

Condensing of self-organizing groups

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Frankfurt a. M. 25.01.2010

Abstract. *Condensing phenomena for systems biology, ecology and sociology present in real life different complex behaviors. Based on local interaction between agents, we present another result of the Energy-based model presented by [20]. We involve an additional condition providing the total condensing (also called consensus) of a discrete positive measure.*

Key words: *Condensing; consensus; random move; self-organizing groups; collective intelligence; stochastic modeling.*

AMS Subject Classifications: 81T80; 93A30; 37M05; 68U20

1 Introduction

The dynamics of autonomous agents systems, which lead to condensing or subcondensing is an interesting and omnipresent modeling approach in the applied sciences. They describe the behaviors of self-organizing populations of multi-agents. For example swarming and schooling display interesting and beautiful social aggregation. Flying or swimming in a groups is conditioned by applying strong local rules. These rules, also called natural intelligence, lead to condensing in order to keep energy, provide security and outwit predators. For more details we refer to [2, 3, 6, 10, 16]. Biology presents another condensing phenomena, like condensation of cells. That one cell becomes a brain cell and another one a liver cell may depend on physical and biological factors. Moreover, the spread of cancer tumor can be seen not only as a diffusion problem but also as a condensing phenomenon. Cancer development and the dynamics of the immune system have been a significant focus of mathematical modeling in recent decades, see for example [14, 17]. In social science, the opinion dynamics studied by Hegselmann and Krause in [11, 12, 13], where opinion formation within an interacting group, leads to consensus, polarization or fragmentation and presents another application of condensing.

We believe that the dynamics of groups have many forms and various applications. The natural intelligence of each member of the group motivates it permanently to look for other positions. Following a gradient trajectory or the barycenter rule on his neighborhood or any local rule, the agent moves locally considering all other positions and neighbors on his neighborhood. Presently, we are particularly interested to extend the Energy-based model for condensing of particles (in academic point of view) to reach total condensing, where the final state with a global zero-energy stabilizes the nonlinear system. Moreover, total condensing is an interesting case with a stochastic construction. Thus, our goal here is to give and to test an additional and local condition, which may almost lead to the total condensing of a collection of particles. We simulate our experiments in one and two dimensional Euclidean metric spaces.

2 Energy function in continuous metric space

Let (X, d) be a metric space with metric d . We shall assume that all bounded subsets of X are compact. Hence, X is locally compact and we may use Radon measures. Let $M_+(X)$ be the set of nonnegative Radon measures on X . We shall however for simplicity deal with discrete measures and discrete time only. A measure is given as

$$m := \sum_{x \in S(m)} m(x) \delta_x, \quad (1)$$

where $S(m)$ denote the support of m and δ_x the Kronecker symbol. Note that $S(m)$ is discrete. For such a measure, the energy map E , defined by [20] given as

$$\begin{aligned} E &: M_+(X) \longrightarrow \mathbb{R}^+ \\ E(m) &= \sum_{d(x,y) \leq \varepsilon} m(x)m(y)d^2(x,y). \end{aligned} \quad (2)$$

is in general a not continuous function of m . The following example illustrates this:

Example 2.1. For $X = \mathbb{R}$, $S(m) = \{1, 2\}$, $\varepsilon = 1$ and a given m with

$$m = \sum_{x \in \{1,2\}} m(x) \delta_x = \delta_1 + \delta_2, \quad (3)$$

it follows that $E(m) = 2$. Now let m_j be a sequence of positive measures defined as

$$m_j = \delta_1 + \delta_{2+\frac{1}{j}}, \quad (4)$$

it follows $\lim_j m_j = m$ and $E(m_j) = 0$, and

$$E(\lim_j m_j) = 2 \neq \lim_j E(m_j) = 0. \quad (5)$$

Hence, from (5), it follows that the map E with the definition (2) is not continuous in m .

In order to obtain an energy function which depends continuously on m , we extend the definition (2) to the following:

$$\begin{aligned} E &: M_+(X) \longrightarrow \mathbb{R}^+ \\ E(m) &= \int_X \int_X \varphi(x,y) d^2(x,y) m(dx) m(dy) \\ &= \sum_{x,y} m(x)m(y) \varphi(x,y) d^2(x,y), \end{aligned} \quad (6)$$

where φ is a continuous function, which satisfies:

$$\begin{aligned} \varphi &: X \times X \longrightarrow [0, 1]; \\ \varphi(x,y) &:= \begin{cases} 1, & \text{if } d(x,y) \leq \varepsilon, \\ 0, & \text{if } d(x,y) \geq \varepsilon + \theta. \end{cases} \end{aligned} \quad (7)$$

For $\varepsilon > 0$ and $\theta > 0$. These parameters will be fixed throughout this paper. The function φ will be called **intensity function**.

Example 2.2.

1. For $\varepsilon, \theta > 0$, define the continuous function ψ as

$$\begin{aligned} \psi & : [0, \infty) \longrightarrow [0, 1]; \\ \psi(x) & := \begin{cases} 1, & \text{if } 0 \leq x \leq \varepsilon, \\ 0, & \text{if } x \geq \varepsilon + \theta. \end{cases} \end{aligned}$$

Then $\varphi : (x, y) \mapsto \varphi(x, y) = \psi(d(x, y))$ is an intensity function.

