ *as* $\ell \to \infty$.

(ii) $\mu_i(\alpha_{i,\ell}) + \lambda_\ell = 0$. *In particular,* $u_{\alpha_{i,\ell}}$ *is degenerate.*

(iii) u_α *is nondegenerate for* $\alpha \in (\alpha_{i,\ell} - \varepsilon_{i,\ell}, \alpha_{i,\ell} + \varepsilon_{i,\ell})$, $\alpha \neq \alpha_{i,\ell}$.

(iv) For $\varepsilon \in (0, \varepsilon_{i,\ell})$ *the Morse index of* $u_{\alpha_{i,l}+\varepsilon}$ *is strictly larger than the Morse index of* $u_{\alpha_{i,l}-\varepsilon}$.

With the help of Corollary 2.1.5 and an abstract bifurcation result in [77], we will derive our second main result on the bifurcation of nonradial solutions from the branch $\alpha \mapsto u_\alpha$.

Theorem 2.1.6. *Let $2 < p < \frac{2N}{N-2}$, and let $K \in \mathbb{N}$, $i \in \{1, \dots, K\}$ be fixed. Then the points $\alpha_{i,\ell}$, $\ell \geq \ell_i$ are bifurcation points for nonradial solutions of (2.1.1).*

More precisely, for every $\ell \geq \ell_i$, there exists a sequence $(\alpha_n, u^n)_n$ in $(0,\infty) \times C^2(\overline{\mathbf{B}})$ with the following properties:

(i) $\alpha_n \to \alpha_{i,\ell}$, *and* $u^n \to u_{\alpha_{i,\ell}}$ *in* $C^2(\overline{\mathbf{B}})$.

(ii) For every $n \in \mathbb{N}$, u^n is a nonradial solution of (2.1.1) with $\alpha = \alpha_n$ having precisely K nodal domains $\Omega_1, \dots, \Omega_K$ such that $0 \in \Omega_1$, Ω_1 is homeomorphic to a ball and $\Omega_2, \dots, \Omega_K$ are homeomorphic to annuli.

Here, $\ell_i \in \mathbb{N} \cup \{0\}$ and the values $\alpha_{i,\ell}$ are given in Corollary 2.1.5.

As mentioned above, Theorem 2.1.6 will be derived from Corollary 2.1.5 and variational bifurcation theory. For this we reformulate (2.1.1) as a bifurcation equation in the Hilbert space $H_0^1(\mathbf{B})$ and show that, as a consequence of Corollary 2.1.5, the crossing number of an associated operator family is nonzero at the points $\alpha_{i,\ell}$. Thus the main theorem in [77] applies and yields that the points $\alpha_{i,\ell}$, $\ell \geq \ell_i$ are bifurcation points for solutions of (2.1.1) along the branch $\alpha \mapsto u_\alpha$. To see that bifurcation of *nonradial* solutions occurs, it suffices to note that the solutions u_α are radially nondegenerate for $\alpha > 0$, i.e., the kernel of L^α does not contain radial functions. A proof of the latter fact can be found in [**5**, Theorem 1.7], and it also follows from results in [**137**].

Since Corollary 2.1.5 is a rather direct consequence of Theorem 2.1.3, the major part of this paper is concerned with the proofs of Proposition 2.1.1 and Theorem 2.1.3. It is not difficult to see that, via the transformation given in (2.1.3), the Hénon equation (2.1.1) transforms into a family of problems depending on the new parameter $\gamma = \frac{N-2}{N+\alpha}$ which admits a well-defined limit problem as $\gamma \to 0^+$ given by (2.1.4). It is then necessary to choose a proper function space which allows to apply the implicit function theorem at $\gamma = 0$, and this yields the convergence statement in Proposition 2.1.1. The idea of the proof of Theorem 2.1.3 is similar, as we use the same transformation (up to scaling) to rewrite the α-dependent eigenvalue problem (2.1.7) as a γ-dependent eigenvalue problem on the interval $[0, \infty)$. We shall then see that (2.1.11) arises as the limit of the transformed eigenvalue problems as $\gamma \to 0^+$. In order to obtain C^1-expansions of eigenvalue curves, we wish to apply the implicit function theorem again at the point $\gamma = 0$. Here a major difficulty arises in the case where $p \in (2, 3]$, as the map $U \mapsto |U|^{p-2}$ fails to be differentiable between standard function spaces. We overcome this problem by restricting this map to the subset of C^1-functions on $[0, \infty)$ having only a finite number of simple zeros and by considering its differentiability with respect to a weighted uniform L^1-norm, see Sections 2.3 and 2.4.

It seems instructive to compare the transformations used in the present paper with the ones used in [**5**, **102**]. Transforming a radial solution u of (2.1.1) by setting $w(\tau) = (\frac{2}{2+\alpha})^{\frac{2}{p-2}} u(\tau^{\frac{2}{2+\alpha}})$ for $\tau \in (0, 1)$ leads to the problem

$$-(t^{M-1}w')' = t^{M-1}|w|^{p-2}w \quad \text{in } (0, 1), \qquad w'(0) = w(1) = 0 \tag{2.1.12}$$

with $M = M(\alpha) = \frac{2(N+\alpha)}{2+\alpha}$. Via this transformation, the associated weighted singular eigenvalue problem (2.1.7) corresponds to the even more singular eigenvalue equation

$$-(t^{M-1}\psi')' - (p-1)t^{M-1}|w|^{p-2}\psi = t^{M-3}\hat{\nu}\psi \qquad \text{in } (0, 1), \tag{2.1.13}$$

which is considered in M-dependent function spaces in [**5**]. In principle, it should be possible to carry out our approach also via these transformations, but we found it easier to find appropriate parameter-independent function spaces in the framework we use here. We stress again that finding parameter-independent function spaces is essential for the application of the implicit function theorem.

The paper is organized as follows. In Section 2.2, we first recall some known results on radial solutions of (2.1.1) and properties of the associated linearized operators. We then study the asymptotic behavior of the functions u_α as $\alpha \to \infty$ and prove Proposition 2.1.1. Section 2.3 is devoted to the proofs of Theorem 2.1.3 and Corollary 2.1.5. In Section 2.4 we prove, in particular, the differentiability of the map $U \mapsto |U|^{p-2}$ for $p \in (2, 3]$ in a suitable functional setting. In Section 2.5, we prove the bifurcation result stated in Theorem 2.1.6. Finally, in Section 2.6, we discuss the analogues of our main results in the case $N = 2$.

2.2 The limit shape of sign changing radial solutions of (2.1.1) as $\alpha \to \infty$

This section is devoted to the asymptotics of branches of sign changing radial solutions of (2.1.1) as $\alpha \to \infty$. In particular, we will prove Proposition 2.1.1. As before, we let $K \in \mathbb{N}$ be fixed, and we first recall a result on the existence, uniqueness and radial Morse index of a radial solution u_α of (2.1.1) with K nodal domains.

Theorem 2.2.1. *For every $p > 2$ and $\alpha > \alpha_p$, equation (2.1.1) has a unique radial solution $u_\alpha \in C^2(\overline{\mathbf{B}})$ with precisely K nodal domains such that $u_\alpha(0) > 0$. Furthermore, the linearized* *operator*

$$L^\alpha : H^2(\mathbf{B}) \cap H_0^1(\mathbf{B}) \to L^2(\mathbf{B}), \qquad L^\alpha \varphi := -\Delta\varphi - (p-1)|x|^\alpha |u_\alpha|^{p-2}\varphi$$

is a Fredholm operator of index zero having the following properties for every $\alpha \geq 0$:

(i) u_α is radially nondegenerate in the sense that the kernel of L^α does not contain radial functions.

(ii) u_α has radial Morse index K in the sense that L^α has precisely K negative eigenvalues corresponding to radial eigenfunctions in $H^2(\mathbf{B}) \cap H_0^1(\mathbf{B})$.

Theorem 2.2.1 is merely a combination of results in [**106**] and [**5**]. More precisely, the existence and uniqueness of u_α is proved in [**106**]. Note that the operator L^α is a compact perturbation of the isomorphism $-\Delta : H^2(\mathbf{B}) \cap H_0^1(\mathbf{B}) \to L^2(\mathbf{B})$, which implies that it is a Fredholm operator of index zero. A proof of the radial nondegeneracy and radial Morse index can be found in [**5**, Theorem 1.7]. We remark here that the radial nondegeneracy can also be deduced from results in [**137**].

Remark 2.2.2. *(i) Since equation (2.1.1) remains invariant under a change of sign $u \mapsto -u$, it follows from Theorem 2.2.1 that for every $p > 2$ and $\alpha > \alpha_p$, equation (2.1.1) has precisely two radial solutions $\pm u_\alpha \in C^2(\overline{\mathbf{B}})$ with precisely K nodal domains.*
*(ii) In [**106**] it is also shown that for $p \geq \frac{2N+2\alpha}{N-2}$, the trivial solution is the only radial solution of equation* (2.1.1).

Next we recall that, in the radial variable, u_α solves

$$-u_{rr} - \frac{N-1}{r}u_r = r^\alpha |u|^{p-2}u, \quad r \in (0,1), \qquad u'(0) = u(1) = 0. \tag{2.2.1}$$

Inspired by Byeon and Wang [**27**], we transform equation (2.2.1), considering

$$U_\alpha : [0,\infty) \to \mathbb{R}, \qquad U_\alpha(t) = (N+\alpha)^{-\frac{2}{p-2}}\, u_\alpha(e^{-\frac{t}{N+\alpha}}).$$

By direct computation, we see that U_α is a bounded solution of the problem

$$-(e^{-\gamma t}U')' = e^{-t}|U|^{p-2}U \quad \text{in } I := [0,\infty), \qquad U(0) = 0 \tag{2.2.2}$$

with $\gamma = \gamma(\alpha) = \frac{N-2}{N+\alpha}$. Moreover, U_α has precisely $K-1$ zeros in $(0,\infty)$ and satisfies $\lim\limits_{t\to\infty} U_\alpha(t) > 0$, which implies that $(-1)^{K-1}U_\alpha'(0) > 0$. Considering the limit $\alpha \to \infty$ in (2.2.1) corresponds to sending $\gamma \to 0$ in (2.2.2), which leads to limit problem

$$-U'' = e^{-t}|U|^{p-2}U \quad \text{in } I, \qquad U(0) = 0. \tag{2.2.3}$$

We first note the following facts regarding (2.2.3).

Proposition 2.2.3. *Let $p > 2$. The problem (2.2.3) admits a unique bounded solution $U_\infty \in C^2(\overline{I})$ with precisely $K-1$ zeros in $(0,\infty)$ and $U_\infty'(0) > 0$.*

Proof. The existence of a bounded solution of (2.2.3) with precisely $K-1$ zeros in $(0,\infty)$ has been proved by Naito [**107**, Theorem 1]. To prove uniqueness, we first note that every solution U of (2.2.3) is concave on intervals where $U > 0$ and convex on intervals where $U < 0$. From this we deduce that every bounded solution U with finitely many zeros has a limit

$$\ell(U) = \lim_{t\to\infty} U(t) \neq 0.$$

Next, we let U_1, U_2 be bounded solutions of (2.2.3) with precisely $K-1$ zeros in $(0,\infty)$. Moreover, we let $\kappa = \frac{\ell(U_1)}{\ell(U_2)}$, $c_\kappa := \ln|\kappa|^{p-2}$ and consider

$$\tilde{U}_2 : [c_\kappa, \infty) \to \mathbb{R}, \qquad \tilde{U}_2(t) = \kappa U_2(t - c_\kappa).$$

Then $\tilde{U}_2$ solves the equation in (2.2.3) on $[c_\kappa, \infty)$ and satisfies $\tilde{U}_2(c_\kappa) = 0$. By construction we have

$$\lim_{t\to\infty} U_1(t) = \lim_{t\to\infty} \tilde{U}_2(t),$$

and thus the local uniqueness result at infinity given in [**107**, Proposition 3.1] implies that

$$U_1(t) = \tilde{U}_2(t) \qquad \text{for } t \geq \max\{0, c_\kappa\}.$$

Since U_1 and $\tilde{U}_2$ have $K-1$ zeros in $(0,\infty)$, (c_κ,∞), respectively and $U_1(0) = \tilde{U}_2(c_\kappa) = 0$, it follows that $c_\kappa = 0$, hence $\kappa = 1$ and therefore $U_1 \equiv U_2$. The uniqueness of U_∞ thus follows. □

In the following, it is more convenient to work with the parameter $\gamma = \frac{N-2}{N+\alpha} \in (0, \frac{N-2}{N})$ in place of α. Hence, from now on, we will write U_γ in place of U_α. We also set $U_0 := (-1)^{K-1} U_\infty$, so that

$$\lim_{t\to\infty} U_0(t) > 0. \tag{2.2.4}$$

We wish to consider (2.1.4) and (2.2.2) in suitable spaces of continuous functions. For $\delta \geq 0$, we let $C_\delta(\bar{I})$ denote the space of all functions $v \in C(\bar{I})$ such that

$$\|v\|_{C_\delta} := \sup_{t\geq 0} e^{\delta t} |v(t)| < \infty.$$

More generally, for an integer $k \geq 0$, we let $C^k_\delta(\bar{I})$ denote the space of all functions $v \in C^k(\bar{I})$ such that $v^{(j)} \in C_\delta(\bar{I})$ for $j = 1, \dots, k$. Then $C^k_\delta(\bar{I})$ is a Banach space with norm

$$\|v\|_{C^k_\delta} := \sum_{j=0}^{k} \|v^{(j)}\|_{C_\delta}.$$

We note the following.

Lemma 2.2.4. *Let $k > \ell \geq 0$ and $\delta_1 > \delta_2 \geq 0$. Then the embedding $C^k_{\delta_1}(I) \hookrightarrow C^\ell_{\delta_2}(I)$ is compact.*

Proof. This is a straightforward consequence of the Arzelà-Ascoli theorem. □

For the remainder of this section, we fix $\delta = \frac{2}{N}$ and consider the spaces

$$E := \{v \in C^2(\bar{I}) \ : \ v(0) = 0,\ v' \in C^1_\delta(I)\} \qquad \text{and} \qquad F := C_\delta(I).$$

As note above, F is a Banach space with norm $\|\cdot\|_F = \|\cdot\|_{C_\delta}$.

Moreover, for every $v \in E$ we have

$$|v(t)| \leq \left| \int_0^t v'(s)\, ds \right| \leq \|v'\|_{C^1_\delta} \int_0^t e^{-\frac{2s}{N}}\, ds \leq \frac{N}{2} \|v'\|_{C^1_\delta} \qquad \text{for all } t \geq 0$$

and therefore $\|v\|_{L^\infty(I)} \leq \frac{N}{2}\|v'\|_{C^1_\delta}$. Hence we may endow E with the norm

$$v \mapsto \|v\|_E := \|v\|_{L^\infty(I)} + \|v'\|_{C^1_\delta}.$$

Since C^1_δ is a Banach space, it easily follows that E is a Banach space as well. We also note that

$$\lim_{t\to\infty} v(t) = \int_0^\infty v'(s)\, ds \quad \text{exists for every } v \in E. \tag{2.2.5}$$

Lemma 2.2.5. *Let $p > 2$, $\gamma \in [0, \frac{N-2}{N}]$, and let $U \in C^2(\bar{I})$ be a bounded nontrivial solution of* (2.2.2). *Then $U \in E$, and $\lim_{t\to\infty} U(t) \neq 0$.*

Proof. Since U is bounded, we have

$$|(e^{-\gamma t}U')'| \le e^{-t}|U|^{p-1} \le Ce^{-t} \qquad \text{for } t \ge 0$$

with a constant $C > 0$. Furthermore, there exists a sequence $t_n \to \infty$ with $U'(t_n) \to 0$ as $n \to \infty$. Consequently,

$$e^{-\gamma t}|U'(t)| = \lim_{n\to\infty}\left|\int_t^{t_n}(e^{-\gamma s}U'(s))'\,ds\right| \le \lim_{n\to\infty} C\int_t^{t_n} e^{-s}\,ds = Ce^{-t}$$

and therefore $|U'(t)| \le Ce^{(\gamma-1)t} \le Ce^{-\frac{2}{N}t}$ for $t \ge 0$. Since we can write (2.2.2) as

$$-U'' + \gamma U' = e^{(\gamma-1)t}|U|^{p-2}U, \tag{2.2.6}$$

it follows that $|U''(t)| \le |\gamma||U'(t)| + e^{(\gamma-1)t}|U(t)|^{p-1} \le C'e^{-\frac{2}{N}t}$ for $t \ge 0$ with a constant $C' > 0$, hence $U \in E$.

It remains to show that $\lim_{t\to\infty} U(t) \neq 0$. For this we consider the nonincreasing function $m(t) := \sup_{s\ge t}|U(s)|$. Using (2.2.2) and the fact that $U \in E$, we find that

$$e^{-\gamma t}|U'(t)| = \left|\int_t^\infty e^{-s}|U(s)|^{p-2}U(s)\,ds\right| \le e^{-t}m^{p-1}(t) \qquad \text{for } t \ge 0$$

and therefore

$$|U(t)| = \left|\int_t^\infty U'(s)\,ds\right| \le \int_t^\infty e^{(\gamma-1)s}m^{p-1}(s)\,ds \le \frac{m^{p-1}(t)}{1-\gamma}e^{(\gamma-1)t} \qquad \text{for } t \ge 0.$$

Consequently,

$$m(t) = \sup_{s\ge t}|U(s)| \le \sup_{s\ge t}\Big(\frac{m^{p-1}(s)}{1-\gamma}e^{(\gamma-1)s}\Big) = \frac{m^{p-1}(t)}{1-\gamma}e^{(\gamma-1)t}$$

and hence $m(t) = 0$ or $m^{p-2}(t) \ge (1-\gamma)e^{(1-\gamma)t} \ge 1-\gamma$ for $t \ge 0$. Since $m(0) \neq 0$ as $U \not\equiv 0$, we conclude by continuity of m that $m^{p-2}(t) \ge 1-\gamma$ for all $t \ge 0$. Together with (2.2.5), this shows that $\lim_{t\to\infty} U(t) \neq 0$. □

We intend to use the implicit function theorem to show that $U_\gamma \to U_0$ in E as $\gamma \to 0$. This requires uniqueness and nondegeneracy properties as given in the following two lemmas.

Lemma 2.2.6. *Let $p > 2$, $\gamma \in (0, \frac{N-2}{N+\alpha_p})$ and let $\tilde{U} \in E$ be a solution of* (2.2.2) *with precisely $K-1$ zeros in $(0,\infty)$ and $\lim_{t\to\infty}\tilde{U}(t) > 0$. Then $\tilde{U} = U_\gamma$.*

Proof. Let $\alpha > 0$ be the unique value such that $\gamma = \gamma(\alpha) = \frac{N-2}{N+\alpha}$, and consider the function

$$u : [0,1] \to \mathbb{R}, \qquad u(r) = \begin{cases} (N+\alpha)^{\frac{2}{p-2}}\tilde{U}(-(N+\alpha)\ln r), & r > 0, \\ (N+\alpha)^{\frac{2}{p-2}}\lim_{t\to\infty}\tilde{U}(t), & r = 0. \end{cases}$$

Since $\tilde{U} \in E$, the latter limit exists. We then have $u \in C^2((0,1]) \cap C([0,1])$, and u solves equation (2.2.1) on $(0,1)$. Moreover, we have $u'(r) = -(N+\alpha)^{\frac{p}{p-2}} \frac{\tilde{U}'(-(N+\alpha)\ln r)}{r}$ for $r \in (0,1]$ and therefore

$$\lim_{r\to 0} \frac{u'(r)}{r} = -(N+\alpha)^{\frac{2}{p-2}} \lim_{t\to\infty} e^{\frac{2t}{N+\alpha}} \tilde{U}'(t).$$

Since $\frac{2}{N+\alpha} < \frac{2}{N}$ and $\tilde{U} \in E$, we deduce that $\lim\limits_{r\to 0} \frac{u'(r)}{r} = 0$. From equation (2.2.1) it then also follows that $\lim\limits_{r\to 0} u''(r)$ exists, and that u also satisfies the boundary conditions in (2.2.1). Moreover, we have $u(0) > 0$ since $\lim\limits_{t\to\infty} \tilde{U}(t) > 0$ by assumption. The uniqueness result in Theorem 2.2.1 then yields that u is equal to u_α. Transforming back, we conclude that $\tilde{U} = U_\gamma$. □

Lemma 2.2.7. *Let $p > 2$ and $\gamma \in [0, \frac{N-2}{N+\alpha_p})$. Then the solution U_γ of problem (2.2.2) is nondegenerate in the sense that the equation*

$$-(e^{-\gamma t} v')' - (p-1)e^{-t}|U_\gamma|^{p-2} v = 0 \quad \textit{in } [0,\infty), \qquad v(0) = 0$$

has no bounded nontrivial solution.

Proof. We consider the auxiliary function $w := U_\gamma' + \frac{\gamma-1}{p-2} U_\gamma$, which, by direct computation, solves the linearized equation

$$-(e^{-\gamma t} w')' - (p-1)e^{-t}|U_\gamma|^{p-2} w = 0 \qquad \text{in } [0,\infty). \tag{2.2.7}$$

Moreover, we have $\lim\limits_{t\to\infty} w'(t) = 0$ since $U_\gamma \in E$ by Lemma 2.2.5. Suppose by contradiction there exists a bounded function $v \in C^2([0,\infty))$, $v \not\equiv 0$ satisfying

$$-(e^{-\gamma t} v')' - (p-1)e^{-t}|U_\infty|^{p-2} v = 0 \quad \text{in } [0,\infty), \qquad v(0) = 0. \tag{2.2.8}$$

Sturm comparison with w yields that v can only have finitely many zeros in I. Let $t_0 > 0$ denote the largest zero of w in $[0,\infty)$. Since v is bounded, there exists a sequence $(t_n)_n \subset [t_0,\infty)$ such that $t_n \to \infty$ and $v'(t_n) \to 0$ as $n \to \infty$. From (2.2.7) and (2.2.8), we deduce that

$$-\int_{t_0}^{\infty} (e^{-\gamma t} v')' w = \int_{t_0}^{\infty} e^{-t} |U_\infty|^{p-2} v w = -\int_{t_0}^{\infty} (e^{-\gamma t} w')' v.$$

Since $\lim\limits_{n\to\infty} e^{-\gamma t_n} v'(t_n) = \lim\limits_{n\to\infty} e^{-\gamma t_n} w'(t_n) = 0$, integration by parts yields

$$\begin{aligned} -e^{-\gamma t_0} v'(t_0) w(t_0) &= \lim_{n\to\infty} e^{-\gamma t_n} v'(t_n) w(t_n) - e^{-\gamma t_0} v'(t_0) w(t_0) \\ &= \lim_{n\to\infty} e^{-\gamma t_n} w'(t_n) v(t_n) - e^{-\gamma t_0} w'(t_0) v(t_0) = 0, \end{aligned}$$

which implies $v'(t_0) = 0$ or $w(t_0) = 0$. In the first case we then have $v \equiv 0$ and the proof is finished. In the other case it also follows that there exists $c \neq 0$ such that $cw'(t_0) = v'(t_0)$, which implies $v \equiv cw$. This contradicts $v(0) = 0 \neq U_\infty'(0) = w(0)$. □

We may now state a continuation result for the map $\gamma \mapsto U_\gamma$ which in particular implies Proposition 2.1.1.

Proposition 2.2.8. *Let $p > 2$. There exists $\varepsilon_0 > 0$ such that the map $(0, \frac{N-2}{N+\alpha_p}) \to E$, $\gamma \mapsto U_\gamma$ extends to a C^1-map $g : (-\varepsilon_0, \frac{N-2}{N+\alpha_p}) \to E$ with $g(0) = U_0$.*

Proof. We consider the map

$$G : \left(-\infty, \frac{N-2}{N+\alpha_p}\right) \times E \to F, \qquad G(\gamma, U) = -U'' + \gamma U' - e^{(\gamma-1)t}|U|^{p-2}U.$$

Since $e^{(\gamma-1)t} \leq e^{-\frac{2}{N}t}$ for $\gamma < \frac{N-2}{N+\alpha_p}$, G is well-defined and of class C^1. Moreover, by definition of U_γ we have

$$G(\gamma, U_\gamma) = 0 \qquad \text{for } \gamma \in \left[0, \frac{N-2}{N+\alpha_p}\right). \tag{2.2.9}$$

We first show that the linear map

$$L_\gamma := d_U G(\gamma, U_\gamma) : E \to F, \qquad L\varphi = -\varphi'' + \gamma\varphi' - (p-1)e^{(\gamma-1)t}|U_\gamma|^{p-2}\varphi \tag{2.2.10}$$

is an isomorphism for $\gamma \in [0, \frac{N-2}{N+\alpha_p})$. For this, we first note that

$$\text{the map } E \to F,\ \varphi \mapsto -\varphi'' + \gamma\varphi' \text{ is an isomorphism.} \tag{2.2.11}$$

Indeed, if $\varphi \in E$ satisfies $-\varphi'' + \gamma\varphi' = 0$, then $-\varphi' + \gamma\varphi$ is constant and $\varphi(0) = 0$, hence $\varphi(t) = c(e^{\gamma t} - 1)$ for $t \in I$ with a constant $c \in \mathbb{R}$. Since $\varphi \in E \subset L^\infty(I)$, we conclude that $\varphi \equiv 0$.

Moreover, if $f \in F$ is given and $\varphi : I \to \mathbb{R}$ is defined by

$$\varphi(t) := \int_0^t \int_s^\infty e^{\gamma(s-\sigma)} f(\sigma)\, d\sigma ds,$$

we have $-\varphi'' + \gamma\varphi' = f$ and $\varphi(0) = 0$. Furthermore,

$$\begin{aligned} |\varphi'(t)| = \left|\int_t^\infty e^{\gamma(t-\sigma)} f(\sigma)\, d\sigma\right| &\leq \int_t^\infty |f(\sigma)|\, d\sigma \\ &\leq \|f\|_F \int_t^\infty e^{-\frac{2}{N}s}\, ds \leq \frac{N}{2}\|f\|_F\, e^{-\frac{2}{N}t} \end{aligned}$$

for $t \geq 0$ and therefore $\varphi \in E$. We thus infer (2.2.11).

Next, we note that the linear map $E \to F$, $\varphi \mapsto e^{(\gamma-1)(\cdot)}|U_0|^{p-2}\varphi$ is compact, since the embedding $E \hookrightarrow C_0(I)$ is compact by Lemma 2.2.4 and the map $C_0(I) \to F$, $\varphi \mapsto e^{(\gamma-1)(\cdot)}|U_0|^{p-2}\varphi$ is continuous. By (2.2.11), we therefore deduce that L is Fredholm of index zero. Since the equation $L_\gamma v = 0$ only has the trivial solution $v = 0$ in E by Lemma 2.2.7, we conclude that L_γ is an isomorphism, as claimed. We now apply the implicit function theorem to the map G in the point $(0, U_0)$. This yields $\varepsilon_0 > 0$ and a differentiable map $\tilde{g} : (-\varepsilon_0, \varepsilon_0) \to E$ with $\tilde{g}(0) = U_0$ and $G(\gamma, \tilde{g}(\gamma)) = 0$ for $\gamma \in (-\varepsilon_0, \varepsilon_0)$.

Next we claim that

$$U_\gamma = \tilde{g}(\gamma) \text{ for } \gamma \in [0, \varepsilon_0). \tag{2.2.12}$$

Indeed, let $v_\gamma := \tilde{g}(\gamma) \in E$ for $\gamma \in (-\varepsilon_0, \varepsilon_0)$. By the continuity of $\tilde{g} : (-\varepsilon_0, \varepsilon_0) \to E$ and (2.2.5), the function

$$(-\varepsilon_0, \varepsilon_0) \to \mathbb{R}, \qquad \gamma \mapsto m_\gamma := \lim_{t\to\infty} v_\gamma(t)$$

is also continuous, and it is nonzero for $\gamma \in [0, \varepsilon_0)$ by Lemma 2.2.5. Moreover, by construction we have $v_0 = U_0$ and therefore $m_0 > 0$. It then follows that

$$m_\gamma > 0 \qquad \text{for all } [0, \varepsilon_0). \tag{2.2.13}$$

By Lemma 2.2.6, we thus only need to prove that v_γ has $K-1$ zeros in $(0,\infty)$ for $\gamma \in [0,\varepsilon_0)$. This is true for $\gamma = 0$ since $v_0 = U_0$. Moreover, the number of zeros of v_γ remains constant for $\gamma \in [0,\varepsilon_0)$. Indeed, as a solution of (2.2.2), v_γ cannot have double zeros, and the largest zero t_γ of v_γ in $[0,\infty)$ remains locally bounded for $\gamma \in [0,\varepsilon_0)$ since

$$m_\gamma = \int_{t_\gamma}^{\infty} v_\gamma'(s)\,ds \le \|v_\gamma\|_E \int_{t_\gamma}^{\infty} e^{-\frac{2}{N}s}\,ds \le \frac{N}{2}\|v_\gamma\|_E\, e^{-\frac{2}{N}t_\gamma}$$

and therefore $t_\gamma \le -\frac{N}{2}\ln\frac{2m_\gamma}{N\|v_\gamma\|_E}$. This finishes the proof of (2.2.12).

By a continuation argument based on (2.2.10), an application of the implicit function theorem in points (γ, U_γ) for $\gamma > 0$ and the same continuity considerations as above, we then see that the map

$$g : (-\varepsilon_0, \frac{N-2}{N+\alpha_p}) \to E, \qquad g(\gamma) = \begin{cases} \tilde g(\gamma), & \gamma \in (-\varepsilon_0, 0),\\ U_\gamma, & \gamma \in [0, \dfrac{N-2}{N+\alpha_p}) \end{cases}$$

is of class C^1. The proof is thus finished. □

Since $U_0 = (-1)^{K-1}U_\infty$, we have now completed the proof of Proposition 2.1.1.

Remark 2.2.9. *Using the function g and $\varepsilon_0 > 0$ from Proposition 2.2.8, it is convenient to define*

$$U_\gamma := g(\gamma) \quad \textit{for } \gamma \in (-\varepsilon_0, 0).$$

With this definition, it follows from Proposition 2.2.8 that the map $(-\varepsilon_0, \frac{N-2}{N+\alpha_p}) \to E, \gamma \mapsto U_\gamma$ is of class C^1.

Moreover, implicit differentiation of (2.2.2) at $\gamma = 0$ shows that $V = \partial_\gamma\big|_{\gamma=0} U_\gamma$ is given as the unique bounded solution of the problem

$$-V'' - (p-1)e^{-t}|U_0|^{p-2}V = U_0' - te^{-t}|U_0|^{p-2}U_0 \quad \textit{in } [0,\infty), \qquad V(0) = 0. \tag{2.2.14}$$

2.3 Spectral asymptotics

This section is devoted to the proofs of Theorem 2.1.3 and Corollary 2.1.5. We fix $p > 2$, and we start by recalling some results from [**5**] on the eigenvalue problem (2.1.5) and its relationship to the Morse index of u_α. Recall that we consider (2.1.5) in weak sense. More precisely, we say that $\varphi \in H_0^1(\mathbf{B})$ is an eigenfunction of (2.1.5) corresponding to the eigenvalue $\lambda \in \mathbb{R}$ if

$$q_\alpha(\varphi,\psi) = \lambda \int_{\mathbf{B}} \frac{\varphi(x)\psi(x)}{|x|^2}dx \qquad \text{for all } \psi \in H_0^1(\mathbf{B}), \tag{2.3.1}$$

where

$$q_\alpha : H_0^1(\mathbf{B}) \times H_0^1(\mathbf{B}) \to \mathbb{R}, \qquad q_\alpha(v,w) := \int_{\mathbf{B}} \Big(\nabla v \cdot \nabla w - (p-1)|x|^\alpha |u_\alpha|^{p-2} vw\Big)\,dx \tag{2.3.2}$$

is the quadratic form associated with the operator L^α. Note that the RHS of (2.3.1) is well-defined for $\varphi, \psi \in H_0^1(\mathbf{B})$ by Hardy's inequality.

Lemma 2.3.1. *(see [**5**, Prop. 4.1 and 5.1]) Let $\alpha > \alpha_p$. Then we have:*

(i) *The Morse index of u_α is given as the number of negative eigenvalues of* (2.1.5)*, counted with multiplicity. Moreover, every eigenfunction $v \in H_0^1(\mathbf{B})$ of* (2.1.5) *corresponding to a nonpositive eigenvalue is contained in $L^\infty(\mathbf{B}) \cap C^2(\mathbf{B} \setminus \{0\})$.*

(ii) *Let $\varphi \in H_0^1(\mathbf{B})$ be an eigenfunction of* (2.1.5) *corresponding to the eigenvalue $\lambda \in \mathbb{R}$. Then there exists a number $\ell_0 \in \mathbb{N} \cup \{0\}$, spherical harmonics Y_ℓ of degree ℓ and functions $\varphi_\ell \in H^1_{0,rad}(\mathbf{B})$, $\ell = 1, \dots, \ell_0$ with the property that*

$$\varphi(x) = \sum_{\ell=0}^{\ell_0} \varphi_\ell(x) Y_\ell\left(\frac{x}{|x|}\right) \qquad \textit{for } x \in \mathbf{B}.$$

Moreover, for every $\ell \in \{1, \dots, \ell_0\}$, we either have $\varphi_\ell \equiv 0$, or φ_ℓ is an eigenfunction of (2.1.7) *corresponding to the eigenvalue $\mu = \lambda - \lambda_\ell$.*

Regarding the reduced weighted eigenvalue problem (2.1.7), we also recall the following.

Lemma 2.3.2. *(see [**5**, p.19 and Prop. 3.7])*
Let $\alpha > \alpha_p$. Then 0 is not an eigenvalue of (2.1.7)*, and the negative eigenvalues of* (2.1.7) *are simple and given by*

$$\mu_j(\alpha) := \inf_{\substack{W \subset H^1_{0,rad}(\mathbf{B}) \\ dim\, W = j}} \max_{v \in W \setminus \{0\}} \frac{\int_{\mathbf{B}} |\nabla v|^2 - (p-1)|x|^\alpha |u_\alpha|^{p-2}|v|^2\, dx}{\int_{\mathbf{B}} |x|^{-2}|v|^2\, dx}, \qquad j = 1, \dots, K. \tag{2.3.3}$$

Here we point out that Theorem 2.2.1(i) already implies that zero is not an eigenvalue of (2.1.7). We also note that Proposition 2.1.2 now merely follows by combining Lemma 2.3.1 and Lemma 2.3.2.

We now turn to the proof of Theorem 2.1.3. For this we transform the radial eigenvalue problem (2.1.7). Note that, if we write an eigenfunction $\psi \in H^1_{0,rad}(\mathbf{B})$ as a function of the radial variable $r = |x|$, it solves

$$-\psi'' - \frac{N-1}{r}\psi' - (p-1)r^\alpha |u_\alpha(r)|^{p-2}\psi(r) = \frac{\mu}{r^2}\psi \qquad \text{in } (0,1), \qquad \psi(1) = 0.$$

We transform this problem by considering again $I := (0, \infty)$ and setting

$$\nu = \frac{1}{(N+\alpha)^2}\mu_j(\alpha), \qquad \Psi(t) = (N+\alpha)\psi(e^{-\frac{t}{N+\alpha}}) \quad \text{for } t \in \overline{I}. \tag{2.3.4}$$

This gives rise to the eigenvalue problem

$$\begin{cases} -(e^{-\gamma t}\Psi')' - (p-1)e^{-t}|U_\gamma(t)|^{p-2}\Psi = \nu e^{-\gamma t}\Psi & \text{in } I, \\ \Psi(0) = 0, \quad \Psi \in L^\infty(I) \end{cases} \tag{2.3.5}$$

with $\gamma = \gamma(\alpha) = \frac{N-2}{N+\alpha} \in (0, \frac{N-2}{N+\alpha_p})$ as before. Here, we have added the condition $\Psi \in L^\infty(I)$ since we focus on eigenfunctions corresponding to negative eigenvalues, and in this case eigenfunctions $\psi \in H^1_{0,rad}(\mathbf{B})$ of (2.1.7) are bounded by Lemma 2.3.2. In the following, we also consider the case $\gamma = 0$ in (2.3.5), which corresponds to the linearization of (2.2.3) at U_0:

$$\begin{cases} -\Psi'' - (p-1)e^{-t}|U_0(t)|^{p-2}\Psi = \nu\Psi & \text{in } I, \\ \Psi(0) = 0, \quad \Psi \in L^\infty(I). \end{cases} \tag{2.3.6}$$

We note that for $\gamma \in [0, \frac{N-2}{N+\alpha})$ and every solution Ψ of (2.3.5) there exists a sequence $t_n \to \infty$ with $\Psi'(t_n) \to 0$, which implies that

$$e^{-\gamma t}\Psi'(t) = \int_t^\infty -(e^{-\gamma s}\Psi')'(s)\,ds = \int_t^\infty \big(\nu e^{-\gamma s} + (p-1)e^{-s}|U_\gamma(s)|^{p-2}\big)\Psi(s)\,ds \qquad (2.3.7)$$

for $t \geq 0$. We also note that problem (2.3.5) can be rewritten as

$$\begin{cases} -\Psi'' + \gamma\Psi' - (p-1)e^{(\gamma-1)t}|U_\gamma(t)|^{p-2}\Psi = \nu\Psi & \text{in } I, \\ \Psi(0) = 0, \quad \Psi \in L^\infty(I). \end{cases} \qquad (2.3.8)$$

We need the following estimate in terms of the space $C^2_\delta(I)$ defined in Section 2.2.

Lemma 2.3.3. *Let $\nu_\diamond < 0$, $\gamma_\diamond \in (0, \frac{N-2}{N+\alpha_p})$, and let $\delta = \frac{1}{2}\big(\sqrt{1-2\nu_\diamond} - 1\big) > 0$. Then there exists a constant $C = C(\nu_\diamond, \gamma_\diamond) > 0$ such that for every solution $\Psi \in L^\infty(I)$ of the equation*

$$-\Psi'' + \gamma\Psi' - (p-1)e^{(\gamma-1)t}|U_\gamma(t)|^{p-2}\Psi = \nu\Psi \qquad (2.3.9)$$

with $\nu \leq \nu_\diamond$ and $\gamma \in [0, \gamma_\diamond]$ we have $\Psi \in C^2_\delta(I)$ with $\|\Psi\|_{C^2_\delta} \leq C\|\Psi\|_{L^\infty(I)}$.

Proof. Since $\|U_\gamma\|_{L^\infty(I)}$ remains uniformly bounded for $\gamma \in [0, \gamma_\diamond]$ by Proposition 2.2.8, there exists $t_0 = t_0(\nu_\diamond, \gamma_\diamond) > 0$ such that

$$(p-1)e^{(\gamma-1)t}|U_\gamma(t)|^{p-2} \leq -\frac{\nu_\diamond}{2} \quad \text{for } t \geq t_0,\ \gamma \in [0, \gamma_\diamond].$$

Let Ψ be a bounded solution of (2.3.9) on I. Then Ψ solves the differential inequality

$$\Psi'' - \gamma\Psi' + \frac{\nu_\diamond}{2}\Psi \geq 0 \qquad \text{in the open set } U_\Psi := \{t \in (t_0, \infty) \,:\, \Psi(t) > 0\}. \qquad (2.3.10)$$

For fixed $\varepsilon > 0$, we consider the function

$$t \mapsto \varphi_\varepsilon(t) := C_\Psi e^{-\delta t} + \varepsilon e^{\delta t} \qquad \text{with } C_\Psi := e^{\delta t_0}\|\Psi\|_{L^\infty(I)}.$$

By (2.3.10) and the definition of δ, the function $v_\varepsilon := \varphi_\varepsilon - \Psi$ satisfies

$$\begin{aligned} v_\varepsilon'' - \gamma v_\varepsilon' + \frac{\nu_\diamond}{2}v_\varepsilon &\leq (\delta^2 + \frac{\nu_\diamond}{2})\varphi_\varepsilon + \gamma\delta C_\Psi e^{-\delta t} - \gamma\delta\varepsilon e^{\delta t} \leq (\delta^2 + |\gamma|\delta + \frac{\nu_\diamond}{2})\varphi_\varepsilon \\ &\leq (\delta^2 + \delta + \frac{\nu_\diamond}{2})\varphi_\varepsilon = 0 \qquad \text{in } U_\Psi. \end{aligned}$$

This implies that v_ε cannot attain a negative minimum in the set (t_0, ∞). Moreover, by definition of v_ε we have

$$v_\varepsilon(t_0) \geq 0 \qquad \text{and} \qquad \lim_{t\to\infty} v_\varepsilon(t) = \infty.$$

Consequently, we have $v_\varepsilon \geq 0$ and therefore $\Psi \leq \varphi_\varepsilon$ on $[t_0, \infty)$. Replacing Ψ by $-\Psi$ in the argument above, we find that $|\Psi| \leq \varphi_\varepsilon$ on $[t_0, \infty)$. By considering the limit $\varepsilon \to 0$, we deduce that

$$|\Psi(t)| \leq C_\Psi e^{-\delta t} = C\|\Psi\|_{L^\infty(I)}e^{-\delta t} \qquad \text{for } t \geq t_0 \text{ with } C := e^{\delta t_0}.$$

Since the same inequality obviously holds for $t \in [0, t_0)$, we conclude that

$$|\Psi(t)| \leq C\|\Psi\|_{L^\infty(I)}e^{-\delta t} \qquad \text{for } t \geq 0.$$

Finally, using (2.3.7) and (2.3.9), we also get that

$$|\Psi'(t)| \leq C\|\Psi\|_{L^\infty(I)}e^{-\delta t} \quad \text{and} \quad |\Psi''(t)| \leq C\|\Psi\|_{L^\infty(I)}e^{-\delta t} \qquad \text{for } t \geq 0$$

after making $C > 0$ larger if necessary. The proof is thus finished. □

Proposition 2.3.4. *For $\gamma \in [0, \frac{N-2}{N+\alpha_p})$, the eigenvalue problem (2.3.5) admits precisely K negative eigenvalues $\nu_1(\gamma) < \nu_2(\gamma) < \cdots < \nu_K(\gamma) < 0$ characterized variationally by*

$$\nu_j(\gamma) = \inf_{\substack{W \subset H_0^1(I) \\ \dim W = j}} \max_{\Psi \in W \setminus \{0\}} \frac{\int_0^\infty e^{-\gamma t}\Psi'^2 - (p-1)e^{-t}|U_\gamma|^{p-2}\Psi^2\, dt}{\int_0^\infty e^{-\gamma t}\Psi^2\, dt} \qquad \textit{for } j = 1, \ldots, K. \tag{2.3.11}$$

Proof. Let $\gamma \in [0, \frac{N-2}{N+\alpha_p})$. We first show that

$$\nu_K(\gamma) < 0. \tag{2.3.12}$$

For $\gamma > 0$, this follows by Lemma 2.3.2. Indeed, in (2.3.3) we may, by density, replace $H^1_{0,rad}(\mathbf{B})$ by the space of radial functions in $C_c^\infty(\mathbf{B} \setminus \{0\})$, and this space corresponds to the dense subspace $C_c^\infty(I) \subset H_0^1(I)$ after the transformation (2.3.4). To show (2.3.12) in the case $\gamma = 0$, we use the auxiliary function $w := U_0' - \frac{1}{p-2}U_0$, which, by direct computation, solves the linearized equation $-w'' - (p-1)e^{-t}|U_0|^{p-2}w = 0$ in $(0, \infty)$. It is clear that w has a zero between any two zeros of U_0 on $[0, \infty)$. Moreover, letting $t_* > 0$ denote the largest zero of U_0, we find that the numbers

$$w(t_*) = U_0'(t_*) \qquad \text{and} \qquad \lim_{t\to\infty} w(t) = -\frac{1}{p-2}\lim_{t\to\infty} U_0(t)$$

have opposite sign, hence w also has a zero in (t_*, ∞). Since U_0 has $K-1$ zeros in $(0, \infty)$ and $U_0(0) = 0$, we infer that w has at least K zeros in $(0, \infty)$. From this, it is standard to deduce that $\nu_K(0) < 0$. We thus have proved (2.3.12).

Next we note that eigenfunctions Ψ of (2.3.5) corresponding to an eigenvalue $\nu_j(\gamma) < 0$ have precisely $j-1$ zeros in I. Indeed, this follows from standard Sturm-Liouville theory since any such eigenfunction decays exponentially as $t \to \infty$ together with their first and second derivatives by Lemma 2.3.3. It also follows that $\nu_j(\gamma)$ is simple in this case, i.e., the corresponding eigenspace is one-dimensional.

In the case $\gamma > 0$, the claim now follows from Lemma 2.3.2, which guarantees that $\nu_1(\gamma), \ldots, \nu_K(\gamma)$ are precisely the negative eigenvalues of (2.3.8). It remains to show that (2.3.6) has precisely K negative eigenvalues given by (2.3.11) in the case $\gamma = 0$. Since the essential spectrum of the linearized operator $L_0 : H^2(I) \cap H_0^1(I) \to L^2(I)$, $L_0\Psi = -\Psi'' - (p-1)e^{-t}|U_0(t)|^{p-2}\Psi$ is given by $[0, \infty)$, standard compactness arguments show that $\nu_j(0)$ is an eigenvalue of (2.3.6) whenever $\nu_j(0) < 0$. Suppose by contradiction that $\nu_{K+1}(0) < 0$, and let v be a corresponding eigenfunction. Then v has K zeros in $(0, \infty)$, and $\lim_{t\to\infty} v(t) = \lim_{t\to\infty} v'(t) = 0$ as $t \to \infty$ by Lemma 2.3.3. By Sturm comparison, it then follows that w has at least $K+1$ zeros in $(0, \infty)$. On the other hand, since

$$\Big(e^{-t}|U_0|^{p-2} + \frac{1}{(p-2)^2}\Big)U_0 = -U_0'' + \frac{1}{(p-2)^2}U_0 = -w' - \frac{1}{p-2}w,$$

U_0 has a zero between any two zeros of w. This contradicts the fact that U_0 has precisely $K-1$ zeros in $(0, \infty)$. We thus conclude that (2.3.6) admits precisely K negative eigenvalues given by (2.3.11) in the case $\gamma = 0$. □

We may now deduce the continuous dependence of the negative eigenvalues of (2.3.5).

Lemma 2.3.5. *For $j = 1, \ldots, K$, the function $\nu_j : [0, \frac{N-2}{N+\alpha_p}) \to (-\infty, 0)$ is continuous.*

Proof. Let $\gamma_0 \in [0, \frac{N-2}{N+\alpha_p})$, and let $(\gamma_n)_n \subset [0, \frac{N-2}{N+\alpha_p})$ be a sequence with $\gamma_n \to \gamma_0$. Recall that $U_{\gamma_n} \to U_{\gamma_0}$ uniformly on $[0, \infty)$ as $n \to \infty$ by Proposition 2.2.8. We fix $j \in \{1, \dots, K\}$ and consider the space $W \subset H_0^1(I)$ spanned by the first j eigenfunctions of (2.3.5) in the case $\gamma = \gamma_0$. Moreover, we let $\mathcal{M} := \{\Psi \in W \;:\; \int_0^\infty \Psi^2 dt = 1\}$. Since $\nu_j(\gamma_0) < 0$, $\mathcal{M}$ is a compact subset of $C^2_\delta(I)$ for some $\delta > 0$ by Lemma 2.3.3. From this we deduce that

$$\int_0^\infty \left(e^{-\gamma_n t}\Psi'^2 - (p-1)e^{-t}|U_{\gamma_n}|^{p-2}\Psi^2\right)dt \;\to$$
$$\int_0^\infty \left(e^{-\gamma_0 t}\Psi'^2 - (p-1)e^{-t}|U_0|^{p-2}\Psi^2\right)dt \quad \text{and}$$
$$\int_0^\infty e^{-\gamma_n t}\Psi^2\,dt \;\to\; \int_0^\infty e^{-\gamma_0 t}\Psi^2\,dt \qquad \text{as } n \to \infty \text{ uniformly in } \psi \in \mathcal{M},$$

and this implies that

$$\begin{aligned}\limsup_{n\to\infty} \nu_j(\gamma_n) &\le \limsup_{n\to\infty} \max_{\Psi\in\mathcal{M}} \frac{\int_0^\infty \left(e^{-\gamma_n t}\Psi'^2 - (p-1)e^{-t}|U_{\gamma_n}|^{p-2}\Psi^2\right)dt}{\int_0^\infty e^{-\gamma_n t}\Psi^2\,dt} \\ &= \max_{\Psi\in\mathcal{M}} \frac{\int_0^\infty \left(e^{-\gamma_0 t}\Psi'^2 - (p-1)e^{-t}|U_0|^{p-2}\Psi^2\right)dt}{\int_0^\infty e^{-\gamma_0 t}\Psi^2\,dt} = \nu_j(\gamma_0).\end{aligned}$$

To show that $\liminf\limits_{n\to\infty} \nu_j(\gamma_n) \ge \nu_j(\gamma_0)$, we argue by contradiction and assume that, after passing to a subsequence, we have

$$\nu_j(\gamma_n) \to \sigma_j < \nu_j(\gamma_0). \tag{2.3.13}$$

Passing again to a subsequence, we may then also assume that

$$\nu_k(\gamma_n) \to \sigma_k \le \sigma_j < 0 \qquad \text{for } k = 1, \dots, j. \tag{2.3.14}$$

Let, for $k = 1, \dots, j$, the function $\Psi_{k,n}$ denote an eigenfunction of (2.3.5) corresponding to the eigenvalue $\nu_k(\gamma_n)$ such that $\|\Psi_{k,n}\|_{L^\infty(I)} = 1$. Since eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the weighted scalar product $(v, w) \mapsto \int_0^\infty e^{-\gamma_n t} vw\,dt$, we may assume that

$$\int_0^\infty e^{-\gamma_n t}\Psi_{k,n}\Psi_{\ell,n}\,dt = 0 \qquad \text{for } k, \ell \in \{1, \dots, j\},\, k \ne \ell. \tag{2.3.15}$$

By Lemma 2.3.3 and (2.3.14), there exists $\delta > 0$ such that $\|\Psi_{k,n}\|_{C^2_\delta} \le C$ for all $n \in \mathbb{N}$, $k \in \{1, \dots, j\}$. By Lemma 2.2.4, we may therefore pass to a subsequence again such that

$$\Psi_{k,n} \to \Psi_k \qquad \text{uniformly in } I,$$

where $\Psi_k \in C^2_\delta(I)$ is a solution of

$$-(e^{\gamma_0 t}\Psi')' - (p-1)e^{-t}|U_0(t)|^{p-2}\Psi = \sigma_k e^{-\gamma_0 t}\Psi \qquad \text{in } I, \qquad \Psi_k(0) = 0 \tag{2.3.16}$$

for $k = 1, \dots, j$. Moreover, since the sequences $(\Psi_{k,n})_n$, $k = 1, \dots, j$ are uniformly bounded in $C^2_\delta(I)$, we may pass to the limit in (2.3.15) to get that

$$\int_0^\infty e^{\gamma_0 t}\Psi_k\Psi_\ell\,dt = 0 \qquad \text{for } k, \ell \in \{1, \dots, j\},\, k \ne \ell. \tag{2.3.17}$$

Consequently, for $\gamma = \gamma_0$, the problem (2.3.5) has j eigenvalues $\sigma_1, \dots, \sigma_j$ (counted with multiplicity) in $(-\infty, \nu_j(\gamma_0))$. This contradicts Proposition 2.3.4. The proof is finished. ◻

Next, we wish to derive some information on the derivative $\partial_\gamma \nu_j(\gamma)$ of the negative eigenvalues of (2.3.5) as $\gamma \to 0^+$. We intend to derive this information via the implicit function theorem applied to the map $G : \left(-\varepsilon_0, \frac{N-2}{N+\alpha_p}\right) \times \tilde{E} \times \mathbb{R} \to \tilde{F} \times \mathbb{R}$ defined by

$$G(\gamma, \Psi, \nu) = \begin{pmatrix} -\Psi'' + \gamma\Psi' - (p-1)e^{(\gamma-1)t}|U_\gamma|^{p-2}\Psi - \nu\Psi \\ \int_0^\infty \Psi^2\,dt - 1, \end{pmatrix} \tag{2.3.18}$$

Here, ε_0 is given in Proposition 2.2.8, so that $(-\varepsilon_0, \frac{N-2}{N+\alpha_p}) \to C_0^1(I)$, $\gamma \mapsto U_\gamma$ is a well defined C^1-map by Remark 2.2.9. Moreover, $\tilde{E}$ and $\tilde{F}$ are suitable spaces of functions on I chosen in a way that eigenfunctions and eigenvalues of (2.3.8) and (2.3.6) correspond to zeros of this map. However, in the case $p \in (2, 3]$, the function $|\cdot|^{p-2}$ is not differentiable at zero and therefore it is not a priori clear how $\tilde{E}$ and $\tilde{F}$ need to be chosen to guarantee that G is of class C^1. In particular, spaces of continuous functions will not work in this case, so we need to introduce different function spaces.

For $\delta > 0$ and $1 \le r < \infty$, we let $L^r_\delta(I)$ denote the space of all functions $f \in L^r_{loc}(I)$ such that

$$\|f\|_{r,\delta} := \sup_{t\ge 0} e^{\delta t}[f]_{t,r} < \infty, \quad \text{where} \quad [f]_{t,r} := \left(\int_t^{t+1} |f(s)|^r\,ds\right)^{\frac{1}{r}} = \|f\|_{L^r(t,t+1)}.$$

The completeness of L^r-spaces readily implies that the spaces $L^r_\delta(I)$ are also Banach spaces. We will need the following observation:

Lemma 2.3.6. *Let $\delta > 0$ and $f \in L^1_\delta(I)$. Then we have*

$$\int_t^\infty e^{\mu s}|f(s)|\,ds \le C_{\mu,\delta}\|f\|_{1,\delta}\, e^{(\mu-\delta)t} \tag{2.3.19}$$

for $\mu < \delta$, $t \ge 0$ with $C_{\mu,\delta} := \frac{\max\{1,e^\mu\}}{1-e^{\mu-\delta}}$, and

$$\int_0^t e^{\mu s}|f(s)|\,ds \le D_{\delta,\mu}\|f\|_{1,\delta}\, e^{(\mu-\delta)t} \tag{2.3.20}$$

for $\mu > \delta$, $t \ge 0$ with $D_{\delta,\mu} := \frac{e^{2\mu-\delta}}{e^{\mu-\delta}-1}$.

Proof. Let $f \in L^1_\delta(I)$ and $t \ge 0$. If $\mu < \delta$, we have

$$\begin{aligned}\int_t^\infty e^{\mu s}|f(s)|\,ds &= \sum_{\ell=0}^\infty \int_{t+\ell}^{t+\ell+1} e^{\mu s}|f(s)|\,ds \le \max\{1,e^\mu\}\sum_{\ell=0}^\infty e^{\mu(t+\ell)}[f]_{t+\ell,1} \\ &\le \max\{1,e^\mu\}\|f\|_{1,\delta}\sum_{\ell=0}^\infty e^{(\mu-\delta)(t+\ell)} = C_{\mu,\delta}\|f\|_{1,\delta}\,e^{(\mu-\delta)t},\end{aligned}$$

and in the case $\mu > \delta$ we have

$$\begin{aligned}\int_0^t e^{\mu s}|f(s)|\,ds &\le \sum_{\ell=0}^{\lfloor t\rfloor}\int_\ell^{\ell+1} e^{\mu s}|f(s)|\,ds \le \sum_{\ell=0}^{\lfloor t\rfloor} e^{\mu(\ell+1)}[f]_{\ell,1} \\ &\le e^\mu\|f\|_{1,\delta}\sum_{\ell=0}^{\lfloor t\rfloor} e^{(\mu-\delta)\ell} = e^\mu\|f\|_{1,\delta}\frac{e^{(\mu-\delta)(\lfloor t\rfloor+1)}-1}{e^{\mu-\delta}-1} \le D_{\delta,\mu}\|f\|_{1,\delta}\,e^{(\mu-\delta)t}\end{aligned}$$

with $C_{\mu,\delta}$ and $D_{\delta,\mu}$ given above. □

Next, for $\delta > 0$, we define the function space

$$W^2_\delta(I) := \left\{ u \in C^1_\delta(\bar{I}) \cap W^{2,1}_{loc}(\bar{I}) \ : \ u(0) = 0,\ u'' \in L^1_\delta(I) \right\}$$

and endow this space with the norm

$$\|u\|_{W^2_\delta} := \|u\|_{C^1_\delta} + \|u''\|_{1,\delta}$$

We first note that

$$u'(t) = -\int_t^\infty u''(s)\, ds \qquad \text{for } u \in W^2_\delta(I) \text{ and } t \geq 0.$$

Lemma 2.3.7. *$W^2_\delta(I)$ is a Banach space.*

Proof. Consider a Cauchy sequence $(u_n)_n$ in $W^2_\delta(I)$. Then we have

$$u_n \to u \quad \text{in } C^1_\delta(\bar{I}) \qquad \text{and} \qquad u''_n \to v \quad \text{in } L^1_\delta(I). \tag{2.3.21}$$

Moreover, we have

$$u'(t) = \lim_{n\to\infty} u'_n(t) = -\lim_{n\to\infty} \int_t^\infty u''_n(s) ds = -\int_t^\infty v(s)\, ds \qquad \text{for all } t > 0, \tag{2.3.22}$$

since

$$\int_t^\infty |u''_n(s) - v(s)|\, ds \leq C_{0,\delta} \|u'' - v\|_{1,\delta}\, e^{-\delta t} \to 0 \qquad \text{as } n \to \infty$$

by (2.3.19). From (2.3.22) we deduce that $u'' = v \in L^1_\delta(I)$ in weak sense. Then it follows from (2.3.21) that $u_n \to u$ in $W^2_\delta(I)$. □

The following simple lemma is essential.

Lemma 2.3.8. *Let $\delta, \gamma, \mu \geq 0$ satisfy $\delta < \sqrt{\frac{\gamma^2}{4} + \mu^2} - \frac{\gamma}{2}$. Then the map $W^2_\delta(I) \to L^1_\delta(I)$, $T\Psi = -\Psi'' + \gamma\Psi' + \mu^2\Psi$ is an isomorphism.*

Proof. Let $\lambda := \sqrt{\frac{\gamma^2}{4} + \mu^2}$. Any solution of the equation $-\Psi'' + \gamma\Psi' + \mu^2\Psi = 0$ is given by $\Psi(t) = Ae^{(\frac{\gamma}{2}-\lambda)t} + Be^{(\frac{\gamma}{2}+\lambda)t}$ with suitable $A, B \in \mathbb{R}$. If $\Psi \in W^2_\delta(I)$, then Ψ is bounded and therefore $B = 0$. Moreover, $A = 0$ since $\Psi(0) = 0$, and therefore $\Psi \equiv 0$. Hence T has zero kernel.

For $g \in L^1_\delta(I)$, a solution of $-\Psi'' + \gamma\Psi' + \mu^2\Psi = g$ is given by

$$\Psi(t) = \frac{1}{2\lambda} e^{(\frac{\gamma}{2}+\lambda)t} \int_t^\infty e^{-(\frac{\gamma}{2}+\lambda)s} g(s)\, ds + \frac{1}{2\lambda} e^{(\frac{\gamma}{2}-\lambda)t} \int_0^t e^{(-\frac{\gamma}{2}+\lambda)s} g(s)\, ds.$$

By (2.3.19) and (2.3.20), we have

$$\left| e^{(\frac{\gamma}{2}+\lambda)t} \int_t^\infty e^{-(\frac{\gamma}{2}+\lambda)s} g(s)\, ds \right| \leq C_{-(\frac{\gamma}{2}+\lambda),\delta} \|g\|_{1,\delta}\, e^{-\delta t},$$
$$\left| e^{(\frac{\gamma}{2}-\lambda)t} \int_0^t e^{(-\frac{\gamma}{2}+\lambda)s} g(s)\, ds \right| \leq D_{-\frac{\gamma}{2}+\lambda,\delta} \|g\|_{1,\delta}\, e^{-\delta t}$$

for $t \geq 0$. Hence $\Psi \in C_\delta(I)$. Since

$$\Psi'(t) = \frac{\frac{\gamma}{2} + \lambda}{2\lambda} e^{(\frac{\gamma}{2}+\lambda)t} \int_t^\infty e^{-(\frac{\gamma}{2}+\lambda)s} g(s)\, ds + \frac{\frac{\gamma}{2} - \lambda}{2\lambda} e^{(\frac{\gamma}{2}-\lambda)t} \int_0^t e^{(-\frac{\gamma}{2}+\lambda)s} g(s)\, ds \tag{2.3.23}$$

it also follows that $\Psi' \in C_\delta(I)$. Additionally, we have $\Psi'' = \mu^2\Psi + \gamma\Psi' - g \in L^1_\delta$. By adding a multiple of the function $t \mapsto e^{(\frac{\gamma}{2}-\lambda)t}$, we can ensure that $\Psi(0) = 0$ and therefore $\Psi \in W^2_\delta(I)$. We conclude that T is an isomorphism. □

From now on, we fix $\gamma_\diamond \in (0, \frac{N-2}{N+\alpha_p})$. By Proposition 2.3.4 and Lemma 2.3.5, we have

$$\nu_\diamond := \sup_{0\le\gamma\le\gamma_\diamond} \nu_K(\gamma) < 0. \tag{2.3.24}$$

Moreover, we fix

$$\delta := \min\left\{\frac{\sqrt{1-2\nu_\diamond}-1}{2}\,,\ \frac{1}{2}\left(\sqrt{\frac{\gamma_\diamond^2}{4}-\nu_\diamond}-\frac{\gamma_\diamond}{2}\right),\ \frac{2}{N}\right\} \tag{2.3.25}$$

for the remainder of this section. By Lemma 2.3.3 and since $\delta \le \frac{1}{2}\big(\sqrt{1-2\nu_\diamond}-1\big)$, there exists $C > 0$ such that

$$\|\Psi\|_{C^2_\delta(I)} \le C\|\Psi\|_{L^\infty(I)} \tag{2.3.26}$$

for every eigenfunction of (2.3.8) corresponding to $\gamma \in [0,\gamma_\diamond]$ and $\nu = \nu_j(\gamma)$, $j = 1,\dots,k$.

We consider the spaces $E_\delta := W^2_\delta(I)$ and $F_\delta := L^1_\delta(I)$. The key observation of this section is the following.

Proposition 2.3.9. *Let $\varepsilon_0 > 0$ be given by Proposition 2.2.8, so that $(-\varepsilon_0,\gamma_\diamond) \to C^1_0(I)$, $\gamma \mapsto U_\gamma$ is a well defined C^1-map by Remark 2.2.9. Moreover, let the map*

$$G : (-\varepsilon_0,\gamma_\diamond) \times E_\delta \times \mathbb{R} \to F_\delta \times \mathbb{R}$$

be defined by (2.3.18). Then G is of class C^1 with

$$\begin{aligned}
\partial_\gamma G(\gamma,\Psi,\nu) &= \begin{pmatrix} \Psi' - (p-1)e^{(\gamma-1)t}|U_\gamma|^{p-2}\left(t + (p-2)\frac{U_\gamma\partial_\gamma U_\gamma}{|U_\gamma|^2}\right)\Psi \\ 0 \end{pmatrix},\\
\partial_\nu G(\gamma,\Psi,\nu) &= \begin{pmatrix} -\Psi \\ 0 \end{pmatrix}\\
\text{and}\quad d_\Psi G(\gamma,\Psi,\nu)\varphi &= \begin{pmatrix} -\varphi'' + \gamma\varphi' - (p-1)e^{(\gamma-1)t}|U_\gamma|^{p-2}\varphi - \nu\varphi \\ \int_0^\infty \Psi\varphi\,dt \end{pmatrix} \quad \text{in } F_\delta\times\mathbb{R}
\end{aligned}$$

for $\varphi \in E_\delta$.

We postpone the somewhat lengthy proof of this proposition to the next section and continue the main argument first. We fix $j \in \{1,\dots,K\}$. For $\gamma \ge 0$, we let $\Psi_{\gamma,j}$ denote an eigenfunction of the eigenvalue problem (2.3.8) corresponding to the eigenvalue $\nu_j(\gamma)$. We thus have

$$-\Psi''_{\gamma,j} + \gamma\Psi'_{\gamma,j} - (p-1)e^{(\gamma-1)t}|U_\gamma(t)|^{p-2}\Psi_{\gamma,j} = \nu_j(\gamma)\Psi_{\gamma,j} \text{ in } [0,\infty), \quad \Psi_{\gamma,j}(0) = 0,$$
$$\Psi_{\gamma,j} \in L^\infty(I).$$

By (2.3.26) we have $\Psi_{\gamma,j} \in E_\delta$. Moreover, we can assume $\int_0^\infty \Psi^2_{\gamma,j}\,dt = 1$, so that

$$G(\gamma,\Psi_{\gamma,j},\nu_j(\gamma)) = 0.$$

To apply the implicit function theorem to G at the point $(\gamma,\Psi_{\gamma,j},\nu_j(\gamma))$, we need the following property.

Proposition 2.3.10. *Let $\gamma \in [0,\gamma_\diamond]$. Then the map*

$$\begin{aligned}
L := d_{\Psi,\nu}G(\gamma,\Psi_{\gamma,j},\nu_j(\gamma)) &: E_\delta\times\mathbb{R} \to F_\delta\times\mathbb{R}\\
(\varphi,\rho) &\mapsto \begin{pmatrix} -\varphi'' + \gamma\varphi' - (p-1)e^{(\gamma-1)t}|U_\gamma|^{p-2}\varphi - \nu_j(\gamma)\varphi - \rho\Psi_{\gamma,j} \\ \int_0^\infty \Psi_{\gamma,j}\varphi\,dt \end{pmatrix}
\end{aligned}$$

is an isomorphism.

Proof. Since, by definition,

$$\delta < \sqrt{\frac{\gamma_\diamond^2}{4} - \nu_\diamond} - \frac{\gamma_\diamond}{2} \le \sqrt{\frac{\gamma^2}{4} - \nu_j(\gamma)} - \frac{\gamma}{2},$$

we may apply Lemma 2.3.8 with $\mu = \sqrt{-\nu_j(\gamma)}$. Hence the map $E_\delta \to F_\delta$, $\varphi \mapsto -\varphi'' + \gamma\varphi' - \nu_j(\gamma)\varphi$ is an isomorphism. Since the linear map $E_\delta \to F_\delta$, $\varphi \mapsto (p-1)e^{(\gamma-1)t}|U_\gamma|^{p-2}\varphi$ is compact, the map

$$\begin{aligned} T : E_\delta &\to F_\delta \\ \varphi &\mapsto -\varphi'' + \gamma\varphi' - (p-1)e^{(\gamma-1)t}|U_\gamma|^{p-2}\varphi - \nu_j(\gamma)\varphi \end{aligned}$$

is a Fredholm operator of index zero. The kernel of this map is one dimensional, since it consists of eigenfunctions corresponding to $\nu_j(\gamma)$. Hence the codimension of the image of T is one, and we claim that $\Psi_{\gamma,j}$ is not contained in the image of T. Otherwise, there exists $\varphi \in E_\delta$ such that $-\varphi'' + \gamma\varphi' - (p-1)e^{(\gamma-1)t}|U_\gamma|^{p-2}\varphi - \nu_j(\gamma)\varphi = \Psi_{\gamma,j}$. Multiplying with $\Psi_{\gamma,j}$ and integrating by parts then yields

$$\begin{aligned} 0 < \int_0^\infty e^{-\gamma t}\Psi_{\gamma,j}^2\,dt &= \int_0^\infty (-(e^{-\gamma t}\varphi')' \\ &\quad - (p-1)e^{-t}|U_\gamma(t)|^{p-2}\varphi - \nu_j(\gamma)e^{-\gamma t}\varphi)\Psi_{\gamma,j}\,dt \\ &= \int_0^\infty (-(e^{-\gamma t}\Psi')' \\ &\quad - (p-1)e^{-t}|U_\gamma(t)|^{p-2}\Psi - \nu_j(\gamma)e^{-\gamma t}\Psi)\varphi\,dt = 0, \end{aligned}$$

a contradiction. It follows that

$$E_\delta = \operatorname{range} T \oplus \operatorname{span}\{\Psi_{\gamma,j}\}. \tag{2.3.27}$$

We now show that L is an isomorphism. First assume $L(\varphi,\rho) = 0$ for some $(\varphi,\rho) \in E_\delta \times \mathbb{R}$, i.e.,

$$\begin{aligned} -\varphi'' + \gamma\varphi' - (p-1)e^{(\gamma-1)t}|U_\gamma|^{p-2}\varphi - \nu_j(\gamma)\varphi &= \rho\Psi_{\gamma,j} \quad \text{in } F_\delta \\ \text{and} \qquad \int_0^\infty \Psi_{\gamma,j}\varphi\,dt &= 0. \end{aligned}$$

Since $\Psi_{\gamma,j} \notin \operatorname{range} T$, the first equality yields $\rho = 0$. But then φ itself is an eigenfunction and therefore $\varphi = c\Psi_{\gamma,j}$ for some $c \in \mathbb{R}$. The second equality then yields $c = 0$, and thus $(\varphi,\rho) = (0,0)$. Hence L is injective.

Now let $(g,\sigma) \in F_\delta \times \mathbb{R}$. By (2.3.27) there exist $g_0 \in \operatorname{range} T$, $\kappa \in \mathbb{R}$ such that $g = g_0 + \kappa\Psi_{\gamma,j}$. Since $g_0 \in \operatorname{range} T$, there exists a solution $\varphi_0 \in E_\delta$ of

$$-\varphi'' + \gamma\varphi' - (p-1)e^{(\gamma-1)t}|U_\gamma|^{p-2}\varphi - \nu_j(\gamma)\varphi = g_0 \qquad \text{in } I.$$

Furthermore, for any $\eta \in \mathbb{R}$, $\varphi_0 + \eta\Psi_{\gamma,j} \in E_\delta$ is also a solution. Taking $\eta = \sigma - \int_0^\infty \Psi_{\gamma,j}\varphi_0\,dt$ yields

$$\int_0^\infty \Psi_{\gamma,j}(\varphi_0 + \eta\Psi_{\gamma,j})\,dt = \sigma.$$

Consequently, we have

$$L(\varphi_0 + \eta\Psi_{\gamma,j}, -\kappa) = \begin{pmatrix} g \\ \sigma \end{pmatrix}.$$

Hence L is surjective. ∎

With the help of Propositions 2.3.9 and 2.3.10, we may now apply the implicit function theorem to G at $(\gamma, \Psi_{\gamma,j}, \nu_j(\gamma))$. This yields the following result.

