

Projections of Tropical Varieties and an Application to Small Tropical Bases

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Abstract

The main topic of this thesis is the description of projections of tropical varieties and the construction of tropical bases by means of those projections. We present a tropical version of the Eisenbud-Evans theorem, use so-called regular projections and combine them with elimination theory. As an application of mixed fiber polytopes we obtain a description of the image of a tropical variety. For tropical curves we deduce some bounds on the complexity of their images.

Tropical geometry is a relatively new area which has its origin in the early seventies in the work of Bergman [Ber71] and in the middle eighties in the work of Bieri and Groves [BG84]. Given an extension field K of a valuated field k Bieri and Groves define for a finite set $\{a_1, \dots, a_n\} \subset K$ the Bieri-Groves set $\Delta_K^v(a_1, \dots, a_n)$ as the set of all vectors $(w(a_1), \dots, w(a_n)) \in \mathbb{R}^n$ where w runs through all valuations of K extending the valuation v of k . If v is the trivial valuation this is just the logarithmic limit set of Bergman, [Ber71]. Today tropical geometry is the geometry of the tropical semiring $(\mathbb{R}_\infty, \min, +)$, $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$, and the Bieri-Groves set $\Delta_K^v(a_1, \dots, a_n)$ is related to a tropical variety of a prime ideal $P \triangleleft k[x_1, \dots, x_n]$ as follows. First note that the tropical variety $\mathcal{T}(I)$ of an ideal $I \triangleleft k[x_1, \dots, x_n]$ is the set of all points w such that the minimum $\min\{v(c_\alpha) + w \cdot \alpha\}$ is attained at least twice for each $f = \sum_\alpha c_\alpha x^\alpha \in I$. Then for a prime ideal P it holds:

$$\Delta_K^v(x_1, \dots, x_n) = \mathcal{T}(P),$$

where K is the quotient field of $k[x_1, \dots, x_n]/P$, see [EKL06]. We remark that Bieri-Groves sets are closely related to the Bieri-Strebel invariant Σ_A which is defined in terms of finiteness properties of the $\mathbb{Z}[x_1, \dots, x_n]$ -modul A , see [BS80, BS81]. For instance when $A = \mathbb{Z}[x_1, \dots, x_n]/P$, where P is a prime ideal, and $K = \text{Quot}(A)$ is the quotient field then the complement Σ_A^c can be computed using

$$\Sigma_A^c = \bigcup_{v: \mathbb{Q} \rightarrow \mathbb{R}_\infty} [\Delta_K^v(x_1, \dots, x_n)],$$

where $[\Delta]$ is the central projection of a subset $\Delta \subset \mathbb{R}^n$ to the unit sphere \mathbb{S}^{n-1} .

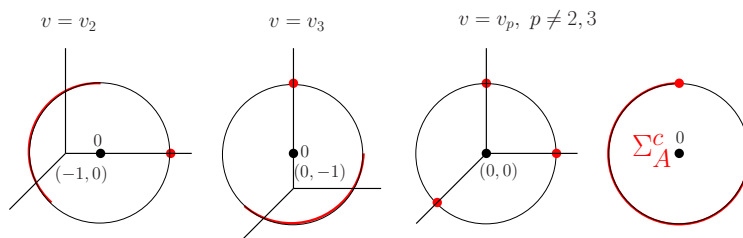


FIGURE 1. The Bieri-Strebel invariant

For example if $P = \langle 2x + 3y + 1 \rangle \triangleleft \mathbb{Q}[x, y]$ then we have the union of the projections of $\Delta_K^{v_p}(x, y)$ over all p -adic valuations v_p of \mathbb{Q} and we get the set $\Sigma_A = \{(s_1, s_2) \in \mathbb{S}^1 \mid s_1 > 0, s_2 > 0\}$, see Figure 1.

So tropical geometry relates algebraic geometric problems with discrete geometric problems. There are many examples of such correspondences. A main result is for example due to Mikhalkin, see [Mik06], who counts the number of plane curves of given degree and genus through a given number of points. This can be done classically or tropically and gives us the Gromov-Witten invariants. Another example of the correspondence is due to Katz, T. Markwig and H. Markwig. They compare the classical j -invariant with its tropical counterpart, see [KMM08].

In this thesis we obtain a tropical version of the Eisenbud-Evans Theorem which states that every algebraic variety in \mathbb{R}^n is the intersection of n hypersurfaces, see [EE73]. We find out that in the tropical setting every tropical variety $\mathcal{T}(I)$ can be written as an intersection of only $(n + 1)$ tropical hypersurfaces. So we get a finite generating system of I such that the corresponding tropical hypersurfaces intersect to the tropical variety, a so-called tropical basis.

THEOREM 0.1. *Let $I \triangleleft K[x_1, \dots, x_n]$ be a prime ideal generated by the polynomials f_1, \dots, f_r . Then there exist $g_0, \dots, g_n \in I$ such that*

$$(1) \quad \mathcal{T}(I) = \bigcap_{i=0}^n \mathcal{T}(g_i)$$

and thus $\mathcal{G} := \{f_1, \dots, f_r, g_0, \dots, g_n\}$ is a tropical basis for I of cardinality $r + n + 1$.

Tropical bases are discussed by Bogart, Jensen, Speyer, Sturmfels and Thomas in [BJS⁺07] where it is shown that tropical bases of linear polynomials of a linear ideal have to be very large. We do not restrict the tropical basis to consist of linear polynomials and therefore we get a shorter tropical basis. But the degrees of our polynomials can be very large. The main ingredient to get a short tropical basis is the use of projections, in particular geometrically regular projections, see [BG84]. Together with the fact that preimages of projections of tropical varieties are themselves tropical varieties of a certain elimination ideal - in fact, they are tropical hypersurfaces - we get the desired result.

THEOREM 0.2. *Let $I \triangleleft K[x_1, \dots, x_n]$ be an m -dimensional prime ideal and $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$ be a rational projection. Then $\pi^{-1}(\pi(\mathcal{T}(I)))$ is a tropical variety, namely*

$$(2) \quad \pi^{-1}(\pi(\mathcal{T}(I))) = \mathcal{T}(J \cap K[x_1, \dots, x_n]).$$

Here J is the ideal in $K[x_1, \dots, x_n, \lambda_1, \dots, \lambda_{n-m-1}]$ derived from the ideal I by

$$J = \langle \tilde{f} \in R : \tilde{f} = f(x_1 \prod_{j=1}^{n-m-1} \lambda_j^{u_1^{(j)}}, \dots, x_n \prod_{j=1}^{n-m-1} \lambda_j^{u_n^{(j)}}) \text{ for some } f \in I \rangle.$$

where $u^{(1)}, \dots, u^{(l)} \in \mathbb{Z}^n$ generate the kernel of π . We show that this elimination ideal is a principal ideal which yields a polynomial in our tropical basis.

A nice side effect is the lifting of points in the tropical variety of an elimination ideal to points of the tropical variety of the original ideal:

THEOREM 0.3 (Tropical Extension Theorem). *Let $I \triangleleft K[x_0, \dots, x_n]$ be an ideal and $I_1 = I \cap K[x_1, \dots, x_n]$ be its first elimination ideal. For any $w \in \mathcal{T}(I_1)$ there exists a point $\tilde{w} = (w_0, \dots, w_n) \in \mathbb{R}^{n+1}$ with $w_i = \tilde{w}_i$ for $1 \leq i \leq n$ and $\tilde{w} \in \mathcal{T}(I)$.*

The advantage of our method is that we find our polynomials by projections and therefore we can use the results of Gelfand, Kapranov and Zelevinsky [**GKZ90**, **GKZ08**], of Esterov and Khovanskii [**Est08**, **EK08**], and of Sturmfels, Tevelev and Yu [**ST08**, **STY07**, **SY08**]. These results involve mixed fiber polytopes of a projection $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$ which are a generalisation of fiber polytopes. Fiber polytopes are Minkowski sums of certain fibers of the projection of a polytope onto its image. With mixed fiber polytopes we get the structure and combinatorics of the image of a tropical variety and therefore the structure of the polynomials in our tropical basis. Here we use the fact that every cell in a transversal tropical variety $\mathcal{T}(f_1) \cap \dots \cap \mathcal{T}(f_k)$ is dual to a cell of a mixed subdivision of the newton polytope $\text{New}(f_1 \cdot \dots \cdot f_k)$.