2. The following function is an intensity function for every $\varepsilon > 0$ and $\theta > 0$:

$$\begin{aligned} \varphi & : \mathbb{R} \times \mathbb{R} \longrightarrow [0, 1]; \\ \varphi(x, y) & := \begin{cases} 1, & \text{if } |x - y| \leq \varepsilon, \\ 0, & \text{if } |x - y| \geq \varepsilon + \theta. \\ \left(\frac{\varepsilon + \theta - |x - y|}{\theta}\right) & \text{otherwise.} \end{cases} \end{aligned}$$

3 Condensing model

Let $M_E(X)$ be the set of discrete and nonnegative measure with $E(m) < \infty$. As in [20] a pair $(a, a^*) \in X \times X$ operates on $M_E(X)$ as

$$m^*(x) = (a, a^*, m)(x) = \begin{cases} m(x); & \text{if } x \notin \{a, a^*\}, \\ 0; & \text{if } x = a, \\ m(a) + m(a^*); & \text{if } x = a^*. \end{cases}$$

Note that if $m \in M_E(X)$ then $m^* \in M_E(X)$. The energy of a point $a \in X$ with respect to $m \in M_E(X)$ is

$$\begin{aligned} e & : X \times M_E(X) \longrightarrow \mathbb{R}^+ \\ e(a, m) & = \sum_y m(y) \varphi(a, y) d^2(a, y). \end{aligned} \tag{8}$$

Lemma 3.1. For $a, a^* \in X$, $m \in M_E(X)$ and $m^* := (a, a^*, m)$, we have

$$E(m) - E(m^*) = 2m(a) [e(a, m) - e(a^*, m^*)]. \tag{9}$$

Proof. For simplicity, let us denote by I_m the following term:

$$I_m := \sum_{\{x, y\} \cap \{a, a^*\} = \emptyset} m(x) m(y) \varphi(x, y) d^2(x, y),$$

and let us compute the energy of m :

$$\begin{aligned} E(m) & = \sum_{x, y} m(x) m(y) \varphi(x, y) d^2(x, y) \\ & = I_m + 2 \sum_{x, y} m(a) m(y) \varphi(a, y) d^2(a, y) \\ & \quad + 2 \sum_{x, y} m(a^*) m(y) \varphi(a^*, y) d^2(a^*, y) - 2m(a^*) m(a) \varphi(a^*, a) d^2(a^*, a) \\ & = I_m + 2m(a) e(a, m) + 2m(a^*) e(a^*, m) - 2m(a^*) m(a) \varphi(a^*, a) d^2(a^*, a). \end{aligned} \tag{10}$$

Similarly for m^* (by replacing m by m^*), we get

$$E(m^*) = I_{m^*} + 2m^*(a)e(a, m^*) + 2m(a^*)e(a^*, m^*) - 2m^*(a^*)m^*(a)\varphi(a^*, a)d^2(a^*, a). \quad (11)$$

Note that

$$m^*(a) = 0 \quad \text{and} \quad I_m = I_{m^*}, \quad (12)$$

and in addition we have:

$$e(a^*, m^*) = e(a^*, m) - m(a)\varphi(a^*, a)d^2(a^*, a). \quad (13)$$

Therefore, from (11)–(13) it follows:

$$E(m^*) = I_{m^*} + 2m^*(a)e(a, m^*). \quad (14)$$

From (10) and (14), it follows that

$$E(m) - E(m^*) = 2m(a)[e(a, m) - e(a^*, m^*)], \quad (15)$$

which prove the result of the lemma.

In order to extend the move notion to a continuous metric space, we use the intensity function to introduce an additional necessary condition. Recall that in the case of a finite metric space, we required for (m, m^*) to be condensing

- (i) $e(a, m) - e(a^*, m^*) > 0$ and
- (ii) $d(a, a^*) \leq \varepsilon$.

Now we require a^* minimizing the function $y \rightarrow e(y, m^*)\varphi(a, y)$.

Definition 3.1 (Model).

A pair (m, m^*) is called condensing, if there is $(a, a^*) \in S(m) \times X$ such that

- (i) $d(a, a^*) \leq \varepsilon + \theta$,
- (ii) $m^* = (a, a^*, m)$ and
- (iii) $e(a^*, m^*)\varphi(a, a^*) < e(y, m^*)\varphi(a, y)$, $d(a, y) \leq \varepsilon + \theta$, $\forall y$.

A sequence of non negative measures m^1, m^2, \dots is called condensing, if for every i the pair (m^i, m^{i+1}) is condensing.

It is important to note that a singular admissible move according to definition 3.1 satisfies the so called minimality condition: If a moves to a^* then $e(a, m) > e(a^*, m^*)$. In other words the particle a moves where the local energy smaller, in this case it moves where the energy is minimal. Our goal is to construct a condensing sequence, with vanishing energy (stable state) at the limit state. The following corollary is useful:

Corollary 3.1. *If m^1, m^2, \dots is a singularly condensing sequence, then*

$$\lim_i E(m^i) = \ell \geq 0. \quad (16)$$

Proof. The proof follows immediately from definition 3.1 and lemma 3.1.

