Precise Values for Critical Fields in Quantum Electrodynamics*

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A careful investigation of different corrections to binding energies of electrons in almost critical fields is performed. We investigate quantitatively the influence of the nuclear charge parameters, nuclear mass, degree of ionization on the value of the critical charge of the nucleus. Rather qualitative arguments are given to establish the contribution of the quantumelectrodynamic corrections, which are found to be small. Some phenomenological modifications of QED are quantitatively investigated and found to be of negligible influence on the value of the critical field. For heavy ion collisions with $Z_1+Z_2>Z_{cr}$ the critical separations between ions are given as results of precise solutions of the relativistic two coulomb center problem. Corrections due to electron-electron interaction are considered. We find (with present theoretical accuracy) $Z_{cr}=173\pm2$, in the heavy ion collisions $R_{cr}(U-U)=34.7\pm2$ fm and $R_{cr}(U-Cf)=47.7\pm2$ fm. We shortly consider the possibility of spontaneous muon production in muonic supercritical fields.

1. Introduction

If the binding energy of a state is greater than $2m_e c^2$, while this state is vacant spontaneous free positron creation occurs and the vacuum becomes charged [1–3]. The external electromagnetic potential for which the spontaneous positron creation may occur is called critical. In order to investigate the positron creation in a heavy ion collision it is necessary to know the precise least nuclear charge which produces the corresponding critical potential.

The solution of the Dirac equation in the case of a pure Coulomb potential leads to the Sommerfeld finestructure formula which allows for $1s_{1/2}$ electrons only solutions for $Z<\alpha^{-1}=\hbar c/e^2=137.03602$. Beyond this value of Z no $1s_{1/2}$ state is found in the discrete spectrum.

For extended nuclei the Dirac equation was first solved by Pomeranchuk and Smorodinsky [4] (Their work suffered from various numerical errors) and by Pieper and Greiner [5]. The nuclear charge distribution was given by a homogeneous charge sphere with a radius

$$R=r_0 A^{1/3} \tag{1.1}$$

where $r_0=1.2$ fm. The atomic mass A has been approximated for superheavy elements by

$$A=0.00733\,Z^2+1.3\,Z+63.6\,. \tag{1.2}$$

The $1s_{1/2}$ state was found for values of $Z\leqq169$. At this point the binding energy of the $1s_{1/2}$ state reached $2m_e c^2$. For the $2p_{1/2}$ electron the critical value was found to be $Z=182$. If the critical value of Z could be increased by more than 15 units due to nuclear properties, the electron–electron interaction and quantumelectrodynamical effects the associated experiments [1, 2] may become impossible. The exact theoretical and experimental determination of the critical charge seems also to be a valuable test for relativistic quantum mechanics and quantum electrodynamics of strong fields. By a precise measurement of electronic transition energies in superheavy atoms or quasiatoms formed during the collision of heavy ions with $Z_1+Z_2\sim Z_{cr}$, it may be possible to decide if selfenergy- and vacuum-polarization corrections are described well within the frame of the generalized Hartree-Fock equation given by Reinhard [6] and also Rafelski and Müller [7]

$$\begin{aligned}(-i\boldsymbol{\alpha}\nabla-e\gamma_0{}^{\mathrm{ext}}+\gamma_0 m)\,\Phi_q(\boldsymbol{x})+\frac{e^2}{8\pi}\gamma_0\Bigg\{\int \mathrm{d}^3z\,\gamma^{\mathrm{K}}\frac{\sum\limits_p \bar{\Phi}_p(\boldsymbol{z})\gamma_{\mathrm{K}}\Phi_p(\boldsymbol{z})}{|\boldsymbol{x}-\boldsymbol{z}|}\Phi_q(\boldsymbol{x})\\ -\int \mathrm{d}^3z\sum_p \gamma^{\mathrm{K}}\Phi_p(\boldsymbol{x})\bar{\Phi}_p(\boldsymbol{z})\gamma_{\mathrm{K}}\Phi_q(\boldsymbol{z})\,\frac{F\left(|\boldsymbol{x}-\boldsymbol{z}|,\varepsilon_p-\varepsilon_q,\mathrm{sg}(p)\right)}{|\boldsymbol{x}-\boldsymbol{z}|}\Bigg\}=\varepsilon_q\Phi_q(\boldsymbol{x})\end{aligned} \tag{1.3}$$

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with

$$F(|\boldsymbol{x}-\boldsymbol{z}|, \varepsilon_p-\varepsilon_q, \mathrm{sg}(p)) = -\frac{2}{\pi}\int_0^\infty \mathrm{d}q' \frac{\sin(q'|\boldsymbol{x}-\boldsymbol{z}|)}{\varepsilon_p-\varepsilon_q+\mathrm{sg}(p)(q'-i\alpha)}, \tag{1.4}$$

$$\mathrm{sg}(p) = \begin{cases} +1 & \text{if } p>F, \\ -1 & \text{if } p<F, \end{cases}$$

F means the Fermi energy.

