Dissipative spin hydrodynamics from quantum field theory

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Zusammenfassung

Einleitung

In dieser Arbeit leiten wir dissipative Spinhydrodynamik aus Quantenfeldtheorie her. Eine Motivation hierfür sind Messungen der Polarisation von Lambda Hyperonen in nichtzentralen Schwerionenkollisionen. Diese Polarisation entsteht, da das System über einen hohen Bahndrehimpuls verfügt, der in Spinpolarisation umgewandelt wird. Dabei richtet sich die Spins der Teilchen entlang der Fluidvortizität aus. Während thermodynamische Rechnungen im lokalen Gleichgewicht die sogenannte globale, also die über den Impuls gemittelte, Polarisation sehr gut beschreiben [1–7], weisen Berechnungen der lokalen, d.h. impulsabhängigen, Polarisation mit demselben Ansatz [8] eine umgekehrte Winkelabhängigkeit im Vergleich zu den Messungen [9] auf. Vor Kurzem wurden vielversprechende Arbeiten veröffentlicht, in denen die Beachtung von Beiträgen der Scherspannung zum Polarisationsvektor in der Lage zu sein scheint, eine Übereinstimmung zwischen Theorie und Experiment herzustellen [10–15]. In den bisherigen Arbeiten zu diesem Thema wurde von lokalem Gleichgewicht ausgegangen, dissipative Effekte wurden vernachlässigt. Dabei ist nicht bekannt, wie groß diese Effekte sind und welche Rolle sie für die Polarisation in Schwerionenkollisionen spielen. Ziel dieser Arbeit ist es, solche dissipativen Effekte zu studieren. Spin ist eine Quanteneigenschaft von Materie, weshalb wir von Quantenfeldtheorie ausgehen. Mit Hilfe der Wignerfunktion leiten wir dann eine kinetische Theorie her. Da wir in dieser Arbeit Teilchen mit Spin $1/2$ betrachten, ist der Ausgangspunkt hierfür die Diracgleichung. Im nächsten Schritt benutzen wir dann die Momentenmethode, um dissipative Spinhydrodynamik von der Boltzmanngleichung herzuleiten.

Schwerionenkollisionen stellen eine einzigartige Möglichkeit dar, das sogenannte Quarkgluonenplasma (QGP) zu studieren. Letzteres ist ein Zustand elementarer Teilchen, die über die starke Wechselwirkung wechselwirken, bei sehr hoher Energie, in dem diese nicht wie in unserer täglichen Umgebung zu Hadronen gebunden sind, sondern ungebunden auftreten und sich wie ein Fluid verhalten. Das Phasendiagram stark wechselwirkender Materie ist aktueller Forschungsgegenstand [16–18] und Schwerionenkollisionen helfen, dieses zu verstehen. Insbesondere geben Messungen der Polarisation, von denen auf Eigenschaften des Systems wie Vortizität und andere Gradienten geschlossen werden kann, Aufschluss über das Verhalten des QGP [19, 20]. Da das QGP sich sehr gut durch Hydrodynamik beschreiben lässt, bietet es sich an, Spinfreiheitsgrade zu konventioneller Hydrodynamik hinzuzufügen, um Polarisationseffekte zu beschreiben.

Zur Beschreibung von Polarisationseffekten spielt der relativistische Spinvektor eine wichtige Rolle. Der nichtrelativistische Spinvektoroperator ist proportional zu den Pauli-Matrizen. Eine Möglichkeit, diesen in einer relativistischen Theorie zu verallgemeinern, ist, einen Vierervektoroperator zu definieren, dessen räumliche Komponenten im Teilchenruhesystem gleich dem nichtrelativistischen Spinvektor sind. Dies ist der sogannte Pauli-Lubanski-Vektor [21].

Der Erwartungswert des Pauli-Lubanski-Vektors stellt die Polarisation des Systems dar [21]. In paritätsverletzenden Zerfällen werden die Tochterteilchen bevorzugt entlang der Polarisation im Ruhesystem des zerfallenden Teilchens emittiert. Aus diesem Grund lässt sich die Polarisation von Lambda-Hyperonen in Schwerionenkollisionen über der Impulsverteilung ihrer Zerfallsprodukte bestimmen. Die sogenannte globale Polarisation erhält man nach Integration über alle Impulse. Diese ist parallel zum globalen Drehimpuls des Systems und wurde in Refs. [22–25] gemessen. Hydrodynamische Berechnungen in lokalem Gleichgewicht unter der Annahme, dass die Polarisation durch die thermische Vortizität bestimmt ist, haben die Resultate dieser Messungen korrekt vorhergesagtbeschreiben [1–3]. Auf der anderen Seite beschreiben die gleichen Modelle die lokale (impulsabhängige) Polarisation entlang der longitudinalen Richtung [9] nicht [8], obwohl aktuell vierversprechende Fortschritte gemacht wurden [10–15]. Um dissipative Effekte auf die Spindynamik, beispielsweise für Anwendung bei Lambda-Polarisation, zu beschreiben, verwenden wir dissipative Hydrodynamik.

Um dissipative Hydrodynamik zu formulieren, ist die Herleitung von der mikroskopischen Theorie, die durch kinetische Theorie gegeben ist, eine häufig verwendete Methode. Dabei unterscheidet man Theorien erster Ordnung, in denen dissipative Ströme direkt durch Gradienten ausgedrückt werden, und Theorien zweiter Ordnung, in denen die dissipativen Ströme dynamisch behandelt werden und ihren eigenen Bewegungsgleichungen folgen. Erstere [26] wurden lange Zeit für akausal und instabil gehalten [27], wobei kürzlich entdeckt wurde, dass dies im Allgemeinen nicht der Fall sein muss [28–31]. Dagegen sind zweitere als kausale und stabile Theorien etabliert. Ihre Herleitung aus kinetischer Theorie erfolgt üblicherweise durch die sogenannte Momentenmethode, die in Refs. [32, 33] entwickelt und in Refs. [34–36] verbessert wurde. Diese wird in dieser Arbeit verwendet, um dissipative Spinhydrodynamik zweiter Ordnung herzuleiten.

Der Spintensor und das Lokalisierungsproblem

Konventionelle Hydrodynamik basiert auf den Erhaltungsgleichungen des Ladungsstroms und des Energieimpulstensors. Um diesen Formalismus um Spinfreiheitsgrade zu erweitern, muss man zusätzlich die Erhaltung des Gesamtdrehimpulses berücksichtigen, welcher die Summe aus Bahndrehimpuls und Spintensor darstellt [37–39]. Der Spintensor selbst ist im Allgemeinen nicht erhalten, sondern seine Divergenz ist proportional zum antisymmetrischen Anteil des Energieimpulstensors. Hierbei ist zu beachten, dass in einer relativistischen Theorie die Aufspaltung des Gesamtdrehimpulses in Bahn- und Spinanteil nicht eindeutig ist. Dies resultiert in der sogenannten Pseudoeichfreiheit des Energieimpuls- und Spintensors [40]. Pseudoeichtransformationen ändern die Form letzterer Tensoren, nicht aber Erhaltungsgleichung des Gesamtdrehimpulses. Weitherhin lassen sie den globalen Drehimpuls, der als der über eine raumartige Hyperfläche integrierte Gesamtdrehimpuls definiert ist, invariant. Wir diskutieren verschiedene Wahlmöglichkeiten der Pseudoeichung für freie Dirac-Felder, basierend auf Ref. [41].

Die sogenannten kanonischen Ströme werden mit Hilfe des Noethertheorems aus der Dirac-Lagrangedichte hergeleitet. Sie haben die Eigenschaft, dass der kanonische Energieimpulstensor für freie Felder nicht symmetrisch ist. Hieraus folgt, dass der kanonische Spintensor selbst in nicht wechselwirkenden Systemen nicht erhalten ist. Dies ist nicht konsistent mit der Interpretation des Spintensors als Spindichte, da sich letztere im physikalischen Bild nur durch Wechselwirkungen ändern sollte. Zudem ist der globale Spin in diesem Fall kein Lorentztensor, da er als Hyperflächenintegral des nichterhaltenen Spintensors definiert ist.

In Einsteins Allgemeiner Relativitätstheorie ist der Energieimpulstensor durch die Variation der Lagrangedichte nach dem metrischen Tensor definiert und daher per Definition symmetrisch. Die Pseudoeichung, in der der Energieimpulstensor dem auf diese Weise definierten entspricht, ist als Belinfanteform bekannt [42]. In dieser Pseudoeichung verschwindet der Spintensor und kann deshalb ebenfalls nicht als Spindichte interpretiert werden.

Ein Set aus symmetrischem Energieimpulstensor und erhaltenem, nichtverschwindendem Spintensor ist in der Hilgevoord-Wouthuyson(HW)-Pseudoeichung [43, 44] gegeben. Die erhaltenen Ströme können in diesem Fall mit Hilfe des Noethertheorems aus der Klein-Gordon-Langrangedichte für Spinoren hergeleitet werden, wobei die Gültigkeit der Diracgleichung als Zusatzbedingung gefordert wird. Da der Spintensor in diesem Fall erhalten ist, ist der globale Spin ein Lorentztensor. Zudem erfüllt er die sogenannte Frenkelbedingung, was darauf hinweist, dass sich der HW Spin im Teilchenruhesystem mit dem nichtrelativistischen Spin identifizieren lässt.

Zwei weitere Pseudoeichungen stellen die De-Groot-van-Leeuwen-van-Weert(GLW)- [45] und alternativen Klein-Gordon(KG)-Ströme dar, wobei Letztere aus einer modifizierten Klein-Gordon-Langrangedichte für Spinoren hergeleitet werden können. Beide Pseudoeichungen weisen starke Ähnlichkeit mit der HW-Pseudoeichung auf und führen zu demselben globalen Spin.

Ein wichtiges Werkzeug zur Formulierung einer kinetischen Theorie ist der Wigneroperator, der aus der Zweipunktfunktion berechnet wird, indem eine Fouriertransformation der relativen Ortskoordinate der Zweipunktfunktion durchgeführt wird. In der Konsequenz hängt die Wignerfunktion von einer Ortskoordinate (der Schwerpunktskoordinate der Zweipunktfunktion) und einer Impulskoordinate (aus der Fouriertransformation) ab, analog zu einer klassischen Phasenraumdichte in kinetischer Theorie. Zudem lassen sich aus der Diracgleichung Bewegungsgleichungen für den Wigneroperator herleiten. Wir zerlegen den Wigneroperator in einer Basis der Generatoren der Cliffordalgebra und spalten die Bewegungsgleichung des Wigneroperators [45] in ihre einzelnen Komponenten auf. Alle diskutierten Energieimpuls- und Spintensoren können dann durch die Komponenten des Wigneroperators ausgedrückt werden.

Eine Möglichkeit, einen relativistischen Spinvektor zu definieren, ist über die räumlichen Komponenten des globalen Spins. Benutzt man hierfür den kanonischen Spintensor, entspricht der so definierte Spinvektor dem nichtrelativistischen in jedem Bezugssystem, was zu einer nicht kovarianten Beschreibung führt. Auf der anderen Seite erhält man mit dem HW Spintensor einen Spinvektor, der nur im Teilchenruhesystem dem kanonischen (und damit auch dem nichtrelativistischen) Spin entpricht, und gleichzeitig Teil eines Lorentztensors ist. Daher kann der so definierte HW Spin als kovariante Verallgemeinerung des kanonischen gesehen werden.

Weiterhin kann ein relativistischer Spinvektor als der sogenannte Pauli-Lubanski-Vektoroperator definiert werden, der vom Impulsoperator und dem globalen Drehimpulsoperator abhängt und damit per Definition unabhängig von der Pseudoeichung ist. Wenn der Wigneroperator benutzt wird, kann der Pauli-Lubanski-Vektor auch über die Impulsvariable des Wigneroperators statt des Impulsoperators definiert werden [46]. Obwohl diese beiden Definitionen nicht identisch sind, führen sie zur selben Interpretation des Spinvektors, da es equivalent ist, den Spin im Ruhesystem des Impulsoperators oder der Impulsvariable der Wignerfunktion zu definieren. Letztere Form des Pauli-Lubanski-Vektors ist ebenfalls unabängig von der Pseudoeichung. Wird die HW-Form verwendet, wird dieser durch die Tensorkomponente des Wigneroperators ausgedrückt, mit der kanonischen Form durch die Axialvektorkomponente. Da der Pauli-Lubanski-Vektor in direkter Verbindung zu der Polarisation z.B. in Schwerionenkollisionen steht, ist das Ziel dieser Arbeit, die Ensemblemittelwerte dieser Komponenten des Wigneroperators zu berechnen.

Der physikalische Hintergrund der Pseudoeichtransformationen liegt in der Beliebigkeit, in einer relativistischen Theorie einen Schwerpunkt zu definieren, oder in anderen Worten, ein sich drehendes Teilchen zu lokalisieren [47, 48]. Jede Wahl der Pseudoeichung entspricht dabei einer anderen Aufspaltung des Gesamtdrehimpulses in Bahn- und Spinanteil. Wählt man eine bestimmten Definition der Position des rotierenden Teilchens, kann sein Gesamtdrehimpuls in einen externen Anteil der Bewegung dieser Ortskoordinate und den internen Anteil der Rotation um diese aufgespalten werden. Auf der anderen Seite kann der totale Drehimpuls auch in Generatoren von Boosts und Rotationen gespalten werden, wobei diese Aufteilung vom Bezugssystem abhängt. Das sogenannte Trägheitszentrum ist als die über die Energiedichte gemittelte Position definiert, und damit ebenfalls abhängig vom Bezugssystem. Eine kovariante Verallgemeinerung ist das sogenannte Zentroid, definiert als das Trägheitszentrum in einem festen Bezugssystem, welches sich mit einer gegebenen Vierergeschwindigkeit bewegt. In der Belinfante Pseudoeichung verschwindet der interne Drehimpuls, und der Gesamtdrehimpuls ist durch den Bahndrehimpuls gegeben. Andererseits lässt sich bei Benutzung des kanonischen Spins eine Positions als Zentrum der Rotation festlegen, die dem Trägheitszentrum entspricht. Der interne Drehimpuls hierbei ist nicht kovariant, jedoch nicht identisch mit dem kanonischen Spin selbst. Wählt man das Teilchenruhesystem als Referenzsystem, entpricht das Zentroid dem Massenschwerpunkt. In diesem Fall verschwinden die internen Anteil der Boost Generatoren und die externen Anteil der Rotationsgeneratoren im Gesamtdrehimpuls, und der interne Anteil der Rotationsgeneratoren ist gerade durch den Pauli-Lubanski-Vektor gegeben. Der interne Drehimpuls ist in diesem Fall identisch zum HW-Spin.

Der zum kanonischen Spin gehörende Positionsoperator hat die Eigenschaft, dass der Betrag seiner Zeitableitung gerade die Lichtgeschwindigkeit ist. Die Bewegung eines freien Dirac-Teilchens setzt sich nämlich aus einer geradlinigen Bewegung und der sogenannten Zitterbewegung zusammen, wobei letztere nicht messbar ist, und die Summe dieser Geschwindigkeiten ergibt immer Lichtgeschwindigkeit [49]. Die nicht messbare und damit unphysikalische Zitterbewegung wird durch die Verwendung des Trägheitszentrums als Positionsoperator herausgemittelt. Wird die Masse hier Null gesetzt, ist der durch den zugehörigen internen Drehimpuls definierte Spinvektor immer parallel zum Impuls, und die Nichtkovarianz des Trägheitszentrums im masselosen Limes führt zu dem sogenannten Side-Jump Effect [50–54]. Auf der anderen Seite kann für massive Teilchen immer ein kovarianter Positionsoperator gefunden werden, der dem Massenschwerpunkt entspricht und zum HW Spin gehört. Dieser ist nur im Teilchenruhesystem gleich dem Trägheitszentrumsoperator, und stellt daher die in diesem System gemittelte Ortskoordinate dar.

Kinetische Theorie mit Spin aus Quantenfeldtheorie: Nichtlokale Kollisionen

Um in mikroskopischen Kollisionen Bahndrehimpuls in Spin umzuwandeln, ist ein nichtlokaler Kollisionsterm notwendig. Dies ist im nichtrelativistischen Fall bereits bekannt [55, 56]. In diesem Kapitel leiten wir einen nichtlokalen Kollsionsterm für relativistische Quantensysteme mit Hilfe des Wignerfunktionsformalismus und einer Entwicklung in der Planck-Konstante $\hbar$ bzw. in Gradienten der Wignerfunktion her [57, 58]. Die Wignerfunktion W ist als das Ensemblemittel des Wigneroperators definiert. Betrachtet man die Diracgleichung, die um einen allgemeinen Wechselwirkungsterm ergänzt ist, werden die kinetischen Gleichungen der Wignerfunktion entsprechend modifiziert. Die Wignerfunktion gehorcht dann einer Boltzmann-artigen Bewegungsgleichung von der Form [45]

$$p \cdot \partial W = C,$$

die einen allgemeinen Kollisionsterm C enthält. Zudem führen die Wechselwirkungsterme zu einer Modifikation der Massenschale. Nun wird der Kollisionsterm in der Boltzmanngleichung für die Wignerfunktion explizit berechnet, wobei wir uns an dem Formalismus aus Ref. [45] orientieren. Hierzu berechnen wir das

Ensemblemittel, indem wir die Spur über die nichtwechselwirkenden initialen n-Teilchenzustände bilden. Hierbei nehmen wir an, dass die Dichte gering genug ist, dass Korrelationen zwischen Zuständen vernachlässigt werden können, dies ist der sogenannte Stoßzahlansatz. Unter der Annahme binärer Kollisionen berücksichtigen wir zudem nur Zweiteilchenzustände. Der resultierende Kollisionsterm hängt nun von der initialen Wignerfunktion ab. Um eine geschlossene Bewegungsgleichung für die wechselwirkende Wignerfunktion zu erhalten, wird die initiale Wignerfunktion im Kollisionsterm durch diese ersetzt, wobei Korrekturterme bei geringer Dichte vernachlässigt werden können. Weiterhin hängt die Wignerfunktion im Kollisionsterm von allen Positionen des gesamten Raumes ab, der Kollisionsterm ist vollkommen nichtlokal. Wir entwickeln die Ortsabhängigkeit der Wignerfunktion im Kollisionsterm auf der rechten Seite der Boltzmanngleichung um die Ortskoordinate x der Wignerfunktion auf der linken Seite der Boltzmanngleichung, und berücksichtigen nichtlokale Korrekturen erster Ordnung in Gradienten der Wignerfunktion. Zudem entwickeln wir auf beiden Seiten der Boltzmanngleichung die Wignerfunktion selbst in Ordnungen von $\hbar$, was ebenfalls einer Gradiententenentwicklung entspricht. Wieder wird die Wignerfunktion in die Generatoren der Cliffordalgebra zerlegt. Man kann zeigen, dass sich unter der Annahme, dass Polarisationseffekte nicht zur nullten Ordnung in $\hbar$ auftreten, alle erhaltenen Strömen durch die zum Impuls parallele Komponente der Vektrokomponente $\bar{\mathcal{F}}$ und die Axialvektorkomponente $\mathcal{A}^\mu$ der Wignerfunktion ausdrücken lassen. Daher werden die Boltzmann-artigen Bewegungsgleichungen und Massenschalenbedingungen für diese beiden Größen betrachtet.

Es ist vorteilhaft, eine skalare Verteilungsfunktion f im erweiterten Phasenraum zu definieren, die zusätzlich zu Ort und Impuls auch von einer kontinuierlichen Spinvariablen abhängt und dieselben Informationen wie $\bar{\mathcal{F}}$ und $\mathcal{A}^\mu$ enthält. Letztere Größen lassen sich durch Integration von f über die Spinvariable berechnen. Auch die skalare Verteilungsfunktion folgt einer Boltzmann-artigen Bewegungsgleichung

$$p \cdot \partial f = \mathfrak{C}[f]$$

und einer modifizierten Massenschalenbedingung. Der Kollisionsterm $\mathfrak{C}[f]$ auf der rechten Seite der Boltzmanngleichung setzt sich aus einem lokalen und einem nichtlokalen Anteil zusammen, wobei ersterer Beiträge sowohl nullter als auch erster Ordnung enthält, letzterer allerdings nur solche erster Ordnung.

Zunächst wird der lokale Kollisionsterm berechnet. Dieser ist immer on-shell. Wir drücken alle Komponenten der Wignerfunktion durch die Verteilungsfunktion im erweiterten Phasenraum aus. Nach Anwendung des optischen Theorems erhalten wir einen lokalen Kollisionsterm, der sich als Summe von zwei Beitragen zusammensetzt. Der erste Summand entspricht Kollisionen, in denen sowohl Spin als auch Impuls ausgetauscht werden, während zweiterer Kollisionen beschreibt, in denen nur Spin, aber kein Impuls ausgetauscht wird. Da in der Herleitung die Annahme geringer Dichte genutzt wurde, enthält der Kollisionterm keine Pauli-Blocking-Faktoren, was im Gleichgewicht zu Boltzmann- statt Fermi-Dirac-Verteilungen führen wird. Falls die Verteilungsfunktionen nicht vom Spin abhängen, reduziert sich der berechnete Kollisionsterm auf die Form, die aus der klassischen Boltzmanngleichung bekannt ist. Für spinabhangige Verteilungsfunktionen hat der erste Summand des lokalen Kollisionsterm allerdings noch nicht die Struktur von Gewinn- und Verlusttermen, die man erwarten würde. Aus diesem Grund machen wir Gebrauch von der Freiheit, sowohl den Kollisionsterm als auch die Verteilungsfunktion im Phasenraum umzudefinieren, ohne die Form der Boltzmanngleichung oder physikalische Größen zu beeinflussen, da man Letztere immer nach Integration über die Spinvariable erhalt. Mit Hilfe dieser Eigenschaft bringen wir den Kollisionsterm in eine Form, die die vertraute Struktur von Gewinn- und Verlustterm aufweist, und nach Integration über die Spinvariable mit der alten Form des Kollisionsterm identisch ist. Die Verteilungsfunktion bleibt dabei bis zur ersten Ordnung in $\hbar$ unverändert.

Im nächsten Schritt berechnen wir den nichtlokalen Kollisionsterm. Dieser setzt sich aus verschiedenen Beiträgen zusammen. Zunachst kann man on-shell und off-shell Terme unterscheiden. Letztere treten auf beiden Seiten der Boltzmanngleichung in identischer Form auf, sodass sie einander wegheben und eine Boltzmanngleichung auf der Massenschale übrig bleibt. Der nichtlokale Kollisionsterm in dieser Gleichung enthält unter anderem Impulsableitungen von Matrixelementen des Kollisionsoperators, welche in der Näherung geringer Dichte vernachlässigt werden können. Alle weiteren nichtlokalen Beitrage lassen sich als Verschiebung Δ^μ der Ortskoordinate der Verteilungsfunktionen im Kollisionsterm ausdrücken,

$$\begin{aligned}
\tilde{\mathfrak{C}}_{\text{on-shell}}[f(x,p,\mathfrak{s})] \quad = \quad & \int d\Gamma_1 d\Gamma_2 d\Gamma' \, \widetilde{\mathcal{W}} \, [f(x+\Delta_1-\Delta,p_1,\mathfrak{s}_1) f(x+\Delta_2-\Delta,p_2,\mathfrak{s}_2) \\
& - f(x,p,\mathfrak{s}) f(x+\Delta'-\Delta,p',\mathfrak{s}')] \\
& + \int d\Gamma_2 \, dS_1(p) \, \mathfrak{W} f(x+\Delta_1,p,\mathfrak{s}_1) f(x+\Delta_2,p_2,\mathfrak{s}_2) \; ,
\end{aligned}$$

wobei $\int d\Gamma$ und $\int dS$ jeweils bedeutet, dass über den erweiterten Phasenraum bzw. die Spinvariable integriert wird. Dies bedeutet, dass die Teilchen in der Kollision nicht im zentralen Kollisionspunkt auftreffen bzw.

emittiert werden, sondern geringfügig zu diesem verschoben. In anderen Worten, sie besitzen einen endlichen Bahndrehimpuls um den Mittelpunkt der Kollision, welcher in Spinpolarisation umgewandelt werden kann.

Um die Gleichgewichtsverteilung zu finden, für die der Kollisionsterm verschwindet, wählen wir den üblichen Ansatz, in dem der Exponent der Boltzmannverteilungsfunktion durch aus der Summe aus den Kollisionsinvarianten Ladung, Viererimpuls und Gesamtdrehimpuls, jeweils multipliziert mit Lagrangemultiplikatoren, besteht [3, 37, 59]. Physikalisch stellen die Lagrangemultiplikatoren für Ladung, Viererimpuls und Gesamtdrehimpuls jeweils das chemische Potential, die thermische Fluidgeschwindigkeit β^μ und das sogenannte Spinpotential dar. Während der Kollisionsterm zur nullten Ordnung aufgrund von Ladungs- und Viererimpulserhaltung verschwindet, wird er zur ersten Ordnung unter Verwendung der Gesamtdrehimpulserhaltung nur Null, wenn das chemische Potential konstant ist, die thermische Fluidgeschwindigkeit die sogenannte Killingbedingung $\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0$ erfüllt und das Spinpotential gleich der thermische Vortizitat $-1/2(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu)$ ist. Dies sind die Bedingungen für globales statt nur lokalem Gleichgewicht.

Im erweiterten Phasenraum lässt sich auf intiutive Weise ein Entropiestrom definieren. Dieser erfüllt ein H-Theorem, d.h. die Entropieproduktion ist semipositiv. Zudem ist die Entropie im oben definierten Gleichgewicht erhalten.

Spin- und Energieimpulstensoren für wechselwirkende Systeme

Die vorherige Diskussion der Pseudoeichtransformation für freie Felder wird nun für wechselwirkende Felder generalisiert [41,57]. Wir betrachten diesmal eine Lagrangedichte, die sich aus der freien Dirac-Lagrangedichte und einem allgemeinen Wechselwirkungsterm zusammensetzt. Unter der Annahme, dass die Wechselwirkungsterme nur von (adjungierten) Spinoren, nicht aber von ihren Ableitungen abhängen, erhält man aus dem Noethertheorem die formal gleichen Ausdrücke wie im nichtwechselwirkenden Fall. Im Allgemeinen verschwindet die wechselwirkende Lagrangedichte nach Anwendung der Bewegungsgleichung nicht und trägt zum Energieimpulstensor bei, und zudem können die Ströme off-shell Beiträge enthalten. Solche Effekte werden allerdings unter der Annahme geringer Dichte vernachlässigt. Unter Anwendung der Boltzmanngleichung berechnet man die hydrodynamischen Bewegungsgleichungen des Energieimpuls- und Spintensors. Ersterer ist erhalten, da der Viererimpuls eine Kollisionsinvariante ist. Auf der anderen Seite ist die Divergenz des Spintensors proportional zum antisymmetrischen Teil des Energieimpulstensors, welcher sich aus zwei Beträgen zusammensetzt. Der erste ist proportional zum Phasenraumintegral des Dipolmomenttensors $\Sigma_{\mathfrak{s}}^{\mu\nu}$ über den Kollisionsterm und damit genau dann Null, wenn der Dipolmomenttensor eine Kollisionsinvariante ist oder der Kollisionsterm verschwindet. Allerdings enthält der antisymmetrische Anteil des kanonischen Energieimplustensors auch einen Beitrag, der nichts mit der (Nicht-) Erhaltung von Spin in Kollisionen zu tun hat. Dieser Beitrag bleibt auch in globalem Gleichgewicht, wenn der Kollisionsterm verschwindet, bestehen. Dies ist nicht konsistent mit dem physikalischen Bild einer Spindichte, die sich nur ändert, bis ein globales Gleichgewicht erreicht wird.

Um die HW Ströme im wechselwirkenden Fall zu erhalten, werden die Pseudoeichtransformationen um einen Wechselwirkungsterm ergänzt. In ähnlicher Weise lassen sich die Pseudoeichtransformationen für die GLW und KG Ströme für wechselwirkende Felder verallgemeinern. Bis zur ersten Ordnung in $\hbar$ sind diese drei Pseudoeichungen auch im wechselwirkenden Fall identisch.

In den hydrodynamischen Bewegungsgleichungen der HW-Tensoren,

$$\begin{aligned} \partial_\mu T^{\mu\nu}_{HW} &= \int d\Gamma\, p^\nu\, \mathfrak{C}[f] = 0\,, \\ \hbar\, \partial_\lambda S^{\lambda,\mu\nu}_{HW} &= \int d\Gamma\, \frac{\hbar}{2} \Sigma_{\mathfrak{s}}^{\mu\nu}\, \mathfrak{C}[f] = T^{\nu\mu}_{HW} - T^{\mu\nu}_{HW}\,, \end{aligned}$$

ist die Erhaltung des Energieimpulstensors $T^{\mu\nu}_{HW}$ wieder mit der Eigenschaft des Viererimpulses, in Kollisionen erhalten zu sein, verbunden. Gleichzeitig ist der antisymmetrische Teil des Energieimpulstensors, der die Nichterhaltung des Spintensors $S^{\lambda,\mu\nu}_{HW}$ beschreibt, ausschließlich durch die Nichterhaltung des Dipolmomenttensors in Kollisionen gegeben. Der zusätzliche Term, der im kanonischen Energieimpulstensor auftrat, erscheint hier nicht mehr. Das heißt, dass der HW-Spintensor genau dann nicht erhalten ist, wenn der mikroskopische Kollisionsterm nichtlokal ist, bis das System globales Gleichgewicht erreicht. Im globalen Gleichgewicht verschwindet der Kollisionsterm und damit auch der antisymmetrische Anteil des HW-Energieimpulstensors. Solange dieser Zustand nicht erreicht ist, ist die Dynamik des Systems aufgrund des nichtlokalen Kollisionsterm dissipativ. Unter Verwendung der zuvor bestimmten globalen Gleichgewichtsfunktion wird der HW-Spintensor berechnet. Dieser hängt im globalen Gleichgewicht von der Teilchendichte, der Fluidgeschwindigkeit und der thermischen Vortizität ab.

Im nichtrelativistischen Limes stimmt der antisymmetrische Teil des HW-Energieimpulstensors mit der bekannten Form des Spannungstensors in Ref. [55] überein. Zudem wird der nichtrelativistische Limes mit mikropolaren Fluiden [60] verglichen. Dabei kann man den Mittelwert des mikroskopischen Spinvektors $\mathfrak{s}^\mu$ mit dem internen Drehimpuls und den HW-Energieimpulstensor mit dem Spannungstensor identifizieren. Wir finden auch einen mikroskopischen Ausdruck für den sogenannten Kopplungsspannungstensor.

Wir betrachten auch die Wignerfunktion und Energieimpuls- und Spintensoren in der Anwesenheit elektromagnetischer Felder. Diese modifizieren die Bewegungsgleichung der Wignerfunktion [61,62]. Der kanonische Energieimpuls- und Spintensor, die man aus der mit Einchfeldern wechselwirkenden Dirac-Lagrangedichte herleitet, sind nicht eichinvariant. Wir führen eine HW-Pseudoeichtransformation für die fermionischen und eine Belinfante-Pseudoeichtransformation für die elektromagnetischen Anteile durch. Wie erwartet, ist danach der elektromagnetische Anteil des Energieimpulstensors symmetrisch, während der elektromagnetische Anteil des Spintensors verschwindet. Der fermionische Anteil des Energieimpulstensors ist dagegen nicht symmetrisch. Aus diesem Grund ist der HW-Spintensor in Anwesenheit elektromagnetischer Felder nicht erhalten. Die globale Spindichte im Impulsraum folgt stattdessen den sogenannten Matthison-Papapetrou-Dixon-Gleichungen [63].

Spinhydrodynamik zweiter Ordnung aus der Momentenmethode

Wir leiten nun dissipative Spinhydrodynamik zweiter Ordnung aus der Momentenmethode her, wobwi wir dem Formalismus aus Refs. [34–36] folgen. Hierfür nutzen wir die zuvor diskutierte HW-Form der erhaltenen Ströme. Zur zweiten Ordnung in $\hbar$ besitzt der Energieimpulstensor einen nichtsymmetrischen Anteil, der durch Wechselwirkungen bedingt ist und zur Nichterhaltung des Spintensors führt. Dieser Wechselwirkungsanteil des Energieimpulstensors hängt von der Nichtlokalität des Kollsionsterm Δ^μ ab. Wir definieren die Fluidgeschwindigkeit als zeitartigen Eigenvektor des Nichtwechselwirkungsanteils des Energieimpulstensors, was als Landau-Bezugssystem bekannt ist. Nun spalten wir in den mikroskopischen Ausdrücken für den Ladungsstrom, des Energieimpulstensor und den Spintensor jeweils die Impulsvariable in einen Anteil parallel und orthogonal zur Fluidgeschwindigkeit. Produkte aus zwei Impulsen werden in parallele, orthogonale und spurfreie orthogonale Teile gespalten. Während wie üblich der Ladungsstrom durch die Teilchendichte und den Diffusionsstrom und der Energieimpulstensor durch die Energiedichte, den thermodynamischen Druck, den viskosen Druck und den Scherspannungstensor beschrieben werden, treten im Spintensor neue den Spintransport charakterisierende Ströme auf. Dies sind die Spinenergie, der Spindruck, die Spindiffusion und die Spinspannung. Insgesamt kommen durch den Spintensor, der ein Rang-3-Tensor und antisymmetrisch in den letzten beiden Indizes ist, 24 Freiheitsgrade zu den üblichen 14 des Ladungsstroms und symmetrischen Energieimpulstensors hinzu. Die Erhaltungsgesetze liefern allerdings insgesamt nur 11 Gleichungen. Die fehlenden Gleichungen, die notwendig sind, um das System zu lösen, werden im Folgenden aus der Boltzmanngleichung hergeleitet.

Die Herleitung von Hydrodynamik aus kinetischer Theorie basiert üblicherweise auf der Entwickulng um einen lokalen Gleichgewichtszustand, welcher durch das Verschwinden des Kollisionsterms definiert ist, während die linke Seite der Boltzmanngleichung in diesem Zustand von Null verschieden ist. In unserem Fall scheint dies problematisch, da die Bedingungen des Verschwindens des nichtlokalen Kollisionsterms die des globalen Gleichgewichts sind, ein lokales Gleichgewicht in diesem Sinn also zunächst nicht existiert. Aus diesem Grund führen wir eine Skalenseparierung ein, die die Definition eines näherungsweisen lokalen Gleichgewichtzustands erlaubt. Hierbei ist zu beachten, dass eine hydrodynamische Entwicklung in kinetischer Theorie üblicherweise auf der Separierung von drei Skalen beruht. Einerseits muss die Wechselwirkungsreichweite viel kleiner als die mittlere freie Weglänge zwischen Kollisionen sein. Dies ist notwendig, damit der sogenannte Stoßzahlansatz gilt, die Teilchen also als frei zwischen Kollisionen angesehen werden können, was die Voraussetzung für die Anwendbarkeit kinetischer Theorie ist. Weiterhin muss die mittlere freie Weglänge viel kleiner als die sogenannte hydrodynamische Skala sein, wobei letztere ein typische Längenskala ist, auf der sich hydrodynamische Größen hinreichend ändern. Die Relation dieser beiden Skalen wird Knudsenzahl genannt und stellt den kleinen Parameter dar, in dem in der Entwicklung um lokales Gleichgewicht entwickelt wird. Üblicherweise wird angenommen, dass alle Gradienten hydrodynamischer Größen von der Ordnung der inversen hydrodynamischen Skala sind. Dies ist jedoch nicht notwendig, da die thermische Vortizität nicht in der Standard-Boltzmanngleichung auftritt und deshalb im Prinzip beliebig groß sein kann, ohne dass die die Entwicklung der Verteilungsfunktion in dieser Gleichung um lokales Gleichgewicht beeinträchtigt. Aus diesem Grund können wir eine neue Skala einführen, die mit der inversen Vortizität assoziiert ist und kleiner als die hydrodynamische Skala sein kann. Temperaturgradienten könnten im Prinzip als von der gleichen Größenordnung wie die Vortizität angenommen werden, in unserem Fall verzichten wir allerdings hierauf, da wir annehmen, weit genug vom Rand das Systems entfernt zu sein. Der lokale Kollisionsterm ist proportional zur inversen mittleren freien Weglänge. Mit der Nichtlokalität des Kollisionsterm führen wir eine weitere Skala ein, die den typischen Abstand der Teilchen zum Zentrum der Kollision beschreibt. Zur Wahrung der Validität des Stoßzahlansatzes sollte dieser Abstand nicht größer als die mittlere freie Weglänge sein. Aus der expliziten Berechnung des nichtlokalen Kollisionsterm wissen wir außerdem, dass dieser Abstand von der Ordnung $\hbar$ ist, genau wie die Polarisation selbst. Wir definieren die sogenannte Quantenknudsenzahl als die Relation zwischen der Nichtlokalität und der hydrodynamischen Skala. Damit die zuvor durchgeführte Entwicklung in $\hbar$ und Gradienten der Wignerfunktion anwendbar ist, muss diese Zahl hinreichend klein sein. In dieser Entwicklung treten allerdings alle Gradienten auf, also muss auch die Vortizitätsskala viel größer als die Nichtlokalität sein, kann jedoch trotzdem kleiner als die hydrodynamische Skala sein. Wir nehmen daher an, dass die Vortizitätsskala viel kleiner als die hydrodynamische Skala ist und die Relation der Nichtlokalität zur Vortizitätsskala von der gleichen Größenordnung wie die Relation der mittleren freien Weglänge zur hydrodynamischen Skala ist. Das heißt, wir können diese beiden Skalenseparierungen jeweils zur Entwicklung in Quanten- und Nichtgleichgewichtskorrekturen nutzen und gleichzeit im "lokalen Gleichgewicht" Korrekturen der Größenordnung der Nichtlokalität im Vergleich zur hydrodynamischen Skala vernachlässigen, da diese Relation viel kleiner ist als die anderen beiden. In diesem Fall verschwindet der nichtlokale Kollisionsterm bis auf Korrekturen dieser Ordnung bei einer lokalen Gleichgewichtsverteilung, in der das Spinpotential gleich der thermischen Vortizität ist. Die Differenz aus Spinpotential und thermischer Vortizität ist somit dissipativ. Diese Definition eines lokalen Gleichgewichtszustands ist verschieden von derjenigen ohne Spin und mit lokalem Kollisionsterm, da bei letzterer der Kollsionsterm exakt und ohne weitere Annahmen verschwindet. Das Problem des lokalen Gleichgewichts bei nichtlokalem Kollisionsterm lässt sich verstehen, indem man beachtet, dass die Quantenknudsenzahl für eine kinetische Beschreibung nicht größer als die Knudsenzahl sein kann. Deshalb wäre es inkonsistent, Spineffekte zu betrachten, ohne Dissipation zu berücksichtigen. Dieses Problem wird durch die zusätzliche Vortizitätsskala gelöst. Nun wird die Verteilungsfunktion um ihren Gleichgewichtswert entwickelt. Die Abweichungen vom Gleichgewicht werden in einer orthogonalen Basis irreduzibler Momente der Verteilungsfunktion ausgedrückt, wobei die Spinfreiheitsgrade durch die Spinmomente gegeben sind. Damit lassen sich auch die Komponenten des Spintensors in Gleichgewichtsanteile und dissipative Anteile aufspalten. Da im Phasenraumintegral nur die zum Impuls orthogonalen Anteile der Spinavariable beitragen, sind nicht alle Tensorkomponenten der Spinmomente unabhängig. Daher kann man jeweils die Komponente parallel zur Fluidgeschwindigkeit durch die orthogonalen Komponenten ausdrücken und es genügt, Bewegungsgleichungen für letztere herzuleiten.

Durch Einführung der lokalen Gleichgewichtsform der Verteilungsfuntion werden dem System neue Freiheitsgrade in Form der dynamischen Lagrangemultiplikatoren der Kollisionsinvarianten hinzugefügt. Dabei ist ihre physikalische Interpretation zunächst nicht klar, sondern wird durch die Wahl des hydrodynamischen Bezugssystems (wie erwähnt hier das Landau-System) und der sogenannten Anpassungsbedingungen festgelegt. Diese definieren die Aufteilung der Verteilungsfunktion in Gleichgewichts- und dissipative Anteile und reduzieren die Freiheitsgrade wieder. Da in Anwesenheit von Spinfreiheitsgraden ein zusätzliches thermodynamisches Potential für die Erhaltung des Gesamtdrehimpulses, nämlich das Spinpotential, eingeführt wurde, fordern wir auch eine Anpassungsbedingung an den Gesamtdrehimpuls. Durch diese Bedingung wird eine bestimmte Kombination der Spinmomente Null gesetzt. Aus den Erhaltungsgleichungen für den Ladungsstrom, den Energieimpulstensor und den Gesamtdrehimpulstensor leiten wir nun unter Verwendung der Anpassungsbedingungen Bewegungsgleichungen für die thermodynamischen Potentiale her.

Die Bewegungsgleichungen für die dissipativen Spinmomente werden direkt aus der Boltzmanngleichung hergeleitet. Man erhält ein exaktes Gleichungssystem, in dem die Bewegungsgleichungen für unendlich viele Spinmomente miteinander gekoppelt sind. Wir leiten die Bewegungsgleichungen der Spinmomente bis einschließlich vierten Tensorrangs her. Für höhere Spinmomente kann man zeigen, dass die asymptotischen Lösungen zur ersten Ordnung in Gradienten verschwinden. Im Folgenden werden wir nur die Bewegungsgleichungen der Spinmomente bis Tensorrang drei verwenden, da höhere Momente nicht in den Erhaltungsgesetzen auftreten. Die kinetischen Gleichungen der Spinmomente und die der spinunabhängigen Momente, wobei letztere in Ref. gefunden werden können, sind nur über die Kollisionsterme gekoppelt. Um das Gleichungssystem zu schließen, wird im Folgenden eine Trunkierung angewendet werden.

Zunächst diskutieren wir jedoch die Kollisionsterme. Diese können als Linearkombination von Momenten der Verteilungsfunktion ausgedrückt werden. Im ersten Schritt linearisieren wir hierzu den Kollisionsterm, d.h. wir berücksichtigen nur Beträge jeweils bis zur ersten Ordnung in dissipativen Gradienten und $\hbar$. Zudem vernachlässigen wir Ordnungen, die sowohl linear in der Ableitung eines dissipativen Gradienten als auch in $\hbar$ sind. Mit diesen Näherungen setzt sich der Kollisionsterm aus zwei verschiedenen Arten von Beiträgen zusammen. Wir betrachten in dieser Arbeit effektive Wechselwirkungen, die entweder skalar oder vektorwertig im Limes niedriger Impulse sind. In diesem Fall vereinfachen sich die Kollisionsterme. An dieser Stelle stellt man fest, dass die Bewegungsgleichungen für die Spinmomente von denen für die üblichen Momente entkoppeln.

Tatsächlich würden die Bewegungsgleichungen der üblichen Momente nur im Fall von paritätsverletzenden Wechselwirkungen Modifikationen durch die Spinmomente im Kollisionsterm erhalten, was wiederum zu Änderungen durch Polarisationseffekte im Navier-Stokes-Limes, etwa für die Scherspannung, führen würde. In unserem Fall lässt sich der erste Beitrag zu den Kollisionstermen der Spinmomente als Linearkombination von Spinmomenten schreiben. Der anderen Beiträge hängen nicht von den dissipativen Spinmomenten ab, sondern jeweils von der Gleichgewichtsverteilung und der Differenz zwischen Spinpotential und thermischer Vortizität bzw. von der thermischen Scherspannung.

Um nun ein geschlossenes System von Bewegungsgleichungen zu erhalten, wenden wir die sogenannte 14+24-Momentennäherung an. Das heißt, wir drücken die Spinmomente, die nicht in den Erhaltungsgleichungen auftreten, durch jene aus, die es tun. Dies sind 14 Momente aus dem Ladungsstrom und dem Energieimpulstensor und 28 aus dem Spintensor. An dieser Stelle spielt die Pseudoeichung eine Rolle, da die explizite Form des Spintensors verwendet wird. In unserem Fall behandeln wir also die Komponenten des HW-Spintensors dynamisch und drücken die anderen Spinmomente durch diese aus. Hätten wir eine andere Pseudoeichung, etwa die kanonische, gewählt, wären die dynamischen Spinmomente andere, daher würde sich auch die Dynamik des Systems ändern. Da die Bewegungsgleichungen des HW-Spintensors (im Gegensatz zum kanonischen) Erhaltungsgleichungen sind, ist jedoch anzunehmen, dass diese Spinmomente auf hydrodynamischen Skalen die entscheidenden Beträge geben. Wir verwenden eine Relation, die es uns erlaubt, die Spinmomente jeweils durch eine Summe über die anderen Spinmomente auszudrücken, und nähern die Summe dann, indem wir nur die Terme mit Komponenten des HW-Spintensors berücksichtigen. Die zur Fluidgeschwindigkeit parallelen Spinmomente werden zudem durch die orthogonalen ausgedrückt. Weiterhin verwenden wir die Anpassungsbedingungen, um einige Komponenten des Spintensors in Abhängigkeit von anderen zu schreiben, wodurch diese nicht mehr dynamisch behandelt werden müssen. Schließlich invertieren wir die Matrizen, die die Spinmomente im ersten Teil der Kollisionsterme multiplizieren, und setzen alle bisherigen Ergebnisse mit den erwähnten Näherungen in die allgemeinen Bewegungsgleichungen ein. Nach einer einfachen Rechnung erhalten wir somit ein geschlossenes Gleichungssystem für die dynamischen Spinmomente.

Der Pauli-Lubanski-Vektor kann nun durch die dynamischen Spinmomente ausgedrückt werden. Wird der Impuls der Wignerfunktion mit dem Teilchenimpuls identifiziert, entspricht dieser im Teilchenruhesystem der Polarisation, die in Schwerionenkollisionen gemessen wird. Die sogenannte lokale Polarisation ist diejenige, die vom vom Impuls abhängt. Diese enthält Beiträge von allen dynamischen Spinmomenten. Die globale Polarisation erhält man, indem man die lokale über den Impuls integriert.

Schließlich berechnen wir den Navier-Stokes Limes der dissipativen Momente, der sich ergibt, wenn man nur Beiträge bis zur ersten Ordnung in dissipativen Gradienten berücksichtigt. Wie erwähnt werden die üblichen dissipativen Momente unter für paritätserhaltende Wechselwirkungen bis zur ersten Ordnung in $\hbar$ nicht durch Spineffekte modifiziert. Allerdings werden ihre Navier-Stokes-Werte durch die Anwesenheit der Vortizität als Größe nullter Ordnung beeinflusst. Die relevanten Gleichungen erhält man, indem in den Bewegungsgleichungen für den Teilchendiffusionsstrom und den Scherspannungsstensor aus Ref. [35] nur Terme erster Ordnung berücksichtigt werden (wobei die Vortizität nicht als erste Ordnung gezählt wird) oder alternativ durch eine Chapman-Enskog-Entwicklung unter Berücksichtigung der Größenordnungsschemas. Die Gleichungen lassen sich nun jeweils nach dem Teilchendiffusionsstrom und dem Scherspannungsstensor lösen, indem Ansätze ähnlich wie in Ref. [64] verwendet werden. Wir finden, dass der Teilchendiffusionsstrom entlang des Vortizitätsvektors unverändert von der Vortizität bleibt, während derjenige orthogonal zum Vortizitätsvektor reduziert wird. Dieser Effekt ist größer, je größer der Betrag der Vortizität ist und je langsamer sich der Diffusionsstrom seinem Navier-Stokes-Wert annähert. Für kleine Werte von Relaxationszeit und Vortizität verschwinden anisotrope Effekte entlang des Vortizitätsvektors und nur ein Term proportional zur Kontraktion des Vortizitätstensors mit dem Diffusionsstrom trägt zur Modifizierung des Diffusionsstroms durch Vortizität bei. Im anderen Limes sehr großer Relaxationszeit und Vortizität ist das System maximal anisotrop und nur den Diffusionsstrom entlang der Vortizität ist von Null verschieden. In ähnlicher Weise wird der Scherspannungstensor im Navier-Stokes-Limes durch Vortizität modifiziert.

Wir berechnen auch den Navier-Stokes-Limes der Spinmomente, wobei wir Gradienten der Spinmomente als zweite Ordnung zählen. Auch diese werden durch die Vortizität verändert. Wir finden zudem, dass die lokale Polarisation im Navier-Stokes-Limes einen Beitrag proportional zur Scherspannung erhält, der durch den nichtlokalen Kollisionsterm entsteht. Scherspannungsbeiträge wurden kürzlich als wichtig zur Beschreibung der lokalen Lambdapolarisation in Schwerionenkollisionen erachtet [10–15].

Schlussfolgerungen und Ausblick

In dieser Arbeit haben wir dissipative Spinhydrodynamik aus Quantenfeldtheorie hergeleitet. Im ersten Schritt haben wir eine kinetische Theorie formuliert, in der der nichtlokale Kollisionsterm verantwortlich für die Umwandlung von Vortizität in Spinpolarisation ist. Dieser Kollisionsterm verschwindet nur im globalen Gleichgewicht und das Spinpotential ist in diesem Fall gleich der thermischen Vortizität. Zudem haben wir ein Set von Energieimpuls- und Spintensor mit klarer physikalischer Interpretation gefunden, das durch eine HW Pseudoeichtransformation aus den kanonischen Strömen hergeleitet werden kann. In diesem Fall ist der Spintensor genau dann nicht erhalten, wenn der Kollisionsterm nichtlokal ist und das System noch keinen Gleichgewichtszustand erreicht hat. Schließlich haben wir durch Anwendung der Momentenmethode dissipative Spinhydrodynamik aus kinetischer Theorie hergeleitet. Das Resultat sind geschlossene Bewegungsgleichungen für die dynamischen Spinmomente, die durch die Komponenten des HW Spintensors gegeben sind.

Die hier präsentierte Arbeit kann in vielfältiger Weise weitergeführt werden. Um die hergeleiteten Bewegungsgleichungen numerisch implementieren zu können, sollte zunächst ihre Kausalität und Stabilität analysiert werden. Ist beides gegeben, können sie angewendet werden, um Spindynamik in Schwerionenkollisionen zu berechnen und mit experimentellen Daten zu vergleichen. Zudem können die hier hergeleiteten Gleichungen noch verbessert werden, indem statt eine explizite Trunkierung der Momentenentwicklung zu verwenden, die Momente resummiert und in Knudsenzahlen entwickelt werden, wie in Ref. [35] vorgestellt. Eine andere Möglichkeit wäre, in den allgemeinen Bewegungsgleichungen der Spinmomente allgemeinere Anpassungsbedingungen und eine allgemeinere Definition der Fluidgeschwindigkeit zu verwenden und eine Theorie erster Ordnung aufzustellen, da kürzlich gezeigt wurde, dass Theorien erster Ordnung dann kausal und stabil sein können [28–31]. Weiterhin kann diese Arbeit ausgeweitet werden, indem allgemeinere Wechselwirkungen oder andere Pseudoeichungen betrachtet werden. Ebenfalls von Interesse ist die Inklusion elektromagnetischer Felder, um dissipative Spin-Magnetohydrodynamik herzuleiten. Schließlich ist es relevant, die Rechnungen dieser Arbeit für Spin-1-Teilchen zu wiederholen, was beispielsweise zur Beschreibung von ϕ-Polarisation in Schwerionenkollisionen angewendet werden kann.

Contents

Chapter 1

Introduction

1.1 Motivation

The study of spin hydrodynamics is, at the same time, an old and very recent issue. Early works consider nonrelativistic fluids with spin degrees of freedom. Already in 1959 it was argued by Dahler that an antisymmetric part of the stress tensor leads to interactions between internal angular momentum/spin and fluid flow [65, 66]. A few years later, the derivation of nonrelativistic dissipative fluid dynamics with spin from kinetic theory was performed in Ref. [55], pointing out the essentiality of a nonlocal collision term for a nonsymmetric stress tensor and hence the conversion between spin and orbital angular momentum, leading to the alignment of polarization with vorticity, the so-called Barnett effect [67]. In a follow-up paper [56], also the form of the nonrelativistic nonlocal collision term was given. Along similar lines, spin kinetic theory with nonlocal effects has also been studied, e.g., in Refs. [68–71] and spin transport in Refs. [72–74]. Nonrelativistic fluids with spin find various applications, e.g., in micropolar fluids [60], spintronics [75–78] and chiral active fluids [79, 80].

On the other hand, for a long time relativistic spin hydrodynamics seemed to be of limited interest. This changed with the measurements of Lambda polarization in relativistic heavy-ion collisions [22–25]. The polarization along the global angular momentum of the system obtained through those measurements was found to be well described by relativistic hydrodynamics in local equilibrium, assuming that spin is aligned with the thermal fluid vorticity [1–7]. However, later measurements of the longitudinal polarization [9] show the opposite angular dependence than the hydrodynamic calculations under that assumption [8]. This disagreement caused a lot of efforts in various directions [10–15, 81–87], one of them being the investigation of relativistic hydrodynamics with dynamical spin degrees of freedom. First developments putting forward this idea can be found in Refs. [37–39], where the spin tensor was promoted to a dynamical variable in addition to the energy-momentum tensor and charge current. In the last few years, many further works considered the derivation of relativistic spin hydrodynamics from different approaches, e.g., effective action [88–91], entropy-current analysis [92–94], holographic duality [95], or linear-response theory [96].

Another convenient and often used approach is to first derive a kinetic theory and use it as a starting point for the formulation of hydrodynamics. Since spin is a quantum feature, it is not sufficient to only consider classical kinetic theory, but a quantum framework is required. However, in a quantum theory one has to deal with the issue that position and momentum of a particle cannot be simultaneously determined, and hence the definition of a phase-space distribution is not straightforward. The standard tool to overcome this problem in quantum kinetic theory is the Wigner function [97], defined from the two-point function through the so-called Wigner transformation and following equations of motion which can be directly derived from the Lagrangian density [45, 61, 98]. The Wigner function is constructed in way that it depends on a position variable which is the central coordinate of the two-point function, and a momentum variable, being the Fourier transform of the relative coordinate of the two-point function. It is a so-called quasi-probability distribution, meaning that macroscopic currents can be obtained as phase-space integrals of the Wigner function, while the latter itself does not have the interpretation of a probability density, since it can assume negative values. Although there exist different approaches to define quantum (quasi-) probability distributions, the Wigner-function formalism turned out to be the most successful one, see, e.g., Refs. [99–101] for reviews. Recently, it has been widely applied, e.g., to obtain nonrelativistic spin kinetic theory and hydrodynamics [102–104], chiral kinetic theory [105–115], and relativistic spin kinetic theory both in the free-streaming limit [62, 116–122] and including particle collisions [57, 58, 123–125].

In the formulation of relativistic spin hydrodynamics, one finds an issue which is not present in the nonrelativistic case. As the splitting of the total angular momentum into spin and orbital part suffers from ambiguities in a relativistic theory, the spin tensor is not uniquely defined. This results in the so-called pseudo-gauge freedom of the definition of the hydrodynamic currents [40], see Refs. [41, 126, 127] for recent reviews. The problem of choosing a "physical" set of tensors is long-standing and not yet solved. Already in 1940 Belinfante and Rosenfeld proposed a symmetric energy-momentum tensor with vanishing spin tensor [42,128,129], traditionally believed to be the physical one since it couples to gravity in Einstein's classical general relativity. However, in later works a gravitational theory allowing for a nonsymmetric energy-momentum tensor and a nonvanishing spin tensor has been developed, known as Einstein-Cartan theory [40,130,131]. Studies on the spin tensor and hydrodynamics with polarization in classical gravitational physics can be also found in Refs. [63,132,133]. Also in quantum field theory, the splitting into spin and orbital angular momentum has a long history, starting in 1948 with a pioneering work by Pryce [47] and continued, e.g., in Refs. [134–137]. More recently, this issue has been extensively discussed in optics [138–144], see in particular Ref. [145] for a discussion about the connections to particle physics, and in relation to the spin of the nucleon, see Ref. [146] for a detailed review. A nonzero spin tensor for Dirac fields, which is conserved in the absence of interactions, was found by Hilgevoord and Wouthuysen (HW) in Refs. [43, 44]. Recently, such spin tensor has been used to derive ideal spin hydrodynamics [147, 148] and first-order dissipative spin hydrodynamics in the relaxation time approximation [149, 150] from the Wigner function. However, a formulation of dissipative spin hydrodynamics from kinetic theory including a nonlocal collision term [57,58] and hence allowing for spin alignment with vorticity has been missing up to now.

The aim of this thesis is to provide a complete and consistent derivation of second-order dissipative relativistic spin hydrodynamics from quantum field theory. We will proceed in two main steps. The first one is the formulation of spin kinetic theory from quantum field theory using the Wigner-function formalism and performing an expansion in powers of the Planck constant $\hbar$. The essential ingredient here is the nonlocal collision term. We will find that the nonlocality of the collision term arises at first order in $\hbar$ and is responsible for the spin alignment with vorticity, as it allows for conversion between spin and orbital angular momentum. In the second step, this kinetic theory is used as the starting point to derive hydrodynamics including spin degrees of freedom. The so-called canonical form of the conserved currents follows from Noether's theorem. Applying an HW pseudo-gauge transformation, we obtain a spin tensor and energy-momentum tensor with obvious physical interpretation. Promoting all components of the HW tensors to be dynamical, we derive second-order dissipative spin hydrodynamics. The additional equations of motion for the dissipative currents are obtained from kinetic theory generalizing the method of moments [35] to include spin degrees of freedom.

1.2 Overview over this thesis

This thesis is organized as follows. In the remainder of Chapter 1, we discuss the quark-gluon plasma and relativistic heavy-ion collisions. Furthermore, we review the theoretical concepts related to polarization measurements in relativistic heavy-ion collisions. In particular, we introduce the relativistic spin vector, or Pauli-Lubanski vector, and show its relation to the angular distribution of the emitted daughter particles in a parity-violating decay. The latter is measurable in heavy-ion collisions in the case of weakly decaying Lambda hyperons. Hence, the first chapter provides a possible application of the framework derived in the remainder of this thesis to experiment. In the end of this chapter, we introduce the most important concepts of relativistic hydrodynamics. Chapter 2 is devoted to the study of a crucial quantity needed for the formulation of spin hydrodynamics, namely the spin tensor. First, we introduce the concept of pseudo-gauge transformations and discuss several choices of pseudo-gauge for free Dirac fields. Then, we aim at developing a physical intuition of the interpretation of the spin tensor in a certain pseudo-gauge as a spin density. To this end, we study the various pseudo-gauges regarding their connections to the Wigner operator, the Pauli-Lubanski vector and the relativistic center of inertia. In Chapter 3, we derive a Boltzmann-like equation for the distribution function in enlarged phase-space, which depends on a continuous spin variable and includes the dynamics of both the scalar and the axial-vector component of the interacting Wigner function, the latter describing the polarization. Using methods from quantum field theory, we then derive both the local and nonlocal collision term. The nonlocality is characterized by a displacement of the colliding particles with respect to the center of the microscopic collision. We show that, requiring that the collision term vanishes and using the conservation of total angular momentum, spin gets aligned with vorticity in global equilibrium. Furthermore, we discuss the issue of defining local equilibrium with spin and introduce an entropy current fulfilling an H-theorem. In Chapter 4, we revisit the energy-momentum and spin tensors in different pseudo-gauges, extending the considerations of Chapter 2 to the interacting case. We find that the HW spin tensor yields a clear physical interpretation, as it is conserved for free fields or for local collisions, but not in the presence of nonlocal collisions until global equilibrium is reached. Furthermore, we find agreement with known results in the nonrelativistic limit. The effects of electromagnetic fields on the hydrodynamic equations of motion are also studied. Finally, in Chapter 5, we derive dissipative spin hydrodynamics from the Boltzmann equation. After defining spin-dependent moments of the distribution function and explaining our power-counting scheme, we obtain exact equations of motion for these spin moments from the Boltzmann equation. This set of coupled equations is closed by making use of an explicit truncation of the moment expansion of the distribution function. We derive the closed set of second-order equations of motion and determine the Pauli-Lubanski vector in terms of the spin moments. Finally, we calculate the Navier-Stokes limit, taking into account our power-counting scheme. Details of calculations and lengthy coefficients are delegated to the appendix.

1.3 Notations and conventions

We use the following notation and conventions: $a \cdot b \equiv a^\mu b_\mu$, $a_{[\mu} b_{\nu]} \equiv a_\mu b_\nu - a_\nu b_\mu$, $a_{(\mu} b_{\nu)} \equiv a_\mu b_\nu + a_\nu b_\mu$, and $\epsilon^{0123} = -\epsilon_{0123} = 1$. Repeated indices are summed over. Throughout this thesis we work in flat space-time with Minkowski metric $g_{\mu\nu} = \text{diag}(+,-,-,-)$. Spatial three-vectors are indicated by boldface letters, $\mathbf{a}$, and unit three-vectors by a hat and boldface letter, $\hat{\mathbf{a}}$. Operators and states in Hilbert space are denoted by a hat (without boldface) $\hat{\mathcal{O}}$ and the notation $|\cdot\rangle$, respectively. If a symbol has been introduced as an operator $\hat{\mathcal{O}}$ and is later used without the hat and with lower indices, $\mathcal{O}_{rs}$, we refer to the overlap of the respective operator with two single-particle states with spin projection r and s onto the quantization direction, respectively. If the same symbol is used without hat and without spin indices, $\mathcal{O}$, it denotes the ensemble average of the respective operator. In order to avoid divergences, it is always implied that operators are normal-ordered before calculating their expectation values or ensemble averages, without denoting it explicitly to simplify the notation. Spinor operators are denoted by ψ without hat. Furthermore, we use the symbol Tr for traces in Hilbert space, while traces of matrices $\in \mathbb{C}^{n\times n}$ are indicated by the symbol tr. The symbol $\star$ denotes the rest frame of the momentum p^μ. If momentum-averaged quantities carry a $\star$ index, it is understood that the averaging is done over the quantity for each momentum calculated in the rest frame of the respective momentum. On the other hand, we use the symbol $*$ for the complex conjugate and the symbol $\dagger$ for the hermitian conjugate. Functional dependences of operators and functions on the space-time coordinate x and/or the momentum p are omitted when there is no risk of confusion. Partial derivatives ∂^μ without further index always refer to the space-time coordinate x_μ. We use natural units, however, we keep the Planck constant $\hbar$ explicit since it serves as our power-counting parameter. The formal $\hbar$ expansion performed in this work should be seen as a gradient expansion of the Wigner function. For this reason, powers of $\hbar$ appearing in normalizations without an accompanying gradient do not participate in the power counting. Furthermore, we note that our expansion scheme requires derivatives of the Wigner function to be small, which is not equivalent to derivatives of spinor (or photon) fields being small. Hence the gradient expansion can only be applied after performing the Wigner transformation of the two-point function.

1.4 The quark-gluon plasma and relativistic heavy-ion collisions

Among the four interactions between elementary-matter particles, the strong force, described by quantum chromodynamics (QCD), nowadays is the least well understood. Although the QCD Lagrangian is well known, solving QCD is highly challenging. The reason for this lies in the nonabelian nature of QCD, resulting in a running coupling, i.e., an energy dependence of the coupling parameter. With increasing energies, the coupling becomes weaker and weaker, and quarks reach an asymptotically free state. This property is known as asymptotic freedom [151, 152]. On the other hand, at low energies the coupling becomes stronger, such that quarks cannot appear in free states, but always form hadrons. This phenomenon is called confinement and makes a perturbative study of QCD, unless at very high energies, impossible.

While the quarks appear in our daily environment only in their confined state, it is desirable to be able to study their properties at higher energies. In fact, up to now there are many unknowns in the phase diagram of strongly interacting matter, shown in Figure 1.1, see, e.g., Refs. [16–18] for reviews. We know that, at lower temperature and baryon chemical potential, the constituents build a hadron gas. At high temperature and high baryon chemical potential, there exists a state called quark-gluon plasma (QGP), where quarks and gluons are interacting, but not bound in hadrons. Consequently, there has to be a transition between the two phases. We also know from Lattice calculations that, at zero baryon chemical potential, the nature of this transition is a crossover phase transition. Furthermore, it is believed that for higher baryon chemical the phase transition is of first order and hence somewhere on the line of phase transitions a second-order critical point is exists. However, the detailed structure of the QCD phase diagram, in particular the location of the critical point, the order of the phase transition for higher chemical potential, possible inhomogeneous phases

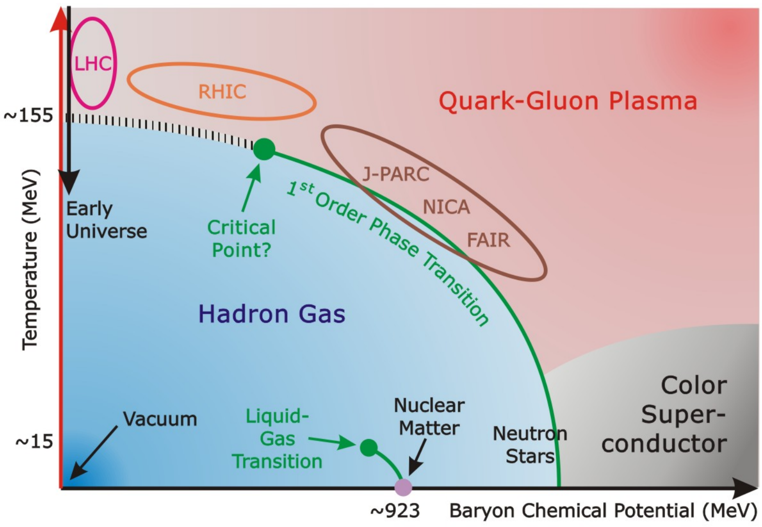


Figure 1.1: Phase diagram of QCD. Figure from Ref. [153]

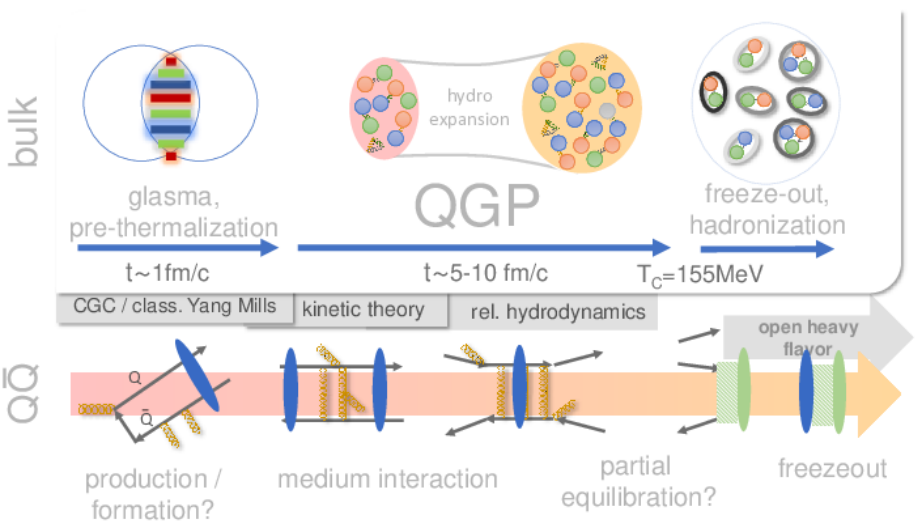


Figure 1.2: Stages of a heavy-ion collision, figure from Ref. [154]

and the onset of chiral-symmetry restoration are still under active investigation.

While a QGP was present in Nature in the early Universe and today can be found in neutron stars, the only known way to produce and experimentally access it on Earth are relativistic heavy-ion collisions. In these experiments, atomic nuclei are accelerated to energies much higher than their rest masses. When they collide, in the first stage a so-called glasma is created, a strongly interacting and unthermalized state. After a short time the QGP is formed. It was found that this stage of a heavy-ion collision is very well described by kinetic theory at early and relativistic hydrodynamics at later times, when the QGP behaves like a nearly perfect fluid. Finally, the quarks and gluons are again confined to hadrons at freeze-out. An overview over the different stages of a heavy-ion collision is shown in Figure 1.2.

Quarks are fermions carrying a spin of 1/2. As pointed out in Section 1.1, a fluid with spin degrees of freedom can be polarized through an interplay between the particle spins and the motion of the medium. Therefore, from measurements of the polarization, one can directly infer properties of the QGP, namely vorticity and other gradients of the system [19,20]. While in standard approaches of kinetic theory polarization effects are ignored by averaging and summing over spin states directly in the cross section, it is essential to keep the spin degrees of freedom in the kinetic equations in order to understand polarization measurements in heavy-ion collisions. These measurements will be further discussed in Section 1.7. Before, we will introduce some theoretical concepts related to spin.

1.5 The relativistic spin vector

In the following three sections we introduce the relativistic spin vector and show how its components can be related to the angular distribution of the emitted daughter particles in a parity-violating decay, mainly following Ref. [21]. We then point out how this relation can be used to experimentally access particle polarization considering the example of hyperon polarization in heavy-ion collisions.

We start from the definition of the nonrelativistic spin-vector operator $\mathbf{S}$, which is in first quantization proportional to the Pauli matrices σ^i [21],

$$\mathbf{S}_{\rm nr} \equiv \frac{\hbar}{2}\boldsymbol{\sigma}. \tag{1.1}$$

One possibility to generalize this definition in a covariant way in quantum field theory is introducing an antisymmetric tensor operator $\hat{S}^{\mu\nu}$ such that its expectation value for a momentum state

$$|p,s\rangle \equiv a_s^\dagger(p)|0\rangle, \tag{1.2}$$

where $a_s^\dagger(p)$ is a creation operator, in its rest frame is given by

$$S^{i0}_{\star,rs} = 0, \tag{1.3a}$$
$$S^{ij}_{\star,rs} \equiv \epsilon^{ijk} S^k_{{\rm nr},rs}. \tag{1.3b}$$

Here, we defined the expectation value for a general operator $\hat{\mathcal{O}}$

$$\mathcal{O}_{rs} \equiv \langle p,r|\hat{\mathcal{O}}|p,s\rangle \tag{1.4}$$

and the rest-frame expectation value

$$\mathcal{O}_{\star,rs} \equiv \langle p_\star,r|\hat{\mathcal{O}}|p_\star,s\rangle \tag{1.5}$$

with $p^\mu_\star \equiv (m,\mathbf{0})$. For an arbitrary state, Eq. (1.3a) can be generalized through the so-called Frenkel condition [155]

$$p_\mu S^{\mu\nu}_{rs} = 0\ . \tag{1.6}$$

Another way of obtaining a covariant spin is the definition of an axial vector $\hat{S}^\mu$ such that its rest-frame expression is identical to the nonrelativistic spin vector

$$S^\mu_{\star,rs} \equiv (0, S_{{\rm nr},rs})\,, \tag{1.7}$$

which for a state in a general frame means

$$p\cdot S_{rs} = 0. \tag{1.8}$$

This condition is fulfilled by the so-called Pauli-Lubanski operator [21, 156]

$$\hat{w}^\mu \equiv -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\hat{P}_\nu \hat{J}_{\alpha\beta} \tag{1.9}$$

where $\hat{P}_\mu$ is the momentum operator with

$$\hat{P}^\mu|p,s\rangle = p^\mu|p,s\rangle \tag{1.10}$$

and $\hat{J}_{\mu\nu}$ is the total angular-momentum operator. The commutation relations of the Pauli-Lubanski operator follow from the Poincaré algebra [21]

$$[\hat{w}^\mu,\hat{w}^\nu] = -i\hbar\,\epsilon^{\mu\nu\alpha\beta}\hat{w}_\alpha\hat{P}_\beta. \tag{1.11}$$

Hence, when acting on a state at rest, the spatial components follow the commutation relation

$$[\hat{w}^i,\hat{w}^j] = -i\hbar\,\epsilon^{ijk0}\hat{w}_k m. \tag{1.12}$$

Thus, defining the relativistic spin-vector operator as

$$\hat{S}^\mu = \frac{\hat{w}^\mu}{m}, \tag{1.13}$$

yields the same commutation relations for the relativistic spin operator in the rest frame as those for the nonrelativistic spin operator given by the Pauli matrices in Eq. (1.1),

$$[\hat{S}_i,\hat{S}_j]|p_\star,s\rangle = i\hbar\,\epsilon_{ijk}\hat{S}_k|p_\star,s\rangle. \tag{1.14}$$

In the remainder of this chapter we will point out how to obtain the spin vector from experiment in the case of hyperon polarization in heavy-ion collisions. Furthermore, in the following chapters we will derive the form of the spin vector from the theoretical point of view, using methods of quantum field theory, kinetic theory, and hydrodynamics.

1.6 Polarization in parity-violating processes

In non-central relativistic nucleus-nucleus collisions the large orbital angular momentum of the system leads to a polarization of the quarks in the QGP [1, 23]. Consequently, also the produced hyperons aquire a polarization. This polarization is experimentally accessible since the weak decay of hyperons is self-analyzing, i.e., the daughter particles are preferably emitted along the polarization of the hyperon in its own rest frame. In this section, we illustrate how the alignment of the momenta of the final particles along the hyperon polarization arises from the theoretical point of view [21]. The following two subsections will be devoted to the experimental methods to measure hyperon polarization in the cases of Lambda polarization.

In order to describe the polarization in a decay process, it is convenient to choose the helicity rest frame of the decaying particle. Imagine we have a massive particle with momentum p and helicity λ in the lab frame. We find the helicity rest frame by two steps:

1. Rotate the coordinate system in space to align the z-axis with the three-momentum $\mathbf{p}$.
2. Boost the coordinate system along the particle velocity $\mathbf{v}$ to the rest frame of the particle.

Consequently the particle will have a spin component $s_z = \lambda$ in the helicity rest frame.

We will use this frame to define a relativistic density matrix. In quantum mechanics, the density matrix ρ contains the information about the state of the system, i.e., by definition the ensemble average in a system described by the quantum states with spin σ

$$|\psi^i\rangle = \sum_{r=-\sigma}^{\sigma} c_r^i |\sigma r\rangle \tag{1.15}$$

(here expanded in the basis of eigenstates of the z-component of the spin operator $\hat{S}$ with $s_z = r$) weighted by $\mathfrak{p}^i$ of any operator $\hat{O}$ is given by,

$$\langle \hat{O} \rangle \equiv \mathrm{Tr}(\hat{\rho}\,\hat{O}) = \sum_i \mathfrak{p}^i \langle \psi^i | \hat{O} | \psi^i \rangle = \mathrm{tr}(O\rho), \tag{1.16}$$

where $\hat{\rho}$ is the density operator and the matrix (O_{rs}) is given by the matrix elements of the corresponding operator

$$O_{rs} \equiv \langle \sigma r | \hat{O} | \sigma s \rangle, \tag{1.17}$$

see Eq. (1.4) for $\sigma = 1/2$ and a basis of momentum states. Inserting Eqs. (1.15) and (1.17) into Eq. (1.16) we conclude that

$$\rho_{rs} = \sum_i \mathfrak{p}^i c_r^i (c_s^i)^*. \tag{1.18}$$

In the relativistic case, we define the density matrix such that it is equal to the non-relativistic density matrix (1.18) in the helicity rest frame of a state with momentum p. Expressing the relativistic state in a basis of helicity eigenstates

$$|\psi^i; p\rangle = \sum_{\lambda=-\sigma}^{\sigma} c_\lambda^i |p, \lambda\rangle \tag{1.19}$$

we obtain the relativistic density matrix

$$\rho_{\lambda\lambda'} = \sum_i \mathfrak{p}^i c_\lambda^i (c_{\lambda'}^i)^*. \tag{1.20}$$

In the case of spin $\sigma = 1/2$ we can decompose the density matrix in the basis of Pauli matrices σ^i

$$\rho = \mathcal{P}^0 + \boldsymbol{\mathcal{P}} \cdot \boldsymbol{\sigma}. \tag{1.21}$$

Using Eq. (1.16) and the relation $\mathrm{tr}(\sigma^i \sigma^j) = 2\delta^{ij}$, we find that the coefficients $\mathcal{P}^i$ are related to the spin matrices σ^i in Eq. (1.1) as

$$\boldsymbol{\mathcal{P}} = \frac{1}{2}\mathrm{tr}(\rho\,\boldsymbol{\sigma})\ . \tag{1.22}$$

Since the density matrix ρ_{rs} is identical to the density matrix from quantum mechanics (1.18) in the helicity rest frame, we have from Eqs. (1.7) and (1.16)

$$\langle \hat{\rho}\,\hat{S}^i \rangle_{p,\star} = \hbar \mathcal{P}_\star^i\ , \tag{1.23}$$

where $\langle\cdot\rangle_{p,\star}$ denotes the ensemble average over all particles with momentum p in their rest frame, and for the momentum-averaged polarization $\bar{\boldsymbol{\mathcal{P}}}$

$$\langle \hat{\rho}\,\hat{S}^i\rangle_\star = \hbar\bar{\mathcal{P}}^i_\star\,, \tag{1.24}$$

with $\langle\cdot\rangle_\star$ the ensemble average of the whole system evaluated for each particle in its own rest frame. This means that the coefficients $\mathcal{P}^i$ give the rest-frame polarization.