THEOREM 0.4. *Let $I = \triangleleft K[x_1, \dots, x_n]$ an m -dimensional ideal, generated by generic polynomials f_1, \dots, f_{n-m} , $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$ a projection and ψ a projection presented by a matrix with a rowspace equal to the kernel of π . Then up to affine isomorphisms, the cells of the dual subdivision of $\pi^{-1}\pi\mathcal{T}(I)$ are of the form*

$$\sum_{i=1}^p \Sigma_{\psi}(C_{i1}^{\vee}, \dots, C_{ik}^{\vee}) \text{ for some } p \in \mathbb{N}.$$

Here $k = n - m$ and F_1, \dots, F_p are faces of $\mathcal{T}(f_1) \cap \dots \cap \mathcal{T}(f_k)$ and the dual cell of $F_i \subseteq U = \mathcal{T}(f_1) \cup \dots \cup \mathcal{T}(f_k)$ is given by $F_i^{\vee} = C_{i1}^{\vee} + \dots + C_{ik}^{\vee}$ with faces C_{i1}, \dots, C_{ik} of $\mathcal{T}(f_1), \dots, \mathcal{T}(f_k)$.

The dual cells $C_{i1}^{\vee} + \dots + C_{ik}^{\vee}$ are all mixed cells of the induced subdivision of $\text{New}(f_1) + \dots + \text{New}(f_k)$, i.e. $\dim(C_{ij}) \geq 1$. So for a geometrically regular projection we obtain the cells with $p = 1$ as the mixed fiber polytopes of the fulldimensional mixed cells of the subdivided Newton polytope $\text{New}(f_1 \cdot \dots \cdot f_k)$.

In case that we project regularly a tropical curve, i.e. a 1-dimensional tropical variety, we want to find the number of $(n-1)$ -cells of the above form with $p > 1$, i.e. the cells which are dual to vertices of $\pi(\mathcal{T}(I))$ which are the intersection of the images of two non-adjacent 1-cells of $\mathcal{T}(I)$. Vertices of this type are called selfintersection points. We derive bounds on their number:

THEOREM 0.5. *As a lower bound for the number of selfintersection points of tropical curves in \mathbb{R}^n , $n \geq 3$, we get:*

- (1) *There exist a tropical line $L_n \subset \mathbb{R}^n$ and a projection $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^2$ such that L_n has*

$$\sum_{i=1}^{n-2} i = \binom{n-1}{2}$$

selfintersection points.

- (2) *There exist a tropical curve $\mathcal{C} \subset \mathbb{R}^n$ which is a transversal intersection of $n-1$ tropical hypersurfaces of degrees d_1, \dots, d_{n-1} and a projection $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^2$ such that \mathcal{C} has at least*

$$(d_1 \cdot \dots \cdot d_{n-1})^2 \cdot \binom{n-1}{2}$$

selfintersection points.

A caterpillar is a certain simple type of a tropical line and for this type we get:

THEOREM 0.6. *As an upper bound we get:*

The image of a tropical line L_n in \mathbb{R}^n which is a caterpillar can have at most

$$\sum_{i=1}^{n-2} i = \binom{n-1}{2}$$

selfintersection points.

The tropical line constructed in the proof of Theorem 0.5 (1) is also a caterpillar and so the upper bound of Theorem 0.6 is tight.

For a general tropical curve the number of selfintersection points can be bounded using Bernstein's Theorem. Let MV_Λ denote a relative mixed volume. Then we obtain the following upper bound.

THEOREM 0.7. *Let $\pi(\mathcal{T}(I)) = \mathcal{T}(f)$ be the image of a tropical curve. Then the number of selfintersection points of $\pi(\mathcal{T}(I))$ is bounded above by*

$$\min \left(\text{vol}(\text{New}(f)), MV_\Lambda \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right) \right).$$

Thesis Overview. This thesis is structured as follows. Chapter 1 provides the fundamental concept of tropical geometry. Tropical polynomials and tropical hypersurfaces are defined. Then we describe the duality of the tropical hypersurfaces and the subdivided Newton polytopes of the corresponding tropical polynomials. Finally mixed cells of a union of tropical hypersurfaces are defined.

Chapter 2 starts from the classical algebraic viewpoint and describes the connection to the tropical viewpoint via valuations of fields, the so-called tropicalization. Then we define tropical varieties and tropical bases and explain some properties of tropical varieties like the concavity condition. Finally we present a few known results about linear tropical bases.

Chapter 3 introduces projections and describes the projection techniques of Bieri and Groves. Then the main results of this thesis, namely that preimages of projections of tropical varieties of prime ideals are themselves tropical hypersurfaces (Theorem 0.2) and that short tropical bases exist and are constructible (Theorem 0.1), are proved. We also calculate an example to illustrate the algorithm. Then we apply the results of Gelfand, Kapranov and Zelevinsky to the linear case which describe the polynomials in the short tropical basis.

Chapter 4 is a review of the concepts of fiber polytopes, secondary polytopes and mixed fiber polytopes.

Chapter 5 describes the Newton polytopes of the polynomial corresponding to the projection of a tropical variety. First some properties of tropical varieties are introduced. These include transversal intersections as well as proper and complete intersections. Newton-nondegeneracy of a system of polynomials is defined. The lifting of this properties from a prime ideal I to the ideal J used in Chapter 3 is explained. So under some assumptions this leads to a method to analyse the Newton polytopes and their subdivisions.

Chapter 6 studies the images of tropical curves. In particular the existence and number of new vertices of the projection, so-called selfintersection points, are treated. We give lower and upper bounds for their number.

Published contents. Some results of this thesis are published in the article [HT09] and in the extended conference abstract, which can be found in [tro07]. The present work contains these results in Chapter 3 but is extended with further results and examples.

Zusammenfassung

Das Hauptthema dieser Arbeit ist die Beschreibung von Projektionen von tropischen Varietäten und die Konstruktion tropischer Basen mithilfe von Projektionen. Wir zeigen eine tropische Version des Eisenbud-Evans-Theorems, benutzen sogenannte reguläre Projektionen und kombinieren diese mit Eliminationstheorie. Wendet man gemischte Faserpolytope an, bekommt man eine Beschreibung des Bildes einer tropischen Varietät. Für tropische Kurven entwickeln wir Schranken für die Komplexität des Bildes.

Tropische Geometrie ist ein relativ junges Gebiet, welches seinen Ursprung Anfang der siebziger Jahre in den Arbeiten von Bergman [Ber71] und Mitte der Achtziger in den Arbeiten von Bieri and Groves [BG84] hat. Für eine Körpererweiterung K eines bewerteten Körpers k definieren Bieri und Groves für eine endliche Menge $\{a_1, \dots, a_n\} \subset K$ die Bieri-Groves-Menge $\Delta_K^v(a_1, \dots, a_n)$ als die Menge aller Vektoren $(w(a_1), \dots, w(a_n)) \in \mathbb{R}^n$, wobei w alle Bewertungen von K , die die Bewertung v von k erweitern, durchläuft. Wenn v die triviale Bewertung ist, erhält man das „logarithmic limit set“ von Bergman, [Ber71]. Heutzutage ist tropische Geometrie die Geometrie des tropischen Semiringes $(\mathbb{R}_\infty, \min, +)$, $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$. Wir ordnen die Bieri-Groves-Menge $\Delta_K^v(a_1, \dots, a_n)$ einer tropischen Varietät eines Primideals $P \triangleleft k[x_1, \dots, x_n]$ wie folgt zu. Die tropische Varietät $\mathcal{T}(I)$ eines Ideals $I \triangleleft k[x_1, \dots, x_n]$ ist die Menge aller Punkte w , so dass das Minimum $\min\{v(c_\alpha) + w \cdot \alpha\}$ für jedes $f = \sum_\alpha c_\alpha x^\alpha \in I$ mindestens zweimal angenommen wird. Dann gilt für ein Primideal P :

$$\Delta_K^v(x_1, \dots, x_n) = \mathcal{T}(P)$$

wobei K der Quotientenkörper von $k[x_1, \dots, x_n]/P$ ist, siehe [EKL06]. Wir merken an, dass die Bieri-Groves Mengen eng mit den Bieri-Strebel-Invarianten Σ_A verwandt sind, welche über Endlichkeitseigenschaften des $\mathbb{Z}[x_1, \dots, x_n]$ -Moduls A definiert sind, siehe [BS80, BS81]. Ist der Modul beispielsweise $A = \mathbb{Z}[x_1, \dots, x_n]/P$ mit einem Primideal P und $K = \text{Quot}(A)$ der Quotientenkörper, dann kann das Komplement Σ_A^c mittels

$$\Sigma_A^c = \bigcup_{v: \mathbb{Q} \rightarrow \mathbb{R}_\infty} [\Delta_K^v(x_1, \dots, x_n)]$$

berechnet werden, wobei $[\Delta]$ die Zentralprojektion einer Teilmenge $\Delta \subset \mathbb{R}^n$ auf die Einheitssphäre \mathbb{S}^{n-1} ist.