This paper is organized as follows. In the next section we discuss the one center problem in the frame of the Dirac equation. We present several numerical results concerning the influence of different parameters on the critical potential. In Sect. 3 we solve the two center problem and approximate the electron-electron interaction and obtain numerical results for critical radii. In Sect. 4 we then consider the quantum electrodynamical corrections for one and two center potentials. In the following section we discuss the possibility of muonic supercritical states and show, that they cannot be created in heavy ion collisions. We close our discussion of the possible effects which could change the value of the critical nuclear charge discussing several phenomenological modifications of classical electrodynamics and QED in Section 6. We show that no essential change of the critical quantities is to be expected from such modifications if the values of the phenomenological parameters are restricted by present experiments. In the last section we summarize our results and give a short outlook on future calculations.

2. Z_{cr} within the Framework of the Dirac Equation

In our calculations we use a realistic nuclear charge distribution of a Fermi type with the surface thickness parameter t fixed usually to 2.2 fm and the half density parameter c which was derived from the equivalent radius

$$R_{eq} = r_0 A^{1/3} \tag{2.1}$$

with $r_0 = 1.2$ fm. The nuclear mass A was chosen according to Eq. (1.2) of Pieper and Greiner [4]. In order to take into account the electronic interaction we first employed a Thomas-Fermi-approximation [8]. We found numerical solutions for the $1s_{1/2}$ state for $Z \leqq 172$ ($c = 9.234$ fm), the binding energy for $Z = 172$ is -1018.139 keV. That means a shift by 1.5 units of the critical value for positron creation when including the electron-electron interaction in the Thomas-Fermi approximation. Numerically it is possible to find bound state wavefunctions up to the region $Z = Z_{cr} - 0.01$. In Fig. 1 one can see the

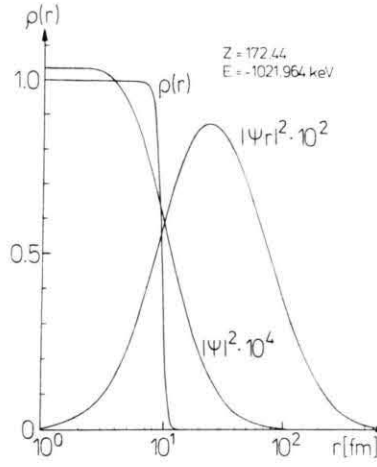

Fig. 1. The radial density $|\psi r|^2$ of the $1s_{1/2}$ electron and the nuclear charge distribution is shown for an assumed nucleus with $Z = 172.44$. The binding energy differs only by 44 eV from the critical value $-2\,m\,c^2$.

radial density of the $1s_{1/2}$ electron of $Z = 172.44$ ($c = 9.9$ fm, $t = 2.2$ fm, $E = -1021.964$ keV) in comparison to the nuclear charge density assumed. A very remarkable fact is the relatively localized appearance of the wave function. For the diving velocity defined as $v_d = \mathrm{d}E/\mathrm{d}Z|_{Z=Z_{cr}}$ we obtain 30.1 keV/unit charge.

In Fig. 2 the maximum of the radial density $|\psi r|^2$ of the $1s_{1/2}$ electron as function of the nuclear charge is shown. For $Z \to Z_{cr}$ it reaches a value of about 25 fm.

In a high energy collision of heavy naked nuclei with $Z = Z_1 + Z_2 > 137$ one could assume because of compression of nuclear matter that it is possible to observe positron creation for example in a Pb + Pb ($Z_1 + Z_2 = 164 < Z_{cr}$) collision or with even lighter projectiles. In Fig. 3 we show the binding energy of the $1s_{1/2}$ electron as a function of Z where we have assumed as charge distribution two concentric superposed homogeneously charged spheres with in-

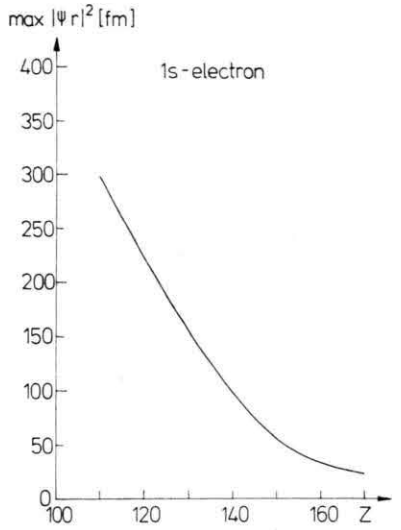

Fig. 2. The maximum of $|\psi r|^2$ of the $1s_{1/2}$ electron as function of Z.