We consider a parity-violating decay process of a spin-1/2 particle C into a spin-1/2 particle D and a spin-0 particle F, $C \to D + F$ in the helicity rest frame of particle C. Particle D is emitted with momentum $\mathbf{p}_D = (p_D, \theta_D, \phi_D)$, in spherical coordinates. One can show [21] that the solid-angle distribution of D is given by

$$\frac{dN}{d\Omega_D}(\theta_D, \phi_D) = \frac{1}{4\pi}\left(1 + \alpha\mathcal{P}^y_C \sin\theta_D \sin\phi_D\right), \tag{1.25}$$

where we chose the coordinate system in a way that the y-component of the polarization appears in the solid-angle distribution and α is a parameter depending on the properties of the decaying particle, which has to be determined from experiment. If particle C is produced in a parity-conserving process, it will be polarized perpendicular to the production plane spanned by $\mathbf{e}_x$ and $\mathbf{e}_z$ in the helicity rest frame, i.e., by definition the polarization is parallel to $\mathbf{e}_y$. We conclude [22]

$$\begin{aligned}\frac{dN}{d\Omega^\star_D} &= \frac{1}{4\pi}\left(1 + \alpha\mathcal{P}^{\star y}_C \hat{p}^y_D\right)\\ &= \frac{1}{4\pi}\left(1 + \alpha\boldsymbol{\mathcal{P}}^\star_C \cdot \hat{\mathbf{p}}_D\right)\\ &= \frac{1}{4\pi}\left(1 + \alpha|\boldsymbol{\mathcal{P}}^\star_C| \cos\xi^\star\right),\end{aligned} \tag{1.26}$$

where $\hat{\mathbf{p}}_D$ is the unit vector of the momentum of particle D in the rest frame of C and $\xi^\star$ is the angle between this momentum and the rest-frame polarization. Equation (1.26) is an important result as it provides a way to determine the rest-frame polarization $\boldsymbol{\mathcal{P}}^\star_C$ of the decaying particle by measuring the angular distribution of the daughter particles.

1.7 Lambda polarization in heavy-ion collisions

In heavy-ion collisions, the produced Lambda hyperons decay weakly as $\Lambda \to p + \pi^-$. Since this process is driven by weak interactions, it violates parity. Therefore, we can make use of Eq. (1.26). Experimentally one determines the average of the proton momenta along a certain direction. Hence, one calculates [22]

$$\langle\hat{\mathbf{p}}_D \cdot \hat{\boldsymbol{\zeta}}\rangle_{\Lambda,e} = \left\langle \int d\Omega^\star_D \frac{dN}{d\Omega^\star_D}\hat{\mathbf{p}}_D \cdot \hat{\boldsymbol{\zeta}} \right\rangle_e = \frac{\alpha}{3}\boldsymbol{\mathcal{P}}^\star_\Lambda \cdot \hat{\boldsymbol{\zeta}}, \tag{1.27}$$

where $\langle\cdot\rangle_{\Lambda,e}$ denotes the average over all Lambdas and events and $\hat{\boldsymbol{\zeta}}$ is an arbitrary unit three-vector. In the first equality, we explicitly inserted the integration over all particles and therefore the remaining averaging is done over the number of events, denoted by $\langle\cdot\rangle_e$.

1.7.1 Global polarization

The so-called global polarization $\bar{\boldsymbol{\mathcal{P}}}$ [22–25] is obtained by integrating over all possible momenta. By symmetry, it has to be proportional to the global angular momentum $\mathbf{J}_s$ of the system [1,5,8],

$$\bar{\boldsymbol{\mathcal{P}}} = \bar{\mathcal{P}}\hat{\mathbf{J}}_s. \tag{1.28}$$

From Eq. (1.27) we have

$$\boldsymbol{\mathcal{P}}^\star_\Lambda \cdot \hat{\mathbf{J}}_s = \frac{3}{\alpha}\langle\hat{\mathbf{p}}_D \cdot \hat{\mathbf{J}}_s\rangle_{\Lambda,e} = \frac{3}{\alpha}\langle\cos\xi^\star_J\rangle_{\Lambda,e}, \tag{1.29}$$

where $\xi^\star_J$ is the angle between the proton momentum in the hyperon rest frame and the angular momentum of the system. Defining the unit three-vector $\hat{\mathbf{b}}$ to be perpendicular to both the angular momentum of the system and the beam direction $\hat{\mathbf{p}}_b$, Eq. (1.29) can also be written as

$$\boldsymbol{\mathcal{P}}^\star_\Lambda \cdot \hat{\mathbf{J}}_s = \frac{3}{\alpha}\langle\hat{\mathbf{p}}_D \cdot (\hat{\mathbf{b}} \times \hat{\mathbf{p}}_b)\rangle_{\Lambda,e} = -\frac{3}{\alpha}\langle\sin(\phi^\star_D - \psi_{RP})\sin(\theta^\star_D)\rangle_{\Lambda,e}, \tag{1.30}$$

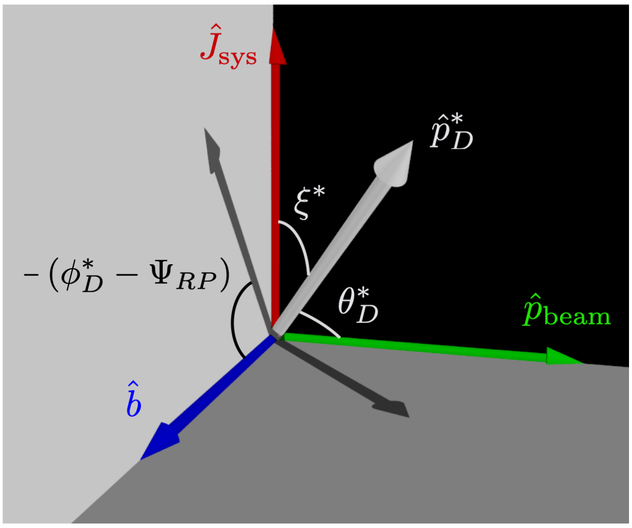


Figure 1.3: Vectors and angles for Lambda polarization. Figure from Ref. [19]

where $\theta^\star_D$ and $\phi^\star_D$ are the angles between the proton momentum and $\hat{\mathbf{p}}_b$ and $\hat{\mathbf{b}}$, respectively. Furthermore ψ_{RP} is the angle between $\hat{\mathbf{p}}_b$ and the x-axis of the coordinate system and describes the orientation of the reaction plane. Thus $\phi^\star_D - \psi_{RP}$ is the angle between $\hat{\mathbf{b}}$ and the projection of $\hat{\mathbf{p}}_D$ onto the plane spanned by $\hat{\mathbf{b}}$ and $\hat{\mathbf{J}}_s$. The geometry of the problem is sketched in Figure 1.3.

In order to obtain the global polarization $\bar{\mathcal{P}}$ defined in Eq. (1.28), which is due to the momentum-averaging not sensitive to $\theta^\star_D$, but only to $\phi^\star_D$, we integrate Eq. (1.30) over $\theta^\star_D$ and obtain the average polarization [22]

$$\bar{\mathcal{P}}_\Lambda = -\frac{8}{\pi\alpha}\langle\sin(\phi^\star_D - \psi_{RP})\rangle_{\Lambda,e}. \tag{1.31}$$

However, the experimentally measured orientation of the reaction plane $\psi^{(1)}_{EP}$ differs from the actual value ψ_{RP}. In order to correct this, one introduces an experimentally determined resolution factor $R^{(1)}_{EP}$. The final formula which relates the global polarization to experimentally accessible quantities then reads [22]

$$\bar{\mathcal{P}}_\Lambda = -\frac{8}{\pi\alpha R^{(1)}_{EP}}\langle\sin(\phi^\star_D - \psi^{(1)}_{EP})\rangle_{\Lambda,e}. \tag{1.32}$$

For Lambda hyperons, the decay constant α is given by $\alpha_\Lambda \approx 0.75$ [157].

The measurement of the global Lambda polarization by the STAR collaboration became famous for the evidence that "the fluid produced in heavy-ion collisions is by far the most vortical system ever observed" [23]. The basis of the estimate of the vorticity ω used in Ref. [23] is the formula

$$\omega = \frac{T}{\hbar}\left(\bar{P}_\Lambda + \bar{P}_{\bar{\Lambda}}\right), \tag{1.33}$$

with T the temperature and $\bar{P}_{\bar{\Lambda}}$ the global polarization of anti-Lambdas. Equation (1.33) was derived in Ref. [5] from the thermodynamic approach, generalizing the global-equilibrium form of the density matrix to local equilibrium. The phase-space density of the spin vector was in this case obtained as [2, 3]

$$s^\mu = -\frac{\hbar^2}{8m}(1 - n_F)\epsilon^{\mu\nu\alpha\beta}p_\nu\varpi_{\alpha\beta}, \tag{1.34}$$

where $\varpi_{\alpha\beta}$ is the thermal vorticity and n_F is the Fermi-Dirac distribution. Integrating this quantity over the freeze-out hypersurface, one obtains the polarization vector measured in heavy-ion collisions. The predictions for the global Lambda polarization based on Eq. (1.34) were found to be in good agreement with the measurements [1–7].

Figure 1.4 shows the measurements of the global Lambda polarization as a function of the collision energy. With increasing beam energy, the global polarization decreases. An important observation is that hyperons and anti-hyperons are polarized along the same direction, indicating that the polarization emerges from properties of the fluid flow instead of electromagnetic interactions. Furthermore, results of simulations using hydrodynamic or transport approaches are shown. Measurements and simulations agree both in magnitude and decreasing trend.

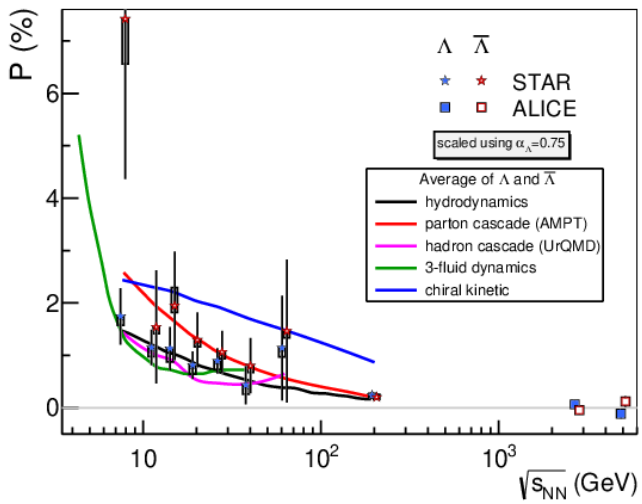


Figure 1.4: Global Lambda polarization measured by STAR [22–24] and ALICE [25] in comparison with simulations using different hydrodynamic/transport models [6, 158–160]. Figure from Ref. [19]

1.7.2 Local polarization

In contrast to the global polarization, the so-called local (i.e., momentum-differential) polarization [8, 9] can have a component orthogonal to the system angular momentum. Choosing $\hat{\zeta} = \hat{\mathbf{p}}_b$ in Eq. (1.27) we obtain the longitudinal polarization [9]

$$\mathcal{P}^z_\Lambda \equiv \boldsymbol{\mathcal{P}}^\star_\Lambda \cdot \hat{\mathbf{p}}_b = \frac{1}{\alpha} \frac{\langle \cos\theta^\star_D \rangle_{\Lambda,e}}{\langle \cos^2\theta^\star_D \rangle_{\Lambda,e}}, \tag{1.35}$$

where now the average is only over hyperons with the same momentum angle and we did not replace $\langle \cos^2\theta^\star_D \rangle_{\Lambda,e}$ by the theoretical value 1/3, as in Ref. [9] it is determined experimentally in order to account for the nonperfect acceptance of the detector, which depends on the pseudorapidity. The left plot in Figure 1.5 shows $\langle \cos\theta^\star_D \rangle_{\Lambda,e}$ as a function of the azimuthal hyperon emission angle relative to the event plane.

While there is good agreement between calculations based on Eq. (1.34) and experimental data for global Lambda polarization, it fails to describe the local polarization, since theoretical calculations [8] show the opposite angular dependence compared to measurements [9], see Figure 1.5. This problem became famous as the "spin sign puzzle". In the past couple of years, many works were attempting to solve this issue. Recently, promising progress has been made by including corrections from shear to the spin vector, which seems to be able to restore the agreement between theory and experiment [10–15].

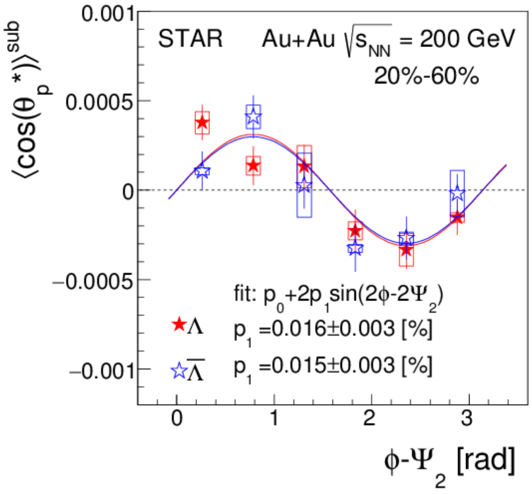
The common assumption of the mentioned calculations of the Pauli-Lubanski vector is local equilibrium, i.e., dissipative effects are neglected. However, nothing is known so far about the magnitude of these effects, making it difficult to justify this assumption. In this work, we aim at developing a framework which can be used to study the effects of dissipation on the spin vector with possible application for heavy-ion collisions. Away from local equilibrium, i.e., when dissipative effects are included, it is not possible to calculate the exact form of the density matrix. Hence, one needs to employ some approximations. In the next section, we will discuss an effective theory which turned out to be very successful for the description of heavy-ion collisions: relativistic hydrodynamics.

1.8 Relativistic dissipative hydrodynamics

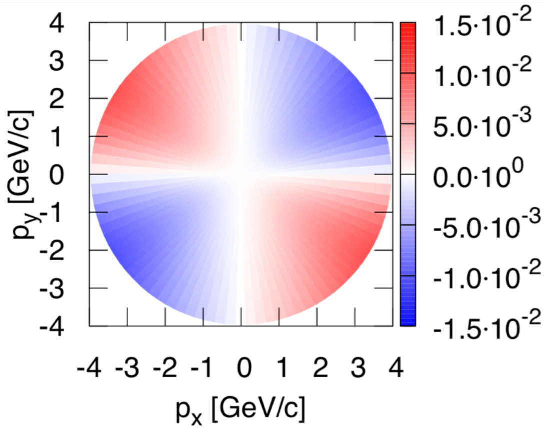
The basic equations of motion in conventional[1] hydrodynamics are the conservation laws for the charge currents N^μ and the symmetric energy-momentum tensor $T^{\mu\nu}$

$$\begin{aligned} \partial \cdot N &= 0, \\ \partial_\mu T^{\mu\nu} &= 0, \end{aligned} \tag{1.36}$$

[1] Here and in the following, we use the words "conventional" or "usual" in the context of hydrodynamics meaning the standard theory, where spin or other quantum effects do not play any role.



(a) Measured angular dependence of $\mathcal{P}^z_\Lambda$ [9]



(b) Angular dependence of $\mathcal{P}^z_\Lambda$ from hydrodynamic calculations [8]

Figure 1.5: Comparison between experiment and theory for local Lambda polarization

see, e.g., Ref. [45]. While in the ideal case the above system of equations is closed, this situation changes when taking into account dissipative effects. As in the latter case all 4 components of the charge current and 10 components of the energy-momentum tensor are nonzero, one has to provide 9 further equations of motion to cover all degrees of freedom of the system. These equations can be derived either from an effective approach, such as an analysis of the entropy current, or from a microscopic theory. The latter is commonly taken to be kinetic theory, where the distribution function $f(x,p)$ contains the microscopic information. Its dynamics is described by the Boltzmann equation

$$p \cdot \partial f(x,p) = C[f], \tag{1.37}$$

where x is a position variable, p is the particle momentum, and $C[f]$ is the collision term.

The derivation of dissipative hydrodynamics from kinetic theory without spin degrees of freedom has been discussed in the literature for many years. One traditional way is the so-called Chapman-Enskog expansion [26], based on a power-expansion of the distribution function around local equilibrium up to first order in gradients. This is shortly reviewed in Appendix E. One obtains the so-called Navier-Stokes equations, relating dissipative currents directly to gradients of hydrodynamic fields, e.g., for the bulk viscous pressure Π, which is a component of the energy-momentum tensor, one has

$$\Pi = -\,\zeta\theta. \tag{1.38}$$

In this equation, the left-hand side is a dissipative component of a conserved current, while the right-hand side is proportional to the gradient of the fluid velocity u^μ, $\theta \equiv \partial \cdot u$. The hydrodynamic equations of motion resulting from such an expansion were for a long time believed to be unstable and acausal in the relativistic case [27]. Recently, it was found that this issue can be fixed by certain choices of the hydrodynamic frame and the matching conditions which define the hydrodynamic fields [28–31].

An established formulation of stable and causal hydrodynamics is the so-called second-order theory, where the dissipative currents are independent dynamical variables following their own equations of motion. Only on long time scales, they relax to their Navier-Stokes values. This means, e.g., instead of using a relation of the form (1.38), Π is treated dynamically with a kinetic equation of the form

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta + \dots\,, \tag{1.39}$$

where τ_Π is the relaxation time and the dot denotes the time derivative in the frame comoving with the fluid. The dots on the right-hand side of this equation stand for terms of second order in derivatives and/or dissipative quantities, which are taken into account in this approach. A possible derivation of such theories from the microscopic theory relies on an expansion of the distribution function in moments of the momentum variable, the so-called method of moments, introduced by Grad [32] and generalized to the relativistic case by Israel and Stewart in a seminal paper [33]. The Boltzmann equation (1.37) implies the equations of motion of all moments

$$\mathcal{I}^{\mu_1\cdots\mu_l} \equiv \int d^4p\, \delta(p^2-m^2) p^{\mu_1}\cdots p^{\mu_l} f(x,p), \tag{1.40}$$

where due the conservation of charge and four-momentum in a microscopic collision the kinetic equations for $\mathcal{I}^{\mu}$ and $\mathcal{I}^{\mu\nu}$ reproduce the conservation laws in Eqs. (1.36). Israel and Stewart obtained the additional 9 equations needed to close the system from the equation of motion for $\mathcal{I}^{\mu\nu\lambda}$, ignoring the dynamics of all other moments. Their method has been improved in Refs. [34–36] by deriving the additional equations of motion directly from the Boltzmann equation and considering an expansion of the distribution function in moments which are irreducible with respect to the Little group of the fluid velocity and hence form an orthogonal basis. Furthermore, a small power-counting parameter in form of the so-called Knudsen number, defined as the ratio of the mean-free path between particle collisions and a hydrodynamic scale characterizing the variation of dissipative quantities, was introduced in Ref. [35], which was used in the Chapman-Enskog expansion, but not in the Israel-Stewart formulation. The approach of Refs. [35, 36], and in particular the Knudsen number and the power counting will be discussed in more detail later in this thesis in Chapter 5. The framework of Refs. [35, 36] also has been extended to include electromagnetic fields [64, 161] and anisotropic effects [162, 163].

Second-order relativistic viscous hydrodynamics has been extremely successfully applied to describe the dynamics of the QGP and compared to experimental data in heavy-ion collisions, see e.g. Refs. [164–166]. Thus, it is a very useful tool to extract the properties of the QGP. It seems appealing to use this approach also to study spin and polarization effects in heavy-ion collisions. In the next chapter, we will introduce an important ingredient for the formulation of spin hydrodynamics: the spin tensor.

Chapter 2

The spin tensor and the localization problem

Hydrodynamics is based on the conservation laws for the hydrodynamic fields. Without spin degrees of freedom, these are the charge current and the energy-momentum tensor. Including spin effects requires to consider an additional conserved current: the total angular momentum, which is the sum of the orbital angular momentum and the spin tensor. In this chapter, which is based on Ref. [41], we derive microscopic expressions for the conserved currents for free Dirac fields from quantum field theory. In general, the spin tensor is not conserved, but its divergence is proportional to the antisymmetric part of the energy-momentum tensor. We explain that the definition of the spin and energy-momentum tensor is not unique and introduce the concept of pseudo-gauge transformations. Then, we discuss various choices of pseudo-gauge, which can be grouped in three categories: nonzero spin tensor and nonsymmetric energy-momentum tensor, zero spin tensor and symmetric energy-momentum tensor, and nonzero spin tensor and symmetric energy-momentum tensor. We point out why the last set of tensors provides the most intuitive interpretation of the spin tensor as a spin density. Furthermore, we introduce the free Wigner function and show how it can be used to express the currents in the different pseudo-gauges. We then point out the connection between the spin tensor and the experimentally accessible Pauli-Lubanski vector introduced in the previous chapter. Finally, we point out that the pseudo-gauge freedom is related to the nonuniqueness of the center of mass in a relativistic theory. For this reason, the choice of a set of tensors corresponds to a "localization" of the theory. In this chapter, we consider free fields, the generalization to interacting fields will be made in Chapter 4.

2.1 Conserved currents and pseudo-gauge transformations

We consider a free Dirac field $\psi(x)$ with mass m and the action

$$A = \int d^4x \, \mathcal{L}(x). \tag{2.1}$$

The Lagrangian $\mathcal{L}$ is left general for the moment. According to Noether's theorem, the invariance of the action under continuous symmetries implies conserved currents. The energy-momentum tensor $\hat{T}^{\mu\nu}$ is defined as the conserved current due the invariance of the action (2.1) under infinitesimal spacetime translations

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu, \tag{2.2}$$

with ξ^μ a constant parameter. The conservation law reads

$$\partial_\mu \hat{T}^{\mu\nu} = 0. \tag{2.3}$$

For a general Lagrangian containing at most second-order derivatives of spinor fields, the energy-momentum tensor is given by [45, 167]

$$\hat{T}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \partial^\nu \psi + (\partial^\nu \bar{\psi}) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \partial_\mu \psi)} \overleftrightarrow{\partial}^\rho \partial^\nu \psi - (\partial^\nu \bar{\psi}) \overleftrightarrow{\partial}^\rho \frac{\partial \mathcal{L}}{\partial(\partial_\rho \partial_\mu \bar{\psi})} - g^{\mu\nu} \mathcal{L} \tag{2.4}$$

with $\overleftrightarrow{\partial}^\mu \equiv \overrightarrow{\partial}^\mu - \overleftarrow{\partial}^\mu$ and $\bar{\psi}$ being the adjoint spinor field. We define the global four-momentum operator as the integration of the energy-momentum tensor over a three-dimensional space-like hypersurface Σ_λ,

$$\hat{P}^\mu = \int_\Sigma d\Sigma_\lambda \, \hat{T}^{\lambda\mu}. \tag{2.5}$$

It is important to note that $\hat{P}^\mu$ transforms covariantly under Lorentz transformations, as $\hat{T}^{\mu\nu}$ is conserved. In fact, for any tensor $\hat{B}^{\lambda\mu_1\cdots\mu_n}$, the quantity defined as

$$\hat{B}^{\mu_1\cdots\mu_n} = \int_\Sigma d\Sigma_\lambda \hat{B}^{\lambda\mu_1\cdots\mu_n}$$

transforms as a tensor only if $\partial_\lambda \hat{B}^{\lambda\mu_1\cdots\mu_n} = 0$ and suitable boundary conditions are fulfilled [168]. To prove this, one has to show that the integration in $\hat{B}^{\mu_1\cdots\mu_n}$ is invariant under the choice of the hypersurface. Consider a region of spacetime enclosed between two space-like hypersurfaces Σ_1, Σ_2 corresponding to two different values of the parameter $\mathfrak{t}$ used for the foliation of the spacetime, $\mathfrak{t}_1$, $\mathfrak{t}_2$, respectively ($\mathfrak{t}$ can be e.g. x^0). Using the divergence theorem and the fact that $\hat{B}^{\mu_1\cdots\mu_n}$ vanishes at the boundary, we have

$$\int_{\Sigma_1} d\Sigma_\lambda \hat{B}^{\lambda\mu_1\cdots\mu_n} - \int_{\Sigma_2} d\Sigma_\lambda \hat{B}^{\lambda\mu_1\cdots\mu_n} = \int_V d^4x\, \partial_\lambda \hat{B}^{\lambda\mu_1\cdots\mu_n},$$

where V is the four-dimensional volume. The right-hand side of the equation above vanishes if $\partial_\lambda \hat{B}^{\lambda\mu_1\cdots\mu_n} = 0$.

Furthermore, we remark that the four components of $\hat{P}^\mu$ can be identified with the four generators of spacetime translations [167].

We now consider infinitesimal Lorentz four-rotations

$$x^\mu \to x'^\mu = x^\mu + \zeta^{\mu\nu} x_\nu, \tag{2.6}$$

with $\zeta^{\mu\nu} = -\zeta^{\nu\mu}$ constant. The spinor field transforms according to

$$\delta_T \psi = \psi'(x') - \psi(x) = \frac{1}{2}\zeta_{\mu\nu} f^{\mu\nu}\psi(x), \tag{2.7}$$

with $f^{\mu\nu} = -\frac{i}{2}\sigma^{\mu\nu}$ and

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu], \tag{2.8}$$

where γ^μ are the Dirac matrices. If the action is invariant under the transformations (2.6) we obtain from Noether's theorem the conservation of the total angular momentum tensor

$$\hat{J}^{\lambda,\mu\nu} = x^\mu \hat{T}^{\lambda\nu} - x^\nu \hat{T}^{\lambda\mu} + \hbar\, \hat{S}^{\lambda,\mu\nu}, \tag{2.9}$$

i.e.,

$$\partial_\lambda \hat{J}^{\lambda,\mu\nu} = 0. \tag{2.10}$$

The third term in Eq. (2.9) is called the spin tensor and can be obtained from the Lagrangian as

$$\hbar\, \hat{S}^{\lambda,\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \psi)} f^{\mu\nu}\psi - \bar{\psi} f^{\mu\nu} \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \bar{\psi})} + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \partial_\lambda \psi)} \overleftrightarrow{\partial}^\rho f^{\mu\nu}\psi + \bar{\psi}\overleftrightarrow{\partial}^\rho f^{\mu\nu} \frac{\partial \mathcal{L}}{\partial(\partial_\rho \partial_\lambda \bar{\psi})}. \tag{2.11}$$

Inserting Eq. (2.9) into (2.10), we obtain

$$\hbar\, \partial_\lambda \hat{S}^{\lambda,\mu\nu} = \hat{T}^{[\nu\mu]}. \tag{2.12}$$

Thus the spin tensor is conserved if and only if the energy-momentum tensor is symmetric. However, we stress that in general Eq. (2.4) will yield an antisymmetric part of $\hat{T}^{\mu\nu}$. The total angular momentum

$$\hat{J}^{\mu\nu} = \int_\Sigma d\Sigma_\lambda\, \hat{J}^{\lambda,\mu\nu} \tag{2.13}$$

transforms as a tensor under Lorentz transformations due to the conservation law (2.10). Its six independent components are the generators of the Lorentz transformations [167]. However, the global spin

$$\hat{S}^{\mu\nu} \equiv \int_\Sigma d\Sigma_\lambda\, \hat{S}^{\lambda,\mu\nu} \tag{2.14}$$

does not transform covariantly under Lorentz transformations for $\hat{T}^{[\mu\nu]} \neq 0$, as in this case the integral depends on the choice of the hypersurface according to Eq. (2.12).

The global momentum (2.5) and angular momentum (2.13) as well as the conservation laws (2.3) and (2.10) are invariant under the so-called pseudo-gauge transformations [40, 42, 128, 129]

$$\hat{T}'^{\mu\nu} = \hat{T}^{\mu\nu} + \frac{\hbar}{2}\partial_\lambda(\hat{\Phi}^{\lambda,\mu\nu} + \hat{\Phi}^{\nu,\mu\lambda} + \hat{\Phi}^{\mu,\nu\lambda}), \tag{2.15a}$$

$$\hat{S}'^{\lambda,\mu\nu} = \hat{S}^{\lambda,\mu\nu} - \hat{\Phi}^{\lambda,\mu\nu} + \hbar\,\partial_\rho \hat{Z}^{\mu\nu\lambda\rho}. \tag{2.15b}$$

Here we introduced the so-called superpotentials $\hat{\Phi}^{\lambda,\mu\nu}$ and $\hat{Z}^{\mu\nu\lambda\rho}$, which are arbitrary differentiable operators that fulfill $\hat{\Phi}^{\lambda,\mu\nu} = -\hat{\Phi}^{\lambda,\nu\mu}$ and $\hat{Z}^{\mu\nu\lambda\rho} = -\hat{Z}^{\nu\mu\lambda\rho} = -\hat{Z}^{\mu\nu\rho\lambda}$. Defining $\hat{J}'^{\lambda,\mu\nu} = x^\mu \hat{T}'^{\lambda\nu} - x^\nu \hat{T}'^{\lambda\mu} + \hbar\,\hat{S}'^{\lambda,\mu\nu}$ one can prove the validity of Eqs. (2.3), (2.10) and Eqs. (2.5), (2.13) for the primed quantities using the divergence theorem under the assumption of vanishing boundary terms.

As already mentioned, the dynamical quantities in hydrodynamics are the densities, hence it is expected that the choice of pseudo-gauge affects the evolution of the system. This will become clearer in Chapter 5. In the following sections, we will discuss various choices of pseudo-gauge and see that they are related to different physical interpretations. Since pseudo-gauge transformations change $\hat{S}^{\mu\nu}$, but not $\hat{J}^{\mu\nu}$, they can be seen as different splittings of the total angular momentum into orbital and spin part.

2.2 Spin tensor and pseudo-gauge choices for free Dirac fields

2.2.1 Canonical pseudo-gauge

The most straightforward choice for a Lagrangian of noninteracting fields with spin 1/2 is the Dirac Lagrangian

$$\mathcal{L}_D(x) = \frac{i\hbar}{2}\,\bar{\psi}(x)\gamma^\mu \overleftrightarrow{\partial}_\mu \psi(x) - m\bar{\psi}(x)\psi(x)\,. \tag{2.16}$$

The Lagrangian (2.16) leads to the equations of motion

$$(i\hbar\gamma^\mu\partial_\mu - m)\psi(x) = 0, \tag{2.17a}$$

$$\bar{\psi}(x)(i\hbar\gamma^\mu\overleftarrow{\partial}_\mu + m) = 0. \tag{2.17b}$$

Using the Lagrangian (2.16) in Eqs. (2.4) and (2.11), respectively, we obtain the so-called canonical energy-momentum tensor

$$\hat{T}^{\mu\nu}_C = \frac{i\hbar}{2}\bar{\psi}\gamma^\mu\overleftrightarrow{\partial}^\nu\psi - g^{\mu\nu}\mathcal{L}_D \tag{2.18}$$

and the canonical spin tensor

$$\begin{aligned}\hat{S}^{\lambda,\mu\nu}_C &= \frac{1}{4}\bar{\psi}\{\gamma^\lambda, \sigma^{\mu\nu}\}\psi \\ &= -\frac{1}{2}\epsilon^{\lambda\mu\nu\alpha}\bar{\psi}\gamma_\alpha\gamma_5\psi.\end{aligned} \tag{2.19}$$

In order to calculate the global charges, we note that due to the conservation of the energy-momentum tensor (2.3), the integration in Eq. (2.5) is independent of the choice of the hypersurface. Hence, we can choose the hyperplane at constant x^0 without loss of generality, and obtain under the assumption of vanishing boundary terms and using $\mathcal{L}_D = 0$ after applying the Dirac equation

$$\hat{P}^\mu = i\hbar\int d^3x\,\psi^\dagger\partial^\mu\psi \tag{2.20}$$

and similarly from Eq. (2.13)

$$\begin{aligned}\hat{J}^{\mu\nu} &= \int d^3x\left[\frac{i\hbar}{2}x^{[\mu}\left(\psi^\dagger\overleftrightarrow{\partial}^{\nu]}\psi\right) + \frac{\hbar}{2}\psi^\dagger\sigma^{\mu\nu}\psi - \frac{\hbar}{4}\bar{\psi}[\gamma^0,\sigma^{\mu\nu}]\psi\right] \\ &= \int d^3x\,\psi^\dagger\left[i\hbar(g^\nu_i x^\mu\partial^i - g^\mu_i x^\nu\partial^i) + \frac{i\hbar}{2}x^{[\mu}g^{\nu]}_0\partial^0 - \frac{1}{2}\gamma^0\left(i\hbar\gamma^i\partial_i - m\right)g^{[\nu}_0 x^{\mu]} + \frac{\hbar}{2}\sigma^{\mu\nu} - \frac{i\hbar}{2}\gamma^0 g^{0[\mu}\gamma^{\nu]}\right]\psi \\ &= \int d^3x\,\psi^\dagger\left[i\hbar(x^\mu\partial^\nu - x^\nu\partial^\mu) + \frac{\hbar}{2}\sigma^{\mu\nu}\right]\psi,\end{aligned} \tag{2.21}$$

where in the second line we integrated the fourth term by parts after using the adjoint Dirac equation and in the last term used Eq. (A.3d). Furthermore, in the last step we used the Dirac equation. In the canonical splitting the contribution from the energy-momentum tensor (2.18) to the total angular momentum is identified with the orbital angular momentum and the tensor (2.19) with the spin. However, this splitting is not unique, and different possibilities can be obtained by applying pseudo-gauge transformations (2.15) with certain superpotentials to the canonical currents. An important feature of the canonical splitting is the fact that the canonical energy-momentum tensor in Eq. (2.18) is not symmetric for free fields. This means that if one interprets $\hat{S}^{\lambda,\mu\nu}$ as a spin density, there is always conversion between spin and orbital angular momentum even in the absence of interactions. This is not consistent with the physical picture. As will become clear later in Section 2.5, the reason for this conversion is the unphysical oscillation of the position of a free Dirac particle called "Zitterbewegung". As a consequence of the asymmetry of the energy-momentum tensor, the canonical global spin (2.14) is not Lorentz covariant, as it depends on the choice of the hyperplane. In the next sections, we will discuss several possibilities to overcome this problem.

2.2.2 Belinfante pseudo-gauge

In Einstein's general relativity, the energy-momentum tensor is obtained from the variation of the Lagrangian with respect to the metric tensor. By definition this energy-momentum tensor is symmetric and hence cannot coincide with the canonical one (2.18) for Dirac fields. Although this is not a reason to reject the canonical set of tensors, since also nonsymmetric energy-momentum tensors can be used in curved space-time in Einstein-Cartan theory in the presence of torsion [40,130,131], there is a possibility to define a symmetric total energy-momentum tensor, which couples to Einstein's gravity. This was first found by Belinfante and Rosenfeld [42,128,129]. The Belinfante energy-momentum tensor is obtained by using

$$\begin{aligned} \hat{\Phi}_B^{\lambda,\mu\nu} &= \hat{S}_C^{\lambda,\mu\nu}, \\ \hat{Z}_B^{\mu\nu\lambda\rho} &= 0 \end{aligned} \tag{2.22}$$

in the pseudo-gauge transformation (2.15). We find

$$\begin{aligned} \hat{T}_B^{\mu\nu} &= \frac{i\hbar}{4}\bar{\psi}\gamma^{(\mu}\overleftrightarrow{\partial}^{\nu)}\psi - g^{\mu\nu}\mathcal{L}_D, \\ \hat{S}_B^{\lambda,\mu\nu} &= 0. \end{aligned} \tag{2.23}$$

The spin tensor in the Belinfante version thus vanishes, which means that the spin contribution to the total angular momentum is hidden in the orbital part proportional to the energy-momentum tensor $\hat{T}_B^{\mu\nu}$. If this tensor is treated as the dynamical one, one cannot introduce an associated dynamical spin tensor with the interpretation of a spin density. However, the information about the spin degrees of freedom need to be taken into account if one intends to consistently describe the full system. One possibility is to keep the nonconservation of the canonical spin tensor

$$\hbar\,\partial_\lambda \hat{S}_C^{\lambda,\mu\nu} = -\hat{T}_C^{[\mu\nu]} \tag{2.24}$$

as an independent constraint [126]. We note that this equation was used to obtain Eqs. (2.23), so we need to ensure its validity when using the Belinfante set of tensors.

Alternatively, it is also possible to obtain a spin tensor, which we call the Belinfante-Rosenfeld spin[1], by splitting the total angular momentum into the orbital part (i.e., the part related to the canonical energy-momentum tensor) and the spin part $\hat{\bar{S}}_B^{\lambda,\mu\nu}$, as originally proposed in Refs. [42,128,129],

$$\begin{aligned} \hat{J}_B^{\lambda,\mu\nu} &= x^{[\mu}\hat{T}_B^{\nu]\lambda} \\ &\equiv x^{[\mu}\hat{T}_C^{\nu]\lambda} + \hbar\,\hat{\bar{S}}_B^{\lambda,\mu\nu}. \end{aligned} \tag{2.25}$$

This splitting of the total angular momentum can be obtained from the canonical one by using the following pseudo-gauge potentials,

$$\begin{aligned} \hat{\bar{\Phi}}_B^{\lambda,\mu\nu} &= 0, \\ \hbar\,\hat{\bar{Z}}_B^{\mu\nu\lambda\rho} &= \frac{1}{2}(x^\mu \hat{S}_C^{\rho,\lambda\nu} - x^\nu \hat{S}_C^{\rho,\lambda\mu}). \end{aligned} \tag{2.26}$$

[1]This should be distinguished from the vanishing spin tensor in Eq. (2.23), to which we refer as Belinfante spin tensor.

By integrating by parts one can show that the Belinfante-Rosenfeld spin tensor leads to the same global spin as the canonical spin tensor:

$$\hat{S}^{\mu\nu}_B \equiv \int d^3x\, \hat{S}^{0,\mu\nu}_B = \hat{S}^{\mu\nu}_C . \tag{2.27}$$

Consequently, the problem of the global spin not being a tensor is not solved in the Belinfante-Rosenfeld version.

2.2.3 Hilgevoord-Wouthuysen pseudo-gauge

In order to be consistent with the physical picture, one searches for a spin tensor which is conserved for free fields. This will lead to a covariant global spin. We will see that there is a choice of such spin tensor with the global spin being identical to the canonical spin in a certain frame. A natural choice for massive fields is the particle rest frame, i.e., we impose the Frenkel condition (1.6). It was found by Hilgevoord and Wouthuysen that a global spin fulfilling this condition can be obtained by applying Noether's theorem to the Klein-Gordon Lagrangian and in addition requiring that the spinors are solutions of the Dirac equation [43, 44]. As the Dirac equation implies the Klein-Gordon equation, the symmetries of the Klein-Gordon Lagrangian

$$\mathcal{L}_{KG} = \frac{1}{2m}(\hbar^2 \partial_\mu \bar{\psi} \partial^\mu \psi - m^2 \bar{\psi}\psi) \tag{2.28}$$

should in particular hold for Dirac spinors. Since this Lagrangian is Poincaré-invariant, we obtain from Noether's theorem the conservation of the Hilgevoord-Wouthuysen (HW) energy-momentum tensor

$$\hat{T}^{\mu\nu}_{HW} = \frac{\hbar^2}{2m}\left\{(\partial^\nu\bar{\psi})\partial^\mu\psi + (\partial^\mu\bar{\psi})\partial^\nu\psi - g^{\mu\nu}\left[(\partial_\lambda\bar{\psi})\partial^\lambda\psi - \frac{m^2}{\hbar^2}\bar{\psi}\psi\right]\right\}, \tag{2.29}$$

and the total angular momentum with the spin tensor

$$\hat{S}^{\lambda,\mu\nu}_{HW} = \frac{i\hbar}{4m}\bar{\psi}\sigma^{\mu\nu}\overleftrightarrow{\partial}^\lambda\psi. \tag{2.30}$$

In contrast to the canonical energy-momentum tensor (2.18), which is derived from the Dirac Lagrangian, the HW energy-momentum tensor (2.29) is symmetric for free fields. This implies that the HW spin tensor is conserved and the global spin

$$\hat{S}^{\mu\nu}_{HW} \equiv \int d^3x\, \hat{S}^{0,\mu\nu}_{HW} \tag{2.31}$$

is a Lorentz tensor. Using the Dirac equation, we obtain

$$\begin{aligned}
\hat{S}^{\mu\nu}_{HW} &= \frac{i\hbar}{4m}\int d^3x\left[-(\partial_0\bar{\psi})\sigma^{\mu\nu}\psi + \psi^\dagger\sigma^{\mu\nu}\gamma^0\partial_0\psi + 2ig^{0[\mu}\psi^\dagger\gamma^{\nu]}\partial_0\psi\right] \\
&= \frac{1}{4m}\int d^3x\left[(i\hbar\partial_i\bar{\psi}\gamma^i + m\bar{\psi})\gamma^0\sigma^{\mu\nu}\psi + \psi^\dagger\sigma^{\mu\nu}(-i\hbar\gamma^i\partial_i + m)\psi - 2\hbar g^{0[\mu}\psi^\dagger\gamma^{\nu]}\partial_0\psi\right] \\
&= \frac{1}{2}\int d^3x\left(\psi^\dagger\sigma^{\mu\nu}\psi + \frac{i\hbar}{2m}\psi^\dagger[\gamma^i,\sigma^{\mu\nu}]\partial_i\psi - \frac{\hbar}{m}g^{0[\mu}\psi^\dagger\gamma^{\nu]}\partial_0\psi\right) \\
&= \frac{1}{2}\int d^3x\left(\psi^\dagger\sigma^{\mu\nu}\psi + \frac{\hbar}{m}\psi^\dagger\gamma^{[\mu}\partial^{\nu]}\psi\right),
\end{aligned} \tag{2.32}$$

where we used Eq. (A.3d) and integrated by parts in the second step. For the expectation value with respect to particle states at rest we find

$$\begin{aligned}
S^{i0}_{HW,rs,\star} &= \left\langle p_\star, r\left|\frac{i\hbar}{4m}\int d^3x\,\left(\bar{\psi}\sigma^{i0}\partial^0\psi - (\partial^0\bar{\psi})\sigma^{i0}\psi\right)\right|p_\star, s\right\rangle \\
&= \left\langle p_\star, r\left|\frac{\hbar}{4}\int d^3x\,\left(\bar{\psi}\sigma^{i0}\gamma^0\psi + \bar{\psi}\gamma^0\sigma^{i0}\psi\right)\right|p_\star, s\right\rangle \\
&= 0,
\end{aligned} \tag{2.33}$$

where we used the Dirac equation. Since by definition $\mathbf{p}_\star = 0$, $S^{\mu\nu}_{HW,rs}$ is antisymmetric, and both objects are covariant, we conclude

$$p_\mu S^{\mu\nu}_{HW,rs} = 0 \tag{2.34}$$

in any frame, consistent with the Frenkel condition (1.6). This means that using the HW spin tensor corresponds to defining the spin in the rest frame.

In order to find the pseudo-gauge transformation to the HW currents, we make use of the Gordon decomposition [169]

$$\bar{\psi}\gamma^\mu\psi = \frac{i\hbar}{2m}\left[\bar{\psi}\overleftrightarrow{\partial}^\mu\psi - i\left(\bar{\psi}\sigma^{\mu\nu}\partial_\nu\psi + \partial_\nu\bar{\psi}\sigma^{\mu\nu}\psi\right)\right]. \tag{2.35}$$

This relation follows directly from the Dirac equation (2.17), which implies after multiplication with γ^λ

$$i\hbar\partial^\lambda\psi = -\hbar\sigma^{\lambda\mu}\partial_\mu\psi + m\gamma^\lambda\psi, \tag{2.36a}$$

$$-i\hbar\partial^\lambda\bar{\psi} = -\hbar\partial_\mu\bar{\psi}\sigma^{\lambda\mu} + m\bar{\psi}\gamma^\lambda. \tag{2.36b}$$

With the help of Eqs. (2.36) we can rewrite the HW energy-momentum tensor as

$$\hat{T}^{\mu\nu}_{HW} = \frac{i\hbar}{2}\bar{\psi}\gamma^\mu\overleftrightarrow{\partial}^\nu\psi + \frac{i\hbar^2}{2m}\left[(\partial_\lambda\bar{\psi})\sigma^{\lambda\mu}\partial^\nu\psi + (\partial^\nu\bar{\psi})\sigma^{\mu\lambda}\partial_\lambda\psi\right] - g^{\mu\nu}\frac{i\hbar^2}{4m}\partial_\lambda(\bar{\psi}\sigma^{\lambda\rho}\overleftrightarrow{\partial}_\rho\psi). \tag{2.37}$$

Here we made use of the Klein-Gordon and Dirac equations to obtain

$$\begin{aligned}\mathcal{L}_{KG} &= \frac{1}{2m}(\hbar^2\partial_\mu\bar{\psi}\partial^\mu\psi - m^2\bar{\psi}\psi)\\ &= \frac{\hbar^2}{4m}\partial_\mu\left[(\partial^\mu\bar{\psi})\psi + \bar{\psi}\partial^\mu\psi\right]\\ &= \frac{i\hbar^2}{4m}\partial_\mu(\bar{\psi}\sigma^{\mu\nu}\overleftrightarrow{\partial}_\nu\psi).\end{aligned} \tag{2.38}$$

Analogously we can write the HW spin tensor

$$\hat{S}^{\lambda,\mu\nu}_{HW} = \hat{S}^{\lambda,\mu\nu}_{C} - \frac{\hbar}{4m}\left(\bar{\psi}\sigma^{\mu\nu}\sigma^{\lambda\rho}\partial_\rho\psi + \partial_\rho\bar{\psi}\sigma^{\lambda\rho}\sigma^{\mu\nu}\psi\right). \tag{2.39}$$

Defining the translational gravitational moment density

$$\hat{M}^{\lambda\mu\nu} \equiv \frac{i\hbar}{4m}\bar{\psi}\sigma^{\mu\nu}\overleftrightarrow{\partial}^\lambda\psi \equiv \hat{S}^{\lambda,\mu\nu}_{HW} \tag{2.40}$$

we find that the pseudo-gauge potentials for the HW currents are given by [131, 170, 171]

$$\begin{aligned}\hat{\Phi}^{\lambda,\mu\nu}_{HW} &= \hat{M}^{[\mu\nu]\lambda} - g^{\lambda[\mu}\hat{M}_\rho{}^{\nu]\rho},\\ \hat{Z}^{\mu\nu\lambda\rho}_{HW} &= -\frac{1}{8m}\bar{\psi}(\sigma^{\mu\nu}\sigma^{\lambda\rho} + \sigma^{\lambda\rho}\sigma^{\mu\nu})\psi.\end{aligned} \tag{2.41}$$

In order to prove that these transformations lead to the currents (2.29) and (2.30) we note that

$$\hat{T}^{\mu\nu}_{HW} = \hat{T}^{\mu\nu}_{C} - \hbar\partial_\lambda\left(\hat{M}^{\nu\mu\lambda} + g^{\nu[\mu}\hat{M}_\rho{}^{\lambda]\rho}\right) \tag{2.42}$$

with $\mathcal{L}_D = 0$ and

$$\begin{aligned}\partial_\lambda\left(\hat{M}^{\nu\mu\lambda} + g^{\nu[\mu}\hat{M}_\rho{}^{\lambda]\rho}\right) =& \frac{i\hbar}{4m}\partial_\lambda\left(\bar{\psi}\sigma^{\mu\lambda}\overleftrightarrow{\partial}^\nu\psi + g^{\nu[\mu}\bar{\psi}\sigma^{\lambda]\rho}\overleftrightarrow{\partial}_\rho\psi\right)\\ =& \frac{i\hbar}{4m}[(\partial_\lambda\bar{\psi})\sigma^{\mu\lambda}\partial^\nu\psi - (\partial_\lambda\partial^\nu\bar{\psi})\sigma^{\mu\lambda}\psi + \bar{\psi}\sigma^{\mu\lambda}\partial_\lambda\partial^\nu\psi - (\partial^\nu\bar{\psi})\sigma^{\mu\lambda}\partial_\lambda\psi\\ &- (\partial^\nu\bar{\psi})\sigma^{\mu\rho}\partial_\rho\psi + (\partial^\nu\partial_\rho\bar{\psi})\sigma^{\mu\rho}\psi - \bar{\psi}\sigma^{\mu\rho}\partial^\nu\partial_\rho\psi + (\partial_\rho\bar{\psi})\sigma^{\mu\rho}\partial^\nu\psi\\ &+ g^{\nu\mu}\partial_\lambda(\bar{\psi}\sigma^{\lambda\rho}\overleftrightarrow{\partial}_\rho\psi)]\\ =& \frac{i\hbar}{4m}[2(\partial_\lambda\bar{\psi})\sigma^{\mu\lambda}\partial^\nu\psi - 2(\partial^\nu\bar{\psi})\sigma^{\mu\lambda}\partial_\lambda\psi + g^{\nu\mu}\partial_\lambda(\bar{\psi}\sigma^{\lambda\rho}\overleftrightarrow{\partial}_\rho\psi)],\end{aligned} \tag{2.43}$$

as well as

$$\begin{aligned}\hat{S}^{\lambda,\mu\nu}_{HW} =& \frac{1}{4}\bar{\psi}\{\gamma^\lambda,\sigma^{\mu\nu}\}\psi + \frac{i\hbar}{4m}\left(\bar{\psi}\overleftrightarrow{\partial}^{[\nu}\sigma^{\mu]\lambda}\psi - g^{\lambda[\nu}\sigma^{\mu]\rho}\overleftrightarrow{\partial}_\rho\psi\right) - \frac{\hbar}{4m}\left[(\partial_\rho\bar{\psi})\sigma^{\lambda\rho}\sigma^{\mu\nu}\psi + \bar{\psi}\sigma^{\mu\nu}\sigma^{\lambda\rho}\partial_\rho\psi\right]\\ &+ \frac{\hbar}{8m}\bar{\psi}[\sigma^{\mu\nu},\sigma^{\lambda\rho}]\overleftrightarrow{\partial}_\rho\psi\\ =& \frac{1}{4}\bar{\psi}\{\gamma^\lambda,\sigma^{\mu\nu}\}\psi + \frac{i\hbar}{4m}\left(\bar{\psi}\overleftrightarrow{\partial}^{[\nu}\sigma^{\mu]\lambda}\psi - g^{\lambda[\nu}\sigma^{\mu]\rho}\overleftrightarrow{\partial}_\rho\psi\right)\\ &- \frac{1}{4m}\left[(m\bar{\psi}\gamma^\lambda + i\hbar\partial^\lambda\bar{\psi})\sigma^{\mu\nu}\psi + \bar{\psi}\sigma^{\mu\nu}(m\gamma^\lambda - i\hbar\partial^\lambda)\psi\right] + \frac{i\hbar}{4m}\left(\bar{\psi}\overleftrightarrow{\partial}^{[\mu}\sigma^{\nu]\lambda}\psi + g^{\lambda[\nu}\bar{\psi}\sigma^{\mu]\rho}\overleftrightarrow{\partial}_\rho\psi\right)\\ =& \frac{i\hbar}{4m}\bar{\psi}\sigma^{\mu\nu}\overleftrightarrow{\partial}^\lambda\psi,\end{aligned} \tag{2.44}$$

where we used Eq. (A.3e).

2.2.4 De Groot, van Leeuwen and van Weert pseudo-gauge

The spin and energy-momentum tensors introduced in Ref. [45] by de Groot, van Leeuwen, and van Weert (GLW) and used to formulate spin hydrodynamics in Refs. [37, 38, 150] also correspond to defining the spin in the rest frame as they yield the same global spin as in the HW version. The pseudo-gauge transformation from the canonical tensors is given by

$$\begin{aligned} \hat{\Phi}^{\lambda,\mu\nu}_{GLW} &= -\frac{i\hbar}{4m}\bar{\psi}(\sigma^{\mu\lambda}\overleftrightarrow{\partial}^{\nu} - \sigma^{\nu\lambda}\overleftrightarrow{\partial}^{\mu})\psi, \\ \hat{Z}^{\mu\nu\lambda\rho}_{GLW} &= 0. \end{aligned} \tag{2.45}$$

Using the Gordon decomposition in Eq. (2.35) or Eqs. (2.36) and following similar steps as above, we obtain

$$\hat{T}^{\mu\nu}_{GLW} = -\frac{\hbar^2}{4m}\bar{\psi}\overleftrightarrow{\partial}^{\mu}\overleftrightarrow{\partial}^{\nu}\psi, \tag{2.46a}$$

$$\hat{S}^{\lambda,\mu\nu}_{GLW} = \frac{i\hbar}{4m}\left(\bar{\psi}\sigma^{\mu\nu}\overleftrightarrow{\partial}^{\lambda}\psi + \partial_\rho\epsilon^{\mu\nu\lambda\rho}\bar{\psi}\gamma^5\psi\right). \tag{2.46b}$$

This choice of currents is to our knowledge not directly related to a Lagrangian.

2.2.5 Alternative Klein-Gordon pseudo-gauge

Instead of the Lagrangian (2.28) one can also derive a conserved spin tensor and a symmetric energy-momentum tensor from the Klein-Gordon Lagrangian in the form

$$\mathcal{L}'_{KG} = \frac{1}{2m}\left\{-\frac{\hbar^2}{2}\left[(\partial^2\bar{\psi})\psi + \bar{\psi}\partial^2\psi\right] - m^2\bar{\psi}\psi\right\}, \tag{2.47}$$

which generates the same equations of motion. Using Noether's theorem in the form of Eqs. (2.4) and (2.11), as well as the squared Dirac equation, we obtain [41]

$$\hat{T}^{\mu\nu}_{KG} = -\frac{\hbar^2}{4m}\bar{\psi}\overleftrightarrow{\partial}^{\mu}\overleftrightarrow{\partial}^{\nu}\psi, \tag{2.48a}$$

$$\hat{S}^{\lambda,\mu\nu}_{KG} = \hat{S}^{\lambda,\mu\nu}_{HW}. \tag{2.48b}$$

Comparing Eqs. (2.48) to Eqs. (2.46), we see that these currents can be obtained from the canonical ones by a pseudo-gauge transformation with

$$\hat{\Phi}^{\lambda,\mu\nu}_{KG} = \frac{i\hbar}{4m}\bar{\psi}(\sigma^{\lambda\mu}\overleftrightarrow{\partial}^{\nu} - \sigma^{\lambda\nu}\overleftrightarrow{\partial}^{\mu})\psi, \tag{2.49a}$$

$$\hat{Z}^{\mu\nu\lambda\rho}_{KG} = -\frac{i}{4m}\epsilon^{\mu\nu\lambda\rho}\bar{\psi}\gamma^5\psi. \tag{2.49b}$$

2.3 The Wigner operator

For complicated systems, where the density matrix cannot be directly calculated, it is advantageous to express energy-momentum and spin tensor in terms of the Wigner operator, which is for free Dirac fields defined as [45, 61]

$$\hat{W}_{\alpha\beta}(x,p) = \int \frac{d^4y}{(2\pi\hbar)^4}e^{-\frac{i}{\hbar}p\cdot y}\bar{\psi}_\beta(x_1)\,\psi_\alpha(x_2), \tag{2.50}$$

with $x_1 \equiv x + y/2$, $x_2 \equiv x - y/2$. In particular, since the Wigner operator is a function of a space-time and a momentum variable analogously to the classical phase space, its ensemble average will be useful to derive a kinetic theory in the next chapter. The equation of motion for the Wigner operator follows from the Dirac equation for free fields (2.17a) and (2.17b) [45, 61, 172],

$$\left[\gamma\cdot\left(p + i\frac{\hbar}{2}\partial\right) - m\right]\hat{W}(x,p) = 0, \tag{2.51}$$

Applying the operator $\gamma \cdot [p + (i\hbar/2)\partial] + m$ to Eq. (2.51) and separating real and imaginary parts we obtain

$$\left(p^2 - m^2 - \frac{\hbar^2}{4}\partial^2\right) \hat{W}(x,p) = 0, \tag{2.52}$$

$$p \cdot \partial \, \hat{W}(x,p) = 0, \tag{2.53}$$

respectively. Equation (2.52) can be interpreted as a modification of the on-shell condition, while Eq. (2.53) is a kinetic equation analogous to the Boltzmann equation.

For convenience we decompose the Wigner operator in terms of a basis of the generators of the Clifford algebra

$$\hat{W} = \frac{1}{4}\left(\hat{\mathcal{F}} + i\gamma^5\hat{\mathcal{P}} + \gamma^\mu\hat{\mathcal{V}}_\mu + \gamma^5\gamma^\mu\hat{\mathcal{A}}_\mu + \frac{1}{2}\sigma^{\mu\nu}\hat{\mathcal{S}}_{\mu\nu}\right), \tag{2.54}$$

where the coefficients are given by

$$\hat{\mathcal{F}} = \mathrm{tr}(\hat{W}), \tag{2.55a}$$

$$\hat{\mathcal{P}} = -i\,\mathrm{tr}(\gamma^5\hat{W}), \tag{2.55b}$$

$$\hat{\mathcal{V}}^\mu = \mathrm{tr}(\gamma^\mu\hat{W}), \tag{2.55c}$$

$$\hat{\mathcal{A}}^\mu = \mathrm{tr}(\gamma^\mu\gamma^5\hat{W}), \tag{2.55d}$$

$$\hat{\mathcal{S}}^{\mu\nu} = \mathrm{tr}(\sigma^{\mu\nu}\hat{W}). \tag{2.55e}$$

Substituting Eq. (2.54) into Eq. (2.51) and separating real and imaginary parts, we obtain the equations of motion for the coefficient functions

$$p \cdot \hat{\mathcal{V}} - m\hat{\mathcal{F}} = 0, \tag{2.56a}$$

$$\frac{\hbar}{2}\partial \cdot \hat{\mathcal{A}} + m\hat{\mathcal{P}} = 0, \tag{2.56b}$$

$$p^\mu\hat{\mathcal{F}} - \frac{\hbar}{2}\partial_\nu\hat{\mathcal{S}}^{\nu\mu} - m\hat{\mathcal{V}}^\mu = 0, \tag{2.56c}$$

$$-\frac{\hbar}{2}\partial^\mu\hat{\mathcal{P}} + \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}p_\nu\hat{\mathcal{S}}_{\alpha\beta} + m\hat{\mathcal{A}}^\mu = 0, \tag{2.56d}$$

$$\frac{\hbar}{2}\partial^{[\mu}\hat{\mathcal{V}}^{\nu]} - \epsilon^{\mu\nu\alpha\beta}p_\alpha\hat{\mathcal{A}}_\beta - m\hat{\mathcal{S}}^{\mu\nu} = 0, \tag{2.56e}$$

for the real part, and

$$\hbar\partial \cdot \hat{\mathcal{V}} = 0, \tag{2.57a}$$

$$p \cdot \hat{\mathcal{A}} = 0, \tag{2.57b}$$

$$\frac{\hbar}{2}\partial^\mu\hat{\mathcal{F}} + p_\nu\hat{\mathcal{S}}^{\nu\mu} = 0, \tag{2.57c}$$

$$p^\mu\hat{\mathcal{P}} + \frac{\hbar}{4}\epsilon^{\mu\nu\alpha\beta}\partial_\nu\hat{\mathcal{S}}_{\alpha\beta} = 0, \tag{2.57d}$$

$$p^{[\mu}\hat{\mathcal{V}}^{\nu]} + \frac{\hbar}{2}\epsilon^{\mu\nu\alpha\beta}\partial_\alpha\hat{\mathcal{A}}_\beta = 0 \tag{2.57e}$$

for the imaginary part.

The energy-momentum tensors in the different pseudo-gauges introduced in Section 2.2 can be expressed in terms of the Wigner-operator coefficients with the help of Eqs. (2.55)

$$\hat{T}^{\mu\nu}_C = \int d^4p\, p^\nu\hat{\mathcal{V}}^\mu, \tag{2.58a}$$

$$\hat{T}^{\mu\nu}_B = \frac{1}{2}\int d^4p\,(p^\nu\hat{\mathcal{V}}^\mu + p^\mu\hat{\mathcal{V}}^\nu), \tag{2.58b}$$

$$\hat{T}^{\mu\nu}_{HW} = \frac{1}{m}\int d^4p\left(p^\mu p^\nu + \frac{\hbar^2}{4}\partial^\mu\partial^\nu - \frac{\hbar^2}{4}g^{\mu\nu}\partial^2\right)\hat{\mathcal{F}}, \tag{2.58c}$$

$$\hat{T}^{\mu\nu}_{GLW} = \frac{1}{m}\int d^4p\, p^\mu p^\nu\hat{\mathcal{F}}, \tag{2.58d}$$

and

$$\hat{S}_C^{\lambda,\mu\nu} = -\frac{1}{2}\epsilon^{\lambda\mu\nu\rho}\int d^4p\,\hat{\mathcal{A}}_\rho, \tag{2.59a}$$

$$\hat{S}_B^{\lambda,\mu\nu} = \frac{1}{2}x^{[\mu}\int d^4p\,\left(p^{\nu]}\hat{\mathcal{V}}^\lambda - p^{\lambda]}\hat{\mathcal{V}}^\nu\right), \tag{2.59b}$$

$$\hat{S}_{HW}^{\lambda,\mu\nu} = \frac{1}{2m}\int d^4p\,p^\lambda\hat{\mathcal{S}}^{\mu\nu}, \tag{2.59c}$$

$$\hat{S}_{GLW}^{\lambda,\mu\nu} = \frac{1}{2m}\int d^4p\,\left(p^\lambda\hat{\mathcal{S}}^{\mu\nu} - \hbar\epsilon^{\lambda\mu\nu\alpha}\frac{1}{2}\partial_\alpha\hat{\mathcal{P}}\right). \tag{2.59d}$$

Some details of the calculation of Eqs. (2.58) are shown in Appendix B, Eqs. (2.59) are obtained analogously. We remark that in the HW spin tensor Eq. (2.59c) the Dirac equation has not yet been used as a subsidiary condition. This will be done when expressing $\hat{\mathcal{S}}^{\mu\nu}$ in terms of $\hat{\mathcal{A}}^\mu$ through Eq. (2.56d) in Chapter 4, as this relation follows from the Dirac equation.

2.4 Spin vectors and physical interpretation of pseudo-gauges

We now return to the possible definitions of a relativistic spin vector introduced in Section 1.5. Let us first discuss the Frenkel theory and its relation to the different definitions of spin. We note that the various pseudo-gauges introduced so far yield only two different (nonzero) expressions for the global spin (2.14). From the canonical or equivalently the Belinfante-Rosenfeld spin tensor we obtain the global spin

$$\hat{S}_C^{i0} = 0\,, \tag{2.60a}$$

$$\hat{S}_C^{ij} = \frac{1}{2}\int d^3x\,\psi^\dagger\sigma^{ij}\psi = \frac{1}{2}\epsilon^{ijk}\int d^3x\,\psi^\dagger\mathfrak{S}^k\psi\,, \tag{2.60b}$$

$$\hat{S}_C^k \equiv \frac{\hbar}{2}\epsilon^{ijk}\hat{S}_C^{ij} = \frac{\hbar}{2}\int d^3x\,\psi^\dagger\mathfrak{S}^k\psi\,, \tag{2.60c}$$

where we defined $\mathfrak{S}^k \equiv \text{diag}(\sigma^k,\sigma^k)$ and in the last equality introduced the canonical spin vector $\hat{\boldsymbol{S}}_C$. As already mentioned and as also can be clearly seen from Eq. (2.60a), $\hat{S}_C^{\mu\nu}$ is not a tensor. Comparing Eq. (2.60c) to Eq. (1.1) we find that the canonical spin vector $\hat{\boldsymbol{S}}_C$ corresponds to the nonrelativistic spin in any frame. Furthermore, it is not part of a covariant object and the Frenkel condition (1.6) is not fulfilled. Instead, in the canonical case the spin is always defined in the lab frame, as in this frame the canonical global spin takes the form of the right-hand sides of Eqs. (1.3).

On the other hand, we obtain for the HW/GLW/KG global spin (2.32)

$$\hbar\,\hat{S}_{HW}^{ij} = \epsilon^{ijk}\hat{S}_{HW}^k \tag{2.61}$$

with

$$\hat{S}_{HW}^k = \hbar\int d^3x\,\psi^\dagger\left(\frac{1}{2}\mathfrak{S}^k + \frac{\hbar}{2m}\epsilon^{kln}\gamma^l\partial^n\right)\psi. \tag{2.62}$$

Here, we used that the contribution from the last term in Eq. (2.46b) vanishes after integrating by parts. When considering the expectation value for states at rest, the second term in Eq. (2.62) vanishes, implying that

$$S_{HW,rs,\star}^k = S_{C,rs,\star}^k, \tag{2.63}$$

i.e., the HW spin vector coincides with the nonrelativistic spin vector in the rest frame. Furthermore, the HW spin fulfills the Frenkel condition, see Eq. (2.34). We can hence see the HW/GLW/KG global spin as a covariant generalization of the canonical global spin.

Alternatively we may use the spin four-vector defined through the Pauli-Lubanski vector in Eq. (1.13), which by definition is independent of the pseudo-gauge. Inserting Eq. (2.21) into Eq. (1.9) and considering the expectation value for one-particle states, we find that the orbital contribution to the total angular momentum cancels due to the antisymmetry of the epsilon tensor and we are left with

$$S_{rs}^\mu = -\frac{\hbar}{4m}\epsilon^{\mu\nu\alpha\beta}p_\nu\int d^3x\,\langle p,r|\,\psi^\dagger\sigma_{\alpha\beta}\psi\,|p,s\rangle\,. \tag{2.64}$$

From Eq. (2.32) we furthermore obtain

$$S^{\mu}_{rs} = -\frac{\hbar}{2m}\epsilon^{\mu\nu\alpha\beta} p_\nu S_{HW,rs,\alpha\beta} \tag{2.65}$$

and using the Frenkel condition the inverse relation

$$\hbar\, S^{\mu\nu}_{HW,rs} = -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta} p_\alpha S_{rs,\beta}. \tag{2.66}$$

So far, we considered expectation values of single-particle states. In many-particle systems like the QGP, it is more natural to work with the Wigner function, which is related to the ensemble average rather than to single momentum states. In this case, we define a spin vector analogously to the Pauli-Lubanski vector, but using the momentum variable p of the Wigner operator in Eq. (2.50). It should be noted that this variable is in general not identical to the ensemble average of the momentum operator if more than one momentum state is considered. In order to find a spin vector which depends on the Wigner-function momentum, we define the spin density $\hat{s}^{\mu\nu}_{HW/C}$ in momentum space for the HW and canonical pseudo-gauge, respectively, such that

$$\hat{S}^{\mu\nu}_{HW/C} = \int d^4p\, \hat{s}^{\mu\nu}_{HW/C}(p). \tag{2.67}$$

In particular we obtain from Eq. (2.59c)

$$\hat{s}^{\mu\nu}_{HW} = \frac{p^0}{2m}\int d^3x\, \hat{\mathcal{S}}^{\mu\nu}. \tag{2.68}$$

Furthermore, from Eq. (2.57c) we find

$$p_\mu \hat{s}^{\mu i}_{HW} = \frac{p^0}{2m}\int d^3x\, p_\mu \hat{\mathcal{S}}^{\mu i} = -\frac{\hbar}{4m}p^0\int d^3x\, \partial^i \hat{\mathcal{F}} = 0 \tag{2.69}$$

and

$$p_\mu \hat{s}^{\mu 0}_{HW} = \frac{p^0}{2m}\int d^3x\, p_\mu \hat{\mathcal{S}}^{\mu 0} = -\frac{\hbar}{4m}\int d^3x\, p\cdot\partial\hat{\mathcal{F}} = \frac{1}{2m}\int d^3x\, p_\mu p_\nu \hat{\mathcal{S}}^{\mu\nu} = 0, \tag{2.70}$$

where we used that $\hat{\mathcal{S}}^{\mu\nu}$ is antisymmetric and assumed that boundary terms vanish. Therefore for free fields defining the spin in the rest frame of the momentum operator is equivalent to defining the spin in the rest frame of the momentum variable p^μ, i.e., $\hat{s}^{\mu\nu}_{HW}$ satisfies a Frenkel condition of the form

$$p_\mu \hat{s}^{\mu\nu}_{HW} = 0. \tag{2.71}$$

This justifies the definition of the Pauli-Lubanski vector [46]

$$\begin{aligned}\hat{\Pi}^\mu(p) &\equiv -\frac{\hbar}{2m}\epsilon^{\mu\nu\alpha\beta} p_\nu \hat{s}_{HW,\alpha\beta}(p)\\ &= -\frac{\hbar}{2m}\epsilon^{\mu\nu\alpha\beta} p_\nu \int d\Sigma^\lambda\, p_\lambda \hat{\mathcal{S}}_{\alpha\beta}(x,p)\\ &= -\hbar\frac{p^0}{2m}\epsilon^{\mu\nu\alpha\beta} p_\nu \int d^3x\, \hat{\mathcal{S}}_{\alpha\beta}(x,p).\end{aligned} \tag{2.72}$$

Using

$$\hat{s}^{\mu\nu}_C \equiv -\frac{1}{2}\epsilon^{\lambda\mu\nu\rho}\int d\Sigma_\lambda\, \hat{\mathcal{A}}_\rho, \tag{2.73}$$

which follows from Eq. (2.59a), one can also prove that

$$\begin{aligned}\hat{\Pi}^\mu(p) &= -\frac{\hbar}{2m}\epsilon^{\mu\nu\alpha\beta} p_\nu \hat{s}_{C,\alpha\beta}(p)\\ &= \frac{\hbar}{2m}\int d\Sigma_\lambda\, p^\lambda \hat{\mathcal{A}}^\mu(x,p)\\ &= \hbar\frac{p^0}{2m}\int d^3\, x\, \hat{\mathcal{A}}^\mu(x,p),\end{aligned} \tag{2.74}$$

where from the first to the second step we used Eq. (2.57b). The respective last lines of Eqs. (2.72) and (2.74) are identical due to Eq. (2.56d), given that boundary terms can be neglected. The relations (2.72) and (2.74) are of great importance since they directly connect components of the Wigner function to the measurable expectation value of the polarization vector. In the following chapters we will show a way to calculate $\langle\hat{\mathcal{A}}^\mu\rangle$ in the presence of dissipation and thus determine the effects of dissipation on the polarization in heavy-ion collisions. Note that, although $\hat{s}_{C,\alpha\beta}$ is not a Lorentz tensor, the expression (2.74) is covariant since the noncovariant parts of the spin tensor cancel when contracted with the epsilon tensor. Indeed, making use of the antisymmetry of the epsilon tensor, we can rewrite Eq. (2.72)

$$\hat{\Pi}^\mu \equiv -\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta}p_\nu \hat{j}_{\alpha\beta}, \tag{2.75}$$

where the total angular-momentum density, which is defined through $\hat{J}^{\mu\nu} = \int d^4p\, \hat{j}^{\mu\nu}$, is independent of the choice of pseudo-gauge. The inverse relation

$$\hbar\, \hat{s}^{\mu\nu}_{HW} = -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}p_\alpha \hat{\Pi}_\beta \tag{2.76}$$

holds only for the HW spin, since only this choice fulfills the Frenkel condition (2.71).

2.5 Pseudo-gauge transformations and the relativistic center of inertia

The physical origin of the freedom to choose a pseudo-gauge lies in the ambiguity of defining the "position" (i.e., the center of inertia) of a relativistic spinning particle [47]. A different definition of the center of inertia is equivalent to a different splitting between spin and orbital angular momentum. In this section we will discuss this equivalence and relate the different pseudo-gauges to their respective definition of the center of inertia following Ref. [47], see also Refs. [48,135]. For simplicity, we consider classical fields, the corresponding relations in quantum field theory follow from the same reasoning. In this section the quantities which have been introduced as operators in other sections, but do not carry a hat here, correspond to the classical counterparts of the operators.