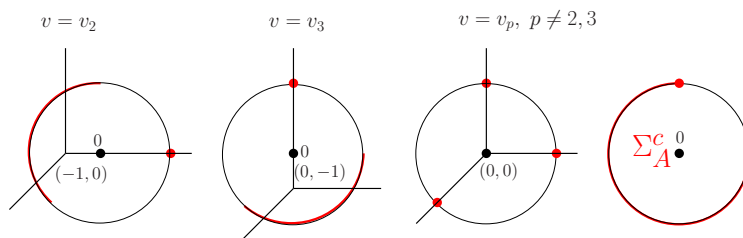


FIGURE 2. Die Bieri-Strebel-Invariante

Ist zum Beispiel $P = \langle 2x + 3y + 1 \rangle \triangleleft \mathbb{Q}[x, y]$, dann haben wir die Vereinigung von allen Projektionen von $\Delta_A^{v_p}(x, y)$ über alle p -adischen Bewertungen v_p von \mathbb{Q} und wir bekommen die Menge $\Sigma_A = \{(s_1, s_2) \in \mathbb{S}^{n-1} \mid s_1 > 0, s_2 > 0\}$, siehe Abbildung 2.

Tropische Geometrie stellt also den algebraisch geometrischen Problemen diskrete geometrische Probleme gegenüber. Es gibt viele Beispiele solcher Korrespondenzen. Ein wichtiges Resultat erzielte zum Beispiel Mikhalkin, siehe [Mik06], welcher die Anzahl ebener Kurven eines gegebenen Grades und Geschlechts durch eine feste Anzahl gegebener Punkte berechnet. Diese Anzahl kann man klassisch oder tropisch berechnen und sie führt auf die Berechnung von Gromov-Witten-Invarianten. Ein anderes Beispiel stammt von Katz, T. Markwig and H. Markwig. Sie vergleichen die klassische j -Invariante mit ihrer tropischen Version, siehe [KMM08].

Wir leiten eine tropische Version des Eisenbud-Evans Theorems her, welches besagt, dass jede algebraische Varietät im \mathbb{R}^n der Durchschnitt von n Hyperflächen ist, siehe [EE73]. Wir zeigen, dass tropisch höchstens $n+1$ tropische Hyperflächen nötig sind um eine tropische Varietät $\mathcal{T}(I)$ als Durchschnitt darzustellen. Also finden wir ein endliches Erzeugendensystem von I , so dass der Durchschnitt der zugehörigen Hyperflächen die tropische Varietät ergibt, eine sogenannte tropische Basis.

THEOREM 0.1. *Sei $I \triangleleft K[x_1, \dots, x_n]$ ein Primideal, erzeugt von den Polynomen f_1, \dots, f_r . dann existieren $g_0, \dots, g_n \in I$, so daß*

$$(3) \quad \mathcal{T}(I) = \bigcap_{i=0}^n \mathcal{T}(g_i)$$

und somit ist $\mathcal{G} := \{f_1, \dots, f_r, g_0, \dots, g_n\}$ eine tropische Basis für I der Kardinalität $r + n + 1$.

Tropische Basen werden beispielsweise von Bogart, Jensen, Speyer, Sturmfels und Thomas in [BJS⁺07] behandelt. Sie haben festgestellt, dass tropische Basen linearer Ideale, die aus linearen Polynomen bestehen, sehr groß sein können. Wir beschränken uns nicht auf den Fall einer tropischen Basis aus linearen Polynomen und deshalb erhalten wir eine kürzere tropische Basis. Dafür werden aber die Grade der Polynome umso größer. Einen anderen Zugang zu tropischen Basen via Gröbner-Basen findet sich im sogenannten konstanten Koeffizienten-Fall. Er wird in der Dissertation von A. Jensen behandelt, siehe [Jen07].

Der wichtigste Baustein um kurze tropische Basen zu finden ist der Gebrauch von Projektionen, vor allem von geometrisch regulären Projektionen, siehe [BG84]. Zusammen mit der Tatsache, dass Urbilder von Projektionen tropischer Varietäten selbst wieder tropische Varietäten von bestimmten Eliminationsidealen sind, bekommen wir das erwünschte Ergebnis.

THEOREM 0.2. *Sei $I \triangleleft K[x_1, \dots, x_n]$ ein m -dimensionales Primideal und $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$ eine rationale Projektion. Dann ist $\pi^{-1}(\pi(\mathcal{T}(I)))$ eine tropische Varietät. Es gilt nämlich*

$$(4) \quad \pi^{-1}(\pi(\mathcal{T}(I))) = \mathcal{T}(J \cap K[x_1, \dots, x_n]).$$

Hier ist J das von I durch

$$J = \langle \tilde{f} \in R : \tilde{f} = f(x_1 \prod_{j=1}^{n-m-1} \lambda_j^{u_1^{(j)}}, \dots, x_n \prod_{j=1}^{n-m-1} \lambda_j^{u_n^{(j)}}) \text{ für ein } f \in I \rangle.$$

abgeleitete Ideal in $K[x_1, \dots, x_n, \lambda_1, \dots, \lambda_{n-m-1}]$, wobei $u^{(1)}, \dots, u^{(l)} \in \mathbb{Z}^n$ den Kern von π erzeugen. Wir zeigen, dass dieses Eliminationsideal ein Hauptideal ist und wir somit ein Polynom unserer tropischen Basis erhalten.

Ein schöner Nebeneffekt ist die Hochhebung von Punkten in der tropischen Varietät eines Eliminationideals zu Punkten in der tropischen Varietät des Ausgangsideals:

THEOREM 0.3 (Tropisches Erweiterungstheorem). *Sei $I \triangleleft K[x_0, \dots, x_n]$ ein Ideal und $I_1 = I \cap K[x_1, \dots, x_n]$ sein erstes Eliminationsideal. Dann existiert für jeden Punkt $w \in \mathcal{T}(I_1)$ ein Punkt $\tilde{w} = (w_0, \dots, w_n) \in \mathbb{R}^{n+1}$ mit $w_i = \tilde{w}_i$ für $1 \leq i \leq n$ und $\tilde{w} \in \mathcal{T}(I)$.*

Der Vorteil unserer Methode ist, dass wir unsere Polynome durch Projektionen gefunden haben und somit die Ergebnisse von Gelfand, Kapranov und Zelevinsky [GKZ90, GKZ08], Esterov und Khovanskii [Est08, EK08] und von Sturmfels, Tevelev und Yu [ST08, STY07, SY08] anwenden können. Ihre Ergebnisse benutzen gemischte Faserpolytope einer Projektion $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$, welche eine Verallgemeinerung von Faserpolytopen sind. Faserpolytope sind Minkowskisummen bestimmter Fasern der Projektion eines Polytops auf sein Bild. Mithilfe gemischter Faserpolytope bekommen wir die Struktur und Kombinatorik des Bildes einer tropischen Varietät und damit die Struktur der Polynome in unserer tropischen Basis. Hierzu benutzen wir, dass jede Zelle einer tropischen Varietät $\mathcal{T}(f_1) \cap \dots \cap \mathcal{T}(f_k)$ dual zu einer Zelle einer gemischten Unterteilung des Newtonpolytops $\text{New}(f_1 \cdot \dots \cdot f_k)$ ist.