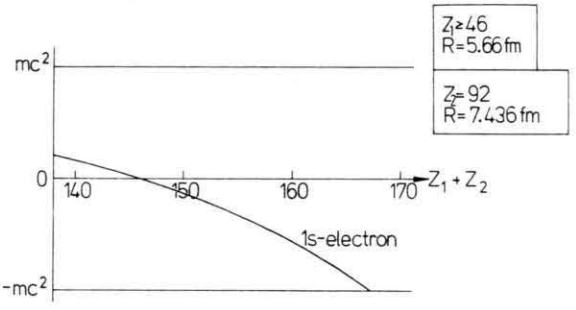

Fig. 3. The energy eigenvalue of the $1s_{1/2}$ electron as function of Z in a high energy heavy ion collision. The nuclear charge distribution consists of superposed homogeneous charge spheres of the collision partners with increasing charge in the smaller nucleus. Diving occurs at $Z=167$.

creasing charge in the smaller particle. Diving then occurs at $Z=167$. Thus the effect of the nuclear compression may shift the critical value of Z down by at most 5 units of charge.

Even under the assumption that superheavy nuclei with $Z\sim Z_{cr}$ are stable it is not possible to determine the critical value better than two units because of the unknown nuclear charge parameters. However, measuring several transition energies one will be able to attempt a best fit of nuclear charge distribution as it is done today in the muonic atom experiments.

For $Z>137$ one can always define for each Z a critical radius for positron creation (we may compress the nucleus in a Gedanken Experiment till the critical value of R is reached). In Fig. 4a the critical equivalent radius of a fermi charge distribution with $t=2.2$ fm as function of Z is shown. We note that a change in the nuclear radius of 10% corresponds to a shift in the diving point by one unit. As the equivalent radius is determined from the atomic mass, we can also plot the A dependence of the energy of the $1s_{1/2}$ state – see Figure 4b. In

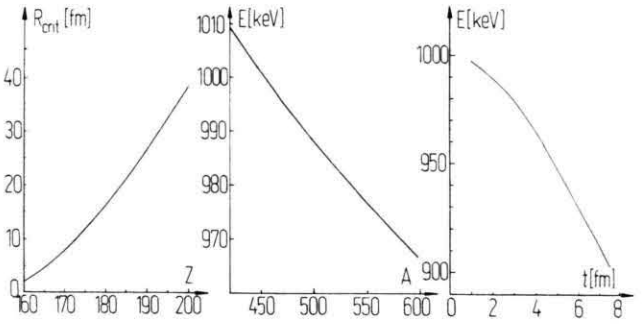

Fig. 4. a) The critical equivalent radius of a fermi charge distribution with $t=2.2$ fm as function of Z. b) and c) Energies of the $1s_{1/2}$ state as function of the atomic mass number A and of the surface thickness t of the nucleus for $Z=171$.

Fig. 4c one can further see the energy as function of the surface thickness parameter t, where c was fixed to 9.218 fm and $Z=171$. With the value of the diving velocity $v_d=30.1$ keV/charge unit it is easy to recover the corresponding effect on the shift in the value of Z_{cr}. The determination of Z_{cr} seems to be uncertain by one unit due to uncertainties in the nuclear charge distribution. As an example of a numerical solution to an overdimensioned nucleus we show the radial density $|\psi r|^2$ of the $1s_{1/2}$ electron for $Z=184$ ($_{92}U+_{92}U$) in Fig. 5 with a assumed nuclear radius corresponding to $c=20.3$ fm. The maximum of the electronic charge distribution lies at about 40 fm.

If one uses the relativistic Hartree-Fock-Slater formalism as a more realistic model for the electronic interaction one can find bound state solutions for all electrons up to $Z=173$. Thus, the Thomas-Fermi approximation seems to underestimate the electron-electron interaction at the critical value. For the description of the HFS-ansatz we refer to Reference [9]. The selfconsistency criterion has been reached in our numerical code when the potential which was generated from the electronic density satisfied the condition

$$\left|\frac{V_{old}-V_{new}}{V_{old}}\right|<\delta=10^{-6}$$

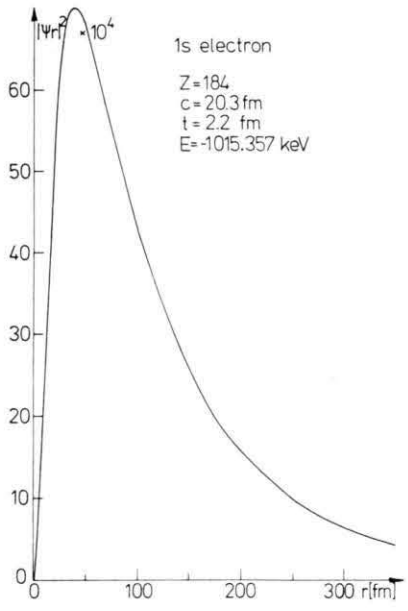

Fig. 5. The radial density $|\psi r|^2$ of the K electron of an overdimensioned nucleus with $Z = 184$. Such a big nucleus can be understood as an approximation of the resulting effect of two colliding U ions separated by about 20 fm.

which is one order of magnitude better than in Reference [9]. The binding energy of the $1s_{1/2}$ electron of $Z = 173$ was found to be -1006.175 keV with the maximum of $|\psi r|^2$ at 23.545 fm. In Table 1 the HFS energies of the first three electron shells of

Table 1. The binding energies (eV) of the electrons of the first 3 shells calculated with a relativistic Hartree-Fock-Slater program of the noble gas $Z = 172$ ($c = 9.243$ fm, $t = 2.2$ fm) are shown. In the last column the Thomas-Fermi results are shown.

