Can Quantum Theory be presented as a Classical Ensemble Theory? *, §

WILHELM OCHS

Sektion Physik der Universität München, 8000 München, Germany

(Z. Naturforsch. **26 a**, 1740—1753 [1971]; received 1 July 1971)

We examine the possibility of reformulating quantum theory (QT) as a deterministic ensemble theory which (a) interprets observables as objective properties of physical systems and (b) coincides with QT in all quantitative statements. As will be demonstrated, such an *Ensemble-Quantum-Theory* (EQT) can only be constructed if (1) one accepts a modified observable-concept, and (2) as long as the theory of measurement is left out of account. A correct treatment of the measuring process is impossible within such an EQT. Consequently, there exist no *Hidden-Variable Theories* with the properties (a) and (b).

1. Introduction

In this paper we examine the possibility of a deterministic reformulation of quantum theory (QT) which has the following features:

(a) Like in classical physics, the "observables" of a physical system should be interpretable as objective properties of the system, i.e. they should always have, independently of an eventual measurement, a definite value which can also be ascertained by a measurement;

(b) the deterministic reformation should exactly reproduce all quantitative statements of QT;

(c) the problem of the "reduction of the state operator", still not satisfactorly solved, should find, in the frame of the theory considered, a simple, formal solution by basing the reduction of the state operator entirely on the revision of the statistical "macro-description" of the system required by the increase of formation provided by "reading the scale".

This problem originated in the *Differential-Space Quantum Theory* (DSQ)[1–5], an attempt by Wiener and Siegel to reformulate QT in analogy to statistical mechanics as a classical ensemble theory with the above properties: In DSQ every quantum state is represented by an ensemble of virtual systems in dispersion-free micro-states in which every observable has a definite value. All micro-states evolve deterministicly, and an appropriate probability distribution ensures that the ensemble mean values coincide with the expectation values of QT for all observables. The reduction of the state operator at the end of a measurement results from the reduction of the ensemble corresponding to the increase of information by *reading the scale* and does not require an additional postulate inside DSQ.

Now, an analysis of DSQ[6] reveals serious discrepances between the physical ideas of this theory and its mathematical formalism so that not even the properties (a) and (b) can be considered as realized in DSQ. But since this "failure" of DSQ proves nothing about the possibility of such a theory, the problem remains whether an *Ensemble-Quantum-Theory* (EQT) is possible which realizes the physical ideas of DSQ, in particular the features (a) to (c), while avoiding its deficiencies[7]. To solve this problem, one must first precisely formulate the physical concepts of such a "DSQ-like" EQT, and from these a formal system of axioms must be extracted establishing the mathematical structure of EQT as far as implied by the physical concepts. Finally, one must check whether this axiom system is free of contradictions.

Reprints request to Dr. W. OCHS, Sektion Physik, Lehrstuhl Prof. SÜSSMANN, *D-8000 München 13*, Schellingstraße 2—8.

* Part of the authors thesis, Frankfurt (Main) 1968.

§ The author gratefully acknowledges that this work was supported in part by the Deutsche Forschungsgemeinschaft.

1 N. WIENER and A. SIEGEL, Phys. Rev. **91**, 1551 (1953).

2 N. WIENER and A. SIEGEL, Nuovo Cim. Suppl. **2**, 982 [1955].

3 A. SIEGEL and N. WIENER, Phys. Rev. **101**, 429 [1956].

4 A. SIEGEL, The Differential-Space Theory of Quantum Systems, in: Differential-Space, Quantum Systems and Prediction, ed. by N. WIENER, Cambridge (Mass.) 1966.

5 W. OCHS, Helv. Phys. Acta **43**, 668 (1970).

6 W. OCHS, Helv. Phys. Acta **43**, 686 [1970].

7 According to the characteristics (a) to (c) of EQT, none of the *Hidden-Variables Theories* developped by D. BOHM and his co-workers belongs to the class of EQTs and will therefore not be considered in this paper; this applies in particular to the theory of BOHM and BUB[8] which disagrees with all of these three characteristics of EQT.

8 D. BOHM and J. BUB, Rev. Mod. Phys. **38**, 453 [1966].

This program was initiated in a previous paper[9] where we investigated the possibility to associate to every physical system a micro-state in which every observable has a definite and measurable value, and to represent all quantum states by *Gibbsian ensembles* of such micro-states. As has been shown by many authors[10–13], this first step in developping an EQT already leads to a contradiction, if the observable-concept of QT is retained unchanged. Hence we introduced in [9] a new, more operative concept of observables[14] which takes more account of the measuring device:

According to QT, the state operator W_2 resulting from an incomplete measurement of the first kind[16] is in general not uniquely determined by the value of the measured observable, but can also depend on the quantum state W_0 of the object before the measurement[15, 17]. According as the coherence of the state vectors in the eigenspaces of the measured observable is destroyed by the measuring device, the former state W_0 will leave more or less traces in the state operator W_2. In the ideal case of a *strictly conservativ* ($\mathbf{A}=a$)-filter, the coherence in all eigenspaces of $\mathbf{A}$ is completely maintained, and W_2 has the form

$$W_2 = [\mathrm{Tr}(W_0 P_a)]^{-1} P_a W_0 P_a \,; \tag{1.1}$$

in this case the traces of W_0 are strongest. In the other extreme of a *completely separating* $\mathbf{A}$-instrument, we have (from the viewpoint of the interaction) in fact a complete measurement and W_2 takes the form

$$W_2 = [\mathrm{Tr}(W_0 P_a)]^{-1} \sum_{|i\rangle\langle i| \leqq P_a} |i\rangle\langle i|\, W_0 \,|i\rangle\langle i| \,; \tag{1.2}$$

here the coherence is completely destroyed in all eigenspaces and only the poor reading-mechanism makes the measuring device an $\mathbf{A}$-instrument. Between these two extremes there are in principle instruments with an arbitrary *separation-character*[15].

These considerations led us to the following new, weakened concept of an observable[9]:

An EQT-observable is, in general, not completely determined by an associated operator but can be characterized furthermore by the separation-character of the $\mathbf{A}$-instrument. This "more detailed" observable-concept obviously also implies a new mathematical representation of the EQT-observables. Following the above considerations, we associate to every observable $\mathbf{A}$ exactly one pair (A, α) composed of the operator A, which QT also associates to the $\mathbf{A}$-instrument, and of an *A-finer*[19] *partition of unity*[18] α which indicates the separation-character of the $\mathbf{A}$-measuring device[20]. This new observable concept has important consequences:

a) In the representation of EQT-observables there only occur operators with a discrete spectrum. QT-observables $\mathbf{X}$ with a more general range (like the position observable) are replaced in EQT by a family of discrete, coarser observables $f(\mathbf{X})$ which, depending on the respective measuring device, conform to the actually measured quantity.

[9] W. Ochs, Z. Naturforsch. **25**, 1546 [1970].

[10] F. Kamber, Nachr. Akad. Wiss. Göttingen, Math.-Nat. Kl., Heft 10 (1964).

[11] N. Zierler and M. Schlessinger, Duke Math. J. **32**, 251 [1965].

[12] J. S. Bell, Rev. Mod. Phys. **38**, 447 [1966].

[13] S. Kochen and E. P. Specker, J. Math. Mech. **17**, 59 [1967].

[14] The new observable-concept of [9] has been inspired 1. by the critical view on the "innocence" of condition (B) in [12], and 2. by the discussion of the various forms of the projection-postulate of QT in part C of [15].

[15] G. Süssmann, Abh. Bayr. Akad. Wiss., Math.-Nat. Kl., Heft 88 [1958].

[16] For the notion of the *measurement of the first* (and *second*) *kind*, see: W. Pauli, Die allgemeinen Prinzipien der Wellenmechanik, in: Handbuch der Physik, ed. by S. Flügge, Vol. V/1, Berlin 1958. Because measurements of the first kind can be used to prepare eigenstates of the measured observable we will also call them *preparative* measurements.

[17] J. M. Jauch, Foundations of Quantum Mechanics, Reading (Mass.) 1968.

[18] A *partition of unity* (PU) is a set $\alpha = \{Q_i \mid i \in K_\alpha\}$ of projection operators of a Hilbert space with the properties

$$(\forall\, i, j \in K_\alpha)\; Q_i\, Q_j = \delta_{ij}\, Q_i \,, \sum_{i \in K_\alpha} Q_i = 1 \,.$$

In the following, we consider the sets α as ordered in an arbitrary way; and since EQT confines itself to separable Hilbert spaces, we can presume $K_\alpha = \{1, \ldots, N\}$ in case of $|\alpha| = N$, and $K_\alpha = \{1, 2, \ldots\} = \mathbb{N}$ in case of $|\alpha| = \infty$.

[19] Let A be an arbitrary self-adjoint operator with the purely discrete spectrum $\mathbf{a}$ and the spectral representation $A = \sum_{a \in \mathbf{a}} a\, P_a$. A PU $\alpha = \{Q_i \mid i \in K_\alpha\}$ is called *A-finer*, if

$$(\forall\, i \in K_\alpha)\; (\exists\, a \in \mathbf{a})\; Q_i \leqq P_a \,.$$

To each observable $\mathbf{A}$ with discrete range, we have accordingly in EQT two distinguished representations of the associated operator A: (1) the *spectral representation* $A = \sum_{a \in \mathbf{a}} a\, P_a$ and (2) the *separation-representation* $A = \sum_{i \in K_\alpha} \sigma_A(i)\, Q_i$; here σ_A is a surjective map of K_α onto the spectrum $\mathbf{a}$ with the property $Q_i \leqq P_a \rightleftharpoons \sigma_A(i) = a$.

