Properties of conically stable polynomials and imaginary projections

Dissertation
zur Erlangung des Doktorgrades
der Naturwissenschaften

vorgelegt beim Fachbereich 12, Informatik und Mathematik
der Johann Wolfgang Goethe-Universität
in Frankfurt am Main

von
Stephan Gardoll

Frankfurt 2022
(D30)

vom Fachbereich 12, Informatik und Mathematik, der

Johann Wolfgang Goethe-Universität als Dissertation angenommen.

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Datum der Disputation: 03.03.2023

Acknowledgement

First of all I want to thank my supervisor Thorsten Theobald for his guidance and steady support while I was completing my doctoral degree. I am grateful for our joint work on this interesting topic. His ideas as well as his feedback and patience have been important cornerstones for my academic career and the write-up of this thesis.

Furthermore, I want to thank all former and current members of the working group for the pleasant time we had during shared projects, events or at lunch: Tomáš Bajbar, Aenne Benjes, Giulia Codenotti, Mareike Dressler, Thorsten Jörgens, Lukas Katthän, Sebastian Manecke, Helen Naumann, Raman Sanyal, Mahsa Sayyary and Benjamin Schröter. I enjoyed the moments we shared.

Aside from my working group, I also want to thank Jaro Eichler, Solveig Plomer, Karim el Salim and Frederic Matter for the support and all the shared memories. A special thanks goes to Oliver Gebhard who started studying math with me from the very first day. I also want to thank Adrian Baumann for all the fruitful discussions and valuable feedback on this thesis.

I also want to thank Papri Dey for the fruitful discussions which highly influenced this thesis. Furthermore, I want to thank Cordian Riener for fruitful discussions and welcoming us to Norway several times. Another thanks goes to the anonymous reviewers of our papers for their valuable feedback onto articles relevant to this thesis.

I am grateful for working with such a great e-learning team: Gülcan Aksu, Riccardo Arconada, Kai Brobeil, Lisa Bruder, Lars Dawidowski, Hendrik Edelmann, Arjun Gautam, Romina Ghasemizadeh, Felix Höfer, Hamayak Howsepjan, Kasra Khani, Lydia Langner, Patrick Magura, Sara Naumann, Andrea Stolz, Lukas Walter, Max Weidemann and Jennifer Winter. A special thanks goes to Jasper Forth. He has been a pillar of the e-learning team. I also want to thank Ralph Neininger who has officially been in charge of the e-learning department and has supported me in administrative matters.

I want to thank the administrative staff of the Goethe University as well as the HRZ Medientechnik and studiumdigitale for the steady support regarding administrative matters and lecture recordings. A special thanks goes to Silke Schultz for her support in many different administrative aspects.

Apart from academia, I want to thank my parents for preparing me for my academic way as well as supporting me on the way. A special thanks goes to my wife Melanie Gardoll for her love and for being the most valuable person in my life. She motivated, encouraged and supported me the whole time. I also thank my kids for motivating and encouraging me. Another thanks goes Tony Müller for being a very good friend.

Contents

1. Introduction

Already in the 18th century, Newton [85] and Maclaurin [75] drew essential connections between roots of polynomials, their coefficients and combinatorics based on the so-called Newton-inequalities. Expressed in a modern notion, we call a sequence of real numbers $A = a_0, \ldots, a_n$ log-concave if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $i \in [n-1] := \{1, \ldots, n-1\}$. The most popular log-concave sequence is the n-th row of Pascal's triangle $\binom{n}{0}, \ldots, \binom{n}{n}$. Its log-concavity may be seen by

$$\frac{\binom{n}{i}^2}{\binom{n}{i-1} \cdot \binom{n}{i+1}} = \frac{(i+1)(n-i+1)}{i(n-i)} = \frac{i+1}{i}\frac{n-i+1}{n-i} > 1 \ \forall i \in [n-1].$$

Now, given a polynomial $f = \sum_{i=0}^{n} a_i x^i \in \mathbb{R}[x]$ with nonnegative coefficients, its coefficients $a_0, \ldots, a_n$ form a log-concave sequence if f is real-rooted. This result goes back to Newton [85] and may be considered as the oldest pillar of the connection between a polynomial's real-rootedness and combinatorics. The key idea of a modern proof of this result [26, Lemma 1.1] is as follows: Instead of showing the log-concavity of $a_0, \ldots, a_n$ directly, the log-concavity of $b_0, \ldots, b_n$ with $a_j = \binom{n}{j} b_j$ is shown by applying the Gauß-Lucas Theorem. Afterwards the claim follows for $a_0, \ldots, a_n$ as well since the term-wise product of a positive log-concave sequence and another log-concave sequence is a log-concave sequence.

Nowadays real-rootedness of polynomials plays a crucial role in combinatorics since generating polynomials in this area are often real-rooted. The generating polynomial of a finite sequence $A = a_0, \ldots, a_n$ is given by $p_A(x) = a_0 + a_1 x + \ldots + a_n x^n$. Furthermore, a sequence $(a_k)_{k=0}^{\infty}$ is called a Pólya frequency sequence if all the minors of the infinite matrix with a_{i-j} in position (i, j) are nonnegative. In 1952, it was shown by Aissen, Endrei, Schoenberg and Whitney [2, 37] that a finite sequence A of nonnegative numbers is a Pólya frequency sequence if and only if the generating polynomial p_A is real-rooted. Thus, finite Pólya frequency sequences are a special case of log-concave sequences. Due to the connection between a polynomial's roots and those sequences, various important results in combinatorics were derived or expressed in terms of the roots of a polynomial [22, 23, 100, 99].

For a random variable X with values in $\{0, \ldots, n\}$, let $a_j := \mathbb{P}(X = j)$ and $p_X(t) = a_0 + a_1 t + \ldots + a_n t^n$ be the partition function of X. Bender showed in [9] that the asymptotic normality of the coefficients of p_X may be deduced from the real-rootedness of p_X. This result was used to prove the asymptotic normality of various discrete sequences [9, 10, 29].

Further connections to combinatorics are given by polynomials associated to graphs. For a graph G, let $\chi(G, k)$ be the chromatic number of G, i.e. the number of proper vertex colorings of G using k colors. There is a unique polynomial $p_G(k)$, called the chromatic polynomial of G, which coincides with $\chi(G, k)$ when evaluated for any integer $k \geq 0$. Although p_G does not only have real roots in general, its real roots are quite well-researched. Every integer $k \geq 0$ which is smaller than the chromatic number gives a real root for p_G. Due to a recent breakthrough of Huh and Katz [54, 55], it is known that the coefficients of the chromatic polynomial also form a log-concave sequence.

Furthermore, the matching generating polynomial is real-rooted [50] and the real roots of the independence polynomial are negative and dense in $(-\infty, 0]$ [24].

Additional interest into real-rooted polynomials is generated by characteristic polynomials since $\det(xI_n - A)$ is real-rooted if A is a symmetric or Hermitian matrix. Through this, real-rootedness of a polynomial forms further connections to almost every mathematical discipline, including graphs, systems of first-order differential equations [102], multivariate analysis [58], dynamical systems [97], statistics and machine learning [11].

Due to the great importance of real-rooted polynomials in various fields, there have been recent approaches to find, or at least approximate, real roots of a polynomial using neural networks [4, 34, 40, 48]. There has also been an approach to study the dynamics of the roots of a polynomial under differentiation using nonlinear and nonlocal parabolic equations [3].

However, while real-rootedness of a polynomial f with nonnegative coefficients implies that the coefficients of f form a log-concave sequence, we can not conclude that a polynomial is real-rooted if it has nonnegative coefficients that form a log-concave sequence as the example $f(x) = 1 + x + x^2$ shows. Therefore several attempts were made to generalize real-rootedness such that every polynomial whose coefficients form a log-concave sequence falls into this more general class of polynomials. Of course, the connection to log-concave sequences is not the only one that naturally asks for a generalization of real-rootedness, and thus, the amount of generalizations of real-rooted polynomials has become quite vast. A brief overview of some generalization concepts is given in Section 2.

Among those concepts, the focus of this thesis is set to conically stable polynomials. A polynomial $f \in \mathbb{C}[\mathbf{z}]$ is called conically stable if the imaginary parts of the roots of f do not lie in the relative interior of some real cone. In this thesis and the related articles [32, 36, 42], we derive the following results for conically stable polynomials and related concepts:

1. We generalize operators that preserve the stability of polynomials to the case of conic stability. This includes the preservation under taking directional derivatives (see 4.1), a conic version of the Lieb-Sokal Lemma (see Theorem 4.3), the preservation under passing over to the initial form (see Theorem 4.12 and Theorem 4.13) and a collection of operators that preserve psd-stability (see Lemma 4.6 and Theorem 4.8).
2. Combinatorial criteria for psd-stable binomials (see Theorem 4.15) and psd-stable polynomials of determinants (see Theorem 4.19) as well as general structural results on psd-stable polynomials (see Theorem 4.14) are developed by us and lead to our combinatorial conjecture for general psd-stable polynomials (see Conjecute 4.21).
3. Our classification of the imaginary projection of complex conics (see Theorem 5.1 and Theorem 5.2) generalizes a previously known classification for the case of real conics.
4. We investigate the imaginary projection of complex quadratic polynomials in n variables with hyperbolic initial form (see Theorem 3.16 and Corollary 5.5)

and families of bivariate polynomials of arbitrary degree whose imaginary projection is the full space (see Theorem 5.6). Furthermore, for any $k > 0$, we derive an explicit construction for complex polynomials whose imaginary projection's complement consists of exactly k bounded strictly convex components (see Theorem 5.7).

5. We derive explicit spectrahedral descriptions for hyperbolicity cones for the cases of determinantal and quadratic polynomials (see Theorem 6.1 and Theorem 6.4).
6. Our sufficient criterion certifies the K-stability of determinantal and quadratic polynomials based on the feasibility of a semi-definite program (see Theorem 6.8 and Theorem 6.9). Furthermore, our methods may be applied to scaled versions of either the cone or the polynomial (see Theorem 6.11).

This thesis is based on [32, 36, 42, see also Appendix A-C] and structured as follows: Firstly, a brief overview of several concepts which generalize real-rootedness is given in Section 2. This section also introduces necessary definitions and notations. In Section 3, the notion of conic stability is introduced together with a brief discussion of the up-to-date study of conic stability and the connection to other generalizations of real-rootedness. Additionally, we introduce the notion of the imaginary projection in this section and clarify its relevance for conic stability. Section 4 focuses on establishing preservation operators for conically stable polynomials as well as formulating combinatorial criteria for conic stability. In Section 5, new results for the imaginary projection of complex polynomials of high degree are presented. This section also includes a full classification of quadratic complex polynomials based on the roots of their initial forms. Section 6 treats determinantal representations of polynomials as well as quadratic polynomials in order to establish a sufficient criterion for conic stability. Each of the sections from Section 4 to Section 6 concludes with a brief summary and a short outlook regarding possible further research directions based on the results of the corresponding articles. In Section 7 the contribution of the author to the related articles is stated. Finally, this thesis concludes with a German summary in Section 8.

2. Preliminaries

In this section we give a brief overview of some of the generalizations of real-rootedness of polynomials. A main focus is on stable polynomials and generalizations thereof. This includes hyperbolic polynomials as they may be seen as a generalization of homogeneous stable polynomials. Furthermore, we briefly discuss determinantal representations and their connections to stable polynomials.

First of all, let us fix the following notation. $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers. For a field $\mathbb{K}$ (usually the real or complex numbers), $\mathbb{K}^n$ denotes the n-dimensional vector space over $\mathbb{K}$ given by $\mathbb{K}^n = \mathbb{K} \times \ldots \times \mathbb{K}$. $\mathbb{P}^n$ and $\mathbb{P}^n_{\mathbb{R}}$ denote the projective spaces over the complex or real numbers of dimension n. $\mathbb{R}_{>0}$ ($\mathbb{R}_{<0}$) is the subset of positive (negative) real numbers and $\mathbb{R}_{\geq 0} = \mathbb{R}_{>0} \cup \{0\}$ ($\mathbb{R}_{\leq 0} = \mathbb{R}_{<0} \cup \{0\}$). For a complex number $a \in \mathbb{C}$, let $\mathsf{Re}(a)$ and $\mathsf{Im}(a)$ denote the real and the imaginary part of a. Thus, we may write $a = \mathsf{Re}(a) + \mathsf{i}\mathsf{Im}(a)$. For a subset $A \subseteq \mathbb{R}^n$, $\overline{A}$ denotes the *topological closure* of A with respect to (w.r.t.) the euclidean topology and $A^{\mathsf{c}} = \{\mathbf{x} \in \mathbb{R}^n : x \notin A\}$ denotes the *complement* of A. The *algebraic degree* of A is given by the degree of its closure with respect to the Zariski topology. Let $\mathbb{N}$ denote the natural numbers, $\mathbb{Z}$ the set of integer numbers and $\mathbb{Z}_{\geq 0} = \mathbb{N} \cup \{0\}$. For vectors we will use bold letters while matrices are denoted by capital letters. For $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, $\mathbf{v} \geq \mathbf{w}$ denotes an entry-wise inequality, i.e. $v_j \geq w_j$, and for matrices $V \succ 0$ ($V \succeq 0$) denotes that V is positive (semi-)definite. A complex matrix $A \in \mathbb{C}^{n \times n}$ is called *Hermitian* if A equals its conjugate transpose. We denote the set of all Hermitian matrices of size $n \times n$ by Herm_n. $\mathbb{K}[z]$ denotes the set of univariate polynomials with coefficients in $\mathbb{K}$. $\mathbb{K}[\mathbf{z}]$ denotes the set of multivariate polynomials with coefficients in $\mathbb{K}$ with the variables $z_1, \ldots, z_n$. For a polynomial $f = \sum_\alpha c_\alpha \mathbf{z}^\alpha \in \mathbb{K}[\mathbf{z}]$, the *degree* of f is given by $\deg(f) = \max\{|\alpha| : c_\alpha \neq 0\}$. The *degree of variable z_j in f* is analogously given by $\deg_j(f) = \max\{|\alpha_j| : c_\alpha \neq 0\}$. A polynomial is called *homogeneous* if all monomials of f have the same degree. For a polynomial $f \in \mathbb{C}[\mathbf{z}]$ of degree d, let $f^h(z_0, \mathbf{z}) = z_0^d \cdot f\left(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}\right) \in \mathbb{C}[z_0, \mathbf{z}]$ be the homogenization of f and for a homogeneous polynomial $f \in \mathbb{C}[z_0, \mathbf{z}]$, let $f(1, \mathbf{z})$ be the de-homogenization of f. By $\mathrm{in}(f)$ we denote the *initial form* given by $\mathrm{in}(f) = f^h(0, \mathbf{z})$. Thus, $\mathrm{in}(f)$ consists of the monomials of f with maximal degree only. Furthermore, for a polynomial $f \in \mathbb{C}[\mathbf{z}]$, $\mathcal{V}(f) = \{\mathbf{z} \in \mathbb{C}^n : f(\mathbf{z}) = 0\}$ denotes the variety of f which is given by the roots of f. We call f *real-rooted* if $\mathcal{V}(f) \subseteq \mathbb{R}^n$. Let $\partial_j f$ denote the j-th partial derivative of f and $\partial_{\mathbf{v}} f$ the directional derivative of f in direction $\mathbf{v} \in \mathbb{R}^n$.

Stability is the first generalization of real-rootedness we consider, since some of the other generalizations build upon the notion of stable polynomials instead of generalizing real-rootedness directly.

2.1. **Stable polynomials.** Let $\mathcal{H} = \{z \in \mathbb{C} : \mathsf{Im}(z) > 0\}$ be the open upper halfplane of $\mathbb{C}$. We call a univariate polynomial $f \in \mathbb{C}[z]$ *stable* if $f(z) \neq 0$, whenever $z \in \mathcal{H}$, i.e. f is not allowed to have roots which are contained in the open upper half-plane of the complex plane. Obviously, the concept of stability generalizes the concept of real-rootedness as the real line is not contained in the open upper half-plane $\mathcal{H}$. In the multivariate cases we define stability analogously as follows.

Definition 2.1. $f \in \mathbb{C}[\mathbf{z}]$ is called *stable* if $f(\mathbf{z}) \neq 0$ whenever $\mathbf{z} \in \mathcal{H}^n$. Furthermore, f is called *real stable* if the coefficients of f are real.

Equivalently, $f \in \mathbb{C}[\mathbf{z}]$ is stable if for every root $\mathbf{z}$ of f, there is an index $j \in [n]$ with $\mathsf{Im}(z_j) \leq 0$. Since stable polynomials generalize real-rootedness, they also enjoy many connections to similar branches of mathematics, including optimization [101], differential equations [17], probability theory [18], matroid theory [20, 31], applied algebraic geometry [103], theoretical computer science [76, 77] and statistical physics [16]. See also the surveys of Pemantle [92] and Wagner [104].

When studying stable polynomials, one of the starting points is the preservation of stability, i.e. operators which return a stable polynomial when applied onto a stable polynomial. Formally, let $T : V \subseteq \mathbb{C}[\mathbf{z}] \to \mathbb{C}[\mathbf{z}]$ be a operator. Then T is called a *stability preserver* if either $T(f)$ is stable or $T(f) \equiv 0$ for any stable polynomial $f \in \mathbb{C}[\mathbf{z}]$. Although some of the stability preservers may not look promising on their own at first glance, there were many stunning results derived by combining them. The first set of stability preservers is given by the following collection:

Proposition 2.2. [104, Lemma 2.4] *Let $f \in \mathbb{C}[\mathbf{z}]$ be stable.*
a) Permutation*: $f(z_{\sigma(1)}, \dots, z_{\sigma(n)})$ is stable for every permutation $\sigma : [n] \to [n]$.*
b) Scaling*: $c \cdot f(a_1 z_1, \dots, a_n z_n)$ is stable or zero for every $c \in \mathbb{C}$ and $\mathbf{a} \in \mathbb{R}^n_{>0}$.*
c) Diagonalization*: $f(\mathbf{z})\big|_{z_j = z_i} \in \mathbb{C}[z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n]$ is stable or zero for every $i, j \in [n],\ i \neq j$.*
d) Specialization*: $f(b, z_2, \dots, z_n) \in \mathbb{C}[z_2, \dots, z_n]$ is stable or zero for every $b \in \mathbb{C}$ with $\mathsf{Im}(b) \geq 0$.*
e) Inversion*: $z_1^{\deg_1(f)} \cdot f(-z_1^{-1}, z_2, \dots, z_n)$ is stable.*
f) Differentiation*: $\partial_j f(\mathbf{z})$ is stable or zero for every $j \in [n]$.*

The statements of Proposition 2.2 a)-d) are rather obvious. e) follows due to the identity $\mathsf{Im}\left(\frac{1}{z}\right) = \mathsf{Im}\left(\frac{\overline{z}}{z \cdot \overline{z}}\right)$, which implies that the imaginary parts of z and z^{-1} have different signs. f) follows by the Gauß-Lucas Theorem 2.3.

Theorem 2.3. [78, Theorem 6.1] *Let $f \in \mathbb{C}[z]$ and let $z_1, \dots, z_k$ denote the roots of f. Then all roots of f' lie in conv$\{z_1, \dots, z_k\}$, the convex hull of the roots of f.*

The Gauß-Lucas-Theorem 2.3 may be considered as a generalization of Rolle's Theorem which implies that for a polynomial f of degree d with d distinct real roots $x_1 < \ldots < x_d$, each root x of f' lies in one of the intervals given by $[x_j, x_{j+1}]$ for some $j \in [d]$ and thus, $x \in [x_1, x_d] = \text{conv}\{x_1, \dots, x_d\}$.

By combining the stability preservers of Proposition 2.2 in a suitable way, Brändén derived a stunning connection to matroid theory and jump systems, a generalization of matroids, [20] as follows:

A matroid on a finite groundset E is given by $\mathcal{M} = (E, \mathcal{I})$, where $\mathcal{I}$ is a collection of subsets of E such that

(1) $\emptyset \in \mathcal{I}$,
(2) if $A \subseteq B,\ B \in \mathcal{I}$, then also $A \in \mathcal{I}$,
(3) the set $\mathcal{B}$ of inclusion-maximal elements of $\mathcal{I}$ respects the exchange axiom: If $A, B \in \mathcal{B},\ x \in A \setminus B$, then there is $y \in B \setminus A$ such that $A \setminus \{x\} \cup \{y\} \in \mathcal{B}$.

$\mathcal{B}$ is called the set of bases of the matroid $\mathcal{M}$. For further reading into matroid theory, we refer to [44, 88, 105]. The concept of matroids was further generalized to jump systems [19] as follows:

For $\alpha, \beta \in \mathbb{Z}^n$, the *steps* from α to β are given by the set

$$\text{St}(\alpha, \beta) = \{\sigma \in \mathbb{Z}^n : \ |\sigma| = 1,\ |\alpha + \sigma - \beta| = |\alpha - \beta| - 1\},$$

where $|\sigma| = \sum_{j=1}^n |\sigma_j|$. Thus, the steps from α to β are exactly those integer vectors which have one non-zero entry with absolute value 1 and taking the step decreases the distance to β by exactly one. Now, a collection $\mathcal{F} \subseteq \mathbb{Z}^n$ is called a *jump system* if for every $\alpha, \beta \in \mathcal{F}$ and $\sigma \in \text{St}(\alpha, \beta)$ we either have $\alpha + \sigma \in \mathcal{F}$ or there is $\tau \in \text{St}(\alpha + \sigma, \beta)$ such that $\alpha + \sigma + \tau \in \mathcal{F}$. In other words, taking a step σ from α to β we either remain in $\mathcal{F}$ or get back into $\mathcal{F}$ after taking another step towards β. This property is also known as the *Two-Steps Axiom*. Further, the *support* of a polynomial $f = \sum_\alpha c_\alpha \mathbf{z}^\alpha$ is given by $\text{supp}(f) = \{\alpha : c_\alpha \neq 0\}$. Brändén's Theorem now establishes a connection between a polynomial's support and jump systems.

Theorem 2.4. [20, Theorem 3.2] *Let $f \in \mathbb{C}[\mathbf{z}]$ be stable. Then supp(f) is a jump system.*

In the indirect proof of this Theorem, a suitable sequence of the preservers given by Proposition 2.2 e) and f) is applied to cut the polynomial down to one which is stable and of a specific structure. Then by applying Proposition 2.2 b) and c), a sequence of univariate polynomials is constructed whose limit is of the form $z^d + c$ with $d \geq 3$ and thus, this limit has a root in $\mathcal{H}$.

If one assumes further structure, the support may even form the set of bases of a matroid. We say that a polynomial $f \in \mathbb{C}[\mathbf{z}]$ is *multi-affine* if the degree of any variable is at most 1.

Corollary 2.5. [20, Corollary 3.3] *Let $f \in \mathbb{C}[\mathbf{z}]$ be stable, multi-affine and homogeneous. Then supp(f) is the set of bases of some matroid.*

Although this is stated as a corollary of Brändén's result here, it has been proven by Choe et. al. [31] before. In [20], Brändén showed that the backward direction of Corollary 2.5 is not true by proving that there is no multi-affine and homogeneous stable polynomial whose support gives the set of bases of the Fano Matroid F_7. Recently, the result of Brändén was extended by Rincón, Vinzant and Yu [96] for the case of stable binomials.

Theorem 2.6. [96, Proposition 4.3] *Let $f = c_\alpha \mathbf{z}^\alpha + c_\beta \mathbf{z}^\beta \in \mathbb{C}[\mathbf{z}]$ with c_α, c_β and $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$ be stable and let $\mathbf{z}^\alpha$ and $\mathbf{z}^\beta$ not have a common factor. Then one of the following holds,*

a) $\{\alpha, \beta\} = \{0, \mathbf{e}_i\}$ *for some* $i \in [n]$,
b) $\{\alpha, \beta\} = \{\mathbf{e}_i, \mathbf{e}_j\}$ *for some* $i, j \in [n]$ *and* $\frac{c_\alpha}{c_\beta} \in \mathbb{R}_{\geq 0}$ *or*
c) $\{\alpha, \beta\} = \{0, \mathbf{e}_i + \mathbf{e}_j\}$ *for some* $i, j \in [n]$ *and* $\frac{c_\alpha}{c_\beta} \in \mathbb{R}_{<0}$.

Another famous preserver of stable polynomials is given by the so-called Lieb-Sokal Lemma [73].

Lemma 2.7. [73, Lemma 2.3] *Let $g(\mathbf{z}) + y f(\mathbf{z}) \in \mathbb{C}[\mathbf{z}, y]$ be stable and assume $\deg_j(f) \leq 1$. Then $g(\mathbf{z}) - \partial_j f(\mathbf{z}) \in \mathbb{C}[\mathbf{z}]$ is stable or identically zero.*

The key ingredient for the proof of the Lieb-Sokal Lemma is given by the Hurwitz-Theorem. It states that if a sequence of polynomials is non-vanishing on some open set in $\mathbb{R}^n$, then this either is true for the limit polynomial as well or the limit is identically zero. The Hurwitz-Theorem is applied on a regular basis in the research areas related to roots of polynomials. Thus, the appearance of the phrase 'or (identically) zero' in the statements of this thesis usually refers to the usage of the Hurwitz-Theorem for some boundary cases within its proof.

Theorem 2.8. [66, Par. 5.3.4] *Let $\{f_k\}$ be a sequence of polynomials non-vanishing in a connected open set $U \subseteq \mathbb{R}^n$, and assume it converges to a function f uniformly on compact subsets of U. Then f is either non-vanishing on U or it is identically zero.*

Although the Lieb-Sokal Lemma initially was just one out of a sequence of Lemmas showing some properties related to ferromagnetism, it then became popular in the study of stable polynomials - especially after it was used as a cornerstone to prove the following full characterization of all linear operators that preserve stability shown by Borcea and Brändén [16, 104]. For $\kappa \in \mathbb{Z}^n_{\geq 0}$, let $\mathbb{C}_\kappa[\mathbf{z}]$ denote all polynomials $f \in \mathbb{C}[\mathbf{z}]$

such that $\alpha_j \leq \kappa_j$ for all α with $c_\alpha \neq 0$, i.e. the set $\mathbb{C}_\kappa[\mathbf{z}]$ consists of such complex polynomials whose degree is bounded by κ only.

Theorem 2.9. [16, Theorem 1.1] *Let $\kappa \in \mathbb{Z}^n_{\geq 0}$ and $T : \mathbb{C}_\kappa[\mathbf{z}] \to \mathbb{C}[\mathbf{z}]$ be a linear operator. Then T preserves stability if and only if either*

(1) T has range of dimension at most one and is of the form

$$T(f) = \alpha(f)P,$$

where α is a linear functional on $\mathbb{C}_\kappa[\mathbf{z}]$ and P is a stable polynomial, or

(2) $G_T(\mathbf{z}, \mathbf{w}) \in \mathbb{C}[\mathbf{z}, \mathbf{w}]$ is stable.

Here, $G_T(\mathbf{z}, \mathbf{w})$ is the *algebraic symbol* of T defined as follows: For $T : \mathbb{C}_\kappa[\mathbf{z}] \to \mathbb{C}_\gamma[\mathbf{z}]$, let $G_T(\mathbf{z}, \mathbf{w}) \in \mathbb{C}_{\kappa \bigoplus \gamma}[\mathbf{z}, \mathbf{w}]$ be the polynomial defined by

$$G_T(\mathbf{z}, \mathbf{w}) = T\left[(\mathbf{z} + \mathbf{w})^\kappa\right] = \sum_{\alpha \leq \kappa} \binom{\kappa}{\alpha} T(\mathbf{z}^\alpha)\mathbf{w}^{\kappa - \alpha}$$

with $\binom{\kappa}{\alpha} = \binom{\kappa_1}{\alpha_1} \cdot \dots \cdot \binom{\kappa_n}{\alpha_n}$. The following example shall clarify the application of Theorem 2.9.

Example 2.10. Let $T := \partial_{z_j}$. Note that in this special case, we write ∂_{z_j} instead of ∂_j to emphasize that the operator is applied to $\mathbf{z}$ only, and does not directly affect $\mathbf{w}$. Then the algebraic symbol of T is given by

$$\begin{aligned} G_T(\mathbf{z}, \mathbf{w}) = G_{\partial_{z_j}}(\mathbf{z}, \mathbf{w}) &= \sum_{\alpha \leq \kappa} \binom{\kappa}{\alpha} \partial_{z_j}(\mathbf{z}^\alpha)\mathbf{w}^{\kappa-\alpha} = \partial_{z_j} \sum_{\alpha \leq \kappa} \binom{\kappa}{\alpha} \mathbf{z}^\alpha \mathbf{w}^{\kappa - \alpha} \\ &= \partial_{z_j} (\mathbf{z} + \mathbf{w})^\kappa = (\mathbf{z} + \mathbf{w})^{\kappa'} \end{aligned}$$

with $\kappa'_i = \kappa_i$ if $i \neq j$ and $\kappa'_j = \kappa_j - 1$. We have $(\mathbf{z} + \mathbf{w})^{\kappa'} \neq 0$ if all coordinates of $\mathbf{z}$ and $\mathbf{w}$ have positive imaginary part. Thus, T preserves stability.

Another preserver with a combinatorial aspect was established by Rincón, Vinzant and Yu in 2020 [96]. The initial form of a polynomial f with respect to a linear functional $\mathbf{w}$ in the dual space $(\mathbb{R}^n)^*$ is defined as

$$\mathrm{in}_\mathbf{w}(f) = \sum_{\alpha \in S_\mathbf{w}} c_\alpha \mathbf{z}^\alpha, \text{ where } S_\mathbf{w} = \{\alpha \in \mathrm{supp}(f) : \langle \mathbf{w}, \alpha \rangle = \max_{\beta \in \mathrm{supp}(f)} \langle \mathbf{w}, \beta \rangle\}.$$

In the special case of $w = (1, \dots, 1)$, we recover the initial form defined before by $\mathrm{in}(f) = f^h(0, \mathbf{z})$. Thus, we dismiss the $\mathbf{w}$ in this case and speak of the initial form of f denoted by $\mathrm{in}(f)$.

Rincón, Vinzant and Yu showed that for a stable polynomial f with real coefficients, the initial form w.r.t. $\mathbf{w}$, $\mathrm{in}_\mathbf{w}(f)$, is stable as well. For the combinatorial aspect, consider the Newton-Polytope $\mathrm{New}(f)$ of f which is given by the convex hull of the support of f. Then there is some face F of $\mathrm{New}(f)$ such that the restriction of the support onto $\mathrm{supp}(f) \cap F$ gives the support of $\mathrm{in}_\mathbf{w}(f)$ (see also Figure 2). For a stable polynomial $f \in \mathbb{C}[\mathbf{z}]$, passing over to the initial form preserves stability as follows:

Theorem 2.11. [96, Proposition 2.2 and Proposition 4.1] *Let $f \in \mathbb{C}[\mathbf{z}]$ be stable and either homogeneous or real stable and $\mathbf{w} \in (\mathbb{R}^n)^* \setminus \{\mathbf{0}\}$. Then $\mathrm{in}_\mathbf{w}(f)$ is also stable.*

The proof of this theorem is based on the notion of positive hyperbolicity which we consider in subsection 2.4.3 and makes use of a machinery involving methods of tropical geometry.

2.2. **Hyperbolic polynomials.** Hyperbolic polynomials represent another class which is closely related to real-rootedness and stable polynomials.

Definition 2.12. A homogeneous polynomial $f \in \mathbb{R}[\mathbf{z}]$ is called *hyperbolic* in direction $\mathbf{e} \in \mathbb{R}^n$ if $f(\mathbf{e}) \neq 0$ and for every $\mathbf{x} \in \mathbb{R}^n$, the univariate polynomial $t \mapsto f(\mathbf{x} + t\mathbf{e})$ has only real roots.

The k-th elementary symmetric polynomial in $\mathbb{R}[\mathbf{z}]$ is given by the sum of all multi-affine monomials induced by the subsets of $[n]$ of size k, i.e. $\sigma_k(\mathbf{z}) = \sum_{S \subseteq [n],\ |S|=k} \mathbf{z}^S$. The elementary symmetric polynomials pose a popular class of hyperbolic polynomials.

There are several connections to stable polynomials. First of all, hyperbolicity can be seen as a generalization of stability for homogeneous polynomials since a homogeneous polynomial $f \in \mathbb{R}[\mathbf{z}]$ is stable if and only if it is hyperbolic w.r.t. all $\mathbf{e} \in \mathbb{R}^n_{\geq 0}$ [60, Theorem 3.5]. Note that hyperbolic polynomials have closures similar to stable polynomials as they are closed under taking derivatives [8, 45] which can be shown by applying Rolle's Theorem.

Furthermore, hyperbolicity follows a conical structure. If a polynomial f is hyperbolic w.r.t. $\mathbf{e} \in \mathbb{R}^n$, the same is true for $\lambda\mathbf{e}$ for all $\lambda \in \mathbb{R} \setminus \{0\}$. This can be easily seen since the roots of $t \mapsto f(\mathbf{x} + t\lambda\mathbf{e})$ are scaled versions of the roots of $t \mapsto f(\mathbf{x} + t\mathbf{e})$ with scaling factor $\frac{1}{\lambda}$. This leads to the definition of hyperbolicity cones:

Definition 2.13. Let $f \in \mathbb{R}[\mathbf{z}]$ be hyperbolic w.r.t. $\mathbf{e} \in \mathbb{R}^n$. Then $C(\mathbf{e}) = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x} + t\mathbf{e}) = 0 \Rightarrow t < 0\}$ is the hyperbolicity cone of f w.r.t. $\mathbf{e}$.

It is well known that $C(\mathbf{e})$ forms an open convex cone, and further, for any $\mathbf{e}' \in C(\mathbf{e})$, f is also hyperbolic w.r.t. $\mathbf{e}'$ and $C(\mathbf{e}) = C(\mathbf{e}')$ [45].

Originally, the interest in hyperbolic polynomials was motivated by partial differential equations [5, 57, 71] since hyperbolic polynomials appear as characteristic polynomials in hyperbolic partial differential equations. Recent further studies were motivated by hyperbolic programming [46, 79, 95] which is an efficiently solvable convex optimization problem which is also rich in theoretical structure. A hyperbolic program is a convex optimization problem where a linear function is minimized constrained by equations which are either linear or define a hyperbolicity cone of some hyperbolic polynomial $f \in \mathbb{C}[\mathbf{z}]$. Thus, hyperbolic programming is an actual generalization of linear programming. Semi-definite programming is another convex optimization problem which generalizes linear programming. In semi-definite programming a linear function is minimized with respect to semi-definite constraints. Whether hyperbolic programming is an actual generalization of semi-definite programming as well is still an open question also known as the generalized Lax-Conjecture [6, 7, 53, 67, 82, see also Conjecture 2.15].

2.3. **Determinantal representations.** Determinantal representations have various connections to the previously mentioned areas. First of all they are closely connected to the Lax conjecture as the feasibility regions of semi-definite programms are given by sets which are defined through inequalities based on determinantal representations

[53]. Furthermore, determinantal representations imply real stability of the underlying polynomial [15].

Definition 2.14. A polynomial $f \in \mathbb{C}[\mathbf{z}]$ is a *determinantal polynomial* if there are matrices $A_0, A_1, \ldots, A_n$ with

$$f(\mathbf{z}) = \det\left(A_0 + \sum_{j=1}^{n} z_j A_j\right).$$

We say $\det\left(A_0 + \sum_{j=1}^{n} z_j A_j\right)$ is the *determinantal representation* of f.

The matrices $A_0, \ldots, A_n$ are often assumed to be real symmetric or Hermitian. If the constant coefficient matrix A_0 is positive definite or the identity, then the determinantal polynomial is called a *definite* or *monic* determinantal polynomial, respectively. Helton, McCullough and Vinnikov as well as Quarez showed that every polynomial $f \in \mathbb{R}[\mathbf{z}]$ with $f(\mathbf{0}) \neq 0$ has a determinantal representation of the form $f(\mathbf{z}) = \det\left(A_0 + \sum_{j=1}^{n} z_j A_j\right)$ with real symmetric matrices $A_0, \ldots, A_n$ [52, 93]. Both results come with large matrix sizes which exceed the degree or the number of variables of the polynomial. We call a polynomial $f \in \mathbb{R}[\mathbf{z}]$ *real zero* if the univariate polynomial $t \mapsto f(t \cdot \mathbf{z})$ has only real roots for any $\mathbf{z} \in \mathbb{R}^n$. In [53] and [83] connections to real zero polynomials were exploited to show that several classes of polynomials have monic determinantal representations.

Let $S \subseteq \mathbb{R}^n$, we say S is a *spectrahedron* if $S = \left\{\mathbf{x} \in \mathbb{R}^n : A_0 + \sum_{j=1}^{n} x_j A_j \succeq 0\right\}$ for some symmetric $A_0, \ldots, A_n$. Thus, spectrahedra are given by the positive semi-definite regions of determinantal polynomials. Further we call $S \subseteq \mathbb{R}^n$ *spectrahedral* if S has a representation as a spectrahedron. Since spectrahedra pose the feasibility sets of positive semi-definite programs, we may express the generalized Lax-Conjecture as follows.

Conjecture 2.15. [6, Conjecture 1.5] *Every hyperbolicity cone is spectrahedral.*

The opposed statement, that every spectrahedral cone is also a hyperbolicity cone has been proven in 2005 by Lewis, Parrilo and Ramana [72, Proposition 2]. If special requirements are met, determinantal representations result in real stable polynomials as shown by Borcea and Brändén [15].

Theorem 2.16. [15, Proposition 2.4] *Let $A_1, \ldots, A_n$ be positive semi-definite $d \times d$ matrices and A_0 a Hermitian $d \times d$ matrix. Then*

$$f(\mathbf{z}) = \det\left(A_0 + \sum_{j=1}^{n} z_j A_j\right)$$

is real stable or identically zero.

The Hurwitz-Theorem 2.8 was used to reduce the proof of Theorem 2.16 onto the case of positive definite matrices $A_1, \ldots, A_n$. Using properties of positive definite matrices, we end up with the characteristic polynomial of a symmetric matrix, which of course, is real-rooted.

2.4. **Generalizations of stable polynomials.**

2.4.1. *Half-plane property.* Similar to stability which is avoiding roots in the upper half-plane, we can also consider polynomials that don't have roots in an arbitrary half-plane. Thus, let $H \subseteq \mathbb{C}$ be an open half-plane such that the boundary of H contains the origin. We say a polynomial $f \in \mathbb{C}[\mathbf{z}]$ has the *half-plane property*, or equivalently is *H-stable* if none of the roots of f are contained in H^n [20]. Obviously, this generalizes stability with stability as the special case when $H = \mathcal{H}$. Polynomials with the half-plane property have been studied even more thoroughly than stable polynomials [1, 30, 20, 31, 47, 63, 106]. The reason for this is not only due to stable polynomials being a part of them, but also due to another prominent special case for $H = \{z \in \mathbb{C} : \mathsf{Re}(z) > 0\}$: The so called Hurwitz-stability, which has further applications in the numerical analysis as well as the design of control systems [12, 49, 59]. Two of the most prominent results here are the Routh-Hurwitz-Criterion [56] which is a criterion in control system theory that certifies or neglects stability of a dynamical system, and Stodola's Criterion, which states that the coefficients of a univariate Hurwitz-stable polynomial $f \in \mathbb{R}[z]$ need to have the same sign [41]. Further, several of the preservation operators for stable polynomials preserve polynomials with the half-plane property as well. This includes preservation under taking partial derivatives as well as the jump system result of Brändén. The reason for this being, that it is merely a rotation from one open half plane whose boundary contains the origin to another one. Let $f \in \mathbb{C}[\mathbf{z}]$ be H-stable for some half-plane $H = \{z \in \mathbb{C} : e^{\mathrm{i}\theta} z : \mathsf{Im}(z) > 0\}$ for some real θ. Then $f(e^{-\mathrm{i}\theta} z_1, \ldots, e^{-\mathrm{i}\theta} z_n)$ is stable and has the same support as f. Scaling and inversion (Proposition 2.2) on the other hand, do not generalize to preservers of general H-stable polynomials. While $z \mapsto z^{-1}$ does not change the sign of the real part and thus, gives rise to a preserver of Hurwitz-stability similar to Proposition 2.2 e), the same is not true for the half-plane given by $H = \{z \in \mathbb{C} : \mathsf{Im}(z) > \mathsf{Re}(z)\}$.

2.4.2. *Lorentzian polynomials.* Lorentzian polynomials are a recent generalization of homogeneous stable polynomials introduced by Brändén and Huh [21]. Let H_n^d be the space of homogeneous polynomials of degree d in n variables with real coefficients. Further let $\mathring{\mathrm{L}}_n^2$ be the open subset of H_n^2 of homogeneous quadratic polynomials with positive coefficients which have the Lorentzian signature. Here, a quadratic form f is said to have the *Lorentzian signature* if f can be expressed as $f(\mathbf{z}) = \mathbf{z}^T A \mathbf{z}$ such that the quadratic matrix A has exactly one positive eigenvalue and $n-1$ negative eigenvalues. For degrees larger than 2, the open subset $\mathring{\mathrm{L}}_n^d$ of H_n^d is defined by setting $\mathring{\mathrm{L}}_n^d = \{f \in \mathrm{H}_n^d : \partial_j f \in \mathring{\mathrm{L}}_n^{d-1}\ \forall j \in [n]\}$, i.e. the set containment is defined in a recursive way such that f belongs to $\mathring{\mathrm{L}}_n^d$ if all partial derivatives of f belonged to $\mathring{\mathrm{L}}_n^{d-1}$. The polynomials in $\mathring{\mathrm{L}}_n^d$ are called *strictly Lorentzian* and the limits of strictly Lorentzian polynomials are called *Lorentzian*. We call $S \in \mathbb{N}^n$ *M-convex* if for any index i and $\alpha, \beta \in S$ with $\alpha_i > \beta_i$, there is an index j with $\alpha_j < \beta_j$ and $\alpha - e_i + e_j,\ \beta + e_i - e_j \in S$. Let M_n^d be the subset of H_n^d such that the support of every polynomial in M_n^d forms a M-convex set. Based on this, we now define the closures L_n^2 and L_n^d as follows: Let L_n^2 be the set of all quadratic forms with nonnegative coefficients and at most 1 positive eigenvalue and let L_n^d be defined recursively by $\mathrm{L}_n^d = \{f \in \mathrm{M}_n^d : \partial_j f \in \mathrm{L}_n^{d-1}\ \forall j \in [n]\}$.

Brändén and Huh [21, Theorem 2.25] have shown that L_n^d indeed, is the set of Lorentzian polynomials of degree d in n variables and that the set of homogeneous stable polynomials is a subset of them. The class of Lorentzian polynomials turned out to be rather rich in structure since, by definition, these polynomials are preserved under taking partial derivatives as well as under other operators stated by Brändén and Huh. Furthermore, they showed that the classes of Lorentzian polynomials and homogeneous polynomials with nonnegative coefficients whose coefficients form a strongly log-concave sequence coincide [21, Theorem 2.30].

Recently, due to the rich structure of Lorentzian polynomials, several new connections were established [27, 38, 81]. Furthermore, Brändén and Leake introduced an even further generalization by studying conically Lorentzian polynomials [28].

2.4.3. *Positively hyperbolic varieties.* Another recent generalization of stable polynomials is given by positively hyperbolic varieties [96]. To state the definition, we need the positive Grassmannian. The *positive Grassmannian* $Gr_+(c,n)$ consists of all c-dimensional linear subspaces of $\mathbb{R}^n$ such that they can be represented as the row space of a matrix in $\mathbb{R}^{c\times n}$, all of whose maximal minors are positive. Furthermore, a variety is called *equidimensional* if all of its irreducible parts have the same dimension. An equidimensional variety $X \subseteq \mathbb{C}^n$ of co-dimension $c \leq n-1$ is then called *positively hyperbolic* if for every linear subspace L in the positive Grassmannian $Gr_+(c,n)$ and every $x \in X$, we have $\mathsf{Im}(x) \notin L \setminus \{\mathbf{0}\}$. The connection to stable polynomials has been investigated by Rincón, Vinzant and Yu and led to the following Proposition.

Proposition 2.17. [96, Proposition 2.2] *Let $\mathcal{V}(f)$ be the hypersurface defined by $f=0$. If $\mathcal{V}(f)$ is positively hyperbolic, then f is stable. Furthermore, if f is either real or homogeneous, then the stability of f is equivalent to the positive hyperbolicity of $\mathcal{V}(f)$.*

They further showed that reducing $\mathcal{V}(f)$ to $\mathcal{V}(\mathrm{in}_{\mathbf{w}}(f))$ preserves positive hyperbolicity and therefore due to Proposition 2.17 derived Theorem 2.11 for the cases of homogeneous polynomials as well as polynomials with real coefficients.

3. Conically stable polynomials

In this subsection we finally define conically stable polynomials. They represent the generalization of stable polynomials which we focus on in this thesis. The notion of conic stability has been introduced by Jörgens and Theobald [60]. Let $K \subseteq \mathbb{R}^n$ be a closed and convex cone. Then, $f \in \mathbb{C}[\mathbf{z}]$ is K-stable if the imaginary part of any root of f does not lie in the relative interior of K. Here, for a convex set $S \subseteq \mathbb{R}^n$, the relative interior is given by $\mathrm{relint}(S) = \{\mathbf{x} \in S : \forall \mathbf{y} \in S \text{ exists } \lambda > 0 \text{ such that } \lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in S\}$.

Definition 3.1. Let $K \subseteq \mathbb{R}^n$ be a closed convex cone. A polynomial $f \in \mathbb{C}[\mathbf{z}]$ is called *K-stable* if $f(\mathbf{z}) \neq 0$ whenever $\mathsf{Im}(\mathbf{z}) \in \mathrm{relint}\, K$.

First of all, note that the assumptions for K to be convex and closed are natural. If a polynomial is K-stable w.r.t. an arbitrary cone K, then its also $\overline{\mathrm{conv}(K)}$-stable [60, Corollary 3.6], where $\mathrm{conv}(\cdot)$ denotes the convex hull. For the special case of the positive orthant $K = (\mathbb{R}_{\geq 0})^n$, we recover the notion of stable polynomials. Stable polynomials do not only appear as a special case of conic stable polynomials. Every conic stable polynomial has a fundamental connection to a univariate stable polynomial as follows:

Lemma 3.2. [60, Lemma 3.4] *A polynomial $f \in \mathbb{C}[\mathbf{z}]$ is K-stable if and only if the univariate polynomial $t \mapsto f(\mathbf{x} + t\mathbf{y})$ is stable for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with $\mathbf{y} \in \operatorname{relint} K$.*

Note that the proof in [60, Lemma 3.4] also works without the assumption of full-dimensionality made there. This connection has been exploited by Jörgens and Theobald who used the interlacing theory connected to stable polynomials to prove a generalized version of the Hermite-Kakeya-Obreschkoff Theorem [60, Theorem 4.3]. Since we do not dive further into this aspect, we refer to [60, 94] for further reading on the interlacing theory of polynomials.

Although K-stability seems to be a natural generalization of stable polynomials, not much is known about possible preservers for K-stable polynomials or combinatorial conditions for polynomials in order to be K-stable. The only results known so far are the generalizations of the differentiation and specialization results of Proposition 2.2.

Theorem 3.3. [6, Lemma 1.1] *Let $f \in \mathbb{C}[\mathbf{z}]$ be homogeneous and K-stable. Then $\partial_{\mathbf{v}} f(\mathbf{z})$ is either K-stable or identically zero.*

Lemma 3.4. [60, Fact 3.8] *Let $K = K_1 \times K_2 \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be a cone. If $f(\mathbf{z}_1, \mathbf{z}_2)$ is K-stable, then $f(\mathbf{a} + \mathrm{i}\mathbf{b}, \mathbf{z}_2)$ is K_2-stable for every $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{b} \in \operatorname{relint} K_1$.*

3.1. **Psd-stability.** In this subsection we introduce psd-stability as a special case of K-stability where K is set to be the cone of real positive semi-definite matrices. The importance of this class comes in naturally as the cone of positive semi-definite matrices generalizes the positive orthant [13] and thus, psd-stability may be considered as a generalization of the notion of stable polynomials in its own way. For a formal definition, we need to fix the following notation (see also[32] or Appendix C):

Let $\mathcal{S}_n^{\mathbb{C}}$ denote the vector space of complex matrices of size $n \times n$ and $\mathcal{S}_n$ the space of the real ones. Further, denote the cones of real positive semi-definite and positive definite matrices of size $n \times n$ by $\mathcal{S}_n^+$ and $\mathcal{S}_n^{++}$. Let $\mathbb{C}[Z]$ be the ring of polynomials on the symmetric matrix variables $Z = (z_{ij})$. To be more precise, we consider $\mathbb{C}[Z]$ to be the vector space that is generated by monomials of the form $Z^\alpha = \prod_{1 \le i,j \le n} z_{ij}^{\alpha_{ij}}$ with some nonnegative symmetric matrix α of size $n \times n$ whose diagonal entries are integers and whose off-diagonal entries are half-integers. By identifying z_{ij} and z_{ji}, polynomials in $\mathbb{C}[Z]$ can also be interpreted as polynomials in the ring $\mathbb{C}[\{z_{ij} : 1 \le i \le j \le n\}]$. For example consider the monomial

$$Z^{\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}} = z_{12}^{\frac{1}{2}} z_{21}^{\frac{1}{2}} = z_{12},$$

which may be interpreted as a polynomial in the polynomial ring $\mathbb{C}[Z]$ over the vector space $\mathcal{S}_2^{\mathbb{C}}$ as well as a polynomial in the polynomial ring $\mathbb{C}[z_{11}, z_{12}, z_{22}]$ over $\mathbb{C}^3$. Now psd-stability may be defined as a special case of K-stability as follows.

Definition 3.5. Let $f \in \mathbb{C}[Z]$. We say f is *psd-stable* (stable w.r.t. to the cone of positive semi-definite matrices) if $f(M) \neq 0$, whenever $\mathsf{Im}(M) \in \mathcal{S}_n^{++}$.

Note that a polynomial $f \in \mathbb{C}[Z]$ is psd-stable if and only if f does not have a root in the *Siegel upper half-space* $\mathcal{H}_{\mathcal{S}} = \{A \in \mathcal{S}_n^{\mathbb{C}} : \mathsf{Im}(A) \in \mathcal{S}_n^{++}\}$. The Siegel upper half-space plays an essential role in algebraic geometry as well as number theory as it appears as the domain of modular forms [25, 86, 98].

Furthermore, since psd-stability is defined for polynomials over the space of complex symmetric matrices, we need to generalize the notion of the support. For a polynomial $f = \sum_\alpha c_\alpha Z^\alpha \in \mathbb{C}[Z]$, $\operatorname{supp}(f)$ is the collection of symmetric exponent matrices such that the corresponding coefficient is non-zero, i.e. $\operatorname{supp}(f) = \{\alpha \in \mathcal{S}_n : c_\alpha \neq 0\}$. The variables z_{jj} are called *diagonal variables* and the variables z_{ij} with $i \neq j$ are called *off-diagonal variables*. Based on this, we call a monomial a *diagonal monomial* if the corresponding exponent matrix α is diagonal, that is $\alpha_{ij} = 0$ if $i \neq j$ or an *off-diagonal monomial* if $\alpha_{jj} = 0$ for all $j \in [n]$. By convention, we consider the constant monomial to be a diagonal one. To clarify this notation, we consider the following example:

Example 3.6. [32, Example 2.12] Let

$$f(Z) = \det(Z) = Z^{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}} - Z^{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}} = z_{11}z_{22} - z_{12}^2 \in \mathbb{C}[Z]$$

be a polynomial over $\mathcal{S}_2^{\mathbb{C}}$. The monomial $z_{11}z_{22}$ is diagonal as its exponent matrix is diagonal while $-z_{12}^2$ is an off-diagonal monomial since the diagonal of its exponent matrix is zero.

A prime example of psd-stable polynomials is given by the determinants of $n \times n$ symmetric matrix variables.

Lemma 3.7. [32, Lemma 2.13] $f(Z) = \det(Z)$ *is psd-stable.*

The proof is quite similar to the proof of Theorem 2.16. We use the same technique for an indirect proof to construct a characteristic polynomial of a symmetric matrix with an imaginary root which then leads to a contradiction. Note that for $n \geq 3$, the polynomial $f(Z) = \det(Z)$ does not only involve diagonal and off-diagonal monomials but also such monomials that mix diagonal and off-diagonal variables.

3.2. **Imaginary projections.** Another notion introduced by Jörgens, Theobald and de Wolff in [62] which helps to develop a new geometric perspective onto K-stable polynomials is the so-called imaginary projection. This is done by shifting from a complex setting into a real setting by focusing on the imaginary part of every point in the variety. The two different perspectives become quite clear in the case of psd-stability. In the complex setting, we do not want f to have roots in the siegel upper half-space $\mathcal{H}_{\mathcal{S}}$ and in the real setting, we do not want the imaginary projection of f to have an actual intersection with the relative interior of the psd-cone $\mathcal{S}_n^+$.

Definition 3.8. Let $f \in \mathbb{C}[\mathbf{z}]$ be a complex polynomial. Then the *imaginary projection* $\mathcal{I}(f)$ is the projection of the variety $\mathcal{V}(f)$ onto its imaginary part

$$\mathcal{I}(f) = \{\mathbf{y} \in \mathbb{R}^n : \exists \mathbf{x} \in \mathbb{R}^n \text{ s.t. } \mathbf{x} + \mathrm{i}\mathbf{y} \in \mathcal{V}(f)\}.$$

Note that the imaginary projection is neither open nor closed in general. We define the boundary $\partial\mathcal{I}(f)$ of the imaginary projection of f as the intersection of the closure of the imaginary projection with the closure of its complement, i.e. $\partial\mathcal{I}(f) = \overline{\mathcal{I}(f)} \cap \overline{\mathcal{I}(f)^{\mathsf{c}}}$. Jörgens and Theobald revealed the connection between conic stable polynomials, hyperbolic polynomials and the imaginary projection in the following Theorem:

Theorem 3.9. [60, Theorem 3.5] *Let $f \in \mathbb{C}[\mathbf{z}]$ be homogeneous. Then the following are equivalent*

(1) f is K-stable.
(2) $\mathcal{I}(f) \cap \operatorname{relint} K = \emptyset$.
(3) f is hyperbolic with respect to every point in $\operatorname{relint} K$.

Due to Theorem 3.9, we get a new point of view onto conically stable polynomials as we may express conic stability in terms of the imaginary projection. Further, we may consider conic stability as a generalization of hyperbolicity as for every hyperbolic polynomial f there is a cone K such that f is K-stable. The converse is true for homogeneous polynomials only. Furthermore, a hyperbolic polynomial f is K-stable with K being any of its hyperbolicity cones. This connection goes even further as expressed in terms of the imaginary projection in the following Theorem by Jörgens and Theobald.

Theorem 3.10. [61, Theorem 1.1] *Let $f \in \mathbb{R}[\mathbf{z}]$ be homogeneous. Then the hyperbolicity cones of f coincide with the components of $\mathcal{I}(f)^{\mathsf{c}}$.*

Based on this result, Jörgens and Theobald derived upper bounds on the amount of components in the complement of the imaginary projection.

The research interest in imaginary projections is not only based on connections to K-stable or hyperbolic polynomials but also motivated by the goal to study general convexity phenomena in algebraic geometry [13]. In the broader context of convexity phenomena in algebraic geometry, imaginary projections align with amoeba and co-amoeba. All of them are of mapping images of the variety $\mathcal{V}(f)$ of a complex polynomial $f \in \mathbb{C}[\mathbf{z}]$ in the algebraic torus $(\mathbb{C}^*)^n$ such that the complement of these images consists of finitely many convex components [39, 43, 62]. While imaginary projections are given by the projection on the imaginary part, amoebas are defined by taking the logarithm of the absolute value entry-wise, and co-amoebas use the argument map instead. Thus, for a polynomial $f \in \mathbb{C}[\mathbf{z}]$, we have the *amoeba* and *co-amoeba* given by

$$\mathcal{A}(f) = \{(\log(|z_1|), \ldots, \log(|z_n|)) : \mathbf{z} \in \mathcal{V}(f) \cap (\mathbb{C}^*)^n\}$$

and

$$\operatorname{co}\mathcal{A}(f) = \{\arg(\mathbf{z}) = (\arg(z_1), \ldots, \arg(z_n)) : \mathbf{z} \in \mathcal{V}(f) \cap (\mathbb{C}^*)^n\}.$$

The structure as well as occurrences of the amoeba [35, 43, 80, 89, 90] and the co-amoeba [39, 87] have been studied thoroughly. In [62] Jörgens, Theobald and de Wolff initiated the study of the imaginary projection.