Orbital	Number of electrons	Hartree-Fock-Slater energies	Thomas-Fermi energies
$1s_{1/2}$	2	976583	1018139
$2s_{1/2}$	2	245903	262196
$2p_{1/2}$	2	524698	564852
$2p_{3/2}$	4	73629	80776
$3s_{1/2}$	2	66548	73045
$3p_{1/2}$	2	88905	98621
$3p_{3/2}$	4	23411	26254
$3d_{3/2}$	4	21793	24864
$3d_{5/2}$	6	18521	20882

the noble gas with $Z = 172$ ($A = 504$, $c = 9.243$ fm) are listed and can be compared with the corresponding Thomas-Fermi results. The Thomas-Fermi binding energies are in each case greater than the HFS ones. The sum of the energy eigenvalues for $Z = 172$ was found to be 4705.411 keV; the contribution from the exchange term is 75.065 keV. We would like to mention, that for all elements from $Z = 47$ to $Z = 173$ the nuclear charge distribution, the nuclear potential, the binding energies, the innerelectronic potential energy, the electronic exchange energy, the electronic density, the resulting potential, the principal maximum of $|\psi r|^2$ of each electron and the maxima in the total density have been computed [10]. In Fig. 6 one can see for example for $Z = 170$ the total radial density of the neutral atom. The maxima are at 25, 910, 2839, 6409 and 13997 fm.

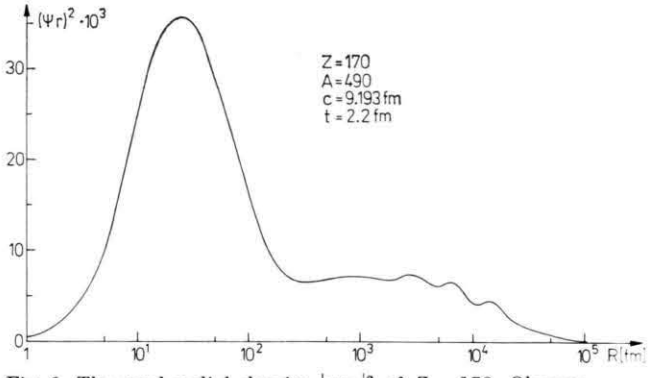

Fig. 6. The total radial density $|\psi r|^2$ of $Z = 170$. Observe the big principal maximum in comparison with those atoms with lower Z which are shown by Fricke and others [9]. Note half logarithmic scale.

Critical potentials will be first experimentally realized in a heavy ion collision where the ion beam can be assumed to be highly ionized ($20 - 25$ electrons removed) [11]. During the collision further ionization will occur – more than 100 electrons can be missing in the quasimolecule. For the neutral atom with $Z = 170$ the relativistic HFS energies for the $1s_{1/2}$ and the $2p_{3/2}$ electrons are -919.638 keV and -71.566 keV. The change of the binding energy of the $1s_{1/2}$ electron due to ionization is shown in Figure 7. In our calculations the outer electrons were removed step by step. Only for a degree of ionization d greater than 100 the influence on the diving point seems to be important. The change of the $2p_{3/2} - 1s_{1/2}$ transition energy is less than 200 eV for $d < 100$ and is less than 3 keV for $d < 150$. In order to create positrons it is necessary to have at least one vacancy in the K shell [3]. If one assumes only one vacancy in the K shell of $Z = 172$ the binding energy of the $1s_{1/2}$ electron would rise by 13.666 keV. We conclude noting that the effects of the electron configuration of a neutral atom with

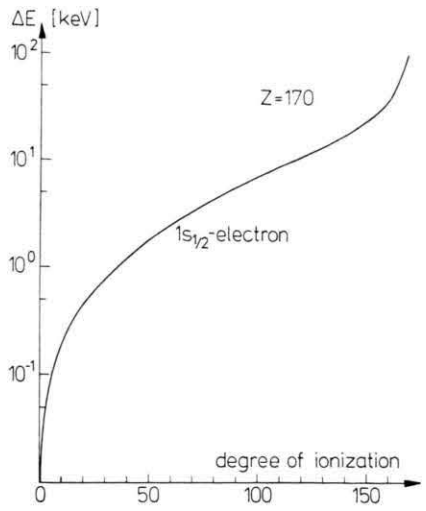

Fig. 7. The change of the relativistic HFS-binding energy of the $1s_{1/2}$ electron of $Z = 170$ as function of the degree of ionization.