Corollary 2.3.11. *There exist $\varepsilon_1 \in (0, \varepsilon_0)$ and, for $j = 1, \dots, K$, C^1-maps $h_j : (-\varepsilon_1, \gamma_\diamond) \to \mathbb{R}$ with the property that*

$$h_j(\gamma) = \nu_j(\gamma) \qquad \text{for } j = 1, \dots, K,\ \gamma \in [0, \gamma_\diamond) \tag{2.3.28}$$

and

$$h_j'(0) = -(p-1) \int_0^\infty \Big(te^{-t} |U_0|^{p-2} \Psi_{0,j}^2 + (p-2) e^{-t} |U_0|^{p-4} U_0 (\partial_\gamma\big|_{\gamma=0} U_\gamma) \Psi_{0,j}^2 \Big)\, dt \tag{2.3.29}$$

for $j = 1, \dots, K$.

Proof. By Propositions 2.3.9, 2.3.10 and the implicit function theorem applied to the map G at $(0, \Psi_{0,j}, \nu_j(0))$, there exists $\varepsilon_1 \in (0, \varepsilon_0)$ and C^1-maps $g_j : (-\varepsilon_1, \varepsilon_1) \to F_\delta \times \mathbb{R}$ with the property that $g_j(0) = (\Psi_{0,j}, \nu_j(0))$ and $G(\gamma, g_j(\gamma)) = 0$ for $\gamma \in (-\varepsilon_1, \varepsilon_1)$. Let h_j denote the second component of g_j. Since

$$\nu_1(0) = h_1(0) < \nu_2(0) = h_2(0) < \cdots < \nu_K(0) = h_K(0) < 0,$$

we may, after making ε_1 smaller if necessary, assume that also

$$h_1(\gamma) < h_2(\gamma) < \cdots < h_K(\gamma) < 0 \qquad \text{for } \gamma \in (0, \varepsilon_1).$$

Since, by construction, the values $h_j(\gamma)$ are eigenvalues of (2.3.5) and the negative eigenvalues of (2.3.5) are precisely given by (2.3.11), the equality (2.3.28) follows for $\gamma \in (0, \varepsilon_1)$. Using Propositions 2.3.9, 2.3.10 and applying the implicit function theorem at $(\gamma, \Psi_{\gamma,j}, \nu_j(\gamma))$, the functions h_j may be extended as C^1-functions to $(-\varepsilon_1, \gamma_\diamond)$ such that (2.3.28) holds for $(0, \gamma_\diamond)$. Moreover, (2.3.29) is a consequence of implicit differentiation of the equation $G(\gamma, g_j(\gamma)) = 0$. □

We may now complete the

Proof of Theorem 2.1.3. We first note that – since $U_0 := (-1)^{K-1} U_\infty$ – the eigenvalue problem (2.1.11) coincides with (2.3.6), and it has precisely K negative eigenvalues $\nu_j^* := \nu_j(0)$, $j = 1, \dots, K$ by Proposition 2.3.4. To prove the expansions (2.1.10), we fix $j \in \{1, \dots, K\}$. By Remark 2.1.4 and Corollary 2.3.11, the constant c_j^* appearing in (2.1.10) is given by $c_j^* = 2N\nu_j^* + (N-2)h_j'(0)$. Now Corollary 2.3.11 yields the expansions

$$\nu_j(\gamma) = \nu_j^* + \gamma h_j'(0) + o(\gamma) \qquad \text{and} \qquad \partial_\gamma \nu_j(\gamma) = h_j'(0) + o(1) \qquad \text{as } \gamma \to 0^+. \tag{2.3.30}$$

Writing $\gamma = \gamma(\alpha) = \frac{N-2}{N+\alpha}$ as before and recalling (2.3.4), we thus have

$$\begin{aligned}
\mu_j(\alpha) &= (N+\alpha)^2 \nu_j(\gamma(\alpha)) = (N+\alpha)^2 \left(\nu_j^* + \frac{N-2}{N+\alpha} h_j'(0) + o\left(\frac{1}{\alpha}\right) \right) \\
&= \nu_j^* \alpha^2 + \big[2N\nu_j^* + (N-2)h_j'(0)\big]\alpha + o(\alpha) = \nu_j^* \alpha^2 + c_j^* \alpha + o(\alpha)
\end{aligned}$$

and

$$\begin{aligned}
\mu_j'(\alpha) &= 2(N+\alpha)\nu_j(\gamma(\alpha)) - (N-2)[\partial_\gamma \nu_j](\gamma(\alpha)) \\
&= 2(N+\alpha)\left(\nu_j^* + \frac{N-2}{N+\alpha} h_j'(0) + o\left(\frac{1}{\alpha}\right) \right) - (N-2)(h_j'(0) + o(1)) \\
&= 2\nu_j^* \alpha + 2N\nu_j^* + (N-2)h_j'(0) + o(1) = 2\nu_j^* \alpha + c_j^* + o(1) \quad \text{as } \alpha \to \infty.
\end{aligned}$$

□

We may also complete the

Proof of Theorem 2.1.5. By Theorem 2.1.3 we have

$$\mu_i'(\alpha) = 2\alpha v_i^* + c_i^* + o(1) \qquad \text{as } \alpha \to \infty$$

for $i = 1, \dots, K$. Since the values v_i^* are negative, we may thus fix $\alpha_* > 0$ such that

$$\mu_i'(\alpha) < 0 \qquad \text{for } \alpha \geq \alpha_*,\ i = 1, \dots, K. \tag{2.3.31}$$

We now fix $i \in \{1, \dots, K\}$. Then there exists a minimal positive integer ℓ_i such that

$$\mu_i(\alpha_*) + \lambda_\ell > 0 \qquad \text{for } \ell \geq \ell_i.$$

Moreover, since $\mu_i(\alpha) \to -\infty$ as $\alpha \to \infty$ by Theorem 2.1.3, there exists, for every $\ell \geq \ell_i$, precisely one value $\alpha_{i,\ell} \in (\alpha_*, \infty)$ such that

$$\mu_i(\alpha_{i,\ell}) + \lambda_\ell = 0.$$

Fix such a value $\alpha_{i,\ell}$ and put $\delta_{i,\ell} = \alpha_{i,\ell} - \alpha_*$. Since the curves $\alpha \mapsto \mu_j(\alpha)$, $j = 1, \dots, K$ are bounded on the interval $[\alpha_*, \alpha_{i,\ell} + \delta_{i,\ell}]$, it follows that the set

$$N_{i,\ell} := \begin{Bmatrix} (j, \ell') \in \{1, \dots, K\} \times (\mathbb{N} \cup \{0\}) \ : \\ \mu_j(\alpha) + \lambda_{\ell'} = 0 \text{ for some } \alpha \in [\alpha_*, \alpha_{i,\ell} + \delta_{i,\ell}] \end{Bmatrix}$$

is finite. Combining this fact with (2.3.31), we find $\varepsilon_{i,\ell} \in (0, \delta_{i,\ell})$ such that

$$\mu_j(\alpha) + \lambda_{\ell'} \neq 0 \quad \text{for } \alpha \in (\alpha_{i,\ell} - \varepsilon_{i,\ell}, \alpha_{i,\ell} + \varepsilon_{i,\ell}) \setminus \{\alpha_{i,\ell}\}, j = 1, \dots, K \text{ and } \ell' \in \mathbb{N} \cup \{0\}.$$

From Proposition 2.1.2, it then follows that u_α is nondegenerate for $\alpha \in (\alpha_{i,\ell} - \varepsilon_{i,\ell}, \alpha_{i,\ell} + \varepsilon_{i,\ell})$, $\alpha \neq \alpha_{i,\ell}$. Finally, it also follows from Proposition 2.1.2 and (2.3.31) that

$$m(u_{\alpha_{i,\ell}+\varepsilon}) - m(u_{\alpha_{i,\ell}-\varepsilon}) = \sum_{(j,\ell') \in M_{i,\ell}} d_{\ell'} > 0 \qquad \text{for } \varepsilon \in (0, \varepsilon_{i,\ell}),$$

where $M_{i,\ell} \subset \{1, \dots, K\} \times (\mathbb{N} \cup \{0\})$ is the set of pairs (j, ℓ') with $\mu_j(\alpha_{i,\ell}) + \lambda_{\ell'} = 0$ and, as before, $d_{\ell'}$ is the dimension of the space of spherical harmonics of degree ℓ'. Here we note that $M_{i,\ell} \neq \emptyset$ since it contains (i, ℓ). □

2.4 Differentiability of the map G

In this section, we give the proof of Proposition 2.3.9, which we restate here in a slightly more general form. As before, we fix $p > 2$ and $\gamma_\diamond \in [0, \frac{N-2}{N+\alpha_p})$.

Proposition 2.4.1. *Let $\varepsilon_0 \in (0, \frac{1}{2})$ be given by Proposition 2.2.8, so that the map $(-\varepsilon_0, \gamma_\diamond) \to C_0^1(I)$, $\gamma \mapsto U_\gamma$ is well defined and differentiable by Remark 2.2.9. Let, furthermore, $\delta \in (0, \frac{2}{N})$, and let the map*

$$G : (-\varepsilon_0, \gamma_\diamond) \times W_\delta^2(I) \times \mathbb{R} \to L_\delta^1(I) \times \mathbb{R}$$

be defined by (2.3.18). Then G is of class C^1 with

$$d_\gamma G(\gamma, \Psi, \nu) = \begin{pmatrix} \Psi' - (p-1)e^{(\gamma-1)t}|U_\gamma|^{p-2}\Big(t + (p-2)\frac{U_\gamma \partial_\gamma U_\gamma}{|U_\gamma|^2}\Big)\Psi \\ 0 \end{pmatrix},$$

$$d_\nu G(\gamma, \Psi, \nu) = \begin{pmatrix} -\Psi \\ 0 \end{pmatrix}$$

and $$d_\Psi G(\gamma, \Psi, \nu)\varphi = \begin{pmatrix} -\varphi'' + \gamma\varphi' - (p-1)e^{(\gamma-1)t}|U_\gamma|^{p-2}\varphi - \nu\varphi \\ \int_0^\infty \Psi_0 \varphi \, dt \end{pmatrix}.$$

The remainder of this section is devoted to the proof of this proposition. We first note that, by Lemma 2.2.5, U_γ has a finite number of simple zeros and satisfies $\lim_{t\to\infty} |U_\gamma(t)| > 0$ for $\gamma \in (-\varepsilon_0, \gamma_\diamond)$. The key step in the proof of Proposition 2.4.1 is the following lemma.

Lemma 2.4.2. *Let $q > 0$, and let $\mathcal{U} \subset C_0^1(I)$ be the open subset of functions $u \in C_0^1(I)$ which have a finite number of simple zeros and satisfy $\lim_{t\to\infty} |u(t)| > 0$. Then the nonlinear map*

$$h_q : \mathcal{U} \to L_0^1(I), \qquad u \mapsto |u|^q$$

is of class C^1 with

$$h_q'(u)w = q|u|^{q-2}uw \ \in \ L_0^1(I) \qquad \text{for } u \in \mathcal{U}, w \in C_0^1(I).$$

Here we identify $|u|^{q-2}u$ with $\operatorname{sgn}(u)$ in the case $q = 1$.

Proof. We only consider the case $q \in (0, 1)$. The proof in the case $q = 1$ is similar but simpler, and the proof in the case $q > 1$ is standard. We first prove

Claim 1. *If $1 \le r < \frac{1}{1-q}$, then the map $\sigma_q : \mathcal{U} \to L_0^r(I)$, $\sigma_q(u) = |u|^{q-2}u$ is well defined and continuous.*

To see this, we note that, by definition of $\mathcal{U}$, for every $u \in \mathcal{U}$ we have

$$\kappa_u := \sup\left\{\frac{|\{|u| \le \tau\} \cap (t, t+1)|}{\tau} : \tau > 0,\ t \ge 0\right\} < \infty. \tag{2.4.1}$$

More generally, if $K \subset \mathcal{U}$ is a compact subset (with respect to $\|\cdot\|_{C_0^1}$), we also have that

$$\kappa_K := \sup_{u\in K} \kappa_u < \infty.$$

As a consequence of (2.4.1), we have

$$\begin{aligned}\int_t^{t+1} |\sigma_q(u)|^r\,dx = \int_t^{t+1} |u|^{(q-1)r}\,dx &= \int_0^\infty |(t, t+1) \cap \{|u|^{(q-1)r} \ge s\}|\,ds \\ &= \int_0^\infty |(t, t+1) \cap \{|u| \le s^{\frac{1}{(q-1)r}}\}|\,ds \\ &\le \int_0^\infty \min\{1, \kappa_u\, s^{\frac{1}{(q-1)r}}\}\,ds < \infty\end{aligned}$$

for every $u \in \mathcal{U}$ and $t \ge 0$, since $\frac{1}{(q-1)r} < -1$ by assumption. Hence $\sigma_q(u) \in L_0^r(I)$ for every $u \in \mathcal{U}$, so the map σ_q is well defined. To see the continuity of σ_q, let $(u_n)_n \subset \mathcal{U}$ be a sequence such that $u_n \to u \in \mathcal{U}$ as $n \to \infty$ with respect to the C_0^1-norm. We then consider the compact set $K := \{u_n, u \ : \ n \in \mathbb{N}\}$. For given $\varepsilon > 0$, we fix $c \in (0, 1)$ sufficiently small such that

$$c^{(q-1)r+1} < \frac{\varepsilon}{2^r \kappa_K \left(\frac{2^{1+(q-1)r}}{1+(q-1)r}\right)}. \tag{2.4.2}$$

Since $u_n \to u$ uniformly on $[0, \infty)$, it is easy to see that

$$\sup_{t\ge 0} \int_t^{t+1} 1_{\{|u|>c\}} \big|\sigma_q(u_n) - \sigma_q(u)\big|^r\,dx \to 0 \qquad \text{as } n \to \infty. \tag{2.4.3}$$

Moreover, there exists $n_0 \in \mathbb{N}$ with the property that

$$\{|u| \le c\} \subset \{|u_n| \le 2c\} \qquad \text{for } n \ge n_0.$$

Consequently, setting $v_n := |u_n|^{(q-1)r}$ for $n \geq n_0$ and $v := |u|^{(q-1)r}$, we find that

$$
\begin{aligned}
&\sup_{t\geq 0}\int_t^{t+1} 1_{\{|u|\leq c\}}\Big|\sigma_q(u_n)-\sigma_q(u)\Big|^r\,dx\\
&\leq 2^{r-1}\int_{\{|u|\leq c\}\cap(t,t+1)}\Big(|u_n|^{(q-1)r}+|u|^{(q-1)r}\Big)\,dx\\
&\leq 2^{r-1}\Big(\int_{\{|u_n|\leq 2c\}\cap(t,t+1)}|u_n|^{(q-1)r}\,dx+\int_{\{|u|\leq c\}\cap(t,t+1)}|u|^{(q-1)r}\,dx\Big)\\
&= 2^{r-1}\Big(\int_{\{v_n\geq (2c)^{(q-1)r}\}\cap(t,t+1)}v_n\,dx+\int_{\{v\geq c^{(q-1)r}\}\cap(t,t+1)}v\,dx\Big)\\
&= 2^{r-1}\Big(\int_{(2c)^{(q-1)r}}^{\infty}|\{v_n\geq s\}\cap(t,t+1)|\,ds\\
&+(2c)^{(q-1)r}|\{v_n\geq (2c)^{(q-1)r}\}\cap(t,t+1)|\\
&+\int_{c^{(q-1)r}}^{\infty}|\{v\geq s\}\cap(t,t+1)|\,ds+c^{(q-1)r}|\{v_n\geq c^{(q-1)r}\}\cap(t,t+1)|\Big)\\
&= 2^{r-1}\Big(\int_{(2c)^{(q-1)r}}^{\infty}|\{|u_n|\leq s^{\frac{1}{(q-1)r}}\}\cap(t,t+1)|\,ds\\
&+(2c)^{(q-1)r}|\{|u_n|\leq 2c\}\cap(t,t+1)|\\
&+\int_{c^{(q-1)r}}^{\infty}|\{|u|\leq s^{\frac{1}{(q-1)r}}\}\cap(t,t+1)|\,ds+c^{(q-1)r}|\{|u|\leq c\}\cap(t,t+1)|\Big)\\
&\leq 2^r\kappa_K\Big(\int_{(2c)^{(q-1)r}}^{\infty}s^{\frac{1}{(q-1)r}}\,ds+(2c)^{1+(q-1)r}\Big)\\
&= 2^r\kappa_K\Big(-\frac{(2c)^{(q-1)r+1}}{\frac{1}{(q-1)r}+1}+(2c)^{1+(q-1)r}\Big)\\
&= 2^r\kappa_K\Big(\frac{2^{1+(q-1)r}}{1+(q-1)r}\Big)c^{(q-1)r+1}<\varepsilon \qquad \text{for } n\geq n_0
\end{aligned}
$$

by (2.4.2). Combining this with (2.4.3) yields

$$\limsup_{n\to\infty}\|\sigma_q(u_n)-\sigma_q(u)\|_{r,0}^r=\limsup_{n\to\infty}\sup_{t\geq 0}[\sigma_q(u_n)-\sigma_q(u)]_{t,r}^r\leq\varepsilon.$$

Since $\varepsilon>0$ was given arbitrarily, we conclude that

$$\|\sigma_q(u_n)-\sigma_q(u)\|_{r,0}^r\to 0 \qquad \text{as } n\to\infty.$$

Hence Claim 1 follows.

Next, we let $u\in\mathcal{U}$ and $w\in C_0^1(I)$ with $\|w\|_{L^\infty(I)}<1$. For $\tau\in\mathbb{R}\setminus\{0\}$ we then have

$$\frac{1}{\tau}\Big(h_q(u+\tau w)-h_q(u)\Big)=I_\tau+J_\tau \quad \text{in } L_0^1(I)$$

with

$$I_\tau(x)=1_{\{|u|>|\tau|\}}\frac{|u+\tau w|^q-|u(x)|^q}{\tau},\quad J_\tau=1_{\{|u|\leq|\tau|\}}\frac{|u+\tau w|^q-|u|^q}{\tau}$$

Note that

$$I_\tau(x)=q\int_0^1 1_{\{|u|>|\tau|\}}(x)\sigma_q(u(x)+\rho\tau w(x))w(x)\,d\rho.$$

Hence

$$\Big[I_\tau-q\sigma_q(u)w\Big](x)=q\int_0^1\Big[\sigma_q(u+\rho\tau w)w-\sigma_q(u)w\Big](x)\,d\rho$$

$$-q\int_0^1 \Big[1_{\{|u|\le|\tau|\}}\sigma_q(u+\rho\tau w)w\Big](x)d\rho$$

where

$$\begin{aligned}&\int_t^{t+1}\Big|\int_0^1\Big[\sigma_q(u+\rho\tau w)w-\sigma_q(u)w\Big](x)\,d\rho\Big|dx\\ &\le \|w\|_{L^\infty(I)}\sup_{0\le\rho\le 1}\|\sigma_q(u+\rho\tau w)-\sigma_q(u)\|_{1,0}\quad \text{for } t\ge 0\end{aligned}$$

and, by Hölder's and Jensen's inequality,

$$\begin{aligned}&\int_t^{t+1}\Big|\Big[1_{\{|u|\le|\tau|\}}\int_0^1\sigma_q(u+\rho\tau w)wd\rho\Big](x)\Big|dx\\ &\le |\{|u|\le\tau\}\cap(t,t+1)|^{1/r'}\|w\|_{L^\infty(I)}\Big(\int_0^1\int_t^{t+1}|\sigma_q(u+\rho\tau w)|^r dxd\rho\Big)^{1/r}\\ &\le |\{|u|\le\tau\}|^{1/r'}\|w\|_{L^\infty(I)}\sup_{0\le\rho\le 1}\|\sigma_q(u+\rho\tau w)\|_{r,0}\qquad\text{for } t\ge 0.\end{aligned}$$

Combining these two estimates with Claim 1 and (2.4.1), we deduce that

$$\|I_\tau-q\sigma_q(u)w\|_{1,0}\longrightarrow 0\qquad \text{as } \tau\longrightarrow 0. \tag{2.4.4}$$

Next we estimate

$$\begin{aligned}\int_t^{t+1}|J_\tau|dx &\le \frac{1}{|\tau|}\int_t^{t+1}1_{\{|u|\le|\tau|\}}\Big(|u+\tau w|^q+|u|^q\Big)\,dx\\ &=|\tau|^{q-1}\int_t^{t+1}1_{\{|u|\le|\tau|\}}\Big|\frac{u}{\tau}+w\Big|^q+\Big|\frac{u}{\tau}\Big|^q\,dx\\ &\le |\tau|^{q-1}(2^q+1)|\{u|\le|\tau|\}\cap(t,t+1)|\le\kappa_K|\tau|^q(2^q+1)\quad\text{for } t\ge 0\end{aligned}$$

and therefore

$$\|J_\tau\|_{1,0}\longrightarrow 0\qquad\text{as }\tau\longrightarrow 0. \tag{2.4.5}$$

Combining (2.4.4) and (2.4.5), we deduce the existence of

$$h_q'(u)w=\lim_{\tau\to\infty}\frac{1}{\tau}\Big(h_q(u+\tau w)-h_q(u)\Big)=\sigma_q(u)w\qquad\text{in } L^1_0(I).$$

Together with Claim 1, this yields that h_q is of class C^1, as claimed. □

We may now complete the

Proof of Proposition 2.4.1. The C^1-regularity of G follows easily once we have seen that the map

$$H:(-\varepsilon_0,\gamma_\diamond)\times W^2_\delta(I)\longrightarrow L^1_\delta(I),\qquad (\gamma,\Psi)\mapsto e^{(\gamma-1)t}|U_\gamma|^{p-2}\Psi$$

is of class C^1. Note that we can write $H=H_3\circ H_2\circ H_1$ with

$$\begin{aligned}&H_1:(-\varepsilon_0,\gamma_\diamond)\times W^2_\delta(I)\longrightarrow(-\varepsilon_0,\gamma_\diamond)\times L^\infty(I)\times C^1_0(I),\qquad (\gamma,\Psi)\mapsto(\gamma,\Psi,U_\gamma)\\ &H_2:(-\varepsilon_0,\gamma_\diamond)\times L^\infty(I)\times\mathcal{U}\longrightarrow(-\varepsilon_0,\gamma_\diamond)\times L^\infty(I)\times L^1_0(I),\\ &\qquad(\gamma,\Psi,v)\mapsto(\gamma,\Psi,|v|^{p-2})\\ &H_3:(-\varepsilon_0,\gamma_\diamond)\times L^\infty(I)\times L^1_0(I)\longrightarrow L^1_\delta(I),\qquad(\gamma,\psi,v)\mapsto e^{(\gamma-1)(\cdot)}v\psi\end{aligned}$$

The C^1-regularity of H_1 is a consequence of Proposition 2.2.8, and the C^1-regularity of H_2 is a consequence of Lemma 2.4.2. Finally, the C^1-regularity of H_3 is easy to check since $e^{(\gamma-1)t}\le e^{-\delta t}$ for $\gamma<\gamma_\diamond$. Hence we conclude that H is of class C^1, and this finishes the proof. □

2.5 Bifurcation of almost radial nodal solutions

In this section, we prove the bifurcation result stated in Theorem 2.1.6.

Proof of Theorem 2.1.6. The proof relies on Corollary 2.1.5 and a result by Kielhöfer [**77**]. To adapt our problem to the setting of [**77**], we consider the Hilbert space $E := L^2(\mathbf{B})$, $D := H^2(\mathbf{B}) \cap H_0^1(\mathbf{B})$, fix $\alpha := \alpha_{i,\ell}$ as in the assumption and consider the map $G : (-\alpha, \infty) \times D \to E$ given by

$$[G(\lambda, u)] = -\Delta(u + u_{\alpha+\lambda}) - |x|^{\alpha+\lambda}|u + u_{\alpha+\lambda}|^{p-2}(u + u_{\alpha+\lambda}).$$

Then G is continuous with $G(\lambda, 0) = 0$ for $\lambda > -\alpha$. Moreover, the Fréchet derivative $A(\lambda) := G_u(\lambda, 0)$, given by

$$A(\lambda)\varphi = -\Delta\varphi - (p-1)|x|^{\alpha+\lambda}|u_{\alpha+\lambda}|^{p-2}\varphi,$$

exists for $\lambda > -\alpha$ and coincides with the linearized operator $L^{\alpha+\lambda}$ from (2.1.2). Hence it is a Fredholm operator of index zero having an isolated eigenvalue 0.

Furthermore, there is a differentiable potential $g : \mathbb{R} \times D \to \mathbb{R}$ such that $g_u(\lambda, u)h = (G(\lambda, u), h)_{L^2}$ for all $h \in D$ in a neighborhood of $(0, 0)$, given by

$$g(\lambda, u) = \int_{\mathbf{B}} \Big(\frac{1}{2}|\nabla(u + u_{\alpha+\lambda})|^2 - \frac{|x|^{\alpha+\lambda}}{p}|u + u_{\alpha+\lambda}|^p\Big)\, dx.$$

To apply the main theorem in [**77**], we need to ensure that the crossing number of the operator family $A(\lambda)$ through $\lambda = 0$ is nonzero. This is a consequence of Corollary 2.1.5(iii), which implies that the number of negative eigenvalues of the linearized operator $L^{\alpha+\varepsilon} = A(\varepsilon)$ is strictly larger than that of $L^{\alpha-\varepsilon} = A(-\varepsilon)$ for small $\varepsilon > 0$.

Therefore, [**77**, Theorem, p.4] implies that $(0, 0)$ is a bifurcation point for the equation $G(\lambda, u) = 0$, $(\lambda, u) \in \mathbb{R} \times D$, i.e. there exists a sequence $((\lambda_n, v_n))_n \subset \mathbb{R} \times D \setminus \{0\}$ such that

$$G(\lambda_n, v_n) = 0 \quad \text{for all } n, \qquad (\lambda_n, v_n) \to (0, 0) \quad \text{in } \mathbb{R} \times D \text{ as } n \to \infty.$$

Setting $\alpha_n := \alpha + \lambda_n$, $u^n := v_n + u_{\alpha_n}$ we conclude

$$-\Delta u^n - |x|^{\alpha_n}|u^n|^{p-2}u^n = G(\lambda_n, v_n) = 0,$$

i.e. u^n is a solution of (2.1.1). Moreover, $u^n \to u_\alpha$ in D. We may therefore deduce by elliptic regularity – using the fact that the RHS of (2.1.1) is Hölder continuous in x and u – that the sequence $(u^n)_n$ is bounded in $C^{2,\rho}(\overline{\mathbf{B}})$ for some $\rho > 0$, and from this we deduce that $u^n \to u_\alpha \in C^2(\overline{\mathbf{B}})$. Since u_α is radially symmetric with precisely K nodal domains, there exist $r_0 := 0 < r_1 < \cdots < r_K := 1$ such that, for $i = 1, \ldots, K$,

$$u_\alpha(x) = 0,\ (-1)^i \partial_r u^n(x) > 0 \quad \text{for } |x| = r_i \qquad \text{and}$$

$$(-1)^{i-1} u_\alpha(x) > 0 \text{ for } r_{i-1} < |x| < r_i,$$

where ∂_r denotes the derivative in the radial direction. Consequently, there exist $\varepsilon, \delta > 0$ such that, after passing to a subsequence,

$$(-1)^{i+1} u^n(x) > \varepsilon \quad \text{for } r_{i-1} + \delta < |x| < r_i - \delta,\ n \in \mathbb{N}$$

and

$$(-1)^i \partial_r u^n(x) > 0 \quad \text{for } r_i - \delta < |x| < r_i + \delta,\ n \in \mathbb{N}.$$

We conclude that for $i = 1, \dots, K-1$ and each direction $w \in \mathbb{S}^{N-1}$ the function

$$(r_i - \delta, r_i + \delta) \longrightarrow \mathbb{R}, \quad t \mapsto u^n(tw)$$

has precisely one zero, which we denote by $r_{i,n}(w)$. In particular, the nodal domains of u^n are given by

$$\Omega_1 := \left\{ x \in \mathbf{B} : |x| < r_{1,n}\left(\frac{x}{|x|}\right) \right\} \quad \text{and}$$

$$\Omega_i := \left\{ x \in \mathbf{B} : r_{i-1,n}\left(\frac{x}{|x|}\right) < |x| < r_{i,n}\left(\frac{x}{|x|}\right) \right\}$$

for $i = 2, \dots K$. Consequently, $0 \in \Omega_1$, Ω_1 is homeomorphic to a ball, and $\Omega_2, \dots, \Omega_K$ are homeomorphic to annuli. Finally, we note that $u^n = v_n + u_{\alpha_n}$ is nonradial, since $v_n \not\equiv 0$ and u_{α_n} is the *unique* radial solution of (2.1.1) with $\alpha = \alpha_n$ and with K nodal domains. □

2.6 Remarks on the case $N = 2$

In the following, we discuss the case $N = 2$. In this case an additional technical difficulty appears, whereas the main results simplify considerably. We first note that, in contrast to the case $N \geq 3$, the transformation considered in (2.1.3) immediately yields a solution of the limit problem. Indeed, we have the following analogue of Proposition 2.1.1:

Proposition 2.6.1. *Let $N = 2$, $p > 2$, $K \in \mathbb{N}$ and $\alpha > 0$. Then the unique radial solution u_α of* (2.1.1) *with K nodal domains and $u_\alpha(0) > 0$ satisfies*

$$U_\infty(t) = (2+\alpha)^{-\frac{2}{p-2}} u_\alpha(e^{-\frac{t}{2+\alpha}}) \qquad \textit{for } t \in [0,\infty),$$

where $U_\infty \in C^2([0,\infty))$ is the unique bounded solution of the problem

$$-U'' = e^{-t}|U|^{p-2}U \quad \textit{in } [0,\infty), \qquad U(0) = 0$$

with $U'(0) > 0$ and with precisely $K-1$ zeros in $(0,\infty)$.

This fact follows by direct calculation and may also be interpreted as a consequence of the following property of the twodimensional Hénon equation observed in [**102**]: The unique radial solution u_α of (2.1.1) with K nodal domains and $u_\alpha(0) > 0$ is given as

$$u_\alpha(x) = \left(\frac{\alpha+2}{2}\right)^{\frac{2}{p-2}} w_0(|x|^{\frac{\alpha}{2}} x)$$

where w_0 is the unique radial solution of the Lane-Emden equation

$$\begin{cases} -\Delta w = |w|^{p-2}w & \text{in } \mathbf{B}, \\ \quad w = 0 & \text{on } \partial\mathbf{B}, \end{cases}$$

with K nodal domains and $w(0) > 0$.

Similarly, one may expect that the eigenvalue expansions given in Theorem 2.1.3 simplify in the case $N = 2$. However, a technical difficulty arises in the ansatz (2.1.6), as, for functions in $H_0^1(\mathbf{B})$, the quantity $\int_{\mathbf{B}} |x|^{-2}\psi^2 dx$ is finite in the case $N \geq 3$ due to Hardy's inequality, but not in the case $N = 2$. Therefore, as noted in [**5**], it is necessary to replace $H_0^1(\mathbf{B})$ by the space

$$\mathcal{H} := H_0^1(\mathbf{B}) \cap \left\{ v \in L^2(\mathbf{B}) : \int_{\mathbf{B}} \frac{v^2}{|x|^2}\, dx < \infty \right\}$$

and to consider the eigenvalue problem

$$L^\alpha \varphi = \frac{\lambda}{|x|^2}\varphi, \qquad \varphi \in \mathcal{H} \tag{2.6.1}$$

in place of (2.1.5). By [**5**, Prop. 4.1 and 5.1], Lemma 2.3.1 continues to hold in the case $N = 2$ for $\alpha > 0$ with (2.1.5) replaced by (2.6.1). Moreover, considering $\mathcal{H}_{rad} := \mathcal{H} \cap H^1_{0,rad}(\mathbf{B})$ and the radial eigenvalue problem

$$L^\alpha \psi = \frac{\mu}{|x|^2}\psi, \qquad \psi \in \mathcal{H}_{rad} \tag{2.6.2}$$

in place of (2.1.7), it again follows from [**5**, Prop. 3.7] that (2.6.2) admits precisely K negative eigenvalues

$$\mu_1(\alpha) < \mu_2(\alpha) < \cdots < \mu_K(\alpha) < 0 \tag{2.6.3}$$

given by

$$\mu_j(\alpha) := \inf_{\substack{W \subset \mathcal{H}_{rad} \\ \dim W = j}} \max_{v \in W \setminus \{0\}} \frac{\int_{\mathbf{B}} |\nabla v|^2 - (p-1)|x|^\alpha |u_\alpha|^{p-2}|v|^2\, dx}{\int_{\mathbf{B}} |x|^{-2}|v|^2\, dx}, \qquad j = 1, \ldots, K \tag{2.6.4}$$

for $\alpha > 0$. Furthermore, Proposition 2.1.2 continuous to hold by [**5**, Prop. 1.3 and 1.4] in this setting. Next we note that if $\psi \in \mathcal{H}_{rad}$ is an eigenfunction of (2.6.2) corresponding to $\mu_j(\alpha)$, then the function $\Psi(t) := (2+\alpha)\psi(e^{-\frac{t}{2+\alpha}})$ satisfies (2.3.6) with $\nu = \frac{1}{(2+\alpha)^2}\mu_j(\alpha)$. We recall that we have already shown in Proposition 2.3.4 that (2.3.6) admits precisely K negative eigenvalues. Indeed, the eigenvalue problem (2.3.6), which does not depend on the dimension N, is included in the statement of Proposition 2.3.4 by considering $\gamma = 0$, $N \geq 3$. In sum, this yields the following more explicit version of Theorem 2.1.3:

Theorem 2.6.2. *Let $N = 2$, $p > 2$ and $\alpha > 0$. Let $\nu_1^* < \nu_2^* < \cdots < \nu_K^* < 0$ denote the negative eigenvalues of the eigenvalue problem*

$$\begin{cases} -\Psi'' - (p-1)e^{-t}|U_\infty(t)|^{p-2}\Psi = \nu\Psi & \text{in } [0,\infty), \\ \quad \Psi(0) = 0, \quad \Psi \in L^\infty(0,\infty), \end{cases} \tag{2.6.5}$$

with U_∞ given in Proposition 2.1.1. Then the negative eigenvalues of (2.6.2) are given by

$$\mu_i(\alpha) = (2+\alpha)^2 \nu_i^* \qquad \text{for } \alpha > 0. \tag{2.6.6}$$

In particular, the curves μ_i, $i = 1, \ldots, K$ are strictly decreasing on $(0,\infty)$.

From Theorem 2.6.2, we now deduce that Corollary 2.1.5 also holds in the case $N = 2$, and from this we may then derive the bifurcation result stated in Theorem 2.1.6 in the case $N = 2$, $2 < p < \infty$.

CHAPTER 3

Spiraling Solutions of Nonlinear Schrödinger Equations

In this chapter, we present our results on spiraling solutions of nonlinear Schrödinger equations as outlined in Section 1.3. Up to minor changes, the contents have appeared in **[P2]**.

3.1 Introduction

This paper is concerned with a new class of solutions to the stationary nonlinear Schrödinger equation

$$-\Delta v + qv = |v|^{p-2}v \qquad \text{in } \mathbb{R}^N, \tag{3.1.1}$$

where $p > 2$ and $q \geq 0$ is a constant. Since the case $q > 0$ is equivalent to $q = 1$ by rescaling, we only consider the cases $q = 1$ and $q = 0$ in the following.

For subcritical exponents p (i.e., $p < \frac{2N}{N-2}$ if $N \geq 3$) and $q = 1$, there is a vast literature on solutions of (3.1.1) in $H^1(\mathbb{R}^N)$, which decay expontially at infinity, see e.g. the monographs [**7**, **80**, **125**, **126**, **136**] and the references therein.

In the present paper, we focus on solutions with only partial decay. These solutions are less understood, but have attracted considerable attention in recent years.

To be more precise, let us write $\bar{x} = (x, t) \in \mathbb{R}^N$ with $x \in \mathbb{R}^{N-1}$ and $t \in \mathbb{R}$. We shall consider solutions $v : \mathbb{R}^N \to \mathbb{R}$ satisfying

$$\lim_{|x|\to\infty} v(x,t) = 0 \quad \text{uniformly in } t. \tag{3.1.2}$$

A trivial class of solutions satisfying (3.1.2) is the class of solutions that are axially symmetric with respect to the axis $\{(0_{\mathbb{R}^{N-1}}, t) \;:\; t \in \mathbb{R}\} \subset \mathbb{R}^N$ and that in addition are t-invariant, i.e., solutions having the form $v(x,t) = \tilde{v}(x)$, where $\tilde{v}$ is a radial solution of (3.1.1) in $\mathbb{R}^{N-1}$ satisfying $\tilde{v}(x) \to 0$ as $|x| \to \infty$. Here and in the following, axial symmetry is always understood with respect to the t-axis.

In a seminal paper, Dancer [**43**] constructed, for $q = 1$, nontrivial, t-periodic axially symmetric solutions of (3.1.1) by means of bifurcation theory. The solutions found in [**43**] are positive, and they bifurcate from the unique family of t-invariant axially symmetric positive solutions of (3.1.1).

It is natural to ask whether, for a given *positive* solution of (3.1.1), the decay property (3.1.2) enforces axial symmetry up to translations. As shown in the following theorem by Farina, Malchiodi and Rizzi in [**56**], this is true for positive solutions which are periodic in t.

Theorem 3.1.1. *[**56**, Special case of Theorem 2]*
Let $p > 2$, $q = 1$, and let $v \in C^2(\mathbb{R}^N)$ be a bounded positive solution of (3.1.1) satisfying the

uniform decay property (3.1.2). Suppose moreover that v is periodic in t, i.e., there exists $\tau \in \mathbb{R}$ with

$$v(x, t+\tau) = v(x,t) \qquad \text{for all } (x,t) \in \mathbb{R}^N \text{ with some constant } \tau > 0.$$

Then, up to translations in the x-variable, v is axially symmetric.

Let us also briefly discuss the case $q = 0$ in (3.1.1). In this case, for subcritical p, it is known that (3.1.1) does not admit positive solutions (see [**63**, Theorem 1.1]), and it also does not admit solutions of any sign in $H^1(\mathbb{R}^N)$ (by Pohozaev's identity, see e.g. [**136**, Appendix B]). The latter property is related to the fact that, in this case, equation (3.1.1) remains invariant under the rescaling transformation $v \mapsto \kappa^{\frac{2}{p-2}} v(\kappa\,\cdot)$.

In the present paper, we discuss solutions of (3.1.1) - (3.1.2) with periodicity in t, but without axial symmetry. By Theorem 3.1.1 and the remarks above, such solutions have to change sign. As far as we know, solutions of this type have not been studied yet with the exception of the t-independent case where $v(x,t) = \tilde{v}(x)$ for some nonradial sign-changing solution $\tilde{v}$ of (3.1.1) in $\mathbb{R}^{N-1}$ with $\tilde{v}(x) \to 0$ as $|x| \to \infty$.

In this context, we briefly recall some existence results on nonradial sign-changing solutions of (3.1.1) in $\mathbb{R}^N$ for $q = 1$ with exponential decay in all variables. In [**10**], solutions of this type were found for $N = 4$ or $N \geq 6$ by a careful application of the Fountain Theorem within the space of functions in $H^1(\mathbb{R}^N)$ that are invariant under the action of the group $O(m) \times O(m) \times O(N-2m)$, with $N \geq 2m+1$. The case $N = 5$ was considered subsequently by a related argument in [**88**]. More recently, in [**8**, **105**], nonradial sign-changing solutions to (3.1.1) with no symmetry and with dihedral symmetry respectively, have been constructed with the Lyapunov-Schmidt reduction method in any dimension $N \geq 2$.

In the following, we restrict our attention to the case $N = 3$ and consider the special class of *spiraling solutions* of the nonlinear Schrödinger equation

$$-\Delta v + qv = |v|^{p-2} v \qquad \text{in } \mathbb{R}^3, \tag{3.1.3}$$

i.e., solutions that are invariant under the action of a screw motion.

To be more precise, let $\lambda > 0$. We call a function $v : \mathbb{R}^3 \to \mathbb{R}$ λ−spiraling if for any $\theta \in \mathbb{R}$,

$$v(R_\theta x, t + \lambda\theta) = v(x,t) \qquad \text{for } x \in \mathbb{R}^2,\ t \in \mathbb{R}, \tag{3.1.4}$$

where $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the counter-clockwise rotation with angle θ in $\mathbb{R}^2$. Notice that λ−spiraling functions are $2\lambda\pi$-periodic in t. Hence, the parameter λ represents the rotational slope of the underlying screw motion, and $2\lambda\pi$ is the associated turn-around shift.

Our work is partly inspired by the papers [**46**] resp. [**39**] where spiraling solutions have been constructed for the classical and fractional Allen-Cahn equation, respectively. Without going into detail, we mention the well known fact that, despite its similar looking form, the Allen-Cahn equation $-\Delta u = u - u^3$ differs significantly from the nonlinear Schrödinger equation (3.1.3) with regard to the variational framework and the shape of solutions.

In cylindrical coordinates $(x,t) = (r\cos\varphi, r\sin\varphi, t)$ with $(r,\varphi,t) \in [0,\infty) \times \mathbb{R} \times \mathbb{R}$, λ−spiraling functions have the form

$$v(r,\varphi,t) = u\left(r, \varphi - \frac{t}{\lambda}\right)$$

with a function $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ which is 2π-periodic in the second variable. Also, in these coordinates the equation (3.1.3) reads as

$$-v_{rr} - \frac{v_r}{r} - \frac{v_{\varphi\varphi}}{r^2} - v_{tt} + q\,v = |v|^{p-2} v$$

so that the equation for u has the form

$$-u_{rr} - \frac{u_r}{r} - \Big(\frac{1}{\lambda^2} + \frac{1}{r^2}\Big)u_{\theta\theta} + q\,u = |u|^{p-2}u. \tag{3.1.5}$$

It is convenient to transform equation (3.1.5) further to planar euclidean coordinates $x = (x_1, x_2)$, where $r = |x|$ and $\theta = \arcsin\frac{x_2}{|x|}$. This leads to the problem

$$\begin{cases} -\Delta u - \dfrac{1}{\lambda^2}[x_1\partial_{x_2} - x_2\partial_{x_1}]^2 u + q\,u = |u|^{p-2}u & \text{on } \mathbb{R}^2, \\ \qquad\qquad u(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{3.1.6}$$

Observe that radial solutions of (3.1.6) correspond to axially symmetric and t-invariant solutions of (3.1.3). By Theorem 3.1.1, every positive solution of (3.1.6) is radial. On the other hand, nonradial solutions of (3.1.6) correspond to solutions of (3.1.3) which are $2\lambda\pi$-periodic in t but neither axially symmetric nor t-invariant. We therefore restrict our attention to nodal (i.e., sign-changing) solutions of (3.1.6).

We study problem (3.1.6) using variational methods, and hence we first introduce some notation related to its variational structure.

We write $\partial_\theta := x_1\partial_{x_2} - x_2\partial_{x_1}$ for the angular derivative and consider the space

$$H := \Big\{u \in H^1(\mathbb{R}^2) \;:\; \int_{\mathbb{R}^2} |\partial_\theta u|^2 dx < \infty\Big\}. \tag{3.1.7}$$

For $\lambda > 0$, we endow H with the λ-dependent scalar product

$$\langle u, v\rangle_\lambda := \int_{\mathbb{R}^2}\Big(\nabla u \cdot \nabla v + \frac{1}{\lambda^2}(\partial_\theta u)(\partial_\theta v) + uv\Big)dx \tag{3.1.8}$$

and consider the Hilbert space $(H, \langle\cdot,\cdot\rangle_\lambda)$.

Let $E_\lambda : H \to \mathbb{R}$ be the energy functional associated to (3.1.6) in the case $q = 1$, defined by

$$E_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^2}\Big(|\nabla u|^2 + \frac{1}{\lambda^2}|\partial_\theta u|^2 + |u|^2\Big)dx - \frac{1}{p}\int_{\mathbb{R}^2}|u|^p\,dx. \tag{3.1.9}$$

By standard arguments, E_λ is of class C^1, and critical points of E_λ are weak solutions of (3.1.6).

By definition, a least energy nodal solution of (3.1.6) is a minimizer of E_λ within the class of sign-changing solutions of (3.1.6). Our first main result is concerned with least energy nodal solutions and reads as follows.

Theorem 3.1.2. *Let $p > 2$ and $q = 1$. For every $\lambda > 0$ there exists a least energy nodal solution of* (3.1.6)*. Furthermore, there exist $0 < \lambda_0 \le \Lambda_0 < \infty$ with the following properties:*

(i) For $\lambda < \lambda_0$, every least energy nodal solution of (3.1.6) *is radial.*

(ii) For $\lambda > \Lambda_0$, every least energy nodal solution of (3.1.6) *is nonradial.*

Theorem 3.1.2 establishes a symmetry breaking phenomenon for least energy nodal solutions which occurs within a finite range of parameters $\lambda \in [\lambda_0, \Lambda_0]$. We are not aware of any other setting where such a transition from radiality to nonradiality has been observed for least energy nodal solutions. The main difficulty when dealing with least energy radial nodal solutions of the equation $-\Delta u + u = |u|^{p-2}u$ in $\mathbb{R}^2$ is given by the fact that so far neither uniqueness (up to sign) nor nondegeneracy is known. Hence, in order to prove the first part of Theorem 3.1.2, we have to follow an approach which does not rely on these properties. In fact, a more general radiality result for solutions of (3.1.6) with small $\lambda > 0$ can be obtained by combining uniform elliptic L^∞-estimates with Poincaré type inequalities in the angular variable. More precisely, we have the following.

Theorem 3.1.3. *Let $p > 2$ and $q = 1$.*

i. If $u \in H$ is a nontrivial weak solution of (3.1.6) for some $\lambda > 0$ satisfying $\lambda < \left(\frac{1}{(p-1)|u|_\infty^{p-2}}\right)^{\frac{1}{2}}$, then u is a radial function.

ii. For every $c > 0$, there exists $\lambda_c > 0$ with the property that every weak solution $u \in H$ of (3.1.6) for some $\lambda \in (0, \lambda_c)$ with $E_\lambda(u) \leq c$ is radial.

The first part of Theorem 3.1.2 turns out to be a consequence of Theorem 3.1.3(ii) and uniform (in λ) energy estimates for least energy nodal solutions of (3.1.6) in the case $p > 2$, $q = 1$, see Section 3.5 below.

While least energy nodal solutions are particularly interesting from a variational point of view, Theorem 3.1.2(i) and Theorem 3.1.3(ii) show that, in order to detect nonradial sign-changing solutions of (3.1.6) for small values $\lambda > 0$, we have to pass to higher energy levels. A natural class of nonradial nodal solutions of (3.1.6) is the class of odd solutions with respect to a hyperplane reflection.

If we consider the hyperplane $\{x_1 = 0\}$, then any such solution corresponds to a solution of the boundary value problem

$$\begin{cases} -\Delta u - \dfrac{1}{\lambda^2}[x_1\partial_{x_2} - x_2\partial_{x_1}]^2 u + q\,u = |u|^{p-2}u & \text{on } \mathbb{R}^2_+, \\ u = 0 & \text{on } \partial\mathbb{R}^2_+ \end{cases} \tag{3.1.10}$$

in the half space $\mathbb{R}^2_+ := \{x \in \mathbb{R}^2 \ : \ x_1 > 0\}$. Moreover, by odd reflection and transformation of coordinates, any such solution u gives rise to a λ−spiraling nodal solution $v : \mathbb{R}^3 \to \mathbb{R}$ of (3.1.3) with the property that

$$v(0,t) = 0 = v(R_t(0,x_2), \lambda t) \qquad \text{for all } t, x_2 \in \mathbb{R}.$$

Consequently, v vanishes on a helicoid, i.e. the condition $u = 0$ on $\partial\mathbb{R}^2_+$ implies that v is zero on the set $\{(x \sin t, x \cos t, \lambda t) : t, x \in \mathbb{R}\}$.

Weak solutions of (3.1.10) correspond to critical points of the C^1-functional $E_\lambda^+ : H^+ \to \mathbb{R}$ defined by

$$E_\lambda^+(u) := \frac{1}{2}\int_{\mathbb{R}^2_+}\left(|\nabla u|^2 + \frac{1}{\lambda^2}|\partial_\theta u|^2 + qu^2\right)dx - \frac{1}{p}\int_{\mathbb{R}^2_+}|u|^p\,dx, \tag{3.1.11}$$

where

$$H^+ := \left\{u \in H^1_0(\mathbb{R}^2_+) \ : \ \int_{\mathbb{R}^2_+}|\partial_\theta u|^2 dx < \infty\right\}. \tag{3.1.12}$$

By trivial extension, we regard H^+ as a closed subspace of H, see Section 3.3 below for details.

Our main result for (3.1.10) reads as follows.

Theorem 3.1.4. *Let $p > 2$, $q \in \{0, 1\}$ and $\lambda > 0$.*

(i) (Existence) Problem (3.1.10) admits a positive least energy solution.

(ii) *(Symmetry) Any positive solution u of (3.1.10) is symmetric with respect to reflection at the x_1-axis and decreasing in the angle $|\theta|$ from the x_1-axis. In particular, u takes its maximum on the x_1-axis.*

(iii) *(Asymptotics) If $q = 1$ and $\lambda_k \geq 1$ are given with $\lambda_k \to +\infty$ as $k \to \infty$ and u_k is a positive least energy solution of (3.1.10) with $\lambda = \lambda_k$, then, after passing to a subsequence, there exists a sequence of numbers $\tau_k > 0$ with*

$$\tau_k \to +\infty, \qquad \frac{\tau_k}{\lambda_k} \to 0 \qquad \text{as } k \to \infty$$

such that the translated functions $w_k \in H^1(\mathbb{R}^2)$, $w_k(x) = u_k(x_1 + \tau_k, x_2)$ satisfy

$$w_k \to w_\infty \qquad \text{strongly in } H^1(\mathbb{R}^2),$$

where w_∞ is the unique positive radial solution of

$$-\Delta w_\infty + w_\infty = |w_\infty|^{p-2} w_\infty, \qquad w_\infty \in H^1(\mathbb{R}^2). \tag{3.1.13}$$

Similarly as defined for the equation (3.1.6), a least energy solution of (3.1.10) is, by definition, an energy minimizer within the class of nontrivial solutions of (3.1.10). More specifically, least energy solutions will be characterized as minimizers of E_λ^+ w.r.t. the associated Nehari manifold and attain the mountain pass level

$$c_\lambda = \inf_{u \in H^+ \setminus \{0\}} \sup_{t \geq 0} E_\lambda^+(tu), \tag{3.1.14}$$

see Section 3.3 below. We also point out that the uniqueness of a positive radial solution to (3.1.13) was shown by Kwong [**78**].

Remark 3.1.5. *(i) Let $p > 2$ and $q = 1$. As a consequence of Theorem 3.1.4, the energy of the least energy nodal solution of (3.1.6), as considered in Theorem 3.1.2, tends to $2c_\infty$ as $\lambda \to \infty$, where c_∞ is the least energy of nontrivial solutions of the limit problem (3.1.13). This fact is the key ingredient in the proof of Theorem 3.1.2(ii).*

*(ii) The existence result for (3.1.10) for $p > 2$ and $q \in \{0, 1\}$ relies on compact embeddings. More precisely, we will prove in Section 3.2 below that the space H is compactly embedded into $L^\rho(\mathbb{R}^2)$ for $\rho \in (2, \infty)$, which readily implies that the space H^+ is compactly embedded in $L^\rho(\mathbb{R}^2_+)$ for $\rho \in (2, \infty)$. With the help of these embeddings and by applying the symmetric mountain pass theorem (see Theorem 6.5 in [**125**]), we may also prove, for any $\lambda > 0$, the existence of a sequence of pairs of solutions $\pm u_j$ whose sequence of energies is unbounded.*

The existence and symmetry parts of Theorem 3.1.4 extend to a larger class of semilinear equations, see Section 3.3 below. Next, we shall see that the case $q = 0$ in (3.1.10) arises naturally when considering the asymptotics of positive least energy solutions of (3.1.10) in the case $q = 1$ when $\lambda \to 0$. We shall see that these solutions concentrate at the origin as $\lambda \to 0$. More precisely, we have the following.

Theorem 3.1.6. *Let $(\lambda_k)_k$ be sequence of numbers $\lambda_k \leq 1$ such that $\lambda_k \to 0$ as $k \to \infty$. Moreover, let $u_k \in H^+$ be a positive least energy solution of* (3.1.10) *with $q = 1$, and let $v_k \in H^+$ be defined by $v_k(x) = \lambda_k^{\frac{2}{p-2}} u_k(\lambda_k x)$.*

Then, after passing to a subsequence, we have $v_k \longrightarrow v^*$ in H^+, where v is a positive least energy solution of the problem

$$\begin{cases} -\Delta v^* - [x_1\partial_{x_2} - x_2\partial_{x_1}]^2 v^* = |v^*|^{p-2}v^* & \text{on } \mathbb{R}^2_+, \\ v = 0 & \text{on } \partial\mathbb{R}^2_+ \end{cases} \tag{3.1.15}$$

Remark 3.1.7. *The statements given in Theorems 3.1.4(i) and 3.1.6 remain valid when the underlying half space $\mathbb{R}^2_+$ is replaced by the cone*

$$C_\alpha := \{x \in \mathbb{R}^2 \,:\, x_1 > 0, \quad \arcsin\frac{x_2}{|x|} < \alpha\}.$$

In particular, in the case where $\alpha = \frac{\pi}{2j}$ for some positive integer j, successive reflection yields solutions with precisely $2j$ nodal domains.

The paper is organized as follows. Section 3.2 sets up the functional analytic framework and provides some preliminary results. In particular, we shall prove the compactness of the embedding $H \hookrightarrow L^\rho(\mathbb{R}^2)$ for $\rho \in (2, \infty)$, and we establish the existence of least energy nodal solutions for problem (3.1.6). In Section 3.3, we study the symmetry and existence of ground state solutions for a generalization of problem (3.1.10). In Section 3.4 we discuss the asymptotics of least energy solutions to (3.1.10) as $\lambda \to \infty$ and as $\lambda \to 0$ and prove Theorems 3.1.4 and 3.1.6. Finally, Section 3.5 is devoted to the proofs of Theorem 3.1.2 and Theorem 3.1.3. In the appendix, we prove a result on uniform L^∞-bounds for weak solutions of (3.1.6) in the case $q = 1$.

3.2 Preliminary results

In the following, all functions are assumed to be real-valued. We consider the space H defined in (3.1.7) with the λ-dependent scalar product defined in (3.1.8) with $\|\cdot\|_\lambda$ denoting the corresponding norm. The space $(H, \langle\cdot,\cdot\rangle_\lambda)$ is a Hilbert space and clearly, all the norms $\|\cdot\|_\lambda$, $\lambda > 0$, are equivalent.

For easier distinction from the norms on H, for $\rho \in [1, \infty]$, we will use the notation $|\cdot|_\rho$ to denote the standard norm on $L^\rho(\mathbb{R}^2)$.

Recall also that we have set $\partial_\theta := [x_1\partial_{x_2} - x_2\partial_{x_1}]$ for the angular derivative operator. We first note the following.

Lemma 3.2.1. *For any $\lambda > 0$, the space $C_c^\infty(\mathbb{R}^2)$ of test functions is dense in $(H, \langle\cdot,\cdot\rangle_\lambda)$.*

Proof. The argument is essentially the same as the one proving the density of $C_c^\infty(\mathbb{R}^2)$ in $H^1(\mathbb{R}^2)$, see e.g. the proof of Theorem 9.2 in [**23**]. We only sketch it briefly. Let W denote the subspace of functions in H which vanish outside a bounded subset of $\mathbb{R}^2$. By a straightforward cut-off argument, W is dense in H. Moreover, for a given function $u \in W$, it is well known that a sequence of mollifications $u_n \in C_c^\infty(\mathbb{R}^2)$ of u converges to u in the H^1-norm. Moreover, since there is a compact set $K \subset \mathbb{R}^2$ with the property that every u_n, $n \in \mathbb{N}$ vanishes in $\mathbb{R}^2 \setminus K$, the convergence in the H^1-norm also implies convergence in $\|\cdot\|_\lambda$. This shows the claim. □

Next, we consider the radial averaging operator

$$\begin{aligned} &L^1_{loc}(\mathbb{R}^2) \to L^1_{loc}(\mathbb{R}^2), \;\; u \mapsto u^\#, \\ &\text{with} \quad u^\#(x) := \frac{1}{2\pi}\int_{S^1} u(|x|\omega)\,d\omega \quad \text{for a.e. } x \in \mathbb{R}^2. \end{aligned} \tag{3.2.1}$$

We note that, as a consequence of Jensen's inequality, the averaging operator extends to a continuous linear map $L^\rho(\mathbb{R}^2) \to L^\rho(\mathbb{R}^2)$ for every $\rho \in [1,\infty]$ with

$$|u^\#|_\rho \le |u|_\rho \qquad \text{for every } u \in L^\rho(\mathbb{R}^2). \tag{3.2.2}$$

Moreover, since $u^\# \in C^1_c(\mathbb{R}^2)$ for $u \in C^1_c(\mathbb{R}^2)$ and

$$\|u^\#\|_\lambda = \|u^\#\|_{H^1(\mathbb{R}^2)} \le \|u\|_{H^1(\mathbb{R}^2)} \le \|u\|_\lambda \quad \text{for } \lambda > 0,$$

the operator $u \mapsto u^\#$ extends to a continuous linear map $H \to H$.

We need the following angular Poincaré type estimates.

Lemma 3.2.2.

(i) For any $u \in H$,

$$|u|_2^2 \le |\partial_\theta u|_2^2 + |u^\#|_2^2.$$

In particular, any $u \in H$ with $u^\# \equiv 0$ satisfies $|u|_2^2 \le |\partial_\theta u|_2^2$.

(ii) Let $\theta_0 \in (0,\pi)$ and consider the cone

$$C_{\theta_0} := \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \;:\; r > 0, |\theta| < \theta_0\}.$$

If $u \equiv 0$ on $\mathbb{R}^2 \setminus C_{\theta_0}$, then we have

$$|u|_2 \le \frac{2\theta_0}{\pi}|\partial_\theta u|_2.$$

Proof. (i) By Lemma 3.2.1, it suffices to prove the claim for $u \in C^\infty_c(\mathbb{R}^2)$.

We first assume that $u^\# \equiv 0$. In this case we have, in polar coordinates,

$$|u|_2^2 = \int_0^\infty r \int_0^{2\pi} |u(r,\theta)|^2 \, d\theta dr,$$

where the function $\theta \mapsto u(r,\theta)$ is 2π-periodic and satisfies $\int_0^{2\pi} u(r,\theta)\, d\theta = 0$ for every $r > 0$. Consequently, by *Wirtinger's inequality* for periodic functions,

$$\int_0^{2\pi} |u(r,\theta)|^2 \, d\theta \le \int_0^{2\pi} |\partial_\theta u(r,\theta)|^2 \, d\theta \qquad \text{for every } r > 0,$$

which implies that

$$|u|_2^2 \le \int_0^\infty r \int_0^{2\pi} |\partial_\theta u(r,\theta)|^2 \, d\theta dr = |\partial_\theta u|_2^2.$$

If $u \in C^\infty_c(\mathbb{R}^2)$ is arbitrary, we may apply the above argument to the function $u - u^\#$. Since $(u-u^\#)^\# = 0$ and $\langle u - u^\#, u^\# \rangle_{L^2(\mathbb{R}^2)} = 0$, we get that

$$|u|_2^2 - |u^\#|_2^2 = |u - u^\#|_2^2 \le |\partial_\theta (u - u^\#)|_2^2 = |\partial_\theta u|_2^2,$$

as claimed.

(ii) Let $u \in H$ with $u \equiv 0$ on $\mathbb{R}^2 \setminus C_{\theta_0}$. By Lemma 3.2.1, there exists a sequence $(u_n)_n$ in $C^\infty_c(\mathbb{R}^2)$ with $u_n \to u$.

We fix $r_0 > 0$ and we let $\rho \in C^\infty([0,\infty))$ be a function with $0 \le \rho \le 1$, $\rho \equiv 0$ on $[0, r_0]$ and $\rho \equiv 1$ on $[2r_0, \infty)$. Moreover, we let $\theta' \in (\theta_0, \pi)$ and $\psi \in C_c^\infty(\mathbb{R})$ be a function with $0 \le \psi \le 1$, $\psi \equiv 1$ in $[-\theta_0, \theta_0]$ and $\psi \equiv 0$ in $\mathbb{R} \setminus [-\theta', \theta']$. Next we define, in polar coordinates,

$$\varphi_0, \varphi_1 \in L^\infty(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2), \qquad \varphi_0(r,\theta) = \rho(r), \quad \varphi_1(r,\theta) = \rho(r)\psi(\theta).$$

Setting $v_n := u_n \varphi_1$ for $n \in \mathbb{N}$, it is then easy to see that

$$v_n \longrightarrow u\varphi_1 = u\varphi_0 \qquad \text{in } H, \tag{3.2.3}$$

where the last equality follows since $u \equiv 0$ on $\mathbb{R}^2 \setminus C_{\theta_0}$. Moreover, we have, in polar coordinates,

$$|v_n|_2^2 = \int_0^\infty r \int_{-\pi}^{\pi} |v_n(r,\theta)|^2 \, d\theta dr,$$

where the function $\theta \mapsto v_n(r,\theta)$ is of class C^1 and satisfies $v_n(r,\theta) = 0$ for $\theta \notin [-\theta', \theta']$, $r > 0$. Using again a classical Wirtinger type inequality (see section 1.7 in [**47**]),

$$\int_{-\pi}^{\pi} |v_n(r,\theta)|^2 \, d\theta \le \left(\frac{2\theta'}{\pi}\right)^2 \int_{-\pi}^{\pi} |\partial_\theta v_n|^2(r,\theta) \, d\theta \qquad \text{for every } r > 0,$$

which implies that

$$|v_n|_2^2 \le \left(\frac{2\theta'}{\pi}\right)^2 \int_0^\infty r \int_{-\pi}^{\pi} |\partial_\theta v_n|^2(r,\theta) \, d\theta dr = \left(\frac{2\theta'}{\pi}\right)^2 |\partial_\theta v_n|_2^2 \tag{3.2.4}$$

for every $n \in \mathbb{N}$.

Using (3.2.3), we may thus pass to the limit in (3.2.4) to obtain the inequality

$$|u\varphi_0|_2^2 \le \left(\frac{2\theta'}{\pi}\right)^2 |\varphi_0 \partial_\theta u|_2^2,$$

which yields that

$$\|u\|_{L^2(\mathbb{R}^2 \setminus B_{2r_0}(0))} \le \frac{2\theta'}{\pi} \|\partial_\theta u\|_{L^2(\mathbb{R}^2)}.$$

Since $r_0 > 0$ and $\theta' > \theta_0$ were chosen arbitrarily, the claim follows. □

Next we note embedding properties of the space H.

Lemma 3.2.3. *For every $\lambda > 0$, $(H, \langle \cdot, \cdot \rangle_\lambda)$ is a Hilbert space canonically embedded in $H^1(\mathbb{R}^2)$. Moreover, H is compactly embedded in $L^\rho(\mathbb{R}^2)$ for all $\rho \in (2, \infty)$.*

Proof. We have

$$\|u\|_{H^1(\mathbb{R}^2)} \le \|u\|_\lambda \qquad \text{for all } \lambda > 0, v \in H,$$

which implies that H is a Hilbert space contained in $H^1(\mathbb{R}^2)$. By standard Sobolev embeddings, H is thus embedded in $L^\rho(\mathbb{R}^2)$ for all $\rho \in [2, \infty)$. It remains to show that these embeddings are compact for $\rho > 2$.

Let $(u_n)_n$ be a sequence in H with $u_n \rightharpoonup 0$ in H, and suppose by contradiction that $u_n \not\to 0$ in $L^\rho(\mathbb{R}^2)$ for some $\rho > 2$.

Since, $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^2)$, it follows from Lions' Lemma [**86**, Lemma I.1] and Rellich's Theorem that, after passing to a subsequence, there exists a sequence $x^n \in \mathbb{R}^2$ with $|x^n| \to \infty$ and such that

$$v_n \rightharpoonup v \ne 0 \qquad \text{in } H^1(\mathbb{R}^2) \tag{3.2.5}$$

for the functions $v_n \in H^1(\mathbb{R}^2)$, $v_n = u_n(\cdot + x^n)$.

Let $r_n := |x^n|$. Passing to a subsequence, we may assume that the limits

$$a := \lim_{n\to\infty} \frac{x_1^n}{r_n}, \qquad b := \lim_{n\to\infty} \frac{x_2^n}{r_n}$$

exist, whereas $a^2 + b^2 = 1$. For every $R > 0$, we then have

$$\begin{aligned}
\lambda^2 \|u_n\|_\lambda^2 &\geq \int_{\mathbb{R}^2_+} |x_1 \partial_{x_2} u_n - x_2 \partial_{x_1} u_n|^2 dx \\
&= \int_{\mathbb{R}^2} |(x_1 + x_1^n)\partial_{x_2} v_n - (x_2 + x_2^n)\partial_{x_1} v_n|^2 dx \\
&\geq \int_{B_R(0)} |(x_1 + x_1^n)\partial_{x_2} v_n - (x_2 + x_2^n)\partial_{x_1} v_n|^2 dx \\
&= r_n^2 \int_{B_R(0)} \left| \frac{x_1 + x_1^n}{r_n} \partial_{x_2} v_n - \frac{x_2 + x_2^n}{r_n} \partial_{x_1} v_n \right|^2 dx \\
&\geq r_n^2 \Big(\int_{B_R(0)} |a\partial_{x_2} v_n - b\partial_{x_1} v_n|^2 dx - \sup_{x\in B_R(0)} \left| \frac{x_1 + x_1^n}{r_n} - a \right| \|\partial_{x_2} v_n\|^2_{L^2(B_R(0))} \\
&\quad - \sup_{x\in B_R(0)} \left| \frac{x_2 + x_1^n}{r_n} - b \right| \|\partial_{x_1} v_n\|^2_{L^2(B_R(0))} \Big) \\
&\geq r_n^2 \Big(\int_{B_R(0)} |a\partial_{x_2} v_n - b\partial_{x_1} v_n|^2 dx + o(1) \Big) \\
&\geq r_n^2 \Big(\int_{B_R(0)} |a\partial_{x_2} v - b\partial_{x_1} v|^2 dx + o(1) \Big),
\end{aligned}$$

where in the last step we used the fact that

$$a\partial_{x_2} v_n - b\partial_{x_1} v_n \rightharpoonup a\partial_{x_2} v - b\partial_{x_1} v \qquad \text{in } L^2(B_R(0))$$

and the weak lower semicontinuity of the L^2-norm. The boundedness of $(u_n)_n$ in H now implies that

$$\int_{B_R(0)} [a\partial_{x_2} v - b\partial_{x_1} v]^2 dx = 0 \qquad \text{for every } R > 0,$$

and thus

$$\int_{\mathbb{R}^2} |a\partial_{x_2} v - b\partial_{x_1} v|^2 dx = 0. \tag{3.2.6}$$

Since $a^2 + b^2 = 1$, if we had $a = 0$ or $b = 0$ it would follow that

$$\int_{\mathbb{R}^2} |\partial_{x_2} v|^2 \, dx = 0 \quad \text{or} \quad \int_{\mathbb{R}^2} |\partial_{x_1} v|^2 \, dx.$$

The fact that $v \in L^2(\mathbb{R}^2)$ would imply $v \equiv 0$, contradicting (3.2.5). If, on the other hand, $a, b \neq 0$, (3.2.6) implies that $\partial_{x_1} v = \frac{a}{b}\partial_{x_2} v$ in $L^2(\mathbb{R}^2)$. Thus v satisfies $\partial_\beta v = 0$ with $\beta = (1, -\frac{a}{b})$, which again implies $v \equiv 0$ and thus contradicts (3.2.5). The proof is finished. □

Lemma 3.2.4. *The embedding $H \hookrightarrow L^2(\mathbb{R}^2)$ is not compact.*

Proof. Let $\psi \in C_c^\infty((1,2)) \setminus \{0\}$. After trivially extending ψ to $\mathbb{R}$, for $n \in \mathbb{N}$ consider the functions

$$u_n(r,s) = \frac{1}{\sqrt{r}} \psi(r - n)$$

so that

$$\operatorname{supp} u_n \subset \left\{x \in \mathbb{R}^2_+ : n+1 < |x| < n+2\right\}.$$

Clearly, $u_n \rightharpoonup 0$ in H, but

$$|u_n|_2^2 = 2\pi \int_0^\infty \psi(r-n)^2\, dr = 2\pi \int_0^\infty \psi(r)^2\, dr > 0$$

so $u_n \not\to 0$ in $L^2(\mathbb{R}^2)$. □

In the following, we fix $p > 2$ and $q = 1$ in (3.1.6), i.e., we consider the equation

$$\begin{cases} -\Delta u - \dfrac{1}{\lambda^2}[x_1\partial_{x_2} - x_2\partial_{x_1}]^2 u + u = |u|^{p-2}u & \text{on } \mathbb{R}^2, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{3.2.7}$$

Here and in what follows, for a given $\lambda > 0$, a function $u \in H$ will be called a weak solution of (3.2.7) if

$$\langle u, v\rangle_\lambda = \int_{\mathbb{R}^2} |u|^{p-2}uv\, dx \qquad \text{for all } v \in H.$$

As a consequence of Lemma 3.2.3 and standard arguments in the calculus of variations, we see that for $\lambda > 0$, the energy functional

$$E_\lambda : H \to \mathbb{R}, \quad E_\lambda(u) := \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{p}\int_{\mathbb{R}^2} |u|^p\, dx$$

is of class C^1 and critical points of E_λ are weak solutions of (3.2.7).

We note the following uniform boundedness property of weak solutions of (3.1.6).

Lemma 3.2.5. *Fix $\lambda > 0$ and let $u \in H$ be a weak solution of*

$$-\Delta u - \frac{1}{\lambda^2}\partial_\theta^2 u + u = |u|^{p-2}u \quad \text{in } \mathbb{R}^2. \tag{3.2.8}$$

Then $u \in L^\infty(\mathbb{R}^2)$. Moreover, there exist constants $\sigma, C > 0$, depending on $p > 2$ but not on u and λ, such that

$$|u|_\infty \le C\|u\|_{H^1(\mathbb{R}^2)}^\sigma. \tag{3.2.9}$$

The fact that the constants C and σ in (3.2.9) do not depend on λ is of key importance in the proofs of Theorems 3.1.2(i) and Theorem 3.1.3(ii). The proof of Lemma 3.2.5 follows by a Moser iteration scheme based on uniform estimates which do not depend on $\lambda > 0$. We include the details in the appendix, see Lemma 3.6.1 below.

Remark 3.2.6. *If $f : \mathbb{R} \to \mathbb{R}$ is a C^1-function with $f(0) = 0$ and $u \in H \cap L^\infty(\mathbb{R}^2)$, it is easy to see that also $f(u) = f \circ u \in H \cap L^\infty(\mathbb{R}^2)$ with $\nabla f(u) = f'(u)\nabla u$ and $\partial_\theta f(u) = f'(u)\partial_\theta u$.*

By Lemma 3.2.5, this observation applies, in particular, to weak solutions $u \in H$ of (3.2.8).