THEOREM 0.4. *Sei $I = \triangleleft K[x_1, \dots, x_n]$ ein m -dimensionales Ideal, erzeugt von generisch gewählten Polynomen f_1, \dots, f_{n-m} , $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$ eine Projektion und ψ eine Projektion, dargestellt durch eine Matrix mit einem Zeilenraum, der gleich dem Kern von π ist. Dann haben bis auf affine Isomorphismen der Zellen die Zellen der dualen Unterteilung von $\pi^{-1}\pi\mathcal{T}(I)$ die Form*

$$\sum_{i=1}^p \Sigma_{\psi}(C_{i1}^{\vee}, \dots, C_{ik}^{\vee}) \text{ für ein } p \in \mathbb{N}.$$

Hierbei ist $k = n - m$ und F_1, \dots, F_p sind Seiten von $\mathcal{T}(f_1) \cap \dots \cap \mathcal{T}(f_k)$ und die duale Zelle von $F_i \subseteq U = \mathcal{T}(f_1) \cup \dots \cup \mathcal{T}(f_k)$ ist durch $F_i^\vee = C_{i1}^\vee + \dots + C_{ik}^\vee$ mit Seiten C_{i1}, \dots, C_{ik} von $\mathcal{T}(f_1), \dots, \mathcal{T}(f_k)$ gegeben.

Die dualen Zellen $C_{i1}^\vee + \dots + C_{ik}^\vee$ sind alles gemischte Zellen der induzierten Unterteilung des Newtonpolytopes $\text{New}(f_1 \cdot \dots \cdot f_k)$, d.h. $\dim(C_{ij}) \geq 1$. Für eine geometrisch reguläre Projektion bekommen wir also die Zellen mit $p = 1$ als gemischte Faserpolytope der volldimensionalen gemischten Zellen des unterteilten Newtonpolytops $\text{New}(f_1 \cdot \dots \cdot f_k)$.

Im Fall der regulären Projektion einer tropischen Kurve, also einer eindimensionalen tropischen Varietät, wollen wir die Anzahl der $(n - 1)$ -Zellen von der obigen Form mit $p > 1$ finden, also der Zellen, die dual zu Ecken von $\pi(\mathcal{T}(I))$ sind, welche Schnitt von Bildern zweier nicht benachbarter 1-Zellen von $\mathcal{T}(I)$ sind. Ecken dieses Typs bezeichnen wir als Selbstschnittpunkte. Wir leiten Schranken für ihr Anzahl her:

THEOREM 0.5. *Als untere Schranke für die Anzahl an Selbstschnittpunkten einer tropischen Kurve in \mathbb{R}^n , $n \geq 3$, bekommen wir:*

- (1) *Es existiert eine tropische Gerade $L_n \subset \mathbb{R}^n$ und eine Projektion $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^2$, so dass L_n*

$$\sum_{i=1}^{n-2} i = \binom{n-1}{2}$$

Selbstschnittpunkte besitzt.

- (2) *Es existiert eine tropische Kurve $C \subset \mathbb{R}^n$, welche transversaler Schnitt von $n - 1$ tropischen Hyperflächen vom Grade d_1, \dots, d_{n-1} ist und eine Projektion $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^2$, so dass C mindestens*

$$(d_1 \cdot \dots \cdot d_{n-1})^2 \cdot \binom{n-1}{2}$$

Selbstschnittpunkte besitzt.

Ein Caterpillar („Raupe“) ist ein bestimmter einfacher Typ einer tropischen Gerade und für diesen Typ erhalten wir:

THEOREM 0.6. *Das Bild einer tropischen Geraden L_n in \mathbb{R}^n , welche ein Caterpillar ist kann höchstens*

$$\sum_{i=1}^{n-2} i = \binom{n-1}{2}$$

Selbstschnittpunkte besitzen.

Die tropische Gerade, die in dem Beweis von Theorem 0.5 (1) konstruiert wird, ist auch ein Caterpillar. Somit ist die obere Schranke aus Theorem 0.6 scharf. Für eine allgemeine tropische Kurve kann die Anzahl der Selbstschnittpunkte mithilfe des Theorems von Bernstein von oben beschränkt werden. MV_Δ bezeichne ein relatives gemischtes Volumen. Dann erhalten wir die folgende obere Schranke:

THEOREM 0.7. *Sei $\pi(\mathcal{T}(I)) = \mathcal{T}(f)$ das Bild einer tropischen Kurve. Dann ist die Anzahl der Selbstschnittpunkte von $\pi(\mathcal{T}(I))$ von oben durch*

$$\min \left(\text{vol}(\text{New}(f)), \text{MV}_\Lambda \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right) \right).$$

beschränkt.

Gliederung der Dissertation. Diese Arbeit ist wie folgt strukturiert. Kapitel 1 stellt die grundlegenden Konzepte der tropischen Geometrie dar. Tropische Polynome und tropische Hyperflächen werden definiert. Die Dualität der tropischen Hyperflächen und der unterteilten Newtonpolytope der entsprechenden tropischen Polynome wird beschrieben. Gemischte Zellen einer Vereinigung von tropischen Hyperflächen werden definiert.

Kapitel 2 beginnt mit der klassischen algebraischen Sichtweise und beschreibt die Verbindung zu der tropischen Sichtweise mithilfe Bewertungen von Körpern, der sogenannten Tropisierung. Tropische Varietäten und tropische Basen werden definiert. Wir stellen einige Eigenschaften tropischer Varietäten wie die Konkavitätsbedingung vor und fassen bekannte Resultate über lineare tropische Basen zusammen.

Kapitel 3 führt Projektionen ein und beschreibt die Projektionstechnik von Bieri and Groves. Die Hauptresultate dieser Arbeit werden bewiesen. Und zwar dass Urbilder von Projektionen tropischer Varietäten von Primidealen selbst tropische Hyperflächen sind (Theorem 0.2) und die Existenz und Konstruktion von kurzen tropischen Basen (Theorem 0.1) werden gezeigt. Wir berechnen ein Beispiel um den Algorithmus darzustellen. Dann wenden wir die Ergebnisse von Gelfand, Kapranov und Zelevinky auf den linearen Fall an, welche die Polynome in der kurzen tropischen Basis beschreiben.

Kapitel 4 ist ein Überblick über die Konzepte des Faserpolytops, des Sekundärpolytops und des gemischten Faserpolytops.

Kapitel 5 beschreibt das Newtonpolytop des Polynoms, welches zu der Projektion der tropischen Varietät gehört. Erst werden einige Eigenschaften von tropischen Varietäten vorgestellt. Diese beinhalten transversale Schnitte, eigentliche und vollständige Schnitte. Newton-Nichtdegeneriertheit eines Systems von Polynomen wird definiert. Das Liften dieser Eigenschaften von einem Primideal I zu dem Ideal J aus Kapitel 3 wird erklärt. Dies führt uns unter einigen Voraussetzungen zu einer Methode um die Newtonpolytope und ihre Unterteilungen zu analysieren.

Kapitel 6 studiert die Bilder tropischer Kurven. Im Speziellen werden die Existenz und Anzahl neuer Ecken der Projektion, sogenannter Selbstschnittpunkte, behandelt. Wir geben untere und obere Schranken für ihr Anzahl an.

Veröffentlichte Inhalte. Einige Ergebnisse dieser Arbeit sind in dem Artikel [HT09] und in der erweiterten Konferenzzusammenfassung, zu finden in [tro07], veröffentlicht. Kapitel 3 beinhaltet die Ergebnisse des Artikels, erweitert sie aber mit weiteren Resultaten und Beispielen.

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CHAPTER 1

Introduction to tropical geometry

In this chapter we revisit some basics of tropical geometry. You can find them in several articles, for example in [RGST05]. We give a pure tropical approach, that means we define the tropical analogue for the classical polynomial and the classical hypersurface, which is the zero set of a polynomial. We analyse the structure of tropical hypersurfaces which are dual to certain subdivisions of Newton polytopes of the corresponding polynomials. We describe the cells of the union of two tropical hypersurfaces and introduce mixed cells.

1. The tropical semiring

The main algebraic structure in tropical geometry is the tropical semiring:

DEFINITION 1.1. *The tropical semiring \mathbb{T} is the triple $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ with the following tropical addition and multiplication:*

$$\begin{aligned} a \oplus b &:= \min(a, b) \\ a \odot b &:= a + b \end{aligned}$$

We observe that these operations are commutative, associative and distributive. Furthermore there exist neutral elements, ∞ is the additive and 0 is the multiplicative neutral element. But there is no tropical subtraction, so in this case we simply speak of a semiring. For example it holds $3 \oplus 2 = 2$, $3 \odot 2 = 5$ and $1 \oplus \infty = 1$

Addition is an idempotent operation, $a \oplus a = a$. In many articles (for example [Vig07]) the tropical multiplication is defined as a maximum and not as a minimum. But this gives us no differences in the analysed structures (just a change of sign).