$Z \sim Z_{cr}$ (which determines in general the chemical behaviour) are negligible with regard to the diving point. An estimate of the magnetic energy contribution to the binding energy following along the lines of Fricke et al.[12] (which is for $_{100}$Fm greater than the quantumelectrodynamic corrections) increases the diving point by about 0.5 units.

3. Two Center Calculations

The electronic levels in the molecular system with two Coulomb centers Z_1 and Z_2 are obtained solving the Two-Center-Dirac equation

$$(c\,\boldsymbol{\alpha}\,\boldsymbol{p} + \beta\, m\, c^2 + V_1(\boldsymbol{r}-\boldsymbol{R}) + V_2(\boldsymbol{r}+\boldsymbol{R}))\,\psi = E\,\psi\,. \tag{3.1}$$

V_1 and V_2 are obtained from finite size charge distributions of the collision partners. The eigenstates and eigenenergies are obtained by diagonalization of Eq. (3.1) in a suitable basis[13] in the adiabatic approximation as function of the two Coulomb center distances R. For critical systems the asymptotic spectrum of the united system is reached only when the two nuclei come very close ($R <$ K-shell radius of the united system ~ 30 fm). As no real Two-Center-Hartree-Fock calculations are available as yet, we estimate the electron screening effects by shifting the function $E(R)$ according to the results of the HFS-calculations of the asymptotic limits $R \to 0$ and $R \to \infty$. For the lowest levels this approximation should be better than few per cents. In Fig. 8 the function $E(R)$ is shown for the undercritical $_{53}\mathrm{I} + _{79}\mathrm{Au}$ collision. The dashed lines demonstrate the HFS results. Only for higher levels ($n > 2$) a drastic change in the correlation diagram due to electron-electron interaction are found. However, for our purpose, the outlined approximation is fully sufficient.

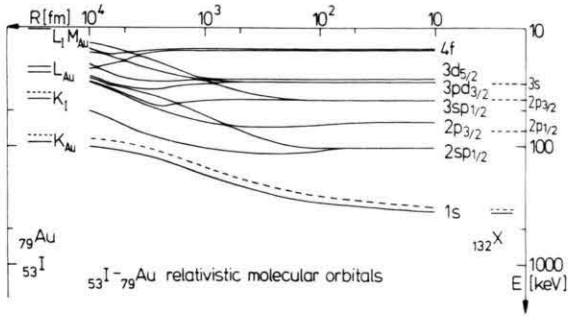

Fig. 8. The electronic energies of the lowest levels of the molecular system with the two Coulomb centers $_{53}$I and $_{79}$Au as function of the distance R. The dashed lines in the asymptotic limits denote the HFS-results. The dashed curve demonstrate an approximation of the Two-Center-Hartree-Fock-Dirac result due to linear interpolation of the asymptotic HFS values and the function $E(R)$ of Müller and others[13].

4. Quantum Electrodynamic Corrections

The most unknown facts in the determination of the critical Z are the quantumelectrodynamic corrections such as vacuumpolarization and selfenergy. Calculations of the associated energy shifts are carried out commonly in a perturbation expansion where the small parameter is $Z\alpha$. This approach yields satisfactory results for all known elements[12, 14]. Even for Fm ($Z = 100$) one gets for the binding energy of the $1s_{1/2}$ electron ($E \sim 142$ keV) only a shift of 484 eV due to selfenergy and -154 eV due to vacuumpolarization[12]. Nevertheless it is, of course, questionable to extend these calculations to atoms with $Z\alpha > 1$. The QED corrections for strong fields are under investigation within the frame of Equation (1.3)[15]. One knows from heavy muonic atoms that in strong fields, but for $Z\alpha < 1$ vacuumpolarization corrections of higher than linear order are only a per cent effect compared to the effect of the Uehling potential[16–18] which we will evaluate further below. It is our present strong belief, that use of the Uehling potential is justified as long as $Z < Z_{cr}$. For $Z > Z_{cr}$ however,

one has to take into account the fact, that also a real charge is included in the vacuum. For further discussions we refer to References [2, 3]. The Uehling potential for radial symmetric charge distribution is given by [19]

$$V_{\mathrm{VP}} = -e^2 \frac{4\,\alpha}{3} \frac{\lambda\!\!\!^{-}_{\mathrm{e}}}{2} \int_0^\infty \varrho(r') \left(Z_1(|r-r'|) - Z_1(r+r')\right) \mathrm{d}r \qquad (4.1)$$

with the structure function

$$Z_1(|r|) = \int_1^\infty \exp\left\{-\frac{2}{\lambda\!\!\!^{-}_{\mathrm{e}}}|r|\,\xi\right\} \cdot \left(1+\frac{1}{2\,\xi^2}\right) \frac{(\xi^2-1)^{1/2}}{\xi^2} \frac{1}{\xi}\,\mathrm{d}\xi \qquad (4.2)$$