As there already are results on the number of components of the amoeba and the co-amoeba as well as their respective complements, Jörgens and Theobald investigated the structure of the imaginary projection. They have shown in [61] that there is no bound to the number of bounded, strictly convex components of the complement of the imaginary projection of a real polynomial $f \in \mathbb{R}[\mathbf{z}]$.

Theorem 3.11. [61, Theorem 1.3] *Let $n \geq 2$. For any $k > 0$ there exists a polynomial $f \in \mathbb{R}[\mathbf{z}]$ such that f has at least k strictly convex, bounded components in the complement of $\mathcal{I}(f)$.*

Aside from the convexity result, they derived the following transformation properties such that the study of imaginary projections of arbitrary complex polynomials may be reduced to studying imaginary projections of corresponding normal forms:

Lemma 3.12. [62, Lemma 5.1] *Let $f \in \mathbb{C}[\mathbf{z}]$ and $A \in \mathbb{R}^{n\times n}$ be an invertible matrix. Then $\mathcal{I}(f(A\mathbf{z})) = A^{-1}\mathcal{I}(f(\mathbf{z}))$.*

Lemma 3.13. [62, Lemma 5.2] *A real translation $\mathbf{z} \mapsto \mathbf{z} + \mathbf{a}$, $\mathbf{a} \in \mathbb{R}^n$ does not change the imaginary projection of a polynomial. An imaginary translation $\mathbf{z} \mapsto \mathbf{z} + \mathrm{i}\mathbf{a}, \mathbf{a} \in \mathbb{R}$, shifts the imaginary projection of a polynomial into direction $-\mathbf{a}$.*

Definition 3.14. Let $f \in \mathbb{C}[z_1, z_2]$ be quadratic. We say that f is the defining polynomial of a complex conic, or short, a *complex conic* if its total degree equals two. A complex conic f is called a *real conic* if all coefficients of f are real.

Based on Lemma 3.12 and Lemma 3.13, Jörgens, Theobald and de Wolff derived normal forms for real conics such that up to the action of G_2, every real conic is equivalent to to one of the following polynomials.

(i) $z_1^2 + z_2^2 - 1$ (ellipse),	*(v)* $z_1^2 - z_2^2$ (pair of crossing lines),
(ii) $z_1^2 - z_2^2 - 1$ (hyperbola),	*(vi)* $z_1^2 - 1$ (parallel lines/one line z_1^2),
(iii) $z_1^2 + z_2$ (parabola),	*(vii)* $z_1^2 + z_2^2$ (isolated point),
(iv) $z_1^2 + z_2^2 + 1$ (empty set),	*(viii)* $z_1^2 + 1$ (empty set).

Here, the group G_n is given by the semi-direct product of the complex translations $\mathbb{C}^n$ and the real general linear group $\mathrm{GL}_n(\mathbb{R})$, i.e. $G_n = \mathbb{C}^n \rtimes \mathrm{GL}_n(\mathbb{R})$. By Lemma 3.12 and Lemma 3.13, letting G_n act on the imaginary projection $\mathcal{I}(f)$ of a complex polynomial $f \in \mathbb{C}[\mathbf{z}]$ does not affect the topology induced by the imaginary projection in $\mathbb{R}^n$, i.e. the number of components in $\mathcal{I}(f)$ as well as the number of components in $\mathcal{I}(f)^{\mathsf{c}}$, the complement of the imaginary projection, is not changed under the action of G_n.

Theorem 3.15. [62, Theorem 5.3] *Let $f \in \mathbb{R}[z_1, z_2]$ be a real conic. For the normal forms (i)–(viii) from above, the imaginary projections $\mathcal{I}(f) \subseteq \mathbb{R}^2$ are as follows:*

(i) $\mathcal{I}(f) = \mathbb{R}^2$,	*(v)* $\mathcal{I}(f) = \{\mathbf{y} \in \mathbb{R}^2 : y_1^2 = y_2^2\}$,
(ii) $\mathcal{I}(f) = \{-1 \leq y_1^2 - y_2^2 < 0\} \cup \{\mathbf{0}\}$,	*(vi)* $\mathcal{I}(f) = \{\mathbf{y} \in \mathbb{R}^2 : y_1 = 0\}$,
(iii) $\mathcal{I}(f) = \mathbb{R}^2 \setminus \{(0, y_2) : y_2 \neq 0\}$,	*(vii)* $\mathcal{I}(f) = \mathbb{R}^2$,
(iv) $\mathcal{I}(f) = \{\mathbf{y} \in \mathbb{R}^2 : y_1^2 + y_2^2 - 1 \geq 0\}$,	*(viii)* $\mathcal{I}(f) = \{\mathbf{y} \in \mathbb{R}^2 : y_1 = \pm 1\}$.

See Figure 1 for a visualization of Theorem 3.15. Afterwards, Jörgens, Theobald and de Wolff, even generalized this result to the case of $n \geq 3$ variables and stated normal forms for this case together with the computed imaginary projection for those normal forms. Up to the action of G_n, every quadratic polynomial $f \in \mathbb{R}[\mathbf{z}]$ is equivalent to one of the following normal forms [62]:

$$\begin{array}{llll} & \text{(I)} & \sum_{j=1}^{p} z_j^2 - \sum_{j=p+1}^{r} z_j^2 & \left(1 \leq p \leq r,\, r \geq 1,\, p \geq \frac{r}{2}\right), \\ (1) & \text{(II)} & \sum_{j=1}^{p} z_j^2 - \sum_{j=p+1}^{r} z_j^2 + 1 & \left(0 \leq p \leq r,\, r \geq 1\right), \\ & \text{(III)} & \sum_{j=1}^{p} z_j^2 - \sum_{j=p+1}^{r} z_j^2 + z_{r+1} & \left(1 \leq p \leq r,\, r \geq 1,\, p \geq \frac{r}{2}\right). \end{array}$$

We say a quadratic polynomial $f \in \mathbb{C}[\mathbf{z}]$ is of type X if it is equivalent to the normal form of type X up to the action of G_n. Jörgens, Theobald and de Wolff derived the imaginary projection for these normal forms as stated in the subsequent Theorem.

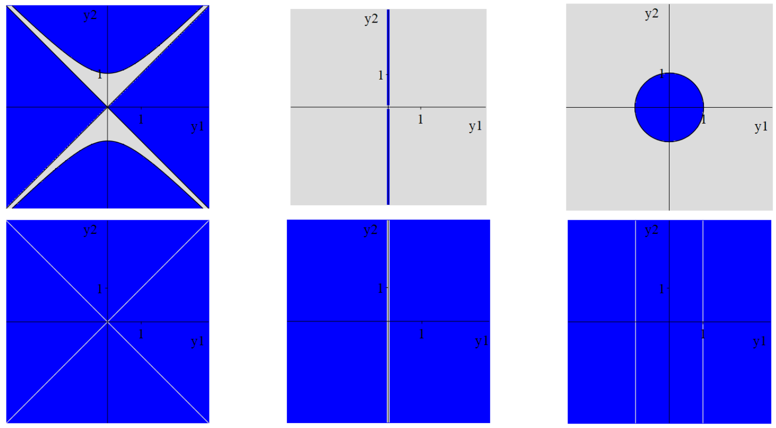


FIGURE 1. The blue area represents the imaginary projection's complement for all cases with $\mathcal{I}(f) \neq \mathbb{R}^2$ of the classification given by Theorem 3.15.

Theorem 3.16. [62, Theorem 5.4] *Let $n \geq r \geq 3$ and $f \in \mathbb{R}[\mathbf{z}]$ be a quadratic polynomial.*

(1) If f is of type (I), then

$$\mathcal{I}(f) = \begin{cases} \mathbb{R}^n & \text{if } \frac{r}{2} \leq p \leq r-1 \text{ or } p = r, \\ \left\{\mathbf{y} \in \mathbb{R}^n : y_r^2 \leq \sum_{j=1}^{r-1} y_j^2\right\} & \text{if } p = r-1. \end{cases}$$

(2) If f is of type (II), then

$$\mathcal{I}(f) = \begin{cases} \mathbb{R}^n & \text{if } p = 0 \text{ or } 1 < p < r-1, \\ \left\{\mathbf{y} \in \mathbb{R}^n : y_1^2 - \sum_{j=2}^{r} y_j^2 \leq 1\right\} & \text{if } p = 1, \\ \left\{\mathbf{y} \in \mathbb{R}^n : \sum_{j=1}^{r-1} y_j^2 > y_r^2\right\} \cup \{\mathbf{0}\} & \text{if } p = r-1, \\ \left\{\mathbf{y} \in \mathbb{R}^n : \sum_{j=1}^{r} y_j^2 \geq 1\right\} & \text{if } p = r. \end{cases}$$

(3) If $n > r$ and f is of type (III), then

$$\mathcal{I}(f) = \mathbb{R}^n \setminus \{(0, \ldots, 0, y_{r+1}, \ldots, y_n) : y_{r+1} \neq 0, y_{r+2}, \ldots, y_n \in \mathbb{R}\}.$$

Further, as they only encountered polynomials $f \in \mathbb{C}[\mathbf{z}]$ whose imaginary projection is open if and only if $\mathcal{I}(f) = \mathbb{R}^n$, they added the following open question about possible other occurrences of open imaginary projections apart from the full space.

Question 3.17. [62, Open Problem 3.4] *Let $f \in \mathbb{C}[\mathbf{z}]$. Is $\mathcal{I}(f)$ open if and only if $\mathcal{I}(f) = \mathbb{R}^n$?*

4. Combinatorics and Preservation of conically stable polynomials

4.1. **Related work.** This work [32, see also Appendix C] was co-written with G. Codenotti and T. Theobald. We were inspired by the immense amount of preservers for stable polynomials as collected in [20, 31, 104] and stated in Subsection 2.1 as well as the necessary combinatorial conditions given by the jump system property [20] and the extensions made for stable binomials [96].

Our ultimate goal is to study and state a variety of generalizations of preservers for stable polynomials to the case of K-stable polynomials as well as the invention of new preservers for K-stable polynomials. A special focus is on the preservation of psd-stability. Furthermore, we invent new combinatorial conditions for the psd-stability of a given polynomial.

4.2. **Preservation of conically stable polynomials.** We start with preservers for conic stability for general cones. The first one is given by the preservation of K-stable polynomials under taking directional derivatives.

Lemma 4.1. [32, Lemma 3.1] *Let $f \in \mathbb{C}[\mathbf{z}]$ be K-stable and $\mathbf{v} \in K$. Then $\partial_{\mathbf{v}} f$ is either K-stable or identically zero.*

The proof is based on the the connection to univariate derivatives by $\partial_{\mathbf{v}} f(\mathbf{z}) = \frac{d}{dt} f(\mathbf{z} + t\mathbf{v})\big|_{t=0}$ in order to use the Gauß-Lucas Theorem 2.3. Note that this also widens the amount of operators that preserve stable polynomials as they are not only preserved under taking partial derivatives but taking directional derivatives $\partial_{\mathbf{v}}$ with $\mathbf{v} \in \mathbb{R}^n_{\geq 0}$. For $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ let $\operatorname{pos}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \{\lambda_1 \mathbf{v}_1 + \cdots + \lambda_k \mathbf{v}_k : \lambda_j \geq 0\}$ be the positive hull of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. This leads to another preserver which generalizes the specialization property of Proposition 2.2 as follows:

Lemma 4.2. [32, Lemma 3.2] *Let $f \in \mathbb{C}[\mathbf{z}]$ be K-stable, $\mathbf{a} \in \mathbb{C}^n$ and $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(k)} \in \mathbb{R}^n$. Further set $K' = \operatorname{pos}(\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(k)})$ and assume that $\mathsf{Im}(\mathbf{a}) + K' \subseteq K$. Then the polynomial $g \in \mathbb{C}[\mathbf{z}]$ defined by*

$$g(z_1, \ldots, z_k) \;=\; f\left(\mathbf{a} + \sum_{j=1}^{k} z_j \mathbf{v}^{(j)}\right)$$

is stable or the zero polynomial.

Although, there has been a generalization of the specialization operator to the case of K-stability given by Lemma 3.4 already, our preserver can be applied to a larger variety of cones and establishes a connection to the usual notion of stability.

Our final result for general cones K is given by a generalization of the Lieb-Sokal-Lemma 2.7 to the case of K-stable polynomials. To formulate the result, we need the notion of the directional degree as a generalization of the degree w.r.t. a variable. Let $\mathbf{v} \in \mathbb{R}^n$. Then we call $\rho_{\mathbf{v}}(f)$ *the degree of f in direction* $\mathbf{v}$, defined by the degree of the univariate polynomial $f(\mathbf{w} + t\mathbf{v}) \in \mathbb{C}[t]$ for some generic $\mathbf{w} \in \mathbb{C}^n$. The directional degree matches with the intuition for the usual notion of degree in the sense that f survives taking $\rho_{\mathbf{v}}(f)$ directional derivatives into direction $\mathbf{v}$ but vanishes when taking $\rho_{\mathbf{v}}(f) + 1$. Now, the Lieb-Sokal-Lemma may be generalized as follows:

Theorem 4.3. [32, Theorem 3.4] *Let K' be given by $K' = K \times \mathbb{R}_{\geq 0}$ and $g(\mathbf{z}) + y f(\mathbf{z}) \in \mathbb{C}[\mathbf{z}, y]$ be K'-stable and such that $\rho_{\mathbf{v}}(f) \leq 1$ for some $\mathbf{v} \in K$. Then $g - \partial_{\mathbf{v}} f$ is K-stable or $g - \partial_{\mathbf{v}} f \equiv 0$.*

We build on Lemma 4.1 in combination with the following Lemma 4.4 for the proof of Theorem 4.3.

Lemma 4.4. [32, Lemma 3.5] *Let $f, g \in \mathbb{C}[\mathbf{z}]$, where $f \not\equiv 0$ and K-stable and let $K' = K \times \mathbb{R}_{\geq 0}$. Then $g + yf \in \mathbb{C}[\mathbf{z}, y]$ is K'-stable if and only if*

$$\mathsf{Im}\left(\frac{g(\mathbf{z})}{f(\mathbf{z})}\right) \geq 0 \quad \textit{for all } \mathbf{z} \in \mathbb{C}^n \textit{ with } \mathsf{Im}(\mathbf{z}) \in \operatorname{relint} K.$$

Note that in the same way as the preservation under taking directional derivatives, our generalized version of the Lieb-Sokal Lemma also broadens the amount of preservers of stable polynomials as the original Lieb-Sokal Lemma may be extended to directional derivatives w.r.t. directions that lie in the positive orthant rather than only partial derivatives. We stated this refined version of the Lieb-Sokal Lemma as follows:

Corollary 4.5 (Refined Lieb-Sokal Lemma)**.** [32, Coroallary 3.6] *Let $g(\mathbf{z}) + yf(\mathbf{z}) \in \mathbb{C}[\mathbf{z}, y]$ be stable and assume $\rho_{\mathbf{v}}(f) \leq 1$ for some $\mathbf{v} \in \mathbb{R}^n_{\geq 0}$. Then $g(\mathbf{z}) - \partial_{\mathbf{v}} f(\mathbf{z}) \in \mathbb{C}[\mathbf{z}]$ is stable or identically 0.*

Further generalizations for the remaining operators preserving stable polynomials are not possible for general cones K. When restricted to simplicial cones its possible to apply existing preservers for stable polynomials to K-stable polynomials by applying a suitable transformation $T \in \mathrm{GL}_n(\mathbb{R})$ since its only a transformation between the positive orthant and any other simplicial cone.

4.3. **Preservation of psd-stable polynomials.** In this subsection we focus on psd-stable polynomials, i.e. $K = \mathcal{S}_n^+$. As a brief reminder, let $\mathbb{C}[Z]$ be the space of polynomials on symmetric matrix variables. We say $f \in \mathbb{C}[Z]$ is psd-stable (stable with respect to the cone of positive semi-definite matrices) if $f(Z) \neq 0$, whenever $\mathsf{Im}(Z) \in \mathcal{S}_n^{++}$. For $f \in \mathbb{C}[Z]$, let $f_{\mathrm{Diag}} \in \mathbb{C}[Z]$ denote the polynomial obtained from f by substituting all off-diagonal variables with 0. We refer back to Subsection 3.1 for further notations and constructions. Among the possible choices for the cone K, the psd-cone $\mathcal{S}_n^+$ is one of the most natural ones since psd-stability generalizes the notion of stable polynomials directly. Thus, its not surprising that it is rather rich in structure which enables us to find several elementary operators preserving psd-stability. We stated a first collection in the following:

Lemma 4.6 (Elementary preservers for psd-stability)**.** [32, Lemma 4.1] *Let $f \in \mathbb{C}[Z]$ be psd-stable.*

a) **Diagonalization***: The polynomial $Z \mapsto f_{\mathrm{Diag}}(Z)$ is psd-stable.*

b) **Transformation***: Let $S \in \mathrm{GL}_n(\mathbb{R})$, then $f(SZS^{-1})$ and $f(SZS^T)$ are psd-stable.*

c) **Minorization***: For $J \subseteq [n]$, let Z_J be the symmetric $|J| \times |J|$ submatrix of Z with index set J. Then $f(Z_J)$, the polynomial on $\mathcal{S}^{\mathbb{C}}_{|J|}$ obtained from f by setting to zero all variables with at least one index outside of J, is psd-stable or zero.*

d) **Specialization***: For a fixed index $i \in [n]$, let $\hat{Z}_i$ be any matrix obtained from Z by assigning real values to z_{ij}, z_{ji} for all indices $j \neq i$ and a value from $\mathcal{H}$ to z_{ii}. Then $f(\hat{Z}_i)$, viewed as polynomial on $\mathcal{S}^{\mathbb{C}}_{n-1}$, is psd-stable or zero.*

e) **Reduction***: For $i, j \in [n]$, let $\bar{Z}_{ij}$ be any matrix obtained from Z by choosing real values for $z_{ik} = z_{ki}$ for $k \neq i$ and setting $z_{ii} := z_{jj}$. Then $f(\bar{Z}_{ij})$, viewed as polynomial on $\mathcal{S}^{\mathbb{C}}_{n-1}$, is psd-stable or zero.*

f) **Permutation**: *Let $\pi : [n] \to [n]$ be a permutation. Then $f((Z_{\pi(j),\pi(k)})_{1\leq j,k\leq n})$ is a psd-stable polynomial on $\mathcal{S}_n^{\mathbb{C}}$.*

g) **Differentiation**: *$\partial_V f(Z)$ is psd-stable or zero for $V \in \mathcal{S}_n^+$.*

Lemma 4.6 b) to f) are derived by basic operations that preserve positive semidefiniteness of the imaginary part of Z and thus, preserve psd-stability. In some cases the proof uses the Hurwitz-Theorem for boundary cases and thus, getting the zero polynomial is also a possibility. Lemma 4.6 g) is a consequence of Lemma 4.1 for the case of $K = \mathcal{S}_n^+$. Lemma 4.6 a) follows due to the fact that for $Z = X + \mathrm{i}Y$ with $Y \succ 0$, the diagonal values of Y have to be positive. Note that Lemma 4.6 a) is not merely an operator to preserve psd-stability of a polynomial but also connects psd-stability with the notion of stability since for a diagonal matrix having a positive imaginary part is equivalent to all diagonal variables being in $\mathcal{H}$.

The subsequent Corollary may be deduced by exploiting Lemma 4.6 a). This Corollary will have a deeper meaning for later structural and combinatorial results on psd-stable polynomials since we are able to use results for stable polynomials due to the connection established here.

Corollary 4.7. [32, Corollary 4.2] *Let $f \in \mathbb{C}[Z]$ be psd-stable. Then:*

a) *The polynomial $(z_{11}, z_{22}, \ldots, z_{nn}) \mapsto f_{\mathrm{Diag}}(Z)$ is stable in $\mathbb{C}[z_{11}, z_{22}, \ldots, z_{nn}]$.*

b) *If $f(0) = 0$, i.e., if f does not have a constant term, then there is a monomial in f consisting only of diagonal variables of Z.*

c) *If f is homogeneous, then*
   - c1) *the sum of the coefficients of all diagonal monomials of f is non-zero.*
   - c2) *all non-zero coefficients of diagonal monomials of f have the same phase.*

While on the one hand, Corollary 4.7 c) gives two necessary conditions for the psd-stability of the underlying polynomial which were derived by applying Corollary 4.7 a) and using results for stable polynomials, Corollary 4.7 b), on the other hand, is not only an easy-to-check necessary condition for psd-stability but will be a pillar of further structural results presented in subsection 4.4.

A more advanced operation that also preserves psd-stability of a polynomial $f \in \mathbb{C}[Z]$ is given by the following inversion operator:

Theorem 4.8 (Psd-stability preservation under inversion)**.** [32, Theorem 4.3] *If $f(Z) \in \mathbb{C}[Z]$ is psd-stable, then the polynomial $\det(Z)^{\deg(f)} \cdot f(-Z^{-1})$ is psd-stable.*

The factor $\det(Z)^{\deg(f)}$ ensures that the result after applying the inversion operator is still a polynomial. Obviously, as $\det(Z)$ is psd-stable by Lemma 3.7, the same is true for $\det(Z)^{\deg(f)}$. Thus, it remains to show that $f(-Z^{-1})$ is non-zero if $\mathsf{Im}(Z) \in \mathcal{S}_n^{++}$. To show this, note that for any eigenvalue λ of $C = A + \mathrm{i}B$ with $B \succ 0$ and a corresponding eigenvector $\mathbf{v}$, we have

$$\lambda = \frac{\mathbf{v}^H \lambda \mathbf{v}}{\mathbf{v}^H \mathbf{v}} = \frac{\mathbf{v}^H A \mathbf{v}}{\mathbf{v}^H \mathbf{v}} + \mathrm{i}\frac{\mathbf{v}^H B \mathbf{v}}{\mathbf{v}^H \mathbf{v}}.$$

Since $B \succ 0$, we get $\lambda \in \mathcal{H}$ and thus, C is invertible. As a consequence of this, we derive the following lemma which enables us to finally prove Theorem 4.8.

Lemma 4.9. [32, Lemma 4.6] *Let $A, B \in \mathcal{S}_n$ with $B \succ 0$. Then the symmetric matrix $C := A + iB$ is invertible and the imaginary part matrix of C^{-1} is negative definite.*

Our final result of this subsection concerns the preservation of psd-stable polynomials when passing over to an initial form.

For $f = \sum_{\alpha \in S} c_\alpha Z^\alpha \in \mathbb{C}[Z]$, the initial form of f is defined with respect to some functional W in the dual space $(\mathcal{S}_n)^*$. It is defined as

$$\mathrm{in}_W(f) = \sum_{\alpha \in S_W} c_\alpha Z^\alpha,$$

where $S_W := \{\alpha \in S : \langle W, \alpha \rangle_F = \max_{\beta \in S} \langle W, \beta \rangle_F\}$ and $\langle \cdot, \cdot \rangle_F$ is the Frobenius product. The following example shows that passing over to the initial form for general $W \in (\mathcal{S}_n)^*$ does not preserve psd-stability.

Example 4.10. [32, Example 4.8] Let f be given by the determinant of the symmetric matrix variables Z of size 3, i.e.

$$f(Z) = \det \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{12} & z_{22} & z_{23} \\ z_{13} & z_{23} & z_{33} \end{pmatrix} = z_{11}z_{22}z_{33} + 2z_{12}z_{13}z_{23} - z_{11}z_{23}^2 - z_{22}z_{13}^2 - z_{33}z_{12}^2.$$

Clearly, f is psd-stable by Lemma 3.7. Taking the initial form w.r.t.

$$W = \frac{1}{4} \begin{pmatrix} 4 & 1 & 6 \\ 1 & 4 & 6 \\ 6 & 6 & 0 \end{pmatrix},$$

we end up with $\mathrm{in}_W(f) = 2z_{12}z_{13}z_{23} - z_{11}z_{23}^2 - z_{22}z_{13}^2$. Obviously, $\mathrm{in}_W(f)$ vanishes for every diagonal matrix as every monomial contains at least one off-diagonal variable. Thus $\mathrm{i}I_3$ is a root of $\mathrm{in}_W(f)$ which ensures that $\mathrm{in}_W(f)$ is not psd-stable.

As a consequence of Example 4.10, the natural question now arises of whether there are subsets S of the space of real symmetric matrices $\mathcal{S}_n$ such that passing over to the initial form with respect to $W \in S$ preserves psd-stability. For $\lambda > 0$ and matrices $W \in \mathcal{S}_n$, let λ^W denote the operation given by $(\lambda^W)_{ij} := \lambda^{w_{ij}}$. Furthermore, for two matrices $A, B \in \mathcal{S}_n$ let $A \circ B$ denote the *Hadamard product* of A and B with $(A \circ B)_{ij} = a_{ij} \cdot b_{ij}$.

Lemma 4.11. [32, Lemma 4.9] *Let $f \in \mathbb{C}[Z]$ be psd-stable and let $W \in \mathcal{S}_n$ be such that there exists some $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$, λ^W is positive definite. Then $\mathrm{in}_W(f)$ is psd-stable.*

Lemma 4.11 is proven by constructing a suitable sequence of psd-stable polynomials which converge to the initial form of f due to the application of the fact that the Hadamard product preserves the positive-definitness of its arguments. Then the claim follows by the Hurwitz' Theorem 2.8.

Lemma 4.11 enables us to show that the psd-stability of a given polynomial is preserved when passing over to the initial form w.r.t. real positive definite matrices as formalized in the following theorem.

Theorem 4.12. [32, Theorem 4.10] *Let $f \in \mathbb{C}[Z]$ be psd-stable and $W \in \mathcal{S}_n$ be positive definite, then $\mathrm{in}_W(f)$ is psd-stable.*

For the proof of Theorem 4.12, let $W^{\circ k}$ denote the k-fold Hadamard product of W with the convention that $W^{\circ 0}$ is the all-ones matrix. Then $\exp[W] := \sum_{k=0}^{\infty} \frac{W^{\circ k}}{k!}$ is positive definite if W is positive definite. Thus the claim follows by Lemma 4.11 due to

$$\exp[\ln(\lambda) \cdot W] = (e^{w_{ij} \ln(\lambda)})_{ij} = (\lambda^{w_{ij}})_{ij} = \lambda^W \succ 0$$

for $W \succ 0$ and $\lambda > 1$.

Along the way to Theorem 4.12, we slightly generalized Theorem 2.11, which concerns the preservation of stable polynomials when passing over to the initial form, to the case of complex polynomials as follows.

Theorem 4.13. [32, Theorem 2.5] *Let $f \in \mathbb{C}[\mathbf{z}]$ be stable and $\mathbf{w} \in (\mathbb{R}^n)^* \setminus \{\mathbf{0}\}$. Then $in_{\mathbf{w}}(f)$ is stable.*

Theorem 4.13 has been shown independently in [68, Proposition 2.6]. The proof of Theorem 4.13 is similar to the proof of Lemma 4.11 as it also consists of the construction of a sequence of stable polynomials converging to the initial form in order to apply Hurwitz' Theorem 2.8.

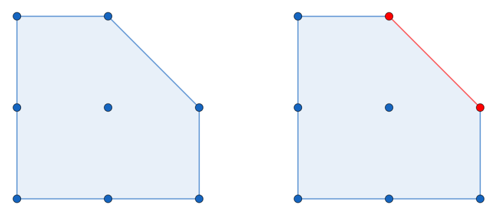


FIGURE 2. The left figure shows the support and Newton-Polytope of $f(z_1, z_2) = (z_1+1)(z_2+1)(z_1+z_2+1) = z_1^2 z_2 + z_1^2 + z_1 z_2^2 + 3z_1 z_2 + 2z_1 + z_2^2 + 2z_2 + 1$, which is stable by Theorem 2.16. The right figure shows the support and Newton-Polytope of $\mathrm{in}_{\mathbf{w}}(f)$ for $\mathbf{w} = (1, 1)$ in red.

4.4. **Combinatorial criteria for psd-stability.** In this subsection we consider structural results for psd-stable polynomials which then will lead to combinatorial criteria for the support of psd-stable polynomials as well as results for specific classes of psd-stable polynomials. First of all, we state the Structure Theorem (Theorem 4.14) which mirrors results concerning the completion problem of positive (semi-)definite real symmetric and Hermitian complex matrices [33, 70] naturally since psd-stability is connected to these concepts.

Theorem 4.14 (Structure Theorem). [32, Theorem 5.1] *Let $f \in \mathbb{C}[Z]$ be psd-stable. If an off-diagonal variable z_{ij} $(i < j)$ occurs in f, then the corresponding diagonal variables z_{ii} and z_{jj} must also occur in f.*

4.4.1. *Psd-stable binomials.* The Structure Theorem has immediate consequences for psd-stable binomials whose monomials do not share a factor. They either consist of two diagonal monomials or a diagonal and an off-diagonal monomial. Further, a possible common factor of the monomials of a psd-stable binomial should be a diagonal monomial, since the Structure Theorem applies to it as it may be considered a psd-stable monomial on its own. Further investigations lead to the main result of this subsection: The full classification of the support of psd-stable binomials analogous to the classification of stable binomials given by Rincón, Vinzant and Yu [96].

Theorem 4.15. [32, Theorem 5.5] *Every psd-stable binomial is of one of the following forms:*

a) Only diagonal variables appear in f and f satisfies the conditions of Theorem 2.6: $f(Z) = Z^\gamma(c_1 Z^{\alpha_1} + c_2 Z^{\alpha_2})$ with $|\alpha_1 - \alpha_2| \leq 2$ and at least one of α_1, α_2 is non-zero,

b) $f(Z) = Z^\gamma(c_1 z_{ii} z_{jj} + c_2 z_{ij}^2)$ with $i < j$ and $\frac{c_1}{c_2} \in \mathbb{R}$,

where $c_1, c_2 \neq 0$ and Z^γ is a diagonal monomial.

Note that Theorem 4.15 a) exactly mirrors Theorem 2.6 for stable binomials since a psd-stable polynomial $f \in \mathbb{C}[Z]$ may be considered as a stable polynomial if only diagonal variables appear in f by Corollary 4.7. We have an appearance of off-diagonal variables in case b) only. Further, the appearance of off-diagonal variables is restricted to homogeneous binomials of degree 2 whose monomials do not have a common factor as appearances in other cases would contradict the Structure Theorem. The proof of the Classification Theorem 4.15 relies on the preservers of psd-stable polynomials given by Lemma 4.6 and the Structure Theorem. Those techniques are used to construct a root $S \in \mathcal{S}_n^{\mathbb{C}}$ with $\mathsf{Im}(S) \in \mathcal{S}_n^{++}$ given by

$$S = \begin{pmatrix} s+i & t & \cdots & t \\ t & s+i & \ddots & \vdots \\ \vdots & \ddots & \ddots & t \\ t & \cdots & t & s+i \end{pmatrix} \quad \text{with } s, t \in \mathbb{R} \tag{2}$$

for all binomials which do not belong to one of the two classes given by Theorem 4.15. This construction relied on the preservation of psd-stability under taking directional derivatives into directions of positive semi-definite matrices and the notion of non-mixed polynomials since binomials are not closed under taking such a derivative. To formalize this, we introduce the following notation (see also [32]):

For $1 \leq i \neq j \leq n$, let B_{ii} be the matrix which has 1 in entry (i,i) and zero otherwise, and let B_{ij} be the matrix which has $1/2$ in entry (i,j) and (j,i) and zero otherwise. Then, for a polynomial $f = \sum_\alpha c_\alpha Z^\alpha \in \mathbb{C}[Z]$ and its equivalent version $\tilde{f} = \sum_\alpha c_\alpha \prod_{k=1}^n z_{kk}^{\alpha_{kk}} \prod_{k<l} z_{kl}^{2\alpha_{kl}}$ in $\mathbb{C}[\{z_{kl} | 1 \leq k \leq l \leq n\}]$, we have the identities $\frac{\partial f}{\partial B_{ii}}\big|_{z_{lk}:=z_{kl}} = \frac{\partial \tilde{f}}{\partial z_{ii}}$ and $\frac{\partial f}{\partial B_{ij}}\big|_{z_{lk}:=z_{kl}} = \frac{1}{2}\frac{\partial \tilde{f}}{\partial z_{ij}}$ as symbolic expressions. To see this, it suffices to observe that for $i < j$ and a monomial $f(Z) = z_{ij}^{\alpha_{ij}} z_{ji}^{\alpha_{ji}} \in \mathbb{C}[Z]$, we have $\tilde{f} = z_{ij}^{2\alpha_{ij}}$ and

$$\frac{\partial}{\partial B_{ij}} f(Z) = \frac{1}{2}\alpha_{ij} z_{ij}^{\alpha_{ij}-1} z_{ji}^{\alpha_{ji}} + \frac{1}{2}\alpha_{ji} z_{ij}^{\alpha_{ij}} z_{ji}^{\alpha_{ji}-1} \in \mathbb{C}[Z].$$

Substituting z_{ji} by z_{ij} gives $\frac{\partial}{\partial B_{ij}} f(Z)\big|_{z_{ji}:=z_{ij}} = \alpha_{ij} z_{ij}^{2\alpha_{ij}-1} = \frac{1}{2}\frac{\partial}{\partial z_{ij}}\tilde{f}$.

Now, consider the example $f(Z) = \det(Z) = z_{11}z_{22} - z_{12}^2$ and $V \in \mathcal{S}_2$ with all entries being 1. Then

$$\partial_V f(Z) = z_{11} + z_{22} - 2z_{12},$$

which shows that binomials are not closed under taking directional derivatives. Due to this, we invented the notion of non-mixed polynomials as a class of polynomials which contains binomials while still maintaining a structure similar to binomials, and being closed under taking directional derivatives.

Definition 4.16. We call a polynomial $f \in \mathbb{C}[Z]$ *non-mixed* if every monomial that occurs in f either consists only of diagonal variables or only of off-diagonal variables. We always write such a non-mixed polynomial as $f = \sum_{\alpha \in A} c_\alpha Z^\alpha + \sum_{\beta \in B} c_\beta Z^\beta$, where A refers to the exponent matrices of diagonal monomials and B refers to the exponent matrices of off-diagonal monomials.

Obviously, psd-stable binomials are psd-stable non-mixed polynomials and since those are closed under taking directional derivatives, they play an essential role in the construction of the roots given in (2).

4.4.2. *Polynomials of determinants.* In this subsection, we construct the polynomials of determinants as a class of psd-stable polynomials which generalizes stable polynomials in their own way. They take determinants of blocks $Z_1, \ldots, Z_k$ of symmetric matrix variables as arguments instead of univariate variables.

Definition 4.17. Suppose that the symmetric matrix of variables Z is a block diagonal matrix with blocks $Z_1, \ldots, Z_k$. Then

i) $f \in \mathbb{C}[Z]$ is a *polynomial of determinants* if f is of the form $f(Z_1, \ldots, Z_k) = \sum_\alpha c_\alpha \det(Z)^\alpha$, where $\det(Z)^\alpha$ is defined as $\det(Z)^\alpha = \det(Z_1)^{\alpha_1} \cdots \det(Z_k)^{\alpha_k}$. We say a polynomial of determinants $f(Z_1, \ldots, Z_k) = \sum_\alpha \det(Z)^\alpha$ is written in *standard form* if the largest possible determinantal monomial is factored out, i.e., $f(Z_1, \ldots, Z_k) = \det(Z)^\gamma \sum_\beta c_\beta \det(Z)^\beta = \det(Z)^\gamma \tilde{f}(Z)$, and all $c_\beta \neq 0$.

ii) Let $f(Z_1, \ldots, Z_k) = \sum_\alpha c_\alpha \det(Z)^\alpha$ be a polynomial of determinants. Then the *determinantal support* is defined by $\mathrm{supp}_{\det}(f) = \{\alpha \in \mathbb{Z}^k_{\geq 0} : c_\alpha \neq 0\}$.

Note that stable polynomials pose a special case of psd-stable polynomials of determinants: If Z is diagonal, i.e. if every block is of size 1, then psd-stable polynomials of determinants are stable in $\mathbb{C}[z_{11}, \ldots, z_{nn}]$ and the notion of the determinantal support equals the usual support. Due to this connection, the jump system result of Brändén can be generalized to the case of psd-stable polynomials of determinants.

Corollary 4.18. [32, Corollary 5.10] *Let $f(Z_1, \ldots, Z_k) = \sum_\alpha c_\alpha \det(Z)^\alpha$ be a psd-stable polynomial of determinants. Then the determinantal support of f forms a jump system.*

Corollary 4.18 is proven rather directly. By using the diagonalization for a psd-stable polynomial given by Lemma 4.6 a), we have a stable polynomial by Corollary 4.7 and thus, can apply the result of Brändén as stated in Theorem 2.4. The following theorem is our main result on psd-stable polynomials of determinants and shows that their structure is even more evolved:

Theorem 4.19. [32, Theorem 5.11] *Let $f(Z_1, \ldots, Z_k) = \det(Z)^\gamma \sum_{\beta \in B} c_\beta \det(Z)^\beta = \det(Z)^\gamma \tilde{f}(Z)$ be a psd-stable polynomial of determinants in standard form. Then any block Z_i appearing in $\tilde{f}$ (that is, any Z_i such that there is $\beta \in B$ with $\beta_i > 0$) has size $d_i \leq 2$.*

Further, for any matrix Z_i which has size exactly 2, let $C_i = \max_{\beta \in B} \beta_i$. Then if $\beta \in B$, then also $\beta + c\mathbf{e}_i \in B$ for all $-\beta_i \leq c \leq C_i - \beta_i$.

The first part is a consequence of the combination of Lemma 4.6 a), Corollary 4.7 and Theorem 2.4 since then blocks of size greater than 2 in $\tilde{f}$ would lead to distances of greater than 2 in the support of $\tilde{f}_{\mathrm{Diag}}$. The second result is derived by a close observation of points required in the support of $\tilde{f}_{\mathrm{Diag}}$ such that taking at most 2 consecutive steps brings us back into the support of $\tilde{f}_{\mathrm{Diag}}$.

4.4.3. *Support of general psd-stable polynomials.* The support of a stable polynomial forms a jump system by Theorem 2.4. The connection established by Lemma 4.6 a) and Corollary 4.7 implies various combinatorial results for specific classes of psd-stable polynomials. Thus, it is natural to ask for a general combinatoric framework for the support of psd-stable polynomials. Obviously, the jump system property does not hold as the monomials of $f(Z) = \det(Z) = z_{11}z_{22} - z_{12}^2$ have a distance of 4 already. Therefore we need a new notion of steps specifically designed for the case of psd-stability. Let us express usual steps from the jump system property as follows: We may call them *linear* or *double* steps and instead of considering them as points in $\mathbb{Z}^n$ with norm 1, we consider them as the multiplication of a monomial Z^α with $z_{ij}^{\pm 1}$ or $z_{ij}^{\pm 1}z_{kl}^{\pm 1}$. Additionally, we define a *transposition step* as the multiplication of a monomial Z^α with $z_{ij}z_{kl}z_{ik}^{-1}z_{jl}^{-1}$ for some indices $i, j, k, l \in [n]$. Note that for $f(Z) = z_{11}z_{22} - z_{12}^2$ its exactly such a transposition step between the two monomials of f. This is no coincidence as the following Lemma shows:

Lemma 4.20. [32, Lemma 5.12] *Any two monomials in the support of the symmetric determinant* $\det(Z)$ *are linked by a sequence of transposition steps decreasing the distance between the monomials which never leave the support.*

Thus, in the following we combine the structures found: On one side, diagonal monomials of a psd-stable polynomial $f \in \mathbb{C}[Z]$ are connected by the jump system result of Theorem 2.4 due to Lemma 4.6 a), while on the other side, non-diagonal monomials seem to respect some kind of determinantal structure similar to Lemma 4.20. This gives rise to the following conjecture:

Conjecture 4.21. [32, Conjecture 5.13] *For any monomial Z^β appearing in a psd-stable polynomial, there is a diagonal monomial Z^α appearing in f which can be reached by a sequence of linear, double and transposition steps which decrease the distance from β to α and which never leave the support of f.*

As a foundation for Conjecture 4.21, we have proven Conjecture 4.21 for the classes of psd-stable binomials, psd-stable non-mixed polynomials and psd-stable polynomials of determinants. Further, the conjecture holds for psd-stable lpm polynomials. Those have been introduced in [14] as polynomials of the form $f(Z) = \sum_{J \subseteq [n]} c_J \det(Z_J)$, where Z_J denotes the square submatrix of Z induced by $J \subseteq [n]$.

4.5. **Conclusion.** Altogether, we found several preservers for K-stable polynomials for a general cone K (Lemma 4.1 and Lemma 4.2) as well as for the special case of psd-stability (Lemma 4.6 and Theorem 4.8). Furthermore, we showed that passing over to the initial form preserves stability and psd-stability (Theorem 4.12 and Theorem 4.13). We further developed several structural results for psd-stable polynomials, namely the Structure Theorem 4.14, the binomial classification given by Theorem 4.15 as well as combinatoric results for polynomials of determinants in Theorem 4.19, and finally stated our combinatorial Conjecture 4.21. This constitutes a good starting point for more in-depth studies into operators that preserve K-stability and possibly also into a characterization of operators that preserve K-stability analogous to the one for stable polynomials given by Theorem 2.9. We have already generalized one of the key-ingredients of Theorem 2.9, the Lieb-Sokal Lemma, to the conic case.

The class of psd-stable polynomials appears to be rather restricted since the currently known classes of psd-stable polynomials seem to be extremely dense as determinants and lpm polynomials or very restrictive as irreducible non-mixed homogeneous psd-stable polynomials, which are of degree at most 2 with a very special selection of allowed monomials. It would be interesting to see whether every psd-stable polynomial already belongs to one of the classes discussed in this section, or if there are others that extend the amount of classes of psd-stable polynomials. If all psd-stable polynomials belonged to the classes we discussed, this would automatically prove our Conjecture 4.21 as well. In any case, it will be interesting to see whether our Conjecture 4.21 turns out to be true in the end.

5. Imaginary projections: Complex versus real Coefficients

5.1. **Related work.** This work [42, see also Appendix B] was co-written with M. Sayyary and T.Theobald. We were inspired by the classification of the imaginary projection of real conics developed by Jörgens, Theobald and de Wolff in [62]. Our first attempt to study the imaginary projection of complex conics tried to use the classification of complex conics given by Newstead in [84]. It turned out to be unpractical as the operations he used to derive the classes do not align with the action of $G_n = \mathbb{C}^n \rtimes \mathrm{GL}_n(\mathbb{R})$ on complex conics, which preserves the topology of the underlying conic's imaginary projection. Since the imaginary projection of a polynomial $f \in \mathbb{C}[\mathbf{z}]$ is closely connected to the imaginary projection of its initial form, we invented our own classification on conics based on the roots of the initial form and described the imaginary projection for the normal forms of each class. Furthermore, we investigated whether the complement components of the imaginary projection are spetrahedral and thus added results towards the generalized Lax Conjecture [6, 7, 53, 67, 82]. Additionally, we gave realization results for the occurrence of a specific amount of strictly convex components in the imaginary projection's complement which is of immense interest when considering imaginary projections under the light of general convexity phenomena in algebraic geometry [35, 39, 43, 80, 87, 89, 90].

5.2. **The main classification of complex conics.** In this subsection we focus on the classification of complex conics we developed while looking for a classification of complex conics such that the orbit of any complex conic under the action of G_2 belongs to one class completely. It turns out that a classification based on the initial form of the underlying conic satisfies this condition. For a complex conic $f \in \mathbb{C}[z_1, z_2]$, its initial form $\mathrm{in}(f)$ has its roots in $\mathbb{P}^1$ and we classify the complex conics by the structure of the roots which may be grouped into two main cases which depend on whether the complex conic has a hyperbolic initial form or not. Thus, the classification distinguishes between the following 5 cases:

Hyperbolic initial form	Non-hyperbolic initial form
(1a) A double real root	(2a) A double non-real root
(1b) Two distinct real roots	(2b) One real and one non-real root
	(2c) Two distinct non-real roots

Note that the real dimension of G_n is given by $n^2 + 2n$. Thus, in the case of $n = 2$, it is 8. Since the real dimension of the set of complex conics is 10, there are infinitely many orbits of complex conics under the action of G_2 on the set of complex conics. In the following theorem we state a representative for every orbit.

Theorem 5.1 (**Normal Form Classification**)**.** [42, Theorem 5.5]
With respect to the group G_2, there are infinitely many orbits for the complex conic sections with the following representatives.

(1a) (1a.1) $f = z_1^2 + \gamma$
(1a.2) $f = z_1^2 + \gamma z_2$

(1b) $f = z_1 z_2 + \gamma$

(2a) (2a.1) $f = (z_1 - \mathrm{i}z_2)^2 + \gamma$
(2a.2) $f = (z_1 - \mathrm{i}z_2)^2 + \gamma z_2$

(2b) $f = z_2(z_1 - \alpha z_2) + \gamma$

(2c) (2c.1) $f = z_1^2 + z_2^2 + \gamma$
(2c.2) $f = (z_1 - \mathrm{i}z_2)(z_1 - \alpha z_2) + \gamma$

for some $\gamma, \alpha \in \mathbb{C}$ such that to avoid overlapping we assume $\gamma \neq 0$ in (1a.2) *and* (2a.2)*, $\alpha \notin \mathbb{R}$ in* (2b) *and* (2c.2)*, and finally $\alpha \neq \pm\mathrm{i}$ in* (2c.2)*.*

In the following we will refer to those representatives as *normal forms*. By studying the normal forms given by Theorem 5.1, we are able to derive the following topological properties of the complement of the imaginary projection of those normal forms: Letting G_2 act on the set of complex conics and using Lemma 3.12 and Lemma 3.13 we are able to extend those results to their respective classes.

Theorem 5.2 (**Topological Classification**)**.** [42, Theorem 5.1]
Let $f \in \mathbb{C}[z_1, z_2]$ be a complex conic. For the above five cases, the set $\mathcal{I}(f)^{\mathsf{c}}$ is

(1a) the union of one, two, or three unbounded components.

(1b) the union of four unbounded components.

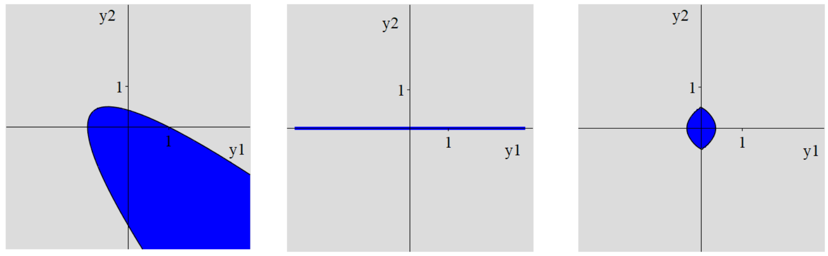
(2a) empty.

(2b) empty, a single point, or a line segment.

(2c) empty or one bounded component, possibly open.

In particular, the components of $\mathcal{I}(f)^{\mathsf{c}}$ are spectrahedral in all the first four classes. This is not true in general for the last class (2c)*.*

Note that Theorem 5.2 answers Question 3.17. In the case (2b) it is possible that the complement of the imaginary projection consists of a single point and thus, the imaginary projection is an open set which is not the full space in this case. This is not the only difference appearing compared to the case of real conics (see also Figure 3):



(1a) (2b) (2c)

FIGURE 3. The pictures show cases in the classification of the imaginary projection of complex conics which do not appear in the real case. The blue area represents the imaginary projection's complement.

In the case of real conics, it is not possible for the complement of the imaginary projection to consist of only one unbounded component. Also the appearance of a bounded component whose interior is empty in the complement of the imaginary projection is impossible for real conics. Further, Theorem 5.2 shows that the complement of the imaginary projection of a complex conic $f \in \mathbb{C}[z_1, z_2]$ is unbounded if and only if the initial form of f is hyperbolic. If f does not have a hyperbolic initial form, the complement of its imaginary projection is always bounded. Theorem 5.2 is connected to the generalized Lax Conjecture 2.15 since the unbounded components of the complement of the imaginary projection of a complex conic $f \in \mathbb{C}[z_1, z_2]$ in the cases (1a) and (1b) reflect hyperbolicity cones of the respective initial form $\mathrm{in}(f)$ of f by Theorem 3.10. Thus, the hyperbolicity cones of these polynomials are also spectrahedra by Theorem 5.2.

The proof of Theorem 5.2 is rather straightforward for the cases (1a) to (2b). For those classes we apply the following techniques: Since every complex variable $\mathbf{z}$ may be expressed as the sum of two real variables $\mathbf{x}, \mathbf{y}$ by $\mathbf{z} = \mathbf{x} + \mathrm{i}\mathbf{y}$, we also may express a complex polynomial $f \in \mathbb{C}[z_1, z_2]$ as $f(\mathbf{z}) = \mathsf{Re}(f)(\mathbf{x}, \mathbf{y}) + \mathrm{i} \cdot \mathsf{Im}(f)(\mathbf{x}, \mathbf{y})$ with $\mathsf{Re}(f), \mathsf{Im}(f) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$. Thus, instead of computing the imaginary projection of f directly, it suffices to determine all values $\mathbf{y}$ such that the polynomial system

$$\mathsf{Re}(f)(\mathbf{x}, \mathbf{y}) = 0 \quad \text{and} \quad \mathsf{Im}(f)(\mathbf{x}, \mathbf{y}) = 0$$

has a real solution for $\mathbf{x}$. We use this technique to compute $\mathcal{I}(f)$ for the normal forms of the classes (1a) to (2b) and extend the results to any complex conic belonging to the case (1a) to (2b) by applying Lemma 3.12 and Lemma 3.13. For class (2c), we show that the complement of the imaginary projection does not contain unbounded components and at most one bounded component in an indirect proof. We assume the existence of either at least two bounded or one unbounded component and lead those assumptions to a contradiction by using a continuity argument. Further, we give the example of the harmless-looking conic $f = z_1^2 + \mathrm{i} z_2^2 + \frac{\mathrm{i}}{4}$ which belongs to the class (2c) and has its imaginary projection given by

$$\mathcal{I}(f) = \{\mathbf{y} \in \mathbb{R}^2 : -64y_1^8 - 128y_1^4y_2^4 - 64y_2^8 - 80y_1^4y_2^2 + 48y_2^6 + y_1^4 - 12y_2^4 + y_2^2 \leq 0\} \setminus \{\mathbf{0}\}.$$

While in the case of a real conic $f \in \mathbb{R}[z_1, z_2]$, the boundary of the imaginary projection $\mathcal{I}(f)$ is of algebraic degree 2, this example shows that this is not true for the case of a complex conic $f \in \mathbb{C}[z_1, z_2]$ in general. In general, the boundary of the imaginary projection of a complex conic f is not necessarily algebraic and the algebraic degree of the irreducible components in the Zariski closure of $\partial\mathcal{I}(f)$ is not restricted by 2 but might rise to 8 which is a sharp bound due to the example above.

5.3. **Higher dimensional complex quadratics.** Already the case of complex conics shows that the structure of the imaginary projection of $f \in \mathbb{C}[z_1, z_2]$ is well-behaved if the initial form of f is hyperbolic. This extends to complex quadratics in arbitrarily many variables $\mathbf{z} = (z_1, ..., z_n)$. Thus, we compute the normal forms of every quadratic $f \in \mathbb{C}[\mathbf{z}]$ with hyperbolic initial form. This leads to the following Lemma:

Lemma 5.3. [42, Lemma 4.1] *Under the action of G_n, any quadratic polynomial $f \in \mathbb{C}[\mathbf{z}]$ with hyperbolic initial form can be transformed to one of the following normal forms:*

(1) $z_1^2 + \alpha z_2 + r z_3 + \gamma$,

(2) $\sum_{i=1}^{j} z_i^2 - z_{j+1}^2 + \alpha z_{j+2} + r z_{j+3} + \gamma$ *for some* $j = 1, \ldots, n-1$,

such that terms containing z_k do not appear for $k > n$, and $\alpha, r, \gamma \in \mathbb{C}$.

Based on the normal forms of Lemma 5.3, we are able to compute the imaginary projection for any quadratic polynomial $f \in \mathbb{C}[\mathbf{z}]$ with a hyperbolic initial form as follows:

Theorem 5.4. [42, Theorem 4.5] *Let $n \geq 3$ and $f \in \mathbb{C}[\mathbf{z}]$ be a quadratic polynomial with a hyperbolic initial form. Up to the action of G_n, the imaginary projection $\mathcal{I}(f)$ is either $\mathbb{R}^n$, $\mathbb{R}^n \setminus \{(0, \ldots, 0, y_n) \in \mathbb{R}^n : y_n \neq 0\}$, or otherwise we can write f as $f = \sum_{i=1}^{n-1} z_i^2 - z_n^2 + \gamma$ for some $\gamma \in \mathbb{C}$ such that $|\gamma| = 1$ and we get*

$$\mathcal{I}(f) = \begin{cases} \left\{\mathbf{y} \in \mathbb{R}^n \ : \ y_n^2 < \sum_{i=1}^{n-1} y_i^2\right\} \cup \{\mathbf{0}\} & \text{if } \gamma = 1, \\ \left\{\mathbf{y} \in \mathbb{R}^n \ : \ y_n^2 - \sum_{i=1}^{n-1} y_i^2 \leq 1\right\} & \text{if } \gamma = -1, \\ \left\{\mathbf{y} \in \mathbb{R}^n \ : \ y_n^2 - \sum_{i=1}^{n-1} y_i^2 \leq \frac{1}{2}(1 - \mathsf{Re}(\gamma))\right\} \setminus \{\mathbf{0}\} & \text{if } \gamma \notin \mathbb{R}. \end{cases}$$

For the proof, we again apply the methods we used for the case of complex conics before. We divide the normal forms given by Lemma 5.3 into 5 cases based on the parameters α, γ, r which appear in Lemma 5.3 and compute the imaginary projection for the normal forms given in Lemma 5.3. Instead of computing the imaginary projection of $f \in \mathbb{C}[\mathbf{z}]$ directly, we determine all $\mathbf{y} \in \mathbb{R}^n$ such that the system $\mathsf{Re}(f)(\mathbf{x}, \mathbf{y}) = 0$, $\mathsf{Im}(f)(\mathbf{x}, \mathbf{y}) = 0$ has a real solution for $\mathbf{x}$. In one of the cases it is necessary to apply an orthogonal transformation in order to succeed.

Surprisingly, the boundary of the imaginary projection of a complex quadratic $f \in \mathbb{C}[\mathbf{z}]$ in n variables with hyperbolic initial form is better behaved than complex conics whose initial form is not hyperbolic. A complex quadratic with hyperbolic initial form always has an imaginary projection such that the irreducible components of its boundary are of algebraic degree 2. Based on the normal forms given by Theorem 5.4, we are able to describe the topology of the imaginary projection of general quadratic polynomials with hyperbolic initial form as stated in the following Corollary:

Corollary 5.5. [42, Corollary 4.6] *Let $f \in \mathbb{C}[\mathbf{z}]$ be a quadratic polynomial with hyperbolic initial form. Then*

(1) *the complement $\mathcal{I}(f)^{\mathsf{c}}$ is either empty or it consists of*
 - *one, two, three, or four unbounded components,*
 - *two unbounded components and a single point.*

(2) *the complement of the closure $\overline{\mathcal{I}(f)}^{\mathsf{c}}$ is either empty or unbounded.*

(3) *the algebraic degrees of the irreducible components in $\partial\mathcal{I}(f)$ are at most two.*

5.4. **Further results.** Our techniques may not be applied to complex polynomials $f \in \mathbb{C}[\mathbf{z}]$ of higher degree in general. Already the polynomial $f = z_1^3 + z_2^3 - 1$ leads to a system $\mathsf{Re}(f)(\mathbf{x}, \mathbf{y}) = 0$, $\mathsf{Im}(f)(\mathbf{x}, \mathbf{y}) = 0$ where both x_1 and x_2 appear with degree higher than 1 and thus, it can not be solved efficiently by our methods. Although our methods do not apply in general, we were still able to find a class of bivariate polynomials whose imaginary projection is the full space $\mathbb{R}^2$.