Our goal is to prove the existence of condensing sequences converging to a measure m , such that $E(m) = 0$. Therefore, we define the so called effectively condensing sequences: A singularly condensing sequence m^1, m^2, \dots is called **effectively condensing sequence** if there exists $c > 0$ such that

- (i) $m^{i+1} = (a, a^*, m^i)$,
- (ii) $E(m^i) - E(m^{i+1}) \geq c\alpha(m^i)$,

where $\alpha(m^i) = \max_y \{\varphi(y, a)d^2(y, a) | y \in S(m^i)\}$. From (14) and (15) and

$$c = 2 \min_a \{m(a) | a \in S(m^i)\},$$

follow the existence of the effectively condensing sequence.

Lemma 3.2. *Suppose that m^1, m^2, \dots is an effectively condensing.*

$$\text{if } \lim_{i \rightarrow \infty} E(m^i) = E(m), \text{ then } \alpha(m) = 0. \quad (17)$$

Proof. Consider $\eta > 0$, from $\lim_{i \rightarrow \infty} E(m^i) = E(m)$, exists i_0 such that for all $i > i_0$, it follows

$$2c\alpha(m^i) \leq E(m^i) - E(m) \geq \eta, \quad (18)$$

for large i , we get $\lim_i \alpha(m^i) = 0$ and still $\alpha(m) = 0$. ■

Theorem 3.1. *If (X, d) is a compact metric space, then every effectively condensing sequence is finite such and it exists $k \geq 0$ such that $E(m^k) = 0$.*

Proof. Since X is compact, then $\cup_i S(m^i)$ is relative compact and there exists a subsequence m^{i_j} of m^i such that $\lim_j m^{i_j} = m$ and $\lim_j E(m^{i_j}) = E(m)$. From lemma 3.2 it follows that $\lim_j \alpha(m^{i_j}) = 0$ and $\alpha(m) = 0$. Since the sequence m^1, m^2, \dots is effectively condensing, and from definition 3.1 there exists k such that $\alpha(m^k) = 0$. Therefore, for all $x, y \in S(m^k)$, it follows

$$d(x, y) = 0 \quad \text{or} \quad d(x, y) \geq \varepsilon + \theta \quad (19)$$

Hence, $E(m^k) = 0$ and still m^k is a collection of isolated masses with propriety (19) or a point mass $m = m(X)\delta_a$ for $a \in X$. ■

Remark 3.1. *We remarked that even if the energy of the limit measure vanishes, the results are non necessary a singleton (total condensing). The limit measure is a collection of segregated subgroups. Total condensing of particles as physical phenomena is subject of many scientists by studying consensus dynamics of opinions, such the model proposed by [11]. Thus, we construct in the next section an additional condition to reach total condensing.*

4 Reaching total condensing (consensus)

An interesting phenomenon of condensing sequences is the state of total condensing, in another context consensus. Under the Hegselmann and Krause (HK) model [13], one of the main questions is the consensus of the opinions under bounded confidence, in dependence of the initial distribution and a suitable choice of the confidence, for more details, we refer to [11, 12, 9, 5].

The HK-model motivated us to propose the energy-based model, which basically has similar behaviors on convex metric spaces. Moreover, we give an additional condition, which may cause, in the case of convex metric spaces, total condensing.

Consider a convex metric space (X, d) with a metric d and let m be a positive measure in $M_+(X)$ with support $S(m)$. We require the following initial condition:

$$\forall x \in S(m) : e_\varepsilon(x, m) > 0.$$

Furthermore, we define a usual move based on the energy given as:

- (i) $m^* = (a, a^*, m)$ and
- (ii) a^* minimizing $y \rightarrow e(y, m^*)\varphi(a, y)$, with $d(a, y) \leq \varepsilon + \theta$,

and suppose for a move candidate a there is a set of positions¹ $N_{\varepsilon+\theta}^*(a)$ minimizing the energy function. Let us now apply the average rule of the HK-model: We build the barycenter of all mass points neighbors of a , namely over the set $N_{\varepsilon+\theta}(a)$ and set $b = \text{Barycenter}\{x|x \in N_{\varepsilon+\theta}(a)\}$ ². The following algorithm present the move based on the energy with the additional condition:

0. Fix $\varepsilon, \theta, r > 0$;
1. Fix a move candidate $a \in S(m)$;
2. Compute the barycenter of $N_{\varepsilon+\theta}(a)$;
3. Approximate $N_{\varepsilon+\theta}^*$, and $N_r(b)$;
4. If $N_{\varepsilon+\theta}^*(a) \cap N_r(b) \neq \emptyset$;
Additional condition:
5. Chose randomly $a^* \in N_{\varepsilon+\theta}^*(a) \cap N_r(b)$ for the confidence $r > 0$;
6. Move a to a^* ;
7. Else if $N_{\varepsilon+\theta}^*(a) \cap N_r(b) = \emptyset$ do nothing.