Based on this we can define tropical Laurent polynomials in n unknowns in this semiring.

The monomials are products $a \odot x_1^{\odot i_1} \odot \dots \odot x_n^{\odot i_n}$ with $i_1, \dots, i_n \in \mathbb{Z}$ and $a \in \mathbb{T}$, and the polynomials are linear combinations of monomials with only a finite number of coefficients $\neq \infty$.

$$\begin{aligned} p(x_1, \dots, x_n) &= a \odot x_1^{i_1} \odot x_2^{i_2} \odot \dots \odot x_n^{i_n} \oplus \\ &\quad b \odot x_1^{j_1} \odot x_2^{j_2} \odot \dots \odot x_n^{j_n} \oplus \dots \\ (5) \quad &= \min\{a + i_1 x_1 + \dots + i_n x_n, b + j_1 x_1 + \dots + j_n x_n, \dots\} \end{aligned}$$

(Here x_j^* is an abbreviation for $x_j^{\odot*}$.) The set of exponent vectors of the monomials with non-infinity coefficients is called the *support* of p .

We want to consider polynomials as formal expressions and not as functions. As functions, two tropical polynomials can be the same although they are not the same formal polynomials. For example $f_1 := x^2 \oplus x \oplus 0$ and $f_2 := x^2 \oplus 0$ give the same function but are not the same polynomials.

2. Tropical hypersurfaces

The tropical analogue to the zero set of a classical polynomial is the tropical hypersurface.

DEFINITION 1.2. *Let $p(x_1, \dots, x_n)$ be a tropical polynomial. Then the tropical hypersurface defined by p is the set*

$$\mathcal{T}(p) := \{w \in \mathbb{R}^n \mid \text{the minimum in (5) is achieved twice in } w\}$$

An equivalent definition is given by a condition on the initial forms.

DEFINITION 1.3. *For a tropical polynomial $p = \bigoplus_{i=1}^k a_i \odot x^{\alpha_i}$ and a point $w \in \mathbb{R}^n$ define the initial form as the tropical sum*

$$in_w(p) = \bigoplus a_i \odot x^{\alpha_i}$$

where the sum goes over all i such that the minimum in (5) is attained at the monomial $a_i \odot x^{\alpha_i}$.

Then we have the following alternative description of a tropical hypersurface.

PROPOSITION 1.4. *The tropical hypersurface is the set*

$$\mathcal{T}(p) = \{w \in \mathbb{R}^n \mid in_w(p) \text{ is not a monomial}\}.$$

There is another description of the tropical hypersurface of a tropical polynomial where p is considered as a function (see [EKL06]).

PROPOSITION 1.5 (Einsiedler-Kapranov-Lind). *If $p \neq \infty$ then $\mathcal{T}(p)$ is equal to the non-linear locus of p , i.e. $\mathcal{T}(p)$ is the set of all points where p is not linear, i.e. not differentiable. For $p = \infty$ the tropical hypersurface is \mathbb{R}^n .*

EXAMPLE 1.6. For two examples of tropical curves in the plane see Figure 1 and for an example of a tropical hypersurface in 3-space see Figure 2.

3. Newton polytopes and their subdivisions

It turns out that the Newton polytope of a (tropical) polynomial is useful to analyse the structure of a tropical hypersurface.

DEFINITION 1.7. *Let f be a tropical polynomial in the unknowns x_1, \dots, x_n with terms $x_1^{i_1} \odot \dots \odot x_n^{i_n}$. Then the Newton polytope of f is the convex hull*

$$\text{New}(f) := \text{conv}\{(i_1, \dots, i_n) \in \mathbb{R}^n : x_1^{i_1} \odot \dots \odot x_n^{i_n} \text{ a term in } p\}$$

$$p := 0 \odot x_1 \oplus 1 \odot x_2 \oplus 2 \quad q := 1 \oplus 0 \odot x_1 \oplus 0 \odot x_2 \oplus 1 \odot x_1^2 \oplus 0 \odot x_1 \odot x_2 \oplus 1 \odot x_2^2$$



FIGURE 1. A tropical line and a tropical quadratic curve

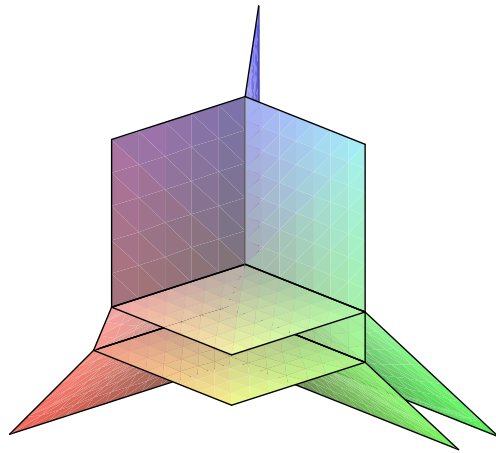


FIGURE 2. A two-dimensional tropical hypersurface

EXAMPLE 1.8. Let $f = 2 \odot x^2 \oplus x^2 \odot y^3 \oplus 3 \odot x \odot y^2 \oplus 1$. Then the Newton polytope is a quadrangle, see Figure 3.

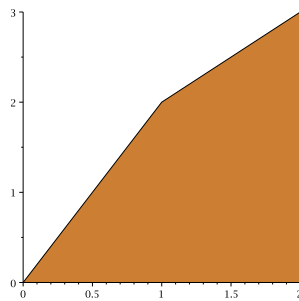


FIGURE 3. The Newton polytope of f

DEFINITION 1.9. For two polyhedra P_1 and P_2 we define the Minkowski sum as the set

$$P_1 + P_2 := \{p_1 + p_2 : p_1 \in P_1, p_2 \in P_2\}$$

The Minkowski sum of two polyhedra is again a polyhedron.

For two polynomials f_1, f_2 the Newton polytope of the tropical multiplication is the Minkowski sum of the two Newton polytopes:

$$\text{New}(f_1 \odot f_2) = \text{New}(f_1) + \text{New}(f_2)$$

EXAMPLE 1.10. Let $f_1 := 0 \oplus 0 \odot x \oplus 0 \odot y$ and $f_2 = 1 \oplus 0 \odot y \oplus 1 \odot x^2$. Then the Minkowski sum of the Newton polytopes is shown in Figure 4.

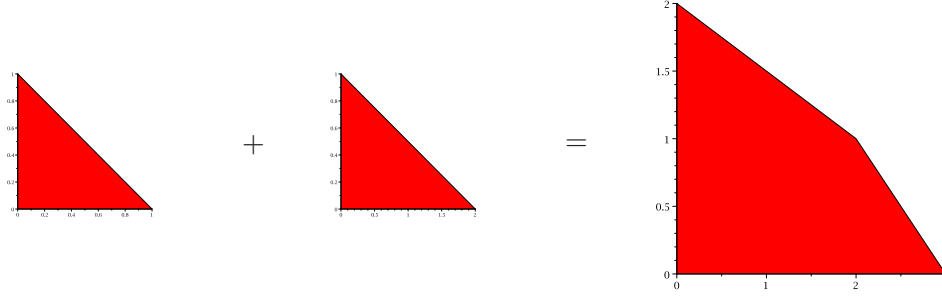


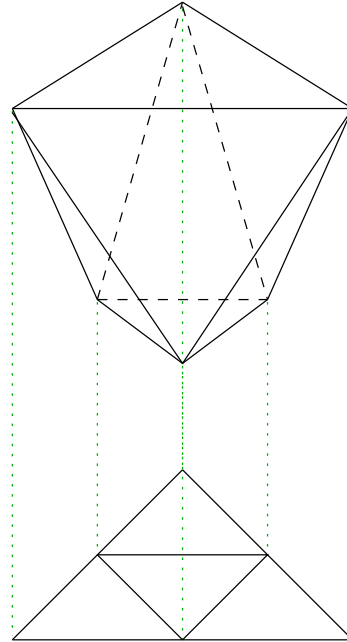
FIGURE 4. The Minkowski sum of two Newton polytopes

DEFINITION 1.11. For a tropical polynomial f the extended Newton polytope is defined by:

$$\text{conv}\{(i_1, \dots, i_n, c_i) : c_i \odot x_1^{i_1} \odot \dots \odot x_n^{i_n} \text{ a monomial in } f\}$$

The projection of the lower convex hull onto the first n coordinates gives a subdivision of the Newton polytope. This subdivision, called *privileged subdivision*, is also denoted by $\text{New}(f)$. Sometimes one denotes the coefficients to indicate the lifting.