Theorem 5.6. [42, Theorem 3.4] *Let $f \in \mathbb{C}[z_1, z_2]$ be a complex bivariate polynomial of total degree d such that its initial form has no real roots in $\mathbb{P}^1$. If d is odd then the imaginary projection $\mathcal{I}(f)$ is $\mathbb{R}^2$.*

For the proof we use our methods to divide $f(\mathbf{z}) = 0$ into a system of real equations and make a suitable substitution to solve it.

We want to conclude this subsection with a complex analogue to Theorem 3.11. By going over to complex polynomials, we may break the symmetry of the imaginary projection with respect to the origin (see [61, Remark 3.5]) and therefore gain more flexibility when constructing a polynomial with a specific structure in its imaginary projection. For $k > 0$, we use a cyclic construction to build up a bivariate polynomial $f \in \mathbb{C}[z_1, z_2]$ such that there are exactly k strictly convex, bounded components in $\mathcal{I}(f)^{\mathsf{c}}$.

Theorem 5.7. [42, Theorem 7.1] *For any $k > 0$ there exists a polynomial $f \in \mathbb{C}[z_1, z_2]$ of degree $2\lceil \frac{k}{4} \rceil + 2$ such that $\mathcal{I}(f)^{\mathsf{c}}$ consists of exactly k strictly convex bounded components.*

5.5. **Conclusion.** We invented a new classification for complex conics and gave a full description of the structure of the imaginary projection for each class of this classification. Furthermore, we extended our results to the cases of families of bivariate complex polynomials with an arbitrary degree as well as complex quadratics with hyperbolic initial form in arbitrarily many variables. Thus, the variety of polynomials whose imaginary projection's structure is known was extended significantly due to our research. Still, this is just the tip of the iceberg and the structure of the imaginary projection of many complex polynomials remains unknown. Even for the cases of bivariate cubics it seems to require completely new methods in order to come up with a classification of their imaginary projections. Since we encountered curves of algebraic degree 8 in the boundary of the imaginary projection in the case of complex conics already, it is to be expected that even higher degrees appear for irreducible components of the boundary of the imaginary projection for complex bivariate cubics. We experienced that quadratic polynomials with hyperbolic initial form behaved better than those whose initial form is not hyperbolic. Thus, we expect that this will be the same for polynomials of higher degrees and therefore focusing on the imaginary projection of bivariate cubics with hyperbolic initial form seems to be a reasonable next step.

The final question concerns Theorem 5.7. Our construction method probably increased the degree of the required polynomial in Theorem 5.7 artificially. Thus, it would be interesting to know whether it is possible to realize the k strictly convex bounded components with a polynomial of degree lower than $2\lceil \frac{k}{4} \rceil + 2$.

6. Conic stability of polynomials and positive maps

6.1. **Related work.** This work [36, see also Appendix A] was co-written with P. Dey and T. Theobald. We were inspired by the various necessary and sufficient conditions that are known for stable polynomials or polynomials with the half-plane property (see for example [20, 104] or or Subsection 2.1). We successfully attempted to combine results about the connection of the K-stability of a polynomial $f \in \mathbb{C}[\mathbf{z}]$ and its hyperbolicity cones given by Jörgens and Theobald [61] with positive maps as they preserve spectrahedrality under certain conditions [64]. In order to exploit this, we first needed to determine conic components with a spectrahdral description in the complement of the imaginary projection $\mathcal{I}(f)$ of a given polynomial which poses a question that is closely connected to the generalized Lax Conjecture [6, 7, 53, 67, 82]. Then, as there are spectrahedral descriptions known for most of the common cones, we build up a semi-definite program based on those spectrahedral descriptions. In case of feasibility of the semi-definite program, we get a positive semi-definite block matrix C, the so called *Choi matrix*, which then certifies $K \subseteq \mathcal{I}(f)^{\mathsf{c}}$. This implies the K-stability of f and enables us to determine further determinantal representations which pose a research field on their own [52, 53, 93].

6.2. **Conic components in the complement of the imaginary projection.** In this subsection we focus on determining conic components in the complement of $\mathcal{I}(f)$ for a given polynomial $f \in \mathbb{C}[\mathbf{z}]$. We focus on polynomials with a determinantal representation and quadratic polynomials. In this section we assume the cone K to be convex, closed and full-dimensional. We begin with determinantal polynomials, i.e. we consider polynomials of the form

$$f(\mathbf{z}) = \det\left(A_0 + A_1 z_1 + \ldots + A_n z_n\right) \tag{3}$$

with $A_0, \ldots, A_n$ being Hermitian matrices of size $d \times d$. The homogeneous determinantal polynomial $f(\mathbf{z}) = \det\left(A_1 z_1 + \ldots + A_n z_n\right)$ is hyperbolic with respect to some $\mathbf{e} \in \mathbb{R}^n$ if $A_1 e_1 + \ldots + A_n e_n \succ 0$ holds. Further the set

$$\{\mathbf{z} \in \mathbb{R}^n : A_1 z_1 + \ldots + A_n z_n \succ 0\}$$

as well as its negative are hyperbolicity cones of f [72, Prop. 2]. If f is irreducible, then these are the only two hyperbolicity cones (see [69]). If f is not irreducible, then there are more, as all irreducible factors of f are hyperbolic and therefore the amount of hyperbolicity cones rises. The following theorem establishes a connection between the hyperbolicity cones of $\mathrm{in}(f)$ and $\mathcal{I}(f)$.

Theorem 6.1. [36, Theorem 3.1] *If f is a degree d determinantal polynomial of the form* (3) *and there exists an $\mathbf{e} \in \mathbb{R}^n$ with $\sum_{j=1}^n A_j e_j \succ 0$, then $\mathrm{in}(f)$ is hyperbolic and every hyperbolicity cone of $\mathrm{in}(f)$ is contained in $\mathcal{I}(f)^{\mathsf{c}}$.*

The proof of Theorem 6.1 first assumes irreducibility of $\mathrm{in}(f)$ and then shows that the two hyperbolicity cones of $\mathrm{in}(f)$, namely

$C_1 = \{\mathbf{x} \in \mathbb{R}^n : A_1x_1 + \ldots + A_nx_n \succ 0\}$ and $C_2 = \{\mathbf{x} \in \mathbb{R}^n : A_1x_1 + \ldots + A_nx_n \prec 0\}$,

are contained in $\mathcal{I}(f)^{\mathsf{c}}$. This is done by expressing the univariate polynomial $t \mapsto f(\mathbf{x} + t\mathbf{e})$ as the characteristic polynomial of a Hermitian matrix which ensures that it is real-rooted. The reducible case is covered as the hyperbolicity cones of $\mathrm{in}(f)$ are intersections of the hyperbolicity cones of the irreducible factors of $\mathrm{in}(f)$ then.

Now, we consider quadratic polynomials of the form

$$f(\mathbf{z}) = \mathbf{z}^T A\mathbf{z} + \mathbf{b}^T\mathbf{z} + c \tag{4}$$

with $A \in \mathcal{S}_n,\ \mathbf{b} \in \mathbb{R}^n,\ c \in \mathbb{R}$. Our goal here is to determine the cases which give rise to spectrahedral components in the complement of $\mathcal{I}(f)$. Starting with the homogeneous case $f = \mathbf{z}^T A\mathbf{z}$, it is well-known that f is hyperbolic if and only if A or $-A$ has signature $(n-1, 1)$ [45]. We assume that A has signature $(n-1, 1)$ by possibly multiplying A with -1. By Theorem 3.16, we know that the normal form $f(\mathbf{z}) = \sum_{j=1}^{n-1} z_j^2 - z_n^2$ has its imaginary projection given by $\mathcal{I}(f) = \left\{\mathbf{y} \in \mathbb{R}^n : y_n^2 \leq \sum_{j=1}^{n-1} y_j^2\right\}$. Therefore, we derive the two full-dimensional cones given by

$$\left\{\mathbf{y} \in \mathbb{R}^{n-1} \times \mathbb{R}_{>0} : \sum_{j=1}^{n-1} y_j^2 < y_n^2\right\} \text{ and } \left\{\mathbf{y} \in \mathbb{R}^{n-1} \times \mathbb{R}_{<0} : \sum_{j=1}^{n-1} y_j^2 < y_n^2\right\}$$

as the only two unbounded components in $\mathcal{I}(f)^{\mathsf{c}}$. Due to Lemma 3.12 we know that the identity $\mathcal{I}(f(T\mathbf{z})) = T^{-1}\mathcal{I}(f(\mathbf{z}))$ holds for an invertible transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ and the imaginary projection of a polynomial $f \in \mathbb{C}[\mathbf{z}]$. Thus, by applying a suitable transformation, we get the following result for general homogeneous quadratic polynomials:

Lemma 6.2. [36, Lemma 3.2] *For a quadratic form $f = \mathbf{z}^T A\mathbf{z} \in \mathbb{R}[\mathbf{z}]$ with A having signature $(n-1, 1)$, the components C in the complement of $\mathcal{I}(f)$ are given by the two components of the set*

$$\{\mathbf{y} \in \mathbb{R}^n \ : \ \mathbf{y}^T A\mathbf{y} < 0\}, \tag{5}$$

and the closures of these components are spectrahedra.

The homogeneous case (type (I) in (1)) has been treated already. It remains to consider the non-homogeneous case. By Theorem 3.16 we know that there are unbounded components in the complement of the imaginary projection of quadratic polynomials of type (II) in (1) if either $p = 1$ or $p = r - 1$. Quadratic polynomials of type (III) do not contain unbounded components in the complement of their imaginary projection. By Theorem 3.16 and [36, Theorem 3.4] we know that the imaginary projections of the normal forms for the two relevant cases are given by

$$\mathcal{I}(f) \ = \ \begin{cases} \{\mathbf{y} \in \mathbb{R}^n \ : \ y_1^2 - \sum_{j=2}^{r} y_j^2 \leq 1\} & \text{if } p = 1\,, \\ \{\mathbf{y} \in \mathbb{R}^n \ : \ \sum_{j=1}^{r-1} y_j^2 > y_r^2\} \cup \{\mathbf{0}\} & \text{if } p = r - 1\,. \end{cases}$$

For a polynomial $f \in \mathbb{C}[\mathbf{z}]$ there is a bijection between the set of unbounded components of $\mathcal{I}(f)^{\mathsf{c}}$ with full-dimensional recession cone and hyperbolicity cones of $\mathrm{in}(f)$.

In the real cases $\mathcal{I}(f)^{\mathsf{c}}$ is empty if $\mathrm{in}(f)$ is not hyperbolic. Due to these facts, we were able to derive the following essential lemma:

Lemma 6.3. [36, Lemma 3.5] *Let $n \geq 3$ and $f \in \mathbb{R}[\mathbf{z}]$ be quadratic of the form* (4).

If f is of type (II) with $p = 1$, then $\mathcal{I}(f)^{\mathsf{c}}$ does not have connected components whose closures contain full-dimensional cones.

If f is of type (II) with $p = n-1$ then every full-dimensional cone which is contained in $\mathcal{I}(f)^{\mathsf{c}}$ is contained in the closure of a hyperbolicity cone of $\mathrm{in}(f)$.

Finally, we are able to actually complete the description of conic components of quadratic polynomials by stating the following theorem:

Theorem 6.4. [36, Theorem 3.6] *Let $n \geq 3$ and $f \in \mathbb{R}[\mathbf{z}]$ be quadratic of the form* (4) *and of type (II) with $p = n - 1$. Then there exists a linear form $\ell(\mathbf{z})$ in $\mathbf{z}$ such that $-\ell(\mathbf{z})^{n-2}\mathrm{in}(f)$ has a determinantal representation. In particular, the closure of each unbounded component of $\mathcal{I}(f)^{\mathsf{c}}$ is a spectrahedral cone.*

We use earlier considerations for the proof of Theorem 6.4 as the initial form of the normal form $g(\mathbf{z}) = \sum_{j=1}^{n-1} z_j^2 - z_n^2 + 1$, given by $\mathrm{in}(g) = \sum_{j=1}^{n-1} z_j^2 - z_n^2$, is a quadratic polynomial of type (I) of course. The two unbounded components of $\mathcal{I}(\mathrm{in}(g))$ are given by the Lorentz cone and its negative which are both spectrahedral cones. For any polynomial f of type (II) with $p = n-1$, it is just a linear transformation $T \in \mathrm{GL}_n(\mathbb{R})$ from $\mathrm{in}(f)$ to $\mathrm{in}(g)$. The claimed linear form then is given by the n-th entry of $T\mathbf{z}$, i.e. $l(\mathbf{z}) = (T\mathbf{z})_n$. Here, for $\mathrm{in}(f) = \mathbf{z}^T A\mathbf{z}$, T may be computed by an LDL^T decomposition of A.

Note that Theorem 6.4 is closely connected to the generalized Lax Conjecture 2.15 as it recovers the well-known fact that hyperbolicity cones defined by homogeneous quadratic polynomials f are spectrahedral [83]. Further, in comparison to other constructions of determinantal representations from Helton, McCullough and Vinnikov [52] or Quarez [93], our construction results in a determinantal representation of rather small size. We consider the following example for clarification:

Example 6.5. [36, Example 3.8] Consider $f(z_1, z_2, z_3, z_4) = -15z_1^2 - 12z_1z_4 + z_2^2 + z_3^2 = \mathbf{z}^T A\mathbf{z}$ with

$$A = \begin{pmatrix} -15 & 0 & 0 & -6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -6 & 0 & 0 & 0 \end{pmatrix}.$$

For $\ell(\mathbf{z}) = 4z_1 + 2z_4$, a representation from Theorem 6.4 is

$$-\ell(\mathbf{z})^2 \cdot f(\mathbf{z}) = \det \begin{pmatrix} 4z_1 + 2z_4 & 0 & 0 & z_1 + 2z_4 \\ 0 & 4z_1 + 2z_4 & 0 & z_2 \\ 0 & 0 & 4z_1 + 2z_4 & z_3 \\ z_1 + 2z_4 & z_2 & z_3 & 4z_1 + 2z_4 \end{pmatrix}.$$

Note that $-l(\mathbf{z})^2 \cdot f(\mathbf{z})$ is a polynomial of degree 4 in 4 variables and the sizes of the symmetric matrices in the determinantal description are 4×4. Although not directly applicable here, the construction of Quarez (see [93, Theorem 4.4]) would result in a determinantal description with symmetric matrices of size 30×30 for a polynomial of degree 4 in 4 variables.

6.3. **Conic stability and positive maps.** In this subsection we use the determinantal descriptions we derived for unbounded components of the complement of the imaginary projection earlier as an ingredient for our sufficient criterion that certifies K-stability in the case of a spectrahedral cone K. The criterion may be considered as an instance of the containment problem of spectrahedra [51, 64, 65]. We use the determinantal descriptions of the spectrahedral complement components together with a spectrahedral description of the cone K. Then, the containment of K in a conic component of the complement of the imaginary projection is certified by the existence of a positive map which maps the spectrahedral descriptions of both sets onto each other. We refer to [91] for further reading into positive maps.

Definition 6.6. Given two linear subspaces $\mathcal{U} \subseteq \mathsf{Herm}_k$ and $\mathcal{V} \subseteq \mathsf{Herm}_l$ (or $\mathcal{U} \subseteq \mathcal{S}_k$ and $\mathcal{V} \subseteq \mathcal{S}_l$), a linear map $\Phi : \mathcal{U} \to \mathcal{V}$ is called *positive* if $\Phi(U) \succeq 0$ for any $U \in \mathcal{U}$ with $U \succeq 0$.

The following proposition from [64] will clarify the connection between positive maps and the containment problem of spectrahedra. In order to actually state the proposition, we use the following notation: For the homogeneous linear matrix pencil $U(\mathbf{x}) = \sum_{j=1}^n U_j x_j$ and $V(\mathbf{x}) = \sum_{j=1}^n V_j x_j$ with hermitian matrices of size $k \times k$ and $l \times l$, respectively, we define the spectrahedra $S_U := \{x \in \mathbb{R}^n : U(\mathbf{x}) \succeq 0\}$, and $S_V := \{x \in \mathbb{R}^n : V(\mathbf{x}) \succeq 0\}$. Further, let $\mathcal{U} = \operatorname{span}(U_1, \dots, U_n) \subseteq \mathcal{S}_k$ and $\mathcal{V} = \operatorname{span}(V_1, \dots, V_n) \subseteq \mathcal{S}_l$.

Let $\Phi_{UV} : \mathcal{U} \to \mathcal{V}$ be the linear map defined by $\Phi_{UV}(U_i) := V_i,\ 1 \leq i \leq n$.

Proposition 6.7. [64, Theorem 4.3] *Let $U_1, \dots, U_n \in \mathsf{Herm}_k$ be linearly independent and $V_1, \dots, V_n \in \mathsf{Herm}_l$ (or, $U_1, \dots, U_n \in \mathcal{S}_k$ and $V_1, \dots, V_n \in \mathcal{S}_l$, respectively) and $S_U \neq \emptyset$. If the the semi-definite feasibility problem given by*

$$C = (C_{ij})_{i,j=1}^k \succeq 0 \text{ and } V_p = \sum_{i,j=1}^k (U_p)_{ij} C_{ij} \text{ for } p = 1, \dots, n \tag{6}$$

has a solution with Hermitian (respectively symmetric) matrix C, then $\mathcal{S}_U \subseteq U_V$.

Note that the linear mapping Φ_{UV} is well defined if $U_1, \dots, U_n$ are linearly independent. But since we do not necessarily need Φ_{UV} for our purpose, we do not need to satisfy this precondition. Thus, if we consider a polynomial $f \in \mathbb{C}[\mathbf{z}]$ such that there is at least one spectrahedral component in $\mathcal{I}(f)^{\mathsf{c}}$ together with a spectrahedral cone K, we may certify the K-stability of f by showing the existence of a block matrix $C \succ 0$ in (6), the so-called *Choi matrix*, which implicates the existence of a positive map Φ that maps the underlying pencils of those spectrahedra onto each other and thereby, certifies their containment. Determining C leads to a semi-definite feasibility problem which certifies the K-stability of f if its feasibility region is non-empty.

Since we have already seen spectrahedral descriptions for the unbounded components in the complement of the imaginary projection for several classes of polynomials in the previous subsection, we now state some spectrahedral descriptions for common cones in the following before we will finally state the main result of this section.

Subsequently, we will denote the linear matrix pencil of the cone K by $M(\mathbf{x}) = \sum_{j=1}^n M_j x_j$, i.e. $K = \{\mathbf{x} \in \mathbb{R}^n : M(\mathbf{x}) \succeq 0\}$. For the notion of the usual stability, K

is the positive orthant and thus, we may express K as the positive semi-definiteness region of the linear matrix pencil given by

$$M^{\geq 0}(\mathbf{x}) \;=\; \sum_{j=1}^{n} M_j^{\geq 0} x_j$$

with $M_j^{\geq 0} = E_{jj}$, where E_{ij} is the matrix with a one in position (i,j) and zeros elsewhere.

In the case of psd-stability, K is the cone of positive semi-definite matrices and can be expressed as a spectrahedron quite naturally by using the linear matrix pencil

$$M^{\mathrm{psd}}(X) \;=\; \sum_{i,j=1}^{n} M_{ij}^{\mathrm{psd}} x_{ij}$$

with symmetric matrix variables $X = (x_{ij})$ and $M_{ij}^{\mathrm{psd}} = \frac{1}{2}(E_{ij} + E_{ji})$, i.e., $M^{\mathrm{psd}}(X)$ is the matrix pencil $M^{\mathrm{psd}}(X) = (x_{ij})_{ij}$ in the symmetric matrix variables x_{ij}.

Now, we finally state our main result:

Theorem 6.8. [36, Theorem 4.3] *Let* $f = \det(A_0 + \sum_{j=1}^{n} A_j z_j)$ *with Hermitian matrices* $A_0, \ldots, A_n$ *be a degree* d *determinantal polynomial of the form* (3) *such that* $\mathrm{in}(f)$ *is irreducible and there exists* $\mathbf{e} \in \mathbb{R}^n$ *with* $\sum_{j=1}^{n} A_j e_j \succ 0$. *Let* $M(\mathbf{x}) = \sum_{j=1}^{n} M_j x_j$ *with symmetric* $l \times l$*-matrices be the pencil of the cone* K. *If there exists a Hermitian block matrix* $C = (C_{ij})_{i,j=1}^{l}$ *with blocks* C_{ij} *of size* $d \times d$ *and*

$$C = (C_{ij})_{i,j=1}^{l} \;\succeq\; 0, \quad \forall p = 1, \ldots, n \;:\; \sigma A_p \;=\; \sum_{i,j=1}^{l} (M_p)_{ij} C_{ij} \tag{7}$$

for some $\sigma \in \{-1, 1\}$, *then* f *is* K*-stable. Deciding whether such a block matrix* C *exists is a semi-definite feasibility problem.*

For the proof it suffices to note that $\mathrm{in}(f)$ is hyperbolic and thus, it is sufficient to show that K lies in the closure of one of its hyperbolicity cones which are given by the positive semi-definiteness regions of $A_+^h(\mathbf{x}) = \sum_{j=1}^{n} A_j x_j \succ 0$ as well as $A_-^h(\mathbf{x}) = \sum_{j=1}^{n} A_j x_j \prec 0$. Finally, the claim follows by applying the Khatri-Rao product. We have $A_+^h(\mathbf{x}) = \sum_{i,j=1}^{l} (M(\mathbf{x}))_{ij} C_{ij}$. Since $M(\mathbf{x})$ and C are positive semi-definite, this holds for their Khatri-Rao product

$$M(\mathbf{x}) * C \;:=\; ((M(\mathbf{x}))_{ij} \otimes C_{ij})_{i,j=1}^{l} \;=\; ((M(\mathbf{x}))_{ij} C_{ij})_{i,j=1}^{l}$$

due to Liu [74]. Altogether, since

$$A^h(\mathbf{x}) \;=\; (I \cdots I)(M(\mathbf{x}) * C) \begin{pmatrix} I \\ \vdots \\ I \end{pmatrix},$$

$A^h(\mathbf{x})$ is positive semi-definite as well. Thus, every $\mathbf{x} \in K$ also belongs to $\mathcal{I}(f)^{\mathsf{c}}$ and the K-stability of f is certified. The argument for A_-^h is analogous.

In the case of stable polynomials, Theorem 6.8 gives a new proof to Theorem 2.16 as the non-diagonal blocks of C vanish and thus, condition (7) specializes to $A_p = C_{pp}$ for all $p = 1, \ldots, n$. Thus, stability is certified if $A_1, ..., A_n$ are positive semi-definite.

Note that Theorem 6.8 is not a necessary criterion as may be seen by the following adapted example from [51, Example 3.1, 3.4] and [64, Section 6.1]. Consider the polynomial

$$f \;=\; \det\begin{pmatrix} z_1 + z_3 & z_2 \\ z_2 & -z_1 + z_3 \end{pmatrix} \;=\; z_3^2 - z_1^2 - z_2^2$$

whose underlying matrix pencil is also one of the matrix pencils defining the Lorentz cone. As such a polynomial its obviously stable with respect to the Lorentz cone but this can not be certified by Theorem 6.8.

In order to treat quadratic polynomials as well, we need the linear matrix pencil we derived for $f(\mathbf{z}) = \mathbf{z}^T A\mathbf{z}$ in the proof of Theorem 6.4, which is given by

$$F(\mathbf{z}) := \sum_{p=1}^{n} F_p x^p \;:=\; \left(\begin{array}{ccc|c} & & & (T\mathbf{z})_1 \\ & (T\mathbf{z})_n I & & \vdots \\ & & & (T\mathbf{z})_{n-1} \\ \hline (T\mathbf{z})_1 & \cdots & (T\mathbf{z})_{n-1} & (T\mathbf{z})_n \end{array}\right) \;\succ\; 0\,, \tag{8}$$

where T is as in that proof. Now, we may state our second main result for quadratic polynomials.

Theorem 6.9. [36, Theorem 4.7] *Let $n \geq 3$ and f be a quadratic polynomial of the form* (4)*, let f be of type (II) with A having signature* $(n-1, 1)$ *and* $\operatorname{in}(f)$ *be irreducible. Let $M(\mathbf{z})$ be a matrix pencil for the cone K, and let T and $F(\mathbf{z}) := \sum_{p=1}^{n} F_p z^p$ be defined as in* (8) *w.r.t.* $\operatorname{in}(f)$*. If there exists a block matrix $C = (C_{ij})_{i=1}^{l}$ with blocks C_{ij} of size $d \times d$ and*

$$C = (C_{ij})_{i,j=1}^{l} \;\succeq\; 0, \quad \forall p = 1, \ldots, n \;:\; \sigma F_p \;=\; \sum_{i,j=1}^{l} (M_p)_{ij} C_{ij} \tag{9}$$

for some $\sigma \in \{-1, 1\}$, then f is K-stable. Deciding whether such a block matrix C exists is a semi-definite feasibility problem.

The proof is similar to the proof of Theorem 6.8 and also involves the application of the Khatri-Rao product in the same manner. This technique may also be used to directly derive determinantal descriptions for the initial forms of quadratic polynomials which satisfy (9) (see [36, Theorem 4.8 and Corollary 4.9]).

The previously stated criteria do not capture all K-stable determinantal or quadratic polynomials and thus, we derived another criterion which applies to scaled versions of the cone in order to extend the range of captured polynomials. The key idea behind this approach is that if $\mathcal{I}(f)^{\mathsf{c}} \cap K \neq \emptyset$, then there should be a scaled version of K such that this scaled version is contained in $\mathcal{I}(f)^{\mathsf{c}}$. To actually find this scaled version, we intersect the cone K with a hyperplane H which does not pass the origin. The containment then follows due to the existence of a Choi-matrix for the semi-definite program which constitutes a scaled version of (6) and is based on the determinantal representations of the intersections between the underlying spectrahedra and the hyperplane H. This is formalized as follows:

Proposition 6.10. [64, Proposition 6.2] *Let $A(\mathbf{x})$ and $B(\mathbf{x})$ be monic linear matrix pencils of size $k \times k$ and $l \times l$, respectively, and such that $S_A := \{\mathbf{x} \in \mathbb{R}^n : A(\mathbf{x}) \succeq 0\}$ is bounded. Then there exists a constant $\nu > 0$ such that for the scaled spectrahedron νS_A the inclusion $\nu S_A \subseteq S_B$ is certified by the system*

$$C = (C_{ij})_{i,j=1}^{k} \succeq 0, \quad \forall p = 1, \ldots, n \ : \ B_p = \sum_{i,j=1}^{k} \left(\frac{1}{\nu} A_p\right)_{ij} C_{ij}.$$

This led to the following scaled version of our Theorem. Here, we assume for notational convenience that the cutting hyperplane is $H = \{\mathbf{x} \in \mathbb{R}^n : x_1 = 1\}$ and $M_1 = I_n$. Due to this, we have that the first unit vector $\mathbf{e}_1$ lies in the interior of K.

Theorem 6.11. [36, Theorem 5.2] *Let $f \in \mathbb{R}[\mathbf{z}]$ and $M(\mathbf{z}) = \sum_{j=1}^{n} M_j z_j$ with symmetric matrices of size $l \times l$ and assume that $K \cap H$ is bounded. Let $N(\mathbf{z})$ be the matrix pencil of a spectrahedral, conic set contained in $\mathrm{cl}(\mathcal{I}(f)^{\mathsf{c}})$, and assume that $N_1 = I_n$ as well.*

Then there exists a constant $\nu > 0$ such that $f_\nu(z_1, \ldots, z_n) := f(z_1, \nu z_2, \ldots, \nu z_n)$ is K-stable and such that the K-stability of f_ν is certified by the system

$$C = (C_{ij})_{i,j=1}^{l} \succeq 0, \quad \forall p = 1, \ldots, n \ : \ \nu N_p = \sum_{i,j=1}^{l} (M_p)_{ij} C_{ij},$$

where the variable matrix C is a block matrix with $l \times l$ blocks.

As a consequence, f is $\hat{K}$-stable with respect to $\hat{K} = \mathrm{cone}(\{1\} \times \nu(K \cap H))$, where the multiplication of ν with the set $K \cap H$ is done in the $(n-1)$-dimensional space with variables $\mathbf{z}' = (z_2, \ldots, z_n)$ and $\mathrm{cone}(\cdot)$ denotes the conic hull.

The proof is written in a manner similar to the proofs of Theorem 6.8 and Theorem 6.9 and additionally relies onto Proposition 6.10.

Note that Theorem 6.11 may be used in both directions. We may either scale the cone K such that K fits into a component of the imaginary projection's complement or we may scale f such that a unbounded component of the imaginary projection's complement of our scaled version of f contains K. Thus, we either certify K'-stability for f for a scaled cone K' or K-stability for f_ν, the scaled version of f.

6.4. **Conclusion.** In this section we considered sufficient criteria in order to certify K-stability of a given polynomial $f \in \mathbb{C}[\mathbf{z}]$. Our methods were applied on determinantal and quadratic polynomials as well as scaled versions of either cones or polynomials. Having a spectrahedral cone as a component of the complement of the imaginary projection was an essential precondition for all of these cases. Thus, it would be interesting to determine further classes of polynomials which also satisfy this precondition in order to apply similar techniques to certify their K-stability. Developing our main result towards necessity would also pose an interesting research topic as it would give rise to a characterization of K-stable polynomials that has yet not been established in such generality.

7. Author's contribution

This thesis is based on the articles [32, 36, 42] and the author's contributions to these articles are stated in this section.

- **Combinatorics and preservation of conically stable polynomials**, co-written with G. Codenotti and T. Theobald, submitted and currently under review, conference-version was accepted for DMD 2022, preprint available at arXiv: 2206.10913

The general notation and setup of the article was developed jointly. The author contributed to the preservation of stability when going over to initial forms (Theorem 4.12 and Theorem 4.13) largely. Section 3 and 4 were mainly developed with T. Theobald. Theorem 5.1 was developed by G. Codenotti primarily. The results regarding binomials and non-mixed polynomials including Theorem 5.5 were joint work of the authors. The notion and the results concerning polynomials of determinants were developed together by all authors. This includes Theorem 5.11. Conjecture 5.13 as well as the evidence given for the conjecture was developed jointly.

- **Imaginary projections: Complex versus real coefficients**, co-written with M. Sayyary and T. Theobald, submitted and currently under review, preprint available at arXiv: 2107.08841

The work was initiated by T. Theobald and the author to answer open questions that emerged after supervising a master student's thesis. Based on jointly developed initial steps, T. Theobald provided the general setup to study the imaginary projections of polynomials distinguished by their initial forms having a hyperbolic initial form or not. The author contributed some group theoretical aspects to this setup. Theorem 3.4 was developed by M. Sayyary primarily. Section 4.1 was joint work of the authors while Section 4.2 was joint work of M. Sayyary and T. Theobald. The author contributed to the main classification Theorem (see Theorem 5.1 and Theorem 5.5) by working out the formal derivation of this theorem together with the other authors. Section 5 and 6 were joint work of the authors as well. Theorem 7.1 was developed by M. Sayyary primarily.

- **Conic stability of polynomials and positive maps**, co-written with P. Dey and T.Theobald, published in: Journal of Pure and Applied Algebra 225(7):106610 (2021)

This work was initiated by T. Theobald and the author and then developed into a slightly different direction under the influence of P. Dey. The author contributed to Theorem 3.4 and Theorem 3.6 by carrying out the technical calculations. T. Theobald delivered the initial idea behind the Theorems 4.3 and 4.8, which then were developed by all authors together. The author contributed to the idea behind Theorem 5.2 and derived example 5.3. The authors jointly worked out the formal derivation of the theorems and their proofs.

8. Deutsche Zusammenfassung

8.1. **Einführung.** Bereits im 18. Jahrhundert haben Newton [85] und Maclaurin [75] erste Beziehungen zwischen den Nullstellen sowie Koeffizienten von Polynomen und kombinatorischen Objekten festgestellt. In unserer modernen mathematischen Sprache würden wir diesen Zusammenhang wie folgt ausdrücken. Wir nennen eine Folge $A = a_0, \ldots a_n$ log-konkav, falls $a_i^2 \geq a_{i-1} \cdot a_{i+1}$ für alle $i \in [n-1]$ gilt. Die bekannteste log-konkave Folge ist mit $\binom{n}{0}, \ldots, \binom{n}{n}$ durch die n-te Zeile des Pascal'schen Dreiecks gegeben.

Newton [85] hat bewiesen, dass die Koeffizienten $a_0, \ldots, a_n$ eines Polynoms $f = \sum_{j=0}^{n} a_j x^j$ eine log-konkave Folge bilden, falls alle Nullstellen von f reell und die Koeffizienten von f nicht-negativ sind. Auch heute ist die Fragestellung nach der Lage der Nullstellen äußerst präsent in der Kombinatorik, da insbesondere im Kontext von Graphen Polynome auftreten, deren Nullstellen entweder reell sind oder die Lage der Nullstellen bestimmte Eigenschaften des assoziierten Graphen implizieren. Allerdings sind die Auswirkungen der Lage der Nullstellen von Polynomen nicht auf die Kombinatorik oder die dort auftretenden Graphen beschränkt. Alleine dadurch, dass das charakteristische Polynom $\det(xI_n - A)$ einer symmetrischen oder hermiteschen Matrix A nur reelle Nullstellen hat, existieren weitere Verbindungen zu Systemen von Differentialgleichungen erster Ordnung [102], multivariater Analysis [58], dynamischen Systemen [97] sowie der Statistik und dem Maschinellen Lernen [11].

Oftmals sind die Zusammenhänge zwischen Polynomen und anderen Gebieten keine Äquivalenzen, sondern lediglich Implikationen. Dies trifft auch auf den Zusammenhang zwischen Polynomen mit reellen Nullstellen und log-konkaven Folgen zu. Beispielsweise ist die Folge der Koeffizienten von $f(x) = x^2 + x + 1$ log-konkav, aber f hat keine reelle Nullstelle. Aus diesem Grund wurden in den letzten Jahren und Jahrzehnten verschiedene Verallgemeinerungen von Polynomen mit reellen Nullstellen untersucht. Der Fokus dieser Doktorarbeit liegt auf der konischen Stabilität, die wiederum eine Verallgemeinerung des folgenden Stabilitätsbegriffes darstellt.

Es sei $\mathcal{H} = \{\mathbf{z} \in \mathbb{C} : \mathsf{Im}(\mathbf{z}) > 0\}$ die obere offene Halbebene der komplexen Zahlenebene. Wir nennen ein multivariates Polynom mit komplexen Koeffizienten $f \in \mathbb{C}[\mathbf{z}] = \mathbb{C}[z_1, \ldots z_n]$ *stabil*, falls $f(\mathbf{z}) \neq 0$ für alle $\mathbf{z} \in \mathbb{C}^n$ mit $\mathbf{z} \in \mathcal{H}^n$. Mit anderen Worten, wir nennen f stabil, falls die Imaginärteile aller Nullstellen außerhalb des positiven Orthanten liegen. Da $\mathcal{H}^n \cap \mathbb{R}^n = \emptyset$, ist jedes Polynom, dessen Nullstellen reell sind, auch stabil. Der Stabilitätsbegriff für Polynome wurde bereits ausgiebig untersucht [20, 31, 104]. Eine zentrale Rolle bei diesen Untersuchungen haben die *stabilitätserhaltenden Operatoren* eingenommen. Dabei handelt es sich um Operatoren, die, wenn sie auf ein stabilies Polynom f angewandt werden, wieder ein stabiles Polynom liefern. Folgende stabilitätserhaltende Operatoren aus [104] bilden die Grundlage für viele weitere Resulte im Bezug auf stabile Polynome.

Proposition 8.1. [104, Lemma 2.4] $f \in \mathbb{C}[\mathbf{z}]$ *sei stabil.*
a) Permutation*:* $f(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$ *ist stabil für jede Permutation* $\sigma : [n] \to [n]$.
b) Skalierung*:* $c \cdot f(a_1 z_1, \ldots, a_n z_n)$ *ist stabil oder identisch* 0 *für* $c \in \mathbb{C}$ *sowie* $\mathbf{a} \in \mathbb{R}^n_{>0}$.
c) Diagonalisierung*:* $f(\mathbf{z})\big|_{z_j = z_i} \in \mathbb{C}[z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n]$ *ist stabil oder identisch* 0 *für alle* $i, j \in [n]$.
d) Spezialisierung*:* $f(b, z_2, \ldots, z_n) \in \mathbb{C}[z_2, \ldots, z_n]$ *ist stabil oder identisch* 0 *für alle* $b \in \mathbb{C}$ *mit* $\mathsf{Im}(b) \geq 0$.
e) Invertierung*:* $z_1^{\deg_1(f)} \cdot f(-z_1^{-1}, z_2, \ldots, z_n)$ *ist stabil.*
f) Differentiation*:* $\partial_j f(\mathbf{z})$ *ist stabil oder identisch* 0 *für alle* $j \in [n]$.

Basierend auf diesen stabilitätserhaltenden Operatoren gelang Brändén [20] der Beweis faszinierender kombinatorischer Resultate für stabile Polynome. Um diese formal auszudrücken, brauchen wir die folgenden Definitionen. Für $\alpha, \beta \in \mathbb{Z}^n$ sei $\mathrm{St}(\alpha, \beta) = \{\sigma \in \mathbb{Z}^n : |\sigma| = 1,\ |\alpha + \sigma - \beta| = |\alpha - \beta| - 1\}$ die Menge der *Schritte* von α in Richtung β. Ferner nennen wir $\mathcal{F} \subseteq \mathbb{Z}^n$ ein *Sprungsystem*, falls für alle $\alpha, \beta \in \mathcal{F}$ und $\sigma \in \mathrm{St}(\alpha, \beta)$ entweder $\alpha + \sigma \in \mathcal{F}$ gilt oder ein $\tau \in \mathrm{St}(\alpha + \sigma, \beta)$ existiert, sodass $\alpha + \sigma + \tau \in \mathcal{F}$ gilt. $\mathcal{F}$ kann also nur ein Sprungsystem sein, wenn die ganzzahligen Punkte in $\mathcal{F}$ ausreichend nah beieinander liegen. Für ein Polynom $f = \sum_\alpha c_\alpha \mathbf{z}^\alpha \in \mathbb{C}[\mathbf{z}]$ nennen wir $\mathrm{supp}(f) = \{\alpha \in \mathbb{Z}^n : c_\alpha \neq 0\}$ den *Support* von f. Das von Brändén bewiesene Resultat lautet nun wie folgt:

Theorem 8.2. [20, Theorem 3.2] *$f \in \mathbb{C}[\mathbf{z}]$ sei stabil. Dann bildet supp(f) ein Sprungsystem.*

Die Exponentenvektoren der in einem Polynom $f \in \mathbb{C}[\mathbf{z}]$ auftretenden Monome müssen also ausreichend nah beieinander liegen, damit es überhaupt stabil sein kann. Theorem 8.2 liefert somit ein kombinatorisches notwendiges Kriterium für die Stabilität von Polynomen, das sich vergleichweise einfach überprüfen lässt. Im Spezialfall, dass f nicht nur stabil, sondern auch multi-affin und homogen ist, bildet der Support von f die Menge der Basen eines Matroiden. Somit hat Brändèn einen äußerst interessanten Zusammenhang zwischen Polynomen und kombinatorischen Objekten gezeigt.

Weitere stabilitätserhaltende Operatoren, die teilweise ebenfalls eine kombinatorische Bedeutung haben, sind durch das Lieb-Sokal-Lemma [73] oder den Übergang zu Initialformen bezüglich linearer Funktionale $\mathbf{w} \in (\mathbb{R}^n)^* \setminus \{\mathbf{0}\}$ [32, 96] gegeben. Ferner ist es Borcea und Brändén gelungen, eine Charakterisierung aller linearen Operatoren anzugeben, die die Stabilität von Polynomen erhalten [16, 104].

In dieser Arbeit liegt der Fokus nun auf der konischen Stabilität, die den Stabilitätsbegriff verallgemeinert.

Definition 8.3. $K \subseteq \mathbb{R}^n$ sei ein abgeschlossener und konvexer Kegel. Ein Polynom $f \in \mathbb{C}[\mathbf{z}]$ heißt K-stabil, falls $f(\mathbf{z}) \neq 0$ für alle $\mathbf{z} \in \mathbb{C}^n$ mit $\mathsf{Im}(\mathbf{z}) \in \mathrm{relint}\, K$, dem relativen Inneren von K, gilt.

Die Imaginärteile der Nullstellen eines K-stabilen Polynoms f dürfen also nicht im relativen Inneren des Kegels K liegen. Für $K = \mathbb{R}^n_{\geq 0}$ kommen wir wieder zum normalen Stabilitätsbegriff zurück. Die Stabilität eines Polynoms f ist also ein Spezialfall der konischen Stabilität. Auch wenn K nicht der positive Orthant ist, lassen sich Bezüge zur Stabilität herstellen. Jörgens und Theobald haben in [60] gezeigt, dass ein Polynom $f \in \mathbb{C}[\mathbf{z}]$ genau dann K-stabil ist, wenn das univariate Polynom $t \mapsto f(\mathbf{x}+t\mathbf{y})$ stabil für alle $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ mit $\mathbf{y} \in \mathrm{relint}\, K$ ist. Ein besonderes Augenmerk liegt auf dem weiteren Spezialfall des Kegels der positiv semi-definiten Matrizen $\mathcal{S}_n^+$. Wir nennen $f \in \mathbb{C}[Z]$, ein Polynom in symmetischen Matrixvariablen, *psd-stabil* (stabil bezüglich des Kegels der positiv semi-definiten Matrizen), falls $f(M) \neq 0$ für alle positiv definiten Matrizen M gilt.

In [62] haben Jörgens, Theobald und de Wolff den Begriff der Imaginärprojektion eingeführt und entwickelt. Diese eröffnet uns eine geometrische Perspektive auf die konische Stabilität, indem die Frage der K-Stabilität, die eigentlich eine Fragestellung innerhalb eines komplexen Vektorraums darstellt, auf einen reellen Vektorraum projeziert wird. Für ein komplexes Polynom $f \in \mathbb{C}[\mathbf{z}]$ mit der Varietät $\mathcal{V}(f) = \{\mathbf{z} \in \mathbb{C}^n : f(\mathbf{z}) = 0\}$ ist $\mathcal{I}(f)$, die Imaginärprojektion von f, durch die Projektion von $\mathcal{V}(f)$ auf ihren Imaginärteil gegeben, d.h. $\mathcal{I}(f) = \{\mathbf{y} \in \mathbb{R}^n : \exists \mathbf{x} \in \mathbb{R}^n \text{ s.t. } \mathbf{x} + \mathbf{i}\mathbf{y} \in \mathcal{V}(f)\}$. Die Imaginärprojektion eines Polynoms f ermöglicht nicht nur eine alternative Betrachtungsmöglichkeit für die konische Stabilität, sondern stellt weitere Zusammenhänge zu Klassen von Polynomen her, die mit der Frage nach reellen Nullstellen verwandt sind.

Eine solche Klasse ist durch hyperbolische Polynome gegeben. Ein homogenes Polynom $f \in \mathbb{R}[\mathbf{z}]$ heißt *hyperbolisch bezüglich* $\mathbf{e} \in \mathbb{R}^n$, falls $f(\mathbf{e}) \neq 0$ und das univariate Polynom $t \mapsto f(\mathbf{x} + t\mathbf{e})$ für alle $\mathbf{x} \in \mathbb{R}^n$ nur reelle Nullstellen hat. Für diese Klassen von Polynomen und deren Imaginärprojektionen haben Jörgens und Theobald folgenden Zusammenhang gezeigt.

Theorem 8.4. [60, Theorem 3.5] *Es sei $f \in \mathbb{C}[\mathbf{z}]$ homogen. Dann sind äquivalent*

(1) f ist K-stabil.
(2) $\mathcal{I}(f) \cap \operatorname{relint} K = \emptyset$.
(3) f ist hyperbolisch bezüglich $\mathbf{e}$ für alle $\mathbf{e} \in \operatorname{relint} K$.

Dieser Zusammenhang gewinnt an zusätzlicher Bedeutung, wenn wir berücksichtigen, dass hyperbolische Polynome ebenfalls kegelartigen Strukturen unterliegen. Für ein Polynom $f \in \mathbb{C}[\mathbf{z}]$, das hyperbolisch bezüglich $\mathbf{e} \in \mathbb{R}^n$ ist, ist $C(\mathbf{e}) = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x} + t\mathbf{e}) = 0 \Rightarrow t < 0\}$ der Hyperbolizitätskegel von f bezüglich $\mathbf{e}$. Ferner ist f hyperbolisch bezüglich $\mathbf{e}'$ für alle $\mathbf{e}' \in C(\mathbf{e})$. Die Verbindung zwischen Hyperbolizitätskegeln eines hyperbolischen Polynoms und dessen Imaginärprojektion lässt sich wie folgt ausdrücken.

Theorem 8.5. [61, Theorem 1.1] *Es sei $f \in \mathbb{R}[\mathbf{z}]$ homogen. Dann stehen die Hyperbolizitätskegel von f in Bijektion zu den Komponenten des Komplements von $\mathcal{I}(f)$.*

Durch Theorem 8.5 konnten Jörgens und Theobald obere Schranken für die Anzahl an Komponenten im Komplement der Imaginärprojektion homogener Polynome herleiten. Zudem werden hierdurch weitere Verbindungen der konischen Stabilität zur konvexen Geometrie eröffnet. Dies gilt insbesondere auch für Zusammenhänge zu der verallgemeinerten Lax-Vermutung, die besagt, dass jeder Hyperbolizitätskegel ein *Spektraeder* ist. Hierbei handelt es sich bei $S \subseteq \mathbb{R}^n$ um einen Spektraeder, falls symmetrische Matrizen $A_0, \ldots A_n$ existieren, sodass S eine Darstellung der Form

$$S = \{\mathbf{x} \in \mathbb{R}^n : A_0 + A_1 x_1 + \ldots A_n x_n \succeq 0\}$$

besitzt.

Durch die Zusammenhänge zwischen konischer Stabilität und der Imaginärprojektion, wie in Theorem 8.4 beschrieben, wird auch die Untersuchung der Imaginärprojektion als eigenständiger Forschungsgegenstand motiviert. Die Imaginärprojektionen gewinnen durch andere konvexe Phänomene, die in der algebraischen Geometrie auftreten, weiter an Relevanz in der mathematischen Forschung. Hier wurden bereits Amoeben sowie co-Amoeben untersucht, bei denen es sich ebenfalls um Bilder von Varietäten komplexer Polynome handelt. Genau wie bei der Imaginärprojektion bestehen ihre Komplemente aus endlichen vielen konvexen Komponenten. Jörgens, Theobald und de Wolff haben gezeigt, dass sich die Topologie der Imaginärprojektion eines Polynoms $f \in \mathbb{C}[\mathbf{z}]$ unter der Operation der Gruppe $G_n := \mathbb{C}^n \rtimes \mathrm{GL}_n(\mathbb{R})$ nicht verändert. Die Gruppe G_n besteht hierbei aus der Zusammensetzung der reellen invertierbaren Transformationen $\mathrm{GL}_n(\mathbb{R})$ sowie komplexen Translationen $\mathbb{C}^n$. Basierend darauf haben Jörgens, Theobald und de Wolff zunächst die Imaginärprojektionen von reellen Kegelschnitten klassifiziert und ihre Ergebnisse dann auf den allgemeinen Fall von reellen quadratischen Polynomen in n Variablen verallgemeinert.

8.2. **Ergebnisse.** Im Folgenden gehen wir stets davon aus, dass K ein abgeschlossener und konvexer Kegel ist.

8.2.1. *Combinatorics and preservation of conically stable polynomials.* Dieser Artikel [32] wurde gemeinsam mit G. Codenotti und T. Theobald verfasst. Ziel dieser Arbeit ist es, kombinatorische Eigenschaften sowie stabilitätserhaltende Operatoren für den Fall konischer Stabilität zu erforschen. Die Resultate in Proposition 8.1 sowie Theorem 8.2 können als Augangspunkt für diese Untersuchung betrachtet werden.

Für ein Polynom $f \in \mathbb{C}[\mathbf{z}]$ ist die Initialform von $f = \sum_\alpha c_\alpha \mathbf{z}^\alpha$ bezüglich $\mathbf{w} \in (\mathbb{R}^n)^*$ durch die Monome gegeben, deren Exponentenvektor α das Skalarprodukt mit $\mathbf{w}$ maximiert. Formal ist die Initialform $\mathrm{in}_\mathbf{w}(f)$ durch

$$\mathrm{in}_\mathbf{w}(f) = \sum_{\alpha \in S_\mathbf{w}} c_\alpha \mathbf{z}^\alpha, \text{ mit } S_\mathbf{w} = \{\alpha \in \mathrm{supp}(f) : \langle \mathbf{w}, \alpha \rangle = \max_{\beta \in \mathrm{supp}(f)} \langle \mathbf{w}, \beta \rangle\} \tag{10}$$

gegeben. Unser erstes Resultat besagt, dass der Übergang zur Initialform die Stabilität eines Polynoms erhält.

Theorem 8.6. [32, Theorem 2.5] *Es sei $f \in \mathbb{C}[\mathbf{z}]$ stabil und $\mathbf{w} \in (\mathbb{R}^n)^* \setminus \{\mathbf{0}\}$. Dann ist $\mathrm{in}_\mathbf{w}(f)$ stabil.*

Damit verallgemeinern wir ein Resultat von Rincón, Vinzant und Yu, das die gleiche Aussage für den Fall von Polynomen, die entweder reell oder homogen sind, getroffen hat. Ferner können wir die Aussage auf psd-stabile Polynome erweitern, wenn wir uns dabei auf reelle positiv-definite Matrizen einschränken. Für die Definition der Initialform $\mathrm{in}_W(f)$ eines Polynoms $f \in \mathbb{C}[Z]$ ersetzen wir $\mathbf{w} \in (\mathbb{R}^n)^*$ sowie das Skalarprodukt in (10) durch $W \in (\mathcal{S}_n)^*$ sowie das Frobeniusprodukt. Damit können wir das zweite Hauptresultat zu Initialformen wie folgt formulieren:

Theorem 8.7. [32, Theorem 4.10] *$f \in \mathbb{C}[Z]$ sei psd-stabil und $W \in \mathcal{S}_n$ positiv definit. Dann ist $\mathrm{in}_W(f)$ psd-stabil.*

Für den Beweis der Aussage verwenden wir das Hadamard-Produkt, das für Matrizen A, B durch $A \circ B$ mit $(A \circ B)_{ij} = a_{ij} \cdot b_{ij}$ definiert ist, da dieses die positive Definitheit seiner Argumente erhält. Dadurch konstruieren wir eine Folge psd-stabiler Polynome, die gegen die Intialform $\mathrm{in}_W(f)$ konvergiert und somit folgt die Aussage letztlich durch das Theorem von Hurwitz.

Leider lässt sich diese Eigenschaft nicht in voller Breite auf den Fall der psd-Stabilität erweitern wie ein Gegenspiel (siehe Example 4.8 in [32]) zeigt. Die beiden folgenden stabilitätserhaltenden Operatoren können jedoch auf den allgemeinen Fall konischer Stabilität erweitert werden. Beide Operatoren basieren auf Richtungsableitungen. Für ein Polynom $f \in \mathbb{C}[\mathbf{z}]$ und eine Richtung $\mathbf{v} \in \mathbb{R}^n$, heißt $\rho_\mathbf{v}(f)$ der Grad von f in Richtung $\mathbf{v}$ und ist für ein generisches $\mathbf{w} \in \mathbb{R}^n$ durch den Grad des univariaten Polynoms

$f(\mathbf{w}+\mathbf{v}t) \in \mathbb{C}[t]$ gegeben. Dann erhalten die beiden folgenden Operatoren die konische Stabilität eines Polynoms $f \in \mathbb{C}[\mathbf{z}]$.

Lemma 8.8. [32, Lemma 3.1] *$f \in \mathbb{C}[\mathbf{z}]$ sei K-stabil und $\mathbf{v} \in K$. Dann ist $\partial_{\mathbf{v}} f$ K-stabil oder identisch* 0*.*

Theorem 8.9. [32, Theorem 3.4] *Es sei $K' = K \times \mathbb{R}_{\geq 0}$ und $g(\mathbf{z}) + yf(\mathbf{z}) \in \mathbb{C}[\mathbf{z}, y]$ sei K'-stabil sowie $\mathbf{v} \in K$ mit $\rho_{\mathbf{v}}(f) \leq 1$. Dann ist $g - \partial_{\mathbf{v}} f$ K-stabil oder identisch* 0*.*

Beide Operatoren verallgemeinern nicht nur ihr Pendant für stabile Polynome, d.h. partielle Ableitungen bzw. das Lieb-Sokal-Lemma auf den Fall von konischer Stabilität, sondern erweitern das Spektrum bekannter Operatoren, die die Stabilität von Polynomen erhalten.

Für den den Fall von psd-stabilen Polynomen konnten wir zeigen, dass die nachstehenden Operatoren die psd-Stabilität eines Polynoms erhalten.

Theorem 8.10. [32, Lemma 4.1 und Theorem 4.3] *$f \in \mathbb{C}[Z]$ sei psd-stabil.*

a) **Diagonalisierung***: Die Einschränkung von Z auf die diagonalen Variablen, $Z \mapsto f_{\mathrm{Diag}}(Z)$, ist psd-stabil.*

b) **Transformation***: Es sei $S \in \mathrm{GL}_n(\mathbb{R})$, dann sind $f(SZS^{-1})$ sowie $f(SZS^T)$ psd-stabil.*

c) **Minorisierung***: Für $J \subseteq [n]$ sei Z_J die symmetrische $|J| \times |J|$ Untermatrix von Z mit Indexmenge J. $f(Z_J)$ sei das Polynom, das aus $f(Z)$ entsteht, indem alle Variablen, die mindestens einen Index aus $[n] \setminus J$ enthalten, auf* 0 *gesetzt werden. Dann ist $f(Z_J)$ als Polynom über $\mathcal{S}_{|J|}^{\mathbb{C}}$ psd-stabil oder identisch* 0*.*

d) **Spezialisierung***: Für einen festen Index $i \in [n]$, sei $\hat{Z}_i$ die Matrix, die aus Z entsteht, indem den Variablen z_{ij}, z_{ji} für $i \neq j$ reelle Werte sowie der Variablen z_{ii} Werte aus $\mathcal{H}$ zugewiesen werden. Dann ist $f(\hat{Z}_i)$, betrachtet als Polynom über $\mathcal{S}_{n-1}^{\mathbb{C}}$, psd-stabil oder identisch* 0*.*

e) **Reduktion***: Für $i, j \in [n]$ sei $\bar{Z}_{ij}$ die Matrix, die aus Z entsteht, wenn z_{ik}, z_{ki} für $k \neq i$ auf reelle Werte sowie z_{ii} auf z_{jj} gesetzt werden. Dann ist $f(\bar{Z}_{ij})$, betrachtet als Polynom über $\mathcal{S}_{n-1}^{\mathbb{C}}$, psd-stabil oder identisch* 0*.*

f) **Permutation***: $\pi : [n] \to [n]$ sei eine Permutation über $[n]$. Dann ist das Polynom $f((Z_{\pi(j),\pi(k)})_{1 \leq j,k \leq n})$ psd-stabil.*

g) **Differentiation***: Für $V \in \mathcal{S}_n^+$ ist $\partial_V f(Z)$ psd-stabil oder identisch* 0*.*

h) **Invertierung***: Dann ist das Polynom* $\det(Z)^{\deg(f)} \cdot f(-Z^{-1})$ *psd-stabil.*

Insbesondere die Resultate aus Theorem 8.10 a) sowie Theorem 8.10 g) sind im Folgenden wichtig, um strukturelle Eigenschaften psd-stabiler Polynome zu zeigen. Theorem 8.10 a) nimmt eine besondere Bedeutung ein, da $f_{\mathrm{Diag}}(Z)$ für ein psd-stabiles Polynom f auch stabil in den Variablen $z_{11}, \ldots, z_{nn}$ ist und somit eine Brücke zwischen Stabilitäten bezüglich verschiedener Kegel darstellt.