5 Numerical simulation

In this section we perform some numerical tests on the condensing model. Although results we present here pertain academic examples only. The method can be extended to more realistic problems using real parameters and suitable modifications. Moreover, to give sense to the condensing sequence (3.1) two steps are required namely, the computation of local energy given by (8) of a particle x and the comparison procedure with the local energy of a randomly chosen point mass y such that $d(x, y) \leq \varepsilon + \theta$. It is important to note that the particle y minimizes the energy in his neighborhood. We simulate the condensing sequences in two domains, the real line and plane such as:

Test 1. Two tests on the real line without the additional condition.

Test 2. Two tests on the real line with the additional condition.

Test 3. Two tests on the real plane with the additional condition.

¹Note that $N_{\varepsilon+\theta}^*(a)$ is not necessary a subset of $S(m)$.

²The barycenter will be approximated and depends on the gripoints of the domain

5.1 One dimensional metric space

This section presents simulations on the real line. In order to compute the density of the points masses of a measure at each iteration, the space domain is discretized into $N_x = 50$ uniform gridpoints. The computational domain is $X = [0, 1]$ and using the absolute value as a metric, we carry out two tests of condensing sequences of two measures given as:

$$m := \sum_{x \in S(m) \subset [0,1]} m(x) \delta_x, \text{ with } S(m) = \{x_1, \dots, x_{50}\}. \quad (20)$$

We run our code after fixing the order of reactions (the array of 50 indexes is randomly permuted). The following table summarizes the results of the simulations without additional condition on the real line:

Parameter/Sim.	(b)	(a)
ε	0.02	0.02
θ	0.01	0.01
Initial state	50 masses (one)	50 masses ($U(0, 4)$)
Final state	4 isolated masses	4 isolated masses
Number of iterations	175	195

Table 1: Results of simulations in the real line.

- We present in table 5.1 the results of the first test on the real line without the additional conditions. The limit measure is a collection of subgroups. In these simulations, we show four subgroups with different distributions and different iteration numbers.
- The figures 1 present the changing processes of one dimensional density functions of the condensing of two measures. On the left, we show some iterations of simulation (b) and on the right those of simulation (a). The initial states are chosen differently, for the case (b), we generate a random distribution of particles on $[0, 1]$ and for the case (a), we put an on the uniform gridpoints equal masses. The density is computed on fixed gridpoints N_x . For each simulation, the limit state is constituted from a $\varepsilon + \theta$ isolated masses with different densities.
- For the same initial data, we simulate the condensing measures above with the additional condition. Figure 2 show the condensing process. For both simulations, the finale states represent a total condensing.
- It is important to note that the additional condition requires no isolating aspect for each agent. It may lead in most cases to the total condensing state.