2.5.1 External and internal components of angular momentum

One can always split the total angular momentum in a relativistic theory into an external component $L^{\mu\nu}_X \equiv X^{[\mu}P^{\nu]}$ and an internal component $S^{\mu\nu}_X$,

$$J^{\mu\nu} = L^{\mu\nu}_X + S^{\mu\nu}_X . \tag{2.77}$$

Then the rotation about X^μ is described by $S^{\mu\nu}_X$. In a quantum theory, one can define a decomposition such that $S^{\mu\nu}_X$ corresponds to the spin. However, this is not necessarily the case for all decompositions of the form (2.77), and $S^{\mu\nu}_X$ can also be defined in the classical case. Furthermore, one can also define a splitting of the total angular momentum into generators of boosts K^μ_n and generators of rotations J^μ_n. Such decomposition depends on a time-like normalized four-vector n^μ corresponding to the four-velocity of the frame used to define the generators,

$$J^{\mu\nu} = K^{[\mu}_n n^{\nu]} - \epsilon^{\mu\nu\alpha\beta}n_\alpha J_{n\beta}, \tag{2.78}$$

where $K^\mu_n \equiv J^{\mu\nu}n_\nu$ and $J^\mu_n \equiv -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}n_\nu J_{\alpha\beta}$. Equations (2.77) and (2.78) can be combined to yield

$$J^{\mu\nu} = (K^{[\mu}_{n,L_X} + K^{[\mu}_{n,S_X})n^{\nu]} - \epsilon^{\mu\nu\alpha\beta}n_\alpha(J_{n,L_X\beta} + J_{n,S_X\beta}) \tag{2.79}$$

with

$$\begin{aligned} K^\mu_{n,L_X} &\equiv (P\cdot n)X^\mu - (X\cdot n)P^\mu, \\ K^\mu_{n,S_X} &\equiv S^{\mu\nu}_X n_\nu, \\ J^\mu_{n,L_X} &\equiv -\epsilon^{\mu\nu\alpha\beta}X_\nu P_\alpha n_\beta, \\ J^\mu_{n,S_X} &\equiv -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}n_\nu S_{X\alpha\beta}. \end{aligned} \tag{2.80}$$

A special case is when the internal angular momentum $S^{\mu\nu}_X$ depends only on the generators of rotations J^μ_n in the frame characterized by the four-velocity n^μ. This is achieved by imposing

$$S^{\mu\nu}_X n_\nu = 0, \tag{2.81}$$

since then the contribution from the boost generators to the internal part in the second line of Eqs. (2.80) vanishes. If the frame vector is chosen to characterize the rest frame, $n^\mu \equiv p^\mu_\star/m$, we recover in Eq. (2.81) the Frenkel condition (1.6). If Eq. (2.81) holds, we can invert the last line of Eqs. (2.80). The internal angular momentum in terms of the rotation generators then reads

$$S^{\mu\nu}_X = -\epsilon^{\mu\nu\alpha\beta} n_\alpha J_{n,S_X\beta}. \tag{2.82}$$

2.5.2 Center of inertia and centroids

We define the center of inertia q^μ as the mean position weighted with the energy density T^{00} of the system,

$$P^0 q^\mu \equiv \int d^3x\, x^\mu T^{00} = x^0 P^\mu + L^{\mu 0}, \tag{2.83}$$

where $T^{\mu\nu}$ is the energy-momentum tensor in some pseudo-gauge and

$$L^{\mu\nu} \equiv J^{\mu\nu} - S^{\mu\nu} \equiv \int d^3x \left(x^\mu T^{0\nu} - x^\nu T^{0\mu} \right). \tag{2.84}$$

Using the conservation equation (2.3), which holds also for any classical $T^{\mu\nu}$, we obtain

$$\frac{d}{dt}(P^0 q^\mu) = \int d^3x\, \left(g^{\mu 0} T^{00} - x^\mu \partial_i T^{i0} \right) = \int d^3x\, \left(g^{\mu 0} T^{00} + g^\mu_i T^{i0} \right). \tag{2.85}$$

Since $dP^0/dt = 0$, the time derivative of the center of inertia is given by

$$\frac{d}{dt} q^\mu = \frac{1}{P^0} \int d^3x\, T^{\mu 0}. \tag{2.86}$$

According to the relativistic center-of-mass theorem, the center of inertia moves along a straight line, i.e., the second derivative of q^μ with respect to the time t should vanish. In order to be consistent with this theorem, any proper choice of pseudo-gauge should fulfill the requirement

$$\partial_\nu T^{[\mu\nu]} = 0. \tag{2.87}$$

If this relation holds, taking the time derivative of Eq. (2.86) implies the center-of-mass theorem, since then

$$\frac{d^2}{dt^2} q^\mu = \frac{1}{P^0} \int d^3x\, \partial_0 T^{\mu 0} = \frac{1}{P^0} \int d^3x\, \partial_\nu T^{\mu\nu} = \frac{1}{P^0} \int d^3x\, \partial_\nu T^{\nu\mu} = 0, \tag{2.88}$$

where in the second step we added a vanishing boundary term and in the third step used Eq. (2.87). For all pseudo-gauges with symmetric energy-momentum tensor the condition (2.87) is trivial. For the canonical energy-momentum tensor one can prove it using Eq. (2.12) and the total antisymmetry of the canonical spin tensor. Thus the relativistic center-of-mass theorem is respected by all pseudo-gauges introduced in Section 2.2.

In order to find a covariant generalization of the center of inertia (2.83), we introduce the so-called centroid q^μ_n, given by the center of inertia in a generic frame moving with four velocity n^μ,

$$q^\mu_n = \frac{1}{P\cdot n}(x^0_n P^\mu + L^{\mu\nu} n_\nu). \tag{2.89}$$

Here we defined $x^0_n \equiv x \cdot n$, which is the time in that frame. On the other hand, we obtain from the first line of Eq. (2.80)

$$X^\mu = \frac{1}{P\cdot n}\left[(X\cdot n) P^\mu + K^\mu_{n,L_X} \right]. \tag{2.90}$$

Choosing the centroid as reference point, $X^\mu \equiv q^\mu_n$ and $L^{\mu\nu} \equiv L^{\mu\nu}_{q_n}$, implies that the external angular momentum is identical to the orbital angular momentum in the frame n^μ,

$$K^\mu_{n,L_q} = L^{\mu\nu} n_\nu. \tag{2.91}$$

In this case, the internal angular momentum is identified with the spin, $S^{\mu\nu} = S^{\mu\nu}_{q_n}$. Furthermore, the condition (2.81) for the spin tensor is necessary to guarantee the consistency with the center-of-mass theorem

(2.88) and to ensure that the centroid behaves as a vector under Lorentz transformations, as it can be used to express q_n^μ through the total momentum and angular momentum, which are both conserved,

$$q_n^\mu = \frac{1}{P \cdot n}(x_n^0 P^\mu + J^{\mu\nu} n_\nu). \tag{2.92}$$

Choosing $q_n^0 = q^0 = x^0$ we find

$$x_n^0 = x^0 \frac{P \cdot n}{P^0} - \frac{1}{P^0} L^{0\nu} n_\nu \tag{2.93}$$

and hence

$$q_n^\mu = \frac{x^0 P^\mu}{P^0} - \frac{L^{\nu 0} n_\nu P^\mu}{P^0 (P \cdot n)} + \frac{L^{\mu\nu} n_\nu}{P \cdot n}. \tag{2.94}$$

This equation describes the worldline of q_n parametrized by the time coordinate x^0. In the following we will discuss the relations between the various pseudo-gauges and choices of the frame vector n^μ.

2.5.3 Vanishing spin tensor: Belinfante pseudo-gauge

Since the spin tensor vanishes in the Belinfante case, $L_B^{\mu\nu} = J^{\mu\nu}$, we have

$$P^0 q^\mu = x^0 P^\mu + J^{\mu 0} \tag{2.95}$$

and the centroid is given by Eq. (2.92).

2.5.4 Center of inertia as reference point: Canonical pseudo-gauge

The canonical spin tensor is related to using the internal angular momentum with the center of inertia as reference point. Due to the total antisymmetry of the canonical spin tensor, it fulfills condition (2.81) in the frame with $n^\mu = (1, \mathbf{0})$,

$$S_C^{i0} = 0 \tag{2.96}$$

and we obtain from Eq. (2.83)

$$P^0 q^\mu = x^0 P^\mu + L_C^{\mu 0} = x^0 P^\mu + J^{\mu 0}. \tag{2.97}$$

We now define a global spin as

$$S_q^{\mu\nu} \equiv J^{\mu\nu} - q^{[\mu} P^{\nu]}, \tag{2.98}$$

fulfilling, as the canonical spin, $S_q^{i0} = 0$ in any frame and not being a tensor. With the definition of the spatial components of the total angular momentum $J^{ij} \equiv \epsilon^{ijk} J^k$ one finds

$$\boldsymbol{S}_q = \mathbf{J} - \mathbf{q} \times \mathbf{P}. \tag{2.99}$$

In order to make the transition from a classical to a quantum framework, we promote P^μ and q^μ to operators. Consider the operator analogue of Eq. (2.98) acting on a single-particle state with momentum p^μ. Inserting Eqs. (2.21) and (2.97) into Eq. (2.98) and using Eq. (1.10), we obtain

$$\hbar\, \hat{S}_q^{\mu\nu} = \int d^3x\, \psi^\dagger \left(\frac{\hbar}{2} \sigma^{\mu\nu} - \frac{\hbar}{2p^0} p^{[\nu} \sigma^{\mu]0} \right) \psi. \tag{2.100}$$

Using Eq. (2.8) and the fact that γ^i and γ^0 anticommute, the spatial components of Eq. (2.100) read

$$\hbar\, \hat{S}_q^{ij} = \int d^3x\, \psi^\dagger \left(\frac{\hbar}{2} \sigma^{ij} - \frac{i\hbar}{2p^0} p^{[j} \gamma^{i]} \gamma^0 \right) \psi. \tag{2.101}$$

In the first term of this equation, we now insert $1 = (m^2 + \mathbf{p}^2)/(p^0)^2$ and in the second term use the Dirac equation (2.17a) in the form $(p \cdot \gamma - m)\psi = 0$, which is valid if the operator ψ acts on a single-particle state with momentum p^μ. We find

$$\begin{aligned} \hbar\, \hat{S}_q^{ij} =& \int d^3x\, \frac{\hbar}{2(p^0)^2} \psi^\dagger \left(m^2 \sigma^{ij} - imp^{[j} \gamma^{i]} + \mathbf{p}^2 \sigma^{ij} - ip^k p^{[j} \gamma^{i]} \gamma^k \right) \psi \\ =& \epsilon^{ijk} \int d^3x\, \frac{\hbar}{2(p^0)^2} \psi^\dagger \left(m^2 \mathfrak{S}^k + im\, \epsilon^{klm} p^l \gamma^m + \frac{1}{2} \epsilon^{lmn} p^k p^l \sigma^{mn} \right) \psi, \end{aligned} \tag{2.102}$$

where we in the second term used Eq. (A.1a) and in the third term used Eq. (A.1b) and $\gamma^k\gamma^j = -\gamma^j\gamma^k + 2g^{jk}$ to obtain

$$\epsilon^{ijk}\epsilon^{lmn}p^k p^l \sigma^{mn} = 2p^{[i}\sigma^{j]k}p^k + 2\sigma^{ij}\mathbf{p}^2 = 2ip^k p^{[i}\gamma^{j]}\gamma^k + 2\sigma^{ij}\mathbf{p}^2. \tag{2.103}$$

Therefore, the spin vector (2.99) takes the form [47]

$$\hat{\boldsymbol{S}}_q = \int d^3x \, \frac{\hbar}{2(p^0)^2}\psi^\dagger \left[m^2\boldsymbol{\mathfrak{S}} + im\,\mathbf{p}\times\boldsymbol{\gamma} + (\mathbf{p}\cdot\boldsymbol{\mathfrak{S}})\mathbf{p}\right]\psi, \tag{2.104}$$

if the action on a single-particle state with momentum p^μ is considered. It should be noted that the second term in Eq. (2.101) distinguishes $\hat{S}^{ij}_q$ from the canonical global spin $\hat{S}^{ij}_C$. The physical interpretation of difference between these two spin operators will be further discussed in Section 2.5.6.

2.5.5 Center of mass as reference point: HW, GLW, and KG pseudo-gauges

For massive particles, there is a preferred reference frame for defining physical quantities in terms of the Poincaré generators in a covariant way. This is the frame comoving with the particle, i.e., moving with the four-velocity $n^\mu_\star \equiv P^\mu/\sqrt{P^2}$. The simplest example is the invariant mass as the proper inertia

$$M \equiv P^0_\star = P\cdot n_\star. \tag{2.105}$$

Furthermore, the proper centroid, called the center of mass, is given by inserting $n^\mu_\star$ into Eq. (2.89)

$$q^\mu_\star = \frac{1}{M}\left(\tau_\star P^\mu + \frac{1}{M}J^{\mu\nu}P_\nu\right), \tag{2.106}$$

where we defined the proper time $\tau_\star \equiv x^0_\star$ already imposed $P_\mu S^{\mu\nu}_\star = 0$ in order to remove the contribution from the boost generators to the internal angular momentum about the center of mass and to make this definition a Lorentz vector. In a similar way, one defines the spin of a massive particle as the proper internal angular momentum, i.e., one chooses the reference vector $n^\mu_\star$ and the reference point $q^\mu_\star$ in Eqs. (2.80),

$$\begin{aligned} K^\mu_{\star,L} &= Mq^\mu_\star - \tau_\star P^\mu, \\ K^\mu_{\star,S} &= 0, \\ J^\mu_{\star,L} &= 0, \\ J^\mu_{\star,S} &= -\frac{1}{2M}\epsilon^{\mu\nu\alpha\beta}P_\nu S_{\star\alpha\beta}. \end{aligned} \tag{2.107}$$

We recognize the Pauli-Lubanski vector $w^\mu = mJ^\mu_{\star,S}$ in the last equation. This shows that it indeed is equal to the generator of rotations defined in the center-of-mass frame. We also notice that the contribution from the external angular momentum to generators of rotations automatically vanishes when choosing this reference frame.

The spin defined in this way is identical to $\hbar S^{\mu\nu}_{HW}$. This can be easily seen in the rest frame, where $S^{i0}_{HW} = 0 = S^{i0}_\star$, since both fulfill the Frenkel condition, and $S^{ij}_{HW} = J^{ij} = S^{ij}_\star$, since the orbital contribution proportional to P^i or P^j vanishes in that frame. As both objects are Lorentz tensors, they are identical in any frame. In particular, the total angular momentum can be written as the sum of the HW global spin and the orbital angular momentum with respect to the center of mass,

$$\hbar\, S^{\mu\nu}_{HW} \equiv J^{\mu\nu} - q^{[\mu}_\star P^{\nu]}. \tag{2.108}$$

Written in this form, the covariant nature of $S^{\mu\nu}_{HW}$ becomes apparent. We then obtain the HW spin vector

$$\boldsymbol{S}_{HW} = \mathbf{J} - \mathbf{q}_\star \times \mathbf{P}, \tag{2.109}$$

reducing in the rest frame to

$$\boldsymbol{S}_{HW\star} = \mathbf{J} = \boldsymbol{S}_C. \tag{2.110}$$

In a quantum framework, P^μ and $q^\mu_\star$ are promoted to operators. Considering the quantized global spin (2.108) acting on a single-particle state with momentum p^μ and inserting Eqs. (2.21) and (2.106), we have

$$\begin{aligned} \hbar\hat{S}^{\mu\nu}_{HW} &= \int d^3x\,\psi^\dagger\left\{x^{[\mu}p^{\nu]} + \frac{\hbar}{2}\sigma^{\mu\nu} + \frac{1}{m^2}p^{[\mu}\left[p\cdot xp^{\nu]} + \left(x^{\nu]}p^\lambda - p^{\nu]}x^\lambda + \frac{\hbar}{2}\sigma^{\nu]\lambda}\right)p_\lambda\right]\right\}\psi \\ &= \frac{\hbar}{2}\int d^3x\,\psi^\dagger\left[\sigma^{\mu\nu} + \frac{i}{2m^2}(p^{[\mu}\gamma^{\nu]}\gamma^\lambda - \gamma^\lambda\gamma^{[\nu}p^{\mu]})p_\lambda\right]\psi \\ &= \frac{\hbar}{2}\int d^3x\,\psi^\dagger\left(\sigma^{\mu\nu} + \frac{i}{m}p^{[\mu}\gamma^{\nu]}\right)\psi, \end{aligned} \tag{2.111}$$

where we used $\gamma^\lambda\gamma^\nu = -\gamma^\nu\gamma^\lambda + 2g^{\nu\lambda}$ and the Dirac equation. We hence recover Eq. (2.32) acting on a single-particle state, see also Ref. [47]. The quantized HW spin vector defined through Eq. (2.109) is then identical to the one given in Eq. (2.62), as it should.

We remark that there is another possibility to define a reference point, which is the mean position [47]

$$\tilde{q}^\mu \equiv \frac{1}{P^0+M}(P^0 q^\mu + M q_\star^\mu). \tag{2.112}$$

The internal angular momentum around this point is given by

$$S_{\tilde{q}}^{\mu\nu} \equiv J^{\mu\nu} - \tilde{q}^{[\mu}P^{\nu]} \tag{2.113}$$

which is not a Lorentz tensor and corresponds to the spin vector

$$\boldsymbol{S}_{\tilde{q}} = \mathbf{J} - \tilde{\mathbf{q}} \times \mathbf{P}. \tag{2.114}$$

Quantizing the global spin (2.113) by making P^μ and $\tilde{q}^\mu$ operators and inserting Eqs. (2.21), (2.97), (2.106), and (2.112) yields

$$\begin{aligned}
\hbar\, \hat{S}_{\tilde{q}}^{\mu\nu} &= \int d^3x\, \psi^\dagger \left[x^{[\mu}p^{\nu]} + \frac{\hbar}{2}\sigma^{\mu\nu} + \frac{1}{(p^0+m)}p^{[\mu}\left(x^{\nu]}p^0 + \frac{\hbar}{2}\sigma^{\nu]0} + x^{\nu]}m + \frac{\hbar}{2m}\sigma^{\nu]\lambda}p_\lambda\right)\right]\psi \\
&= \int d^3x\, \psi^\dagger \left[\frac{\hbar}{2}\sigma^{\mu\nu} + \frac{1}{(p^0+m)}p^{[\mu}\left(\frac{\hbar}{2p^0}\sigma^{\nu]0} + \frac{\hbar}{2m}\sigma^{\nu]\lambda}p_\lambda\right)\right]\psi
\end{aligned} \tag{2.115}$$

Following similar steps as in Eq. (2.102), we obtain for the spatial components

$$\begin{aligned}
\hbar\, \hat{S}^{ij} &= \int d^3x\, \frac{\hbar}{2}\psi^\dagger \left[\sigma^{ij} + \frac{i}{p^0+m}p^{[i}\left(\frac{m}{p^0}\gamma^{j]} + \frac{1}{p^0}\gamma^{j]}\gamma^k p^k + \gamma^{j]}\right)\right]\psi \\
&= \int d^3x\, \frac{\hbar}{2}\psi^\dagger \left[\frac{m}{p^0}\sigma^{ij} + \frac{\mathbf{p}^2}{p^0(p^0+m)}\sigma^{ij} + ip^{[i}\left(\frac{1}{p^0}\gamma^{j]} + \frac{1}{p^0(p^0+m)}\gamma^{j]}\gamma^k p^k\right)\right]\psi \\
&= \epsilon^{ijk}\int d^3x\, \frac{\hbar}{2p^0}\psi^\dagger \left(m\mathfrak{S}^k + i\,\epsilon^{klm}p^l\gamma^m + \frac{1}{2(p^0+m)}\epsilon^{lmn}p^k p^l \sigma^{mn}\right)\psi,
\end{aligned} \tag{2.116}$$

where we used $(p^0)^2 = \mathbf{p}^2 + m^2$, $\gamma^\lambda\gamma^\nu = -\gamma^\nu\gamma^\lambda + 2g^{\nu\lambda}$, the Dirac equation, and Eq. (2.103). We conclude that the quantum version of the spin vector (2.114) is given by [47]

$$\hat{\boldsymbol{S}}_{\tilde{q}} = \int d^3x\, \frac{\hbar}{2p^0}\psi^\dagger \left[m\,\boldsymbol{\mathfrak{S}} + i\,\mathbf{p}\times\boldsymbol{\gamma} + \frac{1}{p^0+m}(\mathbf{p}\cdot\boldsymbol{\mathfrak{S}})\mathbf{p}\right]\psi, \tag{2.117}$$

where again acting on a one-particle state with momentum $\mathbf{p}$ is implied. We note that Eq. (2.117) can be identified with the spin vector introduced by Foldy and Wouthuysen in Ref. [173].

2.5.6 Position operators, Dirac oscillation, and side jumps

Historically, early works on the splitting of the total angular momentum of a Dirac particle into spin and orbital part approached the problem from a different point of view, not being aware of the concept of pseudo-gauge transformations at that time. Instead, one aimed at finding position and spin operators with apparent physical interpretation [134–136, 173, 174]. This issue caused a plethora of works over many decades until recent days, see, e.g., Refs. [175–178]. Since the ideas behind such considerations also shed light on the physical interpretation of the different pseudo-gauges through their relation with the relativistic center of inertia, we will sketch some concepts in the following.

As already pointed out by Dirac himself [49], the result of any exact measurement of the velocity of a quantum particle $\dot{\mathbf{x}}$ would be necessarily the speed of light, as, in order to obtain an exact velocity, the position has to be completely determined. Due to the uncertainty relation, this would imply an infinite momentum. Indeed, the time derivative of the canonical position operator

$$\hat{\mathbf{x}} \equiv \int d^3x\, \psi^\dagger \mathbf{x}\, \psi \tag{2.118}$$

leads to a velocity identical to the speed of light. However, we know that massive particles do not move with the speed of light. The reason is that any measurement of velocity is an average during the motion between two spacetime points. One can show [49] that the motion of a free Dirac particle can be decomposed into a linear part and an oscillatory part, known as "Zitterbewegung". The latter is not measurable, and thus not physical. The total angular momentum can be written as

$$\hat{\mathbf{J}} = \hat{\mathbf{x}} \times \mathbf{p} + \hat{\boldsymbol{S}}_C, \tag{2.119}$$

where $\hat{J}^{ij} \equiv \epsilon^{ijk}\hat{J}^k$, $\hat{J}^{ij}$ is identical to the spatial components of Eq. (2.21), $\hat{\boldsymbol{S}}_C$ is given in (2.60c), and we consider the action of the operators on a one-particle state with momentum p. Since the operator $\hat{\mathbf{x}}$ moves with the speed of light, it cannot be defined as a mean position, and in particular cannot serve as proper reference point for the definition of an orbital angular momentum. As a consequence, it is not possible to find an average position operator such that the total angular momentum can be written in the form (2.77) with $S_X^{\mu\nu} = S_C^{\mu\nu}$.

In order to remove the unphysical Zitterbewegung, one aims at finding a position operator corresponding to a mean position, averaging out the oscillatory motion. As pointed out earlier in this chapter, a redefinition of the position is possible as long as, at the same time, the spin is redefined while keeping the total angular momentum fixed. For instance, consider the position operator introduced in Ref. [47],

$$\hat{\mathbf{q}} \;\; = \;\; \int d^3x\, \psi^\dagger \left\{ \mathbf{x} + \frac{\hbar}{2(p^0)^2}\left[(\mathbf{p}\times\boldsymbol{\mathfrak{S}}) + im\boldsymbol{\gamma}\right] \right\} \psi, \tag{2.120}$$

where the action on a single-particle state is again implied. This position operator corresponds to the spin operator $\hat{\boldsymbol{S}}_q$ in Eq. (2.104) in the sense that the total angular momentum can be written as the sum of the orbital angular momentum with the position defined through Eq. (2.120) and the spin given by Eq. (2.104). In order to see this, we consider

$$\begin{aligned}
\hat{\mathbf{q}} \times \mathbf{p} + \hat{\boldsymbol{S}}_q &= \int d^3x\, \psi^\dagger \left\{ \mathbf{x}\times\mathbf{p} + \frac{\hbar}{2(p^0)^2}\left[(\mathbf{p}\times\boldsymbol{\mathfrak{S}}) + im\boldsymbol{\gamma}\right]\times\mathbf{p} + \frac{\hbar}{2(p^0)^2}\left[m^2\boldsymbol{\mathfrak{S}} + im\,\mathbf{p}\times\boldsymbol{\gamma} + (\mathbf{p}\cdot\boldsymbol{\mathfrak{S}})\mathbf{p}\right] \right\}\psi \\
&= \int d^3x\, \psi^\dagger \left[\mathbf{x}\times\mathbf{p} + \frac{\hbar}{2(p^0)^2}\left(\mathbf{p}^2\boldsymbol{\mathfrak{S}} + m^2\boldsymbol{\mathfrak{S}}\right)\right]\psi \\
&= \int d^3x\, \psi^\dagger \left(\mathbf{x}\times\mathbf{p} + \frac{\hbar}{2}\boldsymbol{\mathfrak{S}} \right)\psi \\
&\equiv \hat{\mathbf{J}},
\end{aligned} \tag{2.121}$$

with $\hat{J}^{ij}$ given by the spatial components of Eq. (2.21). The position operator (2.120) gives the average of the canonical position, moving with the velocity $\mathbf{p}/p^0$, as one would expect, and hence has a clearer physical interpretation than the canonical position. Furthermore, it has the advantage of having a smooth massless limit at the expense of not being covariant, since the time-averaging of the particle motion is frame-dependent. In fact the side-jump phenomenon discussed in Refs. [50–54] originates from the usages of the position operator (2.120) in the massless limit. One can see from Eq. (2.104) that in this limit the spin becomes aligned with the particle momentum, as it should,

$$\hat{\boldsymbol{S}}_{q,m=0} = \int d^3x\, \frac{\hbar}{2}\psi^\dagger \frac{\mathbf{p}\cdot\boldsymbol{\mathfrak{S}}}{|\mathbf{p}|}\frac{\mathbf{p}}{|\mathbf{p}|}\,\psi = \hbar\,\lambda\frac{\mathbf{p}}{|\mathbf{p}|}, \tag{2.122}$$

if this operator acts on a one-particle state $|\mathbf{p},\lambda\rangle$ with helicity $\lambda = \pm 1/2$. Therefore, introducing $\bar{n}^\mu \equiv (1,\mathbf{0})$, we can write the global spin in the massless case in any frame as

$$S^{\mu\nu}_{q,m=0} = \lambda \frac{1}{p\cdot\bar{n}}\epsilon^{\mu\nu\alpha\beta}p_\alpha \bar{n}_\beta, \tag{2.123}$$

being identical to the global spin in Ref. [51]. We will show in the following how the shift in the position due to a Lorentz transformation, the so-called side-jump effect [50–54], emerges. The total angular momentum transforms as a tensor under a Lorentz transformation Λ,

$$J^{\mu\nu} \to J'^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta J^{\alpha\beta}. \tag{2.124}$$

Since $\bar{n}^\mu$ by definition has vanishing spatial components in any frame and therefore is no Lorentz vector, the spin (2.123) does not transform covariantly under Lorentz transformations, in contrast to the total angular momentum. Instead, after the transformation the form of the global spin is still given by Eq. (2.123). Hence we have

$$\begin{aligned}
S^{\mu\nu}_{q,m=0} \to S'^{\mu\nu}_{q,m=0} &= \lambda \frac{1}{p'\cdot\bar{n}} \epsilon^{\mu\nu\alpha\beta} p'_\alpha \bar{n}_\beta \\
&= \lambda \frac{1}{p'\cdot n'} \epsilon^{\mu\nu\alpha\beta} p'_\alpha n'_\beta - \mathfrak{D}^{[\mu} p'^{\nu]} \\
&= \Lambda^\mu_\alpha \Lambda^\nu_\beta S^{\alpha\beta}_{q,m=0} - \mathfrak{D}^{[\mu} p'^{\nu]}
\end{aligned} \tag{2.125}$$

with $p'^\mu \equiv \Lambda^\mu_\nu p^\nu$ and $n'^\mu \equiv \Lambda^\mu_\nu \bar{n}^\nu$. We see that a Lorentz transformation of the global spin involves an anomalous contribution in form of the term $\mathfrak{D}^{[\mu} p'^{\nu]}$. Imposing the covariance of the total angular momentum given by Eq. (2.124),

$$\begin{aligned}
J'^{\mu\nu} &= q'^{[\mu} p'^{\nu]} + \hbar\, S'^{\mu\nu}_{q,m=0} \\
&= \Lambda^\mu_\alpha \Lambda^\nu_\beta J^{\alpha\beta}
\end{aligned} \tag{2.126}$$

yields the following transformation behaviour of the center of inertia q^μ

$$q^\mu \to q'^\mu = \Lambda^\mu_\nu q^\nu + \hbar \mathfrak{D}^\mu . \tag{2.127}$$

We solve Eq. (2.125) for the anomalous shift $\mathfrak{D}^\mu$ by contracting with $\bar{n}_\nu$, where $\mathfrak{D}^\mu$ is chosen to be purely spatial in the frame at rest with the observer after the Lorentz transformation, i.e., $\bar{n}\cdot\mathfrak{D} = 0$. The result reads in this frame

$$\mathfrak{D}^\mu = \lambda \frac{\epsilon^{\mu\nu\alpha\beta} p'_\nu n'_\alpha \bar{n}_\beta}{(p'\cdot\bar{n})(p'\cdot n')}, \tag{2.128}$$

equivalently to what was found in Ref. [51]. We will discuss the physical consequences of this shift in kinetic theory in Chapter 5.

On the other hand, in Ref. [47] the following position operator was introduced,

$$\hat{\mathbf{q}}_\star = \int d^3x\, \psi^\dagger \left\{ \mathbf{x} + \frac{i\hbar}{2m} \left[\boldsymbol{\gamma} - \frac{1}{(p^0)^2} (\boldsymbol{\gamma}\cdot\mathbf{p})\mathbf{p} \right] \right\} \psi, \tag{2.129}$$

which is the spatial part of a four-vector, identical to $\hat{\mathbf{q}}$ in the particle rest frame. In other words, this position operator corresponds to the mean position calculated in that frame. For massive particles, this frame choice is always possible, providing a covariant concept of spin and position, which is missing in the massless case. The position operator (2.129) belongs to the HW spin (2.62) in the sense that the corresponding orbital angular momentum and the HW spin vector (2.62) add up to the total angular momentum,

$$\begin{aligned}
\hat{\mathbf{q}}_\star \times \mathbf{p} + \hat{\boldsymbol{S}}_{HW} &= \int d^3x\, \psi^\dagger \left[\left(\mathbf{x} + \frac{i\hbar}{2m}\boldsymbol{\gamma} \right) \times \mathbf{p} + \frac{\hbar}{2}\boldsymbol{\mathfrak{S}} - \frac{i\hbar}{2m}\boldsymbol{\gamma}\times\mathbf{p} \right] \psi \\
&= \int d^3x\, \psi^\dagger \left(\mathbf{x}\times\mathbf{p} + \frac{\hbar}{2}\boldsymbol{\mathfrak{S}} \right) \psi \\
&\equiv \hat{\mathbf{J}}.
\end{aligned} \tag{2.130}$$

Chapter 3

Spin kinetic theory from quantum field theory: nonlocal collisions

Kinetic theory is a commonly used starting point for the derivation of hydrodynamics from a microscopic theory. In this chapter we derive a spin kinetic theory with nonlocal collisions from quantum field theory using the Wigner-function formalism. The material of this chapter is based on Refs. [57,58]. Starting from the Dirac Lagrangian with a general interaction term, we obtain a modified mass-shell condition and a Boltzmann-like equation of motion for the Wigner function. The latter involves a nonlocal collision term, being a functional of Wigner functions evaluated at all space-time points. We expand this collision term in gradients of the Wigner function, taking into account nonlocal corrections at first order. Then, we combine the dynamics of the relevant components of the Wigner function into one scalar Boltzmann equation by introducing a distribution function which depends not only on the space-time coordinate x and the momentum coordinate p, but also on a spin variable $\mathfrak{s}$ and is exactly related to the Wigner function. We show that off-shell terms cancel on both sides of the Boltzmann equation and the on-shell Boltzmann equation attains a familiar form including a gain and loss term. The nonlocality results in space-time shifts of the distribution functions in the Boltzmann equation with respect to the center of the collision, allowing for transfer of orbital angular momentum to spin and vice versa during a collision process. We show that, taking into account the conservation of total angular momentum, this leads to spin alignment with vorticity in equilibrium, defined from the condition that the collision term vanishes. This equilibrium state corresponds to global rather than local equilibrium. Finally, we show that our collision term fulfills an H-theorem with an intuitive definition of the entropy current. The Boltzmann equation derived in this chapter builds the basis of the equations of motion for the dissipative currents derived in Chapter 5.

3.1 Interacting Wigner function and quantum transport

We define the Wigner function for spin-1/2 particles as the ensemble average of the Wigner operator in Eq. (2.50) [45,61,98],

$$W_{\alpha\beta}(x,p) = \int \frac{d^4y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}p\cdot y} \left\langle \bar{\psi}_\beta\left(x_1\right) \psi_\alpha\left(x_2\right)\right\rangle \ . \tag{3.1}$$

Since we are interested in the effects of collisions in a kinetic-theory description, we modify the free Dirac Lagrangian in Eq. (2.16) by adding a general interaction Lagrangian $\mathcal{L}_I$, i.e., we consider a Lagrangian of the form

$$\mathcal{L}_D = \bar{\psi}\left(\frac{i\hbar}{2}\gamma\cdot\overleftrightarrow{\partial} - m\right)\psi + \mathcal{L}_I \ . \tag{3.2}$$

Here $\mathcal{L}_I$ is thought of as an effective interaction. In the case of gauge-field interactions, Eq. (3.1) is modified by a so-called gauge link and one has to ensure gauge invariance of observables, see Chapter 4.4. Defining $\rho \equiv -(1/\hbar)\partial\mathcal{L}_I/\partial\bar{\psi}$, one obtains the equation of motion

$$\left(i\hbar\gamma\cdot\partial - m\right)\psi(x) = \hbar\rho(x) \ . \tag{3.3}$$

This implies the following transport equation for the Wigner function [45],

$$\left[\gamma\cdot\left(p + i\frac{\hbar}{2}\partial\right) - m\right] W_{\alpha\beta} = \hbar\,\mathcal{C}_{\alpha\beta} \ , \tag{3.4}$$

with

$$\mathcal{C}_{\alpha\beta} \equiv \int \frac{d^4y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}p\cdot y} \left\langle \bar{\psi}_\beta(x_1)\rho_\alpha(x_2)\right\rangle \,. \tag{3.5}$$

We derive a Boltzmann-like equation of motion for the Wigner function by acting with the operator $\gamma\cdot(p+i\,\frac{\hbar}{2}\partial)+m$ onto Eq. (3.4) and taking the imaginary part,

$$p\cdot\partial\, W_{\alpha\beta}(x,p) = C_{\alpha\beta}(x,p) \tag{3.6}$$

with

$$C_{\alpha\beta} \;=\; \frac{i}{2}\int \frac{d^4y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}p\cdot y} \left\langle \left[\bar{\rho}\,(x_1)\,(-i\hbar\gamma\cdot\overleftarrow{\partial}+m)\right]_\beta \psi_\alpha\,(x_2) - \bar{\psi}_\beta\,(x_1)\left[(i\hbar\gamma\cdot\partial+m)\rho\,(x_2)\right]_\alpha \right\rangle \,. \tag{3.7}$$

Furthermore, from the real part we obtain the modified on-shell condition

$$\left(p^2-m^2-\frac{\hbar^2}{4}\partial^2\right)W_{\alpha\beta}(x,p) \;=\; \hbar\,\delta M_{\alpha\beta}(x,p) \tag{3.8}$$

with

$$\delta M_{\alpha\beta} \;\equiv\; \frac{1}{2}\int \frac{d^4y}{(2\pi\hbar)^4}\, e^{-\frac{i}{\hbar}p\cdot y} \left\langle \left[\bar{\rho}(x_1)(i\hbar\gamma\cdot\overleftarrow{\partial}+m)\right]_\beta \psi_\alpha(x_2) + \bar{\psi}_\beta\,(x_1)\left[(-i\hbar\gamma\cdot\partial+m)\rho\,(x_2)\right]_\alpha \right\rangle \,. \tag{3.9}$$

In the following we will consider only the positive-energy part of the Wigner function, however, the presented formalism can be extended to include antiparticles without changing any conclusion. Hence we imply a factor $\theta(p_0)$ in all terms of the form $\delta(p^2-m^2)$ without writing it explicitly in order to keep the notation compact.

Let us consider the equation of motion (3.6). While the left-hand side depends only on the Wigner function, the dependence of the right-hand side on W is not yet apparent. Since we intend to have a closed equation of motion for the Wigner function, we will in the following express Eq. (3.7) as a functional of W. The first step is to calculate the ensemble average, which for convenience is done by performing the trace over the noninteracting initial n-particle states [45]

$$|p_1,\ldots,p_n;r_1,\ldots,r_n\rangle_{\text{in}} \equiv a^\dagger_{\text{in},r_1}(p_1)\cdots a^\dagger_{\text{in},r_n}(p_n)|0\rangle \,. \tag{3.10}$$

In Appendix C.1 we show the calculation of the ensemble average of the collision term. In the derivation we make the assumption that, as needed in kinetic theory, the system is sufficiently dilute, such that we can neglect initial correlations (molecular chaos) and take into account only binary collisions, i.e., we consider only two-particle states. Then Eq. (3.7) becomes [45]

$$\begin{aligned} C_{\alpha\beta} = \frac{1}{2(4\pi\hbar m^2)^2} &\sum_{r_1,r_2,s_1,s_2} \int d^4x_1 d^4x_2 d^4p_1 d^4p_2 d^4q_1 d^4q_2 \\ &\times {}_{\text{in}}\!\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\right|\hat{\Phi}_{\alpha\beta}(p)\left|p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle_{\text{in}} \\ &\times\prod_{j=1}^{2}\exp\left(\frac{i}{\hbar}q_j\cdot x_j\right)\bar{u}_{s_j}\left(p_j+\frac{q_j}{2}\right)W_{\text{in}}(x+x_j,p_j)u_{r_j}\left(p_j-\frac{q_j}{2}\right) \,, \end{aligned} \tag{3.11}$$

with

$$\begin{aligned} \hat{\Phi}_{\alpha\beta}(p) \equiv \frac{i}{2}\int \frac{d^4y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}p\cdot y} \bigg\{ &\left[\hat{P}_\mu,\bar{\rho}\left(\frac{y}{2}\right)\gamma^\mu\right]_\beta \psi_\alpha\left(-\frac{y}{2}\right) + m\bar{\rho}_\beta\left(\frac{y}{2}\right)\psi_\alpha\left(-\frac{y}{2}\right) \\ &- \bar{\psi}_\beta\left(\frac{y}{2}\right)\left[\gamma^\mu\rho\left(-\frac{y}{2}\right),\hat{P}_\mu\right]_\alpha - m\bar{\psi}_\beta\left(\frac{y}{2}\right)\rho_\alpha\left(-\frac{y}{2}\right)\bigg\} \end{aligned} \tag{3.12}$$

and the momentum-dependent spinor fields $u_r(p)$. The Boltzmann-like equation (3.6) with the collision kernel (3.11) is still not closed, since the left-hand side depends on the interacting Wigner function, while $C_{\alpha\beta}$ is a functional of the initial Wigner function W_{in}. This can be solved by approximating

$$W = W_{\text{in}} + \ldots \,. \tag{3.13}$$

This equality holds up to corrections of higher order in density, which are not taken into account, as in this work we study dilute systems [45]. The inverted relation (3.13) is now used to replace $W_{\rm in}$ in the collision term by W.

Moreover, the Wigner functions in Eq. (3.11) do not only depend on the coordinate x of the Wigner function on the left-hand side of Eq. (3.6), but on all possible positions $x + x_j$. In other words, the collision term is fully nonlocal. In usual kinetic theory, one assumes that the Wigner function varies so slowly in space and time on the microscopic scale corresponding to the interaction range, that this nonlocality can be completely ignored [45]. Here we make a similar assumption, however, we take into account the first-order gradient correction in a Taylor-expansion of $W(x + x_j, p_j)$ around x (equivalent to the first-order correction in $\hbar$) , i.e.,

$$W(x + x_j, p_j) = W(x, p_j) + x_j \cdot \partial W(x, p_j) \,. \tag{3.14}$$

Using Eqs. (3.13) and (3.14) in Eq. (3.11) and performing the integrations over d^4x_1 and d^4x_2, we arrive at

$$\begin{aligned} C_{\alpha\beta} = \frac{(2\pi\hbar)^6}{(2m)^4} \sum_{r_1,r_2,s_1,s_2} \int d^4p_1 d^4p_2 d^4q_1 d^4q_2 \; {}_{\rm in}\!\left\langle p_1 - \frac{q_1}{2}, p_2 - \frac{q_2}{2}; r_1, r_2 \middle| \hat{\Phi}_{\alpha\beta}(p) \middle| p_1 + \frac{q_1}{2}, p_2 + \frac{q_2}{2}; s_1, s_2 \right\rangle_{\rm in} \\ \times \prod_{j=1}^{2} \bar{u}_{s_j}\left(p_j + \frac{q_j}{2}\right) \left\{ W(x, p_j)\delta^{(4)}(q_j) - i\hbar \left[\partial^\mu_{q_j} \delta^{(4)}(q_j)\right] \partial_\mu W(x, p_j) \right\} u_{r_j}\left(p_j - \frac{q_j}{2}\right) \,. \end{aligned} \tag{3.15}$$

This is the starting point for the explicit calculation of the collision term in the Boltzmann equation (3.6). Following Refs. [57, 62, 118, 119] we employ an expansion in powers of $\hbar$ for the Wigner function, i.e., we search for solutions of the form

$$W = W^{(0)} + \hbar W^{(1)} + \hbar^2 W^{(2)} + \mathcal{O}(\hbar^3) \,. \tag{3.16}$$

We notice that, since gradients are always accompanied by factors of $\hbar$, such an expansion is also a gradient expansion. Moreover, we stress that the gradient expansion of the nonlocal term has to be considered as an $\hbar$ expansion, as Eq. (3.15) shows. As we consider corrections up to first order in $\hbar$, we thus will insert only the zeroth-order part of the Wigner function into the nonlocal collision term.

Analogously to the decomposition of the Wigner operator (2.54), we write for the Wigner function

$$W = \frac{1}{4}\left(\mathcal{F} + i\gamma^5 \mathcal{P} + \gamma \cdot \mathcal{V} + \gamma^5\gamma \cdot \mathcal{A} + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right) \,. \tag{3.17}$$

In the presence of interactions, Eqs. (2.56) are modified according to

$$\begin{aligned} p \cdot \mathcal{V} - m\mathcal{F} &= \hbar D_\mathcal{F} \,, &\text{(3.18a)}\\ \frac{\hbar}{2}\partial \cdot \mathcal{A} + m\mathcal{P} &= -\hbar D_\mathcal{P} \,, &\text{(3.18b)}\\ p^\mu \mathcal{F} - \frac{\hbar}{2}\partial_\nu \mathcal{S}^{\nu\mu} - m\mathcal{V}^\mu &= \hbar D^\mu_\mathcal{V} \,, &\text{(3.18c)}\\ -\frac{\hbar}{2}\partial^\mu \mathcal{P} + \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} p_\nu \mathcal{S}_{\alpha\beta} + m\mathcal{A}^\mu &= -\hbar D^\mu_\mathcal{A} \,, &\text{(3.18d)}\\ \frac{\hbar}{2}\partial^{[\mu}\mathcal{V}^{\nu]} - \epsilon^{\mu\nu\alpha\beta} p_\alpha \mathcal{A}_\beta - m\mathcal{S}^{\mu\nu} &= \hbar D^{\mu\nu}_\mathcal{S} \,, &\text{(3.18e)} \end{aligned}$$

and Eqs. (2.57) become

$$\begin{aligned} \hbar\partial \cdot \mathcal{V} &= 2\hbar C_\mathcal{F} \,, &\text{(3.19a)}\\ p \cdot \mathcal{A} &= \hbar C_\mathcal{P} \,, &\text{(3.19b)}\\ \frac{\hbar}{2}\partial^\mu \mathcal{F} + p_\nu \mathcal{S}^{\nu\mu} &= \hbar C^\mu_\mathcal{V} \,, &\text{(3.19c)}\\ p^\mu \mathcal{P} + \frac{\hbar}{4}\epsilon^{\mu\nu\alpha\beta}\partial_\nu \mathcal{S}_{\alpha\beta} &= -\hbar C^\mu_\mathcal{A} \,, &\text{(3.19d)}\\ p^{[\mu}\mathcal{V}^{\nu]} + \frac{\hbar}{2}\epsilon^{\mu\nu\alpha\beta}\partial_\alpha \mathcal{A}_\beta &= -\hbar C^{\mu\nu}_\mathcal{S} \,. &\text{(3.19e)} \end{aligned}$$

Here we defined $D_i = \mathrm{Re}\,\mathrm{tr}\,(\tilde{\Gamma}_i \mathcal{C})$, $C_i = \mathrm{Im}\,\mathrm{tr}\,(\tilde{\Gamma}_i \mathcal{C})$, $i = \mathcal{F}, \mathcal{P}, \mathcal{V}, \mathcal{A}, \mathcal{S}$, $\tilde{\Gamma}_\mathcal{F} = 1$, $\tilde{\Gamma}_\mathcal{P} = -i\gamma_5$, $\tilde{\Gamma}_\mathcal{V} = \gamma^\mu$, $\tilde{\Gamma}_\mathcal{A} = \gamma^\mu\gamma^5$, $\tilde{\Gamma}_\mathcal{S} = \sigma^{\mu\nu}$. Taking traces of Eqs. (3.8) and (3.6) with the respective Dirac matrices, one can derive Boltzmann-like equations of motion and modified on-shell relations for each coefficient function in Eq. (3.17). In the following, we will show that, using Eqs. (3.18), all components which appear in the charge current, the energy-momentum tensor, or the spin tensor can be expressed in terms of a scalar distribution function and an axial-vector current. The latter are then determined from their equations of motion.

Here we make the assumption that all polarization effects are at least of first order in $\hbar$, since we describe a system without large initial polarization. In this case, only particle scatterings in the presence of a nonzero fluid vorticity are responsible for the finite polarization of the medium. We conclude that the zeroth-order contribution of the axial-vector current vanishes, $\mathcal{A}^{(0)\mu}=0$, since this quantity describes polarization [62]. This also means that $\mathcal{S}^{(0)\mu\nu}=0$ from Eq. (3.18e) and $\mathcal{P}^{(0)}=0$ from Eq. (3.18b). Hence, there are no pseudoscalar quantities at zeroth order, which implies that also the collision terms with pseudoscalar quantum numbers cannot have contributions at the zeroth order, $D_{\mathcal{P}}^{(0)}=C_{\mathcal{P}}^{(0)}=0$. Consequently we have from Eqs. (3.18b) and (3.19b)

$$\mathcal{P}=\mathcal{O}(\hbar^2)\,,\quad p\cdot\mathcal{A}=\mathcal{O}(\hbar^2)\,. \tag{3.20}$$

Furthermore, if the polarization vanishes at zeroth order in $\hbar$, we also have that the axial-vector $D_{\mathcal{A}}^{(0)\mu}$ and the antisymmetric tensors $D_{\mathcal{S}}^{(0)\mu\nu}$ and $C_{\mathcal{S}}^{(0)\mu\nu}$ are zero.

Considering the vectors at our disposal at zeroth order, we find that the only possibility is that the leading-order vector current is proportional to p^μ, i.e.,

$$D_{\mathcal{V}}^\mu=p^\mu\delta V+\mathcal{O}(\hbar)\,, \tag{3.21}$$

where δV is a scalar function. Using this relation in Eq. (3.18c), we find

$$\mathcal{V}^\mu=\frac{1}{m}p^\mu\bar{\mathcal{F}}+\mathcal{O}(\hbar^2)\,, \tag{3.22}$$

with $\bar{\mathcal{F}}\equiv\mathcal{F}-\hbar\delta V$. We note that this definition can be extended to any order in $\hbar$ by setting

$$\bar{\mathcal{F}}\equiv\frac{m}{p^2}\,\mathrm{tr}(p\cdot\gamma\,W)\,. \tag{3.23}$$

The form of the conserved currents in the presence of interactions will be discussed in Chapter 4. Here we already anticipate that they depend on $\mathcal{V}^\mu$ and $\mathcal{S}^{\mu\nu}$. Hence, we conclude from the above discussion that we will be able to express the relevant quantities in terms of $\bar{\mathcal{F}}$ and $\mathcal{A}^\mu$. For this reason, we will focus on the equations of motion and mass-shell constraints for these two quantities. First, we obtain from Eqs. (3.18a), (3.18c), and (3.19e) the modified on-shell condition for the vector component

$$\left(p^2-m^2\right)\mathcal{V}^\mu=\hbar p^\mu D_{\mathcal{F}}+\hbar m D_{\mathcal{V}}^\mu+\mathcal{O}(\hbar^2)\,. \tag{3.24}$$

It then follows from Eq. (3.22) that

$$\left(p^2-m^2\right)\bar{\mathcal{F}}=\hbar\,\delta M_F+\mathcal{O}(\hbar^2)=\hbar m\left(D_{\mathcal{F}}+\frac{m}{p^2}\,p\cdot D_{\mathcal{V}}\right)+\mathcal{O}(\hbar^2)\,, \tag{3.25}$$

where we can write δM_F at any order in $\hbar$ in terms of δM, which is given in Eq. (3.9), as

$$\delta M_F=\frac{m}{p^2}\mathrm{tr}(p\cdot\gamma\,\delta M)\,. \tag{3.26}$$

The modified on-shell condition for $\mathcal{A}^\mu$ is derived from Eqs. (3.18d) and (3.18e) and reads

$$\left(p^2-m^2\right)\mathcal{A}^\mu=\hbar\,\delta M_A^\mu+\mathcal{O}(\hbar^2) \tag{3.27}$$

with

$$\delta M_A^\mu=\mathrm{tr}(\gamma^\mu\gamma^5\delta M)\,. \tag{3.28}$$

Since δM_A^μ has an axial-vector quantum number, it is itself of order $\mathcal{O}(\hbar)$, such that the right-hand side of Eq. (3.27) is actually of second order in $\hbar$. In other words, the on-shell condition for $\mathcal{A}^\mu$ is not modified by interactions up to first order in $\hbar$, in contrast to the mass-shell condition for $\bar{\mathcal{F}}$.

Furthermore, we obtain the following Boltzmann equations up to corrections of order $\mathcal{O}(\hbar^2)$ from Eqs. (3.19a) and (3.19e),

$$p\cdot\partial\bar{\mathcal{F}} = m\,C_F\,, \tag{3.29a}$$

$$p\cdot\partial\mathcal{A}^\mu = m\,C_A^\mu\,, \tag{3.29b}$$

where we used Eqs. (3.20) and (3.22) and defined $C_F = 2C_{\mathcal{F}}$ and $C_A^\mu \equiv -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}p_\nu C_{\mathcal{S}\alpha\beta}$. With the help of Eqs. (3.6), (3.23), and (3.29) one can express the collision terms on the right-hand sides of Eqs. (3.29) in terms of C given in Eq. (3.15) as

$$C_F = \frac{1}{p^2}\mathrm{tr}(p\cdot\gamma\, C)\ , \tag{3.30a}$$

$$C_A^\mu = \frac{1}{m}\mathrm{tr}(\gamma^\mu\gamma^5 C)\ . \tag{3.30b}$$

Using these expressions, in the remainder of this chapter we will derive the explicit form of C_F and C_A^μ from Eq. (3.11). We will then see that we obtain a closed set of equations of motion which determine $\bar{\mathcal{F}}$ and $\mathcal{A}^\mu$. A method to solve this system of equation will be presented in Chapter 5 .

3.2 Spin in phase space

In spin kinetic theory, the definition of the classical distribution function has to be generalized to include spin degrees of freedom. There exist two distinct classes of approaches to do so [179]. One is the definition of a matrix-valued distribution function with discrete spin indices, as the density matrix in Eq. (1.20). A different approach is to define an enlarged phase-space with a continuous spin variable. The direction of the three-vector components of the latter in the particle rest frame can be interpreted as the spin orientation. Such framework has been used, e.g., in Refs. [57, 59, 103, 150, 180, 181]. In the following calculation, we will find it convenient to use the second approach. This is advantageous since it allows one to express the hydrodynamic currents as moments of a distribution function which appears classical and, at the same time, carries the quantum information from the Wigner function. As we will see, this leads to a natural interpretation for the conservation laws and the collisional invariants [57]. The full dynamics of Eqs. (3.29) can then be collected in one single Boltzmann-like equation. Applying the method of moments, one can use this equation as the starting point to derive dissipative hydrodynamic equations of motion with spin, see Section 5.

We define the distribution function in this phase-space at any order in $\hbar$ as

$$\mathfrak{f}(x,p,\mathfrak{s}) \equiv \frac{1}{2}\left[\bar{\mathcal{F}}(x,p) - \mathfrak{s}\cdot\mathcal{A}(x,p)\right]\ . \tag{3.31}$$

Next, we introduce the definition of the covariant measure

$$\int dS(p)\cdots = \frac{1}{\Gamma}\int d^4\mathfrak{s}\,\delta(\mathfrak{s}\cdot\mathfrak{s}+\sigma^2)\delta(p\cdot\mathfrak{s})\cdots\ , \tag{3.32}$$

where Γ and σ are constants, which will be fixed by requiring

$$\bar{\mathcal{F}} = \int dS(p)\,\mathfrak{f}(x,p,\mathfrak{s}),$$

$$\mathcal{A}^\mu = \int dS(p)\,\mathfrak{s}^\mu\mathfrak{f}(x,p,\mathfrak{s}). \tag{3.33}$$

From

$$2\int dS(p)\,\mathfrak{f}(x,p,s) = \bar{\mathcal{F}}\int dS(p)\ - \mathcal{A}_\mu\int dS(p)\,\mathfrak{s}^\mu = \frac{1}{\Gamma}\frac{2\pi}{\sqrt{p^2}}\sigma\bar{\mathcal{F}} \tag{3.34}$$

we obtain

$$\Gamma = \frac{\pi}{\sqrt{p^2}}\sigma. \tag{3.35}$$

Furthermore we find

$$\begin{aligned} 2\int dS(p)\,\mathfrak{s}^\mu\mathfrak{f}(x,p,\mathfrak{s}) &= \bar{\mathcal{F}}\int dS(p)\,\mathfrak{s}^\mu - \mathcal{A}_\nu\int dS(p)\,\mathfrak{s}^\mu\mathfrak{s}^\nu \\ &= \bar{\mathcal{F}}\alpha_1 p^\mu - \mathcal{A}_\nu\alpha_2 g^{\mu\nu}. \end{aligned} \tag{3.36}$$

Here, we used that the integrals can only depend on the momentum and introduced the constants α_1 and α_2. In the last step, we used $p_\mu\mathcal{A}^\mu = 0$. From

$$p_\mu\int dS(p)\,\mathfrak{s}^\mu = 0,$$

$$p_\mu p_\nu\int dS(p)\,\mathfrak{s}^\mu\mathfrak{s}^\nu = 0,$$

$$\int dS(p)\,\mathfrak{s}^2 = -2\sigma^2 \tag{3.37}$$

we obtain $\alpha_1 = 0$ and $\alpha_2 = -\frac{2}{3}\sigma^2$. Inserting this result into Eq. (3.36) and then comparing to second requirement in Eqs. (3.33), we conclude

$$\sigma^2 \;=\; 3 \tag{3.38}$$

and hence

$$\int dS(p) \equiv \sqrt{\frac{p^2}{3\pi^2}} \int d^4\mathfrak{s}\, \delta(\mathfrak{s}\cdot\mathfrak{s}+3)\delta(p\cdot\mathfrak{s}) \tag{3.39}$$

with

$$\int dS(p) \;=\; 2\,, \tag{3.40a}$$

$$\int dS(p)\,\mathfrak{s}^\mu \;=\; 0\,, \tag{3.40b}$$

$$\int dS(p)\,\mathfrak{s}^\mu\mathfrak{s}^\nu \;=\; -2\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\,. \tag{3.40c}$$

Up to first order in $\hbar$ the vector and tensor component $\mathcal{V}^\mu$ and $\mathcal{S}^{\mu\nu}$ of the Wigner function can be expressed with the help of Eqs. (3.18c), (3.18e), and (3.22) as

$$\begin{aligned}\mathcal{V}^\mu(x,p) &= \int dS(p)\,\left(\frac{1}{m}p^\mu + \frac{\hbar}{2m}\partial_\nu\Sigma^{\mu\nu}_{\mathfrak{s}}\right)\mathfrak{f}(x,p,\mathfrak{s}) + \mathcal{O}(\hbar^2)\,,\\ \mathcal{S}^{\mu\nu}(x,p) &= \int dS(p)\,\left(\Sigma^{\mu\nu}_{\mathfrak{s}} + \frac{\hbar}{2m^2}\partial^{[\mu}p^{\nu]}\right)\mathfrak{f}(x,p,\mathfrak{s}) + \mathcal{O}(\hbar^2)\,,\end{aligned} \tag{3.41}$$

where we defined

$$\Sigma^{\mu\nu}_{\mathfrak{s}} \equiv -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}p_\alpha\mathfrak{s}_\beta\,. \tag{3.42}$$

Combining Eqs. (3.25) and (3.27), we find a modified on-shell condition for the spin-dependent distribution

$$\left(p^2 - m^2\right)\mathfrak{f}(x,p,\mathfrak{s}) = \hbar\,\mathfrak{M}(x,p,\mathfrak{s}) + \mathcal{O}(\hbar^2) \tag{3.43}$$

with

$$\mathfrak{M}(x,p,\mathfrak{s}) = \frac{1}{2}\left[\delta M_F(x,p) - \mathfrak{s}\cdot\delta M_A(x,p)\right]\,. \tag{3.44}$$

We solve Eq. (3.43) by making use of the quasi-particle approximation, i.e., we take $\mathfrak{f}$ to be of the form

$$\mathfrak{f}(x,p,\mathfrak{s}) = m\delta(p^2 - M^2)f(x,p,\mathfrak{s}) \tag{3.45}$$

with $f(x,p,\mathfrak{s})$ a function without singularity at $p^2 = M^2 \equiv m^2 + \hbar\delta m^2$, where $\delta m^2(x,p,\mathfrak{s})$ is a correction to the mass-shell condition for free particles emerging from interactions. Assuming that $f(x,p,\mathfrak{s})$ has no singularity at $p^2 = m^2$, i.e., $(p^2-m^2)\delta(p^2-m^2)f(x,p,\mathfrak{s}) = 0$, we can Taylor-expand the delta function to first order in $\hbar$ and thus find a relation between δm^2 and $\mathfrak{M}$,

$$\hbar\,\mathfrak{M}(x,p,\mathfrak{s}) = \hbar\,\delta m^2(x,p,\mathfrak{s})\delta(p^2-m^2)mf(x,p,\mathfrak{s}) + \mathcal{O}(\hbar^2)\,, \tag{3.46}$$

where we used $(p^2-m^2)\delta'(p^2-m^2) = -\delta(p^2-m^2)$. We note that δm^2 depends on $\mathfrak{s}$ at least at first order in $\hbar$, as this contribution is related to spin degrees of freedom.

From Eqs. (3.29) and (3.31) we obtain the following Boltzmann equation for $\mathfrak{f}$,

$$p\cdot\partial\,\mathfrak{f}(x,p,\mathfrak{s}) = m\,\mathfrak{C}\,, \tag{3.47}$$

where we defined the collision kernel

$$\mathfrak{C} \equiv \frac{1}{2}(C_F - \mathfrak{s}\cdot C_A)\,. \tag{3.48}$$

We will show in the next sections that up to first order in $\hbar$ this collision term can by written as the sum of a local and a nonlocal contribution, denoted by subscripts l and nl, respectively, i.e.,

$$\mathfrak{C} = \mathfrak{C}^{(0)}_l + \hbar\left\{\mathfrak{C}^{(1)}_l + \mathfrak{C}^{(1)}_{nl}\right\} \equiv \mathfrak{C}_l + \hbar\mathfrak{C}^{(1)}_{nl}\,. \tag{3.49}$$

As mentioned before, the zeroth-order contribution is purely local [123], while the first-order contribution has both local and nonlocal parts.

3.3 Local collisions

In this section we explicitly calculate the local part of the collision term, i.e., the term $\sim \delta^{(4)}(q_i)$ in the second line of Eq. (3.15). In order to do so, we follow the steps outlined in Ref. [45], see also Refs. [182–185]. The calculation of the matrix element of $\hat{\Phi}$ appearing in this equation, with $\hat{\Phi}$ given by Eq. (3.12), is shown in App. C.2. Considering Eq. (C.19) with $q_i = 0$, we hence have for the local contribution to Eq. (3.15) [45]

$$(2\pi\hbar)^6{}_{\text{in}}\langle p_1, p_2; r_1, r_2|\hat{\Phi}|p_1, p_2; s_1, s_2\rangle_{\text{in}} = \sum_{rs} u_r(p)\bar{u}_s(p) w^{rs}_{r_1 r_2 s_1 s_2}(p_1, p_2, p) \tag{3.50}$$

with

$$\begin{aligned} w^{rs}_{r_1 r_2 s_1 s_2}(p_1, p_2, p) = 2\Bigg\{ &\sum_{r'} \int dP' \delta(p + p' - p_1 - p_2) \langle p, p'; r, r'|t|p_1, p_2; s_1, s_2\rangle \langle p_1, p_2; r_1, r_2|t^\dagger|p, p'; s, r'\rangle \\ &+ \Big[i\pi\hbar p^0 \delta^{(3)}(\mathbf{p} - \mathbf{p}_1) \left(\langle p, p_2; r, r_2|t|p, p_2; s_1, s_2\rangle \delta_{r_1 s} - \langle p, p_2; r_1, r_2|t^\dagger|p, p_2; s, s_2\rangle \delta_{r s_1} \right) + (1 \leftrightarrow 2) \Big] \Bigg\} \\ &\times \delta(p^2 - m^2) , \end{aligned} \tag{3.51}$$

where the symbol $(1 \leftrightarrow 2)$ denotes the exchange of the indices 1 and 2, $dP \equiv d^4p\, \delta(p^2 - m^2)$, and

$$\langle p, p'; r, r'|t|p_1, p_2; s_1, s_2\rangle \equiv -\sqrt{\frac{(2\pi\hbar)^7}{2}}\, \bar{u}_r(p)_{\text{out}}\langle p'; r'|\rho(0)|p_1, p_2; s_1, s_2\rangle_{\text{in}} \tag{3.52}$$

is the conventional scattering amplitude due to the interaction ρ, which can be computed using standard techniques from quantum field theory [45, 156]. With this knowledge we can calculate the local part of Eq. (3.49). In order to do so, we first insert Eq. (3.50) into Eq. (3.15), and then Eq. (3.15) into Eqs. (3.30). Eventually, we use Eqs. (3.30) in Eq. (3.48) to obtain the following local part of the collision kernel

$$\mathfrak{C}_l = \frac{1}{8m^4} \sum_{r_1, r_2, s_1, s_2} \int d^4p_1 d^4p_2 \sum_{r', s'} h_{s'r'}(p, \mathfrak{s}) w^{r's'}_{r_1 r_2 s_1 s_2}(p_1, p_2, p) \prod_{j=1}^{2} \bar{u}_{s_j}(p_j) W(x, p_j) u_{r_j}(p_j) , \tag{3.53}$$

where we also made use of the relations

$$p^\mu \delta_{sr} \equiv \frac{1}{2} \bar{u}_s(p)\gamma^\mu u_r(p) , \tag{3.54}$$

$$n^\mu_{sr}(p) \equiv \frac{1}{2m} \bar{u}_s(p)\gamma^5\gamma^\mu u_r(p) , \tag{3.55}$$

and the definition

$$h_{sr}(p, \mathfrak{s}) \equiv \delta_{sr} + \mathfrak{s} \cdot n_{sr}(p) . \tag{3.56}$$

We note that the local term is always on-shell, as can be seen from the factor $\delta(p^2 - m^2)$ in Eq. (3.51). The delta function appears due to the difference $G(p) - G^\star(p) = 2\pi i\hbar^2 \delta(p^2 - m^2)$, with

$$G(p) = -\frac{\hbar^2}{p^2 - m^2 + i\epsilon p^0} , \tag{3.57}$$

in the first line of Eq. (C.20) when $q_i = 0$ is considered. Inserting the Clifford decomposition (3.17) for the Wigner function in Eq. (3.53), we arrive at

$$\begin{aligned} \mathfrak{C}_l = \frac{1}{32m^2} &\sum_{r_1, r_2, s_1, s_2} \int d^4p_1 d^4p_2 \sum_{r', s'} h_{s'r'}(p, \mathfrak{s}) w^{r's'}_{r_1 r_2 s_1 s_2}(p_1, p_2, p) \\ &\times \prod_{j=1}^{2} \left[\mathcal{F}(x, p_j)\delta_{s_j r_j} + \frac{1}{m} p \cdot \mathcal{V}(x, p_j)\delta_{s_j r_j} + n_{s_j r_j}(p_j) \cdot \mathcal{A}(x, p_j) + \frac{1}{2} \Sigma^{\mu_j \nu_j}_{s_j r_j}(p_j) \mathcal{S}_{\mu_j \nu_j}(x, p_j) \right] , \end{aligned} \tag{3.58}$$

where we defined

$$\Sigma^{\mu\nu}_{rs}(p) \equiv \frac{1}{2m} \bar{u}_r(p)\sigma^{\mu\nu} u_s(p) = \frac{1}{m} \epsilon^{\mu\nu\alpha\beta} p_\alpha n_{rs\beta}(p) . \tag{3.59}$$

We now replace $\mathcal{F}$ and $\mathcal{A}^\mu$ by Eqs. (3.33) and $\mathcal{S}^\mu$ and $\mathcal{V}^\mu$ by Eqs. (3.41), respectively. Furthermore, we make use of (3.45), the relations $p_\mu \Sigma^{\mu\nu}_{\mathfrak{s}} = p_\mu \Sigma^{\mu\nu}_{sr} = 0$, and

$$\Sigma^{\mu\nu}_{sr} \Sigma_{\mathfrak{s}\mu\nu} = 2\frac{p^2}{m^2}\, \mathfrak{s} \cdot n_{sr} . \tag{3.60}$$

Then Eq. (3.58) takes the form

$$\mathfrak{C}_l = \frac{1}{8} \sum_{r_1,r_2,s_1,s_2} \int d\Gamma_1\, d\Gamma_2 \sum_{r',s'} h_{s'r'}(p,\mathfrak{s}) w^{r's'}_{r_1r_2s_1s_2}(p_1,p_2,p) \prod_{j=1}^{2} h_{s_jr_j}(p_j,\mathfrak{s}_j) f(x,p_j,\mathfrak{s}_j) \tag{3.61}$$

with

$$\int d\Gamma \equiv \int d^4p\, \delta(p^2-m^2) \int dS(p)\ , \tag{3.62}$$

where the momenta p_i are on-shell since mass-shell corrections to the free Wigner function in the collision term are of higher order in the density and hence neglected in our framework. Inserting Eq. (3.51) into (3.61), the collision term becomes

$$\begin{aligned}
\mathfrak{C}_l[f] \ = \ & \Bigg[\frac{1}{4} \sum_{r_1,r_2,s_1,s_2} \sum_{r,r',s} \int d\Gamma_1\, d\Gamma_2\, dP'\, h_{sr}(p,\mathfrak{s}) \delta^{(4)}(p+p'-p_1-p_2) \langle p,p';r,r'|t|p_1,p_2;s_1,s_2\rangle \\
& \times \langle p,p_2;r_1,r_2|t^\dagger|p,p';s,r'\rangle \prod_{j=1}^{2} h_{s_jr_j}(p_j,\mathfrak{s}_j) f(x,p_j,\mathfrak{s}_j) \\
& + \frac{i\pi\hbar}{8} \sum_{r_2,s_1,s_2} \sum_{r,s} \int d\Gamma_2\, dS_1(p)\, h_{sr}(p,\mathfrak{s}) \langle p,p_2;r,r_2|t|p,p_2;s_1,s_2\rangle \\
& \times h_{s_1s}(p,\mathfrak{s}_1) f(x,p,\mathfrak{s}_1) h_{s_2r_2}(p_2,\mathfrak{s}_2) f(x,p_2,\mathfrak{s}_2) \\
& - \frac{i\pi\hbar}{8} \sum_{r_2,s_1,s_2} \sum_{r,s} \int d\Gamma_2\, dS_1(p)\, h_{sr}(p,\mathfrak{s}) \langle p,p_2;s_1,r_2|t^\dagger|p,p_2;s,s_2\rangle \\
& \times h_{rs_1}(p,\mathfrak{s}_1) f(x,p,\mathfrak{s}_1) h_{s_2r_2}(p_2,\mathfrak{s}_2) f(x,p_2,\mathfrak{s}_2) \\
& + \frac{i\pi\hbar}{8} \sum_{r_1,s_1,s_2} \sum_{r,s} \int d\Gamma_1\, dS_2(p)\, h_{sr}(p,\mathfrak{s}) \langle p,p_2;r,r_1|t|p,p_1;s_2,s_1\rangle \\
& \times h_{s_1r_1}(p_1,\mathfrak{s}_1) f(x,p_1,\mathfrak{s}_1) h_{s_2r}(p,\mathfrak{s}_2) f(x,p,\mathfrak{s}_2) \\
& - \frac{i\pi\hbar}{8} \sum_{r_1,s_1,s_2} \sum_{r,s} \int d\Gamma_1\, dS_2(p)\, h_{sr}(p,\mathfrak{s}) \langle p,p_1;s_2,r_1|t^\dagger|p,p_1;s,s_1\rangle \\
& \times h_{s_1r_1}(p_1,\mathfrak{s}_1) f(x,p_1,\mathfrak{s}_1) h_{r\mathfrak{s}_2}(p,\mathfrak{s}_2) f(x,p,\mathfrak{s}_2) \Bigg] \delta(p^2-m^2).
\end{aligned} \tag{3.63}$$

We notice that the second and fourth and third and fifth terms are identical, respectively. Now we relabel the indices in the third term $s \to s_1$, $s_1 \to r$, $r \to s$ to obtain

$$\mathfrak{C}_l = \delta(p^2-m^2)\, \mathfrak{C}_{\text{on-shell},l}[f]\ , \tag{3.64}$$

where

$$\begin{aligned}
\mathfrak{C}_{\text{on-shell},l}[f] =& \frac{1}{4} \sum_{r_1,r_2,s_1,s_2} \sum_{r,r',s} \int d\Gamma_1\, d\Gamma_2\, dP'\, h_{sr}(p,\mathfrak{s}) \delta^{(4)}(p+p'-p_1-p_2) \\
& \times \langle p,p';r,r'|t|p_1,p_2;s_1,s_2\rangle \langle p,p_2;r_1,r_2|t^\dagger|p,p';s,r'\rangle \prod_{j=1}^{2} h_{s_jr_j}(p_j,\mathfrak{s}_j)\, f(x,p_j,\mathfrak{s}_j) \\
& + i\, \frac{\pi\hbar}{4} \sum_{r_2,s_1,s_2} \sum_{r,s} \int d\Gamma_2\, dS_1(p)\, h_{s_2r_2}(p_2,\mathfrak{s}_2)\, f(x,p,\mathfrak{s}_1) f(x,p_2,\mathfrak{s}_2) \\
& \times \left[h_{sr}(p,\mathfrak{s}) h_{s_1s}(p,\mathfrak{s}_1) \langle p,p_2;r,r_2|t|p,p_2;s_1,s_2\rangle - h_{s_1s}(p,\mathfrak{s}) h_{sr}(p,\mathfrak{s}_1) \langle p,p_2;r,r_2|t^\dagger|p,p_2;s_1,s_2\rangle\right]
\end{aligned} \tag{3.65}$$

is the local collision term on the mass shell. With the help of the identity

$$\sum_{s'} n^\mu_{rs'}(p) n^\nu_{s's}(p) = -\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{m^2}\right) \delta_{rs} + \frac{i}{m} \epsilon^{\mu\nu\alpha\beta} p_\alpha n_{rs\beta}(p)\ , \tag{3.66}$$

we obtain

$$\begin{aligned}
\sum_s \left[h_{sr}(p,\mathfrak{s}) h_{s_1s}(p,\mathfrak{s}_1) - h_{s_1s}(p,\mathfrak{s}) h_{sr}(p,\mathfrak{s}_1)\right] \ &= \ \mathfrak{s}_\mu \mathfrak{s}_{1\nu} \sum_s \left[n^\mu_{sr}(p) n^\nu_{s_1s}(p) - n^\mu_{s_1s}(p) n^\nu_{sr}(p)\right] \\
&= \ -i\, \frac{2}{m} \mathfrak{s}_\mu \mathfrak{s}_{1\nu} \epsilon^{\mu\nu\alpha\beta} p_\alpha n_{s_1r\beta}(p)\ ,
\end{aligned} \tag{3.67}$$

which simplifies Eq. (3.65) as

$$\begin{aligned}
\mathfrak{C}_{\text{on-shell},l}[f] &= \frac{1}{4}\sum_{r_1,r_2,s_1,s_2}\sum_{r,r',s}\int d\Gamma_1\, d\Gamma_2\, dP'\, h_{sr}(p,\mathfrak{s})\delta^{(4)}(p+p'-p_1-p_2) \\
&\quad \times\langle p,p';r,r'|t|p_1,p_2;s_1,s_2\rangle\langle p,p_2;r_1,r_2|t^\dagger|p,p';s,r'\rangle\prod_{j=1}^{2}h_{s_jr_j}(p_j,\mathfrak{s}_j)f(x,p_j,\mathfrak{s}_j) \\
&+ i\,\frac{\pi\hbar}{8}\sum_{r_2,s_1,s_2}\sum_{r,s}\int d\Gamma_2\, dS_1(p)\, h_{s_2r_2}(p_2,\mathfrak{s}_2)\, f(x,p,\mathfrak{s}_1)f(x,p_2,\mathfrak{s}_2) \\
&\quad \times\left[h_{sr}(p,\mathfrak{s})h_{s_1s}(p,\mathfrak{s}_1)+h_{s_1s}(p,\mathfrak{s})h_{sr}(p,\mathfrak{s}_1)\right]\langle p,p_2;r,r_2|t-t^\dagger|p,p_2;s_1,s_2\rangle \\
&+ \frac{\pi\hbar}{4m}\sum_{r_2,s_1,s_2}\sum_{r}\int d\Gamma_2\, dS_1(p)\, h_{s_2r_2}(p_2,\mathfrak{s}_2)\, f(x,p,\mathfrak{s}_1)f(x,p_2,\mathfrak{s}_2) \\
&\quad \times\mathfrak{s}_\mu\mathfrak{s}_{1\nu}\epsilon^{\mu\nu\alpha\beta}p_\alpha n_{s_1r\beta}(p)\langle p,p_2;r,r_2|t+t^\dagger|p,p_2;s_1,s_2\rangle\;.
\end{aligned} \tag{3.68}$$

We can combine the first two terms in this equation by making use of the optical theorem [45]

$$i\pi\hbar\langle p,p_1;r,r_1|t-t^\dagger|p,p_1;s,s_1\rangle = -\sum_{r',r_1'}\int dP'dP_1'\langle p,p_1;r,r_1|t|p',p_1';r',r_1'\rangle\langle p',p_1';r',r_1'|t^\dagger|p,p_1;s,s_1\rangle\;. \tag{3.69}$$

With this, the collision term becomes

$$\begin{aligned}
\mathfrak{C}_{\text{on-shell},l}[f] =&\frac{1}{4}\sum_{r_1,r_2,s_1,s_2}\sum_{r,r',s}\int d\Gamma_1\, d\Gamma_2\, dP'\, h_{sr}(p,\mathfrak{s})\delta^{(4)}(p+p'-p_1-p_2) \\
&\times\langle p,p';r,r'|t|p_1,p_2;s_1,s_2\rangle\langle p,p_2;r_1,r_2|t^\dagger|p,p';s,r'\rangle\prod_{j=1}^{2}h_{s_jr_j}(p_j,\mathfrak{s}_j)f(x,p_j,\mathfrak{s}_j) \\
&-\frac{1}{8}\sum_{r_2,s_1,s_2}\sum_{r,s,r',r_1'}\int d\Gamma_2\, dP'\, dP_1'\, dS_1(p)\, h_{s_2r_2}(p_2,\mathfrak{s}_2)\, f(x,p,\mathfrak{s}_1)f(x,p_2,\mathfrak{s}_2) \\
&\times\left[h_{sr}(p,\mathfrak{s})h_{s_1s}(p,\mathfrak{s}_1)+h_{s_1s}(p,\mathfrak{s})h_{sr}(p,\mathfrak{s}_1)\right]\langle p,p_2;r,r_2|t|p',p_1';r',r_1'\rangle\langle p',p_1';r',r_1'|t^\dagger|p,p_2;s_1,s_2\rangle \\
&+\frac{\pi\hbar}{4m}\sum_{r_2,s_1,s_2}\sum_{r}\int d\Gamma_2\, dS_1(p)\, h_{s_2r_2}(p_2,\mathfrak{s}_2)\, f(x,p,\mathfrak{s}_1)f(x,p_2,\mathfrak{s}_2) \\
&\times\,\mathfrak{s}_\mu\mathfrak{s}_{1\nu}\epsilon^{\mu\nu\alpha\beta}p_\alpha n_{s_1r\beta}(p)\langle p,p_2;r,r_2|t+t^\dagger|p,p_2;s_1,s_2\rangle\;.
\end{aligned} \tag{3.70}$$

Finally, in order to write our result in a more compact form, we insert factors of one in the form $1 = (1/2)\int dS(p)$. This yields

$$\mathfrak{C}_{\text{on-shell},l}[f] \equiv \mathfrak{C}_{\text{p+s}}[f] + \mathfrak{C}_{\text{s}}[f]\;, \tag{3.71}$$

with

$$\mathfrak{C}_{\text{p+s}}[f] \equiv \int d\Gamma_1\, d\Gamma_2\, d\Gamma'\, dS_1'(p)\,\mathcal{W}\left[f(x,p_1,\mathfrak{s}_1)f(x,p_2,\mathfrak{s}_2)-f(x,p,\mathfrak{s}_1')f(x,p',\mathfrak{s}')\right]\;, \tag{3.72a}$$

$$\mathfrak{C}_{\text{s}}[f] \equiv \int d\Gamma_2\, dS_1(p)\,\mathfrak{W}\, f(x,p,\mathfrak{s}_1)f(x,p_2,\mathfrak{s}_2)\;, \tag{3.72b}$$

where

$$\begin{aligned}
\mathcal{W} \equiv& \frac{1}{32}\sum_{s,r,s_1'}\left[h_{ss_1'}(p,\mathfrak{s}_1')h_{s_1'r}(p,\mathfrak{s})+h_{ss_1'}(p,\mathfrak{s})h_{s_1'r}(p,\mathfrak{s}_1')\right]\sum_{s',r',s_1,s_2,r_1,r_2}h_{s'r'}(p',\mathfrak{s}')\,h_{s_1r_1}(p_1,\mathfrak{s}_1) \\
&\times h_{s_2r_2}(p_2,\mathfrak{s}_2)\langle p,p';r,r'|t|p_1,p_2;s_1,s_2\rangle\langle p_1,p_2;r_1,r_2|t^\dagger|p,p';s,s'\rangle\delta^{(4)}(p+p'-p_1-p_2)
\end{aligned} \tag{3.73}$$

and

$$\mathfrak{W} \equiv \frac{\pi\hbar}{4m}\sum_{s_1,s_2,r,r_2}\epsilon_{\mu\nu\alpha\beta}\mathfrak{s}^\mu\mathfrak{s}_1^\nu p^\alpha n_{s_1r}^\beta(p)\, h_{s_2r_2}(p_2,\mathfrak{s}_2)\langle p,p_2;r,r_2|t+t^\dagger|p,p_2;s_1,s_2\rangle\;. \tag{3.74}$$

The collision term in Eq. (3.71) is written as the sum of $\mathfrak{C}_{\text{p+s}}[f]$, which describes collisions with both momentum- and spin-exchange, and $\mathfrak{C}_{\text{s}}[f]$, which accounts for collisions with spin exchange, but without momentum exchange. If the distribution functions are independent of the spin variables, i.e., $f(x,p,\mathfrak{s}) \equiv f(x,p)$, the collision term familiar from the standard Boltzmann equation, with averaging and summation over spins done directly in the cross section, is recovered. We remark that Pauli–blocking factors are absent due to the low-density approximation, where Fermi statistics become unimportant [45]. This means that the equilibrium distribution functions will be of the Boltzmann form. As we aim at describing spin effects, we will consider spin-dependent distribution functions in the following. Let us have a closer look at $\mathfrak{C}_{\rm p+s}[f]$ in (3.71) in this case. While the term $\sim f(x,p_1,\mathfrak{s}_1)f(x,p_2,\mathfrak{s}_2)$ has an obvious interpretation as a gain term for particles with momentum p and spin $\mathfrak{s}$, the term $\sim f(x,p,\mathfrak{s}_1')f(x,p',\mathfrak{s}')$ does not yet look like a loss term, as it depends on the spin variable $\mathfrak{s}_1'$, and not $\mathfrak{s}$.