where $\lambda\!\!\!^{-}_{\mathrm{e}}$ is the electron Compton wavelength. We have used the following constants $\lambda\!\!\!^{-}_{\mathrm{e}} = 386.1592$ fm, $\alpha^{-1} = 137.03602$ and $e^2 = h\,c\,\alpha = 1.4399784$ MeV fm. In this calculation we have made no other perturbation expansion. The structure function as well as the resulting potential are integrated numerically. The attractive vacuumpolarization potential was added to the Coulomb potential for extended nuclei before integrating the Dirac equation. We establish that our code calculates V_{VP} with an accuracy better than 2%. For the $1s_{1/2}$ electrons in $Z = 171$ we found $\Delta E = -11.834$ keV. However, for the known elements selfenergy is the dominant quantum electrodynamic correction for K electrons. For example for $_{100}$Fm the Lambshift effect is by a factor 3 larger than the effect of the Uehling potential [12, 14] because electrons are still relatively far from the inducing nuclear charge distribution. The influence of the vacuum fluctuation on electronic binding energies of superheavy elements was earlier estimated by Fricke [20] in a questionable model calculation. A shift of the diving point of 5 units results in his calculations. Our finding is considerably smaller. Our calculation is based on a formula given by Erickson [21]. The energy shift agrees for $Z = 80$ with the unperturbive result of Desiderio and Johnson [22]. For $Z = 173$ we found $\Delta E = 2.9$ keV this is equivalent to a change of Z_{cr} by $\Delta Z = 0.1$. The selfenergy effects are of comparable size to the vacuum polarization shift and of opposite sign. The use of the perturbation expansion of Erickson is certainly not completely justified. More careful evaluation along the lines of Desiderio and Johnson is in preparation. So far we attribute an uncertainty of one unit due to the perturbative determination of QED effects.

5. Supercritical Muonic States?

Beside the electrons an atom may contain muons. These particles are fermions, too, and are subject to exactly the same interactions as the electrons, from which they only differ by mass. Consequently, one would expect the phenomena described above to happen in the very same way in superheavy muonic atoms, and more easily observable since the muonic K shell is usually (in all non-muonic atoms) unoccupied. But the muonic K shell radius lies for atoms with $Z > 100$ inside the nucleus (see Fig. 9)

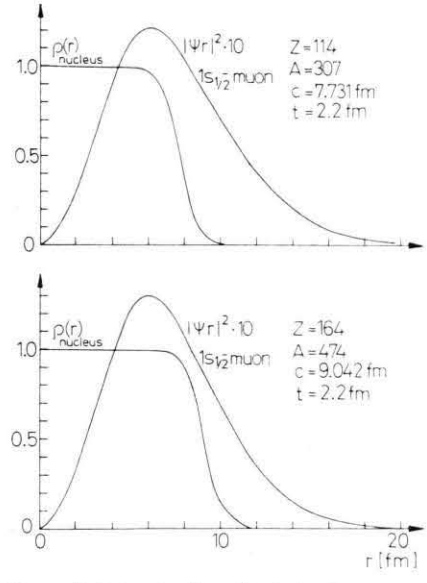

Fig. 9. The radial density $|\psi r|^2$ of the $1s_{1/2}$ muons of the presumable stable superheavy elements a) $Z = 114$ and b) $Z = 164$ and the corresponding nuclear charge distribution. Only the Coulomb potential for extended nuclei and reduced mass effects are considered but no QED corrections are included. The K-shell radius lies inside the nucleus.

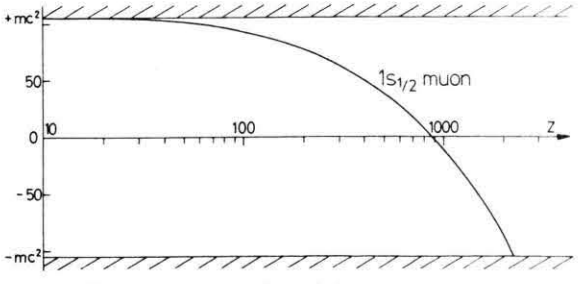

Fig. 10. The energy eigenvalue of the $1s_{1/2}$ muon as function of Z. Diving occurs at the utopic value of $Z = 2200$, where we have assumed $A = 2.5\,Z$, $R_{\mathrm{eq}} = 1.2\,A^{1/3}$ and $t = 2.2$ fm.

and therefore the binding energies increase very slowly compared to the electronic ones. The energy eigenvalue of the $1s_{1/2}$ muon as function of Z is shown in Figure 10. The diving point we found at the utopical value of $Z=2200$ where we have assumed $A=2.5\,Z$, $R_{eq}=1.2\,A^{1/3}$ and $t=2.2$ fm. Even in the case of very high compression of the nuclear charge density in a high energy collision for example $U+U$ the energy of the $1s_{1/2}$ muon would reach the lower continuum only for nuclear radii $R<1$ fm, see Fig. 11, where we show the $1s_{1/2}$