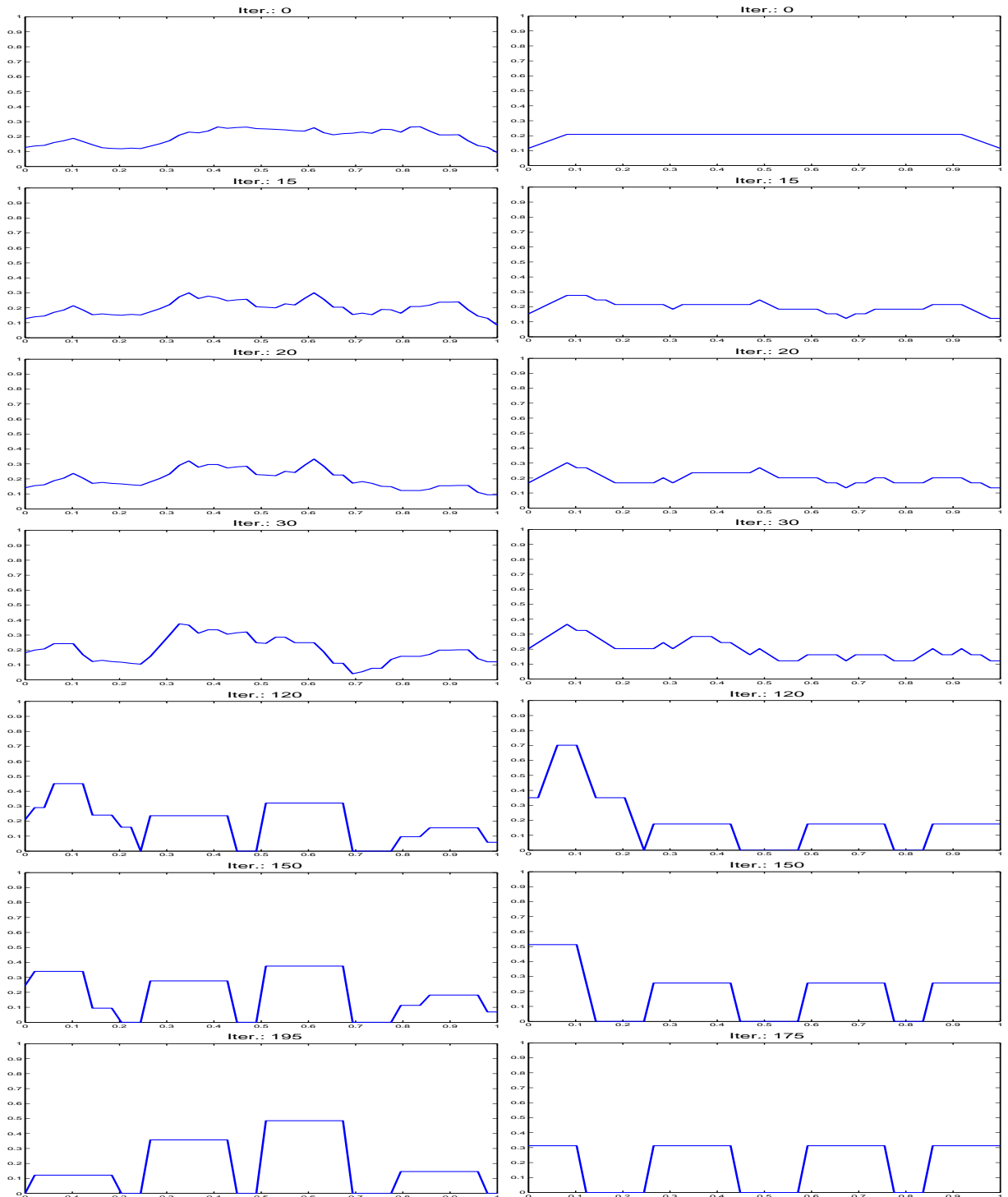


Figure 1: One dimensional density function of the condensing measures on the real line for two simulation, without additional condition (sim. (a) and (b)).

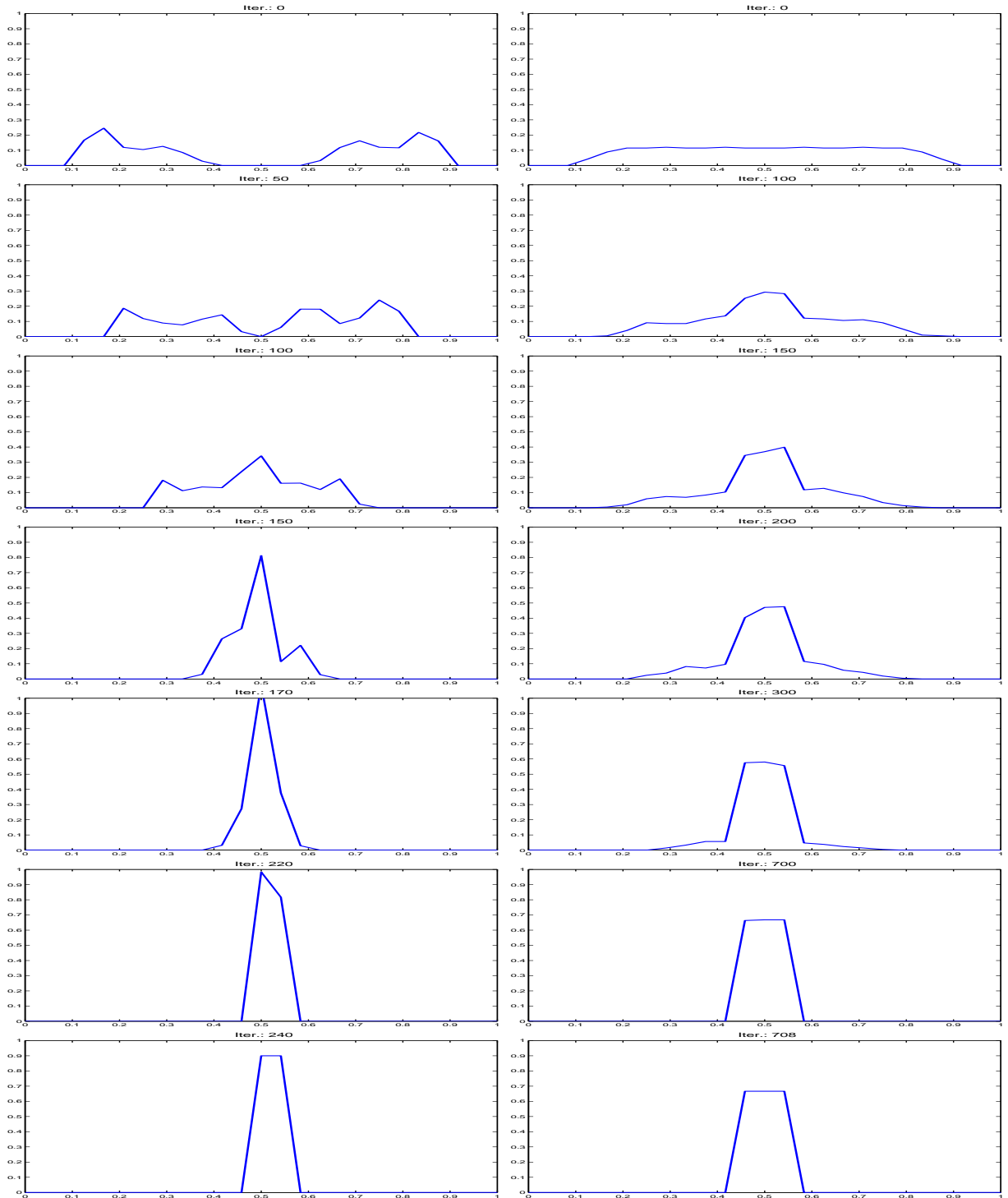


Figure 2: Density of the condensing measures on the real line (sim. (a)).