The tropical hypersurface is dual to that subdivision.



PROPOSITION 1.12. *There is a bijection between the positive dimensional cells of $\text{New}(f)$ and the open cells of $\mathcal{T}(f)$ such that*

- (1) *Every k -dimensional cell C of the polyhedral complex $\mathcal{T}(f)$ corresponds to a $(n - k)$ -dimensional cell C^\vee of $\text{New}(f)$ such that the affine spaces underlying C^\vee and C are orthogonal.*
- (2) *If $C \subset \overline{D}$ then $D^\vee \subset \overline{C^\vee}$*
- (3) *C is unbounded if and only if $C^\vee \subseteq \partial \text{New}(p)$.*

PROOF. The bijection is given by mapping a cell C of $\mathcal{T}(f)$ to the Newton polytope of $\text{in}_w(f)$ where w lies in the relative interior of C . □

So given a fixed (undivided) Newton polytope there are only finitely many ways to subdivide it. This means there are only finitely many combinatorial types of tropical hypersurfaces with a given support of the corresponding polynomial.

EXAMPLE 1.13. A subdivided Newton polytope and its dual

$$f := 3 \odot x_1^2 \oplus 2 \odot x_1 \odot x_2 \oplus 3 \odot x_2^2 \oplus 0$$

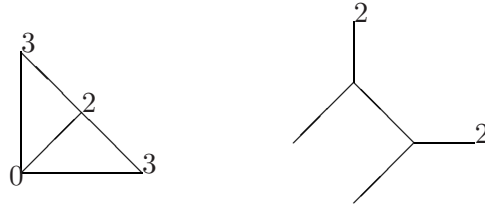


FIGURE 5. A Newton polytope and its dual

EXAMPLE 1.14. One can recognize the type of a tropical hypersurface by the subdivided Newton polytope, see Figure 6.

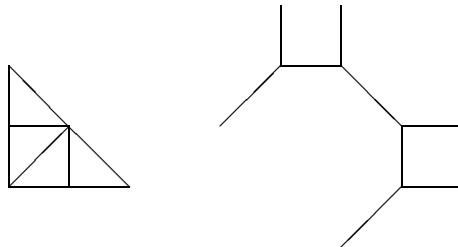


FIGURE 6. Another subdivided newton polytope and its dual

One can also describe a bijection between the vertices of $\text{New}(f)$ and the connected components of $\mathbb{R}^n \setminus \mathcal{T}(f)$:

PROPOSITION 1.15 ([EKL06]). *Let $f = \bigoplus_{a \in \mathcal{A}} c_a \odot x^a$ be a tropical polynomial with support $\mathcal{A} \subset \mathbb{Z}^d$. The connected components of $\mathbb{R}^n \setminus \mathcal{T}(f)$ are in bijection with vertices of the subdivided Newton polytope $\text{New}(f)$, such that if $a = (a_1, \dots, a_n)$ is a vertex of $\text{New}(f)$ then the corresponding component is the following subset of \mathbb{R}^n :*

$$\{u \in \mathbb{R}^n \mid \min_{b \in \mathbb{Z}^d} \{c_b + u \cdot b\} = c_a + u \cdot a \text{ and this minimum is unique}\}$$

The unbounded components of $\mathbb{R}^n \setminus \mathcal{T}(f)$ correspond to vertices on the boundary of $\text{New}(f)$.

4. Some properties of tropical hypersurfaces

In the following we show that the union of two tropical hypersurfaces is again a tropical hypersurface and remark that tropical hypersurfaces fulfill the balancing condition.

The union of tropical hypersurfaces. The union of two tropical hypersurfaces is the tropical hypersurface of the product of the corresponding polynomials.

PROPOSITION 1.16. *For two arbitrary tropical polynomials p_1, p_2 it holds*

$$\mathcal{T}(p_1) \cup \mathcal{T}(p_2) = \mathcal{T}(p_1 \odot p_2)$$

PROOF. To see this note that a point x is in $\mathcal{T}(p_1)$ if the minimum is attained twice, say at the terms $c_\alpha \odot x^\alpha$ and $c_\beta \odot x^\beta$. But then the minimum in the product $p_1 \odot p_2$ is then attained twice, too, at the terms $c_\alpha \odot x^\alpha \odot d_\gamma \odot x^\gamma$ and $c_\beta \odot x^\beta \odot d_\gamma \odot x^\gamma$, where $d_\gamma \odot x^\gamma$ is a term where the minimum in p_2 is attained.

To see the other direction assume that $x \notin \mathcal{T}(p_1) \cup \mathcal{T}(p_2)$. But then the minimum is attained only once in p_1 at the term $c_\alpha \odot x^\alpha$ and once in p_2 at the term $d_\beta \odot x^\beta$. It follows that the minimum in the product is also only attained once, namely at the product of the terms $c_\alpha \odot x^\alpha$ and $d_\beta \odot x^\beta$. \square

Per induction follows

$$\mathcal{T}(p_1) \cup \dots \cup \mathcal{T}(p_r) = \mathcal{T}(p_1 \odot \dots \odot p_r)$$

The union of tropical hypersurfaces is therefore dual to a subdivision of the Minkowski sum of the corresponding Newton polytopes of the polynomials p_1, \dots, p_r .

EXAMPLE 1.17. Let $p_1 := 0 \odot x \oplus 0 \odot y \oplus 0 \odot z \oplus 0$ and $p_2 := 0 \odot x \oplus 1 \odot y \oplus 1$. Then Figure 7 shows the corresponding tropical hypersurfaces of $p_1, p_2, p_1 \odot p_2$.

We can get the privileged subdivision of $\text{New}(p_1 \odot \dots \odot p_r)$ by lifting the Newton polytopes of p_1, \dots, p_r with their coefficients and then projecting the Minkowski sum of these extended Newton polytopes. Then every cell C in this subdivision

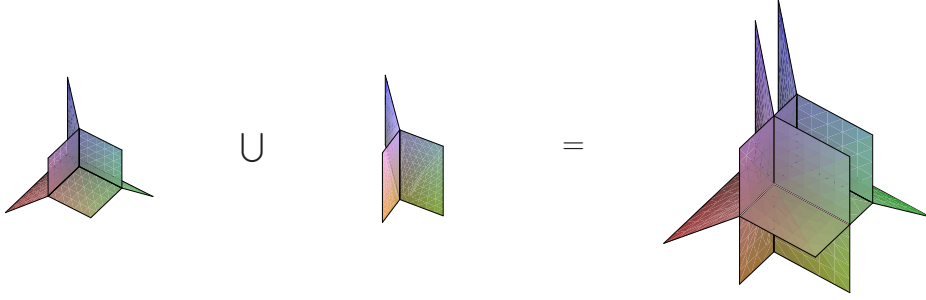


FIGURE 7. The union of two tropical hypersurfaces

is a Minkowski sum of cells C_1, \dots, C_r of $\text{New}(p_1), \dots, \text{New}(p_r)$. So we can define the *type* of C as

$$\text{type}(C) = (\dim(C_1), \dots, \dim(C_r))$$

Clearly $\dim(C_1) + \dots + \dim(C_r) \geq \dim(C)$. Cells of type (d_1, \dots, d_r) with $d_i \geq 1$ are called *mixed*.

Because all cells of $\text{New}(p_i)$ with positive dimension are dual to a cell in $\mathcal{T}(p_i)$ the following hold, see for example [ST09], [Vig07].

PROPOSITION 1.18. *A cell C of the union $\mathcal{T}(p_1) \cup \dots \cup \mathcal{T}(p_r)$ is in the intersection $\mathcal{T}(p_1) \cap \dots \cap \mathcal{T}(p_r)$ iff it is mixed.*

EXAMPLE 1.19. Let $p_1 = 0 \oplus 0 \odot x \oplus 0 \odot y \oplus 2 \odot x \odot y$ and $p_2 = 1 \oplus 0 \odot x \oplus 3 \odot y \oplus 1 \odot x \odot y$. Figure 8 shows the privileged subdivisions and the tropical hypersurface of $p_1 \odot p_2$. We can see that the intersection $\mathcal{T}(p_1) \cap \mathcal{T}(p_2)$ corresponds to the mixed cells of $\text{New}(p_1 \odot p_2)$.