Next we note that every nontrivial solution of (3.2.7) is contained in the Nehari manifold

$$\mathcal{N}_\lambda := \{u \in H \setminus \{0\} : E_\lambda'(u)u = 0\}.$$

Let

$$\alpha_\lambda := \inf_{u \in \mathcal{N}_\lambda} E_\lambda(u) > 0, \tag{3.2.10}$$

then every minimizer is a critical point and hence a solution (cf. [**128**] and Theorem 3.3.5 below). It is easy to see that such a minimizer is positive and thus radial by Theorem 3.1.1. Therefore, $\alpha = \alpha_\lambda$ does not depend on λ.

Hence we now focus on sign-changing solutions. Consider

$$\begin{aligned}\mathcal{M}_\lambda &:= \left\{u \in H : u^+ \not\equiv 0, u^- \not\equiv 0,\ E'_\lambda(u)u^+ = E'_\lambda(u)u^- = 0\right\}\\ &= \left\{u \in H \setminus \{0\} : u^+, u^- \in \mathcal{N}_\lambda\right\}\end{aligned}$$

and set

$$\beta_\lambda := \inf_{u \in \mathcal{M}_\lambda} E_\lambda(u). \tag{3.2.11}$$

Proposition 3.2.7. *The value β_λ is positive. Moreover, every minimizer $u \in \mathcal{M}_\lambda$ of (3.2.11) is a critical point of E_λ and hence a sign-changing solution of* (3.2.7).

The proof of Proposition 3.2.7 follows the same argument as in the proof of Proposition 3.1 in [**11**].

We also remark that $\beta_\lambda \geq 2\alpha > 0$ in view of (3.2.10) and the fact that for any $u \in H$,

$$E_\lambda(u) = E_\lambda(u^+) + E_\lambda(u^-) \quad \text{and} \quad E'_\lambda(u)u = E'_\lambda(u^+)u^+ + E'_\lambda(u^-)u^-.$$

We say that a function $u \in H$ is a *least energy nodal solution* of (3.2.7) if u is a sign-changing solution of (3.2.7) such that $E_\lambda(u) = \beta_\lambda$. The following lemma yields the existence of a least energy nodal solution.

Lemma 3.2.8. *There exists $u \in \mathcal{M}_\lambda$ such that $E_\lambda(u) = \beta_\lambda$.*

Proof. We proceed similarly as in [**30**]. Let $(u_n)_n \subset \mathcal{M}_\lambda$ be a minimizing sequence. Note that for any $u \in \mathcal{M}_\lambda$ we have

$$E_\lambda(u_n) = \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^2} \left(|\nabla u|^2 + \frac{1}{\lambda^2}|\partial_\theta u|^2 + u^2\right) dx,$$

which implies that E_λ is coercive on $\mathcal{M}_\lambda$. This yields that $(u_n)_n$ is bounded and we may therefore pass to a subsequence such that

$$u_n \rightharpoonup u \quad \text{in } H.$$

We then also have $u_n^\pm \rightharpoonup u^\pm$ in H, and the compact embedding $H \hookrightarrow L^p$ implies

$$\int_{\mathbb{R}^2} |u^\pm|^p \, dx = \lim_{n\to\infty} \int_{\mathbb{R}^2} |u_n^\pm|^p \, dx = C\|u_n^\pm\|_\lambda^2 \geq C' > 0.$$

Hence $u^\pm \not\equiv 0$.

Next, we show that $u_n^\pm \to u^\pm$ in H. Arguing by contradiction, assume first that $\|u^+\|_\lambda^2 < \liminf_{n\to\infty} \|u_n^+\|_\lambda^2$. Then

$$E'_\lambda(u^+)u^+ = \|u^+\|_\lambda^2 - \|u^+\|_p^p < \liminf_{n\to\infty} \left(\|u_n^+\|_\lambda^2 - \|u_n^+\|_p^p\right) = 0.$$

Hence the characterization of $\mathcal{N}_\lambda$ yields the existence of $a \in (0,1)$ such that $au^+ \in \mathcal{N}_\lambda$. A similar argument yields $bu^- \in \mathcal{N}_\lambda$ for some $0 < b \leq 1$. Thus, $au^+ + bu^- \in \mathcal{M}_\lambda$ and we estimate

$$\begin{aligned}\beta_\lambda \leq & E_\lambda(au^+ + bu^-)\\ < & \liminf_{n\to\infty} E_\lambda(au_n^+ + bu_n^-) = \liminf_{n\to\infty} \left(E_\lambda(au_n^+) + E_\lambda(bu_n^-)\right)\\ \leq & \liminf_{n\to\infty} (E_\lambda(u_n^+) + E_\lambda(u_n^-)) = \liminf_{n\to\infty} E_\lambda(u_n)\\ = & \beta_\lambda,\end{aligned}$$

which is a contradiction. Thus, after passing to a subsequence if necessary and using the uniform convexity of $(H, \|\cdot\|_\lambda)$, we conclude that $u_n^+ \to u^+$ strongly in H. In particular, $u^+ \in \mathcal{N}_\lambda$. Proceeding similarly, we prove that $u_n^- \to u^-$ strongly in H and that $u^- \in \mathcal{N}_\lambda$ and consequently, $u \in \mathcal{M}_\lambda$ with $E_\lambda(u) = \beta_\lambda$. □

Summarizing the previous results, we have the following.

Corollary 3.2.9. *Let $p > 2$. For every $\lambda > 0$ there exists a least energy nodal solution to* (3.2.7)*, i.e. a sign-changing solution $u \in H$ such that $E_\lambda(u) = \beta_\lambda$.*

Remark 3.2.10. *We may also consider the more general equation*

$$\begin{cases} -\Delta u - \dfrac{1}{\lambda^2}[x_1\partial_{x_2} - x_2\partial_{x_1}]^2 u + u = f(u) & \text{on } \mathbb{R}^2, \\ u(x) \to 0 & \text{for } |x| \to \infty, \end{cases} \tag{3.2.12}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. In order to extend our results, consider the following conditions:

(A_1) *There exists $C > 0$ such that $|f(t)| \le C(|t| + |t|^p)$ for $t \in \mathbb{R}$*

(A_2) $t \mapsto \dfrac{f(t)}{t}$ *is strictly increasing on $\mathbb{R} \setminus \{0\}$ and* $\lim\limits_{t\to 0} \dfrac{f(t)}{t} \le 0$, $\lim\limits_{t\to\pm\infty} \dfrac{f(t)}{t} = \infty$.

Under these assumptions, it can be shown that the results of this section, concerned with problem (3.2.7)*, continue to hold true for* (3.2.12).

3.3 Existence and symmetry of odd solutions

This section is devoted to the study of solutions of the problem (3.1.10), which correspond, by odd reflection, to solutions of (3.1.6) with hyperplane antisymmetry. In particular, we shall prove Parts (i) and (ii) of Theorem 3.1.4.

Consider the space H^+ defined in (3.1.12). For fixed $\lambda > 0$ and $q \in \{0, 1\}$, we endow H^+ with the λ-dependent scalar product

$$\langle u, v\rangle_{\lambda,q} \mapsto \int_{\mathbb{R}^2_+} \Big(\nabla u \cdot \nabla v + \frac{1}{\lambda^2}(\partial_\theta u)(\partial_\theta v) + q\, uv\Big) dx,$$

and we let $\|\cdot\|_{\lambda,q}$ denote the corresponding norm. Observe that any $u \in H^+$ can be extended to an element of H either trivially or by odd reflection. Therefore, Lemma 3.2.2 and 3.2.3 immediately yield the following.

Corollary 3.3.1. *(i) Any $u \in H^+$ satisfies*

$$|u|_2^2 \le \int_{\mathbb{R}^2_+} |\partial_\theta u|^2 dx. \tag{3.3.1}$$

In particular, the norms $\|\cdot\|_{\lambda,0}$ and $\|\cdot\|_{\lambda,1}$ are equivalent on H^+, and H^+ is a Hilbert space with either of these norms. Moreover, we have a continuous embedding $H^+ \hookrightarrow H^1(\mathbb{R}^2_+)$.

(ii) The space H^+ is compactly embedded in $L^\rho(\mathbb{R}^2_+)$ for $\rho > 2$.

Remark 3.3.2. *(i) Similar statements are also true, when the underlying space is the cone C_{θ_0} described in Lemma 3.2.2.*

(ii) As in Lemma 3.2.4, we see that the embedding $H^+ \hookrightarrow L^2(\mathbb{R}^2_+)$ is not compact.

First, we establish the symmetry of postive weak solutions of (3.1.10) as a consequence of the following.

Theorem 3.3.3. *Let $\lambda > 0$, and let $f \in C^1([0,\infty))$ satisfy*

$$f'(t) \le C\Big(t^{\sigma_1} + t^{\sigma_2}\Big) \quad \text{for} \quad t \ge 0 \tag{3.3.2}$$

with constants $\sigma_1, \sigma_2 > 0$. Moreover, let $u \in H^+ \cap L^\infty(\mathbb{R}^2)$ be a positive weak solution of the problem

$$\begin{cases} -\Delta u - \dfrac{1}{\lambda^2}\partial_\theta^2 u = f(u) & \text{on } \mathbb{R}^2_+, \\ \qquad\qquad u = 0 & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{3.3.3}$$

Then u is symmetric with respect to the x_1-axis and decreasing with respect to the angle $|\theta|$ from the x_1-axis.

Remark 3.3.4. *Theorem 3.3.3 in particular applies in the case where the nonlinearity f is given by $f(t) = -qt + |t|^{p-2}t$ for some $p \in (2,\infty)$, $q \in \{0,1\}$. In this case, Lemma 3.2.5 and Remark 3.6.2 below imply that every weak solution $u \in H^+$ of (3.3.2) is bounded. Hence we deduce the statement of Theorem 3.1.4(ii).*

Proof of Theorem 3.3.3. For simplicity, we assume $\lambda = 1$. We shall argue by the method of rotating planes. For $\theta \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$, set $e_\theta := (\cos\theta, \sin\theta)$,

$$T_\theta := \{x \in \mathbb{R}^2 : x \cdot e_\theta = 0\} \quad \text{and} \quad \Sigma_\theta := \{x \in \mathbb{R}^2_+ : x \cdot e_\theta < 0\}.$$

Given a positive solution $u \in H^+ \cap L^\infty(\mathbb{R}^2_+)$ of (3.3.3), consider the functions $u_\theta, w_\theta : \Sigma_\theta \to \mathbb{R}$ defined by

$$u_\theta(x) = u(x - 2(x \cdot e_\theta)e_\theta) \quad \text{and} \quad w_\theta := u_\theta - u$$

and extend them trivially outside Σ_θ.

A direct calculation shows that w_θ satisfies

$$\begin{aligned} -\Delta w_\theta - \partial_\theta^2 w_\theta = c_\theta(x) w_\theta \qquad & \text{in } \Sigma_\theta \\ w_\theta = 0 \qquad & \text{on } T_\theta \\ w_\theta > 0 \qquad & \text{on } \partial\Sigma_\theta \setminus T_\theta, \end{aligned} \tag{3.3.4}$$

where

$$c_\theta(x) = \int_0^1 f'\big((1-t)u(x) + tu_\theta(x)\big)\, dt.$$

Consider the set

$$\Theta^+ := \Big\{\theta \in \Big(0, \frac{\pi}{2}\Big) : w_\theta \ge 0 \text{ in } \Sigma_\theta\Big\}$$

which is clearly a closed set in $(0, \frac{\pi}{2})$.

We claim that Θ^+ is non-empty. To prove this claim, we proceed as follows. Observe first that $w_\theta^- := \min\{w_\theta, 0\} \in H^+$. Moreover, using (3.3.2), we have that for $x \in \Sigma_\theta$ with $w_\theta^-(x) < 0$,

$$\begin{aligned} c_\theta(x) \le & C\int_0^1 \Big[\big((1-t)u(x) + tu_\theta(x)\big)^{\sigma_1} + \big((1-t)u(x) + tu_\theta(x)\big)^{\sigma_2}\Big] dt \\ \le & C\Big[u^{\sigma_1}(x) + u^{\sigma_2}(x)\Big]. \end{aligned} \tag{3.3.5}$$

Also, the boundary conditions imply $w_\theta^- \equiv 0$ on $\partial\Sigma_\theta$, and testing the equation (3.3.4) against w_θ^- yields

$$\begin{aligned} |\nabla w_\theta^-|_2^2 + |\partial_\theta w_\theta^-|_2^2 =& \int_{\mathbb{R}^2} c_\theta(x)(w_\theta^-)^2\,dx \\ \leq& C\int_{\mathbb{R}^2}\Big[u^{\sigma_1}+u^{\sigma_2}\Big](w_\theta^-)^2\,dx \\ \leq& C_0|w_\theta^-|_2^2 \end{aligned} \tag{3.3.6}$$

with $C_0 = C\big(|u|_\infty^{\sigma_1} + |u|_\infty^{\sigma_2}\big)$. Therefore, by Lemma 3.2.2(ii),

$$\frac{\pi}{2\theta}|w_\theta^-|_2 \leq |\partial_\theta w_\theta^-|_2 \leq \sqrt{C_0}|w_\theta^-|_2.$$

Consequently, $w_\theta^- \equiv 0$ provided that $0 < |\theta| < \frac{\sqrt{C_0}\pi}{2}$ and this proves the claim.

Next, we claim that Θ^+ is also open in $(0, \frac{\pi}{2})$. To see this, let $\theta_0 \in \Theta^+$. Since $w_{\theta_0} \not\equiv 0$ by (3.3.4), the strong maximum principle implies that $w_{\theta_0} > 0$ in Σ_{θ_0}.

Fix $\rho > 2$ such that $\tau_i := \frac{\sigma_i\rho}{\rho-2} > 2$ for $i = 1, 2$. By Lemma 3.2.3, there exists $\kappa_\rho > 0$ such that

$$|w|_\rho^2 \leq \kappa_\rho\Big(|\nabla w|_2^2 + |\partial_\theta w|_2^2\Big) \qquad \text{for all } w \in H^+.$$

Moreover, we may choose a compact set $D \subset \Sigma_{\theta_0}$ such that

$$\|u\|^{\sigma_1}_{L^{\tau_1}(\Sigma_{\theta_0}\setminus D)} + \|u\|^{\sigma_2}_{L^{\tau_2}(\Sigma_{\theta_0}\setminus D)} < \frac{1}{2\kappa_\rho C},$$

where $C > 0$ is the constant in (3.3.5).

On the other hand, by continuity of the family w_θ w.r.t. θ there exists a neighborhood $N \subset (0, \frac{\pi}{2})$ of θ_0 with the property that for all $\theta \in N$,

$$w_\theta > 0 \quad \text{in } D \quad \text{and} \quad \|u\|^{\sigma_1}_{L^{\tau_1}(\Sigma_\theta\setminus D)} + \|u\|^{\sigma_2}_{L^{\tau_2}(\Sigma_\theta\setminus D)} < \frac{1}{2\kappa_\rho C}.$$

From (3.3.6) and Hölder's inequality, it follows that

$$\begin{aligned} |w_\theta^-|_\rho^2 &\leq \kappa_\rho\Big(|\nabla w_\theta^-|_2^2 + |\partial_\theta w_\theta^-|_2^2\Big) \leq \kappa_\rho C\int_{\mathbb{R}^2}\Big[u^{\sigma_1}+u^{\sigma_2}\Big](w_\theta^-)^2\,dx \\ &\leq \kappa_\rho C\Big(\|u\|^{\sigma_1}_{L^{\tau_1}(\Sigma_{\theta_0}\setminus D)} + \|u\|^{\sigma_2}_{L^{\tau_2}(\Sigma_{\theta_0}\setminus D)}\Big)|w_\theta^-|_\rho^2 \leq \frac{1}{2}|w_\theta^-|_\rho^2 \end{aligned}$$

for any $\theta \in N$.

Consequently, $w_\theta^- \equiv 0$ for $\theta \in N$ and this proves the claim.

Since Θ^+ is an open, closed and nonempty subset of $\big(0, \frac{\pi}{2}\big)$, we conclude that $\Theta^+ = \big(0, \frac{\pi}{2}\big)$. In the same manner, we see that

$$\Theta^- := \Big\{\theta \in \Big(-\frac{\pi}{2}, 0\Big) : w_\theta \geq 0 \text{ in } \Sigma_\theta\Big\} = (-\frac{\pi}{2}, 0)$$

Consequently u is decreasing with respect to the angle $|\theta|$ from the x_1-axis.

Finally, a continuity argument also shows that $w_\theta \geq 0$ in Σ_θ for $\theta \in \{\pm\frac{\pi}{2}\}$, which, in particular, forces the symmetry of u with respect to reflection at the x_1-axis. □

Next, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (A_1) and (A_2) as in Remark 3.2.10 and set $F(t) = \int_0^t f(s)\,ds$. We consider the energy functional

$$E_\lambda^+ : H^+ \to \mathbb{R}, \quad E_\lambda^+(u) := \frac{1}{2}\|u\|_{\lambda,0}^2 - \int_{\mathbb{R}^2_+} F(u)\,dx$$

Again, standard arguments in the calculus of variations show that E_λ^+ is of class C^1, and critical points of E_λ^+ are solutions of the associated Euler-Lagrange equation

$$\begin{cases} -\Delta u - \frac{1}{\lambda^2}\partial_\theta^2 u = f(u) & \text{on } \mathbb{R}^2_+, \\ u = 0 & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{3.3.7}$$

As in Section 3.2 we consider the associated Nehari manifold

$$\mathcal{N}_\lambda^+ := \left\{u \in H^+ \setminus \{0\} : [E_\lambda^+]'(u)u = 0\right\}$$

and set

$$c_\lambda := \inf_{u \in \mathcal{N}^+} E_\lambda^+(u). \tag{3.3.8}$$

This is the ground state energy in the sense that $E_\lambda^+(u) \geq c_\lambda$ for every nontrivial solution of (3.3.7).

Theorem 3.3.5. *Let $p > 2$, $\lambda > 0$, and assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the assumptions (A_1) and (A_2) listed in Remark 3.2.10. Then*

$$c_\lambda = \inf_{u \in H^+ \setminus \{0\}} \sup_{t \geq 0} E_\lambda^+(tu). \tag{3.3.9}$$

Moreover, problem (3.3.7) *admits a ground state solution, i.e., a solution $v \in H^+ \setminus \{0\}$ such that $E_\lambda^+(v) = c_\lambda$.*

Proof. The proof essentially follows the lines of the proof of [**128**, Theorem 20], see also [**84**, Section 4]. We note here that (A_1) and (A_2) ensure that the assumptions in [**128**, Theorem 20] are satisfied. Indeed, (A_2) implies that for any $R > 0$ there exists $t_R > 0$ such that $f(t) \geq Rt$ for $t \geq t_R$. Thus

$$F(t) = \int_0^t f(s)\, ds \geq \int_{t_R}^t Rs\, ds = \frac{R}{2}(t^2 - t_R^2)$$

for $t \geq t_R$. It follows that

$$\lim_{t \to \infty} \frac{F(t)}{t^2} = \infty,$$

i.e. assumption (iv) in [**128**, Theorem 20] is satisfied. Consequently, the proof given there can be carried through similarly, with some simplifications because the compact embedding $H^+ \hookrightarrow L^p(\mathbb{R}^2_+)$ replaces arguments based on compactness modulo translations in the periodic setting of [**128**, Theorem 20]. □

Remark 3.3.6. *(i) The statement of Theorem 3.1.4(i) is a special case of Theorem 3.3.5, since the nonlinearity $t \mapsto f(t) = -qt + |t|^{p-2}t$ satisfies conditions (A_1) and (A_2) if $q \in \{0, 1\}$ and $p \in (2, \infty)$.*

*(ii) Under the assumptions of Theorem 3.3.5, it can be shown that ground state solutions cannot change sign, see [**128**, Remark 17].*

3.4 Asymptotics of least energy odd solutions

In this section we fix $p \in (2, \infty)$, $q = 1$, and we study the asymptotics of least energy solutions to (3.1.10) in the case $q = 1$ as $\lambda \to \infty$ and as $\lambda \to 0$. In particular, we shall complete the proofs of Theorem 3.1.4(iii) and of Theorem 3.1.6. We will use the notation introduced in the previous section in the special case of the nonlinearity $t \mapsto f(t) = -t + |t|^{p-2}t$ which satisfies conditions (A_1) and (A_2). By the definition of the mountain pass value in (3.3.8) and the fact that $E^+_{\lambda_1} \geq E^+_{\lambda_2}$ for $0 < \lambda_1 < \lambda_2 < \infty$, we infer that the function

$$(0,\infty) \longrightarrow (0,\infty), \qquad \lambda \mapsto c_\lambda$$

is decreasing, and therefore the limits

$$c_0 := \lim_{\lambda\to 0} c_\lambda \qquad \text{and} \qquad c_\infty := \lim_{\lambda\to\infty} c_\lambda \tag{3.4.1}$$

exist in $[0,\infty]$. Next we note that

$$\sup_{t\geq 0} E^+_\lambda(tv) = E^+_\lambda(t^\lambda_v v) = \left(\frac{1}{2} - \frac{1}{p}\right)\frac{\|v\|_{\lambda,1}^{\frac{2p}{p-2}}}{|v|_p^{\frac{2p}{p-2}}} \quad \text{for every } v \in H^+ \setminus \{0\} \tag{3.4.2}$$

with

$$t^\lambda_v = \left(\frac{\|v\|^2_{\lambda,1}}{|v|^p_p}\right)^{\frac{1}{p-2}}.$$

We start by considering the asymptotics of least energy solutions to (3.1.10) as $\lambda \to \infty$.

3.4.1. The limit $\lambda \to \infty$. Consider the limit energy functional

$$E_* : H^1(\mathbb{R}^2) \longrightarrow \mathbb{R}, \qquad E_*(v) = \frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla v|^2 + v^2\right)\,dx - \frac{1}{p}\int_{\mathbb{R}^2}|v|^p\,dx.$$

Similarly as in (3.4.2), for $v \in H^1(\mathbb{R}^2) \setminus \{0\}$ we have

$$\sup_{t\geq 0} E_*(tv) = E_*(t_v v) = \left(\frac{1}{2} - \frac{1}{p}\right)\frac{\|v\|_{H^1(\mathbb{R}^2)}^{\frac{2p}{p-2}}}{|v|_p^{\frac{2p}{p-2}}} \tag{3.4.3}$$

with $t_v = \left(\frac{\|v\|^2_{H^1(\mathbb{R}^2)}}{|v|^p_p}\right)^{\frac{1}{p-2}}$.

Observe that for every $v \in H^1(\mathbb{R}^2)$ with $E_*'(v)v = 0$ we have $t_v = 1$ and hence

$$\sup_{t\geq 0} E_*(tv) = E_*(v).$$

Define

$$\hat{c}_\infty := \inf_{v\in H^1(\mathbb{R}^2)\setminus\{0\}} \sup_{t\geq 0} E_*(tv) \tag{3.4.4}$$

and let w_∞ denote the unique positive radial solution (see [**78**]) of the problem

$$-\Delta w_\infty + w_\infty = |w_\infty|^{p-2}w_\infty, \qquad w_\infty \in C^2(\mathbb{R}^2) \cap H^1(\mathbb{R}^2). \tag{3.4.5}$$

Since $E_*'(w_\infty)w_\infty = 0$, $t_{w_\infty} = 1$ and hence

$$\sup_{t\geq 0} E_*(tw_\infty) = E_*(w_\infty). \tag{3.4.6}$$

The following result provides a variational characterization of the limit c_∞, defined in (3.4.1), in terms of $\hat{c}_\infty$ and w_∞.

Lemma 3.4.1.

$$c_\infty = \hat{c}_\infty = E_*(w_\infty). \tag{3.4.7}$$

Proof. We first prove the second equality in (3.4.7). Since the proof is standard, we only sketch the argument. By (3.4.6), we have $\hat{c}_\infty \le E_*(w_\infty)$. On the other hand, using Schwarz symmetrization and (3.4.3), it is easy to see that

$$\hat{c}_\infty = \inf_{v \in H^1_{rad}(\mathbb{R}^2)\setminus\{0\}} \sup_{t\ge 0} E_*(tv).$$

Proceeding as in Theorem 20 and Remark 17 in [**128**] and using the compactness of the embedding $H^1_{rad}(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$, one can prove that $\hat{c}_\infty$ is attained at a positive radial solution of (3.4.5). By uniqueness, we then deduce that $\hat{c}_\infty = E_*(w_\infty)$.

Next, we prove the first equality in (3.4.7). Identifying $v \in H^+$ with its trivial extension in H, we see that $E^+_\lambda(v) = E_\lambda(v) \ge E_*(v)$ for any $v \in H^+$ and any $\lambda > 0$. Hence $c_\lambda \ge \hat{c}_\infty$ for any $\lambda > 0$ by (3.3.9) and (3.4.4). Taking the limit as $\lambda \to \infty$, we obtain that $c_\infty \ge \hat{c}_\infty$.

To see the opposite inequality, we let $v \in H^1(\mathbb{R}^2) \setminus \{0\}$ be arbitrary. Let $t_v > 0$ be as in (3.4.3), which implies that

$$0 = \frac{\partial_t\big|_{t_v} E_*(tv)}{t_v} = \|v\|^2_{H^1(\mathbb{R}^2)} - t_v^{p-2}\int_{\mathbb{R}^2} |v|^p\,dx.$$

From this we find that

$$\|v\|^2_{H^1(\mathbb{R}^2)} < (2t_v)^{p-2}\int_{\mathbb{R}^2} |v|^p\,dx.$$

Since $C^\infty_c(\mathbb{R}^2)$ is dense in $H^1(\mathbb{R}^2)$, there exists a sequence $\psi_n \in C^\infty_c(\mathbb{R}^2)$ such that $\|v - \psi_n\|_{H^1(\mathbb{R}^2)} \to 0$ as $n \to \infty$, and

$$\|\psi_n\|^2_{H^1(\mathbb{R}^2)} < (2t_v)^{p-2}\int_{\mathbb{R}^2} |\psi_n|^p\,dx \qquad \text{for all } n \in \mathbb{N}.$$

This implies that

$$\sup_{t\ge 0} E_*(t\psi_n) = \sup_{0\le t\le 2t_v} E_*(t\psi_n) \to \sup_{0\le t\le 2t_v} E_*(tv) = E_*(t_v v) \qquad \text{as } n \to \infty. \tag{3.4.8}$$

Next, we fix $n \in \mathbb{N}$ and choose $y_n \in \mathbb{R}^2$ such that $\tilde{\psi}_n \in C^\infty_c(\mathbb{R}^2_+) \subset H^+$ for the function $\tilde{\psi}_n : \mathbb{R}^2_+ \to \mathbb{R}$, $\tilde{\psi}_n(x) = \psi_n(x - y_n)$. Then there exists $t_n > 2t_v$ such that

$$\|\psi_n\|^2_{\lambda,1} = \|\psi_n\|^2_{H^1(\mathbb{R}^2_+)} + \frac{1}{\lambda^2}\|\partial_\theta\psi_n\|^2_{L^2(\mathbb{R}^2_+)} < (2t_n)^{p-2}\int_{\mathbb{R}^2} |\psi_n|^p\,dx \qquad \text{for all } \lambda \ge 1.$$

Using the fact that

$$\frac{t^2}{\lambda^2}\int_{\mathbb{R}^2_+} |\partial_\theta\psi_n|^2\,dx \to 0 \quad \text{as } \lambda \to \infty \text{ uniformly in } t \in [0, t_n],$$

we find that

$$\begin{aligned} c_\infty &= \lim_{\lambda\to\infty} c_\lambda \le \lim_{\lambda\to\infty} \sup_{t\ge 0} E^+_\lambda(t\tilde{\psi}_n) = \lim_{\lambda\to\infty} \sup_{0\le t\le t_n} E^+_\lambda(t\tilde{\psi}_n) \\ &= \sup_{0\le t\le t_n} E_*(t\tilde{\psi}_n) = \sup_{t\ge 0} E_*(t\tilde{\psi}_n) = \sup_{t\ge 0} E_*(t\psi_n), \end{aligned} \tag{3.4.9}$$

Combining (3.4.8) and (3.4.9), it follows that

$$c_\infty \le E_*(t_v v) = \sup_{t\ge 0} E_*(tv).$$

Since $v \in H^1(\mathbb{R}^2) \setminus \{0\}$ was arbitrary, we conclude that $c_\infty \le \hat{c}_\infty$. This completes the proof of the theorem. ◻

Now we are in a position to prove Theorem 3.1.4.

Proof of Theorem 3.1.4. The existence statement in (i) is a direct consequence of Theorem 3.3.5, whereas the symmetry property stated in Theorem 3.1.4 (ii) is a special case of Theorem 3.3.3.

Next, we prove the asymptotics in (iii). In what follows, the functions in H^+ are extended trivially outside $\mathbb{R}^2_+$. Assume that $1 \le \lambda_k \to \infty$ and, for every $k \in \mathbb{N}$, let $u_k \in H^+$ denote a positive least energy solution of (3.1.10) for $\lambda = \lambda_k$. Observe that for $k \in \mathbb{N}$,

$$\|u_k\|^2_{\lambda_k,1} = |u_k|_p^p$$

and

$$c_1 \ge c_{\lambda_k} = E^+_{\lambda_k}(u_k) = \Big(\frac{1}{2} - \frac{1}{p}\Big)\|u_k\|^2_{\lambda_k,1} = \Big(\frac{1}{2} - \frac{1}{p}\Big)|u_k|_p^p \ge c_\infty > 0.$$

Since

$$\|u_k\|^2_{H^1_0(\mathbb{R}^2_+)} \le \|u_k\|^2_{\lambda_k,1} \qquad \text{for every } k \in \mathbb{N},$$

we conclude that $(u_k)_k$ is bounded in $H^1_0(\mathbb{R}^2_+) \subset H^1(\mathbb{R}^2)$. Moreover, $|u_k|_p$ remains bounded away from zero. From Lions' Lemma [**86**, Lemma I.1] and Theorem 3.3.3, it thus follows that, after passing to a subsequence, there exists a sequence of numbers $\tau_k \in (0,\infty)$ such that $w_k \rightharpoonup w \ne 0$ in $H^1(\mathbb{R}^2)$ for the functions $w_k := u_k(\cdot + (\tau_k, 0))$. Observe that $w \ge 0$ a.e. in $\mathbb{R}^2$.

We first claim that

$$\tau_k \to \infty \qquad \text{as } k \to \infty. \tag{3.4.10}$$

Indeed, suppose by contradiction that $(\tau_k)_k$ contains a bounded subsequence. Then we may again pass to a subsequence with the property that

$$u_k \rightharpoonup u \ne 0 \qquad \text{in } H^1_0(\mathbb{R}^2_+),$$

where $u \ge 0$ a.e. in $\mathbb{R}^2_+$. For $\varphi \in C^\infty_c(\mathbb{R}^2_+)$ and $R > 0$ with $\operatorname{supp}\varphi \subset B_R(0)$ we then have

$$\frac{1}{\lambda_k^2}\int_{\mathbb{R}^2_+}(\partial_\theta u_k)(\partial_\theta \varphi)dx \le \frac{R^2}{\lambda_k^2}\|\nabla u_k\|_{L^2(R^2_+)}\|\nabla\varphi\|_{L^2(R^2_+)} \to 0 \qquad \text{as } k \to \infty$$

and thus

$$\int_{\mathbb{R}^2_+}\Big(\nabla u\cdot\nabla\varphi + u\varphi - u^{p-1}\varphi\Big)dx = \lim_{k\to\infty}\Big(\langle u_k,\varphi\rangle_{\lambda_k,1} - \int_{\mathbb{R}^2_+} u_k^{p-1}\varphi\,dx\Big) = 0.$$

Hence $u \in H^1_0(\mathbb{R}^2_+)$ is a nontrivial nonnegative weak solution of the problem

$$-\Delta u + u = u^{p-1} \quad \text{in } \mathbb{R}^2_+, \qquad u = 0 \quad \text{on } \partial\mathbb{R}^2_+$$

which contradicts a classical nonexistence result of Esteban and Lions in [**54**]. Thus (3.4.10) is true.

We now claim that

$$\frac{\tau_k}{\lambda_k} \to 0 \qquad \text{as } k \to \infty. \tag{3.4.11}$$

Before proving the claim, observe that by weak lower semicontinuity,

$$\tau_k^{-2}\int_{\mathbb{R}^2_+}|\partial_\theta u_k|^2dx = \tau_k^{-2}\int_{\mathbb{R}^2_+}|x_1\partial_{x_2}u_k - x_2\partial_{x_1}u_k|^2dx$$

$$
\begin{aligned}
&=\tau_k^{-2}\int_{\mathbb{R}^2}|(x_1+\tau_k)\partial_{x_2}w_k - x_2\partial_{x_1}w_k|^2dx \geq \int_{B_R(0)}\Big|\frac{x_1+\tau_k}{\tau_k}\partial_{x_2}w_k - \frac{x_2}{\tau_k}\partial_{x_1}w_k\Big|^2dx\\
&\geq \int_{B_R(0)}|\partial_{x_2}w|^2dx + o(1) \qquad \text{for every } R>0, \qquad (3.4.12)
\end{aligned}
$$

whereas for $R > 0$ large enough,

$$\int_{B_R(0)}|\partial_{x_2}w|^2dx > 0$$

since $w \in H_0^1(\mathbb{R}^2_+)$ is not identically zero.

Now, in order to prove (3.4.11), assume by contradiction that, passing to a subsequence,

$$\frac{\tau_k}{\lambda_k} \to d \in (0,\infty] \qquad \text{as } k \to \infty.$$

In the case where $d = \infty$ the estimate (3.4.12) implies that

$$\frac{1}{\lambda_k^2}\int_{\mathbb{R}^2_+}|\partial_\theta u_k|^2dx \to \infty \qquad \text{as } k\to\infty$$

and therefore

$$\|u_k\|_{\lambda_k,1} \to \infty \qquad \text{as } k\to\infty$$

which contradicts the fact that $\|u_k\|_{\lambda_k,1}$ is bounded in k.

Therefore we have $d < \infty$ and from (3.4.12),

$$\liminf_{k\to\infty}\frac{1}{\lambda_k^2}\int_{\mathbb{R}^2_+}|\partial_\theta u_k|^2dx \geq d^2\int_{\mathbb{R}^2}|\partial_{x_2}w|^2dx. \qquad (3.4.13)$$

Notice that in this case, $w \in H^1(\mathbb{R}^2)$ is a weak solution of

$$-\Delta w + d^2\partial_{x_2x_2}w + w = w^{p-1} \qquad \text{on } \mathbb{R}^2. \qquad (3.4.14)$$

Indeed, let $\varphi \in C_c^\infty(\mathbb{R}^2)$ and let $\varphi_k \in C_c^\infty(\mathbb{R}^2_+)$ be defined by

$$\varphi_k(x_1,x_2) = \varphi(x_1-\tau_k, x_2)$$

for k sufficiently large. We then have

$$
\begin{aligned}
&\frac{1}{\lambda_k^2}\int_{\mathbb{R}^2_+}(\partial_\theta u_k)(\partial_\theta\varphi_k)dx\\
=&\frac{(d^2+o(1))}{\tau_k^2}\int_{\mathbb{R}^2_+}(x_1\partial_{x_2}u_k - x_2\partial_{x_1}u_k)(x_1\partial_{x_2}\varphi_k - x_2\partial_{x_1}\varphi_k)dx\\
=&(d^2+o(1))\int_{\mathbb{R}^2}\Big(\frac{x_1+\tau_k}{\tau_k}\partial_{x_2}w_k - \frac{x_2}{\tau_k}\partial_{x_1}w_k\Big)\Big(\frac{x_1+\tau_k}{\tau_k}\partial_{x_2}\varphi - \frac{x_2}{\tau_k}\partial_{x_1}\varphi\Big)dx\\
=&d^2\int_{\mathbb{R}^2}\partial_{x_2}w\partial_{x_2}\varphi dx + o(1) \qquad \text{as } k\to\infty
\end{aligned}
$$

and therefore

$$\int_{\mathbb{R}^2_+}\Big(\nabla w\cdot\nabla\varphi + d^2\partial_{x_2}w\partial_{x_2}\varphi + w\varphi - w^{p-1}\varphi\Big)dx$$

$$
\begin{aligned}
&= \lim_{k\to\infty} \int_{\mathbb{R}^2_+} \Big(\nabla u_k \cdot \nabla \varphi_k + \frac{1}{\lambda_k^2}(\partial_\theta u_k)(\partial_\theta \varphi_k) + u_k\varphi_k - u_k^{p-1}\varphi_k\Big)dx \\
&= \lim_{k\to\infty} \Big(\langle u_k, \varphi\rangle_{\lambda_k,1} - \int_{\mathbb{R}^2_+} u_k^{p-1}\varphi_k dx\Big) = 0.
\end{aligned}
$$

Hence w satisfies (3.4.14) in this case. By (3.4.13) and weak lower semicontinuity, this implies that

$$
\begin{aligned}
\sup_{t\ge 0}\Big(E_*(tw) + \frac{t^2 d^2}{2}\int_{\mathbb{R}^2} |\partial_{x_2} w|^2 dx\Big) &= \Big(\frac{1}{2}-\frac{1}{p}\Big)\Big(\|w\|^2_{H^1(\mathbb{R}^2)} + d^2 \int_{\mathbb{R}^2} |\partial_{x_2} w|^2 dx\Big) \\
\le \Big(\frac{1}{2}-\frac{1}{p}\Big) \lim_{k\to\infty} \|u_k\|^2_{\lambda_k,1} &= \lim_{k\to\infty} E_{\lambda_k}(u_k) = \lim_{k\to\infty} c_{\lambda_k,1} = c_\infty.
\end{aligned}
$$

On the other hand, we have

$$
c_\infty \le \sup_{t\ge 0} E_*(tw) < \sup_{t\ge 0}\Big(E_*(tw) + t^2 d^2 \int_{\mathbb{R}^2} |\partial_{x_2} w|^2 dx\Big).
$$

Combining these inequalities yields a contradiction. Hence (3.4.11) holds.

The same argument as above with $d = 0$ yields that $w \ge 0$ is a solution of the limit problem

$$
-\Delta w + w = w^{p-1} \qquad \text{in } \mathbb{R}^2
$$

and by uniqueness we have $w = w_\infty$ after adding a finite translation to the sequence τ_k if necessary.

We finish the proof by showing that $w_k \to w$ strongly in $H^1(\mathbb{R}^2)$. Indeed, by weak lower semicontinuity,

$$
\begin{aligned}
c_\infty &= \Big(\frac{1}{2}-\frac{1}{p}\Big)\|w\|^2_{H^1(\mathbb{R}^2)} \le \Big(\frac{1}{2}-\frac{1}{p}\Big) \liminf_{k\to\infty} \|w_k\|^2_{H^1(\mathbb{R}^2)} \\
&= \Big(\frac{1}{2}-\frac{1}{p}\Big) \liminf_{k\to\infty} \|u_k\|^2_{H^1(\mathbb{R}^2_+)} \le \Big(\frac{1}{2}-\frac{1}{p}\Big) \lim_{k\to\infty} \Big(\|u_k\|^2_{\lambda_k,1}\Big) \\
&= \lim_{k\to\infty} c_{\lambda_k} = c_\infty.
\end{aligned}
$$

Hence equality holds in all steps. Since $H^1(\mathbb{R}^2)$ is uniformly convex, this shows that $w_k \to w$ strongly in $H^1(\mathbb{R}^2)$, as claimed and this completes the proof of the theorem. □

3.4.2. The limit $\lambda \to 0$. Next we consider the asymptotics of least energy solutions to (3.1.10) in the case $q = 1$ as $\lambda \to 0$. To find a suitable limit problem, we consider the transformed Dirichlet problem

$$
\begin{cases}
-\Delta v - \partial_\theta^2 v + \lambda^2 v = |v|^{p-2} v & \text{in } \mathbb{R}^2_+, \\
v = 0 & \text{on } \partial\mathbb{R}^2_+.
\end{cases}
\tag{3.4.15}
$$

Weak solutions $v \in H^+$ of (3.4.15) are critical points of the associated energy functional given by

$$
J_\lambda : H^+ \to \mathbb{R}, \quad J_\lambda(v) = \frac{1}{2}\left(|\nabla v|_2^2 + |\partial_\theta v|_2^2 + \lambda^2 |v|_2^2\right) - \frac{1}{p}\|v\|_p^p.
$$

These notions can be related to the original problem as follows: For $\lambda > 0$, consider the transformation

$$
H^+ \ni u \mapsto v \in H^+, \quad v(x) = \lambda^{\frac{2}{p-2}} u(\lambda x)
$$

so that

$$J_\lambda(v) = \lambda^{\frac{4}{p-2}} E^+_\lambda(u). \tag{3.4.16}$$

Moreover, u is a (least energy) solution of (3.1.10) if and only if v is a (least energy) solution of (3.4.15).

In order to prove Theorem 3.1.6, let $(\lambda_k)_k$ be sequence of numbers $\lambda_k \leq 1$ such that $\lambda_k \to 0$ as $k \to \infty$ and let $u_k \in H^+$ be positive least energy solutions of (3.1.10) for $\lambda = \lambda_k$.

For any $k \in \mathbb{N}$, set

$$v_k(x) = \lambda_k^{\frac{2}{p-2}} u_k(\lambda_k x), \quad v_k \in H^+.$$

Lemma 3.4.2. *The sequence $(v_k)_k$ is bounded in H^+.*

Proof. By Corollary 3.3.1, it suffices to show that there exists $C > 0$ such that

$$\|v_k\|_{1,0} \leq C \qquad \text{for all } k \in \mathbb{N}.$$

By the remarks above, v_k is a least energy solution of the transformed problem (3.4.15) with $\lambda = \lambda_k$. Multiplying this equation with v_k and integrating by parts yields

$$\|v_k\|^2_{1,0} + \lambda_k^2 |v_k|_2^2 = |v_k|_p^p \qquad \text{for all } k \in \mathbb{N}. \tag{3.4.17}$$

Moreover, we have

$$J_{\lambda_k}(v_k) = \inf_{v \in H^+ \setminus \{0\}} \sup_{t \geq 0} J_{\lambda_k}(tv).$$

Fix $\varphi \in C_c^\infty(\mathbb{R}^2_+) \setminus \{0\}$. Since v_k is a least energy solution of (3.4.15) for $\lambda = \lambda_k \leq 1$, we have

$$J_{\lambda_k}(v_k) \leq \sup_{t \geq 0} J_{\lambda_k}(t\varphi) \leq \sup_{t \geq 0} J_1(t\varphi) =: C_0$$

where, clearly, C_0 is independent of k.

We can then use (3.4.17) to get

$$J_{\lambda_k}(v_k) = \left(\frac{1}{2} - \frac{1}{p}\right)\left(\|v_k\|^2_{1,0} + \lambda_k^2 |v_k|_2^2\right) \geq \left(\frac{1}{2} - \frac{1}{p}\right)\|v_k\|^2_{1,0}$$

and hence

$$\|v_k\|^2_{1,0} \leq \frac{C_0}{\frac{1}{2} - \frac{1}{p}} \qquad \text{for all } k \in \mathbb{N}.$$

□

As a consequence of Lemma 3.4.2, we can pass to a subsequence and assume

$$v_k \rightharpoonup v^* \quad \text{in } H^+.$$

Lemma 3.4.3. *The weak limit v^* is a nontrivial weak solution of (3.1.15).*

Proof. Since every v_k is a weak solutions of (3.1.10), for any test function $\varphi \in C_c^\infty(\mathbb{R}^2_+)$ we have

$$\int_{\mathbb{R}^2_+} (\nabla v_k \cdot \nabla \varphi + \partial_\theta v_k \partial_\theta \varphi)\, dx = \int_{\mathbb{R}^2_+} |v_k|^{p-2} v_k \varphi\, dx - \lambda_k^2 \int_{\mathbb{R}^2_+} v_k \varphi\, dx.$$

Besides, since $v_k \rightharpoonup v^*$ weakly in H^+ and $\lambda_k \to 0^+$ as $k \to \infty$,

$$\int_{\mathbb{R}^2_+} (\nabla v_k \cdot \nabla \varphi + \partial_\theta v_k \partial_\theta \varphi)\, dx - \lambda_k^2 \int_{\mathbb{R}^2_+} v_k \varphi\, dx \to \int_{\mathbb{R}^2_+} (\nabla v^* \cdot \nabla \varphi + \partial_\theta v^* \partial_\theta \varphi)\, dx,$$

and

$$\int_{\mathbb{R}^2_+} |v_k|^{p-2} v_k \varphi \, dx \longrightarrow \int_{\mathbb{R}^2+} |v^*|^{p-2} v^* \varphi \, dx$$

as a consequence of the compact embedding $H^+ \hookrightarrow L^p(\mathbb{R}^2_+)$. It then follows that $v^* \in H^+$ is a weak solution of

$$-\Delta v^* - \partial_\theta^2 v^* = |v^*|^{p-2} v^* \quad \text{in } \mathbb{R}^2_+.$$

Next, we prove that $v^* \not\equiv 0$. To do so, first observe that the embedding $H^+ \hookrightarrow L^p$ yields

$$C := \inf_{u \in H^+ \setminus \{0\}} \frac{\|u\|_{1,0}}{|u|_p} \in (0, \infty).$$

Thus, the above comments, together with the fact that $|u|_2^2 \le |\partial_\theta u|_2^2 \le \|u\|_{1,0}^2$ for $u \in H^+$ (see Corollary 3.3.1), imply that

$$C^2 = \inf_{u \in H^+ \setminus \{0\}} \frac{\|u\|_{1,0}^2}{|u|_p^2} \le \inf_{u \in H^+ \setminus \{0\}} \frac{\|u\|_{1,0}^2 + \lambda_k^2 |u|_2^2}{|u|_p^2} \le 2 \inf_{u \in H^+ \setminus \{0\}} \frac{\|u\|_{1,0}^2}{|u|_p^2} = 2C^2.$$

Recalling also that

$$J_{\lambda_k}(v_k) = \inf_{u \in H^+ \setminus \{0\}} \left(\frac{1}{2} - \frac{1}{p}\right) \left(\frac{\|u\|_{1,0}^2 + \lambda_k^2 |u|_2^2}{|u|_p^2}\right)^{\frac{p}{p-2}},$$

we thus have

$$\left(\frac{1}{2} - \frac{1}{p}\right) C^{\frac{2p}{p-2}} \le J_{\lambda_k}(v_k) \le \left(\frac{1}{2} - \frac{1}{p}\right) \left(2C^2\right)^{\frac{p}{p-2}} \qquad \text{for all } k \in \mathbb{N}. \tag{3.4.18}$$

Now assume by contradiction that $v^* = 0$, i.e., $v_k \rightharpoonup 0$ weakly in H^+. The compact embedding $H^+ \hookrightarrow L^p$ implies $v_k \longrightarrow 0$ in L^p, and therefore $\|v_k\|_{1,0} \longrightarrow 0$ by (3.4.17). Hence also $|v_k|_2 \longrightarrow 0$ by Corollary 3.3.1. We then deduce that

$$J_{\lambda_k}(v_k) = \left(\frac{1}{2} - \frac{1}{p}\right) (\|v_k\|_{1,0}^2 + \lambda_k^2 |v_k|_2^2) \longrightarrow 0,$$

which contradicts (3.4.18). We conclude that $v^* \ne 0$, as claimed. □

We will now use Γ-convergence to finish the proof of Theorem 3.1.6:

Proof of Theorem 3.1.6. It remains to prove that v^* is a least energy solution of (3.1.15), and that $v_k \longrightarrow v^*$ strongly in H^+ as $k \longrightarrow \infty$.

To deduce these properties from Γ-convergence theory, we consider the space $X := H^+ \setminus \{0\}$ endowed with the weak topology (induced by $\|\cdot\|_{1,0}$). Consider the functionals $F_k, F : X \longrightarrow [0, \infty]$ defined by

$$F_k(u) := \frac{(\|u\|_{1,0}^2 + \lambda_k^2 |u|_2^2)^{\frac{p}{p-2}}}{|u|_p^{\frac{2p}{p-2}}} \quad \text{and} \quad F(u) := \frac{\|u\|_{1,0}^{\frac{2p}{p-2}}}{|u|_p^{\frac{2p}{p-2}}}.$$

Then we have

$$F(u) \le F_k(u) \qquad \text{for every } k \in \mathbb{N} \text{ and } u \in H^+.$$

Let $(\tilde{u}_k)_k \subset X$ be an arbitrary sequence such that $\tilde{u}_k \to \tilde{u}$ in X (recall that X has the weak topology of H^+). The compact embedding $H^+ \hookrightarrow L^p(\mathbb{R}^2_+)$ and the weak lower semicontinuity of $\|\cdot\|_{1,0}$ imply

$$F(\tilde{u}) \le \liminf_{k\to\infty} F(\tilde{u}_k) \le \liminf_{k\to\infty} F_k(\tilde{u}_k).$$

On the other hand, for any $\tilde{u} \in X$, the constant sequence $\tilde{u}_k := \tilde{u}$ satisfies that $\tilde{u}_k \to \tilde{u}$ in X and

$$F(\tilde{u}) = \lim_{k\to\infty} F_k(\tilde{u}_k).$$

We conclude that $F_k \xrightarrow{\Gamma} F$. Since,

$$F_k(v_k) = \inf_{u\in X} F_k(u)$$

and $v_k \to v$ in X, it follows from [**41**, Corollary 7.20] that

$$F(v) = \inf_{u\in X} F(u) = \lim_{k\to\infty} F_k(v_k). \tag{3.4.19}$$

Consequently,

$$\begin{aligned}\left(\frac{1}{2}-\frac{1}{p}\right)\frac{\|v\|_{1,0}^{\frac{2p}{p-2}}}{|v|_p^{\frac{2p}{p-2}}} &= \inf_{u\in H^+\setminus\{0\}}\left(\frac{1}{2}-\frac{1}{p}\right)\frac{\|u\|_{1,0}^{\frac{2p}{p-2}}}{|u|_p^{\frac{2p}{p-2}}}\\ &= \inf_{u\in H^+\setminus\{0\}}\sup_{t\ge 0}\left(\frac{t^2}{2}\|u\|_{1,0}^2-\frac{t^p}{p}|u|_p^p\right),\end{aligned}$$

and this implies that v is a least energy solution of (3.1.15). Moreover, since $v_k \to v$ in $L^p(\mathbb{R}^2_+)$ by the compact embedding $H^+ \hookrightarrow L^p(\mathbb{R}^2_+)$, it follows from (3.4.19) and the definition of the functionals F_k and F that

$$\|v\|_{1,0}^2 = \lim_{k\to\infty}\left(\|v_k\|_{1,0}^2+\lambda_k^2|v_k|_2^2\right) \ge \limsup_{k\to\infty}\|v_k\|_{1,0}^2 \ge \liminf_{k\to\infty}\|v_k\|_{1,0}^2 \ge \|v\|_{1,0}^2.$$

Consequently, we have

$$\|v_k\|_{1,0} \to \|v\|_{1,0} \qquad \text{as } k\to\infty,$$

and the uniform convexity of $(H^+, \|\cdot\|_{1,0})$ implies that $v_k \to v$ strongly in H^+ as $k \to \infty$. □

3.5 Radial versus nonradial least energy nodal solutions

In this section we complete the proofs of Theorem 3.1.2 and Theorem 3.1.3. Given the assumptions of Theorem 3.1.2, the existence of a least energy nodal solution of (3.1.6) for every $\lambda > 0$ is a direct consequence of Corollary 3.2.9.

We will now first prove Theorem 3.1.2(ii), which will be a consequence of Lemma 3.4.1 and a result in [**135**].

We recall that, as in Section 3.4.1 and Section 3.2, the energy functionals $E_*, E_\lambda : H \to \mathbb{R}$ are defined by

$$E_*(v) := \frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla v|^2+|v|^2\right)dx - \frac{1}{p}\int_{\mathbb{R}^2}|v|^p dx$$

and

$$E_\lambda(v) = E_*(v) + \frac{1}{\lambda^2}\int_{\mathbb{R}^2}|\partial_\theta v|^2 dx$$

for $v \in H$. Moreover, as in Section 3.2, we consider the λ-dependent scalar product $\langle\cdot,\cdot\rangle_\lambda$ defined in (3.1.8) on H and the corresponding norm $\|\cdot\|_\lambda$. In particular, we shall use $\|\cdot\|_1$ given by

$$\|u\|_1^2 = \int_{\mathbb{R}^2}\left(|\nabla u|^2+|\partial_\theta u|^2+|u|^2\right)dx \qquad \text{for } u \in H.$$

Proposition 3.5.1. *There exists $\varepsilon_* > 0$ such that for every $\lambda > 0$ and every radial nodal solution $u \in H$ of (3.1.6) we have*

$$E_*(u) = E_\lambda(u) > 2c_\infty + \varepsilon_*,$$

where c_∞ is given in (3.4.1).

Proof. First observe that $E_*(u) = E_\lambda(u)$ for every radial function $u \in H$. Moreover, if u is a radial nodal solution of (3.1.6), then u also solves the limit problem (3.1.13). By [**135**, Theorem 1.5], and the variational characterization of c_∞ given (3.4.4) and (3.4.7), there exists $\varepsilon_* > 0$ with the property that $E_*(u) > 2c_\infty + \varepsilon_*$ for every nodal solution of (3.1.13). This proves the claim. □

Proof of Theorem 3.1.2(ii) (completed). Let $\varepsilon_* > 0$ be given by Proposition 3.5.1. By (3.4.1), there exists $\Lambda_0 > 0$ with the property that

$$c_\lambda < c_\infty + \frac{\varepsilon_*}{2} \qquad \text{for every } \lambda > \Lambda_0.$$

Consequently, for $\lambda > \Lambda_0$, problem (3.1.10) admits a nontrivial solution $u \in H^+$ with $E^+_\lambda(u) < c_\infty + \frac{\varepsilon_*}{2}$. By odd reflection, we may extend u to a nodal solution of (3.1.6) with $E_\lambda(u) < 2c_\infty + \varepsilon_*$. Proposition 3.5.1 therefore implies that the least energy nodal solutions of (3.1.6) cannot be radial. □

Next, we complete the proof of Theorem 3.1.3, which we restate here for the reader's convenience.

Theorem 3.5.2. *Let $p > 2$.*

i. *If $u \in H$ is a nontrivial weak solution of*

$$-\Delta u - \frac{1}{\lambda^2}\partial_\theta^2 u + u = |u|^{p-2}u \quad \text{in } \mathbb{R}^2 \tag{3.5.1}$$

for some $\lambda > 0$ satisfying $\lambda < \left(\frac{1}{(p-1)|u|_\infty^{p-2}}\right)^{\frac{1}{2}}$, then u is a radial function.

ii. *For every $c > 0$, there exists $\lambda_c > 0$ with the property that every weak solution $u \in H$ of (3.5.1) for some $\lambda \in (0, \lambda_c)$ with $E_\lambda(u) \le c$ is radial.*

Proof. (i) Let $u \in H$ be a nontrivial weak solution of (3.5.1) for some $\lambda > 0$, and let, as before, $u^\#$ denote the radial average of u as defined in (3.2.1). It is easy to see that, for every $k \in \mathbb{N}$, the function $u^\# \in H$ is a weak solution of

$$-\Delta u^\# + u^\# = \left(|u|^{p-2}u\right)^\# \quad \text{in } \mathbb{R}^2.$$

Consequently we have, in weak sense,

$$-\Delta(u - u^\#) - \frac{1}{\lambda^2}\partial_\theta^2(u - u^\#) + (u - u^\#) = |u|^{p-2}u - \left(|u|^{p-2}u\right)^\# \quad \text{in } \mathbb{R}^2.$$

Testing this equation against $u - u^\#$ yields

$$\begin{aligned}\frac{1}{\lambda^2}|\partial_\theta u|_2^2 &= \frac{1}{\lambda^2}|\partial_\theta(u - u^\#)|_2^2 \le |\nabla(u - u^\#)|_2^2 + \frac{1}{\lambda^2}|\partial_\theta(u - u^\#)|_2^2 + |u - u^\#|_2^2\\ &= \int_{\mathbb{R}^2}\left(|u|^{p-2}u - \left(|u|^{p-2}u\right)^\#\right)(u - u^\#)\,dx \le \left||u|^{p-2}u - \left(|u|^{p-2}u\right)^\#\right|_2 |u - u^\#|_2\end{aligned}$$

$$\leq \Big| |u|^{p-2}u - \big(|u|^{p-2}u\big)^{\#} \Big|_2 \, |\partial_\theta u|_2, \tag{3.5.2}$$

where we used Lemma 3.2.2 in the last step. Moreover, $|u|^{p-2}u \in H$ by Remark 3.2.6, and therefore

$$\begin{aligned} \Big| |u|^{p-2}u - \big(|u|^{p-2}u\big)^{\#} \Big|_2 &\leq \big|\partial_\theta\big(|u|^{p-2}u\big)\big|_2 = (p-1)\big||u|^{p-2}\partial_\theta u\big|_2 \\ &\leq (p-1)|u|_\infty^{p-2}|\partial_\theta u|_2, \end{aligned} \tag{3.5.3}$$

again by Lemma 3.2.2. Combining (3.5.2) and (3.5.3), we obtain that

$$\frac{1}{\lambda^2}|\partial_\theta u|_2^2 \leq (p-1)|u|_\infty^{p-2}|\partial_\theta u|_2^2$$

which implies that $\partial_\theta u \equiv 0$ if $\lambda < \left(\frac{1}{(p-1)|u|_\infty^{p-2}}\right)^{\frac{1}{2}}$. The proof of (i) is thus finished.

(ii) Let $c > 0$ be given, and let $u \in H$ be a nontrivial weak solution of (3.5.1) for some $\lambda > 0$ with $E_\lambda(u) \leq c$. Since $E_\lambda(u) = \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|_\lambda^2$, it then follows that

$$\|u\|_{H^1(\mathbb{R}^2)}^2 \leq \|u\|_\lambda^2 = \frac{2p}{p-2}E_\lambda(u) \leq \frac{2pc}{p-2}$$

and therefore

$$|u|_\infty \leq C\|u\|_{H^1(\mathbb{R}^2)}^\sigma \leq C\Big(\frac{2pc}{p-2}\Big)^{\frac{\sigma}{2}} =: \mu_c$$

by Lemma 3.2.5 with the constants $C, \sigma > 0$ given there. Hence, if

$$\lambda < \lambda_c := \Big(\frac{1}{(p-1)\mu_c^{p-2}}\Big)^{\frac{1}{2}},$$

then also $\lambda < \left(\frac{1}{(p-1)|u|_\infty^{p-2}}\right)^{\frac{1}{2}}$ and therefore u is radial by (i). The proof is finished. □

Next we provide uniform energy estimates for least energy nodal solutions of (3.5.1).

Lemma 3.5.3. *Let $p > 2$. There exist constants $c, C > 0$ with the property that*

$$c \leq E_\lambda(u) \leq C \tag{3.5.4}$$

for every $\lambda > 0$ and every least energy nodal solution $u \in H$ of (3.5.1).

Proof. The lower bound is obtained by choosing $c = \hat{c}_\infty$ as defined in (3.4.4), since

$$E_\lambda(u) = \sup_{t\geq 0} E_\lambda(tu) \geq \sup_{t\geq 0} E_*(tu) \geq \hat{c}_\infty$$

for every $\lambda > 0$ and every nontrivial solution $u \in H$ of (3.5.1).

For the upper bound, we first remark that the existence of radial nodal solutions of (3.1.13) is well known, see for instance Theorems 4 and 5 in [**124**]. Let $\hat{u} \in H^1(\mathbb{R}^2)$ be a fixed radial nodal solution of (3.1.13) and set $C = E_*(\hat{u})$. For every $\lambda > 0$, the function $\hat{u} \in H$ is then also a nodal solution of (3.5.1), and therefore

$$E_\lambda(u) \leq E_\lambda(\hat{u}) = E_*(\hat{u}) = C$$

for every least energy nodal solution $u \in H$ of (3.5.1). □

The proof of Theorem 3.1.2 is now completed by deriving Part (i) of this theorem as follows: Let $C > 0$ be given by Lemma 3.5.3, and let $u \in H$ be a least energy solution of (3.5.1) for some $\lambda > 0$. Then we have $E_\lambda(u) \leq C$. Applying Theorem 3.5.2 with $c = C$ and considering $\lambda_0 := \min\{\lambda_c, \Lambda_0\}$ with $\Lambda_0 > 0$ given as in Theorem 3.1.2(ii), we then deduce that $0 < \lambda_0 \leq \Lambda_0$, and u is radial if $\lambda < \lambda_0$. The proof of Theorem 3.1.2(i) is thus finished.

3.6 Appendix

We give the proof of Lemma 3.2.5, which we restate here for the reader's convenience.

Lemma 3.6.1. *Let $\lambda > 0$ and let $u \in H$ be a weak solution of*

$$-\Delta u - \frac{1}{\lambda^2}\partial_\theta^2 u + u = |u|^{p-2}u \quad \text{in } \mathbb{R}^2. \tag{3.6.1}$$

Then $u \in L^\infty(\mathbb{R}^2)$. Furthermore, there exist constants $C, \sigma > 0$, depending on $p > 2$ but not on u and λ, such that

$$|u|_\infty \le C\|u\|_{H^1(\mathbb{R}^2)}^\sigma. \tag{3.6.2}$$

Proof. The proof is based on Moser iteration, cf. Appendix B in [**125**] and the references therein.

We fix $L, s \ge 2$ and consider auxiliary functions $h, g \in C^1([0,\infty))$ defined by

$$h(t) := s\int_0^t \min\{\tau^{s-1}, L^{s-1}\}\,d\tau \qquad \text{and} \qquad g(t) := \int_0^t [h'(\tau)]^2\,d\tau$$

We note that

$$h(t) = t^s \quad \text{for } t \le L \qquad \text{and} \qquad g(t) \le tg'(t) = t(h'(t))^2 \quad \text{for } t \ge 0, \tag{3.6.3}$$

since the function $t \mapsto h'(t) = s\min\{t^{s-1}, L^{s-1}\}$ is nondecreasing. We shall now show that $w := u^+ \in L^\infty(\mathbb{R}^2)$, and that $\|w\|_\infty$ is bounded by the r.h.s. of (3.6.2). Since we may replace u with $-u$, the claim will then follow.

We note that $w \in H$ and $\varphi := g(w) \in H$ with

$$\nabla w = \mathbf{1}_{\{u>0\}}\nabla u, \quad \nabla\varphi = g'(w)\nabla w, \qquad \partial_\theta w = \mathbf{1}_{\{u>0\}}\partial_\theta u, \quad \partial_\theta\varphi = g'(w)\partial_\theta w.$$

This follows from the boundedness of g' and the estimate $g(t) \le s^2 t^{2s-1}$ for $t \ge 0$. Testing (3.6.1) with φ gives

$$\int_{\mathbb{R}^2}\Big(\nabla u\cdot\nabla\varphi + \frac{1}{\lambda^2}(\partial_\theta u\,\partial_\theta\varphi) + u\varphi\Big)dx = \int_{\mathbb{R}^2}|u|^{p-2}u\varphi\,dx,$$

from where we estimate,

$$\begin{aligned}
&\int_{\mathbb{R}^2}\Big(|\nabla h(w)|^2 + \frac{1}{\lambda^2}(\partial_\theta h(w))^2 + wg(w)\Big)dx\\
&= \int_{\mathbb{R}^2}\Big(g'(w)\Big(|\nabla w|^2 + \frac{1}{\lambda^2}(\partial_\theta w)^2\Big) + ug(w)\Big)dx\\
&= \int_{\mathbb{R}^2}|u|^{p-2}ug(w)\,dx\\
&\le \int_{\mathbb{R}^2} w^p(h'(w))^2\,dx.
\end{aligned} \tag{3.6.4}$$

Here we used (3.6.3) in the last step. We now fix $r > 1$ with $\frac{(p-2)r}{r-1} \ge 2$ and $q > 4r$. Combining (3.6.4) with Sobolev embeddings, we obtain the inequality

$$\frac{1}{c_0}|h(w)|_q^2 - |h(w)|_2^2 + \int_{\mathbb{R}^2} wg(w)\,dx \le \int_{\mathbb{R}^2} w^p(h'(w))^2\,dx \tag{3.6.5}$$

with a constant $c_0 = c_0(q) > 0$. Since

$$h(t) = t^s, \quad h'(t) = st^{s-1} \quad \text{and} \quad g(t) = s^2 \int_0^t \tau^{2s-2}\, d\tau = \frac{s^2}{2s-1} t^{2s-1} \quad \text{for } t \le L,$$

we may let $L \longrightarrow \infty$ in (3.6.5) and apply Lebesgue's theorem to obtain

$$\frac{1}{c_0}|w^s|_q^2 + \Big(\frac{s^2}{2s-1} - 1\Big)|w^s|_2^2 \le s^2 \int_{\mathbb{R}^2} w^{p+2s-2}\, dx \le s^2 |w|^{p-2}_{\frac{(p-2)r}{r-1}} |w|^{2s}_{2rs}$$

Since $s \ge 2$, we have $\frac{s^2}{2s-1} \ge 1$, and we thus obtain the inequality

$$|w|_{sq} \le (c_1 s)^{\frac{1}{s}} |w|_{2rs} \qquad \text{with } c_1 := \Big(c_0 |w|^{p-2}_{\frac{r(p-2)}{r-1}}\Big)^{\frac{1}{2}}. \tag{3.6.6}$$

Next we note that the choice of r and q only depends on p but not on $s \ge 2$. We may therefore consider $s = s_n = \rho^n$ for $n \in \mathbb{N}$ with $\rho := \frac{q}{2r} > 2$, so that

$$2s_1 r = q \qquad \text{and} \qquad 2s_{n+1} r = q s_n \quad \text{for } n \in \mathbb{N}.$$

Iteration of (3.6.6) then gives

$$|w|_{\rho^n q} = |w|_{s_n q} \le |w|_q \prod_{j=1}^n (c_1 \rho^j)^{\rho^{-j}} \le c_1^{\frac{\rho}{\rho-1}} c_2 |w|_q \quad \text{for all } n$$

with

$$c_2 := \rho^{\sum_{j=1}^\infty j\rho^{-j}} < \infty.$$

It follows that

$$|w|_\infty = \lim_{n\to\infty} |w|_{\rho^n q} \le c_1^{\frac{\rho}{\rho-1}} c_2 |w|_q. \tag{3.6.7}$$

Moreover, by Sobolev embeddings, we have

$$c_1 \le c_1' \|w\|^{\frac{p-2}{2}}_{H^1(\mathbb{R}^2)} \le c_1' \|u\|^{\frac{p-2}{2}}_{H^1(\mathbb{R}^2)} \qquad \text{and} \qquad |w|_q \le \tilde{c}\|w\|_{H^1} \le \tilde{c}\|u\|_{H^1(\mathbb{R}^2)}$$

with constants $c_1', \tilde{c} > 0$ depending only on p, r and q. It thus follows from (3.6.7) that

$$|w|_\infty \le C \|u\|^{\frac{(p-2)\rho}{2(\rho-1)}+1}_{H^1(\mathbb{R}^2)} \qquad \text{with} \qquad C := c_2 (c_1')^{\frac{\rho}{\rho-1}} \tilde{c}.$$

The proof is thus finished. □

Remark 3.6.2. *Let $\lambda > 0$ and $p \in (2, \infty)$. By a variant of the Moser iteration argument given above, we can also show that every weak solution $u \in H^+$ of*

$$-\Delta u - \frac{1}{\lambda^2}\partial_\theta^2 u = |u|^{p-2}u \quad \text{in } \mathbb{R}^2_+, \qquad u = 0 \quad \text{on } \partial\mathbb{R}^2_+ \tag{3.6.8}$$

satisfies $u \in L^\infty(\mathbb{R}^2_+)$. To see this, we replace, with the help of Corollary 3.3.1 and (3.6.8), the inequalities (3.6.4) and (3.6.5) by

$$\frac{1}{c}|h(w)|_q^2 \le \|h(w)\|^2_{\lambda,0} = \int_{\mathbb{R}^2_+} |u|^{p-2} u g(w)\, dx \le \int_{\mathbb{R}^2_+} w^p \big(h'(w)\big)^2\, dx$$

with a constant $c > 0$ depending on q and λ. We can then complete the argument as above, noting that in this case the constants depend on $\lambda > 0$.

CHAPTER 4

Rotating Waves in Nonlinear Media and Critical Degenerate Sobolev Inequalities

In this chapter, we present our results on Rotating Waves as discussed in Section 1.4. Up to minor changes, the subsequent content has appeared in **[P3]**.

4.1 Introduction

Within a standard model, the analysis of wave propagation in an ambient medium with nonlinear response leads to the study of a nonlinear wave equation of the type

$$\partial_t^2 v - \Delta v + mv = f(v) \qquad \text{in } \mathbb{R} \times \Omega, \tag{4.1.1}$$

in an ambient domain $\Omega \subset \mathbb{R}^N$ with mass parameter $m \geq 0$ and nonlinear response function f. In the case $m = 0$, (4.1.1) is the classical nonlinear wave equation, while the case $m > 0$ is also known as a nonlinear Klein-Gordon equation. For nonlinearities of the form $f(v) = g(|v|^2)v$ with a real-valued function g, standing wave solutions can be found by the ansatz

$$v(t,x) = e^{-ikt}u(x), \qquad k > 0 \tag{4.1.2}$$

with a real-valued function u. Depending on the frequency parameter k, this reduces (4.1.1) either to a stationary nonlinear Schrödinger or a nonlinear Helmholtz equation (see e.g. [**55**] for more details). The resulting stationary nonlinear Schrödinger equation has been studied extensively in the past four decades by variational methods, see e.g. the monograph [**7**] and the references therein. Due to a lack of a direct variational framework, the nonlinear Helmholtz equation requires a different approach and has been studied more recently e.g. in [**32**, **55**, **69**, **92**, **93**] by dual variational methods and bifurcation theory.

Clearly, the amplitude $|v|$ of a solution v of (4.1.1) given by the ansatz (4.1.2) remains time-independent. As a consequence, the analysis of standing wave solutions does not lead to a full understanding of (4.1.1) from a dynamical point of view and should be complemented, in particular, by the study of non-stationary real-valued time-periodic solutions, travelling wave solutions and scattering solutions. We stress that the ansatz (4.1.2) does not give rise to non-stationary real-valued time-periodic solutions since the nonlinearity of the problem does not allow to pass to real and imaginary parts.

In the case where $\Omega = \mathbb{R}^N$ and $f(v)$ in (4.1.1) is replaced by $q(x)f(v)$ with a compactly supported weight function q, spatially localized real-valued time-periodic solutions, also called breathers, have attracted increasing attention recently, see e.g. [**73**, **94**] and the references therein. In the case where Ω is a radial domain, a further interesting type of real-valued time-periodic solution is given by *rotating wave solutions*. In particular, if Ω is a bounded radial domain and (4.1.1) is complemented with the Dirichlet boundary condition $v = 0$ on $\mathbb{R} \times \partial\Omega$, the existence of rotating waves and their variational characterization arises as a natural question which, up to our knowledge, has not been addressed systematically so far.