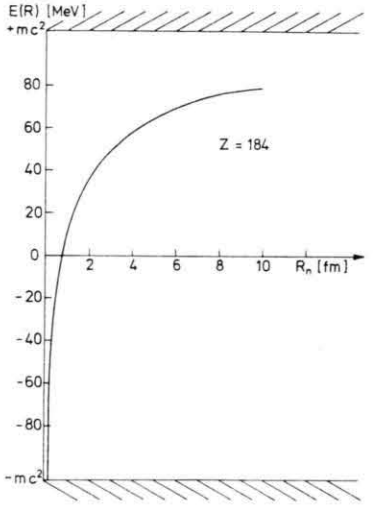

Fig. 11. The energy eigenvalue of the $1s_{1/2}$ muon of $Z=184$ $(U+U)$ as function of the nuclear radius.

muonic state in a $Z=184$ atom as a function of the nuclear radius. An experimental creation of positively charged particles with $m>m_e$ and only electromagnetic interaction is therefore impossible. Nevertheless one can learn much from the muonic atom measurements when investigating the properties of superheavy nuclei, which because the calculations of Refs. [23, 24] and [25] should be stable in the regions around $Z=114$ and probably around $Z=164$. The cross section for capture of muons in atoms is proportional to Z^4 or Z^5. Thus an experiment may be easier done even with a low concentration of superheavy elements.

6. Phenomenological Modification of QED and Electrodynamics

The nonlinear electromagnetic field theories of Born-Infeld type which are described by the following class of Hamiltonian densities $H(n)$ if the magnetic field $\boldsymbol{B}=0$,

$$H(n)=\frac{E_0^2(n)}{2n}\left[\left(1+\frac{D^2}{E_0^2(n)}\right)^n-1\right]. \quad (6.1)$$

D is the electric displacement, $E_0(n)$ is a parameter of the theory, are not able to prevent the diving of the $1s_{1/2}$ state [26]. These nonlinear field theories have the remarkable property that they lead to a finite selfenergy value [26] of a point charge for $n<0.75$ and that they reduce to the Maxwellian Hamiltonian density whenever $D^2\ll E_0^2$. From high precision measurements in muonic atoms [27, 28] it results that one has to choose $E_{max}>2\cdot10^{20}$ Volt/cm in order to avoid disagreements with experimental data. With this maximal electric field strength the diving point is shifted by no more than 2 units [29–31]. In general limiting electromagnetic field theories are not capable to prevent diving because the potential which is responsible for the binding energy of the electrons has no upper bound.

In the frame of limiting potential theories based on the Lagrange function

$$\mathfrak{L}=\tfrac{1}{4}f_{Kl}f^{Kl}V(A_\varrho A^\varrho)-4\pi j^K\Phi_K(A_i,A^i) \quad (6.2)$$

with

$$f_{Kl}=A_{l/K}-A_{K/l},\ \Phi_K=\int_{A_K(\infty)}^{A_K}\sqrt{V}V(A_\varrho A^\varrho)\,\mathrm{d}A_K$$

it would be possible that the binding energy of the electrons is less than $2mc^2$ for each Z [32]. This Lagrangian density leads to the field equation

$$\frac{\partial}{\partial x^K}[\sqrt{V}f^{Kl}]+\frac{V'}{2\sqrt{V}}(A_\varrho A^\varrho)_{|K}f^{Kl}-\frac{1}{2}f_{mn}f^{mn}\frac{V'}{\sqrt{V}}A^l=4\pi j^l. \quad (6.3)$$

Due to the continuity equation one has the condition

$$\frac{\partial}{\partial x^K}\frac{\partial\mathfrak{L}_F}{\partial A_K}=0 \quad (6.4)$$

with

$$\mathfrak{L}_F=\tfrac{1}{4}f_{Kl}f^{Kl}V(A_\varrho A^\varrho). \quad (6.5)$$

In the case of purely electrostatic fields

$$A_K=(0,0,0-\varphi) \quad (6.6)$$

the field equation has the solution

$$\Phi=\int_{\varphi_\infty}^{\varphi}\sqrt{V(-\varphi^2)}\,\mathrm{d}\varphi=\int\varrho(\boldsymbol{r}')\frac{\mathrm{d}^3r'}{|\boldsymbol{r}-\boldsymbol{r}'|} \quad (6.7)$$

where φ can be obtained from special choices of V. With the parametrization ($\delta = \pm 1$)

$$V(A_\varrho A^\varrho) = \left[1 + \delta \frac{A_\varrho A^\varrho}{\varphi_0^2}\right]^n \qquad (6.8)$$