5.2 Two dimensional metric space

The two dimensional domain $X = [0, 1]^2$ is discretized into $Nx \times Nx$ (in this case $Nx = 20$) uniform gridpoints. We carry out two tests of condensing sequences of two measures given as:

$$m := \sum_{x \in S(m) \subset [0,1]^2} m(x) \delta_x, \quad \text{with } S(m) = \{x_1, \dots, x_{441}\}. \quad (21)$$

The following table summarizes the results of the simulations on the real plane for the measure above:

Parameter/Sim.	(a)	(b)
ε	0.1	0.1
θ	0.001	0.001
Initial state	441 masses (one)	441 masses ($U(0, 4)$)
Final state	condensing	condensing
Number of iterations	more than 9600	more than 35000

Table 2: Results of two simulations on real planes.

- Table 5.2 presents the result of the condensing process on the real plane of two simulation. Note that the total condensing behavior is clearly shown.
- For $(\varepsilon, \theta) = (0.1, 0.001)$. Figure 3 present the two dimensional density function of a measure. In order to show the condensing of particles (also consensus), we plot the corresponding contour in figure 4. These simulations need 9600 and 35000 time iterations respectively until the limit state. Our code stops, when the total energy is negligible (compared with a fixed tolerance). It is clearly shown in figures 3 and 4 that the particles build mass point with total density in only one point of the computation domain. This result will be different if we perform another simulation, even if we use the same data for the initial measure.
- Figures 3 and 4 shows respectively the densities and the contour plots of the same experiment.
- Figure 5 present curves of the decreasing energy function along the condensing process. The finale energy vanishes, which means that the finale state is stable.