The main purpose of the present paper is to provide such a systematic study. While we mainly focus on the case where $\Omega = \mathbf{B}$ is the unit ball in $\mathbb{R}^N$, we will also address the case where Ω is an annulus or a general Riemannian model with boundary, see Sections 4.5 and 4.6 below. Specifically, we study the case of a focusing nonlinearity of the form $f(v) = |v|^{p-2}v$, which leads to the superlinear problem

$$\begin{cases} \partial_t^2 v - \Delta v + mv = |v|^{p-2}v & \text{in } \mathbb{R} \times \mathbf{B} \\ v = 0 & \text{on } \mathbb{R} \times \partial\mathbf{B} \end{cases} \tag{4.1.3}$$

for $N \geq 2$, where $2 < p < 2^*$ and $m > -\lambda_1(\mathbf{B})$. Here, $\lambda_1(\mathbf{B})$ denotes the first Dirichlet eigenvalue of $-\Delta$ on $\mathbf{B}$ and 2^* denotes the critical Sobolev exponent given by $2^* = \frac{2N}{N-2}$ for $N \geq 3$ and $2^* = \infty$ for $N = 2$. The ansatz for time-periodic rotating solutions of (4.1.3) is given by

$$v(t, x) = u(R_{\alpha t}(x)) \tag{4.1.4}$$

where, for $\theta \in \mathbb{R}$, we let $R_\theta \in O(N)$ denote a planar rotation in $\mathbb{R}^N$ with angle θ, so the constant $\alpha > 0$ in (4.1.4) is the angular velocity of the rotation. Without loss of generality, we may assume that

$$R_\theta(x) = (x_1 \cos\theta + x_2 \sin\theta, -x_1 \sin\theta + x_2 \cos\theta, x_3, \dots, x_N) \qquad \text{for } x \in \mathbb{R}^N,$$

so R_θ is the rotation in the x_1-x_2-plane with fixed point set $\{0_{\mathbb{R}^2}\} \times \mathbb{R}^{N-2}$. In the following, we call a function u on the unit ball x_1-x_2-*nonradial* if it is not R_θ-invariant for at least one angle $\theta \in \mathbb{R}$. If the profile function u in (4.1.4) is x_1-x_2-nonradial, then the corresponding solution v can be interpreted as a rotating wave in a medium with nonlinear response given by the right hand side of (4.1.3). The ansatz (4.1.4) reduces (4.1.3) to

$$\begin{cases} -\Delta u + \alpha^2 \partial_\theta^2 u + mu = |u|^{p-2}u & \text{in } \mathbf{B} \\ u = 0 & \text{on } \partial\mathbf{B} \end{cases} \tag{4.1.5}$$

where $\partial_\theta = x_1\partial_{x_2} - x_2\partial_{x_1}$ denotes the associated angular derivative operator. We point out that a seemingly closely related equation, with the term $\alpha^2\partial_\theta^2 u$ replaced by $-\alpha^2\partial_\theta^2 u$, arises in an ansatz for solutions of nonlinear Schrödinger equations in $\mathbb{R}^3$ with invariance with respect to screw motion, see **[P2]** and also [**46**] for a related work on Allen-Cahn equations. Note, however, that the positive sign of the term $\alpha^2\partial_\theta^2 u$ results in a drastic change of the nature of the problem, as the operator $-\Delta + \alpha^2\partial_\theta^2$ loses uniform ellipticity in $\mathbf{B}$ if $\alpha \geq 1$. This also distinguishes the study of (4.1.5) from the related study of rotating solutions to nonlinear Schrödinger equations, where the angular velocity α appears within a first order term which does not affect the ellipticity of the associated Schrödinger operator, see e.g. [**83**, **118**] and the references therein.

If a solution u of (4.1.5) satisfies $\partial_\theta u \equiv 0$ in $\mathbf{B}$, then u solves the classical stationary nonlinear Schrödinger equation $-\Delta u + mu = |u|^{p-2}u$ in $\mathbf{B}$ with Dirichlet boundary conditions on $\partial\mathbf{B}$, so it satisfies (4.1.5) with $\alpha = 0$. If, in addition, u is positive, then u has to be a radial function as a consequence of the symmetry result of Gidas, Ni and Nirenberg [**61**]. Thus, the ansatz (4.1.4) then merely gives rise to a radial stationary solution of (4.1.3). We mention here that radially symmetric non-stationary solutions of (4.1.1) in $\Omega = \mathbf{B}$ were first studied by Ben-Naoum and Mahwin [**13**] for sublinear nonlinearities and more recently by Chen and Zhang [36–38]. In this problem, the spectral properties of the radial wave operator lead to delicate assumptions on the dimension as well as the ratio between the radius of the ball and the period length. The main purpose of the present paper is to analyze for which range of parameters α, m and p ground state solutions of (4.1.5) exist and to distinguish under which assumptions on α, m and p they are radial or x_1-x_2-nonradial and therefore correspond to rotating waves via the ansatz (4.1.4).

By a ground state solution of (4.1.5), we mean a solution characterized as a minimizer of the minimization problem for

$$\mathscr{C}_{\alpha,m,p}(\mathbf{B}) := \inf_{u \in H_0^1(\mathbf{B}) \setminus \{0\}} R_{\alpha,m,p}(u), \tag{4.1.6}$$

where, for $m \in \mathbb{R}$, $\alpha \geq 0$ and $p \in [2, 2^*)$, we consider the associated Rayleigh quotient $R_{\alpha,m,p}$ given by

$$R_{\alpha,m,p}(u) = \frac{\int_{\mathbf{B}} \left(|\nabla u|^2 - \alpha^2 |\partial_\theta u|^2 + m u^2\right)\, dx}{\left(\int_{\mathbf{B}} |u|^p \, dx\right)^{\frac{2}{p}}}, \qquad u \in H_0^1(\mathbf{B}) \setminus \{0\}. \tag{4.1.7}$$

As we shall see in Remark 4.4.3 below, this minimization problem is only meaningful for $0 \leq \alpha \leq 1$, since for every $p \in [2, 2^*)$ and $m \in \mathbb{R}$ we have

$$\mathscr{C}_{\alpha,m,p}(\mathbf{B}) = -\infty \quad \text{for } \alpha > 1.$$

Moreover, for every $p \in [2, 2^*)$ and $m \in \mathbb{R}$,

$$\text{the function } \alpha \mapsto \mathscr{C}_{\alpha,m,p}(\mathbf{B}) \text{ is continuous and nonincreasing on } [0,1]. \tag{4.1.8}$$

In the case $0 < \alpha < 1$, the operator $-\Delta + \alpha^2 \partial_\theta^2$ is uniformly elliptic, as can be seen by writing the operator in polar coordinates as

$$-\Delta + \alpha^2 \partial_\theta^2 = -\Delta_r u - \frac{1}{r^2} \Delta_{\mathbb{S}^{N-1}} u + \alpha^2 \partial_\theta^2 u, \tag{4.1.9}$$

where $\Delta_{\mathbb{S}^{N-1}}$ denotes the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^{N-1}$. In this case the existence of minimizers of $R_{\alpha,m,p}$ on $H_0^1(\mathbf{B}) \setminus \{0\}$ follows by a standard compactness and weak lower semicontinuity argument. However, even in this case it is difficult to decide in general whether minimizers are radial or nonradial functions. This is due to competing effects. Firstly, the additional term $-\alpha^2 \|\partial_\theta u\|^2_{L^2(\mathbf{B})}$ favours x_1-x_2-nonradial functions as energy minimizers. On the other hand, the Pólya-Szegö inequality yields $\int_{\mathbf{B}} |\nabla u^*|^2 \, dx \leq \int_{\mathbf{B}} |\nabla u|^2 \, dx$, where u^* denotes the (radial) Schwarz symmetrization of a function $u \in H_0^1(\mathbf{B})$.

Since $R_{\alpha,m,p}(u) = R_{0,m,p}(u)$ for every radial function $u \in H_0^1(\mathbf{B}) \setminus \{0\}$ and every $\alpha \in [0,1]$, a sufficient condition for the x_1-x_2-nonradiality of all ground state solutions is the inequality

$$\mathscr{C}_{\alpha,m,p}(\mathbf{B}) < \mathscr{C}_{0,m,p}(\mathbf{B}). \tag{4.1.10}$$

In particular, we will be interested in proving this inequality for α close to 1. We point out that the borderline case $\alpha = 1$ differs significantly from the case $0 \leq \alpha < 1$, as the differential operator $-\Delta + \partial_\theta^2$ is no longer uniformly elliptic on $\mathbf{B}$. In fact, it follows from the representation (4.1.9) in the case $\alpha = 1$ that the operator $-\Delta + \partial_\theta^2$ fails to be uniformly elliptic in a neighborhood of the great circle $\{x \in \partial \mathbf{B} : x_3 = \cdots = x_N = 0\}$ (which equals $\partial \mathbf{B}$ in the case $N = 2$). We shall see in this paper that the minimization problem in the case $\alpha = 1$ is essentially governed by a degenerate anisotropic critical Sobolev inequality in the half space. The corresponding critical exponent in this Sobolev inequality is given by

$$2_1^* := \frac{4N+2}{2N-3}.$$

The relevance of this exponent is indicated by our first main result which yields the following characterization.

Theorem 4.1.1. *Let $m > -\lambda_1(\mathbf{B})$ and $p \in (2, 2^*)$.*

(i) If $\alpha \in (0,1)$, then there exists a ground state solution of (4.1.5).

(ii) We have

$$\mathscr{C}_{1,m,p}(\mathbf{B}) = 0 \quad \textit{for } p > 2_1^*, \qquad \textit{and} \quad \mathscr{C}_{1,m,p}(\mathbf{B}) > 0 \quad \textit{for } p \le 2_1^*. \tag{4.1.11}$$

Moreover, for any $p \in (2_1^*, 2^*)$, there exists $\alpha_p \in (0,1)$ with the property that

$$\mathscr{C}_{\alpha,m,p}(\mathbf{B}) < \mathscr{C}_{0,m,p}(\mathbf{B}) \qquad \textit{for } \alpha \in (\alpha_p, 1]$$

and therefore every ground state solution of (4.1.5) is x_1-x_2-nonradial for $\alpha \in (\alpha_p, 1)$.

The following new degenerate Sobolev inequality is an immediate consequence of the special case $m = 0$, $\alpha = 1$ in Theorem 4.1.1.

Corollary 4.1.2.

$$\Big(\int_{\mathbf{B}} |u|^{2_1^*}\,dx\Big)^{\frac{2}{2_1^*}} \le \frac{1}{\mathscr{C}_{1,0,p}(\mathbf{B})} \int_{\mathbf{B}} \big(|\nabla u|^2 - |\partial_\theta u|^2\big)dx \qquad \textit{for } u \in H_0^1(\mathbf{B}).$$

Moreover, the exponent 2_1^* is optimal in the sense that no such inequality holds for $p > 2_1^*$.

Theorem 4.1.1 yields symmetry breaking of ground states for suitable parameter values of p, α and m, but the precise parameter range giving rise to this symmetry breaking remains largely open. To shed further light on this question, we state the following result which establishes uniqueness and radial symmetry of ground state solutions for α close to zero and every $m \ge 0$, $2 < p < 2^*$.

Theorem 4.1.3. *Let $m \ge 0$ and $2 < p < 2^*$. Then there exists $\alpha_0 > 0$ such that*

$$\mathscr{C}_{\alpha,m,p}(\mathbf{B}) = \mathscr{C}_{0,m,p}(\mathbf{B}) \qquad \textit{for } \alpha \in [0, \alpha_0).$$

Moreover, for $\alpha \in [0,\alpha_0)$, there is, up to sign, a unique ground state solution of (4.1.5) which is a radial function.

Combining Theorems 4.1.1 and 4.1.3, we find that, for fixed $p > 2_1^*$, symmetry breaking of ground state solutions occurs when passing a critical parameter $\alpha = \alpha(p)$ which lies in the intervall $[\alpha_0, \alpha_*]$. However, so far it remains unclear whether symmetry breaking also occurs in the case $p \le 2_1^*$. Before stating a partial answer to this question for $2 < p < 2_1^*$, we first note that symmetry breaking does not occur in the linear case $p = 2$. More precisely, we shall observe in Section 4.4 below that

$$\mathscr{C}_{\alpha,m,2}(\mathbf{B}) = \mathscr{C}_{0,m,2}(\mathbf{B}) = \lambda_1(\mathbf{B}) + m \qquad \text{for all } \alpha \in [0,1],\ m \in \mathbb{R}.$$

Moreover, every Dirichlet eigenfunction of (4.1.6) is radial in this linear case. On the other hand, for every p strictly greater than 2, symmetry breaking occurs for sufficiently large values of the parameter m, as the following result shows.

Theorem 4.1.4. *Let $\alpha \in (0,1)$ and $2 < p < 2^*$. Then there exists $m_0 > 0$ with the property that (4.1.10) holds for $m \geq m_0$ and therefore every ground state solution of (4.1.5) is x_1-x_2-nonradial for $m \geq m_0$.*

Next, we discuss the limit case $\alpha = 1$ in the minimization problem (4.1.6). We may study this limit case based on Corollary 4.1.2, but we need to look for minimizers in a space larger than $H_0^1(\mathbf{B})$. More precisely, we let $\mathcal{H}$ be given as the closure of $C_c^1(\mathbf{B})$ in

$$\left\{ u \in L^{2_1^*}(\mathbf{B}) : \|u\|_{\mathcal{H}}^2 := \int_{\mathbf{B}} \left(|\nabla u|^2 - |\partial_\theta u|^2 \right) \, dx < \infty \right\}$$

with respect to the norm $\|\cdot\|_{\mathcal{H}}$. We then have the following result, which complements Theorems 4.1.1 and 4.1.4 in the case $\alpha = 1$.

Theorem 4.1.5. *Let $2 < p < 2_1^*$ and $\alpha = 1$.*

(i) For every $m > -\lambda_1(\mathbf{B})$, there exists a ground state solution of (4.1.5).

(ii) There exists $m_0 > 0$ with the property that (4.1.10) holds for $m \geq m_0$ and therefore every ground state solution $u \in \mathcal{H}$ of (4.1.5) is x_1-x_2-nonradial for $m \geq m_0$.

The critical case $\alpha = 1$, $p = 2_1^*$ remains largely open, but we have a partial result on the existence of ground state solutions which relates problem (4.1.5) to a degenerate Sobolev inequality of the form

$$\|u\|_{L^{2_s^*}(\mathbb{R}_+^N)} \leq C \Big(\int_{\mathbb{R}_+^N} \sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2 \, dx \Big)^{1/2} \tag{4.1.12}$$

in the half space

$$\mathbb{R}_+^N := \left\{ x \in \mathbb{R}^N : x_1 > 0 \right\}.$$

This inequality seems new and of independent interest, and it is the key ingredient in the proof of Theorem 4.1.1. Our main result related to this half space inequality is the following.

Theorem 4.1.6. *Let $s > 0$ and set $2_s^* := \frac{4N+2s}{2N-4+s}$. Then we have*

$$\mathcal{S}_s(\mathbb{R}_+^N) := \inf_{u \in C_c^1(\mathbb{R}_+^N)} \frac{\displaystyle\int_{\mathbb{R}_+^N} \sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2 \, dx}{\left(\int_{\mathbb{R}_+^N} |u|^{2_s^*} \, dx \right)^{\frac{2}{2_s^*}}} > 0. \tag{4.1.13}$$

Moreover, the value $\mathcal{S}_s(\mathbb{R}_+^N)$ is attained in $H_s \setminus \{0\}$, where H_s denotes the closure of $C_c^1(\mathbb{R}_+^N)$ in the space

$$\left\{ u \in L^{2_s^*}(\mathbb{R}_+^N) : \|u\|_{H_s}^2 := \int_{\mathbb{R}_+^N} \sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2 \, dx < \infty \right\} \tag{4.1.14}$$

with respect to the norm $\|\cdot\|_{H_s}$.

Here, distributional derivatives are considered in (4.1.14). Several remarks regarding Theorem 4.1.6 are in order. First, we point out that the criticality of the exponent $2_s^* := \frac{4N+2s}{2N-4+s}$ in 4.1.6 corresponds to the fact that the quotient in (4.1.13) is invariant under an anisotropic rescaling given by $u \mapsto u_\lambda$ for $\lambda > 0$ with $u_\lambda(x) := u(\lambda x_1, \lambda x_2, \dots, \lambda x_{N-1}, \lambda^{1+\frac{s}{2}} x_N)$. This invariance leads to a lack of compactness, and we have to apply concentration-compactness methods to deduce the existence of minimizers. We further note that the existence of minimizers in the half space problem is in striking contrast to the case $s = 0$ which is excluded in Theorem 4.1.6. Indeed, the case $s = 0$ corresponds to the classical Sobolev inequality which only admits extremal functions in the entire space $\mathbb{R}^N$.

We have already noted that the case $s = 1$ in Theorem 4.1.6 is of key importance in the proof of Theorem 4.1.1. The more general case $s \in (0, 2]$ arises in a similar way when (4.1.5) is studied in Riemannian models with boundary in place of $\mathbf{B}$, and we will discuss this case in Section 4.6 below. We point out that the setting of Riemannian models includes hypersurfaces of revolution with boundary in $\mathbb{R}^{N+1}$, and that the particular case of a hemisphere corresponds to the case $s = 2$. The latter is no surprise in view of the recent work of Taylor [**130**] and Mukherjee [**103, 104**], who studied the problem of rotating solutions on the unit sphere. In particular, their work relies on degenerate Sobolev embeddings on the unit sphere where also the value $2_2^* = \frac{2(N+1)}{N-1}$ appears as a critical exponent. In fact, our approach allows to use the case $s = 2$ in Theorem 4.1.6 and the corresponding inequality in $\mathbb{R}^N$ (see Theorem 4.2.1 below) to give new proofs of these degenerate Sobolev embeddings which does not rely on Fourier analytic and pseudodifferential arguments as in [**130**].

Next we remark that degenerate Sobolev type inequalities have been studied extensively in the context of Grushin operators which take the form

$$\mathcal{L} = \Delta_x + c|x|^{2s}\Delta_y$$

on $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^k$, where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^k$ and $s > 0$. For a comprehensive survey of the properties of these operators, see e.g. [**70**]. In particular, an associated Sobolev type inequality of the type

$$\|u\|_{L^{\frac{2m+2k(s+1)}{m+k(s+1)-2}}(\mathbb{R}^N)} \leq C\Big(\int_{\mathbb{R}^N} |\nabla_x u|^2 + c|x|^{2s}|\nabla_y u|^2 \, d(x,y)\Big)^{1/2}, \qquad u \in C_c^1(\mathbb{R}^N) \tag{4.1.15}$$

has been established. Here, the associated critical exponent is related to the homogeneous dimension in the context of more general weighted Sobolev inequalities. We also mention symmetry results for positive entire solutions to semilinear problems involving $\mathcal{L}$ in [**100**], as well as the existence of extremal functions on $\mathbb{R}^N$ shown in [**12**] and [**99**].

We point out that the restriction of inequality (4.1.15) to the half space coincides with the inequality (4.1.12) in the case $N = 2$. On the other hand, for $N \geq 3$, the inequality (4.1.12) is not associated to a Grushin operator in the sense above. Nonetheless, it is worth noting that for $m = N - 1$, $k = 1$ and $s = \frac{1}{2}$, the critical exponents coincide.

More closely related to Theorem 4.1.6 in the case $N \geq 3$ is [**58**, Theorem 1.7] where a more general family of Grushin type operators and their associated inequalities has been considered. However, the inequality (4.1.12) associated to (4.1.10) is a limit case which is not part of the family of inequalities considered in [**58**, Theorem 1.7].

Coming back to the existence of ground state solutions of (4.1.5) in the critical case $\alpha = 1$, $p = 2_1^*$, we state the following result.

Theorem 4.1.7. *If*

$$\mathscr{C}_{1,m,2_1^*}(\mathbf{B}) < 2^{\frac{1}{2}-\frac{1}{2_1^*}} \mathcal{S}_1(\mathbb{R}_+^N) \tag{4.1.16}$$

for some $m > -\lambda_1(\mathbf{B})$, then the value $\mathscr{C}_{1,m,2_1^*}(\mathbf{B})$ is attained in $\mathcal{H} \setminus \{0\}$ by a ground state solution of (4.1.5). Moreover, there exists $\varepsilon > 0$ with the property that (4.1.16) holds for every $m \in (-\lambda_1(\mathbf{B}), -\lambda_1(\mathbf{B}) + \varepsilon)$.

Here, the factor $2^{\frac{1}{2}-\frac{1}{2_1^*}}$ is due to the scaling properties of a more general quotient related to (4.1.13), see Remark 4.2.3(ii) below.

The paper is organized as follows. We first study the degenerate Sobolev inequality (4.1.12) and hence prove Theorem 4.1.6 in Section 4.2. This is subsequently used in Section 4.3 to prove the second part of Theorem 4.1.1. In Section 4.4 we then discuss the properties of ground state solutions of (4.1.5) in detail and give the proofs of Theorems 4.1.3 and 4.1.4. This also includes the degenerate case $\alpha = 1$ and the proof of Theorem 4.1.5. Section 4.5 is then devoted to the properties of rotating waves when $\mathbf{B}$ is replaced by an annulus. In this case, our methods give rise to an analogue of Theorem 4.1.1 with more explicit conditions for x_1-x_2-nonradiality of ground states. In Section 4.6 we discuss how the general degenerate Sobolev inequality (4.1.12) can be used to give an analogue of Theorem 4.1.1 for Riemannian models. Finally, in the appendix, we prove uniform L^∞-bounds for weak solutions of (4.1.5) in the case $\alpha = 1$.

4.2 A family of degenerate Sobolev inequalities

In this section, we give the proof of Theorem 4.1.6. More precisely, in the first part of the section, we prove the corresponding degenerate Sobolev inequality

$$\Big(\int_{\mathbb{R}^N} |u|^{2_s^*}\,dx\Big)^{\frac{2}{2_s^*}} \le C \int_{\mathbb{R}^N} \Big(\sum_{i=1}^{N-1} |\partial_i u|^2 + |x_1|^s |\partial_N u|^2\Big)\,dx \qquad \text{for } u \in C_c^1(\mathbb{R}^N) \tag{4.2.1}$$

in the entire space with a constant $C > 0$, from which the positivity of $\mathcal{S}_s(\mathbb{R}^N_+)$ in (4.1.13) follows.

In the second part of the section, we then prove the existence of minimizers of the quotient in (4.1.13) in the larger space H_s defined in Theorem 4.1.6.

4.2.1. Degenerate Sobolev inequality on $\mathbb{R}^N$. The first step in the proof of (4.2.1) is the following key inequality.

Lemma 4.2.1. *Let $\alpha > 0$ and $p > 2$ be given. Then we have*

$$\int_{\mathbb{R}^N} |u|^p\,dx \le \kappa\Big(\int_{\mathbb{R}^N} |x_1|^\alpha |u|^q\,dx\Big)^{\frac{2}{2+\alpha}} \Big(\int_{\mathbb{R}^N} |\partial_1 u|^2\,dx\Big)^{\frac{\alpha}{2+\alpha}} \qquad \text{for } u \in C_c^1(\mathbb{R}^N) \tag{4.2.2}$$

with

$$q = \frac{p(2+\alpha) - 2\alpha}{2} \qquad \text{and} \qquad \kappa > 0. \tag{4.2.3}$$

Proof. Let $u \in C_c^1(\mathbb{R}^N)$. By Hölder's inequality, we have

$$\int_{\mathbb{R}^N} |u|^p\,dx \le \Big(\int_{\mathbb{R}^N} |x_1|^{s\sigma'} |u|^{r\sigma'}\,dx\Big)^{\frac{1}{\sigma'}} \Big(\int_{\mathbb{R}^N} |x_1|^{-s\sigma} |u|^{(p-r)\sigma}\,dx\Big)^{\frac{1}{\sigma}} \tag{4.2.4}$$

for $s > 0$, $\sigma \in (1,\infty)$ and $r \in (0,p)$. It is convenient to write $s = \frac{t}{\sigma}$ and $m = (p-r)\sigma$, then (4.2.4) becomes

$$\int_{\mathbb{R}^N} |u|^p\,dx \le \Big(\int_{\mathbb{R}^N} |x_1|^{\frac{t}{\sigma-1}} |u|^{p\sigma' - \frac{m}{\sigma-1}}\,dx\Big)^{\frac{1}{\sigma'}} \Big(\int_{\mathbb{R}^N} |x_1|^{-t} |u|^m\,dx\Big)^{\frac{1}{\sigma}} \tag{4.2.5}$$

for $t > 0$, $\sigma \in (1,\infty)$ and $m \in (0, p\sigma)$. If, more specifically,

$$t \in (0,1), \quad \sigma \in (1,\infty), \quad m \in (1, p\sigma), \quad \tau > 1 \quad \text{and} \quad \theta \in (0,1), \tag{4.2.6}$$

we may integrate by parts and use Hölder's inequality to get

$$
\begin{aligned}
\int_{\mathbb{R}^N} |x_1|^{-t}|u|^m\,dx &= -\frac{m}{1-t}\int_{\mathbb{R}^N} x_1|x_1|^{-t}|u|^{m-1}\partial_1 u\,dx\\
&\le c\int_{\mathbb{R}^N} |x_1|^{1-t}|u|^{m-1}|\partial_1 u|\,dx \le c\Big(\int_{\mathbb{R}^N} |x_1|^{2(1-t)}|u|^{2(m-1)}\,dx\Big)^{\frac12}\Big(\int_{\mathbb{R}^N} |\partial_1 u|^2\,dx\Big)^{\frac12}\\
&\le c\Big(\int_{\mathbb{R}^N} |x_1|^{2(1-t)\tau}|u|^{2\theta(m-1)\tau}\,dx\Big)^{\frac{1}{2\tau}}\Big(\int_{\mathbb{R}^N} |u|^{2(1-\theta)(m-1)\tau'}\,dx\Big)^{\frac{1}{2\tau'}}\Big(\int_{\mathbb{R}^N} |\partial_1 u|^2\,dx\Big)^{\frac12}.
\end{aligned}
\tag{4.2.7}
$$

We now restrict our attention to values

$$1 > t > \frac{2\sigma-2}{2\sigma-1} \tag{4.2.8}$$

and choose, specifically,

$$\tau = \frac{t}{2(1-t)(\sigma-1)} \tag{4.2.9}$$

which satisfies $\tau > 1$ by (4.2.8) and $2(1-t)\tau = \frac{t}{\sigma-1}$. Therefore (4.2.7) reduces to

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |x_1|^{-t}|u|^m\,dx \\
&\le c\Big(\int_{\mathbb{R}^N} |x_1|^{\frac{t}{\sigma-1}}|u|^{2\theta(m-1)\tau}\,dx\Big)^{\frac{1}{2\tau}}\Big(\int_{\mathbb{R}^N} |u|^{2(1-\theta)(m-1)\tau'}\,dx\Big)^{\frac{1}{2\tau'}}\Big(\int_{\mathbb{R}^N} |\partial_1 u|^2\,dx\Big)^{\frac12}.
\end{aligned}
\tag{4.2.10}
$$

Next we define

$$m := \frac{p(\sigma-1)(\tau-1)+\sigma p-1}{2\tau(\sigma-1)+1}+1 = \frac{p(\sigma-1)(\tau-1)+\sigma p+2\tau(\sigma-1)}{2\tau(\sigma-1)+1} \tag{4.2.11}$$

and

$$\theta = \frac{(m-1)-\frac{p}{2\tau'}}{m-1}. \tag{4.2.12}$$

A short computation shows that these values are chosen such that the conditions

$$2\theta(m-1)\tau = p\sigma' - \frac{m}{\sigma-1} \qquad \text{and} \qquad 2(1-\theta)(m-1)\tau' = p \tag{4.2.13}$$

hold for the exponents in (4.2.10). In order to use the inequalities with these values of θ and m, we have to ensure that these values are admissible in the sense of (4.2.6). By definition, we have $m > 1$. Moreover, we note that $m < \sigma p$ since

$$\sigma \ge 1 \ge \frac{1}{2\tau'}+\frac{1}{p}, \qquad \text{i.e.,} \qquad p(\tau-1)+2\tau \le 2\sigma p\tau,$$

and hence

$$p(\sigma-1)(\tau-1)+\sigma p+2\tau(\sigma-1) \le \sigma p\big(2\tau(\sigma-1)+1\big).$$

Hence $m \in (1,\sigma p)$, as required. Moreover, we have $\theta < 1$ by definition. To see that $\theta > 0$, we note that, since $p > 2$, we have $\tau' > 1 \ge \frac{p}{2(p-1)} \ge \frac{p}{2(\sigma p-1)}$ and therefore

$$2\Big(p(\sigma-1)\tau+\tau'(\sigma p-1)\Big) > p\Big(2\tau(\sigma-1)+1\Big),$$

which shows that

$$2(m-1)\tau' = 2\frac{p(\sigma-1)\tau+\tau'(\sigma p-1)}{2\tau(\sigma-1)+1} > p.$$

Consequently, $\theta > 0$, and thus $\theta \in (0,1)$, as required in (4.2.6). So we may consider these values of τ, m and θ in (4.2.5) and (4.2.10). With (4.2.13), this yields the inequalities

$$\int_{\mathbb{R}^N} |u|^p\,dx \le \Big(\int_{\mathbb{R}^N} |x_1|^{\frac{t}{\sigma-1}}|u|^q\,dx\Big)^{\frac{1}{\sigma'}}\Big(\int_{\mathbb{R}^N} |x_1|^{-t}|u|^m\,dx\Big)^{\frac{1}{\sigma}}$$

and

$$\int_{\mathbb{R}^N} |x_1|^{-t}|u|^m\,dx \le c\Big(\int_{\mathbb{R}^N} |x_1|^{\frac{t}{\sigma-1}}|u|^q\,dx\Big)^{\frac{1}{2\tau}}\Big(\int_{\mathbb{R}^N} |u|^p\,dx\Big)^{\frac{1}{2\tau'}}\Big(\int_{\mathbb{R}^N} |\partial_1 u|^2\,dx\Big)^{\frac{1}{2}}$$

with

$$q := 2\theta(m-1)\tau = p\sigma' - \frac{m}{\sigma-1}. \tag{4.2.14}$$

Combining these inequalities yields

$$\int_{\mathbb{R}^N} |u|^p\,dx \le c\Big(\int_{\mathbb{R}^N} |x_1|^{\frac{t}{\sigma-1}}|u|^q\,dx\Big)^{\frac{1}{\sigma'}+\frac{1}{2\tau\sigma}}\Big(\int_{\mathbb{R}^N} |u|^p\,dx\Big)^{\frac{1}{2\tau'\sigma}}\Big(\int_{\mathbb{R}^N} |\partial_1 u|^2\,dx\Big)^{\frac{1}{2\sigma}}$$

and therefore

$$\int_{\mathbb{R}^N} |u|^p\,dx \le c\Big(\int_{\mathbb{R}^N} |x_1|^{\frac{t}{\sigma-1}}|u|^q\,dx\Big)^{\frac{2\sigma\tau-2\tau+1}{2\sigma\tau-\tau+1}}\Big(\int_{\mathbb{R}^N} |\partial_1 u|^2\,dx\Big)^{\frac{\tau}{2\sigma\tau-\tau+1}}. \tag{4.2.15}$$

To obtain (4.2.2), it is convenient to set $\alpha := \frac{t}{\sigma-1} > 0$, noting that the admissibility condition (4.2.8) translates to

$$\frac{1}{\sigma-1} > \alpha > \frac{2}{2\sigma-1}. \tag{4.2.16}$$

Note that, if $\alpha > 0$ is given, we always find $\sigma \in (1,\infty)$ with the property that (4.2.16) holds. Moreover, the exponents in (4.2.15) then satisfy

$$\frac{\tau}{2\sigma\tau-\tau+1} = \frac{\alpha}{2+\alpha}, \qquad \frac{2\sigma\tau-2\tau+1}{2\sigma\tau-\tau+1} = \frac{2}{2+\alpha},$$

so (4.2.15) becomes

$$\int_{\mathbb{R}^N} |u|^p\,dx \le c\Big(\int_{\mathbb{R}^N} |x_1|^{\alpha}|u|^q\,dx\Big)^{\frac{2}{2+\alpha}}\Big(\int_{\mathbb{R}^N} |\partial_1 u|^2\,dx\Big)^{\frac{\alpha}{2+\alpha}}.$$

This is already the inequality in (4.2.2). So it only remains to show that the two definitions of q given in (4.2.14) and (4.2.14) are consistent, i.e., we have the identity

$$2\theta(m-1)\tau = \frac{p(2+\alpha)-2\alpha}{2}$$

The latter follows by a somewhat tedious but straightforward computation, so the proof of the lemma is complete. □

We may now complete the proof of the main result of this section, given as follows.

Theorem 4.2.2. *Let $s > 0$ and $2_s^* = \frac{4N+2s}{2N-4+s}$ as in Theorem 4.1.6. Then inequality (4.2.1) holds with some constant $C > 0$.*

We remark that this may be proven by combining the previous results with a suitable adaption of the inequality on the halfspace given in [**58**, Theorem 1.7] to the setting of the entire space $\mathbb{R}^N$. For the convenience of the reader, we give a self-contained proof.

Proof. In the following, the letter $c > 0$ stands for a constant which may change from line to line. Let $\alpha = \frac{s}{2(N-1)}$. Then Lemma 4.2.1 yields

$$\int_{\mathbb{R}^N} |u|^{2_s^*}\,dx \le \kappa\Big(\int_{\mathbb{R}^N} |x_1|^\alpha |u|^{q_s}\,dx\Big)^{\frac{2}{2+\alpha}}\Big(\int_{\mathbb{R}^N} |\partial_1 u|^2\,dx\Big)^{\frac{\alpha}{2+\alpha}} \qquad \text{for } u \in C_c^1(\mathbb{R}^N)$$

with $q_s := \frac{N(2_s^*+2)}{2(N-1)}$. To estimate the term $\int_{\mathbb{R}^N} |x_1|^\alpha |u|^{q_s}\,dx$, we define, for $i = 1,\dots,N$, the functions $a_i \in C_c(\mathbb{R}^{N-1})$ by

$$a_i(\hat{x}_i) := \int_{\mathbb{R}} |u|^{\frac{q(N-1)}{N}-1}|\partial_i u|\,dx_i$$

where

$$\hat{x}_i := (x_1,\dots,x_{i-1},x_{i+1},\dots,x_N) \in \mathbb{R}^{N-1} \qquad \text{for } x \in \mathbb{R}^N \text{ and } i = 1,\dots,N.$$

Integrating the derivative $\partial_i |u|^{\frac{q_s(N-1)}{N}}$ in the x_i-direction, we find that $|u(x)|^{\frac{q_s(N-1)}{N}} \le c a_i(\hat{x}_i)$ for all $x \in \mathbb{R}^N$, $i = 1,\dots,N$ and therefore

$$|u(x)|^{q_s(N-1)} \le c \prod_{i=1}^N a_i(\hat{x}_i) \qquad \text{for } x \in \mathbb{R}^N.$$

Applying Gagliardo's Lemma [**60**, Lemma 4.1] to the functions $a_1^{\frac{1}{N-1}},\dots,a_{N-1}^{\frac{1}{N-1}}$ and $x \mapsto |x_1|^\alpha a_N^{\frac{1}{N-1}}(x)$, we thus find that

$$\begin{aligned}
\int_{\mathbb{R}^N} |x_1|^\alpha |u|^{q_s}\,dx &\le c\left(\int_{\mathbb{R}^{N-1}} |x_1|^{(N-1)\alpha} a_N(\hat{x}_N)d\hat{x}_N \prod_{i=1}^{N-1}\int_{\mathbb{R}^{N-1}} a_i(\hat{x}_i)d\hat{x}_i\right)^{\frac{1}{N-1}}\\
&= c\left(\int_{\mathbb{R}^N} |x_1|^{\frac{s}{2}}|u|^{\frac{q_s(N-1)}{N}-1}|\partial_N u|dx \prod_{i=1}^{N-1}\int_{\mathbb{R}^N} |u|^{\frac{q_s(N-1)}{N}-1}|\partial_i u|dx\right)^{\frac{1}{N-1}}\\
&\le c\left(\int_{\mathbb{R}^N} |u|^{2\frac{q_s(N-1)}{N}-2}\,dx\right)^{\frac{N}{2(N-1)}}\left(\int_{\mathbb{R}^N} |x_1|^s|\partial_N u|^2\,dx \prod_{i=1}^{N-1}\int_{\mathbb{R}^N} |\partial_i u|^2\,dx\right)^{\frac{1}{2(N-1)}}.
\end{aligned}$$

Since $\frac{2(N-1)q_s}{N} - 2 = 2_s^*$, we conclude that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |u|^{2_s^*}\,dx \le c\Big(\int_{\mathbb{R}^N} |x_1|^\alpha|u|^{q_s}\,dx\Big)^{\frac{2}{2+\alpha}}\Big(\int_{\mathbb{R}^N} |\partial_1 u|^2\,dx\Big)^{\frac{\alpha}{2+\alpha}}\\
&\le c\left(\left(\int_{\mathbb{R}^N} |u|^{2_s^*}\,dx\right)^{\frac{N}{2(N-1)}}\left(\int_{\mathbb{R}^N} |x_1||\partial_N u|^2\,dx \prod_{i=1}^{N-1}\int_{\mathbb{R}^N} |\partial_i u|^2\,dx\right)^{\frac{1}{2(N-1)}}\right)^{\frac{2}{2+\alpha}}\\
&\times\Big(\int_{\mathbb{R}^N} |\partial_1 u|^2\,dx\Big)^{\frac{\alpha}{2+\alpha}}\\
&= c\left(\int_{\mathbb{R}^N} |u|^{2_s^*}\,dx\right)^{\frac{N}{2(N-1)+\frac{s}{2}}}\left(\int_{\mathbb{R}^N} |x_1||\partial_N u|^2\,dx \prod_{i=2}^{N-1}\int_{\mathbb{R}^N} |\partial_i u|^2\,dx\right)^{\frac{1}{2(N-1)+\frac{s}{2}}}\\
&\times\Big(\int_{\mathbb{R}^N} |\partial_1 u|^2\,dx\Big)^{\frac{1+\frac{s}{2}}{2(N-1)+\frac{s}{2}}}
\end{aligned}$$

and therefore

$$\begin{aligned}&\Big(\int_{\mathbb{R}^N}|u|^{2_s^*}\,dx\Big)^{\frac{N-2+\frac{s}{2}}{2(N-1)+\frac{s}{2}}}\\ \leq c&\Big(\int_{\mathbb{R}^N}|x_1||\partial_N u|^2\,dx\prod_{i=2}^{N-1}\int_{\mathbb{R}^N}|\partial_i u|^2\,dx\Big)^{\frac{1}{2(N-1)+\frac{s}{2}}}\Big(\int_{\mathbb{R}^N}|\partial_1 u|^2\,dx\Big)^{\frac{1+\frac{s}{2}}{2(N-1)+\frac{s}{2}}}.\end{aligned}$$

Finally, Young's inequality gives

$$\begin{aligned}\Big(\int_{\mathbb{R}^N}|u|^{2_s^*}\,dx\Big)^{\frac{2}{2_s^*}}&\leq c\Big(\int_{\mathbb{R}^N}|x_1||\partial_N u|^2\,dx\prod_{i=2}^{N-1}\int_{\mathbb{R}^N}|\partial_i u|^2\,dx\Big)^{\frac{2}{2N+s}}\Big(\int_{\mathbb{R}^N}|\partial_1 u|^2\,dx\Big)^{\frac{2+s}{2N+s}}\\ &\leq c\Big(\int_{\mathbb{R}^N}|x_1||\partial_N u|^2\,dx+\sum_{i=1}^{N-1}\int_{\mathbb{R}^N}|\partial_i u|^2\,dx\Big).\end{aligned}$$

□

In particular, this implies

$$\mathcal{S}_s(\mathbb{R}^N_+)=\inf_{u\in C_c^1(\mathbb{R}^N_+)}\frac{\int_{\mathbb{R}^N_+}\sum_{i=1}^{N-1}|\partial_i u|^2+x_1^s|\partial_N u|^2\,dx}{\Big(\int_{\mathbb{R}^N_+}|u|^{2_s^*}\,dx\Big)^{\frac{2}{2_s^*}}}>0$$

and thus the first part of Theorem 4.1.6.

Remark 4.2.3. ***(Optimality and Variants)***

(i) The exponent 2_s^* in (4.1.13) is optimal in the sense that

$$\inf_{u\in C_c^1(\mathbb{R}^N)}\frac{\int_{\mathbb{R}^N}\Big(\sum_{i=1}^{N-1}|\partial_i u|^2+|x_1|^s|\partial_N u|^2\Big)\,dx}{\|u\|^2_{L^p(\mathbb{R}^N)}}=0\qquad\text{for } p\neq 2_s^*.\tag{4.2.17}$$

This follows by considering the rescaling $u\mapsto u_\lambda$, $\lambda>0$ with

$$u_\lambda(x):=u(\lambda x_1,\lambda x_2,\dots,\lambda x_{N-1},\lambda^{1+\frac{s}{2}}x_N).$$

Indeed, for $u\in C_c^1(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N_+}\Big(\sum_{i=1}^{N-1}|\partial_i u_\lambda|^2+x_1^s|\partial_N u_\lambda|^2\Big)dx=\lambda^{-\frac{2N+s-4}{2}}\int_{\mathbb{R}^N_+}\Big(\sum_{i=1}^{N-1}|\partial_i v|^2+x_1^s|\partial_N u|^2\Big)dx$$

and, for $1<p<\infty$,

$$\Big(\int_{\mathbb{R}^N_+}|u_\lambda|^p\,dx\Big)^{\frac{2}{p}}=\lambda^{-\frac{2}{p}(N+\frac{s}{2})}\Big(\int_{\mathbb{R}^N_+}|u|^p\,dx\Big)^{\frac{2}{p}}.$$

Since $\frac{2N+s-4}{2}=\frac{2}{p}(N+\frac{s}{2})$ if and only if $p=2_s^*$, (4.2.17) follows.

(ii) *For $\kappa > 0$, $u \in C_c^1(\mathbb{R}^N)$, we may consider a rescaled function of the form*

$$v(x) = u\left(x_1, \dots, x_{N-1}, \frac{x_n}{\sqrt{\kappa}}\right).$$

Comparing the associated quotients then yields

$$\inf_{u \in C_c^1(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} \left(\sum_{i=1}^{N-1} |\partial_i u|^2 + \kappa |x_1|^s |\partial_N u|^2 \right) dx}{\|u\|^2_{L^{2_s^*}(\mathbb{R}^N)}} = \kappa^{\frac{1}{2} - \frac{1}{2_s^*}} \mathcal{S}_s(\mathbb{R}^N_+). \tag{4.2.18}$$

In the special case $\kappa = 2$, this quotient will appear later when we connect $\mathscr{C}_{1,m,2_1^*}(\mathbf{B})$ and $\mathcal{S}_1(\mathbb{R}^N_+)$, in particular in the proof of Theorem 4.1.7.

Recalling the space H_s defined in Theorem 4.1.6, we see that Theorem 4.2.2 immediately implies that H_s is continuously embedded into $L^{2_s^*}(\mathbb{R}^N_+)$.

4.2.2. Existence of minimizers. In the following, we fix $s > 0$ and study minimizing sequences for

$$\mathcal{S} := \mathcal{S}_s(\mathbb{R}^N_+) = \inf_{u \in H_s \setminus \{0\}} \frac{\int_{\mathbb{R}^N_+} \left(\sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2 \right) dx}{\left(\int_{\mathbb{R}^N_+} |u|^{2_s^*} \, dx \right)^{\frac{2}{2_s^*}}} > 0.$$

First, consider the following classical lemma due to Lions [**85**], which we give in the form presented in [**125**]:

Lemma 4.2.4. (Concentration-Compactness Lemma)
Suppose $(\mu_n)_n$ is a sequence of probability measures on $\mathbb{R}^N$. Then, after passing to a subsequence, one of the following three conditions holds:

(i) ***(Compactness)*** *There exits a sequence $(x_n)_n \subset \mathbb{R}^N$ such that for any $\varepsilon > 0$ there exists $R > 0$ such that*

$$\int_{B_R(x_n)} d\mu_n \geq 1 - \varepsilon.$$

(ii) ***(Vanishing)*** *For all $R > 0$ it holds that*

$$\lim_{n \to \infty} \left(\sup_{x \in \mathbb{R}^N} \int_{B_R(x)} d\mu_n \right) = 0.$$

(iii) ***(Dichotomy)*** *There exists $\lambda \in (0,1)$ such that for any $\varepsilon > 0$ there exists $R > 0$ and $(x_n)_n \subset \mathbb{R}^N$ with the following property: Given $R' > R$ there are nonnegative measures μ_n^1, μ_n^2 such that*

$$\begin{aligned} &0 \leq \mu_n^1 + \mu_n^2 \leq \mu_n \\ &\operatorname{supp} \mu_n^1 \subset B_R(x_n), \quad \operatorname{supp} \mu_n^2 \subset \mathbb{R}^N \setminus B_{R'}(x_n) \\ &\limsup_{n \to \infty} \left(\left| \lambda - \int_{\mathbb{R}^N} d\mu_n^1 \right| + \left| (1 - \lambda) - \int_{\mathbb{R}^N} d\mu_n^2 \right| \right) \leq \varepsilon. \end{aligned}$$

A characterization of minimizing sequences in the sense of measures is given in the following lemma, which is a straightforward adaption of [**125**, Lemma 4.8]:

Lemma 4.2.5. (Concentration-Compactness Lemma II)
Let $s > 0$ and suppose $u_n \rightharpoonup u$ in H_s and $\mu_n := \left(\sum_{i=1}^{N-1} |\partial_i u_n|^2 + x_1^s|\partial_N u_n|^2\right) dx \rightharpoonup \mu$, $\nu_n := |u_n|^{2_s^*}dx \rightharpoonup \nu$ weakly in the sense of measures where μ and ν are finite measures on $\mathbb{R}^N_+$. Then:

(i) There exists an at most countable set J, a set $\{x^j : j \in J\} \subset \mathbb{R}^N_+$ and $\{\nu^j : j \in J\} \subset (0,\infty)$ such that

$$\nu = |u|^{2_s^*}dx + \sum_{j\in J} \nu^j \delta_{x^j}.$$

(ii) There exists a set $\{\mu^j : j \in J\} \subset (0,\infty)$ such that

$$\mu \geq \left(\sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s|\partial_N u|^2\right) dx + \sum_{j\in J} \mu^j \delta_{x^j}$$

where

$$\mathcal{S}(\nu^j)^{\frac{2}{2_s^*}} \leq \mu^j$$

for $j \in J$. In particular, $\sum_{j\in J} (\nu^j)^{\frac{2}{2_s^*}} < \infty$.

Our main result then states that $\mathcal{S}$ is attained in H_s and completes the proof of Theorem 4.1.6.

Theorem 4.2.6. *Let $s > 0$ and suppose $(u_n)_n$ is a minimizing sequence for*

$$\mathcal{S} = \inf_{u\in H_s\setminus\{0\}} \frac{\int_{\mathbb{R}^N_+} \left(\sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s|\partial_N u|^2\right) dx}{\left(\int_{\mathbb{R}^N_+} |u|^{2_s^*}\, dx\right)^{\frac{2}{2_s^*}}}$$

with $\|u_n\|_{L^{2_s^*}} = 1$. Then, up to translations orthogonal to x_1 and anisotropic scaling, $(u_n)_n$ is relatively compact in H_s.

Proof. For $r > 0$ we define the family of rectangles

$$Q_r := \left\{(0,r^2) \times \left(y + (-r^2,r^2)^{N-2} \times (-r^{2+s}, r^{2+s})\right) : y \in \mathbb{R}^{N-1}\right\}.$$

It is important to note that for $R > 0$, with respect to the transformation

$$\tau_R(x) = (R^2x_1, R^2x_2, \ldots, R^2x_{N-1}, R^{2+s}x_N), \tag{4.2.19}$$

these sets satisfy

$$\tau_R(Q_r) = Q_{rR}.$$

Moreover, the functions

$$Q_n(r) := \sup_{E\in Q_r} \int_E |u_n|^{2_s^*}\, dx$$

are continuous on $[0,\infty)$ and satisfy

$$\lim_{r\to 0} Q_n(r) = 0, \quad \lim_{r\to\infty} Q_n(r) = 1.$$

Hence we may choose $A_n > 0$, $y_n \in \mathbb{R}^{N-1}$ such that the rescaled sequence $v_n \in H_s$ given by

$$v_n(x) := A_n^{\frac{2N-4+s}{2}} u_n(A_n^2 x_1, A_n^2(x_2 + (y_n)_1), \dots, A_n^{2+s}(x_N + (y_n)_{N-1})$$

satisfies

$$Q_n(1) = \sup_{E \in Q_1} \int_E |v_n|^{2_s^*}\, dx = \int_{(0,1)\times(-1,1)^{N-1}} |v_n|^{2_s^*}\, dx = \frac{1}{2}.$$

After passing to a subsequence, we may assume $v_n \rightharpoonup v$ in H_s and in $L^{2_s^*}(\mathbb{R}^N_+)$. We now consider the measures

$$\mu_n := \left(\sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2 \right) dx, \quad \nu_n := |v_n|^{2_s^*}\, dx$$

and apply Lemma 4.2.4 to $(\nu_n)_n$, where we note that μ_n and ν_n are initially measures on $\mathbb{R}^N_+$ but can trivially be extended to $\mathbb{R}^N$. By our normalization, vanishing cannot occur. We assume that we have dichotomy and thus let $\lambda \in (0,1)$ be as in Lemma 4.2.4(iii). Then, considering a sequence $\varepsilon_n \downarrow 0$, for any $n \in \mathbb{N}$ there exist $R_n > 0$, $x_n \in \mathbb{R}^N_+$ as well as nonnegative measures ν_n^1, ν_n^2 on $\mathbb{R}^N_+$ such that

$$\begin{aligned}
&0 \le \nu_n^1 + \nu_n^2 \le \nu_n \\
&\operatorname{supp} \nu_n^1 \subset \mathbb{R}^N_+ \cap B_{R_n}(x_n), \quad \operatorname{supp} \nu_n^2 \subset \mathbb{R}^N_+ \setminus B_{2R_n^{\frac{2+s}{2}}+1}(x_n) \\
&\left| \lambda - \int_{\mathbb{R}^N_+} d\nu_n^1 \right| + \left| (1-\lambda) - \int_{\mathbb{R}^N_+} d\nu_n^2 \right| \le 2\varepsilon_n
\end{aligned}$$

and thus

$$\limsup_{n\to\infty} \left(\left| \lambda - \int_{\mathbb{R}^N_+} d\nu_n^1 \right| + \left| (1-\lambda) - \int_{\mathbb{R}^N_+} d\nu_n^2 \right| \right) = 0.$$

From the proof of the Lemma 4.2.4 (see [**125**]) we can assume $R_n \to \infty$ and, in particular, $R_n \ge 1$.

For $r > 0$, let the anisotropic scaling τ_r be defined as in (4.2.19). We crucially note that

$$B_{R_n}(0) \subset \tau_{\sqrt{R_n}}(B_1(0))$$

and

$$\mathbb{R}^N_+ \setminus B_{2R_n^{\frac{2+s}{2}}+1}(0) \subset \mathbb{R}^N_+ \setminus \tau_{\sqrt{R_n}}(B_2(0)).$$

We take $\varphi \in C_c^\infty(B_2(0))$ with $0 \le \varphi \le 1$ and $\varphi \equiv 1$ in $B_1(0)$. For $n \in \mathbb{N}$, let $\varphi_n(x) := \varphi(\tau^{-1}_{\sqrt{R_n}}(x - x_n))$, so that

$$\varphi_n \equiv 1 \quad \text{on } x_n + \tau_{\sqrt{R_n}}(B_1(0)), \quad \varphi_n \equiv 0 \quad \text{on } \mathbb{R}^N \setminus (x_n + \tau_{\sqrt{R_n}}(B_2(0))),$$

and thus, in particular,

$$\varphi_n \equiv 1 \quad \text{on } \operatorname{supp} \nu_n^1, \quad \varphi_n \equiv 0 \quad \text{on } \operatorname{supp} \nu_n^2.$$

Note that

$$|\partial_1 v_n|^2 + x_1^s |\partial_2 v_n|^2 \ge \left(|\partial_1 v_n|^2 + x_1^s |\partial_2 v_n|^2\right) \left(\varphi_n^2 + (1-\varphi_n)^2\right).$$

We have

$$\left(\int_{\mathbb{R}^N_+}\left(\sum_{i=1}^{N-1}|\partial_i(\varphi_n v_n)|^2+x_1^s|\partial_N(\varphi_n v_n)|^2\right)dx\right)^{\frac12}$$
$$\le\left(\int_{\mathbb{R}^N_+}\varphi_n^2\left(\sum_{i=1}^{N-1}|\partial_i v_n|^2+x_1^s|\partial_N v_n|^2\right)dx\right)^{\frac12}+\left(\int_{\mathbb{R}^N_+}v_n^2\left(\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2\right)dx\right)^{\frac12}$$

and analogously for $(1-\varphi_n)$ instead of φ_n. Squaring and adding these estimates gives

$$\begin{aligned}
&\|\varphi_n v_n\|_{H_s}^2+\|(1-\varphi_n)v_n\|_{H_s}^2\\
&\le\int_{\mathbb{R}^N_+}\left(\sum_{i=1}^{N-1}|\partial_i v_n|^2+x_1^s|\partial_N v_n|^2\right)dx+2\int_{\mathbb{R}^N_+}v_n^2\left(\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2\right)dx\\
&\quad+4\left(\int_{\mathbb{R}^N_+}\left(\sum_{i=1}^{N-1}|\partial_i v_n|^2+x_1^s|\partial_N v_n|^2\right)dx\right)^{\frac12}\left(\int_{\mathbb{R}^N_+}v_n^2\left(\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2\right)dx\right)^{\frac12}.
\end{aligned}$$

Setting

$$\begin{aligned}
\beta_n:=&\,2\int_{\mathbb{R}^N_+}v_n^2\left(\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2\right)dx\\
&+4\left(\int_{\mathbb{R}^N_+}\left(\sum_{i=1}^{N-1}|\partial_i v_n|^2+x_1^s|\partial_N v_n|^2\right)dx\right)^{\frac12}\left(\int_{\mathbb{R}^N_+}v_n^2\left(\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2\right)dx\right)^{\frac12}
\end{aligned}$$

we thus have

$$\int_{\mathbb{R}^N_+}\left(\sum_{i=1}^{N-1}|\partial_i v_n|^2+x_1^s|\partial_N v_n|^2\right)dx\ge\|\varphi_n v_n\|_{H_s}^2+\|(1-\varphi_n)v_n\|_{H_s}^2-\beta_n.$$

Next, we define the anisotropic annulus

$$A_n:=x_n+\tau_{\sqrt{R_n}}(B_2(0))\setminus\tau_{\sqrt{R_n}}(B_1(0))$$

and consider $\delta>0$. Using Young's inequality and the fact that any derivative of φ_n vanishes outside of A_n, we can estimate

$$\beta_n\le\delta\int_{\mathbb{R}^N_+}\left(\sum_{i=1}^{N-1}|\partial_i v_n|^2+x_1^s|\partial_N v_n|^2\right)dx+C(\delta)\int_{A_n}v_n^2\left(\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2\right)dx.$$

Note that

$$\begin{aligned}
\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2&=R_n^{-2}\sum_{i=1}^{N-1}|[\partial_i\varphi](\tau_n(x))|^2+x_1^sR_n^{-2-s}|[\partial_N\varphi](\tau_n(x))|^2\\
&=R_n^{-2}\left(\sum_{i=1}^{N-1}|[\partial_i\varphi]|^2+(\cdot)_1^s|[\partial_N\varphi|^2\right)\circ\tau^{-1}_{\sqrt{R_n}},
\end{aligned}$$

and thus

$$\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2\le CR_n^{-2}$$

for some $C > 0$ independent of n. This gives

$$\int_{A_n} v_n^2 \left(\sum_{i=1}^{N-1} |\partial_i \varphi_n|^2 + x_1^s |\partial_N \varphi_n|^2 \right) dx \le C R_n^{-2} \|v_n\|_{L^2(A_n)}^2.$$

Using Hölder's inequality then further yields

$$\begin{aligned} R_n^{-1} \|v_n\|_{L^2(A_n)} &\le R_n^{-1} |A_n|^{\frac{2}{2N+s}} \|v_n\|_{L^{2_s^*}(A_n)} \le C \|v_n\|_{L^{2_s^*}(A_n)} \\ &\le C \left(\int_{\mathbb{R}^N} d\nu_n - \left(\int_{\mathbb{R}^N} d\nu_n^1 + \int_{\mathbb{R}^N} d\nu_n^2 \right) \right)^{\frac{1}{2_s^*}} \longrightarrow 0 \end{aligned}$$

as $n \longrightarrow \infty$. Here we used

$$|A_n| = |\tau_{\sqrt{R_n}}(B_2(x_n))| - |\tau_{\sqrt{R_n}}(B_1(x_n))| = R_n^{\frac{2N+s}{2}} \big(|B_2(0)| - |B_1(0)| \big).$$

Overall, we find that, for any $\delta > 0$,

$$\limsup_{n \to \infty} \beta_n \le \delta \sup_n \|v_n\|_H^2,$$

and since $(v_n)_n$ remains bounded in H_s, we conclude

$$\begin{aligned} &\int_{\mathbb{R}_+^N} \left(\sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2 \right) dx \ge \|\varphi_n v_n\|_{H_s}^2 + \|(1-\varphi_n) v_n\|_{H_s}^2 - \beta_n \\ &\ge \mathcal{S} \left(\|\varphi_n v_n\|_{L^{2_s^*}(\mathbb{R}_+^N)}^2 + \|(1-\varphi_n) v_n\|_{L^{2_s^*}(\mathbb{R}_+^N)}^2 \right) + o(1) \\ &\ge \mathcal{S} \left(\left(\int_{B_{R_n}(x_n)} d\nu_n \right)^{\frac{2}{2_s^*}} + \left(\int_{\mathbb{R}_+^N \setminus B_{R_n'}(x_n)} d\nu_n \right)^{\frac{2}{2_s^*}} \right) + o(1) \\ &\ge \mathcal{S} \left(\left(\int_{\mathbb{R}_+^N} d\nu_n^1 \right)^{\frac{2}{2_s^*}} + \left(\int_{\mathbb{R}_+^N} d\nu_n^2 \right)^{\frac{2}{2_s^*}} \right) + o(1) \ge \mathcal{S} \left(\lambda^{\frac{2}{2_s^*}} + (1-\lambda)^{\frac{2}{2_s^*}} \right) + o(1). \end{aligned}$$

But since $\lambda \in (0,1)$, we have $\lambda^{\frac{2}{2_s^*}} + (1-\lambda)^{\frac{2}{2_s^*}} > 1$ and thus

$$\begin{aligned} \mathcal{S} &= \lim_{n \to \infty} \int_{\mathbb{R}_+^N} \left(\sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2 \right) dx \\ &\ge \liminf_{n \to \infty} \left(\mathcal{S} \left(\lambda^{\frac{2}{2_s^*}} + (1-\lambda)^{\frac{2}{2_s^*}} \right) + o(1) \right) > \mathcal{S}, \end{aligned}$$

a contradiction. Hence we cannot have dichotomy.

Since we are therefore in case (i) of the Lemma 4.2.4, there exists a sequence $(x_n)_n$ such that for any $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ with

$$\int_{B_R(x_n)} d\nu_n \ge 1 - \varepsilon.$$

Since we normalized so that

$$\int_{(0,1)\times(-1,1)^{N-1}} |v_n|^{2_s^*} dx = \frac{1}{2},$$

we must have $(0,1)\times(-1,1)^{N-1}\cap B_R(x_n)\neq\varnothing$ if $\varepsilon<\frac{1}{2}$. By making R larger if necessary, we can thus assume

$$\int_{B_R(0)} d\nu_n \geq 1-\varepsilon.$$

In particular, we may therefore pass to a subsequence such that $\nu_n \rightharpoonup \nu$ weakly in the sense of measure, where ν is a finite measure on $\mathbb{R}^N_+$. By weak lower (and upper) semicontinuity (of measures), we then have

$$\int_{\mathbb{R}^N_+} d\nu = 1.$$

By Lemma 4.2.5 we may now assume

$$\mu_n \rightharpoonup \mu \geq \sum_{i=1}^{N-1}\left(|\partial_i v|^2 + x_1^s|\partial_N v|^2\right)dx + \sum_{j\in J}\mu^j\delta_{x^j} \quad \text{and} \quad \nu_n \rightharpoonup |v|^{2^*_s}dx + \sum_{j\in J}\nu^j\delta_{x^j}$$

for points $x^j\in\mathbb{R}^N_+$ and positive μ^j, ν^j satisfying $\mathcal{S}(\nu^j)^{\frac{2}{2^*_s}}\leq\mu^j$. We have

$$\begin{aligned}
\mathcal{S}+o(1) = \|v_n\|^2_{H_s} &= \int_{\mathbb{R}^N_+} d\mu_n \geq \int_{\mathbb{R}^N_+} d\mu + o(1) \geq \|v\|^2_{H_s} + \sum_{j\in J}\mu^j + o(1)\\
&\geq \mathcal{S}\left(\|v\|^2_{L^{2^*_s}(\mathbb{R}^N_+)} + \sum_j (\nu^j)^{\frac{2}{2^*_s}}\right) + o(1)\\
&\geq \mathcal{S}\left(\|v\|^{2^*_s}_{L^{2^*_s}(\mathbb{R}^N_+)} + \sum_j \nu^j\right)^{\frac{2}{2^*_s}} + o(1)\\
&= \mathcal{S}\left(\int_{\mathbb{R}^N_+} d\nu\right)^{\frac{2}{2^*_s}} + o(1) = S + o(1)
\end{aligned} \tag{4.2.20}$$

as $n\to\infty$. In the second inequality, we used the fact that the map $t\mapsto t^{\frac{2}{2^*_s}}$ is strictly concave and hence subadditive. Moreover, the strict concavity implies that equality can only hold, if at most one of the terms $\|v\|^{2^*_s}_{L^{2^*_s}(\mathbb{R}^N_+)}$ and ν^j, $j\in J$ is nonzero.

Claim: $\nu^j = 0$ for all j.

Assuming that this is false, we have $\nu_n \rightharpoonup \delta_{x^1}$ for some $x^1\in\overline{\mathbb{R}^N_+}$. By our normalization and weak lower semicontinuity (of measures), $x^1\notin Q := (0,1)\times(-1,1)^{N-1}$ since

$$\delta_{x^1}(Q) \leq \liminf_{n\to\infty}\nu_n(Q) = \frac{1}{2}.$$

Moreover, if $\operatorname{dist}(x^1,Q)>0$, there exists $\varepsilon>0$ such that $B_\varepsilon(x_1)\cap Q\neq\varnothing$ and thus

$$1 = \delta_{x^1}(B_\varepsilon(x^1)) \leq \liminf_{n\to\infty}\nu_n(B_\varepsilon(x^1)) \leq \frac{1}{2},$$

which is a contradiction. Hence it only remains to consider the case $x^1\in\partial Q$. Due to the normalization

$$\sup_{E\in Q_1}\int_E |v_n|^{2^*_s}\,dx = \int_{(0,1)\times(-1,1)^{N-1}} |v_n|^{2^*_s}\,dx = \frac{1}{2},$$

we have $x^1\notin((0,y)+Q)$ for all $y\in\mathbb{R}^{N-1}$, so x^1 must be of the form $x^1=(1,y)$ or $(0,y)$ for some $y\in(-1,1)^{N-1}$. The latter case can be excluded, since, for $\varepsilon\in(0,\frac{1}{2})$,

$$\delta_{x^1}(B_\varepsilon(0,y)) \leq \liminf_{n\to\infty}\nu_n(B_\varepsilon(0,y)) \leq \liminf_{n\to\infty}\nu_n((0,y)+Q) \leq \frac{1}{2}.$$

After a translation orthogonal to the x_1-direction, we may therefore assume $x^1 = (1, 0, \dots, 0)$ and first note that $v \equiv 0$ and hence $\mu \geq \mathcal{S}\delta_{x^1}$ by (4.2.20). On the other hand,

$$\int_{\mathbb{R}^N} d\mu \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} d\mu_n = \mathcal{S},$$

whence we conclude $\mu = \mathcal{S}\delta_{x^1}$.

For any $0 < \delta < \frac{1}{2}$, $B_\delta := B_\delta(x_1)$ is a continuity set of $\nu = \delta_{x_1}$, hence

$$\nu_n(B_\delta) \longrightarrow 1$$

and similarly

$$\mu_n(B_\delta) \longrightarrow S$$

as $n \longrightarrow \infty$. In particular, for fixed $\varepsilon > 0$ we find $n_0 = n_0(\varepsilon, \delta)$ such that

$$\int_{B_\delta} |v_n|^{2_s^*}\, dx \geq 1 - \varepsilon, \qquad \mathcal{S} - \varepsilon \leq \int_{B_\delta} \left(\sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2 \right) dx \leq \mathcal{S} + \varepsilon$$

for $n \geq n_0$. Furthermore,

$$\frac{1}{1+\delta} \int_{B_\delta} \left(\sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2 \right) dx \leq \int_{B_\delta} \sum_{i=1}^{N} |\partial_i v_n|^2\, dx$$

and

$$\int_{B_\delta} \sum_{i=1}^{N} |\partial_i v_n|^2\, dx \leq \frac{1}{1-\delta} \int_{B_\delta} \left(\sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2 \right) dx$$

imply

$$\frac{\mathcal{S} - \varepsilon}{1+\delta} \leq \int_{B_\delta} \sum_{i=1}^{N} |\partial_i v_n|^2\, dx \leq \frac{\mathcal{S} + \varepsilon}{1-\delta}$$

for $n \geq n_0$. It is important to note that the weak convergence $\nu_n \rightharpoonup \delta_{x^1}$ implies that, for any $t \in (0, \delta)$ and $q \in (2_s^*, 2^*)$, we have

$$\begin{aligned} 1 = \liminf_{n\to\infty} \int_{B_t} |v_n|^{2_s^*}\, dx &\leq |B_t|^{1-\frac{2_s^*}{q}} \liminf_{n\to\infty} \left(\int_{B_t} |v_n|^q\, dx \right)^{\frac{2_s^*}{q}} \\ &\leq |B_t|^{1-\frac{2_s^*}{q}} \liminf_{n\to\infty} \left(\int_{B_\delta} |v_n|^q\, dx \right)^{\frac{2_s^*}{q}}. \end{aligned}$$

In particular, this implies

$$\liminf_{n\to\infty} \left(\int_{B_\delta} |v_n|^q\, dx \right)^{\frac{2_s^*}{q}} \geq |B_t|^{\frac{2_s^*}{q}-1}, \tag{4.2.21}$$

and since $t \in (0, \delta)$ was arbitrary, we conclude that $\|v_n\|_{L^q(B_\delta)} \longrightarrow \infty$ as $n \longrightarrow \infty$ for any $q \in (2_s^*, 2^*)$.

Now let $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that $\varphi \equiv 1$ on $B_1(0)$ and $\varphi \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$, and set

$$\varphi_\delta(x) := \varphi\left(\frac{x - x^1}{\delta} \right)$$

so that $\varphi_\delta \equiv 1$ on $B_\delta(x^1)$, $\varphi \equiv 0$ on $\mathbb{R}^N \setminus B_{2\delta}(x^1)$. Then, by Sobolev's inequality

$$\left(\int_{\mathbb{R}^N_+} |\varphi_\delta v_n|^q \, dx\right)^{\frac{2}{q}} \le C_q \left(\int_{\mathbb{R}^N_+} \sum_{i=1}^N |\partial_i(\varphi_\delta v_n)|^2 \, dx + \int_{\mathbb{R}^N_+} |\varphi_\delta v_n|^2 \, dx\right). \tag{4.2.22}$$

Note that (4.2.21) implies that the left hand side goes to infinity as $n \to \infty$ since

$$\int_{B_\delta} |v_n|^q \, dx \le \int_{\mathbb{R}^N} |\varphi_\delta v_n|^q \, dx.$$

On the other hand,

$$\int_{\mathbb{R}^N_+} |\varphi_\delta v_n|^2 \, dx \le |B_{2\delta}|^{1-\frac{2}{2^*_s}} \left(\int_{B_{2\delta}} |v_n|^{2^*_s} \, dx\right)^{\frac{2}{2^*_s}} \le |B_2|^{1-\frac{2}{2^*_s}},$$

and, noting that $\nabla\varphi_\delta(x) = \delta^{-1}[\nabla\varphi](\frac{x-x^1}{\delta})$,

$$\begin{aligned}
\left(\int_{\mathbb{R}^N_+} \sum_{i=1}^N |\partial_i(\varphi_\delta v_n)|^2 \, dx\right)^{\frac{1}{2}} &\le \left(\int_{\mathbb{R}^N_+} \varphi_\delta^2 \sum_{i=1}^N |\partial_i v_n|^2 \, dx\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^N_+} v_n^2 \sum_{i=1}^N |\partial_i \varphi_\delta|^2 \, dx\right)^{\frac{1}{2}} \\
&\le \left(\int_{B_{2\delta}} \sum_{i=1}^N |\partial_i v_n|^2 \, dx\right)^{\frac{1}{2}} + \sqrt{N}\delta^{-1}\|\nabla\varphi\|_\infty \left(\int_{B_{2\delta}\setminus B_\delta} |v_n|^2\right)^{\frac{1}{2}} \\
&\le \sqrt{\frac{\mathcal{S}+\varepsilon}{1-2\delta}} + \sqrt{N}\delta^{-1}\|\nabla\varphi\|_\infty |B_{2\delta}\setminus B_\delta|^{\frac{1}{2}-\frac{1}{2^*_s}} \left(\int_{B_{2\delta}\setminus B_\delta} |v_n|^{2^*_s}\right)^{\frac{1}{2^*_s}} \\
&\le \sqrt{\frac{\mathcal{S}+\varepsilon}{1-2\delta}} + \sqrt{N}\delta^{-1}\|\nabla\varphi\|_\infty |B_{2\delta}\setminus B_\delta|^{\frac{1}{2}-\frac{1}{2^*_s}}.
\end{aligned}$$

This implies that the right hand side of (4.2.22) remains bounded as $n \to \infty$, a contradiction.