For this reason, we search for a redefined collision term and a redefined distribution function which describe the same physics as the old ones, while having an apparent interpretation as gain and loss terms. In order to achieve this, we note that all quantities in the enlarged phase-space are not measurable, but obtain their physical meaning only after integration over $\mathfrak{s}$. Thus we can redefine collision term and distribution function

$$\begin{aligned} \mathfrak{f}(x,p,\mathfrak{s}) &\rightarrow \tilde{\mathfrak{f}}(x,p,\mathfrak{s}), \\ \mathfrak{C}[\mathfrak{f}(x,p,\mathfrak{s})] &\rightarrow \tilde{\mathfrak{C}}[\tilde{\mathfrak{f}}(x,p,\mathfrak{s})], \end{aligned} \tag{3.75}$$

with

$$\begin{aligned} p^\mu\partial_\mu \tilde{\mathfrak{f}}(x,p,\mathfrak{s}) &= \tilde{\mathfrak{C}}[\tilde{\mathfrak{f}}(x,p,\mathfrak{s})], \\ \int dS(p)\,\tilde{\mathfrak{f}}(x,p,\mathfrak{s}) &= \int dS(p)\,\mathfrak{f}(x,p,\mathfrak{s}), \\ \int dS(p)\,\mathfrak{s}^\mu\tilde{\mathfrak{f}}(x,p,\mathfrak{s}) &= \int dS(p)\,\mathfrak{s}^\mu\mathfrak{f}(x,p,\mathfrak{s}), \\ \int dS(p)\,\tilde{\mathfrak{C}}[\tilde{\mathfrak{f}}(x,p,\mathfrak{s})] &= \int dS(p)\,\mathfrak{C}[\mathfrak{f}(x,p,\mathfrak{s})], \\ \int dS\,\mathfrak{s}^\mu\tilde{\mathfrak{C}}[\tilde{\mathfrak{f}}(x,p,\mathfrak{s})] &= \int dS(p)\,\mathfrak{s}^\mu\mathfrak{C}[\mathfrak{f}(x,p,\mathfrak{s})] \end{aligned} \tag{3.76}$$

without changing the physical content, since after the integration the new quantities are equivalent to the old ones.

So now we want to eliminate the $\mathfrak{s}_1$ integration in Eq. (3.72a) while replacing $\mathfrak{s}_1$ by $\mathfrak{s}$ in the new collision term $\tilde{\mathfrak{C}}_{\rm p+s}[\tilde{f}]$. We will perform this procedure separately for C_F and C_A^μ, which both appear in Eq. (3.48). While the calculation for C_F is straightforward, the one for C_A^μ is slightly more complicated and will be shown in the following. When computing the integral $\int dS(p)\,\mathfrak{s}^\alpha\,\mathfrak{C}_{\rm p+s}[f]$, we find terms of the form

$$\begin{aligned} &\frac{1}{2}\int dS(p)dS_1'(p)\,\mathfrak{s}^\alpha\left[\mathfrak{s}_\mu n^\mu_{s_1'r}(p)\mathfrak{s}_{1\nu}' n^\nu_{ss_1'}(p)+\mathfrak{s}_\mu n^\mu_{ss_1'}\mathfrak{s}_{1\nu}'(p)n^\nu_{s_1'r}(p)\right]\mathfrak{s}_1'^\beta \\ =&\int dS(p)dS_1'(p)\mathfrak{s}^\alpha\mathfrak{s}_\mu\mathfrak{s}_{1\nu}'\left(-g^{\mu\nu}+\frac{p^\mu p^\nu}{m^2}\right)\delta_{rs}\mathfrak{s}_1'^\beta \\ =&-\int dS\,dS_1'\,\mathfrak{s}^\alpha\mathfrak{s}_\mu\mathfrak{s}_1'^\mu\mathfrak{s}_1'^\beta\delta_{rs} \\ =&2\int dS\,\mathfrak{s}^\alpha\mathfrak{s}^\beta\delta_{rs} \\ =&-4\left(g^{\alpha\beta}-\frac{p^\alpha p^\beta}{p^2}\right)\delta_{rs}\ , \end{aligned} \tag{3.77}$$

where we used Eq. (3.66) in the first and Eq. (3.40c) in the last step. Using this and similar properties for the other terms, one can show that $\tilde{\mathfrak{C}}_{\rm p+s}$, which is obtained from $\mathfrak{C}_{\rm p+s}$ by applying

$$\sum_{s_1'}\int dS_1'(p)\left[h_{ss_1'}(p,\mathfrak{s}_1')h_{s_1'r}(p,\mathfrak{s})+h_{ss_1'}(p,\mathfrak{s})h_{s_1'r}(p,\mathfrak{s}_1')\right]\longrightarrow 4h_{sr}(p,\mathfrak{s})\ , \tag{3.78}$$

is equivalent to the original collision term in the sense of Eqs. (3.76).

Due to the linearity of $\mathfrak{f}$ in $\mathfrak{s}^\mu$, Eq. (3.31), and since by a Taylor expansion $\tilde{f}$ can be taken to be linear in $\mathfrak{s}$ up to first order in $\hbar$ as well, as $\mathcal{A}^{(0)\mu}=0$, any change in the distribution function has to be of second order in $\hbar$. This can be proven by inserting Taylor expansions of both f and $\tilde{f}$ into the left- and right-hand sides of the second and third equation in Eqs. (3.76). Hence, we only transform the collision term and ignore the changes in f without loss of generality in the applied power-counting scheme. If one wants to include higher orders in $\hbar$, this can be done by applying the transformation also to the distribution function. With the help of Eq. (3.78) and the identities $\int dS(p)\,\mathfrak{s}^\mu = 0$ and $\int dS(p)\,\mathfrak{s}^\mu\mathfrak{s}^\nu\mathfrak{s}^\lambda = 0$ we finally obtain the following collision term, which fulfills all requirements,

$$\tilde{\mathfrak{C}}_{\text{p+s}}[f] \equiv \int d\Gamma_1\, d\Gamma_2\, d\Gamma'\, \widetilde{\mathcal{W}}\big[f(x,p_1,\mathfrak{s}_1)f(x,p_2,\mathfrak{s}_2) - f(x,p,\mathfrak{s})f(x,p',\mathfrak{s}')\big]\;, \tag{3.79}$$

with

$$\begin{aligned}\widetilde{\mathcal{W}} \equiv\;& \delta^{(4)}(p+p'-p_1-p_2)\,\frac{1}{8}\sum_{s,r} h_{sr}(p,\mathfrak{s}) \sum_{s',r',s_1,s_2,r_1,r_2} h_{s'r'}(p',\mathfrak{s}')\, h_{s_1r_1}(p_1,\mathfrak{s}_1)\, h_{s_2r_2}(p_2,\mathfrak{s}_2)\\ &\times\langle p,p';r,r'|t|p_1,p_2;s_1,s_2\rangle\langle p_1,p_2;r_1,r_2|t^\dagger|p,p';s,s'\rangle\;.\end{aligned} \tag{3.80}$$

Considering $\mathfrak{C}_{\text{s}}[f]$, we find that this term corresponds to collisions without momentum exchange, but with spin exchange: $(p,\mathfrak{s}_1),(p_2,\mathfrak{s}_2)\to(p,\mathfrak{s}),(p_2,\mathfrak{s}')$ [45]. Here, the distribution functions $f(x,p,\cdot)$ and $f(x,p',\cdot)$ are associated with particles before *and* after the collision, in other words, they contribute to *both* the gain and the loss term. As can be seen from Eq. (3.74), the sign of $\mathfrak{W}$ is flipped when $\mathfrak{s}^\mu$ and $\mathfrak{s}_1^\nu$ are exchanged. This means that a net gain of particles with $(p,\mathfrak{s})$ corresponds to a net loss of particles with $(p,\mathfrak{s}_1)$. Thus, $\mathfrak{C}_{\text{s}}[f]$ contains both gain and loss terms and has already the expected structure without the need to introduce any transformation.

3.4 Nonlocal collisions

The nonlocal collision term is given by the second term in the second line of Eq. (3.15), which depends on space-time derivatives of the Wigner function. In contrast to the local term, which contains $\delta^{(4)}(q_j)$, due to the factor $\partial^\mu_{q_j}\delta^{(4)}(q_j)$ the momentum p of the Wigner function in the nonlocal term does not lie on the mass-shell. After integrating by parts in the q_j integrals, we can write the nonlocal collision term in Eq. (3.15) as the sum of two contributions,

$$\mathfrak{C}^{(1)}_{nl} = \mathfrak{C}^{(1)}_{nl,1} + \mathfrak{C}^{(1)}_{nl,2}\;. \tag{3.81}$$

The first one comes from acting with the q_j-derivative on the spinors, i.e.,

$$\begin{aligned}\mathfrak{C}^{(1)}_{nl,1} =\;& \frac{i}{8m^4}\sum_{r_1,r_2,s_1,s_2}\int d^4p_1 d^4p_2 d^4q_1 d^4q_2\,\delta^{(4)}(q_1)\delta^{(4)}(q_2)\\ &\times \text{tr}\left[\left(\frac{1}{p^2}\,p\cdot\gamma - \frac{1}{m}\mathfrak{s}\cdot\gamma\gamma^5\right)(2\pi\hbar)^6\;{}_{\text{in}}\!\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\middle|\hat{\Phi}(p)\middle|p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle_{\text{in}}\right]\\ &\times\left\{\bar{u}_{s_1}\left(p_1+\frac{q_1}{2}\right)W(x,p_1)u_{r_1}\left(p_1-\frac{q_1}{2}\right)\partial^\mu_{q_2}\left[\bar{u}_{s_2}\left(p_2+\frac{q_2}{2}\right)\partial_\mu W(x,p_2)u_{r_2}\left(p_2-\frac{q_2}{2}\right)\right]\right.\\ &\left.+\,\partial^\mu_{q_1}\left[\bar{u}_{s_1}\left(p_1+\frac{q_1}{2}\right)\partial_\mu W(x,p_1)u_{r_1}\left(p_1-\frac{q_1}{2}\right)\right]\bar{u}_{s_2}\left(p_2+\frac{q_2}{2}\right)W(x,p_2)u_{r_2}\left(p_2-\frac{q_2}{2}\right)\right\}\;.\end{aligned} \tag{3.82}$$

Since in the above equation the matrix element of $\hat{\Phi}$ contributes the same mass-shell delta function as in the local case, this term is always on-shell,

$$\mathfrak{C}^{(1)}_{nl,1} = \delta(p^2-m^2)\,\mathfrak{C}^{(1)}_{\text{on-shell},nl,1}\;. \tag{3.83}$$

On the other hand, the second term in Eq. (3.81) contains q_j-derivatives acting on the matrix element of $\hat{\Phi}$,

$$\begin{aligned}\mathfrak{C}^{(1)}_{nl,2} =\;& \frac{i}{8m^4}\sum_{r_1,r_2,s_1,s_2}\int d^4p_1 d^4p_2 d^4q_1 d^4q_2\,\delta^{(4)}(q_1)\delta^{(4)}(q_2)\\ &\times\left\{\bar{u}_{s_1}\left(p_1+\frac{q_1}{2}\right)W(x,p_1)u_{r_1}\left(p_1-\frac{q_1}{2}\right)\bar{u}_{s_2}\left(p_2+\frac{q_2}{2}\right)\partial_\mu W(x,p_2)u_{r_2}\left(p_2-\frac{q_2}{2}\right)\partial^\mu_{q_2}\right.\\ &\left.+\,\bar{u}_{s_1}\left(p_1+\frac{q_1}{2}\right)\left[\partial_\mu W(x,p_1)\right]u_{r_1}\left(p_1-\frac{q_1}{2}\right)\bar{u}_{s_2}\left(p_2+\frac{q_2}{2}\right)W(x,p_2)u_{r_2}\left(p_2-\frac{q_2}{2}\right)\partial^\mu_{q_1}\right\}\\ &\times\text{tr}\left[\left(\frac{1}{p^2}\,p\cdot\gamma - \frac{1}{m}\mathfrak{s}\cdot\gamma\gamma^5\right)(2\pi\hbar)^6\;{}_{\text{in}}\!\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\middle|\hat{\Phi}(p)\middle|p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle_{\text{in}}\right]\;.\end{aligned} \tag{3.84}$$

We show in App. C.4 that one can split $\mathfrak{C}^{(1)}_{nl,2}$ into an on-shell and an off-shell contribution,

$$\begin{aligned}\mathfrak{C}^{(1)}_{nl,2} &= \mathfrak{C}^{(1)}_{\text{off-shell}} + \delta(p^2-m^2)\,\mathfrak{C}^{(1)}_{\text{on-shell},2} \\ &= \mathfrak{C}^{(1)}_{\text{off-shell}} + \delta(p^2-m^2)\left(\mathfrak{C}^{(1)}_{\text{on-shell},2,1} + \mathfrak{C}^{(1)}_{\text{on-shell},2,2}\right) .\end{aligned} \tag{3.85}$$

Furthermore, we prove that the off-shell contribution $\mathfrak{C}^{(1)}_{\text{off-shell}}$ and the off-shell part on the left-hand side of the Boltzmann equation (3.47) exactly cancel each other when Eq. (3.45) is inserted. The explicit form of $\mathfrak{C}^{(1)}_{\text{on-shell},2,1}$ is also derived in App. C.4. This term results in an identical shift of the position variables of all distribution functions in the collision term. We will see later that its contribution vanishes when an expansion around equilibrium is considered. For this reason, this term has been dropped in Refs. [57, 58], however, here we will keep it for the sake of generality until we come to the discussion of equilibrium. Finally, also $\mathfrak{C}^{(1)}_{\text{on-shell},2,2}$ is calculated in App. C.4, see Eq. (C.40). It depends on momentum derivatives of matrix elements, which are neglected, since we assume that the scattering amplitude is constant over scales of order of the interaction range defining the scattering nonlocality. This is consistent with the low-density approximation, see e.g. Ref. [186]. In conclusion, we are lead to consider the following on-shell Boltzmann equation for the distribution function $f(x,p,\mathfrak{s})$,

$$\delta(p^2-m^2)\,p\cdot\partial f(x,p,\mathfrak{s}) = \delta(p^2-m^2)\,\mathfrak{C}_{\text{on-shell}}[f] \tag{3.86}$$

with

$$\mathfrak{C}_{\text{on-shell}}[f] \equiv \mathfrak{C}_{\text{on-shell},l}[f] + \hbar\left(\mathfrak{C}^{(1)}_{\text{on-shell},nl,1}[f] + \mathfrak{C}^{(1)}_{\text{on-shell},2,1}[f]\right) \tag{3.87}$$

and $\mathfrak{C}_{\text{on-shell},l}$ being the local term calculated in the previous section.

The contribution which remains to be computed is $\mathfrak{C}^{(1)}_{nl,1}$. As this term is already linear in gradients of the Wigner function and we consider only contributions up to first order, in our scheme it depends only on the zeroth-order Wigner function. Hence, in the calculation we take into account that, as in our framework the zeroth-order parts of $\mathcal{A}^\mu$ and $\mathcal{S}^{\mu\nu}$ vanish, respectively, the only first-order contributions come from $\mathcal{F}$ and $\mathcal{V}^\mu$ when inserting the Wigner function into the nonlocal collision term. With the help of the spinor identities (C.23) one evaluates the relevant terms in Eq. (3.82) as

$$\begin{aligned}& i\partial^\mu_{q_j}\left[\bar{u}_{s_j}\left(p_j+\frac{q_j}{2}\right)\partial_\mu W(x,p_j)u_{r_j}\left(p_j-\frac{q_j}{2}\right)\right]_{q_j=0} \\ =\;& i\left[\partial^\mu_{q_j}\bar{u}_{s_j}\left(p_j+\frac{q_j}{2}\right)u_{r_j}\left(p_j-\frac{q_j}{2}\right)\partial_\mu\mathcal{F}^{(0)}(x,p_j) + \partial^\mu_{q_j}\bar{u}_{s_j}\left(p_j+\frac{q_j}{2}\right)\gamma^\alpha u_{r_j}\left(p_j-\frac{q_j}{2}\right)\partial_\mu\mathcal{V}^{(0)}_\alpha(x,p_j)\right]_{q_j=0} \\ =\;& \frac{1}{p^0_j+m}\,p_{j\nu}\Sigma^{\mu\nu}_{s_jr_j}(p_\star)\partial_\mu f^{(0)}(x,p_j) \\ =\;& \frac{1}{p^0_j+m}\,\left[\mathbf{p}_j\times\mathbf{n}_{s_jr_j}(p_j)\right]\cdot\boldsymbol{\nabla}f^{(0)}(x,p_j) \\ =\;& -\frac{1}{2(p^0_j+m)}\int dS_j(p_j)\,h_{s_jr_j}(p_j,\mathfrak{s}_j)(\mathbf{p}_j\times\mathfrak{s}_j)\cdot\boldsymbol{\nabla}f^{(0)}(x,p_j) .\end{aligned} \tag{3.88}$$

In order to write the result (3.88) in a compact way, we define the space-time shift

$$\Delta^\mu \equiv -\frac{\hbar}{2m(p\cdot\hat{t}+m)}\,\epsilon^{\mu\nu\alpha\beta}p_\nu\hat{t}_\alpha\mathfrak{s}_\beta , \tag{3.89}$$

where $\hat{t}^\mu$ is the time-like unit vector which is $(1,\mathbf{0})$ in the frame where p^μ is measured. Following analogous steps as in the derivation of Eq. (3.72), we obtain for Eq. (3.82)

$$\begin{aligned}\hbar\,\mathfrak{C}^{(1)}_{\text{on-shell},nl,1}[f] \;=\;& \int d\Gamma_1\,d\Gamma_2\,d\Gamma'\,dS_1'(p)\,\mathcal{W} \\ &\times\Big[f(x,p_2,\mathfrak{s}_2)\Delta_1\cdot\partial f(x,p_1,\mathfrak{s}_1) + f(x,p_1,\mathfrak{s}_1)\Delta_2\cdot\partial f(x,p_2,\mathfrak{s}_2) \\ &\quad -f(x,p',\mathfrak{s}')\Delta_1'\cdot\partial f(x,p,\mathfrak{s}_1') - f(x,p,\mathfrak{s}_1')\Delta'\cdot\partial f(x,p',\mathfrak{s}')\Big] \\ &+\frac{\pi\hbar}{4m}\sum_{r,r_2,s_1,s_2}\int d\Gamma_2\,dS_1(p)\,h_{s_2r_2}(p_2,\mathfrak{s}_2)\,\mathfrak{s}_\mu\mathfrak{s}_{1\nu}\epsilon^{\mu\nu\alpha\beta}p_\alpha n_{s_1r\beta}\langle p,p_2;r,r_2|t+t^\dagger|p,p_2;s_1,s_2\rangle \\ &\times[f(x,p,\mathfrak{s}_1)\Delta_2\cdot\partial f(x,p_2,\mathfrak{s}_2) + f(x,p_2,\mathfrak{s}_2)\Delta_1\cdot\partial f(x,p,\mathfrak{s}_1)] .\end{aligned} \tag{3.90}$$

Similarly, we find from Eq. (C.39) and the definition of Δ in Eq. (3.89) that $\mathfrak{C}^{(1)}_{\text{on-shell},2,1}$, after applying a transformation (3.76), can be written as

$$\hbar\, \tilde{\mathfrak{C}}^{(1)}_{\text{on-shell},2,1}[f] = -\int d\Gamma_1\, d\Gamma_2\, d\Gamma'\, \widetilde{\mathcal{W}}\, \Delta\cdot\partial \Big[f(x,p_2,\mathfrak{s}_2) f(x,p_1,\mathfrak{s}_1) - f(x,p',\mathfrak{s}') f(x,p,\mathfrak{s}) \Big], \tag{3.91}$$

where we reinserted spin integrations in order to be able to combine this term with the other parts of the collision term in the next step. Namely, noting that $\Delta\cdot\partial f(x,p,\mathfrak{s})$ is the first-order contribution of the Taylor expansion of $f(x+\Delta,p,\mathfrak{s})$, we can, after applying the transformation (3.76) also to Eq. (3.90), collect all contributions and write the final form of the total collision term up to first order in $\hbar$ as

$$\begin{aligned} \tilde{\mathfrak{C}}_{\text{on-shell}}[f(x,p,\mathfrak{s})] \;\; = \;\; & \int d\Gamma_1 d\Gamma_2 d\Gamma'\, \widetilde{\mathcal{W}}\, [f(x+\Delta_1-\Delta,p_1,\mathfrak{s}_1) f(x+\Delta_2-\Delta,p_2,\mathfrak{s}_2) \\ & - f(x,p,\mathfrak{s}) f(x+\Delta'-\Delta,p',\mathfrak{s}')] \\ & + \int d\Gamma_2\, dS_1(p)\, \mathfrak{W} f(x+\Delta_1,p,\mathfrak{s}_1) f(x+\Delta_2,p_2,\mathfrak{s}_2)\; . \end{aligned} \tag{3.92}$$

Alternatively, we can redefine the position $\bar{x} \equiv x - \Delta$ and rewrite Eq. (3.92) in the form

$$\begin{aligned} \tilde{\mathfrak{C}}_{\text{on-shell}}[f(\bar{x}+\Delta,p,\mathfrak{s})] = & \int d\Gamma_1 d\Gamma_2 d\Gamma'\, \widetilde{\mathcal{W}}\, [f(\bar{x}+\Delta_1,p_1,\mathfrak{s}_1) f(\bar{x}+\Delta_2,p_2,\mathfrak{s}_2) - f(\bar{x}+\Delta,p,\mathfrak{s}) f(\bar{x}+\Delta',p',\mathfrak{s}')] \\ & + \int d\Gamma_2\, dS_1(p)\, \mathfrak{W} f(\bar{x}+\Delta_1,p,\mathfrak{s}_1) f(\bar{x}+\Delta_2,p_2,\mathfrak{s}_2)\; , \end{aligned} \tag{3.93}$$

where the difference between the last lines in Eqs. (3.92) and (3.93) is of second order in $\hbar$ and hence can be neglected.

We can interpret Eq. (3.93) as follows: There is a displacement of the positions of incoming and outgoing particles from the geometric center of the collision $\bar{x}^\mu$, given by a space-like distance Δ^μ. As a consequence, the sum of the orbital angular momenta of the incoming particles differs from the one of the outgoing particles. This difference in orbital angular momentum is transferred into spin angular momentum in the nonlocal collision. As we will see in Section 3.5, this mechanism leads to the alignment of spin polarization with vorticity in equilibrium [57].

At this point, we remark that nonlocal collision terms have been studied for the spinless and nonrelativistic case in Refs. [187–190], where the nonlocality also emerges from quantum effects and results in space-time shifts of the distribution function in the collision term, and in a classical, relativistic approach [191]. In those works, although due to the absence of spin the conservation of total angular momentum does not play any role, the nonlocal collision terms lead to modifications of the usual hydrodynamic equations. We will see in which situations such modifications appear also in our approach in Chapter 5.

It may be also useful to compare the results of this section to the study of the side-jump effect for massless particles in Ref. [51], the latter being in some sense also a nonlocal kinetic theory. However, we will see in the following that the two concepts describe different physical phenomena.

In Ref. [51], instead of using the Wigner-function formalism, the phase-space formulation of kinetic theory is formally obtained by applying momentum and position operators to single-particle states, and integrating over all possible momenta. As explained in Chapter 2.5.6 the position and spin operators in that reference correspond to $\hat{q}^\mu$ and $\hat{S}^{\mu\nu}_q$ in their massless limits, in particular, their expectation values are not identical to the position and momentum variables of the Wigner function. While both approaches yield a valid kinetic-theory description, one has to be careful with identifications between quantities in the two formalisms.

Let us have a closer look at the origin of the side-jump effect due to the anomalous transformation behaviour of the $S^{\mu\nu}_q$ with the shift $\mathfrak{D}^\mu$ given in Eq. (2.128). Consider a binary particle scattering $p_{1i} + p_{2i} \rightarrow p_{1f} + p_{2f}$. We assume that the collision is local in the center-of-momentum frame, the so-called "no-jump frame". In this frame, the incoming particles meet at one collision point, and the outgoing particles are produced at the same point. On the other hand, looking at the collision from a frame boosted in a direction parallel to the initial momenta, we should calculate the shift $\mathfrak{D}^\mu$ in Eq. (2.128) due to the Lorentz transformation for each particle. The positions of the incoming particles are not shifted, $\mathfrak{D}^\mu_{1i} = \mathfrak{D}^\mu_{2i} = 0$, as the spatial components of n'^μ and the three-momenta are parallel. However, the momenta of the outgoing particles are not parallel to the spatial components of n'^μ any longer. Therefore, the shifts $\mathfrak{D}^\mu_{1f}$ and $\mathfrak{D}^\mu_{2f}$ are nonzero. In other words, the positions where the outgoing particles are emitted are displaced by $\mathfrak{D}^\mu_{1f}$ and $\mathfrak{D}^\mu_{2f}$, respectively, from the collision point of the initial particles. This is what we understand by the side-jump effect. It is important to note that this effect occurs independently of the microscopic properties of the collision term. Instead, its origin is the usage of a noncovariant position operator, which is unavoidable in the massless case. If, on the other hand, we were to build a kinetic theory for massive spin-1/2 particles with the approach of Ref. [51], we would, consistently with what we discussed so far, use the covariant position defined through the HW position operator given in Eq. (2.129). In this case, the locality of the collision in one frame implies its locality in all frames.

The side-jump effect should be distinguished from the nonlocality in Eq. (3.92). The latter is a property of the collision term, while the position x^μ in the Wigner function is always covariant. This nonlocality can be seen as a finite impact parameter in all reference frames. There exists no “no-jump frame”. For massless particles, one could introduce such a nonlocal collision term as well, leading to a nonlocality even in the "no-jump frame". The side-jump effect would occur on top of this.

3.5 Equilibrium

In this section we study the conditions under which the collision term vanishes, implying that the system reaches equilibrium. In the standard form of the local-equilibrium distribution function its exponent is a linear combination of the conserved quantities charge, momentum, and total angular momentum. It reads [3, 37, 59]

$$f_{eq}(x,p,\mathfrak{s}) = \frac{1}{(2\pi\hbar)^3}\exp\left[\alpha(x) - \beta(x)\cdot p + \frac{\hbar}{4}\Omega_{\mu\nu}(x)\Sigma_{\mathfrak{s}}^{\mu\nu}\right] , \tag{3.94}$$

where α is the chemical potential, $\beta^\mu \equiv u^\mu/T$ (u^μ is the fluid velocity and T the temperature, respectively) and $\Omega^{\mu\nu}$ is the spin potential [37, 192]. Here, we absorbed the orbital part of the angular momentum into the definition of $\beta^\mu(x)$ [3] . First, inserting $f_{eq}^{(0)}$ from Eq. (3.94) into Eq. (C.39), we find

$$\begin{aligned}\mathfrak{C}^{(1)}_{\text{on-shell},2,1}[f^{(0)}_{eq}] =& \frac{1}{(2\pi\hbar)^3}\frac{1}{8m(p^0+m)}\sum_{r,s,r',r_1,r_2}\left[p_\nu \Sigma^{\mu\nu}_{sr}(p_\star) + \epsilon^{\nu\lambda\mu 0}p_\nu \mathfrak{s}_\lambda \delta_{sr}\right]\\ &\times \int dP_1\, dP_2\, dP'\, \delta^{(4)}(p+p'-p_1-p_2)\langle p,p';r,r'|t|p_1,p_2;r_1,r_2\rangle\langle p_1,p_2;r_1,r_2|t^\dagger|p,p';s,r'\rangle\\ &\times \partial_\mu\beta_\lambda(p_1^\lambda + p_2^\lambda - p^\lambda - p'^\lambda)e^{-\beta\cdot(p_1+p_2)}\\ =&0\ .\end{aligned} \tag{3.95}$$

Therefore, when a gradient expansion around equilibrium is considered, the contribution from $\mathfrak{C}^{(1)}_{\text{on-shell},2,1}$ to the collision term vanishes up to first order in gradients. Then the collision term becomes

$$\begin{aligned}\tilde{\mathfrak{C}}_{\text{on-shell}}[f(x,p,\mathfrak{s})] \quad =& \quad \int d\Gamma_1 d\Gamma_2 d\Gamma'\, \widetilde{\mathcal{W}}\,[f(x+\Delta_1,p_1,\mathfrak{s}_1)f(x+\Delta_2,p_2,\mathfrak{s}_2) - f(x+\Delta,p,\mathfrak{s})f(x+\Delta',p',\mathfrak{s}')]\\ &+\int d\Gamma_2\, dS_1(p)\,\mathfrak{W} f(x+\Delta_1,p,\mathfrak{s}_1)f(x+\Delta_2,p_2,\mathfrak{s}_2)\ .\end{aligned} \tag{3.96}$$

We insert Eq. (3.94) also here and expand to first order in $\hbar$. The zeroth order vanishes due to the conservation of energy-momentum and charge. The first order reads

$$\begin{aligned}\tilde{\mathfrak{C}}_{\text{on-shell}}[f_{\text{eq}}] \quad =& \quad -\int d\Gamma' d\Gamma_1 d\Gamma_2\, \widetilde{\mathcal{W}}\, e^{-\beta\cdot(p_1+p_2)}\Big[-\partial_\mu\alpha\,(\Delta_1^\mu + \Delta_2^\mu - \Delta^\mu - \Delta'^\mu)\\ &+\partial_\mu\beta_\nu\,\left(\Delta_1^\mu p_1^\nu + \Delta_2^\mu p_2^\nu - \Delta^\mu p^\nu - \Delta'^\mu p'^\nu\right) - \frac{\hbar}{4}\Omega_{\mu\nu}\left(\Sigma^{\mu\nu}_{\mathfrak{s}_1} + \Sigma^{\mu\nu}_{\mathfrak{s}_2} - \Sigma^{\mu\nu}_{\mathfrak{s}} - \Sigma^{\mu\nu}_{\mathfrak{s}'}\right)\Big]\\ &-\int d\Gamma_2\, dS_1(p) dS'(p_2)\,\mathfrak{W}\, e^{-\beta\cdot(p+p_2)}\Big\{-\partial_\mu\alpha\,(\Delta_1^\mu + \Delta_2^\mu - \Delta^\mu - \Delta'^\mu)\\ &+\partial_\mu\beta_\nu\,[(\Delta_1^\mu - \Delta^\mu)p^\nu + (\Delta_2^\mu - \Delta'^\mu)p_2^\nu] - \frac{\hbar}{4}\Omega_{\mu\nu}(\Sigma^{\mu\nu}_{\mathfrak{s}_1} + \Sigma^{\mu\nu}_{\mathfrak{s}_2} - \Sigma^{\mu\nu}_{\mathfrak{s}} - \Sigma^{\mu\nu}_{\mathfrak{s}'})\Big\}\ .\end{aligned} \tag{3.97}$$

We note that the antisymmetric parts of the terms containing Δ in the second and fourth line can be expressed by the microscopic orbital angular momentum tensor $\mathfrak{L}^{\mu\nu} \equiv \Delta^{[\mu}p^{\nu]}$. Since similar terms proportional to the spin angular momentum $(\hbar/2)\Sigma^{\mu\nu}_{\mathfrak{s}}$ are also present, it is natural to consider the conservation of the total angular momentum $\mathcal{J}^{\mu\nu} \equiv \mathfrak{L}^{\mu\nu} + (\hbar/2)\Sigma^{\mu\nu}_{\mathfrak{s}}$ of the particles in a microscopic collision,

$$\mathcal{J}^{\mu\nu} + \mathcal{J}'^{\mu\nu} = \mathcal{J}^{\mu\nu}_1 + \mathcal{J}^{\mu\nu}_2 . \tag{3.98}$$

With this knowledge, we find that the collision term vanishes for any $\widetilde{\mathcal{W}}$, $\mathfrak{W}$ if:

$$\partial_\mu \alpha = 0 \,, \tag{3.99a}$$
$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0 \,, \tag{3.99b}$$
$$\Omega_{\mu\nu} = \varpi_{\mu\nu} \equiv -\frac{1}{2}\partial_{[\mu}\beta_{\nu]} = \text{const.}\,. \tag{3.99c}$$

These are the conditions for global (and not just local) equilibrium. The condition (3.99c) was already obtained from statistical quantum field theory [193] and is confirmed here from kinetic theory. We also note that in previous works condition (3.99c) was found in the massless case [51], and for massive particles in the presence of an electromagnetic field [62, 121].

3.6 Entropy and H-theorem

A meaningful collision term should fulfill an H-theorem, i.e., it should be possible to define an entropy current such that the entropy production is always larger than zero or equal to zero. If the latter is the case, the system is in equilibrium. The aim of this section is to give an intuitive expression of the entropy current, show that its production rate is nonnegative and verify the equilibrium conditions found in the previous section from the vanishing of the entropy production. The proof of the H-theorem is analogous to the one for the standard collision term in the Boltzmann equation, which can be found, e.g., in Ref. [45].

First, we define the entropy current as, see also Ref. [59],

$$H^\mu \equiv -\int dP\, dS(p)\, p^\mu f\, [\ln(f) - 1]. \tag{3.100}$$

Using the Boltzmann equation (3.86), we obtain

$$\partial_\mu H^\mu \;\;=\;\; -\int dP dS(p)\; \tilde{\mathfrak{C}}_{\text{on-shell}}[f]\, \ln(f). \tag{3.101}$$

For the sake of notational simplicity, we suppress the functional dependence of the distribution function, i.e., here and in the following we denote $f \equiv f(x, p, \mathfrak{s}) \equiv f(\bar{x} + \Delta, p, \mathfrak{s})$, $f' \equiv f(\bar{x} + \Delta', p', \mathfrak{s}')$, etc, where $\bar{x}$ is defined before Eq. (3.93). There are two contributions to the collision term (3.93), we start with considering the spin-exchange term $\mathfrak{C}_\text{s}$. We note that we can write any distribution function $f(p,\mathfrak{s}) = f^{(0)}(p) + g(p, \hbar\mathfrak{s})$ with g being of at least first order in $\hbar$ according to our assumption of first-order polarization. Actually this assumption is necessary for the interpretation of $\mathfrak{f}$ as a distribution function, since for $\mathcal{A}^\mu$ of the same order as $\mathcal{F}$ we could not guarantee that $\mathfrak{f} > 0$. So we can expand

$$\ln f = \ln f^{(0)}(x,p) + \frac{g(x,p,\mathfrak{s})}{f^{(0)}(x,p)} - \frac{1}{2}\left(\frac{g(x,p,\mathfrak{s})}{f^{(0)}(x,p)}\right)^2 + \mathcal{O}(\hbar^3). \tag{3.102}$$

Furthermore we expand g in powers of $\mathfrak{s}$ as $g(x,p,\mathfrak{s}) = \hbar g^\mu(x,p)\mathfrak{s}_\mu + \mathcal{O}(\hbar^2)$ and notice that all terms of even powers in $\mathfrak{s}^\mu$ and $\mathfrak{s}_1^\mu$ do not contribute to the integral as $\mathfrak{W}$ is linear in both. We obtain

$$\begin{aligned}\int dP dS(p)\; \mathfrak{C}_\text{s}[f]\, \ln(f) \;\;&=\;\; \hbar^2 \int dP dS(p) d\Gamma_2 dS_1(p)\, \mathfrak{W}\, g(x,p)\cdot\mathfrak{s}_1\, f_2\, \frac{g(x,p)\cdot\mathfrak{s}}{f^{(0)}(x,p)} + \mathcal{O}(\hbar^4) \\ &=\;\; \mathcal{O}(\hbar^4)\,,\end{aligned} \tag{3.103}$$

where in the last step we used that $\mathfrak{W}$ is antisymmetric under exchange of $\mathfrak{s}$ and $\mathfrak{s}_1$. We conclude that the pure spin-exchange term does not contribute to the entropy production up to $\mathcal{O}(\hbar^3)$.

On the other hand we obtain for $\mathfrak{C}_{\text{p+s}}$ the following change in entropy

$$\begin{aligned}\int dP dS\; \mathfrak{C}_{\text{p+s}}[f]\, \ln(f) = &\int [d\Gamma]\, \widetilde{\mathcal{W}}_{f_1 f_2 \to f f'} \ln(f)(f_1 f_2 - f f') \\ =&\frac{1}{2}\int [d\Gamma]\, \Big[\widetilde{\mathcal{W}}_{f_1 f_2\to f f'} \ln(f)\, f_1 f_2 + \widetilde{\mathcal{W}}_{f_1 f_2 \to f' f} \ln(f')\, f_1 f_2 - \widetilde{\mathcal{W}}_{f f' \to f_1 f_2} \ln(f_1)\, f_1 f_2 \\ &-\widetilde{\mathcal{W}}_{f' f \to f_2 f_1} \ln(f_2)\, f_1 f_2\Big] \\ =&\frac{1}{2}\int [d\Gamma]\, \ln\left(\frac{f f'}{f_1 f_2}\right) \widetilde{\mathcal{W}}_{f_1 f_2 \to f f'} f_1 f_2,\end{aligned} \tag{3.104}$$

where we defined $[d\Gamma] \equiv d\Gamma d\Gamma' d\Gamma_1 d\Gamma_2$ and added the subscripts $f_1 f_2 \to f f'$ on $\widetilde{\mathcal{W}}$ for clarification. To obtain this result, we relabelled variables p, $\mathfrak{s}$, and $\Delta(p, \mathfrak{s})$ accordingly and used that $\widetilde{\mathcal{W}}_{f' f \to f_2 f_1}$ is symmetric under exchange of all variables. Due to this symmetry, we also have

$$\int [d\Gamma] \widetilde{\mathcal{W}}_{f' f \to f_2 f_1} (f_1 f_2 - f f') = 0. \tag{3.105}$$

Adding this to the above equation, we obtain

$$\partial_\mu H^\mu = \frac{1}{2} \int [d\Gamma] \left(\frac{f f'}{f_1 f_2} - \ln \frac{f f'}{f_1 f_2} - 1 \right) \widetilde{\mathcal{W}}_{f_1 f_2 \to f f'} f_1 f_2. \tag{3.106}$$

The transition probability $\widetilde{\mathcal{W}}_{f_1 f_2 \to f f'}$ and the distribution functions are always larger than zero (under the assumption of small polarization). Furthermore, we have $x - \ln x - 1 > 0$ for $x \neq 1$ and $x - \ln x - 1 = 0$ for $x = 1$. Therefore the entropy production is semipositive and vanishes if

$$\ln f(\bar{x} + \Delta, p, \mathfrak{s}) + \ln f(\bar{x} + \Delta', p', \mathfrak{s}') - \ln f(\bar{x} + \Delta_1, p_1, \mathfrak{s}_1) - \ln f(\bar{x} + \Delta_2, p_2, \mathfrak{s}_2) = 0 \tag{3.107}$$

in the integral (3.106). We again use that the charge, momentum, and total angular momentum are collisional invariants and, as in the previous section, introduce the Lagrange multipliers α, β^μ, and $\Omega^{\mu\nu}$,

$$\ln f(\bar{x} + \Delta, p, \mathfrak{s}) = \alpha(\bar{x} + \Delta) - \beta(\bar{x} + \Delta) \cdot p + \frac{\hbar}{4} \Omega_{\mu\nu}(\bar{x} + \Delta) \Sigma_{\mathfrak{s}}^{\mu\nu} . \tag{3.108}$$

Now we expand $\beta^\mu(\bar{x}+\Delta) = \beta^\mu(\bar{x}) + \Delta_\nu \partial^\nu \beta^\mu(\bar{x}) + \mathcal{O}(\hbar^2)$ and similarly for all other distribution functions. In order to fulfill Eq. (3.107) at first order, we need $\alpha =$ const. Furthermore, we obtain by using the conservation of total angular momentum in Eq. (3.98) that β^μ has to fulfill the Killing condition and $\Omega^{\mu\nu} = \varpi^{\mu\nu}$. We also note that these conditions are sufficient to fulfill Eq. (3.107) at any order, as the Killing condition implies that $\varpi^{\mu\nu}$ is constant. Hence, the conditions for the entropy production to vanish are identical to the equilibrium conditions derived in the previous section.

We conclude the findings of this chapter in the following way: In order to see spin alignment with vorticity, one has to take into account the nonlocality of the collision term, while a local collision term leads to spin diffusion effects and a vanishing polarization in equilibrium. The nonlocal collision term vanishes under the conditions of global equilibrium and the spin potential is then equal to the thermal vorticity. In Chapter 5, we will provide a more detailed interpretation of this result by considering the ordering of scales in the problem. Furthermore, we will use the achievements of this chapter to derive spin hydrodynamics from kinetic theory. Before, we will discuss the conserved currents in the presence of interactions in the next chapter.

Chapter 4

Spin and energy-momentum tensors for interacting systems

In this chapter, which is based on Ref. [194], we study the different energy-momentum and spin tensors related through pseudo-gauge transformations, which were introduced in Chapter 2 for free fields, in the interacting case. We find that the pseudo-gauge potentials are modified by the interaction term, and importantly, that the HW energy-momentum tensor, which is symmetric for free fields, acquires an antisymmetric part from the nonlocal collision term. The equations of motion for the spin tensor and energy-momentum tensor derived in this chapter provide, together with the Boltzmann equation studied in the previous chapter, the starting point to derive dissipative spin hydrodynamics in the next chapter. Furthermore, we show that the HW currents lead to well-known results in the nonrelativistic limit and compare them to the stress tensor and the internal angular momentum of micropolar fluids. Finally, we study the effects of gauge fields on the HW currents and their equations of motion.

4.1 Pseudo-gauge transformations in presence of a general interaction

4.1.1 Canonical currents

Let us first consider the canonical energy-momentum and spin tensor discussed for the free case in Section 2.2.1. Here, we use the Dirac Lagrangian (3.2) with a general interaction and assume that the interaction Lagrangian is a function of (adjoint) spinors only, but not of their derivatives, $\mathcal{L}_I = \mathcal{L}_I(\psi, \bar{\psi})$. Then, the canonical tensors are on the operator level formally still given by Eqs. (2.18) and (2.19). Using Eqs. (3.33) and (3.45), the ensemble averages can be expressed in terms of the Wigner function as, cf. Eqs. (2.58a), (2.59a),

$$
\begin{aligned}
T_C^{\mu\nu} &= \int d^4p\, p^\nu \mathcal{V}^\mu \\
&= \int dP\, dS(p)\, p^\nu \left[p^\mu + \frac{\hbar}{2}\Sigma_{\mathfrak{s}}^{\mu\lambda}\partial_\lambda + \frac{\hbar^2}{4m^2}\partial_\lambda(\partial^\mu p^\lambda - \partial^\lambda p^\mu)\right] f(x,p,\mathfrak{s}) + \frac{\hbar^2}{m}\int d^4p\, p^\nu D_{\mathcal{V}}^{(1)\mu} + \mathcal{O}(\hbar^3) \\
&= \int dP\, dS(p)\, p^\nu \left(p^\mu + \frac{\hbar}{2}\Sigma_{\mathfrak{s}}^{\mu\lambda}\partial_\lambda\right) f(x,p,\mathfrak{s}) + \mathcal{O}(\hbar^2), \\
S_C^{\lambda,\mu\nu} &= -\frac{1}{2}\,\epsilon^{\lambda\mu\nu\alpha}\int d^4p\, \mathcal{A}_\alpha \\
&= -\frac{m}{2}\,\epsilon^{\lambda\mu\nu\alpha}\int dP\, dS(p)\, \mathfrak{s}_\alpha f(x,p,\mathfrak{s}) \\
&= \frac{1}{2}\int dP\, dS(p)\, \left(p^\lambda \Sigma_{\mathfrak{s}}^{\mu\nu} + p^\mu \Sigma_{\mathfrak{s}}^{\nu\lambda} + p^\nu \Sigma_{\mathfrak{s}}^{\lambda\mu}\right) f(x,p,\mathfrak{s}),
\end{aligned}
\tag{4.1}
$$

where we used $dP \equiv d^3p/(2p^0)$, Eqs. (3.18c), (3.18d), (3.22), and in the last equality the Schouten identity (A.2). We remark that in Eqs. (4.1) we do not take into account the term proportional to $g^{\mu\nu}$ in the energy-momentum tensor and off-shell terms in the currents. Although such terms are in general not zero in the presence of interactions, they can be neglected in the low-density limit considered in this work [45].

The corresponding hydrodynamic equations are obtained from Eq. (3.86) as [41]

$$\hbar\,\partial_\mu T^{\mu\nu}_C = \hbar\int d\Gamma\, p^\nu\, \mathfrak{C}[f] + \mathcal{O}(\hbar^3) = 0, \tag{4.2}$$

$$\begin{aligned}\hbar\,\partial_\lambda S^{\lambda,\mu\nu}_C &= \int d\Gamma\,\frac{\hbar}{2}\left(\Sigma^{\mu\nu}_{\mathfrak{s}}\,\mathfrak{C}[f] + p^{[\mu}\Sigma^{\nu]\lambda}_{\mathfrak{s}}\partial_\lambda f(x,p,\mathfrak{s})\right)\\ &= T^{[\nu\mu]}_C,\end{aligned} \tag{4.3}$$

where we drop the index "on-shell" and the tilde symbol on $\mathfrak{C}$ from now on. Equation (4.2) relates the conservation of energy-momentum to p^μ being a collisional invariant. We can use this equation in the second line of Eq. (4.1) to obtain

$$T^{\mu\nu}_C = \int dP\,dS(p)\,p^\nu\left[p^\mu\left(1-\frac{\hbar^2}{4m^2}\partial^2\right) + \frac{\hbar}{2}\Sigma^{\mu\lambda}_{\mathfrak{s}}\partial_\lambda\right] f(x,p,\mathfrak{s}) + \frac{\hbar^2}{m}\int d^4p\,p^\nu D^{(1)\mu}_{\mathcal{V}} + \mathcal{O}(\hbar^3). \tag{4.4}$$

Comparing the antisymmetric part of this equation to Eq. (4.3), we see that the nonconservation of $\Sigma^{\mu\nu}_{\mathfrak{s}}$ in a collision comes only from the interaction contribution $D^\mu_{\mathcal{V}}$. However, as the right-hand side of Eq. (4.3) contains the additional term $p^{[\mu}\Sigma^{\nu]\lambda}_{\mathfrak{s}}\partial_\lambda f$, the divergence of the canonical spin tensor is not proportional to the change of spin during a collision, given by the term $\Sigma^{\mu\nu}_{\mathfrak{s}}\,\mathfrak{C}[f]$. This problem of the canonical set of tensors becomes even more apparent when we plug the global-equilibrium solution (3.94) into Eq. (4.1) or (4.3). We then obtain an antisymmetric part of the canonical energy-momentum tensor even if the interaction contribution $D^\mu_{\mathcal{V}}$ is not taken into account [41],

$$T^{[\mu\nu]}_{C,eq} = \frac{1}{(2\pi\hbar)^3}\frac{\hbar^2}{2}\int dP\,p^{[\nu}\varpi^{\mu]\lambda}p^\rho\varpi_{\lambda\rho}e^{\alpha-\beta\cdot p} + \mathcal{O}(\hbar^3). \tag{4.5}$$

Due to the nonzero antisymmetric part of the energy-momentum tensor, it follows from Eq. (4.3) that the spin tensor is not conserved, even in the case of global equilibrium where $\mathfrak{C}[\mathfrak{f}] = 0$, if the system is rigidly rotating. This is not consistent with the interpretation of $S^{\lambda,\mu\nu}_C$ as a spin density, as the physical picture is that spin changes only due to particle scatterings until global equilibrium is reached. In order to have a meaningful physical interpretation of the spin tensor as the spin density of the system, we aim at finding a pseudo-gauge transformation which exactly removes this additional term and relates the antisymmetric part of the energy-momentum tensor only to the conversion of spin and orbital angular momentum into one another. In the next sections, we will see that this is solved by using the HW or KG currents, which we will derive in the presence of a general interaction by a pseudo-gauge transformation from the canonical currents (4.1).

4.1.2 HW currents

We will see in Section 4.2 that the following extension of the HW pseudo-gauge transformation (2.41) for the free case to the interacting case yields a physically meaningful result,

$$\begin{aligned}\hat{\Phi}^{\lambda,\mu\nu}_{HW} &= \hat{M}^{[\mu\nu]\lambda} - g^{\lambda[\mu}\hat{M}_\rho{}^{\nu]\rho} + \hat{Q}^{\lambda\mu\nu},\\ \hat{Z}^{\mu\nu\lambda\rho}_{HW} &= -\frac{1}{8m}\bar{\psi}(\sigma^{\mu\nu}\sigma^{\lambda\rho} + \sigma^{\lambda\rho}\sigma^{\mu\nu})\psi,\end{aligned} \tag{4.6}$$

with

$$\hat{Q}^{\lambda\mu\nu} \equiv -\frac{\hbar}{4m}\bar{\rho}\gamma^\lambda\sigma^{\mu\nu}\psi - \frac{\hbar}{4m}\bar{\psi}\sigma^{\mu\nu}\gamma^\lambda\rho. \tag{4.7}$$

We remark that, performing a Wigner transformation and taking the ensemble average, we can write, cf. Eqs. (2.40) and (2.59c),

$$M^{\lambda\mu\nu} \equiv \frac{1}{2m}\int d^4p\,p^\lambda\mathcal{S}^{\mu\nu}. \tag{4.8}$$

In order to find the expressions for the HW currents in the interacting case, we note that with the identity (A.3a) the interacting Dirac equation (3.3) and its adjoint can be written as

$$i\hbar\partial^\lambda\psi = -\,\hbar\sigma^{\lambda\mu}\partial_\mu\psi + m\gamma^\lambda\psi + \hbar\gamma^\lambda\rho, \tag{4.9a}$$

$$-i\hbar\partial^\lambda\bar{\psi} = -\,\hbar\partial_\mu\bar{\psi}\sigma^{\lambda\mu} + m\bar{\psi}\gamma^\lambda + \hbar\bar{\rho}\gamma^\lambda. \tag{4.9b}$$

Using Eqs. (4.9) we generalize the Gordon decomposition (2.35) for interacting fields as

$$\bar\psi\gamma^\mu\psi = \frac{i\hbar}{2m}\left[\bar\psi\overleftrightarrow{\partial}^\mu\psi - i\left(\bar\psi\sigma^{\mu\nu}\partial_\nu\psi + \partial_\nu\bar\psi\sigma^{\mu\nu}\psi\right)\right] - \frac{\hbar}{2m}\left(\bar\psi\gamma^\lambda\rho + \bar\rho\gamma^\lambda\psi\right). \tag{4.10}$$

Now we insert Eq. (4.6) into the pseudo-gauge transformation (2.15) and apply the identity (4.10). We obtain for the spin tensor

$$\begin{aligned}
\hat S^{\lambda,\mu\nu}_{HW} &= \frac{1}{4}\bar\psi\{\gamma^\lambda,\sigma^{\mu\nu}\}\psi + \frac{i\hbar}{4m}\left(\bar\psi\overleftrightarrow{\partial}^{[\nu}\sigma^{\mu]\lambda}\psi - \bar\psi g^{\lambda[\nu}\sigma^{\mu]\rho}\overleftrightarrow{\partial}_\rho\psi\right)\\
&\quad - \frac{\hbar}{4m}\left[(\partial_\rho\bar\psi\sigma^{\lambda\rho} - \bar\rho\gamma^\lambda)\sigma^{\mu\nu}\psi + \bar\psi\sigma^{\mu\nu}(\sigma^{\lambda\rho}\partial_\rho\psi - \gamma^\lambda\rho)\right] + \frac{\hbar}{8m}\bar\psi[\sigma^{\mu\nu},\sigma^{\lambda\rho}]\overleftrightarrow{\partial}_\rho\psi\\
&= \frac{1}{4}\bar\psi\{\gamma^\lambda,\sigma^{\mu\nu}\}\psi + \frac{i\hbar}{4m}\left(\bar\psi\overleftrightarrow{\partial}^{[\nu}\sigma^{\mu]\lambda}\psi - \bar\psi g^{\lambda[\nu}\sigma^{\mu]\rho}\overleftrightarrow{\partial}_\rho\psi\right)\\
&\quad - \frac{1}{4m}\left[(m\bar\psi\gamma^\lambda + i\hbar\partial^\lambda\bar\psi)\sigma^{\mu\nu}\psi + \bar\psi\sigma^{\mu\nu}(m\gamma^\lambda - i\hbar\partial^\lambda)\psi\right] + \frac{i\hbar}{4m}\left(\bar\psi\overleftrightarrow{\partial}^{[\mu}\sigma^{\nu]\lambda}\psi + \bar\psi g^{\lambda[\nu}\bar\psi\sigma^{\mu]\rho}\overleftrightarrow{\partial}_\rho\psi\right)\\
&= \frac{i\hbar}{4m}\bar\psi\sigma^{\mu\nu}\overleftrightarrow{\partial}^\lambda\psi,
\end{aligned} \tag{4.11}$$

where we also used Eq. (A.3e). Therefore, the spin tensor in terms of the Wigner function reads

$$S^{\lambda,\mu\nu}_{HW} = \frac{1}{2m}\int d^4p\, p^\lambda\mathcal{S}^{\mu\nu}, \tag{4.12}$$

which has the same form as in the non-interacting case. For the energy-momentum tensor, let us first consider the following term appearing in Eq. (2.15) after inserting Eq. (4.6),

$$\begin{aligned}
&T^{\mu\nu}_C - \hbar\partial_\lambda\left(M^{\nu\mu\lambda} + g^{\nu[\mu}M_\rho{}^{\lambda]\rho}\right)\\
&= \int d^4p\, p^\nu\mathcal{V}^\mu - \frac{\hbar}{2m}\int d^4p\,\partial_\lambda\left(p^\nu\mathcal{S}^{\mu\lambda} + g^{\nu[\mu}\mathcal{S}^{\lambda]\rho}p_\rho\right)\\
&= \frac{1}{m}\int d^4p\left[p^\nu\left(p^\mu\mathcal{F} - \hbar D^\mu_\mathcal{V}\right) + \frac{\hbar^2}{4}\partial^\nu\left(\partial^\mu\mathcal{F} - 2C^\mu_\mathcal{V}\right) - \frac{\hbar^2}{4}g^{\mu\nu}\left(\partial^2\mathcal{F} - 2\,\partial\cdot C_\mathcal{V}\right)\right],
\end{aligned} \tag{4.13}$$

where we used Eqs. (3.18c) and (3.19c). With the identity (A.3b) we furthermore calculate

$$\begin{aligned}
&\partial_\lambda(Q^{\lambda\mu\nu} + Q^{\nu\mu\lambda} + Q^{\mu\nu\lambda})\\
&= -\frac{\hbar}{4m}\partial_\lambda\left\langle\left[\bar\rho\left(2ig^{\nu[\mu}\gamma^{\lambda]} + \epsilon^{\lambda\mu\nu\alpha}\gamma^5\gamma_\alpha\right)\psi + \bar\psi\left(-2ig^{\nu[\mu}\gamma^{\lambda]} + \epsilon^{\lambda\mu\nu\alpha}\gamma^5\gamma_\alpha\right)\rho\right]\right\rangle\\
&= -\frac{\hbar}{m}\partial_\lambda g^{\nu[\mu}\mathrm{Im}\left\langle\bar\psi\gamma^{\lambda]}\rho\right\rangle + \frac{\hbar}{2m}\epsilon^{\lambda\mu\nu\alpha}\partial_\lambda\mathrm{Re}\left\langle\bar\psi\gamma_\alpha\gamma^5\rho\right\rangle\\
&= -\frac{\hbar}{m}\int d^4p\left(g^{\mu\nu}\partial\cdot C_\mathcal{V} - \partial^\nu C^\mu_\mathcal{V} - \frac{1}{2}\epsilon^{\lambda\mu\nu\alpha}\partial_\lambda D_{\mathcal{A}\alpha}\right),
\end{aligned} \tag{4.14}$$

where we performed a Wigner transformation in the last step. Combining Eqs. (4.13) and (4.14) yields

$$T^{\mu\nu}_{HW} = \frac{1}{m}\int d^4p\left[p^\nu\left(p^\mu\mathcal{F} - \hbar D^\mu_\mathcal{V}\right) + \frac{\hbar^2}{4}\partial^\nu\partial^\mu\mathcal{F} - \frac{\hbar^2}{4}g^{\mu\nu}\partial^2\mathcal{F} + \frac{\hbar^2}{4}\epsilon^{\lambda\mu\nu\alpha}\partial_\lambda D_{\mathcal{A}\alpha}\right]. \tag{4.15}$$

We see that there is an antisymmetric part of the HW energy-momentum which is at least of second order in $\hbar$ and vanishes for vanishing interactions. Thus, the HW spin tensor is conserved for free fields, but in general not in the presence of interactions.

4.1.3 GLW and KG currents

In the interacting case, the GLW energy-momentum and spin tensors introduced in Ref. [45] and discussed in Section 2.2.4 can be obtained from a pseudo-gauge transformation with

$$\begin{aligned}
\Phi^{\lambda,\mu\nu}_{GLW} &= \frac{1}{2m}\int d^4p\, p^{[\mu}\mathcal{S}^{\nu]\lambda},\\
Z^{\mu\nu\lambda\rho}_{GLW} &= 0,
\end{aligned} \tag{4.16}$$

which is formally not changed in comparison to the free case in Eq. (2.45). The result is

$$\begin{aligned}
T^{\mu\nu}_{GLW} =& \int d^4p \left(p^\nu \mathcal{V}^\mu + \frac{\hbar}{2m} p^\nu \partial_\lambda \mathcal{S}^{\lambda\mu} \right) \\
=& \frac{1}{m} \int d^4p\, p^\nu \left(p^\mu \mathcal{F} - \hbar D^\mu_{\mathcal{V}} \right) ,
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
S^{\lambda,\mu\nu}_{GLW} =& -\frac{1}{2} \epsilon^{\lambda\mu\nu\alpha} \int d^4p\, \mathcal{A}_\alpha - \frac{1}{2m} \int d^4p\, p^{[\nu} \mathcal{S}^{\mu]\lambda} \\
=& \frac{1}{2m} \int d^4p \left[p^\lambda \mathcal{S}^{\mu\nu} - \hbar \epsilon^{\lambda\mu\nu\alpha} \left(\frac{1}{2} \partial_\alpha \mathcal{P} - D_{\mathcal{A}\alpha} \right) \right] .
\end{aligned} \tag{4.18}$$

Here, we used Eq. (3.18c) in the second equality and Eq. (3.18d) in the last equality. As $\mathcal{P}$ and $D^\alpha_{\mathcal{A}}$ are of first order in $\hbar$, the HW and GLW currents are identical up to first order in $\hbar$ and can be used equivalently at this level of accuracy.

The GLW spin tensor (4.18) contains a contribution proportional to $\partial_\alpha \mathcal{P}$, the physical interpretation of which is not clear. We note that this term is removed by introducing the pseudo-gauge potential $Z^{\mu\nu\lambda\rho}$ in the KG pseudo-gauge transformation given in Eq. (2.49b),

$$Z^{\mu\nu\lambda\rho}_{KG} \equiv \frac{1}{4m} \epsilon^{\mu\nu\lambda\rho} \int d^4p\, \mathcal{P} . \tag{4.19}$$

The KG pseudo-gauge transformation from the free case (2.49) can be used to obtain a set of tensors which contains an energy-momentum tensor identical to the GLW version, Eq. (4.17), and reduces to Eqs. (2.48) for vanishing interactions. To see this for the spin tensor, we use Eqs. (4.9) or the Gordon decomposition (4.10) in the canonical spin tensor

$$\begin{aligned}
\hat{S}^{\lambda,\mu\nu}_C =& \frac{1}{4} \bar{\psi} \{ \gamma^\lambda, \sigma^{\mu\nu} \} \psi \\
=& \frac{1}{4m} \left(-i\hbar \partial^\lambda \bar{\psi} + \hbar \partial_\rho \bar{\psi} \sigma^{\lambda\rho} - \hbar \bar{\rho} \gamma^\lambda \right) \sigma^{\mu\nu} \psi + \frac{1}{4m} \bar{\psi} \sigma^{\mu\nu} \left(i\hbar \partial^\lambda \psi + \hbar \sigma^{\lambda\rho} \partial_\rho \psi - \hbar \gamma^\lambda \rho \right) \\
=& \frac{i\hbar}{4m} \bar{\psi} \sigma^{\mu\nu} \overleftrightarrow{\partial}^\lambda \psi - \frac{\hbar}{4m} \left(\bar{\rho} \gamma^\lambda \sigma^{\mu\nu} \psi + \bar{\psi} \sigma^{\mu\nu} \gamma^\lambda \rho \right) + \frac{\hbar}{4m} \partial_\rho \bar{\psi} (i\gamma^5 \epsilon^{\lambda\mu\nu\rho} + i g^{\lambda[\mu} \sigma^{\nu]\rho} - i g^{\rho[\mu} \sigma^{\nu]\lambda} - g^{\lambda[\nu} g^{\mu]\rho}) \psi \\
&+ \frac{\hbar}{4m} \bar{\psi} (i\gamma^5 \epsilon^{\mu\lambda\rho\nu} + i g^{\mu[\lambda} \sigma^{\rho]\nu} - i g^{\nu[\lambda} \sigma^{\rho]\mu} - g^{\mu[\rho} g^{\lambda]\nu}) \partial_\rho \psi \\
=& \frac{i\hbar}{4m} \bar{\psi} \sigma^{\mu\nu} \overleftrightarrow{\partial}^\lambda \psi - \frac{\hbar}{4m} \left(\bar{\rho} \gamma^\lambda \sigma^{\mu\nu} \psi + \bar{\psi} \sigma^{\mu\nu} \gamma^\lambda \rho \right) + \partial_\rho \frac{i\hbar}{4m} \epsilon^{\lambda\mu\nu\rho} \bar{\psi} \gamma^5 \psi \\
&+ \frac{\hbar}{4m} \left[\partial_\rho \bar{\psi} \gamma^\rho g^{\lambda[\mu} \gamma^{\nu]} \psi + g^{\lambda[\mu} \bar{\psi} \gamma^{\nu]} \gamma^\rho \partial_\rho \psi \right] - \frac{i\hbar}{4m} \bar{\psi} \left(\sigma^{\lambda\nu} \overleftrightarrow{\partial}^\mu - \sigma^{\lambda\mu} \overleftrightarrow{\partial}^\nu \right) \psi \\
=& \hat{S}^{\lambda,\mu\nu}_{KG} + \hat{\Phi}^{\lambda,\mu\nu}_{KG} - \hbar\, \partial_\rho \hat{Z}^{\mu\nu\lambda\rho}_{KG} ,
\end{aligned} \tag{4.20}$$

with

$$\begin{aligned}
\hat{S}^{\lambda,\mu\nu}_{KG} =& \frac{i\hbar}{4m} \bar{\psi} \sigma^{\mu\nu} \overleftrightarrow{\partial}^\lambda \psi - \frac{\hbar}{4m} \left(\bar{\rho} \gamma^\lambda \sigma^{\mu\nu} \psi + \bar{\psi} \sigma^{\mu\nu} \gamma^\lambda \rho \right) + \frac{i\hbar}{4m} g^{\lambda[\mu} \left(\bar{\rho} \gamma^{\nu]} \psi - \bar{\psi} \gamma^{\nu]} \rho \right) \\
=& \frac{i\hbar}{4m} \bar{\psi} \sigma^{\mu\nu} \overleftrightarrow{\partial}^\lambda \psi + \frac{\hbar}{2m} \epsilon^{\lambda\mu\nu\rho} \mathrm{Re}\, \bar{\psi} \gamma_\rho \gamma^5 \rho ,
\end{aligned} \tag{4.21}$$

where in Eq. (4.20) we used Eqs. (A.3b) and (A.3c), as well as Eqs. (4.9) in the last step. Therefore we have

$$S^{\lambda,\mu\nu}_{KG} = \frac{1}{2m} \int d^4p \left(p^\lambda \mathcal{S}^{\mu\nu} + \hbar \epsilon^{\lambda\mu\nu\rho} D_{\mathcal{A}\rho} \right) . \tag{4.22}$$

The form of $T^{\mu\nu}_{KG}$ is obviously identical to Eq. (4.17), since it is independent of $Z^{\mu\nu\lambda\rho}_{KG}$.

4.2 Choice of energy-momentum and spin tensor, equations of motion

Due to the mentioned advantages that the HW tensors have compared to the canonical tensors regarding their physical interpretation, we choose to use the HW tensors in the following. We remark that, since we work up to first order in $\hbar$, in this case the GLW and KG currents are identical and could be used with the same results. To summarize, our energy-momentum tensor (4.15) and spin tensor (4.12) reads up to first order in $\hbar$ [57]

$$T^{\mu\nu}_{HW} = \int dP\, dS(p)\, p^\mu p^\nu f(x,p,\mathfrak{s}) + \mathcal{O}(\hbar^2)\ , \tag{4.23a}$$

$$S^{\lambda,\mu\nu}_{HW} = \int dP\, dS(p) p^\lambda \left(\frac{1}{2}\Sigma^{\mu\nu}_{\mathfrak{s}} - \frac{\hbar}{4m^2} p^{[\mu}\partial^{\nu]}\right) f(x,p,\mathfrak{s}) + \mathcal{O}(\hbar^2)\ , \tag{4.23b}$$

where we used Eq. (3.18e) with $D^{\mu\nu}_{S} \sim \mathcal{O}(\hbar)$ and Eqs. (3.33). The equations of motion for these tensors are obtained using the Boltzmann equation (3.47) and performing a transformation (3.76). They read

$$\partial_\mu T^{\mu\nu}_{HW} = \int d\Gamma\, p^\nu\, \mathfrak{C}[f] = 0\ , \tag{4.24a}$$

$$\hbar\, \partial_\lambda S^{\lambda,\mu\nu}_{HW} = \int d\Gamma\, \frac{\hbar}{2}\Sigma^{\mu\nu}_{\mathfrak{s}}\, \mathfrak{C}[f] + \mathcal{O}(\hbar^3) = T^{[\nu\mu]}_{HW}\ . \tag{4.24b}$$

In order to obtain Eq. (4.24b), we used Eq. (4.24a) to show that the derivative of the second term in Eq. (4.23b) vanishes. It should be noted that the right-hand side of Eq. (4.24a) vanishes due to the energy-momentum conservation in a binary collision. Furthermore, the nonzero contribution to the right-hand side of Eq. (4.24b) comes only from the conversion between spin and orbital angular momentum in collisions; the additional term, which was present in the canonical case in Eq. (4.3) is removed. This means that the antisymmetric part of the HW energy-momentum tensor is nonzero in the presence of nonlocal collisions, as these allow for the conversion between spin and orbital angular momentum. We note that up to $\mathcal{O}(\hbar)$, the HW energy-momentum tensor is symmetric and the antiysmmetric contribution arises at higher orders. On the other hand, if the collision term is local, the HW energy-momentum tensor is symmetric at any order, since in this case there is no orbital angular momentum in a collision and hence the dipole-moment tensor itself is a collision invariant [59]. Then, the right-hand side of Eq. (4.24b) is zero.

In global equilibrium the collision term vanishes and, as a consequence, the HW energy-momentum tensor is symmetric. However, out of global equilibrium, the nonzero nonlocal collision term always leads to dissipation. In this sense, one can assume "ideal spin hydrodynamics" only if the nonlocality of a microscopic collision is neglected. Inserting Eq. (3.94) into Eq. (4.23) we obtain for the spin tensor in global equilibrium

$$\begin{aligned} S^{\lambda,\mu\nu}_{HW,eq}(x) &= \frac{\hbar}{2m^2}\int dP\, p^\lambda \left[-\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} p_\rho \varpi_{\alpha\beta}\epsilon^{\alpha\beta\gamma}{}_\sigma\, p_\gamma + p^{[\mu}(\partial^{\nu]}\beta^\rho)p_\rho\right] f^{(0)}_{eq}(x,p) + \mathcal{O}(\hbar^2) \\ &= \frac{\hbar}{2}\varpi^{\mu\nu}\int dP\, p^\lambda f^{(0)}_{eq}(x,p) + \mathcal{O}(\hbar^2) \\ &= \frac{\hbar}{2} n_0 u^\lambda \varpi^{\mu\nu} + \mathcal{O}(\hbar^2)\ , \end{aligned} \tag{4.25}$$

where $f^{(0)}_{eq}$ is the zeroth order in $\hbar$ of Eq. (3.94) and $n_0 \equiv \int dP\, p\cdot u f^{(0)}_{eq}(x,p)$ is the zeroth-order particle density in equilibrium. In order to obtain Eq. (4.25), we first used that $\partial_\mu \alpha = 0$ and $\Omega^{\mu\nu} = \varpi^{\mu\nu}$ =const. in global equilibrium, then computed the spin integrals with the help of Eqs. (3.40) and in the second step contracted the epsilon tensors and used that $\varpi_{\mu\nu} \equiv -(1/2)\partial_{[\mu}\beta_{\nu]} = \partial_\nu\beta_\mu$. This form of the HW spin tensor in equilibrium corresponds to the "phenomenological" form of the spin tensor used in Refs. [37, 38].

Since the HW spin tensor is conserved in equilibrium, we expect that its dynamics survives on hydrodynamic time scales and its components will give the most important contributions to a moment expansion of the distribution function (in contrast to, e.g. the components of the canonical spin tensor, which is not conserved even in global equilibrium.) As will become clear later in Chapter 5.7, the choice of dynamical quantities in hydrodynamics is different in different pseudo-gauges, hence the pseudo-gauge choice affects the evolution of the system.

4.3 Nonrelativistic limit

In this section, we show that the HW tensors in combination with the non-local collisions yield the correct non-relativistic limit [57]. In order to do so, we consider the limit $p^\mu \to m(1,\mathbf{v})$ with particle velocity $\mathbf{v}$. The dipole-moment tensor (3.42) then reads

$$\Sigma^{ij}_{\mathfrak{s}} = -\frac{1}{m}(\epsilon^{ij0k}p_0\mathfrak{s}_k + \epsilon^{ijk0}p_k\mathfrak{s}_0) = -\frac{1}{m}\left(\epsilon^{ijk}p_0\mathfrak{s}_k - \epsilon^{ijk}\frac{p_k}{p^0}\mathfrak{s}\cdot\mathbf{p}\right) \to \epsilon^{ijk}\mathfrak{s}^k + \mathcal{O}(v^2). \tag{4.26}$$

With this, the spatial components of the antisymmetric part of the energy-momentum tensor according to Eq. (4.24b) are given by

$$\begin{aligned} T^{[ji]}_{HW} &\simeq \frac{\hbar}{2}\int dP\, dS\, \epsilon^{ijk}\mathfrak{s}^k \mathfrak{C}[f] \\ &= \frac{\hbar}{2}\epsilon^{ijk}\int dP\, dS\, \mathfrak{s}^k p\cdot\partial f(x,p,\mathfrak{s}) \\ &\simeq m\epsilon^{ijk}\partial^0\left\langle\frac{\hbar}{2}\mathfrak{s}^k\right\rangle_{v\mathfrak{s}} + m\epsilon^{ijk}\partial^l\left\langle v^l\frac{\hbar}{2}\mathfrak{s}^k\right\rangle_{v\mathfrak{s}} . \end{aligned} \tag{4.27}$$

Here, we used the Boltzmann equation and defined

$$\langle a(x,\mathbf{v},\mathfrak{s})\rangle_{v\mathfrak{s}} \equiv \frac{m^2}{2\pi\sqrt{3}}\int d^3v\, d^3\mathfrak{s}\, \delta\left(\mathfrak{s}^2-3\right) a(x,\mathbf{v},\mathfrak{s}) f(x,\mathbf{v},\mathfrak{s}) \tag{4.28}$$

for any function $a(x,\mathbf{v},\mathfrak{s})$. The result in Eq. (4.27) agrees up to a constant (which can be absorbed into the definition of the distribution function) with Eq. (12.11) in Ref. [69].