Unser allgemeinstes strukturelles Resultat ist das folgende Struktur-Theorem für psd-stabile Polynome.

Theorem 8.11. [32, Theorem 5.1] *$f \in \mathbb{C}[Z]$ sei psd-stabil und z_{ij} für $i < j$ eine Variable, die in f auftaucht. Dann müssen auch die entsprechenden diagonalen Variablen z_{ii} und z_{jj} in f auftauchen.*

Mithilfe des Struktur-Theorems sowie Theorem 8.10 a) und g) konnten wir alle psd-stabilen Binome klassifizieren. Hierbei nennen wir ein Monom Z^α *diagonal*, falls nur diagonale Variablen in Z^α auftreten und wir nennen ein Monom Z^β *nicht-diagonal*, falls keine diagonalen Variablen in Z^β auftreten.

Theorem 8.12. [32, Theorem 5.5] *Jedes psd-stabile Binom $f \in \mathbb{C}[Z]$ hat eine der folgenden Formen:*

a) f besteht nur aus diagonalen Variablen und ist von der Form $f(Z) = Z^\gamma(c_1 Z^{\alpha_1} + c_2 Z^{\alpha_2})$ mit $|\alpha_1 - \alpha_2| \leq 2$, wobei mindestens einer der Exponentenvektoren α_1, α_2 verschieden von 0 sein muss,

b) $f(Z) = Z^\gamma(c_1 z_{ii} z_{jj} + c_2 z_{ij}^2)$ mit $i < j$ sowie $\frac{c_1}{c_2} \in \mathbb{R}$,

wobei $c_1, c_2 \neq 0$ gilt und Z^γ ein diagonales Monom ist.

Theorem 8.12 zeigt, dass psd-stabile Binome entweder ohnehin schon stabil sind (Fall a)) oder, falls nicht-diagonale Variablen z_{ij}, $i < j$ auftreten, dann in nur sehr eingeschränkter Form (Fall b)). Ein psd-stabiles Binom kann höchstens eine nicht-diagonale Variable enthalten. Diese tritt nur bei homogenen Binomen und auch dann immer nur im Grad von exakt 2 auf.

Neben Binomen haben wir mit Polynomen von Determinanten eine weitere interessante Klasse psd-stabiler Polynome erforscht. Ein Polynom f heißt *Polynom von Determinanten*, falls Z die Gestalt einer diagonalen Blockmatrix mit den Blöcken $Z_1, \ldots, Z_k$ hat und f von der Form $f(Z) = \sum_\alpha c_\alpha \det(Z)^\alpha$ ist. Genauso wie bei vektorwertigen stabilen Polynomen, bilden auch die Exponentenvektoren eines Polynoms von Determinanten ein Sprungsystem [32, Corollary 5.10]. Wir sagen, ein Polynom von Determinanten f ist in Standardform, falls das größtmöglichste Determinantenmonom faktorisiert wurde. Für ein psd-stabiles Polynom von Determinanten in Standardform sind die Größen, die die Blöcke von Z aufweisen dürfen, beschränkt, wie in folgendem Theorem beschrieben:

Theorem 8.13. [32, Theorem 5.11] *Es sei $f(Z_1, \ldots, Z_k) = \det(Z)^\gamma \sum_{\beta \in B} c_\beta \det(Z)^\beta = \det(Z)^\gamma \tilde{f}(Z)$ ein psd-stabiles Polynom von Determinanten in Standardform. Dann hat jeder auftretende Block, d.h. jeder Block Z_i für den es ein β mit $\beta_i > 0$ gibt, eine Größe von $d_i \leq 2$. Für jeden Block Z_i, der tatsächlich Größe 2 hat, sei $C_i = \max_{\beta \in B} \beta_i$. Für $\beta \in B$ ist dann auch $\beta + c\mathbf{e}_i \in B$ für alle $-\beta_i \leq c \leq C_i - \beta_i$.*

Inspiriert durch die Sprungsystemeigenschaft von stabilen Polyonmen, haben wir folgende Notation von Transpositionsschritten entworfen. Analog zu linearen einfachen oder doppelten Sprüngen, die auch als Multiplikation eines Monoms Z^α mit Variablen $z_{ij}^{\pm 1}$ bzw. $z_{ij}^{\pm 1} z_{kl}^{\pm 1}$ interpretiert werden können, ist ein Transpositionsschritt auch durch die Multiplikation eines Monoms mit Variablen gegeben. Konkret nennen wir die Multiplikation eines Monoms Z^α mit $z_{ij} z_{kl} z_{ik}^{-1} z_{jl}^{-1}$ für die Indizes $i, j, k, l \in [n]$ einen *Transpositionsschritt*. Wir konnten zeigen, dass für $f(Z) = \det(Z)$, die Determinante symmetrischer Matrixvariablen, gilt, dass alle ihre Monome durch Transpositionsschritte verbunden sind [32, Lemma 5.12]. Kombinieren wir die vorigen Resultate mit der Tatsache, dass diagonale Monome psd-stabiler Polynome wegen Theorem 8.10 a) durch ein Sprungsystem verbunden sind, ergibt sich die Vermutung, dass allgemeine psd-stabile Polynome ebenfalls einer kombinatorischen Struktur unterliegen müssen, die wir wie folgt formalisiert haben:

Vermutung 8.14. [32, Conjecture 5.13] *Für jedes Monom Z^β eines psd-stabilen Polynoms f existiert ein diagonales Monom Z^α in f, sodass Z^α von Z^β durch die Anwendung einer Folge von einfachen und doppelten linearen sowie Transpositionsschritten erreicht werden kann und die Anwendung jedes einzelnen Schrittes die Distanz zum Zielmonom Z^α verringert und stets zu einem anderen Monom führt, dessen Exponentenmatrix im Support von f liegt.*

Als Grundlage für diese Vermutung haben wir diese für alle uns bekannten Klassen psd-stabiler Polynome bewiesen. Dies beinhaltet Determinanten, Binome, ungemischte Polynome, die als eine Verallgemeinerung von Binomen betrachtet werden können, Polynome von Determinanten und lpm-Polynome, die kürzlich von Blekherman et. al. [14] entworfen wurden.

8.2.2. *Imaginary projections: Complex versus real coefficients.* Dieser Artikel [42] wurde gemeinsam mit M. Sayyary und T. Theobald verfasst. Die Arbeit wurde durch [62] inspiriert. In [62] haben Jörgens, Theobald und de Wolff die Imaginärprojektionen reeller Polynome untersucht und eine vollständige Beschreibung der Imaginärprojektion für den Fall quadratischer Polynome mit reellen Koeffizienten angegeben. Wir haben diese Resultate nun auf den komplexen Fall verallgemeinert. Hierbei treten bereits im quadratischen Fall interessante Phänomene auf, die im Fall von reellen quadratischen Polynomen nicht auftreten.

In dieser Arbeit haben wir zunächst eine eigene Klassifikation komplexer Kegelschnitte, d.h. quadratischer Polynome in $\mathbb{C}[z_1, z_2]$, entwickelt, da bekannte Klassifikationen wie die von Newstead [84] nicht mit der Operation der Gruppe $G_n = \mathbb{C}^n \rtimes \mathrm{GL}_n(\mathbb{R})$, die die Topologie der Imaginärprojektion eines Polynoms erhält, vereinbar sind. Die von uns entwickelte Klassifikation unterscheidet die Kegelschnitte danach, ob ihre Initialform hyperbolisch ist oder nicht, und teilt die Polynome dann gemäß der Nullstellen ihrer Initialform wie folgt ein:

Hyperbolische Initialform	Nicht-hyperbolische Initialform
(1a) Doppelte Nullstelle in $\mathbb{R}$	(2a) Doppelte Nullstelle in $\mathbb{C} \setminus \mathbb{R}$
(1b) 2 verschiedene Nullstellen in $\mathbb{R}$	(2b) Jeweils eine Nullstelle in $\mathbb{R}$ und $\mathbb{C} \setminus \mathbb{R}$
	(2c) Zwei verschiedene Nullstellen in $\mathbb{C} \setminus \mathbb{R}$

Um für diese Klassen von Kegelschnitten die Imaginärprojektion zu berechnen, haben wir die Operation der Gruppe G_2 zur Bestimmung der Normalformen genutzt, für die wir dann die Imaginärprojektion bestimmt haben. Die Imaginärprojektion eines allgemeinen Kegelschnittes ergibt sich dann aus der Imaginärprojektion der jeweiligen Normalform unter der Operation von G_2. Da die reelle Dimension der Menge der komplexen Kegelschnitte 10 beträgt und die der Gruppe G_2 lediglich 8, erhalten wir unendlich viele Orbits von komplexen Kegelschnitten unter der Operation von G_2.

Theorem 8.15 (Klassifizierung der Normalformen). [42, Theorem 5.5] *Unter der Operation der Gruppe G_2 existieren unendlich viele Orbits komplexer Kegel- schnitte mit den folgenden Normalformen als Representanten.*

(1a) (1a.1) $f = z_1^2 + \gamma$
(1a.2) $f = z_1^2 + \gamma z_2$

(1b) $f = z_1 z_2 + \gamma$

(2a) (2a.1) $f = (z_1 - \mathrm{i} z_2)^2 + \gamma$
(2a.2) $f = (z_1 - \mathrm{i} z_2)^2 + \gamma z_2$

(2b) $f = z_2(z_1 - \alpha z_2) + \gamma$

(2c) (2c.1) $f = z_1^2 + z_2^2 + \gamma$
(2c.2) $f = (z_1 - \mathrm{i} z_2)(z_1 - \alpha z_2) + \gamma$

Um eine Überlappung der Klassen zu vermeiden, setzen wir $\gamma \neq 0$ in (1a.2) *und* (2a.2)*, $\alpha \notin \mathbb{R}$ in* (2b) *und* (2c.2)*, sowie $\alpha \neq \pm\mathrm{i}$ in* (2c.2) voraus.

Durch die Untersuchung der Normalformen aus Theorem 8.15 konnten wir folgende topologische Resultate für die Imaginärprojektionen von komplexen Kegelschnitten gewinnen.

Theorem 8.16 (Topologische Klassifizierung). [42, Theorem 5.1]
$f \in \mathbb{C}[z_1, z_2]$ sei ein komplexer Kegelschnitt. Dann gilt für die obigen Fälle, dass $\mathcal{I}(f)^{\mathsf{c}}$

(1a) die Vereinigung von bis zu drei unbeschränkten Komponenten ist.

(1b) die Vereinigung von vier unbeschränkten Komponenten ist.

(2a) leer ist.

(2b) leer, ein einzelner Punkt oder ein Geradensegment ist.

(2c) leer oder eine beschränkte, möglicherweise offene Komponente ist.

Insbesondere sind die Komponenten von $\mathcal{I}(f)^{\mathsf{c}}$ in den ersten vier Klassen Spektraeder. Dies gilt im Allgemeinen nicht für die letzte Klasse (2c)*.*

Diese Ergebnisse weisen enorme Unterschiede zum reellen Fall auf, da es in diesem nicht möglich ist, dass das Komplement der Imaginärprojektion lediglich aus einer beschränkten Komponente besteht. Ferner ist es nur im komplexen Fall möglich, dass die Imaginärprojektion offen ist, obwohl sie nicht den gesamten Raum $\mathbb{R}^2$ ausmacht. Dies beantwortet eine offene Frage von Jörgens, Theobald und de Wolff [62]. Auch das Auftreten einer beschränkten Komponente, deren Inneres leer ist, ist für das Komplement der Imaginärprojektion eines reellen Kegelschnitts nicht möglich. Ein anderer Unterschied ist durch die Randkurven gegeben. Im Fall von reellen Kegelschnitten sind die Ränder von Imaginärprojektionen stets algebraisch vom Grad 2. Im komplexen Fall haben wir herausgefunden, dass die Ränder von Imaginärprojektionen weder algebraisch sein müssen noch ist der algebraische Grad der irreduziblen Komponenten des Zariski Abschlusses von $\partial\mathcal{I}(f)$ durch 2 beschränkt. Tatsächlich sind hier im Fall (2c) Grade von bis zu 8 möglich.

Wir haben unsere Resultate auch auf den Fall allgemeiner quadratischer Polynome mit hyperbolischer Initialform wie folgt ausgeweitet:

Lemma 8.17. [42, Lemma 4.1] *Unter der Operation von G_n kann jedes quadratische Polynom $f \in \mathbb{C}[\mathbf{z}]$ mit hyperbolischer Initialform in eine der folgenden Normalformen gebracht werden:*

(1) $z_1^2 + \alpha z_2 + r z_3 + \gamma$,

(2) $\sum_{i=1}^{j} z_i^2 - z_{j+1}^2 + \alpha z_{j+2} + r z_{j+3} + \gamma$ *für ein* $j = 1, \ldots, n-1$,

wobei $\alpha, r, \gamma \in \mathbb{C}$ *und Terme, die* z_k *für* $k > n$ *enthalten, nicht auftreten.*

Analog zum Fall von Kegelschnitten konnten wir nun diese Normalformen untersuchen.

Theorem 8.18. [42, Theorem 4.5] *Es sei* $n \geq 3$ *und* $f \in \mathbb{C}[\mathbf{z}]$ *ein quadratisches Polynom mit hyperbolischer Initialform. Unter der Operation von* G_n *lässt sich* f *so umformen, dass die Imaginärprojektion* $\mathcal{I}(f)$ *durch* $\mathbb{R}^n$, $\mathbb{R}^n \setminus \{(0, \ldots, 0, y_n) \in \mathbb{R}^n : y_n \neq 0\}$ *gegeben ist, oder* f *von der Form* $f = \sum_{i=1}^{n-1} z_i^2 - z_n^2 + \gamma$ *mit* $\gamma \in \mathbb{C}$ *ist, sodass* $|\gamma| = 1$ *gilt und wir*

$$\mathcal{I}(f) = \begin{cases} \left\{\mathbf{y} \in \mathbb{R}^n \ : \ y_n^2 < \sum_{i=1}^{n-1} y_i^2\right\} \cup \{\mathbf{0}\} & \textit{falls } \gamma = 1, \\ \left\{\mathbf{y} \in \mathbb{R}^n \ : \ y_n^2 - \sum_{i=1}^{n-1} y_i^2 \leq 1\right\} & \textit{falls } \gamma = -1, \\ \left\{\mathbf{y} \in \mathbb{R}^n \ : \ y_n^2 - \sum_{i=1}^{n-1} y_i^2 \leq \frac{1}{2}(1 - \mathsf{Re}(\gamma))\right\} \setminus \{\mathbf{0}\} & \textit{falls } \gamma \notin \mathbb{R}. \end{cases}$$

erhalten.

Basierend auf diesem Resultat konnten wir die Topologie der Imaginärprojektion für diese Fälle untersuchen. Besonders interessant ist, dass die algebraische Struktur von $\partial\mathcal{I}(f)$ in diesem Fall dem Fall reeller Kegelschnitte gleicht. Wir haben stets algebraische Ränder vom Grad höchstens zwei.

Korollar 8.19. [42, Corollary 4.6] $f \in \mathbb{C}[\mathbf{z}]$ *sei ein quadratisches Polynom mit hyperbolischer Initialform. Dann*

(1) ist $\mathcal{I}(f)^{\mathsf{c}}$ *entweder leer oder besteht aus*
  - *bis zu* 4 *unbeschränkten Komponenten,*
  - *oder zwei unbeschränkten Komponenten und einem einzelnen Punkt.*

(2) ist das Komplement des Abschlusses $\overline{\mathcal{I}(f)}^{\mathsf{c}}$ *entweder leer oder unbeschränkt.*

(3) ist der algebraische Grad der irreduziblen Komponenten von $\partial\mathcal{I}(f)$ *höchstens zwei.*

Weitere Resultate dieses Artikels zeigen, dass es Klassen bivariater Polynome von beliebigen ungeraden Grad gibt, deren Imaginärprojektion stets der gesamte Raum $\mathbb{R}^2$ ist [42, Theorem 3.4]. Ferner haben wir eine Konstruktion angegeben, die für festes $k > 0$ ein Polynom konstruiert, sodass das Komplement der Imaginärprojektion stets aus exakt k strikt konvexen, beschränkten Komponenten besteht [42, Theorem 7.1].

8.2.3. *Conic stability of polynomials and positive maps.* Diese Arbeit [36] wurde gemeinsam mit P. Dey und T. Theobald verfasst. Sie wurde inspiriert durch zahlreiche notwendige und hinreichende Bedingungen für die Stabilität von Polynomen [20, 104]. Der Fokus dieser Arbeit liegt auf der Entwicklung eines hinreichenden Kriteriums für den allgemeineren Fall der konischen Stabilität. In diesem Abschnitt gehen wir davon aus, dass der Kegel K volldimensional, konvex und abgeschlossen ist.

Für die Entwicklung des hinreichenden Kriteriums war das Finden von Klassen von Polynomen, die mindestens einen Hyperbolizitätskegel C besitzen, für den eine Darstellung als Spektraeder bekannt ist, ein wichtiger Zwischenschritt.

Dann kann die K-Stabilität von $f \in \mathbb{C}[\mathbf{z}]$ auf die Frage, ob der Spektraeder C den Spektraeder K enthält, zurückgeführt werden. Dieser Ansatz kann schließlich zu hinreichenden Bedingung für die K-Stabilität von f ausgeweitet werden, da $C \subseteq \mathcal{I}(f)^{\mathsf{c}}$ nach Theorem 8.4 gilt und für gängige Kegel wie den positiven Orthanten, den Lorentz-Kegel oder den psd-Kegel eine Darstellung als Spektraeder vorliegt. Für die beiden folgenden Klassen von Polynomen ist es uns gelungen, Darstellungen als Spektraeder für die zugehörigen Hyperbolizitätskegel anzugeben.

Die erste Klasse sind Polynome mit einer *Determinantendarstellung*, d.h. Polynome die von der Form

$$f(\mathbf{z}) = \det\left(A_0 + A_1 z_1 + \cdots + A_n z_n\right) \tag{11}$$

mit $A_0, \ldots A_n \in \mathbb{C}^{d\times d}$ hermitesch sind. Im irreduziblen Fall liegen die beiden Hyperbolizitätskegel direkt durch

$$\{\mathbf{z} \in \mathbb{R}^n : A_1 z_1 + \cdots + A_n z_n \succ 0\}$$

sowie dessen Negatives vor [72]. Die zweite Klasse sind quadratische Polynome von der Form

$$f(\mathbf{z}) = \mathbf{z}^T A \mathbf{z} + b^T \mathbf{z} + c,$$

wobei $A \in \mathcal{S}_n,\ b \in \mathbb{R}^n,\ c \in \mathbb{R}$ sind. Ferner fordern wir, dass f entweder homogen ist und A die Signatur $(n-1,1)$ oder $(1,n-1)$ hat oder f durch die Operation von G_n in die Form $f(\mathbf{z}) = \sum_{j=1}^{n-1} z_j^2 - z_n^2 + 1$ gebracht werden kann. In allen anderen Fällen weisen reelle quadratische Polynome keine kegelartigen Strukturen im Komplement ihrer Imaginärprojektion auf, sodass in diesen Fällen ohnehin keine konische Stabilität vorliegen könnte.

Liegt ein entsprechender Hyperbolizitätskegel für $f \in \mathbb{C}[\mathbf{z}]$ vor, kann die K-Stabilität nun durch das Lösen eines semi-definiten Zulässigkeitsproblems verifiziert werden. Hierfür wird zusätzlich zu der Darstellung der Hyperbolizitätskegel als Spektraeder auch eine Darstellung des Kegels K als Spektraeder benötigt. Im Folgenden werden wir die Darstellung von K als Spektraeder im Allgemeinen durch $M(\mathbf{x}) = \sum_{j=1}^{n} M_j x_j$, i.e. $K = \{\mathbf{x} \in \mathbb{R}^n : M(\mathbf{x}) \succeq 0\}$ ausdrücken. In den konkreten Fällen, dass K entweder der positive Orthant oder der Kegel der positiv semi-definiten Matrizen ist, können folgende explizite Beschreibungen verwendet werden: Falls K der positive Orthant ist, können wir K durch

$$K = \left\{ \mathbf{x} \in \mathbb{R}^n : M^{\geq 0}(\mathbf{x}) \ = \ \sum_{j=1}^{n} M_j^{\geq 0} x_j \succeq 0 \right\}$$

als Spektraeder ausdrücken. Hierbei ist $M_j^{\geq 0} = E_{jj}$, wobei die Matrix E_{ij} mit einer Eins an Position (i,j) und Nullen an allen anderen Positionen gegeben ist. Falls K der

Kegel der positiv semi-definiten Matrizen sein sollte, so können wir K durch

$$K = \left\{ X \in \mathbb{R}^{n \times n} : M^{\text{psd}}(X) \;=\; \sum_{i,j=1}^{n} M_{ij}^{\text{psd}} x_{ij} \succeq 0 \right\}$$

mit symmetrischen Matrix-Variablen $X = (x_{ij})$ und $M_{ij}^{\text{psd}} = \frac{1}{2}(E_{ij} + E_{ji})$ ausdrücken.

Die fehlende Komponente, die letztlich auch das semi-definite Problem definiert, ist durch positive Abbildung gegeben, die wie folgt definiert sind:

Definition 8.20. $\mathcal{U} \subseteq \mathsf{Herm}_k$ und $\mathcal{V} \subseteq \mathsf{Herm}_l$ (bzw. $\mathcal{U} \subseteq \mathcal{S}_k$ and $\mathcal{V} \subseteq \mathcal{S}_l$) seien zwei lineare Vektorräume. Die Abbildung $\Phi : \mathcal{U} \to \mathcal{V}$ heißt *positiv*, falls $\Phi(U) \succeq 0$ für alle $U \in \mathcal{U}$ mit $U \succeq 0$ gilt.

Folgendes Resultat von Kellner, Theobald und Trabandt stellt den Zusammenhang zwischen positiven Abbildungen und der Problemstellung, ob ein Spektraeder einen anderen enthält, her:

Proposition 8.21. [64, Theorem 4.3] *Es seien $U_1, \ldots, U_n \in \mathsf{Herm}_k$ und $V_1, \ldots, V_n \in \mathsf{Herm}_l$ (bzw. $U_1, \ldots, U_n \in \mathcal{S}_k$ und $V_1, \ldots, V_n \in \mathcal{S}_l$) linear unabhängig sowie $S_U = \{\mathbf{x} \in \mathbb{R}^n : \sum_{j=1}^n U_j x_j \succeq 0\} \neq \emptyset$ und $S_V = \{\mathbf{x} \in \mathbb{R}^n : \sum_{j=1}^n V_j x_j \succeq 0\}$. Ferner sei $\mathcal{U} = span(U_1, \ldots, U_n)$, $\mathcal{V} = span(V_1, \ldots, V_n)$ sowie $\Phi_{UV} : \mathcal{U} \to \mathcal{V}$ mit $\Phi_{UV}(U_i) := V_i$ gegeben. Falls das semi-definite Zulässsigkeitsprolem bestehend aus*

$$C = (C_{ij})_{i,j=1}^{k} \succeq 0 \text{ und } V_p = \sum_{i,j=1}^{k} (U_p)_{ij} C_{ij} \text{ mit } p = 1, \ldots, n$$

eine Lösung mit hermitescher (beziehungsweise symmetrischer) Matrix C besitzt, gilt $S_U \subseteq S_V$.

Durch die Existenz einer positiv semi-definiten Blockmatrix C, der sogenannten *Choi-Matrix*, wird verifiziert, dass der Spektraeder S_U im Spektraeder S_V enthalten ist. In unserem Fall bedeutet das, dass der Kegel K in einem der Hyperbolizitätskegel von f und damit in $\mathcal{I}(f)^{\mathsf{c}}$ enthalten ist, was letztlich die K-Stabilität von f impliziert.

Theorem 8.22. [36, Theorem 4.3] *Es sei $f = \det(A_0 + \sum_{j=1}^n A_j z_j)$ mit hermitschen Matrizen $A_0, \ldots, A_n$ ein Polynom vom Grad d von der Form* (11)*, sodass* $\operatorname{in}(f)$ *irreduzibel ist und ein $\mathbf{e} \in \mathbb{R}^n$ mit $\sum_{j=1}^n A_j e_j \succ 0$ existiert. Ferner sei für K die Darstellung als Spektraeder durch $K = \left\{ \mathbf{x} \in \mathbb{R}^n : M(\mathbf{x}) = \sum_{j=1}^n M_j x_j \right\}$ mit symmetrischen $l \times l$-Matrizen gegeben. Falls eine hermitesche Blockmatrix $C = (C_{ij})_{i,j=1}^l$ mit den Blöcken C_{ij} der Größe $d \times d$ existiert, die*

$$C = (C_{ij})_{i,j=1}^{l} \;\succeq\; 0, \quad \forall p = 1, \ldots, n \;:\; \sigma A_p \;=\; \sum_{i,j=1}^{l} (M_p)_{ij} C_{ij}$$

für $\sigma \in \{-1, 1\}$ erfüllt, dann ist f K-stabil. Die Entscheidung darüber, ob eine solche Blockmatrix C existiert führt zu einem semi-definiten Zulässigkeitsproblem.

Zusätzlich zu Theorem 8.22 haben wir analoge Resultate für die entsprechenden Darstellungen für quadratische Polynome [36, Theorem 4.7, Theorem 4.8] sowie für den Fall von skalierten Versionen des Kegels bzw. des Polynoms [36, Theorem 5.2] entwickelt. Für weitere Details verweisen wir auf Appendix A Kapitel 4.

8.3. **Fazit.** Die zahlreichen Publikationen im Bereich von Verallgemeinerungen von Polynomen mit reellen Nullstellen sowie die vielen Querverbindungen in andere Bereiche der Mathematik und darüber hinaus zeigen deutlich die Relevanz dieses Themenbereichs innerhalb der mathematischen Forschung. Dies wird nicht zuletzt durch die Verleihung der Fields-Medaille an June Huh, einen Forscher, der mit den Lorentz-Polynomen eine andere Verallgemeinerung von Polynomen mit reellen Nullstellen mitentwickelt hat, deutlich.

In dieser Arbeit haben wir offene Fragen zu Imaginärprojektionen [62] beantwortet und die entsprechenden Klassifikationen weiterentwickelt und ergänzt. Zudem haben wir erste grundlegende Operatoren aufgezeigt, die die konische Stabilität von Polynomen erhalten sowie das Spektrum der stabilitätserhaltenden Operatoren für die Stabilität erweitert. Ferner haben wir Verbindungen zur Kombinatorik untersucht, deren Strukturen zu notwendigen Kriterien für die K-Stabilität geführt haben. Abgerundet wird dies durch hinreichende Kriterien für die K-Stabilität von Polynomen, die auf einem Ansatz mittels semi-definiter Programmierung basieren. Durch die Forschung im Rahmen dieser Arbeit haben sich jedoch auch zahlreiche weitere offene Fragen ergeben, sodass das Forschungspotential im Bereich der K-stabilen Polynome noch lange nicht ausgeschöpft ist. Zu den interessantesten Fragestellungen gehören

- Gibt es weitere Klassen von psd-stabilen Polynomen, die nicht durch die hier behandelten Klassen abgedeckt werden, und lässt sich Vermutung 8.14 für diese ebenfalls beweisen?
- Lassen sich die kombinatorischen Strukturen von (psd-) stabilen Polynomen auf andere Kegel abgesehen vom psd-Kegel und dem postiven Orthanten übertragen?
- Gibt es weitere interessante Phänomene von Imaginärprojektionen komplexer Polynome, die bisher noch nicht beobachtet wurden?
- Gibt es weitere Klassen von Polynomen für deren Hyperbolizitätskegel eine Beschreibung als Spektraeder vorliegt und lässt sich für diese Klassen eine hinreichende Bedingung für deren K-Stabilität analog zu Theorem 8.22 zeigen?

Wir sind überzeugt, dass die Einblicke, die durch unsere Forschung gewonnen wurden, hilfreich sein werden, und als Basis für die weitere Forschung in diesem Bereich dienen können. Ferner wird die Beantwortung der obigen Fragen einen positiven Einfluss auf verwandte Themengebiete haben und wir empfehlen daher eine ausführliche Beleuchtung dieser offenen Fragen.

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Appendix A. Conic stability of polynomials and positive maps

CONIC STABILITY OF POLYNOMIALS AND POSITIVE MAPS

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Abstract. Given a proper cone $K \subseteq \mathbb{R}^n$, a multivariate polynomial $f \in \mathbb{C}[\mathbf{z}] = \mathbb{C}[z_1, \ldots, z_n]$ is called K-stable if it does not have a root whose vector of the imaginary parts is contained in the interior of K. If K is the non-negative orthant, then K-stability specializes to the usual notion of stability of polynomials.

We study conditions and certificates for the K-stability of a given polynomial f, especially for the case of determinantal polynomials as well as for quadratic polynomials. A particular focus is on psd-stability. For cones K with a spectrahedral representation, we construct a semidefinite feasibility problem, which, in the case of feasibility, certifies K-stability of f. This reduction to a semidefinite problem builds upon techniques from the connection of containment of spectrahedra and positive maps.

In the case of psd-stability, if the criterion is satisfied, we can explicitly construct a determinantal representation of the given polynomial. We also show that under certain conditions, for a K-stable polynomial f, the criterion is at least fulfilled for some scaled version of K.

arXiv:1908.11124v2 [math.AG] 28 Aug 2020

1. Introduction

Recently, there has been wide-spread research interest in stable polynomials and the geometry of polynomials, accompanied by a variety of new connections to other branches of mathematics (including combinatorics [6], differential equations [4], optimization [36], probability theory [5], applied algebraic geometry [40], theoretical computer science [28, 29] and statistical physics [3]). See also the surveys of Pemantle [32] and Wagner [41]. Stable polynomials are strongly linked to matroid theory [6], as delta-matroids arise from support sets of stable polynomials.

In this paper, we concentrate on the generalized notion of K-stability as introduced in [20]. Given a proper cone $K \subseteq \mathbb{R}^n$, a polynomial $f \in \mathbb{C}[\mathbf{z}] = \mathbb{C}[z_1, \ldots, z_n]$ is called *K-stable* if $\mathcal{I}(f) \cap \operatorname{int} K = \emptyset$, where $\operatorname{int} K$ is the interior of K and $\mathcal{I}(f)$ denotes the imaginary projection of f (as formally defined in Section 2). Note that $(\mathbb{R}_{\geq 0})^n$-stability coincides with the usual stability, and stability with respect to the positive semidefinite cone on the space of symmetric matrices is denoted as *psd-stability*. In the case of a homogeneous polynomial, K-stability of f is equivalent to the containment of $\operatorname{int} K$ in a hyperbolicity cone of f (see Section 2), which also provides a link to hyperbolic programming.

Here, we study conditions and certificates for the K-stability of a given polynomial $f \in \mathbb{C}[\mathbf{z}]$, especially for the case of determinantal polynomials of the form $f(\mathbf{z}) = \det(A_0 + A_1 z_1 + \cdots + A_n z_n)$ with symmetric or Hermitian matrices $A_0, \ldots, A_n$ as well as for quadratic polynomials. A particular focus is on psd-stability.

Specifically, for cones K with a spectrahedral representation we construct a semidefinite feasibility problem, which, in the case of non-emptiness, certifies K-stability of

f. This reduction to a semidefinite problem builds upon two ingredients. Firstly, we characterize certain conic components in the complement of the imaginary projection of the (not necessarily homogeneous) polynomial f. Secondly, the sufficient criterion employs techniques from [23] on containment problems of spectrahedra and positive maps in order to check whether $\operatorname{int} K \subseteq \mathcal{I}(f)^{\mathsf{c}}$. For the special case of usual stability, we will recover the well-known determinantal stability criterion of Borcea and Bränden (see Proposition 2.6 and Remark 4.4) and thus obtain, as a byproduct, an alternative proof of that statement.

In the case of psd-stability, if the sufficient criterion is satisfied, we can explicitly construct a determinantal representation of the given polynomial, see Corollary 4.9. To this end, the determinantal criterion for psd-stability from [20] can be seen as a special case of our more general results. The procedure enables to check and certify the conic stability for a large subclass of polynomials.

Moreover, we show that under certain preconditions, there always exists a positive scaling factor such that the sufficient criterion applies to a scaled version of the polynomial (or, equivalently, a scaled version of the cone). See Theorem 5.2.

The paper is structured as follows. Section 2 provides relevant background on imaginary projections, conic stability and determinantal representations. In Section 3, we study the conic components in the complement of the imaginary projection for the relevant classes of polynomials. Section 4 develops the sufficient criterion for K-stability based on the techniques from positive maps. The scaling result is contained in Section 5, and Section 6 concludes the paper.

Acknowledgments. Part of this work was done while the first and the third author were visiting the Simons Institute for the Theory of Computing within the semester program "Geometry of Polynomials". They are grateful for the inspiring atmosphere there. Thanks to Bernd Sturmfels for encouraging us to work jointly on this topic and to the anonymous referees for very valuable comments.

The first author would like to gratefully acknowledge the financial support through a Simons-Berkeley postdoctoral fellowship and the third author the support through DFG grant TH 1333/7-1.

2. Preliminaries

Throughout the text, bold letters will denote n-dimensional vectors unless noted otherwise.

2.1. **Imaginary projections and conic stability.**

For a polynomial $f \in \mathbb{C}[\mathbf{z}]$, define its *imaginary projection* $\mathcal{I}(f)$ as the projection of the variety of f onto its imaginary part, i.e.,

$$\mathcal{I}(f) = \{\mathsf{Im}(\mathbf{z}) = (\mathsf{Im}(z_1), \ldots, \mathsf{Im}(z_n)) : f(\mathbf{z}) = 0\}, \tag{1}$$

where $\mathsf{Im}(\cdot)$ denotes the imaginary part of a complex number [22].

Let $\mathcal{S}_d, \mathcal{S}_d^+$ and $\mathcal{S}_d^{++}$ denote the set of symmetric $d \times d$ matrices as well as the subsets of positive semidefinite and positive definite matrices. Moreover, let Herm_d be the set of all Hermitian $d \times d$-matrices.

We consider the following generalization of stability. Let K be a *proper* cone in $\mathbb{R}^n$, that is, a full-dimensional, closed and pointed convex cone in $\mathbb{R}^n$.

Definition 2.1. A polynomial $f \in \mathbb{C}[\mathbf{z}]$ is called *K-stable*, if $f(\mathbf{z}) \neq 0$ whenever $\mathsf{Im}(\mathbf{z}) \in \operatorname{int} K$.

If $f \in \mathbb{C}[Z]$ on the symmetric matrix variables $Z = (z_{ij})_{n\times n}$ is $\mathcal{S}_n^+$-stable, then f is called *positive semidefinite-stable* (for short, *psd-stable*).

A stable or K-stable polynomial with real coefficients is called *real stable* or *real K-stable*, respectively.

Remark 2.2. 1. A set of the form $\mathbb{R}^n + iC$, where C is an open convex cone, is called a *Siegel domain (of the first kind).* Siegel domains provide an important concept in function theory of several complex variables and harmonic analysis, see the books [19, 33, 35].

2. The *Siegel upper half-space* (or *Siegel upper half-plane*) $\mathcal{H}_g$ of degree g (or genus g) is defined as

$$\mathcal{H}_g \ = \{A \in \mathbb{C}^{g\times g} \text{ symmetric } \ : \mathsf{Im}(A) \text{ is positive definite}\},$$

where $\mathsf{Im}(A) = (\mathsf{Im}(a_{ij}))_{g\times g}$ (see, e.g., [38, §2]). The Siegel upper half-space occurs in algebraic geometry and number theory as the domain of modular forms. Using that notation, psd-stability can be viewed as stability with respect to the Siegel upper half-space.

A form (i.e., a homogeneous polynomial) $f \in \mathbb{R}[\mathbf{z}]$ is *hyperbolic* in direction $\mathbf{e} \in \mathbb{R}^n$ if $f(\mathbf{e}) \neq 0$ and for every $\mathbf{x} \in \mathbb{R}^n$ the univariate polynomial $t \mapsto f(\mathbf{x}+t\mathbf{e})$ has only real roots. The cone $C(\mathbf{e}) = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}+t\mathbf{e}) = 0 \Rightarrow t < 0\}$ is called the *hyperbolicity cone* of f with respect to $\mathbf{e}$. This cone $C(\mathbf{e})$ is convex, f is hyperbolic with respect to every point $\mathbf{e}' \in C(\mathbf{e})$ and $C(\mathbf{e}) = C(\mathbf{e}')$ (see [11]).

Let f be a hyperbolic polynomial and $C(\mathbf{e})$ denote the hyperbolicity cone containing $\mathbf{e}$. By definition of K-stability, a homogeneous polynomial f is hyperbolic w.r.t. every point $\mathbf{e}' \in C(\mathbf{e})$ if and only if f is $(\operatorname{cl} C(\mathbf{e}))$-stable, where cl denotes the topological closure of a set. The following theorem in [20] reveals the connection between K-stable polynomials and hyperbolic polynomials.

Theorem 2.3. *For a homogeneous polynomial $f \in \mathbb{R}[\mathbf{z}]$, the following are equivalent.*

(1) *f is K-stable.*
(2) $\mathcal{I}(f) \cap \operatorname{int} K = \emptyset$.
(3) *f is hyperbolic w.r.t. every point in* $\operatorname{int} K$.

By [21], the hyperbolicity cones of a homogeneous polynomial f coincide with the components of $\mathcal{I}(f)^{\mathsf{c}}$, where $\mathcal{I}(f)^{\mathsf{c}}$ denotes the complement of $\mathcal{I}(f)$. This implies:

Corollary 2.4. *A hyperbolic polynomial $f \in \mathbb{R}[\mathbf{z}]$ is K-stable if and only if* $\operatorname{int} K \subseteq C(\mathbf{e})$ *for some hyperbolicity direction $\mathbf{e}$ of f.*

Proof. This follows from the observation that a hyperbolic polynomial $f \in \mathbb{R}[\mathbf{z}]$ is K-stable if and only if $\operatorname{int} K \subseteq \mathcal{I}(f)^{\mathsf{c}}$. □

It is shown in [21] that the number of hyperbolicity cones of a homogeneous polynomial $f \in \mathbb{R}[\mathbf{z}]$ is at most 2^d for $d \leq n$ and at most $2\sum_{k=0}^{n-1}\binom{d-1}{k}$ for $d > n$.

2.2. **Determinantal representations.** A *determinantal polynomial* is a polynomial of the form $f(\mathbf{z}) = \det(A_0 + \sum_{j=1}^n A_j z_j)$. For our purposes, we always assume that the matrices $A_0, \ldots, A_n$ are Hermitian unless stated otherwise. If the constant coefficient matrix A_0 is positive definite or the identity, then the determinantal polynomial is called *definite* or *monic* determinantal polynomial, respectively. Helton, McCullough and Vinnikov showed that every polynomial $p \in \mathbb{R}[\mathbf{z}]$ with $p(0) \neq 0$ has a symmetric determinantal representation of the form $p(\mathbf{z}) = \det(A_0 + \sum_{j=1}^n A_j z_j)$ with real symmetric matrices $A_0, \ldots, A_n$ ([17, Theorem 14.1], see also Quarez [34, Theorem 4.4] and, for the earlier result of a not necessarily symmetric determinantal representation, Valiant [37] and its exposition in Bürgisser et al. [9]). Note that A_0 is not necessarily positive definite and not even necessarily positive semidefinite.

In [18] and [30], it was shown that several classes of polynomials have monic determinantal representations due to the connection to real zero polynomials. Here, a polynomial $f \in \mathbb{R}[\mathbf{z}]$ is called *real zero*, if the mapping $t \mapsto f(t \cdot \mathbf{z})$ has only real roots. Brändén has constructed a real zero polynomial for which A_0 cannot be taken to be positive definite in a determinantal representation [7]. Recently, Dey and Pillai [10] added a complete characterization of the quadratic case by also using the connection to real zero polynomials.

Proposition 2.5 ([10])**.** *A quadratic polynomial $f(\mathbf{z}) = \mathbf{z}^T A\mathbf{z} + \mathbf{b}^T\mathbf{z} + 1 \in \mathbb{R}[\mathbf{z}]$ is a real zero polynomial if and only if $Q/(1,1) := A - \frac{1}{4}\mathbf{b}\mathbf{b}^T$ is negative semidefinite. The polynomial $f(\mathbf{z})$ has a monic determinantal representation if and only if at least one of the following conditions holds:*

- *A is negative semidefinite.*
- *$Q/(1,1)$ is negative semidefinite and* $\operatorname{rank}(Q/(1,1)) \leq 3$.

2.3. **Real stable polynomials.** As specified in the Introduction and Section 2.1, a real polynomial f is real stable if it is real K-stable with respect to the non-negative orthant $K = \mathbb{R}^n_+$. This holds true if and only if for every $\mathbf{e} \in \mathbb{R}^n_{>0}$ and $\mathbf{x} \in \mathbb{R}^n$, the univariate polynomial $t \mapsto f(t\mathbf{e}+\mathbf{x})$ is real-rooted. Indeed, a particular prominent class of real stable polynomials is generated from determinantal polynomials as follows.

Proposition 2.6. ([2, Thm. 2.4]) Let $A_1, \ldots, A_n$ be positive semidefinite $d \times d$-matrices and A_0 be a Hermitian $d \times d$-matrix. Then

$$f(\mathbf{z}) \ = \ \det(A_0 + \sum_{j=1}^n A_j z_j)$$

is real stable or the zero polynomial.

It is also known a real polynomial $f \in \mathbb{R}[\mathbf{z}]$ is real stable if and only if the (unique) homogenization polynomial w.r.t. the variable z_0 is hyperbolic w.r.t. every vector $\mathbf{e} \in \mathbb{R}^{n+1}$ such that $e_0 = 0$ and $e_j > 0$ for all $1 \leq j \leq n$ (see [4]).

Example 2.7. The class of homogeneous stable polynomials is contained in the following class of Lorentzian polynomials, see [8, 15]. Let $f \in \mathbb{R}[\mathbf{z}]$ be homogeneous of degree $d \geq 2$ with only positive coefficients. f is called *strictly Lorentzian* if

- $d = 2$ and the *Hessian* $\mathbb{H}(f) = (\partial_i \partial_j f)_{i,j=1}^n$ is non-singular and has exactly one positive eigenvalue (i.e., $\mathbb{H}(f)$ has the *Lorentzian signature* $(1, n-1)$, which expresses that f has one positive eigenvalue and $n-1$ negative eigenvalues [15]),

- or $d>2$ and for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = d-2$, the α-th derivative $\partial^\alpha f$ is strictly Lorentzian.

By convention, in degrees 0 and 1, every polynomial with only positive coefficients is strictly Lorentzian. Limits of strictly Lorentzian polynomials are called *Lorentzian*.

Concerning psd-stability, the following variant of Proposition 2.6 is known.

Proposition 2.8. ([20, Thm. 5.3]) Let $A = (A_{ij})_{n\times n}$ be a Hermitian block matrix with $n \times n$ blocks of size $d \times d$. If A is positive semidefinite and A_0 a Hermitian $d \times d$-matrix, then the polynomial $f(Z) = \det(A_0 + \sum_{i,j=1}^n A_{ij} z_{ij})$ on the set of symmetric $n \times n$-matrices is psd-stable or identically zero.

Determinantal representations of complex polynomials which are stable with respect to the unit ball of symmetric matrices have been studied in [13, 14].

In the present paper, for cones K with a spectrahedral representation, we derive a semidefinite problem, which, in the case of feasibility, certifies K-stability of f. For the case of psd-stability, if that criterion is satisfied, we can explicitly construct the determinantal representation of Proposition 2.8. In this respect, the criterion from Proposition 2.8 can be seen as a special case of our treatment.

The following examples serve to pinpoint some relationships between stable, psd-stable and determinantal polynomials.

Example 2.9. a) A quadratic determinantal polynomial does not need to be stable in order to be psd-stable (with respect to a suitable ordering identification between the variables z_i and the matrix variables z_{jk}). Namely, the determinantal polynomial

$$f(z_1, z_2, z_3) = (z_1 + z_3)^2 - z_2^2 = (z_1 + z_3 - z_2)(z_1 + z_3 + z_2)$$

is not stable, because $(1, 2, 1) \in \mathcal{I}(f) \cap \mathbb{R}^3_{>0}$. However, in the matrix variables $Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}$, the polynomial $f(Z) = f(z_1, z_2, z_3)$ is psd-stable. To see this, observe that by the arithmetic-geometric mean inequality, every $\mathbf{y} \in \mathcal{I}(f) = \{\mathbf{y} \in \mathbb{R}^3 : y_1 + y_3 = y_2 \text{ or } y_1 + y_3 = -y_2\}$ satisfies

$$\det \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix} = \det \begin{pmatrix} y_1 & \pm(y_1 + y_3) \\ \pm(y_1 + y_3) & y_3 \end{pmatrix} = y_1 y_3 - (y_1 + y_3)^2 \le 0$$

and thus $\mathbf{y} \notin \operatorname{int} \mathcal{S}_2^+$.

b) An example of a non-psd-stable determinantal polynomial on 2×2-matrices, i.e., with matrix variables $Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix}$, is $f(Z) = \det \operatorname{Diag}(z_{11}, z_{12}, z_{22}) = z_{11} z_{12} z_{22}$. Namely, since $\mathcal{I}(f) = \{X \in \mathcal{S}_2 : x_{11} x_{12} x_{22} = 0\}$, we have $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{I}(f)$ and thus $\mathcal{I}(f) \cap \operatorname{int} \mathcal{S}_2^+ \neq \emptyset$.

c) Another example of a non-psd-stable determinantal polynomial on 2×2-matrices is the determinant of the spectrahedral representation of the open Lorentz cone $g(\mathbf{z}) = \det \begin{pmatrix} z_1 + z_3 & z_2 \\ z_2 & z_1 - z_3 \end{pmatrix} = z_1^2 - z_2^2 - z_3^2$, where the same variable identification as in a) is used. Note that $g(\mathbf{z}) = 0$ for $\mathbf{z} = (1+2i, 1+i, \sqrt{-3+2i})$ and $\begin{pmatrix} 2 & 1 \\ 1 & \alpha \end{pmatrix} \in \operatorname{int} \mathcal{S}_2^+$ for $\alpha = \mathsf{Im}(\sqrt{-3+2i}) > 1$. Hence, g is not psd-stable.

3. Conic components in the complement of the imaginary projection

To prepare for the conic stability criteria for determinantal and quadratic polynomials, we characterize particular conic components in the complement of the imaginary projection for these classes. Denote by $X \succ 0$ the positive definiteness of a matrix X.

First consider a determinantal polynomial

$$f(\mathbf{z}) \ = \ \det(A_0 + A_1 z_1 + \cdots + A_n z_n) \tag{2}$$

with $A_0, \ldots, A_n \in \mathsf{Herm}_d$. Note that if $A_0 = I$, then the homogenization of f w.r.t. a variable z_0 is hyperbolic w.r.t. $\mathbf{e} = (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$. Moreover, for a homogeneous determinantal polynomial $f = \det(\sum_{j=1}^n A_j z_j)$, if there exists an $\mathbf{e} \in \mathbb{R}^n$ with $\sum_{j=1}^n A_j e_j \succ 0$, then f is hyperbolic w.r.t. $\mathbf{e}$, and the set

$$\{\mathbf{z} \in \mathbb{R}^n \ : \ A_1 z_1 + \cdots + A_n z_n \succ 0\}$$

as well as its negative are hyperbolicity cones of f, see [26, Prop. 2]. If f is irreducible, then these are the only two hyperbolicity cones (see [25]), whereas in the reducible case there can be more (cf. Section 2.1). Let $A(\mathbf{z})$ be the linear matrix pencil $A(\mathbf{z}) = A_0 + \sum_{j=1}^n A_j z_j$. The *initial form* of f, denoted by $\operatorname{in}(f)$, is defined as $\operatorname{in}(f)(\mathbf{z}) = f_h(0, \mathbf{z})$, where f_h is the homogenization of f w.r.t. the variable z_0.

Theorem 3.1. *If f is a degree d determinantal polynomial of the form* (2) *and there exists an $\mathbf{e} \in \mathbb{R}^n$ with $\sum_{j=1}^n A_j e_j \succ 0$, then* $\operatorname{in}(f)$ *is hyperbolic and every hyperbolicity cone of* $\operatorname{in}(f)$ *is contained in $\mathcal{I}(f)^{\mathsf{c}}$.*

Proof. Let $f = \det(A_0 + \sum_{j=1}^n A_j z_j)$ with $A_0, \ldots, A_n \in \mathsf{Herm}_d$. Since f is of degree d, it holds $\operatorname{in}(f) = \det(\sum_{j=1}^n A_j z_j)$. Then $\sum_{j=1}^n A_j e_j \succ 0$ implies that $\operatorname{in}(f)$ is hyperbolic.

First we assume that $\operatorname{in}(f)$ is irreducible. By the precondition $\sum_{j=1}^n A_j e_j \succ 0$, the initial form $\operatorname{in}(f)$ has exactly the two hyperbolicity cones $C_1 = \{\mathbf{x} \in \mathbb{R}^n \ : \ \sum_{j=1}^n A_j x_j \succ 0\}$ and $C_2 = \{\mathbf{x} \in \mathbb{R}^n \ : \ \sum_{j=1}^n A_j x_j \prec 0\}$.

First we show that $C_1 \subseteq \mathcal{I}(f)^{\mathsf{c}}$. For every $\mathbf{x} \in \mathbb{R}^n$, we have

$$f(\mathbf{x} + t\mathbf{e}) \ = \ \det(A_0 + \sum_{j=1}^n A_j x_j + t \sum_{j=1}^n A_j e_j).$$

Since $\sum_{j=1}^n A_j e_j \succ 0$, we obtain

$$f(\mathbf{x} + t\mathbf{e}) \ = \ \det(\sum_{j=1}^n A_j e_j) \det\left((\sum_{j=1}^n A_j e_j)^{-1/2} (A_0 + \sum_{j=1}^n A_j x_j)(\sum_{j=1}^n A_j e_j)^{-1/2} + tI \right).$$

Since $A_0 + \sum_{j=1}^n A_j x_j$ is Hermitian, all the roots of $t \mapsto f(\mathbf{x} + t\mathbf{e})$ are real. Hence, there cannot be a non-real vector $\mathbf{a} + i\mathbf{e}$ with $f(\mathbf{a} + i\mathbf{e}) = 0$, because otherwise setting $\mathbf{x} = \mathbf{a}$ would give a non-real solution to $t \mapsto f(\mathbf{x} + t\mathbf{e})$. Thus, there is a connected component C' in $\mathcal{I}(f)^{\mathsf{c}}$ containing C_1. The case $C_2 \subseteq \mathcal{I}(f)^{\mathsf{c}}$ is symmetric, since $-\mathbf{e} \in C_2$.

To cover also the case of reducible $\mathrm{in}(f)$, it suffices to observe that for reducible $\mathrm{in}(f) = \prod_{j=1}^k h_j$ with irreducible $h_1, \ldots, h_k$, every hyperbolicity cone C of $\mathrm{in}(f)$ is of the form $C = \bigcap_{j=1}^k C_j$ with some hyperbolicity cones C_j of h_j, $1 \le j \le k$. □

Quadratic polynomials. Now let $f \in \mathbb{R}[\mathbf{z}]$ be a quadratic polynomial of the form

$$f \;=\; \mathbf{z}^T A\mathbf{z} + \mathbf{b}^T\mathbf{z} + c \tag{3}$$

with $A \in \mathcal{S}_n$, $\mathbf{b} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. We show that those components of $\mathcal{I}(f)^{\mathsf{c}}$ which are cones, can be described in terms of spectrahedra, as made precise in the following.

First recall the situation of a homogeneous quadratic polynomial $f = \mathbf{z}^T A\mathbf{z}$. By possibly multiplying A with -1, we can assume that the number of positive eigenvalues of A is at least the number of negative eigenvalues. In this setting, it is well known that a non-degenerate quadratic form $f \in \mathbb{R}[\mathbf{z}]$ is hyperbolic if and only if A has signature $(n-1, 1)$ [11].

Specifically, for the normal form

$$f(\mathbf{z}) \;=\; \sum_{j=1}^{n-1} z_j^2 - z_n^2,$$

we have $\mathcal{I}(f) \;=\; \{\mathbf{y} \in \mathbb{R}^n \;:\; y_n^2 \le \sum_{j=1}^{n-1} y_j^2\}$ (see [22]). Hence, there are two unbounded components in the complement $\mathcal{I}(f)^{\mathsf{c}}$, both of which are full-dimensional cones, and these two components are

$$\{\mathbf{y} \in \mathbb{R}^{n-1} \times \mathbb{R}_+ \;:\; \sum_{j=1}^{n-1} y_j^2 < y_n^2\} \;\text{ and }\; \{\mathbf{y} \in \mathbb{R}^{n-1} \times \mathbb{R}_- \;:\; \sum_{j=1}^{n-1} y_j^2 < y_n^2\}.$$

For a general homogeneous quadratic form, this generalizes as follows.

Lemma 3.2. *For a quadratic form $f = \mathbf{z}^T A\mathbf{z} \in \mathbb{R}[\mathbf{z}]$ with A having signature $(n-1, 1)$, the components C of the complement of $\mathcal{I}(f)$ are given by the two components of the set*

$$\{\mathbf{y} \in \mathbb{R}^n \;:\; \mathbf{y}^T A\mathbf{y} < 0\}\,, \tag{4}$$

and the closures of these components are spectrahedra.

The proof makes use of the following property from [22].

Proposition 3.3. *Let $g \in \mathbb{C}[\mathbf{z}]$ and $T \in \mathbb{R}^{n\times n}$ be an invertible matrix. Then, $\mathcal{I}(g(T\mathbf{z})) = T^{-1}\mathcal{I}(g(\mathbf{z}))$.*

Proof of Lemma 3.2. Since $-A$ has Lorentzian signature, there exists $S \in \mathrm{GL}(n, \mathbb{R})$ with $A_I \;:=\; S^T AS \;=\; \mathrm{Diag}(1, \ldots, 1, -1)$. Observing

$$\mathcal{I}(f(S\mathbf{z})) \;=\; \mathcal{I}(\mathbf{z}^T A_I\mathbf{z}) \;=\; \{\mathbf{y} \in \mathbb{R}^n \;:\; y_n^2 \le \sum_{j=1}^{n-1} y_j^2\} \;= \{\mathbf{y} \in \mathbb{R}^n \;:\; \mathbf{y}^T A_I\mathbf{y} \ge 0\}\,,$$

Proposition 3.3 then gives

$$\mathcal{I}(f(\mathbf{z})) \;=\; S \cdot \mathcal{I}(f(S\mathbf{z})) \;=\; \{S \cdot \mathbf{y} \in \mathbb{R}^n \;:\; \mathbf{y}^T A_I\mathbf{y} \ge 0\} \;=\; \{\mathbf{y} \in \mathbb{R}^n \;:\; \mathbf{y}^T A\mathbf{y} \ge 0\}\,.$$

□

For the general, not necessarily homogeneous case, recall that every quadric in $\mathbb{R}^n$ is affinely equivalent to a quadric given by one of the following polynomials,

$$\begin{array}{lll} \text{(I)} & \sum_{j=1}^{p} z_j^2 - \sum_{j=p+1}^{r} z_j^2 & (1 \le p \le r,\, r \ge 1,\, p \ge \frac{r}{2}), \\ \text{(II)} & \sum_{j=1}^{p} z_j^2 - \sum_{j=p+1}^{r} z_j^2 + 1 & (0 \le p \le r,\, r \ge 1), \\ \text{(III)} & \sum_{j=1}^{p} z_j^2 - \sum_{j=p+1}^{r} z_j^2 + z_{r+1} & (1 \le p \le r,\, r \ge 1,\, p \ge \frac{r}{2}). \end{array}$$

We refer to [1] as a general background reference for real quadrics. We say that a given quadratic polynomial $f \in \mathbb{R}[\mathbf{z}]$ is of type X if it can be transformed to the normal form X by an affine real transformation.

The homogeneous case, case (I), has already been treated, and by [22], it is known that in case (III), the imaginary projection does not contain a full-dimensional component in $\mathcal{I}(f)^{\mathsf{c}}$.