(limiting potentials results if $\delta = +1$) one finds for $n = -2$, $\delta = +1$

$$\varphi = \varphi_0 \tanh(Z e/\varphi_0 r) . \qquad (6.9)$$

If one chooses

$$\varphi_0 = 2 m_e c^2/e \qquad (6.10)$$

it follows $e\varphi \leqq e\varphi_0 = 2 m_e c^2$. As the binding energy of the $1s_{1/2}$ state has to be smaller than $e\varphi$, that means that the binding energy of an electron cannot reach the critical value $2 m c^2$ in such a theory. These theories are not gauge invariant and are ruled out by present experiments in atomic physics. Solution of the Dirac equation

$$\{\gamma^K[p_K - (e/c) A_K] - m_e c\}\psi = 0 \qquad (6.11)$$

leads for the $2p_{3/2} - 1s_{1/2}$ transition energy in lead ($Z = 82$) to 64.531 keV which differs of about 14 keV from the usual value for the Coulomb potential. Because of precise measurements [33] in electronic $_{100}$Fm one is forced to set $e\varphi_0 \geqq 1000\, m_e c^2$. This lower bound of limiting potentials increases by a factor of 10^4 in order to avoid discrepancies in the calculation of transition matrix elements in nuclei. Therefore it can be expected that there exists no observable shift of the diving point due to the limiting potential theories.

The coupling of the potential of Eq. (6.11) to the Dirac field is not unique. In the case of a pure electrostatic potential the Dirac equation for mixed scalar and vector (Coulomb) potential can be written ($\hbar = c = 1$)

$$[\boldsymbol{\alpha}\,\boldsymbol{p} + \beta(m + V_2) - (E - V_1)]\,\psi = 0 . \qquad (6.12)$$

It was shown that this equation admits analytic solutions [34] for any $V_1, V_2 \sim 1/|r|$. Choosing

$$V_1 = -\alpha/r , \quad V_2 = -\alpha'/r \qquad (6.13)$$

and $\alpha = -\alpha'$ the energy spectrum becomes

$$E = m\left[1 - \frac{\frac{1}{2}(\alpha/n)^2}{1 + \frac{1}{4}(\alpha/n)^2}\right] . \qquad (6.14)$$

For $\alpha \to \infty$ the energies approach the negative energy continuum and no positron creation would appear. However, the limits on the scalar coupling constant are set by atomic precisions experiments [27, 28, 33]. One obtains the maximal size of the scalar coupling constant $\alpha' = 2.5 \cdot 10^{-8}$ per unit charge. These scalar potentials can be understood as the effect of the exchange of a massless or almost massless scalar boson analogous to the suggestion of Sundaresan and Watson [16] of a massive (8 MeV) scalar boson coupled to the $\mu^+\mu^-$-field in order to explain present discrepancies in muonic experiments [27, 28]. Any displacement of electronic binding energies in superheavy elements due to different forms of potential couplings can therefore be ruled out.

Considering the Heisenberg field equation [35]

$$\gamma_\nu \partial_\nu \psi \pm l^2 \gamma_\nu \gamma_5 \psi(\bar{\psi} \gamma_\nu \gamma_5 \psi) = 0 \qquad (6.15)$$

and the associated one particle equation with electromagnetic interaction

$$\gamma_\nu(i\partial_\nu - e A_\nu \pm l^2 \bar{\psi} \gamma_\nu \psi)\psi - m\psi = 0 \qquad (6.16)$$

small corrections to the electronic energies cannot be excluded [36]. The introduction of the density-dependent term of the form ψ^3 would correspond to energy shifts $\Delta E \sim l^2 \int \psi^4 \, d^3x$ where l is a parameter with a dimension of a length. Concerning the present discrepancies in muonic atoms [27, 28] the maximal value of l can be determined to be $l \sim 0.024 \text{ fm}/Z$. From numerical solution of Eq. (6.16) follows an upper bound to a change of the diving point of half a unit from this kind of self-interaction [36, 37].

7. Conclusions and Outlook

In this paper we have discussed quantitatively all contributions which would affect the value of the critical nuclear charge or critical separation in heavy ion collisions. Most of these contributions are negligible – the important ones being a) electron-electron (HF-effects) interactions (± 0.5 units), b) uncertainty in the charge distribution (± 1.0 units), c) magnetic energy ($+0.5 \pm 0.25$ units), d) vacuum polarization and selfenergy (± 1 units) shifts. Therefore we believe that the most realistic value for the critical nuclear charge is $Z = 173 \pm 2$ where the uncertainty mainly comes from the still unprecise value for the QED effects. The critical separation in heavy ion collision was found to be $R(U-U) = 34.7 \pm 2$ fm, $R(U-Cf) = 47.7 \pm 2$ fm. The uncertainty corresponds to an assumed 60 keV uncertainty in the determination of the molecular

state energy due to the estimates of the HF- and QED effects. The uncertainty in the nuclear charge distribution is negligible in this case. We would like to stress again that our calculations always include the finite nuclear size and are exact solutions of the relativistic Dirac equation.

In the future very tedious calculations are necessary to compute more precisely all the aforementioned effects, such as vacuum polarization, self-energy, magnetic energy in the frame of the two center Reinhard equation. Two preliminary steps are under way: The two center code is being developed towards a Hartree-Fock code and the Reinhard equation is being used in the one center case[15]. The present calculations will however allow, when compared with the experiment, to determine whether our understanding of supercritical fields and quantum electrodynamics is adequate.

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