Furthermore, we compare to the results obtained for micropolar fluids [60]. Here we are interested in Eq. (2.2.9) in Ref. [60], in the absence of external fields given by

$$\rho\left(\partial^0 + u^j\partial^j\right)\ell^i = \partial^j C^{ji} + \epsilon^{ijk}T^{jk}, \tag{4.29}$$

where ρ is the mass density, $\boldsymbol{\ell}$ is the internal angular momentum, C^{ji} is the so-called couple stress tensor and T^{jk} is the conventional stress tensor. With the help of the continuity equation

$$\partial^0\rho + \partial^i(\rho u^i) = 0 \tag{4.30}$$

we can write

$$\partial^0(\rho l^i) = \partial^j(C^{ij} - \rho\, u^j l^i) + \epsilon^{ijk}T^{jk}. \tag{4.31}$$

From the comparison of this equation to

$$\frac{\hbar}{2}m\partial^0\left\langle\mathfrak{s}^i\right\rangle_{v\mathfrak{s}} = -\frac{\hbar}{2}\partial^j\left\langle\mathfrak{s}^i p^j\right\rangle_{v\mathfrak{s}} - \epsilon^{ijk}T^{jk}_{HW}, \tag{4.32}$$

which is derived from Eq. (4.27), we identify

$$\begin{aligned} \frac{\hbar}{2}m\left\langle\mathfrak{s}^i\right\rangle_{v\mathfrak{s}} &= \rho\,\ell^i, \\ C^{ji} &= -\left\langle\frac{\hbar}{2}\mathfrak{s}^i p^j\right\rangle_{v\mathfrak{s}} + \frac{\hbar}{2}m\left\langle\mathfrak{s}^i\right\rangle_{v\mathfrak{s}} u^j, \\ T^{ij} &= -T^{ij}_{HW}. \end{aligned} \tag{4.33}$$

The internal angular momentum is thus proportional to the average of the microscopic spin vector, and the couple stress tensor has the form of a "stress" arising from the "coupling" between momentum and spin.

4.4 Including electromagnetic fields

So far, we have considered a general effective collision term, but no gauge-field interactions. However, it is also of interest to study the spin tensor and the energy-momentum tensor of a system interacting through classical electromagnetic fields. This is important for the derivation of spin magneto-hydrodynamics. In this section, we perform a pseudo-gauge transformation to obtain the KG currents in the presence of electromagnetic fields. (The HW and GLW currents can be computed analogously.) For simplicity, we treat these elecromagnetic fields as classical and set the collision term to zero. We also absorb the electromagnetic charge e in the definition of the gauge potential and set $\mu_0 = 1$.

In the presence of gauge fields, the Wigner function has to be complemented with a gauge link

$$U(x_1,x_2) = \exp\left[-\frac{i}{\hbar}y^\mu\int_{-1/2}^{1/2} dt\, \mathbb{A}_\mu\left(x+ty\right)\right] \tag{4.34}$$

in order to ensure gauge invariance, where $\mathbb{A}^\mu(x)$ is the electromagnetic gauge potential. The definition of the Wigner function is then given by [61,98],

$$W_{\alpha\beta}(x,p) = \int \frac{d^4y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}p\cdot y} \left\langle \bar{\psi}_\beta\left(x_1\right) U(x_1,x_2) \psi_\alpha\left(x_2\right)\right\rangle \, , \tag{4.35}$$

and obeys the equation of motion

$$\left[\gamma\cdot\left(\pi + \frac{i\hbar}{2}\mathcal{D}\right)\right] W = 0\, . \tag{4.36}$$

Here, we defined the operators

$$\mathcal{D}^\mu \equiv \partial^\mu - j_0\left(\frac{\hbar}{2}\partial\cdot\partial_p\right) F^{\mu\nu}\partial_{p\nu} \tag{4.37}$$

and

$$\pi^\mu \equiv p^\mu - \frac{\hbar}{2} j_1\left(\frac{\hbar}{2}\partial\cdot\partial_p\right) F^{\mu\nu}\partial_{p\nu}, \tag{4.38}$$

where $j_0(x)\equiv \sin x/x$ and $j_1(x)\equiv(\sin x - x\cos x)/x^2$ are spherical Bessel functions and $F^{\mu\nu}\equiv\partial^{[\mu}\mathbb{A}^{\nu]}$ is the electromagnetic field-strength tensor. The space-time derivatives in the arguments of the spherical Bessel functions act only on the field-strength tensor, but not on the Wigner function. The equations of motion for the Wigner-function components read in this case [61,62]

$$\begin{aligned}
\pi^\mu\mathcal{V}_\mu - m\mathcal{F} &= 0\, , & (4.39a)\\
\frac{1}{2}\hbar\mathcal{D}^\mu\mathcal{A}_\mu + m\mathcal{P} &= 0\, , & (4.39b)\\
\pi_\mu\mathcal{F} - \frac{1}{2}\hbar\mathcal{D}^\nu\mathcal{S}_{\nu\mu} - m\mathcal{V}_\mu &= 0\, , & (4.39c)\\
-\frac{1}{2}\hbar\mathcal{D}_\mu\mathcal{P} + \frac{1}{2}\epsilon_{\mu\beta\nu\sigma}\pi^\beta\mathcal{S}^{\nu\sigma} + m\mathcal{A}_\mu &= 0\, , & (4.39d)\\
\frac{1}{2}\hbar(\mathcal{D}_\mu\mathcal{V}_\nu - \mathcal{D}_\nu\mathcal{V}_\mu) - \epsilon_{\mu\nu\alpha\beta}\pi^\alpha\mathcal{A}^\beta - m\mathcal{S}_{\mu\nu} &= 0 & (4.39e)
\end{aligned}$$

for the real parts, and

$$\begin{aligned}
\frac{\hbar}{2}\mathcal{D}^\mu\mathcal{V}_\mu &= 0\, , & (4.40a)\\
\pi^\mu\mathcal{A}_\mu &= 0\, , & (4.40b)\\
\frac{1}{2}\hbar\mathcal{D}_\mu\mathcal{F} + \pi^\nu\mathcal{S}_{\nu\mu} &= 0\, , & (4.40c)\\
\pi_\mu\mathcal{P} + \frac{1}{4}\hbar\epsilon_{\mu\beta\nu\sigma}\mathcal{D}^\beta\mathcal{S}^{\nu\sigma} &= 0\, , & (4.40d)\\
(\pi_\mu\mathcal{V}_\nu - \pi_\nu\mathcal{V}_\mu) + \frac{1}{2}\hbar\epsilon_{\mu\nu\alpha\beta}\mathcal{D}^\alpha\mathcal{A}^\beta &= 0 & (4.40e)
\end{aligned}$$

for the imaginary parts. These equations were used in Ref. [62] to derive a kinetic theory with spin and electromagnetic interactions. The canonical energy-momentum and spin tensor of the whole system are given by [62]

$$\begin{aligned}
T_C^{\mu\nu} &= \int d^4p\, \mathcal{V}^\mu\left(p^\nu + \mathbb{A}^\nu\right) - F^{\mu\lambda}\partial^\nu\mathbb{A}_\lambda + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta},\\
S_C^{\lambda,\mu\nu} &= -\frac{1}{2}\epsilon^{\lambda\mu\nu\rho}\int d^4p\,\mathcal{A}_\rho - \frac{1}{\hbar}F^{\lambda[\mu}\mathbb{A}^{\nu]}.
\end{aligned} \tag{4.41}$$

We note that these currents are not gauge-invariant. Thus, we search for a set of tensors which, as in the previous sections, follows physically meaningful equations of motion, and in addition is gauge invariant. Furthermore, we would like to have a transparent splitting into matter and gauge-field part. At this point, we remark that both issues have been discussed in the literature in detail in connection with the spin of the nucleon, see, e.g., Ref. [146] for a review and Ref. [127] for a review with connections to relativistic heavy-ion collisions. However, those works do not consider the HW/KG form of the fermion currents.

In order to achieve the mentioned properties, we generalize the KG transformation to the interacting case by replacing the partial derivative with a covariant derivative in Eq. (2.49). (We could also choose to use the

HW pseudo-gauge transformation, which would yield the same result up to $\mathcal{O}(\hbar)$.) For the gauge-field part we perform a Belinfante transformation, as this is known to yield a gauge-invariant result [146]. Thus the pseudo-gauge potentials are given as

$$\Phi^{\lambda,\mu\nu} = \frac{1}{2m}\int d^4p\,\mathcal{S}^{\lambda[\mu}p^{\nu]} - \frac{1}{\hbar}F^{\lambda[\mu}\mathbb{A}^{\nu]},$$
$$Z^{\mu\nu\lambda\rho} = \frac{1}{4m}\epsilon^{\mu\nu\lambda\rho}\int d^4p\,\mathcal{P}. \tag{4.42}$$

We obtain for the spin tensor

$$\begin{aligned}S^{\lambda,\mu\nu}_{KG} =& \frac{1}{2m}\epsilon^{\lambda\mu\nu\rho}\int d^4p\left(\frac{1}{2}\epsilon_{\rho\alpha\beta\gamma}\pi^\alpha\mathcal{S}^{\beta\gamma} - \frac{\hbar}{2}\mathcal{D}_\rho\mathcal{P}\right) - \frac{1}{2m}\int d^4p\,\mathcal{S}^{\lambda[\mu}p^{\nu]} + \frac{\hbar}{4m}\epsilon^{\mu\nu\lambda\rho}\partial_\rho\int d^4p\,\mathcal{P}\\ =& \frac{1}{2m}\int d^4p\,p^\lambda\mathcal{S}^{\mu\nu},\end{aligned} \tag{4.43}$$

where we used Eq. (4.39d) and assumed vanishing boundary terms. Furthermore, we have for the energy-momentum tensor

$$\begin{aligned}T^{\mu\nu}_{KG} =& \int d^4p\,\mathcal{V}^\mu\left(p^\nu + \mathbb{A}^\nu\right) - F^{\mu\lambda}\partial^\nu\mathbb{A}_\lambda + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} + \frac{\hbar}{2}\partial_\lambda\Bigg(\frac{1}{2m}\int d^4p\,\mathcal{S}^{\lambda[\mu}p^{\nu]} - \frac{1}{\hbar}F^{\lambda[\mu}\mathbb{A}^{\nu]}\\ &+ \frac{1}{2m}\int d^4p\,\mathcal{S}^{\nu[\mu}p^{\lambda]} - \frac{1}{\hbar}F^{\nu[\mu}\mathbb{A}^{\lambda]} + \frac{1}{2m}\int d^4p\,\mathcal{S}^{\mu[\nu}p^{\lambda]} - \frac{1}{\hbar}F^{\mu[\nu}\mathbb{A}^{\lambda]}\Bigg)\\ =& \frac{1}{m}\int d^4p\left(\pi^\mu\mathcal{F} - \frac{\hbar}{2}\mathcal{D}_\lambda\mathcal{S}^{\lambda\mu}\right)p^\nu + \int d^4p\,\mathcal{V}^\mu\mathbb{A}^\nu - F^{\mu\lambda}\partial^\nu\mathbb{A}_\lambda + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\\ &+ \frac{\hbar}{2m}\partial_\lambda\int d^4p\,\mathcal{S}^{\lambda\mu}p^\nu - F^{\lambda\mu}\partial_\lambda\mathbb{A}^\nu - \mathbb{A}^\nu\int d^4p\,\mathcal{V}^\mu\\ =& \frac{1}{m}\int d^4p\left(p^\mu p^\nu\mathcal{F} + \frac{\hbar}{2}F^\nu{}_\lambda\mathcal{S}^{\lambda\mu}\right) - F^{\mu\lambda}F^\nu{}_\lambda + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta},\end{aligned} \tag{4.44}$$

where we used Eq. (4.39c), the Maxwell equation $\partial_\mu F^{\mu\nu} = J^\nu$ with the charge current

$$J_\mu \equiv \int d^4p\,\mathcal{V}_\mu \tag{4.45}$$

and again assumed that boundary terms vanish. We note that in Eqs. (4.43) and Eq. (4.44) we did not employ any expansion in powers of $\hbar$, however, higher than linear orders in momentum derivatives appearing in π^μ and $\mathcal{D}^\mu$ vanish after integrating by parts. The set of tensors obtained in this way is gauge invariant and the antisymmetric part of the energy-momentum tensor reads

$$T^{[\mu\nu]}_{KG} = \frac{\hbar}{2m}\int d^4p\,\mathcal{S}^{\lambda[\mu}F^{\nu]}{}_\lambda. \tag{4.46}$$

We separate the currents into fluid parts and electromagnetic parts as

$$\begin{aligned}T^{\mu\nu}_f &= \frac{1}{m}\int d^4p\left(p^\mu p^\nu\mathcal{F} + \frac{\hbar}{2}F^\nu{}_\lambda\mathcal{S}^{\lambda\mu}\right),\\ T^{\mu\nu}_{em} &= -F^{\mu\lambda}F^\nu{}_\lambda + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta},\\ S^{\lambda,\mu\nu}_f &= \frac{1}{2m}\int d^4p\,p^\lambda\mathcal{S}^{\mu\nu},\\ S^{\lambda,\mu\nu}_{em} &= 0.\end{aligned} \tag{4.47}$$

In other words, we treat only the fluid spin as independent dynamical variable and the spin of the electromagnetic fields as a background contribution, which is absorbed into the energy-momentum tensor by a Belinfante pseudo-gauge transformation. The equation of motion for the fluid energy-momentum tensor reads

$$
\begin{aligned}
\partial_\mu T_f^{\mu\nu} =& \frac{1}{m}\int d^4p\, p^\mu p^\nu F_{\mu\lambda}\partial_p^\lambda \mathcal{F} + \frac{\hbar}{6m}\int d^4p\, p^\mu p^\nu (\partial^\alpha F^{\rho\lambda})\partial_{p\lambda}\partial_{p\alpha}\mathcal{S}_{\rho\mu} \\
&+ \frac{\hbar}{2m}(\partial_\mu F^\nu{}_\lambda)\int d^4p\, S^{\lambda\mu} + \frac{\hbar}{2m}\int d^4p\, F^\nu{}_\lambda \partial_\mu \mathcal{S}^{\lambda\mu} \\
=& -\frac{1}{m}\int d^4p\, F^{\mu\nu}\left(p_\mu \mathcal{F} + \frac{\hbar}{2}\mathcal{D}^\lambda \mathcal{S}_{\mu\lambda}\right) + \frac{\hbar}{6m}\int d^4p\, (\mathcal{S}_{\rho\mu}\partial^\nu F^{\rho\mu} + \mathcal{S}_{\rho\mu}\partial^\mu F^{\rho\nu}) - \frac{\hbar}{4m}(\partial^\nu F^{\lambda\mu})\int d^4p\, S_{\lambda\mu} \\
=& -F^{\mu\nu}J_\mu, \qquad (4.48)
\end{aligned}
$$

where we used Eqs. (4.39c) and (4.40c), and the Maxwell relation $\partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} + \partial^\lambda F^{\mu\nu} = 0$. Furthermore, we have

$$
\partial_\mu T_{em}^{\mu\nu} = F^{\mu\nu}J_\mu = -\partial_\mu T_f^{\mu\nu}, \qquad (4.49)
$$

and

$$
\hbar\,\partial_\lambda S_f^{\lambda,\mu\nu} = -\frac{\hbar}{2m}\int d^4p\, \mathcal{S}^{\lambda[\mu}F^{\nu]}{}_\lambda = T_f^{[\nu\mu]}. \qquad (4.50)
$$

As expected, the total energy-momentum tensor is conserved, but not the spin tensor.

It is interesting to observe the similarity between the above derived currents and those obtained in Ref. [63] for fluids with polarization, the former being exact in $\hbar$ and the latter purely classical, where $\mathcal{S}^{\mu\nu}$ plays the role of the dipole moment [62]. In particular, we find from Eq. (4.50) that $s_{HW}^{\mu\nu}$ defined in Eq. (2.68) follows the Matthison-Papapetrou-Dixon equation [63, 133]

$$
\begin{aligned}
m\frac{d}{d\tau_\star}s_{HW}^{\mu\nu} =& \frac{1}{2m}\int d\Sigma_\lambda p^\lambda p^\rho \partial_\rho \mathcal{S}^{\mu\nu} \\
=& -\frac{1}{2m}\int d\Sigma_\lambda p^\lambda \mathcal{S}^{\rho[\mu}F^{\nu]}{}_\rho \\
=& -s_{HW}^{\rho[\mu}F^{\nu]}{}_\rho\,, \qquad (4.51)
\end{aligned}
$$

where $\tau_\star \equiv x_\star^0$ is again the proper time.

If one neglects the nonlocality of the collision term, the distribution function (3.94) is a local-equilibrium solution for any (space-time dependent) α, β^μ, and $\Omega^{\mu\nu}$. In this case, the energy-momentum tensor, charge current, and spin tensor can be expressed in terms of these dynamical fields through Eqs. (4.43), (4.44) and (4.45), respectively. The system of equations of motion (4.48) and (4.50), together with the conservation of charge and Maxwell's equations, is then closed and provides the starting point to solve ideal spin magneto-hydrodynamics. On the other hand, if we want to include more degrees of freedom in the form of dissipative currents, as is needed in the presence of nonlocal collisions, we have to derive additional equations to close the system. This can be done, e.g., by using the method of moments as detailed in the next chapter. In the presence of electromagnetic fields, the Boltzmann equation features additional terms describing the motion of a spinning particle and its spin precession in an electromagnetic field. Combining the results of Ref. [62] to a Boltzmann equation for the scalar distribution function yields

$$
\begin{aligned}
&\Big\{p^\mu \partial_{x\mu} + \left[F^{\mu\nu}p_\nu + \frac{1}{2}\hbar(\partial^\mu \tilde{F}^{\alpha\beta})p_\alpha \mathfrak{s}_\beta\right]\partial_{p\mu} + F^{\mu\nu}\mathfrak{s}_\nu \partial_{\mathfrak{s}\mu} \\
&+ \frac{1}{2}\hbar\left[(\partial^\lambda \tilde{F}^{\mu\nu})p_\nu - (\partial^\lambda \tilde{F}^{\nu\rho})p_\nu \mathfrak{s}_\rho \mathfrak{s}^\mu\right]\partial_{p\lambda}\partial_{\mathfrak{s}\mu}\Big\}\mathfrak{f}(x,p,\mathfrak{s}) = \mathfrak{C}[\mathfrak{f}]. \qquad (4.52)
\end{aligned}
$$

However, in the next chapter we will focus on effects of vorticity and nonlocal collisions in dissipative spin hydrodynamics and hence do not consider electromagnetic fields. The extension to the latter can be done starting from the equations derived in this section and is left for future work.

Chapter 5

Second-order spin hydrodynamics from the method of moments

In the previous chapters we found that, in the presence of nonlocal collisions, the spin dynamics is dissipative. In this case, the conservation laws are not sufficient to determine the evolution of the system, as the dissipative currents contribute degrees of freedom not covered by the conservation equations. In order to close the system of equations, in this chapter we apply the method of moments [35,36] to derive dissipative equations of motion from the Boltzmann equation, where the material of is based on Ref. [195]. We work up to first order in $\hbar$. In the first step, we decompose the charge current, the energy-momentum tensor, and the spin tensor with respect to the fluid velocity. Their components are then treated as the dynamical variables of the theory. At this point, a pseudo-gauge dependence enters the theory, since the choice which quantities are promoted to be dynamical depends on the form of the energy-momentum and spin tensors. We introduce a power-counting scheme considering the vorticity as an equilibrium quantity on a scale independent of the scale characterizing the dissipative gradients. This allows for the definition of an approximate local-equilibrium state where the spin potential $\Omega^{\mu\nu}$ is dynamical. Then, we define spin-dependent moments, which we use in addition to the conventional moments to expand the distribution function around its equilibrium form. The hydrodynamic fields appearing in the local-equilibrium distribution function are fixed requiring matching conditions, where we introduce six new matching conditions in comparison to the spinless case due to the presence of the dynamical spin potential. The equations of motion for the latter are derived from the conservation of total angular momentum. From the Boltzmann equation we obtain an infinite set of exact equations of motion for the spin moments. In order to close this system of equations, we make use of the "14+24-moment approximation", expressing the moments which do not appear in the conserved currents in terms of those which do appear. Under the assumption of a parity-conserving interaction, the collision terms in the kinetic equations of the spin moments are independent of the usual moments, and we obtain a closed system of 24 dynamical equations for the same number of spin moments. We also point out how the Pauli-Lubanski vector can be expressed in terms of those moments. Finally, we discuss the Navier-Stokes limits of both the usual and the spin moments, considering vorticity as a zeroth-order quantity.

5.1 Currents and equations of motion in spin hydrodynamics

Spin hydrodynamics is based on the equations of motion for the dynamical quantities of the theory: the charge current N^μ, the energy-momentum tensor $T^{\mu\nu}$, and the spin tensor $S^{\lambda,\mu\nu}$, where the forms of the latter two have been derived in Chapter 4. For the reasons pointed out in Section 4.2, we choose to work in the HW pseudo-gauge in the following, dropping the index HW on the currents for the sake of simplicity. This means that we use the form of the currents given in Eqs. (4.23)

$$\begin{aligned} N^\mu &= \langle p^\mu \rangle \ , \\ T^{\mu\nu} &= \langle p^\mu p^\nu \rangle + \hbar T_i^{\mu\nu} \ , \\ S^{\lambda,\mu\nu} &= \frac{1}{2} \left\langle p^\lambda \Sigma_{\mathfrak{s}}^{\mu\nu} \right\rangle - \frac{\hbar}{4m^2} \partial^{[\nu} \left\langle p^{\mu]} p^\lambda \right\rangle \ , \end{aligned} \tag{5.1}$$

where we defined the phase-space average

$$\langle \cdots \rangle \equiv \int d\Gamma \, (\cdots) f(x, p, \mathfrak{s}). \tag{5.2}$$

We note that this definition should be distinguished from the ensemble average of an operator, indicated by the same angular brackets around a symbol with a hat, as used in previous chapters. The interaction contribution $T_i^{\mu\nu}$ to the energy-momentum tensor in Eq. (5.1) is of second order in $\hbar$, see Eq. (4.15), and therefore beyond our level of accuracy in the conservation law of $T^{\mu\nu}$. However, this term has to be taken into account in the equations of motion of the spin tensor, since it provides the nonconservation of the latter. This happens due to nonlocal collisions, which, as we have seen in Chapter 3, make the energy-momentum tensor nonsymmetric and are responsible for the conversion between spin and orbital angular momentum.

For convenience we display the equations of motion of the currents given in Eqs. (4.24) again,

$$\begin{aligned}\partial_\mu N^\mu &= 0 \;, \\ \partial_\mu T^{\mu\nu} &= 0 \;, \\ \partial_\lambda S^{\lambda,\mu\nu} &= \frac{1}{2}\int d\Gamma\, \Sigma_{\mathfrak{s}}^{\mu\nu} \mathfrak{C}[f] \equiv T_i^{[\nu\mu]} \;,\end{aligned} \tag{5.3}$$

where we also included the conservation of the charge current N^μ. The conservation of the total angular momentum $\mathcal{J}^{\mu\nu} \equiv \Delta^{[\mu} p^{\nu]} + (\hbar/2)\Sigma_{\mathfrak{s}}^{\mu\nu}$ in a microscopic collision can by used to specify the interaction contribution to the energy-momentum tensor,

$$\hbar\, T_i^{\mu\nu} = \int d\Gamma \Delta^\mu p^\nu \mathfrak{C}[f] \;. \tag{5.4}$$

Here we used that the ν-component of the energy-momentum tensor is proportional to p^ν. This can be seen from the explicit form of the HW energy-momentum tensor at order $\mathcal{O}(\hbar^2)$ in Eq. (4.15), where the last term is of order $\mathcal{O}(\hbar^3)$.

In order to simplify the calculations, the currents in Eq. (5.1) are decomposed with respect to the fluid velocity u^μ. In this work we define the fluid velocity as the normalized timelike eigenvector of the free part $T_{ni}^{\mu\nu} \equiv T^{\mu\nu} - \hbar\, T_i^{\mu\nu}$ of the energy-momentum tensor with eigenvalue ϵ,

$$T_{ni}^{\mu\nu} u_\nu = \epsilon u^\mu \;. \tag{5.5}$$

This frame choice is usually referred to as Landau frame. We then decompose the momenta p^μ in the explicit forms of the currents in Eq. (5.1) into components parallel and orthogonal to the fluid velocity,

$$p^\mu = E_p u^\mu + p^{\langle\mu\rangle}, \tag{5.6}$$

where $E_p \equiv p\cdot u$ and $p^{\langle\mu\rangle} \equiv \Delta^{\mu\nu} p_\nu$, with $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ being the projector orthogonal to the fluid velocity. In addition, we divide products of momenta into parallel, orthogonal, and traceless orthogonal components with the help of the traceless projector $\Delta^{\mu\nu}_{\alpha\beta} \equiv (1/2)\Delta^{(\mu}_{\alpha}\Delta^{\nu)}_{\beta} - (1/3)\Delta^{\mu\nu}\Delta_{\alpha\beta}$, introducing the notation $p^{\langle\mu}p^{\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta} p^\alpha p^\beta$. We then arrive at the following form of the decomposed currents,

$$\begin{aligned}N^\mu &= n u^\mu + n^\mu \;, \\ T_{ni}^{\mu\nu} &= \epsilon u^\mu u^\nu - \Delta^{\mu\nu}(P_0+\Pi) + \pi^{\mu\nu} \;, \\ S^{\lambda,\mu\nu} &= u^\lambda \tilde{\mathfrak{N}}^{\mu\nu} + \Delta^\lambda_\alpha \tilde{\mathfrak{P}}^{\alpha\mu\nu} + 2u_{(\alpha}\tilde{\mathfrak{H}}^{\lambda)\mu\nu\alpha} + \tilde{\mathfrak{Q}}^{\lambda\mu\nu} + \frac{\hbar}{2m}\partial^{[\nu}\left[\epsilon_0 u^{\mu]} u^\lambda - \Delta^{\mu]\lambda}(P_0+\Pi) + \pi^{\mu]\lambda}\right] \;,\end{aligned} \tag{5.7}$$

where we made use of the usual hydrodynamic definitions, i.e. the particle density $n \equiv \langle E_p\rangle$, the particle diffusion $n^\mu \equiv \langle p^{\langle\mu\rangle}\rangle$, the energy density $\epsilon \equiv \langle E_p^2\rangle$, the thermodynamic pressure P_0, the bulk viscous pressure Π with $P_0 + \Pi \equiv -(1/3)\langle \Delta^{\mu\nu} p_\mu p_\nu\rangle$, and the shear-stress tensor $\pi^{\mu\nu} \equiv \langle p^{\langle\mu}p^{\nu\rangle}\rangle$. Furthermore, we introduced new hydrodynamic quantities characterizing spin transport and given by

$$\tilde{\mathfrak{N}}^{\mu\nu} \equiv -\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta} u_\alpha \langle E_p^2\, \mathfrak{s}_\beta\rangle \;, \tag{5.8a}$$

$$\tilde{\mathfrak{P}}^{\alpha\mu\nu} \equiv -\frac{1}{6m}\epsilon^{\alpha\mu\nu\beta}\langle \Delta^{\rho\sigma} p_\rho p_\sigma\, \mathfrak{s}_\beta\rangle \;, \tag{5.8b}$$

$$\tilde{\mathfrak{H}}^{\lambda\mu\nu\alpha} \equiv -\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta}\langle E_p p^{\langle\lambda\rangle}\mathfrak{s}_\beta\rangle \;, \tag{5.8c}$$

$$\tilde{\mathfrak{Q}}^{\lambda\mu\nu} \equiv -\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta}\langle p^{\langle\lambda}p_{\alpha\rangle}\,\mathfrak{s}_\beta\rangle \;. \tag{5.8d}$$

which are dual to the *spin-energy tensor*

$$\mathfrak{N}^{\mu\nu} \equiv -\frac{1}{2m}u^\mu\langle E_p^2\,\mathfrak{s}^\nu\rangle \;, \tag{5.9a}$$

the *spin-pressure tensor*

$$\mathfrak{P}^{\mu} \equiv -\frac{1}{6m} \langle \Delta^{\rho\sigma} p_\rho p_\sigma \, \mathfrak{s}^{\mu} \rangle \;, \tag{5.9b}$$

the *spin-diffusion tensor*

$$\mathfrak{H}^{\lambda\mu} \equiv -\frac{1}{2m} \langle E_p p^{\langle\lambda\rangle} \mathfrak{s}^{\mu} \rangle \;, \tag{5.9c}$$

and the *spin-stress tensor*

$$\mathfrak{Q}^{\lambda\mu\nu} \equiv -\frac{1}{2m} \langle p^{\langle\mu} p^{\nu\rangle} \, \mathfrak{s}^{\lambda} \rangle \;. \tag{5.9d}$$

The HW spin tensor has 24 degrees of freedom, as it is antisymmetric in the last two indices. In the decomposition in Eq. (5.7), 3 of these degrees of freedom originate from the spin-energy tensor, 3 from the spin-pressure tensor, 9 from the spin-diffusion tensor and 9 from the spin-stress tensor. On the other hand, the set of tensors in Eqs. (5.9) has more than 24 degrees of freedom. However, as will be shown later, the number of dynamical degrees of freedom is reduced to 24 by matching conditions and constraints. Hence, our total number of dynamical degrees of freedom in the system is 14+24=38 in the dissipative case. In this case the system of equations of motion (5.3) does not provide sufficient information to fix all these degrees of freedom. Therefore, in the following we will use the method of moments to derive additional equations of motion for the dissipative currents from the Boltzmann equation (3.86), closing the set of moment equations [35].

5.2 Power-counting scheme

The derivation of hydrodynamics from kinetic theory is usually based on the notion of local equilibrium, defined as an artificial state where the collision term vanishes, but the left-hand side of the Boltzmann equation does not. The local-equilibrium distribution function serves as the starting point for a power-expansion, requiring that gradients of hydrodynamic variables are sufficiently small. In our case, we meet a difficulty with this definition, since, as we showed in Chapter 3.5, the nonlocal collision term vanishes identically only in global equilibrium. In other words, a zero nonlocal collision term implies that also the left-hand side of the Boltzmann equation vanishes. In this section we overcome this problem by introducing a specific power-counting scheme allowing for the concept of (approximate) local equilibrium in the presence of spin and nonlocal collisions. Then we expand the distribution function $f(x, p, \mathfrak{s})$ around this artificial equilibrium state.

Let us first discuss the separation of scales in our framework. In order for hydrodynamics to be a valid description of the system, one imposes an ordering of scales

$$l_{\text{int}} \ll l_{\text{mfp}} \ll l_{\text{hydro}}. \tag{5.10}$$

The first inequality tells us that the mean free path between collisions is much larger than the interaction range. This scale separation is necessary for treating particles as free between collisions, implying that correlations between particles do not play any role. We call this assumption molecular chaos. It ensures that the kinetic theory derived in Chapter 3 is applicable. Furthermore, we require the second inequality to be valid for the gradient expansion around local equilibrium. The small power-counting parameter in this expansion is the Knudsen number Kn$\equiv l_{\text{mfp}}/l_{\text{hydro}}$, where l_{hydro} characterizes a scale over which the usual dissipative hydrodynamic quantities vary significantly,

$$\frac{1}{m} p \cdot \partial f_{0p} \sim \frac{1}{l_{\text{hydro}}} f_{0p}. \tag{5.11}$$

Here

$$f_{0p} \equiv \frac{1}{(2\pi\hbar)^3} e^{-\beta_0 u \cdot p + \alpha_0} \tag{5.12}$$

is the local-equilibrium distribution function given by the zeroth order of Eq. (3.94). The index 0 on α and $\beta \equiv 1/T$ denotes the arbitrary space-time dependence of these quantities in contrast to their global-equilibrium counterparts. From Eq. (5.11) we obtain

$$\begin{aligned} \partial_\mu \alpha_0 &= \mathcal{O}(l_{\text{hydro}}^{-1}), \\ \frac{1}{\beta_0} \partial_{(\mu} \beta_0 u_{\nu)} &= \mathcal{O}(l_{\text{hydro}}^{-1}). \end{aligned} \tag{5.13}$$

On the other hand, the antisymmetric part of the derivative of the thermal velocity does not appear in Eq. (5.11). Hence, we can introduce another scale l_v, which is not related to l_{hydro},

$$\frac{1}{\beta_0}\partial_{[\mu}\beta_0 u_{\nu]} = \mathcal{O}(l_v^{-1}), \tag{5.14}$$

see also Ref. [94] for a similar concept. Introducing $\nabla^\mu \equiv \Delta^\mu_\nu \partial^\nu$ and $\dot{A} \equiv u\cdot\partial A \equiv dA/d\tau$, as well as the expansion scalar $\theta \equiv \nabla\cdot u$, the shear stress $\sigma^{\mu\nu} \equiv \nabla^{\langle\mu}u^{\nu\rangle}$, and the vorticity tensor $\omega^{\mu\nu} \equiv (1/2)\nabla^{[\mu}u^{\nu]}$, we can decompose the Eqs. (5.13) with respect to the fluid velocity u^μ. First, we contract Eq. (5.13) with $u^\mu u^\nu$, finding $\dot{\beta}_0 = \mathcal{O}(l_{\text{hydro}}^{-1})$. Moreover, contracting Eq. (5.13) with $\Delta^{\mu\nu}_{\alpha\beta}$ and $\Delta^{\mu\nu}$, respectively, yields

$$\begin{aligned}\sigma_{\alpha\beta} &= \mathcal{O}(l_{\text{hydro}}^{-1}),\\ \theta &= \mathcal{O}(l_{\text{hydro}}^{-1}).\end{aligned} \tag{5.15}$$

Finally, we contract with $\Delta^\mu_\alpha u^\nu$ and obtain

$$\nabla_\alpha\beta_0 + \beta_0\dot{u}_\alpha = \mathcal{O}(l_{\text{hydro}}^{-1}). \tag{5.16}$$

On the other hand, contracting Eq. (5.14) with $\Delta^\mu_\alpha\Delta^\nu_\beta$ gives

$$\omega_{\alpha\beta} = \mathcal{O}(l_v^{-1}). \tag{5.17}$$

This means that we treat vorticity independently of the other hydrodynamic gradients, and in particular we allow the former to be much larger than the latter. In principle the expression $\nabla_\alpha\beta_0 - \beta_0\dot{u}_\alpha$ could be defined on the same scale as vorticity, since it appears in Eq. (5.14) instead of Eq. (5.13). However, in the following we will describe a system with both small temperature gradient and small acceleration, $(1/\beta_0)\nabla_\alpha\beta_0 = \mathcal{O}(l_{\text{hydro}}^{-1})$, $\dot{u}_\alpha = \mathcal{O}(l_{\text{hydro}}^{-1})$. Such a situation is met if one considers an expansion around a rigidly rotating equilibrium state sufficiently far away from the boundary. Therefore we have

$$\nabla_\alpha\beta_0 - \beta_0\dot{u}_\alpha = \mathcal{O}(l_{\text{hydro}}^{-1}). \tag{5.18}$$

The standard (local) collision term is of the order of the mean free path

$$\frac{1}{m}C_l[f] \sim \frac{1}{l_{\text{mfp}}}f. \tag{5.19}$$

In our case, the nonlocality of the collision term (3.72) introduces another scale Δ, characterizing the typical separation between the colliding particles. This scale is also microscopic, and, for the assumption of molecular chaos to be fulfilled, much smaller than the mean free path,

$$\Delta \lesssim l_{\text{int}} \ll l_{\text{mfp}}. \tag{5.20}$$

We note from Eq. (3.89) that Δ is of order $\hbar$. Furthermore, as in Chapter 3, we exclude the case of a zeroth-order initial polarization, such that the polarization is of the same order as Δ. In order to ensure the applicability of the semiclassical expansion of the Wigner function and the collision term in Chapters 3 and 4, we impose

$$\Delta\,\partial f \sim \frac{\hbar}{m}\partial f \ll f. \tag{5.21}$$

Comparing this relation to Eq. (5.11) we find that

$$\Delta\,\partial f \ll \frac{l_{\text{hydro}}}{m}p\cdot\partial f \tag{5.22}$$

is necessary for consistency. This means that the so-called quantum Knudsen number

$$\kappa \equiv \frac{\Delta}{l_{\text{hydro}}} \ll 1 \tag{5.23}$$

has to be sufficiently small, consistent with Eqs. (5.10), (5.20). However, in contrast to Eq. (5.11), Eq. (5.21) contains not only dissipative gradients, but also vorticity. Although we require

$$\Delta \ll l_v, \tag{5.24}$$

we can describe situations where l_v is smaller than $l_{\rm hydro}$.

Consider a system with $l_v \ll l_{\rm hydro}$ in a way that

$$\frac{\Delta}{l_v} \sim \frac{l_{\rm mfp}}{l_{\rm hydro}} \tag{5.25}$$

is fulfilled. In such situation we can define "local equilibrium" through the separation of the scales $l_{\rm mfp}$ and $l_{\rm hydro}$, based on the assumption that on the larger scale it is not possible to "see" the nonlocality of the collision term. In this case, we introduce a local-equilibrium distribution function of the form (3.94),

$$f_{eq}(x,p,\mathfrak{s}) = f_{0p}\left(1+\frac{\hbar}{4}\Omega_{\mu\nu}\Sigma_{\mathfrak{s}}^{\mu\nu}\right) + \mathcal{O}(\hbar^2), \tag{5.26}$$

where f_{0p} is given in Eq. (5.12). Inserting this distribution function into Eq. (3.92), the collision term vanishes up to negligible contributions of order $\Delta/l_{\rm hydro}$. At this point, we remind that $\Omega_{\mu\nu}$ is the Lagrange multiplier of the *total* angular momentum, where we absorb the orbital part into the definition of $\beta_0 u^\mu$:

$$\beta_0 u^\mu = b^\mu - \Omega^{\mu\nu} x_\nu. \tag{5.27}$$

Here b^μ is the Lagrange multiplier for the linear momentum. Inserting Eq. (5.26) into the nonlocal collision term and following similar steps as in Chapter 3.5, we neglect the symmetric part of $\partial_\mu(\beta_0 u_\nu)$ in the ideal limit, since its order of magnitude is $\Delta/l_{\rm hydro}$ and hence much smaller than the usual dissipative corrections. On the other hand, we take into account the antisymmetric part of $\partial_\mu(\beta_0 u_\nu)$, even in the ideal approximation, as this contribution is nonvanishing even in global equilibrium. The order of magnitude of this contribution to the nonlocal collision term is Δ/l_v. Since we define the local-equilibrium distribution as the one that leads to a vanishing nonlocal collision term up to corrections of order $\mathcal{O}(\Delta/l_{\rm hydro})$, we have the following conditions for the Lagrange multipliers,

$$\frac{1}{\beta_0}\Delta^\lambda \partial_\lambda b_\mu \sim \frac{1}{\beta_0}\Delta^\lambda x^\nu \partial_\lambda \Omega_{\mu\nu} \sim \mathcal{O}(\Delta/l_{\rm hydro}) \,. \tag{5.28}$$

It follows that

$$\partial^\nu \beta_0 u^\mu = \Omega^{\mu\nu} + \mathcal{O}\left(\frac{1}{l_{\rm hydro}}\right), \tag{5.29}$$

meaning that the spin potential equals the thermal vorticity at the leading order, where the equality is exact in global equilibrium.

It is convenient to decompose $\Omega^{\mu\nu}$ as

$$\Omega^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} u_\alpha \omega_{0\beta} + u^{[\mu}\kappa_0^{\nu]}, \tag{5.30}$$

where

$$\kappa_0^\mu \equiv -\Omega^{\mu\nu} u_\nu \tag{5.31}$$

and

$$\omega_0^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu \Omega_{\alpha\beta}. \tag{5.32}$$

The power-counting explained above implies that

$$\begin{aligned} \frac{\omega_0^\mu}{\beta_0} &= \mathcal{O}\left(\frac{1}{l_v}\right), \\ \frac{\kappa_0^\mu}{\beta_0} &= \mathcal{O}\left(\frac{1}{l_{\rm hydro}}\right), \end{aligned} \tag{5.33}$$

and

$$\beta_0 \omega^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} u_\alpha \omega_{0\beta} + \mathcal{O}\left(\frac{1}{l_{\rm hydro}}\right). \tag{5.34}$$

We remark that due to our power-counting $\Omega_{\mu\nu}$ depends on $\beta_\mu \equiv \beta_0 u_\mu$ at the leading order. On the other hand, $\Omega_{\mu\nu}$ and b_μ are independent of each other.

The definition of local equilibrium as defined above should be contrasted with the definition of local equilibrium in standard kinetic theory. In the latter case, the collision term vanishes exactly and without further assumption when the local-equilibrium distribution function is inserted. This is not possible anymore when the nonlocality of a microscopic collision is taken into account. The reason lies in the separation of scales necessary for a kinetic description. In fact, the inequality (5.20) implies that

$$\kappa \lesssim \mathrm{Kn}. \tag{5.35}$$

Due to the inequality (5.35), it would be inconsistent to take into account polarization, which is of order κ, but not dissipation, which is of order Kn. This problem is solved by introducing the independent vorticity scale l_v and assuming the relation (5.25).

5.3 Expansion around equilibrium

After defining local equilibrium, we can expand the distribution function around this state up to first order as

$$f_{p\mathfrak{s}} = f_{0p}\left(1 + \frac{\hbar}{4}\Omega_{\mu\nu}\Sigma_{\mathfrak{s}}^{\mu\nu}\right) + \delta f_{p\mathfrak{s}} \tag{5.36}$$

with

$$\delta f_{p\mathfrak{s}} \equiv f_{0p}\left(\phi_p + \mathfrak{s}\cdot\zeta_p\right). \tag{5.37}$$

Here we introduced ζ_p^μ characterizing the deviations from local equilibrium of the spin degrees of freedom and ϕ_p for the deviations of the spin-independent part of the distribution function. We expand the latter in moments of the distribution function identically to what was done in Ref. [35], see Appendix E for details. Furthermore, extending the formalism of Ref. [35] to include spin degrees of freedom, we make use of the following expansion of ζ_p^μ in terms of moments

$$\zeta_p^\mu = \sum_{l=0}^{\infty} \eta_p^{\mu,\langle\mu_1\cdots\mu_l\rangle} p_{\langle\mu_1}\cdots p_{\mu_l\rangle}. \tag{5.38}$$

In the next step, we expand the energy-dependent coefficients $\eta_p^{\mu,\langle\mu_1\cdots\mu_l\rangle}$ in terms of polynomials of order n in E_p,

$$\eta_p^{\mu,\langle\mu_1\cdots\mu_l\rangle} = \sum_{n\in\mathbb{S}_l}\left(g^\mu_\nu - \frac{p^\mu p_\nu}{m^2}\right) e_n^{\nu,\langle\mu_1\cdots\mu_l\rangle}\mathcal{P}_{pn}^{(l)}, \tag{5.39}$$

where $\mathbb{S}_l$ denotes the set of indices of the dynamical spin moments, $e_n^{\nu,\langle\mu_1\cdots\mu_l\rangle}$ are momentum-independent coefficients and

$$\mathcal{P}_{pn}^{(l)} \equiv \sum_{r=0}^{n} a_{nr}^{(l)} E_p^r\,. \tag{5.40}$$

The coefficients $a_{nr}^{(l)}$ are chosen in way that the orthogonality relation

$$2\int dP\,\frac{w^{(l)}}{(2l+1)!!}\left(\Delta^{\alpha\beta}p_\alpha p_\beta\right)^l f_{0p}\mathcal{P}_{pm}^{(l)}\mathcal{P}_{pn}^{(l)} = \delta_{mn} \tag{5.41}$$

is fulfilled. Here, we defined $w^{(l)} \equiv (-1)^l I_{(2l)l}$, with

$$I_{nq}(\alpha_0,\beta_0) \equiv \frac{1}{(2q+1)!!}\left\langle E_p^{n-2q}(-\Delta^{\alpha\beta}p_\alpha p_\beta)^q\right\rangle_0 \tag{5.42}$$

cf. Ref. [35], and

$$\langle\cdots\rangle_0 \equiv \int d\Gamma\,(\cdots) f_{eq}(x,p,\mathfrak{s})\,. \tag{5.43}$$

Now ζ_p^μ can be written as

$$\begin{aligned}
\zeta_p^\mu &= \sum_{l=0}^{\infty}\sum_{n=0}^{N_l}\left(\bar{d}_n^{\mu,\langle\mu_1\cdots\mu_l\rangle}\mathcal{P}_{pn}^{(l)} p_{\langle\mu_1}\cdots p_{\mu_l\rangle} - \bar{e}_{n,\langle\mu_1\cdots\mu_l\rangle}\mathcal{P}_{pn}^{(l+1)} p^{\langle\mu}p^{\mu_1}\cdots p^{\mu_l\rangle}\right)\\
&= \sum_{l=0}^{\infty}\sum_{n=0}^{N_l}\left(\bar{d}_n^{\mu,\langle\mu_1\cdots\mu_l\rangle}\mathcal{P}_{pn}^{(l)} p_{\langle\mu_1}\cdots p_{\mu_l\rangle} - \bar{e}_n^{\langle\mu_1\cdots\mu_l\rangle} g^{\mu\mu_{l+1}}\mathcal{P}_{pn}^{(l+1)} p_{\langle\mu_1}\cdots p_{\mu_{l+1}\rangle}\right)\\
&= \sum_{l=0}^{\infty}\sum_{n=0}^{N_l} d_n^{\mu,\langle\mu_1\cdots\mu_l\rangle}\mathcal{P}_{pn}^{(l)} p_{\langle\mu_1}\cdots p_{\mu_l\rangle},
\end{aligned} \tag{5.44}$$

where the last term was again expanded in a series of irreducible tensors and the appearing terms were resummed with different coefficients $\bar{d}_n^{\mu,\langle\mu_1\cdots\mu_l\rangle}$ and $\bar{e}_{n,\langle\mu_1\ldots\mu_l\rangle}$. Here we expanded the E_p-dependence in the second term in Eq. (5.44) for convenience in terms of $\mathcal{P}_{pn}^{(l+1)}$. One finds

$$
\begin{aligned}
\left\langle \mathcal{P}_{pn}^{(l)} \mathfrak{s}^\mu p^{\langle\mu_1}\ldots p^{\mu_l\rangle} \right\rangle_\delta &= \sum_{j=0}^{\infty} \sum_{m=0}^{N_j} \int d\Gamma\, \mathcal{P}_{pn}^{(l)} \mathfrak{s}^\mu p^{\langle\mu_1}\ldots p^{\mu_l\rangle} \mathfrak{s}_\lambda d_m^{\lambda,\langle\nu_1\ldots\nu_j\rangle} p_{\langle\nu_1}\ldots p_{\nu_j\rangle} \mathcal{P}_{pm}^{(j)} f_{0p} \\
&= -2 \sum_{j=0}^{\infty} \sum_{m=0}^{N_j} \int dP\, \mathcal{P}_{pn}^{(l)} p^{\langle\mu_1}\ldots p^{\mu_l\rangle} d_m^{\mu,\langle\nu_1\ldots\nu_j\rangle} p_{\langle\nu_1}\ldots p_{\nu_j\rangle} \mathcal{P}_{pm}^{(j)} f_{0p} \\
&= -2 \frac{l!}{(2l+1)!!} \sum_{m=0}^{N_j} \int dP\, \mathcal{P}_{pn}^{(l)} \mathcal{P}_{pm}^{(l)} d_m^{\mu,\langle\mu_1\ldots\mu_l\rangle} \left(\Delta^{\alpha\beta} p_\alpha p_\beta\right)^l f_{0p} \\
&= -\frac{l!}{w^{(l)}} d_n^{\mu,\langle\mu_1\ldots\mu_l\rangle}
\end{aligned} \tag{5.45}
$$

with

$$
\langle\cdots\rangle_\delta \equiv \langle\cdots\rangle - \langle\cdots\rangle_0. \tag{5.46}
$$

In Eq. (5.45) we made use of $\zeta_p \cdot p = 0$, the orthogonality of the irreducible tensors (D.7), and of the polynomials $\mathcal{P}_{pn}^{(l)}$ (5.41). We define the spin moments

$$
\tau_n^{\mu,\mu_1\cdots\mu_l} \equiv \langle E_p^n\, \mathfrak{s}^\mu p^{\langle\mu_1}\ldots p^{\mu_l\rangle}\rangle_\delta \tag{5.47}
$$

and use them to write $\eta_p^{\mu,\langle\mu_1\cdots\mu_l\rangle}$ as

$$
\eta_p^{\mu,\langle\mu_1\cdots\mu_l\rangle} = \sum_{n=0}^{N_l} \mathcal{H}_{pn}^{(l)} \tau_n^{\mu,\mu_1\cdots\mu_l}, \tag{5.48}
$$

where

$$
\mathcal{H}_{pn}^{(l)} = \frac{w^{(l)}}{l!} \sum_{m=n}^{N_l} a_{mn}^{(l)} \mathcal{P}_{pm}^{(l)}. \tag{5.49}
$$

In summary we can express the distribution function in the following form,

$$
f_{p\mathfrak{s}} = f_{0p}\left[1 + \frac{\hbar}{4}\Omega_{\mu\nu}\Sigma_{\mathfrak{s}}^{\mu\nu} + \sum_{l=0}^{\infty}\sum_{n=0}^{N_l} \mathcal{H}_{pn}^{(l)} \left(\rho_n^{\mu_1\cdots\mu_l} + \mathfrak{s}_\mu \tau_n^{\mu,\mu_1\cdots\mu_l}\right) p_{\langle\mu_1}\ldots p_{\mu_l\rangle}\right] \tag{5.50}
$$

with

$$
\rho_n^{\mu_1\cdots\mu_l} \equiv \langle E_p^n\, p^{\langle\mu_1}\ldots p^{\mu_l\rangle}\rangle_\delta \tag{5.51}
$$

being the conventional spin-independent moments [35] (see also Appendix E).

With the help of the local-equilibrium distribution function (5.26), we separate the components of the spin tensor in Eq. (5.7) into equilibrium and nonequilibrium contributions,

$$
\begin{aligned}
\mathfrak{N}^{\mu\nu} &\equiv \mathfrak{n}_0^{\mu\nu} - \frac{1}{2m} u^\mu \mathfrak{n}^\nu, \\
\mathfrak{P}^\mu &\equiv \mathfrak{p}_0^\mu - \frac{1}{6m}\left(m^2 \mathfrak{p}^\mu - \mathfrak{n}^\mu\right), \\
\mathfrak{H}^{\lambda\mu} &\equiv \mathfrak{h}_0^{\lambda\mu} - \frac{1}{2m}\mathfrak{h}^{\lambda\mu}, \\
\mathfrak{Q}^{\lambda\mu\nu} &\equiv -\frac{1}{2m}\mathfrak{q}^{\lambda\mu\nu},
\end{aligned} \tag{5.52}
$$

with

$$
\begin{aligned}
\mathfrak{n}_0^{\mu\nu} &\equiv -\frac{1}{2m} u^\mu \langle E_p^2\, \mathfrak{s}^\nu\rangle_0, \\
\mathfrak{p}_0^\mu &\equiv -\frac{1}{6m}\langle \Delta^{\rho\sigma} p_\rho p_\sigma\, \mathfrak{s}^\mu\rangle_0, \\
\mathfrak{h}_0^{\lambda\mu} &\equiv -\frac{1}{2m}\langle E_p p^{\langle\lambda\rangle} \mathfrak{s}^\mu\rangle_0
\end{aligned} \tag{5.53}
$$

being the nondissipative parts and

$$\mathfrak{n}^{\nu} \equiv \tau_2^{\nu}, \qquad \mathfrak{p}^{\mu} \equiv \tau_0^{\mu}, \qquad \mathfrak{h}^{\lambda\mu} \equiv \tau_1^{\mu,\lambda}, \qquad \mathfrak{q}^{\lambda\mu\nu} \equiv \tau_0^{\lambda,\mu\nu} \tag{5.54}$$

being the dissipative parts.

Since only three components of ζ_p^{μ} are independent, some components of the spin moments can be expressed in terms of the others. More specifically, the condition $p \cdot \zeta_p = 0$ relates the components parallel to u^{μ} to those orthogonal to u^{μ} in the spin moments, i.e.,

$$u_{\mu}\tau_r^{\mu,\mu_1\cdots\mu_n} = \int d\Gamma E_p^r (u \cdot \mathfrak{s}) p^{\langle\mu_1} \cdots p^{\mu_n\rangle} \delta f = -\int d\Gamma E_p^{(r-1)} p^{\langle\nu\rangle} \mathfrak{s}_{\nu} p^{\langle\mu_1} \cdots p^{\mu_n\rangle} \delta f \,. \tag{5.55}$$

Therefore, it is sufficient to derive equations of motion only for the orthogonal components. Once the latter are known, they can be used to compute the parallel components.

5.4 Matching conditions and equations of motion for hydrodynamic variables

In the definition of the local-equilibrium distribution function (5.26), the Lagrange multipliers α_0, β_0, u^{μ}, and $\Omega^{\mu\nu}$ are treated dynamically, adding additional degrees of freedom to the system. A priori, the physical interpretation of these thermodynamic fields is not clear, but is fixed by choosing a thermodynamic frame, see Eq. (5.5), and requiring so-called matching conditions. The latter define the splitting of the total distribution function into local-equilibrium and dissipative parts. The matching conditions serve as a supplementary set of equations, making the system of equations closed again. Here, we choose to work with the following matching conditions defining α_0 and β_0,

$$\begin{aligned} n &= n_0 \equiv \langle E_p \rangle_0, \\ \epsilon &= \epsilon_0 \equiv \langle E_p^2 \rangle_0. \end{aligned} \tag{5.56}$$

Moreover, we define the spin potential $\Omega^{\mu\nu}$ near equilibrium by imposing additional matching conditions of the form

$$u_{\lambda} J^{\lambda,\mu\nu} = u_{\lambda} J_{eq}^{\lambda,\mu\nu} \tag{5.57}$$

with $J^{\lambda,\mu\nu} \equiv x^{\mu}T^{\lambda\nu} - x^{\nu}T^{\lambda\mu} + \hbar S^{\lambda,\mu\nu}$ being the total angular-momentum tensor in Eq. (2.9). The above condition is selected in analogy with the Landau frame choice in Eq. (5.5), resulting in a total angular-momentum density identical to its equilibrium value in the fluid rest frame. Due to the conservation of the total angular-momentum tensor, the associated global charge, defined in Eq. (2.13), transforms covariantly under Lorentz transformations, see Chapter 2. This would be different if we made use of the spin tensor in the matching conditions, since this quantity is in general not conserved. We now plug Eqs. (5.5) and (5.7) into Eq. (5.57) and obtain

$$-u_{\lambda} x^{[\mu} T_i^{\nu]\lambda} + \tilde{\mathfrak{N}}^{\mu\nu} + 2\tilde{\mathfrak{H}}_{\lambda}^{\ \mu\nu\lambda} = \tilde{\mathfrak{n}}_0^{\mu\nu} + 2\tilde{\mathfrak{h}}_{0\lambda}^{\ \ \mu\nu\lambda}. \tag{5.58}$$

Note that in the above equality we neglected derivatives of Π and $\pi^{\mu\nu}$ originating from inserting Eq. (5.7), as they would lead to contributions of second-order derivatives of dissipative quantities in the equations of motion, and hence not considered in second-order dissipative hydrodynamics. Contraction with $\epsilon_{\alpha\beta\mu\nu}$ and then with u^{α} and $\Delta^{\alpha}_{\mu}\Delta^{\beta}_{\nu}$, respectively, yields the following conditions

$$\begin{aligned} \mathfrak{n}^{\langle\mu\rangle} - 2u_{\lambda}\mathfrak{h}^{\mu\lambda} &= 2m u_{\alpha} \epsilon^{\mu\alpha\rho\nu} u^{\lambda} x_{\rho} T_{i\lambda\nu} \,, \\ \mathfrak{h}^{[\nu\langle\mu\rangle]} &= m \Delta^{\mu}_{\alpha} \Delta^{\nu}_{\beta} \epsilon^{\alpha\beta\rho\tau} u^{\lambda} x_{\rho} T_{i\lambda\tau}, \end{aligned} \tag{5.59}$$

with $A^{[\mu\langle\nu\rangle]} \equiv A^{[\mu}{}_{\alpha}\Delta^{\nu]\alpha}$. This in turn implies

$$\mathfrak{h}^{[\lambda\mu]} = -\frac{1}{2} u^{[\lambda}\tau_2^{\mu]} - m\epsilon^{\lambda\mu\alpha\beta} u^{\rho} x_{\alpha} T_{i\rho\beta} \,. \tag{5.60}$$

At this point we should mention that one could also choose other matching conditions and a frame not identical to the Landau frame. For instance, following Ref. [28], it would be possible to search for coefficients a_r and b_r in a matching condition of the form

$$\sum_{r\in\mathbb{S}_0} a_r \tau_r^{[\mu} u^{\nu]} + \sum_{n\in\mathbb{S}_1} b_n \tau_n^{[\mu,\nu]} = 0, \tag{5.61}$$

rendering the first-order theory derived from the equations of motion, which we will obtain in the next section, causal and stable [28–30]. However, in the following we will derive a second-order theory using the Landau frame and matching conditions in Eqs. (5.59).

The conservation equations (5.3) imply the following equations of motion,

$$\dot{\alpha}_0 = \frac{1}{D_{20}} \left\{ -I_{30}(n_0\theta + \partial \cdot n) + I_{20}[(\epsilon_0 + P_0 + \Pi)\theta - \pi^{\mu\nu}\sigma_{\mu\nu}] \right\}, \tag{5.62a}$$

$$\dot{\beta}_0 = \frac{1}{D_{20}} \left\{ -I_{20}(n_0\theta + \partial \cdot n) + I_{10}[(\epsilon_0 + P_0 + \Pi)\theta - \pi^{\mu\nu}\sigma_{\mu\nu}] \right\}, \tag{5.62b}$$

$$\dot{u}^\mu = \frac{1}{\epsilon_0 + P_0} \left(\nabla^\mu P_0 - \Pi \dot{u}^\mu + \nabla^\mu \Pi - \Delta^\mu_\alpha \partial_\beta \pi^{\alpha\beta} \right) \tag{5.62c}$$

with

$$D_{nq} \equiv I_{(n+1)q} I_{(n-1)q} - I_{nq}^2. \tag{5.63}$$

These equations are identical to those obtained in Ref. [35], unaffected by spin contributions up to first order in $\hbar$. Furthermore, using the conservation of total angular momentum in Eqs. (5.3), we derive the kinetic equations of the spin potential. We obtain

$$\begin{aligned} T^{[\nu\mu]} =& \partial_\lambda \left(u^\lambda \tilde{\mathfrak{N}}^{\mu\nu} + \Delta^\lambda_\alpha \tilde{\mathfrak{P}}^{\alpha\mu\nu} + 2u_{(\alpha}\tilde{\mathfrak{H}}^{\lambda)\mu\nu\alpha} + \tilde{\mathfrak{Q}}^{\lambda\mu\nu} \right) \\ =& \theta \tilde{\mathfrak{n}}_0^{\mu\nu} + \dot{\tilde{\mathfrak{n}}}_0^{\mu\nu} + \nabla_\alpha \tilde{\mathfrak{P}}^{\alpha\mu\nu} - (\theta u_\alpha + \dot{u}_\alpha) \tilde{\mathfrak{P}}^{\alpha\mu\nu} + 2(\nabla_\lambda u_\alpha)\tilde{\mathfrak{H}}^{\lambda\mu\nu\alpha} + 2\theta \tilde{\mathfrak{h}}_{0\alpha}{}^{\mu\nu\alpha} + 2u_\alpha \partial_\lambda \tilde{\mathfrak{H}}^{\lambda\mu\nu\alpha} \\ &+ 2\dot{\tilde{\mathfrak{h}}}_{0\alpha}{}^{\mu\nu\alpha} + \partial_\lambda \tilde{\mathfrak{Q}}^{\lambda\mu\nu}, \end{aligned} \tag{5.64}$$

where we inserted the matching conditions Eq. (5.60). The unit vector $\hat{t}^\mu$ in Eq. (3.89) defines the frame from which the parameters of the collisions are observed. In the following, we choose this frame equal to the fluid rest frame, $\hat{t}^\mu = u^\mu$. It follows from Eq. (5.4) that $u_\lambda T_i^{\lambda\nu} = 0$, making the last term in Eq. (5.60) vanish. We then replace the equilibrium quantities in Eq. (5.64) by their explicit expressions,

$$\begin{aligned} \tilde{\mathfrak{n}}_0^{\mu\nu} =& \frac{\hbar}{4m^2} \epsilon^{\mu\nu\alpha\beta} u_\alpha \omega_{0\beta} \langle E_p^3 \rangle_0, \\ \tilde{\mathfrak{h}}_0^{\lambda\mu\nu\alpha} =& -\frac{1}{3m^2}\frac{\hbar}{4} \left(\Delta^{\mu\lambda}\Omega^{\nu\alpha} + \Delta^{\nu\lambda}\Omega^{\alpha\mu} + \Delta^{\alpha\lambda}\Omega^{\mu\nu} \right) \langle E_p(m^2 - E_p^2) \rangle_0, \\ \mathfrak{p}_0^\mu =& \frac{\hbar}{4m^2}\frac{1}{3} \omega_0^\mu \left\langle E_p \Delta^{\alpha\beta} p_\alpha p_\beta \right\rangle_0 . \end{aligned} \tag{5.65}$$

Using

$$\begin{aligned} \frac{d}{d\tau}\langle E_p^3 \rangle_0 =& 2\int dP\, E_p^3 \left(-E_p \dot{\beta}_0 + \dot{\alpha}_0 \right) f_{0p} = -I_{40}\dot{\beta}_0 + I_{30}\dot{\alpha}_0, \\ \frac{d}{d\tau}\langle E_p(m^2 - E_p^2) \rangle_0 =& 2\int dP\, E_p \Delta^{\alpha\beta} p_\alpha p_\beta \left(-E_p \dot{\beta}_0 + \dot{\alpha}_0 \right) f_{0p} = 3I_{41}\dot{\beta}_0 - 3I_{31}\dot{\alpha}_0, \\ \nabla_\lambda \langle E_p(m^2 - E_p^2) \rangle_0 =& 2\int dP\, E_p \Delta^{\alpha\beta} p_\alpha p_\beta \left(-E_p \nabla_\lambda \beta_0 + \nabla_\lambda \alpha_0 \right) f_{0p} = 3I_{41}\nabla_\lambda \beta_0 - 3I_{31}\nabla_\lambda \alpha_0, \end{aligned} \tag{5.66}$$

and contracting Eq. (5.64) with $\epsilon_{\mu\nu\alpha\beta}u^\beta$ and u_μ, respectively, we obtain

$$\begin{aligned} \frac{\hbar}{m^2}\dot{\omega}_0^{\langle\alpha\rangle} =& -\frac{2}{I_{30} - 2I_{31}} \left\{ \left[\frac{\hbar}{2m^2}(\theta I_{30} - I_{40}\dot{\beta}_0 + I_{30}\dot{\alpha}_0) + \frac{\hbar}{m^2}(I_{41}\dot{\beta}_0 - I_{31}\dot{\alpha}_0) - \frac{2\hbar}{3m^2} I_{31}\theta \right] \omega_0^\alpha \right. \\ &+ \frac{1}{3m} \left[u_\lambda \nabla^\alpha \left(m^2 \tau_0^\lambda - \tau_2^\lambda \right) - u_\lambda \left(m^2 \tau_0^\lambda - \tau_2^\lambda \right) \dot{u}^\alpha + \theta \left(m^2 \tau_0^{\langle\alpha\rangle} - \tau_2^{\langle\alpha\rangle} \right) \right] \\ &- \frac{\hbar}{m^2} \epsilon^{\langle\alpha\rangle\lambda\mu\nu} \kappa_{0\nu} \left(-I_{41} u_\mu \nabla_\lambda \beta_0 + I_{31} u_\mu \nabla_\lambda \alpha_0 - 3I_{31} u_\mu \dot{u}_\lambda \right) - \frac{\hbar}{m^2} \epsilon^{\langle\alpha\rangle\lambda\mu\nu} I_{31} u_\mu \nabla_\lambda \kappa_{0\nu} \\ &- \frac{\hbar}{2m^2} I_{31} (\sigma^{\alpha\lambda} + \omega^{\alpha\lambda}) \omega_{0\lambda} - \frac{1}{m} \Delta^\alpha_\beta \nabla_\lambda \tau_1^{\rho,(\lambda} \Delta^{\beta)}_\rho + \frac{1}{m} \tau_1^{(\langle\alpha\rangle,\nu)} \dot{u}_\nu + \frac{1}{m} u_\beta \nabla_\lambda \tau_0^{[\beta,\alpha]\lambda} \\ &\left. - \frac{1}{m} u_\beta \tau_0^{\beta,\alpha\lambda} \dot{u}_\lambda + 2\epsilon^{\alpha\beta\mu\nu} \hbar T_{i\mu\nu} u_\beta \right\}, \end{aligned} \tag{5.67}$$

and

$$\frac{\hbar}{m^2}\dot{\kappa}_0^{\langle\mu\rangle} = -\frac{1}{I_{31}}\left\{\frac{\hbar}{4m^2}\left(I_{30}-I_{31}\right)\epsilon^{\mu\nu\alpha\beta}\dot{u}_\alpha\omega_{0\beta}u_\nu + \frac{\hbar}{4m^2}\epsilon^{\mu\nu\alpha\beta}u_\nu\left[-I_{31}\nabla_\alpha\omega_{0\beta} + (I_{41}\nabla_\alpha\beta_0 - I_{31}\nabla_\alpha\alpha_0)\omega_{0\beta}\right]\right.$$
$$-\frac{1}{6m}\epsilon^{\mu\nu\alpha\beta}u_\nu\left[\nabla_\alpha\left(m^2\tau_{0\beta}-\tau_{2\beta}\right) - \dot{u}_\alpha\left(m^2\tau_{0\beta}-\tau_{2\beta}\right)\right] - \frac{\hbar}{2m^2}I_{31}(\sigma^{\mu\nu}+\omega^{\mu\nu})\kappa_{0\nu}$$
$$+\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta}u_\alpha\tau_{1(\beta,\lambda)}\left(\frac{\theta}{3}\Delta^\lambda_\nu + \sigma^\lambda{}_\nu + \omega^\lambda{}_\nu\right) + \frac{\hbar}{m^2}\left(\frac{4}{3}I_{31}\theta - I_{41}\dot{\beta}_0 + I_{31}\dot{\alpha}_0\right)\kappa_0^\mu$$
$$\left.-\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta}u_\nu\left(\nabla^\lambda\tau_{0\beta,\alpha\lambda} - \dot{u}^\lambda\tau_{0\beta,\alpha\lambda}\right) + \hbar T_{\mathrm{i}}^{\mu\nu}u_\nu\right\}. \tag{5.68}$$

We can express the last term in Eq. (5.67) with the help of Eq. (5.4) as

$$2\epsilon^{\alpha\beta\mu\nu}u_\beta\int d\Gamma\,\Delta_\mu p_\nu\mathfrak{C}[f] = \frac{\hbar}{m}\int d\Gamma\left[(E_p-m)\mathfrak{s}^{\langle\alpha\rangle} - \frac{E_p}{E_p+m}(u\cdot\mathfrak{s})p^{\langle\alpha\rangle}\right]\mathfrak{C}[f]$$
$$= \frac{\hbar}{m}\int d\Gamma\,(E_p-m)\left[\mathfrak{s}^{\langle\alpha\rangle} - \frac{1}{E_p^2}\sum_{j=0}^{\infty}\left(\frac{m^2}{E_p^2}\right)^j(u\cdot\mathfrak{s})p^{\langle\alpha\rangle}\right]\mathfrak{C}[f]\,. \tag{5.69}$$

Here we made use of the geometric series to write $1/(1-m^2/E_p^2)$ as a polynomial in E_p. Analogously, for the last term in Eq. (5.68) we obtain

$$\int d\Gamma E_p\Delta^\mu = -\frac{\hbar}{2m}\epsilon^{\mu\nu\alpha\beta}u_\alpha\int d\Gamma\,\frac{1}{E_p}(E_p-m)\sum_{j=0}^{\infty}\left(\frac{m^2}{E_p^2}\right)^j p_{\langle\nu\rangle}\mathfrak{s}_\beta\,. \tag{5.70}$$

We conclude that Eqs. (5.67) and (5.68) depend on an infinite sum of moments with negative r. This is not a problem, as in Section 5.7 we will see that such moments can be expressed through those with positive r. Since $m \leq E_p$, the geometric series converges and the sum can be truncated at a finite value of j.

5.5 Equations of motion for spin moments

In this section, the equations of motion for the spin moments $\tau_r^{\langle\mu\rangle,\mu_1\cdots\mu_n}$ are presented. Considering the comoving derivative of Eq. (5.47) we obtain

$$\dot{\tau}_r^{\mu,\langle\mu_1\cdots\mu_n\rangle} = \Delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\frac{d}{d\tau}\int d\Gamma\,E_p^r\,p^{\langle\nu_1}\cdots p^{\nu_n\rangle}\mathfrak{s}^\mu\delta f_{p\mathfrak{s}}. \tag{5.71}$$

Furthermore, inserting Eq. (5.36) into the Boltzmann equation (3.86) yields

$$\delta\dot{f}_{p\mathfrak{s}} = -\dot{f}_{0p}\left(1+\frac{\hbar}{4}\Omega_{\alpha\beta}\Sigma_{\mathfrak{s}}^{\alpha\beta}\right) - \frac{\hbar}{4}f_{0p}\dot{\Omega}_{\alpha\beta}\Sigma_{\mathfrak{s}}^{\alpha\beta} - E_p^{-1}p\cdot\nabla f_{0p}\left(1+\frac{\hbar}{4}\Omega_{\alpha\beta}\Sigma_{\mathfrak{s}}^{\alpha\beta}\right) - E_p^{-1}p\cdot\nabla\delta f_{p\mathfrak{s}} + E_p^{-1}\mathfrak{C}[f], \tag{5.72}$$

where we from now on drop the index "on-shell" on the collision term. This form of the Boltzmann equation is now used to compute the derivative in Eq. (5.71). For the details of the calculation we refer to Appendix D.2. The resulting equations of motion for the spin moments of tensor-rank zero in momentum read

$$\dot{\tau}_r^{\langle\mu\rangle} - \mathfrak{C}_{r-1}^{\langle\mu\rangle} = \frac{\hbar}{2m}\left[\xi_r^{(0)}\theta + \frac{G_{2(r+1)}}{D_{20}}\Pi\theta - \frac{G_{2(r+1)}}{D_{20}}\pi^{\lambda\nu}\sigma_{\lambda\nu} - \frac{G_{3r}}{D_{20}}\partial\cdot n\right]\omega_0^\mu - \frac{\hbar}{4m}I_{(r+1)1}\Delta^\mu_\lambda\nabla_\nu\tilde{\Omega}^{\lambda\nu}$$
$$-\frac{\hbar}{4m}\tilde{\Omega}^{\langle\mu\rangle\nu}\left[I_{(r+1)1}I_\nu - I_{(r+2)1}\frac{\beta_0}{\epsilon_0+P_0}\left(-\Pi\dot{u}_\nu + \nabla_\nu\Pi - \Delta_{\nu\lambda}\partial_\rho\pi^{\lambda\rho}\right)\right]$$
$$+ r\,\dot{u}_\nu\tau_{r-1}^{\langle\mu\rangle,\nu} + (r-1)\sigma_{\alpha\beta}\tau_{r-2}^{\langle\mu\rangle,\alpha\beta} - \Delta^\mu_\lambda\nabla_\nu\tau_{r-1}^{\lambda,\nu} - \frac{1}{3}\left[(r+2)\tau_r^{\langle\mu\rangle} - (r-1)m^2\tau_{r-2}^{\langle\mu\rangle}\right]\theta$$
$$-\frac{\hbar}{4m}I_{(r+1)0}\epsilon^{\mu\nu\alpha\beta}u_\nu\dot{\Omega}_{\alpha\beta}, \tag{5.73}$$

where $I^\mu \equiv \nabla^\mu\alpha_0$,

$$\xi_r^{(0)} \equiv -\frac{1}{m}\left\{I_{(r+1)0} + r\,I_{(r+1)1} + \frac{1}{D_{20}}\left[-G_{3(r+1)}n_0 + G_{2(r+1)}(\epsilon_0+P_0)\right]\right\}, \tag{5.74}$$

$$G_{nm} \equiv I_{n0}I_{m0} - I_{(n-1)0}I_{(m+1)0}. \tag{5.75}$$

Furthermore, we defined the collision integrals

$$\mathfrak{C}_r^{\mu,\langle\mu_1\cdots\mu_n\rangle} \equiv \int d\Gamma\, E_p^r\, p^{\langle\mu_1}\cdots p^{\mu_n\rangle}\mathfrak{s}^\mu \mathfrak{C}[f] \tag{5.76}$$

and used the identity

$$\nabla^\mu P_0 = \frac{n_0}{\beta_0}\nabla^\mu\alpha_0 - \frac{\epsilon_0+P_0}{\beta_0}\nabla^\mu\beta_0, \tag{5.77}$$

where the latter is not changed by spin effects up to $\mathcal{O}(\hbar)$. Moreover, we obtain for the spin moments of tensor-rank one in momentum

$$\begin{aligned}\dot{\tau}_r^{\langle\mu\rangle,\langle\nu\rangle} - \mathfrak{C}_{r-1}^{\langle\mu\rangle,\langle\nu\rangle} =& \frac{\hbar}{2m}\omega_0^\mu\left[\frac{\beta_0}{\epsilon_0+P_0}I_{(r+3)1}\left(-\Pi\dot{u}^\nu + \nabla^\nu\Pi - \Delta^\nu_\lambda\partial_\rho\pi^{\lambda\rho}\right) - I_{(r+2)1}I^\nu\right] - \frac{\hbar}{2m}\beta_0 I_{(r+3)2}\tilde{\Omega}^{\langle\mu\rangle}{}_\lambda\sigma^{\nu\lambda}\\ &- \frac{\hbar}{4m}I_{(r+2)1}\Delta^\mu_\rho\left(\nabla^\nu\tilde{\Omega}^{\rho\lambda}\right)u_\lambda + \omega^\nu{}_\rho\tau_r^{\langle\mu\rangle,\rho} + \frac{1}{3}\left[(r-1)m^2\tau_{r-2}^{\langle\mu\rangle,\nu} - (r+3)\tau_r^{\langle\mu\rangle,\nu}\right]\theta\\ &- \Delta^\nu_\lambda\Delta^\mu_\alpha\nabla_\rho\tau_{r-1}^{\alpha,\lambda\rho} + r\dot{u}_\rho\tau_{r-1}^{\langle\mu\rangle,\nu\rho} + \frac{1}{5}\left[2(r-1)m^2\tau_{r-2}^{\langle\mu\rangle,\lambda} - (2r+3)\tau_r^{\langle\mu\rangle,\lambda}\right]\sigma^\nu{}_\lambda\\ &+ \frac{1}{3}\dot{u}^\nu\left[m^2 r\tau_{r-1}^{\langle\mu\rangle} - (r+3)\tau_{r+1}^{\langle\mu\rangle}\right] - \frac{1}{3}\Delta^\mu_\lambda\nabla^\nu\left(m^2\tau_{r-1}^\lambda - \tau_{r+1}^\lambda\right) + (r-1)\sigma_{\lambda\rho}\tau_{r-2}^{\langle\mu\rangle,\nu\lambda\rho},\end{aligned} \tag{5.78}$$

and for the spin moments of tensor-rank two in momentum

$$\begin{aligned}\dot{\tau}_r^{\langle\mu\rangle,\langle\nu\lambda\rangle} - \mathfrak{C}_{r-1}^{\langle\mu\rangle,\langle\nu\lambda\rangle} =& \frac{\hbar}{2m}\xi_r^{(2)}\tilde{\Omega}^{\langle\mu\rangle\langle\nu}I^{\lambda\rangle} + \frac{\hbar}{2m}I_{(r+3)2}\Delta^\mu_\rho\Delta^{\nu\lambda}_{\alpha\beta}\nabla^\alpha\tilde{\Omega}^{\rho\beta} - \frac{\hbar}{2m}\tilde{\Omega}^{\mu\rho}\beta_0 u_\rho\sigma^{\nu\lambda}I_{(r+4)2}\\ &- \frac{\hbar}{2m}\frac{\beta_0}{\epsilon_0+P_0}I_{(r+4)2}\tilde{\Omega}^{\langle\mu\rangle\langle\nu}\left(-\Pi\dot{u}^{\lambda\rangle} + \nabla^{\lambda\rangle}\Pi - \Delta^{\lambda\rangle}_\alpha\partial_\beta\pi^{\alpha\beta}\right)\\ &+ r\dot{u}_\rho\tau_{r-1}^{\langle\mu\rangle,\nu\lambda\rho} + \frac{2}{5}\left(m^2\tau_{r-1}^{\langle\mu\rangle,\langle\nu} - (r+5)\tau_{r+1}^{\langle\mu\rangle,\langle\nu}\right)\dot{u}^{\lambda\rangle}\\ &- \Delta^\mu_\gamma\Delta^{\nu\lambda}_{\alpha\beta}\nabla_\rho\tau_{r-1}^{\gamma,\alpha\beta\rho} + \Delta^\mu_\rho\frac{2}{5}\Delta^{\nu\lambda}_{\alpha\beta}\nabla^\beta\left(\tau_{r+1}^{\rho,\alpha} - m^2\tau_{r-1}^{\rho,\alpha}\right)\\ &+ \frac{1}{3}\left[(r-1)m^2\tau_{r-2}^{\langle\mu\rangle,\nu\lambda} - (r+4)\tau_r^{\langle\mu\rangle,\nu\lambda}\right]\theta + (r-1)\sigma_{\rho\tau}\tau_{r-2}^{\langle\mu\rangle,\nu\lambda\rho\tau}\\ &+ \frac{2}{7}\left[2(r-1)m^2\tau_{r-2}^{\langle\mu\rangle,\rho\langle\nu} - (2r+5)\tau_r^{\langle\mu\rangle,\rho\langle\nu}\right]\sigma^{\lambda\rangle}_\rho + 2\tau_r^{\langle\mu\rangle,\rho\langle\nu}\omega^{\lambda\rangle}_\rho\\ &+ \frac{2}{15}\left[(r-1)m^4\tau_{r-2}^{\langle\mu\rangle} - (2r+3)m^2\tau_r^{\langle\mu\rangle} + (r+4)\tau_{r+2}^{\langle\mu\rangle}\right]\sigma^{\nu\lambda},\end{aligned} \tag{5.79}$$

with

$$\xi_r^{(2)} \equiv I_{(r+3)2} - \frac{n_0}{\epsilon_0+P_0}I_{(r+4)2}. \tag{5.80}$$

Finally, for the sake of completeness, we also list the equation of motion for the spin moments tensor-rank three in momentum,

$$\begin{aligned}\dot{\tau}_r^{\mu,\langle\nu\lambda\rho\rangle} - \mathfrak{C}_{r-1}^{\mu,\langle\nu\lambda\rho\rangle} =& \frac{3\hbar}{2m}I_{(r+5)3}\tilde{\Omega}^{\mu\langle\nu}\sigma^{\lambda\rho\rangle} + \frac{3}{2}\tau_r^{\mu,\tau\langle\nu\lambda}\omega^{\rho\rangle}_\tau + r\dot{u}_\eta\tau_{r-1}^{\mu,\nu\lambda\rho\eta} + \frac{3}{14}\left[2m^2 r\tau_{r-1}^{\mu,\langle\nu\lambda} - (2r+7)\tau_{r+1}^{\mu,\langle\nu\lambda}\right]\dot{u}^{\rho\rangle}\\ &- \Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}\nabla_\tau\tau_{r-1}^{\mu,\alpha\beta\gamma\tau} - \frac{3}{7}\Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}\nabla^\gamma\left(m^2\tau_{r-1}^{\mu,\alpha\beta} - \tau_{r+1}^{\mu,\alpha\beta}\right)\\ &+ \frac{1}{6}\left[(2r-2)m^2\tau_{r-2}^{\mu,\nu\lambda\rho} - (2r+7)\tau_r^{\mu,\nu\lambda\rho}\right]\theta + (r-1)\sigma_{\tau\eta}\tau_{r-2}^{\mu,\nu\lambda\rho\tau\eta}\\ &+ \frac{1}{3}\left[(2r-2)m^2\tau_{r-2}^{\mu,\eta\langle\nu\lambda} - (2r+5)\tau_r^{\mu,\eta\langle\nu\lambda}\right]\sigma^{\rho\rangle}_\eta\\ &+ \frac{3}{35}\left[(2r-2)m^4\tau_{r-2}^{\mu,\langle\nu} - (4r+3)m^2\tau_r^{\mu,\langle\nu} + (2r+15)\tau_{r+2}^{\mu,\langle\nu}\right]\sigma^{\lambda\rho\rangle}.\end{aligned} \tag{5.81}$$

Since no rank-four tensors appear in the conserved currents, this equation will not be considered in the following. Furthermore, without needing an explicit calculation, we will see that the asymptotic solutions of the spin moments of tensor-rank four in momentum vanish at the first order. We first notice that all contributions from the equilibrium distribution function to the equations of motion for the spin moments of rank four in momentum vanish due to the orthogonality relation (D.7). All other contributions will be linear in one spin moment and one gradient of u^μ, respectively. Hence, there is only one possible tensor structure which can lead to contributions of first order in dissipative quantities,

$$\dot{\tau}_r^{\mu,\langle\nu\lambda\rho\sigma\rangle} - \mathfrak{C}_{r-1}^{\mu,\langle\nu\lambda\rho\sigma\rangle} = \xi^{(4)}\tau_r^{\mu,\eta\langle\nu\lambda\rho}\omega^{\sigma\rangle}{}_\eta + \dots, \tag{5.82}$$

where the factor $\xi^{(4)}$ is of no importance and the dots denote terms of higher orders. In the Navier-Stokes limit, the first term on the left-hand side of Eq. (5.82) vanishes. Furthermore, the collision term is proportional to a linear combination of $\tau_r^{\mu,\nu\lambda\rho\sigma}$, cf. the next section. Therefore, both sides of the equation are linear in $\tau_r^{\mu,\nu\lambda\rho\sigma}$, which immediately yields the trivial solution $\tau_r^{\mu,\nu\lambda\rho\sigma} = 0$.