We conclude $\nu^j = 0$ for all j and hence $\|v\|_{L^{2^*_s}(\mathbb{R}^N_+)} = 1$. Since $L^{2^*_s}(\mathbb{R}^N_+)$ is uniformly convex, this implies $v_n \to v$ in $L^{2^*_s}(\mathbb{R}^N_+)$. Moreover, since $\|v\|^2_{H_s} \ge \mathcal{S}$, weak lower semicontinuity gives $\|v_n\|^2_{H_s} \to \mathcal{S} = \|v\|^2_{H_s}$ and hence $v_n \to v$ in H_s again by uniform convexity of the Hilbert space H_s. This completes the proof. □

Remark 4.2.7. (Existence of minimizers on $\mathbb{R}^N$)
We note that Theorem 4.2.2 implies

$$\mathcal{S}_s(\mathbb{R}^N) := \inf_{u \in C^1_c(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} \sum_{i=1}^{N-1} |\partial_i u|^2 + |x_1|^s |\partial_N u|^2 \, dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*_s} \, dx\right)^{\frac{2}{2^*_s}}} > 0.$$

Consequently, we can look for minimizers in the closure of $C^1_c(\mathbb{R}^N)$ in

$$\left\{u \in L^{2^*_s}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \sum_{i=1}^{N-1} |\partial_i u|^2 + |x_1|^s |\partial_N u|^2 \, dx < \infty\right\}.$$

The previous arguments can then easily be adapted to prove the existence of minimizers of $\mathcal{S}_s(\mathbb{R}^N)$ similar to Theorem 4.2.6.

4.3 A degenerate Sobolev inequality on B

In this section we shall prove the second part of Theorem 4.1.1, namely the properties of $\mathscr{C}_{1,m,p}(\mathbf{B})$ given in (4.1.11).

We first use the scaling properties discussed in Remark 4.2.3(i) to prove the following.

Proposition 4.3.1. *Let $p > 2_1^*$ and $m > -\lambda_1(\mathbf{B})$. Then $\mathscr{C}_{1,m,p}(\mathbf{B}) = 0$, i.e.*

$$\inf_{u \in C_c^1(\mathbf{B})\setminus\{0\}} \frac{\|\nabla u\|_2^2 - \|\partial_\theta u\|_2^2 + m\|u\|_2^2}{\|u\|_p^2} = 0.$$

Proof. Let $\varepsilon > 0$. By (4.2.17), there exists $v \in C_c^1(\mathbb{R}_+^N)$ with the property that

$$\int_{\mathbb{R}_+^N} \Big(\sum_{i=1}^{N-1} |\partial_i v|^2 + 2x_1 |\partial_N v|^2 \Big) dx < \frac{\varepsilon}{2} \Big(\int_{\mathbb{R}_+^N} |v|^p \, dx \Big)^{\frac{2}{p}}.$$

For $\lambda \in (0,1)$, let

$$\tau_\lambda : \mathbf{B} \to \mathbb{R}_+^N, \quad \tau_\lambda(x) = (\lambda^{-2}(x_1+1), \lambda^{-2}x_3, \dots, \lambda^{-2}x_{N-1}, \lambda^{-3}x_2) \tag{4.3.1}$$

and set $u_\lambda := v \circ \tau_\lambda$. If λ is chosen sufficiently small, we have $u \in C_c^1(\mathbf{B})$ and

$$\|\nabla u\|_{L^2(\mathbf{B})}^2 - \|\partial_\theta u\|_{L^2(\mathbf{B})}^2 = \int_{\mathbf{B}} \Big(\sum_{i=1}^{N} |\partial_i u|^2 - |x_1 \partial_2 u - x_2 \partial_1 u|^2 \Big) dx$$

$$= \int_{\mathbf{B}} \left(\sum_{i=1}^{N-1} |\lambda^{-2}[\partial_i v] \circ \tau_\lambda|^2 + |\lambda^{-3}[\partial_N v] \circ \tau_\lambda|^2 - \left| x_1 \lambda^{-3} [\partial_N v] \circ \tau_\lambda - x_2 \lambda^{-2} [\partial_1 v] \circ \tau_\lambda \right|^2 \right) dx$$

$$= \lambda^{2N+1} \int_{\mathbb{R}_+^N} \left(\sum_{i=1}^{N-1} \lambda^{-4} |\partial_i v|^2 + \lambda^{-6} |\partial_N v|^2 - |(\lambda^2 x_1 - 1)\lambda^{-3} \partial_N v - \lambda^3 x_2 \lambda^{-2} \partial_1 v|^2 \right) dx$$

$$= \lambda^{2N-3} \int_{\mathbb{R}_+^N} \left(\sum_{i=1}^{N-1} |\partial_i v|^2 + 2x_1 |\partial_N v|^2 \right) dx$$

$$+ \lambda^{2N-3} \int_{\mathbb{R}_+^N} \left(-\lambda^2 x_1^2 |\partial_N v|^2 - 2x_2 \lambda^2 (\lambda^2 x_1 - 1) \partial_1 v \, \partial_N v + \lambda^6 x_2^2 |\partial_1 v|^2 \right) \, dx,$$

while

$$\|u\|_{L^2(\mathbf{B})}^2 = \lambda^{2N+1} \|v\|_{L^2(\mathbb{R}_+^N)}^2 \qquad \text{and} \qquad \|u\|_{L^p(\mathbf{B})}^2 = \lambda^{\frac{4N+2}{p}} \|v\|_{L^p(\mathbb{R}_+^N)}^2.$$

We conclude that

$$\mathscr{C}_{1,m,p}(\mathbf{B}) \le \frac{\|\nabla u\|_{L^2(\mathbf{B})}^2 - \|\partial_\theta u\|_{L^2(\mathbf{B})}^2 + m\|u\|_{L^2(\mathbf{B})}^2}{\|u\|_{L^p(\mathbf{B})}^2}$$

$$= \lambda^{\frac{p(2N-3)-(4N+2)}{p}} \frac{\int_{\mathbb{R}_+^N} \left(\sum\limits_{i=1}^{N-1} |\partial_i v|^2 + 2x_1 |\partial_N v|^2 \right) dx}{\|v\|_{L^p(\mathbb{R}_+^N)}^2} + o\left(\lambda^{\frac{p(2N-3)-(4N+2)}{p}} \right) < \varepsilon$$

for $\lambda > 0$ small enough, since $p > 2_1^* = \frac{4N+2}{2N-3}$. Recalling that $\varepsilon > 0$ was arbitrary, this yields the claim. □

To prove the second assertion on $\mathscr{C}_{1,m,p}(\mathbf{B})$ in (4.1.11), we now transfer the information given by Theorem 4.1.6 in the case $s = 1$ to the ball $\mathbf{B}$. To this end, we consider the great circle

$$\gamma := \{x \in \partial\mathbf{B} : x_3 = \cdots = x_N = 0\}\,. \tag{4.3.2}$$

We have the following key lemma.

Lemma 4.3.2. *Let $\varepsilon > 0$. Then there exists $\delta > 0$ with the property that, for any $x_0 \in \gamma$,*

$$\frac{\int_{\Omega_{x_0,\delta}} \left(|\nabla u|^2 - |\partial_\theta u|^2\right)\,dx}{\|u\|^2_{L^{2^*_1}(\Omega_{x_0,\delta})}} \geq (1-\varepsilon)\,2^{\frac{1}{2}-\frac{1}{2^*_1}}\mathcal{S}_1(\mathbb{R}^N_+) \qquad \textit{for } u \in C^1_c(\Omega_{x_0,\delta}) \setminus \{0\},$$

where $\mathcal{S}_1(\mathbb{R}^N_+)$ is given in Theorem 4.1.6 and

$$\Omega_{x_0,\delta} := \mathbf{B} \cap B_\delta(x_0) = \{x \in \mathbf{B} : |x - x_0| < \delta\}. \tag{4.3.3}$$

Proof. We may assume $x_0 = e_2 = (0,1,0,\dots,0)$ is the second coordinate vector. We fix $\delta > 0$ and consider a function $u \in C^1_c(\Omega_{e_2,\delta})$ which we extend trivially to a function $u \in C^1_c(\mathbb{R}^N)$. Moreover, we write u in N-dimensional polar coordinates, so we consider $U := [0,1] \times (-\pi,\pi) \times (0,\pi)^{N-2}$ and the function

$$v = u \circ P \,:\, U \longrightarrow \mathbb{R}$$

with $P : U \longrightarrow \mathbb{R}^N$ given by

$$\begin{aligned} P(r,\theta,\vartheta_1,\dots,\vartheta_{N-2}) = (&r\cos\theta\sin\vartheta_1\cdots\sin\vartheta_{N-2},\ r\sin\theta\sin\vartheta_1\cdots\sin\vartheta_{N-2},\\ &r\cos\vartheta_1, r\sin\vartheta_1\cos\vartheta_2,\dots, r\sin\vartheta_1\cdots\sin\vartheta_{N-3}\cos\vartheta_{N-2}, r\sin\vartheta_1\cdots\vartheta_{N-2}) \end{aligned} \tag{4.3.4}$$

We then have

$$\int_{\mathbf{B}} \left(|\nabla u|^2 - |\partial_\theta u|^2\right)\,dx \tag{4.3.5}$$

$$= \int_0^1 \int_{-\pi}^{\pi} \int_0^{\pi} \cdots \int_0^{\pi} \left(|\partial_r u|^2 + \frac{1}{r^2}\sum_{i=1}^{N-2} g_i |\partial_{\vartheta_i} u|^2 + \left(\frac{g_{N-1}}{r^2} - 1\right)|\partial_\theta u|^2 \right) g\, d\vartheta_1\cdots d\vartheta_{N-2}\, d\theta\, dr \tag{4.3.6}$$

with the functions $g, g_i : U \longrightarrow \mathbb{R}$, $i = 1,\dots,N-1$ given by

$$g(r,\theta,\vartheta_1,\dots,\vartheta_{N-2}) = r^{N-1}\prod_{k=1}^{N-2}\sin^{N-1-k}\vartheta_k, \qquad g_i(r,\theta,\vartheta_1,\dots,\vartheta_{N-2}) = \prod_{k=1}^{i-1}\frac{1}{\sin^2\vartheta_k} \tag{4.3.7}$$

In particular, we have $g \leq 1$ and $g_i \geq 1$ in U for $i = 1,\dots,N-1$. Moreover, since $P^{-1}(e_2) = (1,\frac{\pi}{2},\dots,\frac{\pi}{2})$ and $g(1,\frac{\pi}{2},\dots,\frac{\pi}{2}) = 1$, we can choose $\delta > 0$ sufficiently small so that

$$P^{-1}(\Omega_{e_2,\delta}) \subset (0,1) \times (0,\pi)^{N-1} \qquad \text{and} \qquad g \geq (1-\varepsilon) \text{ in } P^{-1}(\Omega_{e_2,\delta}). \tag{4.3.8}$$

Therefore

$$\int_{\mathbf{B}} \left(|\nabla u|^2 - |\partial_\theta u|^2\right)\,dx$$

$$\geq (1-\varepsilon)\int_0^1\int_{-\pi}^{\pi}\int_0^{\pi}\cdots\int_0^{\pi}\left(|\partial_r u|^2+\sum_{i=1}^{N-2}|\partial_{\vartheta_i}u|^2+\frac{(1-r)(1+r)}{r^2}|\partial_\theta u|^2\right)d\vartheta_1\cdots d\vartheta_{N-2}d\theta dr.$$

Noting that

$$\frac{(1-r)(1+r)}{r^2}\geq\frac{(2-\delta)(1-r)}{(1-\delta)^2}\geq 2(1-r)$$

and substituting $t=1-r$ we thus find that

$$\begin{aligned}&\int_{\mathbf{B}}\left(|\nabla u|^2-|\partial_\theta u|^2\right)\,dx\\ \geq&(1-\varepsilon)\int_0^1\int_{-\pi}^{\pi}\int_0^{\pi}\cdots\int_0^{\pi}\left(|\partial_t\tilde{v}|^2+\sum_{i=1}^{N-2}|\partial_{\vartheta_i}\tilde{v}|^2+2t|\partial_\theta\tilde{v}|^2\right)d\vartheta_1\cdots d\vartheta_{N-2}\,d\theta\,dt.\end{aligned}$$

with

$$\tilde{v}:U\to\mathbb{R},\qquad \tilde{v}(t,\vartheta_1,\dots,\vartheta_{N-2},\theta)=u(P(1-t,\vartheta_1,\dots,\vartheta_{N-2},\theta))$$

Note that $u\in C^1_c(\Omega_{e_2,\delta})$ implies, by (4.3.8), that $\tilde{v}$ is compactly supported in $(0,1)\times(0,\pi)^{N-1}\subset\mathbb{R}^N_+$, so we may regard $\tilde{v}$ as a function in $C^1_c(\mathbb{R}^N_+)$ and deduce that

$$\int_{\mathbf{B}}\left(|\nabla u|^2-|\partial_\theta u|^2\right)\,dx\geq(1-\varepsilon)\int_{\mathbb{R}^N_+}\left(\sum_{i=1}^{N-1}|\partial_i\tilde{v}|^2+2x_1|\partial_N\tilde{v}|^2\right)dx.$$

Rather directly, we also find that, by a change of variables,

$$\begin{aligned}\int_\Omega|u|^{2_1^*}\,dx&=\int_U|v|^{2_1^*}g\,d(r,\theta,\vartheta_1,\dots,d\vartheta_{N-2})\leq\int_U|v|^{2_1^*}\,d(r,\theta,\vartheta_1,\dots,d\vartheta_{N-2})\\&=\int_U|\tilde{v}|^{2_1^*}\,d(r,\theta,\vartheta_1,\dots,d\vartheta_{N-2})=\int_{\mathbb{R}^N_+}|\tilde{v}|^{2_1^*}\,dx.\end{aligned}$$

Using (4.2.18) with $\kappa=2$, we conclude that

$$\frac{\int_\Omega\left(|\nabla u|^2-|\partial_\theta u|^2\right)\,dx}{\|u\|^2_{L^{2_1^*}(\Omega)}}\geq(1-\varepsilon)\frac{\int_{\mathbb{R}^N_+}\left(\sum_{i=1}^{N-1}|\partial_i\tilde{v}|^2+2x_1|\partial_N\tilde{v}|^2\right)dx}{\left(\int_{\mathbb{R}^N_+}|\tilde{v}|^{2_1^*}\,dx\right)^{\frac{2}{2_1^*}}}\geq(1-\varepsilon)2^{\frac{1}{2}-\frac{1}{2_1^*}}\mathcal{S}_1(\mathbb{R}^N_+)$$

as claimed. □

We can now prove the main result of this section.

Theorem 4.3.3. *For any $1\leq p\leq 2_1^*$ there exists $C>0$, such that any $u\in C^1_c(\mathbf{B})$ satisfies*

$$\|u\|^2_{L^p(\mathbf{B})}\leq C\int_{\mathbf{B}}\left(|\nabla u|^2-|\partial_\theta u|^2\right)\,dx.$$

Proof. Since $\mathbf{B}$ is bounded, it suffices to consider the case $p=2_1^*$. In the following, $C>0$ denotes a constant independent of u, which may change from line to line. Fix $\varepsilon\in(0,\frac{1}{2})$ and let $\delta>0$ be given as in Lemma 4.3.2. Take points $x_1,\dots x_m\in\gamma$ such that the sets $U_k:=B_\delta(x_k)$ satisfy

$$\gamma\subset\bigcup_{k=1}^m U_k$$

and let $\delta_0 := \operatorname{dist}(\gamma, \mathbf{B} \setminus \bigcup_{k=1}^{m} U_k)$. We then let $U_0 := \left\{x \in \mathbf{B} : \operatorname{dist}(x,\gamma) > \frac{\delta_0}{2}\right\}$ and thus have $\mathbf{B} \subset \bigcup_{k=0}^{m} U_k$. We may then choose a partition of unity $\eta_0, \cdots, \eta_m$ subordinate to this covering. Then

$$\|u\|_{L^{2^*_1}(\mathbf{B})} \leq \sum_{k=0}^{m} \|\eta_k u\|_{L^{2^*_1}(U_k)} \leq C \sum_{k=0}^{m} \left(\int_{U_k} \left(|\nabla(\eta_k u)|^2 - |\partial_\theta(\eta_k u)|^2 \right) dx \right)^{\frac{1}{2}},$$

where we used Lemma 4.3.2 and the fact that $v \mapsto \int_{U_0} \left(|\nabla v|^2 - |\partial_\theta v|^2\right) dx$ induces an equivalent norm on $H^1_0(U_0)$. Note that, for $k = 0, \ldots, m$, we have

$$\begin{aligned}
&\int_{U_k} \left(|\nabla(\eta_k u)|^2 - |\partial_\theta(\eta_k u)|^2\right) dx \\
\leq& 2\left(\int_{U_k} \eta_k^2 \left(|\nabla u|^2 - |\partial_\theta u|^2\right) dx + \int_{U_k} u^2 \left(|\nabla \eta_k|^2 - |\partial_\theta \eta_k|^2\right) dx \right) \\
\leq& C \int_{U_k} \left(|\nabla u|^2 - |\partial_\theta u|^2 + u^2\right) dx,
\end{aligned}$$

with some fixed $C > 0$. We conclude that

$$\|u\|_{L^{2^*_1}(\mathbf{B})} \leq C \sum_{k=0}^{m} \left(\int_{U_k} \left(|\nabla u|^2 - |\partial_\theta u|^2 + u^2\right) dx \right)^{\frac{1}{2}},$$

and thus

$$\|u\|^2_{L^{2^*_1}(\mathbf{B})} \leq C \sum_{k=0}^{m} \int_{U_k} \left(|\nabla u|^2 - |\partial_\theta u|^2 + u^2\right) dx = C \int_{\mathbf{B}} \left(|\nabla u|^2 - |\partial_\theta u|^2 + u^2\right) dx. \tag{4.3.9}$$

In order to complete the proof, we note that Proposition 4.4.1 implies

$$\inf_{u \in C^1_c(\mathbf{B}) \setminus \{0\}} \frac{\int_{\mathbf{B}} \left(|\nabla u|^2 - |\partial_\theta u|^2\right) dx}{\int_{\mathbf{B}} u^2\, dx} = \lambda_1(\mathbf{B}) > 0$$

and hence

$$\int_{\mathbf{B}} u^2\, dx \leq \frac{1}{\lambda_1(\mathbf{B})} \int_{\mathbf{B}} \left(|\nabla u|^2 - |\partial_\theta u|^2\right) dx.$$

In view of (4.3.9) this finishes the proof. □

4.4 The variational setting for and preliminary results on ground state solutions

4.4.1. The variational setting. In this section, we set up the variational framework for (4.1.5) and prove some preliminary estimates for the quantities $\mathscr{C}_{\alpha,0,2}(\mathbf{B})$ and $R_{\alpha,m,p}$ defined in (4.1.6) and (4.1.7). We first show a Poincaré type estimate. Recall here that $\lambda_1(\mathbf{B})$ is the first Dirichlet eigenvalue of $-\Delta$ on the unit ball $\mathbf{B}$.

Proposition 4.4.1. *For $0 \leq \alpha \leq 1$, we have*

$$\mathscr{C}_{\alpha,0,2}(\mathbf{B}) = \inf_{u \in C^1_c(\mathbf{B}) \setminus \{0\}} \frac{\int_{\mathbf{B}} \left(|\nabla u|^2 - \alpha^2 |\partial_\theta u|^2\right)}{\int_{\mathbf{B}} u^2\, dx} = \lambda_1(\mathbf{B}). \tag{4.4.1}$$

Moreover, minimizers are precisely the Dirichlet eigenfunctions of $-\Delta$ on $\mathbf{B}$ corresponding to the eigenvalue $\lambda_1(\mathbf{B})$ and are therefore radial.

Proof. By (4.1.8) and since $\mathscr{C}_{0,0,2}(\mathbf{B}) = \lambda_1(\mathbf{B})$ by the variational characterization of $\lambda_1(\mathbf{B})$, it suffices to prove (4.4.1) in the case $\alpha = 1$. In the following, we let $\{Y_{\ell,k} \,:\, \ell \in \mathbb{N} \cup \{0\},\ k = 1, \dots, d_\ell\}$ denote an L^2-orthonormal basis of $L^2(\mathbb{S}^{N-1})$ of spherical harmonics of degree ℓ. More precisely, we can choose $Y_{\ell,k}$ in such a way that, for every $\ell \in \mathbb{N} \cup \{0\}$, the functions $Y_{\ell,k},\ k = 1 \dots, d_\ell$ form a basis of the eigenspace of the Laplace Beltrami operator $-\Delta_{\mathbb{S}^{N-1}}$ corresponding to the eigenvalue $\ell(\ell + N - 2)$ and such that

$$-\partial_\theta^2 Y_{\ell,k} = \ell_k^2 Y_{\ell,k} \qquad \text{for } k = 1 \dots, d_\ell$$

where $|\ell_k| \le \ell$, see e.g. [**72**]. Let $\varphi \in C_c^1(\mathbf{B})$, and let $\varphi_{\ell,k} \in C^1([0,1])$ be the angular Fourier coefficient functions defined by

$$\varphi_{\ell,k}(r) = \int_{\mathbb{S}^{N-1}} \varphi(r\omega) Y_{\ell,k}(\omega)\, d\omega, \qquad 0 \le r \le 1.$$

For fixed $r \in [0,1]$, we then have the Parseval identities

$$\begin{aligned}
\|\varphi(r\,\cdot)\|_{L^2(\mathbb{S}^{N-1})}^2 &= \sum_{\ell,k} |\varphi_{\ell,k}(r)|^2 \|Y_{\ell,k}\|_{L^2(\mathbb{S}^{N-1})}^2,\\
\|\partial_r\varphi(r\,\cdot)\|_{L^2(\mathbb{S}^{N-1})}^2 &= \sum_{\ell,k} |\partial_r\varphi_{\ell,k}(r)|^2 \|Y_{\ell,k}\|_{L^2(\mathbb{S}^{N-1})}^2,\\
\|\nabla_{\mathbb{S}^{N-1}}\varphi(r\,\cdot)\|_{L^2(\mathbb{S}^{N-1})}^2 &= \sum_{\ell,k} (\ell + N - 2)|\varphi_{\ell,k}(r)|^2 \|Y_{\ell,k}\|_{L^2(\mathbb{S}^{N-1})}^2 \quad \text{and}\\
\|\partial_\theta\varphi(r\,\cdot)\|_{L^2(\mathbb{S}^{N-1})}^2 &= \sum_{\ell,k} \ell_k^2 |\varphi_{\ell,k}(r)|^2 \|Y_{\ell,k}\|_{L^2(\mathbb{S}^{N-1})}^2
\end{aligned}$$

in $L^2(\mathbb{S}^{N-1})$. Here and in the following, we simply write $\sum\limits_{\ell,k}$ in place of $\sum\limits_{\ell=0}^{\infty}\sum\limits_{k=1}^{d_\ell}$. Since $\frac{\ell(\ell+N-2)}{r^2} \ge \ell_k^2$ for $r \in [0,1]$ and every ℓ, k, we estimate that

$$\begin{aligned}
&\int_{\mathbf{B}} \left(|\nabla\varphi|^2 - |\partial_\theta\varphi|^2\right)\, dx\\
&= \int_0^1 r^{N-1} \int_{\mathbb{S}^{N-1}} \left(|\partial_r\varphi(r\omega)|^2 + \frac{1}{r^2}|\nabla_{\mathbb{S}^{N-1}}\varphi(r\omega)|^2 - |\partial_\theta\varphi(r\omega|^2\right) d\omega dr\\
&= \sum_{\ell,k} \|Y_{\ell,k}\|_{L^2(\mathbb{S}^{N-1})}^2 \int_0^1 r^{N-1}\left(|\partial_r\varphi_{\ell,k}(r)|^2 + \left(\frac{\ell(\ell+N-2)}{r^2} - \ell_k^2\right)|\varphi_{\ell,k}(r)|^2\right) dr\\
&\ge \sum_{\ell,k} \|Y_{\ell,k}\|_{L^2(\mathbb{S}^{N-1})}^2 \int_0^1 r^{N-1}|\partial_r\varphi_{\ell,k}(r)|^2 dr\\
&\ge \lambda_1(\mathbf{B}) \sum_{\ell,k} \|Y_{\ell,k}\|_{L^2(\mathbb{S}^{N-1})}^2 \int_0^1 r^{N-1}|\varphi_{\ell,k}(r)|^2 dr\\
&= \lambda_1(-\Delta, \mathbf{B}) \int_0^1 r^{N-1}\|\varphi(r\,\cdot)\|_{L^2(\mathbb{S}^{N-1})}^2\, dr = \lambda_1(\mathbf{B}) \int_{\mathbf{B}} |\varphi|^2\, dx.
\end{aligned}$$

Clearly, equality holds if and only if $\varphi_{\ell,k} \equiv 0$ for $\ell \ge 1$ and φ_0 corresponds to a first eigenfunction of the Dirichlet Laplacian on $\mathbf{B}$. □

Corollary 4.4.2.

(i) We have $\mathscr{C}_{\alpha,m,2}(\mathbf{B}) = \mathscr{C}_{0,m,2}(\mathbf{B}) = \lambda_1(\mathbf{B}) + m$ *for* $\alpha \in [0,1]$, $m \in \mathbb{R}$.

(ii) For $\alpha \in [0,1]$, $m > -\lambda_1(\mathbf{B})$, $2 \le p < 2^*$ and $u \in H_0^1(\mathbf{B}) \setminus \{0\}$ we have $R_{\alpha,m,p}(u) > 0$.

Proof. (i) This follows immediately from Proposition 4.4.1.
(ii) For $\alpha \in [0,1]$, $m \ge -\lambda_1(\mathbf{B})$, $2 \le p < 2^*$ and $u \in H_0^1(\mathbf{B}) \setminus \{0\}$ we have $R_{\alpha,m,p}(u) > R_{1,-\lambda_1(\mathbf{B}),p}(u)$ and

$$R_{1,-\lambda_1(\mathbf{B}),p}(u)\left(\int_{\mathbf{B}} |u|^p\,dx\right)^{\frac{2}{p}} = \left(\int_{\mathbf{B}} |u|^p\,dx\right)^{-\frac{2}{p}} \int_{\mathbf{B}} \left(|\nabla u|^2 - |\partial_\theta u|^2 - \lambda_1(\mathbf{B})u^2\right) dx \ge 0$$

by Proposition 4.4.1. □

Remark 4.4.3. ***(The case** $\alpha > 1$**)***
It is natural to ask what happens for $\alpha > 1$. In fact, in this case, the infimum $\mathscr{C}_{\alpha,m,p}(\mathbf{B})$ in (4.1.6) satisfies

$$\mathscr{C}_{\alpha,m,p}(\mathbf{B}) = -\infty \qquad \text{for every } m \in \mathbb{R},\ p \in [2,\infty). \tag{4.4.2}$$

To see this, we fix $\varepsilon \in (0,1)$ and nonzero functions $\varphi \in C_c^1(1-\varepsilon,1))$, $\psi \in C_c^1(\frac{\pi}{2}-\varepsilon,\frac{\pi}{2}+\varepsilon)$. Moreover, we consider the sequence of functions $u_k \in C_c^1(\mathbf{B})$ which, in the polar coordinates from (4.3.4), are given by

$$(r,\theta,\vartheta_1,\dots,\vartheta_{N-2}) \mapsto \varphi(r)\psi(\vartheta_1)\cdots\psi(\vartheta_{N-2})X_k(\theta), \qquad \text{where } X_k(\theta) = \sin(k\theta).$$

Similarly as in (4.3.5), we then find, with $U_\varepsilon := (1-\varepsilon,1)\times(-\pi,\pi)\times(\frac{\pi}{2}-\varepsilon,\frac{\pi}{2}+\varepsilon)^{N-2}$, that

$$\begin{aligned}
&\int_{\mathbf{B}} \left(|\nabla u_k|^2 - \alpha^2|\partial_\theta u_k|^2\right) dx \\
&= \int_{U_\varepsilon} \Big(|\varphi'(r)|^2|X_k(\theta)|^2 \prod_{i=1}^{N-2}|\psi(\vartheta_i)|^2 + \frac{1}{r^2}\sum_{i=1}^{N-2} g_i|\psi'(\vartheta_i)|^2|\varphi(r)|^2|X_k(\theta)|^2 \prod_{\substack{j=1\\ j\ne i}}^{N-2}|\psi(\vartheta_j)|^2 \\
&\quad + \left(\frac{g_{N-1}}{r^2}-\alpha^2\right)|X_k'(\theta)|^2|\varphi(r)|^2\prod_{i=1}^{N-2}|\psi(\vartheta_i)|^2\Big) g\, d(r,\theta,\vartheta_1,\dots,\vartheta_{N-2})
\end{aligned}$$

with the functions $g, g_i : U \to \mathbb{R}$, $i = 1,\dots,N-1$ given in (4.3.7). We may now choose $\varepsilon = \varepsilon(\alpha) > 0$ so small that

$$\frac{1}{2} \le g \le 1 \quad \text{and} \quad \alpha^2 - \frac{g_{N-1}}{r^2} \ge \varepsilon \qquad \text{on } U_\varepsilon.$$

Since also $|X_k| \le 1$ by definition, we estimate

$$\int_{\mathbf{B}} \left(|\nabla u_k|^2 - \alpha^2|\partial_\theta u_k|^2\right) dx \le c - d(k),$$

where

$$c := \int_{U_\varepsilon} \Big(|\varphi'(r)|^2 \prod_{i=1}^{N-2}|\psi(\vartheta_i)|^2 + \frac{1}{r^2}\sum_{i=1}^{N-2} g_i|\psi'(\vartheta_i)|^2|\varphi(r)|^2 \prod_{\substack{j=1\\ j\ne i}}^{N-2}|\psi(\vartheta_j)|^2\Big)\, d(r,\theta,\vartheta_1,\dots,\vartheta_{N-2})$$

and

$$d(k) := \int_{U_\varepsilon}\left(\alpha^2 - \frac{g_{N-1}}{r^2}\right)|X_k'(\theta)|^2|\varphi(r)|^2\prod_{i=1}^{N-2}|\psi(\vartheta_i)|^2 g\, d(r,\theta,\vartheta_1,\dots,\vartheta_{N-2})$$

$$\geq \frac{k^2\varepsilon}{2}\int_{1-\varepsilon}^{1}|\varphi(r)|^2dr\int_{-\pi}^{\pi}\cos^2(k\theta)d\theta\Big(\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon}|\psi(\vartheta)|^2d\vartheta\Big)^{N-2}=\frac{\varepsilon\pi}{2}d_2k^2$$

with $d_2 := \int_{1-\varepsilon}^{1}|\varphi(r)|^2dr\Big(\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon}|\psi(\vartheta)|^2d\vartheta\Big)^{N-2}$. *Hence* $d(k)\to\infty$ *as* $k\to\infty$. *Moreover, for every* $p\in[2,\infty)$ *we have*

$$\int_{\mathbf{B}}|u_k|^p\,dx=\int_{U_\varepsilon}|\varphi(r)|^p|X_k(\theta)|^p\prod_{i=1}^{N-2}|\psi(\vartheta_i)|^p gd(r,\theta,\vartheta_1,\dots,\vartheta_{N-2})\leq d_p$$

with

$$d_p := 2\pi\int_{1-\varepsilon}^{1}|\varphi(r)|^pdr\Big(\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon}|\psi(\vartheta)|^pd\vartheta\Big)^{N-2}<\infty.$$

It thus follows that

$$\frac{\int_{\mathbf{B}}\big(|\nabla u_k|^2-\alpha^2|\partial_\theta u_k|^2+m|u_k|^2\big)\,dx}{\Big(\int_{\mathbf{B}}|u_k|^p\,dx\Big)^{\frac{2}{p}}}\leq\frac{c-d(k)-md_2}{(d_p)^{\frac{2}{p}}}\to-\infty\qquad\text{as } k\to\infty.$$

for every $p\in[2,\infty)$, $m\in\mathbb{R}$. *This shows (4.4.2).*

Consequently, the study of ground state solutions of (4.1.5) *requires a completely different approach in the case* $\alpha>1$. *This is further treated in the forthcoming paper* ***[P4]***.

In the following, we show that, for $\alpha\in[0,1)$ and $2<p<2^*$, the value $\mathscr{C}_{\alpha,m,p}(\mathbf{B})>0$ is attained in $H_0^1(\mathbf{B})\setminus\{0\}$ and that any minimizer gives rise to a weak solution of (4.1.5).

Lemma 4.4.4. *Let* $0\leq\alpha<1$, $2<p<2^*$ *and* $m>-\lambda_1(\mathbf{B})$. *Then the value* $\mathscr{C}_{\alpha,m,p}(\mathbf{B})$ *is positive and attained at a function* $u_0\in H_0^1(\mathbf{B})\setminus\{0\}$. *Moreover, after multiplication by a positive constant,* u_0 *is a weak solution of (4.1.5), and* $u_0\in C^{2,\sigma}(\overline{\mathbf{B}})$ *for some* $\sigma>0$.

Proof. We first note that

$$\int_{\mathbf{B}}\big(|\nabla u|^2-\alpha^2|\partial_\theta u|^2\big)\,dx\geq(1-\alpha^2)\int_{\mathbf{B}}|\nabla u|^2\,dx\qquad\text{for } u\in H_0^1(\mathbf{B}).$$

Since $\alpha\in[0,1)$, it therefore follows from Sobolev embeddings that

$$R_{\alpha,m,p}(u)\geq C_{m,\alpha}\frac{\int_{\mathbf{B}}|\nabla u|^2\,dx}{\Big(\int_{\mathbf{B}}|u|^p\,dx\Big)^{\frac{2}{p}}}\qquad u\in H_0^1(\mathbf{B})\setminus\{0\}\tag{4.4.3}$$

with a constant $C_{m,\alpha}>0$. We take a minimizing sequence $(u_n)_n$ for the Rayleigh quotient $R_{\alpha,m,p}$, normalized such that $\int_{\mathbf{B}}|u|^p\,dx=1$ for all n. By (4.4.3), $(u_n)_n$ remains bounded in $H_0^1(\mathbf{B})$ and we may pass to subsequence that weakly converges to $u_0\in H_0^1(\mathbf{B})$. The compactness of the embedding $H_0^1(\mathbf{B})\hookrightarrow L^p(\mathbf{B})$ and the weak lower semicontinuity of the quadratic form $u\mapsto\int_{\mathbf{B}}\big(|\nabla u|^2-\alpha^2|\partial_\theta u|^2\big)\,dx$ then imply that $\int_{\mathbf{B}}|u_0|^p\,dx=1$ and $R_{\alpha,m,p}(u_0)=\mathscr{C}_{\alpha,m,p}(\mathbf{B})$. Hence $\mathscr{C}_{\alpha,m,p}(\mathbf{B})$ is attained, and $\mathscr{C}_{\alpha,m,p}(\mathbf{B})>0$ by Corollary 4.4.2.

Next, standard variational arguments show that every L^p-normalized minimizer u_0 must be a weak solution of

$$\begin{cases}-\Delta u+\alpha^2\partial_\theta^2u+mu=\mathscr{C}_{\alpha,m,p}(\mathbf{B})|u|^{p-2}u & \text{in } \mathbf{B}\\ u=0 & \text{on } \partial\mathbf{B}.\end{cases}$$

We then conclude that $\big[\mathscr{C}_{\alpha,m,p}(\mathbf{B})\big]^{\frac{1}{p-2}}u_0$ solves (4.1.5). Finally, classical elliptic regularity theory yields $C^{2,\sigma}(\overline{\mathbf{B}})$ since the operator $-\Delta-\alpha^2\partial_\theta$ is uniformly elliptic in $\mathbf{B}$ in the case $0\leq\alpha<1$. □

Definition 4.4.5. *Let $0 \le \alpha < 1$, $2 < p < 2^*$ and $m > -\lambda_1(\mathbf{B})$. A weak solution $u \in H_0^1(\mathbf{B})\setminus\{0\}$ of (4.1.5) such that $R_{\alpha,m,p}(u) = \mathscr{C}_{\alpha,m,p}(\mathbf{B})$ will be called a* ***ground state solution****.*

4.4.2. The degenerate elliptic case $\alpha = 1$.. In the limiting case $\alpha = 1$, problem (4.1.5) becomes degenerate and requires to work in a function space different from $H_0^1(\mathbf{B})$. From Proposition 4.4.1, we deduce that

$$(u,v) \mapsto \langle u,v\rangle_{\mathcal{H}} := \int_{\mathbf{B}} (\nabla u \cdot \nabla v - \partial_\theta u \partial_\theta v)\, dx$$

defines a scalar product on $C_c^1(\mathbf{B})$. The induced norm will be denoted by $\|\cdot\|_{\mathcal{H}}$.

Lemma 4.4.6. *Let $\mathcal{H}$ denote the completion of $C_c^1(\mathbf{B})$ with respect to $\|\cdot\|_{\mathcal{H}}$. Then $\mathcal{H}$ is a Hilbert space which is embedded in $L^p(\mathbf{B})$ for $p \in [2, 2_1^*]$, where $2_1^* := \frac{4N+2}{2N-3}$ as before. Moreover, we have:*

(i) If $1 \le p < 2_1^*$, then the embedding $\mathcal{H} \hookrightarrow L^p(\mathbf{B})$ is compact.

(ii) If $m > -\lambda_1(\mathbf{B})$ and $p \in [2, 2_1^*]$, then the Rayleigh quotient $R_{1,m,p}(u)$ is well defined by (4.1.7) and positive for functions $u \in \mathcal{H} \setminus \{0\}$,

Proof. The embedding $\mathcal{H} \hookrightarrow L^p(\mathbf{B})$ for $p \in [2, 2_1^*]$ is an immediate consequence of Theorem 4.3.3.

To prove (i), we fix $p \in [1, 2_1^*)$, and we let $(u_n)_n \subset \mathcal{H}$ be a bounded sequence. Moreover, we put $B_m := B_{1-1/m}(0) \subset \mathbf{B}$ for $m \ge 2$. Then $u_n^m := \mathbb{1}_{B_m} u_n$ defines a bounded sequence in $H^1(B_m)$ for every $m \ge 2$. After passing to a subsequence, $(u_n^m)_n$ converges in $L^p(B_m)$ by Rellich-Kondrachov. After passing to a diagonal sequence we may therefore assume that there exists $u \in L^p(\mathbf{B})$ with the property that $u_n \to u$ for $m \in \mathbb{N}$. Moreover,

$$\|u - u_n\|_{L^p(\mathbf{B})} \le \|u - u_n\|_{L^p(B_m)} + \|u - u_n\|_{L^{2_1^*}(\mathbf{B}\setminus B_m)} |\mathbf{B} \setminus B_m|^{\frac{1}{p} - \frac{1}{2_1^*}}.$$

Since $\|u - u_n\|_{L^{2_1^*}(\mathbf{B}\setminus B_m)} \le \|u - u_n\|_{L^{2_1^*}(\mathbf{B})}$ remains bounded independently of m and n, this gives

$$\limsup_{n\to\infty} \|u - u_n\|_{L^p(\mathbf{B})} \le C|\mathbf{B} \setminus B_m|^{\frac{1}{p} - \frac{1}{2_1^*}}$$

for some $C > 0$ independent of m, where the right hand side tends to zero as $m \to \infty$. This proves that $u_n \to u$ in $L^p(\mathbf{B})$.

Finally, we note that (ii) is an immediate consequence of Proposition 4.4.1 and the embedding $\mathcal{H} \hookrightarrow L^p(\mathbf{B})$ for $p \in [2, 2_1^*]$. □

Lemma 4.4.6 allows the following definition of a weak solution of (4.1.5) with $\alpha = 1$ in the case where $p \in [2, 2_1^*]$.

Definition 4.4.7. *Let $m > -\lambda_1(\mathbf{B})$, $p \in [2, 2_1^*]$.*

(i) We call $u \in \mathcal{H}$ a ***weak solution*** *of (4.1.5) with $\alpha = 1$ if*

$$\langle u,v\rangle_{\mathcal{H}} = \int_{\mathbf{B}} \left(|u|^{p-2}uv - muv\right)dx \qquad \text{for every } v \in \mathcal{H}.$$

(ii) A weak solution $u \in \mathcal{H}$ of (4.1.5) with $\alpha = 1$ will be called a ***ground state solution*** *if u is a minimizer for $R_{1,m,p}$, i.e., we have $R_{1,m,p}(u) = \mathscr{C}_{1,m,p}(\mathbf{B})$.*

We then have the following existence result which replaces Proposition 4.4.4 in the degenerate elliptic case $\alpha = 1$.

Proposition 4.4.8. *Let $1 < p < 2_1^*$ and $m > -\lambda_1(\mathbf{B})$. Then we have*

$$\mathscr{C}_{1,m,p}(\mathbf{B}) > 0, \tag{4.4.4}$$

and there exists $u_0 \in \mathcal{H} \setminus \{0\}$ with $R_{1,m,p}(u_0) = \mathscr{C}_{1,m,p}(\mathbf{B})$, i.e., u_0 minimizes $R_{1,m,p}$ in $\mathcal{H} \setminus \{0\}$. Furthermore, after multiplication by a positive constant, u_0 is ground state solution of (4.1.5) with $\alpha = 1$ and $u_0 \in C^{2,\sigma}_{loc}(\mathbf{B})$ for some $\sigma > 0$.

Proof. Proving the existence of u_0 is completely analogous to the proof of Lemma 4.4.4, making use of the Rellich-Kondrachov type result stated in Lemma 4.4.6(i).

In order to prove the regularity result, we first note that a Moser iteration scheme can be used to show that $u_0 \in L^\infty(\mathbf{B})$, see Lemma 4.7.1 in the appendix for a detailed proof. For any fixed $s \in (0,1)$ we may then use the fact that the operator $-\Delta + \partial_\theta^2$ is uniformly elliptic in the ball $B_s = \{x \in \mathbb{R}^N : |x| < s\}$ and classical elliptic regularity theory, to show $u_0 \in C^{2,\sigma}_{\mathrm{loc}}(B_s)$. □

Next, we treat the critical case $p = 2_1^*$, and first show that $\mathscr{C}_{1,m,2_1^*}(\mathbf{B})$ is attained, provided it is small enough.

Theorem 4.4.9. *Let $m > -\lambda_1(\mathbf{B})$ such that $\mathscr{C}_{1,m,2_1^*}(\mathbf{B}) < 2^{\frac{1}{2}-\frac{1}{2_1^*}}\mathcal{S}_1(\mathbb{R}^N_+)$. Then the value $\mathscr{C}_{1,m,2_1^*}(\mathbf{B})$ is attained in $\mathcal{H} \setminus \{0\}$.*

In particular, this proves the first part of Theorem 4.1.7. The strategy of the proof is inspired by [**59**] and first requires the following characterization of sequences in $\mathcal{H}$:

Lemma 4.4.10. *Let*

$$Z(v) := \int_{\mathbf{B}} \left(|\nabla v|^2 - |\partial_\theta v|^2 + mv^2\right)\,dx \qquad \textit{and} \qquad N(v) := \int_{\mathbf{B}} |v|^{2_1^*}\,dx \qquad \textit{for } v \in \mathcal{H}.$$

Then we have

$$2^{\frac{1}{2}-\frac{1}{2_1^*}}\mathcal{S}_1(\mathbb{R}^N_+) \le \inf\left\{\liminf_{n\to\infty} Z(w_n) : (w_n)_n \subset \mathcal{H},\ N(w_n) = 1,\ w_n \rightharpoonup 0 \textit{ in } \mathcal{H}\right\}.$$

Proof. Let $(w_n)_n \subset \mathcal{H}$ such that $N(w_n) = 1$, $w_n \rightharpoonup 0$ in $\mathcal{H}$. Let $\varepsilon > 0$ and choose $U_0, \dots, U_m \subset \mathbf{B}$ as in the proof of Theorem 4.3.3, so that

$$\mathbf{B} \subset \bigcup_{k=0}^m U_k.$$

We may then choose functions $\eta_0, \dots, \eta_m \in C_c^2(\mathbf{B})$ such that $\operatorname{supp} \eta_k \subset U_k$ and $\sum_{k=0}^m \eta_k^2 \equiv 1$ on $\mathbf{B}$. Then

$$\begin{aligned}\int_{\mathbf{B}} \left(|\nabla(\eta_k w_n)|^2 - |\partial_\theta \eta_k w_n|^2\right)\,dx &= \int_{\mathbf{B}} \left(\eta_k^2|\nabla w_n|^2 + 2w_n\eta_k\nabla w_n \cdot \nabla\eta_k + w_n^2|\nabla\eta_k|^2\right)\,dx \\ &\quad - \int_{\mathbf{B}} \left(\eta_k^2|\partial_\theta w_n|^2 + 2w_n\eta_k\partial_\theta w_n \cdot \partial_\theta\eta_k + w_n^2|\partial_\theta\eta_k|^2\right)\,dx\end{aligned}$$

and thus

$$\int_{\mathbf{B}} \left(|\nabla w_n|^2 - |\partial_\theta w_n|^2 + mw_n^2\right)\,dx \ge \sum_{k=0}^m \int_{\mathbf{B}} \left(|\nabla(\eta_k w_n)|^2 - |\partial_\theta \eta_k w_n|^2\right)\,dx - C\int_{\mathbf{B}} w_n^2\,dx$$

with a constant $C > 0$ independent of n. Here we used the fact that the mixed terms can be estimated as follows:

$$\int_{\mathbf{B}} w_n^2 \left(|\nabla \eta_k|^2 - |\partial_\theta \eta_k|^2\right) dx \leq 2 \sup_{k \in \{0,\dots,m\}} \|\nabla \eta_k\|_\infty^2 \int_{\mathbf{B}} w_n^2 \, dx$$

$$\begin{aligned} \int_{\mathbf{B}} \eta_k w_n \left(\nabla w_n \cdot \nabla \eta_k - \partial_\theta w_n \, \partial_\theta \eta_k\right) dx &\leq \int_{\mathbf{B}} \eta_k w_n^2 \left|-\Delta \eta_k + \partial_\theta^2 \eta_k\right| dx \\ &\leq \sup_{k \in \{0,\dots,m\}} \left\|-\Delta \eta_k + \partial_\theta^2 \eta_k\right\|_\infty \int_{\mathbf{B}} |w_n|^2 \, dx. \end{aligned}$$

We first note that $w_n \to 0$ in $L^2(\mathbf{B})$, since the embedding $\mathcal{H} \hookrightarrow L^2(\mathbf{B})$ is compact by Lemma 4.4.6(i). Moreover, it is easy to see that $\|\cdot\|_{\mathcal{H}}$ induces an equivalent norm on $H_0^1(U_0)$, which implies that $\eta_0 w_n \rightharpoonup 0$ in $H_0^1(U_0)$. In particular, noting that by $2_1^* < 2^*$ the classical Rellich-Kondrachov theorem implies $\eta_0 w_n \to 0$ in $L^{2_1^*}(\mathbf{B})$, we conclude

$$\liminf_{n\to\infty} \int_{\mathbf{B}} \left(|\nabla(\eta_0 w_n)|^2 - |\partial_\theta(\eta_0 w_n)|^2 + m(\eta_0 w_n)^2\right) dx \geq \liminf_{n\to\infty} \left(\int_{\mathbf{B}} |\eta_0 w_n|^{2_1^*} \, dx\right)^{\frac{2}{2_1^*}}.$$

On the other hand, Lemma 4.3.2 gives

$$\int_{\mathbf{B}} \left(|\nabla(\eta_k w_n)|^2 - |\partial_\theta \eta_k w_n|^2\right) dx \geq (1-\varepsilon) 2^{\frac{1}{2} - \frac{1}{2_1^*}} \mathcal{S}_1(\mathbb{R}_+^N) \left(\int_{\mathbf{B}} |\eta_k w_n|^{2_1^*} \, dx\right)^{\frac{2}{2_1^*}}$$

for $k = 1, \dots, m$ and hence

$$\begin{aligned} &\liminf_{n\to\infty} \int_{\mathbf{B}} \left(|\nabla w_n|^2 - |\partial_\theta w_n|^2 + m w_n^2\right) dx \\ \geq &\liminf_{n\to\infty} \sum_{k=0}^m \int_{\mathbf{B}} \left(|\nabla(\eta_k w_n)|^2 - |\partial_\theta \eta_k w_n|^2\right) dx \\ \geq &(1-\varepsilon) 2^{\frac{1}{2} - \frac{1}{2_1^*}} \mathcal{S}_1(\mathbb{R}_+^N) \liminf_{n\to\infty} \sum_{k=0}^m \left(\int_{\mathbf{B}} |\eta_k w_n|^{2_1^*} \, dx\right)^{\frac{2}{2_1^*}} \\ = &(1-\varepsilon) 2^{\frac{1}{2} - \frac{1}{2_1^*}} \mathcal{S}_1(\mathbb{R}_+^N) \liminf_{n\to\infty} \sum_{k=0}^m \left\|\eta_k^2 w_n^2\right\|_{\frac{2_1^*}{2}} \\ \geq &(1-\varepsilon) 2^{\frac{1}{2} - \frac{1}{2_1^*}} \mathcal{S}_1(\mathbb{R}_+^N) \liminf_{n\to\infty} \left\|\sum_{k=0}^m \eta_k^2 w_n^2\right\|_{\frac{2_1^*}{2}} \\ = &(1-\varepsilon) 2^{\frac{1}{2} - \frac{1}{2_1^*}} \mathcal{S}_1(\mathbb{R}_+^N) \liminf_{n\to\infty} \|w_n\|_{\frac{2_1^*}{2}} \\ &= (1-\varepsilon) 2^{\frac{1}{2} - \frac{1}{2_1^*}} \mathcal{S}_1(\mathbb{R}_+^N). \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\liminf_{n\to\infty} \int_{\mathbf{B}} \left(|\nabla w_n|^2 - |\partial_\theta w_n|^2 + m w_n^2\right) dx \geq 2^{\frac{1}{2} - \frac{1}{2_1^*}} \mathcal{S}_1(\mathbb{R}_+^N)$$

as claimed. □

We may now complete the proof of our main result.

Proof of Theorem 4.4.9. Consider a minimizing sequence $(u_n)_n \subset \mathcal{H}$ for $\mathscr{C}_{1,m,2_1^*}(\mathbf{B})$ with $\|u_n\|_{2_1^*} = 1$. Then $(u_n)_n$ is bounded in $\mathcal{H}$, hence, after passing to a subsequence, we may assume $u_n \rightharpoonup u_0$ in $\mathcal{H}$. We set $v_n := u_n - u_0$ and note that, by Sobolev embeddings,

$$v_n \longrightarrow 0 \quad \text{in } L^q(B_s)$$

for $1 \le q < 2_1^*$ and $0 < s < 1$, where $B_s := \{x \in \mathbb{R}^N : |x| < s\}$. Weak convergence implies

$$\mathscr{C}_{1,m,2_1^*}(\mathbf{B}) = \lim_{n\to\infty} Z(u_n) = Z(u_0) + \lim_{n\to\infty} Z(v_n),$$

whereas the Brezis-Lieb Lemma yields

$$1 = N(u_n) = N(u_0) + N(v_n) + o(1).$$

In particular, the limits

$$T := \lim_{n\to\infty} N(v_n), \qquad M := \lim_{n\to\infty} Z(v_n)$$

exist. If $T = 0$, it follows that $N(u_0) = 1$ and we are finished. For $T > 0$, Lemma 4.4.10 implies

$$M \ge \mathcal{S}_1(\mathbb{R}_+^N) T^{\frac{2}{2_1^*}}$$

and hence

$$\begin{aligned}
\mathscr{C}_{1,m,2_1^*}(\mathbf{B}) &= Z(u_0) + M \ge Z(u_0) + 2^{\frac{1}{2}-\frac{1}{2_1^*}} \mathcal{S}_1(\mathbb{R}_+^N) T^{\frac{2}{2_1^*}} \\
&\ge Z(u_0) + \left(2^{\frac{1}{2}-\frac{1}{2_1^*}} \mathcal{S}_1(\mathbb{R}_+^N) - \mathscr{C}_{1,m,2_1^*}(\mathbf{B})\right) T^{\frac{2}{2_1^*}} + \mathscr{C}_{1,m,2_1^*}(\mathbf{B}) \left(1 - N(u_0)\right)^{\frac{2}{2_1^*}} \\
&\ge Z(u_0) + (2^{\frac{1}{2}-\frac{1}{2_1^*}} \mathcal{S}_1(\mathbb{R}_+^N) - \mathscr{C}_{1,m,2_1^*}(\mathbf{B})) T^{\frac{2}{2_1^*}} + \mathscr{C}_{1,m,2_1^*}(\mathbf{B}) - \mathscr{C}_{1,m,2_1^*}(\mathbf{B}) N(u_0)^{\frac{2}{2_1^*}},
\end{aligned}$$

where we used the inequality $(a-b)^\tau \ge a^\tau - b^\tau$ for $a \ge b \ge 0$ and $0 \le \tau \le 1$. It follows that

$$Z(u_0) + (2^{\frac{1}{2}-\frac{1}{2_1^*}} \mathcal{S}_1(\mathbb{R}_+^N) - \mathscr{C}_{1,m,2_1^*}(\mathbf{B})) T^{\frac{2}{2_1^*}} - \mathscr{C}_{1,m,2_1^*}(\mathbf{B}) N(u_0)^{\frac{2}{2_1^*}} \le 0,$$

and therefore

$$\begin{aligned}
&\int_{\mathbf{B}} \left(|\nabla u_0|^2 - |\partial_\theta u_0|^2 + m u_0^2\right) dx - \mathscr{C}_{1,m,2_1^*}(\mathbf{B}) \left(\int_{\mathbf{B}} |u_0|^{2_1^*} dx\right)^{\frac{2}{2_1^*}} \\
&+ (2^{\frac{1}{2}-\frac{1}{2_1^*}} \mathcal{S}_1(\mathbb{R}_+^N) - \mathscr{C}_{1,m,2_1^*}(\mathbf{B})) T^{\frac{2}{2_1^*}} \le 0.
\end{aligned} \tag{4.4.5}$$

By definition, we have $\int_{\mathbf{B}} \left(|\nabla u_0|^2 - |\partial_\theta u_0|^2 + m u_0^2\right) dx - \mathscr{C}_{1,m,2_1^*}(\mathbf{B}) \left(\int_{\mathbf{B}} |u_0|^{2_1^*} dx\right)^{\frac{2}{2_1^*}} \ge 0$ and since $2^{\frac{1}{2}-\frac{1}{2_1^*}} \mathcal{S}_1(\mathbb{R}_+^N) - \mathscr{C}_{1,m,2_1^*}(\mathbf{B}) > 0$ by assumption, we must have $T = 0$, i.e. $v_n \to 0$ in $L^p(\Omega)$. It follows that $u_0 \not\equiv 0$ and $\int_{\mathbf{B}} |u_0|^{2_1^*} dx = 1$, and (4.4.5) gives

$$\int_{\mathbf{B}} \left(|\nabla u_0|^2 - |\partial_\theta u_0|^2 + m u_0^2\right) dx \le \mathscr{C}_{1,m,2_1^*}(\mathbf{B}) \left(\int_{\mathbf{B}} |u_0|^{2_1^*} dx\right)^{\frac{2}{2_1^*}},$$

which implies that u_0 is a minimizer. □

We note the following consequence of Theorem 4.4.9, which extends (4.4.4) to the critical case.

Corollary 4.4.11. *We have* $\mathscr{C}_{1,m,2_1^*}(\mathbf{B}) > 0$.

Proof. If the value $\mathscr{C}_{1,m,2_1^*}(\mathbf{B})$ is attained in $\mathcal{H} \setminus \{0\}$, then we have $\mathscr{C}_{1,m,2_1^*}(\mathbf{B}) > 0$ by Lemma 4.4.6(ii). If not, we have $\mathscr{C}_{1,m,2_1^*}(\mathbf{B}) \geq \mathcal{S}_1(\mathbb{R}^N_+) > 0$ by Theorem 4.4.9 and Theorem 4.2.2. □

In general, the existence of ground state solutions in the case $\alpha = 1$, $p = 2_1^*$ remains an open problem and might depend on the parameter $m > -\lambda_1(\mathbf{B})$. We have the following partial existence result in the critical case.

Theorem 4.4.12. *There exists* $\varepsilon > 0$*, such that for* $m \in (-\lambda_1(\mathbf{B}), -\lambda_1(\mathbf{B}) + \varepsilon)$ *there exists* $u_0 \in \mathcal{H} \setminus \{0\}$ *such that*

$$R_{1,m,2_1^*}(u_0) = \inf_{u \in \mathcal{H}\setminus\{0\}} R_{1,m,2_1^*}(u),$$

i.e. u_0 *minimizes* $R_{1,m,2_1^*}$*. Furthermore, after multiplication by a positive constant,* u_0 *is a weak solution of*

$$\begin{cases} -\Delta u + \partial_\theta^2 u + mu = |u|^{2_1^*-2}u & \text{in } \mathbf{B}, \\ u = 0 & \text{on } \partial\mathbf{B}, \end{cases}$$

i.e., u_0 *satisfies*

$$\int_{\mathbf{B}} \nabla u \cdot \nabla \varphi - \partial_\theta u \, \partial_\theta \varphi + mu\varphi \, dx = \int_{\mathbf{B}} |u|^{2_1^*-2} u\varphi \, dx$$

for all $\varphi \in \mathcal{H}$.

Proof. For a (necessarily radial) eigenfunction φ_1 of $-\Delta$ on $\mathbf{B}$ corresponding to $\lambda_1(\mathbf{B})$, we have

$$\mathscr{C}_{1,m,2_1^*}(\mathbf{B}) \leq R_{1,m,2_1^*}(\varphi_1) = \frac{(\lambda_1(\mathbf{B}) + m) \int_{\mathbf{B}} \varphi_1^2 \, dx}{\left(\int_{\mathbf{B}} |\varphi_1|^p \, dx\right)^{\frac{2}{p}}}$$

which implies $\mathscr{C}_{1,m,2_1^*}(\mathbf{B}) \to 0$ as $m \to -\lambda_1(\mathbf{B})^+$. In particular, it follows that there exists $\varepsilon > 0$ such that

$$\mathscr{C}_{1,m,2_1^*}(\mathbf{B}) < 2^{\frac{1}{2} - \frac{1}{2_1^*}} \mathcal{S}_1(\mathbb{R}^N_+)$$

holds for $m \in (-\lambda_1(\mathbf{B}), -\lambda_1(\mathbf{B}) + \varepsilon)$. By Theorem 4.4.9, this finishes the proof. □

Note that this completes the proof of Theorem 4.1.7.

4.4.3. Radiality versus x_1-x_2-nonradiality of ground state solutions.. By classical results due to McLeod and Serrin [**97**], Kwong [**78**], Kwong and Li [**79**] (see also references in [**42**]), the problem

$$\begin{cases} -\Delta u + mu = |u|^{p-2}u & \text{in } \mathbf{B} \\ u = 0 & \text{on } \partial\mathbf{B}, \end{cases} \tag{4.4.6}$$

has a unique radial positive solution $u_{rad} \in H_0^1(\mathbf{B})$ which is a minimizer for $\mathscr{C}_{0,m,p}(\mathbf{B})$. Clearly, u_{rad} is also a weak solution of (4.1.5) for every $\alpha > 0$, but it might not be a ground state solution. In fact, we have the following.

Lemma 4.4.13. *Let* $2 < p < 2^*$ *and* $m > -\lambda_1(\mathbf{B})$ *be fixed.*

(i) *The map*

$$[0, 1] \to \mathbb{R}, \qquad \alpha \mapsto \mathscr{C}_{\alpha,m,p}(\mathbf{B})$$

is continuous and nonincreasing.

(ii) Let $\alpha \in (0,1]$, and suppose that $p \leq 2_1^*$ in the case $\alpha = 1$. Then the following properties are equivalent:

$(ii)_1$ $\mathscr{C}_{\alpha,m,p}(\mathbf{B}) < \mathscr{C}_{0,m,p}(\mathbf{B})$.

$(ii)_2$ *Every ground state solution of (4.1.5) is x_1-x_2-nonradial.*

Proof. (i) The monotonicity of $\mathscr{C}_{\alpha,m,p}(\mathbf{B})$ in α follows immediately from the definition. In order to prove continuity, we first consider $\alpha_0 \in (0,1]$ and let $\varepsilon > 0$. Moreover, we let $u_0 \in H_0^1(\mathbf{B}) \setminus \{0\}$ be a function with $R_{\alpha_0,m,p}(u_0) < \mathscr{C}_{\alpha_0,m,p}(\mathbf{B}) + \varepsilon$. For $\alpha \leq \alpha_0$, we then have

$$
\begin{aligned}
\mathscr{C}_{\alpha_0,m,p}(\mathbf{B}) \leq \mathscr{C}_{\alpha,m,p}(\mathbf{B}) &\leq R_{\alpha,m,p}(u_0) \\
&\leq R_{\alpha_0,m,p}(u_0) + (\alpha_0^2 - \alpha^2) \frac{\int_{\mathbf{B}} |\partial_\theta u_0|^2 \, dx}{\left(\int_{\mathbf{B}} |u_0|^p \, dx\right)^{\frac{2}{p}}} \\
&\leq \mathscr{C}_{\alpha_0,m,p}(\mathbf{B}) + (\alpha_0^2 - \alpha^2) \frac{\int_{\mathbf{B}} |\partial_\theta u_0|^2 \, dx}{\left(\int_{\mathbf{B}} |u_0|^p \, dx\right)^{\frac{2}{p}}}
\end{aligned}
$$

which implies that $\limsup_{\alpha \to \alpha_0^-} |\mathscr{C}_{\alpha,m,p}(\mathbf{B}) - \mathscr{C}_{\alpha_0,m,p}(\mathbf{B})| \leq \varepsilon$. This shows the continuity from the left in α_0.

Next we let $\alpha_0 \in [0,1)$ and show continuity from the right in α_0. For this we fix $\delta > 0$ such that $(\alpha_0, \alpha_0 + \delta) \subset (0,1)$. For $\alpha \in (\alpha_0, \alpha_0 + \delta)$, Lemma 4.4.4 implies that the value $\mathscr{C}_{\alpha,m,p}(\mathbf{B})$ is attained at a function $u_\alpha \in H_0^1(\mathbf{B}) \setminus \{0\}$ with $\int_{\mathbf{B}} |u_\alpha|^p \, dx = 1$. Therefore

$$
\begin{aligned}
\mathscr{C}_{\alpha_0,m,p}(\mathbf{B}) \geq \mathscr{C}_{\alpha,m,p}(\mathbf{B}) = R_{\alpha,m,p}(u_\alpha) &= R_{\alpha_0,m,p}(u_\alpha) + (\alpha_0^2 - \alpha^2) \int_{\mathbf{B}} |\partial_\theta u_\alpha|^2 \, dx \\
&\geq \mathscr{C}_{\alpha_0,m,p}(\mathbf{B}) - |\alpha_0^2 - \alpha^2| \int_{\mathbf{B}} |\nabla u_\alpha|^2 \, dx,
\end{aligned}
$$

whence, using the fact that

$$
(1 - \alpha^2) \int_{\mathbf{B}} |\nabla u_\alpha|^2 \, dx \leq \int_{\mathbf{B}} \left(|\nabla u_\alpha|^2 - \alpha^2 |\partial_\theta u_\alpha|^2\right) dx = \mathscr{C}_{\alpha,m,p}(\mathbf{B}) \leq \mathscr{C}_{0,m,p}(\mathbf{B}),
$$

we conclude

$$
\begin{aligned}
\mathscr{C}_{\alpha_0,m,p}(\mathbf{B}) \geq \mathscr{C}_{\alpha,m,p}(\mathbf{B}) &\geq \mathscr{C}_{\alpha_0,m,p}(\mathbf{B}) - \frac{|\alpha_0^2 - \alpha^2|}{1 - \alpha^2} \mathscr{C}_{0,m,p}(\mathbf{B}) \\
&\geq \mathscr{C}_{\alpha_0,m,p}(\mathbf{B}) - \frac{|\alpha_0^2 - \alpha^2|}{1 - (\alpha_0 + \delta)^2} \mathscr{C}_{0,m,p}(\mathbf{B}).
\end{aligned}
$$

This shows the continuity from the right in α_0.

(ii) As noted above, $\mathscr{C}_{0,m,p}(\mathbf{B})$ is attained by a radial positive solution u_{rad} of (4.4.6) and we have $R_{0,m,p}(u_{rad}) = R_{\alpha,m,p}(u_{rad})$. Hence, if $\mathscr{C}_{0,m,p}(\mathbf{B}) = \mathscr{C}_{\alpha,m,p}(\mathbf{B})$, then u_{rad} is also a radial ground state solution of (4.1.5). Hence $(ii)_2$ and (i) imply that $\mathscr{C}_{\alpha,m,p}(\mathbf{B}) < \mathscr{C}_{0,m,p}(\mathbf{B})$. If, conversely, there exists a radial ground state solution u of (4.1.5), then we have

$$
\mathscr{C}_{0,m,p}(\mathbf{B}) \leq R_{0,m,p}(u) = R_{\alpha,m,p}(u) = \mathscr{C}_{\alpha,m,p}(\mathbf{B})
$$

and therefore equality holds by (i). Consequently, the $\mathscr{C}_{\alpha,m,p}(\mathbf{B}) < \mathscr{C}_{0,m,p}(\mathbf{B})$ implies that every ground state solution of (4.1.5) is x_1-x_2-nonradial. □

The second part of this section is devoted to the proof of Theorem 4.1.3, which yields radiality of ground state solutions for α close to zero. For this, we fix $m \geq 0$ and $2 < p < 2^*$. Moreover, we consider a sequence of numbers $\alpha_n \in (0,1)$, $\alpha_n \to 0$ and, for every $n \in \mathbb{N}$, a positive ground state solution $u_n \in H_0^1(\mathbf{B})$ of (4.1.5) with $\alpha = \alpha_n$. Recall that the existence of u_n is proved in Lemma 4.4.4. To prove Theorem 4.1.3, it then suffices to show that

$$u_n = u_{rad} \quad \text{for } n \text{ sufficiently large,} \tag{4.4.7}$$

where u_{rad} is the unique positive solution of (4.4.6). We first claim the following.

Lemma 4.4.14. *$u_n \to u_{rad}$ in $H_0^1(\mathbf{B})$ as $n \to \infty$.*

Proof. We put $v_n := \frac{u_n}{\|u_n\|_{L^p(\mathbf{B})}}$, so v_n is an L^p-normalized minimizer for $\mathscr{C}_{\alpha_n,m,p}(\mathbf{B})$. Then $(v_n)_n$ is bounded in $H_0^1(\mathbf{B})$ by definition of $\mathscr{C}_{\alpha_n,m,p}(\mathbf{B})$. Consequently, we have $v_n \rightharpoonup v_0$ in $H_0^1(\mathbf{B})$ after passing to a subsequence, which implies that $v_n \to v_0$ in $L^p(\mathbf{B})$ and therefore $\int_{\mathbf{B}} |v_0|^p\,dx = 1$. We show that v_0 is minimizer for $\mathscr{C}_{0,m,p}(\mathbf{B})$. Indeed, by weak lower semicontinuity, we have

$$\begin{aligned}\mathscr{C}_{0,m,p}(\mathbf{B}) \leq R_{0,m,p}(v_0) &\leq \liminf_{n\to\infty} R_{0,m,p}(v_n) \leq \lim_{n\to\infty}\Big(R_{\alpha_n,m,p}(v_n) + \alpha_n^2\|\partial_\theta u_n\|_{L^2(\mathbf{B})}^2\Big)\\ &\leq \lim_{n\to\infty} \mathscr{C}_{\alpha_n,m,p}(\mathbf{B}) + \alpha_n\|u_n\|_{H^1(\mathbf{B})}^2 = \mathscr{C}_{0,m,p}(\mathbf{B}),\end{aligned}$$

where we used Lemma 4.4.13 in the last step. Hence v_0 is a minimizer of $\mathscr{C}_{0,m,p}(\mathbf{B})$, and a posteriori we find that

$$\begin{aligned}\|\nabla v_n\|_{L^2(\mathbf{B})}^2 + m\|v_n\|_{L^2(\mathbf{B})}^2 &= R_{\alpha_n,m,p}(v_n) + \alpha_n^2\|\partial_\theta v_n\|_{L^2(\mathbf{B})}^2\\ &\to R_{0,m,p}(v_0) = \|\nabla v_0\|_{L^2(\mathbf{B})}^2 + m\|v_0\|_{L^2(\mathbf{B})}^2 \qquad \text{as } n \to \infty.\end{aligned}$$

By uniform convexity of $H^1(\mathbf{B})$, we thus conclude that $v_n \to v_0$ in $H_0^1(\mathbf{B})$. Next we recall that, as noted in the proof of Lemma 4.4.4, we have

$$u_n := \big[\mathscr{C}_{\alpha_n,m,p}(\mathbf{B})\big]^{\frac{1}{p-2}} v_n \qquad \text{and, by uniqueness,} \qquad u_{rad} := \big[\mathscr{C}_{\alpha_n,m,p}(\mathbf{B})\big]^{\frac{1}{p-2}} v_0.$$

Hence Lemma 4.4.13 implies that $u_n \to u_{rad}$ in $H_0^1(\mathbf{B})$. Although we have proved this only after passing to a subsequence, the convergence of the full sequence now follows from the uniqueness of u_{rad}. The proof is thus finished. □

Next, we improve Lemma 4.4.14 by noting that

$$u_n \to u_{rad} \qquad \text{in } H^2(\mathbf{B}). \tag{4.4.8}$$

This follows in a standard way from Lemma 4.4.14 and standard elliptic regularity theory (see e.g. [**64**, Theorem 8.12]), since $u_n = u_{rad} - u_n \in H_0^1(\mathbf{B})$ is a weak solution of

$$\begin{cases}-\Delta w_n + \alpha_n^2\partial_\theta w_n + m w_n = |v_{rad}|^{p-2}v_{rad} - |v_n|^{p-2}v_n & \text{in } \mathbf{B}\\ \qquad\qquad w_n = 0 & \text{on } \partial\mathbf{B},\end{cases}$$

and the coefficients of the differential operator $-\Delta + \alpha_n^2\partial_\theta$ are uniformly bounded and elliptic in $n \in \mathbb{N}$.

We may now complete the proof of our main result as follows.

Proof of Theorem 4.1.3. To complete the proof of (4.4.7), we consider the map

$$F:(-1,1)\times H^2(\mathbf{B})\cap H_0^1(\mathbf{B})\to L^2(\mathbf{B}),\qquad F(\alpha,u):=-\Delta u+\alpha^2\partial_\theta^2 u+mu-|u|^{p-2}u,$$

and we note that weak solutions of (4.1.5) correspond to zeroes of F. We also note that $F(\alpha,u_{rad})=0$ for all α. We wish to apply the implicit function theorem at $(0,u_{rad})$, so we need to check that

$$[\partial_u F](0,u_{rad})=-\Delta+m-(p-1)|u_{rad}|^{p-2}$$

is invertible as a map $H^2(\mathbf{B})\cap H_0^1(\mathbf{B})\to L^2(\mathbf{B})$. This is equivalent to the nondegeneracy of u_{rad} as a solution of (4.4.6) which is noted e.g. in [**42**, Theorem 4.2] for $m=0$ and in [**1**, Theorem 1.1] in the case $m>0$. Now the implicit function theorem yields $\varepsilon>0$ with the following property: If $u\in H^2(\mathbf{B})\cap H_0^1(\mathbf{B})$ satisfies $\|u-u_{rad}\|_{H^2(\mathbf{B})}<\varepsilon$ and $F(\alpha,u)=0$ for some $\alpha\in(-\varepsilon,\varepsilon)$, then $u=u_{rad}$.