[20] The expression "separation-character" (introduced in [15] for a more precise description of measurements of the first kind) might indicate that the new observable-concept is meaningful only for observables corresponding to measurements of the first kind. But this is not the case. With more general measurements, we also interpret the separation-character of an $\mathbf{A}$-instrument as the extent to which subspaces of the eigenspaces of A are 1-1-correlated to orthogonal instrument-states (even if this correlation can only partially be recorded by the reading mechanism). Loosely speaking, α characterizes the extent to which the $\mathbf{A}$-instrument measures more than it records.

b) By analogy with the new observable-concept, a new concept of macro-states (or quantum states) also emerges in EQT, since macro-states are produced by preparative measurements.

c) As a decisive consequence of the new observable-concept, a new *compatibility-relation* follows: Two EQT-observables $\mathbf{A} \leftrightarrow (A, \alpha)$ and $B \leftrightarrow (B, \beta)$ are *EQT-compatible* if and only if $\alpha = \beta$. Hence two observables are, intuitively speaking, compatible, if (in principle) they can be measured by the same measuring device. Only this extremely narrow compatibility concept makes it possible after all to regard EQT-observables as objective properties of physical systems.

d) From the new observable-concept one finally obtains a more general *projection-formula*, comprising the two extreme cases (1.1) and (1.2): An *ideal preparative measurement* of the observable $\mathbf{A} \leftrightarrow (A, \alpha) = (\sum_{a \in \mathbf{a}} a\, P_a,\ \{Q_i \mid i \in K_\alpha\})$ with the outcome $\mathbf{A} = b$ reduces the state operator W_0 to the operator

$$W_2(b) = [\mathrm{Tr}(W_0 P_0)]^{-1} \sum_{Q_i \leqq P_b} Q_i W_0 Q_i \,. \quad (1.3)$$

As was shown in [9], the new observable-concept makes possible the representation of all macro-states by Gibbsian ensembles of virtual systems whose micro-states are uniquely determined by assigning a value to each EQT-observable.

Starting from that result, in the present paper we analyse the possibility of a complete EQT in which also the temporal evolution is treated and related to an appropriate motion of the ensemble elements. According to the two kinds of temporal change existing in QT[21], we face two different problems: (1) Can the continuous and uniquely determined evolution

$$W(t) = U(t,t')\, W(t')\, U^{\dagger}(t,t') \quad (1.4)$$

of the state operator of a closed system (given by a continuous one-parameter group of unitary operators) be represented by a deterministic and continuous motion of the elements of the t'-ensemble? (2) Can the *reduction of the state operator* in the quantum theory of measurement be explained by a reduction of the corresponding ensemble?

To solve these problems we discuss in Sect. 2 all the physical ideas and hypotheses underlying the EQT. From these hypotheses, we then abstract an axiom system for the mathematical structure of EQT. In Sect. 3 and 4 we examine the compatibility of these axioms, arriving at the following results:

1. Every continuous evolution of a state operator in accordance with Eq. (1.4) can be represented by an invertible and continuous "phase-flow" of the ensemble elements in an appropriate state space.

2. On the other hand, it proves impossible within the scope of EQT to explain the reduction of a state operator by an ensemble-reduction.

2. The General Assumptions of EQT

In this section we shall compile all the postulates which a reformulation of QT must satify in order to realize in a consistent way the physical ideas of DSQ and, in particular, to show the features (a) to (c) of Sect. 1. To begin with, we repeat the general assumptions of DSQ as far as they make precise its physical concepts without anticipating their mathematical realisation. In contrast to DSQ, we presuppose however from the outset the new concepts of observables and macro-states[9] as sketched in the introduction.

(**Po 1**) To each physical system, a complex, separable Hilbert space $\mathscr{H}$ is associated and there exists a bijective map $\mathfrak{g}$ of the set of all proper[22] observables of the system onto the set of all pairs (A, α) composed of the operator A which QT also associates to this observable, and of an A-finer PUα. The map $\mathfrak{g}$ satisfies the ralation

$$\mathfrak{g}(A) = (A, \alpha) \Rightarrow \mathfrak{g}[f(A)] = (f(A), \alpha)$$

for all finite real functions f.

(**Po 2**) Every physical system is in a micro-state in which all observables have definite values[23]. The only possible values of an observable are the eigenvalues of the corresponding operator.

[21] J. v. Neumann, Mathematical Foundations of Quantum Mechanics, Princeton 1955.

[22] By a *proper* observable we understand an observable that can take on at least two different values. In addition we introduce (as *improper* observables) the "absurd" observable $\mathbf{0}$ and the family $c \cdot \mathbf{1}$ of "trivial" observables; we define that every improper observable is uniquely determined by the corresponding operator and is EQT-compatible with all observables.

[23] DSQ supposed in addition that the totality of all observable-values already determines uniquely the micro-state of a system. This means that, in DSQ, all parameters defining the micro-state of a system can be measured — though not simultaneously. We drop this assumption for the EQT as the generalisation (induced by it) does not influence our results.

(**Po 3**) Every observable permits an exact measurement that ascertains the value of the observable existing before (and independent of) the measurement; and at least some proper observables permit *measurements of the first kind* that ascertain the observable-value without changing it.

(**Po 4**) The micro-states of a physical system change deterministicly; in particular, the evolution of the micro-states of a closed system (closed in the sense of QT) is uniquely determined by the structure of the system.

(**Po 5**) Every physical system allows a correspondance between its micro-states and the points of an appropriate metric space Ω that makes Ω a *state-space*. With this correspondance, the temporal evolution of a closed system induces a continuous "phase-flow" in Ω.

(**Po 6**) The existence of incompatible observables limits the available knowledge about the micro-state of a system to the information contained in its macrostate. Accordingly, the EQT describes every macro-state by a *Gibbsian ensemble* of virtual systems in definite micro-states and derives the characteristic dispersions of the measurement results from the probability distribution of the ensemble elements.

(**Po 7**) There exists a bijective map $\mathfrak{h}$ of the set of all macro-states on the set of all pairs (W, ε) composed of the trace-operator[24] W, which QT also associates to this macro-state, and of a W-finer PU ε. The mean value of the observable $\mathbf{A} = \mathfrak{g}^{-1}(A,\alpha)$ in the macro-state $\mathbf{W} = \mathfrak{h}^{-1}(W, \varepsilon)$ is given by

$$\langle \mathbf{A} \rangle_{\mathbf{W}}^{\mathfrak{E}} = \mathrm{Tr}(A\,W)\,.$$

(**Po 8**) To every closed physical system a continuous one-parameter group $\{U_t \mid t \in \mathsf{R}\}$ of unitary operators is associated, governing the temporal evolution of the macro-state (W_t, ε_t) by the formulas[25]

$$W_t = U_t W_0 U_t^\dagger$$

$$\varepsilon_t := u_t(\varepsilon_0) := \mathfrak{d}[U_t](\varepsilon_0) = \{U_t Q_i U_t^\dagger \mid Q_i \in \varepsilon_0\}\,.$$

A nonclosed system with negligible reaction on his environs is characterized by a two-parameter family $\{U_{t,t'} \mid t, t' \in \mathsf{R}\}$ of unitary operators (depending on external conditions) which determines the evolution of the macro-state (W_t, ε_t) by the analogous equations

$$W_t = U_{t,t'}\, W_{t'}\, U_{t,t'}^\dagger\,, \qquad \varepsilon_t = u_{t,t'}(\varepsilon_{t'})\,.$$

(**Po 9**) While the evolution of an EQT-ensemble is uniquely determined, according to (Po 5), by the "phase-flow" of the ensemble elements, the temporal development of the corresponding macro-state is given by (Po 8). Hence the consistency of the EQT requires an exact coordination between both laws of evolution in order to make the equivalence between macro-states and ensembles time-independent.

These are the general postulates of DSQ, already modified through the introduction of the new observable-concept to make possible a consistent formulation of EQT.

In a previous paper[6] we have pointed out three decisive deficiencies of DSQ which must be avoided in EQT:

(1) To every macro-state, DSQ constructs a specific state space. This entails in particular that DSQ can not describe the temporal evolution of an ensemble as a continnous flow of the ensemble elements in one state space.

(2) The relation between the observable-values of a micro-state in DSQ differs critically from the observable-structure of physical systems in QT. Hence the ensemble elements of DSQ can not be interpreted as virtual copies of an original system.

(3) In the treatment of preparative measurements, DSQ arrives at quite a different result than QT. In the description of an ideal preparative measurement of the observable

$$\mathbf{A} = \mathfrak{g}^{-1}\big(\sum_{a \in \mathfrak{a}} a\,P_a,\ \{Q_i \mid i \in K_\alpha\}\big)$$

within QT, two phases can be distinguished[15]: (a) the proper interaction between object and instrument, which is represented by an unitary operator U_{M} in the Hilbert space of the composed system *object* & **A**-*instrument* and described by[26]

[24] A *trace-operator* is a positive, semi-definite, self-adjoint operator W with $\mathrm{Tr}(W) = 1$.

[25] The relation $u(\varepsilon) := \{U Q_i U^\dagger \mid Q_i \in \varepsilon\}$ associates to every unitary operator U a permutation $\mathfrak{d}(U) = u$ of Λ; the correspondance $\mathfrak{d}$ is a homorphism of the group of all unitary transformations of $\mathscr{H}$ into the permutation group of Λ.

[26] The superscripts ^{1,2,3} refer to the object, the instrument and the composed system respectively. Eq. (2.1) describes the 1-1-correlation, characteristic of the ideal measurement of the first kind, between object-states and instrument-states in the quantum state of the composed system at the end of the measurement; Eq. (2.2) imports that objects-eigenstates corresponding to different values of the measured observable produce orthogonal instrument-states. Both equations become more transparent if one consideres pure states in particular.

$$\overset{3}{W}_0 = \overset{1}{W}_0 \times \overset{2}{W}_0 \to \overset{3}{W}_1 = U_{\mathrm{M}} \overset{3}{W}_0 U_{\mathrm{M}}^{\dagger} = \sum_{i,j \in K_\alpha} Q_i \overset{1}{W}_0 Q_j \times \tilde{W}[i,j] \tag{2.1}$$

with

$$\overset{2}{\mathrm{Tr}}(\tilde{W}[i,j]) = \delta_{ij}, \tag{2.2}$$

and (b) the "pointer-reading" of the outcome $\mathbf{A} = b$ resulting in the reduction

$$\overset{3}{W}_1 \to \overset{3}{W}_2(b) = [\mathrm{Tr}(\overset{1}{W}_0 P_b)]^{-1} \sum_{Q_i \leqq P_b} Q_i \overset{1}{W}_0 Q_i \times \tilde{W}[i,i] \tag{2.3}$$

of the state operator $\overset{3}{W}_1$.