By [22], in case (II), unbounded components only exist in the cases $p = 1$ and $p = r - 1$, so we can restrict to these cases. We list these relevant two cases from [22].

Theorem 3.4. *Let $n \ge r \ge 3$ and $f \in \mathbb{R}[\mathbf{z}]$ be a quadratic polynomial. If f is of type* (II)*, then*

$$\mathcal{I}(f) \;=\; \begin{cases} \{\mathbf{y} \in \mathbb{R}^n \;:\; y_1^2 - \sum_{j=2}^{r} y_j^2 \le 1\} & \text{if } p = 1, \\ \{\mathbf{y} \in \mathbb{R}^n \;:\; \sum_{j=1}^{r-1} y_j^2 > y_r^2\} \cup \{\mathbf{0}\} & \text{if } p = r-1. \end{cases} \tag{5}$$

For the proof see [22]. Since the proofs of the case $p = 1$ and of the case $p = r - 1$ differ in some important details, which are not carried out there, we include a proof here for the convenience of the reader.

Proof. Without loss of generality we can assume $r = n$. Writing $z_j = x_j + iy_j$, we have $f(\mathbf{z}) = \sum_{j=1}^{p} z_j^2 - \sum_{j=p+1}^{n} z_j^2 + 1 = 0$ if and only if

$$\sum_{j=1}^{p} x_j^2 - \sum_{j=p+1}^{n} x_j^2 - \sum_{j=1}^{p} y_j^2 + \sum_{j=p+1}^{n} y_j^2 + 1 \;=\; 0 \tag{6}$$

$$\text{and} \sum_{j=1}^{p} x_j y_j - \sum_{j=p+1}^{n} x_j y_j \;=\; 0. \tag{7}$$

Set $\alpha := -\sum_{j=1}^{p} y_j^2 + \sum_{j=p+1}^{n} y_j^2 + 1$, and let $\mathbf{y} \in \mathbb{R}^n$ be fixed. Note that in both cases $p = 1$ and $p = n - 1$, we have $\mathbf{0} \in \mathcal{I}(f)$, since $f(\mathbf{x} + i \cdot \mathbf{0}) = 0$ for $\mathbf{x} = (0, \ldots, 0, 1)$. Hence, we can assume $\mathbf{y} \neq \mathbf{0}$.

Case $p = 1$: Write $\mathbf{x} = (x_1, \mathbf{x}') = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, \mathbf{y}') = (y_1, y_2, \ldots, y_n)$. Observe the rotational symmetry of (6) w.r.t. $\mathbf{x}'$ and $\mathbf{y}'$ and the invariance of the standard scalar product $(\mathbf{x}', \mathbf{y}') \mapsto \sum_{j=2}^{n} x_j y_j$ under orthogonal transformations. Hence, if $((x_1, \mathbf{x}'), (y_1, \mathbf{y}'))$ is a solution of (6) and (7), then for any $T \in \mathrm{SO}(n{-}1)$, the point $((x_1, T\mathbf{x}'), (y_1, T\mathbf{y}'))$ is a solution as well, where $\mathrm{SO}(n{-}1)$ denotes the special orthogonal group of order $n - 1$. Thus, we can assume $y_3 = \cdots = y_n = 0$, and α simplifies to $\alpha = -y_1^2 + y_2^2 + 1$. Solving (7) for x_1 (by assuming, without loss of generality, $y_1 \neq 0$) yields $x_1 = \frac{x_2 y_2}{y_1}$ and substituting this into (6) then

$$0 \;=\; \left(\frac{y_2^2}{y_1^2} - 1\right) x_2^2 - \sum_{j=3}^{n} x_j^2 + \alpha \;=\; \frac{(\alpha - 1)x_2^2}{y_1^2} - \sum_{j=3}^{n} x_j^2 + \alpha.$$

This equation has a real solution $(x_2, \ldots, x_n)$ if and only if $\alpha \geq 0$, which shows $\mathcal{I}(f) = \{\mathbf{y} \in \mathbb{R}^n \ : \ y_1^2 - \sum_{j=2}^n y_j^2 \leq 1\}$.

Case $p = n-1$: Following the same proof strategy, we now write $x = (\mathbf{x}', x_n) = (x_1, x_2, \ldots, x_n)$ and $y = (\mathbf{y}', y_n) = (y_1, y_2, \ldots, y_n)$. Then the symmetry of the problem allows to assume $y_2 = \cdots = y_{n-1} = 0$, and α simplifies to $\alpha = -y_1^2 + y_n^2 + 1$. If $y_1 \neq 0$, solving (7) for x_1 gives $x_1 = \frac{x_n y_n}{y_1}$, and a substitution into (6)

$$0 \ = \ \left(\frac{y_n^2}{y_1^2} - 1\right) x_n^2 + \sum_{j=2}^{n-1} x_j^2 + \alpha \ = \ \frac{(\alpha-1)x_2^2}{y_1^2} + \sum_{j=2}^{n-1} x_j^2 + \alpha.$$

There exists a real solution $(x_2, \ldots, x_n)$ if and only if $\alpha < 1$, which, taking also into account the special case $y_1 = 0$, gives $\mathcal{I}(f) = \{\mathbf{y} \in \mathbb{R}^n \ : \ \sum_{j=1}^{n-1} y_j^2 > y_n^2\} \cup \{\mathbf{0}\}$. ☐

For the inhomogeneous case, we use the following lemma to reduce it to the homogeneous case.

Lemma 3.5. *Let $n \geq 3$ and $f \in \mathbb{R}[\mathbf{z}]$ be quadratic of the form* (3)*.*

If f is of type (II) with $p = 1$, then $\mathcal{I}(f)^{\mathsf{c}}$ does not have connected components whose closures contain full-dimensional cones.

If f is of type (II) with $p = n-1$ then every full-dimensional cone which is contained in $\mathcal{I}(f)^{\mathsf{c}}$ is contained in the closure of a hyperbolicity cone of $\mathrm{in}(f)$*.*

Note, that in particular, that $\mathcal{I}(f)^{\mathsf{c}}$ does not contain a point at all if and only if $\mathrm{in}(f)$ is not hyperbolic.

Proof. If f is of type (II) with $p = 1$, then the statement is a consequence of (5).

Now consider the case that f is of type (II) with $p = n-1$ and let C be full-dimensional cone which is contained in a component of $\mathcal{I}(f)^{\mathsf{c}}$. By [21, Theorem 4.2 and Lemma 4.3], $\mathrm{int}\, C$ is contained in a hyperbolicity cone of $\mathrm{in}(f)$. ☐

Hence, among the quadratic polynomials of type (II), only the ones with $p = n-1$ might possibly be K-stable.

Theorem 3.6. *Let $n \geq 3$ and $f \in \mathbb{R}[\mathbf{z}]$ be quadratic of the form* (3) *and of type (II) with $p = n-1$. Then there exists a linear form $\ell(\mathbf{z})$ in $\mathbf{z}$ such that $-\ell(\mathbf{z})^{n-2}\mathrm{in}(f)$ has a determinantal representation. In particular, the closure of each unbounded component of $\mathcal{I}(f)^{\mathsf{c}}$ is a spectrahedral cone.*

The theorem can be seen as an adaption of the well-known result that hyperbolic quadratic forms have determinantal representations. See, e.g., [39, Section 2] or [30, Example 2.16] for the determinantal representations which underlie that result and which are utilized in the subsequent proof.

Proof. First consider the normal form of type (II) with $p = n-1$,

$$g(\mathbf{z}) \ = \ \sum_{j=1}^{n-1} z_j^2 - z_n^2 + 1\,.$$

By (5), the complement of $\mathcal{I}(g)$ has the two unbounded conic components

$$\{\mathbf{y} \in \mathbb{R}^{n-1} \times \mathbb{R}_+ : \sum_{j=1}^{n-1} y_j^2 \leq y_n^2\} \setminus \{0\} \text{ and } \{\mathbf{y} \in \mathbb{R}^{n-1} \times \mathbb{R}_- : \sum_{j=1}^{n-1} y_j^2 \leq y_n^2\} \setminus \{0\},$$

which (up to the origin) are the open Lorentz cone and its negative. Their closures are exactly the closures of the hyperbolicity cones of the initial form $\mathrm{in}(g)$ of g. It is well-known that the open Lorentz cone has the spectrahedral representation

$$L(\mathbf{z}) := \left(\begin{array}{ccc|c} & & & z_1 \\ & z_n I & & \vdots \\ & & & z_{n-1} \\ \hline z_1 & \cdots & z_{n-1} & z_n \end{array} \right) \succ 0 , \tag{8}$$

and thus we also have $z_n^{n-2} \mathrm{in}(g) = -\det(L(\mathbf{z}))$. Since g results from f by an affine transformation, the initial form $\mathrm{in}(g)$ results from the initial form $\mathrm{in}(f)$ by a linear transformation,

$$\mathrm{in}(g)(T\mathbf{z}) = \mathrm{in}(f)(\mathbf{z})$$

for some matrix $T \in \mathrm{GL}(n, \mathbb{R})$. Hence, we obtain the spectrahedral representation for one of the unbounded conic components in $\mathcal{I}(f)^{\mathsf{c}}$,

$$F(\mathbf{z}) := \left(\begin{array}{ccc|c} & & & (T\mathbf{z})_1 \\ & (T\mathbf{z})_n I & & \vdots \\ & & & (T\mathbf{z})_{n-1} \\ \hline (T\mathbf{z})_1 & \cdots & (T\mathbf{z})_{n-1} & (T\mathbf{z})_n \end{array} \right) \succ 0 ,$$

as well as its negative. Moreover,

$$-\det F(\mathbf{z}) = ((T\mathbf{z})_n)^{n-2} \, \mathrm{in}(f) ,$$

so that $(T\mathbf{z})_n$ provides the desired linear form $\ell(\mathbf{z})$. □

Remark 3.7. Concerning $L(\mathbf{z})$ in (8), by subtracting $\frac{z_j}{z_n}$ times the j-th row from its n-th row for every $j \in \{1, \ldots, n-1\}$, we obtain

$$\det(L(\mathbf{z})) = \det \left(\begin{array}{ccc|c} & & & z_1 \\ & z_n I & & \vdots \\ & & & z_{n-1} \\ \hline 0 & \cdots & 0 & z_n - \frac{1}{z_n} \sum_{i=1}^{n-1} z_i^2 \end{array} \right) = z_n^{n-2} \left(z_n^2 - \sum_{i=1}^{n-1} z_i^2 \right) .$$

By (3), in the proof we have $\mathrm{in}(f) = \mathbf{z}^T A \mathbf{z}$. Let $A = LDL^T$ be an LDL^T decomposition of A with $D = \mathrm{Diag}(d_1, \ldots, d_{n-1}, d_n)$ such that $d_1, \ldots, d_{n-1} > 0$ and $d_n < 0$. Then the variable transformation T in the proof is

$$T = \mathrm{Diag}(\sqrt{d_1}, \ldots, \sqrt{d_{n-1}}, \sqrt{|d_n|}) \cdot L^T$$

and we derive

$$A = T^T \cdot \left(\begin{array}{ccc|c} & & & 0 \\ & I & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & -1 \end{array} \right) \cdot T.$$

Example 3.8. Consider $f(z_1, z_2, z_3, z_4) = -15z_1^2 - 12z_1z_4 + z_2^2 + z_3^2 = \mathbf{z}^T A\mathbf{z}$ with

$$A = \begin{pmatrix} -15 & 0 & 0 & -6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -6 & 0 & 0 & 0 \end{pmatrix}.$$

For $\ell(\mathbf{z}) = 4z_1 + 2z_4$, a representation from Theorem 3.6 is

$$-\ell(\mathbf{z})^2 \cdot f(\mathbf{z}) = \det \begin{pmatrix} 4z_1 + 2z_4 & 0 & 0 & z_1 + 2z_4 \\ 0 & 4z_1 + 2z_4 & 0 & z_2 \\ 0 & 0 & 4z_1 + 2z_4 & z_3 \\ z_1 + 2z_4 & z_2 & z_3 & 4z_1 + 2z_4 \end{pmatrix}.$$

□

Remark 3.9. A quadratic polynomial $f \in \mathbb{R}[\mathbf{z}]$ is of the form (3) and of type (II) with $p = n - 1$ (i.e., $-f$ has Lorentzian signature) if and only if $f \in \mathbb{R}[\mathbf{z}]$ is a real zero polynomial, see for example [10].

Remark 3.10. For the case of homogeneous polynomials, Theorem 3.6 recovers the known fact that hyperbolicity cones defined by homogeneous quadratic polynomials f are spectrahedral [30]. In the affine setting, we can homogenize the type (II) polynomial f w.r.t. variable z_0 and get a quadratic polynomial of type (I) in $n + 1$ variables with $p = n$. Then, using $\mathrm{in}(f_h) = f_h$, Theorem 3.6 recovers that *the rigidly convex sets* (introduced by Helton-Vinnikov [18]) defined by real zero polynomials f are spectrahedra [30].

Remark 3.11. The proof of Theorem 3.6 explicitly explains a technique to compute a suitable linear factor $\ell(\mathbf{z})$ as well as a determinantal representation to get a spectrahedral structure.

4. Conic stability and positive maps

Based on the characterizations of the conic components in the complement of $\mathcal{I}(f)$, we now study the problem whether f is K-stable, in particular, whether it is psd-stable.

In order to decide whether the cone K is contained in one of the components of $\mathcal{I}(f)^{\mathsf{c}}$, observe that in the case of spectrahedral representations of K and of the components of $\mathcal{I}(f)^{\mathsf{c}}$, the problem of K-stability can be phrased as a containment problem for spectrahedra. The theory of positive and completely positive maps (as detailed in [31]) provides a sufficient condition for the containment problem of spectrahedra, see [16, 23, 24].

Definition 4.1. Given two linear subspaces $\mathcal{U} \subseteq \mathsf{Herm}_k$ and $\mathcal{V} \subseteq \mathsf{Herm}_l$ (or $\mathcal{U} \subseteq \mathcal{S}_k$ and $\mathcal{V} \subseteq \mathcal{S}_l$), a linear map $\Phi : \mathcal{U} \to \mathcal{V}$ is called *positive* if $\Phi(U) \succeq 0$ for any $U \in \mathcal{U}$ with $U \succeq 0$.

For $d \geq 1$, define the *d-multiplicity map* Φ_d on the set of all Hermitian $d \times d$ block matrices with symmetric $n \times n$-matrix entries by

$$(A_{ij})_{i,j=1}^d \mapsto \left(\Phi\left(A_{ij}\right)\right)_{i,j=1}^d.$$

The map Φ is called *d-positive* if the d-multiplicity map Φ_d (viewed as a map on a Hermitian matrix space) is a positive map. Φ is called *completely positive* if Φ_d is a positive map for all $d \geq 1$.

Let $U(\mathbf{x}) = \sum_{j=1}^n U_j x_j$ and $V(\mathbf{x}) = \sum_{j=1}^n V_j x_j$ be homogeneous linear pencils with symmetric matrices of size $k \times k$ and $l \times l$, respectively (since the matrices are symmetric, we prefer to denote the variables by $\mathbf{x}$ rather than $\mathbf{z}$). Then the spectrahedra $S_U := \{\mathbf{x} \in \mathbb{R}^n : U(\mathbf{x}) \succeq 0\}$, and $S_V := \{\mathbf{x} \in \mathbb{R}^n : V(\mathbf{x}) \succeq 0\}$ are cones. Further, let $\mathcal{U} = \operatorname{span}(U_1, \ldots, U_n) \subseteq \mathcal{S}_k$ and $\mathcal{V} = \operatorname{span}(V_1, \ldots, V_n) \subseteq \mathcal{S}_l$.

If $U_1, \ldots, U_n$ are linearly independent, then the linear mapping $\Phi_{UV} : \mathcal{U} \to \mathcal{V}$, $\Phi_{UV}(U_i) := V_i$, $1 \leq i \leq n$, is well defined.

Proposition 4.2 ([23])**.** *Let $U_1, \ldots, U_n \subseteq \mathsf{Herm}_k$ (or, $U_1, \ldots, U_n \subseteq \mathcal{S}_k$, respectively) be linearly independent and $S_U \neq \emptyset$. Then for the properties*

(1) *the semidefinite feasibility problem*

$$C = (C_{ij})_{i,j=1}^k \succeq 0 \text{ and } V_p = \sum_{i,j=1}^k (U_p)_{ij} C_{ij} \text{ for } p = 1, \ldots, n \tag{9}$$

has a solution with Hermitian (respectively symmetric) matrix C,

(2) *Φ_{UV} is completely positive,*

(3) *Φ_{UV} is positive,*

(4) *$S_U \subseteq S_V$,*

the implications and equivalences (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longleftrightarrow$ (4) *hold, and if $\mathcal{U}$ contains a positive definite matrix,* (1) $\Longleftrightarrow$ (2).

Note that the statement (1) $\Longrightarrow$ (4) (which does not involve the definition of Φ_{UV}) is also valid without the assumption of linear independence of $U_1, \ldots, U_n$ (see [16, 23]).

So, in case the cone K and the conic components of $\mathcal{I}(f)^{\mathsf{c}}$ can be described in terms of spectrahedra, we can approach the conic stability problem in terms of the block matrix $C \succeq 0$ in (9), the so-called *Choi matrix*, corresponding to an appropriate positive map Φ, which maps the underlying pencils of those spectrahedra onto each other certifying their containment. This sufficient condition is provided by a certain semidefinite feasibility problem whose non-emptiness of its feasible domain thus provides a sufficient criterion for psd-stability.

Moreover, if we know a spectrahedral description of some of the components of $\mathcal{I}(f)^{\mathsf{c}}$ (as in the quadratic case or the determinantal case), the sufficient containment criterion is based on writing a matrix pencil for these components using linear combinations of the matrices of a linear matrix pencil for K. As formalized in Theorem 4.8 and Corollary 4.9, taking the determinant of a matrix pencil for a suitable component of $\mathcal{I}(f)^{\mathsf{c}}$ provides a particular determinantal description for the homogeneous part of the given polynomial f. That description has exactly the structure of the sufficient determinantal criterion for psd-stability and thus provides an elegant determinantal representation that certifies the psd-stability of a homogeneous polynomial f.

Let K be a cone which is given as the positive semidefiniteness region of a linear matrix pencil $M(\mathbf{x}) = \sum_{j=1}^n M_j x_j$ with symmetric $l \times l$-matrices (since K is a cone in $\mathbb{R}^n$, we prefer to denote the variables by $\mathbf{x}$ rather than $\mathbf{z}$). In the case of usual stability, the cone K is the positive semidefiniteness region of the linear matrix pencil

$$M^{\geq 0}(\mathbf{x}) \;=\; \sum_{j=1}^{n} M_j^{\geq 0} x_j \tag{10}$$

with $M_j^{\geq 0} = E_{jj}$, where E_{ij} is the matrix with a one in position (i,j) and zeros elsewhere. In the case of psd-stability, the matrix pencil is

$$M^{\mathrm{psd}}(X) \;=\; \sum_{i,j=1}^{n} M_{ij}^{\mathrm{psd}} x_{ij} \tag{11}$$

with symmetric matrix variables $X = (x_{ij})$ and $M_{ij}^{\mathrm{psd}} = \frac{1}{2}(E_{ij} + E_{ji})$, i.e., $M^{\mathrm{psd}}(X)$ is the matrix pencil $M^{\mathrm{psd}}(X) = (x_{ij})_{ij}$ in the symmetric matrix variables x_{ij}.

Theorem 4.3. *Let $f = \det(A_0 + \sum_{j=1}^n A_j z_j)$ with Hermitian matrices $A_0, \ldots, A_n$ be a degree d determinantal polynomial of the form* (2) *such that* $\mathrm{in}(f)$ *is irreducible and there exists $\mathbf{e} \in \mathbb{R}^n$ with $\sum_{j=1}^n A_j e_j \succ 0$. Let $M(\mathbf{x}) = \sum_{j=1}^n M_j x_j$ with symmetric $l \times l$-matrices be a pencil of the cone K. If there exists a Hermitian block matrix $C = (C_{ij})_{i,j=1}^l$ with blocks C_{ij} of size $d \times d$ and*

$$C = (C_{ij})_{i,j=1}^l \;\succeq\; 0, \quad \forall p = 1, \ldots, n \;:\; \sigma A_p \;=\; \sum_{i,j=1}^{l} (M_p)_{ij} C_{ij} \tag{12}$$

for some $\sigma \in \{-1, 1\}$, then f is K-stable. Deciding whether such a block matrix C exists is a semidefinite feasibility problem.

Note that a necessary condition of K-stability of f is obtained as follows. Fix any vector $\mathbf{v}$ in the interior of the cone K. Then a necessary condition for K-stability is that $\mathbf{v}$ is contained in the complement of $\mathcal{I}(f)$.

Proof. Let C be a block matrix $C = (C_{ij})_{i,j=1}^l$ with $d \times d$-blocks and which satisfies (12) for some $\sigma \in \{-1, 1\}$. The initial form $\mathrm{in}(f)$ is hyperbolic and, by Theorem 3.1, every hyperbolicity cone of $\mathrm{in}(f)$ is contained in $\mathcal{I}(f)^{\mathsf{c}}$. So, in order to show K-stability of f, it suffices to show that K is contained in the closure of a hyperbolicity cone of $\mathrm{in}(f)$, i.e., in the closure of a component of $\mathcal{I}(\mathrm{in}(f))^{\mathsf{c}}$.

As recorded at the beginning of Section 3, since $\mathrm{in}(f)$ is irreducible, $\mathrm{in}(f)$ has exactly two hyperbolicity cones, and these are given by $A^h(\mathbf{x}) = \sum_{j=1}^n A_j x_j \succ 0$ as well as $A^h(\mathbf{x}) = \sum_{j=1}^n A_j x_j \prec 0$.

By Proposition 4.2, if (12) is satisfied, say with $\sigma = 1$, then the spectrahedron given by the matrix pencil $M(\mathbf{x})$ is contained in the closure of $\mathcal{I}(\mathrm{in}(f))^{\mathsf{c}}$. For the service of the reader, we provide an explicit derivation of this step in our setting. Namely, for $\mathbf{x}$ in the spectrahedron defined by $M(\mathbf{x})$, we have

$$A^h(\mathbf{x}) \;=\; \sum_{p=1}^{n} A_p x_p \;=\; \sum_{p=1}^{n} x_p \sum_{i,j=1}^{l} (M_p)_{ij} C_{ij} \tag{13}$$

$$\;=\; \sum_{i,j=1}^{l} (M(\mathbf{x}))_{ij} C_{ij}. \tag{14}$$

Apply the Khatri-Rao product (where the blocks of $M(\mathbf{x})$ are of size 1×1 and the blocks of C are of size $d \times d$). Since $M(\mathbf{x})$ and C are positive semidefinite, the Khatri-Rao product

$$M(\mathbf{x}) * C \ := \ ((M(\mathbf{x}))_{ij} \otimes C_{ij})_{i,j=1}^{l} \ = \ ((M(\mathbf{x}))_{ij} C_{ij})_{i,j=1}^{l}$$

is positive semidefinite as well; see Liu [27], where this property is stated on the space of symmetric positive semidefinite matrices. Since $M(\mathbf{x})$ is a real symmetric pencil, Liu's result carries over to our situation of a Hermitian positive semidefinite matrix C by employing that a Hermitian matrix $Z = X + iY$ with $X \in \mathcal{S}_k$ and Y skew-symmetric is positive semidefinite if and only if the real symmetric matrix

$$\begin{pmatrix} X & -Y \\ Y & Z \end{pmatrix} \ \in \ \mathcal{S}_{2k}$$

is positive semidefinite (see, e.g., [12]).

Altogether, since

$$A^h(\mathbf{x}) \ = \ (I \cdots I)(M(\mathbf{x}) * C) \begin{pmatrix} I \\ \vdots \\ I \end{pmatrix},$$

$A^h(\mathbf{x})$ is positive semidefinite as well. Hence, $\mathbf{x}$ is contained in the spectrahedron defined by $A^h(\mathbf{x})$. Since $A^h(\mathbf{x})$ is the matrix pencil of the closure of a component of $\mathcal{I}(\mathrm{in}(f))^{\mathrm{c}}$, the claim follows. □

Note that the constant coefficient matrix A_0 does not play any role for the criterion in Theorem 4.3. This comes from Theorem 3.1 and its proof, where only the Hermitian property of A_0 matters rather than the exact values of the coefficients themselves.

Remark 4.4. In the special case of usual stability, Theorem 4.3 provides a new proof for Borcea and Brändén's determinantal criterion from Proposition 2.6. Namely, for usual stability, K is given by (10) and thus, a matrix C satisfying the hypothesis of Theorem 4.3 can be viewed as a a block diagonal matrix $C = (C_{ij})_{i=1}^{l}$ with diagonal blocks C_{ii} of size $d \times d$ and vanishing non-diagonal blocks C_{ij} $(i \neq j)$. Since the condition (12) specializes to

$$A_p = C_{pp} \quad \text{for } p = 1, \ldots, n,$$

the stability criterion in Theorem 4.3 is satisfied if and only if the matrices $A_1, \ldots, A_n$ are positive semidefinite.

Remark 4.5. Theorem 4.3 gives a sufficient criterion, but it is not necessary. As a counterexample, consider the following adaption from an example in [16, Example 3.1, 3.4] and [23, Section 6.1]. Let $K \subseteq \mathbb{R}^3$ be the Lorentz cone as given by (8). The polynomial

$$f \ = \ \det \begin{pmatrix} z_1 + z_3 & z_2 \\ z_2 & -z_1 + z_3 \end{pmatrix} \ = \ z_3^2 - z_1^2 - z_2^2$$

(whose underlying matrix pencil provides an alternative matrix pencil for the Lorentz cone) has all its zeroes on the boundary of the Lorentz cone or on its negative. Hence, f is K-stable, but by the results in [16] and [23], the condition (12) is not satisfied.

Example 4.6. i) Let $g(z_1, z_2, z_3) := 31z_1^2+32z_1z_3+8z_3^2-8z_1z_2-16z_2^2$. A determinantal representation of g is given by $\det \begin{pmatrix} 4z_1+2z_3 & z_1+4z_2 \\ z_1+4z_2 & 8z_1+4z_3 \end{pmatrix}$, and at $\mathbf{z} = (0,0,1)^T$, the matrix polynomial is positive definite. Let $M(\mathbf{x})$ denote the linear matrix pencil of the psd cone $\mathcal{S}_2^+$. Then the psd-stability of g follows from Theorem 4.3 and by the matrix

$$C = \begin{pmatrix} 4 & 1 & 0 & 2 \\ 1 & 8 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 4 \end{pmatrix} \succeq 0.$$

ii) Let $f = \sum_{i,j=1}^2 M_{ij}^{\text{psd}} x_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x_{11} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x_{12} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x_{22}$ be the canonical matrix polynomial of the 2×2-psd cone. Clearly, f is psd-stable, and the following consideration shows that this is also recognized by the sufficient criterion. For symmetric 2×2-matrices, the condition in Theorem 4.3 requires to find a block matrix $C \succeq 0$ with 2×2 blocks of size 2×2 such that

$$M_{pq}^{\text{psd}} = \sum_{i,j=1}^{2} (M_{pq}^{\text{psd}})_{ij} C_{ij} \quad \text{for } 1 \le p,q \le 2. \tag{15}$$

This yields $C_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $C_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $C_{12} + C_{21} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ is symmetric, C_{12} must be of the form $\begin{pmatrix} 0 & \gamma \\ \delta & 0 \end{pmatrix}$ with $\gamma, \delta \in \mathbb{R}$. Positive semidefiniteness of C then implies $\delta = 0$, and further, the condition on $C_{12}+C_{21}$ gives $\gamma = 1$. Hence, the matrix

$$C = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

satisfies (15) and thus certifies the psd-stability of f in view of the sufficient criterion in Theorem 4.3.

For quadratic polynomials, we can provide the following criterion. As in the proof of Theorem 3.6, for a homogeneous quadratic polynomial $f(\mathbf{z}) = \mathbf{z}^T A \mathbf{z}$ of signature $(n-1,1)$, we consider

$$F(\mathbf{x}) := \sum_{p=1}^{n} F_p x_p := \left(\begin{array}{ccc|c} & & & (T\mathbf{x})_1 \\ & (T\mathbf{x})_n I & & \vdots \\ & & & (T\mathbf{x})_{n-1} \\ \hline (T\mathbf{x})_1 & \cdots & (T\mathbf{x})_{n-1} & (T\mathbf{x})_n \end{array} \right) \succ 0, \tag{16}$$

where T is as in that proof.

Theorem 4.7. *Let $n \ge 3$ and f be a quadratic polynomial of the form* (3)*, let f be of type (II) with A having signature $(n-1,1)$ and $\text{in}(f)$ be irreducible. Let $M(\mathbf{x})$ be a matrix pencil for the cone K, and let T and $F(\mathbf{x}) := \sum_{p=1}^n F_p x_p$ be defined as in* (16) *w.r.t.* in(f). *If there exists a block matrix* $C = (C_{ij})_{i=1}^{l}$ *with blocks* C_{ij} *of size* $d \times d$ *and*

$$C = (C_{ij})_{i,j=1}^{l} \succeq 0, \quad \forall p = 1, \ldots, n \,:\, \sigma F_p = \sum_{i,j=1}^{l} (M_p)_{ij} C_{ij} \tag{17}$$

for some $\sigma \in \{-1, 1\}$, *then* f *is* K*-stable. Deciding whether such a block matrix* C *exists is a semidefinite feasibility problem.*

Proof. By Theorem 3.6 and its proof, the unbounded components of $\mathcal{I}(f)^{\mathsf{c}}$ which are full-dimensional cones are exactly the hyperbolicity cones of in(f). For $\mathbf{x}$ in the spectrahedron defined by $M(\mathbf{x}) \succeq 0$, we have

$$F(\mathbf{x}) = \sum_{p=1}^{n} F_p x_p = \sum_{p=1}^{n} x_p \sum_{i,j=1}^{l} (M_p)_{ij} C_{ij} = \sum_{i,j=1}^{l} (M(\mathbf{x}))_{ij} C_{ij}.$$

Analogous to the application of the Khatri-Rao product in the proof of Theorem 4.3, this yields $F(\mathbf{x}) \succeq 0$. Hence, f is K-stable. □

Theorem 4.8. *Let* $n \geq 3$ *and* $f(\mathbf{z}) = \mathbf{z}^T A \mathbf{z}$ *be an irreducible homogeneous quadratic polynomial of signature* $(n-1, 1)$, $M(\mathbf{z})$ *be a matrix pencil for the cone* K, *and let* T *and* $F(\mathbf{z}) := \sum_{p=1}^{n} F_p z_p$ *be defined as in* (16). *If there exists a block matrix* $C = (C_{ij})_{i=1}^{l}$ *with blocks* C_{ij} *of size* $d \times d$ *satisfying*

$$C = (C_{ij})_{i,j=1}^{l} \succeq 0, \quad \forall p = 1, \ldots, n \,:\, \sigma F_p = \sum_{i,j=1}^{l} (M_p)_{ij} C_{ij} \tag{18}$$

for some $\sigma \in \{-1, 1\}$, *then there exists a linear form* $\ell(\mathbf{z})$ *such that* $-\ell(\mathbf{z})^{n-2} f$ *has a determinantal representation*

$$-\sigma \ell(\mathbf{z})^{n-2} f = \det(\sum_{p=1}^{n} z_p \sum_{i,j=1}^{l} (M_p)_{ij} C_{ij})$$

with positive semidefinite matrices C_{ij}. *The representation provides a certificate for the* K*-stability of* f.

Proof. The K-stability was shown in Theorem 4.7. By (18) and the definition of $F(\mathbf{z})$, we have

$$\sigma \det F(\mathbf{z}) = \det\big(\sum_{p=1}^{n} z_p \sum_{i,j=1}^{l} (M_p)_{ij} C_{ij}\big).$$

Since $\det F(\mathbf{z}) = -((T\mathbf{z})_n)^{n-2} f$, the choice $\ell(\mathbf{z}) := (T\mathbf{z})_n$ provides the desired representation. This provides a certificate for the K-stability of f. □

Corollary 4.9. *Let* $n \geq 2$ *and* $f(Z)$ *be a homogeneous quadratic polynomial on symmetric* $n \times n$*-variables, in the linearized vector* $\mathbf{z} = (z_1, \ldots, z_N)$ *let* $f = \mathbf{z}^T A \mathbf{z}$ *with* $A \in \mathbb{R}^{N \times N}$ *of signature* $(N-1, 1)$. *If* $M(\mathbf{z})$ *is a matrix pencil for the psd-cone and* C

is a block matrix satisfying (18)*, then for some linear form* $\ell(\mathbf{z})$ *in* $\mathbf{z}$*, the polynomial* $-\ell(\mathbf{z})^{N-2}f$ *has a determinantal representation of the form*

$$-\ell(\mathbf{z})^{N-2}f \;=\; \det\big(\sum_{i,j=1}^{l} C_{ij}z_{ij}\big)$$

with positive semidefinite matrices C_{ij}*. This representation provides a certificate for the psd-stability of* f *in the sense of the sufficient criterion for psd-stability.*

Proof. This is a consequence of Theorem 4.8. □

5. Certifying K-stability with respect to scaled cones

The sufficient criterion does not capture all the cases of K-stable polynomials. Here, we extend our techniques to scaled versions of the cone. To this end, we will reduce a scaled version of the K-stability problem to the situation of the following statement.

Proposition 5.1 (Proposition 6.2 in [23])**.** *Let* $A(\mathbf{z})$ *and* $B(\mathbf{z})$ *be monic linear matrix pencils of size* $k \times k$ *and* $l \times l$*, respectively, and such that* $S_A := \{\mathbf{z} \in \mathbb{R}^n : A(\mathbf{z}) \succeq 0\}$ *is bounded. Then there exists a constant* $\nu > 0$ *such that for the scaled spectrahedron* νS_A *the inclusion* $\nu S_A \subseteq S_B$ *is certified by the system*

$$C = (C_{ij})_{i,j=1}^{k} \;\succeq\; 0, \quad \forall p = 1, \ldots, n \;:\; B_p \;=\; \sum_{i,j=1}^{k} \Big(\frac{1}{\nu}A_p\Big)_{ij} C_{ij}.$$

As before, let K be a proper cone which is given by a linear matrix pencil $M(\mathbf{z}) = \sum_{j=1}^{n} M_j z_j$ with $l \times l$-matrices, and assume that there exists a hyperplane H not passing through the origin and such that $K \cap H$ is bounded. For notational convenience, assume that $H = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,:\, x_1 = 1\}$ and that $M_1 = I_n$. In particular, then the first unit vector $\mathbf{e}^{(1)}$ is contained in the interior of the full-dimensional cone K.

Theorem 5.2. *Let* $f \in \mathbb{R}[\mathbf{z}]$ *and* $M(\mathbf{z})$ *be as described before. Let* $N(\mathbf{z})$ *be the matrix pencil of a spectrahedral, conic set contained in* $\mathrm{cl}(\mathcal{I}(f)^{\mathrm{c}})$*, and assume that* $N_1 = I_n$ *as well.*

Then there exists a constant $\nu > 0$ *such that* $g_\nu(z_1, \ldots, z_n) := f(z_1, \nu z_2, \ldots, \nu z_n)$ *is* K*-stable and such that the* K*-stability of* g *is certified by the system*

$$C = (C_{ij})_{i,j=1}^{l} \succeq 0, \quad \forall p = 1, \ldots, n \;:\; \nu N_p \;=\; \sum_{i,j=1}^{l} (M_p)_{ij}\, C_{ij}, \tag{19}$$

where the variable matrix C *is a block matrix with* $l \times l$ *blocks.*

As a consequence, f *is* $\hat{K}$*-stable with respect to* $\hat{K} = \mathrm{cone}(\{1\} \times \nu(K \cap H))$*, where the multiplication of* ν *with the set* $K \cap H$ *is done in the* $(n-1)$*-dimensional space with variables* $\mathbf{z}' = (z_2, \ldots, z_n)$ *and* cone *denotes the conic hull.*

Since the scaling variable ν occurs linearly in (19), its optimal value can be expressed by a semidefinite program. Further note that the preconditions $M_1 = I_n$ and $N_1 = I_n$ imply that the induced matrix pencils of the conic spectrahedra of $M(\mathbf{z})$ and of $N(\mathbf{z})$ give monic pencils within the hyperplane H.

Proof. Let $N'(\mathbf{z}'), M'(\mathbf{z}')$ be the matrix pencils in the $n-1$ variables $\mathbf{z}' = (z_2, \ldots, z_n)$ defined by

$$N'(\mathbf{z}') = N(\mathbf{z})\Big|_{z_1=1} \text{ and } M'(\mathbf{z}') = M(\mathbf{z})\Big|_{z_1=1}.$$

$N'(\mathbf{z}')$ and $M'(\mathbf{z}')$ are monic linear matrix pencils and the spectrahedron $S_{M'(\mathbf{z}')} = \{\mathbf{z}' = (z_2, \ldots, z_n) \in \mathbb{R}^n : M'(\mathbf{z}') \succeq 0\}$ is bounded. By Proposition 5.1, the inclusion $\nu S_{M'(\mathbf{z}')} \subseteq S_{L'(\mathbf{z}')}$ is certified by the system

$$C = (C_{ij})_{i,j=1}^{l} \succeq 0, \quad \forall p = 1, \ldots, n : N'_p = \sum_{i,j=1}^{l} \left(\frac{1}{\nu} M'_p\right)_{ij} C_{ij} \tag{20}$$

with some block matrix $C = (C_{ij})_{i,j=1}^{l}$. Since $M'_p = M_p$ and $N'_p = N_p$ for $p \geq 1$, this is equivalent to (19).

Moreover, $\nu S_{M'(\mathbf{z}')} \subseteq S_{N'(\mathbf{z}')}$ implies that $\nu S_{M(\mathbf{z})} \subseteq S_{N(\mathbf{z})}$ and also that for any $\mathbf{z}$ with $z_1 = 1$ and $f(\mathbf{z}) = 0$, we have $(1, \frac{z_2}{\nu}, \ldots, \frac{z_n}{\nu}) \notin \operatorname{int} S_{M'(\mathbf{z}')}$, or, equivalently, $g_\nu(\mathbf{z})$ is K-stable. Finally, this also gives the reformulation that f is $\hat{K}$-stable. □

Theorem 5.2 can also be applied to such polynomials f which meet the requirements of the theorem after applying a invertible linear transformation, since those preserve the containment of sets.

Example 5.3. Setting $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix}$, the polynomial $f = \det \begin{pmatrix} z_1 & 2z_2 \\ 2z_2 & z_3 \end{pmatrix}$ $= z_1 z_3 - 4z_2^2$ is not psd-stable. To fit the requirements of Theorem 5.2, let Q be the rotation matrix $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix}$ and consider the rotated versions of the underlying matrix pencils

$$\begin{aligned} N_Q(\mathbf{y}) &= N(Q^{-1}\mathbf{z}) = \frac{1}{\sqrt{2}} \begin{pmatrix} y_1 - y_3 & \sqrt{8} y_2 \\ \sqrt{8} y_2 & y_1 + y_3 \end{pmatrix} \\ \text{and } M_Q(\mathbf{y}) &= M(Q^{-1}\mathbf{z}) = \frac{1}{\sqrt{2}} \begin{pmatrix} y_1 - y_3 & \sqrt{2} y_2 \\ \sqrt{2} y_2 & y_1 + y_3 \end{pmatrix}. \end{aligned}$$

For $N_{Q,\nu}(\mathbf{y}) := N_Q(y_1, \nu y_2, \nu y_3)$ and $M_Q(\mathbf{y})$, (19) leads to the equations

$$C_{11} + C_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_{12} + C_{21} = \begin{pmatrix} 0 & 2\nu \\ 2\nu & 0 \end{pmatrix}, \quad -C_{11} + C_{22} = \begin{pmatrix} -\nu & 0 \\ 0 & \nu \end{pmatrix}.$$

Hence, the set of matrices $C = C_\nu$ satisfying (19) is given by the system

$$C = \frac{1}{2} \begin{pmatrix} 1+\nu & 0 & 0 & 4\lambda\nu \\ 0 & 1-\nu & 4(1-\lambda)\nu & 0 \\ 0 & 4(1-\lambda)\nu & 1-\nu & 0 \\ 4\lambda\nu & 0 & 0 & 1+\nu \end{pmatrix}, \quad C \succeq 0 \quad \text{with } \lambda \in \mathbb{R}. \tag{21}$$

The largest ν satisfying (21) is given by $\nu = \frac{1}{2}$ with $\lambda = \frac{3}{4}$. When rotating back, this certifies the psd-stability of

$$f_{\frac{1}{2}}(\mathbf{z}) := \det\left(N_{Q,\frac{1}{2}}(Q\mathbf{z})\right) = \frac{1}{16} \cdot (3z_1^2 + 10 z_1 z_3 + 3z_3^2 - 16 z_2^2).$$

In addition to that, we obtain that f is $\hat{K}$-stable with respect to the cone

$$\hat{K} \;=\; \left\{ \mathbf{y} \in \mathbb{R}^3 \;:\; \frac{1}{2}\begin{pmatrix} 3y_1 - y_3 & 4y_2 \\ 4y_2 & -y_1 + 3y_3 \end{pmatrix} \succeq 0 \right\}.$$

6. Conclusion and open questions

In this paper, we have shown how techniques from the theory of positive maps and from the containment of spectrahedra can be used to provide a sufficient criterion for the K-stability of a given polynomial f. In particular, we have considered quadratic and determinantal polynomials. Beyond that, our approach generally applies whenever (for a polynomial of arbitrary degree) some spectrahedral components in the complement of $\mathcal{I}(f)$ are known.

It would be interesting to understand whether this or related techniques can be effectively exploited also for classes of polynomials beyond the ones studied in the paper. In particular, with regard to the recent development of a theory of Lorentzian polynomials [8], which provides a superset of the set of homogeneous stable polynomials, it would be of interest to understand the connection of Lorentzian polynomials to conic stability and to the effective methods presented in our paper.

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IMAGINARY PROJECTIONS: COMPLEX VERSUS REAL COEFFICIENTS

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Abstract. Given a multivariate complex polynomial $p \in \mathbb{C}[z_1, \ldots, z_n]$, the imaginary projection $\mathcal{I}(p)$ of p is defined as the projection of the variety $\mathcal{V}(p)$ onto its imaginary part. We focus on studying the imaginary projection of complex polynomials and we state explicit results for certain families of them with arbitrarily large degree or dimension. Then, we restrict to complex conic sections and give a full characterization of their imaginary projections, which generalizes a classification for the case of real conics. That is, given a bivariate complex polynomial $p \in \mathbb{C}[z_1, z_2]$ of total degree two, we describe the number and the boundedness of the components in the complement of $\mathcal{I}(p)$ as well as their boundary curves and the spectrahedral structure of the components. We further show a realizability result for strictly convex complement components which is in sharp contrast to the case of real polynomials.

1. Introduction

Given a polynomial $p \in \mathbb{C}[\mathbf{z}] := \mathbb{C}[z_1, \ldots, z_n]$, the imaginary projection $\mathcal{I}(p)$ as introduced in [20] is the projection of the variety $\mathcal{V}(p) \subseteq \mathbb{C}^n$ onto its imaginary part, that is,

$$\mathcal{I}(p) = \ \{\mathbf{z}_{\text{im}} = ((z_1)_{\text{im}}, \ldots, (z_n)_{\text{im}}) \ : \ \mathbf{z} \in \mathcal{V}(p)\} \ \subseteq \ \mathbb{R}^n, \tag{1}$$

where $(\cdot)_{\text{im}}$ is the imaginary part of a complex number. Recently, there has been wide-spread research interest in mathematical branches which are directly connected to the imaginary projection of polynomials.

As a primary motivation, the imaginary projection provides a comprehensive geometric view for notions of *stability of polynomials* and generalizations thereof. A polynomial $p \in \mathbb{C}[\mathbf{z}]$ is called *stable*, if $p(\mathbf{z}) = 0$ implies $(z_j)_{\text{im}} \leq 0$ for some $j \in [n]$. In terms of the imaginary projection $\mathcal{I}(p)$, we can express the stability of p as the condition $\mathcal{I}(p) \cap \mathbb{R}^n_{>0} = \emptyset$. Stable polynomials have applications in many branches of mathematics including combinatorics ([5] and see [8] for the connection of the imaginary projection to combinatorics), differential equations [3], optimization [34], probability theory [4], and applied algebraic geometry [37]. Further application areas include theoretical computer science [23, 24], statistical physics [2], and control theory [25], see also the surveys [29] and [38].

Recently, various generalizations and variations of the stability notion have been studied, such as stability with respect to a polyball [13, 14], conic stability [9, 18], Lorentzian polynomials [6], or positively hyperbolic varieties [31]. Exemplarily, regarding the conic stability, a polynomial $p \in \mathbb{C}[\mathbf{z}]$ is called K*-stable* for a proper cone $K \subset \mathbb{R}^n$ if $p(\mathbf{z}) \neq 0$, whenever $\mathbf{z}_{\text{im}} \in \operatorname{int} K$, where int is the interior. In terms of the imaginary projection, this condition can be equivalently expressed as $\mathcal{I}(p) \cap \operatorname{int} K = \emptyset$.

This work was supported through DFG grant TH 1333/7-1.

Another motivation comes from the close connection of the imaginary projection to hyperbolic polynomials and hyperbolicity cones [11]. As shown in [19], in case of a real *homogeneous* polynomial p, the components of the complement $\mathcal{I}(p)^{\mathsf{c}}$ coincide with the hyperbolicity cones of p. These concepts play a central role in hyperbolic programming, see [15, 26, 27, 32]. A prominent open question in this research direction is the generalized Lax conjecture, which claims that every hyperbolicity cone is spectrahedral, see [36]. Representing convex sets by spectrahedra is not only motivated by the general Lax conjecture, but also by the question of effective handling convex semialgebraic sets (see, for example, [1, 21]). Recently, the conjecture that every convex semialgebraic set would be the linear projection of a spectrahedron, the "Helton-Nie conjecture", has been disproven by Scheiderer [33].

Moreover, the imaginary projection closely relates to and complements the notions of *amoebas*, as introduced by Gel'fand, Kapranov and Zelevinsky [12], and *coamoebas*. The amoeba $\mathcal{A}(p)$ of a polynomial p is defined as $\mathcal{A}(p) := \{(\ln|z_1|, \ldots, \ln|z_n|) : \mathbf{z} \in \mathcal{V}(p) \cap (\mathbb{C}^*)^n\}$, so it considers the logarithm of the absolute value of a complex number rather than its imaginary part. The coamoeba of a polynomial deals with the phase of a complex number. Each of these three viewpoints of a complex variety gives a set in a real space with the characteristic property that the complement of the closure consists of finitely many *convex* connected components. See [10], [12] and [20] for the convexity properties of amoebas, coamoebas, and imaginary projections, respectively. Due to their convexity phenomenon, these structures provide natural classes in recent developments of convex algebraic geometry.

For amoebas, an exact upper bound on the number of components in the complement is known [12]. For the coamoeba of a polynomial p, it has been conjectured that there are at most $n! \operatorname{vol} \operatorname{New}(p)$ connected components in the complement, where vol denotes the volume and $\operatorname{New}(p)$ the Newton polytope of p, see [10] for more background as well as a proof for the special case $n = 2$. For imaginary projections, a tight upper bound is known in the homogeneous case [19], but for the non-homogeneous case there only exists a lower bound [20].

Currently, no efficient method is known to calculate the imaginary projection for a general real or complex polynomial. For some families of polynomials, the imaginary projection has been explicitly characterized, including complex linear polynomials and real quadratic polynomials, see [20] and [18, Proposition 3.2]. However, since imaginary projections for non-linear complex polynomials exhibit new structural phenomena compared to the real case, even the characterization of the imaginary projection of complex conics had remained elusive so far.

Our primary goal is to reveal fundamental and surprising differences between imaginary projections of *real polynomials* and *complex polynomials*. In fixed degree and dimension, for a polynomial p with non-real coefficients, the algebraic degree of the *boundary* of the imaginary projection $\partial\mathcal{I}(p) := \overline{\mathcal{I}(p)} \cap \overline{\mathcal{I}(p)^{\mathsf{c}}}$ can be higher than the case of real coefficients. Here $(.)^{\mathsf{c}}$ and $\overline{(.)}$ are the complement and Euclidean closure, respectively. These incidences already begin when the degree and dimension are both two. However, the contrast is not only concerning the boundary degrees, but also the arrangements and the strict convexity of the components in $\mathcal{I}(p)^{\mathsf{c}}$.

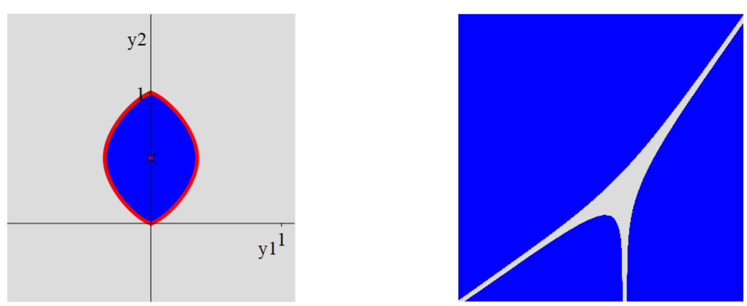
We start with structural results which serve to work out the differences between the case of real and complex coefficients. Our first result is a sufficient criterion on the roots of the *initial form* of an arbitrarily large degree non-real bivariate complex polynomial to have the real plane as its imaginary projection, see Theorem 3.4 and Corollary 3.5.

Next, we characterize the imaginary projections of n-dimensional multivariate complex quadratics with hyperbolic initial form, see Theorem 4.5 and Corollary 4.6.

In the two-dimensional case, although by generalizing from real to complex conics, the bounds on the number of bounded and unbounded components in the complement of the imaginary projections remain unchanged, the possible arrangements of these components, strictness of their convexity, and the algebraic degrees of their boundaries strongly differ. See Corollaries 5.3 and 5.4. For conic sections with real coefficients, it was shown by Jörgens, Theobald, and de Wolff [20] that the boundary $\partial\mathcal{I}(p)$ consists of pieces which are algebraic curves of degree at most two. In sharp contrast to this, for complex polynomials, the boundary may not be algebraic and the degree of its irreducible pieces can go up to 8. For example, despite the simple expression of the polynomial $p = z_1^2 + \mathrm{i}z_2^2 + z_2$, an exact description of $\mathcal{I}(p)$ is

$$\begin{aligned}\mathcal{I}(p) \;=\; \{y \in \mathbb{R}^2 \,:\, & -64y_1^8 - 128y_1^4y_2^4 - 64y_2^8 + 256y_1^4y_2^3 + 256y_2^7 - 272y_1^4y_2^2 \\ & -400y_2^6 + 144y_1^4y_2 + 304y_2^5 - 27y_1^4 - 112y_2^4 + 16y_2^3 \le 0\} \setminus \{(0,1/2)\},\end{aligned} \tag{2}$$

and the describing polynomial in (2) is irreducible over $\mathbb{C}$. In this example, the set $\mathcal{I}(p)^{\mathsf{c}}$ consists of a single convex connected and bounded component. Any polynomial vanishing on the boundary will also vanish on the single point $(0,1/2)$ which is not part of the boundary $\partial\mathcal{I}(p)$. Thus, $\partial\mathcal{I}(p)$ is not algebraic. See Figure 1 for an illustration and we return to this example in Section 3 and at the end of Section 6.



(A) (B)

FIGURE 1. (A) The gray area and its boundary form the imaginary projection $\mathcal{I}(p)$ of $p = z_1^2 + \mathrm{i}z_2^2 + z_2$. The polynomial in (2) vanishes on the red curve, which consists of a single point and another bounded component. The complement $\mathcal{I}(p)^{\mathsf{c}}$ contains the single point and it is bounded by the other component. (B) The amoeba of p is shown in gray.

Since the topology of the imaginary projection in $\mathbb{R}^n$ is invariant under the action of $G_n := \mathbb{C}^n \rtimes \mathrm{GL}_n(\mathbb{R})$, that is the semi-direct product of $\mathrm{GL}_n(\mathbb{R})$ and complex translations, the problem to understand the imaginary projections naturally leads to a polynomial classification problem.

As starting point, recall that under the action of the affine group $\mathrm{Aff}(\mathbb{C}^2)$, there are precisely five orbits for complex conics, with the following representatives:

$$z_1^2 \text{ (one line)}, \qquad z_1^2+1 \text{ (two parallel lines)}, \qquad z_1^2 - z_2 \text{ (parabola)},$$
$$z_1^2+z_2^2 \text{ (two crossing lines)}, \qquad z_1^2+z_2^2-1 \text{ (circle)}.$$

However, the arrangement of the components in $\mathcal{I}(p)^{\mathsf{c}}$ is not invariant under the action of $\mathrm{Aff}(\mathbb{C}^2)$, but only under its restriction to G_2. There are several other related classifications of complex conic sections. Newstead [28] has classified the set of projective complex conics under real linear transformations. However, out of a projective setting his method becomes ineffective as it is based on the arrangements of four intersection points between a conic and its conjugate. On the other hand, by considering the real part and the imaginary part of a complex conic p, under the action of G_2 the classification of conic sections has some relations to the problem of classifying pairs of real conics. Systematic classifications of this kind are mostly done in the projective setting and are well understood. See [7, 22, 30, 35]. However, those classifications rely on the invariance of the number and multiplicity of real intersection points between the two real conics. The drawback here is that under complex translations on p, these numbers are not invariant anymore, except at infinity.

To capture the invariance under G_2, we develop a novel classification based on the initial forms of complex conics. This classification is adapted to the imaginary projection and it is rather fine but coarse enough to allow handling the inherent algebraic degree of 8 in the boundary description of the imaginary projection.

Finally, we show that non-real complex conics can significantly improve a realization result on the complement of the imaginary projections. In [19], for any given integer $k \geq 1$, they present a polynomial p of degree $d = 4\lceil \frac{k}{4} \rceil + 2$ as a product of real conics, such that $\mathcal{I}(p)^{\mathsf{c}}$ has at least k components that are strictly convex and bounded. Using non-real conics, we furnish a degree $d/2+1$ polynomial having exactly k components with these properties. See Theorem 7.1 and Question 7.2.

The paper is structured as follows. Section 2 provides our notation and the necessary background on the imaginary projection of polynomials and contains the classification of the imaginary projection for the case of real conics. Section 3 deals with complex plane curves and provides a highlighting example where the complex versus real coefficients make a remarkable difference in the complexity of the imaginary projection. Moreover, we determine a family of arbitrarily large degree non-real plane curves with a full-space imaginary projection, based on the arrangements of roots of the initial form. In Section 4, we set the degree to be two and let the dimension grow and we classify the imaginary projections of complex quadratics with hyperbolic initial form. In Sections 5 and 6, we restrict the degree and dimension both to be two and we provide a full classification of the imaginary projections for affine complex conics based on their initial forms. Moreover, we determine in which classes the components in the complement of the imaginary projection have a spectrahedral description and also state them explicitly.

Section 5 contains our main classification theorems and the corollaries differentiating the cases of complex and real coefficients. The part where the initial form is hyperbolic is already covered in 4. Each subsection of Section 6 treats one of the remaining classes and explains their spectrahedral structure. In particular, we show that the only class where the components in the complement are not necessarily spectrahedral is the case where the initial form has two distinct non-real roots in $\mathbb{P}^1_{\mathbb{C}}$ such that they do not form a complex conjugate pair. In Section 7, we prove a realization result for strictly convex complement components, which highlights another contrast between the imaginary projections of complex and real polynomials. Section 8 gives some open questions.

2. Preliminaries and background

For a set $S \subseteq \mathbb{R}^n$, we denote by $\overline{S}$ the topological closure of S with respect to the Euclidean topology on $\mathbb{R}^n$ and by S^{c} the complement of S in $\mathbb{R}^n$. The *algebraic degree* of S is the degree of its closure with respect to the Zariski topology. The set of non-negative and the set of strictly positive real numbers are abbreviated by $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ throughout the text. Moreover, bold letters will denote n-dimensional vectors. By $\mathbb{P}^n$ and $\mathbb{P}^n_{\mathbb{R}}$, we denote the n-dimensional complex and real projective spaces, respectively.