Hence Lemma 4.4.8 implies that $u_n=u_{rad}$ for n sufficiently large, which shows (4.4.7), as claimed. □

In the remainder of this section, we show x_1-x_2-nonradial ground states for large m, as claimed in Theorem 4.1.4. We restate this theorem here in an equivalent form.

Theorem 4.4.15. *Let $\alpha\in(0,1)$ and $2<p<2^*$. Then there exists $\varepsilon_0>0$, such that the ground states of*

$$\begin{cases}-\Delta u+\alpha^2\partial_\theta^2 u+\dfrac{1}{\varepsilon^2}u=|u|^{p-2}u & \text{in } \mathbf{B},\\ u=0 & \text{on } \partial\mathbf{B},\end{cases}\tag{4.4.9}$$

are x_1-x_2-nonradial for $\varepsilon\in(0,\varepsilon_0)$. Moreover, if $p<2_1^*$, the same result holds for $\alpha=1$.

Proof. We first treat the case $\alpha\in(0,1)$. In the following, for $u\in H_0^1(\mathbf{B})$ and $\varepsilon>0$, we consider $B_{1/\varepsilon}:=B_{1/\varepsilon}(0)$ and the rescaled function $u_\varepsilon\in H_0^1(B_{1/\varepsilon})$, $u_\varepsilon(x)=u(\varepsilon x)$. A direct computation then shows that

$$\frac{\int_{B_{1/\varepsilon}}\left(|\nabla u_\varepsilon|^2-\alpha^2\varepsilon^2|\partial_\theta u_\varepsilon|^2+u_\varepsilon^2\right)dx}{\left(\int_{B_{1/\varepsilon}}|u_\varepsilon|^p\,dx\right)^{\frac{2}{p}}}=\varepsilon^{2-N+\frac{2N}{p}}R_{\alpha,\frac{1}{\varepsilon^2},p}(u).\tag{4.4.10}$$

As a consequence, we have

$$\mathscr{C}_{\alpha\varepsilon,1,p}(B_{1/\varepsilon}):=\inf_{v\in H_0^1(B_{1/\varepsilon})\setminus\{0\}}\frac{\int_{B_{1/\varepsilon}}\left(|\nabla v|^2-\alpha^2\varepsilon^2|\partial_\theta v|^2+v^2\right)dx}{\left(\int_{B_{1/\varepsilon}}|v|^p\,dx\right)^{\frac{2}{p}}}=\varepsilon^{2-N+\frac{2N}{p}}\mathscr{C}_{\alpha,\frac{1}{\varepsilon^2},p}(\mathbf{B}).$$

It therefore suffices to show that there exists $\varepsilon_0>0$ such that all minimizers for $\mathscr{C}_{\alpha\varepsilon,1,p}(B_{1/\varepsilon})$ in $H_0^1(B_{1/\varepsilon})\setminus\{0\}$ are x_1-x_2-nonradial if $\varepsilon\in(0,\varepsilon_0)$. We argue by contradiction and suppose that there exists a sequence $\varepsilon_n\to0$ and, for every $n\in\mathbb{N}$, a minimizer $v_{\varepsilon_n}\in H_0^1(B_{1/\varepsilon_n})\setminus\{0\}$ for $\mathscr{C}_{\alpha\varepsilon_n,1,p}(B_{1/\varepsilon_n})$ which satisfies

$$\partial_\theta v_{\varepsilon_n}\equiv0\qquad\text{in } B_{1/\varepsilon_n}.\tag{4.4.11}$$

To simplify the notation, we continue writing ε in place of ε_n in the following. From (4.4.11) and the inclusion $H_0^1(B_{1/\varepsilon})\subset H^1(\mathbb{R}^N)$, we then deduce that

$$\mathscr{C}_{\alpha\varepsilon,1,p}(B_{1/\varepsilon})=\frac{\int_{B_{1/\varepsilon}}\left(|\nabla v_\varepsilon|^2+v^2\right)dx}{\left(\int_{B_{1/\varepsilon}}|v|^p\,dx\right)^{\frac{2}{p}}}$$

$$\geq \inf_{v \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} \left(|\nabla v|^2 + v^2\right) dx}{\left(\int_{\mathbb{R}^N} |v|^p \, dx\right)^{\frac{2}{p}}} =: \mathscr{C}_{0,1,p}(\mathbb{R}^N). \tag{4.4.12}$$

We will now derive a contradiction to this inequality by constructing suitable functions in $H_0^1(B_{1/\varepsilon} \setminus \{0\})$ to estimate $\mathscr{C}_{\alpha\varepsilon,1,p}(B_{1/\varepsilon})$. To this end, we first note that the value $\mathscr{C}_{0,1,p}(\mathbb{R}^N)$ is attained by any translation of the unique positive radial solution $\tilde{u}_0 \in H^1(\mathbb{R}^N)$ of the nonlinear Schrödinger equation

$$-\Delta u + u = |u|^{p-2}u \qquad \text{in } \mathbb{R}^N.$$

Now take a radial function $\eta \in C_c^1(\mathbf{B})$ such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ in $B_{1/2}$, and let $u_0(x) := \tilde{u}_0(x - e_1)$ where $e_1 = (1, 0, \dots, 0)$. We then define

$$\eta_\varepsilon, \ w_\varepsilon \in C_c^1(B_{1/\varepsilon}) \qquad \text{by} \qquad \eta_\varepsilon(x) = \eta(\varepsilon x), \quad w_\varepsilon(x) = \eta_\varepsilon(x) u_0(x).$$

Then we have $w_\varepsilon \equiv u_0$ in $B_{1/(2\varepsilon)}$, and

$$\mathscr{C}_{\alpha\varepsilon,1,p}(B_{1/\varepsilon}) \leq \frac{\int_{B_{1/\varepsilon}} \left(|\nabla w_\varepsilon|^2 - \alpha^2\varepsilon^2|\partial_\theta w_\varepsilon|^2 + w_\varepsilon^2\right) dx}{\left(\int_{B_{1/\varepsilon}} |w_\varepsilon|^p \, dx\right)^{\frac{2}{p}}} \tag{4.4.13}$$

$$= \frac{\int_{B_{1/\varepsilon}} \eta_\varepsilon^2 \left(|\nabla u_0|^2 + u_0^2\right) dx}{\left(\int_{B_{1/\varepsilon}} \eta_\varepsilon^p |u_0|^p \, dx\right)^{\frac{2}{p}}} + \frac{\int_{B_{1/\varepsilon}} \left(u_0^2|\nabla_\varepsilon \eta|^2 + 2\eta_\varepsilon u_0 \nabla\eta_\varepsilon \cdot \nabla u_0 - \alpha^2\varepsilon^2\eta_\varepsilon^2|\partial_\theta u_0|^2\right) dx}{\left(\int_{B_{1/\varepsilon}} \eta_\varepsilon^p |u_0|^p \, dx\right)^{\frac{2}{p}}}.$$

We first estimate the second term and note that classical results (see [**14**]) imply that there exist $C_0, \delta_0 > 0$, such that

$$|u_0(x)|, |\nabla u_0(x)| \leq C_0 e^{-\delta_0|x|} \quad \text{for } x \in \mathbb{R}^N. \tag{4.4.14}$$

Noting that $\nabla\eta_\varepsilon \equiv 0$ on $B_{1/(2\varepsilon)}$, this readily implies

$$\int_{B_{1/\varepsilon}} \left(u_0^2|\nabla\eta_\varepsilon|^2 + 2\eta_\varepsilon u_0 \nabla\eta_\varepsilon \cdot \nabla u_0\right) dx \leq C_1 e^{-\frac{\delta_1}{\varepsilon}}$$

for some constants $C_1, \delta_1 > 0$. Moreover, for $\varepsilon \in (0, \frac{1}{2})$ we have

$$\alpha^2\varepsilon^2 \int_{B_{1/\varepsilon}} \eta_\varepsilon^2 |\partial_\theta u_0|^2 \, dx \geq C_2\varepsilon^2 \qquad \text{with} \quad C_2 := \alpha^2 \int_{\mathbf{B}} |\partial_\theta u_0|^2 \, dx > 0,$$

since u_0 is an x_1-x_2-nonradial function. After possibly modifying $C_1, C_2 > 0$, this gives

$$\frac{\int_{B_{1/\varepsilon}} \left(u_0^2|\nabla\eta_\varepsilon|^2 + 2\eta_\varepsilon u_0 \nabla\eta_\varepsilon \cdot \nabla u_0 - \alpha^2\varepsilon^2\eta_\varepsilon^2|\partial_\theta u_0|^2\right) dx}{\left(\int_{B_{1/\varepsilon}} \eta_\varepsilon^p |u_0|^p \, dx\right)^{\frac{2}{p}}} \leq C_1 e^{-\frac{\delta_1}{\varepsilon}} - C_2\varepsilon^2.$$

Next we consider the first term in (4.4.13) and note that

$$\frac{\int_{B_{1/\varepsilon}} \eta_\varepsilon^2 \left(|\nabla u_0|^2 + u_0^2\right) dx}{\left(\int_{B_{1/\varepsilon}} \eta_\varepsilon^p |u_0|^p \, dx\right)^{\frac{2}{p}}} \leq \frac{\int_{\mathbb{R}^N} \left(|\nabla u_0|^2 + u_0^2\right) dx}{\left(\int_{B_{1/(2\varepsilon)}} |u_0|^p \, dx\right)^{\frac{2}{p}}},$$

while (4.4.14) implies

$$\int_{\mathbb{R}^N\setminus B_{1/(2\varepsilon)}} |u_0|^p\,dx \le C_3 e^{-\frac{\delta_2}{\varepsilon}}$$

for some $C_3, \delta_2 > 0$. It thus follows that

$$\begin{aligned}\frac{\int_{B_{1/\varepsilon}} \eta_\varepsilon^2 \left(|\nabla u_0|^2 + u_0^2\right)dx}{\left(\int_{B_{1/\varepsilon}} \eta_\varepsilon^p |u_0|^p\,dx\right)^{\frac{2}{p}}} &\le \frac{\int_{\mathbb{R}^N} \left(|\nabla u_0|^2 + u_0^2\right)dx}{\left(\int_{B_{1/(2\varepsilon)}} |u_0|^p\,dx\right)^{\frac{2}{p}}} \le \frac{\int_{\mathbb{R}^N} \left(|\nabla u_0|^2 + u_0^2\right)dx}{\left(\int_{\mathbb{R}^N} |u_0|^p\,dx - C_3 e^{-\frac{\delta_2}{\varepsilon}}\right)^{\frac{2}{p}}}\\ &\le \frac{\int_{\mathbb{R}^N} \left(|\nabla u_0|^2 + u_0^2\right)dx}{\left(\int_{\mathbb{R}^N} |u_0|^p\,dx\right)^{\frac{2}{p}}} + C_4 e^{-\frac{\delta_2}{\varepsilon}} = \mathscr{C}_{0,1,p}(\mathbb{R}^N) + C_4 e^{-\frac{2\delta_2}{p\varepsilon}}\end{aligned}$$

for $\varepsilon > 0$ sufficiently small with some constant $C_4 > 0$, since u_0 attains $\mathscr{C}_{0,1,p}(\mathbb{R}^N)$. In view of (4.4.12) and (4.4.13), this yields that

$$\mathscr{C}_{0,1,p}(\mathbb{R}^N) \le \mathscr{C}_{\alpha\varepsilon,1,p}(B_{1/\varepsilon}) \le \mathscr{C}_{0,1,p}(\mathbb{R}^N) - C_2\varepsilon^2 + C_1 e^{-\frac{\delta_1}{\varepsilon}} + C_4 e^{-\frac{2\delta_2}{p\varepsilon}},$$

and the right hand side of this inequality is smaller than $\mathscr{C}_{0,1,p}(\mathbb{R}^N)$ if $\varepsilon > 0$ is sufficiently small. This is a contradiction, and thus the claim follows in this case.

In the case $\alpha = 1$, the argument is the same up to replacing $H_0^1(\mathbf{B})$ by $\mathcal{H}$ and by considering the corresponding rescaled function space $\mathcal{H}_\varepsilon$ on $B_{1/\varepsilon}$. Then the contradiction argument can be carried out in the same way, since radial functions in $\mathcal{H}_\varepsilon$ belong to $H_0^1(B_{1/\varepsilon}) \subset H^1(\mathbb{R}^N)$. □

4.5 The case of an annulus

In this section, we consider rotating solutions of (4.1.3) in the case where $\mathbf{B}$ is replaced by an annulus

$$\mathbf{A}_r := \{x \in \mathbb{R}^N : r < |x| < 1\}$$

for some $r \in (0,1)$. The ansatz (4.1.4) then leads to the reduced problem

$$\begin{cases} -\Delta u + \alpha^2 \partial_\theta^2 u + mu = |u|^{p-2}u & \text{in } \mathbf{A}_r,\\ u = 0 & \text{on } \partial\mathbf{A}_r \end{cases} \tag{4.5.1}$$

where $m > -\lambda_1(\mathbf{A}_r)$, $p \in (2, \frac{2N}{N-2})$ and $\partial_\theta = x_{N-1}\partial_{x_N} - x_N\partial_{x_{N-1}}$ as before. Here, $\lambda_1(\mathbf{A}_r)$ denotes the first Dirichlet eigenvalue of $-\Delta$ on $\mathbf{A}_r$. As in (4.1.6), we may then define

$$\mathscr{C}_{\alpha,m,p}(\mathbf{A}_r) := \inf_{u\in H_0^1(\mathbf{A}_r)\setminus\{0\}} R_{\alpha,m,p}(u) \tag{4.5.2}$$

with the Rayleigh quotient $R_{\alpha,m,p}(u)$ given by (4.1.7) for functions $u \in H_0^1(\mathbf{A}_r)$. In the following, a weak solution of (4.5.1) will be called a ground state solution if it is a minimizer for (4.5.2). We then have the following analogue of Theorem 4.1.1.

Theorem 4.5.1. *Let $r \in (0,1)$, $m > -\lambda_1(\mathbf{A}_r)$ and $p \in (2, 2^*)$.*

(i) If $\alpha \in (0,1)$, then there exists a ground state solution of (4.5.1).

(ii) We have

$$\mathscr{C}_{1,m,p}(\mathbf{A}_r) = 0 \quad \text{for } p > 2_1^*, \qquad \text{and} \quad \mathscr{C}_{1,m,p}(\mathbf{A}_r) > 0 \quad \text{for } p \le 2_1^*.$$

Moreover, for any $p \in (2_1^*, 2^*)$, *there exists* $\alpha_p \in (0,1)$ *with the property that*

$$\mathscr{C}_{\alpha,m,p}(\mathbf{A}_r) < \mathscr{C}_{0,m,p}(\mathbf{A}_r) \qquad \text{for } \alpha \in (\alpha_p, 1]$$

and therefore every ground state solution of (4.5.1) is x_1-x_2*-nonradial for* $\alpha \in (\alpha_p, 1]$.

This theorem does not come as a surprise and is proved by precisely the same arguments as Theorem 4.1.1, so we omit the proof. Instead, we now discuss an interesting additional feature of the annulus $\mathbf{A}_r$. Unlike in the case of the ball, we can formulate *explicit* sufficient conditions for the parameters p, α, m and r which guarantee that every ground state solution of (4.5.1) is x_1-x_2-nonradial. This is the content of the following theorem.

Theorem 4.5.2. *Let* $N \geq 2$, $m \geq 0$, $r, \alpha \in (0,1)$ *and assume* $p > \frac{N-1-r^2\alpha^2}{\kappa(r,m)} + 2$ *with*

$$\kappa(r,m) = \begin{cases} mr^2 + \max\left\{\left(\dfrac{N-2}{2}\right)^2, \left(\dfrac{\pi}{1-r}\right)^2 r^{N-1}\right\}, & N \geq 3; \\ mr^2 + \left(\dfrac{\pi}{1-r}\right)^2 r^N, & N = 2. \end{cases} \tag{4.5.3}$$

Then every ground state solution of (4.5.1) *is* x_1-x_2*-nonradial.*

We point out that $\kappa(m,r) \to \infty$ if $m \to \infty$ or $r \to 1^-$. Consequently, for given $p > 2$, ground states of (4.5.1) are nonradial if either m is large or the annulus is thin, i.e. r is close to 1. The proof is based on the following lemma.

Lemma 4.5.3. *Suppose that* $m \geq 0$, *and that there exists a function* $v \in H_0^1(\mathbf{A}_r)$ *satisfying*

$$\int_{\mathbb{S}^{N-1}} v(s,\cdot)\, d\sigma = 0 \qquad \text{for every } s \in (r,1) \tag{4.5.4}$$

and

$$\int_{\mathbf{A}_r} \left(|\nabla v|^2 - \alpha^2 |\partial_\theta v|^2 + mv^2\right) dx - (p-1)\int_{\mathbf{A}_r} |u_0|^{p-2} v^2\, dx < 0. \tag{4.5.5}$$

Then we have

$$\mathscr{C}_{\alpha,m,p}(\mathbf{A}_r) < R_{\alpha,m,p}(u_0), \tag{4.5.6}$$

where $u_0 \in H_0^1(\mathbf{A}_r)$ *is the unique positive radial solution of* (4.5.1).

Here we note that in the case $m = 0$, the uniqueness of the positive radial solution u_0 of (4.5.1) has been first proved by Ni and Nussbaum [**109**]. In the case $m > 0$, the uniqueness is due to Tang [**129**] and Felmer, Martínez and Tanaka [**57**] for $N \geq 3$ and $N = 2$, respectively.

Proof. We argue by contradiction and assume that equality holds in (4.5.6). Then u_0 is a minimizer for $\mathscr{C}_{\alpha,m,p}(\mathbf{A}_r)$, which implies, in particular, that

$$R'_{\alpha,m,p}(u_0)v = 0 \quad \text{and} \quad R''_{\alpha,m,p}(u_0)(v,v) \geq 0. \tag{4.5.7}$$

In the following, we write $R_{\alpha,m,p} = \frac{Z(u)}{N(u)}$ for $u \in H^1(\mathbf{A}_r) \setminus \{0\}$ with

$$Z(u) := \int_{\mathbf{A}_r} \left(|\nabla u|^2 - \alpha^2|\partial_\theta u|^2 + mu^2\right) dx \quad \text{and} \quad N(u) := \left(\int_{\mathbf{A}_r} |u|^p\, dx\right)^{\frac{2}{p}}.$$

The first property in (4.5.7) then implies $N(u_0)Z'(u_0)v = Z(u_0)N'(u_0)v$ and consequently

$$N(u_0)^3 [R_{\alpha,m,p}]''(u_0)(v,v) = N(u_0)^2 Z''(u_0)(v,v) - Z(u_0)N(u_0)N''(u_0)(v,v)$$

for $v \in H_0^1(\mathbf{A}_r)$. Therefore, the second property in (4.5.7) yields

$$Z''(u_0)(v,v) - \frac{Z(u_0)}{N(u_0)} N''(u_0)(v,v) \geq 0.$$

Moreover, noting that u_0 is a weak solution of (4.5.1) and therefore $Z(u_0) = N(u_0)^{\frac{p}{2}}$, we conclude that

$$\begin{aligned}
0 \leq& \frac{1}{2}\left(Z''(u_0)(v,v) - \frac{Z(u_0)}{N(u_0)} N''(u_0)(v,v)\right) \\
=& \int_{\mathbf{A}_r} \left(|\nabla v|^2 - \alpha^2 |\partial_\theta v|^2 + m v^2\right)\, dx - (p-1) \int_{\mathbf{A}_r} |u_0|^{p-2} v^2 \, dx \\
&+ (p-2) N(u_0)^{-\frac{p}{2}} \left(\int_{\mathbf{A}_r} |u_0|^{p-2} u_0 \, v \, dx\right)^2 .
\end{aligned}$$

This, however, contradicts (4.5.5) since $\int_{\mathbf{A}} |u_0|^{p-2} u_0 \, v \, dx = 0$ by (4.5.4). The proof is thus finished. □

Proof of Theorem 4.5.2. Our goal is to construct a function that satisfies the conditions of Lemma 4.5.3. To this end, let μ_1 be the first eigenvalue of the weighted eigenvalue problem

$$\begin{cases} -w_{rr} - \dfrac{N-1}{r} + mw - (p-1)|u_0(r)|^{p-2} w = \dfrac{\mu}{r^2} w & \text{in } (r,1), \\ w(r) = w(1) = 0, \end{cases}$$

and let w the up to normalization unique positive eigenfunction. Moreover, let $Y \in C^\infty(\mathbb{S}^{N-1})$ be a spherical harmonic of degree 1 such that $\partial_\theta^2 Y = -Y$ and set $v(r,\omega) := w(r)Y(\omega)$. Then condition (4.5.4) of Lemma 4.5.3 is satisfied. By construction, v also satisfies

$$-\Delta v + \alpha^2 \partial_\theta^2 v + (m - \alpha^2) v - (p-1)|u_0|^{p-2} v = \frac{\mu_1 + N - 1}{|x|^2} v - \alpha^2 v$$

and testing this equation with v itself then yields

$$\begin{aligned}
&\int_{\mathbf{A}_r} \left(|\nabla v|^2 - \alpha^2 |\partial_\theta v|^2 + m v^2 - (p-1)|u_0|^{p-2} v^2\right)\, dx \qquad (4.5.8)\\
=&(\mu_1 + (N-1)) \int_{\mathbf{A}_r} \frac{v^2}{|x|^2}\, dx - \alpha^2 \int_{\mathbf{A}_r} v^2 \, dx \leq (\mu_1 + (N-1) - r^2\alpha^2) \int_{\mathbf{A}_r} \frac{v^2}{|x|^2}\, dx.
\end{aligned}$$

We recall that μ_1 can be characterized by

$$\mu_1 = \min_{\varphi \in H^1_{0,rad}(\mathbf{A}_r)\setminus\{0\}} \frac{\int_{\mathbf{A}_r} \left(|\nabla \varphi|^2 + m\varphi^2\right)\, dx - (p-1)\int_{\mathbf{A}_r} |u_0|^{p-2}\varphi^2 \, dx}{\int_{\mathbf{A}_r} \frac{\varphi^2}{|x|^2}\, dx}.$$

Taking $\varphi = u_0$ in this quotient, we obtain the estimate

$$\begin{aligned}
\mu_1 \leq& \frac{\int_{\mathbf{A}_r} \left(|\nabla u_0|^2 + m u_0^2\right)\, dx - (p-1)\int_{\mathbf{A}_r} |u_0|^p \, dx}{\int_{\mathbf{A}_r} \frac{u_0^2}{|x|^2}\, dx} \qquad (4.5.9)\\
=& -(p-2) \frac{\int_{\mathbf{A}_r} \left(|\nabla u_0|^2 + m u_0^2\right)\, dx}{\int_{\mathbf{A}_r} \frac{u_0^2}{|x|^2}\, dx} \leq -(p-2)\left(\frac{\int_{\mathbf{A}_r} |\nabla u_0|^2 dx}{\int_{\mathbf{A}_r} \frac{u_0^2}{|x|^2}\, dx} + mr^2\right)
\end{aligned}$$

We now distinguish the cases $N \geq 3$ and $N = 2$. If $N \geq 3$, Hardy's inequality gives

$$\int_{A_r} |\nabla u_0|^2\, dx \geq \left(\frac{N-2}{2}\right)^2 \int_{A_r} \frac{u_0^2}{|x|^2}\, dx. \tag{4.5.10}$$

Alternatively, we may also estimate, since u_0 is radial,

$$\begin{aligned}\int_{A_r} |\nabla u_0|^2\, dx &= \int_r^1 \rho^{N-1}|\partial_r u_0(\rho)|^2 d\rho \geq r^{N-1}\int_r^1 |\partial_r u_0(\rho)|^2 d\rho\\ &\geq \left(\frac{\pi}{1-r}\right)^2 r^{N-1}\int_r^1 u_0^2(\rho)d\rho \geq \left(\frac{\pi}{1-r}\right)^2 r^{N-1}\int_r^1 \rho^{N-3}u_0^2(\rho)d\rho\\ &= \left(\frac{\pi}{1-r}\right)^2 r^{N-1}\int_{A_r}\frac{u_0^2}{|x|^2}\, dx. \end{aligned}\tag{4.5.11}$$

Thus (4.5.9) gives $\mu_1 < -(p-2)\kappa(r,m)$ with $\kappa(r,m)$ given in (4.5.3) for $N \geq 3$. Inserting this into (4.5.8) yields

$$\begin{aligned}&\int_{A_r} \left(|\nabla v|^2 - \alpha^2|\partial_\theta v|^2 + mv^2 - (p-1)|u_0|^{p-2}v^2\right)\, dx\\ &< -(p-2)\kappa + N - 1 - r^2\alpha^2,\end{aligned}$$

i.e., condition (4.5.5) of Lemma 4.5.3 is satisfied if $p > \frac{N-1-r^2\alpha^2}{\kappa} + 2$, which holds by assumption.

Hence v satisfies the assumptions of Lemma 4.5.3, which implies that (4.5.6) holds and therefore every minimizer for (4.5.2) is nonradial. Let u denote such a nonradial ground state solution, and suppose by contradiction that $\partial_\theta u_0 \equiv 0$ The nonradiality of u implies that there exists an isometry $A \in O(N)$ such that $\tilde{u} := u \circ A \in H_0^1(\mathbf{A}_r)$ satisfies $\partial_\theta \tilde{u} \not\equiv 0$. Since A is an isometry, this implies

$$R_{\alpha,m,p}(\tilde{u}) = R_{\alpha,m,p}(u) - \alpha^2 \frac{\int_{A_r} |\partial_\theta \tilde{u}|^2\, dx}{\left(\int_{A_r} |u|^p\, dx\right)^{\frac{2}{p}}} < R_{\alpha,m,p}(u) = \mathscr{C}_{1,m,p}(\mathbf{A}_r),$$

which contradicts (4.5.2). Consequently, we have $\partial_\theta u_0 \not\equiv 0$, which yields that u_0 is x_1-x_2-nonradial. This finishes the proof in the case $N \geq 3$.

It remains to consider the case $N = 2$. In this case, we replace the estimates (4.5.10) and (4.5.11) by

$$\int_{A_r} |\nabla u_0|^2\, dx \geq \left(\frac{\pi}{1-r}\right)^2 r^{N-1}\int_r^1 u_0^2(\rho)d\rho \geq \left(\frac{\pi}{1-r}\right)^2 r^N \int_{A_r}\frac{u_0^2}{|x|^2}\, dx.$$

Combining this with (4.5.9) we again get $\mu_1 < -(p-2)\kappa(r,m)$ with $\kappa(r,m)$ given in (4.5.3) for $N = 2$. We may thus complete the proof as above. □

4.6 Riemannian models

So far we only used the inequality stated in Theorem 4.2.2 in the case $s = 1$. We shall now consider an application for general $s \in (0,2]$ by considering (4.1.3) on a special class of Riemannian manifolds with boundary.

Indeed, consider a Riemannian model, i.e., a Riemannian manifold (M, g) of dimension $N \geq 2$ admitting a pole $o \in M$ and whose metric is (locally) given by

$$ds^2 = dr^2 + (\psi(r))^2 d\Theta^2 \tag{4.6.1}$$

for $r > 0$, $\Theta \in \mathbb{S}^{N-1}$, where $d\Theta^2$ denotes the canonical metric on $\mathbb{S}^{N-1}$ and ψ is a smooth function that is positive on $(0,\infty)$. Moreover, we assume

$$\psi'(0) = 1 \quad \text{and} \quad \psi^{(2k)}(0) = 0 \qquad \text{for } k \in \mathbb{N}_0. \tag{4.6.2}$$

For such a Riemannian model, the associated Laplace-Beltrami operator becomes

$$\Delta_g f = \frac{1}{\psi^{N-1}}\partial_r\left(\psi^{N-1}\partial_r f\right) + \frac{1}{\psi^2}\Delta_{\mathbb{S}^{N-1}} f$$

where $\Delta_{\mathbb{S}^{N-1}}$ denotes the Laplace-Beltrami operator on $\mathbb{S}^{N-1}$. Riemannian models are of independent geometric interest, we refer to [**16**] and the references therein for a broader overview.

In the following, we focus on the case $M = \mathbf{B}$, $o = 0 \in \mathbb{R}^N$ and again study the problem

$$\begin{cases} \partial_t^2 v - \Delta_g v + mv = |v|^{p-2}v & \text{in } M \\ v = 0 & \text{on } \partial M \end{cases} \tag{4.6.3}$$

where $2 < p < \frac{2N}{N-2}$ and $m > -\lambda_1(M))$ with $\lambda_1(M)$ denoting the first Dirichlet eigenvalue of $-\Delta_g$ on M. We stress that the case $\psi(r) = r$ corresponds to the classical flat metric on $\mathbf{B}$ considered in detail in the previous sections. As a further example, the hemisphere of radius $\frac{2}{\pi}$ given by $\mathbb{S}^N_{2/\pi,+} := \{x \in \mathbb{S}^N_{2/\pi} : x_N > 0\}$ can be interpreted as a Riemannian model. Indeed, using polar coordinates $(r,\omega) \in (0,1)\times\mathbb{S}^{N-1}$, a parametrization $\mathbf{B} \to \mathbb{S}^N_{2/\pi,+}$ is given by $(r,\omega) \mapsto \frac{2}{\pi}(\sin(\frac{\pi}{2}r)\,\omega, \cos(\frac{\pi}{2}r))$. This yields a Riemannian model with $\psi(r) = \sin(\frac{\pi}{2}r)$. Similarly, Riemannian models can also be used for spherical caps.

As in the flat case, we restrict our attention to solutions of (4.6.3) of the form $v(t,x) = u(R_{\alpha t}(x))$, where $R_\theta \in O(N+1)$ denote a planar rotation in $\mathbb{R}^N$ with angle θ. We may again assume, without loss of generality, that

$$R_\theta(x) = (x_1\cos\theta + x_2\sin\theta, -x_1\sin\theta + x_2\cos\theta, x_3, \dots, x_N) \qquad \text{for } x \in \mathbb{R}^N,$$

so R_θ is the rotation in the $x_1 - x_2$-plane. This leads to the reduced equation

$$\begin{cases} -\Delta_g u + \alpha^2\partial_\theta^2 u + mu = |u|^{p-2}u & \text{in } M \\ u = 0 & \text{on } \partial M \end{cases} \tag{4.6.4}$$

with the differential operator $\partial_\theta = x_1\partial_{x_2} - x_2\partial_{x_1}$ associated to the Killing vector field $x \mapsto (-x_2, x_1, 0, \dots, 0)$ on M. We may then again study the quotient

$$R^M_{\alpha,m,p} : H^1_0(M)\setminus\{0\} \to \mathbb{R}, \qquad R^M_{\alpha,m,p}(u) := \frac{\int_M \left(|\nabla_g u|^2 - \alpha^2|\partial_\theta u|^2 + mu^2\right)\,dg}{\|u\|^2_{L^p(M)}}$$

and its minimizers, i.e.

$$\mathscr{C}_{\alpha,m,p}(M) := \inf_{u\in C^1_c(\mathbf{B})\setminus\{0\}} R^M_{\alpha,m,p}(u).$$

Analogously to Theorem 4.1.1, we can use the general inequality stated in Theorem 4.2.2 to give the following result.

Theorem 4.6.1. *Let $s \in (0,2]$, and let (M,g) be a Riemannian model with $M = \mathbf{B}$ and associated function $\psi \in C^\infty[0,1)$ satisfying (4.6.2) and*

$$c_1(1-r)^s \le 1 - \psi(r) \le c_2(1-r)^s \qquad \textit{for } r \in (0,1) \textit{ with constants } c_1, c_2 > 0. \tag{4.6.5}$$

Moreover, let $m > -\lambda_1(M)$, and let $2 < p < 2^*$.

(i) If $\alpha \in (0,1)$, then there exists a ground state solution of (4.1.5).

(ii) We have

$$\mathscr{C}_{1,m,p}(M) = 0 \quad \text{for } p > 2_s^*, \qquad \text{and} \quad \mathscr{C}_{1,m,p}(M) > 0 \quad \text{for } p \le 2_s^*. \tag{4.6.6}$$

Moreover, for any $p \in (2_s^*, 2^*)$, there exists $\alpha_p \in (0,1)$ with the property that

$$\mathscr{C}_{\alpha,m,p}(M) < \mathscr{C}_{0,m,p}(M) \qquad \text{for } \alpha \in (\alpha_p, 1]$$

and therefore every ground state solution of (4.6.4) is x_1-x_2-nonradial for $\alpha \in (\alpha_p, 1)$.

Proof. Since the proof is completely parallel to the proof of Theorem 4.1.1, we omit some details and focus our attention on showing where condition (4.6.5) enters. It is again useful to introduce polar coordinates $(r, \theta, \vartheta_1, \dots, \vartheta_{N-2}) \in U := (0,1) \times (-\pi, \pi) \times (0,\pi)^{N-2}$ given by

$$\begin{aligned}(x_1, \dots, x_N) =&(r\cos\theta \sin\vartheta_1 \cdots \sin\vartheta_{N-2},\ r\sin\theta\sin\vartheta_1 \cdots \sin\vartheta_{N-2},\\ & r\cos\vartheta_1, r\sin\vartheta_1\cos\vartheta_2, \dots, r\sin\vartheta_1 \dots \sin\vartheta_{N-3}\cos\vartheta_{N-2}, r\sin\vartheta_1 \dots \vartheta_{N-2}).\end{aligned}$$

In the following, we will abbreviate the coordinates $(\theta, \vartheta_1, \dots, \vartheta_{N-2})$ to Θ for simplicity. In these coordinates, the metric (4.6.1) becomes

$$g = dr^2 + (\psi(r))^2 \left(\sum_{i=1}^{N-2} \left(\prod_{k=1}^{i-1} \sin^2\vartheta_k \right) d\vartheta_k^2 + \left(\prod_{k=1}^{N-1} \sin^2\vartheta_k \right) d\theta^2 \right),$$

and therefore the quadratic form associated to the operator $-\Delta_g + \alpha^2\partial_\theta^2$ is given by

$$\begin{aligned}&\int_M \left(|\nabla_g u|^2 - |\partial_\theta u|^2\right) dg\\ &= \int_U \left(|\partial_r u|^2 + \frac{1}{\psi^2}\sum_{i=1}^{N-2} h_i |\partial_{\vartheta_i} u|^2 + \left(\frac{h_{N-1}}{\psi^2} - 1\right)|\partial_\theta u|^2 \right) |g| \, d(r,\Theta)\end{aligned}$$

for $u \in C_c^1(M)$ with

$$|g|(r,\Theta) = (\psi(r))^{N-1}\prod_{k=1}^{N-2}\sin^{N-1-k}\vartheta_k, \qquad h_i(r,\Theta) = \prod_{k=1}^{i-1}\frac{1}{\sin^2\vartheta_k}.$$

Moreover,

$$\int_M |u|^p \, dg = \int_U g|u|^p \, d(r,\Theta) \qquad \text{for } u \in C_c^1(M) \text{ and } p > 1.$$

Next we note that, as a consequence of (4.6.5), we have

$$|g|(\Theta_0) = 1 \quad \text{and} \quad h^i(\Theta_0) = 1 \qquad \text{for } i = 1, \dots, N \text{ with } \Theta_0 := \left(1, 0, \frac{\pi}{2}, \dots, \frac{\pi}{2}\right). \tag{4.6.7}$$

Moreover, by assumption (4.6.5), the function $\frac{h_{N-1}}{\psi^2} - 1$ satisfies

$$\tilde{c}_1\Big((1-r)^s + \sum_{k=1}^{N-2}(\vartheta_k - \frac{\pi}{2})^2\Big) \le \frac{h_{N-1}}{\psi^2}(r,\Theta) - 1 \le \tilde{c}_2\Big((1-r)^s + \sum_{k=1}^{N-2}(\vartheta_k - \frac{\pi}{2})^2\Big) \tag{4.6.8}$$

for $(r,\theta,\vartheta_1,\dots,\vartheta_{N-2}) \in U_0$ with suitable constants $\tilde c_1, \tilde c_2 > 0$, where

$$U_0 := \left(\frac{1}{2},1\right)\times(-\pi,\pi)\times\left(\frac{\pi}{4},\frac{3}{4}\pi\right)^{N-2} \subset U.$$

We now consider a fixed function $u \in C_c^1(U_0)\setminus\{0\} \subset C_c^1(U)\setminus\{0\}$, which, regarded as a function of polar coordinates, gives rise to a function in $C_c^1(M)$. For $\lambda \in (0,1)$ we consider the map

$$\Lambda_\lambda : U_0 \to U_0, \qquad (r,\Theta) \mapsto \left(1+\lambda(1-r), \lambda^{1+\frac{s}{2}}\theta, \frac{\pi}{2}+\lambda\left(\vartheta_1-\frac{\pi}{2}\right),\dots,\frac{\pi}{2}+\lambda\left(\vartheta_2-\frac{\pi}{2}\right)\right),$$

and we define $u_\lambda := u\circ\Lambda_\lambda^{-1} \in C_c^1(U_0)\setminus\{0\}$ for $\lambda\in(0,1)$.

Using (4.6.7) and (4.6.8), we find that

$$\begin{aligned}\lambda^{-\frac{2N+s}{p}}\left(\int_U g|u_\lambda|^p\,d(r,\Theta)\right)^{\frac{2}{p}} &= \left(\int_U g\circ\Lambda_\lambda|u|^p\,d(r,\Theta)\right)^{\frac{2}{p}}\\ &\to \left(\int_U |u|^p\,d(r,\Theta)\right)^{\frac{2}{p}} =: c_u(p)\end{aligned}$$

as $\lambda\to0^+$ and

$$\begin{aligned}&\limsup_{\lambda\to0^+}\lambda^{2-\frac{s}{2}-N}\int_U\left(|\partial_r u_\lambda|^2+\frac{1}{\psi^2}\sum_{i=1}^{N-2}h_i|\partial_{\vartheta_i}u_\lambda|^2+\left(\frac{h_{N-1}}{\psi^2}-1\right)|\partial_\theta u_\lambda|^2\right)|g|\,d(r,\Theta)\\ =&\limsup_{\lambda\to0^+}\int_U\left(|\partial_r u|^2+\frac{1}{\psi^2}\circ\Lambda_\lambda\sum_{i=1}^{N-2}h_i\circ\Lambda_\lambda|\partial_{\vartheta_i}u|^2+\frac{1}{\lambda^s}\Big(\frac{h_{N-1}}{\psi^2}\circ\Lambda_\lambda-1\Big)|\partial_\theta u|^2\right)|g|\circ\Lambda\,d(r,\Theta)\\ \le& d_u^1+d_u^2\end{aligned}\tag{4.6.9}$$

with

$$d_u^1 := \int_U\left(|\partial_r u|^2+\sum_{i=1}^{N-2}|\partial_{\vartheta_i}u|^2\right)d(r,\Theta)$$

and

$$\begin{aligned}d_u^2 &= \tilde c_2\limsup_{\lambda\to0^+}\int_U\left((1-r)^s+\lambda^{2-s}\sum_{k=1}^{N-2}\big(\vartheta_k-\frac{\pi}{2}\big)^2\right)|\partial_\theta u|^2 d(r,\Theta)\\ &= \begin{cases}\tilde c_2\displaystyle\int_U(1-r)^s|\partial_\theta u|^2\,d(r,\Theta), & s\in(0,2),\\ \tilde c_2\displaystyle\int_U\left((1-r)^2+\sum_{i=1}^{N-2}\big(\vartheta_k-\frac{\pi}{2}\big)^2\right)|\partial_\theta u|^2\,d(r,\Theta), & s=2.\end{cases}\end{aligned}$$

It thus follows that

$$\mathscr{C}_{1,m,p}(M) \le \limsup_{\lambda\to0^+}R^M_{1,m,p}(u_\lambda) = \limsup_{\lambda\to0^+}\frac{\lambda^{N+\frac{s}{2}-2}(d_u^1+d_u^2)+\lambda^{\frac{2N+s}{2}}c_u(2)}{\lambda^{\frac{2N+s}{p}}c_u(p)} = 0 \quad \text{if } p > 2_s^*.$$

This shows the first identity in (4.6.6). To see the second identity in (4.6.6), we argue as in Section 4.3. More precisely, we first note that it is sufficient to consider the case $p = 2_s^*$, and then we show the inequality

$$\left(\int_U g|u|^{2_s^*}\,d(r,\Theta)\right)^{\frac{2}{2_s^*}} \le C\int_U\left(|\partial_r u|^2+\frac{1}{\psi^2}\sum_{i=1}^{N-2}h_i|\partial_{\vartheta_i}u|^2+\left(\frac{h_{N-1}}{\psi^2}-1\right)|\partial_\theta u|^2\right)|g|\,d(r,\Theta)$$

for functions $u \in C_c^1(U_0)$ with a suitable constant $C > 0$. For this, we use Theorem 4.1.6 and the first inequality in (4.6.8). The argument is then completed by using the rotation invariance of the problem and a partition of unity argument to localize the problem. □

Remark 4.6.2. *(i) In the case of a hemisphere mentioned earlier, i.e. $\psi(r) = \frac{2}{\pi}\sin(\frac{\pi}{2}r)$, Theorem 4.6.1 yields nonradial ground state solutions for $p > 2_s^* = \frac{2(N+1)}{N-1}$. Notably, this corresponds to the critical exponent for generalized travelling waves on the sphere $\mathbb{S}^N$ found in [**103**, **104**, **130**]. In fact, our approach based on Theorem 4.1.6 can be used to give an alternative proof for the existence of nontrivial solutions and the embeddings stated in [**130**, Proposition 3.2] and [**104**, Proposition 1.2 + Lemma 1.3].*

(ii) Theorem 4.6.1 leaves open the case $s > 2$. Note that the two-sided estimate (4.6.8) *needs to be analyzed more carefully if $s > 2$ and $N \geq 3$, as the leading order term is then 2 in place of s. In this case, if* (4.6.5) *holds for some $s > 2$, the conclusion will instead be*

$$\mathscr{C}_{1,m,p}(M) = 0 \quad \text{for } p > 2_2^*, \qquad \text{and} \quad \mathscr{C}_{1,m,p}(M) > 0 \quad \text{for } p \leq 2_2^*.$$

For $N = 2$, on the other hand, (4.6.8) *suggests that Theorem 4.6.1 also holds for $s > 2$.*

4.7 Boundedness of solutions

In the proof of the regularity properties of ground states in the case $\alpha = 1$ stated in Proposition 4.4.8, we used the following:

Lemma 4.7.1. *Let $2 < p < 2_1^*$, $m > -\lambda_1$ and let $u \in \mathcal{H}$ be a weak solution of*

$$-\Delta u + \partial_\theta^2 u + mu = |u|^{p-2}u \quad \text{in } \mathbf{B}. \tag{4.7.1}$$

Then $u \in L^\infty(\mathbf{B})$. Furthermore, there exist constants $C = C(N, m), \sigma > 0$ such that

$$|u|_\infty \leq C\|u\|_{\mathcal{H}}^\sigma. \tag{4.7.2}$$

For $m \geq 0$, the constant $C = C(N) > 0$ can be chosen independent of m.

Proof. The proof is based on Moser iteration, cf. [**125**, Appendix B] and the references therein.

We fix $L, s \geq 2$ and consider auxiliary functions $h, g \in C^1([0,\infty))$ defined by

$$h(t) := s\int_0^t \min\{\tau^{s-1}, L^{s-1}\}\, d\tau \qquad \text{and} \qquad g(t) := \int_0^t [h'(\tau)]^2\, d\tau$$

We note that

$$h(t) = t^s \quad \text{for } t \leq L \qquad \text{and} \qquad g(t) \leq tg'(t) = t(h'(t))^2 \quad \text{for } t \geq 0, \tag{4.7.3}$$

since the function $t \mapsto h'(t) = s\min\{t^{s-1}, L^{s-1}\}$ is nondecreasing. We shall now show that $w := u^+ \in L^\infty(\mathbf{B})$, and that $\|w\|_\infty$ is bounded by the right hand side of (4.7.2). Since we may replace u with $-u$, the claim will then follow.

We note that $w \in \mathcal{H}$ and $\varphi := g(w) \in \mathcal{H}$ with

$$\nabla w = \mathbb{1}_{\{u>0\}}\nabla u, \qquad \nabla\varphi = g'(w)\nabla w, \qquad \partial_\theta w = \mathbb{1}_{\{u>0\}}\partial_\theta u, \qquad \partial_\theta\varphi = g'(w)\partial_\theta w.$$

This follows from the boundedness of g' and the estimate $g(t) \leq s^2t^{2s-1}$ for $t \geq 0$. Testing (4.7.1) with φ gives

$$\int_{\mathbf{B}}\Big(\nabla u \cdot \nabla\varphi - (\partial_\theta u\, \partial_\theta\varphi) + mu\varphi\Big)dx = \int_{\mathbf{B}} |u|^{p-2}u\varphi\, dx,$$

from where we estimate

$$\begin{aligned}\int_{\mathbf{B}}\Big(|\nabla h(w)|^2-(\partial_\theta h(w))^2+mwg(w)\Big)dx&=\int_{\mathbf{B}}\Big(g'(w)\Big(|\nabla w|^2-(\partial_\theta w)^2\Big)+mug(w)\Big)dx\\&=\int_{\mathbf{B}}|u|^{p-2}ug(w)\,dx\\&\le\int_{\mathbf{B}}w^p(h'(w))^2\,dx.\end{aligned}\tag{4.7.4}$$

Here we used (4.7.3) in the last step. We now fix $r>1$ with $\frac{(p-2)r}{r-1}\ge 2$ and $q>4r$. Combining (4.7.4) with Proposition 4.4.1 and Theorem 4.3.3, we obtain the inequality

$$|h(w)|_{p^*}^2\le c_0\int_{\mathbf{B}}w^p(h'(w))^2\,dx\tag{4.7.5}$$

with a constant $c_0=c_0(N,m)>0$. Note that for $m\ge 0$, c_0 only depends on N. Since

$$h(t)=t^s,\quad h'(t)=st^{s-1}\quad\text{and}\quad g(t)=s^2\int_0^t\tau^{2s-2}\,d\tau=\frac{s^2}{2s-1}t^{2s-1}\qquad\text{for } t\le L,$$

we may let $L\to\infty$ in (4.7.5) and apply Lebesgue's theorem to obtain

$$|w^s|_{p^*}^2\le c_0s^2\int_{\mathbf{B}}w^{p+2s-2}\,dx\le c_0s^2|w|_{p^*}^{p-2}|w|_{2sq}^{2s},$$

where $q=\frac{p^*}{p^*-p+2}$ is the the conjugated exponent to $\frac{p^*}{p-2}$. This yields

$$|w|_{sp^*}\le(c_1s)^{\frac{1}{s}}|w|_{2sq}\qquad\text{with } c_1:=\Big(c_0|w|_{p^*}^{p-2}\Big)^{\frac12},\tag{4.7.6}$$

whenever $w\in L^{2sq}(\mathbf{B})$. We now consider $s=s_n=\rho^n$ for $n\in\mathbb{N}$ with $\rho:=\frac{p^*}{2q}=\frac{2+p^*-p}{2}>1$, so that

$$2s_1q=p^*\qquad\text{and}\qquad 2s_{n+1}q=s_np^*\quad\text{for } n\in\mathbb{N}.$$

Iteration of (4.7.6) then gives

$$|w|_{\rho^np^*}=|w|_{s_np^*}\le|w|_{p^*}\prod_{j=1}^n(c_1\rho^j)^{\rho^{-j}}\le c_1^{\frac{\rho}{\rho-1}}c_2|w|_{p^*}$$

for all n with

$$c_2:=\rho^{\sum\limits_{j=1}^{\infty}j\rho^{-j}}<\infty.$$

It follows that

$$|w|_\infty=\lim_{n\to\infty}|w|_{\rho^np^*}\le c_1^{\frac{\rho}{\rho-1}}c_2|w|_{p^*}.\tag{4.7.7}$$

Moreover, by Theorem 4.3.3, we have

$$c_1\le c_1'\|w\|_{\mathcal{H}}^{\frac{p-2}{2}}\le c_1'\|u\|_{\mathcal{H}}^{\frac{p-2}{2}}\qquad\text{and}\qquad|w|_q\le\tilde c\|w\|_{\mathcal{H}}\le\tilde c\|u\|_{\mathcal{H}}$$

with constants $c_1',\tilde c>0$ depending only on N. It thus follows from (4.7.7) that

$$|w|_\infty\le C\|u\|_{\mathcal{H}}^{\frac{(p-2)\rho}{2(\rho-1)}+1}\qquad\text{with}\qquad C:=c_2(c_1')^{\frac{\rho}{\rho-1}}\tilde c.$$

The proof is thus finished. □

CHAPTER 5

On the Spectrum of a Mixed-Type Operator with Applications to Rotating Wave Solutions

In this chapter, we present our results on the mixed-type operator appearing in the study of rotating waves for $\alpha > 1$ as outlined in Section 1.5. Up to minor changes, the contents have appeared in **[P4]**.

5.1 Introduction

We consider time-periodic solutions of the nonlinear wave equation

$$\begin{cases} \partial_t^2 v - \Delta v + mv = |v|^{p-2}v & \text{in } \mathbb{R} \times \mathbf{B} \\ v = 0 & \text{on } \mathbb{R} \times \partial \mathbf{B} \end{cases} \tag{5.1.1}$$

where $2 < p < \infty$, $m \in \mathbb{R}$ and $\mathbf{B} \subset \mathbb{R}^2$ denotes the unit disk. In the case $m > 0$, this is also commonly referred to as a nonlinear Klein-Gordon equation. A well-known class of such solutions is given by standing wave solutions, which reduce (5.1.1) either to a stationary nonlinear Schrödinger or a nonlinear Helmholtz equation and have been studied extensively on the whole space $\mathbb{R}^N$, see [**55**, **124**]. Note that this yields complex-valued solutions whose amplitude remains stationary, however, while other types of time-periodic solutions are significantly less well understood. In particular, much less is known about the dynamics of nonlinear wave equations in general bounded domains.

In the one-dimensional setting, which typically describes the forced vibrations of a nonhomogeneous string, the existence of time-periodic solutions satisfying either Dirichlet or periodic boundary conditions has been treated in the seminal works of Rabinowitz [**116**] and Brézis, Coron and Nirenberg [**22**] by variational methods, but the results in higher dimensions are more sparse. On balls centered at the origin, the existence of radially symmetric time-periodic solutions was first studied by Ben-Naoum and Mawhin [**13**] for sublinear nonlinearities and subsequently received further attention, see e.g. the recent works of Chen and Zhang [**36**–**38**] and the references therein.

In this paper, we study *rotating wave solutions* as introduced in **[P3]**, which are time-periodic real-valued solutions of (5.1.1) given by the ansatz

$$v(t,x) = u(R_{\alpha t}(x)), \tag{5.1.2}$$

where $R_\theta \in O(2)$ describes a rotation in $\mathbb{R}^2$ with angle $\theta > 0$, i.e.,

$$R_\theta(x) = (x_1 \cos\theta + x_2 \sin\theta, -x_1 \sin\theta + x_2 \cos\theta) \qquad \text{for } x \in \mathbb{R}^2. \tag{5.1.3}$$

In particular, the constant $\alpha > 0$ in (5.1.2) is the angular velocity of the rotation. Consequently, such solutions can be interpreted as rotating waves in a nonlinear medium. We note that a related ansatz for generalized traveling waves on manifolds has also been considered in [**103**, **104**, **130**], while a class of spiral shaped solutions for a nonlinear Schrödinger equation on $\mathbb{R}^3$ has been treated in **[P2]**.

In the following, we let θ denote the angular variable in two-dimensional polar coordinates and note that the ansatz (5.1.2) reduces (5.1.1) to

$$\begin{cases} -\Delta u + \alpha^2 \partial_\theta^2 u + mu = |u|^{p-2}u & \text{in } \mathbf{B} \\ u = 0 & \text{on } \partial\mathbf{B} \end{cases} \tag{5.1.4}$$

where $\partial_\theta = x_1\partial_{x_2} - x_2\partial_{x_1}$ then corresponds to the angular derivative. Note that this equation has solutions which are independent of θ, but these correspond to stationary and therefore non-rotating solutions of (5.1.1). In the following, our goal is to prove the existence of nonradial, i.e., θ-dependent, solutions of (5.1.4).

In the case $\alpha \leq 1$, this question has been studied in great detail in **[P3]**, where a connection to degenerate Sobolev inequalities is explored. In particular, it has been observed that the ground states, i.e., minimizers of the associated Rayleigh quotient, are nonradial in certain parameter regimes for p and α.

The main purpose of the present paper is the study of nonradial solutions of (5.1.4) for $\alpha > 1$. However, the direct variational methods employed in **[P3]** cannot be extended to this case since the operator

$$L_\alpha := -\Delta + \alpha^2 \partial_\theta^2$$

is neither elliptic nor degenerate elliptic, and the associated Rayleigh quotient becomes unbounded from below, see **[P3]**. Indeed, note that in polar coordinates $(r, \theta) \in (0, 1) \times (-\pi, \pi)$ we have

$$L_\alpha u = -\partial_r^2 u - \frac{1}{r}\partial_r u - \left(\frac{1}{r^2} - \alpha^2\right)\partial_\theta^2 u$$

and hence the operator is in fact of mixed-type for $\alpha > 1$: It is elliptic in the smaller ball $B_{1/\alpha}(0)$ of radius $1/\alpha$, parabolic on the sphere of radius $1/\alpha$ and hyperbolic in the annulus $\mathbf{B} \setminus \overline{B_{1/\alpha}(0)}$. In general, such operators are difficult to deal with via variational methods, and instead results often rely on separate treatments of the different regions of specific type and then gluing the solutions together, see e.g. [**101**, **110**] for more details.

From a functional analytic viewpoint, the quadratic form associated to L_α is strongly indefinite, i.e., it is negative on an infinite-dimensional subspace. Classically, related problems have been treated for operators of the form $-\Delta - E$ on $\mathbb{R}^N$ where $E \in \mathbb{R}$ lies in a spectral gap of the Laplacian. In this direction, we mention the use of a dual variational framework in order to prove the existence of nonzero solutions of a nonlinear stationary Schrödinger equation in [**2**], as well as abstract operator theoretic methods used in [**25**] for a related problem. However, both of these exemplary approaches require specific assumptions regarding spectral properties of the associated operator. Moreover, the sole existence of nonzero solutions to (5.1.4) is insufficient in our case since we are interested in nontrivial *rotating* wave solutions.

In the present case of problem (5.1.4), a main obstruction, in addition to the unboundedness of the spectrum of the linear operator L_α from above and below, is the possible existence of finite accumulation points of this spectrum. As a first step, we therefore analyze the spectrum of L_α in detail, which is closely related to the spectrum of the Laplacian and thus the zeros of Bessel functions. In fact, the Dirichlet eigenvalues of L_α are given by

$$j_{\ell,k}^2 - \alpha^2\ell^2,$$

where $\ell \in \mathbb{N}_0$, $k \in \mathbb{N}$ and $j_{\ell,k}$ denotes the k-th zero of the Bessel function of the first kind J_ℓ. In particular, the structure of the spectrum heavily depends on the asymptotic behavior of the zeros of these Bessel functions. Despite this explicit characterization, it is not clear whether the spectrum of L_α only consists of isolated points. Indeed, known results on the asymptotics of the zeros of Bessel functions turn out to be insufficient to exclude accumulation points or even density in $\mathbb{R}$. In fact, similar spectral issues arise in the study of radially symmetric time-periodic solutions of (5.1.1) on balls $B_a(0)$, where the spectral properties of the radial wave operator are intimately connected to the arithmetic properties of the ratio between the radius $a > 0$ and the period length, see e.g. [**17**, **95**] and the references therein for more details.

This turns out to be a serious obstruction for the use of variational methods and thus necessitates a detailed analysis of the asymptotic behavior of different sequences of zeros. Our first main result then characterizes the spectrum of L_α as follows.

Theorem 5.1.1. *For any $\alpha > 1$ the spectrum of L_α is unbounded from above and below. Moreover, there exists an unbounded sequence $(\alpha_n)_n \subset (1, \infty)$ such that the following properties hold for $n \in \mathbb{N}$:*

(i) The spectrum of L_{α_n} consists of eigenvalues with finite multiplicity.

(ii) There exists $c_n > 0$ such that for each $\ell \in \mathbb{N}_0$, $k \in \mathbb{N}$ we either have $j_{\ell,k}^2 - \alpha_n^2 \ell^2 = 0$ or

$$|j_{\ell,k}^2 - \alpha_n^2 \ell^2| \geq c_n j_{\ell,k}. \tag{5.1.5}$$

(iii) The spectrum of L_{α_n} has no finite accumulation points.

The proof of this result is based on the observation that the formula

$$j_{\ell,k}^2 - \alpha^2 \ell^2 = (j_{\ell,k} + \alpha\ell)\ell \left(\frac{j_{\ell,k}}{\ell} - \alpha \right)$$

implies that for any unbounded sequences $(\ell_i)_i$, $(k_i)_i$, the corresponding sequence of eigenvalues $j_{\ell_i,k_i}^2 - \alpha^2 \ell_i^2$ can only remain bounded if

$$\frac{j_{\ell_i,k_i}}{\ell_i} - \alpha \to 0 \tag{5.1.6}$$

as $i \to \infty$. It turns out that (5.1.6) can only hold if $\ell_i / k_i \to \sigma$, where $\sigma = \sigma(\alpha) > 0$ is uniquely determined and can be characterized via a transcendental equation. This motivates a more detailed investigation of $j_{\sigma k,k}$, $k \in \mathbb{N}$ which gives rise to a new estimate for $j_{\ell,k}$, $\ell \in \mathbb{N}_0$, $k \in \mathbb{N}$, see Lemma 5.3.3 and Remark 5.3.4 below. In order to estimate arbitrary sequences in (5.1.6), we are then forced to restrict the problem to velocities $\alpha = \alpha_n$ such that the associated values $\sigma_n = \sigma(\alpha_n)$ are suitable rational numbers. The fact that such a restriction is necessary is not surprising when compared to similar properties observed for the radial wave operator as mentioned above.

Theorem 5.1.1 then plays a central role in the formulation of a variational framework for (4.1.5) and allows us to recover sufficient regularity properties for L_{α_n}. More specifically, for $\alpha = \alpha_n$ we may then define a suitable Hilbert space $E_{\alpha,m}$ whose norm is related to the quadratic form

$$u \mapsto \int_{\mathrm{B}} \left(|\nabla u|^2 - \alpha^2 |\partial_\theta u|^2 + m u^2 \right) \, dx,$$

see Section 5.5 below for details. The space $E_{\alpha,m}$ admits a decomposition of the form

$$E_{\alpha,m} = E_{\alpha,m}^+ \oplus F_{\alpha,m},$$

where the spaces $E^+_{\alpha,m}$ and $F_{\alpha,m}$ essentially correspond to the eigenspaces of positive and nonpositive eigenvalues of $-\Delta + \alpha^2\partial_\theta^2 + m$, respectively. Crucially, the estimate (5.1.5) and fractional Sobolev embeddings allow us to deduce that $E_{\alpha,m}$ compactly embeds into $L^p(\mathbf{B})$ for $p \in (2,4)$.

We may then find solutions of (5.1.4) as critical points of the associated energy functional $\Phi_{\alpha,m} : E_{\alpha,m} \to \mathbb{R}$ given by

$$\Phi_{\alpha,m}(u) := \frac{1}{2}\int_{\mathbf{B}} \left(|\nabla u|^2 - \alpha^2|\partial_\theta u|^2 + mu^2\right)\, dx - \frac{1}{p}\int_{\mathbf{B}} |u|^p\, dx.$$

Due to the strongly indefinite nature of (5.1.4), $\Phi_{\alpha,m}$ is unbounded from above and below and does not possess a mountain pass structure so, in particular, the classical mountain pass theorem and its variants are not applicable. Instead, we consider the generalized Nehari manifold introduced by Pankov [**112**]

$$\mathcal{N}_{\alpha,m} := \left\{u \in E_{\alpha,m} \setminus F_{\alpha,m} :\ \Phi'_{\alpha,m}(u)u = 0 \text{ and } \Phi'_{\alpha,m}(u)v = 0 \text{ for all } v \in F_{\alpha,m}\right\}.$$

Using further abstract results due to Szulkin and Weth [**128**], we can then show that

$$c_{\alpha,m} = \inf_{u\in\mathcal{N}_{\alpha,m}} \Phi_{\alpha,m}(u)$$

is positive and attained by a critical point of $\Phi_{\alpha,m}$ for $\alpha = \alpha_n$ as in Theorem 5.1.1 and $m \in \mathbb{R}$. In particular, such a minimizer then necessarily has minimal energy among all critical points of $\Phi_{\alpha,m}$, and is therefore referred to as a *ground state solution* or *ground state* of (5.1.4).

In general, it is not clear whether such a ground state is nonradial. Our second main result further states that (5.1.4) has nonradial ground state solutions for certain choices of parameters.

Theorem 5.1.2. *Let $p \in (2,4)$ and let the sequence $(\alpha_n)_n \subset (1,\infty)$ be given by Theorem 5.1.1. Then the following properties hold:*

(i) For any $n \in \mathbb{N}$ and $m \in \mathbb{R}$ there exists a ground state solution of (5.1.4) *for $\alpha = \alpha_n$.*

(ii) For any $n \in \mathbb{N}$ there exists $m_n > 0$ such that the ground state solutions of (5.1.4) *are nonradial for $\alpha = \alpha_n$ and $m > m_n$.*

In fact, we can prove a slightly more general result in the sense that the statement of Theorem 5.1.2 holds whenever the kernel of L_α is finite-dimensional and an inequality of the form (5.1.5) holds. The proof is essentially based on an energy comparison, noting that the minimal energy of the unique positive radial solution can be estimated from below in terms of m. Using a minimax characterization of $c_{\alpha,m}$, we can then show that this ground state energy grows slower than the radial energy as $m \to \infty$.

Throughout the paper, we only consider real-valued solutions and consequently let all function spaces be real. Nonradial complex-valued solutions, on the other hand, can be found much more easily using constrained minimization over suitable eigenspaces. This technique has been applied to a related problem in [**130**]. We point out, however, that the modulus of such solutions is necessarily radial, while Theorem 5.1.2 yields genuinely rotating with nonradial modulus. With our methods, by combining (5.1.2) with a standing wave ansatz, we can also prove the existence of genuinely complex-valued ground states with nonradial modulus, see the appendix of this paper.

The paper is organized as follows. In Section 5.2, we introduce Sobolev spaces via their spectral characterization and collect several known results on the properties of the zeros of Bessel functions. In Section 5.3 we then prove a crucial technical estimate for certain sequences of such zeros. This result is subsequently used in Section 5.4 to investigate the asymptotics of the zeros of Bessel functions in detail and, in particular, prove Theorem 5.1.1. Section 5.5 is then devoted to the rigorous formulation of the variational framework outlined earlier and the proof of Theorem 5.1.2. In Appendix 5.6, we discuss the results for complex-valued solutions mentioned above.

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5.2 Preliminaries

We first collect some general facts on eigenvalues and eigenfunctions of the Laplacian on $\mathbf{B}$, we refer to [**67**] for a more comprehensive overview. Recall that the eigenvalues of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \mathbf{B} \\ u = 0 & \text{on } \partial\mathbf{B} \end{cases}$$

are given by $j_{\ell,k}^2$, where $j_{\ell,k}$ denotes the k-th zero of the Bessel function of the first kind J_ℓ with $k \in \mathbb{N}_0$, $l \in \mathbb{N}$. To each eigenvalue $j_{\ell,k}^2$ correspond two linearly independent eigenfunctions

$$\begin{aligned} \varphi_{\ell,k}(r,\theta) &:= A_{\ell,k}\cos(\ell\theta)J_\ell(j_{\ell,k}r) \\ \psi_{\ell,k}(r,\theta) &:= B_{\ell,k}\sin(\ell\theta)J_\ell(j_{\ell,k}r), \end{aligned} \tag{5.2.1}$$

where the constants $A_{\ell,k}, B_{\ell,k} > 0$ are chosen such that $\|\varphi_{\ell,k}(r,\theta)\|_2 = \|\psi_{\ell,k}(r,\theta)\|_2 = 1$. These functions constitute an orthonormal basis of $L^2(\mathbf{B})$ and we can then characterize Sobolev spaces as follows:

$$H_0^1(\mathbf{B}) := \left\{ u \in L^2(\mathbf{B}) : \|u\|_{H^1}^2 := \sum_{\ell=0}^{\infty}\sum_{k=1}^{\infty} j_{\ell,k}^2 \left(|\langle u, \varphi_{\ell,k}\rangle|^2 + |\langle u, \psi_{\ell,k}\rangle|^2 \right) < \infty \right\}.$$

It can be shown that this is consistent with the usual definition of $H^1(\mathbf{B})$. By classical Sobolev embeddings, $H_0^1(\mathbf{B})$ compactly maps into $L^p(\mathbf{B})$ for any $1 \le p < \infty$.

Similarly, we consider the fractional Sobolev spaces

$$H_0^s(\mathbf{B}) := \left\{ u \in L^2(\mathbf{B}) : \|u\|_{H^s}^2 := \sum_{\ell=0}^{\infty}\sum_{k=1}^{\infty} j_{\ell,k}^{2s} \left(|\langle u, \varphi_{\ell,k}\rangle|^2 + |\langle u, \psi_{\ell,k}\rangle|^2 \right) < \infty \right\}$$

for $s \in (0,1)$. Using interpolation, it can be shown that this is equivalent to the classical definition and $H_0^s(\mathbf{B})$ compactly maps into $L^p(\mathbf{B})$ for $p < \frac{2}{1-s}$, i.e., there exists $C_s > 0$ such that

$$\|u\|_p \le C_s \|u\|_{H_0^s(\mathbf{B})}$$

holds for $u \in H_0^1(\mathbf{B})$.

Next, we collect several results on the properties of zeros Bessel functions, see e.g. [**48**] for a more extensive overview. In the following, we let $j_{\nu,k}$ denote the k-th zero of the Bessel function J_ν, where $\nu \ge 0$, $k \in \mathbb{N}$. By definition, $j_{\nu,k} < j_{\nu,k+1}$.

Proposition 5.2.1. *For each fixed $k \in \mathbb{N}$, $j_{\nu,k}$ is increasing with respect to ν. Moreover, the following properties hold:*

*(i) ([**115**]) We have*

$$\nu + \frac{|a_k|}{2^{\frac{1}{3}}}\nu^{\frac{1}{3}} < j_{\nu,k} < \nu + \frac{|a_k|}{2^{\frac{1}{3}}}\nu^{\frac{1}{3}} + \frac{3}{20}|a_k|^2\frac{2^{\frac{1}{3}}}{\nu^{\frac{1}{3}}}$$

where a_k denotes the k-th negative zero of the Airy function $\mathrm{Ai}(x)$.

*(ii) ([**96**]) For each fixed $k \in \mathbb{N}$ the map*

$$\nu \mapsto \frac{j_{\nu,k}}{\nu}$$

is strictly decreasing on $(0, \infty)$.

*(iii) ([**51**]) For $k \in \mathbb{N}$ it holds that*

$$\pi k - \frac{\pi}{4} < j_{0,k} \leq \pi k - \frac{\pi}{4} + \frac{1}{8\pi(k - \frac{1}{4})}.$$

*(iv) ([**49**]) For each fixed $k \in \mathbb{N}$ the map $\nu \mapsto j_{\nu,k}$ is differentiable on $(0, \infty)$ and*

$$\frac{dj_{\nu,k}}{d\nu} \in \left(1, \frac{\pi}{2}\right)$$

for $\nu \geq 0$.

The zeros of the Airy function can in turn be estimated (see [**21**]) by

$$\left(\frac{3\pi}{8}(4k - 1.4)\right)^{\frac{2}{3}} < |a_k| < \left(\frac{3\pi}{8}(4k - 0.965)\right)^{\frac{2}{3}}$$

for $k \in \mathbb{N}$, which yields the following result:

Corollary 5.2.2. *Let $j_{\nu,k} \in \mathbb{R}$ be defined as above. Then*

$$\nu + \frac{\left(\frac{3\pi}{8}(4k - 2)\right)^{\frac{2}{3}}}{2^{\frac{1}{3}}}\nu^{\frac{1}{3}} < j_{\nu,k} < \nu + \frac{\left(\frac{3\pi}{2}k\right)^{\frac{2}{3}}}{2^{\frac{1}{3}}}\nu^{\frac{1}{3}} + \frac{3}{20}\left(\frac{3\pi}{2}k\right)^{\frac{4}{3}}\frac{2^{\frac{1}{3}}}{\nu^{\frac{1}{3}}}.$$

5.3 Asymptotics of the Zeros of Bessel Functions

In order to study L_α in Section 5.4, we will be particularly interested in the asymptotics of the zeros $j_{\nu,k}$ when the ratio ν/k remains fixed. For this case, we note the following result by Elbert and Laforgia:

Theorem 5.3.1. *([**50**])*
Let $x > -1$ be fixed. Then

$$\lim_{k\to\infty} \frac{j_{xk,k}}{k} =: \iota(x)$$

exists. Moreover, $\iota(x)$ is given by

$$\iota(x) = \begin{cases} \pi, & x = 0, \\ \frac{x}{\sin\varphi} & x \neq 0 \end{cases}$$

where $\varphi = \varphi(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ denotes the unique solution of

$$\frac{\sin\varphi}{\cos\varphi - (\frac{\pi}{2} - \varphi)\sin\varphi} = \frac{x}{\pi}. \tag{5.3.1}$$

Moreover, we note the following properties of a function associated to ι.

Lemma 5.3.2. *The map*

$$f : (0, \infty) \to \mathbb{R}, \qquad f(x) = \frac{\iota(x)}{x}$$

is strictly decreasing and satisfies

$$\lim_{x \to 0} f(x) = \infty, \qquad \lim_{x \to \infty} f(x) = 1.$$

Moreover, its inverse is explicitly given by

$$f^{-1} : (1, \infty) \to \mathbb{R}, \qquad f^{-1}(y) = \frac{\pi}{\sqrt{y^2 - 1} - \left(\frac{\pi}{2} - \arcsin \frac{1}{y}\right)}.$$

Proof. Note that the left hand side of (5.3.1) is strictly increasing with respect to φ, and the right hand side is strictly increasing with respect to x, so that φ is necessarily an increasing function of x. In particular, we then have $f(x) = \frac{\iota(x)}{x} = \frac{1}{\sin \varphi}$ which clearly implies the monotonicity of f.