We remark that formally the equations of motion for the spin moments derived in this section do not explicitly depend on the usual moments $\rho_r^{\mu_1\cdots\mu_l}$ defined in Eq. (5.51) and used for the expansion of the spin-independent part of the distribution function in Eq. (5.50). Also vice versa, the equations of motion for $\rho_r^{\mu_1\cdots\mu_l}$, which can be found in Ref. [35], are formally unaffected by the spin contributions. However, an implicit dependence of the equations of motion for the spin moments on the usual moments and vice versa is given through the collision terms, indicating that interactions couple polarization effects and properties of the fluid flow. We will come back to this later.

The kinetic equations obtained in this section are, except for the truncation of the $\hbar$ expansion at first order, exact. However, a truncation procedure is needed to close this system of equations. If one considers only terms of first order in dissipative gradients, i.e. gradients of $\mathcal{O}(l_{\text{hydro}}^{-1})$, the Navier-Stokes theory is obtained, which is discussed in Section 5.9.2. As mentioned in the introduction, in such theories one usually finds issues with causality and stability of the hydrodynamic equations. For this reason, in the following we will treat the components of the HW spin tensor dynamically, where their time evolution is governed by the second-order equations of motion. A possible way to close the system of equations of motion in this case is separating the slowest microscopic time scales and resumming moments in the transport coefficients [35]. That approach has the advantage of a clear power-counting in small Knudsen numbers. However, in this thesis, we make use of a simpler truncation procedure analogous to Israel-Stewart theory [33], where the moment expansion of the distribution function is directly truncated [36]. This means that in Eq. (5.50) only the moments which are part of the conserved currents are taken into account. As the spin tensor adds 24 further dynamical moments to the 14 of usual Israel-Stewart theory, we call this truncation the "14+24-moment approximation".

5.6 Collision integrals

In this section the collision integrals are written as a linear combination of moments of the distribution function. The first step is the linearization of the collision term. To this end, we insert the distribution function (5.36) into the collision term (3.92) and expand the result up to first order in $\hbar$ and dissipative corrections, respectively. We obtain

$$\begin{aligned}\mathfrak{C}[f] = \int d\Gamma_1 d\Gamma_2 d\Gamma'\, \mathcal{W} f_{0p} f_{0p'} \Bigg\{ &\left[1 + \frac{\hbar}{4}\Omega_{\mu\nu}\Sigma_{\mathfrak{s}_1}^{\mu\nu} + \phi_1 + \mathfrak{s}_1\cdot\zeta_1 - (\partial_\mu\beta_\nu)\Delta_1^\mu p_1^\nu\right] \\ &\times \left[1 + \frac{\hbar}{4}\Omega_{\mu\nu}\Sigma_{\mathfrak{s}_2}^{\mu\nu} + \phi_2 + \mathfrak{s}_2\cdot\zeta_2 - (\partial_\mu\beta_\nu)\Delta_2^\mu p_2^\nu\right] \\ &- \left[1 + \frac{\hbar}{4}\Omega_{\mu\nu}\Sigma_{\mathfrak{s}}^{\mu\nu} + \phi + \mathfrak{s}\cdot\zeta - (\partial_\mu\beta_\nu)\Delta^\mu p^\nu\right]\left[1 + \frac{\hbar}{4}\Omega_{\mu\nu}\Sigma_{\mathfrak{s}'}^{\mu\nu} + \phi' + \mathfrak{s}'\cdot\zeta' - (\partial_\mu\beta_\nu)\Delta'^\mu p'^\nu\right] \\ &+ (\partial_\mu\beta_\nu)\Delta^\mu\left(p_1^\nu + p_2^\nu - p^\nu - p'^\nu\right)\Bigg\} + \dots, \end{aligned} \tag{5.83}$$

where we keep suppressing the index "on-shell" and the tilde symbol on $\mathfrak{C}$ and also drop the tilde symbol on $\mathcal{W}$. Furthermore the dots denote higher-order terms of $\mathcal{O}(\hbar^2)$ or of first order in $\hbar$ and gradients of dissipative corrections, $\mathcal{O}(\hbar\partial\delta f)$. Noting that the last term vanishes due to four-momentum conservation and neglecting furthermore terms of second order in dissipative corrections, $\mathcal{O}(\phi^2,\zeta^2)$, the collision term becomes

$$\begin{aligned}\mathfrak{C}[f] = \int d\Gamma_1 d\Gamma_2 d\Gamma'\, \mathcal{W} f_{0p} f_{0p'} \Big\{ &\phi_1 + \phi_2 - \phi - \phi' + \mathfrak{s}_1\cdot\zeta_1 + \mathfrak{s}_2\cdot\zeta_2 - \mathfrak{s}\cdot\zeta - \mathfrak{s}'\cdot\zeta' \\ &- (\partial_\mu\beta_\nu)\left[\Delta_1^\mu p_1^\nu(1+\phi_2) + \Delta_2^\mu p_2^\nu(1+\phi_1) - \Delta^\mu p^\nu(1+\phi') + \Delta'^\mu p'^\nu(1+\phi)\right] \\ &+ \frac{\hbar}{4}\Omega_{\mu\nu}\left[\Sigma_{\mathfrak{s}_1}^{\mu\nu}(1+\phi_2) + \Sigma_{\mathfrak{s}_2}^{\mu\nu}(1+\phi_1) - \Sigma_{\mathfrak{s}}^{\mu\nu}(1+\phi') - \Sigma_{\mathfrak{s}'}^{\mu\nu}(1+\phi)\right]\Big\} + \dots \end{aligned} \tag{5.84}$$

There are two classes of contributions to the collision term, those in the first line, depending on dissipative corrections ϕ and ζ in the same form as the conventional collision term does, which, as we will see later, contribute to the spin relaxation times, and those in the second and third line, related to the nonlocality and the spin potential, respectively, which will give contrubtions to the Navier-Stokes limit of the spin moments. We first study the former contributions and define

$$\bar{\mathfrak{C}}_{r-1}^{\mu,\langle\mu_1\cdots\mu_n\rangle} \equiv \int [d\Gamma]\, \mathcal{W}\, E_p^{r-1} p^{\langle\mu_1} \cdots p^{\mu_n\rangle} \mathfrak{s}^\mu f_{0p} f_{0p'} \left(\phi_1 + \phi_2 - \phi - \phi' + \mathfrak{s}_1\cdot\zeta_1 + \mathfrak{s}_2\cdot\zeta_2 - \mathfrak{s}\cdot\zeta - \mathfrak{s}'\cdot\zeta'\right) , \tag{5.85}$$

where $[d\Gamma] \equiv d\Gamma d\Gamma' d\Gamma_1 d\Gamma_2$. Using the expansion of the distribution function (5.50) in Eq. (5.85), we obtain

$$\begin{aligned}\bar{\mathfrak{C}}_{r-1}^{\mu,\langle\mu_1\cdots\mu_n\rangle} =&8 \sum_{l=0}^{\infty} \int [dP]\, E_p^{r-1} p^{\langle\mu_1} \cdots p^{\mu_n\rangle} f_{0p} f_{0p'} \Big[-\mathcal{W}_0\, p_{\langle\nu_1} \cdots p_{\nu_l\rangle} \eta_p^{\mu,\langle\nu_1\cdots\nu_l\rangle} + w^\mu \Big(p_{1\langle\nu_1} \cdots p_{1\nu_l\rangle} \lambda_{p_1}^{\langle\nu_1\cdots\nu_l\rangle} \\ &+ p_{2\langle\nu_1} \cdots p_{2\nu_l\rangle} \lambda_{p_1}^{\langle\nu_2\cdots\nu_l\rangle} - p_{\langle\nu_1} \cdots p_{\nu_l\rangle} \lambda_p^{\langle\nu_1\cdots\nu_l\rangle} - p'_{\langle\nu_1} \cdots p'_{\nu_l\rangle} \lambda_{p'}^{\langle\nu_1\cdots\nu_l\rangle} \Big) \\ &+ \bar{w}_1{}^{\mu}{}_{\nu} \eta_{p_1}^{\nu,\langle\nu_1\cdots\nu_l\rangle} p_{1\langle\nu_1} \cdots p_{1\nu_l\rangle} + \bar{w}_2{}^{\mu}{}_{\nu} \eta_{p_2}^{\nu,\langle\nu_2\cdots\nu_l\rangle} p_{2\langle\nu_1} \cdots p_{2\nu_l\rangle} - \bar{w}'{}^{\mu}_{\nu} \eta_{p'}^{\nu,\langle\nu_1\cdots\nu_l\rangle} p'_{\langle\nu_1} \cdots p'_{\nu_l\rangle} \Big] ,\end{aligned} \tag{5.86}$$

where $p_\nu \eta_p^{\nu,\langle\nu_1\cdots\nu_l\rangle} = 0$ was used, see Eq. (5.39). We also introduced

$$\lambda_p^{\langle\nu_1\cdots\nu_l\rangle} \equiv \sum_{n=0}^{N_l} \mathcal{H}_{pn}^l \rho_n^{\nu_1\cdots\nu_l}, \tag{5.87}$$

where details can be found in Ref. [35] and Appendix E, and defined

$$\mathcal{W}_0 \equiv \frac{1}{8} \int [dS]\, \mathcal{W}, \tag{5.88a}$$

$$w^\mu \equiv \frac{1}{8} \int [dS]\, \mathfrak{s}^\mu \mathcal{W}, \tag{5.88b}$$

$$\bar{w}_i^{\mu\nu} \equiv \frac{1}{8} \int [dS]\, \mathfrak{s}^\mu \mathfrak{s}_i^\nu \mathcal{W} , \tag{5.88c}$$

with $[dS] \equiv dS(p) dS'(p') dS_1(p_1) dS_2(p_2)$. We prove in Appendix D.3 that Eqs. (5.88b) and (5.88c) vanish, respectively, if a scalar interaction is considered. Furthermore, in the case of a vector interaction Eq. (5.88b) is zero, too, whereas Eq. (5.88c) is not. However, in the limit of low momentum transfer, Eq. (5.88c) vanishes also for a vector interaction, and only Eq. (5.88a) contributes. In the following, we will describe the case of either a scalar interaction or a vector interaction in the limit of low momentum transfer. Therefore, the terms corresponding to Eqs. (5.88b) and (5.88c) in Eq. (5.86) are neglected. As a remark, we mention that Eq. (5.88b) is nonvanishing only if the interaction violates parity. Then, the equations of motion of standard moments and spin moments become directly coupled, leading to contributions from spin polarization to shear stress and bulk viscous pressure. Such effects will be studied in the future.

On the other hand, in the situation considered here we find

$$\bar{\mathfrak{C}}_{r-1}^{\mu,\langle\mu_1\cdots\mu_n\rangle} =8 \sum_{l=0}^{\infty} \int [dP]\, \mathcal{W}_0\, E_p^{r-1} f_{0p} f_{0p'} p^{\langle\mu_1} \cdots p^{\mu_n\rangle} p_{\langle\nu_1} \cdots p_{\nu_l\rangle} \eta_p^{\mu,\langle\nu_1\cdots\nu_l\rangle}. \tag{5.89}$$

Plugging Eq. (5.48) into Eq. (5.89) yields

$$\bar{\mathfrak{C}}_{r-1}^{\mu,\langle\mu_1\cdots\mu_l\rangle} = -\sum_{m=0}^{\infty} \sum_{n\in\mathbb{S}_m} \left(B_{rn}^{(l)}\right)^{\langle\mu_1\cdots\mu_l\rangle}_{\langle\nu_1\cdots\nu_m\rangle} \tau_n^{\mu,\langle\nu_1\cdots\nu_m\rangle}, \tag{5.90}$$

where

$$\left(B_{rn}^{(l)}\right)^{\langle\mu_1\cdots\mu_l\rangle}_{\langle\nu_1\cdots\nu_m\rangle} \equiv -8 \int [dP] \mathcal{W}_0 f_{0p} f_{0p'} E_p^{r-1} p^{\langle\mu_1} \cdots p^{\mu_l\rangle} \mathcal{H}_{pn}^{(m)} p_{\langle\nu_1} \cdots p_{\nu_m\rangle}. \tag{5.91}$$

One can see that these tensors only depend on the local-equilibrium equilibrium distribution functions at zeroth order. This means that, after performing three of the four momentum integrations, the integrand of the remaining momentum integration depends only on u^μ. Let us consider the following integration in Eq. (5.91),

$$\int [dP] \mathcal{W}_0 f_{0p} f_{0p'} E_p^{r-1} p^{\langle\mu_1} \cdots p^{\mu_l\rangle} \mathcal{H}_{pn}^{(m)} p_{\langle\nu_1} \cdots p_{\nu_m\rangle} \tag{5.92}$$

with $\mathcal{W}$ being a function of p, p', p_1, and p_2 and $\mathcal{H}^{(m)}_{pn}$ expressible as a polynomial of E_p. The only vectors which are left at our disposal after integrating over all momenta except p^μ are u^μ and p^μ. Therefore, the remaining integrals take the form

$$\int dP\, p^{\langle\mu_1\cdots\mu_m\rangle} p_{\langle\nu_1\cdots\nu_n\rangle} F(E_p) \tag{5.93}$$

and Eq. (D.7) is applicable. Hence, we obtain (see also Ref. [35])

$$\bar{\mathfrak{C}}^{\mu,\langle\mu_1\cdots\mu_l\rangle}_{r-1} = -\sum_{n\in\mathbb{S}_l} B^{(l)}_{rn} \mathcal{T}^{\mu,\langle\mu_1\cdots\nu_l\rangle}_n, \tag{5.94}$$

where

$$B^{(l)}_{rn} \equiv \frac{1}{2l+1}\Delta^{\nu_1\cdots\nu_l}_{\mu_1\cdots\mu_l}\left(B^{(l)}_{rn}\right)^{\langle\mu_1\cdots\mu_l\rangle}_{\langle\nu_1\cdots\nu_l\rangle}. \tag{5.95}$$

We now consider the second and third lines in Eq. (5.84). As $\Omega_{\mu\nu} = \varpi_{\mu\nu}$ at the leading order and all contributions linear in $\mathfrak{s}'$, $\mathfrak{s}_1$, or $\mathfrak{s}_2$ are zero after performing the spin integrations, the conservation of total angular momentum can be used to cancel some terms. We find

$$\mathfrak{C}^{\mu,\langle\mu_1\cdots\mu_n\rangle}_{r-1} = \bar{\mathfrak{C}}^{\mu,\langle\mu_1\cdots\mu_n\rangle}_{r-1} + \int [d\Gamma]\,\mathcal{W}\,E_p^{r-1} p^{\langle\mu_1}\cdots p^{\mu_n\rangle}\mathfrak{s}^\mu f_{0p} f_{0p'}\left[-\frac{\hbar}{4}(\Omega_{\alpha\beta}-\varpi_{\alpha\beta})\Sigma^{\alpha\beta}_{\mathfrak{s}} + \frac{1}{2}\partial_{(\beta}\beta_{\alpha)}\Delta^\beta p^\alpha\right]. \tag{5.96}$$

As we will see later, inserting Eq. (5.96) into the equations of motion derived in Section 5.5 results in contributions from the difference between thermal vorticity and spin potential and from thermal shear $\partial_{(\mu}\beta_0 u_{\nu)}/2$ to the spin moments. For the first three irreducible moments in momentum, the collision integrals read

$$\begin{aligned}
\mathfrak{C}^{\mu}_{r-1} &= \bar{\mathfrak{C}}^{\mu}_{r-1} + g^{(0)}_r\left(\tilde{\Omega}^{\mu\nu} - \tilde{\varpi}^{\mu\nu}\right)u_\nu,\\
\mathfrak{C}^{\mu,\nu}_{r-1} &= \bar{\mathfrak{C}}^{\mu,\nu}_{r-1} + g^{(1)}_r\left(\tilde{\Omega}^{\mu\langle\nu\rangle} - \tilde{\varpi}^{\mu\langle\nu\rangle}\right) + h^{(1)}_r(\beta_0\dot{u}_\lambda + \nabla_\lambda\beta_0)\epsilon^{\mu\nu\alpha\lambda}u_\alpha,\\
\mathfrak{C}^{\mu,\nu\lambda}_{r-1} &= \bar{\mathfrak{C}}^{\mu,\nu\lambda}_{r-1} + h^{(2)}_r\beta_0\sigma_\rho^{\ \langle\nu}\epsilon^{\lambda\rangle\mu\alpha\rho}u_\alpha.
\end{aligned} \tag{5.97}$$

Here, the orthogonality relation (D.7) was applied for similar reasons as pointed after Eq. (5.91). Moreover, we inserted $\hat{t}^\mu = u^\mu$ and introduced the following definitions,

$$\begin{aligned}
g^{(0)}_r &\equiv \frac{\hbar}{m}\int [dP]\,\mathcal{W}_0 E_p^r f_{0p} f_{0p'},\\
g^{(1)}_r &\equiv \frac{\hbar}{3m}\int [dP]\,\mathcal{W}_0 E_p^{r-1} f_{0p} f_{0p'}(\Delta^{\alpha\beta}p_\alpha p_\beta),\\
h^{(1)}_r &\equiv -\frac{2\hbar}{3m}\int [dP]\,\frac{1}{E_p+m}\mathcal{W}_0 E_p^{r-1} f_{0p} f_{0p'}(\Delta^{\alpha\beta}p_\alpha p_\beta),\\
h^{(2)}_r &\equiv -\frac{4\hbar}{15m}\int [dP]\,\frac{1}{E_p+m}\mathcal{W}_0 E_p^{r-1} f_{0p} f_{0p'}(\Delta^{\alpha\beta}p_\alpha p_\beta)^2.
\end{aligned} \tag{5.98}$$

It will be found in Section 5.8 that the terms originating from the shear tensor in Eq. (5.97) give contributions to the local, but not to the global polarization. This is similar to the behaviour found in Refs. [10–13].

5.7 Second-order equations of motion in the 14+24-moment approximation

In this section a closed set of equations of motion for all components of the HW spin tensor is derived, using a truncation of the moment expansion of the distribution function in Eq. (5.50). More specifically, in analogy to the 14-moment approximation, we consider only spin moments in the moment expansion which are part of the HW spin tensor, as those appear in the conservation equation of the total angular momentum. This means that we have 24 independent degrees of freedom describing spin transport, hence we refer to this truncation as "14+24-moment approximation", where the first 14 moments are those of usual Israel-Stewart theory. We note that, as we use the HW spin tensor, 24 is the minimal number of moments which should be taken into account in the dissipative case. On the other hand, if we had instead chosen to treat the components of the canonical spin tensor dynamically, they would add only 4 degrees of freedom associated with spin transport. However, the equations of motion of the canonical spin tensor are not conservation laws even in global equilibrium. Hence, there is no reason to believe that its components are the most important on hydrodynamic scales. In conclusion, the pseudo-gauge choice affects the dynamics of the system, and hence will have an impact on observables. This finding is consistent with similar results of other approaches [14, 192, 196]. Even if global charges would be by definition independent of the pseudo-gauge, if they were calculated exactly, in a realistic setup some approximation has to be applied (such as the truncation of the moment expansion in our case). The way an approximation is chosen does depend on the pseudo-gauge, and thus different pseudo-gauges can lead to better or worse descriptions of the system.

We now aim at expressing the moments which are not part of the conservation laws in terms of those appearing in the conservation laws. To this end, we make use of the relation, cf. Ref. [36],

$$\tau_r^{\mu,\mu_1\cdots\mu_l} = \sum_{n\in\mathbb{S}_l} \tau_n^{\mu,\mu_1\cdots\mu_l} \mathfrak{F}_{rn}^{(l)} \tag{5.99}$$

with

$$\mathfrak{F}_{rn}^{(l)} \equiv \frac{2l!}{(2l+1)!!}\int dP\, E_p^r\, \mathcal{H}_{pn}^{(l)} \left(\Delta^{\alpha\beta}p_\alpha p_\beta\right)^l f_{0p}, \tag{5.100}$$

which can be derived by plugging Eq. (5.50) into Eq. (5.47) and employing the orthogonality relation (D.7). While Eq. (5.99) is an exact identity for $r \in \mathbb{S}_l$, it is an approximation for all other values of r. In Eq. (5.99) we now consider only the moments appearing in Eqs. (5.9), i.e., we take $\mathbb{S}_0 = \{0, 2\}$, $\mathbb{S}_1 = \{1\}$, $\mathbb{S}_2 = \{0\}$, and $\mathbb{S}_l$ to be an empty set for $l > 2$. This means, we approximate

$$\begin{aligned}
\tau_r^{\langle\mu\rangle} &\simeq \mathfrak{F}_{r0}^{(0)}\,\mathfrak{p}^{\langle\mu\rangle} + \mathfrak{F}_{r2}^{(0)}\,\mathfrak{n}^{\langle\mu\rangle},\\
\tau_r^{\langle\mu\rangle,\nu} &\simeq \mathfrak{F}_{r1}^{(1)}\,\mathfrak{h}^{\langle\mu\rangle\nu} = \frac{1}{2}\mathfrak{F}_{r1}^{(1)}\,\mathfrak{z}^{\mu\nu},\\
\tau_r^{\langle\mu\rangle,\nu\lambda} &\simeq \mathfrak{F}_{r0}^{(2)}\,\mathfrak{q}^{\langle\mu\rangle\nu\lambda},\\
\tau_r^{\mu,\nu\lambda\rho\cdots} &\simeq 0,
\end{aligned} \tag{5.101}$$

with $\mathfrak{z}^{\nu\mu} \equiv \mathfrak{h}^{(\langle\mu\rangle\nu)}$. In the second line of the above equation the antisymmetric contribution is zero using the matching condition Eq. (5.60). Furthermore, the components of the spin moments parallel to the fluid velocity read

$$\begin{aligned}
u_\mu\tau_r^{\mu} &= -\,\tau^{\mu}_{r-1,\mu}\\
&\simeq -\,\frac{1}{2}\mathfrak{F}_{(r-1)1}^{(1)}\mathfrak{z}^{\mu}{}_{\mu},\\
u_\mu\tau_r^{\mu,\nu} &= -\,\tau^{\mu,\nu}_{r-1\,\mu} - \frac{1}{3}\left(m^2\tau_{r-1}^{\langle\nu\rangle} - \tau_{r+1}^{\langle\nu\rangle}\right)\\
&\simeq -\,\mathfrak{F}_{(r-1)0}^{(2)}\mathfrak{q}^{\mu\nu}{}_{\mu} - \frac{1}{3}\left(m^2\mathfrak{F}_{(r-1)0}^{(0)} - \mathfrak{F}_{(r+1)0}^{(0)}\right)\mathfrak{p}^{\langle\nu\rangle} - \frac{1}{3}\left(m^2\mathfrak{F}_{(r-1)2}^{(0)} - \mathfrak{F}_{(r+1)2}^{(0)}\right)\mathfrak{n}^{\langle\nu\rangle},\\
u_\mu\tau_r^{\mu,\nu\lambda} &= -\,\tau^{\mu,\nu\lambda}_{r-1\;\mu} + \frac{2}{15}\left(m^2\tau^{\mu}_{r-1,\mu} - \tau^{\mu}_{r+1,\mu}\right)\Delta^{\nu\lambda} - \frac{1}{5}\left(m^2\tau_{r-1}^{(\langle\nu\rangle,\lambda)} - \tau_{r+1}^{(\langle\nu\rangle,\lambda)}\right)\\
&\simeq \frac{1}{15}\left(m^2\mathfrak{F}_{(r-1)1}^{(1)} - \mathfrak{F}_{(r+1)1}^{(1)}\right)\mathfrak{z}^{\mu}{}_{\mu}\Delta^{\nu\lambda} - \frac{1}{5}\left(m^2\mathfrak{F}_{(r-1)1}^{(1)} - \mathfrak{F}_{(r+1)1}^{(1)}\right)\mathfrak{z}^{\nu\lambda},
\end{aligned} \tag{5.102}$$

where we used Eq. (5.55). We also note that with the help of the first equality in the matching condition (5.59) $\mathfrak{n}^{\langle\mu\rangle}$ can be expressed as

$$\frac{1}{3}\mathfrak{n}^{\langle\nu\rangle} = -2\mathfrak{q}^{\mu\nu}{}_{\mu} - \frac{2}{3}m^2\mathfrak{p}^{\langle\nu\rangle}\,. \tag{5.103}$$

Therefore, this moment can be expressed in terms of the others and is not a dynamical degree of freedom determined from its equation of motion.

It should be noted that the matrices $B_{rn}^{(l)}$ defined in Eq. (5.95) are invertible. Therefore, defining the matrix

$$\mathfrak{T}^{(l)} \equiv \left(B^{(l)}\right)^{-1}, \tag{5.104}$$

Eq. (5.94) can be inverted for $l \neq 1$, yielding

$$\tau_n^{\mu,\mu_1\cdots\mu_l} = -\sum_{r\in\mathbb{S}_l}\mathfrak{T}_{nr}^{(l)}\,\bar{\mathfrak{C}}_{r-1}^{\mu,\langle\mu_1\cdots\mu_l\rangle}. \tag{5.105}$$

As $\tau_2^{\langle\nu\rangle}$ is determined by the matching conditions, cf. Eq. (5.103), we have to exclude $r=2$ from the set $\mathbb{S}_0$ in the sum in Eq. (5.105) for $l=0$. This happens since a relation analogous to Eq. (5.55) connects the components of $u_\mu \mathfrak{C}^{\mu,\langle\mu_1\cdots\mu_n\rangle}$ to the components of $\mathfrak{C}^{\langle\mu\rangle,\langle\mu_1\cdots\mu_n\rangle}$. Therefore, additional components of the collision integrals are determined by collisional invariants. Furthermore, using the matching conditions in the approximation (5.101), the collision term for $l=1$ can be written as

$$\bar{\mathfrak{C}}_{r-1}^{\mu,\langle\nu\rangle} = -\sum_{n\in\mathbb{S}_1} B_{rn}^{(1)} \tau_n^{\mu,\nu} = -\frac{1}{2}\sum_{n\in\mathbb{S}_1} B_{rn}^{(1)} \mathfrak{F}_{n1}^{(1)} \left(\tau_1^{(\langle\mu\rangle,\nu)} + u^\mu \tau_2^{\langle\nu\rangle}\right). \tag{5.106}$$

Inverting the orthogonal, symmetric part we obtain

$$\tau_1^{(\langle\mu\rangle,\nu)} = -\sum_{r\in\mathbb{S}_1} \mathfrak{T}_{1r}^{(1)} \bar{\mathfrak{C}}_{r-1}^{(\langle\mu\rangle,\langle\nu\rangle)}. \tag{5.107}$$

At this point we note that there are six redundant equations in the system of equations of motion of the spin moments calculated in Section 5.5, reproducing the conservation of total angular momentum, which has been used for the computation of the kinetic equation for $\Omega^{\mu\nu}$, and therefore was already taken into account. This is the reason why we used the matching conditions in Eq. (5.59) to remove the dynamics of certain spin moments, dropping the redundant antisymmetric part of Eq. (5.106) and closing the system of equations of motion.

Multiplying Eqs. (5.73), (5.78) and (5.79) with $\mathfrak{T}_{nr}^{(l)}$, summing over r in each equation and inserting Eq. (5.101), we find with $\mathbb{S}_0=\{0\}$, $\mathbb{S}_1=\{1\}$, and $\mathbb{S}_2=\{0\}$ the kinetic equations for all dynamical spin moments, valid up to second order.[1] For $l=0$ we have

$$\begin{aligned}
\tau_{\mathfrak{p}} \Delta^\mu_\nu \frac{d}{d\tau}\mathfrak{p}^{\langle\nu\rangle} + \mathfrak{p}^{\langle\mu\rangle} =& \mathfrak{e}^{(0)}\left(\tilde{\Omega}^{\mu\nu} - \tilde{\varpi}^{\mu\nu}\right) u_\nu + \left(\mathfrak{K}_{\theta\omega}^{(0)}\theta + \mathfrak{K}_{\theta\omega\Pi}^{(0)}\Pi\theta + \mathfrak{K}_{\pi\sigma\omega}^{(0)}\pi^{\lambda\nu}\sigma_{\lambda\nu} + \mathfrak{K}_{n\omega}^{(0)}\partial\cdot n\right)\omega_0^\mu \\
&+ \left[\mathfrak{K}_{I\Omega}^{(0)} I_\nu + \mathfrak{K}_{\Pi\Omega}^{(0)}\left(-\Pi\dot{u}_\nu + \nabla_\nu\Pi - \Delta_{\nu\lambda}\partial_\rho\pi^{\lambda\rho}\right)\right]\tilde{\Omega}^{\langle\mu\rangle\nu} + \mathfrak{K}_{\nabla\Omega}^{(0)}\Delta^\mu_\lambda\nabla_\nu\tilde{\Omega}^{\lambda\nu} \\
&+ \mathfrak{g}_1^{(0)}\mathfrak{z}^{\mu\nu}F_\nu + \mathfrak{g}_2^{(0)}\sigma_{\alpha\beta}\mathfrak{q}^{\langle\mu\rangle\alpha\beta} - \mathfrak{g}_3^{(0)}\Delta^\mu_\lambda\nabla_\nu\mathfrak{z}^{\lambda\nu} + \mathfrak{g}_4^{(0)}\theta\mathfrak{p}^{\langle\mu\rangle} \\
&- \mathfrak{g}_5^{(0)}\theta\mathfrak{q}^{\nu\mu}{}_\nu + \mathfrak{g}_6^{(0)}\mathfrak{z}^{\mu\nu}I_\nu + \left(\mathfrak{g}_7^{(0)}\mathfrak{p}_\nu + \mathfrak{g}_8^{(0)}\mathfrak{q}^\lambda{}_{\nu\lambda}\right)\left(\sigma^{\nu\mu} + \omega^{\nu\mu}\right) \\
&+ \mathfrak{K}_{\dot{\Omega}}^{(0)}\left(\dot{\omega}_0^{\langle\mu\rangle} - \tilde{\Omega}^{\langle\mu\rangle\nu}\dot{u}_\nu\right) + \mathfrak{g}_9^{(0)}\mathfrak{z}^\nu{}_\nu F^\mu\;.
\end{aligned} \tag{5.108}$$

In the derivation of the above equation we used that the thermodynamic integrals are functions of α_0 and β_0, inserted Eqs. (5.62a), (5.62b) and (5.62c), and defined $F_\nu \equiv \nabla_\nu P_0$. We note that here it would be possible to replace each $\dot{u}^\mu$ by Eq. (5.62c), keeping only terms up to second order. Symmetrizing Eq. (5.78) yields the following kinetic equation for $\mathfrak{z}^{\mu\nu}$,

$$\begin{aligned}
\tau_{\mathfrak{z}}\Delta^\mu_\lambda\Delta^\nu_\rho\frac{d}{d\tau}\mathfrak{z}^{\lambda\rho} + \mathfrak{z}^{\mu\nu} =& \left[\mathfrak{K}_{\omega\Pi}^{(1)}\left(-\Pi\dot{u}^{(\nu} + \nabla^{(\nu}\Pi - \Delta^{(\nu}_\lambda\partial_\rho\pi^{\lambda\rho}\right) - \mathfrak{K}_{\omega I}^{(1)} I^{(\nu}\right]\omega_0^{\mu)} + \mathfrak{K}_{\Omega\sigma}^{(1)}\tilde{\Omega}_\lambda{}^{(\langle\mu\rangle}\sigma^{\nu)\lambda} \\
&- \mathfrak{K}_{\nabla\Omega}^{(1)}\Delta^{(\mu}_\rho\left(\nabla^{\nu)}\tilde{\Omega}^{\rho\lambda}\right)u_\lambda - \tau_{\mathfrak{z}}\omega_\rho{}^{(\nu}\mathfrak{z}^{\mu)\rho} + \mathfrak{g}_1^{(1)}\mathfrak{z}^{\mu\nu}\theta + \mathfrak{g}_2^{(1)}\Delta^{(\nu}_\lambda\Delta^{\mu)}_\tau\nabla_\rho(\Delta^\tau_\alpha\mathfrak{q}^{\alpha,\lambda\rho}) \\
&+ \mathfrak{g}_3^{(1)}(\nabla_\rho u^{(\mu})\mathfrak{z}^{\langle\nu)\rho\rangle} + \mathfrak{g}_4^{(1)}\mathfrak{q}^{(\langle\mu\rangle,\nu)\lambda}F_\lambda + \mathfrak{g}_5^{(1)}\sigma_\lambda{}^{(\nu}\mathfrak{z}^{\mu)\lambda} + \mathfrak{g}_6^{(1)}F^{(\nu}\mathfrak{p}^{\langle\mu\rangle)} + \mathfrak{g}_7^{(1)}\mathfrak{q}^{\rho(\nu}{}_\rho F^{\mu)} \\
&+ \mathfrak{g}_8^{(1)}\Delta^{(\mu}_\lambda\nabla^{\nu)}(\Delta^\lambda_\rho\mathfrak{p}^\rho) + \mathfrak{g}_9^{(1)}\Delta^{(\mu}_\lambda\nabla^{\nu)}\mathfrak{q}^{\rho\lambda}{}_\rho + \mathfrak{g}_{10}^{(1)}(\nabla^{(\nu}u^{\mu)})\mathfrak{z}^\lambda{}_\lambda\;.
\end{aligned} \tag{5.109}$$

The kinetic equation for $\mathfrak{q}^{\langle\lambda\rangle\mu\nu}$ reads

$$\begin{aligned}
\tau_{\mathfrak{q}}\Delta^\mu_\rho\Delta^{\nu\lambda}_{\alpha\beta}\frac{d}{d\tau}\mathfrak{q}^{\langle\rho\rangle\alpha\beta} + \mathfrak{q}^{\langle\mu\rangle\nu\lambda} =& -\mathfrak{d}^{(2)}\beta_0\sigma_\rho{}^{\langle\nu}\epsilon^{\lambda\rangle\mu\alpha\rho}u_\alpha + \mathfrak{K}_{\Omega I}^{(2)}\tilde{\Omega}^{\langle\mu\rangle\langle\nu}I^{\lambda\rangle} + \mathfrak{K}_{\nabla\Omega}^{(2)}\Delta^\mu_\rho\Delta^{\nu\lambda}_{\alpha\beta}\nabla^\alpha\tilde{\Omega}^{\rho\beta} - \mathfrak{K}_{\omega\sigma}^{(2)}\sigma^{\nu\lambda}\omega_0^\mu \\
&- \mathfrak{K}_{\Omega\Pi}^{(2)}\tilde{\Omega}^{\langle\mu\rangle\langle\nu}\left(-\Pi\dot{u}^{\lambda\rangle} + \nabla^{\lambda\rangle}\Pi - \Delta^{\lambda\rangle}_\alpha\partial_\beta\pi^{\alpha\beta}\right) + \mathfrak{g}_1^{(2)}\mathfrak{z}^{\mu\langle\nu}F^{\lambda\rangle} + \mathfrak{g}_2^{(2)}\mathfrak{z}^{\mu\langle\nu}I^{\lambda\rangle} \\
&+ \mathfrak{g}_3^{(2)}\Delta^\mu_\rho\Delta^{\nu\lambda}_{\alpha\beta}\nabla^\beta\mathfrak{z}^{\rho\alpha} + \mathfrak{g}_4^{(2)}\mathfrak{q}^{\langle\mu\rangle,\nu\lambda}\theta + \mathfrak{g}_5^{(2)}\mathfrak{q}^{\langle\mu\rangle,\rho\langle\nu}\sigma^{\lambda\rangle}_\rho + 2\tau_{\mathfrak{q}}\mathfrak{q}^{\langle\mu\rangle,\rho\langle\nu}\omega^{\lambda\rangle}_\rho \\
&+ \mathfrak{g}_6^{(2)}\mathfrak{p}^{\langle\mu\rangle}\sigma^{\nu\lambda} - 6\mathfrak{g}_7^{(2)}\mathfrak{q}^{\rho\mu}{}_\rho\sigma^{\nu\lambda} + \mathfrak{g}_8^{(2)}F^\mu\mathfrak{z}^{\langle\nu\lambda\rangle} + \mathfrak{g}_9^{(2)}\mathfrak{p}^{\langle\nu}\nabla^{\lambda\rangle}u^\mu + \mathfrak{g}_{10}^{(2)}\mathfrak{q}^{\rho\langle\nu}{}_\rho\nabla^{\lambda\rangle}u^\mu\;.
\end{aligned} \tag{5.110}$$

We show the explicit form of the relaxation times and transport coefficients of all equations of motion in Appendix D.4.

[1] In this context, by second order we mean terms of first order in products of dissipative quantities with gradients of the order of the inverse hydrodynamic scale or first-order gradients of dissipative quantities.

In Eq. (5.108) one should insert Eq. (5.67) for $\dot{\omega}_0^\mu$, however, this would make the expression lengthy, thus we do not do it here. The antisymmetric part of the energy-momentum tensor appearing in Eq. (5.67) now has to be written in terms of the dynamical spin moments. The summation in Eqs. (5.69) and (5.70), respectively, is truncated in the 14+24-moment approximation using $\mathbb{S}_1 = \{1\}$. The result is

$$\begin{aligned} 2\epsilon^{\alpha\beta\mu\nu}u_\beta T_{\mu\nu} =& \frac{\hbar}{m}\left[\mathfrak{C}_1^{\langle\alpha\rangle} - m\mathfrak{C}_0^{\langle\alpha\rangle} - \sum_{j\in\mathbb{S}_1} m^{2j}u_\lambda\left(\mathfrak{C}^{\lambda,\alpha}_{-2j-1} - m\mathfrak{C}^{\lambda,\alpha}_{-2j-2}\right)\right], \\ T^{[\mu\nu]}u_\nu =& -\frac{\hbar}{2m}\epsilon^{\mu\nu\alpha\beta}u_\alpha \sum_{j\in\mathbb{S}_1} m^{2j}\left(\mathfrak{C}_{-2j\beta,\nu} - m\mathfrak{C}_{(-2j-1)\beta,\nu}\right). \end{aligned} \tag{5.111}$$

The collision terms appearing in the above equation can be written in terms of the dynamical spin moments with the help of Eqs. (5.90), (5.97), (5.101), and (5.102) .

5.8 Pauli-Lubanski vector

As discussed in Chapter 1, the measureable quantity in heavy-ion collisions is the Pauli-Lubanski vector. Using Eq. (2.74) it can be expressed in terms of the axial-vector component of the Wigner operator $\hat{\mathcal{A}}^\mu$. Taking the ensemble average and choosing the hypersurface of the integration at the freeze-out time, one has [41, 46, 126]

$$\Pi^\mu(p) = \frac{\hbar}{2\mathcal{N}}\int d\Sigma_\lambda p^\lambda dS\, \mathfrak{s}^\mu f(x,p,\mathfrak{s}), \tag{5.112}$$

where $d\Sigma_\lambda$ denotes the integration over the freeze-out hypersurface and we defined

$$\mathcal{N} \equiv \int d\Sigma_\lambda p^\lambda \int dS f(x,p,\mathfrak{s}) \,. \tag{5.113}$$

Replacing the distribution function f in Eq. (5.112) by Eq. (5.50) and employing the 14+24-moment approximation, i.e., inserting Eqs. (5.101) and (5.102), we find we obtain

$$\begin{aligned} \Pi^\mu(p) =& \frac{1}{2\mathcal{N}}\left(g^\mu{}_\nu - \frac{p^\mu p_\nu}{p^2}\right)\int d\Sigma_\lambda p^\lambda f_{0p}\bigg\{-\frac{\hbar}{2m}\tilde{\Omega}^{\nu\rho}\left(E_p u_\rho + p_{\langle\rho\rangle}\right) + \chi_\mathfrak{p}\mathfrak{p}^{\langle\nu\rangle} - 6\chi_\mathfrak{n}\mathfrak{q}^{\rho\nu}{}_\rho + \mathfrak{x}_\mathfrak{n} u^\nu \mathfrak{z}^\lambda{}_\lambda \\ &+ \left[\chi_\mathfrak{z}\mathfrak{z}^{\nu\alpha} + \left(\mathfrak{x}_\mathfrak{q}\mathfrak{q}^{\lambda\alpha}{}_\lambda + \mathfrak{x}_\mathfrak{p}\mathfrak{p}^{\langle\alpha\rangle}\right)u^\nu\right]p_{\langle\alpha\rangle} + \left(\chi_\mathfrak{q}\mathfrak{q}^{\langle\nu\rangle\alpha\beta} + \mathfrak{x}_\mathfrak{z} u^\nu \mathfrak{z}^{\langle\alpha\beta\rangle}\right)p_{\langle\alpha}p_{\beta\rangle}\bigg\}, \end{aligned} \tag{5.114}$$

with

$$\begin{aligned} \chi_\mathfrak{p} &\equiv -2\sum_{n\in\mathbb{S}_0}\mathcal{H}^{(0)}_{pn}\left(\mathfrak{F}^{(0)}_{n0} - 2m^2\mathfrak{F}^{(0)}_{n2}\right), \\ \chi_\mathfrak{n} &\equiv -2\sum_{n\in\mathbb{S}_0}\mathcal{H}^{(0)}_{pn}\mathfrak{F}^{(0)}_{n2}, \\ \chi_\mathfrak{z} &\equiv -\sum_{n\in\mathbb{S}_1}\mathcal{H}^{(1)}_{pn}\mathfrak{F}^{(1)}_{n1}, \\ \chi_\mathfrak{q} &\equiv -2\sum_{n\in\mathbb{S}_2}\mathcal{H}^{(2)}_{pn}\mathfrak{F}^{(2)}_{n0}, \\ \mathfrak{x}_\mathfrak{n} &\equiv \sum_{n\in\mathbb{S}_0}\mathcal{H}^{(0)}_{pn}\mathfrak{F}^{(1)}_{(n-1)1}, \\ \mathfrak{x}_\mathfrak{q} &\equiv 2\sum_{n\in\mathbb{S}_1}\mathcal{H}^{(1)}_{pn}\left[\mathfrak{F}^{(2)}_{(n-1)0} - 2\left(m^2\mathfrak{F}^{(0)}_{(n-1)2} - \mathfrak{F}^{(0)}_{(n+1)2}\right)\right], \\ \mathfrak{x}_\mathfrak{p} &\equiv \frac{2}{3}\sum_{n\in\mathbb{S}_1}\mathcal{H}^{(1)}_{pn}\left[\left(m^2\mathfrak{F}^{(0)}_{(n-1)0} - \mathfrak{F}^{(0)}_{(n+1)0}\right) - 2m^2\left(m^2\mathfrak{F}^{(0)}_{(n-1)2} - \mathfrak{F}^{(0)}_{(n+1)2}\right)\right], \\ \mathfrak{x}_\mathfrak{z} &\equiv \frac{2}{5}\sum_{n\in\mathbb{S}_2}\mathcal{H}^{(2)}_{pn}\left(m^2\mathfrak{F}^{(1)}_{(n-1)1} - \mathfrak{F}^{(1)}_{(n+1)1}\right). \end{aligned} \tag{5.115}$$

The global polarization is obtained by averaging over all momenta, i.e.,

$$\bar{\Pi}^\mu = \frac{\hbar}{2\bar{\mathcal{N}}}\int dP \int d\Sigma_\lambda p^\lambda dS(p)\, \mathfrak{s}^\mu f(x,p,\mathfrak{s}), \tag{5.116}$$

where we defined

$$\bar{\mathcal{N}} \equiv \int dP\, \mathcal{N} \; . \tag{5.117}$$

In the 14+24-moment approximation we find

$$\begin{aligned}
\bar{\Pi}^\mu =& \frac{\hbar}{2N} \int d\Sigma_\lambda \left(-u^\lambda \frac{\hbar}{2m} \omega_0^\mu I_{20} + \frac{\hbar}{2m} \Delta^\lambda_\nu \tilde{\Omega}^{\mu\nu} I_{21} + u^\lambda \tau_1^\mu + \tau_0^{\mu,\lambda} \right) \\
=& \frac{\hbar}{2N} \int d\Sigma_\lambda \Big[-u^\lambda \frac{\hbar}{2m} \omega_0^\mu I_{20} + \frac{\hbar}{2m} \Delta^\lambda_\nu \tilde{\Omega}^{\mu\nu} I_{21} + u^\lambda \left(\mathfrak{y}_\mathfrak{p} \mathfrak{p}^{\langle\mu\rangle} + \mathfrak{y}_\mathfrak{q} \mathfrak{q}^{\rho\mu}{}_\rho - \mathfrak{y}_\mathfrak{z} u^\mu \mathfrak{z}^\rho{}_\rho \right) + \mathfrak{y}_\mathfrak{z} \mathfrak{z}^{\mu\lambda} \\
&+ u^\mu \left(\mathfrak{w}_\mathfrak{p} \mathfrak{p}^{\langle\lambda\rangle} + \mathfrak{w}_\mathfrak{q} \mathfrak{q}^{\rho\lambda}{}_\rho \right) \Big] \; ,
\end{aligned} \tag{5.118}$$

where

$$\begin{aligned}
&\mathfrak{y}_\mathfrak{p} \equiv \mathfrak{F}^{(0)}_{10} - 2m^2 \mathfrak{F}^{(0)}_{12} \; , \qquad \mathfrak{y}_\mathfrak{q} \equiv -6 \mathfrak{F}^{(0)}_{12} \; , \qquad \mathfrak{y}_\mathfrak{z} \equiv \frac{1}{2} \mathfrak{F}^{(1)}_{01} , \\
&\mathfrak{w}_\mathfrak{p} \equiv -\frac{1}{3} \left(m^2 \mathfrak{F}^{(0)}_{-10} - \mathfrak{F}^{(0)}_{10} \right) + m^2 \frac{2}{3} \left(m^2 \mathfrak{F}^{(0)}_{-12} - \mathfrak{F}^{(0)}_{12} \right) \; , \quad \mathfrak{w}_\mathfrak{q} \equiv -\mathfrak{F}^{(2)}_{-10} + 2 \left(m^2 \mathfrak{F}^{(0)}_{-12} - \mathfrak{F}^{(0)}_{12} \right) \; .
\end{aligned} \tag{5.119}$$

In kinetic theory, the momentum p^μ of the distribution function is usually considered as the particle momentum. Although this identification is not accurate, as the momentum variable of the Wigner function is not the momentum of a particle state, this is a reasonable approximation for a system where the quantum momenta cannot be calculated. Then the polarization vector $\boldsymbol{\mathcal{P}}_\star$ entering the angular distribution in Eq. (1.26) can be identified with the Pauli-Lubanski vector in Eq. (5.114) or Eq. (5.118) in the rest frame of p for the local or global polarization, respectively,

$$\boldsymbol{\mathcal{P}}_\star \equiv \boldsymbol{\Pi}_\star \; . \tag{5.120}$$

This yields the connection of our results to the experimental observation of Λ polarization in heavy-ion collisions discussed in Chapter 1.

5.9 Navier-Stokes limit

In the second-order theory considered so far, the spin moments are treated dynamically and follow the equations of motion shown in Section 5.7. On long time scales, they approach their Navier-Stokes values, which will be discussed in the following. In this limit, the Pauli-Lubanski vector can be expressed in terms of the thermodynamic potentials and their derivatives. In particular, it will obtain some contribution from shear, which is proportional to $p_{\langle\alpha} p_{\beta\rangle}$, see also Refs. [10–13] for related results.

5.9.1 Vorticity effects on Navier-Stokes values of spin-independent moments

As already mentioned, for parity-conserving interactions, the standard dissipative currents, i.e., the bulk viscous pressure Π, the particle diffusion n^μ, and the shear-stress tensor $\pi^{\mu\nu}$ are not modified by including spin degrees of freedom up to first order in $\hbar$. However, their explicit expressions are modified by the power-counting introduced in Section 5.2, as vorticity as a nondissipative quantity is counted as of zeroth order in the expansion. In this section, we discuss the Navier-Stokes solutions in the presence of arbitrarily large vorticity. In the Navier-Stokes limit, employing the power-counting explained above, the equations which determine n^μ and $\pi^{\mu\nu}$ read

$$\begin{aligned}
n^\mu - \tau_n \omega^\mu{}_\nu n^\nu =& \kappa_n I^\mu , \\
\pi^{\mu\nu} + 2\tau_\pi \omega_\lambda{}^{\langle\mu} \pi^{\nu\rangle\lambda} =& 2\eta \sigma^{\mu\nu} ,
\end{aligned} \tag{5.121}$$

where τ_n and τ_π are the relaxation times for particle diffusion and shear stress in Eqs. (E.21), respectively, and κ_n and η are given in Eqs. (E.18). Equations (5.121) can be obtained by considering only terms of first order in dissipative gradients (i.e., gradients except vorticity) in the relaxation equations in Ref. [35] or, alternatively, by a Chapman-Enskog expansion of the distribution function which takes into account our power-counting scheme. The latter is explained in Appendix E. The structure of Eqs. (5.121) is similar to the relativistic Navier-Stokes equations in the presence of electromagnetic fields in Ref. [64]. We solve these equations following similar lines. First, we define $\omega \equiv \sqrt{\omega^{\mu\nu}\omega_{\mu\nu}/2}$ and $\hat{\omega}^{\mu\nu} \equiv -\omega^{\mu\nu}/\omega$. Furthermore, we define the unit vector along the vorticity direction $\hat{\omega}^{\mu} \equiv \omega^{\mu}/\omega$ with the vorticity vector

$$\omega^{\mu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_{\nu}\omega_{\alpha\beta} \tag{5.122}$$

and the projector orthogonal to both the fluid velocity and vorticity vector

$$\Xi^{\mu\nu} \equiv g^{\mu\nu} - u^{\mu}u^{\nu} + \hat{\omega}^{\mu}\hat{\omega}^{\nu}. \tag{5.123}$$

Now we make the following ansatz to solve the first equation in (5.121),

$$n^{\mu} = \left(\kappa_{\Delta}\Xi^{\mu\nu} + \kappa_{v}\hat{\omega}^{\mu}\hat{\omega}^{\nu} + \kappa_{\omega}\hat{\omega}^{\mu\nu}\right) I_{\nu}. \tag{5.124}$$

Inserting yields the following set of equations,

$$\begin{aligned}
\kappa_{\Delta} - \tau_n\omega\kappa_{\omega} &= \kappa_n, \\
\kappa_v &= -\kappa_n, \\
\kappa_{\omega} - \tau_n\omega\kappa_{\Delta} &= 0,
\end{aligned} \tag{5.125}$$

with the solution

$$\begin{aligned}
\kappa_v &= -\kappa_n, \\
\kappa_{\omega} &= \frac{\tau_n\omega}{1+(\tau_n\omega)^2}\kappa_n, \\
\kappa_{\Delta} &= \frac{1}{1+(\tau_n\omega)^2}\kappa_n.
\end{aligned} \tag{5.126}$$

We see that the vorticity does not change the particle diffusion along the direction of the vorticity vector, but reduces the particle diffusion orthogonal to the vorticity vector. This leads to an anisotropy of the system and is similar to the effect of a magnetic field. As is intuitive, the effects of vorticity are larger the larger the absolute value of the vorticity is and the slower the dissipative currents relax to their asymptotic solutions. For small $\tau_n\omega$ one can Taylor expand the above solutions up to first order

$$\begin{aligned}
\kappa_v &= -\kappa_n, \\
\kappa_{\omega} &= \tau_n\omega\kappa_n + \mathcal{O}((\tau_n\omega)^2), \\
\kappa_{\Delta} &= \kappa_n + \mathcal{O}((\tau_n\omega)^2).
\end{aligned} \tag{5.127}$$

To this order the change in the particle diffusion orthogonal to the vorticity vector vanishes and only the term proportional to κ_{ω} remains. In other words, anisotropic effects can be neglected. In the other limit $\tau_n\omega \gg 1$ we obtain

$$\begin{aligned}
\kappa_v &= -\kappa_n, \\
\kappa_{\omega} &= \frac{1}{\tau_n\omega}\kappa_n + \mathcal{O}((\tau_n\omega)^{-2}), \\
\kappa_{\Delta} &= \mathcal{O}((\tau_n\omega)^{-2}).
\end{aligned} \tag{5.128}$$

In this case the system is highly anisotropic, only the projection of the particle diffusion along the vorticity vector is nonzero if the vorticity approaches infinity. In both limits κ_{ω} is small compared to κ_n. This coefficient assumes the maximal value of $\kappa_{\omega} = \kappa_n/2$ for $\tau_n\omega = 1$.

We now turn to the slightly more complicated solution for the shear-stress tensor. Due to the analogy of vorticity to a magnetic field, we choose the same ansatz as in Ref. [64],

$$\begin{aligned}
\pi^{\mu\nu} = \Big[& 2\eta_0\Delta^{\mu\nu\alpha\beta} + \eta_1\left(\Delta^{\mu\nu} - \frac{3}{2}\Xi^{\mu\nu}\right)\left(\Delta^{\alpha\beta} - \frac{3}{2}\Xi^{\alpha\beta}\right) - 2\eta_2\Xi^{\alpha(\mu}\hat{\omega}^{\nu)}\hat{\omega}^{\beta} - 2\eta_3\Xi^{\alpha(\mu}\hat{\omega}^{\nu)\beta} \\
& + 2\eta_4\hat{\omega}^{\alpha(\mu}\hat{\omega}^{\nu)}\hat{\omega}^{\beta}\Big]\sigma_{\alpha\beta}.
\end{aligned} \tag{5.129}$$

We have

$$\hat{\omega}_{\lambda}^{\ \langle\mu}\pi^{\nu\rangle\lambda} = 2\eta_0\hat{\omega}_{\lambda}^{\ \langle\mu}\sigma^{\nu\rangle\lambda} - 2\eta_2\hat{\omega}^{\alpha\langle\mu}\hat{\omega}^{\nu\rangle}\hat{\omega}^{\beta}\sigma_{\alpha\beta} - 2\eta_3\left(\Xi^{\beta\langle\mu}\Xi^{\nu\rangle\alpha} + \hat{\omega}^{\alpha\langle\mu}\hat{\omega}^{\nu\rangle\beta}\right)\sigma_{\alpha\beta} - 2\eta_4\Xi^{\alpha\langle\mu}\hat{\omega}^{\nu\rangle}\hat{\omega}^{\beta}\sigma_{\alpha\beta}. \tag{5.130}$$

Inserting this into Eq. (5.121) and collecting terms of the same tensor structure we obtain

$$\begin{aligned} 2\eta_0 + 8\tau_\pi\omega\eta_3 &= 2\eta, \\ 4\eta_3 - 4\tau_\pi\omega\eta_0 &= 0, \\ 4(\eta_3+\eta_4) + 4\tau_\pi\omega\eta_2 &= 0, \\ -4\eta_2 + 4\tau_\pi\omega(4\eta_3+\eta_4) &= 0, \\ \frac{3}{4}\eta_1 - 4\tau_\pi\omega\eta_3 &= 0\,. \end{aligned} \tag{5.131}$$

Equations (5.131) are solved by

$$\begin{aligned} \eta_0 &= \frac{1}{1+4(\tau_\pi\omega)^2}\eta, \\ \eta_1 &= \frac{16}{3}\frac{(\tau_\pi\omega)^2}{1+4(\tau_\pi\omega)^2}\eta, \\ \eta_2 &= (\tau_\pi\omega)^2\left(\frac{1}{1+4(\tau_\pi\omega)^2} - \frac{1}{1+(\tau_\pi\omega)^2}\right)\eta, \\ \eta_3 &= \frac{\tau_\pi\omega}{1+4(\tau_\pi\omega)^2}\eta, \\ \eta_4 &= -\frac{\tau_\pi\omega}{1+(\tau_\pi\omega)^2}\eta. \end{aligned} \tag{5.132}$$

For $\tau_\pi\omega \ll 1$ we have

$$\begin{aligned} \eta_0 &= \eta + \mathcal{O}\left((\tau_\pi\omega)^2\right), \\ \eta_3 &= \tau_\pi\omega\,\eta + \mathcal{O}\left((\tau_\pi\omega)^2\right), \\ \eta_4 &= -\tau_\pi\omega\,\eta + \mathcal{O}\left((\tau_\pi\omega)^2\right), \end{aligned} \tag{5.133}$$

and η_1 and η_2 vanish at this order. In other words, the shear-viscosity coefficient reduces to standard one at zeroth order and to linear order the only vorticity-dependent contribution to the shear is proportional to $\Delta^{\alpha(\mu}\hat{\omega}^{\nu)\beta}$. On the other hand, for $\tau_\pi\omega \gg 1$ the only nonzero contributions up to first order come from

$$\begin{aligned} \eta_1 &= \frac{4}{3}\eta + \mathcal{O}\left((\tau_\pi\omega)^{-2}\right), \\ \eta_3 &= \frac{1}{4\tau_\pi\omega}\eta + \mathcal{O}\left((\tau_\pi\omega)^{-2}\right), \\ \eta_2 &= -\frac{3}{4}\eta + \mathcal{O}\left((\tau_\pi\omega)^{-2}\right). \end{aligned} \tag{5.134}$$

Similar to the particle diffusion, the longer the relaxation time and the stronger the vorticity, the more the usual viscosity η_0 is reduced and the vorticity determines the components of the shear-stress tensor.

5.9.2 Navier-Stokes values of spin moments

The Navier-Stokes limit of the spin moments is obtained by considering only terms up to first order in dissipative gradients in the equations of motion in Section 5.7. As pointed out before, in our power-counting scheme, the vorticity is considered to be of zeroth order in dissipative gradients, since we take into account the possibility of a rotating equilibrium state. Note that we count $\nabla^\nu \tau_r^{\mu,\mu_1\cdots\mu_n}$ as of second order. We obtain the following Navier-Stokes solutions

$$\begin{aligned} \mathfrak{p}_{\rm NS}^{\langle\mu\rangle} &= \mathfrak{p}_{\rm nr}^{\langle\mu\rangle}\,, \\ \mathfrak{z}_{\rm NS}^{\mu\nu} &= \mathfrak{z}_{\rm nr}^{\mu\nu} + \tau_{\mathfrak{z}}\omega_\rho{}^{(\nu}\mathfrak{z}_{\rm NS}^{\mu)\rho}\,, \\ \mathfrak{q}_{\rm NS}^{\langle\mu\rangle\nu\lambda} &= \mathfrak{q}_{\rm nr}^{\langle\mu\rangle\nu\lambda} - 2\tau_{\mathfrak{q}}\mathfrak{q}_{\rm NS}^{\langle\mu\rangle\rho\langle\nu}\omega^{\lambda\rangle}_{\ \ \rho}\,, \end{aligned} \tag{5.135}$$

where the Navier-Stokes values for relaxation to a nonrotating equilibrium state are given as

$$\mathfrak{p}_{\rm nr}^{\langle\mu\rangle} = \mathfrak{e}^{(0)}\left(\tilde{\Omega}^{\mu\nu} - \tilde{\varpi}^{\mu\nu}\right)u_\nu + \mathfrak{K}_{\theta\omega}^{(0)}\theta\omega_0^\mu + \mathfrak{K}_{I\Omega}^{(0)}I_\nu\tilde{\Omega}^{\langle\mu\rangle\nu} + \mathfrak{K}_{\nabla\Omega}^{(0)}\Delta^\mu_\lambda\nabla_\nu\tilde{\Omega}^{\lambda\nu} + \mathfrak{K}_{\dot{\Omega}}^{(0)}\dot{\omega}_0^{\langle\mu\rangle}\,, \tag{5.136a}$$

$$\mathfrak{z}_{\rm nr}^{\mu\nu} = -\mathfrak{K}_{\nabla\Omega}^{(1)}\Delta^{(\mu}_\rho\left(\nabla^{\nu)}\tilde{\Omega}^{\rho\lambda}\right)u_\lambda - \mathfrak{K}_{\omega I}^{(1)}I^{(\nu}\omega_0^{\mu)} + \mathfrak{K}_{\Omega\sigma}^{(1)}\tilde{\Omega}_\lambda{}^{(\langle\mu\rangle}\sigma^{\nu)\lambda}\,, \tag{5.136b}$$

$$\mathfrak{q}_{\rm nr}^{\langle\mu\rangle\nu\lambda} = -\mathfrak{d}^{(2)}\beta_0\sigma_\rho{}^{\langle\nu}\epsilon^{\lambda\rangle\mu\alpha\rho}u_\alpha + \mathfrak{K}_{\Omega I}^{(2)}\tilde{\Omega}^{\langle\mu\rangle\langle\nu}I^{\lambda\rangle} + \mathfrak{K}_{\nabla\Omega}^{(2)}\Delta^{\nu\lambda}_{\alpha\beta}\nabla^\alpha\tilde{\Omega}^{\mu\beta} - \mathfrak{K}_{\omega\sigma}^{(2)}\sigma^{\nu\lambda}\omega_0^\mu\,. \tag{5.136c}$$

The solution of Eqs. (5.135) is obtained analogously to the calculation in the previous section as

$$
\begin{aligned}
\mathfrak{z}^{\mu\nu}_{\rm NS} =& \bigg[2\lambda_0 \Delta^{\mu\nu\alpha\beta} + \lambda_1 \left(\Delta^{\mu\nu} - \frac{3}{2}\Xi^{\mu\nu}\right)\left(\Delta^{\alpha\beta} - \frac{3}{2}\Xi^{\alpha\beta}\right) - 2\lambda_2 \Xi^{\alpha(\mu}\hat{\omega}^{\nu)}\hat{\omega}^{\beta} - 2\lambda_3 \Xi^{\alpha(\mu}\hat{\omega}^{\nu)\beta} \\
&+ 2\lambda_4 \hat{\omega}^{\alpha(\mu}\hat{\omega}^{\nu)}\hat{\omega}^{\beta} + \frac{1}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}\bigg]\mathfrak{z}_{{\rm nr}\alpha\beta} \,, \\
\mathfrak{q}^{\langle\mu\rangle\nu\lambda}_{\rm NS} =& \bigg[2\eta_0 \Delta^{\nu\lambda\alpha\beta} + \eta_1 \left(\Delta^{\nu\lambda} - \frac{3}{2}\Xi^{\nu\lambda}\right)\left(\Delta^{\alpha\beta} - \frac{3}{2}\Xi^{\alpha\beta}\right) - 2\eta_2 \Xi^{\alpha(\nu}\hat{\omega}^{\lambda)}\hat{\omega}^{\beta} - 2\eta_3 \Xi^{\alpha(\nu}\hat{\omega}^{\lambda)\beta} \\
&+ 2\eta_4 \hat{\omega}^{\alpha(\nu}\hat{\omega}^{\lambda)}\hat{\omega}^{\beta}\bigg]\mathfrak{q}^{\langle\mu\rangle}_{{\rm nr}\;\alpha\beta} \,. \qquad (5.137)
\end{aligned}
$$

The coefficients are given in Appendix D.4, see Eq. (D.59). We see that there is a contribution to $\mathfrak{q}^{\langle\mu\rangle\nu\lambda}$ from the shear tensor $\sigma^{\mu\nu}$ in Eq. (5.136c), which is independent of the spin potential. This will give a contribution to the local polarization in Eq. (5.114) in the Navier-Stokes limit, but not to the global polarization in Eq. (5.118), similar to what was found in Refs. [10–13]. Although the form of our shear-dependent contribution to the local polarization is similar to the one of Refs. [10–13], their origin and in general also their coefficients are different. While our result is derived from the nonlocal collision term in the dissipative case, the derivations in Refs. [10–13] assume local equilibrium. On the other hand, in our framework shear-related terms are absent in local equilibrium. In this context, it should be noted that the result derived in Ref. [13] depends on the choice of pseudo-gauge [14]. In particular, the shear-dependent contributions to the polarization in the approach of Ref. [13] vanish in local equilibrium, if the HW pseudo-gauge is used, as is done in this work. It may be speculated that the choice of pseudo-gauge affects the splitting of hydrodynamic quantities into ideal and dissipative contributions, and that in the approach of Ref. [13] with the HW pseudo-gauge shear-dependent contributions to the polarization may be found if dissipative effects are considered. On the other hand, a different choice of pseudo-gauge in our framework may lead to ideal contributions from the shear tensor to polarization. Detailed studies on this and an analysis of the polarization determined through the equations presented here in comparison to the one determined through the results of Refs. [10–13] regarding the agreement with experimental data for Lambda polarization in heavy-ion collisions will certainly be interesting and are left for future work.

Chapter 6

Conclusions and future perspectives

In this thesis, we derived second-order dissipative spin hydrodynamics from quantum field theory, using the Wigner-function formalism to obtain a kinetic theory and, in the next step, the method of moments to derive dissipative equations of motion from the Boltzmann equation. We summarize our most important findings in the following.

In spin kinetic theory, a nonlocal collision term is necessary to see that, in the approach to equilibrium, spin polarization becomes aligned with the fluid vorticity. If in the case of a local collision term the spin is a collisional invariant, changes in the spin density only happen in form of diffusion. On the other hand, taking into account the microscopic nonlocality of the collisions, only the total angular momentum, but not the spin, is a collisional invariant, and orbital angular momentum can be converted into spin. Thus, the fluid is polarized along the vorticity direction. A quantum field-theoretical calculation shows that the dynamics of the Wigner function can be described by an on-shell kinetic equation of Boltzmann form, where the distribution functions depend on an additional phase-space variable for the spin degrees of freedom. The nonlocality of the collision term in this equation is characterized by displacements of the position arguments of the distribution functions with respect to the center of the collision. The nonlocal collision term vanishes identically under the conditions of global equilibrium. In particular in this case the spin potential becomes equal to the thermal vorticity. However, under the conditions of local equilibrium, the nonlocal collision term does not vanish in general. This can be understood considering the ordering of scales in our framework, where the scale characterizing the nonlocality of the collision term is smaller or comparable to the scale defined through dissipative gradients. Therefore, neglecting dissipation while taking into account a nonlocal collision term would be inconsistent. Apart from defining the equilibrium state, the spin-dependent Boltzmann equation also provides the starting point to derive dissipative spin hydrodynamics.

A central role in spin hydrodynamics is played by the spin tensor, which depends on the choice of pseudo-gauge. The canonical form of the spin tensor is derived applying Noether's theorem to the Dirac Lagrangian and is not conserved even for free fields or in global equilibrium, inconsistent with the physical picture of a spin density changing only through interactions until the system reaches equilibrium. On the other hand, the HW spin tensor derived from the Klein-Gordon Lagrangian for spinors is conserved for free fields. In the presence of interactions, the HW pseudo-gauge transformation is modified by the interaction terms. The antisymmetric part of the HW energy-momentum tensor is nonzero only in the presence of nonlocal collisions, yielding the nonconservation of the spin tensor in this case and making the dynamics dissipative. When the collision term vanishes in global equilibrium, the HW spin tensor is conserved. In conclusion, the HW set of tensors yields a clearer physical interpretation of the spin tensor as a spin density than the canonical one. Furthermore, the antisymmetric part of the HW energy-momentum tensor reduces to a well-known form in the nonrelativistic limit.

In order to derive equations of motion for the dissipative spin currents, the method of moments was generalized to include spin degrees of freedom, treating all components of the HW spin tensor as dynamical variables with equations of motion following from the Boltzmann equation. The definition of the local-equilibrium state makes use of an ordering of scales where the vorticity is an equilibrium quantity, which can be larger than the dissipative gradients. As the choice of the dynamical spin moments is pseudo-gauge dependent, the evolution of the system can be different if a different pseudo-gauge is chosen. The equations of motion of the spin moments are coupled with those of the usual moments only through the collision term. In the case of a scalar or a low-momentum vector interaction, the two sets of equations formally decouple from each other. The nonlocal collision term gives contributions proportional to the difference between spin potential and thermal vorticity, as well as from thermal shear, to the Navier-Stokes limit of the Pauli-Lubanski vector. In this limit, the global polarization does not depend on the shear tensor, but the local polarization does. The treatment of vorticity as a zeroth-order quantity leads to modifications of the Navier-Stokes limits of both the usual moments and the spin moments in comparison to their values when expanding around a homogeneous equilibrium state.

The work presented in this thesis can be continued and extended in various ways. The equations of motion for the dynamical spin moments shown in Chapter 5.7 should be analyzed regarding their causality and stability. If they fulfill these requirements, they can be numerically implemented and used, e.g., for the application in relativistic heavy-ion collisions by comparing with experimental data of Lambda polarization. This may shed new light on polarization effects in the quark-gluon plasma.

The kinetic equations of the spin moments derived in this thesis can be improved by applying the DNMR method [35] instead of using the 14+24-moments approximation. That method has the advantage of considering an expansion in a small power-counting parameter, the Knudsen number, which the 14+24-moment approximation lacks. Another possibility is to use the general equations of motion for the spin moments in Chapter 5.5 with a general frame choice and general matching conditions to derive a first-order theory similar to the approach of Refs. [28–31]. First steps towards a stable and causal first-order theory with chirality have recently been performed in Ref. [197]. Furthermore, one can also use the second-order theory presented in this thesis with a different frame-choice and matching conditions, see Refs. [198, 199] for studies in that direction in the spinless case. Another interesting extension of this work is the generalization to interactions which lead to a coupling between the spin moments and the usual moments. In particular, this allows for studies of the modifications of the bulk viscous pressure and shear stress tensor due to spin and parity-violating interactions. Moreover, it may be of interest to repeat the calculation of second-order equations of motion presented here in a different pseudo-gauge and study the effects on the dynamics of the system.