For a polynomial $p \in \mathbb{C}[\mathbf{z}]$, the imaginary projection $\mathcal{I}(p)$ is defined in (1) and its boundary $\overline{\mathcal{I}(p)} \cap \overline{\mathcal{I}(p)^{\mathsf{c}}}$ is denote by $\partial \mathcal{I}(p)$.

Theorem 2.1. [20] *Let $p \in \mathbb{C}[\mathbf{z}]$ be a complex polynomial. The set $\overline{\mathcal{I}(p)}^{\mathsf{c}}$ consists of a finite number of convex connected components.*

We denote by a_{re} and a_{im} the real and the imaginary parts of a complex number $a \in \mathbb{C}$, i.e., a is written in the form $a_{\mathrm{re}} + \mathrm{i}a_{\mathrm{im}}$, such that $a_{\mathrm{re}}, a_{\mathrm{im}} \in \mathbb{R}$. Let $p \in \mathbb{C}[\mathbf{z}]$ be a complex polynomial. After substituting $z_j = x_j + \mathrm{i}y_j$ for all $1 \leq j \leq n$, the complex polynomial can be written in the form

$$p(\mathbf{z}) = p_{\mathrm{re}}(\mathbf{x}, \mathbf{y}) + \mathrm{i}p_{\mathrm{im}}(\mathbf{x}, \mathbf{y}),$$

such that $p_{\mathrm{re}}, p_{\mathrm{im}} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$. We call the real polynomials p_{re} and p_{im}, the *real part* and the *imaginary part* of p, respectively. Thus, finding $\mathcal{I}(p)$ is equivalent to determining the values of $\mathbf{y}$ for which the real polynomial system

$$p_{\mathrm{re}}(\mathbf{x}, \mathbf{y}) = 0 \quad \text{and} \quad p_{\mathrm{im}}(\mathbf{x}, \mathbf{y}) = 0 \tag{3}$$

has real solutions for $\mathbf{x}$.

Definition 2.2. Let $p \in \mathbb{C}[z_1, z_2]$ be a quadratic polynomial, i.e., $p = az_1^2 + bz_1z_2 + cz_2^2 + dz_1 + ez_2 + f$ such that $a, b, c, d, e, f \in \mathbb{C}$. We say that p is the defining polynomial of a complex conic, or shortly, a *complex conic* if its total degree equals two, i.e., at least one of the coefficients a, b, or c is non-zero. A complex conic p is called a *real conic* if all coefficients of p are real.

The following lemma from [20] shows how real linear transformations and complex translations act on the imaginary projection. These are the key ingredients for computing the imaginary projection of every class of conic sections.

Lemma 2.3. *Let $p \in \mathbb{C}[\mathbf{z}]$ and $A \in \mathbb{R}^{n \times n}$ be an invertible matrix. Then*

$$\mathcal{I}(p(A\mathbf{z})) = A^{-1}\mathcal{I}(p(\mathbf{z})).$$

Moreover, a real translation $\mathbf{z} \mapsto \mathbf{z} + \mathbf{a}$, $\mathbf{a} \in \mathbb{R}^n$ does not change the imaginary projection. An imaginary translation $\mathbf{z} \mapsto \mathbf{z} + \mathrm{i}\mathbf{a}$, $\mathbf{a} \in \mathbb{R}^n$ shifts the imaginary projection into the direction $-\mathbf{a}$.

By the previous lemma, to classify the imaginary projection of polynomials we consider their orbits under the action of the group $G_n := \mathbb{C}^n \rtimes \mathrm{GL}_n(\mathbb{R})$, given by real linear transformations and complex translations. Further let $\mathrm{Aff}(\mathbb{K}^n) := \mathbb{K}^n \rtimes \mathrm{GL}_n(\mathbb{K})$ be the general affine group for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The real dimensions of these groups are

$$\dim_{\mathbb{R}}(\mathrm{Aff}(\mathbb{C}^n)) = 2\dim_{\mathbb{R}}(\mathrm{Aff}(\mathbb{R}^n)) = 2(n^2+n), \qquad \dim_{\mathbb{R}}(G_n) = n^2 + 2n.$$

Up to the action of G_2, a real conic $p \in \mathbb{R}[z_1, z_2]$ is equivalent to a conic given by one of the following polynomials.

(*i*) $z_1^2 + z_2^2 - 1$ (ellipse),
(*ii*) $z_1^2 - z_2^2 - 1$ (hyperbola),
(*iii*) $z_1^2 + z_2$ (parabola),
(*iv*) $z_1^2 + z_2^2 + 1$ (empty set),
(*v*) $z_1^2 - z_2^2$ (pair of crossing lines),
(*vi*) $z_1^2 - 1$ (parallel lines/one line z_1^2),
(*vii*) $z_1^2 + z_2^2$ (isolated point),
(*viii*) $z_1^2 + 1$ (empty set).

In [20], a full classification of the imaginary projection for real quadratics was shown. In particular, the following theorem is the classification for real conics. For illustrations of the cases, see Figure 2. The theorem that comes after provides the imaginary projection of some families of real quadratics. Furthermore, they state the subsequent question as an open problem.

Theorem 2.4. *Let $p \in \mathbb{R}[z_1, z_2]$ be a real conic. For the normal forms (i)–(viii) from above, the imaginary projections $\mathcal{I}(p) \subseteq \mathbb{R}^2$ are as follows.*

(*i*) $\mathcal{I}(p) = \mathbb{R}^2$,
(*ii*) $\mathcal{I}(p) = \{-1 \le y_1^2 - y_2^2 < 0\} \cup \{\mathbf{0}\}$,
(*iii*) $\mathcal{I}(p) = \mathbb{R}^2 \setminus \{(0, y_2) : y_2 \neq 0\}$,
(*iv*) $\mathcal{I}(p) = \{\mathbf{y} \in \mathbb{R}^2 : y_1^2 + y_2^2 - 1 \ge 0\}$,
(*v*) $\mathcal{I}(p) = \{\mathbf{y} \in \mathbb{R}^2 : y_1^2 = y_2^2\}$,
(*vi*) $\mathcal{I}(p) = \{\mathbf{y} \in \mathbb{R}^2 : y_1 = 0\}$,
(*vii*) $\mathcal{I}(p) = \mathbb{R}^2$,
(*viii*) $\mathcal{I}(p) = \{\mathbf{y} \in \mathbb{R}^2 : y_1 = \pm 1\}$.

Theorem 2.5. *Let $p \in \mathbb{C}[z_1, \ldots, z_n]$ be $p = \sum_{i=1}^{n-1} z_i^2 - z_n^2 + k$ for $k \in \{\pm 1\}$. Then*

$$\mathcal{I}(p) = \begin{cases} \{\mathbf{y} \in \mathbb{R}^n \ : \ y_n^2 < \sum_{i=1}^{n-1} y_i^2\} \cup \{\mathbf{0}\} & \text{if } k = 1, \\ \{\mathbf{y} \in \mathbb{R}^n \ : \ y_n^2 - \sum_{i=1}^{n-1} y_i^2 \le 1\} & \text{if } k = -1. \end{cases}$$

The following question, which is true for real quadratics $p \in \mathbb{C}[\mathbf{z}]$, was asked in [20, Open problem 3.4]. In Section 6.2, we show that it is not true in general even for complex conics.

Question 2.6. Let $p \in \mathbb{C}[\mathbf{z}]$ be a polynomial. Is $\mathcal{I}(p)$ open if and only if $\mathcal{I}(p) = \mathbb{R}^n$?

We use the *initial form* of p abbreviated by $\mathrm{in}(p)(\mathbf{z}) = p^h(\mathbf{z}, 0)$, where p^h is the homogenization of p. The initial form consists of the terms of p with the maximal total degree. Furthermore, a complex polynomial $p \in \mathbb{C}[\mathbf{z}]$ is called *hyperbolic* w.r.t. $\mathbf{e} \in \mathbb{R}^n$ if the univariate polynomial $t \mapsto p(\mathbf{x} + t\mathbf{e})$ is real-rooted. Note that any hyperbolic polynomial is a, possibly complex, multiple of a real polynomial.

Finally, a *spectrahedron* is a set of the form

$$\{\mathbf{x} \in \mathbb{R}^n \ : \ A_0 + \sum_{j=1}^n A_j x_j \succeq 0\},$$

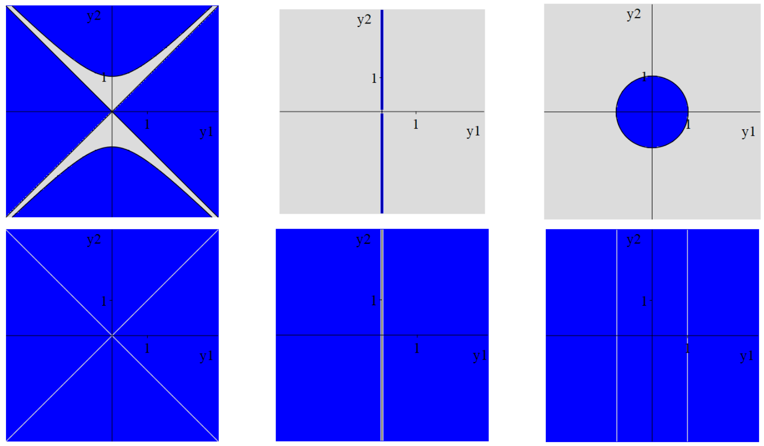


Figure 2. The imaginary projections of the real conic sections and their complements are colored in gray and blue, respectively. The cases (i) and (vii) are skipped, as their imaginary projection is the whole plane.

where $A_1, \ldots, A_n$ are real symmetric matrices of size d. Here, "$\succeq 0$" denotes the positive semidefiniteness of a matrix. We also speak of a spectrahedral set if the set is given by positive definite conditions, i.e., by strict conditions.

3. Imaginary projections of complex plane curves

In this section, we determine the imaginary projection of some families of arbitrarily high degree complex plane curves. Our point of departure is the characterization of real conics in Theorem 2.4. In the following example, which is an affine version of case (B_+) in Newstead's classification [28], we show that by allowing non-real coefficients the imaginary projection of a complex conic can significantly change in terms of the algebraic degree of its boundary. See Corollary 5.3.

Remark 3.1. Recall that the discriminant of a univariate polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$ is given by $\mathrm{Disc}(p) = (-1)^{\frac{1}{2}n(n-1)} \frac{1}{a_n} \mathrm{Res}(p, p')$, where Res denotes the resultant. For a quartic, having negative discriminant implies the existence of a real root. However, a positive discriminant can correspond to either four real roots or none. Let
$P = 8a_2a_4 - 3a_3^2$, $R = a_3^3 + 8a_1a_4^2 - 4a_4a_3a_2$, $D = 64a_4^3a_0 - 16a_4^2a_2^2 + 16a_4a_3^2a_2 - 16a_4^2a_3a_1 - 3a_3^4$.
If $\mathrm{Disc}(p) > 0$, then $p = 0$ has four real roots if $P < 0$ and $D < 0$, and no real roots otherwise. Finally, if the discriminant is zero, the only conditions under which there is no real solution is having $D = R = 0$ and $P > 0$ (see, e.g., [17, Theorem 9.13 (vii)]).

Example 3.2. Let $p = z_1^2 + \mathrm{i}z_2^2 + z_2$. For simplifying the calculations, we use the translation $z_2 \mapsto z_2 + \mathrm{i}/2$ to eliminate the linear term. This turns the equation $p = 0$ into $q := z_1^2 + \mathrm{i}z_2^2 + \mathrm{i}/4 = 0$. Building the real polynomial system as introduced in (3) implies

$$q_{\mathrm{re}} = x_1^2 - 2x_2y_2 - y_1^2 = 0 \quad \text{and} \quad q_{\mathrm{im}} = 4x_2^2 + 8x_1y_1 - 4y_2^2 + 1 = 0.$$

First assume $y_1 \neq 0$. Substituting x_1 from $q_{\mathrm{im}} = 0$ into $q_{\mathrm{re}} = 0$ gives

$$16x_2^4 + (-32y_2^2 + 8)x_2^2 - 128y_1^2y_2x_2 - 64y_1^4 + 16y_2^4 - 8y_2^2 + 1 = 0.$$

We calculate the discriminant of the above equation with respect to x_2. By the previous remark, there is a real solution for x_2 if the discriminant is negative, i.e.,

$$-64y_1^8 - 128y_1^4y_2^4 - 64y_2^8 - 80y_1^4y_2^2 + 48y_2^6 + y_1^4 - 12y_2^4 + y_2^2 < 0.$$

Now we need to check the conditions where the discriminant is zero or positive. To show the positive discriminant implies no real solution for x_2, we rewrite the condition with the substitution $u = y_1^4$:

$$\triangle := -64u^2 + (-128y_2^4 - 80y_2^2 + 1)u - 64y_2^8 + 48y_2^6 - 12y_2^4 + y_2^2 > 0.$$

It is a quadratic polynomial in u with negative leading coefficient. It can only be positive between the two roots for u in $\triangle = 0$. Those are

$$-y_2^4 - \frac{5}{8}y_2^2 + \frac{1}{128} \pm \frac{\sqrt{32768y_2^6 + 3072y_2^4 + 96y_2^2 + 1}}{128}.$$

To obtain $\triangle > 0$, we need to have a solution $u > 0$, i.e., we need to have either $-y_2^4 - \frac{5}{8}y_2^2 + \frac{1}{128} \geq 0$ or otherwise

$$\left(-y_2^4 - \frac{5}{8}y_2^2 + \frac{1}{128}\right)^2 > \frac{32768y_2^6 + 3072y_2^4 + 96y_2^2 + 1}{128^2}.$$

The first inequality implies $y_2^2 \leq \frac{3\sqrt{3}-5}{16}$ and after simplifications the second inequality implies $y_2^2 < 1/4$. The polynomial P from the previous remark for the quartic polynomials evaluates to $4(1 - 4y_2^2)$, which is positive for $y_2^2 < 1/4$. Therefore, for $\triangle > 0$, there is no real solution for x_2. It remains now to consider the case $\triangle = 0$. Since $y_1 \neq 0$, to have $R = -262144y_2y_1^2 = 0$ we need $y_2 = 0$. Substituting $y_2 = 0$ in $D = 0$ implies $-4096y_1^4 - 960 = 0$, which is a contradiction. Therefore, if $y_1 \neq 0$, the imaginary projection of q consists of points $\mathbf{y} \in \mathbb{R}^2$ for which $\triangle \leq 0$.

Now assume $y_1 = 0$. From $q_{\text{im}} = 0$ we can observe that $\mathbf{0} \notin \mathcal{I}(q)$. Thus, assume $y_2 \neq 0$. Solving $q_{\text{re}} = 0$ for x_2 and substituting in $q_{\text{im}} = 0$ implies $x_1^4 - y_2^2(4y_2^2 - 1) = 0$. This equation has a real solution if and only if $-y_2^2(4y_2^2 - 1) \leq 0$. Substituting $y_1 = 0$ in $\triangle$ allows to write $\triangle$ in terms of y_2, which gives $\triangle_{y_2} = -y_2^2(4y_2^2 - 1)^3$. Therefore, the imaginary projection on the y_2-axis is $\{(0, y_2) \in \mathbb{R}^2 : \triangle_{y_2} \leq 0\} \setminus \{(0, 0)\}$. Thus,

$$\mathcal{I}(q) = \{\mathbf{y} \in \mathbb{R}^2 : -64y_1^8 - 128y_1^4y_2^4 - 64y_2^8 - 80y_1^4y_2^2 + 48y_2^6 + y_1^4 - 12y_2^4 + y_2^2 \leq 0\} \setminus \{\mathbf{0}\}.$$

The irreducibility of the polynomial above over $\mathbb{C}$ can be verified for example using MAPLE. For the original polynomial p, this gives the inequality description for $\mathcal{I}(p)$ stated in (2) in the Introduction.

Even in the case of real polynomials, extending the case of real conics by letting the degree or the number of variables be greater than two dramatically increases the difficulty of characterizing the imaginary projection. Let us see one such example of a cubic plane curve, i.e., where we have two unknowns and the total degree is three.

Example 3.3. Let $p \in \mathbb{R}[\mathbf{z}] = \mathbb{R}[z_1, z_2]$ be of the form $p = z_1^3 + z_2^3 - 1$. The similar attempt as before to calculate the imaginary projection $\mathcal{I}(p)$ is to separate the real and the imaginary parts of p according to (3),

$$p_{\text{re}} = x_1^3 - 3x_1y_1^2 + x_2^3 - 3x_2y_2^2 - 1 = 0 \quad \text{and} \quad p_{\text{im}} = 3x_1^2y_1 + 3x_2^2y_2 - y_1^3 - y_2^3 = 0.$$

Despite the simplicity of the polynomial p, one cannot use the previous techniques to find the values of $\mathbf{y} \in \mathbb{R}^2$ such that the above system has real solutions for $\mathbf{x}$. The reason is that both x_1 and x_2 appear in higher degree than one in both equations. The resultant with respect to one of x_1 or x_2 is a univariate polynomial of degree six in the other, where we lack the exact tools to specify the reality of the roots.

In the following theorem, we show that the imaginary projection of a generic complex plane curve of odd degree is the whole plane.

Theorem 3.4. *Let $p \in \mathbb{C}[z_1, z_2]$ be a complex bivariate polynomial of total degree d such that its initial form has no real roots in $\mathbb{P}^1$. If d is odd then the imaginary projection $\mathcal{I}(p)$ is $\mathbb{R}^2$. As a consequence, the imaginary projection of a generic complex bivariate polynomial of odd total degree is $\mathbb{R}^2$.*

Proof. Since the initial form has no real roots, it can be written in the form

$$\mathrm{in}(p) = \prod_{j=1}^{d} (z_1 - \alpha_j z_2),$$

where $\alpha_j \notin \mathbb{R}$ for $1 \leq j \leq d$. Substitute $z_j = x_j + \mathrm{i} y_j$ for $j = 1, 2$ in p and form the polynomial system $p_{\mathrm{re}} = p_{\mathrm{im}} = 0$ as introduced in (3). For any fixed $\mathbf{y} \in \mathbb{R}^2$, both equations are of total degree d in x_1 and x_2. Denote by p_{re}^h and p_{im}^h, the homogenization of these two polynomials by a new variable x_3. Since both, p_{re}^h and p_{im}^h, have odd degree, the number of complex intersection points (counted with multiplicities) is odd while the number of non-real intersection points (counted with multiplicities) is even. Thus, there is a real intersection point in $\mathbb{P}^2_{\mathbb{R}}$. We claim that this intersection point lies in the affine piece where $x_3 = 1$. This implies that for any given $\mathbf{y} \in \mathbb{R}^2$, there exist $x_1, x_2 \in \mathbb{R}$ for which $p_{\mathrm{re}} = p_{\mathrm{im}} = 0$ and therefore completes the proof.

To prove our claim, we show that the two curves defined by $p_{\mathrm{re}}^h = 0$ and $p_{\mathrm{im}}^h = 0$ do not intersect at infinity, i.e., their intersection point has $x_3 \neq 0$. Let us assume that they intersect at infinity and set $x_3 = 0$ in p_{re}^h and p_{im}^h. This substitution turns the complex polynomial $p_{\mathrm{re}}^h + \mathrm{i} p_{\mathrm{im}}^h$ into

$$q := \prod_{j=1}^{d} (x_1 - \alpha_j x_2).$$

Thus, for the two projective curves to intersect at infinity we need to have $q = 0$. Since $\alpha_j \notin \mathbb{R}$ for $1 \leq j \leq d$, the only real solution for x_1 and x_2 is zero. This is a contradiction. $\square$

Corollary 3.5. *Let $p \in \mathbb{C}[z_1, z_2]$ be a complex bivariate polynomial. The imaginary projection $\mathcal{I}(p)$ is $\mathbb{R}^2$ if p has a factor q such that the total degree of q is odd and its initial form has no real roots in $\mathbb{P}^1$.*

Proof. Since for $p_1, p_2 \in \mathbb{C}[\mathbf{z}]$, we have $\mathcal{I}(p_1 \cdot p_2) = \mathcal{I}(p_1) \cup \mathcal{I}(p_2)$, we claim that if there is a factor q in p whose imaginary projection is $\mathbb{R}^2$, then $\mathcal{I}(p) = \mathbb{R}^2$. The result now follows from the previous theorem. $\square$

In the following section, instead of the dimension we set the degree to be two and characterize the imaginary projection for a certain family of quadratic hypersurfaces.

4. Complex quadratics with hyperbolic initial form

As we have seen in Example 3.2, the methods used to compute the imaginary projection of real quadratics is not always useful for complex ones. However, for a certain family, namely the quadratics with hyperbolic initial form, one can build up on the methods for the real case. To classify the imaginary projections of any family of polynomials, Lemma 2.3 suggests bringing them to their proper normal forms.

Lemma 4.1. *Under the action of G_n, any quadratic polynomial $p \in \mathbb{C}[z_1, \ldots, z_n]$ with hyperbolic initial form can be transformed to one of the following normal forms:*

(1) $z_1^2 + \alpha z_2 + r z_3 + \gamma$,

(2) $\sum_{i=1}^{j} z_i^2 - z_{j+1}^2 + \alpha z_{j+2} + r z_{j+3} + \gamma$ *for some* $j = 1, \ldots, n-1$,

such that terms containing z_k do not appear for $k > n$, and $\alpha, r, \gamma \in \mathbb{C}$.

Proof. The initial form $\mathrm{in}(p)$ is a hyperbolic polynomial of degree two. That is, after a real linear transformation it can be either z_1^2 or of the form $\mathbf{z}'^T M \mathbf{z}'$ such that $\mathbf{z}' = (z_1, \ldots, z_{j+1})$ for some $1 \le j \le n-1$ and M is a square matrix of size $j+1$ with signature $(j, 1)$. See [11]. This explains the initial forms in (1) and (2).

Any term λz_j for some $1 \le j \le n$, such that z_j appears in our transformed initial forms, cancels out by one of the translations $z_j \mapsto z_j \pm \frac{\lambda}{2}$ without changing the initial form. Finally, we show that the number of linear terms in the rest of the variables is at most two. Consider the complex linear form $\sum_{j=1}^{n} \lambda_j z_j$. For $1 \le j \le n$, let $\lambda_j = r_j + \mathrm{i} s_j$ such that $r_j, s_j \in \mathbb{R}$. We can now write the sum as $(\sum_{j=1}^{n} r_j z_j) + \mathrm{i}(\sum_{j=1}^{n} s_j z_j)$. If in the real part at least one of the r_j, say, r_1, is non-zero, then a sequence of linear transformations $z_1 \mapsto z_1 - \frac{r_j}{r_1} z_j$ for $j = 2, \ldots, n$, cancels out $\sum_{j=2}^{n} r_j z_j$. Similarly, the complex part reduces to only one term. □

We first focus on the case where $n = 2$. In this case, we explicitly express the unbounded spectrahedral components forming $\mathcal{I}(p)^{\mathsf{c}}$. The following subsection covers part of the proof of Theorem 5.1.

4.1. **Complex conics with hyperbolic initial form.** To match them with our classification of conics in Theorem 5.5, we do a real linear transformation in the case (2) and write them as

$$(1\text{a}.1)\, p = z_1^2 + \gamma, \qquad (1\text{a}.2)\, p = z_1^2 + \gamma z_2 \quad \gamma \neq 0, \qquad (1\text{b})\, p = z_1 z_2 + \gamma,$$

for some $\gamma \in \mathbb{C}$. To find $\mathcal{I}(p)$ for each normal form, we compute the resultant of the two real polynomials, as introduced in (3), with respect to x_i to have a univariate polynomial in x_j, where $i, j \in \{1, 2\}$, and $i \neq j$. Then we use the discriminantal conditions on the univariate polynomials to argue about the real roots.

First consider the normal form (1a.1). If $\gamma_{\mathrm{im}} = 0$, then we have the real conics of the cases (vi) and $(viii)$ in Theorem 2.4. The two real polynomials $p_{\mathrm{re}} = x_1^2 - y_1^2 + \gamma_{\mathrm{re}} = 0$ and $p_{\mathrm{im}} = 2x_1 y_1 + \gamma_{\mathrm{im}} = 0$ form the system (3) here. From $\gamma_{\mathrm{im}} \neq 0$, we need to have $y_1 \neq 0$. Now substituting $x_1 = \frac{-\gamma_{\mathrm{im}}}{2y_1}$ from $p_{\mathrm{im}} = 0$ into $p_{\mathrm{re}} = 0$ and solving for y_1^2 implies $y_1^2 = \frac{1}{2}\left(\gamma_{\mathrm{re}} + \sqrt{\gamma_{\mathrm{re}}^2 + \gamma_{\mathrm{im}}^2}\right)$. Therefore,

$$\mathcal{I}(p) = \begin{cases} \text{A unique line} & \text{if } \gamma \in \mathbb{R}_{\leq 0}, \\ \text{Two parallel lines} & \text{otherwise.} \end{cases} \tag{1a.1}$$

Clearly, the closures of the components in the complement are spectrahedra.

Now consider (1a.2) which is a generalization of the parabola case (iii) in Theorem 2.4, where $\gamma = 1$. Similarly to the previous case, we build the corresponding polynomial system as (3). The discriminantal condition after substituting x_2 from $p_{\mathrm{im}} = 0$ into $p_{\mathrm{re}} = 0$ implies that there exists a real x_1 if and only if $4|\gamma|^2(y_1^2 + \gamma_{\mathrm{im}} y_2) \geq 0$. Hence, $\mathcal{I}(p)^{\mathsf{c}}$ consists of $\mathbf{y} \in \mathbb{R}^2$ such that $y_1^2 + \gamma_{\mathrm{im}} y_2 < 0$. This inequality specifies the open subset of $\mathbb{R}^2$ bounded by the parabola $y_1^2 + \gamma_{\mathrm{im}} y_2 = 0$ and containing its focus. Therefore,

$$\mathcal{I}(p) = \begin{cases} \mathbb{R}^2 \setminus \{(0, y_2) : y_2 \neq 0\} & \text{if } \gamma \in \mathbb{R}, \\ \{\mathbf{y} \in \mathbb{R}^2 : y_1^2 + \gamma_{\mathrm{im}} y_2 \geq 0\} & \text{otherwise.} \end{cases} \tag{1a.2}$$

Notice that this incidence of $\mathcal{I}(p)^{\mathsf{c}}$ consisting of one unbounded component does not occur for real conics. See Corollary 5.4. Further, $\mathcal{I}(p)^{\mathsf{c}}$ for $\gamma \notin \mathbb{R}$ is given by the unbounded spectrahedral set defined by

$$\begin{pmatrix} 1 & y_1 \\ y_1 & -\gamma_{\mathrm{im}} y_2 \end{pmatrix} \succ 0.$$

For the last case (1b) from the corresponding real polynomial system $p_{\mathrm{re}} = p_{\mathrm{im}} = 0$, one can simply check that $\gamma = 0$ implies $\mathcal{I}(p) = \{\mathbf{y} \in \mathbb{R}^2 : y_1 y_2 = 0\}$. Now let $\gamma \neq 0$ and first assume $y_1 y_2 \neq 0$. After the substitution of x_2 from $p_{\mathrm{im}} = 0$ to $p_{\mathrm{re}} = 0$, the discriminantal condition on the quadratic univariate polynomial to have a real x_1 implies

$$\gamma_{\mathrm{re}} - |\gamma| \leq 2 y_1 y_2 \leq \gamma_{\mathrm{re}} + |\gamma|.$$

If $\gamma \in \mathbb{R} \setminus \{0\}$, then $\mathbf{0}$ is the only point with $y_1 y_2 = 0$ that is included in $\mathcal{I}(p)$. If $\gamma \notin \mathbb{R}$, then the union of the two axes except the origin is included in $\mathcal{I}(p)$. Thus,

$$\mathcal{I}(p) = \begin{cases} \text{The union of the two axes } y_1 \text{ and } y_2 & \text{if } \gamma = 0, \\ \{\mathbf{y} \in \mathbb{R}^2 \ : \ 0 < y_1 y_2 \leq \gamma\} \cup \{\mathbf{0}\} & \text{if } \gamma \in \mathbb{R}_{>0}, \\ \{\mathbf{y} \in \mathbb{R}^2 \ : \ \gamma \leq y_1 y_2 < 0\} \cup \{\mathbf{0}\} & \text{if } \gamma \in \mathbb{R} \setminus \mathbb{R}_{\geq 0}, \\ \left\{\mathbf{y} \in \mathbb{R}^2 \ : \ \frac{1}{2}(\gamma_{\mathrm{re}} - |\gamma|) \leq y_1 y_2 \leq \frac{1}{2}(\gamma_{\mathrm{re}} + |\gamma|)\right\} \setminus \{\mathbf{0}\} & \text{if } \gamma \notin \mathbb{R}. \end{cases} \tag{1b}$$

Corollary 4.2. *Let $p \in \mathbb{C}[z_1, z_2]$ be a complex conic with hyperbolic initial form. The complement $\mathcal{I}(p)^{\mathsf{c}}$ of the imaginary projection consists of only unbounded spectrahedral components.*

Proof. We saw this already for the cases (1a.1) and (1a.2). Therefore, we only prove the statement for (1b). There are four unbounded components, namely in each quadrant one, and no bounded component in $\mathcal{I}(p)^{\mathsf{c}}$. The closures of the four unbounded components after setting

$$w = \sqrt{\frac{1}{2}(|\gamma| + \gamma_{\mathrm{re}})} \quad \text{and} \quad u = \sqrt{\frac{1}{2}(|\gamma| - \gamma_{\mathrm{re}})}$$

have the following representations as spectrahedra. In the quadrants $y_1y_2 \geq 0$, they are expressed by $y_1y_2 - \frac{1}{2}(\gamma_{\text{re}} + |\gamma|) \geq 0$, or equivalently, $S_1(y_1,y_2) \succeq 0$ and $S_2(y_1,y_2) \succeq 0$, where

$$S_1(y_1,y_2) = \begin{pmatrix} y_1 & w \\ w & y_2 \end{pmatrix}, \quad S_2(y_1,y_2) = \begin{pmatrix} -y_1 & w \\ w & -y_2 \end{pmatrix}.$$

In the quadrants with $y_1y_2 \leq 0$, they are expressed by $y_1y_2 - \frac{1}{2}(\gamma_{\text{re}} - |\gamma|) \leq 0$, or equivalently, $S_3(y_1,y_2) \succeq 0$ and $S_4(y_1,y_2) \succeq 0$, where

$$S_3(y_1,y_2) = \begin{pmatrix} y_1 & u \\ u & -y_2 \end{pmatrix}, \quad S_4(y_1,y_2) = \begin{pmatrix} -y_1 & u \\ u & y_2 \end{pmatrix}.$$

□

Given a conic q, an explicit description of the components of $\mathcal{I}(q)^{\mathsf{c}}$ can be derived by using those of its normal form p and applying on $\mathbf{y}$ the inverse operations turning q to p. We close this subsection by providing two examples for the cases (1a.2) and (1b) and their corresponding spectrahedral components.

Example 4.3. Let $q(z_1,z_2) = z_1^2 + 2z_1z_2 + z_2^2 + 2\mathrm{i}z_2 + 1$. By applying the transformation A and the translation $\mathbf{w}$ given by

$$A := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{w} := \begin{pmatrix} 0 \\ \mathrm{i}/2 \end{pmatrix},$$

the conic q is transformed to its normal form $p = z_1^2 + 2\mathrm{i}z_2$. Thus, we have

$$\mathcal{I}(p)^{\mathsf{c}} = \left\{ y \in \mathbb{R}^2 : \begin{pmatrix} 1 & y_1 \\ y_1 & -2y_2 \end{pmatrix} \succ 0 \right\} \text{ and } \mathcal{I}(q)^{\mathsf{c}} = \left\{ y \in \mathbb{R}^2 : \begin{pmatrix} 1 & y_1+y_2 \\ y_1+y_2 & -2y_2+1 \end{pmatrix} \succ 0 \right\},$$

such that $\mathcal{I}(q)^{\mathsf{c}}$ is obtained by the inverse transformations for $\mathbf{y}$ in $\mathcal{I}(p)^{\mathsf{c}}$. Figure 4 (1a) illustrates $\mathcal{I}(q)^{\mathsf{c}}$.

Example 4.4. Let $q(z_1,z_2) = z_1^2 - z_2^2 + 2\mathrm{i}$. Applying $A = \frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$ transfers the conic q into $p = z_1z_2 + 2\mathrm{i} = 0$. The value of both u and w introduced in the proof of Corollary 4.2 is 1. By applying A^{-1} to $\mathbf{y}$, the matrices $S_1,\ldots,S_4$ transform to

$$T_1(y_1,y_2) = \begin{pmatrix} -y_1-y_2 & 1 \\ 1 & -y_1+y_2 \end{pmatrix}, \qquad T_2(y_1,y_2) = \begin{pmatrix} y_1+y_2 & 1 \\ 1 & y_1-y_2 \end{pmatrix},$$
$$T_3(y_1,y_2) = \begin{pmatrix} -y_1-y_2 & 1 \\ 1 & y_1-y_2 \end{pmatrix}, \qquad T_4(y_1,y_2) = \begin{pmatrix} y_1+y_2 & 1 \\ 1 & -y_1+y_2 \end{pmatrix}.$$

Thus, the complement of the imaginary projection as shown in Figure 3 is given by

$$\overline{\mathcal{I}(q)^{\mathsf{c}}} = \bigcup_{j=1}^{4} \left\{ \mathbf{y} \in \mathbb{R}^2 : T_j(y_1,y_2) \succeq 0 \right\}.$$

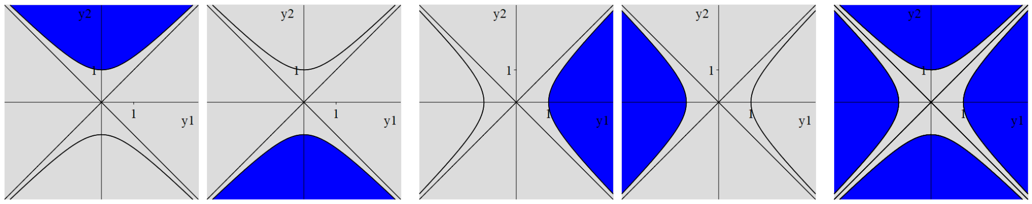


FIGURE 3. The first four pictures represent $T_j(y_1, y_2) \succeq 0$ for $1 \leq j \leq 4$, and the last one shows their union, which gives $\mathcal{I}(q)^{\mathsf{c}}$ for $q = z_1^2 - z_2^2 + 2\mathrm{i}$.

In the example above all four components are strictly convex, which can not occur in the case of real conics. This provides a key ingredient in the proof of Theorem 7.1.

4.2. **Higher dimensional complex quadratics.** We now let the dimension to be at least three and we use the normal forms provided in Lemma 4.1 to show the following classification of the imaginary projection. To avoid redundancy, for each quadratic polynomial we set n to be the largest index of z appearing in its normal form. Since we have already covered the case of conics, we need to consider $n \geq 3$.

Theorem 4.5. *Let $n \geq 3$ and $p \in \mathbb{C}[z_1, \ldots, z_n]$ be a quadratic polynomial with hyperbolic initial form. Up to the action of G_n, the imaginary projection $\mathcal{I}(p)$ is either $\mathbb{R}^n$, $\mathbb{R}^n \setminus \{(0, \ldots, 0, y_n) \in \mathbb{R}^n : y_n \neq 0\}$, or otherwise we can write p as $p = \sum_{i=1}^{n-1} z_i^2 - z_n^2 + \gamma$ for some $\gamma \in \mathbb{C}$ such that $|\gamma| = 1$ and we get*

$$\mathcal{I}(p) = \begin{cases} \left\{\mathbf{y} \in \mathbb{R}^n \ : \ y_n^2 < \sum_{i=1}^{n-1} y_i^2\right\} \cup \{\mathbf{0}\} & \text{if } \gamma = 1, \\ \left\{\mathbf{y} \in \mathbb{R}^n \ : \ y_n^2 - \sum_{i=1}^{n-1} y_i^2 \leq 1\right\} & \text{if } \gamma = -1, \\ \left\{\mathbf{y} \in \mathbb{R}^n \ : \ y_n^2 - \sum_{i=1}^{n-1} y_i^2 \leq \frac{1}{2}(1 - \gamma_{\mathrm{re}})\right\} \setminus \{\mathbf{0}\} & \text{if } \gamma \notin \mathbb{R}. \end{cases}$$

Proof. By real scaling and complex translations, any of the forms in Lemma 4.1 drops into one of the following cases:

(a) $\alpha = r = \gamma = 0$, (b) $\alpha = 1$, and $r = \gamma = 0$, (c) $\alpha \notin \mathbb{R}$, and $r, \gamma = 0$,

(d) $\alpha \notin \mathbb{R}$, $r = 1$, and $\gamma = 0$, (e) $\alpha = r = 0$, and $\gamma \neq 0$.

For the normal form (1) all cases but (d) drop into the conic sections discussed previously. Case (d) is similar for both normal forms (1) and (2). Thus we focus on (2).

The imaginary projection for the cases (a) and (b) are known from the real classification and they are $\mathbb{R}^n$ and $\mathbb{R}^n \setminus \{(0, \ldots, 0, y_n) \in \mathbb{R}^n : y_n \neq 0\}$, respectively. See [20, Theorem 5.4].

In case (c) after building the system (3) and considering two cases, based on whether the real part of α is zero or not, one can then check that $\mathcal{I}(p) = \mathbb{R}^n$ as follows. We have

$$\begin{aligned} p_{\mathrm{re}} &= \sum_{i=1}^{n-2} x_i^2 - x_{n-1}^2 - \sum_{i=1}^{n-2} y_i^2 + y_{n-1}^2 + \alpha_{\mathrm{re}} x_n - \alpha_{\mathrm{im}} y_n, \\ p_{\mathrm{im}} &= 2\sum_{i=1}^{n-2} x_i y_i - 2x_{n-1} y_{n-1} + \alpha_{\mathrm{im}} x_n + \alpha_{\mathrm{re}} y_n. \end{aligned}$$

First assume $\alpha_{\mathrm{re}} = 0$. For any $\mathbf{y} \in \mathbb{R}^n$, the equation $p_{\mathrm{re}} = 0$ has solutions $(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. By substituting any of those solutions in $p_{\mathrm{im}} = 0$ we can solve it for x_n and get a real solution. Now let $\alpha_{\mathrm{re}} \neq 0$. In this case, we substitute x_n from the second equation into the first. For any $\mathbf{y} \in \mathbb{R}^n$, we get $\sum_{i=1}^{n-2}(x_i - r_i)^2 - (x_{n-1} - r_{n-1})^2 = r_n$

for some $r_1, \dots, r_n \in \mathbb{R}$ and therefore, there always exists a real solution $(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$.

Similarly, in the case (d), for any $\mathbf{y} \in \mathbb{R}^n$, there exists a real solution $(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ for $p_{\text{im}} = 0$ and for any $\mathbf{y} \in \mathbb{R}^n$ and any $(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$, there exists a real x_n for $p_{\text{re}} = 0$. Thus $\mathcal{I}(p) = \mathbb{R}^n$ in this case, too.

Now we focus on case (e). Let $p = \sum_{i=1}^{n-1} z_i^2 - z_n^2 + \gamma$ for some $\gamma \in \mathbb{C} \setminus \{0\}$. Building the real system (3) for p yields

$$p_{\text{re}} \;=\; \textstyle\sum_{i=1}^{n-1} x_i^2 - x_n^2 - \sum_{i=1}^{n-1} y_i^2 + y_n^2 + \gamma_{\text{re}}, \qquad p_{\text{im}} \;=\; 2\sum_{i=1}^{n-1} x_i y_i - 2x_n y_n + \gamma_{\text{im}}.$$

We can assume $|\gamma| = 1$. Note that $\{\mathbf{0}\} \in \mathcal{I}(p)$ if and only if $\gamma \in \mathbb{R}$. We can thus exclude the origin in the following calculations. Moreover, Theorem 2.5 shows the cases where $\gamma = \pm 1$. Thus, we need to consider the case $\gamma \notin \mathbb{R}$.

Let T be an orthogonal transformation on $\mathbb{R}^{n-1}$. Invariance of the polynomials $\sum_{j=1}^{n-1} y_j^2$ and $\sum_{j=1}^{n-1} x_j y_j$ under the mapping $(x, y) \mapsto (T(x), T(y))$ implies

$$(y_1, y_2, \dots, y_n) \in \mathcal{I}(p) \qquad \text{if and only if} \qquad (y_1', \dots, y_{n-1}', y_n) \in \mathcal{I}(p),$$

where $(y_1', \dots, y_{n-1}') = T(y_1, \dots, y_{n-1})$. For a given $\mathbf{y} \in \mathcal{I}(p)$, let T be a transformation with the property $T(y_1, \dots, y_{n-1}) = (\sqrt{\sum_{i=1}^{n-1} y_i^2}, 0, \dots, 0)$ and set $(x_1', \dots, x_{n-1}') = T(x_1, \dots, x_{n-1})$. We can now rewrite the simplified polynomial system as

$$p_{\text{re}} \;=\; \textstyle\sum_{i=1}^{n-1} {x_i'}^2 - x_n^2 - {y_1'}^2 + y_n^2 + \gamma_{\text{re}}, \qquad p_{\text{im}} \;=\; 2x_1' y_1' - 2x_n y_n + \gamma_{\text{im}}.$$

First consider $y_1' = 0$. This implies $y_n \neq 0$. Solving $p_{\text{im}} = 0$ for x_n and substituting in $p_{\text{re}} = 0$ implies

$$4y_n^2 \Big(\sum_{i=1}^{n-1} {x_i'}^2\Big) = \left(\gamma_{\text{re}}^2 + \gamma_{\text{im}}^2\right) - \left(2y_n^2 + \gamma_{\text{re}}\right)^2 = 1 - \left(2y_n^2 + \gamma_{\text{re}}\right)^2.$$

This has a real solution for $(x_1', \dots, x_{n-1}')$ if and only if $y_n^2 \le \frac{1-\gamma_{\text{re}}}{2}$. Now assume $y_1' \neq 0$. Observe that if ${y_1'}^2 = y_n^2$ then we always get a real solution. Thus assume $\frac{y_n^2}{{y_1'}^2} - 1 \neq 0$. Solving $p_{\text{im}} = 0$ for x_1' and substituting in $p_{\text{re}} = 0$ implies

$$\left(\frac{y_n^2}{{y_1'}^2} - 1\right)\left(x_n - \frac{\gamma_{\text{im}} y_n}{2{y_1'}^2\left(\frac{y_n^2}{{y_1'}^2} - 1\right)}\right)^2 + \sum_{i=2}^{n-1} {x_i'}^2 + \frac{\left(y_n^2 - {y_1'}^2\right)^2 + \gamma_{\text{re}}\left(y_n^2 - {y_1'}^2\right) - \left(\frac{\gamma_{\text{im}}}{2}\right)^2}{y_n^2 - {y_1'}^2} = 0.$$

If ${y_1'}^2 > y_n^2$, there always is a real solution and otherwise, it has a real solution if and only if $\left(y_n^2 - {y_1'}^2\right)^2 + \gamma_{\text{re}}\left(y_n^2 - {y_1'}^2\right) - \left(\frac{\gamma_{\text{im}}}{2}\right)^2 \le 0$. That is, $y_n^2 - {y_1'}^2 \le \frac{1-\gamma_{\text{re}}}{2}$. To get the imaginary projection of the original system, it is enough to do the inverse transformation T^{-1}. This completes the proof. □

Corollary 4.6. *Let $p \in \mathbb{C}[z_1, \dots, z_n]$ be a quadratic polynomial with hyperbolic initial form. Then*

(1) *the complement $\mathcal{I}(p)^{\mathsf{c}}$ is either empty or it consists of*
   - *one, two, three, or four unbounded components; or*
   - *two unbounded components and a single point.*

(2) *the complement of the closure $\overline{\mathcal{I}(p)}^{\mathsf{c}}$ is either empty or unbounded.*

(3) *the algebraic degrees of the irreducible components in $\partial\mathcal{I}(p)$ are at most two.*

5. The main classification of complex conics

In this section, we give a classification of the imaginary projection $\mathcal{I}(p)$ where $p \in \mathbb{C}[\mathbf{z}] = \mathbb{C}[z_1, z_2]$ is a complex conic as in Definition 2.2. We state our topological classification in terms of the number and boundedness of the components in $\mathcal{I}(p)^{\mathsf{c}}$. In particular, this implies that the number of bounded and unbounded components do not exceed one and four, respectively. Furthermore, $\mathcal{I}(p)^{\mathsf{c}}$ cannot contain both bounded and unbounded components for some complex conic p.

A main achievement of this section is to establish a suitable classification and normal forms of complex conics under the action of the group G_2. There are infinitely many orbits on the set of complex conics under this action, since the real dimension of G_2 is 8 and the set of complex conics has real dimension 10. Each of our normal forms corresponds to infinitely many orbits that share their topology of imaginary projection by Lemma 2.3.

As a consequence of the obstructions in the existing classifications of conics that we discussed in the Introduction, we developed our own classification of conic sections. It is based on the five distinct arrangement possibilities for the roots of the initial form in $\mathbb{P}^1$ that are grouped in two main cases, depending on whether the initial form of the complex conic is hyperbolic or not:

Hyperbolic initial form	Non-hyperbolic initial form
(1a) A double real root	(2a) A double non-real root
(1b) Two distinct real roots	(2b) One real and one non-real root
	(2c) Two distinct non-real roots

Theorem 5.1 (**Topological Classification**). *Let $p \in \mathbb{C}[z_1, z_2]$ be a complex conic. For the above five cases, the set $\mathcal{I}(p)^{\mathsf{c}}$ is*

(1a) *the union of one, two, or three unbounded components.*

(1b) *the union of four unbounded components.*

(2a) *empty.*

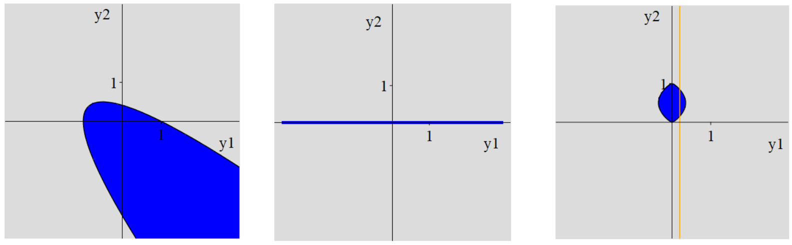
(2b) *empty, a single point, or a line segment.*

(2c) *empty or one bounded component, possibly open.*

In particular, the components of $\mathcal{I}(p)^{\mathsf{c}}$ are spectrahedral in all the first four classes. This is not true in general for the last class (2c)*.*

The following corollary relates the boundedness of the components in $\mathcal{I}(p)^{\mathsf{c}}$ to the hyperbolicity of the initial form $\mathrm{in}(p)$.

Corollary 5.2. *Let $p \in \mathbb{C}[z_1, z_2]$ be a complex conic. Then $\mathcal{I}(p)^{\mathsf{c}}$ consists of unbounded components if and only if the initial form of p is hyperbolic. Otherwise, $\mathcal{I}(p)^{\mathsf{c}}$ is empty or consists of one bounded component. Moreover, if there is a bounded component with non-empty interior, then* $\mathrm{in}(p)$ *has two distinct non-real roots.*



(1a) (2b) (2c)

FIGURE 4. The complements of the imaginary projections are colored in blue. The pictures show cases in the classification of the imaginary projection for complex conics which do not appear for real conics. The orange line in the right figure represents a generic line intersecting the boundary in two points, which is used to prove the non-spectrahedrality of this example in Section 6.

Figure 4 represents the types that do not appear for real coefficients. For instance, the middle picture, labeled as (2b), shows the case where $\mathcal{I}(p)^{\mathsf{c}}$ consists of a bounded component with empty interior. This can not occur if p has only real coefficients. The other two pictures are discussed in the next two corollaries. The following corollary compares the algebraic degrees of the irreducible components in the boundary $\partial\mathcal{I}(p)$. Its proof comes at the end of the next section.

Corollary 5.3. *Let $p \in \mathbb{C}[z_1, z_2]$ be a complex conic.*

(1) *The boundary $\partial\mathcal{I}(p)$ may not be algebraic. The algebraic degree of any irreducible component in its Zariski closure is at most 8. The bound is tight. If $\mathcal{I}(p)^{\mathsf{c}}$ has no bounded components, then $\partial\mathcal{I}(p)$ is algebraic and it consists of irreducible pieces of degree at most two.*

(2) *If all coefficients are real, then $\partial\mathcal{I}(p)$ is algebraic and it consists of irreducible pieces of degree at most two.*

Example 3.2, that is shown in Figure 4 (2c), illustrates an instance where the above contrast appears. The next corollary compares the number and strict convexity of the unbounded components that occur in $\mathcal{I}(p)^{\mathsf{c}}$ when p is a complex or a real conic.

Corollary 5.4. *Let $p \in \mathbb{C}[z_1, z_2]$ be a complex conic.*

(1) *The number of unbounded components in $\mathcal{I}(p)^{\mathsf{c}}$ can be any integer $0 \leq k \leq 4$ and up to 4 of them can be strictly convex.*

(2) *If all coefficients are real, the number of unbounded components in $\mathcal{I}(p)^{\mathsf{c}}$ can be any integer $0 \leq k \leq 4$ except for $k = 1$ and up to 2 of them can be strictly convex.*

The proof follows from Theorems 2.4 and 5.1, together with Example 4.4. The highlighting difference in the previous corollary, i.e., when $\mathcal{I}(p)^{\mathsf{c}}$ has one unbounded component, appears in the first class (1a) where the initial form has a double real root. Example 4.3 provides such an instance and is shown in Figure 4 (1a).

Theorem 5.1 is only proven by the end of Section 6. In the previous section, we discussed the case where p has hyperbolic initial form in details. It remains to consider the case where $\mathrm{in}(p)$ is not hyperbolic. As in Subsection 4.1, we first need to compute proper normal forms and then by Lemma 2.3, it suffices to compute the imaginary projections of those forms for each case.

Theorem 5.5 (**Normal Form Classification**)**.** *With respect to the group G_2, there are infinitely many orbits for the complex conic sections with the following representatives.*

(1a) (1a.1) $p = z_1^2 + \gamma$
(1a.2) $p = z_1^2 + \gamma z_2$

(1b) $p = z_1 z_2 + \gamma$

(2a) (2a.1) $p = (z_1 - \mathrm{i}z_2)^2 + \gamma$
(2a.2) $p = (z_1 - \mathrm{i}z_2)^2 + \gamma z_2$

(2b) $p = z_2(z_1 - \alpha z_2) + \gamma$

(2c) (2c.1) $p = z_1^2 + z_2^2 + \gamma$
(2c.2) $p = (z_1 - \mathrm{i}z_2)(z_1 - \alpha z_2) + \gamma$

for some $\gamma, \alpha \in \mathbb{C}$ such that, to avoid overlapping, we assume $\gamma \neq 0$ in (1a.2) *and* (2a.2)*, $\alpha \notin \mathbb{R}$ in* (2b) *and* (2c.2)*, and finally $\alpha \neq \pm\mathrm{i}$ in* (2c.2)*.*

Proof. By applying a real linear transformation we first map the roots of $\mathrm{in}(p)$ to $(0:1)$ in (1a), to $(1:0)$ and $(0:1)$ in (1b), to $(\mathrm{i}:1)$ in (2a), to $(1:0)$ and $(\alpha, 1)$ such that $\alpha \notin \mathbb{R}$ in (2b), to $(\pm\mathrm{i}:1)$ in (2c.1), and to $(\mathrm{i}:1)$ and $(\alpha:1)$ such that $\alpha \notin \mathbb{R}$ and $\alpha \neq \pm\mathrm{i}$ in (2c.2). Then, similar to the proof of Lemma 4.1, by eliminating some linear terms or the constant by complex translations we arrive at the given normal forms for each case. Since the arrangements of the two roots in $\mathbb{P}^1$ is invariant under the action of G_2, the given five cases lie in different orbits. Note that the orbits of the subcases in each case do not overlap. For the subcases of (1a), in (1a.2), z_1 and z_2 may be transformed to $az_1 + bz_2 + e$ and $cz_1 + dz_2 + f$ with $a, b, c, d \in \mathbb{R}$ and $e, f \in \mathbb{C}$. This leads to $(az_1 + bz_2 + e)^2 + \gamma$. Since z_2^2 does not appear in the normal form of case (1a.2), we get $b = 0$ and thus z_2 can not appear. Further $z_1^2 + \gamma_1$ and $z_1^2 + \gamma_2$ with $\gamma_1 \neq \gamma_2$ belong to different orbits since the previous argument enforces $a = 1, b = 0, e = 0$. The other cases are similar. Thus, for any of the eight normal forms, there are infinitely many orbits corresponding to each $\gamma \in \mathbb{C}$ (and $\alpha \in \mathbb{C}$ in some cases). □

6. Complex conics with non-hyperbolic initial form

We complete the proof of the Topological Classification Theorem 5.1 by treating the case where the complex conic $p \in \mathbb{C}[\mathbf{z}] = \mathbb{C}[z_1, z_2]$ does not have a hyperbolic initial form. In particular, we see that, as previously stated in Corollary 5.2, if the initial form of p is not hyperbolic, then $\mathcal{I}(p)^{\mathsf{c}}$ is empty or consists of one bounded component whose interior is non-empty only if $\mathrm{in}(p)$ has two distinct non-real roots in $\mathbb{P}^1$.

The overall steps in computing the imaginary projection of the cases with non-hyperbolic initial form are as follows. After building up the real polynomial system for the classes (2b) and (2c.1) of Theorem 5.5 as in (3), we use the same techniques as in Subsection 4.1. However, in the case (2a), by the nature of the polynomial system, we directly argue that the imaginary projection is $\mathbb{R}^2$. In the last case (2c.2), we do not explicitly represent the components of $\mathcal{I}(p)^{\mathsf{c}}$. Instead, in Theorem 6.1 we prove that it does not contain any unbounded components and the number of bounded components does not exceed one.

6.1. **A double non-real root (2a).** We show that in this case we have a full space imaginary projection. First consider the normal form (2a.1). We have

$$\begin{array}{rclcl} p_{\mathrm{re}} & = & x_1^2 - x_2^2 + 2y_2x_1 + 2y_1x_2 + \gamma_{\mathrm{re}}x_2 - y_1^2 + y_2^2 - \gamma_{\mathrm{im}}y_2 & = & 0, \\ p_{\mathrm{im}} & = & -2x_1x_2 + 2y_1x_1 - 2y_2x_2 + \gamma_{\mathrm{im}}x_2 + 2y_1y_2 + \gamma_{\mathrm{re}}y_2 & = & 0. \end{array}$$

We prove $\mathcal{I}(p) = \mathbb{R}^2$ by showing that for every given $\mathbf{y} \in \mathbb{R}^2$, these two real conics in $\mathbf{x} = (x_1, x_2)$ have a real intersection point. For any fixed $\mathbf{y} \in \mathbb{R}^2$, the bivariate polynomial p_{re} in $\mathbf{x}$ has the quadratic part $x_1^2 - x_2^2$, and hence, the equation $p_{\mathrm{re}} = 0$ defines a real hyperbola in $\mathbf{x}$ with asymptotes $x_1 = x_2 + c_1$ and $x_1 = -x_2 + c_2$ for some constants $c_1, c_2 \in \mathbb{R}$; possibly the hyperbola degenerates to a union of these two lines. The degree two part of the polynomial p_{im} is given by $-2x_1x_2$ and hence, the equation $p_{\mathrm{im}} = 0$ defines a real hyperbola in $\mathbf{x}$ with asymptotes $x_1 = d_1$ and $x_2 = d_2$ for some constants $d_1, d_2 \in \mathbb{R}$; possibly the hyperbola may degenerate to a union of these two lines. Since the two hyperbolas have a real intersection point, the claim follows. The case (2a.2) is similar.