In the description of the same measuring process in the frame of DSQ (or EQT) [3,6] two analogous phases can be distinguished: (a) the interaction between object and instrument causes a continuous motion of the elements of the $\overset{3}{\mathbf{W}}_0$-ensemble and finally results in a $\overset{3}{\mathbf{W}}_1$-ensemble, in the virtual composed systems of which the value of the object-observable $\mathbf{A}$ is fixed in the instrument; (b) the "pointer-reading" of the result $\mathbf{A} = b$ at the original system induces a reduction of the $\overset{3}{\mathbf{W}}_1$-ensemble onto the subset of all ensemble elements with the property *value of* $\mathbf{A} = b$. In the frame of DSQ (or EQT), the subensemble resulting from this *ensemble-reduction* represents the macro-state of the original system after the measurement. For a quantitative agreement of DSQ (or EQT) and QT, it is therefore necessary and sufficient that the equivalence between ensemble and macro-state is preserved in both phases of the measuring process. But this is not the case in DSQ [6].

In order to exclude these deficiencies in EQT, one must impose, besides above postulates, some additional conditions on EQT. To avoid the first two defects, we introduced in [9] the following postulates:

(**Po 10**) The construction of the state space does not depend on the macro-state of the system.

(**Po 11**) The probability distribution of the elements of an EQT-ensemble is uniquely determined by the corresponding macro-state.

(**Po 12**) The observable-values of a micro-state satisfy the relation

$$\textit{value of } f(A) = f\,(\textit{value of } A)$$

for all finite real functions f.

For a correct description of the measuring process of the first kind in the frame of EQT, it is necessary that the reduction of the $\overset{3}{\mathbf{W}}_1$-ensemble onto the subset of all ensemble elements with the property *value of* $\mathbf{A} = b$ yields a subensemble which is equivalent to the reduced state operator $\overset{3}{W}_2(b)$ given by QT. This leads to the last postulate:

(**Po 13**) For all observables $\mathbf{C}$ of the composed system, the ensemble mean values of the reduced $\overset{3}{\mathbf{W}}_1$-ensemble coincide with the expectation values $\overset{3}{\mathrm{Tr}}(C\overset{3}{W}_2)$ of the reduced state operator $\overset{3}{W}_2$.

Herewith we have compiled all the intended demands on EQT. In order to analyse the purely mathematical problem of compatibility of these postulates, it is advisable to leave their physical meaning completely out of consideration. Accordingly we construct in the following a formal axiom system which determines only the mathematical structure of EQT as far as is implied by the above postulates.

***Axiom* (*1*)** To every complex separable Hilbert space $\mathscr{H}$, a measurable space $(\Omega, \mathscr{L})$ exists with the properties:

(1a) To each pair (A, α) composed of a self-adjoint operator A of $\mathscr{H}$ with discrete spectrum and of an A-finer PU α, a surjective $\mathscr{L}$-measurable map $A^\alpha : \Omega \mapsto \mathbf{a}$ of Ω onto the spectrum $\mathbf{a}$ of A can be associated.

(1b) The relation $f(A^\alpha) = f(A)^\alpha$ holds for all maps A^α and all finite real functions f.

(1c) To each pair (W, ε) composed of a trace-operator W and of a W-finer PU ε, one can associate a probability μ_W^ε on $(\Omega, \mathscr{L})$.

***Axiom* (2)** All pairs [27] (A, α) and (W, ε) from axiom (1) satisfy the relation

$$\int_\Omega A^\alpha \, \mathrm{d}\mu_W^\varepsilon = \mathrm{Tr}(AW).$$

***Axiom* (3)** (3a) There exists an injective homomorphism $\mathfrak{l}$ of the group of all unitary operators of $\mathscr{H}$ into the group of all permutations of Ω, which are $\mathscr{L}$-measurable in both directions.

(3b) All macro-states (W, ε) and all unitary operators U of $\mathscr{H}$ satisfy the relation [25]

$$\mu_{UWU^\dagger}^{u(\varepsilon)}(\,\cdot\,) = \mu_W^\varepsilon[\mathfrak{l}(U)^{-1}\,\cdot\,].$$

(3c.1) Let $\{U_t \mid t \in \mathsf{R}\}$ be an arbitrary one-parameter continuous group of unitary operators and let $\mathscr{F}$ be the Boolean algebra generated by all the sets of the form $\{x \in \Omega \mid A^\alpha(x) = a\}$; then the rela-

[27] According to its envisaged physical interpretation, we call the pairs (A, α) "observables" and the pairs (W, ε) "macro-states". These names have no physical meaning in the context of our axiom system, but are mere nominal definitions.

tion

$$\lim_{|t-t'|\to 0} \mu_W^{\varepsilon}(\mathfrak{l}(U_t)M \,\triangle\, \mathfrak{l}(U_{t'})M) = 0 \tag{2.4}$$

holds for all $M \in \mathscr{F}$ and all (W, ε).

(3c.2) Moreover a metric $\mathrm{d}(x,y)$ exists in Ω with the property

$$\lim_{|t-t'|\to 0} \mathrm{d}(\mathfrak{l}(U_t)\,x\,, \mathfrak{l}\,(U_{t'})\,x) = 0\,, \tag{2.5}$$

μ_W^{ε}-almost everywhere for all (W, ε).

***Axiom* (4)** To at least one observable $(A, \alpha) = (\sum_{a\in \mathbf{a}} a\, P_a, \{Q_i \mid i \in K_\alpha\})$ of $\mathscr{H}$ with $|\,\mathbf{a}\,| \geq 2$ there exists an observable $(\tilde{A}, \tilde{\alpha}) = (\sum_{a\in \mathbf{a}} a\, \tilde{P}_a, \tilde{\alpha})$ and a macro-state (X, ξ) of a second Hilbert space $\overset{2}{\mathscr{H}}$ together with an unitrary operator U_{M} in $\overset{3}{\mathscr{H}} := \mathscr{H} \otimes \overset{2}{\mathscr{H}}$ having the following properties:

(a) All macro-states (W, ε) of $\mathscr{H}$ satisfy the relation

$$V_1 := U_{\mathrm{M}}(W \times X)\, U_{\mathrm{M}}^{\dagger} = \sum_{i,j\in K_\alpha} (Q_i\, W\, Q_j) \times \tilde{W}[i,j]\,,$$

where the $\tilde{W}[i,j]$ are operators of $\overset{2}{\mathscr{H}}$ with the property:

$$\overset{2}{\mathrm{Tr}}\,(\tilde{P}_a\, \tilde{W}[i,j]) = \begin{cases} 1 \text{ if } i = j \text{ and } Q_i \leq P_a \\ 0 \text{ otherwise} \end{cases} .$$

(b) The macro-states[25,28] $(V_1, \zeta) = (U_{\mathrm{M}}\ (W \times X)\ U_{\mathrm{M}}^{\dagger},\ u_{\mathrm{M}}\,(\varepsilon \times \xi))$ of $\overset{3}{\mathscr{H}}$ satisfy the equation

$$\mu_{V_1}^{\zeta}\, \{x \in \overset{3}{\Omega} \mid (A \times 1)^{\alpha\times 1}(x) = (1 \times \tilde{A})^{1\times\tilde{\alpha}}(x)\} = 1\,.$$

(c) For all $b \in \mathbf{a}$ and all observables (C, γ) of $\overset{3}{\mathscr{H}}$, the relation

$$[\mu_{V_1}^{\zeta}\, \{x \mid (1 \times \tilde{A})^{1\times\tilde{\alpha}} = b\}]^{-1} \int_{\{x \mid (1\times\tilde{A})^{1\times\tilde{\alpha}} = b\}} C^{\gamma}\, \mathrm{d}\mu_{V_1}^{\zeta} = \overset{3}{\mathrm{Tr}}\,(C\, V_2(b)) \tag{2.6}$$

holds with $V_2(b) := [\overset{1}{\mathrm{Tr}}\,(W P_b)]^{-1} \sum_{Q_i \leq P_b} (Q_i\, W Q_i) \times \tilde{W}[i,i]\,.$

Axiom (4) compiles some restrictions which the existence of even one measurement of the first kind imposes on the mathematical structure of the EQT. In our formalism, $(\tilde{A}, \tilde{\alpha})$ denotes the distinguished instrument-observable (like a pointer setting or a digital read-out) the value of which *is* the result of the measurement, and (X, ξ) denotes the instrument-state at the start of the measurement. Axiom (4a) concerns the influence of the measurement interaction on the quantum state of the composed system; it requires a 1-1-correlation between the elements Q_i of the PU α of the observable (A, α) to measure and the possible instrument-states $\tilde{W}[i,i]$ at the end of the measurement, and it guarantees that every instrument-state $\tilde{W}[i,i]$ implies the corresponding value of $(\tilde{A}, \tilde{\alpha})$. Axiom (4b) concerns the influence of the interaction on the micro-states of the composed system and postulates that, in accordance with axiom (4a), the value of the measured observable becomes fixed in the instrument. Axiom (4c) concerns the reduction of the macro-state (V_1, ζ) and simply expresses (Po 13) in terms of an ensemble theory.

Obviously, these axioms are not uniquely determined by the above postulates; especially is axiom (3) stronger than necessary. But in the authors opinion, none of the changes in the axiom system which are admissible with regards to our postulates, has any considerable influence on the results of this paper.

In detail, the following relations hold between the four groups of axioms and the thirteen postulates:

axioms	postulates
(1a)	1, 2, 5 and 10
(1b)	1 and 12
(1c) & (2)	6, 7 and 11
(3a)	4 and 8
(3b)	9
(3c)	5
(4a)	3 and 8
(4b)	3
(4c)	13

Each model satisfying these four axioms can be interpreted as the mathematical formalism of a consistent EQT.

3. A Model Realizing the Axioms (1) to (3)

In the following we analyse the compatibility and independence of the axioms laid down in Sect. 2.

[28] If $\varepsilon_1 = \{Q_i \mid i \in K_{\varepsilon_1}\}$ and $\varepsilon_2 = \{R_j \mid j \in K_{\varepsilon_2}\}$ are PUs of $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively, $\{Q_i \times R_j \mid (i,j) \in K\varepsilon_1 \times K\varepsilon_2\}$ is a PU of $\mathscr{H}_1 \otimes \mathscr{H}_2$ which we denote by $\varepsilon_1 \times \varepsilon_2$.

To begin with, we construct a model satisfying the axioms (1), (2) and (3).