In our derivation of dissipative spin hydrodynamics we did not consider electromagnetic fields. These can be included using the generalized Boltzmann equation (4.52) and the form of the conserved currents in Chapter 4.4 to derive dissipative spin magneto-hydrodynamics from the method of moments, see Refs. [64,161] for the spinless case. Furthermore, in this work we restricted ourselves to spin-1/2 particles. However, our framework can be extended to the case of spin-1 particles, which are also of interest, e.g., in heavy-ion collisions for the study of the polarization of ϕ mesons [200, 201]. Therefore, it will be useful to repeat both the derivation of kinetic theory with a nonlocal collision term from quantum field theory and of dissipative hydrodynamics from the Boltzmann equation for particles with spin 1. Also an extension of the (massive) spin-1 theory to non-abelian gauge fields to study gluon dynamics would be of interest. Finally, the combination of both the spin-1/2 and spin-1 theories, including interactions between fermions and gluons, would lead to a deeper understanding of the spin dynamics in the quark-gluon plasma and hence would be desirable.

Appendix A

Useful identities

Properties of three-dimensional Levi-Civita tensor:

$$\epsilon^{ijk}\epsilon^{imn} = \delta^{jm}\delta^{kn} - \delta^{jn}\delta^{km} \tag{A.1a}$$

$$\epsilon^{ijk}\epsilon^{lmn} = \delta^{il}\delta^{jm}\delta^{kn} + \delta^{im}\delta^{jn}\delta^{kl} + \delta^{in}\delta^{jl}\delta^{km} - \delta^{im}\delta^{jl}\delta^{kn} - \delta^{in}\delta^{jm}\delta^{kl} - \delta^{im}\delta^{jl}\delta^{kn} \tag{A.1b}$$

Schouten identity: for any four-vector a^μ one has

$$a^\mu\epsilon^{\nu\lambda\rho\sigma} + a^\nu\epsilon^{\lambda\rho\sigma\mu} + a^\lambda\epsilon^{\rho\sigma\mu\nu} + a^\rho\epsilon^{\sigma\mu\nu\lambda} + a^\sigma\epsilon^{\mu\nu\lambda\rho} = 0. \tag{A.2}$$

Identities involving Dirac matrices γ^μ:

$$\gamma^\lambda\gamma^\mu = g^{\lambda\mu} - i\sigma^{\lambda\mu}, \tag{A.3a}$$

$$\gamma^\lambda\sigma^{\mu\nu} = ig^{\lambda[\mu}\gamma^{\nu]} + \epsilon^{\lambda\mu\nu\rho}\gamma^5\gamma_\rho, \tag{A.3b}$$

$$\sigma^{\mu\nu}\sigma^{\lambda\rho} = g^{\mu[\lambda}g^{\rho]\nu} + ig^{\mu[\lambda}\sigma^{\rho]\nu} - ig^{\nu[\lambda}\sigma^{\rho]\mu} + i\epsilon^{\mu\nu\lambda\rho}\gamma^5, \tag{A.3c}$$

$$[\gamma^\mu, \sigma^{\nu\lambda}] = 2ig^{\mu[\nu}\gamma^{\lambda]}, \tag{A.3d}$$

$$[\sigma^{\mu\nu}, \sigma^{\lambda\rho}] = 2i(g^{\mu\rho}\sigma^{\nu\lambda} + g^{\nu\lambda}\sigma^{\mu\rho} - g^{\mu\lambda}\sigma^{\nu\rho} - g^{\nu\rho}\sigma^{\mu\lambda}), \tag{A.3e}$$

where $\sigma^{\mu\nu} \equiv (i/2)[\gamma^\mu, \gamma^\nu]$.

Appendix B

Energy-momentum and spin tensors from the Wigner function

In this appendix, we show some details of how to express the HW and GLW energy-momentum tensors in terms of the Wigner operator in Eq. (2.58). The calculation of the spin tensors $\hat{S}^{\lambda,\mu\nu}_{HW/GLW/KG}$ is completely analogous. First, $\hat{T}^{\mu\nu}_{GLW}$ in Eq. (2.46a) can be connected to the Wigner operator as follows,

$$
\begin{aligned}
\hat{T}^{\mu\nu}_{GLW}(x) =& \frac{\hbar^2}{4m}\bar{\psi}(x)\overleftrightarrow{\partial}^{\mu}_{x}\overleftrightarrow{\partial}^{\nu}_{x}\psi(x) \\
=& \int d^4y\, \delta^4(y) \frac{\hbar^2}{4m}\bar{\psi}\left(x+\frac{y}{2}\right)\overleftrightarrow{\partial}^{\mu}_{x}\overleftrightarrow{\partial}^{\nu}_{x}\psi\left(x-\frac{y}{2}\right) \\
=& \frac{1}{(2\pi\hbar)^4}\int d^4y \int d^4p\, e^{-\frac{i}{\hbar}y\cdot p}\frac{\hbar^2}{4m}\bar{\psi}\left(x+\frac{y}{2}\right)\overleftrightarrow{\partial}^{\mu}_{x}\overleftrightarrow{\partial}^{\nu}_{x}\psi\left(x-\frac{y}{2}\right) \\
=& \frac{1}{(2\pi\hbar)^4}\int d^4y \int d^4p\, e^{-\frac{i}{\hbar}y\cdot p}\frac{\hbar^2}{m}\left[\left(\partial^{\mu}_{y}\partial^{\nu}_{y}\bar{\psi}\left(x+\frac{y}{2}\right)\right)\psi\left(x-\frac{y}{2}\right)+\bar{\psi}\left(x+\frac{y}{2}\right)\partial^{\mu}_{y}\partial^{\nu}_{y}\psi\left(x-\frac{y}{2}\right)\right. \\
& \left.+\left(\partial^{(\mu}_{y}\bar{\psi}\left(x+\frac{y}{2}\right)\right)\partial^{\nu)}_{y}\psi\left(x-\frac{y}{2}\right)\right] \\
=& \frac{1}{(2\pi\hbar)^4}\int d^4y \int d^4p\, e^{-\frac{i}{\hbar}y\cdot p}\frac{\hbar^2}{m}\left[\left(\frac{i}{\hbar}p^{\mu}\partial^{\nu}_{y}\bar{\psi}\left(x+\frac{y}{2}\right)\right)\psi\left(x-\frac{y}{2}\right)+\frac{i}{\hbar}p^{\mu}\bar{\psi}\left(x+\frac{y}{2}\right)\partial^{\nu}_{y}\psi\left(x-\frac{y}{2}\right)\right] \\
=& \frac{1}{(2\pi\hbar)^4}\int d^4y \int d^4p\, e^{-\frac{i}{\hbar}y\cdot p}\frac{1}{m}p^{\mu}p^{\nu}\bar{\psi}\left(x+\frac{y}{2}\right)\psi\left(x-\frac{y}{2}\right) \\
=& \frac{1}{m}\int d^4p\, p^{\mu}p^{\nu}\hat{\mathcal{F}}(x,p),
\end{aligned} \tag{B.1}
$$

where we integrated by parts in the 4th and 5th step. We obtain the HW energy-momentum tensor (2.29) by comparison to the GLW version. Using

$$
\frac{\hbar^2}{2m}\left[(\partial^{\nu}\bar{\psi})\partial^{\mu}\psi+(\partial^{\mu}\bar{\psi})\partial^{\nu}\psi\right] = \frac{\hbar^2}{2m}\left[\partial^{\mu}\partial^{\nu}(\bar{\psi}\psi)-(\partial^{\mu}\partial^{\nu}\bar{\psi})\psi-\bar{\psi}\partial^{\mu}\partial^{\nu}\psi\right] \tag{B.2}
$$

we find

$$
\begin{aligned}
\hat{T}^{\mu\nu}_{GLW} =& -\frac{\hbar^2}{4m}\bar{\psi}\overleftrightarrow{\partial}^{\mu}\overleftrightarrow{\partial}^{\nu}\psi \\
=& \frac{\hbar^2}{2m}\left[(\partial^{\mu}\bar{\psi})\partial^{\nu}\psi+(\partial^{\nu}\bar{\psi})\partial^{\mu}\psi-\frac{1}{2}\partial^{\mu}\partial^{\nu}(\bar{\psi}\psi)\right] \\
=& \hat{T}^{\mu\nu}_{HW}-\frac{\hbar^2}{4m}\partial^{\mu}\partial^{\nu}(\bar{\psi}\psi)+g^{\mu\nu}\mathcal{L}_{KG}.
\end{aligned} \tag{B.3}
$$

For the term multiplying the metric tensor in the HW energy-momentum tensor (2.29) we obtain

$$\begin{aligned}\hbar^2(\partial_\lambda\bar{\psi})\partial^\lambda\psi - m^2\bar{\psi}\psi =& \frac{\hbar^2}{2}\left[\partial^2(\bar{\psi}\psi) + (\partial^2\bar{\psi})\psi + \bar{\psi}\partial^2\psi\right] - m^2\bar{\psi}\psi \\ =& \frac{\hbar^2}{2}\partial^2(\bar{\psi}\psi) + m^2\bar{\psi}\psi - m^2\bar{\psi}\psi \\ =& \frac{\hbar^2}{2}\partial^2\hat{\mathcal{F}}.\end{aligned} \tag{B.4}$$

We conclude

$$\hat{T}^{\mu\nu}_{HW} = \frac{1}{m}\int d^4p\,\left(p^\mu p^\nu + \frac{\hbar^2}{4}\partial^\mu\partial^\nu - \frac{\hbar^2}{4}g^{\mu\nu}\partial^2\right)\hat{\mathcal{F}}. \tag{B.5}$$

Appendix C

Calculations for the collision term

C.1 Ensemble average of the collision term

Let us consider the ensemble average $\langle \hat{O} \rangle$ of an arbitrary normal-ordered operator $\hat{O}$. In kinetic theory we can take this ensemble average with respect to the initial, free n-particle states in Eq. (3.10). In this case, it has been shown in Ref. [45] that $\langle \hat{O} \rangle$ can be written as

$$\langle \hat{O} \rangle = \sum_{n=0}^{\infty} \frac{4^n}{n!} \sum_{r_1 \cdots r_n} \sum_{r_1' \cdots r_n'} \int dP_1 \cdots dP_n dP_1' \cdots dP_n' \\ \times {}_{\text{in}}\langle\langle p_1, \ldots, p_n; r_1, \ldots, r_n | \hat{O} | p_1', \ldots, p_n'; r_1', \ldots, r_n' \rangle\rangle_{\text{in}} \langle a^\dagger_{\text{in},r_1}(p_1) \cdots a^\dagger_{\text{in},r_n}(p_n) a_{\text{in},r_1'}(p_1') \cdots a_{\text{in},r_n'}(p_n') \rangle , \tag{C.1}$$

where $dP_i \equiv d^4p_i\, \delta(p_i^2 - m^2)$, $dP_i' \equiv d^4p_i'\, \delta(p_i'^2 - m^2)$ and where we defined

$$ {}_{\text{in}}\langle\langle p_1, \ldots, p_n; r_1, \ldots, r_n | \hat{O} | p_1', \ldots, p_n'; r_1', \ldots, r_n' \rangle\rangle_{\text{in}} \\ = \mathfrak{A} \sum_{m=0}^{n} (-1)^m \left(\frac{n!}{m!(n-m)!} \right)^2 {}_{\text{in}}\langle p_1, \ldots, p_m; r_1, \ldots, r_m | p_1', \ldots, p_m'; r_1', \ldots, r_n' \rangle_{\text{in}} \\ \times {}_{\text{in}}\langle p_{m+1}, \ldots, p_n; r_{m+1}, \ldots, r_n | \hat{O} | p_{m+1}', \ldots, p_n'; r_{m+1}', \ldots, r_n' \rangle_{\text{in}} . \tag{C.2}$$

Here the symbol $\mathfrak{A}$ indicates the antisymmetrization with respect to all momenta and spin indices. If we neglect initial correlations, which is also justified for low density, the expectation values of the creation and annihilation operators factorize and read

$$\langle a^\dagger_{\text{in},r_1}(p_1) \cdots a^\dagger_{\text{in},r_n}(p_n) a_{\text{in},r_1'}(p_1') \cdots a_{\text{in},r_n'}(p_n') \rangle = \sum_{\mathfrak{P}} (-1)^{\mathfrak{P}} \prod_{j=1}^{n} \langle a^\dagger_{\text{in},r_j}(p_j) a_{\text{in},r_j'}(p_j') \rangle , \tag{C.3}$$

where $\mathfrak{P}$ denotes the sum over all permutations of primed and unprimed variables with $(-1)^{\mathfrak{P}} = 1$ for even permutations and $(-1)^{\mathfrak{P}} = -1$ for odd permutations.

With this formalism we now want to calculate the ensemble average of the collision operator in Eq. (3.7). To this end, we use the following relation for the field operator in the Heisenberg picture,

$$\psi\left(x - \frac{y}{2}\right) = e^{\frac{i}{\hbar}\hat{P}\cdot x} \psi\left(-\frac{y}{2}\right) e^{-\frac{i}{\hbar}\hat{P}\cdot x} , \tag{C.4}$$

where similar equations apply also to $\bar{\psi}$, ρ, $\bar{\rho}$. Then we obtain for Eq. (3.7)

$$C_{\alpha\beta} = \left\langle e^{\frac{i}{\hbar}\hat{P}\cdot x} \hat{\Phi}_{\alpha\beta}(p) e^{-\frac{i}{\hbar}\hat{P}\cdot x} \right\rangle , \tag{C.5}$$

with $\hat{\Phi}_{\alpha\beta}$ given by Eq. (3.12). The ensemble average in $C_{\alpha\beta}$ is now computed using Eq. (C.1) with the factorization in Eq. (C.3). For the scattering kernel $C_{\alpha\beta}$ neglecting initial correlations, i.e., assuming Eq. (C.3), is called molecular chaos. Furthermore, we consider $n = 2$, which corresponds to binary collisions, since only two-particle states appear. These are eigenstates of the total momentum operator, so Eq. (C.5) becomes

$$C_{\alpha\beta} = 8 \sum_{r_1, r_2, r_1', r_2'} \int dP_1 dP_2 dP_1' dP_2' {}_{\text{in}}\langle p_1, p_2; r_1, r_2 | \Phi(p) | p_1', p_2'; r_1', r_2' \rangle_{\text{in}} \prod_{j=1}^{2} e^{\frac{i}{\hbar}(p_j - p_j')\cdot x} \langle a^\dagger_{\text{in},r_j}(p_j) a_{\text{in},r_j'}(p_j') \rangle . \tag{C.6}$$

The positive-energy part of the initial noninteracting field is given by

$$\psi_{\rm in}(x) = \sqrt{\frac{2}{(2\pi\hbar)^3}} \sum_r \int dP\, e^{-\frac{i}{\hbar}p\cdot x} u_r(p) a_{{\rm in},r}(p) \ . \tag{C.7}$$

This relation can be inverted to obtain

$$\frac{1}{m\sqrt{2(2\pi\hbar)^5}} \int d^4x\, e^{\frac{i}{\hbar}p\cdot x} \bar{u}_r(p)\psi_{\rm in}(x) = 2\,\delta(p^2 - m^2) a_{{\rm in},r}(p) \ . \tag{C.8}$$

Using Eq. (C.8) we can express Eq. (C.6) in terms of the initial Wigner function

$$W_{{\rm in},\alpha\beta}(x,p) = \int \frac{d^4y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}p\cdot y} \left\langle \bar{\psi}_{{\rm in},\beta}(x_1)\, \psi_{{\rm in},\alpha}(x_2) \right\rangle \ , \tag{C.9}$$

with the result given in Eq. (3.11).

C.2 Calculation of the expectation value of $\hat{\Phi}$

The aim of this appendix is to explicitly calculate the scattering-matrix element in Eq. (3.15),

$$_{\rm in}\left\langle p_1 - \frac{q_1}{2}, p_2 - \frac{q_2}{2}; r_1, r_2 \middle| \hat{\Phi}(p) \middle| p_1 + \frac{q_1}{2}, p_2 + \frac{q_2}{2}; s_1, s_2 \right\rangle_{\rm in} \ , \tag{C.10}$$

where the operator $\hat{\Phi}(p)$ is given in Eq. (3.12). First, we insert a completeness relation of free out-states between the field operators and follow the calculation done in Ref. [45]. In particular, we employ the fact that one- and two-particle states are eigenstates of the total momentum operator to write the expectation values as, e.g.,

$$\begin{aligned} _{\rm out}\left\langle p'; r' \middle| \psi\left(-\frac{y}{2}\right) \middle| p_1 + \frac{q_1}{2}, p_2 + \frac{q_2}{2}; s_1, s_2 \right\rangle_{\rm in} =& e^{\frac{i}{2\hbar}(p' - p_1 - q_1/2 - p_2 - q_2/2)\cdot y} \\ & \times {}_{\rm out}\left\langle p'; r' \middle| \psi(0) \middle| p_1 + \frac{q_1}{2}, p_2 + \frac{q_2}{2}; s_1, s_2 \right\rangle_{\rm in} \ , \\ \partial_{y^\mu}\, {}_{\rm in}\left\langle p_1 - \frac{1}{2}u_1, p_2 - \frac{1}{2}u_2; r_1, r_2 \middle| \bar{\rho}\left(\frac{y}{2}\right) \middle| p'; r' \right\rangle_{\rm out} =& -\frac{i}{2\hbar}\left(p_1 - \frac{1}{2}u_1 + p_2 - \frac{1}{2}u_2 - p'\right)_\mu \\ & \times e^{-\frac{i}{2\hbar}(p_1 - u_1/2 + p_2 - u_2/2 - p')\cdot y} \\ & \times {}_{\rm in}\left\langle p_1 - \frac{1}{2}u_1, p_2 - \frac{1}{2}u_2; r_1, r_2 \middle| \bar{\rho}(0) \middle| p'; r' \right\rangle_{\rm out} \ . \end{aligned} \tag{C.11}$$

After performing the y-integration in Eq. (3.12) we find

$$\begin{aligned} & {}_{\rm in}\left\langle p_1 - \frac{q_1}{2}, p_2 - \frac{q_2}{2}; r_1, r_2 \middle| \hat{\Phi}(p) \middle| p_1 + \frac{q_1}{2}, p_2 + \frac{q_2}{2}; s_1, s_2 \right\rangle_{\rm in} \\ = \ & i \sum_{r'} \int dP' \delta^{(4)}(p + p' - p_1 - p_2) \\ \times & \left\{ {}_{\rm out}\left\langle p'; r' \middle| \psi(0) \middle| p_1 + \frac{q_1}{2}, p_2 + \frac{q_2}{2}; s_1, s_2 \right\rangle_{\rm in}\, {}_{\rm in}\left\langle p_1 - \frac{q_1}{2}, p_2 - \frac{q_2}{2}; r_1, r_2 \middle| \bar{\rho}(0) \middle| p'; r' \right\rangle_{\rm out} \right. \\ & \times \left[\gamma\cdot\left(p - \frac{q_1 + q_2}{2}\right) + m\right] \left[\gamma\cdot\left(p + \frac{q_1 + q_2}{2}\right) + m\right] {}_{\rm out}\left\langle p'; r' \middle| \rho(0) \middle| p_1 + \frac{q_1}{2}, p_2 + \frac{q_2}{2}; s_1, s_2 \right\rangle_{\rm in} \\ & \left. - {}_{\rm in}\left\langle p_1 - \frac{q_1}{2}, p_2 - \frac{q_2}{2}; r_1, r_2 \middle| \bar{\psi}(0) \middle| p'; r' \right\rangle_{\rm out} \right\} \ . \end{aligned} \tag{C.12}$$

The matrix element on the right-hand side of this equation is calculated by making use of the general solution of the Dirac equation in the presence of an interaction,

$$\psi(0) = \psi_{\rm in}(0) + \int d^4x\, S_R(-x)\rho(x) \ , \tag{C.13}$$

to replace $\psi(0)$. Here $\psi_{\rm in}$ is defined in Eq. (C.7) and $S_R(x)$ is the retarded Green's function and can be written as a Fourier transform

$$S_R(x) = \frac{1}{(2\pi\hbar)^4}\int d^4p\, \tilde{S}_R(p) e^{-\frac{i}{\hbar}p\cdot x} \,, \tag{C.14}$$

with

$$\tilde{S}_R(p) = -\frac{1}{\hbar}(\gamma\cdot p + m)G(p) \,, \tag{C.15}$$

where $G(p)$ is given in Eq. (3.57). The matrix element of ψ hence reads

$$\begin{aligned}
{}_{\rm out}\left\langle p';r'\middle|\psi(0)\middle|p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle_{\rm in} &= \left[u_{s_1}\left(p_1+\frac{q_1}{2}\right)\delta^{(3)}\left(\mathbf{p}'-\mathbf{p_2}-\frac{\mathbf{q_2}}{2}\right)\delta_{r's_2}-(1\leftrightarrow 2)\right] \\
&\quad\times\frac{p'^0}{\sqrt{2(2\pi\hbar)^3}}+\tilde{S}_R\left(p_1+\frac{q_1}{2}+p_2+\frac{q_2}{2}-p'\right) \\
&\quad\times\, {}_{\rm out}\left\langle p',r'\middle|\rho(0)\middle|p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle_{\rm in} \,,
\end{aligned} \tag{C.16}$$

where we made use of the orthogonality condition $\langle p,r|p',r'\rangle = p^0\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{rr'}$. Inserting Eq. (C.16) into Eq. (C.12) and making use of Eqs. (C.15), (3.57), and furthermore employing

$$(\gamma\cdot k+m)_{\alpha\beta} = \sum_r u_r(k)_\alpha \bar{u}_r(k)_\beta \,, \tag{C.17}$$

we find

$$\begin{aligned}
&{}_{\rm in}\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\middle|\hat{\Phi}(p)\middle|p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle_{\rm in} \\
=&\frac{i}{2}\sum_{r,s}\left\{\frac{1}{\sqrt{2(2\pi\hbar)^3}}\left[\delta^{(3)}\left(\mathbf{p}-\mathbf{p_1}+\frac{\mathbf{q_2}}{2}\right)\delta\left(p^0+\sqrt{\left(\mathbf{p_2}+\frac{\mathbf{q_2}}{2}\right)^2+m^2}-\varepsilon_{p_1}-\varepsilon_{p_2}\right)\delta_{s_1 r}\, u_r\left(p+\frac{q_1+q_2}{2}\right)\right.\right. \\
&\times\bar{u}_s\left(p-\frac{q_1+q_2}{2}\right)\, {}_{\rm in}\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\middle|\bar{\rho}(0)\middle|p_2;s_2\right\rangle_{\rm out} u_s\left(p-\frac{q_1+q_2}{2}\right)+(1\leftrightarrow 2)\Bigg] \\
&-\frac{1}{\sqrt{2(2\pi\hbar)^3}}\left[\delta^{(3)}\left(\mathbf{p}-\mathbf{p_1}-\frac{\mathbf{q_2}}{2}\right)\delta\left(p^0+\sqrt{\left(\mathbf{p_2}-\frac{\mathbf{q_2}}{2}\right)^2+m^2}-\varepsilon_{p_1}-\varepsilon_{p_2}\right)\delta_{sr_1}\, u_r\left(p+\frac{q_1+q_2}{2}\right)\right. \\
&\times\bar{u}_s\left(p-\frac{q_1+q_2}{2}\right)\bar{u}_r\left(p+\frac{q_1+q_2}{2}\right)\, {}_{\rm out}\left\langle p_2;r_2\middle|\rho(0)\middle|p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle_{\rm in}+(1\leftrightarrow 2)\Bigg] \\
&-\hbar\sum_{r'}\int dP'\delta^{(4)}(p+p'-p_1-p_2)\left[G\left(p+\frac{q_1+q_2}{2}\right)-G^\star\left(p-\frac{q_1+q_2}{2}\right)\right] \\
&\times u_r\left(p+\frac{q_1+q_2}{2}\right)\bar{u}_s\left(p-\frac{q_1+q_2}{2}\right)\bar{u}_r\left(p+\frac{q_1+q_2}{2}\right)\, {}_{\rm out}\left\langle p';r'\middle|\rho(0)\middle|p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle_{\rm in} \\
&\left.\times\, {}_{\rm in}\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\middle|\bar{\rho}(0)\middle|p';r'\right\rangle_{\rm out} u_s\left(p-\frac{q_1+q_2}{2}\right)\right\} \,,
\end{aligned} \tag{C.18}$$

where we defined $\varepsilon_p \equiv \sqrt{\mathbf{p}^2+m^2}$. Finally, we obtain with the help of Eq. (3.52)

$$\begin{aligned}
&{}_{\rm in}\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\middle|\hat{\Phi}(p)\middle|p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle_{\rm in} \\
&= \frac{1}{2(2\pi\hbar)^6}\sum_{r,s} u_r\left(p+\frac{q_1+q_2}{2}\right)\bar{u}_s\left(p-\frac{q_1+q_2}{2}\right) w^{rs}_{r_1r_2s_1s_2}(p_1,q_1,p_2,q_2,p) \,,
\end{aligned} \tag{C.19}$$

with

$$
\begin{aligned}
w^{rs}_{r_1 r_2 s_1 s_2}(p_1, q_1, p_2, q_2, p) \;\; = \;\; & 2\sum_{r'} \int dP' \, \frac{1}{i\pi\hbar^2} \left[G\left(p + \frac{q_1+q_2}{2}\right) - G^\star\left(p - \frac{q_1+q_2}{2}\right)\right] \\
& \times \delta^{(4)}(p + p' - p_1 - p_1) \left\langle p + \frac{q_1+q_2}{2}, p'; r, r' \middle| t \middle| p_1 + \frac{q_1}{2}, p_2 + \frac{q_2}{2}; s_1, s_2 \right\rangle \\
& \times \left\langle p_1 - \frac{q_1}{2}, p_2 - \frac{q_2}{2}; r_1, r_2 \middle| t^\dagger \middle| p - \frac{q_1+q_2}{2}, p'; s, r' \right\rangle \\
& - i2\pi\hbar\delta^{(3)}\left(\mathbf{p} - \mathbf{p_1} + \frac{\mathbf{q_2}}{2}\right) \delta\left(p^0 + p_2^0 + \frac{q_2^0}{2} - \varepsilon_{p_1} - \varepsilon_{p_2}\right) \\
& \times \left\langle p_1 - \frac{q_1}{2}, p_2 - \frac{q_2}{2}; r_1, r_2 \middle| t^\dagger \middle| p - \frac{q_1+q_2}{2}, p_2; s, s_2 \right\rangle \delta_{rs_1} + (1 \leftrightarrow 2) \\
& + i2\pi\hbar\delta^{(3)}\left(\mathbf{p} - \mathbf{p_1} - \frac{\mathbf{q_2}}{2}\right) \delta\left(p^0 + p_2^0 - \frac{q_2^0}{2} - \varepsilon_{p_1} - \varepsilon_{p_2}\right) \\
& \times \left\langle p_1 + \frac{q_1}{2}, p_2 + \frac{q_2}{2}; r, r_2 \middle| t \middle| p + \frac{q_1+q_2}{2}, p_2; s_1, s_2 \right\rangle \delta_{r_1 s} + (1 \leftrightarrow 2) \; . \qquad \text{(C.20)}
\end{aligned}
$$

In order to obtain this result, we used that to linear order in $\mathbf{q}_2$ we have $\sqrt{(\mathbf{p_2} \pm \mathbf{q_2}/2)^2 + m^2} = p_2^0 \pm q_2^0/2$. Higher-order terms can be dropped as we only consider zeroth- and first-order terms of a Taylor expansion in q_i in Eq. (3.15). This final form of the scattering-matrix element is used in Eq. (3.15) to compute the collision term explicitly. In order to simplify the notation, we also define

$$
w^{rs}_{r_1 r_2 s_1 s_2}(p_1, p_2, p) \equiv w^{rs}_{r_1 r_2 s_1 s_2}(p_1, q_1 = 0, p_2, q_2 = 0, p) \; . \qquad \text{(C.21)}
$$

C.3 Spinor identities

For any on-shell momentum $p^\mu = (\varepsilon_p, \mathbf{p})$, the corresponding spinor can be written as

$$
u_r(p) = \frac{\gamma \cdot p + m}{\sqrt{2m(\varepsilon_p + m)}} \, u_r(p_\star) \; . \qquad \text{(C.22)}
$$

With this identity, one can derive the following relations to first order in q^μ,

$$
\begin{aligned}
\bar{u}_s\left(p + \frac{q}{2}\right) u_r\left(p - \frac{q}{2}\right) &= 2m\, \delta_{sr} - \frac{i}{2(\varepsilon_p + m)} \, q_\mu p_\nu \Sigma^{\mu\nu}_{sr}(p_\star) \; , \\
\bar{u}_s\left(p + \frac{q}{2}\right) \gamma^\alpha u_r\left(p - \frac{q}{2}\right) &= 2p^\alpha \delta_{sr} + \frac{im}{2(\varepsilon_p + m)} \, q_\mu \Sigma^{\alpha\mu}_{sr}(p_\star) - \frac{i}{\varepsilon_p + m} \epsilon^{\alpha\mu\nu\rho} q_\mu p_\nu n_{sr\rho}(p_\star) \; , \\
\bar{u}_s\left(p + \frac{q}{2}\right) \gamma^5 \gamma^\alpha u_r\left(p - \frac{q}{2}\right) &= 2m n^\alpha_{sr}(p) - \frac{i}{\varepsilon_p + m} \epsilon^{\alpha\mu\nu 0} q_\nu p_\mu \delta_{sr} \; , \qquad \text{(C.23)}
\end{aligned}
$$

where we also made use of Eqs. (3.55), (3.59), and the identity

$$
\gamma^\mu \gamma^\alpha \gamma^\nu = g^{\mu\alpha}\gamma^\nu + g^{\alpha\nu}\gamma^\mu - g^{\nu\mu}\gamma^\alpha - i\epsilon^{\mu\alpha\nu\rho}\gamma_\rho\gamma^5 \; . \qquad \text{(C.24)}
$$

C.4 Calculation of nonlocal collision term

We compute the second contribution of the nonlocal term in Eq. (3.84) to be

$$
\begin{aligned}
m\,\mathfrak{C}^{(1)}_{nl,2} = {} & i\frac{(2\pi\hbar)^6}{8m^4}\sum_{r_1,r_2,s_1,s_2}\int d^4p_1 d^4p_2 d^4q_1 d^4q_2\,\delta^{(4)}(q_1)\delta^{(4)}(q_2) \\
& \times\left\{\partial^\mu_{q_1}\mathrm{Tr}\left[\left(\frac{m}{p^2}\,p\cdot\gamma-\mathfrak{s}\cdot\gamma\gamma^5\right)\,{}_{\text{in}}\!\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\right|\Phi(p)\left|p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle_{\text{in}}\right]\right. \\
& \times\bar{u}_{s_2}(p_2)W(x,p_2)u_{r_2}(p_2)\bar{u}_{s_1}(p_1)\partial_\mu W(x,p_1)u_{r_1}(p_1) \\
& +\partial^\mu_{q_2}\mathrm{Tr}\left[\left(\frac{m}{p^2}\,p\cdot\gamma-\mathfrak{s}\cdot\gamma\gamma^5\right)\,{}_{\text{in}}\!\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\right|\Phi(p)\left|p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle_{\text{in}}\right] \\
& \left.\times\bar{u}_{s_1}(p_1)W(x,p_1)u_{r_1}(p_1)\bar{u}_{s_2}(p_2)\partial_\mu W(x,p_2)u_{r_2}(p_2)\right\} \\
= {} & \frac{i}{16m^4}\sum_{r,s,r_1,r_2,s_1,s_2}\int d^4p_1 d^4p_2 d^4q_1 d^4q_2\,\delta^{(4)}(q_1)\delta^{(4)}(q_2) \\
& \times\left\{\partial^\mu_{q_1}\left[\bar{u}_s\left(p-\frac{q_1}{2}-\frac{q_2}{2}\right)\left(\frac{m}{p^2}\,p\cdot\gamma-\mathfrak{s}\cdot\gamma\gamma^5\right)u_r\left(p+\frac{q_1+q_2}{2}\right)w^{rs}_{r_1r_2s_1s_2}(p_1,q_1,p_2,q_2,p)\right]\right. \\
& \times\bar{u}_{s_2}(p_2)W(x,p_2)u_{r_2}(p_2)\bar{u}_{s_1}(p_1)\partial_\mu W(x,p_1)u_{r_1}(p_1) \\
& +\partial^\mu_{q_2}\left[\bar{u}_s\left(p-\frac{q_1+q_2}{2}\right)\left(\frac{m}{p^2}\,p\cdot\gamma-\mathfrak{s}\cdot\gamma\gamma^5\right)u_r\left(p+\frac{q_1}{2}+\frac{q_2}{2}\right)w^{rs}_{r_1r_2s_1s_2}(p_1,q_1,p_2,q_2,p)\right] \\
& \left.\times\bar{u}_{s_1}(p_1)W(x,p_1)u_{r_1}(p_1)\bar{u}_{s_2}(p_2)\partial_\mu W(x,p_2)u_{r_2}(p_2)\right\} \\
= {} & \frac{i}{16m^4}\sum_{r,s,r_1,r_2,s_1,s_2}\int d^4p_1 d^4p_2\left\{\frac{i}{2(p^0+m)}\left[p_\nu\Sigma^{\mu\nu}_{sr}(p_\star)+\epsilon^{\nu\lambda\mu 0}p_\nu\mathfrak{s}_\lambda\delta_{sr}\right]w^{rs}_{r_1r_2s_1s_2}(p_1,p_2,p)\right. \\
& \left.\times\partial_\mu\bar{u}_{s_1}(p_1)W(x,p_1)u_{r_1}(p_1)\bar{u}_{s_2}(p_2)W(x,p_2)u_{r_2}(p_2)\right\} \\
& +\frac{i}{16m^4}\sum_{r,s,r_1,r_2,s_1,s_2}\int d^4p_1 d^4p_2\left[\frac{m}{p^2}\bar{u}_s(p)\,p\cdot\gamma\,u_r(p)-\mathfrak{s}_\mu\bar{u}_s(p)\gamma^\mu\gamma^5u_r(p)\right] \\
& \times\left\{\left[\partial^\mu_{q_1}w^{rs}_{r_1r_2s_1s_2}(p_1,q_1,p_2,q_2,p)\right]_{q_1=q_2=0}\bar{u}_{s_2}(p_2)W(x,p_2)u_{r_2}(p_2)\bar{u}_{s_1}(p_1)\partial_\mu W(x,p_1)u_{r_1}(p_1)\right. \\
& \left.+\left[\partial^\mu_{q_2}w^{rs}_{r_1r_2s_1s_2}(p_1,q_1,p_2,q_2,p)\right]_{q_1=q_2=0}\bar{u}_{s_1}(p_1)W(x,p_1)u_{r_1}(p_1)\bar{u}_{s_2}(p_2)\partial_\mu W(x,p_2)u_{r_2}(p_2)\right\}\,.
\end{aligned}
\tag{C.25}
$$

Here we made use of Eq. (C.19) in the second step and, in the last step, Eq. (C.21) and the relation

$$
\begin{aligned}
&\partial^\alpha_{q_j}\left[\frac{m}{p^2}p_\mu\bar{u}_s\left(p-\frac{q_1+q_2}{2}\right)\gamma^\mu u_r\left(p+\frac{q_1+q_2}{2}\right)+\mathfrak{s}_\mu\bar{u}_s\left(p-\frac{q_1+q_2}{2}\right)\gamma^5\gamma^\mu u_r\left(p+\frac{q_1+q_2}{2}\right)\right]_{q_1=q_2=0} \\
&=\frac{i}{2(p^0+m)}\left[\frac{m^2}{p^2}p_\nu\Sigma^{\alpha\nu}_{sr}(p_\star)+\epsilon^{\nu\mu\alpha 0}p_\nu\mathfrak{s}_\mu\delta_{sr}\right]\,.
\end{aligned}
\tag{C.26}
$$

We note that the term $m^2/p^2=1$ in this equation, after it is inserted into Eq. (C.25), since then p^μ is on-shell due to the delta function in $w^{rs}_{r_1r_2s_1s_2}(p_1,p_2,p)$, see Eq. (3.51).

When the q_j-derivatives act on $w^{rs}_{r_1r_2s_1s_2}(p_1,q_1,p_2,q_2,p)$ in the last two lines of Eq. (C.25), we obtain different terms, which are computed in the following. For convenience we split $w^{rs}_{r_1r_2s_1s_2}(p_1,q_1,p_2,q_2,p)$, cf. Eq. (C.20), into a gain term,

$$
\begin{aligned}
w^{rs}_{r_1r_2s_1s_2,\text{gain}}(p_1,q_1,p_2,q_2,p) = {} & 2\sum_{r'}\int dP'\,\frac{1}{i\pi\hbar^2}\left[G\left(p+\frac{q_1+q_2}{2}\right)-G^\star\left(p-\frac{q_1+q_2}{2}\right)\right]\delta^{(4)}(p+p'-p_1-p_1) \\
\times\ & \left\langle p+\frac{q_1+q_2}{2},p';r,r'\right|t\left|p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\right|t^\dagger\left|p-\frac{q_1+q_2}{2},p';s,r'\right\rangle\,,
\end{aligned}
\tag{C.27}
$$

and a loss term

$$
\begin{aligned}
w^{rs}_{r_1r_2s_1s_2,\mathrm{loss}}(p_1,q_1,p_2,q_2,p) \quad = \quad & -i2\pi\hbar\delta^{(3)}\left(\mathbf{p}-\mathbf{p_1}+\frac{\mathbf{q_2}}{2}\right)\delta\left(p^0+\sqrt{\left(\mathbf{p_2}+\frac{\mathbf{q_2}}{2}\right)^2+m^2}-\varepsilon_{p_1}-\varepsilon_{p_2}\right)\\
&\times\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\middle|t^\dagger\middle|p-\frac{q_1+q_2}{2},p_2;s,s_2\right\rangle\delta_{rs_1}+(1\leftrightarrow 2)\\
&+i2\pi\hbar\delta^{(3)}\left(\mathbf{p}-\mathbf{p_1}-\frac{\mathbf{q_2}}{2}\right)\delta\left(p^0+\sqrt{\left(\mathbf{p_2}-\frac{\mathbf{q_2}}{2}\right)^2+m^2}-\varepsilon_{p_1}-\varepsilon_{p_2}\right)\\
&\times\left\langle p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};r,r_2\middle|t\middle|p+\frac{q_1+q_2}{2},p_2;s_1,s_2\right\rangle\delta_{r_1s}+(1\leftrightarrow 2)\,. \qquad \text{(C.28)}
\end{aligned}
$$

As mentioned in the main text, in this work we consider terms up to order $\mathcal{O}(\hbar)$, such that we can replace the Wigner functions in Eq. (C.25) by their zeroth-order expressions. As a consequence, terms $\sim W\partial_\mu W$ result in terms $\sim f^{(0)}(x,p)\partial_\mu f^{(0)}(x,p)$, where $f(x,p)$ is the zeroth-order contribution to $f(x,p,\mathfrak{s})$. We also explained in the main text that this contribution is independent of the spin variable. First, let us consider the terms with the q_j-derivative acting on the gain part, Eq. (C.27). From Eq. (C.25) we obtain contributions to these terms which have the form

$$
\begin{aligned}
&\left[f^{(0)}(x,p_2)\partial^\mu f^{(0)}(x,p_1)\partial_{q_1\mu}+f^{(0)}(x,p_1)\partial^\mu f^{(0)}(x,p_2)\partial_{q_2\mu}\right]\left[G\left(p+\frac{q_1+q_2}{2}\right)-G^\star\left(p-\frac{q_1+q_2}{2}\right)\right]_{q_1=q_2=0}\\
&=-\frac{1}{2}\hbar^2\partial^\mu f^{(0)}(x,p_1)f^{(0)}(x,p_2)\partial_{q\mu}\left[\frac{1}{(p+q)^2-m^2-i\epsilon(p^0+q^0)}-\frac{1}{(p-q)^2-m^2+i\epsilon(p^0-q^0)}\right]_{q=0}\\
&=\partial^\mu f^{(0)}(x,p_1)f^{(0)}(x,p_2)\frac{p_\mu}{p^2-m^2}\left[G(p)+G^\star(p)\right]\,. \qquad \text{(C.29)}
\end{aligned}
$$

We see that there is a factor p^2-m^2 in the denominator, making such term an off-shell contribution to the Boltzmann equation. Furthermore, the q_j-derivatives acting on the loss term, Eq. (C.28), also result in an off-shell contribution. The relevant terms read

$$
\begin{aligned}
&\frac{im}{2}\sum_{r,s,r_1,r_2,s_1,s_2}\int dP_1dP_2\,h_{sr}(p,\mathfrak{s}_1)\left[f^{(0)}(x,p_2)\partial_\nu f^{(0)}(x,p)\partial^\nu_{q_1}+f^{(0)}(x,p)\partial_\nu f^{(0)}(x,p_2)\partial^\nu_{q_2}\right]\\
&\times\Big[-i2\pi\hbar\delta^{(3)}\left(\mathbf{p}-\mathbf{p_1}+\frac{\mathbf{q_2}}{2}\right)\delta\left(p^0+p_2^0+\frac{q_2^0}{2}-\varepsilon_{p_1}-\varepsilon_{p_2}\right)\\
&\times\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\middle|t^\dagger\middle|p-\frac{q_1+q_2}{2},p_2-\frac{q_2}{2};s,s_2\right\rangle\delta_{rs_1}+(1\leftrightarrow 2)\\
&+i2\pi\hbar\delta^{(3)}\left(\mathbf{p}-\mathbf{p_1}-\frac{\mathbf{q_2}}{2}\right)\delta\left(p^0+p_2^0-\frac{q_2^0}{2}-\varepsilon_{p_1}-\varepsilon_{p_2}\right)\\
&\times\left\langle p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};r,r_2\middle|t\middle|p+\frac{q_1+q_2}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle\delta_{r_1s}+(1\leftrightarrow 2)\Big]_{q_1=q_2=0}\delta_{r_1s_1}\delta_{r_2s_2}\\
=&\frac{im}{2}\sum_{r,s,r_1,r_2,s_2}\int dP_2h_{sr}(p,\mathfrak{s}_1)\partial_\nu\left[f^{(0)}(x,p_2)f^{(0)}(x,p)\right](\partial^\nu_{q_1}+\partial^\nu_{q_2})\\
&\times\Big[-\frac{i\pi\hbar}{\varepsilon_{p+\frac{q_2}{2}}}\delta\left(p^0+\frac{q_2^0}{2}-\varepsilon_{p+\frac{q_2}{2}}\right)\left\langle p-\frac{q_1-q_2}{2},p_2-\frac{q_2}{2};r_1,r_2\middle|t^\dagger\middle|p-\frac{q_1+q_2}{2},p_2-\frac{q_2}{2};s,s_2\right\rangle\delta_{rr_1}\delta_{r_2s_2}\\
&+\frac{i\pi\hbar}{\varepsilon_{p-\frac{q_2}{2}}}\delta\left(p^0-\frac{q_2^0}{2}-\varepsilon_{p-\frac{q_2}{2}}\right)\left\langle p+\frac{q_1-q_2}{2},p_2+\frac{q_2}{2};r,r_2\middle|t\middle|p+\frac{q_1+q_2}{2},p_2+\frac{q_2}{2};r_1,s_2\right\rangle\delta_{r_1s}\delta_{r_2s_2}\Big]_{q_1=q_2=0}\,.
\end{aligned}
$$
$$\text{(C.30)}$$

Noting that

$$
\partial^\mu_q\,\frac{1}{2\varepsilon_{p+\frac{q}{2}}}\delta\left(p^0+\frac{q^0}{2}-\varepsilon_{p+\frac{q}{2}}\right)\bigg|_{q=0}=\partial^\mu_q\,\delta\left(\left(p^0+\frac{q^0}{2}\right)^2-\varepsilon^2_{p+\frac{q}{2}}\right)\bigg|_{q=0}=p^\mu\delta'(p^2-m^2) \qquad \text{(C.31)}
$$

makes apparent that the above term contains both on-shell and off-shell parts. Collecting the contributions shown above, we obtain for the total off-shell part of the collision kernel

$$
\begin{aligned}
\mathfrak{C}^{(1)}_{\text{off-shell}} &= \frac{i}{2(p^2-m^2)}\, p\cdot\partial \sum_{r,s,r_1,r_2,s_1,s_2} dP_2\, h_{sr}(p,\mathfrak{s}) f^{(0)}(x,p_1) f^{(0)}(x,p_2) \\
&\times \Bigg\{ 2\sum_{r'} \int dP_1 dP'\, \frac{1}{i\pi\hbar^2}\left[G(p)+G^\star(p)\right] \delta^{(4)}(p+p'-p_1-p_1) \\
&\quad \times \langle p,p';r,r'|t|p_1,p_2;s_1,s_2\rangle\langle p_1,p_2;r_1,r_2|t^\dagger|p,p';s,r'\rangle \\
&\quad + i2\pi\hbar\, p^0\delta(p^2-m^2)\left[\langle p,p_2;r_1,r_2|t^\dagger|p,p_2;s,s_2\rangle\delta_{rr_1}\delta_{s_2r_2} + \langle p,p_2;r,r_2|t|p,p_2;r_1,s_2\rangle\delta_{r_1s}\delta_{r_2s_2}\right]\Bigg\} .
\end{aligned}
\tag{C.32}
$$

In the following we prove that the off-shell part Eq. (C.32) cancels with the off-shell part on the left-hand side of the Boltzmann equation (3.47). To this end, we apply the quasiparticle approximation in Eq. (3.45) to the left-hand side of Eq. (3.47). This yields

$$
m\, p\cdot\partial\, \delta(p^2-m^2-\hbar\delta m^2) f(x,p,\mathfrak{s}) = m\delta(p^2-m^2) p\cdot\partial\, f(x,p,\mathfrak{s}) + \hbar\frac{1}{p^2-m^2}\, p\cdot\partial\, \mathfrak{M}^{(0)} , \tag{C.33}
$$

with the zeroth-order correction to the mass shell

$$
\mathfrak{M}^{(0)} = \frac{im}{2} \sum_{r,s,r_1,r_2,s_1,s_2} \int dP_1\, dP_2\, h_{sr}(p,\mathfrak{s})\, \mathfrak{m}^{rs}_{r_1,r_2,s_1,s_2}(p_1,p_2,p) \prod_{j=1}^{2} \delta_{s_jr_j} f^{(0)}(x,p_j) , \tag{C.34}
$$

where

$$
\begin{aligned}
\mathfrak{m}^{r,s}_{r_1,r_2,s_1,s_2}(p_1,p_2,p) &= 2\sum_{r'} \int dP'\, \frac{1}{i\pi\hbar^2}\left[G(p)+G^\star(p)\right] \delta^{(4)}(p+p'-p_1-p_1) \\
&\quad \times\langle p,p';r,r'|t|p_1,p_2;s_1,s_2\rangle\langle p_1,p_2;r_1,r_2|t^\dagger|p,p';s,r'\rangle \\
&\quad + i2\pi\hbar p^0\delta(p^2-m^2)\{\delta^{(3)}(\mathbf{p}-\mathbf{p_1})[\langle p_1,p_2;r_1,r_2|t^\dagger|p,p_2;s,s_2\rangle\delta_{rs_1} \\
&\quad + \langle p_1,p_2;r,r_2|t|p,p_2;s_1,s_2\rangle\delta_{r_1s}] + (1\leftrightarrow 2)\} .
\end{aligned}
\tag{C.35}
$$

Since δM is just the real part of the quantity of which C given by Eq. (3.15) is the imaginary part, see Eqs. (3.9) and (3.6), Eq. (C.34) can be derived following completely analogous steps as in the calculation of the local collision term.

Thus we can conclude from comparing Eq. (C.32) with Eq. (C.34) that up to first order

$$
m\, \mathfrak{C}^{(1)}_{\text{off-shell}} = \frac{1}{p^2-m^2}\, p\cdot\partial\, \mathfrak{M}^{(0)} . \tag{C.36}
$$

This shows that all off-shell contributions cancel on the left- and right-hand sides and only on-shell terms contribute to the Boltzmann equation. The kinetic equation for the distribution function $f(x,p,\mathfrak{s})$ hence reads

$$
\delta(p^2-m^2)\, p\cdot\partial\, f(x,p,\mathfrak{s}) = \delta(p^2-m^2)\mathfrak{C}_{\text{on-shell}}[f] , \tag{C.37}
$$

with

$$
\mathfrak{C}_{\text{on-shell}}[f] \equiv \mathfrak{C}_{\text{on-shell},l}[f] + \hbar\, \mathfrak{C}^{(1)}_{\text{on-shell},nl,1}[f] + \hbar\, \mathfrak{C}^{(1)}_{\text{on-shell},2,1}[f] + \hbar\, \mathfrak{C}^{(1)}_{\text{on-shell},2,2}[f] . \tag{C.38}
$$

In this equation, the term

$$
\begin{aligned}
\mathfrak{C}^{(1)}_{\text{on-shell},2,1}[f] &= -\frac{1}{8m(p^0+m)} \sum_{r,s,r',r_1,r_2} \left(p_\nu \Sigma^{\mu\nu}_{sr}(p_\star) + \epsilon^{\nu\lambda\mu 0} p_\nu \mathfrak{s}_\lambda \delta_{sr}\right) \\
&\times \int dP_1\, dP_2\, dP'\, \delta^{(4)}(p+p'-p_1-p_2)\langle p,p';r,r'|t|p_1,p_2;r_1,r_2\rangle\langle p_1,p_2;r_1,r_2|t^\dagger|p,p';s,r'\rangle \\
&\times \left[\partial_\mu f^{(0)}(x,p_1) f^{(0)}(x,p_2) - \partial_\mu f^{(0)}(x,p') f^{(0)}(x,p)\right]
\end{aligned}
\tag{C.39}
$$

was obtained from the first two lines in the last equality in Eq. (C.25) after properly relabelling indices and using the optical theorem Eq. (3.69). Moreover, the term

$$
\begin{aligned}
\mathfrak{C}^{(1)}_{\text{on-shell},2,2}[f] \;=\;& \frac{1}{4m}\sum_{r_1,r_2,s_1,s_2}\sum_{r,r',s}\int dP_1\, dP_2\, dP'\, h_{sr}(p,\mathfrak{s})\delta^{(4)}(p+p'-p_1-p_2)\delta(p^2-m^2) \\
&\times \left[f(x,p_2)\partial_\nu f(x,p_1)\partial^\nu_{q_1} + f(x,p_1)\partial_\nu f(x,p_2)\partial^\nu_{q_2}\right] \\
&\times \left\langle p+\frac{q_1}{2}+\frac{q_2}{2},p';r,r'\middle|t\middle|p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle \\
&\times \left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\middle|t^\dagger\middle|p-\frac{q_1}{2}-\frac{q_2}{2},p';s,r'\right\rangle \delta_{s_1r_1}\delta_{s_2r_2} \\
&-\frac{m}{16\pi}\sum_{r_2,s_2}\int dP_2\sum_{r,s} h_{sr}(p,\mathfrak{s})\delta_{r_2s_2}\delta(p^2-m^2)\partial_\nu\left[f(x,p_2)f(x,p)\right]\left(\partial^\nu_{q_1}+\partial^\nu_{q_2}\right) \\
&\times i4\pi\hbar\left\langle p+\frac{q_2}{2}-\frac{q_1}{2},p_2-\frac{q_2}{2};r,r_2\middle|t+t^\dagger\middle|p-\frac{q_1}{2}-\frac{q_2}{2},p_2-\frac{q_2}{2};s,s_2\right\rangle
\end{aligned}
\tag{C.40}
$$

is the on-shell contribution from the last three lines in Eq. (C.25). We see that Eq. (C.40) is proportional to momentum derivatives of the scattering amplitude which, as pointed out in Sec. 3.4, are neglected in our approach in consistency with the low-density approximation. Hence, this term does not contribute and we arrive at the Boltzmann equation and the on-shell collision terms given in Eqs. (3.86) and (3.87).

Appendix D

Calculations for the method of moments

D.1 Properties of irreducible tensors

When calculating the equations of motion for the spin moments, see Section 5.5 and Appendix D.2, we use the following identities [202]:

$$p^{\mu} = E_p u^{\mu} + p^{\langle\mu\rangle}, \tag{D.1}$$

$$\Delta^{\alpha\beta} p_\alpha p_\beta = m^2 - E_p^2, \tag{D.2}$$

$$p^{\langle\mu\rangle} p^{\langle\nu\rangle} = p^{\langle\mu} p^{\nu\rangle} + \frac{1}{3}(m^2 - E_p^2)\Delta^{\mu\nu}, \tag{D.3}$$

$$p^{\langle\mu\rangle} p^{\langle\nu\rangle} p^{\langle\lambda\rangle} = p^{\langle\mu} p^{\nu} p^{\lambda\rangle} + \frac{1}{5}(m^2 - E_p^2)\left(p^{\langle\mu\rangle}\Delta^{\nu\lambda} + p^{\langle\nu\rangle}\Delta^{\lambda\mu} + p^{\langle\lambda\rangle}\Delta^{\mu\nu}\right), \tag{D.4}$$

$$\begin{aligned}
p^{\langle\mu\rangle} p^{\langle\nu\rangle} p^{\langle\lambda\rangle} p^{\langle\rho\rangle} &= p^{\langle\mu} p^{\nu} p^{\lambda} p^{\rho\rangle} + \frac{1}{7}\Big(p^{\langle\mu\rangle} p^{\langle\nu\rangle}\Delta^{\lambda\rho} + p^{\langle\mu\rangle} p^{\langle\lambda\rangle}\Delta^{\nu\rho} + p^{\langle\mu\rangle} p^{\langle\rho\rangle}\Delta^{\lambda\nu} + p^{\langle\lambda\rangle} p^{\langle\nu\rangle}\Delta^{\mu\rho} \\
&\quad + p^{\langle\rho\rangle} p^{\langle\nu\rangle}\Delta^{\lambda\mu} + p^{\langle\lambda\rangle} p^{\langle\rho\rangle}\Delta^{\mu\nu}\Big)(m^2 - E_p^2) \\
&\quad - \frac{1}{35}\left(\Delta^{\mu\nu}\Delta^{\lambda\rho} + \Delta^{\mu\lambda}\Delta^{\nu\rho} + \Delta^{\mu\rho}\Delta^{\nu\lambda}\right)(m^2 - E_p^2)^2 \\
&= p^{\langle\mu} p^{\nu} p^{\lambda} p^{\rho\rangle} + \frac{1}{7}\Big(p^{\langle\mu} p^{\nu\rangle}\Delta^{\lambda\rho} + p^{\langle\mu} p^{\lambda\rangle}\Delta^{\nu\rho} + p^{\langle\mu} p^{\rho\rangle}\Delta^{\lambda\nu} + p^{\langle\lambda} p^{\nu\rangle}\Delta^{\mu\rho} + p^{\langle\rho} p^{\nu\rangle}\Delta^{\lambda\mu} \\
&\quad + p^{\langle\lambda} p^{\rho\rangle}\Delta^{\mu\nu}\Big)(m^2 - E_p^2) + \frac{1}{15}\left(\Delta^{\mu\nu}\Delta^{\lambda\rho} + \Delta^{\mu\lambda}\Delta^{\nu\rho} + \Delta^{\mu\rho}\Delta^{\nu\lambda}\right)(m^2 - E_p^2)^2,
\end{aligned} \tag{D.5}$$

$$\begin{aligned}
p^{\langle\mu\rangle} p^{\langle\nu\rangle} p^{\langle\lambda\rangle} p^{\langle\rho\rangle} p^{\langle\sigma\rangle} &= p^{\langle\mu} p^{\nu} p^{\lambda} p^{\rho} p^{\sigma\rangle} + \frac{1}{9}(m^2 - E_p^2)\left(p^{\langle\mu\rangle} p^{\langle\nu\rangle} p^{\langle\lambda\rangle}\Delta^{\rho\sigma} + \text{perm.}\right) \\
&\quad - \frac{1}{63}(m^2 - E_p^2)^2\left(p^{\langle\mu\rangle}\Delta^{\nu\lambda}\Delta^{\rho\sigma} + \text{perm}\right) \\
&= p^{\langle\mu} p^{\nu} p^{\lambda} p^{\rho} p^{\sigma\rangle} + \frac{1}{9}(m^2 - E_p^2)\left(p^{\langle\mu} p^{\nu} p^{\lambda\rangle}\Delta^{\rho\sigma} + \text{perm.}\right) \\
&\quad + \frac{1}{35}(m^2 - E_p^2)^2\left(p^{\langle\mu\rangle}\Delta^{\nu\lambda}\Delta^{\rho\sigma} + \text{perm.}\right),
\end{aligned} \tag{D.6}$$

where "perm" stands for all distinct permutations of indices. We also make use of the orthogonality relation

$$\int dP\, p^{\langle\mu_1\cdots\mu_m\rangle} p_{\langle\nu_1\cdots\nu_n\rangle} F(E_p) = \frac{m!\delta_{mn}}{(2m+1)!!}\Delta^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_m} \int dP\, \left(\Delta^{\alpha\beta} p_\alpha p_\beta\right)^m F(E_p), \tag{D.7}$$

which holds for an arbitrary function $F(E_p)$.

D.2 Calculation of the general equations of motion of the spin moments

In this appendix we show details of the calculation of the equations of motion in Eq. (5.73), (5.78), (5.79), and (5.81) from Eq. (5.71). For the zeroth spin moment we obtain

$$\dot{\tau}_r^{\mu} = r\,\dot{u}_\nu \tau_{r-1}^{\mu,\nu} + \int d\Gamma\, E_p^r\, \mathfrak{s}^{\mu} \delta\dot{f}_{p\mathfrak{s}}, \tag{D.8}$$

where the last term will be evaluated using the Boltzmann equation. To simplify the notation we define $\bar{\Omega}^{\mu\nu} \equiv (\hbar/4)\Omega^{\mu\nu}$. Using

$$m \int dS\, \mathfrak{s}^\mu \bar{\Omega}_{\alpha\beta}\Sigma_{\mathfrak{s}}^{\alpha\beta} = -2\,\tilde{\bar{\Omega}}^{\mu\nu}(E_p u_\nu + p_{\langle\nu\rangle}) \tag{D.9}$$

and

$$\dot{f}_{0p} = f_{0p}(-\dot{\beta}_0 E_p - \beta_0 \dot{u}_\mu p^{\langle\mu\rangle} + \dot{\alpha}_0), \tag{D.10}$$

we obtain by inserting the first term of Eq. (5.72) into the last term of Eq. (D.8)

$$\begin{aligned}
&-\int d\Gamma\, E_p^r\, \mathfrak{s}^\mu \dot{f}_{0p} \left(1 + \bar{\Omega}_{\alpha\beta}\Sigma_{\mathfrak{s}}^{\alpha\beta}\right) \\
&= \frac{2}{m}\tilde{\bar{\Omega}}^{\mu\nu} \int dP\, (E_p^{r+1} u_\nu + E_p^r p_{\langle\nu\rangle}) f_{0p}(-\dot{\beta}_0 E_p - \beta_0 \dot{u}_\lambda p^{\langle\lambda\rangle} + \dot{\alpha}_0) \\
&= \frac{1}{m}\tilde{\bar{\Omega}}^{\mu\nu} u_\nu \left(I_{(r+1)0}\dot{\alpha}_0 - I_{(r+2)0}\dot{\beta}_0\right) + \frac{2}{3m}\beta_0 \tilde{\bar{\Omega}}^{\mu\nu}\dot{u}_\nu \int dP\, E_p^r \Delta_{\alpha\beta} p^\alpha p^\beta \\
&= \frac{1}{m}\tilde{\bar{\Omega}}^{\mu\nu} u_\nu \left(I_{(r+1)0}\dot{\alpha}_0 - I_{(r+2)0}\dot{\beta}_0\right) + \frac{1}{m}\beta_0 \tilde{\bar{\Omega}}^{\mu\nu}\dot{u}_\nu I_{(r+2)1}.
\end{aligned} \tag{D.11}$$

For the second term we have

$$-\int d\Gamma\, E_p^r\, \mathfrak{s}^\mu f_{0p} \dot{\bar{\Omega}}_{\alpha\beta}\Sigma_{\mathfrak{s}}^{\alpha\beta} = \frac{1}{m}\dot{\tilde{\bar{\Omega}}}^{\mu\nu} u_\nu I_{(r+1)0}. \tag{D.12}$$

The third term is given by

$$\begin{aligned}
&-\int d\Gamma\, E_p^{(r-1)}\, \mathfrak{s}^\mu p\cdot\nabla f_{0p} \left(1 + \bar{\Omega}_{\alpha\beta}\Sigma_{\mathfrak{s}}^{\alpha\beta}\right) \\
=&\frac{2}{m}\tilde{\bar{\Omega}}^{\mu\nu} u_\nu \left[\nabla_\lambda \int dP\, E_p^r \left(E_p u^\lambda + p^{\langle\lambda\rangle}\right) f_{0p} - \int dP\, (\nabla_\lambda E_p^r) \left(E_p u^\lambda + p^{\langle\lambda\rangle}\right) f_{0p}\right] \\
&+ \frac{2}{m}\int dP\, E_p^{r-1} p_{\langle\nu\rangle} p\cdot\nabla\tilde{\bar{\Omega}}^{\mu\nu} f_{0p} \\
=&\frac{1}{m}\theta\, \tilde{\bar{\Omega}}^{\mu\nu} u_\nu I_{(r+1)0} - \frac{2}{m}\tilde{\bar{\Omega}}^{\mu\nu} u_\nu r(\nabla_\lambda u_\rho)\int dP\, E_p^{r-1} p^{\langle\rho\rangle} p^{\langle\lambda\rangle} f_{0p} + \frac{2}{m}\frac{1}{3}\int dP\, E_p^{r-1}\Delta_{\alpha\beta}p^\alpha p^\beta \nabla_\nu \tilde{\bar{\Omega}}^{\mu\nu} f_{0p} \\
=&\frac{1}{m}\theta\, \tilde{\bar{\Omega}}^{\mu\nu} u_\nu \left(I_{(r+1)0} + r\, I_{(r+1)1}\right) - \frac{1}{m}\tilde{\bar{\Omega}}^{\mu\nu}\left(I_{(r+1)1}\nabla_\nu\alpha_0 - I_{(r+2)1}\nabla_\nu\beta_0\right) - \frac{1}{m}I_{(r+1)1}\nabla_\nu\tilde{\bar{\Omega}}^{\mu\nu}.
\end{aligned} \tag{D.13}$$

Finally, we obtain for the fourth term

$$\begin{aligned}
-\int d\Gamma\, E_p^{r-1}\mathfrak{s}^\mu p\cdot\nabla\delta f_{p\mathfrak{s}} =& -\theta\tau_r^\mu - \nabla_\nu \tau_{r-1}^{\mu,\nu} + (r-1)(\nabla_\alpha u_\beta)\int d\Gamma\, E_p^{r-2} p^{\langle\alpha\rangle} p^{\langle\beta\rangle}\mathfrak{s}^\mu \delta f_{p\mathfrak{s}} \\
=& -\theta\tau_r^\mu - \nabla_\nu \tau_{r-1}^{\mu,\nu} + (r-1)(\nabla_\alpha u_\beta)\tau_{r-2}^{\mu,\alpha\beta} + \frac{1}{3}\theta\int d\Gamma\, E_p^{r-2}\left(m^2 - E_p^2\right)\mathfrak{s}^\mu\delta f_{p\mathfrak{s}} \\
=& -\theta\tau_r^\mu - \nabla_\nu \tau_{r-1}^{\mu,\nu} + (r-1)\sigma_{\alpha\beta}\tau_{r-2}^{\mu,\alpha\beta} + (r-1)\frac{m^2}{3}\theta\tau_{r-2}^\mu - (r-1)\frac{1}{3}\theta\tau_r^\mu.
\end{aligned} \tag{D.14}$$

The contribution from the collision term is written as

$$\mathfrak{C}_r^{\mu,\langle\mu_1\cdots\mu_n\rangle} \equiv \int d\Gamma\, E_p^r\, p^{\langle\mu_1}\cdots p^{\mu_n\rangle}\mathfrak{s}^\mu \mathfrak{C}[f]. \tag{D.15}$$

Collecting all terms and inserting Eqs. (5.62a) and (5.62b) we obtain

$$\begin{aligned}
\dot{\tau}_r^\mu - \mathfrak{C}_{r-1}^\mu =& \frac{\hbar}{2m}\left[\xi_r^{(0)}\theta + \frac{G_{2(r+1)}}{D_{20}}\Pi\theta - \frac{G_{2(r+1)}}{D_{20}}\pi^{\lambda\nu}\sigma_{\lambda\nu} - \frac{G_{3r}}{D_{20}}\partial\cdot n\right]\omega_0^\mu \\
&- \frac{\hbar}{4m}\tilde{\Omega}^{\mu\nu}\left(I_{(r+1)1}\nabla_\nu\alpha_0 - I_{(r+2)1}\nabla_\nu\beta_0\right) - \frac{\hbar}{4m}I_{(r+1)1}\nabla_\nu\tilde{\Omega}^{\mu\nu} + \frac{\hbar}{4m}I_{(r+2)1}\beta_0\Omega^{\mu\nu}\dot{u}_\nu \\
&+ r\,\dot{u}_\nu\tau_{r-1}^{\mu,\nu} + (r-1)\sigma_{\alpha\beta}\tau_{r-2}^{\mu,\alpha\beta} - \nabla_\nu\tau_{r-1}^{\mu,\nu} - \frac{1}{3}\left[(r+2)\tau_r^\mu - (r-1)m^2\tau_{r-2}^\mu\right]\theta \\
&- \frac{\hbar}{4m}I_{(r+1)0}\epsilon^{\mu\nu\alpha\beta}u_\nu\dot{\Omega}_{\alpha\beta},
\end{aligned} \tag{D.16}$$

with

$$\xi_r^{(0)} \equiv -\left\{ I_{(r+1)0} + r\, I_{(r+1)1} + \frac{1}{D_{20}} \left[-G_{3(r+1)} n_0 + G_{2(r+1)} (\epsilon_0 + P) \right] \right\},$$
$$G_{nm} \equiv I_{n0} I_{m0} - I_{(n-1)0} I_{(m+1)0}. \tag{D.17}$$

Making use of

$$\nabla^\mu P_0 = \frac{n_0}{\beta_0} \nabla^\mu \alpha_0 - \frac{\epsilon_0 + P_0}{\beta_0} \nabla^\mu \beta_0 \tag{D.18}$$

and Eq. (5.62c) we can write

$$\begin{aligned}
\dot{\tau}_r^\mu - \mathfrak{C}_{r-1}^\mu =& \frac{\hbar}{2m} \left[\xi_r^{(0)} \theta + \frac{G_{2(r+1)}}{D_{20}} \Pi \theta - \frac{G_{2(r+1)}}{D_{20}} \pi^{\lambda\nu} \sigma_{\lambda\nu} - \frac{G_{3r}}{D_{20}} \partial \cdot n \right] \omega_0^\mu - \frac{\hbar}{4m} I_{(r+1)1} \nabla_\nu \tilde{\Omega}^{\mu\nu} \\
& - \frac{\hbar}{4m} \tilde{\Omega}^{\mu\nu} \left[I_{(r+1)1} I_\nu - I_{(r+2)1} \frac{\beta_0}{\epsilon_0 + P_0} \left(-\Pi \dot{u}_\nu + \nabla_\nu \Pi - \Delta_{\nu\lambda} \partial_\rho \pi^{\lambda\rho} \right) \right] \\
& + r\, \dot{u}_\nu \tau_{r-1}^{\mu,\nu} + (r-1) \sigma_{\alpha\beta} \tau_{r-2}^{\mu,\alpha\beta} - \nabla_\nu \tau_{r-1}^{\mu,\nu} - \frac{1}{3} \left[(r+2) \tau_r^\mu - (r-1) m^2 \tau_{r-2}^\mu \right] \theta \\
& - \frac{\hbar}{4m} I_{(r+1)0} \epsilon^{\mu\nu\alpha\beta} u_\nu \dot{\Omega}_{\alpha\beta},
\end{aligned} \tag{D.19}$$

where we defined $I^\mu \equiv \nabla^\mu \alpha_0$. The comoving derivatives of u_ν and $\Omega_{\alpha\beta}$ still appearing in this equation could in principle be replaced by Eqs. (5.62c), (5.67), and (5.68), respectively.