6.2. **One real and one non-real root (2b).** This case gives the system of equations

$$\begin{array}{rclcl} p_{\mathrm{re}} & = & -\alpha_{\mathrm{re}}x_2^2 + x_1x_2 + 2\alpha_{\mathrm{im}}y_2x_2 + \alpha_{\mathrm{re}}y_2^2 - y_1y_2 + \gamma_{\mathrm{re}} & = & 0, \\ p_{\mathrm{im}} & = & -\alpha_{\mathrm{im}}x_2^2 + y_2x_1 + y_1x_2 - 2\alpha_{\mathrm{re}}y_2x_2 + \alpha_{\mathrm{im}}y_2^2 + \gamma_{\mathrm{im}} & = & 0. \end{array}$$

First assume $y_2 \neq 0$. By solving the second equation for x_1, substituting the solution into the first equation and clearing the denominator, we get a univariate cubic polynomial in x_2 with non-zero leading coefficient. Since real cubic polynomials always have a real root, this shows that for $\mathbf{y} \in \mathbb{R}^2$ with $y_2 \neq 0$, there is a solution $\mathbf{x} \in \mathbb{R}^2$.

It remains to consider $y_2 = 0$. In this case, the second equation has a real solution in x_2 whenever the corresponding discriminant $y_1^2 + 4\alpha_{\mathrm{im}}\gamma_{\mathrm{im}}$ is non-negative, and if one of these solutions is non-zero, the first equation then gives a real solution for x_1. The special case that in the second equation both solutions for x_2 are zero, can only occur for $y_1 = 0$ and $\gamma_{\mathrm{im}} = 0$. Then the first equation has a real solution for x_1 if and only if $\gamma_{\mathrm{re}} = 0$. Altogether, we obtain

$$\mathcal{I}(p) \;=\; \begin{cases} \mathbb{R}^2 & \text{if } \gamma = 0 \;\text{ or }\; \alpha_{\mathrm{im}}\gamma_{\mathrm{im}} > 0, \\ \mathbb{R}^2 \setminus \{\mathbf{0}\} & \text{if } \gamma \in \mathbb{R} \setminus \{0\}, \\ \mathbb{R}^2 \setminus \{(y_1, 0) \;:\; y_1^2 < -4\alpha_{\mathrm{im}}\gamma_{\mathrm{im}}\} & \text{if } \alpha_{\mathrm{im}}\gamma_{\mathrm{im}} < 0. \end{cases} \tag{2b}$$

Note that when $\gamma \in \mathbb{R} \setminus \{0\}$ then $\mathcal{I}(p)$ is open but not $\mathbb{R}^2$. This answers Question 2.6. See Figure 4 (2b) for the imaginary projection of $p = z_2(z_1 - \mathrm{i}z_2) - \mathrm{i}$ from this class.

6.3. **Two distinct non-real roots (2c).** First we show that in (2c.1), i.e., where the roots of the initial form are complex conjugate, the imaginary projection is one open bounded component. After forming the polynomial system (3), the same methods as those in Subsection 4.1, i.e., taking the resultant of the two polynomials p_{re} and p_{im} with respect to x_2 and checking the discriminantal conditions to have a real x_1, lead to the imaginary projection

$$\mathcal{I}(p) = \left\{ \mathbf{y} \in \mathbb{R}^2 : y_1^2 + y_2^2 \geq \frac{1}{2}(\gamma_{\mathrm{re}} + \sqrt{\gamma_{\mathrm{re}}^2 + \gamma_{\mathrm{im}}^2}) \right\}. \tag{2c.1}$$

In particular, we have $\mathcal{I}(p) = \mathbb{R}^2$ if and only if $\gamma_{\text{im}} = 0$ and $\gamma_{\text{re}} \leq 0$. Hence, in the case of two non-real conjugate roots, $\mathcal{I}(p)^{\mathsf{c}}$ consists of either one or zero bounded component and it is a spectrahedral set.

The subsequent lemma shows that for the case (2c) in general $\mathcal{I}(p)^{\mathsf{c}}$ is either empty or consists of one bounded component.

Lemma 6.1. *Let $p = (z_1 - \alpha z_2)(z_1 - \beta z_2) + dz_1 + ez_2 + f$ with $\alpha, \beta \notin \mathbb{R}$ and $d, e, f \in \mathbb{C}$. Then*

(1) *$\mathcal{I}(p)^{\mathsf{c}}$ has at most one bounded component.*
(2) *$\mathcal{I}(p)^{\mathsf{c}}$ does not have unbounded components.*

Proof. (1) Assume that there are at least two bounded components in $\mathcal{I}(p)^{\mathsf{c}}$. By Lemma 2.3, we can assume without loss of generality that the y_1-axis intersects both components. Solving $p = 0$ for z_1 gives

$$z_1 \;=\; \frac{\alpha+\beta}{2} z_2 - \frac{d}{2} + \sqrt[\mathbb{C}]{\left(\frac{\alpha-\beta}{2}\right)^2 z_2^2 - ez_2 - f}\,. \tag{4}$$

By letting $z_2 \in \mathbb{R}$ we obtain two continuous branches $y_1^{(1)}(z_2)$ and $y_1^{(2)}(z_2)$ satisfying (4). Therefore, the set $\mathcal{I}(p) \cap \{\mathbf{y} \in \mathbb{R}^2 \,:\, y_2 = 0\}$ has at most two connected components. This is a contradiction to our assumption that the y_1-axis intersects the two bounded components in $\mathcal{I}(p)^{\mathsf{c}}$.

For (2), assume that there exists an unbounded component in the complement of $\mathcal{I}(p)$. The convexity implies that it must contain a ray. By Lemma 2.3, we can assume without loss of generality that the ray is the non-negative part of the y_1-axis. Similarly to the proof of (1), we set $y_2 = 0$ and check the imaginary projection on y_1-axis, using the two complex solutions in (4). Since $\alpha \neq \beta$, we have $D := \left(\frac{\alpha-\beta}{2}\right)^2 \neq 0$, where D is the discriminant of $\text{in}(p)$ with z_2 substituted to 1. We consider two cases: $D \notin \mathbb{R}_{>0}$ and $D \in \mathbb{R}_{>0}$. In both cases we get into a contradiction to the assumption that the unbounded component contains the non-negative part of the y_1-axis.

First assume $D \notin \mathbb{R}_{>0}$. For $z_2 \to \pm\infty$, the imaginary part of the radicand is dominated by the imaginary part of the square root of D. Since $D \notin \mathbb{R}_{>0}$ at least one of the two expressions

$$\left(\frac{\alpha+\beta}{2}\right)_{\text{im}} \pm \sqrt{\frac{-D_{\text{re}} + \sqrt{D_{\text{re}}^2 + D_{\text{im}}^2}}{2}}$$

is non-zero. Thus, letting $z_2 \mapsto \pm\infty$, implies $y_1 \mapsto +\infty$ in at least one of the branches.

Now assume $D \in \mathbb{R}_{>0}$. This implies $(\alpha - \beta)/2 \in \mathbb{R}$. Thus $(\alpha + \beta)/2 \notin \mathbb{R}$, since otherwise it contradicts with $\alpha, \beta \notin \mathbb{R}$. In this case, by letting z_2 grow to infinity, the dominating expression for y_1 is $\frac{1}{2}(\alpha + \beta)_{\text{im}} z_2$. Therefore, y_1 converges to $+\infty$ in one of the two branches. In both cases, for some $s > 0$, the ray $\{(y_1, 0) \in \mathbb{R}^2 : y_1 \geq s\}$ lies in the imaginary projection. This completes the proof. $\square$

Before, in Example 3.2 we have shown that the defining polynomial of the imaginary projection can be irreducible of degree 8. The previous lemma enables us to show that $\mathcal{I}(q)^{\mathsf{c}}$ has exactly one bounded component. Note that $\mathbf{0} \in \mathcal{I}(q)^{\mathsf{c}}$. Let B_ϵ be an open ball with center at the origin and radius ϵ. By letting y_1 and y_2 converge to zero, the dominating part of $\triangle$ is $y_1^4 + y_2^2$. Thus, for sufficiently small ϵ, any non-zero point in B_ϵ

has $\triangle > 0$. Therefore, $\mathcal{I}(q)^{\mathsf{c}}$ contains an open ball around the origin. Now the claim follows from Theorems 6.1.

In this example, the imaginary projection is Euclidean closed, i.e., $\overline{\mathcal{I}(q)} = \mathcal{I}(q)$, however, its boundary is not Zariski closed. We claim that the set $\mathcal{I}(q)^{\mathsf{c}}$ is not a spectrahedron. By the characterization of Helton and Vinnikov [16], it suffices to show that $\overline{\mathcal{I}(q)}$ is not rigidly convex. That is, if h is a defining polynomial of minimal degree for the component $\mathcal{I}(q)^{\mathsf{c}}$, then we have to show that a generic line ℓ through the interior of $\mathcal{I}(q)^{\mathsf{c}}$ does not meet the variety $V := \{\mathbf{x} \in \mathbb{R}^2 : h(\mathbf{x}) = 0\}$ in exactly $\deg(h)$ many real points, counting multiplicities. However, this can be checked immediately. For example, the line $y_1 = 1/3$ intersects the variety V in exactly two real points, and any sufficiently small perturbation of the line preserves the number of real intersection points. See Figure 4 (2c).

This completes the proof of Theorem 5.1. We now prove Corollary 5.3 by showing that 8 is an upper bound.

Proof of Corollary 5.3. For the first four classes we have precisely computed the boundaries $\partial\mathcal{I}(p)$ and they are algebraic with irreducible components of degree at most two. It remains to consider the case (2c), more precisely (2c.2), where $p = (z_1 - \mathrm{i}z_2)(z_1 - \alpha z_2) + \gamma$ for some $\alpha, \gamma \in \mathbb{C}$, $\alpha \notin \mathbb{R}$, and $\alpha \neq \pm\mathrm{i}$. Using Remark 3.1, we show that the degrees of the irreducible components in the Zariski closure of $\partial\mathcal{I}(p)$ do not exceed 8. This, together with Example 3.2, completes the proof of (1). We separate the real and the imaginary parts as before.

$$p_{\mathrm{re}} = x_1^2 + ((\alpha_{\mathrm{im}}+1)y_2) - \alpha_{\mathrm{re}}x_2)x_1 - \alpha_{\mathrm{im}}x_2^2 + ((\alpha_{\mathrm{im}}+1)y_1 - 2\alpha_{\mathrm{re}}y_2)x_2 + \alpha_{\mathrm{re}}y_2y_1 + \alpha_{\mathrm{im}}y_2^2 - y_1^2 + \gamma_{\mathrm{re}} = 0,$$

$$p_{\mathrm{im}} = ((\alpha_{\mathrm{im}}+1)x_2 + \alpha_{\mathrm{re}}y_2 - 2y_1)x_1 - \alpha_{\mathrm{re}}x_2^2 + (\alpha_{\mathrm{re}}y_1 + 2\alpha_{\mathrm{im}}y_2)x_2 + \alpha_{\mathrm{re}}y_2^2 - (\alpha_{\mathrm{im}}+1)y_1y_2 - \gamma_{\mathrm{im}} = 0.$$

First we assume $(\alpha_{\mathrm{im}} + 1)x_2 + \alpha_{\mathrm{re}}y_2 - 2y_1 \neq 0$. Solving $p_{\mathrm{im}} = 0$ for x_1 and substituting in $p_{\mathrm{re}} = 0$ returns

$$\begin{aligned}
&\Big(\alpha_{\mathrm{im}}(\alpha_{\mathrm{re}}^2 + (\alpha_{\mathrm{im}}+1)^2)\Big)x_2^4 - \Big((\alpha_1^2+\alpha_2^2+6\alpha_2+1)(-\alpha_1y_2+y_1(\alpha_2+1))\Big)x_2^3 + \Big((\alpha_1^2+5\alpha_2^2+14\alpha_2+5)y_1^2 \\
&-y_1\alpha_1(\alpha_1^2+\alpha_2^2+14\alpha_2+9)y_2 + (4\alpha_1^2+\alpha_2(\alpha_1^2+(\alpha_2-1)^2))y_2^2 + (k_2\alpha_1 - 2k_1 - k_1\alpha_2)\alpha_2 - k_2\alpha_1 - k_1\Big)x_2^2 \\
&+\Big(8(-\alpha_2-1)y_1^3 + 8\alpha_1(\alpha_2+2)y_1^2y_2 - (\alpha_2(\alpha_1^2+\alpha_2^2-\alpha_2-1) + 9\alpha_1^2+1)y_1y_2^2 + \alpha_1(\alpha_1^2+(\alpha_2-1)^2)y_2^3 \\
&+4k_1(\alpha_2+1)y_1 + ((\alpha_1^2-(\alpha_2-1)^2)k_2 - 2k_1\alpha_1(\alpha_2+1))y_2\Big)x_2 + 4y_1^4 - 8\alpha_1y_1^3y_2 + (5\alpha_1^2+(\alpha_2-1)^2)y_1^2y_2^2 \\
&\qquad -\alpha_1(\alpha_1^2+(\alpha_2-1)^2)y_1y_2^3 - 4k_1y_1^2 + 4\alpha_1k_1y_1y_2 - \alpha_1(k_1\alpha_1 + \alpha_2k_2 - k_2)y_2^2 - k_2^2.
\end{aligned}$$

Since $\alpha \notin \mathbb{R}$, the leading coefficient is non-zero. Therefore, we have a quartic univariate polynomial in x_2. The relevant polynomials for the decision of whether this polynomial has a real root for x_2 are P, D and the discriminant Disc from Remark 3.1. By computing these polynomials, we observe that Disc decomposes as $Q_1^2 \cdot q$, where Q_1 is a quadratic polynomial and q is of degree 8 in $\mathbf{y}$. The total degrees of P and D are 2 and 4, respectively.

Now let us assume $(\alpha_{\mathrm{im}} + 1)x_2 + \alpha_{\mathrm{re}}y_2 - 2y_1 = 0$. If $\alpha_{\mathrm{im}} \neq -1$, then substituting $x_2 = \frac{-\alpha_{\mathrm{re}}y_2 + 2y_1}{\alpha_{\mathrm{im}}+1}$ into $p_{\mathrm{im}} = 0$ is the quadratic Q_1. Otherwise, the substitution $\alpha_{\mathrm{im}} = -1$

and $y_1 = \frac{\alpha_{\text{re}} y_2}{2}$ in p_{re} and p_{im}, and setting $s = 2p_{\text{im}} - \alpha_{\text{re}} p_{\text{re}}$ simplifies the original system to

$$\begin{array}{rcccl} p_{\text{re}} & = & \alpha_{\text{re}}^2 y_2^2 - 4\alpha_{\text{re}} x_1 x_2 - 8\alpha_{\text{re}} x_2 y_2 + 4x_1^2 + 4x_2^2 - 4y_2^2 + 4\gamma_{\text{re}} & = & 0, \\ s & = & 2(2\alpha_{\text{re}}^2 x_1 + 3\alpha_{\text{re}}^2 y_2 + 4y_2)x_2 - (\alpha_{\text{re}}^3 y_2^2 + 4\alpha_{\text{re}} x_1^2 + 4\gamma_{\text{re}}\alpha_1 - 4\gamma_{\text{im}}) & = & 0. \end{array}$$

If the coefficient of x_2 in s is non-zero, then solving $s = 0$ for x_2 and substituting in $p_{\text{re}} = 0$ results in a quartic polynomial in x_1 with non-zero leading coefficient. In this case, the polynomials Disc, P, and D from Remark 3.1 are all univariate in y_2. The decomposition of the discriminant in this case consists of the polynomial q after the substitution $y_1 = \frac{\alpha_{\text{re}} y_2}{2}$ and the square of a quadratic polynomial Q_2. The total degrees of P and D are 2 and 4, respectively.

Otherwise, solving $2\alpha_{\text{re}}^2 x_1 + 3\alpha_{\text{re}}^2 y_2 + 4y_2 = 0$ for x_1 and substituting in $s = 0$, results in Q_2. In all the cases that we have discussed above, the degree of none of the irreducible factors appearing in the polynomials that could possibly form the $\partial\mathcal{I}(p)$ exceeds 8. Example 3.2 shows an example where this bound is reached. This completes the proof of (1). (2) follows from Theorem 2.4. □

We have precisely verified the imaginary projections for all the normal forms in Theorem 5.5 except for (2c.2) . In particular, we have shown that if p is not of the class (2c.2), then $\mathcal{I}(p) = \mathbb{R}^2$ if and only if there exist some $\gamma, \alpha \in \mathbb{C}$, and $\alpha \notin \mathbb{R}$ such that p can be transformed to one of the following normal forms.

$$(5) \qquad \begin{cases} (2a) : (z_1 - \mathrm{i}z_2)^2 + \gamma z_2 \quad \text{or} \quad (z_1 - \mathrm{i}z_2)^2 + \gamma & \\ (2b) : z_2(z_1 - \alpha z_2) + \gamma & \text{for} \quad \gamma = 0 \ \text{ or } \ \alpha_{\text{im}}\gamma_{\text{im}} < 0, \\ (2c.1) : z_1^2 + z_2^2 + \gamma & \text{for} \quad \gamma_{\text{im}} = 0 \ \text{ and } \ \gamma_{\text{re}} \leq 0. \end{cases}$$

An example for a complex conic of class (2c.2) where the imaginary projection is $\mathbb{R}^2$ is $p = z_1^2 - 3\mathrm{i}z_1z_2 - 2z_2^2$. The reason is that for any given $(y_1, y_2) \in \mathbb{R}^2$, the polynomial p vanishes on the point $(-y_2 + \mathrm{i}y_1, y_1 + \mathrm{i}y_2)$. Answering the following question completes the verification of complex conics with a full-space imaginary projection.

Question 6.2. Let $p \in \mathbb{C}[z_1, z_2]$ be a complex conic of the form $p = (z_1 - \mathrm{i}z_2)(z_1 - \alpha z_2) + \gamma$ such that $\alpha \notin \mathbb{R}$ and $\alpha \neq \pm\mathrm{i}$. Under which conditions on the coefficients $\gamma, \alpha \in \mathbb{C}$ does $\mathcal{I}(p)$ coincide with $\mathbb{R}^2$?

7. CONVEXITY RESULTS

For the case of complex plane conics, we have shown in Theorem 6.1 that there can be at most one bounded component in the complement of its imaginary projection. An example of such a conic is $z_1^2 + z_2^2 + 1 = 0$, where the unique bounded component is the unit disc, which in particular is strictly convex. In the following theorem, we show that for any $k > 0$, there exists a complex plane curve whose complement of the imaginary projection has exactly k strictly convex bounded components. For the case of real coefficients, only the lower bound of k and no exactness result is known (see [19, Theorem 1.3]).

Allowing non-real coefficients lets us break the symmetry of the imaginary projection with respect to the origin and this enables us to fix the number of components exactly instead of giving a lower bound. Furthermore, using a non-real conic which has four strictly convex unbounded components, illustrated in Figure 3, notably drops the degree of the corresponding polynomial.

Theorem 7.1. *For any $k > 0$ there exists a polynomial $p \in \mathbb{C}[z_1, z_2]$ of degree $2\lceil \frac{k}{4} \rceil + 2$ such that $\mathcal{I}(p)^{\mathsf{c}}$ consists of exactly k strictly convex bounded components.*

Proof. Let R^{φ} be the rotation map and $g : \mathbb{C}^2 \to \mathbb{C}^2$ be defined as

$$g(z_1, z_2) = z_1 z_2 + 2\mathrm{i}.$$

Note that the equation

$$\prod_{j=0}^{m-1} (g \circ R^{\pi j/2m})(z_1, z_2) = 0 \tag{6}$$

where $m = \lceil \frac{k}{4} \rceil$ as before, has $4m$ unbounded components in the complement of its imaginary projection. We need to find a circle that intersects with k of them and does not intersect with the rest $4m - k$ components. By symmetry of the construction of the equation above, the smallest distance between the origin O and each component is the same for all the components. The following picture shows the case $m = 2$.

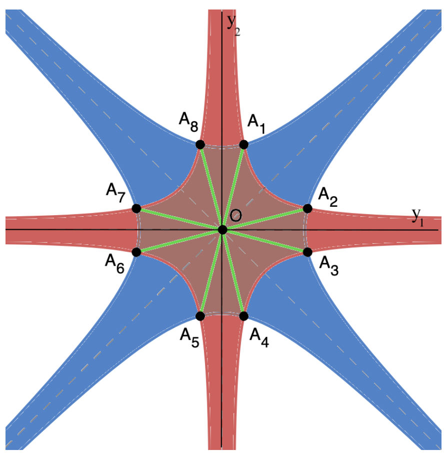


FIGURE 5. The imaginary projection of (6) for $m = 2$ is the union of the imaginary projections for polynomials corresponding to $j = 0$ and $j = 1$.

Let C be the boundary of the imaginary projection of $z_1^2 + z_2^2 + r^2$ where $r = |OA_1|$. The center of C is the origin and it passes through all $4m$ points $A_1, \ldots, A_{4m}$ that minimize the distance from the origin to each component. A sufficiently small perturbation of the center and the diameter can result in a circle C' with center (a, b) and radius s that only intersects the interiors of the first k unbounded components. Now define

$$q := (z_1 - \mathrm{i}a)^2 + (z_2 - \mathrm{i}b)^2 + s^2.$$

By Lemma 2.3 and the fact that the imaginary projection of the multiplication of two polynomials is the union of their imaginary projections, the polynomial

$$p := q \cdot \prod_{j=0}^{m-1} (g \circ R^{\pi j/2m})(z_1, z_2),$$

has exactly k strictly convex bounded components in $\mathcal{I}(p)^{\mathsf{c}}$. □

Although, by generalizing from real to complex coefficients, we improved the degree of the desired polynomial from $d = 4\lceil \frac{k}{4} \rceil + 2$ to $d/2 + 1$, it is not the optimal degree. For instance if $k = 1$, the polynomial $z_1^2 + z_2^2 + 1$ has the desired imaginary projection, while the degree is $2 < 4$. Thus, we can ask the following question.

Question 7.2. For $k > 0$, what is the smallest integer $d > 0$ for which there exists a polynomial $p \in \mathbb{C}[z_1, z_2]$ of degree d such that $\mathcal{I}(p)^{\mathsf{c}}$ consists of exactly k strictly convex bounded components.

8. Conclusion and open questions

We have classified the imaginary projections of complex conics and revealed some phenomena for polynomials with complex coefficients in higher degrees and dimensions. It seems widely open to come up with a classification of the imaginary projections of bivariate cubic polynomials, even in the case of real coefficients. In particular, the maximum number of components in the complement of the imaginary projection for both complex and real polynomials of degree d where $d \geq 3$ is currently unknown. We have shown that in degree two they coincide for real and complex conics, however, this may not be the case for cubic polynomials.

Acknowledgment. We thank the anonymous referees for their helpful comments.

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Appendix C. Combinatorics and Preservation of conically stable polynomials

COMBINATORICS AND PRESERVATION OF CONICALLY STABLE POLYNOMIALS

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Abstract. Given a closed, convex cone $K \subseteq \mathbb{R}^n$, a multivariate polynomial $f \in \mathbb{C}[\mathbf{z}]$ is called K-stable if the imaginary parts of its roots are not contained in the relative interior of K. If K is the non-negative orthant, K-stability specializes to the usual notion of stability of polynomials.

We develop generalizations of preservation operations and of combinatorial criteria from usual stability towards conic stability. A particular focus is on the cone of positive semidefinite matrices (psd-stability). In particular, we prove the preservation of psd-stability under a natural generalization of the inversion operator. Moreover, we give conditions on the support of psd-stable polynomials and characterize the support of special families of psd-stable polynomials.

arXiv:2206.10913v2 [math.CO] 28 Nov 2022

1. Introduction

Multivariate stable polynomials can be seen as a generalization of real-rooted polynomials, and they enjoy many connections to other branches in mathematics, including differential equations [3], optimization [23], probability theory [4], matroid theory [6, 9], applied algebraic geometry [24], theoretical computer science [18, 19] and statistical physics [2]. See also the surveys of Pemantle [20] and Wagner [25].

Classical related notions include *hyperbolic polynomials* [11] or stability with respect to an arbitrary domain (see, e.g., [12] and the references therein). Recently, further variants and generalizations have been developed, including *conic stability* introduced by Jörgens and the third author [14], *Lorentzian polynomials* introduced by Brändén and Huh [7] and *positively hyperbolic varieties* introduced by Rincón, Vinzant and Yu [22].

In this work we focus on the notion of conic stability. Given a closed, convex cone $K \subseteq \mathbb{R}^n$, a polynomial $f \in \mathbb{C}[\mathbf{z}] = \mathbb{C}[z_1, \ldots, z_n]$ is called *K-stable*, if $\mathsf{Im}(\mathbf{z}) \notin \operatorname{relint} K$ for every root $\mathbf{z}$ of f, where $\mathsf{Im}(\mathbf{z})$ denotes the vector of the imaginary parts of the components of $\mathbf{z}$ and $\operatorname{relint} K$ denotes the relative interior of K. Note that $(\mathbb{R}_{\geq 0})^n$-stability coincides with the usual stability. In the case of a homogeneous polynomial, K-stability of f is equivalent to the containment of $\operatorname{relint} K$ in a hyperbolicity cone of f. The notion of K-Lorentzian polynomials recently introduced by Brändén and Leake [8] is, up to scaling, a generalization of homogeneous K-stable polynomials. Stability with respect to the positive semidefinite cone on the space of symmetric matrices is denoted as *psd-stability*. In the homogeneous case such polynomials are also known as Dirichlet-Gårding polynomials [13]. Prominent subclasses of psd-stable polynomials arise from determinantal representations [10]. Blekherman, Kummer, Sanyal et al.

Date: November 29, 2022.

An extended abstract of this work was accepted for presentation at the Workshop Discrete Mathematics Days 2022, Santander.

[1] have constructed a family of psd-stable *lpm-polynomials* (*linear principle minor polynomials*) from multiaffine stable polynomials.

The purpose of the current paper is to initiate the study of generalizations of two prominent research directions in stable polynomials towards conically stable polynomials: preservation operators and combinatorial criteria. In particular, a focus is to understand the transition from the classical stability situation to the conic stability with respect to non-polyhedral cones such as the positive semidefinite cone.

With regard to preservation, stable polynomials have been recognized to remain stable under a number of operations, see the survey [25]. Prominent examples include the inversion operation (see [2]), the preservation under taking partial derivatives (as a consequence of the univariate Gauß-Lucas Theorem), the Lieb-Sokal Lemma ([17, Lemma 2.3], see also [2, Lemma 2.1]) and the celebrated characterization of Borcea and Brändén of linear operators preserving stability [2, Theorem 1.3]. Many of the mentioned applications of stability rely on the preservation properties.

With regard to *combinatorial criteria*, several important combinatorial results have been achieved, which provide effective criteria for the recognition of stable polynomials. A groundbreaking result of Choe, Oxley, Sokal and Wagner states that the support of a multi-affine, homogeneous and stable polynomial $f \in \mathbb{R}[\mathbf{z}] = \mathbb{R}[z_1, \ldots, z_n]$ is the set of bases of a matroid [9, Theorem 7.1]. Brändén [6, Theorem 3.2] proved a generalization of this result for the support of any stable polynomial $f \in \mathbb{R}[\mathbf{z}]$, showing that it forms a jump system, i.e., it satisfies the so-called Two-Steps Axiom. See Section 2 for formal definitions. Recently, Rincón, Vinzant and Yu gave an alternative proof of the matroid result, based on a tropical proof of the auxiliary statement that positive hyperbolicity of a variety is preserved under passing over to the initial form [22, Corollary 4.9].

The proofs of these combinatorial properties strongly rely on the preservation properties of stable polynomials. These preservation properties establish the connection between the combinatorial and the algebraic viewpoint. For example, taking the partial derivative of a polynomial f shifts the support vectors of f by a unit vector in a negative coordinate direction (and some support vectors may disappear). Since stability of a polynomial is preserved under taking partial derivatives, one can use this preserver to argue about the combinatorics of the support. In the univariate case, these considerations are classical for deriving log-concavity of sequences with real-rooted generating functions.

Our contributions. 1. We generalize several preserving operators for usual stability to the conic stability. In particular, we derive a conic version of the Lieb-Sokal Lemma (see Lemma 2.2 and Corollary 3.6).

2. For the case of psd-stability, we can prove the preservation under a natural generalization of the inversion operator. See Theorem 4.3. This generalized inversion operator is specific to the case of psd-stability and exhibits a prominent role of this class. Furthermore, we show that psd-stable polynomials are preserved under taking initial forms with respect to positive definite matrices. See Theorem 4.10.

3. *Combinatorics of psd-stable polynomials.* We prove a necessary criterion on the support of any psd-stable polynomial in Theorem 5.1 and characterize the support of special families of psd-stable polynomials. In particular, we characterize psd-stability of binomials (Theorem 5.5), give a necessary criterion for psd-stability of a larger class containing binomials (Theorem 5.4), and introduce a class of polynomials of

determinants, which satisfies a generalized jump system criterion with regard to psd-stability. Theorem 5.11 characterizes the restrictive structure of psd-stable polynomials of determinants. These results are complemented by an additional conjecture on the support of general psd-stable polynomials. We provide evidence for this conjecture by verifying it for the classes of polynomials treated previously.

The paper is structured as follows. Section 2 collects relevant background on preservers of the usual stability notion as well as an introduction to the notion of K-stability.

In Section 3, we study preservers of conic stability for general and polyhedral cones, including the generalized version of the Lieb-Sokal Lemma. Section 4 treats the case of psd-stability, in particular, the preservation of psd-stability under an inversion operation and under passing over to certain initial forms. Section 5 deals with combinatorial conditions of psd-stable polynomials. Therein, Subsections 5.1 and 5.2 discuss the support of special families of psd-stable polynomials. Subsection 5.3 considers the support of general psd-stable polynomials and also raises a conjecture.

Acknowledgment. This work was supported through DFG grant TH 1333/7-1. The research was primarily done while the first author was with Goethe University, supported by an Early Career Researcher grant of Goethe University.

2. Preliminaries

Let $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ denote the sets of non-negative and of positive real numbers. Further, let $\mathcal{H} := \{z \in \mathbb{C} : \mathsf{Im}(z) > 0\}$ be the *open upper half-plane* of $\mathbb{C}$. Throughout the text, bold letters will denote n-dimensional vectors unless noted otherwise.

In this section, we collect known properties of stable polynomials and then introduce the generalization of stability, namely conic stability, with which the paper is concerned.

2.1. **Stable polynomials.** A polynomial $f \in \mathbb{C}[\mathbf{z}]$ is called *stable* if for every root $\mathbf{z}$ of f, there exists some $j \in [n]$ with $\mathsf{Im}(z_j) \leq 0$. Hence, a univariate real polynomial f is stable if and only if it is real-rooted, because the non-real roots of univariate real polynomials occur in conjugate pairs. The following collection from [25, Lemma 2.4] recalls some elementary operations that preserve stability, where f) can be derived from the Gauß-Lucas Theorem. Denote by $\deg_i$ the degree in the variable z_i.

Proposition 2.1. *Let $f \in \mathbb{C}[\mathbf{z}]$ be stable.*
a) Permutation*: $f(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$ is stable for every permutation $\sigma : [n] \to [n]$.*
b) Scaling*: $c \cdot f(a_1 z_1, \ldots, a_n z_n)$ is stable or zero for every $c \in \mathbb{C}$ and $\mathbf{a} \in \mathbb{R}^n_{>0}$.*
c) Diagonalization*: $f(\mathbf{z})\big|_{z_j = z_i} \in \mathbb{C}[z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n]$ is stable or zero for every $i \neq j \in [n]$.*
d) Specialization*: $f(b, z_2, \ldots, z_n) \in \mathbb{C}[z_2, \ldots, z_n]$ is stable or zero for every $b \in \mathbb{C}$ with $\mathsf{Im}(b) \geq 0$.*
e) Inversion*: $z_1^{\deg_1(f)} \cdot f(-z_1^{-1}, z_2, \ldots, z_n)$ is stable.*
f) Differentiation*: $\partial_j f(\mathbf{z})$ is stable or zero for every $j \in [n]$.*

A prominent linear stability preserver is the Lieb-Sokal Lemma ([17, Lemma 2.3], see also [2, Lemma 2.1] or [25, Lemma 3.2]). It is an essential ingredient in Borcea and

Brändén's full characterization of linear operations preserving stability [2, Theorem 1.1], see also [3, Section 3.2].

Proposition 2.2 (Lieb-Sokal Lemma)**.** *Let $g(\mathbf{z})+yf(\mathbf{z}) \in \mathbb{C}[\mathbf{z}, y]$ be stable and assume $\deg_i(f) \leq 1$. Then $g(\mathbf{z}) - \partial_i f(\mathbf{z}) \in \mathbb{C}[\mathbf{z}]$ is stable or identically zero.*

The following statement due to Hurwitz allows us to obtain (conic) stability statements as limit of statements on compact subsets under a uniform convergence condition.

Proposition 2.3. [15, Par. 5.3.4] *Let $\{f_k\}$ be a sequence of polynomials non-vanishing in a connected open set $U \subseteq \mathbb{R}^n$, and assume it converges to a function f uniformly on compact subsets of U. Then f is either non-vanishing on U or it is identically zero.*

As a consequence of [9, Theorem 6.1], the following necessary condition for homogeneous stable polynomials based on their coefficients applies.

Theorem 2.4. *All nonzero coefficients of a homogeneous stable polynomial $f \in \mathbb{C}[\mathbf{z}]$ have the same phase.*

2.2. **Stability and initial forms.** The initial form $\mathrm{in}_{\mathbf{w}}(f)$ of a polynomial $f(\mathbf{z}) = \sum_{\alpha \in S} c_\alpha \mathbf{z}^\alpha$ with respect to a functional $\mathbf{w}$ in the dual space $(\mathbb{R}^n)^*$ is defined as

$$\mathrm{in}_{\mathbf{w}}(f) = \sum_{\alpha \in S_{\mathbf{w}}} c_\alpha \mathbf{z}^\alpha,$$

where $S_{\mathbf{w}} := \{\alpha \in S \, : \, \langle \mathbf{w}, \alpha \rangle = \max_{\beta \in S} \langle \mathbf{w}, \beta \rangle\}$ and $\langle \cdot, \cdot \rangle$ is the natural dual pairing. That is, we restrict the polynomial f to those monomials whose exponents lie on the face of the Newton polytope of f where the functional $\mathbf{w}$ is maximized.

In the context of their work on positively hyperbolic varieties, Rincón, Vinzant and Yu [22, Proposition 4.1] showed that for polynomials with real coefficients, stability is preserved under taking initial forms. Their proof is based on tropical geometry. For the convenience of the reader, we give here a simplified proof, and at the same time slightly generalize the statement to also cover polynomials with complex coefficients. The observation that the statement is also valid for complex coefficients has independently been derived by Kummer and Sert [16, Proposition 2.6].

Theorem 2.5. *If $f \in \mathbb{C}[\mathbf{z}]$ is stable and $\mathbf{w} \in (\mathbb{R}^n)^* \setminus \{\mathbf{0}\}$, then $\mathrm{in}_{\mathbf{w}}(f)(\mathbf{z})$ is also stable.*

Proof. Let $\varphi := \max \{\langle \alpha, \mathbf{w} \rangle : \alpha \in \mathrm{supp}(f)\}$, and for $\lambda > 0$, define the polynomial $f_\lambda(\mathbf{z}) := \frac{1}{\lambda^\varphi} \cdot f(\lambda^{w_1} z_1, \ldots, \lambda^{w_n} z_n)$, which is stable by Proposition 2.1.

To apply Hurwitz' Theorem to finally achieve stability of the initial form, we need to ensure that f_λ converges uniformly to $\mathrm{in}_{\mathbf{w}}(f)$ on every compact subset $C \subseteq \mathbb{C}^n$. Let $\mu = \max\{\langle \alpha, \mathbf{w} \rangle : \langle \alpha, \mathbf{w} \rangle < \varphi, \alpha \in \mathrm{supp}(f)\}$ and $\delta = \varphi - \mu > 0$. Then

$$\begin{aligned} \lim_{\lambda \to \infty} \sup_{\mathbf{z} \in C} |f_\lambda(\mathbf{z}) - \mathrm{in}_{\mathbf{w}}(f)(\mathbf{z})| &\leq \lim_{\lambda \to \infty} \sup_{\mathbf{z} \in C} \sum\nolimits_{\langle \alpha, \mathbf{w} \rangle < \varphi} \left| \frac{1}{\lambda^\delta} c_\alpha \mathbf{z}^\alpha \right| \\ &= \lim_{\lambda \to \infty} \frac{1}{\lambda^\delta} \sup_{\mathbf{z} \in C} \sum\nolimits_{\langle \alpha, \mathbf{w} \rangle < \varphi} |c_\alpha \mathbf{z}^\alpha| \;=\; 0, \end{aligned}$$

since the norm in the last equality is bounded, given that C is a compact set. □

The discussion of the preservation of conically stable polynomials when passing over to initial forms is continued at the end of Section 4.

2.3. **Combinatorics of stable polynomials.** For $\alpha, \beta \in \mathbb{Z}^n$, the *steps* between α and β are defined as the set

$$\mathrm{St}(\alpha, \beta) := \{\sigma \in \mathbb{Z}^n : |\sigma| = 1, |\alpha + \sigma - \beta| = |\alpha - \beta| - 1\},$$

where $|\sigma| := \sum_{i=1}^n |\sigma_i|$. A collection of points $\mathcal{F} \subseteq \mathbb{Z}^n$ is called a *jump system* if for every $\alpha, \beta \in \mathcal{F}$ and $\sigma \in \mathrm{St}(\alpha, \beta)$ with $\alpha + \sigma \notin \mathcal{F}$ there is some $\tau \in \mathrm{St}(\alpha + \sigma, \beta)$ such that $\alpha + \sigma + \tau \in \mathcal{F}$. In words, if after one step from α towards β we have left the set $\mathcal{F}$, then there must be a second step that takes us back into $\mathcal{F}$. This property is also known as the *Two-Steps Axiom*. The *support* of a complex polynomial $f(\mathbf{z}) = \sum_\alpha c_\alpha \mathbf{z}^\alpha$ is defined as $\mathrm{supp}(f) = \{\alpha \in \mathbb{Z}^n_{\geq 0} : c_\alpha \neq 0\}$, that is, it is the set of all exponent vectors α such that the corresponding coefficient c_α is non-zero in f. The following theorem reveals the connection between stable polynomials and jump systems.

Theorem 2.6 (Brändén [6])**.** *If $f \in \mathbb{C}[\mathbf{z}]$ is stable, then its support is a jump system.*

In [22, Proposition 4.1], the support of stable binomials is explicitly classified as follows. Here, $\mathbf{e}_i$ denotes the i-th unit vector in $\mathbb{R}^n$.

Theorem 2.7. *Let $f = c_\alpha \mathbf{z}^\alpha + c_\beta \mathbf{z}^\beta$ with $c_\alpha, c_\beta \neq 0$ and $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$ be stable and let $\mathbf{z}^\alpha$ and $\mathbf{z}^\beta$ do not have a common factor. Then one of the following holds,*

a) $\{\alpha, \beta\} = \{0, \mathbf{e}_i\}$ for some $i \in [n]$,
b) $\{\alpha, \beta\} = \{\mathbf{e}_i, \mathbf{e}_j\}$ for some $i, j \in [n]$ and $\frac{c_\alpha}{c_\beta} \in \mathbb{R}_{\geq 0}$, or
c) $\{\alpha, \beta\} = \{0, \mathbf{e}_i + \mathbf{e}_j\}$ for some $i, j \in [n]$ and $\frac{c_\alpha}{c_\beta} \in \mathbb{R}_{<0}$.

2.4. **Conic stability.** The following notion of conic stability as introduced in [14] generalizes stability to more general cones. Let K be a closed, convex cone in $\mathbb{R}^n$ and denote by $\mathrm{relint}\, K$ its relative interior.

Definition 2.8. A polynomial $f \in \mathbb{C}[\mathbf{z}]$ is called K-*stable*, if $f(\mathbf{z}) \neq 0$ whenever $\mathsf{Im}(\mathbf{z}) \in \mathrm{relint}\, K$.

Observe that by choosing the cone $K = \mathbb{R}^n_{\geq 0}$, we recover the usual notion of stability. For any closed, convex cone K, conic stability can be characterized through stability of univariate polynomials (see [14, Lemma 3.4], that proof literally also works without the assumption of full-dimensionality made there).

Proposition 2.9 ([14], Lemma 3.4)**.** *A polynomial $f \in \mathbb{C}[\mathbf{z}] \setminus \{0\}$ is K-stable if and only if for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with $\mathbf{y} \in \mathrm{relint}\, K$, the univariate polynomial $t \mapsto f(\mathbf{x} + t\mathbf{y})$ is stable or identically zero.*

Remark 2.10. A homogeneous polynomial $f \in \mathbb{C}[\mathbf{z}]$ is called *hyperbolic* w.r.t. $\mathbf{e} \in \mathbb{R}^n$ if $f(\mathbf{e}) \neq 0$ and the univariate polynomial $t \mapsto f(\mathbf{x} + t\mathbf{e})$ is real rooted. For a full-dimensional cone $K \subset \mathbb{R}^n$, every homogeneous K-stable polynomial is hyperbolic w.r.t. every $\mathbf{e} \in \mathrm{relint}\, K = \mathrm{int}\, K$ by [14, Theorem 3.5] and hence, up to a multiplicative constant every homogeneous K-stable polynomial has real coefficients [11].

2.5. **Positive semidefinite stability.** We introduce the notion of psd-stability, an important special case of conic stability where the cone is chosen to be the positive semidefinite cone.

Denote by $\mathcal{S}_n^{\mathbb{C}}$ the vector space of complex symmetric matrices (rather than Hermitian matrices) and by $\mathcal{S}_n$ the space of real ones. The cones of real positive semidefinite and positive definite matrices are denoted by $\mathcal{S}_n^+$ and $\mathcal{S}_n^{++}$. Let $\mathbb{C}[Z]$ denotes the ring of polynomials on the symmetric matrix variables $Z = (z_{ij})$. More precisely, $\mathbb{C}[Z]$ is the vector space generated by monomials of the form $Z^\alpha = \prod_{1 \le i,j \le n} z_{ij}^{\alpha_{ij}}$ with some non-negative symmetric matrix α whose diagonal entries are integers and whose off-diagonal entries are half-integers. Polynomials in $\mathbb{C}[Z]$ can also be interpreted as polynomials in the polynomial ring $\mathbb{C}[\{z_{ij} | 1 \le i \le j \le n\}]$, by identifying z_{ij} and z_{ji} for $i \neq j$. For example, consider the monomial

$$Z^{\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}} \;=\; z_{12}^{1/2} z_{21}^{1/2} \;=\; z_{12}$$

in the polynomial ring $\mathbb{C}[Z]$ over the vector space $\mathcal{S}_2^{\mathbb{C}}$.

Definition 2.11. *Psd-stability* is defined as $\mathcal{S}_n^+$-stability for polynomials over the vector space $\mathcal{S}_n^{\mathbb{C}}$ of complex symmetric matrices. That is, a polynomial $f \in \mathbb{C}[Z]$ is psd-stable if it has no root $M \in \mathcal{S}_n^{\mathbb{C}}$ such that $\mathsf{Im}(M) \in \mathcal{S}_n^{++}$.

The *support* $\operatorname{supp}(f)$ of a polynomial $f \in \mathbb{C}[Z]$ is the set of all symmetric exponent matrices of the monomials occurring with non-zero coefficients in the polynomial. The variables z_{ii} are called *diagonal variables*, while the variables z_{ij} with $i \neq j$ are the *off-diagonal variables*. We say that a monomial with exponent matrix α is a *diagonal monomial* if $\alpha_{ij} = 0$ for all $i \neq j \in [n]$, and we say that it is an *off-diagonal* monomial if $\alpha_{ii} = 0$ for all $i \in [n]$. By convention, we say that a constant is a diagonal monomial, but not an off-diagonal one.

Example 2.12. Let $f(Z) = \det(Z)$ in the polynomial ring $\mathbb{C}[Z]$ over the vector space $\mathcal{S}_2^{\mathbb{C}}$. Then

$$f(Z) \;=\; z_{11} z_{22} - z_{12}^2 \;=\; Z^{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} - Z^{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}.$$

The monomial $z_{11} z_{22}$ is a diagonal monomial while the other one is an an off-diagonal monomial.

A prime example of psd-stable polynomials are determinants. The proof is included for completeness.

Lemma 2.13. *$f(Z) = \det(Z)$ is psd-stable.*

Proof. Suppose that f is not psd-stable, that is, there exist real symmetric matrices A and B with B positive definite, such that $f(A + iB) = 0$. Then B is invertible and $0 = f(A + iB) = \det(A + iB) = \det(B) \det(B^{-\frac{1}{2}} A B^{-\frac{1}{2}} + i I_n)$, where I_n denotes the identity matrix of size n. Hence, $-i$ is a root of the characteristic polynomial of, and thus an eigenvalue of, $B^{-\frac{1}{2}} A B^{-\frac{1}{2}}$: a contradiction, since a symmetric real matrix has only real eigenvalues. $\square$

Contrary to the usual stability notion, monomials are not necessarily psd-stable. In fact, every monomial with an off-diagonal variable as a factor, is not psd-stable since it evaluates to zero for $Z = i \cdot I_n$.

Psd-stability can be viewed as stability with respect to the *Siegel upper half-space* $\mathcal{H}_{\mathcal{S}} = \{A \in \mathbb{C}^{n \times n}$ symmetric $: \mathsf{Im}(A)$ is positive definite$\}$. The Siegel upper half-space occurs in algebraic geometry and number theory as the domain of modular forms.

3. Preservers for conic stability

We provide generalizations of the stability preservers from Section 2 to conic stability with respect to some closed, convex cone K. Our focus is on general cones and on the subclass of polyhedral cones. A main result in this section is Theorem 3.4, a conic version of the Lieb-Sokal Lemma. In Section 4, the specific case of preservers for psd-stability will be studied.

A conical analogue of property b) from Proposition 2.1, scaling, holds trivially since K is a cone: for any $c \in \mathbb{C}$ and $a \in \mathbb{R}_{\geq 0}$, the polynomial $c \cdot f(az_1, \dots, az_n)$ is K-stable or identically zero. We now study the preservation of conical stability under directional derivatives. For a vector $\mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, denote by $\partial_{\mathbf{v}}$ the directional derivative in direction $\mathbf{v}$, i.e., $\partial_{\mathbf{v}} f(\mathbf{z}) = \frac{d}{dt} f(\mathbf{z} + t\mathbf{v})\big|_{t=0}$.

Lemma 3.1. *Let $f \in \mathbb{C}[\mathbf{z}]$ be K-stable. For $\mathbf{v} \in K$, the polynomial $\partial_{\mathbf{v}} f$ is K-stable or identically zero.*

In the homogeneous case, this statement follows from the concept of a Renegar derivative [21] for hyperbolic polynomials.

Proof. Let f be K-stable and $\mathbf{v} \in K$. Assume that $\partial_{\mathbf{v}} f$ is neither 0 nor K-stable. Then there is some $\mathbf{z} \in \mathbb{C}^n$ such that $\mathsf{Im}(\mathbf{z}) \in \operatorname{relint} K$ and $\partial_{\mathbf{v}} f(\mathbf{z}) = 0$.

To aim at a contradiction to the univariate Gauß-Lucas Theorem, we construct through a substitution in f a univariate polynomial $g \not\equiv 0$, which has a non-real zero. Since $\mathsf{Im}(\mathbf{z}) \in \operatorname{relint}(K)$, there exists some $\varepsilon > 0$ such that $\mathsf{Im}(\mathbf{z}) - \varepsilon\mathbf{v} \in \operatorname{relint} K$. Define the univariate polynomial $g : t \mapsto f(\mathbf{z} - i\varepsilon\mathbf{v} + t\mathbf{v})$. If $g \equiv 0$, then $f(\mathbf{z}) = g(i\varepsilon) = 0$ in contradiction to the K-stability of f. Hence, $g \not\equiv 0$. Since $\mathsf{Im}(\mathbf{z}) - \varepsilon\mathbf{v} \in \operatorname{relint}(K)$ and $\mathbf{v} \in K$, the univariate polynomial g is stable: if it had any root t with $\mathsf{Im}(t) > 0$, $\mathbf{z} - i\varepsilon\mathbf{v} + t\mathbf{v}$ would be a root of f, but its imaginary part $\mathsf{Im}(\mathbf{z}) - \varepsilon\mathbf{v} + \mathsf{Im}(t)\mathbf{v}$ is in the relative interior of the cone K, a contradiction to the conical stability of f. Moreover, g is not constant, because $\partial_{\mathbf{v}} f \not\equiv 0$. Hence, by the Gauß-Lucas Theorem, the derivative g' is stable. Since

$$g'(i\varepsilon) = \frac{\partial}{\partial t} f(\mathbf{z} - i\varepsilon\mathbf{v} + t\mathbf{v})\Big|_{t=i\varepsilon} = \partial_{\mathbf{v}} f(\mathbf{z}) = 0,$$

we obtain a contradiction to the stability of g'. □

There is a natural generalization of property d) in Lemma 2.1 to conic stability.

Lemma 3.2. *Let $f \in \mathbb{C}[\mathbf{z}]$ be K-stable, $\mathbf{a} \in \mathbb{C}^n$ and $\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(k)} \in \mathbb{R}^n$. Further set $K' = \operatorname{pos}\{\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(k)}\}$ and assume that $\mathsf{Im}(\mathbf{a}) + K' \subseteq K$. Then the polynomial $g \in \mathbb{C}[\mathbf{z}]$ defined by*

$$g(z_1, \dots, z_k) \ = \ f(\mathbf{a} + \sum_{j=1}^{k} z_j \mathbf{v}^{(j)})$$

is stable or the zero polynomial.

Setting $K = \mathbb{R}^n_{\geq 0}$, $k = n-1$, $\mathbf{v}^{(j)} = \mathbf{e}^{(j+1)}$ with the $(j+1)$-th unit vector $\mathbf{e}^{(j+1)}$, $1 \leq j \leq n-1$, and $a_2 = \cdots = a_n = 0$ yields Lemma 2.1 d).

Proof. First consider the special case where $\mathsf{Im}(\mathbf{a}) + \operatorname{relint} K' \subseteq \operatorname{relint} K$. Further assume that the polynomial $g \in \mathbb{C}[\mathbf{z}]$ is neither zero nor stable. Then there exists

$\mathbf{w} \in \mathbb{C}^k$ with $\mathsf{Im}(\mathbf{w}) \in \mathbb{R}^k_{>0}$ and $g(\mathbf{w}) = 0$, and thus $f(\mathbf{a} + \sum_{j=1}^k w_j \mathbf{v}^{(j)}) = 0$. Since $\mathsf{Im}(\mathbf{a}) + \sum_{j=1}^k w_j \mathbf{v}^{(j)} \in \mathsf{Im}(\mathbf{a}) + \operatorname{relint} K' \subseteq \operatorname{relint} K$, f is not K-stable, contradiction.

The general case ($\mathsf{Im}(\mathbf{a}) + K' \subseteq K$) follows from Hurwitz' Theorem. ☐

In the rest of this section, we present and prove a generalization of the Lieb-Sokal Lemma (Lemma 2.2) to conic stability. In the usual Lieb-Sokal Lemma, we take a partial derivative of a polynomial which has degree at most 1 in the corresponding variable. To formulate a similar result for arbitrary cones, we take a directional derivative in a direction lying in the cone, since these directional derivatives preserve conic stability by Lemma 3.1. To this end, we need a generalized notion of degree with respect to an arbitrary direction.

Definition 3.3. For $\mathbf{v} \in \mathbb{R}^n$, we call $\rho_{\mathbf{v}}(f)$ the *degree of f in direction* $\mathbf{v}$, defined as the degree of the univariate polynomial $f(\mathbf{w} + t\mathbf{v}) \in \mathbb{C}[t]$ for generic $\mathbf{w} \in \mathbb{C}^n$.

In particular, after taking the directional derivative in direction $\mathbf{v}$ exactly $\rho_{\mathbf{v}}(f) + 1$ times, we obtain the identically zero polynomial. The degree in the direction of a unit vector $\mathbf{e}^{(j)}$ coincides with the univariate degree with respect to the variable j. We can now state the conical version of Lieb-Sokal stability preservation.

Theorem 3.4 (Conic Lieb-Sokal stability preservation)**.** *Let K' be given by $K' = K \times \mathbb{R}_{\geq 0}$ and $g(\mathbf{z}) + yf(\mathbf{z}) \in \mathbb{C}[\mathbf{z}, y]$ be K'-stable and such that $\rho_{\mathbf{v}}(f) \leq 1$ for some $\mathbf{v} \in K$. Then $g - \partial_{\mathbf{v}} f$ is K-stable or $g - \partial_{\mathbf{v}} f \equiv 0$.*

We first establish a connection between a cone K and its lift K' into a higher-dimensional space, which we will use to prove Theorem 3.4.

Lemma 3.5. *Let $f, g \in \mathbb{C}[\mathbf{z}]$, where $f \not\equiv 0$ and K-stable and let $K' = K \times \mathbb{R}_{\geq 0}$. Then $g + yf \in \mathbb{C}[\mathbf{z}, y]$ is K'-stable if and only if*

$$\mathsf{Im}\left(\frac{g(\mathbf{z})}{f(\mathbf{z})}\right) \geq 0 \quad \text{for all } \mathbf{z} \in \mathbb{C}^n \text{ with } \mathsf{Im}(\mathbf{z}) \in \operatorname{relint} K.$$

Proof. Let $g + yf$ be K'-stable. Fix some $\mathbf{z}$ with $\mathsf{Im}(\mathbf{z}) \in \operatorname{relint} K$. By K-stability, we have $f(\mathbf{z}) \neq 0$, and thus we may consider $g(\mathbf{z}) + yf(\mathbf{z})$ as a univariate stable polynomial.

Setting $w = -g(\mathbf{z})/f(\mathbf{z})$, the stability of the univariate polynomial $y \mapsto g(\mathbf{z}) + yf(\mathbf{z})$ implies $\mathsf{Im}(w) \leq 0$. It follows that

$$\mathsf{Im}\left(\frac{g(\mathbf{z})}{f(\mathbf{z})}\right) = \mathsf{Im}(-w) \geq 0.$$

Conversely, suppose $\mathsf{Im}\left(\frac{g(\mathbf{z})}{f(\mathbf{z})}\right) \geq 0$ for all $\mathbf{z} \in \mathbb{C}^n$ with $\mathsf{Im}(\mathbf{z}) \in \operatorname{relint} K$. Assume $g \not\equiv 0$, since otherwise $yf(\mathbf{z})$ would clearly be K'-stable. For $\mathbf{z} \in \mathbb{C}^n$ with $\mathsf{Im}(\mathbf{z}) \in \operatorname{relint} K$, we have for $w \in \mathbb{C}$ with $\mathsf{Im}(w) > 0$ that $\frac{g(\mathbf{z})}{f(\mathbf{z})} \neq -w$. So $g(\mathbf{z}) + wf(\mathbf{z}) \neq 0$ and K'-stability follows. ☐

We can now complete the proof of Theorem 3.4.