Definition: Two PU's $\alpha = \{R_i \mid i \in K_\alpha\}$, $\beta = \{S_j \mid j \in K_\beta\}$ are called equivalent ($\alpha \sim \beta$), if a unitary transformation U in $\mathscr{H}$ exists with the property

$$(\forall\, i \in K_\alpha)\,(\exists\, j \in K_\beta)\; U R_i U^\dagger = S_j .$$

Accordingly we restrain the so far arbitrary indexing of the elements of a PU by the condition

$$\alpha \sim \beta \rightleftharpoons (\exists\, U)\,(\forall\, i \in K_\alpha)\; U R_i U^\dagger = S_i .$$

The equivalence relation $\sim$ induces a partition of Λ in classes of equivalent PU's; we denote the class containing α by $\langle\alpha\rangle$, the set of all equivalence-classes by $\hat{\Lambda}$ and the unit sphere $\{x \in \mathscr{H} \mid \|x\| \leq 1\}$ by $\boldsymbol{H}$.

To every projection operator $Q_r \in \gamma$ we now associate a set

$$N_r^\gamma := \{x \in \boldsymbol{H} \mid \|x\|^4 \leq \langle x \mid Q_1 \mid x\rangle\} \quad \text{if} \quad r = 1 ,$$

$$N_r^\gamma := \{x \in \boldsymbol{H} \mid \sum_{i=1}^{r-1} \langle x \mid Q_i \mid x\rangle < \|x\|^4$$
$$\leq \sum_{i=1}^{r} \langle x \mid Q_i \mid x\rangle\} \quad \text{if} \quad \mathrm{r} > 1 , \tag{3.1}$$

and we introduce the Boolean algebra $\mathscr{F}_{\langle\alpha\rangle}$, and the σ-algebra $\mathscr{L}_{\langle\alpha\rangle} = {}^\sigma\mathscr{F}_{\langle\alpha\rangle}$, generated by all the sets N_r^γ with $r \in K_\gamma$, $\gamma \in \langle\alpha\rangle$.

Let P be an arbitrary elementary projection operator, L_P the eigenspace of P, $\mathfrak{f} : \mathsf{C} \mapsto \mathsf{R}^2$ the usual representation of C as the "complex plane" and m_L the two-dimensional Lebesgue measure. Then all the sets $\mathfrak{f}(N_r^\gamma \cap L_P)$ with $r \in K_\gamma$, $\gamma \in \langle\alpha\rangle$ are concentric circular rings in R^2 and accordingly the function

$$\varphi_P(M) = \frac{1}{\pi}\, m_L\, \{\mathfrak{f}(M \cap L_P)\} \tag{3.2}$$

is a probability on $(\boldsymbol{H}, \mathscr{L}_{\langle\alpha\rangle})$.

Next we choose at random a definite rule[29] which associates to every projection operator $R \neq 0$ a unique partition in elementary projection operators. According to axiom (1c), every macro-state (W, ε) with $\varepsilon = \{R_i \mid i \in K_\varepsilon\}$ has a unique separation-representation[19] $W = \sum_{i \in K_\varepsilon} \lambda_i R_i$ with $\lambda_i \geq 0$, $\sum_{i \in K_\varepsilon} \lambda_i \operatorname{Tr}(R_i) = 1$. Hence, if we apply our rule to all elements $R_i \in \varepsilon$, we obtain a unique ε-dependent representation of W in the form

$$W = \sum_{i \in K_\varepsilon} \sum_{j=1}^{\operatorname{Tr}(R_i)} w_{ij} P_{ij} =: \sum_{r=1}^{\dim \mathscr{H}} w_r P_r \tag{3.3}$$

with $\operatorname{Tr}(P_{ij}) = 1$, $\sum_{j=1}^{\operatorname{Tr}(R_i)} P_{ij} = R_i$, $w_{ij} = \lambda_i$, $\sum_{i \in K_\varepsilon} \sum_{j=1}^{\operatorname{Tr}(R_i)} w_{ij} = 1$. With the help of this representation we can associate to every macro-state (W, ε) a uniquely determined probability

$$\nu^\varepsilon_{W,\langle\alpha\rangle} := \sum_{r=1}^{\dim \mathscr{H}} w_r\, \varphi_{P_r} \tag{3.4}$$

on $(\boldsymbol{H}, \mathscr{L}_{\langle\alpha\rangle})$ and we have thus constructed a probability space $(\boldsymbol{H}, \mathscr{L}_{\langle\alpha\rangle}, \nu^\varepsilon_{W,\langle\alpha\rangle})$ to every macro-state (W, ε) and each element $\langle\alpha\rangle \in \hat{\Lambda}$.

The intended probability spaces can now be defined as

$$(\Omega, \mathscr{L}, \mu^\varepsilon_W) := \bigotimes_{i \in \hat{\Lambda}} (\boldsymbol{H}_i, \mathscr{L}_i, \nu^\varepsilon_{W,i})$$
$$= (\mathop{\mathrm{X}}_{i \in \hat{\Lambda}} \boldsymbol{H}_i, \bigotimes_{i \in \hat{\Lambda}} \mathscr{L}_i, \bigotimes_{i \in \hat{\Lambda}} \nu^\varepsilon_{W,i}) . \tag{3.5}$$

The set Ω consists of all maps $x : \hat{\Lambda} \mapsto \boldsymbol{H}$, the images of which (i.e. the i-th component of the "point" x) are denoted by x_i or $x(i)$. Obviously, the construct (3.5) satisfies axiom (1c).

Next, we associate to every projection operator $E_r \in \gamma$, $\gamma \in \Lambda$ the subset

$$M_r^\gamma := N_r^\gamma \times (\mathrm{X}\, \boldsymbol{H}_i)_{i \in \hat{\Lambda},\, i \neq \langle\gamma\rangle} \tag{3.6}$$

of Ω and define the corresponding characteristic function $E_r^\gamma := \operatorname{Ch}(M_r^\gamma)$. For an arbitrary observable $(A, \alpha) = (\sum_{a \in \mathbf{a}} a\, P_a, \{Q_i \mid i \in K_\alpha\})$, the separation-representation of A reads

$$A = \sum_{a \in \mathbf{a}} a \sum_{Q_i \leq P_a} Q_i = \sum_{i \in K_\alpha} \sigma_A(i)\, Q_i .$$

With the help of this representation we associate to every observable (A, α) an *observable-function*

$$A^\alpha := \sum_{i \in K_\alpha} \sigma_A(i)\, Q_i^\alpha . \tag{3.7}$$

Obviously, A^α is a surjective map of Ω onto the spectrum $\mathbf{a}$ of A and is uniquely determined by (A, α). All functions E_r^γ are $\mathscr{L}$-measurable by definition; hence all observable-functions are, because of $(\forall\, \gamma \in \Lambda)\; K_\gamma \subseteq \mathsf{N}$, also $\mathscr{L}$-measurable. Accordingly, the entities $(\Omega, \mathscr{L})$ and A^α satisfy axiom (1a).

[29] For an example of such a rule, see the last appendix.

By restricting its domain to $\boldsymbol{H}$, an unitary operator U of $\mathscr{H}$ induces an isometric permutation of $\boldsymbol{H}$, which we also denote by U. To every unitary operator U of $\mathscr{H}$ we now associate a transformation $\mathfrak{I}(U)$ in Ω by the definition

$$(\forall\, i \in \hat{\Lambda})\, [\mathfrak{I}(U)\, x](i) := U\, x(i)\,. \tag{3.8}$$

Evidently, $\mathfrak{I}(U)$ is a permutation of Ω for every U.

Finally we define the function

$$\mathrm{d}(x,y) := \sup_{i \in \hat{\Lambda}} \| x_i - y_i \| \tag{3.9}$$

on $\Omega \times \Omega$, $\| \ldots \|$ standing for the norm in $\mathscr{H}$. As one easily checks, d is a metric of Ω.

Theorem 1: The axioms (1) to (3) are compatible. In particular they are realized in the above model formed by Ω, $\mathscr{L}$, μ_W^ε, A^α, $\mathfrak{I}$ and d.[30]

Lemma 1: Each of the axioms (2), (3b) and (3c) is independent of the respective remainder of the axioms (1) to (3).[30]

Lemma 2: In the model constructed above, all macro-states, observables and unitary operators satisfy the relations[30]

(a) $A^\alpha(\mathfrak{I}(U)\, x) = [U^\dagger A\, U]^{u^{-1}(\alpha)}(x)$, and

(b) $\langle\alpha\rangle \neq \langle\beta\rangle \Rightarrow \mu_W^\varepsilon(M_r^\alpha \cap M_s^\beta) = \mu_W^\varepsilon(M_r^\alpha) \cdot \mu_W^\varepsilon(M_s^\beta)$.

From axiom (2), (3b) and lemma (2a) it follows:

$$\begin{aligned} \mathrm{Tr}(U W U^\dagger A) &= \int_\Omega A^\alpha \, \mathrm{d}\mu_{UWU^\dagger}^{u(\varepsilon)} \\ &= \int_\Omega A^\alpha(x)\, \mathrm{d}\mu_W^\varepsilon [\mathfrak{I}(U)^{-1} x] \\ &= \int_\Omega A^\alpha(\mathfrak{I}(U)\, x)\, \mathrm{d}\mu_W^\varepsilon \\ &= \int_\Omega [U^\dagger A U]^{u^{-1}(\alpha)}\, \mathrm{d}\mu_W^\varepsilon\,. \end{aligned} \tag{3.10}$$

Hence in our model a complete symmetry exists between the *Schrödinger-picture* and the *Heisenberg-picture*, in accord with QT.