Next, we note that $y = f(x) = \frac{1}{\sin \varphi}$ implies $\varphi = \arcsin \frac{1}{y}$ and hence

$$\frac{x}{\pi} = \frac{\frac{1}{y}}{\cos\left(\arcsin \frac{1}{y}\right) - (\frac{\pi}{2} - \arcsin \frac{1}{y})\frac{1}{y}}.$$

The identity $\cos(\arcsin(t)) = \sqrt{1 - t^2}$ then gives

$$\frac{x}{\pi} = \frac{\frac{1}{y}}{\sqrt{1 - \frac{1}{y^2}} - (\frac{\pi}{2} - \arcsin \frac{1}{y})\frac{1}{y}} = \frac{1}{\sqrt{y^2 - 1} - \left(\frac{\pi}{2} - \arcsin \frac{1}{y}\right)}$$

and thus the claim follows. □

In order to characterize the eigenvalues of L_α later on, we need more information on the order of convergence in Theorem 5.3.1. To this end, we first recall some ingredients of the proof of this result. By the Watson integral formula [**134**, p. 508], for fixed $k \in \mathbb{N}$ the function $\nu \mapsto j_{\nu,k}$ satisfies

$$\frac{d}{d\nu} j_{\nu,k} = 2 j_{\nu,k} \int_0^\infty K_0(2 j_{\nu,k} \sinh(t)) e^{-2\nu t}\, dt,$$

where K_0 denotes the modified Bessel function of the second kind of order zero. It then follows that the function

$$\iota_k(x) := \frac{j_{kx,k}}{k}$$

satisfies

$$\frac{d}{dx}\iota_k(x) = 2\iota_k \int_0^\infty K_0\left(t 2\iota_k \frac{\sinh\left(\frac{t}{k}\right)}{\left(\frac{t}{k}\right)}\right) e^{-2xt}\, dt =: F_k(\iota_k, x) \tag{5.3.2}$$

for $k \in \mathbb{N}$ and $x \in (-1, \infty)$. In [**50**] it is then shown that ι_k converges pointwise to the solution of

$$\begin{cases} \dfrac{d}{dx}\iota(x) = 2\iota \displaystyle\int_0^\infty K_0\left(t 2\iota\right) e^{-2xt}\, dt =: G(\iota, x) \\ \iota(0) = \pi, \end{cases} \tag{5.3.3}$$

which is precisely given by the function ι discussed in Theorem 5.3.1. Moreover, it is shown that

$$\iota_k(x) < \iota(x) \tag{5.3.4}$$

holds for all $k \in \mathbb{N}$.

We now give a more precise characterization of this convergence in the case $x > 0$.

Lemma 5.3.3. *For any $x > 0$ and $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that*

$$-\exp\left(\left(\frac{1}{3}+\varepsilon\right)x\right)\frac{\pi}{4k} \le \frac{j_{xk,k}}{k} - \iota(x) \le -(1-\varepsilon)\frac{\pi}{4k}$$

holds for $k \ge k_0$.

Proof. Recall that we set $\iota_k(x) = \frac{j_{xk,k}}{k}$ and the functions satisfy

$$\frac{d}{dx}\iota_k = F_k(\iota_k, x)$$
$$\frac{d}{dx}\iota = G(\iota, x)$$

in $(-1,\infty)$ with F_k and G defined in (5.3.2) and (5.3.3), respectively. Now consider $u_k(x) := \iota_k(x) - \iota(x)$ so that

$$\frac{d}{dx}u_k = \frac{F_k(\iota_k, x) - G(\iota, x)}{\iota_k(x) - \iota(x)}u_k(x) = \beta_k(x)u_k(x)$$

where we set

$$\beta_k(x) := \frac{F_k(\iota_k, x) - G(\iota, x)}{\iota_k(x) - \iota(x)}.$$

Note that β_k is well-defined by (5.3.4). In particular, we find that

$$u_k(x) = u_k(0)\exp\left(\int_0^x \beta_k(t)\,dt\right).$$

Next, we note that the monotonicity of K_0 and the fact that $\sinh(t) > t$ holds for $t > 0$ imply

$$F_k(\iota_k, x) = 2\iota_k \int_0^\infty K_0\left(t2\iota_k \frac{\sinh\left(\frac{t}{k}\right)}{\left(\frac{t}{k}\right)}\right)e^{-2xt}\,dt$$
$$< 2\iota \int_0^\infty K_0(t2\iota)\,e^{-2xt}\,dt = G(\iota_k, x)$$

where [**134**, p. 388] implies

$$G(y,x) = 2y\int_0^\infty K_0(t2y)\,e^{-2xt}\,dt = \frac{\arccos\frac{x}{y}}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \qquad \text{if} \left|\frac{x}{y}\right| < 1. \tag{5.3.5}$$

Importantly, the function

$$g : (1,\infty) \mapsto \mathbb{R}, \quad t \mapsto \frac{\arccos\frac{1}{t}}{\sqrt{1-\frac{1}{t^2}}}$$

is strictly increasing. Indeed, we have

$$\begin{aligned} g'(t) &= \frac{1}{t^2(1-\frac{1}{t^2})} - \frac{\arccos\frac{1}{t}}{t^3(1-\frac{1}{t^2})^{\frac{3}{2}}} = \frac{1}{t^2(1-\frac{1}{t^2})}\left(1 - \frac{\arccos\frac{1}{t}}{t\sqrt{1-\frac{1}{t^2}}}\right) \\ &= \frac{1}{t^2-1}\left(1 - \frac{\arccos\frac{1}{t}}{\sqrt{t^2-1}}\right) \end{aligned}$$

and [**138**, Theorem 2 for $b = 1/2$] gives

$$\arccos s < 2\frac{\sqrt{1-s}}{\sqrt{1+s}}$$

for $s \in (0,1)$ so that

$$\frac{\arccos\frac{1}{t}}{\sqrt{t^2-1}} = \frac{\arccos\frac{1}{t}}{t\sqrt{1-\frac{1}{t}}\sqrt{1+\frac{1}{t}}} < \frac{2}{t\sqrt{1+\frac{1}{t}}} = \frac{2}{\sqrt{t^2+t}} < 1$$

holds for $t > 1$, which implies that g' is a positive function. Moreover, g' can be continuously extended by $g'(1) = \frac{1}{3}$ and is decreasing, which implies

$$g'(t) \leq \frac{1}{3} \tag{5.3.6}$$

for $t > 1$.

Noting that Lemma 5.3.2 and the convergence $\iota_k(x) \to \iota(x)$ imply that $\left|\frac{x}{\iota_k(x)}\right| < 1$ holds for sufficiently large k, we may combine the identity (5.3.5) with $\iota_k(x) < \iota(x)$ and the monotonicity properties stated above to deduce $F_k(\iota_k, x) < G(\iota_k, x) < G(\iota, x)$ and hence

$$0 \leq \beta_k(x). \tag{5.3.7}$$

In particular, this yields

$$u_k(x) \leq u_k(0)$$

for $x > 0$.

We now estimate $u_k(0)$. Recall that $\iota(0) = \pi$ and therefore

$$u_k(0) = \frac{j_{0,k}}{k} - \pi,$$

so Proposition 5.2.1(iii) yields

$$-\frac{\pi}{4k} \leq u_k(0) \leq -\frac{\pi}{4k} + \frac{1}{8\pi k(k-\frac{1}{4})}. \tag{5.3.8}$$

In view of $u_k(x) = u_k(0)\exp\left(\int_0^x \beta_k(t)\,dt\right)$, combining the last estimate with (5.3.7) therefore implies

$$u_k(x) \leq -\frac{\pi}{4k} + \frac{1}{8\pi k(k-\frac{1}{4})}$$

for $x > 0$ and hence the upper bound stated in the claim.

It remains to prove the lower bound. To this end, we employ arguments inspired by [**19**] and first note that

$$\sinh(x) \leq x + x^3$$

holds for $x \in (0,1)$, which implies

$$\frac{\sin\left(\frac{t}{k}\right)}{\frac{t}{k}} \leq 1 + \frac{1}{k^{\frac{4}{3}}} \tag{5.3.9}$$

for $k \in \mathbb{N}$ and $0 < t \leq t_k := k^{\frac{1}{3}}$. In the following, we fix $x > 0$ and let $y > x$. Then the monotonicity of K_0 and (5.3.9) yield

$$F_k(y,x) \geq \int_0^{t_k} K_0\left(t\left(1 + \frac{1}{k^{\frac{4}{3}}}\right)\right) e^{-\frac{xt}{y}}\,dt$$

and therefore

$$\begin{aligned} &F_k(y,x) - G(y,x) \\ &\geq \int_0^{t_k} \left[K_0\left(t\left(1 + \frac{1}{k^{\frac{4}{3}}}\right)\right) - K_0(t)\right] e^{-\frac{xt}{y}}\,dt - \int_{t_k}^{\infty} K_0(t) e^{-\frac{xt}{y}}\,dt. \end{aligned} \tag{5.3.10}$$

From

$$K_0(t) \leq K_{\frac{1}{2}}(t) = \sqrt{\frac{\pi}{2t}} e^{-t}$$

we then find

$$\int_{t_k}^{\infty} K_0(t) e^{-\frac{xt}{y}}\,dt \leq \sqrt{\frac{\pi}{2t_k}} e^{-t_k} \int_0^{\infty} e^{-\frac{xt}{y}}\,dt = \sqrt{\frac{\pi}{2t_k}} e^{-t_k} \frac{y}{x}.$$

For any $y_0 > x$ and $\delta \in (0,1)$, we thus find $k_0 \in \mathbb{N}$ such that

$$\int_{t_k}^{\infty} K_0(t) e^{-\frac{xt}{y}}\,dt \leq \frac{\delta}{k^{\frac{4}{3}}} \tag{5.3.11}$$

holds for $|y - y_0| < y_0 - x$ and $k \geq k_0$.

In order to estimate the other term in (5.3.10), we note that for $t \in \mathbb{R}$ there exists $\xi_k \in \left(t, t + \frac{t}{k^{\frac{4}{3}}}\right)$ such that

$$K_0\left(t\left(1 + \frac{1}{k^{\frac{4}{3}}}\right)\right) - K_0(t) = K_0'(\xi_k)\frac{1}{k^{\frac{4}{3}}} = -K_1(\xi_k)\frac{t}{k^{\frac{4}{3}}} \geq -K_1(t)\frac{t}{k^{\frac{4}{3}}}.$$

This implies

$$\begin{aligned} \int_0^{t_k} \left[K_0\left(t\left(1 + \frac{1}{k}\right)\right) - K_0(t)\right] e^{-\frac{xt}{y}}\,dt &\geq -\frac{1}{k^{\frac{4}{3}}} \int_0^{t_k} K_1(t) t e^{-\frac{xt}{y}}\,dt \\ &\geq -\frac{1}{k^{\frac{4}{3}}} \int_0^{\infty} K_1(t) t e^{-\frac{xt}{y}}\,dt, \end{aligned}$$

where [**134**, p. 388] gives

$$\int_0^{\infty} K_1(t) t e^{-\frac{xt}{y}}\,dt \leq \int_0^{\infty} K_1(t) t\,dt = \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right) = \frac{\pi}{2}.$$

Combined with (5.3.11), it thus follows that for any $x > 0$, $y_0 > x$ and $\delta \in (0,1)$ there exists $k_0 \in \mathbb{N}$ such that

$$F_k(y,x) - G(y,x) \geq -\left(\frac{\pi}{2} + \delta\right)\frac{1}{k^{\frac{4}{3}}}$$

holds for $k \geq k_0$ and $|y - y_0| < y_0 - x$.

We now proceed by taking $y_0 = \iota(x)$ and note that there exists $k_0' \in \mathbb{N}$ such that $|\iota_k(x) - \iota(x)| < \iota(x) - x$ holds for $k \geq k_0'$. Combined with (5.3.6), we then conclude that for given $\delta \in (0,1)$ we can find $k_0 \in \mathbb{N}$ such that

$$
\begin{aligned}
F_k(\iota_k, x) - G(\iota, x) &= F_k(\iota_k, x) - G(\iota_k, x) + G(\iota_k, x) - G(\iota, x) \\
&\geq -\left(\frac{\pi}{2} + \delta\right)\frac{1}{k^{\frac{4}{3}}} - \max_{\xi \in (\iota_k(x), \iota(x))} \frac{dG}{dy}(\xi, x)|\iota_k(x) - \iota(x)| \\
&\geq -\left(\frac{\pi}{2} + \delta\right)\frac{1}{k^{\frac{4}{3}}} - \frac{1}{3}|\iota_k(x) - \iota(x)|
\end{aligned}
$$

holds for $k \geq k_0$. It follows that

$$
\beta_k(x) = \frac{F_k(\iota_k, x) - G(\iota, x)}{\iota_k(x) - \iota(x)} \leq \frac{1}{3} + \left(\frac{\pi}{2} + \delta\right)\frac{1}{k^{\frac{4}{3}}} \frac{1}{|\iota_k(x) - \iota(x)|}
$$

and since $|\iota_k(x) - \iota(x)| = |u_k(x)| \geq (\frac{\pi}{4} - \delta)\frac{1}{k}$ holds for sufficiently large k we therefore have

$$
\beta_k(x) \leq \frac{1}{3} + \frac{\frac{\pi}{2} + \delta}{\frac{\pi}{4} - \delta}\frac{1}{k^{\frac{1}{3}}}.
$$

Consequently, we may choose k_0 such that

$$
\beta_k(x) \leq \frac{1}{3} + \varepsilon
$$

holds for $k \geq k_0$. Overall, this yields

$$
1 \leq \exp\left(\int_0^x \beta_k(t)\, dt\right) \leq \exp\left(\left(\frac{1}{3} + \varepsilon\right) x\right)
$$

for $k \geq k_0$. Recalling (5.3.8) and

$$
u_k(x) = u_k(0) \exp\left(\int_0^x \beta_k(t)\, dt\right),
$$

the claim thus follows. □

Remark 5.3.4. *Lemma 5.3.3 improves the bound obtained in [**50**, Theorem 2.1] as follows: For any $\varepsilon, \nu > 0$ there exists $k_0 \in \mathbb{N}$ such that*

$$
j_{\nu,k} < k\,\iota\left(\frac{\nu}{k}\right) - (1 - \varepsilon)\frac{\pi}{4}
$$

holds for $k \geq k_0$.

5.4 Spectral Characterization

For $\alpha > 1$ recall the operator

$$
L_\alpha = -\Delta + \alpha^2 \partial_\theta^2.
$$

If $\varphi \in H_0^1(\mathbf{B})$ is an eigenfunction of $-\Delta$ corresponding to the eigenvalue $j_{\ell,k}^2$, then it follows from the representation (5.2.1) that φ is also an eigenfunction of L_α with

$$
L_\alpha \varphi = (j_{\ell,k}^2 - \alpha^2 \ell^2)\varphi.
$$

Since the eigenfunctions of $-\Delta$ constitute an orthonormal basis of $L^2(\mathbf{B})$, we find that the Dirichlet eigenvalues of L_α are given by

$$\left\{ j_{\ell,k}^2 - \alpha^2 \ell^2 \,:\, \ell \in \mathbb{N}_0,\, k \in \mathbb{N} \right\}.$$

In the following, we wish to study this set in more detail. The following result already shows a stark contrast to the case $\alpha \in [0,1]$.

Proposition 5.4.1. *Let $\alpha > 1$. Then the spectrum of the operator $L_\alpha = -\Delta + \alpha^2 \partial_\theta^2$ is unbounded from above and below.*

Proof. For $\ell \in \mathbb{N}_0$, $k \in \mathbb{N}$ we write

$$j_{\ell,k}^2 - \alpha^2 \ell^2 = (j_{\ell,k} + \alpha \ell)\ell \left(\frac{j_{\ell,k}}{\ell} - \alpha \right)$$

and note that Corollary 5.2.2 implies

$$1 - \alpha + \frac{\left(\frac{3\pi}{8}(4k-2)\right)^{\frac{2}{3}}}{2^{\frac{1}{3}}} \ell^{-\frac{2}{3}} < \frac{j_{\ell,k}}{\ell} - \alpha < 1 - \alpha + \frac{\left(\frac{3\pi}{2}k\right)^{\frac{2}{3}}}{2^{\frac{1}{3}}} \ell^{-\frac{2}{3}} + \frac{3}{20}\left(\frac{3\pi}{2}k\right)^{\frac{4}{3}} \frac{2^{\frac{1}{3}}}{\ell^{\frac{4}{3}}}. \tag{5.4.1}$$

If we choose sequences $(\ell_i)_i$, $(k_i)_i$, such that $\frac{\ell_i}{k_i} \to \infty$, this readily implies $j_{\ell_i,k_i}^2 - \alpha^2 \ell_i^2 \to -\infty$, whereas sequences such that $\frac{\ell_i}{k_i} \to 0$ yield $j_{\ell,k}^2 - \alpha^2 \ell^2 \to \infty$ and thus the claim. ◻

In particular, this proves the first part of Theorem 5.1.1. As noted in the introduction, it is not clear whether the spectrum of L_α only consists of isolated points. Indeed, note that Lemma 5.3.3 suggests that certain subsequences of $j_{\ell,k} - \alpha \ell$ may converge and it is unclear if there exists a subsequence that even converges to zero. In particular, the spectrum of the operator could even be dense in $\mathbb{R}$.

This is excluded by the second part of Theorem 5.1.1 which we restate as follows.

Theorem 5.4.2. *There exists a sequence $(\alpha_n)_n \subset (1,\infty)$ such that the following properties hold for $n \in \mathbb{N}$:*

(i) The spectrum of L_{α_n} consists of eigenvalues with finite multiplicity.

(ii) There exists $c_n > 0$ such that for each $\ell \in \mathbb{N}_0$, $k \in \mathbb{N}$ we either have $j_{\ell,k}^2 - \alpha_n^2 \ell^2 = 0$ or

$$|j_{\ell,k}^2 - \alpha_n^2 \ell^2| \geq c_n j_{\ell,k}. \tag{5.4.2}$$

(iii) The spectrum of L_{α_n} has no finite accumulation points.

Proof. We set $\sigma_n := \frac{1}{n}$ and $\alpha_n := \frac{\iota(\sigma_n)}{\sigma_n}$, where the function ι is given by Theorem 5.3.1. It then suffices to show that there exists $n_0 \in \mathbb{N}$ such that properties (i)-(iii) hold for $n \geq n_0$. In the following, we fix $n \in \mathbb{N}$ and assume that there exists $\Lambda \in \mathbb{R}$ and increasing sequences $(\ell_i)_i$, $(k_i)_i$ such that $j_{\ell_i,k_i}^2 - \alpha_n^2 \ell_i^2 \to \Lambda$ as $n \to \infty$. Note that the case of an eigenvalue with infinite multiplicity, i.e., $j_{\ell_i,k_i}^2 - \alpha_n^2 \ell_i^2 = \Lambda$ for all i, is included here. The identity $j_{\ell,k}^2 - \alpha_n^2 \ell^2 = (j_{\ell,k} + \alpha_n \ell)\ell \left(\frac{j_{\ell,k}}{\ell} - \alpha_n \right)$ then implies that we must have

$$\frac{j_{\ell_i,k_i}}{\ell_i} \to \alpha_n. \tag{5.4.3}$$

Our goal is to show that such sequences can only converge of order $\frac{1}{l_i}$, which will allow us to derive a suitable contradiction.

Firstly, the estimate (5.4.1) implies that there must exist $\sigma \in (0,\infty)$ such that $\frac{\ell_i}{k_i} \to \sigma$. We now claim that for any $\varepsilon > 0$, there exists $i_0 \in \mathbb{N}$ such that

$$(1-\varepsilon)\frac{j_{\sigma k_i,k_i}}{\sigma k_i} < \frac{j_{\ell_i,k_i}}{\ell_i} < (1+\varepsilon)\frac{j_{\sigma k_i,k_i}}{\sigma k_i} \qquad \text{for } i \geq i_0. \tag{5.4.4}$$

To this end, we first assume that $\ell_i < \sigma k_i$ holds. Then Proposition 5.2.1(ii) implies

$$\frac{j_{\ell_i,k_i}}{\ell_i} > \frac{j_{\sigma k_i,k_i}}{\sigma k_i}$$

and, in particular, the lower bound. Moreover, the fact that the function $\nu \mapsto j_{\nu,k}$ is increasing for fixed k yields

$$\frac{j_{\ell_i,k_i}}{\ell_i} \leq \frac{j_{\sigma k_i,k_i}}{\ell_i} = \frac{\sigma k_i}{\ell_i}\frac{j_{\sigma k_i,k_i}}{\sigma k_i}.$$

Noting that $\frac{\sigma k_i}{\ell_i} \to 1$ as $i \to \infty$, we conclude that for any $\varepsilon > 0$, there exists $i_0 \in \mathbb{N}$ such that

$$\frac{j_{\ell_i,k_i}}{\ell_i} < (1+\varepsilon)\frac{j_{\sigma k_i,k_i}}{\sigma k_i}$$

holds for $i \geq i_0$ with $\ell_i < \sigma k_i$. The case $\ell_i \geq \sigma k_i$ can be treated analogously.

Overall, (5.4.4) implies

$$(1-\varepsilon)\frac{\iota(\sigma)}{\sigma} \leq \liminf_{n\to\infty}\frac{j_{\ell_i,k_i}}{\ell_i} \leq \limsup_{n\to\infty}\frac{j_{\ell_i,k_i}}{\ell_i} \leq (1+\varepsilon)\frac{\iota(\sigma)}{\sigma}$$

for arbitrary $\varepsilon > 0$, with the function ι given by Theorem 5.3.1. In particular, (5.4.3) then yields

$$\frac{\iota(\sigma)}{\sigma} = \lim_{n\to\infty}\frac{j_{\ell_i,k_i}}{\ell_i} = \alpha_n$$

and Lemma 5.3.2 thus implies that we must have $\sigma = \sigma_n$ due to our choice of α_n. In particular, it follows that $\frac{\ell_i}{k_i} \to \sigma_n$. We now distinguish two cases:

Case 1: There exists $i_0 \in \mathbb{N}$ such that $\frac{\ell_i}{k_i} \geq \sigma_n$ holds for $i \geq i_0$.
In this case, Proposition 5.2.1(ii) implies

$$\frac{j_{\ell_i,k_i}}{\ell_i} - \alpha_n \leq \frac{j_{\sigma_n k_i,k_i}}{\sigma_n k_i} - \alpha_n = \frac{1}{\sigma_n}\left(\frac{j_{\sigma_n k_i,k_i}}{k_i} - \iota(\sigma_n)\right),$$

so that Lemma 5.3.3 yields

$$\frac{j_{\ell_i,k_i}}{\ell_i} - \alpha_n \leq -\frac{\pi}{8\sigma_n k_i}$$

for $i \geq i_0$, after possibly enlarging i_0. In particular, this implies

$$|j_{\ell_i,k_i} - \alpha_n \ell_i| = \ell_i\left|\frac{j_{\ell_i,k_i}}{\ell_i} - \alpha_n\right| \geq \frac{\pi \ell_i}{8\sigma_n k_i} \geq \frac{\pi}{8}$$

for $i \geq i_0$ and therefore $\liminf_{i\to\infty}(j_{\ell_i,k_i} - \alpha_n \ell_i) \geq \frac{\pi}{8}$.

Case 2: There exists $i_0 \in \mathbb{N}$ such that $\frac{\ell_i}{k_i} < \sigma_n$ holds for $i \geq i_0$.
We may write $\ell_i = \sigma_n k_i - \delta_i$ with $\delta_i > 0$ satisfying $\frac{\delta_i}{k_i} \to 0$ as $i \to \infty$. Then

$$j_{\ell_i,k_i} - \alpha_n \ell_i = j_{(\sigma_n k_i - \delta_i),k_i} - \alpha_n(\sigma_n k_i - \delta_i)$$

$$\begin{aligned} &= (j_{\sigma_n k_i, k_i} - \alpha_n \sigma_n k_i) + (j_{(\sigma_n k_i - \delta_i), k_i} - j_{\sigma_n k_i, k_i}) + \alpha_n \delta_i \\ &= (j_{\sigma_n k_i, k_i} - \alpha_n \sigma_n k_i) + R_{n,i} \delta_i, \end{aligned}$$

where we have set

$$R_{n,i} := \alpha_n - \frac{j_{\sigma_n k_i, k_i} - j_{\sigma_n k_i - \delta_i, k_i}}{\delta_i}.$$

By Lemma 5.3.3 we may further enlarge i_0 to ensure that

$$j_{\sigma_n k_i, k_i} - \alpha_n \sigma_n k_i \geq -\frac{\pi}{4} e^{\sigma_n/3}$$

holds for $i \geq i_0$. Next, Proposition 5.2.1(iv) gives

$$\frac{j_{\sigma_n k_i, k_i} - j_{\sigma_n k_i - \delta_i, k_i}}{\delta_i} \in \left(1, \frac{\pi}{2}\right)$$

and hence

$$\liminf_{i \to \infty} R_{n,i} \geq \alpha_n - \frac{\pi}{2}.$$

Since $\alpha_n = \frac{\iota(\sigma_n)}{\sigma_n} \to \infty$ as $n \to \infty$ by Lemma 5.3.2, this term is positive for sufficiently large n, and it therefore follows that

$$\liminf_{i \to \infty} (j_{\ell_i, k_i} - \alpha_n \ell_i) \geq -\frac{\pi}{4} e^{\sigma_n/3} + \left(\alpha_n - \frac{\pi}{2}\right) \inf_{i \in \mathbb{N}} \delta_i.$$

In order to show that the right hand side is positive, we recall that $\sigma_n = \frac{1}{n}$ and therefore the fact that $\ell_i = \sigma_n k_i - \delta_i$ must be a natural number implies $\delta_i = \frac{n'}{n}$ for some $n' \in \mathbb{N}$ and, in particular, $\inf_i \delta_i = \frac{1}{n}$. Moreover, by Lemma 5.3.2 the associated $\alpha_n = \frac{\iota(\sigma_n)}{\sigma_n}$ is uniquely determined by the equation

$$\pi n = \frac{\pi}{\sigma_n} = \sqrt{\alpha_n^2 - 1} - \left(\frac{\pi}{2} - \arcsin \frac{1}{\alpha_n}\right).$$

Since the right hand side is strictly increasing in α_n and we have

$$\frac{\sqrt{n^2 - 1} - \left(\frac{\pi}{2} - \arcsin \frac{1}{n}\right)}{n} = \sqrt{1 - \frac{1}{n^2}} + \frac{1}{n} \arcsin \frac{1}{n} - \frac{\pi}{2n} \to 1 < \pi$$

as $n \to \infty$, there must exist $n_0 \in \mathbb{N}$ such that $\alpha_n > n$ holds for $n \geq n_0$. We thus have

$$\begin{aligned} -\frac{\pi}{4} e^{\sigma_n/3} + \left(\alpha_n - \frac{\pi}{2}\right) \inf_i \delta_i &\geq -\frac{\pi}{4} e^{\sigma_n/3} + \frac{1}{n} \left(n - \frac{\pi}{2}\right) \\ &= 1 - \pi \left(\frac{1}{2n} + \frac{e^{\frac{1}{3n}}}{4}\right) \to 1 - \frac{\pi}{4} > 0 \end{aligned}$$

as $n \to \infty$. We conclude that after possibly further enlarging n_0,

$$\kappa_n := -\frac{\pi}{4} e^{\sigma_n/3} + \left(\alpha_n - \frac{\pi}{2}\right) \inf_i \delta_i > 0$$

holds for $n \geq n_0$.

Since we may always pass to a subsequence such that one of these two cases holds, we overall find that

$$\liminf_{i \to \infty} (j_{\ell_i, k_i} - \alpha_n \ell_i) \geq \min \left\{\kappa_n, \frac{\pi}{4}\right\} > 0$$

for any sequences $(\ell_i)_i, (k_i)_i$ such that $\frac{\ell_i}{k_i} \longrightarrow \sigma_n = \frac{1}{n}$, provided $n \geq n_0$. In particular, it follows that $j_{\ell,k}^2 - \alpha_n^2 \ell^2 = (j_{\ell,k} - \alpha_n \ell)(j_{\ell,k} + \alpha_n \ell)$ cannot converge to Λ, which implies (i) and (iii).

Moreover, since we considered arbitrary sequences satisfying (5.4.3), we further find that

$$\gamma_n := \lim_{N\to\infty} \inf_{\ell,k\geq N} |j_{\ell,k} - \alpha_n \ell| > 0$$

for $n \geq n_0$. Consequently, taking $N_0 \in \mathbb{N}$ such that $\inf_{\ell,k\geq N} |j_{\ell,k} - \alpha_n \ell| > \frac{\gamma_n}{2}$ holds for $N \geq N_0$ and setting

$$c_n := \min \left\{ \frac{\gamma_n}{2}, \inf_{\substack{\ell,k\leq N_0 \\ j_{\ell,k}\neq \alpha_n \ell}} |j_{\ell,k} - \alpha_n \ell| \right\} > 0$$

yields

$$|j_{\ell,k}^2 - \alpha_n^2 \ell^2| = |j_{\ell,k} - \alpha_n \ell||j_{\ell,k} + \alpha_n \ell| \geq c_n j_{\ell,k}$$

as claimed in (ii). This completes the proof. □

Remark 5.4.3. *(i) The sequence $(\alpha_n)_n$ can be characterized further by noting that*

$$\pi n = \sqrt{\alpha_n^2 - 1} - \left(\frac{\pi}{2} - \arcsin \frac{1}{\alpha_n}\right)$$

implies

$$\alpha_n^2 = 1 + \left(\pi n + \frac{\pi}{2} - \arcsin \frac{1}{\alpha_n}\right)^2.$$

Since $\arcsin \frac{1}{\alpha_n} = O(n^{-2})$, *this implies*

$$\alpha_n^2 \approx 1 + \left(\pi n + \frac{\pi}{2}\right)^2$$

(ii) The methods used above can be further extended to include some additional values of α. If we let $\sigma = \frac{m}{n}$ with $m, n \in \mathbb{N}$, we find that $\inf_i \delta_i = \frac{1}{n}$ *and similar arguments as above then lead to the condition*

$$0 < \sqrt{\frac{1}{n^2} + \frac{\pi^2}{m^2}} - \pi \left(\frac{1}{2n} + \frac{e^{\frac{m}{3n}}}{4}\right).$$

As $n \longrightarrow \infty$, we find that this holds for $m = 1, 2, 3$.

Moreover, we note that numerical computations imply that the result should also hold for $\sigma = 1, 2, 3$.

5.5 Variational Characterization of Ground States

We now return to solutions of (5.1.4). Setting

$$L_{\alpha,m} := -\Delta + \alpha^2 \partial_\theta^2 + m$$

for $\alpha > 1$, $m \in \mathbb{R}$, our first goal is to find a suitable domain for the quadratic form

$$u \mapsto \langle L_{\alpha,m} u, u \rangle_{L^2(\mathbf{B})} = \int_{\mathbf{B}} (L_{\alpha,m} u) u \, dx = \int_{\mathbf{B}} \left(|\nabla u|^2 - \alpha^2 |\partial_\theta u|^2 + m u^2 \right) \, dx.$$

In order to simplify the notation, we set

$$\mathcal{I}^+_{\alpha,m} := \left\{(\ell,k)\in\mathbb{N}_0\times\mathbb{N} : j^2_{\ell,k}-\alpha^2\ell^2+m>0\right\}$$
$$\mathcal{I}^0_{\alpha,m} := \left\{(\ell,k)\in\mathbb{N}_0\times\mathbb{N} : j^2_{\ell,k}-\alpha^2\ell^2+m=0\right\}$$
$$\mathcal{I}^-_{\alpha,m} := \left\{(\ell,k)\in\mathbb{N}_0\times\mathbb{N} : j^2_{\ell,k}-\alpha^2\ell^2+m<0\right\}$$

for $\alpha>1$, $m\in\mathbb{R}$, i.e., the index sets of positive, zero and negative eigenvalues, respectively. Instead of restricting ourselves to the sequence (α_n) given by Theorem 5.1.1, we consider

$$\mathcal{A} := \left\{\alpha>1 : \ |\mathcal{I}^0_{\alpha,0}|<\infty \text{ and } \min_{(\ell,k)\notin\mathcal{I}^0_{\alpha,0}} |j_{\ell,k}-\alpha\ell|>0\right\}.$$

In particular, $\mathcal{A}$ contains the sequence $(\alpha_n)_n$ and is therefore nonempty and unbounded. Moreover, writing $j^2_{\ell,k}-\alpha^2\ell^2=(j_{\ell,k}+\alpha\ell)(j_{\ell,k}-\alpha\ell)$ we find that for any $\alpha\in\mathcal{A}$ there exists $c_\alpha>0$ such that

$$|j^2_{\ell,k}-\alpha^2\ell^2|\geq c_\alpha j_{\ell,k}$$

holds for $(\ell,k)\notin\mathcal{I}^0_{\alpha,0}$.

Lemma 5.5.1. *Let $\alpha\in\mathcal{A}$ and $m\in\mathbb{R}$. Then $\mathcal{I}^0_{\alpha,m}$ is finite and there exists $c_m>0$ such that any $(\ell,k)\notin\mathcal{I}^0_{\alpha,m}$ satisfy*

$$|j^2_{\ell,k}-\alpha^2\ell^2+m|\geq c_m j_{\ell,k}.$$

In particular, the spectrum of $L_{\alpha,m}$ has no finite accumulation points.

Proof. Let $c_\alpha>0$ be given as above. We first note that

$$|j^2_{\ell,k}-\alpha^2\ell^2+m| = (j_{\ell,k}+\alpha\ell)\left|j_{\ell,k}-\alpha\ell+\frac{m}{j_{\ell,k}+\alpha\ell}\right|,$$

so the fact that $\mathcal{I}^0_{\alpha,0}$ is finite by assumption implies that $\mathcal{I}^0_{\alpha,m}$ is finite as well. Moreover, there exist $\ell_0,k_0\in\mathbb{N}$ such that

$$|j^2_{\ell,k}-\alpha_n^2\ell^2+m|\geq (j_{\ell,k}+\alpha\ell)\frac{c_\alpha}{2}$$

holds for all $(\ell,k)\notin\mathcal{I}^0_{\alpha,m}$ with $\ell\geq\ell_0$, $k\geq k_0$. Setting

$$c_m := \min\left\{\frac{c_\alpha}{2}, \min_{\substack{(\ell,k)\notin\mathcal{I}^0_{\alpha,m}\\ \ell\leq\ell_0,k\leq k_0}}\left|j_{\ell,k}-\alpha\ell+\frac{m}{j_{\ell,k}+\alpha\ell}\right|\right\}>0$$

then completes the proof. □

Next, we recall the eigenfunctions $\varphi_{\ell,k},\psi_{\ell,k}$ given in (5.2.1) and set

$$E_{\alpha,m} := \left\{u\in L^2(\mathbf{B}) : \sum_{\ell=0}^\infty\sum_{k=1}^\infty \left|j^2_{\ell,k}-\alpha^2\ell^2+m\right|\left(|\langle u,\varphi_{\ell,k}\rangle|^2+|\langle u,\psi_{\ell,k}\rangle|^2\right)<\infty\right\}$$

for $\alpha\in\mathcal{A}$, $m\in\mathbb{R}$ and endow $E_{\alpha,m}$ with the scalar product

$$\begin{aligned}\langle u,v\rangle_{\alpha,m} := &\sum_{\ell=0}^\infty\sum_{k=1}^\infty \left|j^2_{\ell,k}-\alpha^2\ell^2+m\right|\left(\langle u,\varphi_{\ell,k}\rangle\langle v,\varphi_{\ell,k}\rangle+\langle u,\psi_{\ell,k}\rangle\langle v,\psi_{\ell,k}\rangle\right)\\ &+\sum_{(\ell,k)\in\mathcal{I}^0_{\alpha,m}}\left(\langle u,\varphi_{\ell,k}\rangle\langle v,\varphi_{\ell,k}\rangle+\langle u,\psi_{\ell,k}\rangle\langle v,\psi_{\ell,k}\rangle\right).\end{aligned}$$

In the following, $\|\cdot\|_{\alpha,m}$ denotes the norm induced by $\langle\cdot,\cdot\rangle_{\alpha,m}$.

Remark 5.5.2. *For fixed $\alpha \in \mathcal{A}$, the norm $\|\cdot\|_{\alpha,m}$ is equivalent to $\|\cdot\|_{\alpha,0}$ and $E_{\alpha,m} = E_{\alpha,0}$, i.e., the spaces are equal as sets. Nonetheless, the use of an m-dependent scalar product is useful for the variational methods we will employ below.*

We now consider the following decomposition associated to the eigenspaces of positive, zero and negative eigenvalues of $L_{\alpha,m}$, respectively:

$$E^+_{\alpha,m} := \left\{ u \in E_{\alpha,m} : \int_{\mathbf{B}} u\,\varphi_{\ell,k}\,dx = \int_{\mathbf{B}} u\,\psi_{\ell,k}\,dx = 0 \text{ for } (\ell,k) \in \mathcal{I}^0_{\alpha,m} \cup \mathcal{I}^-_{\alpha,m} \right\}$$
$$E^0_{\alpha,m} := \left\{ u \in E_{\alpha,m} : \int_{\mathbf{B}} u\,\varphi_{\ell,k}\,dx = \int_{\mathbf{B}} u\,\psi_{\ell,k}\,dx = 0 \text{ for } (\ell,k) \in \mathcal{I}^+_{\alpha,m} \cup \mathcal{I}^-_{\alpha,m} \right\}$$
$$E^-_{\alpha,m} := \left\{ u \in E_{\alpha,m} : \int_{\mathbf{B}} u\,\varphi_{\ell,k}\,dx = \int_{\mathbf{B}} u\,\psi_{\ell,k}\,dx = 0 \text{ for } (\ell,k) \in \mathcal{I}^+_{\alpha,m} \cup \mathcal{I}^0_{\alpha,m} \right\}$$

so that, in particular,

$$E_{\alpha,m} = E^+_{\alpha,m} \oplus E^0_{\alpha,m} \oplus E^-_{\alpha,m} = E^+_{\alpha,m} \oplus F_{\alpha,m},$$

where we have set $F_{\alpha,m} := E^0_{\alpha,m} \oplus E^-_{\alpha,m}$. In the following, we will routinely write

$$u = u^+ + u^0 + u^-$$

where $u^+ \in E^+_{\alpha,m}$, $u^0 \in E^0_{\alpha,m}$, $u^- \in E^-_{\alpha,m}$ are uniquely determined. The use of the norm $\|\cdot\|_{\alpha,m}$ allows us to write

$$\langle L_{\alpha,m}u, u\rangle_{L^2(\mathbf{B})} = \int_{\mathbf{B}} \left(|\nabla u|^2 - \alpha^2|\partial_\theta u|^2 + mu^2\right)\,dx = \|u^+\|^2_{\alpha,m} - \|u^-\|^2_{\alpha,m}.$$

Importantly, $E_{\alpha,m}$ has the following embedding properties:

Proposition 5.5.3. *Let $p \in (2,4)$, $\alpha \in \mathcal{A}$ and $m \in \mathbb{R}$. Then $E_{\alpha,m} \subset L^p(\mathbf{B})$ and the embedding*

$$E_{\alpha,m} \hookrightarrow L^p(\mathbf{B})$$

is compact.

Proof. Because of the compact embedding $H^{\frac{1}{2}}(\mathbf{B}) \hookrightarrow L^p(\mathbf{B})$ it is enough to show that the embedding

$$E_{\alpha,m} \hookrightarrow H^{\frac{1}{2}}(\mathbf{B})$$

is well-defined and continuous. We first note that it suffices to consider $u \in E^+_{\alpha,m} \oplus E^-_{\alpha,m}$, since the space $E^0_{\alpha,m}$ is finite–dimensional and only contains smooth functions. By Lemma 5.5.1, there exists $c > 0$ such that

$$|j^2_{\ell,k} - \alpha^2\ell^2 + m| \geq c\,j_{\ell,k}$$

holds for $(\ell,k) \notin \mathcal{I}^0_{\alpha,m}$. This implies

$$\begin{aligned}
\|u\|^2_{H^{\frac{1}{2}}} &= \sum_{\ell=0}^{\infty}\sum_{k=1}^{\infty} j_{\ell,k}\left(|\langle u, \varphi_{\ell,k}\rangle|^2 + |\langle u, \psi_{\ell,k}\rangle|^2\right) \\
&\leq \frac{1}{c}\sum_{\ell=0}^{\infty}\sum_{k=1}^{\infty} |j^2_{\ell,k} - \alpha^2\ell^2 + m|^2\left(|\langle u, \varphi_{\ell,k}\rangle|^2 + |\langle u, \psi_{\ell,k}\rangle|^2\right) \\
&= \frac{1}{c}\|u\|^2_{\alpha,m}
\end{aligned}$$

and thus the claim. □

Remark 5.5.4. *It is natural to ask for the optimal $q > 2$ such that the preceding proposition holds for $p \in (2, q)$. We conjecture that $q = 10$ due to two observations:*

Firstly, $q = 10$ appears in the degenerate elliptic case $\alpha = 1$ treated in **[P3]** *as the critical exponent for Sobolev-type embeddings for the associated degenerate operator. Secondly, this exponent also appears in a Pohožaev-type identity in [**90**] with respect to related semilinear problems involving the Tricomi operator.*

In particular, the map

$$I_p : E_{\alpha,m} \to \mathbb{R}, \qquad I_p(u) := \frac{1}{p}\int_{\mathbf{B}} |u|^p \, dx = \frac{1}{p}\|u\|_p^p$$

is well-defined and continuous for $p \in (2, 4)$. We note the following properties corresponding to the conditions of Theorem 35 in [**128**].

Lemma 5.5.5. *Let $\alpha \in \mathcal{A}$, $m \in \mathbb{R}$ and $p \in (2, 4)$. Then the following properties hold:*

(i) $\frac{1}{2}I_p'(u)u > I_p(u) > 0$ *for all* $u \not\equiv 0$ *and* I_p *is weakly lower semicontinuous.*

(ii) $I_p'(u) = o(\|u\|_{\alpha,m})$ *as* $u \to 0$.

(iii) $\frac{I_p(su)}{s^2} \to \infty$ *uniformly in u on weakly compact subsets of $E_{\alpha,m} \setminus \{0\}$ as $s \to \infty$.*

(iv) I_p' *is a compact map.*

Proof. The properties (i),(ii) and (iv) follow from routine computations and Proposition 5.5.3, while (iii) has essentially been proved in [**128**, Theorem 16], though we can give a slightly simpler argument in this case:

Let $W \subset E_{\alpha,m} \setminus \{0\}$ be a weakly compact subset. We claim that there exists $c > 0$ such that $\|u\|_p \geq c$ holds for $u \in W$. Indeed, if this was false, there would exist a sequence $(u_n)_n \subset W$ such that $u_n \to 0$ in $L^p(\mathbf{B})$. The weak compactness of W and Proposition 5.5.3 would then imply $u_n \rightharpoonup 0$, contradicting the fact that $0 \notin W$. We thus have

$$\frac{I_p(su)}{s^2} = \frac{s^{p-2}}{p}\|u\|_p^p \geq \frac{c^p}{p}s^{p-2}$$

and clearly the right hand side goes to infinity uniformly as $s \to \infty$. □

In the following, we always assume that $p \in (2, 4)$ is fixed and consider the energy functional $\Phi_{\alpha,m} : E_{\alpha,m} \to \mathbb{R}$ given by

$$\begin{aligned}\Phi_{\alpha,m}(u) &:= \frac{1}{2}\|u^+\|_{\alpha,m}^2 - \frac{1}{2}\|u^-\|_{\alpha,m} - I_p(u) \\ &= \frac{1}{2}\int_{\mathbf{B}} \left(|\nabla u|^2 - \alpha^2|\partial_\theta u|^2 + mu^2\right) dx - \frac{1}{p}\int_{\mathbf{B}} |u|^p \, dx.\end{aligned}$$

In particular, any critical point $u \in E_{\alpha,m}$ of $\Phi_{\alpha,m}$ satisfies

$$\int_{\mathbf{B}} |u|^{p-2}u\,\varphi \, dx = \langle u^+, \varphi\rangle_{\alpha,m} - \langle u^-, \varphi\rangle_{\alpha,m} = \int_{\mathbf{B}} u\, L_{\alpha,m}\varphi \, dx$$

and can thus be interpreted as a weak solution of (5.1.4). As outlined in the introduction, we will now characterize ground states of $\Phi_{\alpha,m}$ by considering the generalized Nehari manifold

$$\mathcal{N}_{\alpha,m} := \left\{u \in E_{\alpha,m} \setminus F_{\alpha,m} : \ \Phi_{\alpha,m}'(u)u = 0 \text{ and } \Phi_{\alpha,m}'(u)v = 0 \text{ for all } v \in F_{\alpha,m}\right\}.$$

In particular, $\mathcal{N}_{\alpha,m}$ contains all nontrivial critical points of Φ. Consequently, the value

$$c_{\alpha,m} := \inf_{u \in \mathcal{N}_{\alpha,m}} \Phi_{\alpha,m}(u)$$

is the ground state energy in the sense that any critical point $u \in E_{\alpha,m} \setminus \{0\}$ of $\Phi_{\alpha,m}$ satisfies $\Phi_{\alpha,m}(u) \geq c_{\alpha,m}$. This motivates the following definition.

Definition 5.5.6. *Let $\alpha \in \mathcal{A}$, $m \in \mathbb{R}$ and $p \in (2,4)$. We call a function $u \in E_{\alpha,m}$ a **ground state solution** of* (4.1.5)*, if u is a critical point of $\Phi_{\alpha,m}$ and satisfies $\Phi_{\alpha,m}(u) = c_{\alpha,m}$.*

In order to show that ground state solutions exist, we wish to verify that $\Phi_{\alpha,m}$ satisfies condition (B_2) from [**128**]. To this end, we let $u \in E_{\alpha,m} \setminus F_{\alpha,m}$ and consider

$$\widehat{E}_{\alpha,m}(u) := \{tu + w : \ t \geq 0, \ w \in F_{\alpha,m}\} = \mathbb{R}^+ u \oplus F_{\alpha,m}.$$

Importantly, $u \in \mathcal{N}_{\alpha,m}$ if and only if u is a critical point of $\Phi_{\alpha,m}\big|_{\widehat{E}_{\alpha,m}(u)}$. Moreover, we have $\widehat{E}_{\alpha,m}(u) = \widehat{E}_{\alpha,m}(tu^+)$ for all $t \geq 0$, $u \in E_{\alpha,m} \setminus F_{\alpha,m}$, hence when considering $\widehat{E}_{\alpha,m}(u)$ we may always assume $u \in E^+_{\alpha,m}$. This will be useful in the following.

Lemma 5.5.7. *For each $u \in E_{\alpha,m} \setminus F_{\alpha,m}$ there exists a unique nontrivial critical point $\hat{m}(u)$ of $\Phi_{\alpha,m}\big|_{\widehat{E}_{\alpha,m}}$. Moreover, $\hat{m}(u)$ is the unique global maximum of $\Phi_{\alpha,m}\big|_{\widehat{E}_{\alpha,m}}$.*

Proof. The following argument is essentially taken from [**128**, Proposition 39]. Without loss of generality we may assume $u \in E^+_{\alpha,m}$ and $\|u\|_{\alpha,m} = 1$.

Claim 1: There exists $R > 0$ such that $\Phi_{\alpha,m}(v) \leq 0$ holds for $v \in \widehat{E}_{\alpha,m}$ and $\|v\|_{\alpha,m} \geq R$. Indeed, if this was false there would exist a sequence $(v_n)_n \subset \widehat{E}_{\alpha,m}(u)$ such that $\|v_n\|_{\alpha,m} \to \infty$ and $\Phi_{\alpha,m}(v_n) > 0$. Setting $w_n := \frac{v_n}{\|v_n\|_{\alpha,m}}$ we may pass to a weakly convergent subsequence and note that

$$\begin{aligned} 0 < \frac{\Phi_{\alpha,m}(v_n)}{\|v_n\|^2_{\alpha,m}} &= \frac{1}{2}\|w_n^+\|^2_{\alpha,m} - \frac{1}{2}\|w_n^-\|^2_{\alpha,m} - \frac{1}{p}\frac{\big\|\|v_n\|_{\alpha,m} w_n\big\|_p^p}{\|v_n\|^2_{\alpha,m}} \\ &\leq \|w_n\|^2_{\alpha,m} - \frac{I(\|\|v_n\|_{\alpha,m} w_n)}{\|v_n\|^2_{\alpha,m}} \end{aligned}$$

so that Lemma 5.5.5(iii) implies $0 < \frac{\Phi_{\alpha,m}(v_n)}{\|v_n\|^2_{\alpha,m}} \to -\infty$ if the weak limit is nonzero. Hence we must have $w_n \rightharpoonup 0$. Moreover, the inequality above also implies $\|w_n^+\|_{\alpha,m} \geq \|w_n^-\|_{\alpha,m}$. If $w_n^+ \to 0$, the latter also implies $w_n^- \to 0$ and therefore

$$\|w_n^0\|^2_{\alpha,m} = 1 - \|w_n^+\|_{\alpha,m} - \|w_n^-\|^2_{\alpha,m} \to 1.$$

The fact that $E^0_{\alpha,m}$ is finite-dimensional then implies that w_n^0 converges to a nontrivial function, which contradicts $w_n \rightharpoonup 0$. Hence w_n^+ cannot converge to zero and we may therefore pass to a subsequence such that $\|w_n^+\|_{\alpha,m} \geq \gamma$ holds from some $\gamma > 0$ and all n. However, by definition of $\widehat{E}_{\alpha,m}(u)$ we must have $w_n^+ = u\|w_n^+\|_{\alpha,m}$ and therefore there exists $c > 0$ such that $w_n^+ \to cu$ holds after passing to a subsequence, contradicting $w_n \rightharpoonup 0$. This proves Claim 1.

Next, we note that Lemma 5.5.5 yields $\Phi_{\alpha,m}(tu) = \frac{t^2}{2} + o(t^2)$ as $t \to 0$ and therefore

$$\sup_{\widehat{E}_{\alpha,m}(u)} \Phi_{\alpha,m} > 0.$$

Now Claim 1 implies that any maximizing sequence $(v_n)_n \subset \widehat{E}_{\alpha,m}(u)$ must remain bounded, so we may assume $v_n \rightharpoonup v$ after passing to a subsequence. Moreover, recalling that

$$\Phi_{\alpha,m}(v_n) = \frac{\|v_n^+\|_{\alpha,m}^2}{2} - \frac{\|v_n^-\|_{\alpha,m}^2}{2} - I_p(v_n),$$

we can use that v_n^+ is a multiple of u, while the norm $\|\cdot\|_{\alpha,m}$ and I_p are weakly lower semicontinuous on $E_{\alpha,m}$, making $\Phi_{\alpha,m}$ weakly upper semicontinuous on $\widehat{E}_{\alpha,m}(u)$. It thus follows that $\sup_{\widehat{E}_{\alpha,m}(u)} \Phi_{\alpha,m}$ is attained by a critical point u_0 of $\Phi_{\alpha,m}\big|_{\widehat{E}_{\alpha,m}(u)}$. Noting that $\sup_{t\geq 0} \Phi_{\alpha,m}(tu) > 0$ since $u \in E_{\alpha,m}^+$, it follows that $u_0 \in \mathcal{N}_{\alpha,m}$.

It remains to prove that this is the only critical point of $\Phi_{\alpha,m}\big|_{\widehat{E}_{\alpha,m}(u)}$. To this end, we let $w \in E_{\alpha,m}$ such that $u_0 + w \in \widehat{E}_{\alpha,m}(u)$. Since $\widehat{E}_{\alpha,m}(u) = \widehat{E}_{\alpha,m}(u_0)$, there exists $s \geq -1$ such that $u_0 + w = (1+s)u_0 + v$ for some $v \in F_{\alpha,m}$. Setting

$$\begin{aligned} B(v_1, v_2) &:= \int_{\mathbf{B}} \left(\nabla v_1 \cdot \nabla v_2 - \alpha^2 (\partial_\theta v_1)(\partial_\theta v_2) + m v_1 v_2\right)\, dx \\ &= \langle v_1^+, v_2^+\rangle_{\alpha,m} - \langle v_1^-, v_2^-\rangle_{\alpha,m} \end{aligned}$$

we then have

$$\begin{aligned} \Phi_{\alpha,m}(u_0 + w) - \Phi_{\alpha,m}(u_0) &= \frac{1}{2}\left(B((1+s)u_0 + v, (1+s)u_0 + v) - B(u_0, u_0)\right) \\ &\quad - I_p((1+s)u_0 + v) + I_p(u_0) \\ &= -\frac{\|v^-\|_{\alpha,m}^2}{2} + B\left(u_0, s\left(\frac{s}{2} - 1\right) u_0 + (1+s)v\right) \\ &\quad - I_p((1+s)u_0 + v) + I_p(u_0), \end{aligned}$$

where the fact that $\Phi'_{\alpha,m}(u_0)(\cdot) = B(u_0, \cdot) - I'_p(u_0)(\cdot) = 0$ then implies

$$\begin{aligned} & B\left(u_0, s\left(\frac{s}{2} - 1\right) u_0 + (1+s)v\right) - I_p((1+s)u_0 + v) + I_p(u_0) \\ =& I'_p(u_0)\left(s\left(\frac{s}{2} - 1\right) u_0 + (1+s)v\right) - I_p((1+s)u_0 + v) + I_p(u_0) \\ =& \int_{\mathbf{B}} \left(|u_0|^{p-2}u_0\left(s\left(\frac{s}{2} - 1\right) u_0 + (1+s)v\right) - \frac{1}{p}|(1+s)u_0 + v|^p + \frac{1}{p}|u_0|^p\right) dx \\ <& 0 \end{aligned}$$

by [**127**, Lemma 2.2]. □

We can then give the following existence result.

Proposition 5.5.8. *Let $\alpha \in \mathcal{A}$, $m \in \mathbb{R}$ and $p \in (2,4)$. Then $c_{\alpha,m}$ is positive and attained by a critical point of $\Phi_{\alpha,m}$. In particular,* (5.1.4) *thus has a ground state solution. Moreover,*

$$c_{\alpha,m} = \inf_{w \in E_{\alpha,m}\setminus F_{\alpha,m}} \max_{w \in \widehat{E}_{\alpha,m}(u)} \Phi_{\alpha,m}(w)$$

holds.

Proof. Note that Lemma 5.5.5 and Lemma 5.5.7 imply that $\Phi_{\alpha,m}$ satisfies the conditions of [**128**, Theorem 35]. □

In particular, this implies Theorem 5.1.2(i). Notably, this minimax characterization of $c_{\alpha,m}$ will allow us to compare the ground state energy to the minimal energy among radial solutions, which we estimate in the following.

Lemma 5.5.9. *Let $p > 2$ and $m \geq 0$, where $\lambda_1 > 0$ denotes the first Dirichlet eigenvalue of $-\Delta$ on* **B**. *Then there exists a unique positive radial solution $u_m \in H^1_{0,rad}(\mathbf{B})$ of* (4.1.5), *i.e., satisfying*

$$\begin{cases} -\Delta u + mu = |u|^{p-2}u & \text{in } \mathbf{B} \\ u = 0 & \text{on } \partial\mathbf{B}. \end{cases}$$

Moreover, there exists $c > 0$ such that

$$\beta_m^{rad} := \Phi_{\alpha,m}(u_m) \geq c m^{\frac{2}{p-2}}$$

holds for all $\alpha > 1$ and $m \geq 0$.

Proof. We consider the functional

$$\begin{aligned} J_m &: H^1_{0,rad}(\mathbf{B}) \to \mathbb{R} \\ J_m(u) &:= \frac{1}{2}\int_{\mathbf{B}} \left(|\nabla u|^2 - \alpha^2|\partial_\theta u|^2 + mu^2\right)\, dx - \frac{1}{p}\int_{\mathbf{B}} |u|^p\, dx \end{aligned}$$

which satisfies $J_m(u) = \Phi_{\alpha,m}(u)$ for every $u \in H^1_{0,rad}(\mathbf{B})$ and $\alpha > 1$. For $m \geq 0$ we consider the classical Nehari manifold

$$\mathcal{N}_m^{rad} := \left\{u \in H^1_{0,rad}(\mathbf{B}) \setminus \{0\} \,:\, J_m'(u)u = 0\right\}.$$

Clearly, any nontrivial radial critical point u of $\Phi_{\alpha,m}$ is contained in $\mathcal{N}_m^{rad}$. Moreover, the map

$$(0,\infty) \to \mathbb{R}, \quad t \mapsto J_m(tu)$$

attains a unique maximum $t_u > 0$ for each $u \in H^1_{0,rad}(\mathbf{B}) \setminus \{0\}$ and simple computations yield

$$J_m(t_u u) = \sup_{t\geq 0} J_m(tu) = \left(\frac{1}{2} - \frac{1}{p}\right)\left(\frac{\int_{\mathbf{B}} \left(|\nabla u|^2 + mu^2\right)\, dx}{\left(\int_{\mathbf{B}} |u|^p\, dx\right)^{\frac{2}{p}}}\right)^{\frac{p}{p-2}}$$

and t_u is the unique value $t > 0$ such that $tu \in \mathcal{N}_m$. It can be shown that

$$\beta_m^{rad} := \inf_{u\in\mathcal{N}_m^{rad}} J_m(u)$$

is a critical value of J_m, see e.g. [**128**]. Moreover, the principle of symmetric criticality (see e.g. [**111**]) shows that β_m^{rad} is in fact a critical value of $\Phi_{\alpha,m}$ and attained by a unique positive radial function u_m. This proves the first part of the theorem.

Next, we note that the characterization above gives

$$\begin{aligned} \beta_m^{rad} &= \inf_{u\in H^1_{0,rad}(\mathbf{B})\setminus\{0\}} \sup_{t\geq 0} J_m(tu) \\ &= \inf_{u\in H^1_{0,rad}(\mathbf{B})\setminus\{0\}} \left(\frac{1}{2} - \frac{1}{p}\right)\left(\frac{\int_{\mathbf{B}} \left(|\nabla u|^2 + mu^2\right)\, dx}{\left(\int_{\mathbf{B}} |u|^p\, dx\right)^{\frac{2}{p}}}\right)^{\frac{p}{p-2}}. \end{aligned} \tag{5.5.1}$$

In the following, we assume $m > 0$ and let $B_{\sqrt{m}}$ denote the ball of radius $\sqrt{m}$ centered at the origin. We then consider the function $v_m \in H^1_0(B_{\sqrt{m}})$ given by

$$v_m(x) = m^{-\frac{1}{p-2}} u_m\left(\frac{x}{\sqrt{m}}\right).$$

Then

$$\frac{\int_{\mathbf{B}} \left(|\nabla u_m|^2 + m u_m^2\right)\, dx}{\left(\int_{\mathbf{B}} |u_m^2|^p\, dx\right)^{\frac{2}{p}}} = m^{\frac{2}{p}} \frac{\int_{B_{\sqrt{m}}} \left(|\nabla v_m|^2 + v_m^2\right) dx}{\left(\int_{B_{\sqrt{m}}} |v_m|^p\, dx\right)^{\frac{2}{p}}}$$
$$\geq m^{\frac{2}{p}} \inf_{v \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} \left(|\nabla v|^2 + v^2\right) dx}{\left(\int_{\mathbb{R}^N} |v|^p\, dx\right)^{\frac{2}{p}}}.$$

Setting

$$C_p := \inf_{v \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} \left(|\nabla v|^2 + v^2\right) dx}{\left(\int_{\mathbb{R}^N} |v|^p\, dx\right)^{\frac{2}{p}}} > 0$$

we thus have

$$\frac{\int_{\mathbf{B}} \left(|\nabla u_m|^2 + m u_m^2\right)\, dx}{\left(\int_{\mathbf{B}} |u_m|^p\, dx\right)^{\frac{2}{p}}} \geq C_p m^{\frac{2}{p}}.$$

Therefore (5.5.1) implies

$$\beta_m^{rad} \geq \left(\frac{1}{2} - \frac{1}{p}\right)\left(C_p m^{\frac{2}{p}}\right)^{\frac{p}{p-2}}$$

and hence the claim. □

We will compare the previous estimate for the radial energy with suitable estimates for $c_{\alpha,m}$, starting with the following result.

Lemma 5.5.10. *Let $p \in (2,4)$ and $\alpha \in \mathcal{A}$. Then*

$$c_{\alpha,m} \leq \left(\frac{1}{2} - \frac{1}{p}\right) |\mathbf{B}| \inf_{(\ell,k)\in\mathcal{I}^+_{\alpha,m}} \left(j^2_{\ell,k} - \alpha^2\ell^2 + m\right)^{\frac{p}{p-2}}$$

holds for $m \in \mathbb{R}$.

Proof. By Lemma 5.5.1, there exist $\ell_0, k_0 \in \mathbb{N}$ such that

$$\left(j^2_{\ell_0,k_0} - \alpha^2\ell_0^2 + m\right) = \inf_{(\ell,k)\in\mathcal{I}^+_{\alpha,m}} \left(j^2_{\ell,k} - \alpha^2\ell^2 + m\right)$$

and we set

$$u_0 := \varphi_{\ell_0,k_0} \in E^+_{\alpha,m}.$$

For any $t \geq 0$ and $v \in F_{\alpha,m}$ it then holds that $\int_{\mathbf{B}} u_0 v\, dx = 0$ and therefore

$$\|tu_0 + v\|_p^p \geq |\mathbf{B}|^{1-\frac{p}{2}} \|tu_0 + v\|_2^p = |\mathbf{B}|^{1-\frac{p}{2}} \left(\|tu_0\|_2^2 + \|v\|_2^2\right)^{\frac{p}{2}}$$
$$\geq t^p |\mathbf{B}|^{1-\frac{p}{2}} \|u_0\|_2^p = t^p |\mathbf{B}|^{1-\frac{p}{2}}.$$

This yields

$$\Phi_{\alpha,m}(tu_0 + v) \leq \frac{t^2}{2}\left(j^2_{\ell_0,k_0} - \alpha^2\ell_0^2 + m\right) - \frac{1}{p}\|tu_0 + v\|_p^p$$
$$\leq \frac{t^2}{2}\left(j^2_{\ell_0,k_0} - \alpha^2\ell_0^2 + m\right) - \frac{t^p}{p}|\mathbf{B}|^{1-\frac{p}{2}}.$$

A straightforward computation shows that the right hand side attains a unique global maximum in

$$t^* = \left(j_{\ell_0,k_0}^2 - \alpha^2\ell_0^2 + m\right)^{\frac{1}{p-2}} |\mathbf{B}|^{\frac{1}{2}}$$

and therefore

$$\Phi_{\alpha,m}(tu_0 + v) \leq \left(\frac{1}{2} - \frac{1}{p}\right) |\mathbf{B}| \left(j_{\ell_0,k_0}^2 - \alpha^2\ell_0^2 + m\right)^{\frac{p}{p-2}}.$$

In particular, this gives

$$\max_{w\in\widehat{E}_{\alpha,m}(u_0)} \Phi_{\alpha,m}(w) \leq \left(\frac{1}{2} - \frac{1}{p}\right) |\mathbf{B}| \left(j_{\ell_0,k_0}^2 - \alpha^2\ell_0^2 + m\right)^{\frac{p}{p-2}}$$

and Proposition 5.5.8 then finally implies

$$c_{\alpha,m} = \inf_{w\in E_{\alpha,m}\setminus F_{\alpha,m}} \max_{w\in\widehat{E}_{\alpha,m}(u)} \Phi_{\alpha,m}(w) \leq \left(\frac{1}{2} - \frac{1}{p}\right) |\mathbf{B}| \left(j_{\ell_0,k_0}^2 - \alpha^2\ell_0^2 + m\right)^{\frac{p}{p-2}}$$

as claimed. □

The previous results allow us to deduce the existence of nonradial ground states whenever

$$\left(\frac{1}{2} - \frac{1}{p}\right) |\mathbf{B}| \inf_{(\ell,k)\in\mathcal{I}_{\alpha,m}^+} \left(j_{\ell,k}^2 - \alpha^2\ell^2 + m\right)^{\frac{p}{p-2}} < \beta_m^{rad}$$

holds. To this end, we estimate the growth of the left hand side as $m \longrightarrow \infty$.

Proposition 5.5.11. *Let $\alpha \in \mathcal{A}$. Then there exist constants $C > 0$, $m_0 > 0$ such that*

$$\inf_{(\ell,k)\in\mathcal{I}_{\alpha,m}^+} \left(j_{\ell,k}^2 - \alpha^2\ell^2 + m\right) \leq Cm^{\frac{1}{2}}$$

holds for $m > m_0$.

Proof. By Proposition 5.2.1 we have

$$\ell + \frac{|a_1|}{2^{\frac{1}{3}}}\ell^{\frac{1}{3}} < j_{\ell,1} < l + \frac{|a_1|}{2^{\frac{1}{3}}}\ell^{\frac{1}{3}} + \frac{3}{20}|a_1|^2\frac{2^{\frac{1}{3}}}{\ell^{\frac{1}{3}}}, \tag{5.5.2}$$

where a_1 denotes the first negative zero of the Airy function $\mathrm{Ai}(x)$. In particular, this implies that there exists $\ell_0 \in \mathbb{N}$ such that the map

$$\ell \mapsto j_{\ell,1}^2 - \alpha^2\ell$$

is strictly decreasing for $\ell \geq \ell_0$. Taking $m_0 > \alpha^2\ell_0^2 - j_{\ell_0,1}^2$ we thus find that for any $m > m_0$ there exists $\ell \geq \ell_0$ such that

$$m \in \left(\alpha^2\ell^2 - j_{\ell,1}^2, \alpha^2(\ell+1)^2 - j_{\ell+1,1}^2\right].$$

In the following, we fix such m and ℓ and note that since $j_{\ell,1} < j_{\ell+1,1}$, we have

$$\begin{aligned} 0 < j_{\ell,1}^2 - \alpha\ell^2 - (j_{\ell+1,1}^2 - \alpha(\ell+1)^2) &= j_{\ell,1}^2 - j_{\ell+1,1}^2 + \alpha^2\left((\ell+1)^2 - \ell^2\right) \\ &\leq 2\alpha^2\ell + \alpha^2 \end{aligned}$$

for $\ell \geq \ell_0$, and therefore

$$0 < j_{\ell,1}^2 - \alpha\ell^2 + m \leq 2\alpha^2\ell + \alpha^2.$$

Importantly, (5.5.2) implies that there exists $C = C(\alpha) > 0$ independent of m such that

$$2\alpha^2\ell + \alpha^2 \leq C\left(\alpha^2\ell^2 - j_{\ell,1}^2\right)^{\frac{1}{2}}$$

holds for $\ell \geq \ell_0$, after possibly enlarging ℓ_0. Ultimately, we thus find that

$$0 < j_{\ell,1}^2 - \alpha\ell^2 + m \leq C\left(\alpha^2\ell^2 - j_{\ell,1}^2\right)^{\frac{1}{2}} \leq Cm^{\frac{1}{2}}$$

holds. Since C was independent of m, this completes the proof. □

Theorem 5.1.2(ii) is now a direct consequence of the following more general result.

Theorem 5.5.12. *Let $\alpha \in \mathcal{A}$ and $p \in (2,4)$ be fixed. Then there exists $m_0 > 0$ such that the ground states of* (5.1.4) *are nonradial for $m > m_0$.*

Proof. Lemma 5.5.10 and Proposition 5.5.11 imply that there exist $C > 0$, $m_0 > 0$ such that

$$c_{\alpha,m} \leq \left(\frac{1}{2} - \frac{1}{p}\right)|\mathbf{B}|Cm^{\frac{p}{2(p-2)}}$$

holds for $m > m_0$. On the other hand, Lemma 5.5.9 gives

$$\beta_m^{rad} \geq cm^{\frac{2}{p-2}}$$

with a constant $c > 0$ independent of m. Noting that the assumption $p < 4$ implies $\frac{p}{2(p-2)} < \frac{2}{p-2}$, it follows that

$$c_{\alpha,m} < \beta_m^{rad}$$

holds for $m > m_0$, after possibly enlarging m_0. □

5.6 Complex-valued Solutions

Throughout this section we assume that all functions are complex-valued and that $p > 2$ is fixed. In this case, the eigenspaces

$$V_k := \left\{u \in H_0^1(\mathbf{B}) : \partial_\theta u = iku\right\}$$

are nonempty for $k \in \mathbb{N}$. This observation can be used to find complex-valued solutions of (5.1.4) as stated in the following.

Theorem 5.6.1. *Let $\alpha > 1$, $m > 0$ and $k \in \mathbb{N}$ be chosen such that*

$$m - \alpha^2k^2 > -\lambda_1, \tag{5.6.1}$$

where $\lambda_1 > 0$ denotes the first Dirichlet eigenvalue of $-\Delta$ on $\mathbf{B}$. Then there exists a weak solution $u \in V_k$ of (5.1.4)*. In particular, this solution is nonradial.*

We point out that the solutions found in the preceding theorem cannot be real-valued and are thus distinct from the solutions found in Theorem 5.1.2.

Proof. Inspired by [**130**], the proof is based on a constrained minimization argument for the functional

$$J_{\alpha,m} : H_0^1(\mathbf{B}) \to \mathbb{R},$$
$$J_{\alpha,m}(u) := \frac{1}{2}\int |\nabla u|^2 - \alpha^2|\partial_\theta u|^2 + mu^2\, dx.$$

Importantly, for $u \in V_k$ we have

$$J_{\alpha,m}(u) = \frac{1}{2} \int |\nabla u|^2 + (m - \alpha^2 k^2) u^2 \, dx$$

and our goal is to minimize $J_{\alpha,m}$ on V_k subject to the constraint

$$I(u) := \|u\|_p^p = 1.$$

To this end, we let $(u_n)_n \subset V_k$ be a constrained minimizing sequence, i.e., $I(u_n) = 1$ for all n and

$$\lim_{n\to\infty} J_{\alpha,m}(u_n) = \min_{\substack{u \in V_k \\ I(u)=1}} J_{\alpha,m}(u).$$

Note that V_k is a closed subspace of $H_0^1(\mathbf{B})$ and, by assumption, there exist $c, C > 0$ such that

$$c\|u\|^2_{H_0^1(\mathbf{B})} \le J_{\alpha,m}(u) \le C\|u\|^2_{H^1(\mathbf{B})}$$

holds for $u \in V_k$, which implies that the sequence $(u_n)_n$ remains bounded in $H_0^1(\mathbf{B})$ and we may pass to a weakly convergent subsequence with a weak limit $u_0 \in V_k$. The compact embedding $H_0^1(\mathbf{B}) \hookrightarrow L^p(\mathbf{B})$ then implies $I(u_0) = 1$ whereas weak lower semicontinuity yields $J_{\alpha,m}(u_0) \le \liminf J_{\alpha,m}(u_n)$, i.e., u_0 is a minimizer of $J_{\alpha,m}$ subject to the constraint $I(u_0) = 1$.