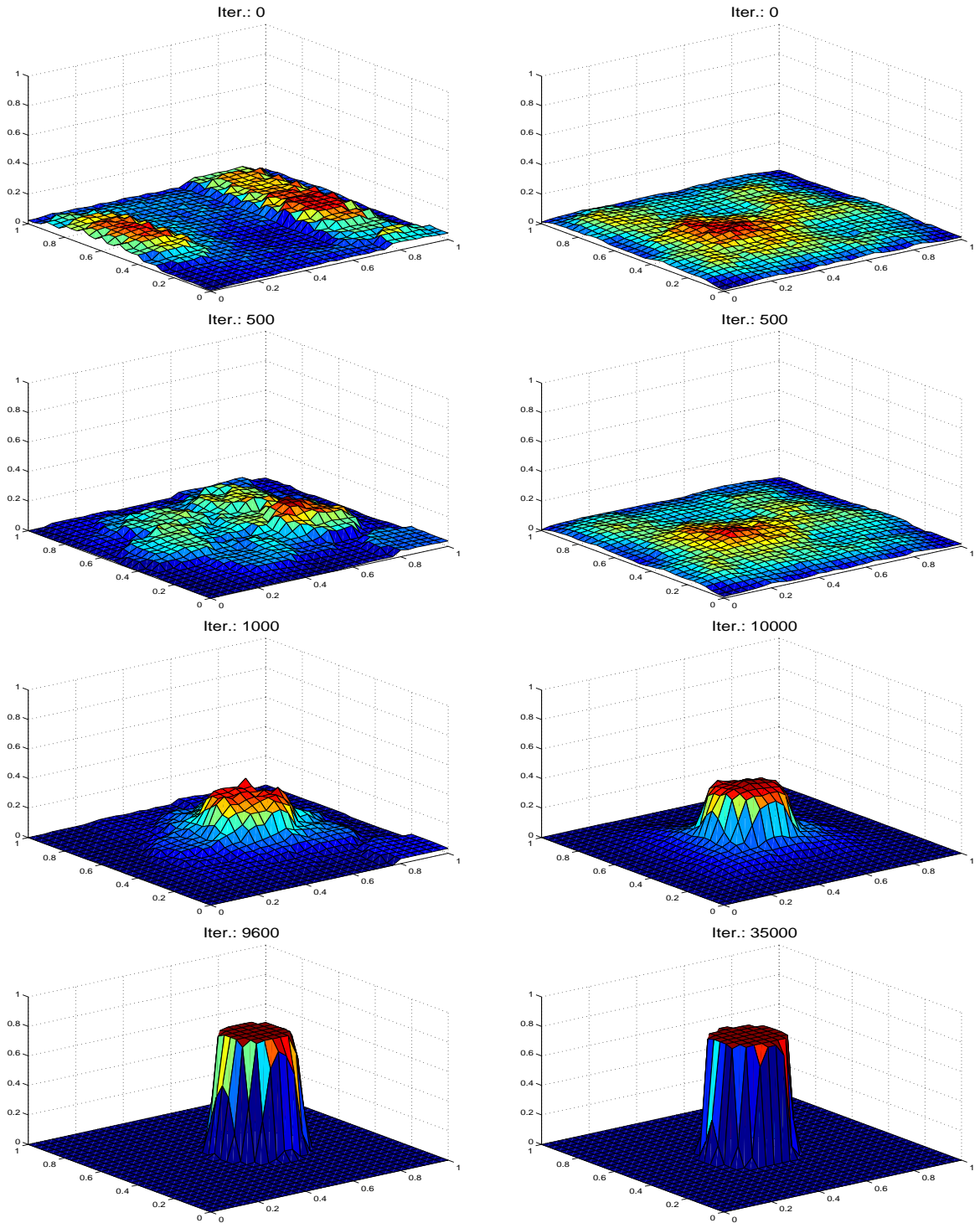


Figure 3: Densities of a condensing sequence in an Euclidian continuous metric space, simulation (a) (left column), (b) (right column).

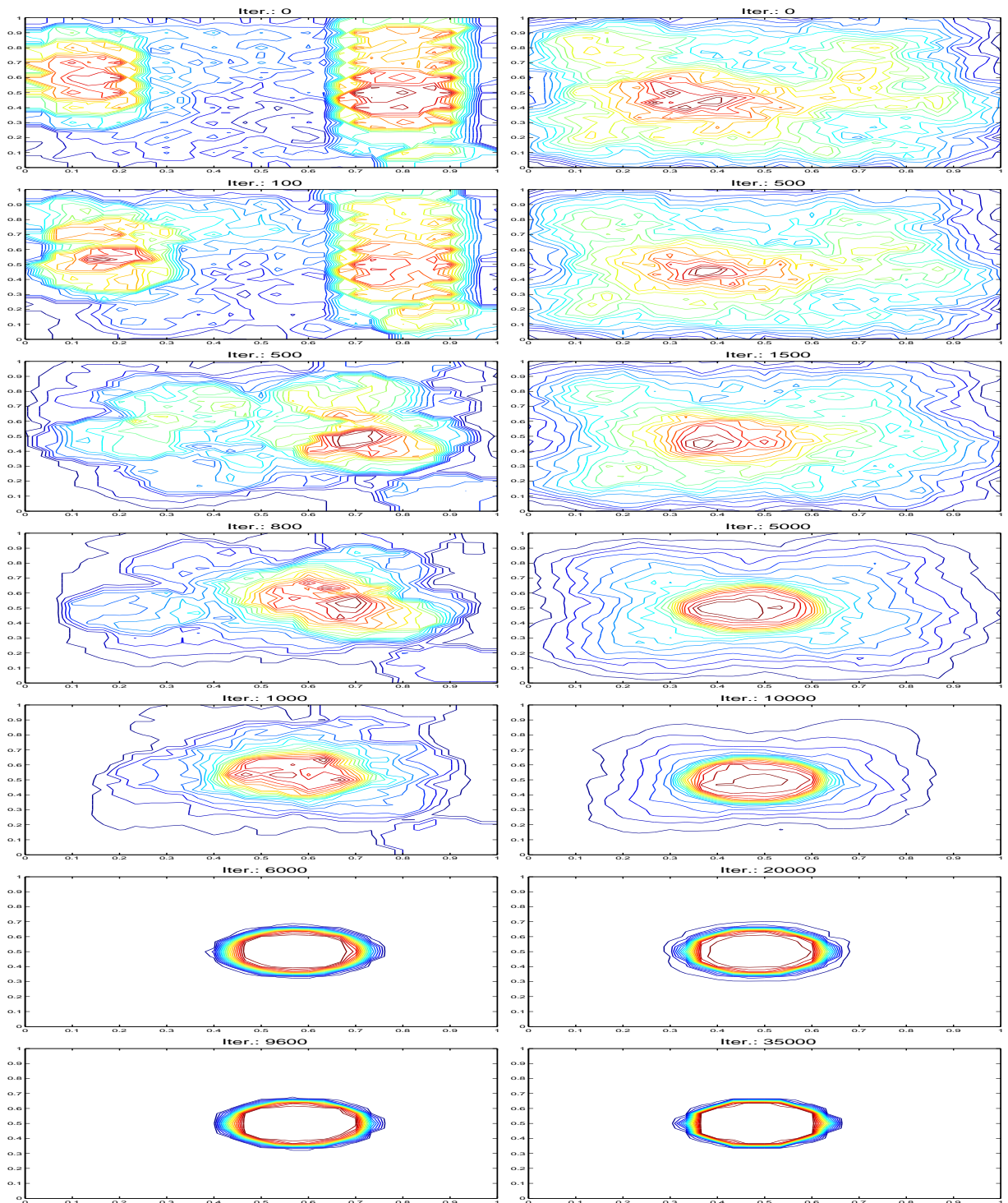


Figure 4: Condensing in an Euclidian continuous metric space, simulation (a) (left column), (b) (right column).