Now we calculate the equation of motion for the first spin moment. We obtain

$$\begin{aligned}
\dot{\tau}_r^{\mu,\langle\nu\rangle} =& \Delta_\lambda^\nu \int d\Gamma\, \mathfrak{s}^\mu r E_p^{r-1} p^{\langle\lambda\rangle} p^{\langle\rho\rangle} \dot{u}_\rho \delta f_{p\mathfrak{s}} + \Delta_\lambda^\nu \int d\Gamma\, \mathfrak{s}^\mu E_p^r p^{\langle\lambda\rangle} \delta \dot{f}_{p\mathfrak{s}} - \Delta_\lambda^\nu \int d\Gamma\, \mathfrak{s}^\mu E_p^{r+1} \dot{u}^\lambda \delta f_{p\mathfrak{s}} \\
=& r \dot{u}_\rho \tau_{r-1}^{\mu,\nu\rho} + \frac{1}{3} \dot{u}^\nu \left[m^2 r \tau_{r-1}^\mu - (r+3) \tau_{r+1}^\mu \right] + \int d\Gamma\, \mathfrak{s}^\mu E_p^r p^{\langle\nu\rangle} \delta \dot{f}_{p\mathfrak{s}}.
\end{aligned} \tag{D.20}$$

Again we evaluate the last term using the Boltzmann equation,

$$\begin{aligned}
& \int d\Gamma\, \mathfrak{s}^\mu E_p^r p^{\langle\nu\rangle} \delta \dot{f}_{p\mathfrak{s}} = \int d\Gamma\, E_p^{r-1} \mathfrak{s}^\mu p^{\langle\nu\rangle} \mathfrak{C}[f] + \frac{2}{m} \int dP\, p^{\langle\nu\rangle} \left(E_p^r p_{\langle\lambda\rangle} + E_p^{r+1} u_\lambda \right) \frac{d}{d\tau} \tilde{\bar{\Omega}}^{\mu\lambda} f_{0p} \\
& + \frac{2}{m} \int dP \left(E_p^{r-1} p_{\langle\lambda\rangle} + E_p^r u_\lambda \right) p^{\langle\nu\rangle} p^{\langle\rho\rangle} \nabla_\rho \tilde{\bar{\Omega}}^{\mu\lambda} f_{0p} - \int d\Gamma\, \mathfrak{s}^\mu E_p^{r-1} p^{\langle\nu\rangle} p^{\langle\rho\rangle} \nabla_\rho \delta f_{p\mathfrak{s}}.
\end{aligned} \tag{D.21}$$

The first term is the collision term. The second term is obtained as

$$\begin{aligned}
-\frac{2}{m} \int dP\, p^{\langle\nu\rangle} E_p^r p_{\langle\lambda\rangle} \frac{d}{d\tau} \tilde{\bar{\Omega}}^{\mu\lambda} f_{0p} =& -\frac{2}{m} \int dP\, p^{\langle\nu\rangle} E_p^r p_{\langle\lambda\rangle} \left[\tilde{\bar{\Omega}}^{\mu\lambda} (\dot{\alpha}_0 - E_p \dot{\beta}_0) + \dot{\tilde{\bar{\Omega}}}^{\mu\lambda} \right] f_{0p} \\
& + \frac{2}{m} \beta_0 \tilde{\bar{\Omega}}^\mu{}_\lambda \dot{u}_\rho \int dP\, p^{\langle\nu\rangle} E_p^r p^{\langle\lambda\rangle} p^{\langle\rho\rangle} f_{0p} \\
=& \frac{1}{m} \Delta_\lambda^\nu \tilde{\bar{\Omega}}^{\mu\lambda} \left(\dot{\alpha}_0 I_{(r+2)1} - \dot{\beta}_0 I_{(r+3)1} \right) - \frac{1}{m} \Delta_\lambda^\nu \dot{\tilde{\bar{\Omega}}}^{\mu\lambda} I_{(r+2)1},
\end{aligned} \tag{D.22}$$

where the last term vanished due to the orthogonality relation (D.7). For the third term we have

$$-\frac{2}{m} \int dP\, p^{\langle\nu\rangle} E_p^{r+1} u_\lambda \frac{d}{d\tau} \tilde{\bar{\Omega}}^{\mu\lambda} f_{0p} = -\frac{1}{m} \tilde{\bar{\Omega}}^{\mu\lambda} u_\lambda \beta_0 \dot{u}^\nu I_{(r+3)1}. \tag{D.23}$$

The fourth term is given by

$$\begin{aligned}
& -\frac{2}{m} \int dP\, E_p^{r-1} p_{\langle\lambda\rangle} p^{\langle\nu\rangle} p^{\langle\rho\rangle} \nabla_\rho \tilde{\bar{\Omega}}^{\mu\lambda} f_{0p} = \frac{2}{m} \beta_0 \tilde{\bar{\Omega}}^{\mu\lambda} (\nabla_\rho u_\alpha) \int dP\, E_p^{r-1} p_{\langle\lambda\rangle} p^{\langle\nu\rangle} p^{\langle\rho\rangle} p^{\langle\alpha\rangle} f_{0p} \\
=& \frac{1}{15} \frac{2}{m} \beta_0 \tilde{\bar{\Omega}}^\mu{}_\lambda (\nabla_\rho u_\alpha) \int dP\, E_p^{r-1} \left(\Delta^{\alpha\nu} \Delta^{\lambda\rho} + \Delta^{\alpha\lambda} \Delta^{\nu\rho} + \Delta^{\alpha\rho} \Delta^{\nu\lambda} \right) \left(\Delta^{\beta\gamma} p_\beta p_\gamma \right)^2 \\
=& \frac{1}{m} \beta_0 I_{(r+3)2} \left(2\, \tilde{\bar{\Omega}}^\mu{}_\lambda \sigma^{\nu\lambda} + \frac{5}{3} \theta \Delta_\lambda^\nu \tilde{\bar{\Omega}}^{\mu\lambda} \right).
\end{aligned} \tag{D.24}$$

For the fifth term we obtain

$$
\begin{aligned}
-\frac{2}{m}\int dP\, E_p^r u_\lambda p^{\langle\nu\rangle}p^{\langle\rho\rangle}\nabla_\rho\tilde{\bar{\Omega}}^{\mu\lambda}f_{0p} =& -\frac{2}{m}\int dP\, E_p^r u_\lambda p^{\langle\nu\rangle}p^{\langle\rho\rangle}\left[\tilde{\bar{\Omega}}^{\mu\lambda}\left(\nabla_\rho\alpha_0 - E_p\nabla_\rho\beta_0\right) + \nabla_\rho\tilde{\bar{\Omega}}^{\mu\lambda}\right]f_{0p} \\
=& \frac{1}{m}\tilde{\bar{\Omega}}^{\mu\lambda}u_\lambda\left(I_{(r+2)1}\nabla^\nu\alpha_0 - I_{(r+3)1}\nabla^\nu\beta_0\right) + \frac{1}{m}I_{(r+2)1}\left(\nabla^\nu\tilde{\bar{\Omega}}^{\mu\lambda}\right)u_\lambda.
\end{aligned} \tag{D.25}
$$

Finally, evaluating the last term yields

$$
\begin{aligned}
-\int d\Gamma\,\mathfrak{s}^\mu E_p^{r-1}p^{\langle\nu\rangle}p^{\langle\rho\rangle}\nabla_\rho\delta f_{p\mathfrak{s}} =& -\nabla_\rho\int d\Gamma\,\mathfrak{s}^\mu E_p^{r-1}p^{\langle\nu\rangle}\left(E_p u^\rho + p^{\langle\rho\rangle}\right)\delta f_{p\mathfrak{s}} \\
&+ (r-1)(\nabla_\rho u_\lambda)\int d\Gamma\,\mathfrak{s}^\mu E_p^{r-2}p^{\langle\nu\rangle}p^{\langle\lambda\rangle}p^{\langle\rho\rangle}\delta f_{p\mathfrak{s}} \\
&- \int d\Gamma\,\mathfrak{s}^\mu E_p^{r-1}\left(E_p\nabla_\rho u^\nu + p^{\langle\lambda\rangle}u^\nu\nabla_\rho u_\lambda\right)p^{\langle\rho\rangle}\delta f_{p\mathfrak{s}} \\
=& -\theta\tau_r^{\mu,\nu} - \nabla_\rho\tau_{r-1}^{\mu,\nu\rho} - \frac{1}{3}\nabla^\nu\left(m^2\tau_{r-1}^\mu - \tau_{r+1}^\mu\right) + \frac{1}{3}u^\nu\theta\left(m^2\tau_{r-1}^\mu - \tau_{r+1}^\mu\right) \\
&+ (r-1)\left(\frac{\theta}{3}\Delta_{\lambda\rho} + \sigma_{\lambda\rho}\right)\left[\tau_{r-2}^{\mu,\nu\lambda\rho} + \frac{1}{5}\Delta^{\nu\lambda}\left(m^2\tau_{r-2}^{\mu,\rho} - \tau_r^{\mu,\rho}\right)\right. \\
&\left.+\frac{1}{5}\Delta^{\nu\rho}\left(m^2\tau_{r-2}^{\mu,\lambda} - \tau_r^{\mu,\lambda}\right) + \frac{1}{5}\Delta^{\lambda\rho}\left(m^2\tau_{r-2}^{\mu,\nu} - \tau_r^{\mu,\nu}\right)\right] \\
&- (\nabla_\rho u^\nu)\tau_r^{\mu,\rho} - (\nabla_\rho u_\lambda)u^\nu\tau_{r-1}^{\mu,\lambda\rho} - \frac{1}{3}\theta u^\nu\left(m^2\tau_{r-1}^\mu - \tau_{r+1}^\mu\right) \\
=& -\frac{4}{3}\theta\tau_r^{\mu,\nu} - \nabla_\rho\tau_{r-1}^{\mu,\nu\rho} - \frac{1}{3}\nabla^\nu\left(m^2\tau_{r-1}^\mu - \tau_{r+1}^\mu\right) - (\sigma^\nu{}_\rho - \omega^\nu{}_\rho)\tau_r^{\mu,\rho} \\
&+ (r-1)\left[\sigma_{\lambda\rho}\tau_{r-2}^{\mu,\nu\lambda\rho} + \frac{\theta}{3}\left(m^2\tau_{r-2}^{\mu,\nu} - \tau_r^{\mu,\nu}\right) + \frac{2}{5}\sigma^\nu{}_\lambda\left(m^2\tau_{r-2}^{\mu,\lambda} - \tau_r^{\mu,\lambda}\right)\right] \\
&- u^\nu(\nabla_\rho u_\lambda)\tau_{r-1}^{\mu,\lambda\rho}.
\end{aligned} \tag{D.26}
$$

Collecting all terms we obtain

$$
\begin{aligned}
\dot{\tau}_r^{\mu,\langle\nu\rangle} - \mathfrak{C}_{r-1}^{\mu,\langle\nu\rangle} =& \frac{1}{m}\Delta^\nu_\lambda\tilde{\bar{\Omega}}^{\mu\lambda}\left[\xi_r^{(1)}\theta + \frac{H_{3(r+2)}}{D_{20}}\partial\cdot n - \frac{H_{2(r+2)}}{D_{20}}\left(\Pi\theta - \pi^{\alpha\beta}\sigma_{\alpha\beta}\right)\right] + \frac{1}{m}\Delta^\nu_\lambda\dot{\tilde{\bar{\Omega}}}^{\mu\lambda}I_{(r+2)1} \\
&+ \frac{1}{m}\tilde{\bar{\Omega}}^{\mu\lambda}u_\lambda\left[\frac{\beta_0}{\epsilon_0 + P_0}I_{(r+3)1}\left(-\Pi\dot{u}^\nu + \nabla^\nu\Pi - \Delta^\nu_\lambda\partial_\rho\pi^{\lambda\rho}\right) - I_{(r+2)1}I^\nu\right] - \frac{2}{m}\beta_0 I_{(r+3)2}\tilde{\bar{\Omega}}^\mu{}_\lambda\sigma^{\nu\lambda} \\
&- \frac{1}{m}I_{(r+2)1}\left(\nabla^\nu\tilde{\bar{\Omega}}^{\mu\lambda}\right)u_\lambda + \omega^\nu{}_\rho\tau_r^{\mu,\rho} + \frac{1}{3}\left[(r-1)m^2\tau_{r-2}^{\mu,\nu} - (r+3)\tau_r^{\mu,\nu}\right]\theta - \Delta^\nu_\lambda\nabla_\rho\tau_{r-1}^{\mu,\lambda\rho} \\
&+ r\dot{u}_\rho\tau_{r-1}^{\mu,\nu\rho} + \frac{1}{5}\left[2(r-1)m^2\tau_{r-2}^{\mu,\lambda} - (2r+3)\tau_r^{\mu,\lambda}\right]\sigma^\nu{}_\lambda + \frac{1}{3}\dot{u}^\nu\left[m^2 r\tau_{r-1}^\mu - (r+3)\tau_{r+1}^\mu\right] \\
&- \frac{1}{3}\nabla^\nu\left(m^2\tau_{r-1}^\mu - \tau_{r+1}^\mu\right) + (r-1)\sigma_{\lambda\rho}\tau_{r-2}^{\mu,\nu\lambda\rho},
\end{aligned} \tag{D.27}
$$

where we defined

$$
\begin{aligned}
H_{nm} &\equiv I_{n0}I_{m1} - I_{(n-1)0}I_{(m+1)1}, \\
\xi_r^{(1)} &\equiv \frac{H_{3(r+2)}}{D_{20}}n_0 - \frac{H_{2(r+2)}}{D_{20}}(\epsilon_0 + P_0) - \frac{5}{3}\beta_0 I_{(r+3)2}.
\end{aligned} \tag{D.28}
$$

Now we calculate the equation of motion for the second spin moment, which is given by

$$\begin{aligned}
\dot{\tau}_r^{\mu,\langle\nu\lambda\rangle} =&\Delta^{\nu\lambda}_{\alpha\beta}\frac{d}{d\tau}\int d\Gamma E_p^r p^{\langle\alpha}p^{\beta\rangle}\mathfrak{s}^\mu \delta f_{p\mathfrak{s}} \\
=&\Delta^{\nu\lambda}_{\alpha\beta}\left(r\dot{u}_\rho\int d\Gamma E_p^{r-1}p^{\langle\alpha}p^{\beta\rangle}p^{\langle\rho\rangle}\mathfrak{s}^\mu\delta f_{p\mathfrak{s}} + \int d\Gamma E_p^r\left(\frac{d}{d\tau}\Delta^{\alpha\beta}_{\rho\tau}\right)p^\rho p^\tau \mathfrak{s}^\mu\delta f_{p\mathfrak{s}} + \int d\Gamma E_p^r p^{\langle\alpha}p^{\beta\rangle}\mathfrak{s}^\mu\delta\dot{f}_{p\mathfrak{s}}\right) \\
=&\Delta^{\nu\lambda}_{\alpha\beta}\left[r\dot{u}_\rho\int d\Gamma E_p^{r-1}\left(p^{\langle\alpha}p^\beta p^{\rho\rangle}+\frac{1}{5}(m^2-E_p^2)p^{\langle(\alpha\rangle}\Delta^{\beta)\rho}\right)\mathfrak{s}^\mu\delta f_{p\mathfrak{s}} - 2\dot{u}^\alpha\int d\Gamma E_p^{r+1}p^{\langle\beta\rangle}\mathfrak{s}^\mu\delta f_{p\mathfrak{s}}\right. \\
&\left. + \int d\Gamma E_p^r p^{\langle\alpha}p^{\beta\rangle}\mathfrak{s}^\mu \delta\dot{f}_{p\mathfrak{s}}\right] \\
=&\Delta^{\nu\lambda}_{\alpha\beta}\left[r\dot{u}_\rho\tau^{\mu,\alpha\beta\rho}_{r-1} + r\frac{2}{5}m^2\dot{u}^\beta\left(\tau^{\mu,\alpha}_{r-1}-\tau^{\mu,\alpha}_{r+1}\right) - 2\dot{u}^\alpha\tau^{\mu,\beta}_{r+1} + \int d\Gamma E_p^r p^{\langle\alpha}p^{\beta\rangle}\mathfrak{s}^\mu\delta\dot{f}_{p\mathfrak{s}}\right] \\
=&r\dot{u}_\rho\tau^{\mu,\nu\lambda\rho}_{r-1} + \frac{2}{5}\left(m^2\tau^{\mu,\langle\nu}_{r-1} - (r+5)\tau^{\mu,\langle\nu}_{r+1}\right)\dot{u}^{\lambda\rangle} + \Delta^{\nu\lambda}_{\alpha\beta}\int d\Gamma E_p^r p^{\langle\alpha}p^{\beta\rangle}\mathfrak{s}^\mu\delta\dot{f}_{p\mathfrak{s}},
\end{aligned} \tag{D.29}$$

where we used

$$p^{\langle\alpha}p^{\beta\rangle} = \left(\Delta^\alpha_\rho\Delta^\beta_\tau - \frac{1}{3}\Delta^{\alpha\beta}\Delta_{\rho\tau}\right)p^\rho p^\tau \tag{D.30}$$

and $\Delta^{\nu\lambda}_{\alpha\beta}u^\alpha = 0 = \Delta^{\nu\lambda}_{\alpha\beta}\Delta^{\alpha\beta}$. The last term is again evaluated with the help of the Boltzmann equation

$$\begin{aligned}
\Delta^{\nu\lambda}_{\alpha\beta}\int d\Gamma E_p^r p^{\langle\alpha}p^{\beta\rangle}\mathfrak{s}^\mu\delta\dot{f}_{p\mathfrak{s}} =& \int d\Gamma\, E_p^{r-1}\mathfrak{s}^\mu p^{\langle\nu}p^{\lambda\rangle}\mathfrak{C}[f] + \frac{2}{m}\int dP\, p^{\langle\nu}p^{\lambda\rangle}\left(E_p^r p_{\langle\rho\rangle} + E_p^{r+1}u_\rho\right)\frac{d}{d\tau}\tilde{\bar{\Omega}}^{\mu\rho}f_{0p} \\
&+\frac{2}{m}\int dP\,\left(E_p^{r-1}p_{\langle\tau\rangle}+E_p^r u_\tau\right)p^{\langle\nu}p^{\lambda\rangle}p^{\langle\rho\rangle}\nabla_\rho\tilde{\bar{\Omega}}^{\mu\tau}f_{0p} \\
&-\Delta^{\nu\lambda}_{\alpha\beta}\int d\Gamma\,\mathfrak{s}^\mu E_p^{r-1}p^{\langle\alpha}p^{\beta\rangle}p^{\langle\rho\rangle}\nabla_\rho\delta f_{p\mathfrak{s}}.
\end{aligned} \tag{D.31}$$

We obtain for the second term

$$\begin{aligned}
-\frac{2}{m}\int dP\, p^{\langle\nu}p^{\lambda\rangle}E_p^r p_{\langle\rho\rangle}\frac{d}{d\tau}\tilde{\bar{\Omega}}^{\mu\rho}f_{0p} =& \frac{2}{m}\beta_0\tilde{\bar{\Omega}}^{\mu\rho}\dot{u}^\tau\int dP\, p^{\langle\nu}p^{\lambda\rangle}E_p^r p_{\langle\rho\rangle}p_{\langle\tau\rangle}f_{0p} \\
=&\frac{2}{m}\beta_0\tilde{\bar{\Omega}}^{\mu\rho}\dot{u}^\tau\frac{2}{15}\Delta^{\nu\lambda}_{\rho\tau}\int(\Delta^{\alpha\beta}p_\alpha p_\beta)^2E_p^r f_{0p} \\
=&\frac{2}{m}\beta_0\tilde{\bar{\Omega}}^{\mu\langle\nu}\dot{u}^{\lambda\rangle}I_{(r+4)2},
\end{aligned} \tag{D.32}$$

where most terms vanish due to the orthogonality relation (D.7). The third term in Eq. (D.31) vanishes for the same reason. The fourth term is given by

$$\begin{aligned}
&-\frac{2}{m}\int dP\, E_p^{r-1}p_{\langle\tau\rangle}p^{\langle\nu}p^{\lambda\rangle}p^{\langle\rho\rangle}\nabla_\rho\tilde{\bar{\Omega}}^{\mu\tau}f_{0p} \\
=&-\frac{2}{m}\int dP\, E_p^{r-1}p^{\langle\nu}p^{\lambda\rangle}p_{\langle\tau}p_{\rho\rangle}\left[\left(-E_p\nabla^\rho\beta_0+\nabla^\rho\alpha_0\right)\tilde{\bar{\Omega}}^{\mu\tau}+\nabla^\rho\tilde{\bar{\Omega}}^{\mu\tau}\right]f_{0p} \\
=&-\frac{2}{m}\tilde{\bar{\Omega}}^{\mu\langle\nu}\left(-I_{(r+4)2}\nabla^{\lambda\rangle}\beta_0 + I_{(r+3)2}\nabla^{\lambda\rangle}\alpha_0\right) - \frac{2}{m}I_{(r+3)2}\Delta^{\nu\lambda}_{\alpha\beta}\nabla^\alpha\tilde{\bar{\Omega}}^{\mu\beta}.
\end{aligned} \tag{D.33}$$

For the fifth term we obtain

$$\begin{aligned}
-\frac{2}{m}\int dP\, E_p^r u_\tau p^{\langle\nu}p^{\lambda\rangle}p^{\langle\rho\rangle}\nabla_\rho\tilde{\bar{\Omega}}^{\mu\tau}f_{0p} =& \frac{2}{m}\tilde{\bar{\Omega}}^{\mu\tau}\beta_0 u_\tau(\nabla^\rho u^\alpha)\int dP\, E_p^r p^{\langle\nu}p^{\lambda\rangle}p_{\langle\rho}p_{\alpha\rangle}f_{0p} \\
=&\frac{2}{m}\tilde{\bar{\Omega}}^{\mu\rho}\beta_0 u_\rho\sigma^{\nu\lambda}I_{(r+4)2}.
\end{aligned} \tag{D.34}$$

Finally, we calculate the last term

$$
\begin{aligned}
&-\Delta^{\nu\lambda}_{\alpha\beta}\int d\Gamma\,\mathfrak{s}^\mu E_p^{r-1}p^{\langle\alpha}p^{\beta\rangle}p^{\langle\rho\rangle}\nabla_\rho\delta f_{p\mathfrak{s}}\\
=&-\Delta^{\nu\lambda}_{\alpha\beta}\nabla_\rho\int d\Gamma\,\mathfrak{s}^\mu E_p^{r-1}p^{\langle\alpha}p^{\beta\rangle}\left(p^{\langle\rho\rangle}+E_pu^\rho\right)\delta f_{p\mathfrak{s}}+(r-1)\Delta^{\nu\lambda}_{\alpha\beta}(\nabla_\rho u_\tau)\int d\Gamma\,\mathfrak{s}^\mu E_p^{r-2}p^{\langle\alpha}p^{\beta\rangle}p^{\langle\rho\rangle}p^{\langle\tau\rangle}\delta f_{p\mathfrak{s}}\\
&+\Delta^{\nu\lambda}_{\alpha\beta}\int d\Gamma\,\mathfrak{s}^\mu E_p^{r-1}\left(-\Delta^\alpha_\sigma\nabla_\rho u^\beta u_\tau-\Delta^\beta_\tau\nabla_\rho u^\alpha u_\sigma\right)p^\sigma p^\tau p^{\langle\rho\rangle}\delta f_{p\mathfrak{s}}\\
=&-\Delta^{\nu\lambda}_{\alpha\beta}\nabla_\rho\int d\Gamma\,\mathfrak{s}^\mu E_p^{r-1}\left[p^{\langle\alpha\rangle}p^{\langle\beta\rangle}-\frac{1}{3}(m^2-E_p^2)\Delta^{\alpha\beta}\right]p^{\langle\rho\rangle}\delta f_{p\mathfrak{s}}-\theta\tau_r^{\mu,\nu\lambda}\\
&+(r-1)\Delta^{\nu\lambda}_{\alpha\beta}(\nabla_\rho u_\tau)\int d\Gamma\,\mathfrak{s}^\mu E_p^{r-2}p^{\langle\alpha\rangle}p^{\langle\beta\rangle}p^{\langle\rho\rangle}p^{\langle\tau\rangle}\delta f_{p\mathfrak{s}}-2\Delta^{\nu\lambda}_{\alpha\beta}(\nabla_\rho u^\beta)\int d\Gamma\,\mathfrak{s}^\mu E_p^{r}p^{\langle\alpha\rangle}p^{\langle\rho\rangle}\delta f_{p\mathfrak{s}}\\
=&-\Delta^{\nu\lambda}_{\alpha\beta}\nabla_\rho\left[\tau_{r-1}^{\mu,\alpha\beta\rho}+\frac{1}{5}m^2\left(\tau_{r-1}^{\mu,\alpha}\Delta^{\beta\rho}+\tau_{r-1}^{\mu,\beta}\Delta^{\alpha\rho}+\tau_{r-1}^{\mu,\rho}\Delta^{\alpha\beta}\right)-\frac{1}{5}\left(\tau_{r+1}^{\mu,\alpha}\Delta^{\beta\rho}+\tau_{r+1}^{\mu,\beta}\Delta^{\alpha\rho}+\tau_{r+1}^{\mu,\rho}\Delta^{\alpha\beta}\right)\right]\\
&+\frac{1}{3}\left(m^2\tau_{r-1}^{\mu,\rho}-\tau_{r+1}^{\mu,\rho}\right)\Delta^{\nu\lambda}_{\alpha\beta}\nabla_\rho\Delta^{\alpha\beta}-\theta\tau_r^{\mu,\nu\lambda}\\
&+(r-1)\Delta^{\nu\lambda}_{\alpha\beta}(\nabla_\rho u_\tau)\left[\tau_{r-2}^{\mu,\alpha\beta\rho\tau}+\frac{1}{7}\left(m^2\tau_{r-2}^{\mu,\alpha\beta}\Delta^{\rho\tau}-\tau_r^{\mu,\alpha\beta}\Delta^{\rho\tau}\right)\right.\\
&\left.+\frac{2}{7}\left(m^2\tau_{r-2}^{\mu,\alpha(\rho}\Delta^{\tau)\beta}-\tau_r^{\mu,\alpha(\rho}\Delta^{\tau)\beta}\right)+\frac{2}{15}\Delta^{\alpha\rho}\Delta^{\beta\tau}\left(m^4\tau_{r-2}^\mu+\tau_{r+2}^\mu-2m^2\tau_1^\mu\right)\right]\\
&-2\Delta^{\nu\lambda}_{\alpha\beta}(\nabla_\rho u^\beta)\tau_r^{\mu,\alpha\rho}-\frac{2}{3}\Delta^{\nu\lambda}_{\alpha\beta}\Delta^{\alpha\rho}(\nabla_\rho u^\beta)\left(m^2\tau_r^\mu-\tau_{r+2}^\mu\right)\\
=&-\Delta^{\nu\lambda}_{\alpha\beta}\nabla_\rho\tau_{r-1}^{\mu,\alpha\beta\rho}+\frac{2}{5}\Delta^{\nu\lambda}_{\alpha\beta}\nabla^\beta\left(\tau_{r+1}^{\mu,\alpha}-m^2\tau_{r-1}^{\mu,\alpha}\right)-\theta\tau_r^{\mu,\nu\lambda}+(r-1)\sigma_{\rho\tau}\tau_{r-2}^{\mu,\nu\lambda\rho\tau}\\
&+(r-1)\frac{1}{7}\theta\left(m^2\tau_{r-2}^{\mu,\nu\lambda}-\tau_r^{\mu,\nu\lambda}\right)+(r-1)\frac{4}{7}\Delta^{\nu\lambda}_{\alpha\beta}\sigma_\rho{}^\beta\left(m^2\tau_{r-2}^{\mu,\alpha\rho}-\tau_r^{\mu,\alpha\rho}\right)\\
&+(r-1)\frac{4}{21}\theta\left(m^2\tau_{r-2}^{\mu,\nu\lambda}-\tau_r^{\mu,\nu\lambda}\right)+\frac{2}{15}(r-1)\sigma^{\nu\lambda}\left(m^4\tau_{r-2}^\mu+\tau_{r+2}^\mu-2m^2\tau_r^\mu\right)\\
&-2\Delta^{\nu\lambda}_{\alpha\beta}\sigma_\rho{}^\beta\tau_r^{\mu,\alpha\rho}-\frac{2}{3}\theta\tau_r^{\mu,\nu\lambda}-2\Delta^{\nu\lambda}_{\alpha\beta}\omega_\rho{}^\beta\tau_r^{\mu,\alpha\rho}-\frac{2}{3}\sigma^{\nu\lambda}\left(m^2\tau_r^\mu-\tau_{r+2}^\mu\right)\\
=&-\Delta^{\nu\lambda}_{\alpha\beta}\nabla_\rho\tau_{r-1}^{\mu,\alpha\beta\rho}+\frac{2}{5}\Delta^{\nu\lambda}_{\alpha\beta}\nabla^\beta\left(\tau_{r+1}^{\mu,\alpha}-m^2\tau_{r-1}^{\mu,\alpha}\right)\\
&+\frac{1}{3}\left[(r-1)m^2\tau_{r-2}^{\mu,\nu\lambda}-(r+4)\tau_r^{\mu,\nu\lambda}\right]\theta+(r-1)\sigma_{\rho\tau}\tau_{r-2}^{\mu,\nu\lambda\rho\tau}\\
&+\frac{2}{7}\left[2(r-1)m^2\tau_{r-2}^{\mu,\rho\langle\nu}-(2r+5)\tau_r^{\mu,\rho\langle\nu}\right]\sigma^{\lambda\rangle}_\rho+2\tau_r^{\mu,\rho\langle\nu}\omega^{\lambda\rangle}_\rho\\
&+\frac{2}{15}\left[(r-1)m^4\tau_{r-2}^\mu-(2r+3)m^2\tau_r^\mu+(r+4)\tau_{r+2}^\mu\right]\sigma^{\nu\lambda}.
\end{aligned}\tag{D.35}
$$

Combining all terms we arrive at the equation of motion

$$
\begin{aligned}
\dot{\tau}_r^{\mu,\langle\nu\lambda\rangle}-\mathfrak{C}_{r-1}^{\mu,\langle\nu\lambda\rangle}=&\frac{2}{m}\xi_r^{(2)}\tilde{\bar{\Omega}}^{\mu\langle\nu}I^{\lambda\rangle}+\frac{2}{m}I_{(r+3)2}\Delta^{\nu\lambda}_{\alpha\beta}\nabla^\alpha\tilde{\bar{\Omega}}^{\mu\beta}-\frac{2}{m}\tilde{\bar{\Omega}}^{\mu\rho}\beta_0u_\rho\sigma^{\nu\lambda}I_{(r+4)2}\\
&-\frac{2}{m}\frac{\beta_0}{\epsilon_0+P_0}I_{(r+4)2}\tilde{\bar{\Omega}}^{\mu\langle\nu}\left(-\Pi\dot{u}^{\lambda\rangle}+\nabla^{\lambda\rangle}\Pi-\Delta^{\lambda\rangle}_\alpha\partial_\beta\pi^{\alpha\beta}\right)\\
&+r\dot{u}_\rho\tau_{r-1}^{\mu,\nu\lambda\rho}+\frac{2}{5}\left(m^2\tau_{r-1}^{\mu,\langle\nu}-(r+5)\tau_{r+1}^{\mu,\langle\nu}\right)\dot{u}^{\lambda\rangle}\\
&-\Delta^{\nu\lambda}_{\alpha\beta}\nabla_\rho\tau_{r-1}^{\mu,\alpha\beta\rho}+\frac{2}{5}\Delta^{\nu\lambda}_{\alpha\beta}\nabla^\beta\left(\tau_{r+1}^{\mu,\alpha}-m^2\tau_{r-1}^{\mu,\alpha}\right)\\
&+\frac{1}{3}\left[(r-1)m^2\tau_{r-2}^{\mu,\nu\lambda}-(r+4)\tau_r^{\mu,\nu\lambda}\right]\theta+(r-1)\sigma_{\rho\tau}\tau_{r-2}^{\mu,\nu\lambda\rho\tau}\\
&+\frac{2}{7}\left[2(r-1)m^2\tau_{r-2}^{\mu,\rho\langle\nu}-(2r+5)\tau_r^{\mu,\rho\langle\nu}\right]\sigma^{\lambda\rangle}_\rho+2\tau_r^{\mu,\rho\langle\nu}\omega^{\lambda\rangle}_\rho\\
&+\frac{2}{15}\left[(r-1)m^4\tau_{r-2}^\mu-(2r+3)m^2\tau_r^\mu+(r+4)\tau_{r+2}^\mu\right]\sigma^{\nu\lambda},
\end{aligned}\tag{D.36}
$$

with

$$
\xi_r^{(2)}\equiv I_{(r+3)2}-\frac{n_0}{\epsilon_0+P_0}I_{(r+4)2}.\tag{D.37}
$$

As we will see, the third spin moments have nonzero first-order solutions. The equations of motion are obtained as

$$
\begin{aligned}
\dot{\tau}_r^{\mu,\langle\nu\lambda\rho\rangle} =& \Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}\frac{d}{d\tau}\int d\Gamma E_p^r p^{\langle\alpha}p^\beta p^{\gamma\rangle}\mathfrak{s}^\mu \delta f_{p\mathfrak{s}} \\
=& \Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}\left[r\dot{u}_\eta \int d\Gamma E_p^{r-1} p^{\langle\alpha}p^\beta p^{\gamma\rangle}p^{\langle\eta\rangle}\mathfrak{s}^\mu \delta f_{p\mathfrak{s}} + \int d\Gamma E_p^r \left(\frac{d}{d\tau}\Delta^{\alpha\beta\gamma}_{\sigma\tau\eta}\right) p^\sigma p^\tau p^\eta \mathfrak{s}^\mu \delta f_{p\mathfrak{s}} \right. \\
& \left. + \int d\Gamma E_p^r p^{\langle\alpha}p^\beta p^{\gamma\rangle}\mathfrak{s}^\mu \delta \dot{f}_{p\mathfrak{s}}\right] \\
=& \Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}\left\{r\dot{u}_\eta \int d\Gamma E_p^{r-1}\left[p^{\langle\alpha}p^\beta p^\gamma p^{\eta\rangle} + \frac{3}{7}p^{\langle\alpha}p^{\beta\rangle}\Delta^{\gamma\eta}\left(m^2 - E_p^2\right)\right]\mathfrak{s}^\mu \delta f_{p\mathfrak{s}} \right. \\
& \left. - \frac{3}{2}\int d\Gamma E_p^r \dot{u}^\alpha u_\sigma \Delta^\beta_\tau \Delta^\gamma_\eta p^\sigma p^\tau p^\eta \mathfrak{s}^\mu \delta f_{p\mathfrak{s}} + \int d\Gamma E_p^r p^{\langle\alpha}p^\beta p^{\gamma\rangle}\mathfrak{s}^\mu \delta \dot{f}_{p\mathfrak{s}}\right\} \\
=& \Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}\left\{r\dot{u}_\eta\left[\tau_{r-1}^{\mu,\alpha\beta\gamma\eta} + \frac{3}{7}\Delta^{\gamma\eta}\left(m^2\tau_{r-1}^{\mu,\alpha\beta} - \tau_{r+1}^{\mu,\alpha\beta}\right)\right] - \frac{3}{2}\dot{u}^\alpha \tau_{r+1}^{\mu,\beta\gamma} + \int d\Gamma E_p^r p^{\langle\alpha}p^\beta p^{\gamma\rangle}\mathfrak{s}^\mu \delta \dot{f}_{p\mathfrak{s}}\right\} \\
=& r\dot{u}_\eta \tau_{r-1}^{\mu,\nu\lambda\rho\eta} + \frac{3}{14}\left[2m^2 r\tau_{r-1}^{\mu,\langle\nu\lambda} - (2r+7)\tau_{r+1}^{\mu,\langle\nu\lambda}\right]\dot{u}^{\rho\rangle} + \Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}\int d\Gamma E_p^r p^{\langle\alpha}p^\beta p^{\gamma\rangle}\mathfrak{s}^\mu \delta \dot{f}_{p\mathfrak{s}},
\end{aligned}
\tag{D.38}
$$

where we used

$$
\Delta^{\alpha\beta\gamma}_{\sigma\tau\eta}p^\sigma p^\tau p^\eta = \left[\frac{1}{2}\Delta^\alpha_\sigma \Delta^\beta_\tau \Delta^\gamma_\eta - \frac{1}{5}\left(\Delta^{\alpha\beta}\Delta_{\sigma\tau}\Delta^\gamma_\eta + \Delta^{\alpha\gamma}\Delta_{\sigma\tau}\Delta^\beta_\eta + \Delta^{\beta\gamma}\Delta_{\sigma\tau}\Delta^\alpha_\eta\right)\right]p^\sigma p^\tau p^\eta. \tag{D.39}
$$

Once more we evaluate the last term using the Boltzmann equation,

$$
\begin{aligned}
\Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}\int d\Gamma E_p^r p^{\langle\alpha}p^\beta p^{\gamma\rangle}\mathfrak{s}^\mu \delta \dot{f}_{p\mathfrak{s}} =& \int d\Gamma\, E_p^{r-1}\mathfrak{s}^\mu p^{\langle\nu}p^\lambda p^{\rho\rangle}\mathfrak{C}[f] + \frac{2}{m}\int dP\, p^{\langle\nu}p^\lambda p^{\rho\rangle}\left(E_p^r p_{\langle\tau\rangle} + E_p^{r+1}u_\tau\right)\frac{d}{d\tau}\tilde{\bar{\Omega}}^{\mu\tau} f_{0p} \\
& + \frac{2}{m}\int dP\,\left(E_p^{r-1}p_{\langle\tau\rangle} + E_p^r u_\tau\right)p^{\langle\nu}p^\lambda p^{\rho\rangle}p^{\langle\eta\rangle}\nabla_\eta \tilde{\bar{\Omega}}^{\mu\tau} f_{0p} \\
& - \Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}\int d\Gamma\, \mathfrak{s}^\mu E_p^{r-1}p^{\langle\alpha}p^\beta p^{\gamma\rangle}p^{\langle\tau\rangle}\nabla_\tau \delta f_{p\mathfrak{s}}.
\end{aligned}
\tag{D.40}
$$

We see immediately that the second, third, and fifth term vanish, respectively, due to the orthogonality relation (D.7). The fourth term is given by

$$
\begin{aligned}
-\frac{2}{m}\int dP\, E_p^{r-1}p_{\langle\tau\rangle}p^{\langle\nu}p^\lambda p^{\rho\rangle}p^{\langle\eta\rangle}\nabla_\eta \tilde{\bar{\Omega}}^{\mu\tau} f_{0p} =& \frac{2}{m}\tilde{\bar{\Omega}}^{\mu\tau}(\nabla^\eta u^\xi)\int dP\, E_p^{r-1}p^{\langle\nu}p^\lambda p^{\rho\rangle}p_{\langle\tau\rangle}p_{\langle\eta\rangle}p_{\langle\xi\rangle} f_{0p} \\
=& \frac{2}{m}\tilde{\bar{\Omega}}^{\mu\tau}(\nabla^\eta u^\xi)\Delta^{\nu\lambda\rho}_{\tau\eta\xi}\frac{2}{35}\int dP\, E_p^{r-1}(\Delta^{\alpha\beta}p_\alpha p_\beta)^3 f_{0p} \\
=& -\frac{6}{m}I_{(r+5)3}\tilde{\bar{\Omega}}^{\mu\langle\nu}\sigma^{\lambda\rho\rangle}.
\end{aligned}
\tag{D.41}
$$

For the last term, we obtain

$$
\begin{aligned}
-\Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}\int d\Gamma\, \mathfrak{s}^\mu E_p^{r-1}p^{\langle\alpha}p^\beta p^{\gamma\rangle}p^{\langle\tau\rangle}\nabla_\tau \delta f_{p\mathfrak{s}} =& -\Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}\nabla_\tau \int d\Gamma\, \mathfrak{s}^\mu E_p^{r-1}p^{\langle\alpha}p^\beta p^{\gamma\rangle}\left(p^{\langle\tau\rangle} + E_p u^\tau\right)\delta f_{p\mathfrak{s}} \\
& + (r-1)(\nabla_\tau u_\eta)\Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}\int d\Gamma\, \mathfrak{s}^\mu E_p^{r-2}p^{\langle\alpha}p^\beta p^{\gamma\rangle}p^{\langle\tau\rangle}p^{\langle\eta\rangle}\delta f_{p\mathfrak{s}} \\
& - \frac{3}{2}\Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}(\nabla_\tau u^\alpha)\int d\Gamma\, \mathfrak{s}^\mu E_p^r p^{\langle\beta\rangle}p^{\langle\gamma\rangle}p^{\langle\tau\rangle}\delta f_{p\mathfrak{s}},
\end{aligned}
\tag{D.42}
$$

where we again used Eq. (D.39). We calculate each term separately. The first one is given by

$$
\begin{aligned}
&-\Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}\nabla_\tau \int d\Gamma\, \mathfrak{s}^\mu E_p^{r-1} p^{\langle\alpha}p^\beta p^{\gamma\rangle}p^{\langle\tau\rangle}\delta f_{p\mathfrak{s}} \\
&= -\Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}\nabla_\tau \int d\Gamma\, \mathfrak{s}^\mu E_p^{r-1} \left[p^{\langle\alpha\rangle}p^{\langle\beta\rangle}p^{\langle\gamma\rangle}p^{\langle\tau\rangle} - \frac{3}{5}(m^2-E_p^2)\Delta^{\alpha\beta}p^{\langle\gamma\rangle}p^{\langle\tau\rangle}\right]\delta f_{p\mathfrak{s}} \\
&= -\Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}\nabla_\tau \int d\Gamma\, \mathfrak{s}^\mu E_p^{r-1} \left[p^{\langle\alpha}p^\beta p^\gamma p^{\tau\rangle} + \frac{3}{7}(m^2-E_p^2)\left(p^{\langle\alpha}p^{\beta\rangle}\Delta^{\gamma\tau} + p^{\langle\alpha}p^{\tau\rangle}\Delta^{\beta\gamma}\right) + \frac{1}{5}\Delta^{\alpha\beta}\Delta^{\gamma\tau}(m^2-E_p^2)^2\right. \\
&\left. -\frac{3}{5}(m^2-E_p^2)\Delta^{\alpha\beta}p^{\langle\gamma}p^{\tau\rangle} - \frac{1}{5}\Delta^{\alpha\beta}\Delta^{\gamma\tau}(m^2-E_p^2)^2\right]\delta f_{p\mathfrak{s}} \\
&= -\Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}\nabla_\tau \left[\tau_{r-1}^{\mu,\alpha\beta\gamma\tau} + \frac{3}{7}\Delta^{\gamma\tau}\left(m^2\tau_{r-1}^{\mu,\alpha\beta} - \tau_{r+1}^{\mu,\alpha\beta}\right) - \frac{6}{35}\Delta^{\alpha\beta}\left(m^2\tau_{r-1}^{\mu,\gamma\tau} - \tau_{r+1}^{\mu,\gamma\tau}\right)\right] \\
&= -\Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}\nabla_\tau \tau_{r-1}^{\mu,\alpha\beta\gamma\tau} - \frac{3}{7}\Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}\nabla^\gamma\left(m^2\tau_{r-1}^{\mu,\alpha\beta} - \tau_{r+1}^{\mu,\alpha\beta}\right).
\end{aligned}
\tag{D.43}
$$

The second term reads

$$
-\Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}\nabla_\tau \int d\Gamma\, \mathfrak{s}^\mu E_p^r p^{\langle\alpha}p^\beta p^{\gamma\rangle}u^\tau \delta f_{p\mathfrak{s}} = -\theta\tau_r^{\mu,\nu\lambda\rho}. \tag{D.44}
$$

For the third term we obtain

$$
\begin{aligned}
&(r-1)(\nabla_\tau u_\eta)\Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}\int d\Gamma\, \mathfrak{s}^\mu E_p^{r-2} p^{\langle\alpha}p^\beta p^{\gamma\rangle}p^{\langle\tau\rangle}p^{\langle\eta\rangle}\delta f_{p\mathfrak{s}} \\
&= (r-1)(\nabla_\tau u_\eta)\Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}\left[\tau_{r-2}^{\mu,\alpha\beta\gamma\tau\eta} + \frac{1}{9}\Delta^{\eta\tau}\left(m^2\tau_{r-2}^{\mu,\alpha\beta\gamma} - \tau_r^{\mu,\alpha\beta\gamma}\right) + \frac{1}{3}\Delta^{\eta\alpha}\left(m^2\tau_{r-2}^{\mu,\beta\gamma\tau} - \tau_r^{\mu,\beta\gamma\tau}\right)\right. \\
&\left. + \frac{1}{3}\Delta^{\tau\alpha}\left(m^2\tau_{r-2}^{\mu,\beta\gamma\eta} - \tau_r^{\mu,\beta\gamma\eta}\right) + \frac{8}{105}\Delta^{\alpha\eta}\Delta^{\beta\tau}\left(m^4\tau_{r-2}^{\mu,\gamma} - 2m^2\tau_r^{\mu,\gamma} + \tau_{r+2}^{\mu,\gamma}\right)\right] \\
&= (r-1)\sigma_{\tau\eta}\tau_{r-2}^{\mu,\nu\lambda\rho\tau\eta} + \frac{1}{3}(r-1)\theta\left(m^2\tau_{r-2}^{\mu,\nu\lambda\rho} - \tau_r^{\mu,\nu\lambda\rho}\right) + \frac{2}{3}(r-1)\left(m^2\tau_{r-2}^{\mu,\eta\langle\nu\lambda} - \tau_r^{\mu,\eta\langle\nu\lambda}\right)\sigma^{\rho\rangle}_\eta \\
&+ \frac{8}{105}(r-1)\left(m^4\tau_{r-2}^{\mu,\langle\nu} - 2m^2\tau_r^{\mu,\langle\nu} + \tau_{r+2}^{\mu,\langle\nu}\right)\sigma^{\lambda\rho\rangle}.
\end{aligned}
\tag{D.45}
$$

Finally, we calculate the fourth term

$$
\begin{aligned}
-\frac{3}{2}\Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}(\nabla_\tau u^\alpha)\int d\Gamma\, \mathfrak{s}^\mu E_p^r p^{\langle\beta\rangle}p^{\langle\gamma\rangle}p^{\langle\tau\rangle}\delta f_{p\mathfrak{s}} &= -\frac{3}{2}\Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}(\nabla_\tau u^\alpha)\left[\tau_r^{\mu,\beta\gamma\tau} + \frac{2}{5}\Delta^{\beta\tau}\left(m^2\tau_r^{\mu,\gamma} - \tau_{r+2}^{\mu,\gamma}\right)\right] \\
&= -\frac{3}{2}\tau_r^{\mu,\tau\langle\nu\lambda}\sigma^{\rho\rangle}_\tau + \frac{3}{2}\tau_r^{\mu,\tau\langle\nu\lambda}\omega^{\rho\rangle}_\tau - \frac{1}{2}\theta\tau_r^{\mu,\nu\lambda\rho} \\
&\quad - \frac{3}{5}\left(m^2\tau_r^{\mu,\langle\nu} - \tau_{r+2}^{\mu,\langle\nu}\right)\sigma^{\lambda\rho\rangle}.
\end{aligned}
\tag{D.46}
$$

Putting all terms together, we obatin the following equations of motion for the third spin moments,

$$
\begin{aligned}
\dot{\tau}_r^{\mu,\langle\nu\lambda\rho\rangle} - \mathfrak{C}_{r-1}^{\mu,\langle\nu\lambda\rho\rangle} =& \frac{6}{m}I_{(r+5)3}\tilde{\bar{\Omega}}^{\mu\langle\nu}\sigma^{\lambda\rho\rangle} + \frac{3}{2}\tau_r^{\mu,\tau\langle\nu\lambda}\omega^{\rho\rangle}_\tau + r\dot{u}_\eta\tau_{r-1}^{\mu,\nu\lambda\rho\eta} + \frac{3}{14}\left[2m^2 r\tau_{r-1}^{\mu,\langle\nu\lambda} - (2r+7)\tau_{r+1}^{\mu,\langle\nu\lambda}\right]\dot{u}^{\rho\rangle} \\
&- \Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}\nabla_\tau\tau_{r-1}^{\mu,\alpha\beta\gamma\tau} - \frac{3}{7}\Delta^{\nu\lambda\rho}_{\alpha\beta\gamma}\nabla^\gamma\left(m^2\tau_{r-1}^{\mu,\alpha\beta} - \tau_{r+1}^{\mu,\alpha\beta}\right) \\
&+ \frac{1}{6}\left[(2r-2)m^2\tau_{r-2}^{\mu,\nu\lambda\rho} - (2r+7)\tau_r^{\mu,\nu\lambda\rho}\right]\theta + (r-1)\sigma_{\tau\eta}\tau_{r-2}^{\mu,\nu\lambda\rho\tau\eta} \\
&+ \frac{1}{3}\left[(2r-2)m^2\tau_{r-2}^{\mu,\eta\langle\nu\lambda} - (2r+5)\tau_r^{\mu,\eta\langle\nu\lambda}\right]\sigma^{\rho\rangle}_\eta \\
&+ \frac{1}{105}\left[(8r-8)m^4\tau_{r-2}^{\mu,\langle\nu} - (16r+47)m^2\tau_r^{\mu,\langle\nu} + (8r+55)\tau_{r+2}^{\mu,\langle\nu}\right]\sigma^{\lambda\rho\rangle}.
\end{aligned}
\tag{D.47}
$$

D.3 Scattering matrix elements

In this appendix some steps of the calculation of the scattering matrix elements in the collision term discussed in Section 5.6 are presented. The definition of the vacuum scattering-matrix element reads [45, 182]

$$
\langle p, p'; r, r'|t|p_1, p_2; s_1, s_2\rangle = \langle p, p'; r, r'| : H_I(0) : |p_1, p_2; s_1, s_2\rangle, \tag{D.48}
$$

with H_I being the effective interaction Hamiltonian. Here, we assume the latter to be of a NJL-type form [203, 204]

$$H_I(x) = G\,\bar{\psi}(x)\Gamma^a\psi(x)\,\bar{\psi}(x)\Gamma_a\psi(x), \tag{D.49}$$

with G being a coupling constant and Γ^a being a linear combination of Dirac matrices, which is consistent with all symmetry requirements. In order to explicitly calculate the matrix elements, we make use of the free-field expansion of the spinors,

$$\psi(x) = \sqrt{\frac{2}{(2\pi\hbar)^3}}\sum_r \int dP\, e^{-\frac{i}{\hbar}p\cdot x}u_r(p)a_r(p)\ , \tag{D.50}$$

and the anticommutation relation of the creation and annihilation operators

$$\{a_r(p), a_s^\dagger(p')\} = p^0\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{rs}. \tag{D.51}$$

We obtain

$$\langle p, p'; r, r'|t|p_1, p_2; r_1, r_2\rangle = \bar{G}\left[\bar{u}_r(p)\Gamma^a u_{r_2}(p_2)\bar{u}_{r'}(p')\Gamma_a u_{r_1}(p_1) - \bar{u}_r(p)\Gamma^a u_{r_1}(p_1)\bar{u}_{r'}(p')\Gamma_a u_{r_2}(p_2)\right], \tag{D.52}$$

where we defined $\bar{G} \equiv 4/(2\pi\hbar)^6 G$. With the help of Eq. (3.80), we then find for Eq. (5.88a) in the main text

$$\begin{aligned}
\int [dS]\,\mathcal{W} =& |\bar{G}|^2\delta^{(4)}(p+p'-p_1-p_2)\ \left[\bar{u}_r(p)\Gamma^a u_{r_2}(p_2)\bar{u}_{r'}(p')\Gamma_a u_{r_1}(p_1) - \bar{u}_r(p)\Gamma^a u_{r_1}(p_1)\bar{u}_{r'}(p')\Gamma_a u_{r_2}(p_2)\right] \\
&\times \left[\bar{u}_r(p)\Gamma^b u_{r_2}(p_2)\bar{u}_{r'}(p')\Gamma_b u_{r_1}(p_1) - \bar{u}_r(p)\Gamma^b u_{r_1}(p_1)\bar{u}_{r'}(p')\Gamma_b u_{r_2}(p_2)\right]^\dagger \\
=& |\bar{G}|^2\delta^{(4)}(p+p'-p_1-p_2)\left\{\mathrm{Tr}\left[\left(\not{p}+m\right)\Gamma^a\left(\not{p}_2+m\right)\Gamma^b\right]\mathrm{Tr}\left[\left(\not{p}'+m\right)\Gamma_a\left(\not{p}_1+m\right)\Gamma_b\right]\right. \\
&- \mathrm{Tr}\left[\left(\not{p}+m\right)\Gamma^a\left(\not{p}_2+m\right)\Gamma_b\left(\not{p}'+m\right)\Gamma_a\left(\not{p}_1+m\right)\Gamma^b\right] \\
&- \mathrm{Tr}\left[\left(\not{p}+m\right)\Gamma^a\left(\not{p}_1+m\right)\Gamma_b\left(\not{p}'+m\right)\Gamma_a\left(\not{p}_2+m\right)\Gamma^b\right] \\
&\left.+ \mathrm{Tr}\left[\left(\not{p}+m\right)\Gamma^a\left(\not{p}_1+m\right)\Gamma^b\right]\mathrm{Tr}\left[\left(\not{p}'+m\right)\Gamma_a\left(\not{p}_2+m\right)\Gamma_b\right]\right\}.
\end{aligned} \tag{D.53}$$

Moreover for Eq. (5.88b) we obtain

$$\begin{aligned}
\int [dS]\,\mathfrak{s}^\alpha\mathcal{W} =& -|\bar{G}|^2\delta^{(4)}(p+p'-p_1-p_2)\,\frac{1}{2m}\bar{u}_s(p)\gamma^5\gamma^\alpha u_r(p) \\
&\times \left[\bar{u}_r(p)\Gamma^a u_{r_2}(p_2)\bar{u}_{r'}(p')\Gamma_a u_{r_1}(p_1) - \bar{u}_r(p)\Gamma^a u_{r_1}(p_1)\bar{u}_{r'}(p')\Gamma_a u_{r_2}(p_2)\right] \\
&\times \left[\bar{u}_s(p)\Gamma^b u_{r_2}(p_2)\bar{u}_{r'}(p')\Gamma_b u_{r_1}(p_1) - \bar{u}_s(p)\Gamma^b u_{r_1}(p_1)\bar{u}_{r'}(p')\Gamma_b u_{r_2}(p_2)\right]^\dagger \\
=& -\frac{|\bar{G}|^2}{2m}\delta^{(4)}(p+p'-p_1-p_2) \\
&\times\left\{\mathrm{tr}\left[\left(\not{p}+m\right)\gamma^5\gamma^\alpha\left(\not{p}+m\right)\Gamma^a\left(\not{p}_2+m\right)\Gamma^b\right]\mathrm{tr}\left[\left(\not{p}'+m\right)\Gamma_a\left(\not{p}_1+m\right)\Gamma_b\right]\right. \\
&- \mathrm{tr}\left[\left(\not{p}+m\right)\gamma^5\gamma^\alpha\left(\not{p}+m\right)\Gamma^a\left(\not{p}_2+m\right)\Gamma_b\left(\not{p}'+m\right)\Gamma_a\left(\not{p}_1+m\right)\Gamma^b\right] \\
&- \mathrm{tr}\left[\left(\not{p}+m\right)\gamma^5\gamma^\alpha\left(\not{p}+m\right)\Gamma^a\left(\not{p}_1+m\right)\Gamma_b\left(\not{p}'+m\right)\Gamma_a\left(\not{p}_2+m\right)\Gamma^b\right] \\
&\left.+ \mathrm{tr}\left[\left(\not{p}+m\right)\gamma^5\gamma^\alpha\left(\not{p}+m\right)\Gamma^a\left(\not{p}_1+m\right)\Gamma^b\right]\mathrm{tr}\left[\left(\not{p}'+m\right)\Gamma_a\left(\not{p}_2+m\right)\Gamma_b\right]\right\},
\end{aligned} \tag{D.54}$$

while evaluating Eq. (5.88c) yields

$$\begin{aligned}\int [dS]\, \mathfrak{s}^{\alpha}\mathfrak{s}'^{\beta}\mathcal{W} =& \frac{|\bar{G}|^2}{(2m)^2}\delta^{(4)}(p+p'-p_1-p_2)\Big\{ \text{tr}\left[(\not{p}+m)\,\gamma^5\gamma^{\alpha}\,(\not{p}+m)\,\Gamma^a \left(\not{p}_2+m\right)\Gamma^b\right] \\ & \times \text{tr}\left[(\not{p}'+m)\,\gamma^5\gamma^{\beta}\,(\not{p}'+m)\,\Gamma_a \left(\not{p}_1+m\right)\Gamma_b\right] \\ & - \text{tr}\left[(\not{p}+m)\,\gamma^5\gamma^{\alpha}\,(\not{p}+m)\,\Gamma^a \left(\not{p}_2+m\right)\Gamma_b\,(\not{p}'+m)\,\gamma^5\gamma^{\beta}\,(\not{p}'+m)\,\Gamma_a \left(\not{p}_1+m\right)\Gamma^b\right] \\ & - \text{tr}\left[(\not{p}+m)\,\gamma^5\gamma^{\alpha}\,(\not{p}+m)\,\Gamma^a \left(\not{p}_1+m\right)\Gamma_b\,(\not{p}'+m)\,\gamma^5\gamma^{\beta}\,(\not{p}'+m)\,\Gamma_a \left(\not{p}_2+m\right)\Gamma^b\right] \\ & + \text{tr}\left[(\not{p}+m)\,\gamma^5\gamma^{\alpha}\,(\not{p}+m)\,\Gamma^a \left(\not{p}_1+m\right)\Gamma^b\right] \text{tr}\left[(\not{p}'+m)\,\gamma^5\gamma^{\beta}\,(\not{p}'+m)\,\Gamma_a \left(\not{p}_2+m\right)\Gamma_b\right] \Big\}. \end{aligned} \tag{D.55}$$

For a scalar interaction, we have $\Gamma^a = 1$. In this case Eqs. (D.54) and (D.55) are zero, which can by shown by explicitly evaluating the traces of Dirac matrices. On the other hand, if we deal with a vector interaction, i.e., $\Gamma^a = \gamma^\mu$, the computation of Eq. (D.54) still yields zero, whereas Eq. (D.55) does not vanish in general. However, if the particle three-momenta in the center-of-mass frame of the collision are sufficiently small, we can neglect contributions of second order in these momenta. In this case Eq. (D.55) vanishes also for the vector interaction, and only Eq. (D.53) gives a nonzero contribution. We also note that Eq. (D.54) is nonzero only for a parity-violating interaction, e.g., $\Gamma^a = (1+\gamma^5)\gamma^\mu$.

D.4 Transport coefficients

The relaxation times and transport coefficients in Eq. (5.108) read

$$\tau_{\mathfrak{p}} = \mathfrak{T}^{(0)}_{00} \; ,$$

$$\mathfrak{e}^{(0)} = \tau_{\mathfrak{p}}\, g^{(0)}_0 \; , \tag{D.56a}$$

$$\mathfrak{K}^{(0)}_{\theta\omega} = \frac{\hbar}{2m}\tau_{\mathfrak{p}}\, \xi^{(0)}_0 \; , \tag{D.56b}$$

$$\mathfrak{K}^{(0)}_{\theta\omega\Pi} = \frac{\hbar}{2m}\frac{G_{21}}{D_{20}}\tau_{\mathfrak{p}} = -\mathfrak{K}^{(0)}_{\pi\sigma\omega} \; , \tag{D.56c}$$

$$\mathfrak{K}^{(0)}_{n\omega} = -\frac{\hbar}{2m}\frac{G_{30}}{D_{20}}\tau_{\mathfrak{p}} \; , \tag{D.56d}$$

$$\mathfrak{K}^{(0)}_{I\Omega} = -\frac{\hbar}{4m} I_{11}\tau_{\mathfrak{p}} = \mathfrak{K}^{(0)}_{\nabla\Omega} \; , \tag{D.56e}$$

$$\mathfrak{K}^{(0)}_{\Pi\Omega} = \frac{\hbar}{4m}\frac{\beta_0 I_{21}}{\epsilon_0 + P_0}\tau_{\mathfrak{p}} \; , \tag{D.56f}$$

$$\mathfrak{K}^{(0)}_{\dot{\Omega}} = -\frac{\hbar}{4m} I_{10}\tau_{\mathfrak{p}} \; , \tag{D.56g}$$

$$\mathfrak{g}^{(0)}_1 = \frac{1}{2}\tau_{\mathfrak{p}}\frac{\partial \mathfrak{F}^{(1)}_{-11}}{\partial \beta_0}\frac{\beta_0}{\epsilon_0 + P_0} \; , \tag{D.56h}$$

$$\mathfrak{g}^{(0)}_2 = -\tau_{\mathfrak{p}}\mathfrak{F}^{(2)}_{-20} \; , \tag{D.56i}$$

$$\mathfrak{g}^{(0)}_3 = \frac{1}{2}\tau_{\mathfrak{p}}\mathfrak{F}^{(1)}_{-11} \; , \tag{D.56j}$$

$$\mathfrak{g}^{(0)}_4 = -\frac{1}{9}\tau_{\mathfrak{p}}\left(7 - 4m^2\mathfrak{F}^{(0)}_{-20} + 8m^4\mathfrak{F}^{(0)}_{-22}\right) \; , \tag{D.56k}$$

$$\mathfrak{g}^{(0)}_5 = \frac{1}{3}\tau_{\mathfrak{p}}\left(8m^2\mathfrak{F}^{(0)}_{-22} - \mathfrak{F}^{(2)}_{-20}\right) \; , \tag{D.56l}$$

$$\mathfrak{g}^{(0)}_6 = -\frac{1}{2}\tau_{\mathfrak{p}}\left(\frac{\partial \mathfrak{F}^{(1)}_{-11}}{\partial \alpha_0} + \frac{\partial \mathfrak{F}^{(1)}_{-11}}{\partial \beta_0}\frac{n_0}{\epsilon_0 + P_0}\right) \; , \tag{D.56m}$$

$$\mathfrak{g}^{(0)}_7 = \frac{1}{3}\tau_{\mathfrak{p}}\left(m^2\mathfrak{F}^{(0)}_{-20} - 1 - 2m^4\mathfrak{F}^{(0)}_{-22}\right) \; , \tag{D.56n}$$

$$\mathfrak{g}^{(0)}_8 = \tau_{\mathfrak{p}}\left(\mathfrak{F}^{(2)}_{-20} - 2m^2\mathfrak{F}^{(0)}_{-22}\right) \; , \tag{D.56o}$$

$$\mathfrak{g}^{(0)}_9 = \frac{1}{2(\epsilon_0 + P_0)}\tau_{\mathfrak{p}}\mathfrak{F}^{(1)}_{-11} \; . \tag{D.56p}$$

Moreover, the relaxation time and transport coefficients in Eq. (5.109) are given by

$$\tau_{\mathfrak{z}} = \mathfrak{T}^{(1)}_{11} \ , \tag{D.57a}$$

$$\mathfrak{K}^{(1)}_{\omega\Pi} = \frac{\hbar}{2m}\frac{\beta_0 I_{41}}{\epsilon_0 + P_0}\tau_{\mathfrak{z}} \ , \tag{D.57b}$$

$$\mathfrak{K}^{(1)}_{\omega I} = \frac{\hbar}{2m}\frac{\beta_0 I_{31}}{\epsilon_0 + P_0}\tau_{\mathfrak{z}} \ , \tag{D.57c}$$

$$\mathfrak{K}^{(1)}_{\Omega\sigma} = -\frac{\hbar}{2m}\beta_0 I_{42}\tau_{\mathfrak{z}} \ , \tag{D.57d}$$

$$\mathfrak{K}^{(1)}_{\nabla\Omega} = \frac{\hbar}{4m} I_{31}\tau_{\mathfrak{z}} \ , \tag{D.57e}$$

$$\mathfrak{g}^{(1)}_{1} = -\frac{4}{3}\tau_{\mathfrak{z}} \ , \tag{D.57f}$$

$$\mathfrak{g}^{(1)}_{2} = -\tau_{\mathfrak{z}} \ , \tag{D.57g}$$

$$\mathfrak{g}^{(1)}_{3} = \frac{1}{5}\tau_{\mathfrak{z}}\left(m^2\mathfrak{F}^{(1)}_{-11} - 1\right) \ , \tag{D.57h}$$

$$\mathfrak{g}^{(1)}_{4} = \tau_{\mathfrak{z}}\frac{1}{\epsilon_0 + P_0} \ , \tag{D.57i}$$

$$\mathfrak{g}^{(1)}_{5} = -\frac{1}{2}\tau_{\mathfrak{z}} \ , \tag{D.57j}$$

$$\mathfrak{g}^{(1)}_{6} = \frac{4m^2}{(\epsilon_0 + P_0)}\tau_{\mathfrak{z}} \ , \tag{D.57k}$$

$$\mathfrak{g}^{(1)}_{7} = \frac{11}{\epsilon_0 + P_0}\tau_{\mathfrak{z}} \ , \tag{D.57l}$$

$$\mathfrak{g}^{(1)}_{8} = -m^2\tau_{\mathfrak{z}} \ , \tag{D.57m}$$

$$\mathfrak{g}^{(1)}_{9} = -2\tau_{\mathfrak{z}} \ , \tag{D.57n}$$

$$\mathfrak{g}^{(1)}_{10} = \frac{1}{6}\tau_{\mathfrak{z}}\left(m^2\mathfrak{F}^{(1)}_{-11} - 1\right) \ . \tag{D.57o}$$

The relaxation time and transport coefficients in Eq. (5.110) read

$$\tau_{\mathfrak{q}} = \mathfrak{T}^{(2)}_{00} \ , \tag{D.58a}$$
$$\mathfrak{d}^{(2)} = \tau_{\mathfrak{q}}\, h^{(2)}_0 \ , \tag{D.58b}$$
$$\mathfrak{K}^{(2)}_{\Omega I} = \frac{\hbar}{2m}\tau_{\mathfrak{q}}\, \xi^{(2)}_0 \ , \tag{D.58c}$$
$$\mathfrak{K}^{(2)}_{\nabla\Omega} = \frac{\hbar}{2m} I_{32}\tau_{\mathfrak{q}} \ , \tag{D.58d}$$
$$\mathfrak{K}^{(2)}_{\omega\sigma} = \frac{\hbar}{2m}\beta_0 I_{42}\tau_{\mathfrak{q}} \ , \tag{D.58e}$$
$$\mathfrak{K}^{(2)}_{\Omega\Pi} = \frac{\hbar}{2m}\frac{\beta_0 I_{42}}{\epsilon_0+P_0}\tau_{\mathfrak{q}} \ , \tag{D.58f}$$
$$\mathfrak{g}^{(2)}_1 = \frac{1}{5}\tau_{\mathfrak{q}}\frac{1}{\epsilon_0+P_0}\left(m^2\mathfrak{F}^{(1)}_{-11} - 5\mathfrak{F}^{(1)}_{11} + m^2\frac{\partial\mathfrak{F}^{(1)}_{-11}}{\partial\beta_0}\beta_0\right) \ , \tag{D.58g}$$
$$\mathfrak{g}^{(2)}_2 = -\frac{m^2}{5}\tau_{\mathfrak{q}}\left(\frac{\partial\mathfrak{F}^{(1)}_{-11}}{\partial\alpha_0} + \frac{n_0}{\epsilon_0+P_0}\frac{\partial\mathfrak{F}^{(1)}_{-11}}{\partial\beta_0}\right) \ , \tag{D.58h}$$
$$\mathfrak{g}^{(2)}_3 = \frac{1}{5}\tau_{\mathfrak{q}}\left(\mathfrak{F}^{(1)}_{11} - m^2\mathfrak{F}^{(1)}_{-11}\right) \ , \tag{D.58i}$$
$$\mathfrak{g}^{(2)}_4 = \frac{1}{3}\tau_{\mathfrak{q}}\left(-m^2\mathfrak{F}^{(2)}_{-20} - 4\right) \ , \tag{D.58j}$$
$$\mathfrak{g}^{(2)}_5 = \frac{2}{7}\tau_{\mathfrak{q}}\left(-2m^2\mathfrak{F}^{(2)}_{-20} - 5\right) \ , \tag{D.58k}$$
$$\mathfrak{g}^{(2)}_6 = \frac{2m^2}{15}\tau_{\mathfrak{q}}\left(2m^4\mathfrak{F}^{(0)}_{-22} - m^2\mathfrak{F}^{(0)}_{-20} - 11\right) \ , \tag{D.58l}$$
$$\mathfrak{g}^{(2)}_7 = \frac{2}{15}\tau_{\mathfrak{q}}\left(4 - m^4\mathfrak{F}^{(0)}_{-22}\right) \ , \tag{D.58m}$$
$$\mathfrak{g}^{(2)}_8 = \frac{1}{5(\epsilon_0+P_0)}\tau_{\mathfrak{q}}\left(m^2\mathfrak{F}^{(1)}_{-11} - 1\right) \ , \tag{D.58n}$$
$$\mathfrak{g}^{(2)}_9 = -\frac{2}{15}\tau_{\mathfrak{q}}\left(4m^2 - m^4\mathfrak{F}^{(2)}_{-20} + 2m^6\mathfrak{F}^{(0)}_{-22}\right) \ , \tag{D.58o}$$
$$\mathfrak{g}^{(2)}_{10} = \frac{2}{5}\tau_{\mathfrak{q}}\left(-3 + m^2\mathfrak{F}^{(2)}_{-20} - 2m^4\mathfrak{F}^{(0)}_{-22}\right) \ . \tag{D.58p}$$

Finally, the coefficients of the Navier-Stokes values in section 5.9.2 are obtained as

$$\lambda_0 = \frac{1}{1+4(\tau_{\mathfrak{z}}\omega)^2} \ , \tag{D.59a}$$
$$\lambda_1 = \frac{16}{3}(\tau_{\mathfrak{z}}\omega)^2\,\lambda_0 \ , \tag{D.59b}$$
$$\lambda_2 = (\tau_{\mathfrak{z}}\omega)^2\,(\lambda_0+\lambda_4) \ , \tag{D.59c}$$
$$\lambda_3 = \tau_{\mathfrak{z}}\omega\,\lambda_0 \ , \tag{D.59d}$$
$$\lambda_4 = -\frac{\tau_{\mathfrak{z}}\omega}{1+(\tau_{\mathfrak{z}}\omega)^2} \ , \tag{D.59e}$$
$$\eta_0 = \frac{1}{1+4(\tau_{\mathfrak{q}}\omega)^2} \ , \tag{D.59f}$$
$$\eta_1 = \frac{16}{3}(\tau_{\mathfrak{q}}\omega)^2\,\eta_0 \ , \tag{D.59g}$$
$$\eta_2 = (\tau_{\mathfrak{q}}\omega)^2\,(\eta_0+\eta_4) \ , \tag{D.59h}$$
$$\eta_3 = \tau_{\mathfrak{q}}\omega\,\eta_0 \ , \tag{D.59i}$$
$$\eta_4 = -\frac{\tau_{\mathfrak{q}}\omega}{1+(\tau_{\mathfrak{q}}\omega)^2} \ . \tag{D.59j}$$

Appendix E

Chapman-Enskog expansion with arbitrary vorticity

We start with a short review of the standard Chapman-Enskog expansion [26,45]. For consistency, we use the definitions and notation introduced in Ref. [35] and used in the main part of this thesis instead of the original ones by Chapman and Enskog. In this appendix, we do not consider spin effects. Then, the Boltzmann equation has the form

$$p \cdot \partial f = C[f], \tag{E.1}$$

with the standard (spin-independent) distribution function f and collision term $C[f]$, the latter given by the local and spin-independent part of Eq. (3.93). As introduced in the main text, the Knudsen number is defined as the ratio of the mean free path and the hydrodynamic scale associated with dissipative gradients, $\mathrm{Kn} \equiv l_{\mathrm{mfp}}/l_{\mathrm{hydro}}$. Multiplying the Boltzmann equation by l_{mfp}/E_p we obtain

$$\mathrm{Kn}\bar{D}f_p + \frac{\mathrm{Kn}}{E_p} p \cdot \bar{\nabla} f_p = \frac{l_{\mathrm{mfp}}}{E_p} C[f], \tag{E.2}$$

where we defined the dimensionless operators $\bar{D} \equiv l_{\mathrm{hydro}} d/d\tau$ and $\bar{\nabla}^\mu \equiv l_{\mathrm{hydro}} \nabla^\mu$, which are of order one in the expansion scheme. As the collision term is of the order of the inverse mean free path, the right-hand side of the above equation is also of order one and one can solve order by order in Kn,

$$f_p = f_{0p} + \mathrm{Kn} f_{1p} + \mathcal{O}(\mathrm{Kn}^2). \tag{E.3}$$

Here we use the notation f_{ip} to distinguish the Chapman-Enskog expansion from the $\hbar$ expansion in the main text. At the zeroth order we have

$$0 = C[f_{0p}], \tag{E.4}$$

which is solved by the local-equilibrium distribution function (5.12). We stress that the contribution from the vorticity $\omega_{\mu\nu}$ to the left-hand side of the Boltzmann equation is zero. Consequently the vorticity does not change the zero on the left-hand side of Eq. (E.4) and can be counted as zeroth order in Kn. At the first order we obtain

$$(\bar{D} f_{0p})_0 + \frac{1}{E_p} p \cdot \bar{\nabla} f_{0p} = \frac{l_{\mathrm{mfp}}}{E_p} C[f_{1p}]. \tag{E.5}$$

In order to solve this equation, we use the moment expansion of the spin-independent distribution function, which for convenience is shortly reviewed here [35]. The dissipative correction to the distribution function ϕ_p in Eq. (5.37) is expanded as

$$\phi_p = \sum_{l=0}^{\infty} \lambda_p^{\langle \mu_1 \cdots \mu_l \rangle} p_{\langle \mu_1} \cdots p_{\mu_l \rangle} \, . \tag{E.6}$$

We then make use of a further expansion of the energy-dependent coefficients

$$\lambda_p^{\langle \mu_1 \ldots \mu_l \rangle} = \sum_{n=0}^{N_l} c_n^{\langle \mu_1 \ldots \mu_l \rangle} \mathcal{P}_{pn}^{(l)}, \tag{E.7}$$

with momentum-independent coefficients $c_n^{\langle\mu_1\cdots\mu_l\rangle}$ and $\mathcal{P}_{pn}^{(l)}$ defined in Eq. (5.40). Following the same reasoning as in Ref. [35] and Chapter 5.3 we can express

$$\lambda_p^{\langle\mu_1\cdots\mu_l\rangle} = \sum_{n=0}^{N_l} \mathcal{H}_{pn}^{(l)} \rho_n^{\mu_1\cdots\mu_l}, \tag{E.8}$$

with

$$\rho_n^{\mu_1\cdots\mu_l} \equiv \langle E_p^n\, p^{\langle\mu_1}...p^{\mu_l\rangle}\rangle_\delta \;, \tag{E.9}$$

and $\mathcal{H}_{pn}^{(l)}$ given in Eq. (5.49). The distribution function then takes the form

$$f_p = f_{0p}\left(1+\sum_{l=0}^{\infty}\sum_{n=0}^{N_l}\mathcal{H}_{pn}^{(l)}\rho_n^{\mu_1\cdots\mu_l}p_{\langle\mu_1}...p_{\mu_l\rangle}\right), \tag{E.10}$$

which is the spin-independent part of the moment expansion (5.50). Inserting Eq. (5.12) into the left-hand side and Eq. (E.10) into the right-hand side of Eq. (E.5) and keeping only terms of first order in Kn in the standard power counting (i.e., considering vorticity to be of order $1/l_{\text{hydro}}$), we obtain after integrating over $\int dP\, E_p^r$, $\int dP\, E_p^r p^{\langle\mu\rangle}$, and $\int dP\, E_p^r p^{\langle\mu}p^{\nu\rangle}$, respectively,

$$\begin{aligned}
\alpha_r^{(0)}\theta &= \sum_n A_{rn}^{(0)}\rho_n, \\
\alpha_r^{(1)}I^\mu &= \sum_n A_{rn}^{(1)}\rho_n^\mu, \\
2\alpha_r^{(2)}\sigma^{\mu\nu} &= \sum_n A_{rn}^{(2)}\rho_n^{\mu\nu}.
\end{aligned} \tag{E.11}$$

Here we defined the coefficients

$$\begin{aligned}
\alpha_r^{(0)} &\equiv (1-r)I_{r1} - I_{r0} - \frac{1}{D_{20}}[G_{2r}(\epsilon_0+P_0) - G_{3r}n_0], \\
\alpha_r^{(1)} &\equiv I_{r+1,1} - \frac{n_0}{\epsilon_0+P_0}I_{r+2,1}, \\
\alpha_r^{(2)} &\equiv I_{r+2,1} + (r-1)I_{r+2,2}
\end{aligned} \tag{E.12}$$

with I_{nq}, D_{nq}, and G_{nm} given in Eqs. (5.42), (5.63), and (5.75), respectively. We also made use of Eqs. (5.62) and the fact that the collision term can be written as [35]

$$C_{r-1}^{\langle\mu_1\cdots\mu_l\rangle} \equiv \int dP\, E_p^{r-1} p^{\langle\mu_1}\cdots p^{\mu_l\rangle} C[f] = -\sum_{n=0}^{N_l} A_{rn}^{(l)}\rho_n^{\mu_1\cdots\mu_l}, \tag{E.13}$$

with

$$\begin{aligned}
A_{rn}^{(l)} \equiv \frac{1}{2l+1}\int [dP]\, \mathcal{W}_0 f_{0p} f_{0p'} E_p^{r-1} p^{\langle\mu_1}\cdots p^{\mu_l\rangle} \\
\times\left(\mathcal{H}_{pn}^{(l)}p_{\langle\mu_1}\cdots p_{\mu_l\rangle} + \mathcal{H}_{p'n}^{(l)}p'_{\langle\mu_1}\cdots p'_{\mu_l\rangle} - \mathcal{H}_{p_1n}^{(l)}p_{1\langle\mu_1}\cdots p_{1\mu_l\rangle} - \mathcal{H}_{p_2n}^{(l)}p_{2\langle\mu_1}\cdots p_{2\mu_l\rangle}\right).
\end{aligned} \tag{E.14}$$

After excluding the zero rows and columns, i.e., $r,n=1,2$ for $l=0$ and $r,n=1$ for $l=1$, respectively, the matrices $A_{rn}^{(i)}$ can be inverted

$$\tau^{(l)} \equiv \left(A^{(l)}\right)^{-1}. \tag{E.15}$$

Multiplying both sides of each equation in Eqs. (E.11) with by $\tau_{0r}^{(0)}$, $\tau_{0r}^{(1)}$, and $\tau_{0r}^{(2)}$, respectively, and summing over r, we obtain the Navier-Stokes values

$$\begin{aligned}
\Pi &= -\zeta\theta, \\
n^\mu &= \kappa_n I^\mu, \\
\pi^{\mu\nu} &= 2\eta\sigma^{\mu\nu}.
\end{aligned} \tag{E.16}$$

Here, we used

$$\rho_0 = -\frac{3}{m^2}\Pi, \qquad \rho_0^\mu = n^\mu, \qquad \rho_0^{\mu\nu} = \pi^{\mu\nu}, \tag{E.17}$$

and defined the coefficients

$$\begin{aligned}
\zeta &\equiv \frac{m^2}{3} \sum_{r=0,\neq 1,2}^{N_0} \tau_{0r}^{(0)} \alpha_r^{(0)}, \\
\kappa_n &\equiv \sum_{r=0,\neq 1}^{N_1} \tau_{0r}^{(1)} \alpha_r^{(1)}, \\
\eta &\equiv \sum_{r=0}^{N_2} \tau_{0r}^{(2)} \alpha_r^{(2)}.
\end{aligned} \tag{E.18}$$

However, if the vorticity is counted as being of different order than the dissipative gradients, the standard power-counting in Eq. (E.5) needs to be reconsidered. In order to do so, we first calculate the moments of the left-hand side of the Boltzmann equation and then identify terms of first order in Kn. Here, we count gradients of $\rho_r^{\mu_1\cdots\mu_l}$ as of second order in Knudsen numbers. For the second term in Eq. (E.2) we obtain the moments

$$\begin{aligned}
\int dP\, E_p^{r-1} p\cdot\nabla f_{1p} &= \mathcal{O}(\mathrm{Kn}^2), \\
\int dP\, E_p^{r-1} p^{\langle\mu\rangle} p\cdot\nabla f_{1p} &= (\nabla_\nu u^\mu)\rho_r^\nu + \mathcal{O}(\mathrm{Kn}^2) = \omega_\nu{}^\mu \rho_r^\nu + \mathcal{O}(\mathrm{Kn}^2), \\
\int dP\, E_p^{r-1} p^{\langle\mu} p^{\nu\rangle} p\cdot\nabla f_{1p} &= 2(\nabla_\lambda u^{\langle\mu})\rho_r^{\nu\rangle\lambda} + \mathcal{O}(\mathrm{Kn}^2) = 2\omega_\lambda{}^{\langle\mu}\rho_r^{\nu\rangle\lambda} + \mathcal{O}(\mathrm{Kn}^2).
\end{aligned} \tag{E.19}$$

We see that there are contributions from vorticity which give terms of first order in Kn. These result in the following changes in the Navier-Stokes equations,

$$\begin{aligned}
\alpha_r^{(0)}\theta &= \sum_n A_{rn}^{(0)} \rho_n, \\
\alpha_r^{(1)} I^\mu &= \sum_n A_{rn}^{(1)} \rho_n^\mu - \omega^\mu{}_\nu \rho_r^\nu, \\
2\alpha_r^{(2)}\sigma^{\mu\nu} &= \sum_n A_{rn}^{(2)} \rho_n^{\mu\nu} + 2\omega_\lambda{}^{\langle\mu}\rho_r^{\nu\rangle\lambda}.
\end{aligned} \tag{E.20}$$

The first equation is not changed compared to the standard power counting. Multiplying the second and third equation, respectively, by $\tau_{0r}^{(l)}$, summing over r and defining

$$\begin{aligned}
\tau_n &\equiv \sum_{r=0,\neq 1}^{N_1} \tau_{0r}^{(1)}, \\
\tau_\pi &\equiv \sum_{r=0}^{N_2} \tau_{0r}^{(2)},
\end{aligned} \tag{E.21}$$

we arrive at Eqs. (5.121).

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