The minimization property then implies that there exists a Lagrange multiplier $K_0 \in \mathbb{R}$ such that

$$\int \nabla u_0 \cdot \nabla \varphi + (m - \alpha^2 k^2) u_0 \varphi \, dx = K_0 \int |u_0|^{p-2} u_0 \varphi \, dx \tag{5.6.2}$$

holds for $\varphi \in V_k$. Taking $\varphi = u_0$, the condition (5.6.1) then implies that K_0 must be positive. We now set

$$E : H_0^1(\mathbf{B}) \to \mathbb{R}, \qquad E(u) := J_{\alpha,m}(u) - K_0 I(u),$$

so that, in particular, u_0 is a nontrivial critical point of $E\big|_{V_k}$.

For $t \in \mathbb{R}$ we then consider the action

$$g_t : H_0^1(\mathbf{B}) \to H_0^1(\mathbf{B}), \qquad [g_t u](x) = e^{-ikt} u(R_t(x)),$$

where R_t was defined in (5.1.3). Note that g_t is an isometry on $H_0^1(\mathbf{B})$ and $L^p(\mathbf{B})$ so that E is invariant with respect to g_t. Moreover, this defines a group action on $H_0^1(\mathbf{B})$ and we have

$$V_k = \{u \in H_0^1(\mathbf{B}) : g_t u = u\}.$$

The principle of symmetric criticality (see e.g. [**111**]) then implies that u_0 is also a critical point of E on $H_0^1(\mathbf{B})$ or, equivalently, (5.6.2) holds for all $\varphi \in H_0^1(\mathbf{B})$. But this means that $K_0^{\frac{1}{p-2}} u_0$ is a weak solution of (5.1.4). □

By construction, the solutions found above are contained in the eigenspaces of the operator ∂_θ, i.e., for any such solution u there exists $k \in \mathbb{N}$ such that $u \in V_k$ and therefore $\partial_\theta u = iku$. However, this implies that $|u|$ is radial.

In the following, we briefly sketch how our methods can be used to find complex-valued solutions u of (5.1.1) (which are not real-valued) such that the modulus $|u|$ is also nonradial. To this end, we combine the ansatz (5.1.2) for rotating solutions with a standing wave ansatz, i.e.,

$$v(t, x) = e^{i\mu t} u(R_t(x))$$

with R_t given by (5.1.3) and $\mu > 0$. This reduces (5.1.1) to the modified problem

$$\begin{cases} -\Delta u + \alpha^2 \partial_\theta^2 u + 2i\mu\partial_\theta u + (m - \mu^2)u = |u|^{p-2}u & \text{in } \mathbf{B} \\ u = 0 & \text{on } \partial\mathbf{B}. \end{cases} \tag{5.6.3}$$

Here, the eigenvalues of the operator

$$L_{\alpha,m,\mu}u := -\Delta u + \alpha^2 \partial_\theta^2 u + 2i\mu\partial_\theta u + (m - \mu^2)u$$

are given by

$$j_{\ell,k}^2 - \alpha^2\ell^2 \pm 2\mu\ell + (m - \mu^2)$$

and the associated eigenfunctions are given by

$$\varphi_{\ell,k}^{\pm}(r,\theta) := e^{\pm i\ell\theta} J_\ell(j_{\ell,k}r), \qquad \ell \in \mathbb{N}_0, k \in \mathbb{N}.$$

This readily implies the following analogue to Lemma 5.5.1:

Lemma 5.6.2. *Let the sequence $(\alpha_n)_n \subset (1,\infty)$ be given by Theorem 5.4.2. Then for any $n \in \mathbb{N}$ and $m \geq 0$ there exist $c_{n,m}, \mu_n > 0$ with the following property:*

If $|\mu| \leq \mu_n$ and ℓ, k are such that $j_{\ell,k}^2 - \alpha^2\ell^2 - 2\mu\ell + (m - \mu^2) \neq 0$ holds, we have

$$|j_{\ell,k}^2 - \alpha^2\ell^2 \pm 2\mu\ell + (m - \mu^2)| \geq c_{n,m} j_{\ell,k}.$$

Proof. Note that

$$j_{\ell,k}^2 - \alpha^2\ell^2 \pm 2\mu\ell = (j_{\ell,k} + \alpha\ell)\left(j_{\ell,k} - \alpha\ell \pm \frac{2\mu\ell}{j_{\ell,k} + \alpha\ell}\right)$$

and for $\alpha = \alpha_n$ Theorem 5.4.2 then implies

$$\left| j_{\ell,k} - \alpha_n\ell \pm \frac{2\mu\ell}{j_{\ell,k} + \alpha_n\ell} \right| \geq c_n - \mu\frac{2l}{j_{\ell,k} + \alpha_n\ell} \geq c_n - \frac{2\mu}{1 + \alpha_n}$$

for sufficiently large ℓ, k. Setting

$$\mu_n := \frac{1 + \alpha_n}{2} c_n,$$

we thus find that

$$\lim_{N\to\infty} \inf_{\ell,k \geq N} \left| j_{\ell,k} - \alpha_n\ell \pm \frac{2\mu\ell}{j_{\ell,k} + \alpha_n\ell} \right| > 0$$

for $\mu < \mu_n$. □

Repeating the arguments of Section 5.5 ultimately gives the following result:

Theorem 5.6.3. *Let $p \in (2,4)$. Then there exists a sequence $(\alpha_n)_n \subset (1,\infty)$ with the following properties:*

(i) For each $n \in \mathbb{N}$ the problem (5.6.3) *has a ground state solution.*

(ii) For each $n \in \mathbb{N}$ there exists $m_n > 0$ such that any ground state u (5.6.3) *with $\alpha = \alpha_n$ and $m > m_n$ has a nonradial modulus, i.e., $|u|$ is nonradial.*

CHAPTER 6

Deutsche Zusammenfassung

6.1 Einleitung

Symmetrien spielen in der mathematischen Beschreibung naturwissenschaftlicher Beobachtungen eine wichtige Rolle. So stehen beispielsweise die Symmetrien eines Naturgesetzes und die zugehörigen Erhaltungssätze in enger Beziehung. Aus analytischer Sicht legen die Symmetrien einer partiellen Differentialgleichung wiederum nahe, dass geeignete Lösungsklassen dieselben Symmetrien besitzen sollten. Es ist daher überraschend, dass diese intuitive Relation zwischen den Symmetrien eines Problems und dessen Lösungen in vielen Fällen nicht vorliegt. Dieses Phänomen wird als *Symmetriebrechung* bezeichnet. In der modernen Physik kommt derartigen Beobachtungen eine zentrale Bedeutung zu, wobei wir auf [**68**, **98**] für weitere Beispielen aus der Quantenfeldtheorie und anderen Teilgebieten verweisen.

In dieser Arbeit untersuchen wir derartige Phänomene im Hinblick auf verschiedene partielle Differentialgleichungen mit *Radialsymmetrie*, also rotationsinvariante Probleme. Genauer sei im Folgenden $\Omega \subset \mathbb{R}^N$ stets ein rotationsinvariantes Gebiet, das heißt, eine offene zusammenhängende Menge, die $R(\Omega) = \Omega$ für alle $R \in O(N)$ erfüllt. Folglich ist Ω ein im Ursprung zentrierter Ball oder Kreisring, das Komplement eines solchen Balles oder der ganze Raum $\mathbb{R}^N$. Ferner sei L ein linearer Differentialoperator zweiter Ordnung, der insofern rotationsinvariant ist, dass $L(u \circ R) = (Lu) \circ R$ für $R \in O(N)$ und $u \in C^2(\mathbb{R}^N)$ gilt. Zusätzlich nehmen wir an, dass L Divergenzform besitzt, das heißt, es existieren C^1-Funktionen $a_{ij} : \Omega \to \mathbb{R}$, $i, j = 1, \dots, N$ und eine Funktion $c : \Omega \to \mathbb{R}$ derart, dass

$$Lu(x) = -\sum_{i,j=1}^{N} \partial_i \left[a_{ij}(x)\partial_j u(x)\right] + c(x)u(x)$$

für $u \in C^2(\Omega)$ gilt. Zunächst nehmen wir ferner an, L sei *elliptisch*, also dass die $N \times N$ Matrix $a_{ij}(x)$ für jedes $x \in \Omega$ positiv definit ist (degeneriert elliptische und elliptisch-hyperbolische Operatoren werden später ebenfalls behandelt). Das Standardbeispiel für L ist dabei durch den (negativen) Laplace-Operator $L = -\Delta$ gegeben.

Unter den obigen Voraussetzungen betrachten wir dann Symmetriebrechung für *semilineare* Gleichungen der Form

$$Lu = f(|x|, u) \quad \text{in } \Omega, \tag{6.1.1}$$

mit entsprechenden Dirichlet-Randbedingungen oder Abklingbedingungen für beschränktes und unbeschränktes Ω. Dabei sei f eine stetig differenzierbare Funktion auf $[0, \infty) \times \mathbb{R}$. In diesem Fall würde die oben diskutierte Intuition darauf hindeuten, dass sich die Radialsymmetrie von Ω, L und f in ähnlicher Form auf die Lösungen übertragen. Insbesondere stellt sich die Frage, welche Klassen von Lösungen von (6.1.1) radialsymmetrisch sein müssen und in welcher strukturellen Beziehung radialsymmetrische und nichtradialsymmetrische Lösungen stehen.

Dabei ist zunächst anzumerken, dass wir im Allgemeinen nicht erwarten können, dass alle Lösungen von (6.1.1) radialsymmetrisch sein müssen. Dies lässt sich schon an simplen linearen Beispielen, wie dem Eigenwertproblem für den Laplace-Operator auf dem Einheitsball, beobachten.

Tatsächlich sind Symmetrieeigenschaften von Lösungen eng mit deren variationeller Charakterisierung verknüpft. Die Tatsache, dass der Zustand eines physikalischen Systems häufig als Minimierer eines geeigneten Wirkungsfunktionals gegeben ist, macht dies umso interessanter. In dieser Arbeit betrachten wir insbesondere das zu (6.1.1) gehörende *Energiefunktional*, gegeben durch

$$E(u) = \frac{1}{2} \int_\Omega \left(\sum_{i,j=1}^{N} a_{ij} \partial_i u \partial_j u + cu^2 \right) dx - \int_\Omega F(x,u)\, dx$$

wobei $F(x,\cdot)$ eine Stammfunktion von $f(x,\cdot)$ sei. In Abhängigkeit der Funktionen a_{ij}, c und f ist dieses Funktional für Funktionen, deren schwache Ableitungen geeignete Integrabilitätsbedingungen erfüllen, wohldefiniert und motiviert die Betrachtung dazu passender Hilberträume. Der vorher genannte Beispielfall $L = -\Delta$ führt beispielsweise auf den klassischen Sobolev-Raum $H_0^1(\Omega)$.

Die Euler-Lagrange-Gleichung zum Funktional E ist genau durch (1.1.1) gegeben, das heißt, die Lösungen von (1.1.1) sind kritische Punkte von E. Insbesondere sind wir dann an Lösungen interessiert, die unter allen kritischen Punkten die Energie E minimieren, sogenannte *Grundzustandslösungen* oder *Grundzustände*. Aufgrund ihrer physikalischen Bedeutung ist es naheliegend, für die Grundzustände strengere Symmetrie-Eigenschaften zu erwarten. In vielen Fällen kann gezeigt werden, dass Grundzustandslösungen ihr Vorzeichen nicht wechseln und für geeignetes f als positiv angenommen werden können. Daher liefert das folgende klassische Resultat von Gidas, Ni und Nirenberg ein zentrales Werkzeug zur Untersuchung der Symmetrie-Eigenschaften von Grundzuständen.

Theorem 6.1.1. ([61])
Sei $\Omega \subset \mathbb{R}^N$ eine offene, im Ursprung zentrierte Kugel und sei $f \in C^1([0,\infty) \times \mathbb{R})$ derart, dass die Funktion $f(\cdot,t)$ für jedes feste $t \in \mathbb{R}$ monoton fallend ist. Sei ferner $u \in C^2(\overline{\Omega})$ eine positive Lösung des Problems

$$\begin{cases} -\Delta u = f(|x|,u) & \text{in } \Omega, \\ u = 0 & \text{auf } \partial\Omega. \end{cases} \tag{6.1.2}$$

Dann ist u radialsymmetrisch, und als Funktion der radialen Variable monoton fallend.

Der Beweis dieses Resultats beruht auf der sogenannten Moving-Planes-Methode, die im Folgenden auf eine Vielzahl von Problemen verallgemeinert wurde. Wir verweisen diesbezüglich beispielsweise auf [**15**, **29**, **33**, **34**, **62**, **75**, **76**, **114**]. Insbesondere liefern die Voraussetzungen dieses Resultats und dessen Varianten folglich Hindernisse für das Vorliegen von Symmetriebrechung von positiven Lösungen eines gegebenen Problems.

Der erste Fall, in dem derartige Resultate nicht gelten, ist durch Gebiete Ω gegeben, die keine passende Konvexitätsbedingung erfüllen. Während Theorem 6.1.1 geeignet auf $\mathbb{R}^N$ erweitert werden kann, existiert zum Beispiel kein analoges Resultat für Kreisringe. Ein weiterer wichtiger Fall sind Nichtlinearitäten f, die die Monotonie-Bedingung verletzen, insbesondere wenn $f(\cdot,t)$ beispielsweise streng monoton wachsend ist. Ferner gelten Symmetrie-Resultate im Sinne von Theorem 6.1.1 im Allgemeinen nicht, wenn vorzeichenwechselnde Lösungen betrachtet werden. Schließlich können wir in vielen Fällen keine

Symmetrie erwarten, wenn der Laplace-Operator in (6.1.2) durch einen anderen rotationsinvarianten Differentialoperator zweiter Ordnung ersetzt wird. Dies hängt unter anderem mit dem Verhalten des Operators bezüglich Translationen zusammen, und führt selbst bei zusätzlich elliptischen Operatoren möglicherweise zu Symmetriebrechung.

Diese Dissertation besteht aus den Arbeiten **[P1]**, **[P2]**, **[P3]** und **[P4]**,deren jeweilige Problemstellungen und Methoden sich teilweise jedoch sehr stark unterscheiden. Im Folgenden geben wir daher jeweils eine kurze Einleitung zum Inhalt jeder Arbeit und diskutieren die jeweiligen Hauptresultate sowie die verwendeten Methoden.

6.2 Symmetriebrechung für die Hénon-Gleichung

Die Hénon-Gleichung

$$\begin{cases} -\Delta u = |x|^{\alpha}|u|^{p-2}u & \text{in } \mathbf{B}, \\ \quad u = 0 & \text{auf } \partial\mathbf{B}, \end{cases} \tag{6.2.1}$$

wurde ursprünglich von Hénon [**71**] zur Beschreibung von Sternhaufen eingeführt. Aus mathematischer Sicht ist sie im Hinblick auf Symmetrie-Eigenschaften besonders interessant, weil sich die oben genannte Moving-Planes-Methode und ihre Erweiterungen nicht auf sie anwenden lassen. Dies legt nahe, dass positive Lösungen existieren, die nicht radialsymmetrisch sind. Tatsächlich gilt sogar mehr: Smets, Willem und Su [**122**] konnten zeigen, dass die Grundzustandslösungen nicht radialsymmetrisch sind, wenn α hinreichend groß ist, während Radialsymmetrie vorliegt, sofern α nahe 0 ist. Das Verhalten dieser Lösungen in verschiedenen Kontexten wurde anschließend weitgehend untersucht, siehe beispielsweise [**4**–**6**, **27**, **28**, **31**, **89**, **113**, **119**, **121**]. Dabei ist insbesondere die Beziehung zwischen radialsymmetrischen und unsymmetrischen Lösungen von Interesse. Genauer fixieren wir im Folgenden stets $K \in \mathbb{N}$, $p > 2$, sowie

$$\alpha > \alpha_p := \max\left\{\frac{(N-2)p-2N}{2}, 0\right\}.$$

Dies stellt sicher, dass $p < 2^*_\alpha = \frac{2N+2\alpha}{N-2}$ gilt, da die Existenz von Lösungen in diesem Bereich bekannt ist, während für größere p aufgrund einer entsprechenden Pohozaev-Identität keine Lösungen existieren können.

Wir betrachten dann radialsymmetrische Lösungen von (6.2.1), die genau K Knotenmengen besitzen, wobei die Knotenmengen einer Funktion $u : \Omega \to \mathbb{R}$ als die Zusammenhangskomponenten der Menge $\{x \in \Omega : u(x) \neq 0\}$ definiert sind. Für die hier betrachteten radialsymmetrischen Lösungen bedeutet dies also, dass die Funktionen genau $K-1$ Nullstellen in der radialen Variable $r = |x| \in (0,1)$ besitzen. Die Existenz einer eindeutigen, klassischen Lösung $u_\alpha \in C^2(\overline{\mathbf{B}})$, die radialsymmetrisch ist, genau K Knotenmengen besitzt und $u_\alpha(0) > 0$ erfüllt, wurde von Nagasaki [**106**] gezeigt.

In der Arbeit **[P1]** studieren wir die Verzweigung nichtradialer Lösungen von diesen radialsymmetrischen Lösungen in Abhängigkeit für $\alpha \to \infty$. Die Frage nach solcher Verzweigung führt auf die Untersuchung des linearisierten Operators

$$L^\alpha : H^2(\mathbf{B}) \cap H^1_0(\mathbf{B}) \to L^2(\mathbf{B}), \qquad \varphi \mapsto -\Delta - (p-1)|x|^\alpha |u_\alpha|^{p-2}\varphi$$

und dessen Eigenwerten. Überlegungen basierend auf dem Satz von der impliziten Funktion suggerieren nämlich, dass Verzweigung nur in solchen Punkten auftreten kann, in denen L^α degeneriert, das heißt wenn L^α den Eigenwert 0 besitzt. Dieses Kriterium der Degeneriertheit ist somit für Verzweigung notwendig, jedoch im Allgemeinen nicht hinreichend. Um tatsächliche Verzweigung zu zeigen, argumentieren wir mit geeigneten Veränderungen des Morse-Index, der durch die Anzahl der negativen Eigenwerte von L^α gegeben ist. Um derartige Änderungen des Morse-Index zu identifizieren, betrachten wir anstelle des klassischen Eigenwertproblems $L^\alpha \varphi = \lambda\varphi$ das gewichtete Eigenwertproblem

$$L^\alpha \varphi = \frac{\lambda}{|x|^2}\varphi,$$

welches von der Darstellung des Laplace-Operators in Polarkoordinaten

$$\Delta u = u_{rr} + \frac{N-1}{r}u_r + \frac{1}{r^2}\Delta_{\mathbb{S}^{N-1}} u$$

nahegelegt wird. Für Funktionen der Form

$$\varphi(x) = \psi(|x|)Y_\ell\left(\frac{x}{|x|}\right),$$

wobei Y_ℓ eine Eigenfunktion des Laplace-Beltrami Operators $\Delta_{\mathbb{S}^{N-1}}$ auf der Einheitssphäre $\mathbb{S}^{N-1}$ zum Eigenwert $-\lambda_\ell$ sei, liefert dies nämlich die Gleichung

$$L^\alpha \psi = \frac{\mu}{|x|^2}\psi$$

mit $\mu = \lambda - \lambda_\ell$. Wichtig ist dabei, dass wir nur noch radiale Funktionen ψ betrachten müssen, und uns somit auf ein eindimensionales Problem reduzieren konnten. Ein wichtiger Teil der Arbeit besteht nun darin, dieses eindimensionale Problem besser zu verstehen. Dies setzt wiederum Informationen zum Verhalten der radialen Lösungen u_α voraus, welche das folgende Resultat für $\alpha \to \infty$ charakterisiert.

Proposition 6.2.1. *Sei $p > 2$, $K \in \mathbb{N}$ und für $\alpha > \alpha_p$ sei u_α die eindeutige radiale Lösung von (6.2.1) mit K Knotenmengen und $u_\alpha(0) > 0$. Wir setzen*

$$U_\alpha : [0,\infty) \to \mathbb{R}, \qquad U_\alpha(t) = (N+\alpha)^{-\frac{2}{p-2}}\, u_\alpha(e^{-\frac{t}{N+\alpha}}).$$

Dann gilt $U_\alpha \to (-1)^{K-1}U_\infty$ gleichmäßig in $[0,\infty)$ für $\alpha \to \infty$, wobei $U_\infty \in C^2([0,\infty))$ die eindeutige beschränkte Lösung des Grenzproblems

$$-U'' = e^{-t}|U|^{p-2}U \quad \text{in } [0,\infty), \qquad U(0) = 0$$

mit $U'(0) > 0$ und genau $K-1$ Nullstellen in $(0,\infty)$ bezeichne.

Die hier verwendete, von Byeon und Wang [**27**] inspirierte Transformation überführt die Hénon-Gleichung 6.2.1 in ein eindimensionales Problem auf $[0,\infty)$, gegeben durch

$$-(e^{-\gamma t}U')' = e^{-t}|U|^{p-2}U \quad \text{in } I := [0,\infty), \qquad U(0) = 0, \tag{6.2.2}$$

mit dem neuen Parameter $\gamma = \frac{N-2}{N+\alpha}$. Das Resultat beruht dann auf einer Anwendung des Satzes von der impliziten Funktion im Punkt $\gamma = 0$ in geeigneten Funktionenräumen. Wir merken ferner an, dass die entsprechenden Argumente im Fall $N = 2$ deutlich einfacher sind.

Die dadurch erhaltene asymptotische Beschreibung der radialen Lösungen ermöglicht uns anschließend eine asymptotische Analyse des gewichteten Eigenwertproblems.

Theorem 6.2.2. *Sei $p > 2$ und $\alpha > \alpha_p$. Dann sind die negativen Eigenwerte von*

$$L^\alpha \varphi = \frac{\lambda}{|x|^2}\varphi$$

durch C^1-Funktionen $(\alpha_p, \infty) \to \mathbb{R}$, $\alpha \mapsto \mu_i(\alpha)$, $i = 1, \dots, K$ gegeben, die

$$\mu_i(\alpha) = v_i^* \alpha^2 + c_i^* \alpha + o(\alpha) \quad \text{und} \quad \mu_i'(\alpha) = 2v_i^*\alpha + c_i^* + o(1) \qquad \text{für } \alpha \to \infty,$$

erfüllen. Dabei sind c_i^*, $i = 1, \dots, K$ Konstanten und $v_1^* < v_2^* < \dots < v_K^* < 0$ sind die negativen Eigenwerte von

$$\begin{cases} -\Psi'' - (p-1)e^{-t}|U_\infty(t)|^{p-2}\Psi = \nu\Psi & \text{in } [0,\infty), \\ \quad \Psi(0) = 0, \quad \Psi \in L^\infty(0,\infty) \end{cases}$$

mit U_∞ wie in Proposition 6.2.1. Insbesondere existiert $\alpha^* > 0$ derart, dass die Funktionen μ_i, $i = 1, \dots, K$ auf $[\alpha^*, \infty)$ strikt monoton fallend sind.

Auch hier verwenden wir eine ähnliche Transformation wie in Proposition 6.2.1, um das gewichtete Eigenwertproblem auf $\mathbf{B}$ in ein eindimensionales Problem zu überführen, welches dann wiederum vom Parameter γ abhängt. Die asymptotische Beschreibung der Eigenwerte ist dann ebenfalls eine Folge des Satzes von der impliziten Funktion, dessen Anwendung hier jedoch deutlich schwieriger ist. Dies liegt zum einen an der Tatsache, dass die Eigenwerte über geeignete Quotienten variationell charakterisiert werden müssen. Zum anderen ist die im Eigenwertproblem auftauchende Abbildung $U \mapsto |U|^{p-2}$ für $p \in (2,3]$ zwischen klassischen Funktionenräumen nicht mehr differenzierbar, was eine delikatere Wahl eines geeigneten Definitionsbereiches erfordert.

Das obige Resultat liefert uns detailliertere Informationen zum asymptotischen Verhalten der Eigenwerte im Hinblick auf Nullstellen und somit potentielle Verzweigungspunkte:

Korollar 6.2.3. *Sei $p > 2$. Dann existiert zu jedem $i \in \{1, \dots, K\}$ ein $\ell_i \in \mathbb{N} \cup \{0\}$, sowie Folgen $\alpha_{i,\ell} \in (\alpha_p, \infty)$, $\varepsilon_{i,\ell} > 0$, $\ell \geq \ell_i$ mit den folgenden Eigenschaften:*

(i) $\alpha_{i,\ell} \to \infty$ *für* $\ell \to \infty$.

(ii) $\mu_i(\alpha_{i,\ell}) + \lambda_\ell = 0$. *Insbesondere ist $u_{\alpha_{i,\ell}}$ degeneriert.*

(iii) *u_α ist nichtdegeneriert für $\alpha \in (\alpha_{i,\ell} - \varepsilon_{i,\ell}, \alpha_{i,\ell} + \varepsilon_{i,\ell})$, $\alpha \neq \alpha_{i,\ell}$.*

(iv) *Für $\varepsilon \in (0, \varepsilon_{i,\ell})$ ist der Morse-Index von $u_{\alpha_{i,l}+\varepsilon}$ strikt größer als der von $u_{\alpha_{i,l}-\varepsilon}$.*

Wir verwenden diese Informationen, um mithilfe eines abstrakten Resultats von Kielhöfer [77] unser folgendes Hauptresultat zur Verzweigung nichtradialer Lösungen zu zeigen.

Theorem 6.2.4. *Sei $2 < p < \frac{2N}{N-2}$ und $K \in \mathbb{N}$, $i \in \{1, \dots, K\}$ fest. Dann sind die Punkte $\alpha_{i,\ell}$ für $\ell \geq \ell_i$ Verzweigungspunkte für nichtradiale Lösungen von (6.2.1).*

Genauer existiert für jedes $\ell \geq \ell_i$ eine Folge $(\alpha_n, u^n)_n$ in $(0,\infty) \times C^2(\overline{\mathbf{B}})$ mit den folgenden Eigenschaften:

(i) *$\alpha_n \to \alpha_{i,\ell}$ und $u^n \to u_{\alpha_{i,\ell}}$ in $C^2(\overline{\mathbf{B}})$.*

(ii) *Für alle $n \in \mathbb{N}$ ist u^n eine nichtradiale Lösung von (6.2.1) mit $\alpha = \alpha_n$ und besitzt genau K Knotenmengen $\Omega_1, \dots, \Omega_K$. Dabei gilt $0 \in \Omega_1$, Ω_1 ist homöomorph zu einem Ball, und $\Omega_2, \dots, \Omega_K$ sind homöomorph zu Annuli.*

Dabei sind $\ell_i \in \mathbb{N} \cup \{0\}$ und $\alpha_{i,\ell}$ durch Korollar 6.2.3 gegeben.

6.3 Spiralförmige Lösungen der Schrödingergleichung

Schrödinger-Gleichungen gehören aufgrund ihrer zentralen Bedeutung für die Physik zu den am meisten studierten partiellen Differentialgleichungen überhaupt und tauchen in vielen Kontexten auf. Insbesondere die nichtlineare stationäre Schrödingergleichung

$$-\Delta v + qv = |v|^{p-2}v \qquad \text{in } \mathbb{R}^N, \tag{6.3.1}$$

motivierte unzählige Resultate über Lösungen mit exponentiellem Abklingverhalten und deren Eigenschaften. Deutlich weniger ist jedoch zu Lösungen bekannt, die nur in einige, aber nicht alle Richtungen abklingen. Solche Lösungen wurden in den vergangenen Jahren zunehmend studiert und tauchen beispielsweise auch als Grenzprobleme nach Reskalierungen ähnlicher Gleichungen in beschränkten Gebieten auf.

In der Arbeit **[P2]** betrachten wir den Fall $N = 3$, $p > 2$ und untersuchen Lösungen der Gleichung

$$-\Delta v + v = |v|^{p-1}v \qquad \text{in } \mathbb{R}^3, \tag{6.3.2}$$

die spiralförmig sind, das heißt, sie sind invariant bezüglich einer Schraubbewegung. Für $\lambda > 0$ ist dies dadurch charakterisiert, dass

$$v(R_\theta x, t + \lambda\theta) = v(x, t) \qquad \text{für } x \in \mathbb{R}^2,\ t \in \mathbb{R},$$

gilt, wobei $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ die Rotation mit Winkel θ entgegen des Uhrzeigersinns bezeichne. Insbesondere sind diese Funktionen in t also 2π-periodisch. Durch einen geeigneten Ansatz kann das Problem dann auf die Gleichung

$$\begin{cases} -\Delta u - \dfrac{1}{\lambda^2}\partial_\theta^2 u + u = |u|^{p-2}u & \text{in } \mathbb{R}^2, \\ \qquad\qquad u(x) \to 0 & \text{für } |x| \to \infty, \end{cases} \tag{6.3.3}$$

reduziert werden, wobei $\partial_\theta := x_1\partial_{x_2} - x_2\partial_{x_1}$ die Winkelableitung bezeichnet. Die Funktion $u : \mathbb{R}^2 \to \mathbb{R}$ kann dabei als das Profil bei $t = 0$, also $v(\cdot, 0)$ interpretiert werden.

Radialsymmetrische Lösungen von (6.3.3) entsprechen dabei jeweils axialsymmetrischen, t-invarianten Lösungen von (6.3.2). Ein Resultat von Farina, Malchiodi und Rizzi [**56**], in dem Symmetrie-Eigenschaften im Sinne der Resultate von Gidas, Ni und Nirenberg wie in Theorem 6.1.1 für Lösungen ohne exponentielles Abklingverhalten studiert werden, impliziert dabei, dass positive Lösungen von (6.3.3) stets radialsymmetrisch sein müssen.

Unser Ziel besteht folglich darin, vorzeichenwechselnde Lösungen zu studieren. Genauer verwenden wir variationelle Methoden, um die Lösungen zu charakterisieren und analysieren, die unter allen vorzeichenwechselnden Lösungen die niedrigste Energie besitzen. Dazu betrachten wir den Raum

$$H := \left\{ u \in H^1(\mathbb{R}^2) \ :\ \int_{\mathbb{R}^2} |\partial_\theta u|^2 dx < \infty \right\}.$$

Für $\lambda > 0$ betrachten wir H als Hilbertraum mit dem λ-abhängigen Skalarprodukt

$$\langle u, v \rangle_\lambda := \int_{\mathbb{R}^2} \left(\nabla u \cdot \nabla v + \frac{1}{\lambda^2}(\partial_\theta u)(\partial_\theta v) + uv \right) dx.$$

Ferner definieren wir das Energiefunktional $E_\lambda : H \to \mathbb{R}$ bezüglich (6.3.3) als

$$E_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla u|^2 + \frac{1}{\lambda^2}|\partial_\theta u|^2 + |u|^2 \right) dx - \frac{1}{p}\int_{\mathbb{R}^2} |u|^p \, dx.$$

Dann ist E_λ ein C^1-Funktional, dessen kritische Punkte genau die schwachen Lösungen von (6.3.3) sind.

Die vorzeichenwechselnde Lösung kleinster Energie definieren wir dann als Minimierer von E_λ bezüglich der Menge aller vorzeichenwechselnder Lösungen von (6.3.3). Unser erstes Hauptresultat charakterisiert diese Lösungen im Hinblick auf Radialsymmetrie:

Theorem 6.3.1. *Sei $p > 2$. Für alle $\lambda > 0$ existiert eine vorzeichenwechselnde Lösung kleinster Energie für* (6.3.3)*. Ferner existieren $0 < \lambda_0 \leq \Lambda_0 < \infty$ mit den folgenden Eigenschaften:*

(i) Für $\lambda < \lambda_0$ ist jede vorzeichenwechselnde Lösung kleinster Energie von (6.3.3) *radialsymmetrisch.*

(ii) Für $\lambda > \Lambda_0$ ist jede vorzeichenwechselnde Lösung kleinster Energie von (6.3.3) *nicht radialsymmetrisch.*

Insbesondere beobachten wir Symmetrie-Brechung für $\lambda \to \infty$. Der Beweis verwendet im Wesentlichen zwei verschiedene Charakterisierungen der Energien für $\lambda \to 0$ und $\lambda \to \infty$. Der Fall $\lambda \to 0$ entspricht (i), und beruht auf der folgenden energetischen Bedingung für die Radialität von Lösungen.

Theorem 6.3.2. *Sei $p > 2$.*

(i) Sei $u \in H$ eine nichttriviale schwache Lösung von (6.3.3) *für ein $\lambda > 0$ mit $\lambda < \left(\frac{1}{(p-1)\|u\|_{L^\infty}^{p-2}}\right)^{\frac{1}{2}}$. Dann ist u radialsymmetrisch.*

(ii) Für $c > 0$ existiert $\lambda_c > 0$ derart, dass jede schwache Lösung $u \in H$ von (6.3.3) *mit $\lambda \in (0, \lambda_c)$ und $E_\lambda(u) \leq c$ notwendigerweise radialsymmetrisch ist.*

Der Beweis von Theorem 6.3.2 basiert auf uniformen L^∞-Abschätzungen für schwache Lösungen von (6.3.3) und einer Poincaré-Ungleichung für die Winkelableitung. Letztere liefert nämlich die Ungleichung

$$\frac{1}{\lambda^2}\|\partial_\theta u\|_{L^2}^2 \leq (p-1)\|u\|_{L^\infty}^{p-2}\|\partial_\theta u\|_{L^2}^2,$$

die bereits (i) impliziert und gemeinsam mit den genannten Abschätzungen schließlich auch (ii) zeigt.

Der Fall $\lambda \to \infty$, der Theorem 6.3.1(ii) entspricht, beruht auf der Konstruktion einer geeigneten Klasse von Lösungen und der Abschätzung deren Energie. Genauer betrachten wir Lösungen von

$$\begin{cases} -\Delta u - \dfrac{1}{\lambda^2}\partial_\theta^2 u + u = |u|^{p-2}u & \text{in } \mathbb{R}^2_+, \\ u = 0 & \text{auf } \partial\mathbb{R}^2_+ \end{cases} \tag{6.3.4}$$

und reflektieren diese negativ an der Hyperebene $\partial\mathbb{R}^2_+$, um eine vorzeichenwechselnde Lösung von (6.3.3) zu erhalten. Die wesentliche Erkenntnis ist dabei, dass Lösungen von (6.3.4) für $\lambda \to \infty$ nach Translationen gegen eine Lösung von

$$-\Delta u + u = |u|^{p-2}u \qquad \text{in } \mathbb{R}^2 \tag{6.3.5}$$

konvergieren. Genauer gilt Folgendes.

Theorem 6.3.3. *Sei $p > 2$ und $\lambda > 0$.*

(i) ***(Existenz)*** *Das Problem* (6.3.4) *besitzt eine Grundzustandslösung.*

(ii) ***(Symmetrie)*** *Jede positive Lösung u von* (6.3.4) *ist bezüglich der Reflektion an der x_1-Achse symmetrisch und im Winkel $|\theta|$ zur x_1-Achse monoton fallend. Insbesondere nimmt u sein Maximum also auf der x_1-Achse an.*

(iii) ***(Asymptotik)*** *Für $\lambda_k \geq 1$ mit $\lambda_k \to +\infty$ für $k \to \infty$ sei u_k jeweils eine positive Grundzustandslösung von* (6.3.4) *mit $\lambda = \lambda_k$. Nach Übergang zu einer Teilfolge existieren dann $\tau_k > 0$ mit*
$$\tau_k \longrightarrow +\infty, \qquad \frac{\tau_k}{\lambda_k} \longrightarrow 0 \qquad \text{für } k \to \infty$$
derart, dass für die translatierten Funktionen $w_k \in H^1(\mathbb{R}^2)$, $w_k(x) = u_k(x_1 + \tau_k, x_2)$ dann
$$w_k \longrightarrow w_\infty \qquad \text{in } H^1(\mathbb{R}^2)$$
gilt, wobei w_∞ die eindeutige radiale Lösung in $H^1(\mathbb{R}^2)$ von (6.3.5) *bezeichne.*

Der Beweis von Theorem 6.3.1(ii) beruht dann auf der Beobachtung, dass die Energie der somit konstruierten Folge von Lösungen von (6.3.3) gegen $2c_\infty$ konvergiert, wobei c_∞ die Energie der Grundzustandslösung von (6.3.5) bezeichne. Im Kontrast dazu ist die minimale Energie von radialen vorzeichenwechselnden Lösungen jedoch nach unten durch $2c_\infty + \varepsilon_*$ für ein festes $\varepsilon_* > 0$ beschränkt und somit für hinreichend großes λ also nicht mehr minimal.

6.4 Rotierende Wellen in nichtlinearen Medien

In der Arbeit **[P3]** betrachten wir zeitperiodische Lösungen der nichtlinearen Wellengleichung
$$\begin{cases} \partial_{tt} v - \Delta v + mv = |v|^{p-2} v & \text{in } \mathbb{R} \times \mathbf{B}, \\ v = 0 & \text{auf } \mathbb{R} \times \partial \mathbf{B} \end{cases} \tag{6.4.1}$$
für $N \geq 2$, wobei $\mathbf{B} \subset \mathbb{R}^N$ die Einheitskugel bezeichnet, $2 < p < 2^*$ und $m > -\lambda_1(\mathbf{B})$. Dabei bezeichnet $\lambda_1(\mathbf{B})$ den ersten Dirichlet-Eigenwert von $-\Delta$ auf $\mathbf{B}$ und 2^* bezeichnet den kritischen Sobolev-Exponenten gegeben durch $2^* = \frac{2N}{N-2}$ für $N \geq 3$ und $2^* = \infty$ für $N = 2$. Für $m > 0$ wird (6.4.1) auch als nichtlineare Klein-Gordon Gleichung bezeichnet. Nichtlineare Wellengleichungen werden typischerweise verwendet, um die Ausbreitung von Wellen in einem nichtlinearen Medium, beispielsweise in der nichtlinearen Optik, zu modellieren.

Wir studieren zeitperiodische Lösungen von (6.4.1) der Form
$$v(t, x) = u(R_{\alpha t}(x)), \tag{6.4.2}$$
wobei $R_\theta \in O(N)$ eine Rotation in einer Ebene in $\mathbb{R}^N$ mit Winkel $\theta \in \mathbb{R}$ bezeichne, das heißt die Konstante $\alpha > 0$ entspricht der Winkelgeschwindigkeit. Ohne Beschränkung der Allgemeinheit können wir annehmen, dass
$$R_\theta(x) = (x_1 \cos\theta + x_2 \sin\theta, -x_1 \sin\theta + x_2 \cos\theta, x_3, \dots, x_N)$$
für $x \in \mathbb{R}^N$ gilt, und R_θ somit die Drehung in der x_1-x_2-Ebene mit Fixpunktmenge $\{0_{\mathbb{R}^2}\} \times \mathbb{R}^{N-2}$ darstellt. Im Folgenden bezeichnen wir eine Funktion u auf $\mathbf{B}$ als x_1-x_2-*nichtradial*, wenn mindestens ein Winkel $\theta \in \mathbb{R}$ existiert, sodass u nicht R_θ-invariant ist. Ist die Profilfunktion

u im Ansatz (6.4.2) x_1-x_2-nichtradial, so kann die entsprechende Lösung v also als rotierende Welle in einem nichtlinearen Medium interpretiert werden.

Der Ansatz (6.4.2) reduziert (6.4.1) zu

$$\begin{cases} -\Delta u + \alpha^2 \partial_\theta^2 u + mu = |u|^{p-2}u & \text{in } \mathbf{B}, \\ u = 0 & \text{auf } \partial\mathbf{B}, \end{cases} \tag{6.4.3}$$

wobei $\partial_\theta = x_1\partial_{x_2} - x_2\partial_{x_1}$ die Ableitung nach dem Winkel in der x_1-x_2-Ebene bezeichne.

Dieser Ableitungsoperator trat auch im Zusammenhang mit spiralförmigen Lösungen nichtlinearer Schrödingergleichungen in Abschnitt 6.3 auf, das negative Vorzeichen führt hier jedoch zu einem völlig andersartigen Verhalten. Insbesondere gilt

$$-\Delta + \alpha^2\partial_\theta^2 = -\Delta_r u - \frac{1}{r^2}\Delta_{\mathbb{S}^{N-1}} u + \alpha^2 \partial_\theta^2 u, \tag{6.4.4}$$

woraus ersichtlich wird, dass der Operator für $\alpha \geq 1$ nich mehr gleichmäßig elliptisch ist.

Es ist zu beachten, dass eine Lösung u von (6.4.3) auch $\partial_\theta u \equiv 0$ in $\mathbf{B}$ erfüllen könnte. In diesem Fall löst u die klassische stationäre nichtlineare Schrödingergleichung $-\Delta u + mu = |u|^{p-2}u$ in $\mathbf{B}$ mit Dirichlet-Randbedingungen auf $\partial\mathbf{B}$, und erfüllt somit (6.4.3) mit $\alpha = 0$. Ist u zudem positiv, so impliziert das klassische Symmetrie-Resultat von Gidas, Ni und Nirenberg [**61**], dass u eine radiale Funktion sein muss. Radiale Lösungen von (6.4.3) erzeugen jedoch rotierende Wellen, die zeitlich konstant und folglich uninteressant sind. Unser Hauptziel ist daher die Existenz positiver Lösungen von (6.4.3), die $\partial_\theta u \equiv 0$ nicht erfüllen.

Genauer untersuchen wir Grundzustandslösungen, die als Minimierer des Rayleigh-Quotienten $R_{\alpha,m,p} : H_0^1(\mathbf{B}) \setminus \{0\} \to \mathbb{R}$ charakterisiert sind. Dieser ist durch

$$R_{\alpha,m,p}(u) = \frac{\int_{\mathbf{B}} \left(|\nabla u|^2 - \alpha^2|\partial_\theta u|^2 + mu^2\right)\, dx}{\left(\int_{\mathbf{B}} |u|^p\, dx\right)^{\frac{2}{p}}} \tag{6.4.5}$$

für $\alpha, m \in \mathbb{R}$ und $p \in [2, 2^*)$ gegeben. Wir betrachten also Funktionen, die das Minimierungsproblem

$$\mathscr{C}_{\alpha,m,p}(\mathbf{B}) := \inf_{u \in H_0^1(\mathbf{B})\setminus\{0\}} R_{\alpha,m,p}(u) \tag{6.4.6}$$

lösen. In diesem Abschnitt beschränken wir uns dabei ferner auf den Fall $\alpha \in [0, 1]$, da

$$\mathscr{C}_{\alpha,m,p}(\mathbf{B}) = -\infty \tag{6.4.7}$$

für jedes $p \in [2, 2^*)$, $m \in \mathbb{R}$ und $\alpha > 1$ gilt. Dies folgt im Wesentlichen aus der Tatsache, dass der in (6.4.4) angegebene Operator in $\mathbf{B} \setminus B_{1/\alpha}(0)$ für $\alpha > 1$ hyperbolisch ist, was die Konstruktion einer geeigneten Folge von Testfunktionen mit negativen Werten ermöglicht. Insbesondere erfordert der Fall $\alpha > 1$ daher komplett andere Methoden, die im späteren Abschnitt 6.5 vorgestellt werden.

Für $0 \leq \alpha < 1$ ist der Operator gleichmäßig elliptisch und die Existenz von Minimierern von $R_{\alpha,m,p}$ auf $H_0^1(\mathbf{B}) \setminus \{0\}$ folgt aus der Kompaktheit der Einbettung $H_0^1(\mathbf{B}) \hookrightarrow L^p(\mathbf{B})$, sowie der schwachen Unterhalbstetigkeit des Zählers von $R_{\alpha,m,p}$. Dabei ist zunächst jedoch völlig unklar, ob Minimierer radial oder nichtradial sind. Um dieser Frage weiter nachzugehen, stellen wir fest, dass die Abbildung

$$\alpha \mapsto \mathscr{C}_{\alpha,m,p}(\mathbf{B}) \tag{6.4.8}$$

für jedes feste $p \in [2, 2^*)$ und $m \in \mathbb{R}$ auf $[0,1]$ stetig und monoton fallend ist. Da für $\alpha \in [0,1]$ und jede radiale Funktion $u \in H_0^1(\mathbf{B}) \setminus \{0\}$ ferner $R_{\alpha,m,p}(u) = R_{0,m,p}(u)$ gilt, liefert die Ungleichung

$$\mathscr{C}_{\alpha,m,p}(\mathbf{B}) < \mathscr{C}_{0,m,p}(\mathbf{B}) \tag{6.4.9}$$

eine hinreichende Bedingung für die x_1-x_2-Nichtradialität aller Grundzustandslösungen.

Unser erstes Ergebnis zeigt, dass die Grundzustände radial sind, sofern α hinreichend klein ist.

Theorem 6.4.1. *Seien $m \geq 0$ und $2 < p < 2^*$. Dann existiert $\alpha_0 > 0$ derart, dass*

$$\mathscr{C}_{\alpha,m,p}(\mathbf{B}) = \mathscr{C}_{0,m,p}(\mathbf{B})$$

für $\alpha \in [0, \alpha_0)$ gilt. Ferner existiert für $\alpha \in [0, \alpha_0)$ eine (bis auf das Vorzeichen) eindeutige Grundzustandslösung von (6.4.3) und diese ist radialsymmetrisch.

Der Beweis basiert auf dem Satz über implizite Funktionen unter Verwendung bekannter Ergebnisse bezüglich der Nichtdegeneriertheit von positiven radialen Lösungen des klassischen Problems

$$\begin{cases} -\Delta u + mu = |u|^{p-2}u & \text{in } \mathbf{B}, \\ u = 0 & \text{auf } \partial\mathbf{B}. \end{cases}$$

Im Folgenden stellt sich also die Frage, ob die Bedingung (6.4.9) für passende Wahlen der Parameter erfüllt sein kann, wobei das vorige Resultat insbesondere die Untersuchung der Ungleichung für α nahe 1 suggeriert. Dabei ist anzumerken, dass sich der Grenzfall $\alpha = 1$ deutlich vom Fall $0 \leq \alpha < 1$ unterscheidet, da der Differentialoperator $-\Delta + \partial_\theta^2$ wie oben angemerkt auf $\mathbf{B}$ nicht mehr gleichmäßig elliptisch ist. Nichtsdestotrotz werden wir die Stetigkeit und Monotonie der Abbildung (6.4.8) verwenden, um Symmetriebrechung anhand des Falls $\alpha = 1$ zu untersuchen.

Wie sich überraschend herausstellt, ist das Minimierungsproblem im Fall $\alpha = 1$ eng mit einer degenerierten anisotropen kritischen Sobolev-Ungleichung auf dem Halbraum verbunden. Der entsprechende kritische Exponent in dieser Sobolev-Ungleichung ist dabei durch

$$2_1^* := \frac{4N+2}{2N-3}$$

gegeben. Die Relevanz dieses Exponenten zeigt unser erstes Hauptresultat.

Theorem 6.4.2. *Sei $m > -\lambda_1(\mathbf{B})$ und $p \in (2, 2^*)$.*

(i) Für $\alpha \in (0,1)$ existiert eine Grundzustandslösung von (6.4.3).

(ii) Es gilt

$$\mathscr{C}_{1,m,p}(\mathbf{B}) = 0 \quad \text{für } p > 2_1^*, \qquad \text{und} \quad \mathscr{C}_{1,m,p}(\mathbf{B}) > 0 \quad \text{für } p \leq 2_1^*. \tag{6.4.10}$$

Außerdem existiert für jedes $p \in (2_1^*, 2^*)$ ein $\alpha_p \in (0,1)$ mit der Eigenschaft, dass

$$\mathscr{C}_{\alpha,m,p}(\mathbf{B}) < \mathscr{C}_{0,m,p}(\mathbf{B})$$

für $\alpha \in (\alpha_p, 1]$ gilt. Insbesondere ist jede Grundzustandslösung von (6.4.3) also x_1-x_2-nichtradial für $\alpha \in (\alpha_p, 1)$.

Der Fall $m = 0$, $\alpha = 1$ liefert insbesondere die folgende neue degenerierte Sobolev-Ungleichung

$$\left(\int_{\mathbf{B}} |u|^{2_1^*}\,dx\right)^{\frac{2}{2_1^*}} \leq \frac{1}{\mathscr{C}_{1,0,p}(\mathbf{B})} \int_{\mathbf{B}} \left(|\nabla u|^2 - |\partial_\theta u|^2\right) dx \qquad \text{für } u \in H_0^1(\mathbf{B}).$$

Außerdem ist der Exponent 2_1^* insofern optimal, dass für $p > 2_1^*$ keine solche Ungleichung gilt.

Ferner merken wir an, dass die Sätze 6.4.2 und 6.4.1 implizieren, dass für festes $p > 2_1^*$ Symmetriebrechung von Grundzustandslösungen vorliegt, wenn eine kritische Schwelle $\alpha = \alpha(p) \in [\alpha_0, \alpha_*]$ überschritten wird. Allerdings ist bisher unklar, ob Derartiges auch im Fall $p \leq 2_1^*$ auftritt. Bevor wir jedoch weitere Symmetriebrechungsresultate für beliebige Werte von p diskutieren, erläutern wir zunächst die Beweisidee für Theorem 6.4.

Der Beweis beruht auf einer genauen Analyse des Quotienten $R_{\alpha,m,p}$ für Funktionen, deren Träger in einer Umgebung des Äquators $\{x \in \partial\mathbf{B} : x_3 = \cdots = x_N = 0\}$ liegt. Wie sich zeigt, ist der Quotient für solche Funktionen nach unten durch

$$\inf_{u \in C_c^1(\mathbb{R}_+^N)} \frac{\displaystyle\int_{\mathbb{R}_+^N} \left(\sum_{i=1}^{N-1} |\partial_i u|^2 + 2x_1 |\partial_N u|^2\right) dx}{\left(\int_{\mathbb{R}_+^N} |u|^{2_1^*}\,dx\right)^{\frac{2}{2_1^*}}} \tag{6.4.11}$$

beschränkt, wobei wir hier den Halbraum

$$\mathbb{R}_+^N := \left\{x \in \mathbb{R}^N : x_1 > 0\right\}$$

betrachten. Im Fall $N = 2$ ist die zu (6.4.11) gehörende Ungleichung auch als Grushin-Ungleichung bekannt, wobei wir auf [**70**] für eine ausführlichere Darstellung von Grushin-Operatoren und -Ungleichungen verweisen. Im Fall $N \geq 3$ hat (6.4.11) zwar denselben kritischen Exponenten wie gewisse Grushin-Ungleichungen und deren Varianten, insbesondere [**58**, Theorem 1.7], wird jedoch von diesen bekannten Ergebnissen nicht abgedeckt.

Dies motiviert das folgende, allgemeinere Resultat.

Theorem 6.4.3. *Sei $s > 0$ und wir setzen $2_s^* := \frac{4N+2s}{2N-4+s}$. Dann gilt*

$$\mathcal{S}_s(\mathbb{R}_+^N) := \inf_{u \in C_c^1(\mathbb{R}_+^N)} \frac{\displaystyle\int_{\mathbb{R}_+^N} \sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2\,dx}{\left(\int_{\mathbb{R}_+^N} |u|^{2_s^*}\,dx\right)^{\frac{2}{2_s^*}}} > 0. \tag{6.4.12}$$

Außerdem wird der Wert $\mathcal{S}_s(\mathbb{R}_+^N)$ in $H_s \setminus \{0\}$ erreicht, wobei H_s den Abschluss von $C_c^1(\mathbb{R}_+^N)$ im Raum

$$\left\{u \in L^{2_s^*}(\mathbb{R}_+^N) : \|u\|_{H_s}^2 := \int_{\mathbb{R}_+^N} \sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2\,dx < \infty\right\} \tag{6.4.13}$$

bezüglich der Norm $\|\cdot\|_{H_s}$ bezeichne.

Ähnlich zur klassischen Sobolev-Ungleichung auf $\mathbb{R}^N$, ist der Quotient in (6.4.12) invariant bezüglich einer anisotropen Reskalierung, was dazu führt, dass die Einbettung $H_s \hookrightarrow L^{2_s^*}(\mathbb{R}_+^N)$ nicht kompakt ist. Folglich müssen auch hier Concentration-Compactness-Methoden zum Beweis der Existenz von Minimierern verwendet werden. Dabei weisen wir jedoch darauf hin, dass der hier betrachtete Fall $s > 0$ sich deutlich vom klassischen Fall $s = 0$ unterscheidet, da die beste Konstante in der Sobolev-Ungleichung bekanntlich nur auf $\mathbb{R}^N$ angenommen wird. Ein wesentlicher Unterschied liegt dabei in der Tatsache, dass das Gewicht x_1^s im Fall $s > 0$ dazu führt, dass der Quotient in x_1-Richtung nicht mehr translationsinvariant ist.

Ferner merken wir an, dass wir für den Beweis von Theorem 6.4.2 nur den Fall $s = 1$ benötigen, und dies ebenso zum Beweis analoger Aussagen auf Kreisringen mit äußerem Radius 1 verwendet werden kann. Allerdings kann die allgemeinere Ungleichung für $s \in (0, 2]$ für die Untersuchung von (6.4.3) auf einer Klasse von abstrakteren Riemannschen Mannigfaltigkeiten mit Rand anstelle von $\mathbf{B}$ verwendet werden. Diese Klasse beinhaltet inbesondere Rotationshyperflächen wie die Halbsphäre, was beispielsweise auf die Ungleichung im Fall $s = 2$ führt.

Wie oben angemerkt liefert Theorem 6.4.2 detaillierte Informationen zur Symmetriebrechung für festes m und großes α und p, allerdings bleiben die Symmetrie-Eigenschaften der Grundzustände für p nahe 2 weiterhin unklar. Deren Eigenschaften können wir zumindest für große $m > 0$ im folgenden Resultat erläutern.

Theorem 6.4.4. *Seien $\alpha \in (0, 1)$ und $2 < p < 2^*$. Dann existiert $m_0 > 0$ mit der Eigenschaft, dass die Bedingung (6.4.9) für $m \geq m_0$ erfüllt ist, und somit jede Grundzustandslösung von (6.4.3) für $m \geq m_0$ x_1-x_2-nichtradial ist.*

Der Beweis basiert auf einer Reskalierung von Funktionen $u \in H_0^1(\mathbf{B})$ durch $u_\varepsilon(x) := u(\varepsilon x)$. Dies liefert eine Funktion in $H_0^1(B_{1/\varepsilon})$, wobei $B_{1/\varepsilon} := B_{1/\varepsilon}(0)$. Dies kann verwendet werden, um das Minimierungsproblem auf $\mathbf{B}$ für radiale Funktionen mit dem klassischen Minimierungsproblem auf dem Ganzraum für

$$\inf_{v \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} \left(|\nabla v|^2 + v^2\right) dx}{\left(\int_{\mathbb{R}^N} |v|^p \, dx\right)^{\frac{2}{p}}}$$

in Verbindung zu bringen. Der zusätzliche Winkelableitungsterm in $R_{\alpha,m,p}$ kann dann geeignet abgeschätzt werden, um zu zeigen, dass die Minimierer für hinreichend kleines ε nicht radial sein können.

Als nächstes diskutieren wir den Grenzfall $\alpha = 1$ im Minimierungsproblem (6.4.6). Da $u \mapsto \left(\int_{\mathbf{B}} \left(|\nabla u|^2 - |\partial_\theta u|^2\right) \, dx\right)^{\frac{1}{2}}$ auf $H_0^1(\mathbf{B})$ keine äquivalente Norm definiert, arbeiten wir stattdessen mit dem größeren Raum $\mathcal{H}(\mathbf{B})$, der als Abschluss von $C_c^1(\mathbf{B})$ in

$$\left\{u \in L^{2_1^*}(\mathbf{B}) : \|u\|^2_{\mathcal{H}(\mathbf{B})} := \int_{\mathbf{B}} \left(|\nabla u|^2 - |\partial_\theta u|^2\right) \, dx < \infty\right\}$$

bezüglich der Norm $\|\cdot\|_{\mathcal{H}(\mathbf{B})}$ gegeben ist. Dies erlaubt uns die Erweiterung der Ergebnisse aus den Theoremen 6.4.2 und 6.4.4 auf den Fall $\alpha = 1$.

Theorem 6.4.5. *Seien $2 < p < 2_1^*$ und $\alpha = 1$.*

(i) Für jedes $m > -\lambda_1(\mathbf{B})$ existiert eine Grundzustandslösung von (6.4.3).

(ii) Es existiert $m_0 > 0$ derart, dass die Bedingung (6.4.9) für $m \geq m_0$ erfüllt ist, und somit jede Grundzustandslösung $u \in \mathcal{H}(\mathbf{B})$ von (6.4.3) für $m \geq m_0$ x_1-x_2-nichtradial ist.

Da die Einbettung $\mathcal{H} \hookrightarrow L^{2_1^*}(\mathbf{B})$ nicht kompakt ist, ist die Existenz von Grundzuständen im kritischen Fall $\alpha = 1$, $p = 2_1^*$ im Allgemeinen offen. Für diesen Fall haben wir folgendes Teilresultat, das diese Frage mit der optimalen Konstante in der Ungleichung (6.4.11) in Verbindung bringt.

Theorem 6.4.6. *Sei $m > -\lambda_1(\mathbf{B})$ derart, dass*

$$\mathscr{C}_{1,m,2_1^*}(\mathbf{B}) < 2^{\frac{1}{2}-\frac{1}{2_1^*}} \mathcal{S}_1(\mathbb{R}^N_+) \tag{6.4.14}$$

erfüllt ist. Dann wird der Wert $\mathscr{C}_{1,m,2_1^*}(\mathbf{B})$ in $\mathcal{H}(\mathbf{B}) \setminus \{0\}$ angenommen, das heißt, es existiert eine Grundzustandslösung von (6.4.3)*. Ferner existiert $\varepsilon > 0$ mit der Eigenschaft, dass die Bedingung* (6.4.14)*) für alle $m \in (-\lambda_1(\mathbf{B}), -\lambda_1(\mathbf{B}) + \varepsilon)$ erfüllt ist.*

Der zusätzliche Faktor $2^{\frac{1}{2}-\frac{1}{2_1^*}}$ in (6.4.14) taucht aufgrund der Skalierungseigenschaften des Quotienten (6.4.12) auf.

6.5 Rotierende Wellen im elliptisch-hyperbolischen Fall

Da die vorherigen Resultate detaillierte Informationen zum Fall $\alpha \leq 1$ liefern, stellt sich die Frage, wie sich der Fall $\alpha > 1$ im Kontrast dazu verhält. In der Arbeit **[P4]** studieren wir dies im zweidimensionalen Fall ausführlich. Dazu erinnern wir daran, dass der Ansatz (6.4.2) die nichtlineare Wellengleichung (6.4.1) auf das Problem

$$\begin{cases} -\Delta u + \alpha^2 \partial_\theta^2 u + mu = |u|^{p-2}u & \text{in } \mathbf{B}, \\ \qquad\qquad\qquad u = 0 & \text{auf } \partial\mathbf{B} \end{cases} \tag{6.5.1}$$

reduzierte, wobei $\partial_\theta = x_1 \partial_{x_2} - x_2 \partial_{x_1}$ die Ableitung nach dem Winkel bezeichne.

Zunächst stellen wir fest, dass die variationelle Struktur in diesem Fall komplett anders ist, da der Quotient $R_{\alpha,m,p}$ nun nicht mehr von unten beschränkt ist, wie wir zuvor angemerkt hatten. Dies ist im Wesentlichen der Tatsache geschuldet, dass der Operator

$$L_\alpha := -\Delta + \alpha^2 \partial_\theta^2$$

für $\alpha > 1$ nicht mehr elliptisch ist. In Polarkoordinaten $(r, \theta) \in (0,1) \times (-\pi, \pi)$ zeigt sich nämlich, dass

$$L_\alpha u = -\partial_r^2 u - \frac{1}{r}\partial_r u - \left(\frac{1}{r^2} - \alpha^2\right) \partial_\theta^2 u$$

gilt und der Term $1/r^2 - \alpha^2$ sein Vorzeichen wechselt. Folglich ist L_α im Ball mit Radius $1/\alpha$ elliptisch, auf der Sphäre mit Radius $1/\alpha$ parabolisch und im Rest hyperbolisch.

Der Operator ist also *elliptisch-hyperbolisch*. Zu derartigen Gleichungen ist im Allgemeinen deutlich weniger bekannt, da die Bereiche verschiedenen Typs meist getrennt untersucht werden müssen, siehe [**101**, **110**]. Als Konsequenz dieses Mangels an verfügbaren Resultaten, ist die Frage nach der Existenz geschweige denn Symmetrie-Eigenschaften von Grundzuständen völlig unklar.

Die Grundlage unserer Untersuchungen bildet eine Charakterisierung des Spektrums von L_α. Dabei verwenden wir die Tatsache, dass in Polarkoordinaten (r, θ) Funktionen der Form

$$(r, \theta) \mapsto J_\ell(j_{\ell,k} r) \times \begin{cases} \cos(\ell\theta) \\ \sin(\ell\theta) \end{cases}, \qquad \ell \in \mathbb{N}_0, k \in \mathbb{N}$$

eine Basis von $L^2(\mathbf{B})$ bilden, wobei J_ℓ die Besselfunktion erster Gattung ℓ-ter Ordnung bezeichne, und $j_{\ell,k}$ deren k-te Nullstelle. Wichtig ist hierbei, dass diese Funktionen jeweils Eigenfunktionen des (negativen) Laplace-Operators $-\Delta$ zum Eigenwert $j_{\ell,k}^2$ darstellen, aber offensichtlich auch Eigenfunktionen des Winkelableitungsoperators ∂_θ^2 sind. Folglich ist jede solche Funktion eine Eigenfunktion von L_α zum Eigenwert $j_{\ell,k}^2 - \alpha^2\ell^2$. Diese Darstellung der Eigenwerte beleuchtet direkt eine der Hauptfragepunkte: Kann eine unbeschränkte Folge (ℓ_i, k_i) existieren, für die $j_{\ell_i,k_i}^2 - \alpha^2\ell_i^2$ beschränkt bleibt? Dies würde die Existenz eines Häufungspunktes im Spektrum implizieren, was die Anwendung variationeller Methoden erheblich erschwert. Besonders problematisch wäre dabei ein Häufungspunkt in 0, aber prinzipiell könnte das Spektrum sogar eine dichte Teilmenge der reellen Zahlen sein. Unser erstes Hauptresultat schließt diese Fälle aus.

Theorem 6.5.1. *Für jedes $\alpha > 1$ ist das Spektrum von L_α nach oben und unten unbeschränkt. Ferner existiert eine unbeschränkte Folge $(\alpha_n)_n \subset (1,\infty)$ mit den folgenden Eigenschaften:*

(i) Das Spektrum von L_{α_n} besteht für $n \in \mathbb{N}$ nur aus Eigenwerten endlicher Vielfachheit.

(ii) Es existiert $c_n > 0$ derart, dass für jedes $\ell \in \mathbb{N}_0$, $k \in \mathbb{N}$ entweder $j_{\ell,k}^2 - \alpha_n^2\ell^2 = 0$ oder

$$|j_{\ell,k}^2 - \alpha_n^2\ell^2| \geq c_n j_{\ell,k} \tag{6.5.2}$$

gilt.

(iii) Das Spektrum von L_{α_n} besitzt keine endlichen Häufungspunkte.

Der Beweis beruht notwendigerweise auf neuen Resultaten zum asymptotischen Verhalten der Nullstellen $j_{\ell,k}$ für $\ell, k \to \infty$. Dazu stellen wir zunächst fest, dass die Formel

$$j_{\ell,k}^2 - \alpha^2\ell^2 = \ell(j_{\ell,k} + \alpha\ell)\left(\frac{j_{\ell,k}}{\ell} - \alpha\right)$$

impliziert, dass eine Folge von Punkten $j_{\ell_i,k_i}^2 - \alpha^2\ell_i^2$ genau dann beschränkt bleiben kann, wenn $j_{\ell_i,k_i}/\ell_i \to \alpha$ gilt. Bekannte Abschätzungen für $j_{\ell,k}$ liefern dann, dass dies nur dann der Fall sein kann, wenn $\ell_i/k_i \to \sigma$ für ein $\sigma > 0$ gilt. Tatsächlich suggeriert ein Resultat von Elbert and Laforgia [**50**], dass für jedes $\alpha > 1$ ein eindeutiges $\sigma > 0$ existiert, für das

$$\frac{j_{\sigma k,k}}{\sigma k} \to \alpha \qquad \text{für } k \to \infty \tag{6.5.3}$$

gilt. Ist $\alpha > 1$ so gewählt, dass σ rational ist, so könnten solche Folgen also potentiell problematisch sein. Folglich untersuchen wir zunächst die Konvergenzordnung in (6.5.3) mithilfe der Watson Formel [**134**]. Dies erlaubt es uns schließlich auszuschließen, dass derartige Folgen zu Häufungspunkten im Spektrum führen. Das Verhalten allgemeiner Folgen j_{ℓ_i,k_i}, die nur $\ell_i/k_i = \sigma + o(1)$ erfüllen, erfordert dann eine genauere Betrachtung und führt letztlich zur Einschränkung auf solche $\alpha > 1$, für die das zugehörige $\sigma > 0$ eine geeignete rationale Zahl ist. Für irrationale Werte von σ ist das Verhalten des Spektrums jedoch offen, was im Hinblick auf ähnliche Phänomene im Zusammenhang mit dem Spektrum des radialen Wellenoperators auf Bällen [**17, 95**] nicht überraschend ist.

Die Charakterisierung des Spektrums von L_α in Theorem 6.5.1 ist von zentraler Bedeutung, da uns dies erlaubt einen Hilbertraum $E_{\alpha,m}$ zu definieren, auf dem die quadratische Form

$$u \mapsto \int_{\mathbf{B}} \left(|\nabla u|^2 - \alpha^2|\partial_\theta u|^2 + mu^2\right)\, dx$$

wohldefiniert ist und mithilfe der zugehörigen Norm ausgedrückt werden kann. Die Abschätzung (6.5.2) kann dabei verwendet werden, um zu zeigen, dass $E_{\alpha,m}$ für $p \in (2,4)$ kompakt in den Raum $L^p(\mathbf{B})$ eingebettet ist.

Folglich können wir für $p \in (2,4)$ somit schwache Lösungen von (6.5.1) als kritische Punkte des Energiefunktionals

$$\Phi_{\alpha,m} : E_{\alpha,m} \to \mathbb{R}, \quad \Phi_{\alpha,m}(u) := \frac{1}{2}\int_{\mathbf{B}} \left(|\nabla u|^2 - \alpha^2|\partial_\theta u|^2 + mu^2\right)\, dx - \frac{1}{p}\int_{\mathbf{B}} |u|^p\, dx$$

charakterisieren. Wie im Fall $\alpha \in [0,1]$ ist das Energiefunktional nach oben und unten jeweils unbeschränkt und besitzt nur die Nullfunktion als lokales Minimum. Allerdings ist $\Phi_{\alpha,m}$ nun im Gegensatz zum Fall $\alpha \in [0,1]$ auf einem unendlichdimensionalen Unterraum strikt negativ und besitzt somit für $\alpha > 1$ keine Mountain-Pass-Geometrie.

Um kritische Punkte zu finden, betrachten wir daher zunächst die Unterräume $E^+_{\alpha,m}$, $E^0_{\alpha,m}$, $E^-_{\alpha,m}$, die jeweils von den Eigenfunktionen zu positiven, Null- und negativen Eigenwerten aufgespannt werden. Hier ist wichtig, dass der Raum $E^0_{\alpha,m}$ für $\alpha = \alpha_n$ gemäß Theorem 6.5.1 endlichdimensional ist.

Wir setzen dann $F_{\alpha,m} := E^0_{\alpha,m} \oplus E^-_{\alpha,m}$ und betrachten die verallgemeinerte Nehari-Mannigfaltigkeit

$$\mathcal{N}_{\alpha,m} := \left\{u \in E_{\alpha,m} \setminus F_{\alpha,m} :\ \Phi'_{\alpha,m}(u)u = 0 \text{ und } \Phi'_{\alpha,m}(u)v = 0 \text{ für alle } v \in F_{\alpha,m}\right\}.$$

Diese wurde ursprünglich von Pankov [**112**] eingeführt, und später von Szulkin und Weth [**128**] weiter untersucht. Genau wie die klassische Nehari-Mannigfaltigkeit, enthält $\mathcal{N}_{\alpha,m}$ alle kritischen Punkte von $\Phi_{\alpha,m}$, und es kann gezeigt werden, dass das Infimum

$$c_{\alpha,m} = \inf_{u \in \mathcal{N}_{\alpha,m}} \Phi_{\alpha,m}(u)$$

positiv ist und von einem kritischen Punkt von $\Phi_{\alpha,m}$ angenommen wird, sofern $\alpha = \alpha_n$ gilt, wobei die Werte α_n durch Theorem 6.5.1 gegeben sind. Folglich bezeichnen wir derartige Minimierer als *Grundzustandslösungen* von (6.5.1).

Wie im Fall $\alpha \leq 1$ müssen wir uns aber auch hier dann die Frage stellen, ob diese Grundzustandslösungen radialsymmetrisch sein könnten, was stationäre Lösungen von (6.4.1) liefern würde. Unser zweites Hauptresultat zeigt, dass die Grundzustandslösungen für geeignete Wahlen der Parameter nicht radialsymmetrisch sind.

Theorem 6.5.2. *Sei $p \in (2,4)$, und die Folge $(\alpha_n)_n \subset (1,\infty)$ sei wie in Theorem 6.5.1 gegeben. Dann gelten folgende Eigenschaften:*

(i) Für jedes $n \in \mathbb{N}$ und $m \in \mathbb{R}$ existiert eine Grundzustandslösung von (6.5.1) für $\alpha = \alpha_n$.

(ii) Für jedes $n \in \mathbb{N}$ existiert ein $m_n > 0$ derart, dass die Grundzustandslösungen von (6.5.1) für $\alpha = \alpha_n$ und $m > m_n$ nicht radialsymmetrisch sind.

Der Beweis dieses Resultats beruht im Wesentlichen auf Abschätzungen der entsprechenden Energien. Dazu bezeichnen wir die Energie der eindeutigen positiven radialsymmetrischen Lösung von (6.5.1) mit β_m^{rad}. Mit einem ähnlichen Reskalierungsargument wie im Beweis von Theorem 6.4.4 können wir dann zeigen, dass eine Konstante $c > 0$ existiert, sodass

$$\beta_m^{rad} \geq cm^{\frac{2}{p-2}}$$

für $\alpha > 1$ und $m \geq 0$ gilt. Andererseits erlaubt uns eine geeignete Minimax-Charakterisierung der Grundzustandsenergie $c_{\alpha,m}$, kombiniert mit einer weiteren Untersuchung der Werte $j_{\ell,k}$ zu zeigen, dass

$$c_{\alpha,m} \leq \left(\frac{1}{2} - \frac{1}{p}\right) |\mathbf{B}| C m^{\frac{p}{2(p-2)}}$$

mit einer Konstante $C > 0$ gilt. Da nach Voraussetzung $p < 4$ gilt, erhalten wir für hinreichend großes $m > 0$ also

$$c_{\alpha,m} < \beta_m^{rad}$$

wie behauptet.

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