Lemma (2b) signifies that two observable-functions A^α, B^β with $\langle\alpha\rangle \neq \langle\beta\rangle$ are independent random variables. But this fact prevents our model from satisfying axiom (4) even approximately: If we choose in axiom (4) $(C,\gamma) = (P_e \times 1, \gamma)$ with $e \in \mathbf{a}$, $e \neq b$, $\langle\gamma\rangle \neq \langle 1 \times \tilde{\alpha}\rangle$, the left side of Eq. (2.5) yields the expression

$$\begin{aligned} &[\mu_{V_1}^\zeta \{(1 \times \tilde{A})^{1\times\tilde{\alpha}} = b\}]^{-1} \int_{\{(1\times\tilde{A})^{1\times\tilde{\alpha}}=b\}} (P_e \times 1)^\gamma \, \mathrm{d}\mu_{V_1}^\zeta \\ &= \frac{\mu_{V_1}^\zeta(\{(1 \times \tilde{A})^{1\times\tilde{\alpha}} = b\} \cap \{(P_e \times 1)^\gamma = 1\})}{\mu_{V_1}^\zeta(\{(1 \times \tilde{A})^{1\times\tilde{\alpha}} = b\})} \\ &= \mu_{V_1}^\zeta(\{(P_e \times 1)^\gamma = 1\}) \\ &= \overset{3}{\mathrm{Tr}}\,(P_e \times 1 \cdot \sum_{i,j \in K_\alpha} Q_i W Q_j \times \tilde{W}[i,j]) \\ &= \sum_{i,j \in K_\alpha} \overset{1}{\mathrm{Tr}}(P_e Q_i W Q_j)\, \overset{2}{\mathrm{Tr}}(\tilde{W}[i,j]) \\ &= \sum_{Q_i \leqq P_e} \overset{1}{\mathrm{Tr}}(Q_i W Q_i) = \overset{1}{\mathrm{Tr}}(P_e W)\,, \end{aligned}$$

which can be made arbitrarily close to one by an appropriate choice of W. On the other hand, the right side of Eq. (2.5) yields

$$[\overset{1}{\mathrm{Tr}}(W P_b)]^{-1}\, \overset{2}{\mathrm{Tr}}(P_e \times 1 \cdot \sum_{Q_i \leqq P_b} Q_i W Q_i \times \tilde{W}[i,i]) = 0$$

for arbitrary W because of $P_e P_b = 0$.

The same result applies all the more to the model of the axioms (1) and (2) constructed in [9], where two observable-functions A^α, B^β are already independent if $\alpha \neq \beta$.

4. The Complete Axiom System

As we will see, the invalidity of axiom (4) is not caused by the special measures constructed in the models of [9] and Sect. 3; but axiom (4) is incompatible with the other axioms. To see this, we add both sides of the equation

$$\int_{\{(1\times\tilde{A})^{1\times\tilde{\alpha}}=b\}} C^\gamma \, \mathrm{d}\mu_{V_1}^\zeta = \mu_{V_1}^\zeta(\{(1 \times \tilde{A})^{1\times\tilde{\alpha}} = b\})\, \mathrm{Tr}(C V_2(b)) \tag{2.6*}$$

over all eigenvalues of A. The addition of the left sides yields

$$\sum_{a\in\mathbf{a}} \int_{\{(1\times\tilde{A})^{1\times\tilde{\alpha}}=a\}} C^\gamma \, \mathrm{d}\mu_{V_1}^\zeta = \int_{\bigcup_{a\in\mathbf{a}} \{(1\times\tilde{A})^{1\times\tilde{\alpha}}=a\}} C^\gamma \, \mathrm{d}\mu_{V_1}^\zeta = \int_{\overset{3}{\Omega}} C^\gamma \, \mathrm{d}\mu_{V_1}^\zeta = \mathrm{Tr}(C V_1) = \mathrm{Tr}\,[C \sum_{i,j\in K_\alpha} Q_i W Q_j \times \tilde{W}[i,j]]\,; \tag{4.1}$$

[30] The proofs are in the appendix.

[31] From axiom (1a) and (1b) results[9]

$$A = \sum_{a\in\mathbf{a}} a\, P_a \Rightarrow A^\alpha(x) = \sum_{a\in\mathbf{a}} a\, P_a^\alpha(x)\,,$$

and therefrom it follows

$$\{x \in \overset{3}{\Omega} \mid (1 \times \tilde{A})^{1\times\tilde{\alpha}}(x) = a)\} = \{x \in \overset{3}{\Omega} \mid (1 \times \tilde{P}_a)^{1\times\tilde{\alpha}}(x) = 1\}.$$

adding the right sides of Eq. (2.6)*, we get[31]

$$\sum_{a\in\mathbf{a}} \mu^{\xi}_{V_1}(\{1\times\tilde{A})^{1\times\tilde{\alpha}} = a\})\,\mathrm{Tr}(CV_2(a))$$

$$= \sum_{a\in\mathbf{a}} \overset{3}{\mathrm{Tr}}[(1\times\tilde{P}_a)\,V_1]\,\overset{3}{\mathrm{Tr}}(CV_2(a))$$

$$= \sum_{a\in\mathbf{a}} [\overset{1}{\mathrm{Tr}}(W P_a)]^{-1}\,\overset{3}{\mathrm{Tr}}[(1\times\tilde{P}_a)\sum_{i,j\in K_\alpha} Q_i W Q_j \times \tilde{W}[i,j]]\,\overset{3}{\mathrm{Tr}}\{C\sum_{Q_i\leq P_a} Q_i W Q_i \times \tilde{W}[i,i]\}$$

$$= \sum_{a\in\mathbf{a}} [\overset{1}{\mathrm{Tr}}(W P_a)]^{-1} \sum_{i,j\in K_\alpha} \overset{1}{\mathrm{Tr}}(Q_i W Q_j)\,\overset{2}{\mathrm{Tr}}(\tilde{P}_a \tilde{W}[i,j]) \sum_{Q_i\leq P_a} \overset{3}{\mathrm{Tr}}\{\mathrm{C}\cdot Q_i W Q_i \times \tilde{W}[i,i]\}$$

$$= \sum_{a\in\mathbf{a}} \sum_{Q_i\leq P_a} \overset{3}{\mathrm{Tr}}[C(Q_i W Q_i \times \tilde{W}[i,i])] = \sum_{i\in K_\alpha} \overset{3}{\mathrm{Tr}}[C(Q_i W Q_i \times \tilde{W}[i,i])]\,. \tag{4.2}$$

A comparison of Eqs. (4.1) and (4.2) finally gives the result

$$\sum_{a\in\mathbf{a}}\{\int_{\{(1\times\tilde{A})^{1\times\tilde{\alpha}}=a\}} C^\gamma\,\mathrm{d}\mu^{\xi}_{V_1} - \mu^{\xi}_{V_1}(\{(1\times A)^{\tilde{1}\times\alpha} \tilde{=} \mathrm{a}\})\,\mathrm{Tr}(CV_2(a))\} = \sum_{i\neq j\in K_\alpha} \overset{3}{\mathrm{Tr}}[C(Q_i W Q_j \times \tilde{W}[i,j])] \tag{4.3}$$

which is in general different from zero. So we arrive at

Theorem 2: The axioms (1), (2), (4a) and (4c) are incompatible.

Corrolary: The postulates (Po 1) to (Po 13) are incompatible and hence a consistent EQT is impossible.

The formal reason for this contradiction is, according to Eq. (4.3), due to the existence of the *interference-terms* in V_1. The incompatibility of axiom (4) with the remaining axioms can also be confirmed intuitively within the physical interpretation of the axiom system: As we have seen in a previous paper[32], every (W,ε)-ensemble in EQT has, in addition to the relation $f(A^\alpha) = f(A)^\alpha$ valid in all of Ω, a specific observable-structure in order to satisfy axiom (2). This ensemble-specific observable-structure results through the concentration of the measure μ^{ε}_{W} on a set of micro-states having this structure. Now, if in an ensemble a certain observable-relation is valid, this relation can obviously not be cancelled by the reduction of this ensemble to a subensemble; but exactly this contradiction is implied by axiom (4), as the following example shows: Let us consider a preparative measurement of the observable[33] $\mathbf{Q} = \mathfrak{g}^{-1}(Q,\alpha)$ (with $Q^2 = Q = Q^\dagger$ and $\alpha = \{R_i \mid i\in K_\alpha\}$) on an object in the macro-state $\mathbf{W} = \mathfrak{h}^{-1}(|\,\varphi\rangle\langle\varphi\,|,\varepsilon)$ with the result $\mathbf{Q} = 1$. Before and after the pointer-reading of the result $\mathbf{Q} = 1$, we have the state-operators

$$V_1 = \sum_{i,j\in K_\alpha} (R_i\,|\,\varphi\rangle\langle\varphi\,|\,R_j)\times\tilde{W}[i,j] \quad \text{and}$$

$$V_2(1) = (\langle\varphi\,|\,Q\,|\,\varphi\rangle)^{-1}\sum_{R_i\leq Q}(R_i\,|\,\varphi\rangle\langle\varphi\,|\,R_i)\times\tilde{W}[i,i].$$

For the observable $\mathbf{C} = \mathfrak{g}^{-1}(C,\gamma)$ with the properties

$$C = \sum_{k,l\in K_\alpha} R_k\,|\,\psi\rangle\langle\psi\,|\,R_l\times\tilde{W}[k,l]\,,\quad \langle\varphi\,|\,\psi\rangle = 0\,,\quad \langle\varphi\,|\,R_r\,|\,\psi\rangle \neq 0\,,\quad R_r\leq Q\,, \tag{4.4}$$

the axioms (2) and (4) yield the following ensemble mean values:

$$\int_\Omega C^\gamma\,\mathrm{d}\mu^{\xi}_{V_1} = \mathrm{Tr}(V_1 C) = 0\,,$$

$$\int_{\{(1\times\tilde{Q})^{1\times\tilde{\alpha}}=1\}} C^\gamma\,\mathrm{d}\mu^{\xi}_{V_1} = \mu^{\xi}_{V_1}(\{(1\times Q)^{1\times\tilde{\alpha}} = \tilde{1}\})\,\mathrm{Tr}[CV_2(1)]$$

$$= \sum_{R_i\leq Q} |\langle\varphi\,|\,R_i\,|\,\psi\rangle|^2\,\overset{2}{\mathrm{Tr}}(X^2) \geq |\langle\varphi\,|\,R_r\,|\,\psi\rangle|^2\,\overset{2}{\mathrm{Tr}}(X^2) > 0\,.$$

32 W. Ochs, Z. Naturforsch. **26a**, 204 [1971].

33 (Po2), (Po3) and (Po12) imply that all observables $f(\mathbf{A})$ can be measured by the same instruments as $\mathbf{A}$; hence (Po3) guarantees the existence of a 0-1-valued observable $\mathbf{Q}$ permitting a measurement of the first kind.