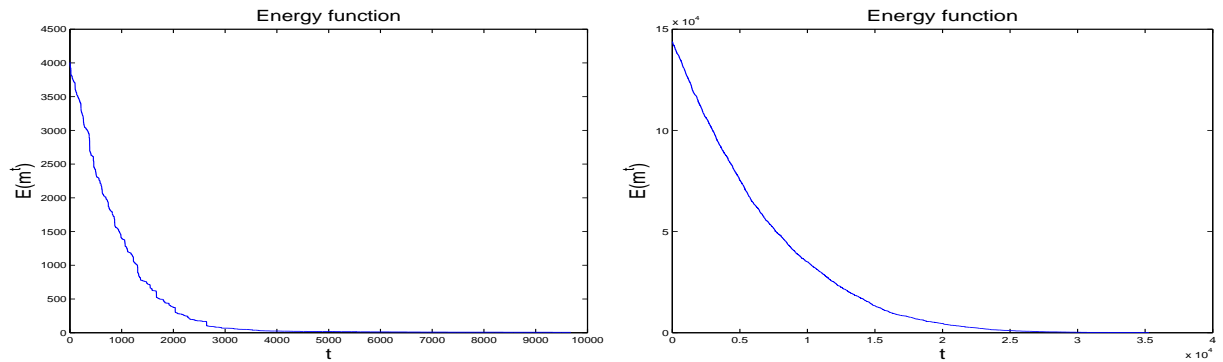


Figure 5: *Condensing in an euclidian continuous metric space, simulation (a) (left column), (b) (right column).*

6 Concluding remarks

We have presented an experimental method, which leads, in discrete time, to total condensing in a Euclidean metric space. Therefore, we believe that No-Isolation of the agents is a necessary condition providing total condensing. The result presented here is one of many variations of the energy-based model, thus it opens many possibilities to analyze other problems. The continuous energy model can explain other interesting phenomena, for example the transformation of continuous masses.

References

- [1] Babak P., Magnússon K. G., and S. Th. Sigurdsson, Dynamics of group formation in collective motion of organisms. *Math. Med. Biol.*, 21:269–292, 2004.
- [2] Barbaro A., Birnir B., and Taylor K., A comparison of simulations of discrete and ode models for the motion of pelagic fish. UCSB Preprint, 2006.
- [3] Barbaro A., Birnir B., and Taylor K., Discrete and Continuous Models of the Dynamics of Pelagic Fish: Orientation-Induced Swarming and Applications to the Capelin. 2007
- [4] Birnir B., Einarsson B., and Sigurdsson S., Simulations and stability of complex fish schools. RH Preprint, 2006.
- [5] Berger R L, A necessary and sufficient condition for reaching a consensus using De Groot’s method. *J. Amer. Statist. Assoc.* 1981.
- [6] Cucker F., Smale S., Emergent Behavior in Flocks. *Automatic Control, IEEE Transactions on*, 2007.
- [7] Czirt’ok A. and Vicsek T., Collective behavior of interacting self-propelled particles. *Physica A*, 281:pp 17–29, 2000.
- [8] Czirt’ok A. and Vicsek T., Collective Motion in Fluctuations and Scaling in Biology. ed. T. Vicsek, chapter 6, pp 177–209. Oxford University Press, Oxford, UK, 2001.
- [9] De Groot M H, Reaching a consensus. *J. Amer. Statist. Assoc.* 69. pp. 118–121. 1974.

- [10] Gazi V., B. Fidan, Y. S. Hanay And M. I. Köksal, Aggregation, Foraging, and Formation Control of Swarms with Non-Holonomic Agents Using Potential Functions and Sliding Mode Techniques. Turk J Elec Engin, VOL.15, NO.2, 2007.
- [11] Hegselmann R., Krause U., Opinion Dynamics and Bounded Confidence, Models, Analysis and Simulation. Journal of Artificial Societies and Social Simulation, 2001.
- [12] Hegselmann R and Krause U, Opinion Dynamics Driven by Various Ways of Averaging. Computational Economics 25. pp. 381–405, 2005.
- [13] Hegselmann R and Krause U, Truth and Cognitive Division of Labour: First Steps Towards a Computer Aided Social Epistemology. Journal of Artificial Societies and Social Simulation vol. 9, no. 3, 2006.
- [14] Ivanenok, Joseph F.; Sievers, Robert K.; Schultz, William W. Modeling of Remote Condensing AMTEC Cells, SPACE NUCLEAR POWER AND PROPULSION: Eleventh Symposium. AIP Conference Proceedings, Volume 301, pp. 1501-1506, 1994.
- [15] Kennedy J. and Eberhard R., Swarm Intelligence, Academic 2001.
- [16] Shen J., Cucker-Smale Flocking under Hierarchical Leadership. Populations and Evolution(q-bio.PE); Quantitative Methods (q-bio.QM), 2007
- [17] Mackenzie D., Mathematical Modeling and Cancer, SIAM News, Volume 37, Number 1, January/February 2004
- [18] Sieveking M., Condensing. In Positive Systems, volume 341 of Lecture Notes in Control and Information Sciences, pages 247–254, 2006.
- [19] Zahri M., Energy based Model for self organizing groups, Proceeding, 1st Hispano-Moroccan Days on Applied Mathematics and Statistics, Vol 1, 411–416, 2008.
- [20] Zahri M., Energy-based model of condensing on metric spaces: Modeling, analysis and simulation of condensing of particles, Book, ISBN-13: 978-3838110813- SVH, 2009.