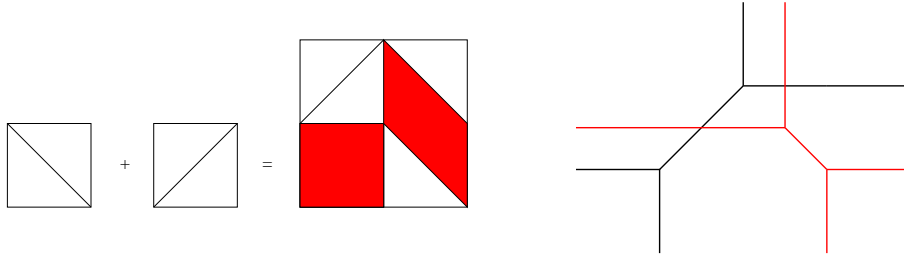


FIGURE 8. The privileged subdivisions with mixed cells

Balancing condition. Let F be a maximal cell of the tropical hypersurface $\mathcal{T}(f) \subset \mathbb{R}^n$. Then its *multiplicity* is defined by the lattice length of the corresponding edge (the dual cell is $n - (n - 1)$ dimensional) in the subdivided Newton polytope.

Let H be any $(n - 2)$ -dimensional cell of $\mathcal{T}(f)$ and v_1, \dots, v_r the primitive lattice vectors in the direction of the maximal cells emanating from H and orthogonal

to H and m_1, \dots, m_r the multiplicities of the corresponding maximal cells.

Then the following *balancing condition* holds.

$$m_1 \cdot v_1 + m_2 \cdot v_2 + \dots + m_r \cdot v_r = 0$$

This holds because of the duality: The $n - 2$ dimensional cell H of $\mathcal{T}(f)$ is dual to a 2-cell H^\vee of the subdivided Newton polytope $\text{New}(f)$. The maximal cells incident with H are dual to the edges of H^\vee and so the vectors orthogonal to these edges and with the length of the corresponding edge sum up to 0.

EXAMPLE 1.20. Let $f = 2 \oplus 0 \odot x \oplus 0 \odot y \oplus 0 \odot x^2 \odot y \oplus 1 \odot x^3 \oplus 1 \odot y^3$. Then we get the subdivision and the corresponding tropical hypersurface shown in Figure 9.

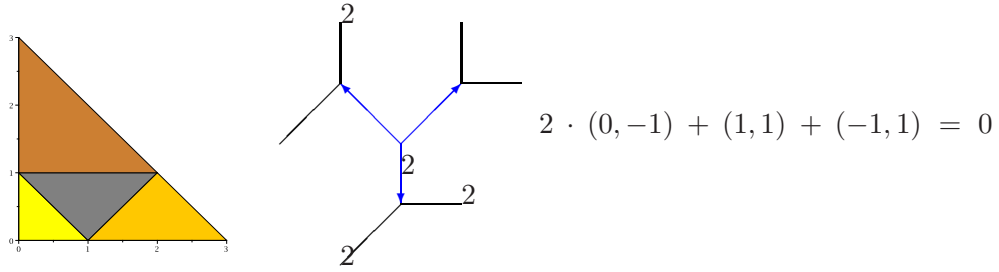


FIGURE 9. The balancing condition

CHAPTER 2

Valuations

This chapter introduces real valuations, i.e. the p -adic valuations and the field of Puiseux series with its natural valuation. The connection between the classical and the tropical viewpoint is described, see for example [JMM08]. We therefore define tropical varieties and tropical bases. The concavity condition, very useful in Chapter 5, is explained. Then we present some known results about tropical linear spaces, see [BJS⁺07].

1. Real valuations

For a field K , a *real valuation* is a map $\text{ord} : K \rightarrow \bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ with

- $K \setminus \{0\} \rightarrow \mathbb{R}$ and
- $0 \mapsto \infty$
- $\text{ord}(ab) = \text{ord}(a) + \text{ord}(b)$ and
- $\text{ord}(a + b) \geq \min\{\text{ord}(a), \text{ord}(b)\}$.

A first example is the p -adic valuation:

EXAMPLE 2.1. $K = \mathbb{Q}$ can be equipped with the p -adic valuation: Every $q \in \mathbb{Q}$ with

$$q = p^s \frac{m}{n}, \quad p \nmid m, p \nmid n, s \in \mathbb{Z}$$

has the valuation

$$\text{ord}(q) = v_p(q) := s.$$

We can extend the valuation map to an algebraic closure \bar{K} and then to \bar{K}^n via

$$\text{ord} : \bar{K}^n \rightarrow \bar{\mathbb{R}}^n, \quad (a_1, \dots, a_n) \mapsto (\text{ord}(a_1), \dots, \text{ord}(a_n)).$$

Puiseux series. Another interesting field is $\mathbb{C}\{\{t\}\}$, the field of *puiseux series*:

The elements are formal power series of the form

$$\sum_{n=m}^{\infty} c_n t^{n/k}$$

where the coefficients c_n are in \mathbb{C} and $c_m \neq 0$. $\mathbb{C}\{\{t\}\}$ is an algebraic closed field. A valuation is given by the order map ord :

$$\begin{aligned} \text{ord} : \mathbb{C}\{\{t\}\} &\rightarrow \mathbb{R} \\ \sum_{n=m}^{\infty} c_n t^{n/k} &\mapsto m/k \end{aligned}$$

To see why the field of Puiseux series is the field commonly used in tropical geometry we want to find a zero of a polynomial in $\mathbb{C}\{\{t\}\}[x]$. Suppose the n -th coefficient a_n of the polynomial

$$p(x) := a_d x^d + a_{d-1} x^{d-1} + \dots + a_1 x + a_0$$

has a puiseux series expansion

$$a_i = c_i \cdot t^{v_i} + \text{terms of higher order in } t$$

and the zero z has the representation

$$z = \gamma \cdot t^w + \text{terms of higher order in } t$$

In the first step we want to find possible numbers $(\gamma, w) \in \mathbb{C} \times \mathbb{Q}$ such that $p(z) = 0$. After inserting the puiseux series expansions in the polynomial we get that all coefficients of the powers of t have to be 0. So the coefficients of the lowest order terms have to cancel each other, that means there must be at least two of them. This forces that the minimum

$$\min\{v_d + dw, v_{d-1} + (d-1)w, \dots, v_2 + 2w, v_1 + w, v_0\}$$

is attained twice. So we are searching for points w in the tropical hypersurface of $\bigoplus_{i=1}^d v_i \odot w^i$.

EXAMPLE 2.2. Let p be the polynomial

$$p = tx^2 + x - t$$

So the minimum $\min\{1 + 2w, w, 1\}$ should be attained twice. So either $w = -1$ or $w = 1$. This gives us two different zeros of p .

For more details about valuations, see for example [End72].

2. Tropical varieties and tropical bases

To explain the connection between the classical algebraic objects like polynomials and the tropical ones we need the notion of tropicalization. Then we give some properties of tropical varieties.

Tropicalization and tropical varieties. Let $f = \sum_{\alpha} c_{\alpha} x^{\alpha}$ be a polynomial in $K[x_1, \dots, x_n]$ where K is a field with a valuation ord . (f can also be a Laurent polynomial if necessary.)