Whereas, according to axiom (2), the non-negative[34] function C^{γ} vanishes $\mu^{\zeta}_{V_1}$-almost everywhere in $\overset{8}{\Omega}$, it is claimed by axiom (4) that C^{γ} is greater than zero on a set of positive measure $\mu^{\zeta}_{V_1}$.

So we arrive at the following conclusion: The reduction of the state-operator, introduced in QT as an independent postulate ,can not be explained by an ensemble-reduction in an EQT; accordingly, the idea of SIEGEL and WIENER[3], to avoid the difficulties inherent in the quantum theory of measurement[35] by constructing an EQT, can not be realized. On the contrary: According to theorem 1, QT can be reformulated as an EQT only as far as QT can be considered to be without problems; and, significantly, the EQT breaks down exactly at the only point where it would be ,,superior'' to QT, if the four axioms were compatible.

To maintain EQT would mean to regard the ensemble-reduction as the genuine description of correcting the macro-state after "reading the scale". In this case, the projection-postulate of QT would be incorrect; but it could still serve as an elegant approximation of the complicated formula for the ensemble-reduction, if the interference-terms are negligible for all really occuring observables C .

In conclusion, we want to demonstrate that the assertion of theorem 2 remains valid also under much weaker assumptions:

a) The validity of theorem 2 does not depend on the special observable-concept of axiom (1), because the projection-postulate (1.3) (related to our new observable-concept) is not vital to the proof while the "parameters" γ and ζ do not even enter the proof of theorem 2 at all. Hence any other observable-concept complying with (Po 2) and (Po 12) (and adhering to the quantitative laws of QT) necessarily leads to the same result.

b) Actually, one does not even need a measurement of the first kind to prove theorem 2, but it suffices to presuppose at least one measurement of the second kind. (In this case, axiom (4b) has to be abolished completely and the axioms (4a) and (4c) must be weakened correspondingly by a more general expresison for $V_2(b)$[36].) Even then, theorem 2 remains valid since the existence of the interference-terms (vital to the proof) is not affected by this generalisation.

c) Finally, the validity of theorem 2 neither depends on the existence of infinitely many observables (of the composed system) nor on the existence of "indecomposable" observables [as that considered in Eq. (4.4)]. For even the restriction on a sufficiently dense, finite set of "product-observables" $(C, \overset{8}{\gamma}) = (\overset{1}{Y} \times \overset{2}{Z}, \overset{1}{\beta} \times \overset{2}{\delta})$ does not make the expression

$$\sum_{\substack{i \neq j \\ i,j \in K_\alpha}} \overset{3}{\mathrm{Tr}} \{ (Y \times Z) (Q_i W Q_j \times \tilde{W}[i,j])$$

$$= \sum_{i \neq j} \overset{1}{\mathrm{Tr}} (Y Q_i W Q_j) \overset{2}{\mathrm{Tr}} (Z \tilde{W}[i,j])$$

vanish in general.

5. Conclusions

In this paper we arrived at the following results:

(1) By an appropriate modification of the observable-concept, all quantum states can be represented by *Gibbsian ensembles* of virtual systems in dispersion-free micro-states which, having definite values for all EQT-observables, obey the (weaker) *quantum ordering* of EQT. Accordingly, these modified observables can be interpreted as objective properties of physical systems.

(2) Moreover, the continuous temporal evolution of the quantum states can be traced to a continuous and deterministic "phase-flow" of the ensemble elements in an appropriate state-space.

(3) On the other hand, every *Hidden-Variable Theory* (HVT) with property (1) arrives in the description of preparative measurements at results different from QT.

Hence a consistent EQT is impossible. This result extends previous "impossibility-theorems[21, 37, 38, 10-13] for HVTs in so far as it excludes HVTs not covered by these theorems[39]. This extension became

[34] According to axiom (4a), we have $C = U_M(|\,\psi\rangle\langle\psi\,| \times X)\, U_M^{\dagger}$; hence the spectra of C and $|\,\psi\rangle\langle\psi\,| \times X$ are equal. On account of the relation *spectrum* $(|\,\psi\rangle\langle\psi\,| \times X) \subseteq [0,1]$ and of axiom (1a), it follows $C\gamma(x) \geqq 0$.

[35] A. FINE, Phys. Rev. **D 2**, 2783 (1970).

[36] B. d'ESPAGNAT, Nuovo Cim. Suppl. **4**, 828 [1966].

[37] J. M. JAUCH and C. PIRON, Helv. Phys. Acta **36**, 827 [1963].

[38] S. GUDDER Proc. Am. Math. Soc. **19**, 319 [1968].

[39] In § 5 of [13], Kochen and Specker considered the general case (including EQT) that QT-observables, in a HVT, can split into several *new* observables, and they tried to prove that this case, too, is already excluded by (Po 12) (corresponding to their Eq. (1.4)). But as is shown by theorem 2 of [9] or theorem 1 of the present paper, this assertion is wrong.

possible only by taking also the theory of measurement into account which supplies additional restrictions on HVTs exceeding the conditions on the observable-structure of micro-states or on the mean values of ensembles of micro-states (exclusively considered in the above cited papers).

As the above results show, every HVT either has to differ quantitatively from QT or must abandon the idea that observables are object-properties.

6. Appendixes

Proof of Theorem 1

To proof theorem 1, one must show that the entities Ω, $\mathscr{L}$, A^α, μ_W^ε, $\mathfrak{l}$ and d defined in Sect. 3 satisfy the axioms (1), (2) and (3). In the construction of the probability spaces $(\Omega, \mathscr{L}, \mu_W^\varepsilon)$ and the observable-functions A^α, we established already that they realize the axioms (1a) and (1c).

In Eq. (3.7) we defined to each observable (A,α) an observable-function

$$A^\alpha = \sum_{i\in K_\alpha} \sigma_A(i)\, Q_i^\alpha \tag{6.1}$$

with the properties

$$\sum_{i\in K_\alpha} Q_i^\alpha(x) = 1\,, \qquad (\forall\, i \neq j \in K_\alpha)\; Q_i^\alpha Q_j^\alpha = 0\,. \tag{6.2}$$

Equations (6.1) and (6.2) yield

$$f(A^\alpha) = \sum_{i\in K_\alpha} f[\sigma_A(i)]\, Q_i^\alpha \tag{6.3}$$

for every finite real function f. On the other hand, the operator $f(A)$ has the spectral-representation $f(A) = \sum_{a\in \mathbf{a}} f(a)\, P_a$; consequently, the observable $(f(A),\alpha)$ has the separation-representation

$$f(A) = \sum_{i\in K_\alpha} \sigma_{f(A)}(i)\, Q_i = \sum_{i\in K_\alpha} f[\sigma_A(i)]\, Q_i \tag{6.4}$$

with the corresponding observable-function

$$f(A)^\alpha = \sum_{i\in K_\alpha} f[\sigma_A(i)]\, Q_i^\alpha\,. \tag{6.5}$$

The Eqs. (6.3) and (6.5) imply $f(A^\alpha) = f(A)^\alpha$, and our model satisfies axiom (1).

Let (A,α) be an observable with the separation-representation $A = \sum_{i\in K_\alpha} \sigma_A(i)\, Q_i$ and let (W,ε) be an arbitrary macro-state with the representation $W = \sum_{r=1}^{\dim\mathscr{H}} w_r\, P_r$, $(\forall\, r)\; \mathrm{Tr}(P_r) = 1$ specified in Eq. (3.3). Then from Eqs. (3.1) to (3.7) results

$$\int_\Omega A^\alpha\, \mathrm{d}\mu_W^\varepsilon = \sum_{i\in K_\alpha} \sigma_A(i) \int_\Omega Q_i^\alpha\, \mathrm{d}\mu_W^\varepsilon = \sum_{i\in K_\alpha} \sigma_A(i)\, \mu_W^\varepsilon(M_i^\alpha)\,, \tag{6.6}$$

$$\mu_W^\varepsilon(M_i^\alpha) = \nu_{W,\langle\alpha\rangle}^\varepsilon(N_i^\alpha) = \sum_{r=1}^{\dim\mathscr{H}} w_r\, \varphi_{P_r}(N_i^\alpha) \tag{6.7}$$

and[40]

$$\begin{aligned}
\varphi_{P_r}(N_i^\alpha) &= \frac{1}{\pi}\, m_L\left(\mathfrak{f}\left[L_{P_r} \cap \left\{x\in \boldsymbol{H} \,\Big|\, \sum_{k=1}^{i-1} \langle x\,|\,Q_k\,|\,x\rangle \leftarrow \|\, x\,\|^4 \leq \sum_{k=1}^{i} \langle x\,|\,Q_k\,|\,x\rangle\right\}\right]\right)\\
&= \frac{1}{\pi}\, m_L\left(\mathfrak{f}\left\{x\in L_{P_r} \,\Big|\, \sum_{k=1}^{i-1} \langle x\,|\,Q_k\,|\,x\rangle \leftarrow \|\, x\,\|^4 \leq \sum_{k=1}^{i} \langle x\,|\,Q_k\,|\,x\rangle\right\}\right)\\
&= \frac{1}{\pi}\, m_L\left(\left\{(x,y)\in \mathsf{R}^2 \,\Big|\, \sum_{k=1}^{i-1} \mathrm{Tr}(P_r Q_k) \leftarrow (x^2+y^2) \leq \sum_{k=1}^{i} \mathrm{Tr}(P_r Q_k)\right\}\right) = \mathrm{Tr}(P_r Q_i)\,.
\end{aligned} \tag{6.8}$$

The Eqs. (6.7) and (6.8) yield

$$\mu_W^\varepsilon(M_i^\alpha) = \sum_{r=1}^{\dim\mathscr{H}} w_r\, \mathrm{Tr}(P_r Q_i) = \mathrm{Tr}(W Q_i)$$

and with Eq. (6.6) we finally arrive at

$$\int_\Omega A^\alpha\, \mathrm{d}\mu_W^\varepsilon = \sum_{i\in K_\alpha} \sigma_A(i)\, \mathrm{Tr}(W Q_i) = \mathrm{Tr}(WA)$$

thus establishing axiom (2).