Proof of Theorem 3.4. We begin by observing that g is K-stable or $g \equiv 0$. Let $\mathbf{v} \in K$ with $\rho_{\mathbf{v}}(f) \leq 1$. If $\partial_{\mathbf{v}} f \equiv 0$, there is nothing to prove. So assume $\partial_{\mathbf{v}} f \not\equiv 0$, and thus implies $f \not\equiv 0$. For a fixed $\mathbf{z} \in \mathbb{C}^n$ with $\mathsf{Im}(\mathbf{z}) \in \operatorname{relint} K$ we may consider $g(\mathbf{z}) + yf(\mathbf{z})$ as a univariate polynomial in y. By Lemma 3.1, the polynomial $f(\mathbf{z}) = \partial_y(g(\mathbf{z}) + yf(\mathbf{z}))$ is K-stable. For $\mathbf{z} \in \mathbb{C}^n$ with $\mathsf{Im}(\mathbf{z}) \in \operatorname{relint} K$, $\mathbf{v} \in K$ and $y \in \mathbb{C}$ with $\mathsf{Im}(y) > 0$, we have $\mathsf{Im}(\mathbf{z} - \frac{1}{y}\mathbf{v}) \in \operatorname{relint} K$, because

$$\mathsf{Im}\left(\mathbf{z} - \frac{1}{y}\mathbf{v}\right) = \mathsf{Im}(\mathbf{z}) - \mathsf{Im}\left(\frac{1}{y}\right)\mathbf{v} = \mathsf{Im}(\mathbf{z}) + \frac{1}{|y|^2}\mathsf{Im}(y) \cdot \mathbf{v} \in \operatorname{relint} K.$$

It follows that $yf(\mathbf{z} - \frac{1}{y}\mathbf{v})$ is K'-stable. Since $\rho_{\mathbf{v}}(f) \leq 1$, there exist polynomials f_0 and f_1 with $\rho_{f_0}(\mathbf{v}), \rho_{f_1}(\mathbf{v}) = 0$ and $f(\mathbf{z}) = f_0(\mathbf{z}) + \langle \mathbf{v}, \mathbf{z} \rangle \cdot f_1(\mathbf{z})$. Thus, the identity

$$yf\left(\mathbf{z} - \frac{1}{y}\mathbf{v}\right) = yf(\mathbf{z}) - \partial_{\mathbf{v}} f(\mathbf{z})$$

implies the K' stability of $yf(\mathbf{z}) - \partial_{\mathbf{v}} f(\mathbf{z})$. Applying Lemma 3.5 twice gives

$$\mathsf{Im}\left(\frac{g(\mathbf{z}) - \partial_{\mathbf{v}} f(\mathbf{z})}{f(\mathbf{z})}\right) = \mathsf{Im}\left(\frac{g(\mathbf{z})}{f(\mathbf{z})}\right) + \mathsf{Im}\left(\frac{-\partial_{\mathbf{v}} f(\mathbf{z})}{f(\mathbf{z})}\right) \geq 0.$$

Using Lemma 3.5 again, the K'-stability of $g(\mathbf{z}) - \partial_{\mathbf{v}} f(\mathbf{z}) + yf(\mathbf{z})$ follows. By specializing to $y = 0$ and using Lemma 2.1, we obtain that $g(\mathbf{z}) - \partial_{\mathbf{v}} f(\mathbf{z})$ is K-stable or $g(\mathbf{z}) - \partial_{\mathbf{v}} f(\mathbf{z}) \equiv 0$. $\square$

Theorem 3.4 not only generalizes the usual Lieb-Sokal Lemma to the case of arbitrary cones, but also extends it to directional derivatives with respect to every direction in the positive orthant. We can formulate this explicitly as the following refined version for the usual stability notion.

Corollary 3.6 (Refined Lieb-Sokal Lemma). *Let $g(\mathbf{z}) + yf(\mathbf{z}) \in \mathbb{C}[\mathbf{z}, y]$ be stable and assume $\rho_{\mathbf{v}}(f) \leq 1$ for some $\mathbf{v} \in \mathbb{R}^n_{\geq 0}$. Then $g(\mathbf{z}) - \partial_{\mathbf{v}} f(\mathbf{z}) \in \mathbb{C}[\mathbf{z}]$ is stable or identically 0.*

4. Preservers for psd-stability

In this section, we restrict to psd-stability. For a complex symmetric matrix $Z \in \mathcal{S}_n^{\mathbb{C}}$, we write $Z = X + iY$ with $X, Y \in \mathcal{S}_n$. After collecting some elementary preservers, our main results of this section are the preservation of psd-stability under an inversion operation (see Theorem 4.3 and Corollary 4.7) and the preservation of psd-stability under taking initial forms with respect to positive definite matrices (see Theorem 4.10).

For a polynomial $f \in \mathbb{C}[Z]$, let $f_{\mathrm{Diag}} \in \mathbb{C}[Z]$ denote the polynomial obtained from f by substituting all off-diagonal variables by 0. For $1 \leq i \neq j \leq n$, let B_{ii} be the matrix which is 1 in entry (i, i) and zero otherwise, and let B_{ij} be the matrix which is 1/2 in entry (i, j) and (j, i) and zero otherwise. Then, for a polynomial $f = \sum_\alpha c_\alpha Z^\alpha \in \mathbb{C}[Z]$ and its equivalent version $\tilde{f} = \sum_\alpha c_\alpha \prod_{k=1}^n z_{kk}^{\alpha_{kk}} \prod_{k<l} z_{kl}^{2\alpha_{kl}}$ in $\mathbb{C}[\{z_{kl} | 1 \leq k \leq l \leq n\}]$, we have the identities $\frac{\partial f}{\partial B_{ii}}\big|_{z_{lk} := z_{kl}} = \frac{\partial \tilde{f}}{\partial z_{ii}}$ and $\frac{\partial f}{\partial B_{ij}}\big|_{z_{lk} := z_{kl}} = \frac{1}{2}\frac{\partial \tilde{f}}{\partial z_{ij}}$ as symbolic expressions. To see this, it suffices to observe that for $i < j$ and a monomial $f(Z) = z_{ij}^{\alpha_{ij}} z_{ji}^{\alpha_{ji}} \in \mathbb{C}[Z]$, we have $\tilde{f} = z_{ij}^{2\alpha_{ij}}$ and

$$\frac{\partial}{\partial B_{ij}} f(Z) = \frac{1}{2}\alpha_{ij} z_{ij}^{\alpha_{ij}-1} z_{ji}^{\alpha_{ji}} + \frac{1}{2}\alpha_{ji} z_{ij}^{\alpha_{ij}} z_{ji}^{\alpha_{ji}-1} \in \mathbb{C}[Z].$$

Substituting z_{ji} by z_{ij} gives $\frac{\partial}{\partial B_{ij}} f(Z)\big|_{z_{ji} := z_{ij}} = \alpha_{ij} z_{ij}^{2\alpha_{ij}-1} = \frac{1}{2}\frac{\partial}{\partial z_{ij}}\tilde{f}$.

Lemma 4.1 (Elementary preservers for psd-stability)**.** *Let $f \in \mathbb{C}[Z]$ be psd-stable.*

a) **Diagonalization**: *The polynomial $Z \mapsto f_{\mathrm{Diag}}(Z)$ is psd-stable.*

b) **Transformation**: *Let $S \in \mathrm{GL}_n(\mathbb{R})$, then $f(SZS^{-1})$ and $f(SZS^T)$ are psd-stable.*

c) **Minorization**: *For $J \subseteq [n]$, let Z_J be the symmetric $|J| \times |J|$ submatrix of Z with index set J. Then $f(Z_J)$, the polynomial on $\mathcal{S}_{|J|}^{\mathbb{C}}$ obtained from f by setting to zero all variables with at least one index outside of J, is psd-stable or zero.*

d) **Specialization**: *For a fixed index $i \in [n]$, let $\hat{Z}_i$ be any matrix obtained from Z by assigning real values to z_{ij}, z_{ji} for all indices $j \neq i$ and a value from $\mathcal{H}$ to z_{ii}. Then $f(\hat{Z}_i)$, viewed as polynomial on $\mathcal{S}_{n-1}^{\mathbb{C}}$, is psd-stable or zero.*

e) **Reduction**: *For $i, j \in [n]$, let $\bar{Z}_{ij}$ be any matrix obtained from Z by choosing real values for $z_{ik} = z_{ki}$ for $k \neq i$ and setting $z_{ii} := z_{jj}$. Then $f(\bar{Z}_{ij})$, viewed as polynomial on $\mathcal{S}_{n-1}^{\mathbb{C}}$, is psd-stable or zero.*

f) **Permutation**: *Let $\pi : [n] \to [n]$ be a permutation. Then $f((Z_{\pi(j),\pi(k)})_{1 \leq j,k \leq n})$ is a psd-stable polynomial on $\mathcal{S}_n^{\mathbb{C}}$.*

g) **Differentiation**: *$\partial_V f(Z)$ is psd-stable or zero for $V \in \mathcal{S}_n^+$.*

Proof. a) Assume f_{Diag} is not psd-stable. Then there are real symmetric matrices A, B with $B \succ 0$ and $f_{\mathrm{Diag}}(A + iB) = 0$. Let A' and B' be the matrices obtained from A and B by setting all off-diagonal variable to zero. In particular, B' is positive definite. Since the only variables occurring in f_{Diag} are the diagonal ones, we have $f(A' + iB') = f_{\mathrm{Diag}}(A + iB) = 0$. Hence, f is not psd-stable.

b) Both transformations $Z \mapsto S^T ZS$ and $Z \mapsto S^{-1}ZS$ preserve the inertia of $\mathsf{Im}(Z)$ and thus also psd-stability.

c) Set $k := |J|$ and assume without loss of generality $J = \{1, \ldots, k\}$. For $\varepsilon > 0$, let g_ε be the polynomial on the space $\mathcal{S}_k$ defined by $g_\varepsilon(Z) := f(\mathrm{Diag}(Z, i\varepsilon I_{n-k}))$, where $\mathrm{Diag}(Z, i\varepsilon I_{n-k})$ is the block diagonal matrix with blocks Z and $i\varepsilon I_{n-k}$. The psd-stability of g implies the psd-stability of g_ε for all $\varepsilon > 0$. Hurwitz' Theorem 2.3 then gives the desired result, because $f(Z_J) = g_0(Z)$.

d) is obvious, e) and f) are similar to c), and g) is the special case of Lemma 3.1 when K is the psd-cone. $\square$

The diagonalization property from Lemma 4.1 plays a central role in the theory of psd-stable polynomials, since it establishes connections to the usual stability notion and also gives further insights into the monomial structure of psd-stable polynomials.

Corollary 4.2. *Let $f \in \mathbb{C}[Z]$ be psd-stable. Then:*

a) *The polynomial $(z_{11}, z_{22}, \ldots, z_{nn}) \mapsto f_{\mathrm{Diag}}(Z)$ is stable in $\mathbb{C}[z_{11}, z_{22}, \ldots, z_{nn}]$.*

b) *If $f(0) = 0$, i.e., if f does not have a constant term, then there is a monomial in f consisting only of diagonal variables of Z.*

c) *If f is homogeneous, then*
   - c1) *the sum of the coefficients of all diagonal monomials of f is nonzero.*
   - c2) *all nonzero coefficients of diagonal monomials of f have the same phase.*

Proof. a) By Lemma 4.1, we know that $f_{\mathrm{Diag}}(Z) \not\equiv 0$ is psd-stable. Now it suffices to observe that $f_{\mathrm{Diag}}(Z) \neq 0$ whenever the diagonal of $\mathsf{Im}(Z)$ has positive entries only.

b) Let $f(0) = 0$. If each monomial in f contains an off-diagonal variable of Z, then $f_{\mathrm{Diag}}(Z) \equiv 0$, in contradiction to the psd-stability of $f_{\mathrm{Diag}}(Z)$.

c1) The claim follows since the sum of the coefficients of all diagonal monomials is given by $f(I_n)$ which cannot be zero due to $f(i \cdot I_n) = i^{\deg(f)} f(I_n) \neq 0$.

c2) The claim follows by combining a) with Theorem 2.4. □

When investigating the combinatorics of psd-stable polynomials in Section 5, we will refer to the following observation, which could also be considered as a special case of specialization. Let $f(Z)$ be psd-stable. For the real matrix variables X and any fixed real matrix $B \succ 0$, the polynomial $f(X + iB)$ does not have any real roots.

As the first main result in this section, we show the following preservation statement under inversion for psd-stability.

Theorem 4.3 (Psd-stability preservation under inversion)**.** *If $f(Z) \in \mathbb{C}[Z]$ is psd-stable, then the polynomial $\det(Z)^{\deg(f)} \cdot f(-Z^{-1})$ is psd-stable.*

Here, the factor $\det(Z)^{\deg(f)}$ serves to ensure that the product is a polynomial again. For the proof of Theorem 4.3, we begin with a technical lemma.

Lemma 4.4. *Let $A, B \in \mathcal{S}_n$ with $B \succ 0$. Then the symmetric matrix $C := A + iB$ is invertible and the imaginary part matrix of the symmetric matrix C^{-1} is negative definite.*

We will use the following elementary computation rules, which can be verified immediately.

Lemma 4.5. *Assume that $C = A + iB$ is invertible, and denote its inverse by $W = U + iV$.*

a) If A is invertible, then $U = (A + BA^{-1}B)^{-1}$.
b) If B is invertible, then $V = (-B - AB^{-1}A)^{-1}$.

We also use the following basic statement on eigenvalues in the proof of Lemma 4.4.

Lemma 4.6. *Let $A, B \in \mathcal{S}_n$ and set $C = A + iB$. If $B \succ 0$ then $\lambda \in \mathcal{H}$ for all eigenvalues λ of C.*

Proof. Let $B \succ 0$, and let λ be an eigenvalue of C with some corresponding eigenvector $\mathbf{v}$. Then

$$\lambda = \frac{\mathbf{v}^H \lambda \mathbf{v}}{\mathbf{v}^H \mathbf{v}} = \frac{\mathbf{v}^H A \mathbf{v}}{\mathbf{v}^H \mathbf{v}} + i \frac{\mathbf{v}^H B \mathbf{v}}{\mathbf{v}^H \mathbf{v}}. \tag{1}$$

Since $B \succ 0$, we have $\frac{\mathbf{v}^H B \mathbf{v}}{\mathbf{v}^H \mathbf{v}} > 0$ and thus $\lambda \in \mathcal{H}$. □

Proof of Lemma 4.4. Let $C = A + iB$ with $A, B \in \mathcal{S}_n$ and $B \succ 0$. Lemma 4.6 gives that C is invertible. The symmetry of C^{-1} is an immediate consequence of the invertibility. Indeed, $C^{-1}C = I$ implies $I = I^T = (C^{-1}C)^T = C^T (C^{-1})^T$. Since C is symmetric, the matrix $(C^{-1})^T$ is the inverse of C, that is, $(C^{-1})^T = C^{-1}$.

By Lemma 4.5, the imaginary part of $W = C^{-1}$ is given by $(-B - AB^{-1}A)^{-1}$. We observe that B^{-1} is positive definite and thus $AB^{-1}A$ is positive semidefinite. Hence, $-B - AB^{-1}A$ is negative definite. Since the inverse of that matrix is negative definite as well, the claim follows. □

We can complete the proof of Theorem 4.3.

Proof of Theorem 4.3. The inverse of a symmetric matrix $C = A+iB$ with positive definite imaginary part B has a negative definite imaginary part, as shown in Lemma 4.4. Thus $f(-C^{-1}) \neq 0$ if $B \succ 0$. Since $\det(Z)$ is a psd-stable polynomial as well as f, the polynomial $\det(Z)^{\deg(f)} f\left(-Z^{-1}\right)$ is psd-stable. Note that the factor $\det(Z)^{\deg(f)}$ ensures that $\det(Z)^{\deg(f)} f\left(-Z^{-1}\right)$ is a polynomial. This directly follows from Cramer's rule, saying $Z^{-1} = \frac{1}{\det(Z)} \cdot \operatorname{adj}(Z)$, where $\operatorname{adj}(Z)$ denotes the adjugate matrix of Z. $\square$

The following is a slight generalization which resembles the existing formulation of the scalar version in Lemma 2.1.

Corollary 4.7. *If Z is a symmetric block diagonal matrix with blocks $Z_1, \ldots, Z_k$ and $f(Z) = f(Z_1, \ldots, Z_k)$ is psd-stable, then $\det(Z_1)^{\deg_{Z_1} f} \cdot f(-Z_1^{-1}, Z_2, \ldots, Z_k)$ is a psd-stable polynomial. Here, $\deg_{Z_1} f$ denotes the total degree of f with respect to the variables from the block Z_1.*

We close the section with a brief discussion and our second main result of this section on the preservation of the psd-stability of a polynomial $f \in \mathbb{C}[Z]$ when passing over to an initial form. For $f = \sum_{\alpha \in S} c_\alpha Z^\alpha \in \mathbb{C}[Z]$, the initial form of f is defined with respect to some functional W in the dual space $\mathcal{S}_n^*$. It is defined as

$$\operatorname{in}_W(f) = \sum_{\alpha \in S_W} c_\alpha Z^\alpha,$$

where $S_W := \{\alpha \in S \,:\, \langle W, \alpha \rangle_F = \max_{\beta \in S} \langle W, \beta \rangle_F\}$ and $\langle \cdot, \cdot \rangle_F$ is the Frobenius product. The following example shows that Theorem 2.5 on stability preservation under taking the initial form for any non-zero functional $\mathbf{w}$ does not generalize to the case of psd-stability.

Example 4.8. The polynomial $f \in \mathbb{C}[Z]$ given by

$$f(Z) = \det \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{12} & z_{22} & z_{23} \\ z_{13} & z_{23} & z_{33} \end{pmatrix} = z_{11}z_{22}z_{33} - z_{11}z_{23}^2 - z_{22}z_{13}^2 - z_{33}z_{12}^2 + 2z_{12}z_{13}z_{23}.$$

is a psd-stable polynomial. However, taking the initial form $\operatorname{in}_W(f)$ for

$$W = \begin{pmatrix} 4 & 4 & 6 \\ 4 & 4 & 6 \\ 6 & 6 & 0 \end{pmatrix}$$

yields $\operatorname{in}_W(f) = -z_{11}z_{23}^2 - z_{22}z_{13}^2 + 2z_{12}z_{13}z_{23}$, which vanishes at $Z = iI_3$. Since the imaginary part of iI_3 is a positive definite matrix, $\operatorname{in}_W(f)$ is not psd-stable.

To answer the natural question of whether psd-stability is preserved by passing over to the initial form with respect to certain symmetric matrices, we show that it is enough for W to be positive definite.

For $\lambda > 0$ and matrices $W \in \mathcal{S}_n$, let λ^W denote the operation given by $(\lambda^W)_{ij} := \lambda^{w_{ij}}$. Furthermore, for two matrices $A, B \in \mathcal{S}_n$ let $A \circ B$ denote the *Hadamard product* of A and B with $(A \circ B)_{ij} = a_{ij} \cdot b_{ij}$. Generalizing the notation $|\cdot|$ for vectors, we write $|\alpha| = \sum_{1 \le i,j \le n} |\alpha_{ij}|$ for an exponent matrix α.

Lemma 4.9. *Let $f \in \mathbb{C}[Z]$ be psd-stable and let $W \in \mathcal{S}_n$ be such that there exists some $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$, λ^W is positive definite. Then* $\mathrm{in}_W(f)$ *is psd-stable.*

Proof. The Schur product theorem states that the Hadamard product of the two positive definite matrices is positive definite. Thus we have $\lambda^W \circ A \succ 0$ for all $A \succ 0$ and $\lambda > \lambda_0$. Let $\varphi = \max\{\langle\alpha, W\rangle_F : \alpha \in \mathrm{supp}(f)\}$ and define the polynomial $f_\lambda(Z) := \frac{1}{\lambda^\varphi} f(\lambda^W \circ Z)$. This is psd-stable for any $\lambda > \lambda_0$, since the positive semi-definiteness of the imaginary part is preserved due to the previous observation. Let $\mu := \max\{\langle\alpha, W\rangle : \alpha \in \mathrm{supp}(f), \langle\alpha, W\rangle < \varphi\}$ and $\delta := \varphi - \mu > 0$. Now, for any compact subset $C \in \mathcal{S}_n$,

$$\begin{aligned}\lim_{\lambda\to\infty} \sup_{Z\in C} |f_\lambda(Z) - \mathrm{in}_W(f)(Z)| &\leq \lim_{\lambda\to\infty} \sup_{Z\in C} \sum\nolimits_{\langle\alpha,W\rangle<\varphi} \left|\frac{1}{\lambda^\delta} c_\alpha Z^\alpha\right| \\ &= \lim_{\lambda\to\infty} \frac{1}{\lambda^\delta} \sup_{Z\in C} \sum\nolimits_{\langle\alpha,W\rangle<\varphi} |c_\alpha Z^\alpha| \;=\; 0,\end{aligned}$$

since the norm in the last equality is bounded. By Hurwitz' Theorem 2.3, $\mathrm{in}_W(f)$ is psd-stable. □

Theorem 4.10. *Let $f \in \mathbb{C}[Z]$ be psd-stable and $W \in \mathcal{S}_n$ be positive definite, then* $\mathrm{in}_W(f)$ *is psd-stable.*

Proof. Let $W \in \mathcal{S}_n$ be positive definite. Then $W^{\circ k}$, the k-fold Hadamard product of W, is positive definite for all $k \geq 1$ and so is $\exp[W] := \sum_{k=0}^\infty \frac{W^{\circ k}}{k!}$, with the convention that $W^{\circ 0}$ is the all-ones matrix. For $\lambda > 1$, we have $\ln(\lambda) \cdot W \succ 0$. Therefore,

$$\exp[\ln(\lambda) \cdot W] = (e^{w_{ij}\ln(\lambda)})_{ij} = (\lambda^{w_{ij}})_{ij} = \lambda^W$$

is positive definite. The claim now follows from Lemma 4.9 with $\lambda_0 = 1$. □

5. Combinatorics of psd-stable polynomials

This section is about combinatorial properties of the support of psd-stable polynomials, inspired by the results in [6, 9, 22] on the support of stable polynomials listed in Sections 1 and 2. Theorem 5.1 gives a necessary condition on the support of any psd-stable polynomial. In Sections 5.1 and 5.2, we characterize psd-stability of binomials and non-mixed polynomials and the class of *polynomials of determinants*. Finally, Section 5.3 discusses some aspects on the support of general psd-stable polynomials, provides a conjecture and verifies this conjecture for some special families of polynomials. We sometimes write both z_{ij} and z_{ji} with some $i \neq j$, but both denote the same variable z_{ij} with $i \leq j$, as explained at the beginning of Section 4.

Theorem 5.1. *If an off-diagonal variable z_{ij} (where $i < j$) occurs in a psd-stable polynomial $f \in \mathbb{C}[Z]$, then the corresponding diagonal variables z_{ii} and z_{jj} must also occur in f.*

This mimics the basic fact about positive semidefinite matrices that if an off-diagonal entry is non-zero, the corresponding diagonal entries must also be non-zero.

Proof. We prove the contrapositive. Suppose without loss of generality that z_{1n} is a variable appearing in f but z_{nn} is not. We can choose an $(n-1) \times (n-1)$ complex symmetric matrix A and $a_{2n}, \ldots, a_{n-1,n} \in \mathbb{C}$ such that $\mathsf{Im}(A)$ is positive definite and such that substituting these values into f gives a non-constant univariate polynomial g in the variable z_{1n}. The second condition is possible because the set $\mathcal{S}_{n-1}^{++} \times \mathbb{C}^{n-2}$ is an open set. Indeed, we can choose all real parts to be zero.

The univariate non-constant polynomial g has a complex root a_{1n}. The assignment $z_{ij} = a_{ij}$ for all $(i,j) \neq (n,n)$ gives therefore a root of f no matter what value we choose for z_{nn}. We now claim that if we assign a value a_{nn} with $\mathsf{Im}(a_{nn})$ positive and large enough, the matrix A' which results from assigning these values to Z has a positive definite imaginary part.

Observe that by Sylvester's criterion of leading principle minors, it is enough to check that the determinant of $\mathsf{Im}(A')$ is strictly positive; the remaining leading principal minors will necessarily be positive because they are minors of $\mathsf{Im}(A)$, which we chose positive definite. Now, by developing the determinant along the last row, we obtain

$$\det \mathsf{Im}(A') = \mathsf{Im}(a_{nn}) \cdot \det \mathsf{Im}(A) + c,$$

where c is a constant. Since $\det \mathsf{Im}(A)$ is positive, we can choose $\mathsf{Im}(a_{nn})$ positive and sufficiently large so that $\det \mathsf{Im}(A')$ is positive. Thus $f(A') = 0$ with $\mathsf{Im}(A')$ positive definite, which proves that f is not psd-stable. □

The argument used in the proof is connected to the 'positive (semi-)definite matrix completion problem', see for example [5, Section 3.5]. In the special case of binomials Theorem 5.1 can be extended as follows.

Lemma 5.2. *Let $f(Z) = c_\alpha Z^\alpha + c_\beta Z^\beta$ be a psd-stable binomial. If the two monomials Z^α and Z^β do not have a common factor, then either both consist only of diagonal variables, or one only of diagonal and the other only of off-diagonal variables.*

Proof. Z^α and Z^β cannot both be off-diagonal monomials, since this contradicts Theorem 5.1. It remains to be shown that neither monomial can contain both diagonal and off-diagonal variables. Suppose towards a contradiction that one of the two monomials did contain both, w.l.o.g. Z^β, and choose j such that $\beta_{jj} > 0$. Since the monomials of f do not share any variable, $\frac{\partial Z^\alpha}{\partial z_{jj}} \equiv 0$, where we use the derivative notation $\frac{\partial}{\partial z_{ij}}$ on the symmetric matrix space as introduced at the beginning of Section 4.

Hence, $g(Z) := \frac{\partial f}{\partial z_{jj}} = c_\beta \beta_{jj} Z^{\beta'}$ is a non-zero monomial with $\beta'_{kl} = \beta_{kl}$ for $(k,l) \neq (j,j)$ and $\beta'_{jj} = \beta_{jj} - 1$, that is, g is a monomial containing an off-diagonal variable. Thus $g(i \cdot I_n) = 0$, which is a contradiction since g is psd-stable by Lemma 3.1. □

5.1. **Binomials and non-mixed polynomials.** We give characterizations of the support of psd-stable binomials. Some of the results will be stated for the family of *non-mixed polynomials*, which includes irreducible binomials thanks to Lemma 5.2.

Definition 5.3. We call a polynomial $f \in \mathbb{C}[Z]$ *non-mixed* if every monomial that occurs in f either consists only of diagonal variables or only of off-diagonal variables. We always write such a non-mixed polynomial as $f = \sum_{\alpha \in A} c_\alpha Z^\alpha + \sum_{\beta \in B} c_\beta Z^\beta$, where A refers to the exponent matrices of diagonal monomials and B refers to the exponent matrices of off-diagonal monomials.

It is useful to consider this larger family because it is closed under directional derivatives while the family of binomials is not. The following two theorems are the main results in this subsection.

Theorem 5.4. *Let $f(Z) = \sum_{\alpha \in A} c_\alpha Z^\alpha + \sum_{\beta \in B} c_\beta Z^\beta$ be a homogeneous non-mixed polynomial of degree $d \geq 3$ and assume $c_\beta \neq 0$ for some $\beta \in B$. Then f is not psd-stable.*

The following theorem is a complete characterization of the support of psd-stable binomials, analogous to the classification of stable binomials from Theorem 2.7.

Theorem 5.5. *Every psd-stable binomial is of one of the following forms:*

a) Only diagonal variables appear in f and f satisfies the conditions of Theorem 2.7: $f(Z) = Z^\gamma(c_1 Z^{\alpha_1} + c_2 Z^{\alpha_2})$ with $|\alpha_1 - \alpha_2| \leq 2$ and at least one of α_1, α_2 is non-zero,

b) $f(Z) = Z^\gamma(c_1 z_{ii} z_{jj} + c_2 z_{ij}^2)$ with $i < j$ and $\frac{c_1}{c_2} \in \mathbb{R}$,

where $c_1, c_2 \neq 0$ and Z^γ is a diagonal monomial.

This theorem shows that the only psd-stable binomials with off-diagonal variables are those described in b): in particular, at most one off-diagonal variable occurs in a psd-stable binomial, and it has degree exactly 2.

The following lemma is a first step towards a proof of the main theorems and shows that the exponents of psd-stable binomials cannot be far apart. The proof relies on taking derivatives in direction $V^{(ij)}$, with $i \neq j$, which denotes the $n \times n$ matrix with $v_{ii} = v_{jj} = v_{ij} = v_{ji} = 1$ and 0 elsewhere. In terms of the basis matrices B_{ij} introduced at the beginning of Section 4, we have $V^{(ij)} = B_{ii} + B_{jj} + 2B_{ij}$.

Lemma 5.6. *Let $f(Z) = c_\alpha Z^\alpha + c_\beta Z^\beta$ be a psd-stable binomial (thus $c_\alpha, c_\beta \neq 0$). Then $||\alpha| - |\beta|| \leq 2$.*

Proof. We may assume that the monomials of f do not have a common factor since this does neither affect $|\alpha - \beta|$ nor $||\alpha| - |\beta||$. By Lemma 5.2, either both monomials are diagonal monomials or w.l.o.g. only Z^β is an off-diagonal monomial. If both monomials are diagonal, the claim follows directly from Theorem 2.6, because psd-stable polynomials involving only diagonal variables are stable polynomials.

Now assume that Z^β is an off-diagonal monomial. Then $|\beta| \leq |\alpha|$ follows from Theorem 5.1 after possibly taking derivatives in direction $V^{(ij)}$ for some z_{ij} appearing in Z^β. It remains to show $|\alpha| \leq |\beta| + 2$. Assume to the contrary that $|\alpha| - |\beta| \geq 3$. Choose i and j with $i < j$ such that z_{ij} occurs in Z^β. Since $\frac{\partial f}{\partial V^{(ij)}}(Z) = \left(\frac{\partial}{\partial z_{ii}} + \frac{\partial}{\partial z_{jj}}\right)(c_\alpha Z^\alpha) + \frac{\partial}{\partial z_{ij}}(c_\beta Z^\beta)$ by the computation rules at the beginning of Section 4, we see that $\frac{\partial f}{\partial V^{(ij)}}(Z)$ has at most two diagonal monomials, each of degree $|\alpha| - 1$, and exactly one off-diagonal monomial of degree $|\beta| - 1$. By applying this procedure consecutively $|\beta|$ times, we obtain a polynomial $g(Z) = \sum_{\alpha'} c_{\alpha'} Z^{\alpha'} + c_{\beta'}$, where $\sum_{\alpha'} c_{\alpha'} Z^{\alpha'}$ is a homogeneous polynomial in diagonal variables of degree $|\alpha| - |\beta| \geq 3$ and $c_{\beta'}$ is a constant. Further $g(Z)$ is psd-stable by Lemma 3.1. Since g does not involve any off-diagonal variables, it is a stable polynomial. This is a contradiction to Theorem 2.6, since the support of g does not satisfy the Two-Steps Axiom. □

In the following, we show that most binomials are not psd-stable by explicitly constructing a root S (of the binomial or a directional derivative of it) whose imaginary part lies in the interior of the psd-cone. This root S will be a symmetric $n \times n$ matrix of the form

$$S = \begin{pmatrix} s+i & t & \cdots & t \\ t & s+i & \ddots & \vdots \\ \vdots & \ddots & \ddots & t \\ t & \cdots & t & s+i \end{pmatrix} \quad \text{with } s,t \in \mathbb{R}. \tag{2}$$

Since $\mathsf{Im}(S) = I_n \succ 0$, any polynomial with root S is not psd-stable.

Lemma 5.7. *Let $f(Z) = c_\alpha Z^\alpha + c_\beta Z^\beta$ be a binomial (thus, $c_\alpha, c_\beta \neq 0$) with $|\alpha| > |\beta| \geq 1$ and such that Z^α and Z^β do not have a common factor. Then f is not psd-stable.*

Proof. Assume towards a contradiction that f is psd-stable. Since $|\alpha| > |\beta| \geq 1$ and Z^α and Z^β do not share a factor, we have that $|\alpha - \beta| \geq 3$. If both monomials were diagonal monomials, psd-stability would imply stability, and $|\alpha - \beta| \geq 3$ would yield a contradiction to Theorem 2.6.

Now assume that Z^β is an off-diagonal monomial. We will show that there are $s, t \in \mathbb{R}$ such that S is a root of f, thus contradicting that f is psd-stable. By Lemma 5.6, the only possibly psd-stable cases are $|\alpha| = |\beta| + 1$ and $|\alpha| = |\beta| + 2$.

First consider the case $|\beta| = 1$. Then $|\alpha| \in \{2, 3\}$. After substituting S, $f = 0$ is of the form

$$(s+i)^a + bt = 0 \quad \text{with} \quad a := |\alpha| \in \{2,3\},\ s,t \in \mathbb{R},\ b \in \mathbb{C} \setminus \{0\}. \tag{3}$$

One may split the real and imaginary part of equation (3) to obtain two real equations, denoted by (Re) and (Im). First let $a = 2$. If $\mathsf{Im}(b) \neq 0$, there is a real solution $s = \frac{\mathsf{Re}(b)}{\mathsf{Im}(b)} + \sqrt{\left(\frac{\mathsf{Re}(b)}{\mathsf{Im}(b)}\right)^2 + 1}$ and $t = \frac{1-s^2}{\mathsf{Im}(b)}$. If $\mathsf{Im}(b) = 0$, the solution $t = \frac{1}{\mathsf{Re}(b)}$ and $s = 0$ may be found. Now let $a = 3$. If $\mathsf{Im}(b) \neq 0$, (Im) implies $t = \frac{1-3s^2}{\mathsf{Im}(b)}$, which then gives $s^3 - 3s + \frac{\mathsf{Re}(b)}{\mathsf{Im}(b)}(1 - 3s^2)$, which has a real solution. If $\mathsf{Im}(b) = 0$, (Im) becomes $3s^2 = 1$, which has the real solutions $s = \pm\frac{1}{\sqrt{3}}$. Substituting these into (Re) gives a linear function in t, which has a real solution as well.

Now consider the case $|\beta| > 1$. Choose i and j with $i < j$ such that the variable z_{ij} occurs in f. Since f is psd-stable, its partial derivative in direction $V^{(ij)}$ is psd-stable by Lemma 3.1. Further $\frac{\partial f}{\partial V^{(ij)}}(Z) = \left(\frac{\partial}{\partial z_{ii}} + \frac{\partial}{\partial z_{jj}}\right)(c_\alpha Z^\alpha) + \frac{\partial}{\partial z_{ij}}(c_\beta Z^\beta)$ is a non-mixed polynomial with the degree of each monomial reduced by 1 and exactly one off-diagonal monomial. Taking $|\beta| - 1$ consecutive derivatives in a similar way, we obtain a non-mixed polynomial of the form $g(Z) = \sum_{\alpha'} c_{\alpha'} Z^{\alpha'} + c_{\beta'} Z^{\beta'}$ with $|\beta'| = 1$ and $|\alpha'| \in \{2, 3\}$. Substituting S into g gives equation (3). Thus neither g nor f can be psd-stable. $\square$

Now we prove Theorem 5.4, which shows that homogeneous non-mixed polynomials of high degree cannot be psd-stable.

Proof of Theorem 5.4. Assume to the contrary that f is a homogeneous non-mixed psd-stable polynomial of degree at least 3. By Remark 2.10, we can assume without loss of generality that all coefficients of f are real.

First assume the degree of f is $d = 3$. We will show a contradiction to psd-stability by explicitly finding a forbidden root of f.

Let $a := \sum_{\alpha \in A} c_\alpha$ and $b := \sum_{\beta \in B} c_\beta$. Note that a and b are real and $a = f(I_n) \neq 0$ by Corollary 4.2 c1). If $b \neq 0$, w.l.o.g. we normalize so that $a = 1$. To obtain the desired forbidden root we look for a solution of the form S introduced above, that is, real solutions s, t for the equation $f(S) = (s+i)^3 + bt^3 = 0$. By splitting the equation into real and imaginary part, we obtain the system

$$\begin{array}{lrcl} (\mathrm{Re}): & (s^3 - 3s) + bt^3 & = & 0, \\ (\mathrm{Im}): & 3s^2 - 1 & = & 0. \end{array}$$

Consider the positive real solution $s^* = \frac{1}{\sqrt{3}}$ of (Im). Plugging this solution into (Re) gives a real cubic in t, which has a real solution t^*.

If instead $b = 0$, we tweak matrix S to S' as follows: let $\beta_0 \in B$ such that $c_{\beta_0} \neq 0$, and let z_{ij} be a variable occurring in the monomial Z^β. Then we let $S'_{ij} = S'_{ji} = (1+\varepsilon)t$ for a small $\varepsilon > 0$. The remaining entries of S' are the same as those in S. Since $b = \sum_\beta c_\beta = 0$, we have that $f(S') = (s+i)^3 + \varepsilon c_{\beta_0} t^3$, and $\varepsilon c_{\beta_0} > 0$, which means we fall into the case above with the coefficient of t^3 non-zero. We have thus constructed solutions violating psd-stability for any such degree 3 polynomial f.

Now let $d > 3$ and assume d is the smallest degree such that there is a polynomial f of the specified form which is psd-stable of degree d. Its partial derivative in any direction $V^{(ij)}$ is psd-stable by Lemma 3.1. If we choose (i,j), $i \neq j$ such that the variable z_{ij} occurs in f, $\frac{\partial f}{\partial V^{(ij)}}$ is a polynomial of the same form of degree $d-1$: since $\frac{\partial f}{\partial V^{(ij)}}(Z) = \left(\frac{\partial}{\partial z_{ii}} + \frac{\partial}{\partial z_{jj}}\right)(\sum_\alpha c_\alpha Z^\alpha) + \frac{\partial}{\partial z_{ij}}(\sum_\beta c_\beta Z^\beta)$, the coefficients of off-diagonal monomials are positive multiples of those of f and therefore there must be a non-zero one. This is a contradiction, since we assumed that d was the smallest degree which a psd-stable polynomial of this form could have. □

We finally have all the tools needed to prove Theorem 5.5, which provides a complete classification of the support of psd-stable binomials.

Proof of Theorem 5.5. Let f be a binomial. Then f can be written in the form $f(Z) = Z^\gamma \tilde{f}(Z)$, where $\tilde{f}(Z) = c_\alpha Z^\alpha + c_\beta Z^\beta$ is an irreducible psd-stable binomial and therefore also a non-mixed polynomial.

If all variables appearing in f are diagonal variables, then f is stable, and by Theorem 2.6 its support has to satisfy the Two-Steps Axiom, which leads to $|\alpha - \beta| \leq 2$ in the case of binomials. Thus, now we can assume the occurrence of an off-diagonal monomial, say Z^β. By the structure Theorem 5.1 and after possibly taking derivatives in direction $V^{(ij)}$ for some z_{ij} appearing in Z^β, we see $|\beta| \leq |\alpha|$.

In the homogeneous case, by Theorem 5.4, we have $\deg(\tilde{f}) \leq 2$. The only possibility is given by $\tilde{f}(Z) = c_1 z_{ii} z_{jj} + c_2 z_{ij}^2$ with $c_1, c_2 \neq 0$ and $i \neq j$, since otherwise we would get a contradiction to the structure Theorem 5.1. Clearly $|\alpha - \beta| \leq 2$ holds. Further we have $\frac{c_1}{c_2} \in \mathbb{R}$ by Remark 2.10. In the non-homogeneous case, i.e., $|\alpha| \neq |\beta|$, Lemma 5.7 implies $\beta = 0$ or $|\beta| > |\alpha|$. $\beta = 0$ is not involving an off-diagonal variable. The case $|\beta| > |\alpha|$ contradicts the earlier observation that $|\beta| \leq |\alpha|$. Therefore, there is no non-homogeneous psd-stable binomial involving off-diagonal variables. □

From Theorem 5.5, we observe that psd-stable binomials cannot contain a monomial which is the product of different off-diagonal variables. This also holds for psd-stable homogeneous non-mixed polynomials.

Theorem 5.8. *Let f be a psd-stable homogeneous non-mixed polynomial of degree* 2. *Then f is of the form $f(Z) = \sum_{\alpha \in A} c_\alpha Z^\alpha + \sum_{i<j} c_{ij} z_{ij}^2$.*

Proof. Let $f(Z)$ be a psd-stable homogeneous non-mixed polynomial of degree 2 and assume to the contrary that there is a monomial $z_{ij}z_{kl}$ in f involving distinct variables, that is, $\{i,j\} \neq \{k,l\}$. Note that the index sets $\{i,j\}$ and $\{k,l\}$ can intersect. The order of the variable matrix must therefore be at least 3.

Consider S' as a modified version of S from (2) with $S'_{ij} := S'_{ji} := t_1,\ S'_{kl} := S'_{lk} := t_2$ for complex t_1 and t_2 and set all other off-diagonal entries of S' to 0 while the diagonal of S is set to some complex value s. Thus, up to a factor, $f(S') = 0$ is of the form

$$s^2 + c_1 t_1^2 + c t_1 t_2 + c_2 t_2^2 = 0 \tag{4}$$

with some constants c_1, c_2, c and $c \neq 0$. Since f is hyperbolic due to its homogeneity and psd-stability, we may assume c_1, c_2, c to be real.

Since f is hyperbolic, the quadratic polynomial g in s, t_1, t_2 on the left hand side of (4) is hyperbolic as well. Hyperbolic quadratic polynomials have signature $(n-1,1)$ or $(1,n-1)$ ([11], see, e.g., also [20]). Since the term s^2 in g comes from a substitution into the terms $z_{11}^2, \ldots, z_{nn}^2$, the representation matrix of g must have signature $(2,1)$. Hence, the lower right 2×2-matrix of the representation matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & \frac{c}{2} \\ 0 & \frac{c}{2} & c_2 \end{pmatrix}$$

has signature $(1,1)$. If at least one of the c_i is positive, then we can choose real values for t_1 and t_2 such that $s^2 + \gamma = 0$ with some $\gamma > 0$, which gives among the two solutions for s one with positive imaginary part. If one of the c_i, say, c_2, is zero and $c_1 \leq 0$, then setting $t_1 = 1$ and $t_2 = \frac{1-c_1}{c}$ gives the solution $s = i$ with positive imaginary part. It remains to consider the case $c_1 < 0$, $c_2 < 0$, in which the signature condition implies $(c/2)^2 > c_1 c_2$. By choosing t_1, t_2 to satisfy $t_1^2 = -\frac{1}{c_1}$, $t_2^2 = -\frac{1}{c_2}$, we obtain

$$t_1^2 t_2^2 \left(\frac{c}{2}\right)^2 \ > \ t_1^2 c_1 t_2^2 c_2 \ = \ \frac{1}{4}(t_1^2 c_1 + t_2^2 c_2)^2,$$

which can formally be viewed as the equality case of the arithmetic-geometric inequality. We can pick the signs of t_1, t_2 such that $c t_1 t_2 > 0$. And the previous inequality implies

$$|c t_1 t_2| > |c_1 t_1^2 + c_2 t_2^2|$$

(and the expression in the argument of the absolute value on the right hand side is negative). Hence, we obtain $s^2 + \gamma = 0$ for some positive γ, which gives among the two solutions for s one with positive imaginary part.

Altogether, we have constructed a zero S' of f with $\mathsf{Im}(S') \succ 0$, which contradicts the psd-stability of f. ∎

5.2. **Polynomials of determinants.** We show that the following class of polynomials of determinants satisfies a generalized jump system criterion with regard to psd-stability. Suppose that the symmetric matrix of variables Z is a diagonal block matrix with blocks $Z_1, \dots, Z_k$. A *polynomial of determinants* is a polynomial in Z of the form $f(Z_1, \dots, Z_k) = \sum_\alpha c_\alpha \det(Z)^\alpha$, where we define $\det(Z)^\alpha = \det(Z_1)^{\alpha_1} \cdots \det(Z_k)^{\alpha_k}$.

We say a polynomial of determinants $f(Z_1, \dots, Z_k) = \sum_\alpha \det(Z)^\alpha$ is written in *standard form* if the largest possible determinantal monomial is factored out, i.e., $f(Z_1, \dots, Z_k) = \det(Z)^\gamma \sum_\beta c_\beta \det(Z)^\beta = \det(Z)^\gamma \tilde{f}(Z)$, and all $c_\beta \neq 0$. We investigate the following notion of support for polynomials of determinants.

Definition 5.9. Let $f(Z_1, \dots, Z_k) = \sum_\alpha c_\alpha \det(Z)^\alpha$ be a polynomial of determinants. Then the *determinantal support* is defined as $\operatorname{supp}_{\det}(f) = \{\alpha \in \mathbb{Z}^k_{\geq 0} : c_\alpha \neq 0\}$.

Note that the determinantal support specializes to the usual support when Z is a diagonal matrix, that is, all Z_i are 1×1 matrices of a single variable. As a corollary of Theorem 2.6, we obtain the following analogue for the determinantal support of psd-stable polynomials of determinants.

Corollary 5.10. *Let $f(Z_1, \dots, Z_k) = \sum_\alpha c_\alpha \det(Z)^\alpha$ be psd-stable. Then the determinantal support of f forms a jump system.*

The next theorem shows that psd-stable polynomials of determinants have a very special structure.

Theorem 5.11. *Let $f(Z_1, \dots, Z_k) = \det(Z)^\gamma \sum_{\beta \in B} c_\beta \det(Z)^\beta = \det(Z)^\gamma \tilde{f}(Z)$ be a psd-stable polynomial of determinants in standard form. Then any block Z_i appearing in $\tilde{f}$ (that is, any Z_i such that there is $\beta \in B$ with $\beta_i > 0$) has size $d_i \leq 2$.*

Further, for any matrix Z_i which has size exactly 2, let $C_i = \max_{\beta \in B} \beta_i$. Then if $\beta \in B$, then also $\beta + c\mathbf{e}_i \in B$ for all $-\beta_i \leq c \leq C_i - \beta_i$.

Proof. Observe that, by construction, a variable in the matrix Z_i does not appear in any other matrix Z_j. This ensures that all vectors in the support of the polynomial $\tilde{f}_{\mathrm{Diag}}(Z)$, which involves only the diagonal variables, are of the form

$$(\underbrace{\beta_1, \dots, \beta_1}_{d_1 \text{ times}}, \dots, \underbrace{\beta_k, \dots, \beta_k}_{d_k \text{ times}}), \tag{5}$$

where $\beta = (\beta_1, \dots, \beta_k) \in B$ is an exponent vector of $\det(Z)$ in $\tilde{f}$ and d_i is the size of the matrix Z_i for each i.

Further, $\tilde{f}_{\mathrm{Diag}}$ is stable and its support is therefore a jump system. Suppose now that some matrix Z_i, say Z_1, has size $d_1 \geq 3$. Since f is in standard form, there are $\beta \in B$ such that $\beta_1 > 0$ and $\beta' \in B$ such that $\beta'_1 = 0$. Then there are corresponding vectors $\alpha = (\underbrace{\beta_1, \dots, \beta_1}_{d_1 \text{ times}}, \beta_2, \dots)$ and $\alpha' = (\underbrace{0, \dots, 0}_{d_1 \text{ times}}, \beta'_2, \dots)$ in the support of $\tilde{f}_{\mathrm{Diag}}$, which is a jump system. Thus $\mathbf{e}_1$ is a valid step from α' to α, but since $\alpha' + \mathbf{e}_1 = (1, 0, \dots, 0, \dots)$ is not of the form (5) it cannot belong to the support of $\tilde{f}_{\mathrm{Diag}}$. Now by definition of a jump system, there must be a step from $\alpha' + \mathbf{e}_1$ to α which is in the support. However, whichever step we take will lead us again to a vector where the first d_1 entries are not all equal, since $d_1 \geq 3$, and thus none of these vectors can be in the support of $\tilde{f}_{\mathrm{Diag}}$, contradicting the fact that it is a jump system. Thus all blocks Z_i in $\tilde{f}$ must have size $d_i \leq 2$.

Now suppose that $d_i = 2$ for some block Z_i, without loss of generality let it be Z_1. Just as before, we know there are $\beta \in B$ such that $\beta_1 > 0$ and $\beta' \in B$ such that $\beta'_1 = 0$; further, If $C_1 = \max_{\beta \in B} \beta_1$, then there is also a vector $\beta'' \in B$ such that $\beta''_1 = C_1$. This implies that in the support of $\tilde{f}_{\text{Diag}}$ there are vectors $\alpha = (\beta_1, \beta_1, \dots)$, $\alpha' = (0, 0, \dots)$ and $\alpha'' = (C_1, C_1, \dots)$. Thus $\alpha - \mathbf{e}_1 = (\beta_1 - 1, \beta_1, \dots)$ is a valid step from α to α'. Just as before, $\alpha - \mathbf{e}_1$ does not belong to the support of $\tilde{f}_{\text{Diag}}$ because it is not of the form of (5). Thus there must be a further step from $\alpha - \mathbf{e}_1$ towards α' which is in the support. The only such step is in the second coordinate, so that (5) is satisfied, and thus $\alpha - \mathbf{e}_1 - \mathbf{e}_2 \in \text{supp}(\tilde{f}_{\text{Diag}})$. This argument can be repeated until we obtain the statement of the theorem. $\square$

5.3. **Considerations on the support of general psd-stable polynomials.** By Theorem 2.6, the support of a stable polynomial defines a jump system. Hence, there cannot be large gaps in the support, that is, if two vectors are in the support and are far apart, there is some other vector of the support between them. The families studied in Subsections 5.1 and 5.2 suggest that a similar phenomenon happens for psd-stability: when there are too-large gaps in the support, the polynomial cannot be psd-stable.

In order to quantify what a *large gap* should be, we make two observations. First, since restricting a psd-stable polynomial in the symmetric matrix variables Z to its diagonal yields a stable polynomial, between two monomials involving only diagonal variables the Two-Steps Axiom holds. A weaker statement is that between any two such monomials there is a sequence of linear and double steps which does not leave the support of the polynomial, where we define a *linear step* from a monomial to be multiplying the monomial by $z_{ij}^{\pm 1}$, a *double step* multiplying by $z_{ij}^{\pm 1} z_{kl}^{\pm 1}$.

Recall from Lemma 2.13 that a prominent example of psd-stable polynomials is the symmetric determinant $\det(Z)$. In the symmetric matrix variables $(z_{ij})_{i \leq j}$, its support has a special structure: it contains all monomials that can be obtained from $z_{11} \cdots z_{nn}$ by transpositions of indices, that is, by successively multiplying the monomial by $z_{ij} z_{kl} z_{ik}^{-1} z_{jl}^{-1}$ for some indices $i, j, k, l \in [n]$. We call such a move on monomials a *transposition step.*

Lemma 5.12. *Any two monomials in the support of the symmetric determinant* $\det(Z)$ *are linked by a sequence of transposition steps decreasing the distance between the monomials which never leave the support.*

Proof. Monomials in $\det(Z)$ are precisely those products of symmetric variables z_{ij} (where $i \leq j$) such that each index $k \in [n]$ appears exactly twice. Indeed, when considering the determinant as a polynomial in n^2 (i.e., non-symmetric) variables, each monomial corresponds to a permutation in the symmetric group S_n, and thus each element of $[n]$ must appear precisely once in the rows and once in the columns index in the monomial. When considering the determinant as a polynomial in the symmetric variables, certain distinct permutations define the same monomial. Observe that the variable z_{ij} appears in the monomial defined by a permutation π if either $i = \pi(j)$ or $j = \pi(i)$. Thus, both a cycle $\sigma = (i_1 i_2 \dots i_k) \in S_k$ and its inverse $(i_1 i_k i_{k-1} \dots i_2)$ yield the monomial $\Pi_j z_{i_j i_{j+1}}$, and in general, two permutations correspond to the same monomial if and only if their cycle decompositions are made of pairwise the same or inverse cycles. Since two such permutations have the same sign, there is no cancelation of monomials in the symmetric determinant $\det(Z)$.

Thus, applying any transposition step to any monomial of $\det(Z)$ will yield another monomial of $\det(Z)$: exchanging $z_{ij}z_{kl}$ with $z_{ik}z_{jl}$ or $z_{il}z_{kj}$ preserves the property that each index appears exactly twice. We now only need to show that, given any two monomials Z^α and Z^β of $\det(Z)$, there exists a transposition step from α to β. Choose a variable z_{ij} such that $z_{ij} \mid Z^\alpha$ but $z_{ij} \nmid Z^\beta$. There must be an index $k \neq j$ such that $z_{ik} \mid Z^\beta$ and an index $l \neq i$ such that $z_{jl} \mid Z^\beta$. Then multiplying Z^α by $z_{ij}^{-1}z_{kl}^{-1}z_{ik}z_{jl}$ is a transposition step, since it decreases the distance to β in the norm $|\cdot|$. □

We conjecture that a property inspired by the structure of the determinant and that of stable polynomials holds for all psd-stable polynomials.

Conjecture 5.13. *For any monomial Z^β appearing in a psd-stable polynomial, there is a diagonal monomial Z^α appearing in f which can be reached by a sequence of linear, double and transposition steps which decrease the distance from β to α and which never leave the support of f.*

Example 5.14. The polynomial

$$\begin{aligned} f(Z) &= (z_{11} + z_{22} - 2z_{12})(z_{11}z_{33} - z_{13}^2) \\ &= z_{11}^2 z_{33} + z_{11}z_{22}z_{33} - 2z_{11}z_{33}z_{12} - z_{11}z_{13}^2 - z_{13}^2 z_{22} + 2z_{12}z_{13}^2 \end{aligned}$$

is psd-stable because it is the product of two psd-stable polynomials: the first one is the derivative in direction $V^{(12)}$ of the 2×2 determinant, the other one is a 2×2 determinant sharing one variable with the first.

This polynomial satisfies Conjecture 5.13: for example, if we choose the monomial $z_{12}z_{13}^2$, with a double step we reach $z_{11}z_{13}^2$, which is also in the support of f, and with a transposition step we reach $z_{11}^2 z_{33}$, a diagonal monomial in the support. Notice that the double step produces a monomial whose exponent vector is closer to the exponent of the final diagonal monomial (with respect to $|\cdot|$). Such a sequence of valid steps can be found for all monomials of f.

As evidence for the conjecture, we observe that it holds for the classes of polynomials we have studied.

Lemma 5.15. *Psd-stable binomials satisfy Conjecture 5.13.*

Proof. By Theorem 5.5, $c_\alpha z_{ii}z_{jj} + c_\beta z_{12}^2$ is the only irreducible psd-stable binomial involving off-diagonal variables. Clearly, it is exactly one transposition step between the both monomials. □

Lemma 5.16. *Psd-stable homogeneous non-mixed polynomials satisfy Conjecture 5.13.*

Proof. Let f be a psd-stable homogeneous non-mixed polynomial. If f does not involve off-diagonal monomials, the claim follows from the jump system property of usual stable polynomials. Thus assume that f involves off-diagonal variables. We have $d := \deg(f) \leq 2$ by Theorem 5.4. In the case of $d = 1$ there is a double step between every two monomials of f, thus assume $d = 2$ and let $c_\beta Z^\beta$ be an off-diagonal monomial of f. By Theorem 5.8, $c_\beta Z^\beta$ is of the form $c_{jk}z_{jk}^2$ for some $j \neq k$. Let $J = \{j, k\}$, then

$f(Z_J)$ is psd-stable by Lemma 4.1 c). By the structure Theorem 5.1, $f(Z_J)$ is of the form

$$f(Z_J) = c_1 z_{jj}^2 + c_2 z_{jj} z_{kk} + c_3 z_{kk}^2 + c_{jk} z_{jk}^2$$

with $c_k \in \mathbb{C}$ such that z_{jj} and z_{kk} both appear. We claim that $c_2 \neq 0$. Assuming $c_2 = 0$ gives $c_1, c_3 \neq 0$. Reducing $f(Z_J)$ to the diagonal contradicts the jump system property and thus we obtain $c_2 \neq 0$. Therefore, the monomial $c_2 z_{jj} z_{kk}$ appears in $f(Z_J)$ and hence also in $f(Z)$. Thus, it is a transposition step from $c_{jk} z_{jk}^2$ to the corresponding diagonal monomial $c_2 z_{jj} z_{kk}$. □

Lemma 5.17. *Psd-stable polynomials of determinants satisfy Conjecture 5.13.*

Proof. Every monomial Z^β in a polynomial of determinants f belongs to a determinantal monomial $\det(Z)^\gamma$ and thus is a product of monomials Z^{β_j} (with multiplicities γ_j) belonging to determinantal blocks $\det(Z_j)$, $1 \leq j \leq k$. Let Z^{α_j} be the diagonal monomial of block $\det(Z_j)$. By Lemma 5.12 there is a sequence of transposition steps from Z^{β_j} to Z^{α_j} which never leaves the support of $\det(Z_j)$ for all $1 \leq j \leq k$. Concatenation of these sequences (with multiplicities γ_j) gives a sequence of transposition steps from Z^β to the diagonal monomial Z^α of $\det(Z)^\gamma$ which never leaves the support of f. □

Another class of psd-stable polynomials which satisfy Conjecture 5.13 are the psd-stable lpm polynomials introduced in [1], which are polynomials of the form $f(Z) = \sum_{J \subseteq [n]} c_J \det(Z_J)$, where Z_J is the square submatrix of Z with index set J. Indeed, every monomial belongs to a square minor of Z, and since every minor has a different index set, there is no cancellation of monomials in the sum. Thus for each summand the Lemma 5.12 holds and it holds for the whole polynomial as well.

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