From the definition of the map $\mathfrak{l}(U)$ it follows immediately, that $\mathfrak{l}$ is an injective homomorphism of the group of all unitary operators of $\mathscr{H}$ into the permutation group of Ω. If $\alpha = \{F_r \,|\, r\in K_\alpha\}$ is an arbitrary PU, U an unitary operator and $\mathfrak{U} = \mathfrak{d}(U)$ the permutation of Λ induced by U, we get

[40] To allow for the fallunterscheidung in Eq. (3.1), we define the notation

$$a \leftarrow b \;\underset{\text{def}}{\Longleftarrow\!=}\; \begin{cases} a < b & a \neq 0\,,\\ & \text{if}\\ a \leqslant b & a = 0\,. \end{cases}$$

$$([UF_rU^\dagger]^{u(\alpha)})^{-1}(1) = M_r^{u(\alpha)} = N_r^{u(\alpha)} \times (\mathbf{X}\,\boldsymbol{H}_i)_{i\in\hat{\Lambda},\, i\neq\langle\alpha\rangle}\,,$$

$$\begin{aligned} N_r^{u(\alpha)} &= \left\{x\in\boldsymbol{H}\,\Big|\,\sum_{i=1}^{r-1}\langle x\,|\,U\,F_i\,U^\dagger\,|\,x\rangle \leftarrow \|\,x\,\|^4 \leq \sum_{i=1}^{r}\langle x\,|\,U\,F_i\,U^\dagger\,|\,x\rangle\right\} \\ &= \left\{x\in\boldsymbol{H}\,\Big|\,\sum_{i=1}^{r-1}\langle U^\dagger x\,|\,F_i\,|\,U^\dagger x\rangle \leftarrow \|\,U^\dagger x\,\|^4 \leq \sum_{i=1}^{r}\langle U^\dagger x\,|\,F_i\,|\,U^\dagger x\rangle\right\} \\ &= \left\{U\,y\in\boldsymbol{H}\,\Big|\,\sum_{i=1}^{r-1}\langle y\,|\,F_i\,|\,y\rangle \leftarrow \|\,y\,\|^4 \leq \sum_{i=1}^{r}\langle y\,|\,F_i\,|\,y\rangle\right\} = U\,N_r^\alpha\,. \end{aligned} \tag{6.9}$$

Equations (6.9) yield

$$\mathfrak{l}(U)^{-1}\,M_r^\alpha = M_r^{u^{-1}(\alpha)}\,, \qquad \mathfrak{l}(U)\,M_r^\alpha = M_r^{u(\alpha)}\,,$$
$$F_r^\alpha\,[\mathfrak{l}(U)\,x] = [U^\dagger F_r\,U]^{u^{-1}(\alpha)}\,(x)\,, \qquad F_r^\alpha\,[\mathfrak{l}(U)^{-1}\,x] = [U\,F_rU^\dagger]^{u(\alpha)}\,(x)\,. \tag{6.10}$$

By construction, the class of all sets M_r^α ($r\in K_\alpha$, $\alpha\in\Lambda$) generates the σ-algebra $\mathscr{L}$ and, according to Eq. (6.10), this generating system is invariant under all transformations $\mathfrak{l}(U)$. Hence every map $\mathfrak{l}(U)$ is $\mathscr{L}$-measurable, which establishes axiom (3a).

If $\alpha_r = \{Q_i^{(r)}\,|\,i\in K_\beta\}$, $r = 1,\ldots,n$, are n arbitrary different PU's with $\langle\alpha_1\rangle = \ldots = \langle\alpha_n\rangle$ and if P is an arbitrary elementary projection operator, we find

$$\begin{aligned} \varphi_{UPU^\dagger}\left(\bigcap_{i=1}^{n} N_{r_i}^{\alpha_i}\right) &= \frac{1}{\pi}\,m_L\left(\mathfrak{f}\left\{L_{UPU^\dagger}\cap\bigcap_{i=1}^{n} N_{r_i}^{\alpha_i}\right\}\right) \\ &= \frac{1}{\pi}\,m_L\left(\mathfrak{f}\left\{x\in L_{UPU^\dagger}\,\Big|\,\max_{i=1,\ldots,n}\left(\sum_{j=1}^{r_i-1}\langle x\,|\,Q_j^{(i)}\,|\,x\rangle\right)\leftarrow\|\,x\,\|^4\leq\min_{i=1,\ldots,n}\left(\sum_{j=1}^{r_i}\langle x\,|\,Q_j^{(i)}\,|\,x\rangle\right)\right\}\right) \\ &= \frac{1}{\pi}\,m_L\left(\left\{(x,y)\in\mathsf{R}^2\,\Big|\,\max_{i=1,\ldots,n}\left(\sum_{j=1}^{r_i-1}\mathrm{Tr}\,(U\,P\,U^\dagger\,Q_j^{(i)})\right)\leftarrow x^2+y^2\leq\min_{i=1,\ldots,n}\left(\sum_{j=1}^{r_i}\mathrm{Tr}\,U\,P\,U^\dagger\,Q_j^{(i)})\right)\right\}\right) \\ &= \frac{1}{\pi}\,m_L\left(\mathfrak{f}\left\{x\in L_P\,\Big|\,\max_{i=1,\ldots,n}\left(\sum_{j=1}^{r_i-1}\langle x\,|\,U^\dagger\,Q_j^{(i)}\,U\,|\,x\rangle\right)\leftarrow\|\,x\,\|^4\leq\min_{i=1,\ldots,n}\left(\sum_{j=1}^{r_i}\langle x\,|\,U^\dagger\,Q_j^{(i)}\,U\,|\,x\rangle\right)\right\}\right) \\ &= \frac{1}{\pi}\,m_L\left(\mathfrak{f}\left\{L_P\cap\bigcap_{i=1}^{n} N_{r_i}^{u^{-1}(\alpha_i)}\right\}\right) = \varphi_P\left(U^{-1}\left\{\bigcap_{i=1}^{n} N_{r_i}^{\alpha_i}\right\}\right). \end{aligned} \tag{6.11}$$

Hence from Eqs. (6.11) and (3.4) results

$$(\forall\,(W,\varepsilon))\,(\forall\,M\in\mathscr{F}_{\langle\alpha\rangle})\;\nu_{UWU^\dagger,\langle\alpha\rangle}^{u(\varepsilon)}\,(M) = \nu_{W,\langle\alpha\rangle}^{\varepsilon}\,(U^{-1}\,M)\,. \tag{6.12}$$

We consider next n different elements $\alpha_{ij}\in\Lambda$ with

$$i = 1,\ldots,e\,; \qquad j = 1,\ldots,m(i)\,; \qquad \sum_{i=1}^{e} m(i) = n\,; \qquad (\forall\,i)\,\langle\alpha_{i1}\rangle = \ldots = \langle\alpha_{im(i)}\rangle\,. \tag{6.13}$$

The Eqs. (6.12), (6.13) and (3.5) yield

$$\mu_{UWU^\dagger}^{u(\varepsilon)}\left(\bigcap_{i,j} M_{r_{ij}}^{\alpha_{ij}}\right) = \prod_{i=1}^{e}\nu_{UWU^\dagger\langle\alpha_{i1}\rangle}^{u(\varepsilon)}\left(\bigcap_{j=1}^{m(i)} N_{r_{ij}}^{\alpha_{ij}}\right) = \prod_{i=1}^{e}\nu_{W,\langle\alpha_{i1}\rangle}^{\varepsilon}\left[U^{-1}\bigcap_{j=1}^{m(i)} N_{r_{ij}}^{\alpha_{ij}}\right] = \mu_W^\varepsilon\left[\mathfrak{l}(U)^{-1}\,(\bigcap_{i,j} M_{r_{ij}}^{\alpha_{ij}})\right]$$

and hence we get

$$\mu_{UWU^\dagger}^{u(\varepsilon)}(M) = \mu_W^\varepsilon\,[\mathfrak{l}(U)^{-1}\,M] \tag{6.14}$$

for all M of the Boolean algebra $F = \bigotimes_{i\in\hat{\Lambda}}\mathscr{F}_i$. Now, both $\mu_{UWU^\dagger}^{u(\varepsilon)}$ and $\mu_W^\varepsilon[\mathfrak{l}(U)^{-1}\,\cdot]$ are measures on $\mathscr{L} = {}^\sigma\mathscr{F}$; and since a measure defined on an algebra $\mathscr{F}$ can be extended in at most one way to a measure on ${}^\sigma\mathscr{F}$, it follows from Eq. (6.14) that

$$\mu_{UWU^\dagger}^{u(\varepsilon)} = \mu_W^\varepsilon\,[\mathfrak{l}(U)^{-1}\,\cdot\,]\,,$$

establishing axiom (3b).

If U_1, U_2 are two unitray operators, P an elementary projection operator and F_r an element of the PU α, we find

$$\begin{aligned}\varphi_P(U_1 N_r^\alpha \triangle U_2 N_r^\alpha) &= \varphi_P(\overline{U_1 N_r^\alpha} \cap U_2 N_r^\alpha) + \varphi_P(U_1 N_r^\alpha \cap \overline{U_2 N_r^\alpha}) \\ &= \varphi_P(\overline{N_r^{u_1(\alpha)}} \cap N_r^{u_2(\alpha)}) + \varphi_P(N_r^{u_1(\alpha)} \cap \overline{N_r^{u_2(\alpha)}})\end{aligned} \tag{6.15}$$

and

$$\begin{aligned}\varphi_P(\overline{N_r^{u_1(\alpha)}} \cap N_r^{u_2(\alpha)}) &= \sum_{\substack{i \neq r \\ i \in K_\alpha}} \varphi_P(N_i^{u_1(\alpha)} \cap N_r^{u_2(\alpha)}), \\ &= \frac{1}{\pi} \sum_{i \neq r} m_L \left\{ (x,y) \in \mathsf{R}^2 \mid \max\left(\sum_{k=1}^{i-1} \mathrm{Tr}(U_1 F_k U_1^\dagger P), \quad \sum_{k=1}^{r-1} \mathrm{Tr}(U_2 F_k U_2^\dagger P) \right) \right. \\ &\left. \leftarrow (x^2+y^2) \leq \min\left(\sum_{k=1}^{i} \mathrm{Tr}(U_1 F U_1^\dagger P), \quad \sum_{k=1}^{r} \mathrm{Tr}(U_2 F U_2^\dagger P) \right) \right\}.\end{aligned} \tag{6.16}$$

Now, if $\{U_t \mid t \in \mathsf{R}\}$ is an arbitrary continuous one-parameter group of unitary operators, it follows immediately from Eqs. (6.15) and (6.16) that

$$\lim_{t \to t_0} \varphi_P(U_t N_r^\alpha \triangle U_{t_0} N_r^\alpha) = 0 \tag{6.17}$$

for all $t_0 \in \mathsf{R}$. Then the Eqs. (6.17), (3.4) and (3.5) yield the relation

$$\lim_{t \to t_0} \mu_W^\varepsilon(\mathfrak{l}(U_t) M_r^\alpha \triangle \mathfrak{l}(U)_{t_0} M_r^\alpha) = 0 \tag{6.18}$$

for all (W, ε) and all sets of the type M_r^α. From Eq. (6.18) and the identity

$$(M_1 \curlyvee M_2) \triangle (N_1 \curlyvee N_2) \subseteq (M_1 \triangle N_1) \cup (M_2 \triangle N_2)$$

we finally arrive at Eq. (2.3).

If U_1, U_2 are two arbitrary unitary operators of $\mathscr{H}$, it follows from Eqs. (3.8) and (3.9) that

$$\begin{aligned}\mathrm{d}(\mathfrak{l}(U_1)x, \mathfrak{l}(U_2)x) &= \sup_{i \in \hat{\Lambda}} \| U_1 x_i - U_2 x_i \| \\ &\leq \sup_{y \in \boldsymbol{H}} \| (U_1 - U_2) y \| = \| U_1 - U_2 \|_{\mathrm{op}},\end{aligned}$$

which proofs Eq. (2.4) for all $x \in \Omega$.

Proof of Lemma 1

To proof this lemma, we have to construct a model for each of the following three groups of properties:

(I) axiom (1) $\wedge$ $\neg$ axiom (2) $\wedge$ axiom (3)

(II) axiom (1) $\wedge$ axiom (2) $\wedge$ axiom (3a) $\wedge$ axiom (3b) $\wedge$ $\neg$ axiom (3c)

(III) axiom (1) $\wedge$ axiom (2) $\wedge$ axiom (3a) $\wedge$ $\neg$ axiom (3b) $\wedge$ axoim (3c).

ad (I): We choose an arbitrary unitary operator $V \neq 1$ and associate, in contrast to the model of theorem 1, to every macro-state (W, ε) a new measure $\chi_W^\varepsilon := \mu_{VWV^\dagger}^{v(\varepsilon)}$. If we let all the other definitions unchanged, the entities Ω, $\mathscr{L}$, A^α, χ_W^ε, $\mathfrak{l}$, d obviously form a model with the properties (I).

ad (II): One obtains a model with the properties (II), if the metric d of our original model is replaced by the trivial metric

$$\tau(x,y) := \begin{cases} 0 & x = y \\ & \text{if} \\ 1 & x \neq y \end{cases}.$$

ad (III): To realize the properties (III), we introduce new permutations $\tilde{\mathfrak{l}}(U)$ of Ω, defining

$$\begin{aligned}(\forall i \in \hat{\Lambda}, i \neq \langle\gamma\rangle)\ [\tilde{\mathfrak{l}}(U)x](i) &= [\mathfrak{l}(U)x](i) = Ux(i) \\ [\tilde{\mathfrak{l}}(U)x](\langle\gamma\rangle) &= x(\langle\gamma\rangle).\end{aligned} \tag{6.19}$$

With $E_r \in \gamma$, the Eqs. (3.6) and (6.19) yield

$$\begin{aligned}\mu_W^\varepsilon(\mathfrak{l}(U)^{-1} M_r^\gamma) &= \mu_W^\varepsilon(M_r^\gamma) = \mathrm{Tr}(W E_r), \\ \mu_{UWU^\dagger}^{u(\varepsilon)}(M_r^\gamma) &= \mathrm{Tr}(U W U^\dagger E_r),\end{aligned}$$

thus cancelling axiom (3b). Since the change from $\mathfrak{l}$ to $\tilde{\mathfrak{l}}$ does not affect any other relation, we so obtain a model with the properties (III).

Proof of Lemma 2

Lemma (2a) results from the Eqs. (3.7) and (6.10). Lemma (2b) follows immediately from definition (3.5).

Rule referred to in Footnote [22]

Such a rule can be constructed e.g. with the help of a certain basis $a_1, a_2, \ldots$ of $\mathscr{H}$: If L_R denotes the eigenspace of the projection operator $R \neq 0$, the sets

$$\begin{aligned}N_0 &= \{x \in L_R \mid \| x \| = 1\}, \\ N_{2i-1} &= \{x \in N_{2i-2} \mid \mathrm{Re}\langle x \mid a_i \rangle = \sup_{y \in N_{2i-2}} \mathrm{Re}\langle y \mid a_i \rangle\},\end{aligned}$$

$$N_{2i} = \{x \in N_{2i-1} \mid \operatorname{Im} \langle x \mid a_i \rangle = \sup_{y \in N_{2i-1}} \operatorname{Im} \langle y \mid a_i \rangle\}$$

form (for $i = 1, 2, \ldots$) a monotonically decreasing sequence of sets, whose intersection consists of exactly one unit element $r_1 \in L_R$. The repetition of this procedure applied to the projection operators

$$R^{(1)} := R - |r_1\rangle\langle r_1|, \ldots, \; R^{(n)} := R^{(n-1)} - |r_n\rangle\langle r_n|$$

obviously yields a unique partition of R in elementary projection operators $|r_i\rangle\langle r_i|$.

Acknowledgements

The author would like to thank Prof. G. SÜSSMANN and Prof. H. DINGES for many helpful discussions.

Modelle kraftfreier Magnetfelder

R. WAGNER *

Lehrstuhl für Theoretische Physik B Technische Universität Braunschweig

(Z. Naturforsch. **26 a**, 1753—1762 [1971]; eingegangen am 1. April 1969)

A method for computing force-free magnetic fields of known anomality $\alpha \neq$ const is described. The procedure is introduced by treating plane fields; thereby it is proved that force-free fields of constant strength are alyaws plane. In general case the surfaces α=const are chosen as coordinate surface of an orthogonal curvilinear coordinate system. In this system the magnetic field is described by a linear partial differential equation which can be solved numerically. Making use of simplifying assumptions about symmetries in the coordinate systems, analytic solutions are found which are extended on constant α.

The formulae derived can be used to decide if to a given geometry a force-free field does exist. Existing fields can be computed immediately. The results are illustrated by examples.

Im Jahre 1951 wies LUNDQUIST [1] erstmals darauf hin, daß die magnetohydrostatischen Gleichungen die Existenz von Magnetfeldern zulassen, in denen der sie erzeugende Strom in Magnetfeldrichtung fließt. In diesen Feldern verschwindet die Lorentz-Kraft, weswegen man von „kraftfreien Magnetfeldern" spricht. Ist $\mathfrak{j}$ die Stromdichte, $\mathfrak{h}$ die magnetische Feldstärke, so gilt also:

$$\mathfrak{j} \times \mathfrak{h} = 0 \tag{1}$$

bzw. wegen $\operatorname{rot} \mathfrak{h} = \mathfrak{j}$,

$$\mathfrak{j} = \operatorname{rot} \mathfrak{h} = \alpha \, \mathfrak{h} . \tag{2}$$

Dabei ist α ein ortsabhängiger Proportionalitätsfaktor. (Behandelt man dynamische Probleme, so ist α auch zeitabhängig. Im folgenden wollen wir uns jedoch auf den statischen Fall beschränken, in dem $\mathfrak{j}$, $\mathfrak{h}$ und daher auch α zeitunabhängig sind.)

Wegen
$$\operatorname{div} \mathfrak{h} = 0 \tag{3}$$
folgt aus (2):
$$(\mathfrak{h}, \operatorname{grad} \alpha) = 0 . \tag{4}$$

Eine besondere Lösungsklasse von (2), (3) sind die stromlosen Magnetfelder. Für sie ist $\alpha \equiv 0$, und man spricht von einem trivialen kraftfreien Magnetfeld. Lösungen für $\alpha \not\equiv 0$ sind in speziellen Koordinatensystemen, d. h. unter Annahme spezieller Symmetrien, seit längerem bekannt [2–9]. Der Fall α = const – eine Lösung von (2) heißt dann ein TRKAL-Feld – wurde von CHANDRASEKHAR und KENDALL [10] 1957 vollständig gelöst. TRKAL-Felder zeichnen sich dadurch aus, daß sie immer divergenzfrei sind, so daß Gl. (3) keine zusätzliche Bedingung liefert. Außerdem sind sie die einzigen kraftfreien Felder, für die $\mathfrak{j} \times \operatorname{rot} \mathfrak{j} = 0$ gilt. Denn einmal sieht man sofort, daß in TRKAL-Feldern diese Bedingung gilt. Ist andererseits die Stromdichte zu ihrer Rotation parallel, so folgt mit (4) wegen

$$\operatorname{rot} \mathfrak{j} = \operatorname{rot} \alpha \, \mathfrak{h} = \alpha \, \mathfrak{j} + \operatorname{grad} \alpha \times \mathfrak{h} ,$$

daß $\operatorname{grad} \alpha \equiv 0$, also α = const gelten muß. In analoger Weise sieht man: Ist $\mathfrak{h}$ ein kraftfreies Magnet-

* Jetzige Adresse: Porz-Lind, Im Linder Bruch 68.

1 S. LUNDQUIST, Ark. Fysik **2**, 361 [1951].
2 R. LÜST u. A. SCHLÜTER, Z. Astrophys. **31**, 263 [1954].
3 A. SCHLÜTER, Z. Naturforsch. **12 a**, 855 [1957].
4 S. CHANDRASEKHAR, Proc. Nat. Acad. Sci. USA **42**, 1 [1956].
5 S. K. MAJUMDAR, Z. Astrophys. **47**, 44 [1959].
6 G. J. BUCK, J. Appl. Phys. **36**, 2231 [1965].
7 R. N. HENRIKSEN, Ap. Letters **1**, 37 [1967].
8 F. G. FREIRE, Amer. J. Phys. **34**, 567 [1962].
9 G. S. MURTY, Ark. Fysik **21**, 203 [1962].
10 S. CHANDRASEKHAR u. P. C. KENDALL, Ap. J. **126**, 457 [1957].