DEFINITION 2.3. *The tropicalization of f is defined as*

$$\begin{aligned} \text{trop}(f) &:= \bigoplus_{\alpha} \text{ord}(c_{\alpha}) \odot x^{\alpha} \\ &= \bigoplus_{\alpha} \text{ord}(c_{\alpha}) \odot x_1^{\alpha_1} \odot \dots \odot x_n^{\alpha_n} \\ &= \min_{\alpha} \{\text{ord}(c_{\alpha}) + \alpha_1 x_1 + \dots + \alpha_n x_n\} \end{aligned}$$

and the tropical hypersurface of f is

$$\begin{aligned} \mathcal{T}(f) &:= \mathcal{T}(\text{trop}(f)) \\ &= \{w \in \mathbb{R}^n : \text{the minimum in } \text{trop}(f) \\ &\quad \text{is attained at least twice in } w\} \\ &= \{w \in \mathbb{R}^n : \text{in}_w(\text{trop}(f)) \text{ is not a monomial}\} \end{aligned}$$

EXAMPLE 2.4. Let $f = 2x + 4y - x^2 + 3y^2$ a polynomial in $\mathbb{Q}[x, y]$ with the 2-adic valuation. Then

$$\begin{aligned} \text{trop}(f) &= 1 \odot x \oplus 2 \odot y \oplus 0 \odot x^2 \oplus 0 \odot y^2 \\ &= \min\{1 + x, 2 + y, 2x, 2y\} \end{aligned}$$

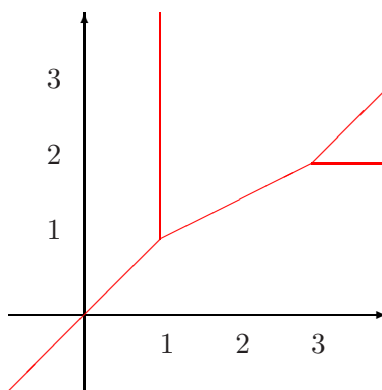


FIGURE 1. The tropical hypersurface $\mathcal{T}(f)$

We generalize the idea of tropical hypersurfaces to ideals:

DEFINITION 2.5. For an ideal $I \triangleleft K[x_1, \dots, x_n]$, the tropical variety of I is defined by

$$\mathcal{T}(I) = \bigcap_{f \in I} \mathcal{T}(f)$$

There is another description of the tropical variety of an ideal:

PROPOSITION 2.6. If the valuation is nontrivial, i.e. there is an $a \neq 0$ with $\text{ord}(a) \neq 0$, then the tropical variety is the topological closure

$$\mathcal{T}(I) = \overline{\text{ord } \mathcal{V}(I)}$$

where $\mathcal{V}(I) \subset (\bar{K}^*)^n$ is the algebraic variety of I .

If we define the initial form $\text{in}_w(f)$ of a (classical) polynomial $f = \sum_{i=1}^k c_i x^{\alpha_i}$ and a point $w \in \mathbb{R}^n$ as the sum of terms which tropicalizes to $\text{in}_w(\text{trop}(f))$ then the initial ideal for an ideal $I \triangleleft K[x_1, \dots, x_n]$ can be defined by

$$\text{in}_w(I) = \langle \text{in}_w(f) \mid f \in I \rangle$$

Now we have again another description.

PROPOSITION 2.7. $\mathcal{T}(I) = \{w \in \mathbb{R}^n \mid \text{in}_w(I) \text{ contains no monomial}\}$

EXAMPLE 2.8. Let I be generated by

$$f_1 := 2 + y - 4x^2y + x^2y^2 + 2xy^2$$

$$f_2 := xyz - 2z + 4xyz^2 - 2 + z^2$$

Then we get the tropical variety in \mathbb{R}^3 of Figure 2.

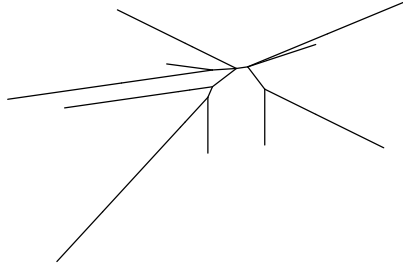


FIGURE 2. Tropical variety $\mathcal{T}(I)$

Polyhedral complexes. In this section we revisit the notion of polyhedral complexes and related subjects which will be used later to describe tropical varieties.

DEFINITION 2.9. A subset Δ of the affine space \mathbb{R}^n is a convex polyhedron if it can be written as a finite intersection of closed affine halfspaces in \mathbb{R}^n , i.e.

$$\Delta = H_1 \cap H_2 \cap \dots \cap H_r.$$

The dimension of Δ is the dimension of the affine subspace spanned by Δ .

Especially the empty set is a convex polyhedron.

DEFINITION 2.10. A face of a polyhedron $\Delta \subset \mathbb{R}^n$ is a subset F of Δ where some linear functional attains its maximum, i.e. there is some $\omega \in \mathbb{R}^n$ such that

$$F = \text{face}_\omega(\Delta) = \{x \in \Delta \mid \omega \cdot x = \max\{\omega \cdot y \mid y \in \Delta\}\},$$

or $F = \emptyset$.

If we set $\omega = 0$ then we see that Δ is always a face of itself.

If we glue several convex polyhedrons together (under certain conditions) we gain a polyhedral complex:

DEFINITION 2.11. A collection \mathcal{P} of convex polyhedrons is a polyhedral complex if

- $\emptyset \in \mathcal{P}$,
- for all $\Delta_1, \Delta_2 \in \mathcal{P}$ hold that $\Delta_1 \cap \Delta_2$ is a face of both Δ_1 and Δ_2 ,
- for all $\Delta \in \mathcal{P}$ and all faces $\Gamma \subset \Delta$ hold $\Gamma \in \mathcal{P}$.

DEFINITION 2.12. A polyhedral complex \mathcal{P} is pure of dimension m if all maximal convex polyhedrons in \mathcal{P} are of dimension m .

Properties of tropical varieties. A tropical variety has several properties.

If P_0, \dots, P_n are prime ideals such that $I \subseteq P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_n$, then these ideals form a chain of length n . The *Krull dimension* of I is the supremum of the length of chains of prime ideals. With this Bieri and Groves showed:

PROPOSITION 2.13 (Bieri-Groves 1984). *Let I be a prime ideal. Then $\mathcal{T}(I)$ is a pure m -dimensional polyhedral complex where $m = \dim(I)$ is the Krull dimension of the ideal.*

Define the local cone of a point x of a polyhedral complex $\Delta \subseteq \mathbb{R}^n$ as the set

$$LC_x(\Delta) := \{x + y \in \mathbb{R}^n : \exists \epsilon > 0 \text{ such that } \{x + \rho y : 0 \leq \rho \leq \epsilon\} \subseteq \Delta\}$$

Then the following holds.

PROPOSITION 2.14 (Bieri-Groves 1984). *$\mathcal{T}(I)$ is totally concave, which means that each convex hull of a local cone of a point x is an affine subspace (see Figure 3).*

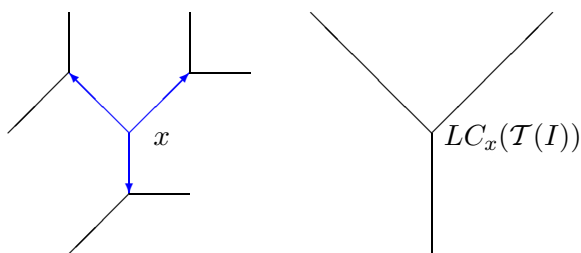


FIGURE 3. The local cone of a point

Tropical prevarieties and tropical bases.

DEFINITION 2.15. Let $f_1, \dots, f_s \in K[x_1, \dots, x_n]$ be polynomials. Then the tropical prevariety defined by f_1, \dots, f_s is the intersection of the tropical hypersurfaces

$$\mathcal{T}(f_1, \dots, f_s) := \bigcap_{i=1}^s \mathcal{T}(f_i)$$

The next example shows that this is not a tropical variety in general:

EXAMPLE 2.16. A tropical prevariety is given by the intersection of the tropical hypersurfaces of the two polynomials

$$f_1 := t + (t + 1)x^2 + (2t^3 - t^4)y^2 + txy$$

$$f_2 := t + (t^{\frac{1}{2}} + t^{\frac{3}{2}})x + t^{\frac{3}{2}}y$$

In this case it is the union of two rays emanating from the point $(\frac{1}{2}, -\frac{1}{2})$. This is not a tropical variety because the concavity condition is not fulfilled.

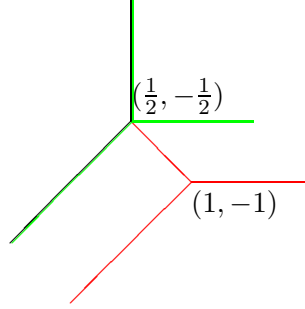


FIGURE 4. The intersection of two tropical hypersurfaces

This motivates the following definition.

DEFINITION 2.17. A basis $\mathcal{F} = \{f_1, \dots, f_s\}$ of I is a tropical basis, if

$$\mathcal{T}(I) = \bigcap_{i=1}^s \mathcal{T}(f_i)$$

PROPOSITION 2.18. Let $I \triangleleft K[x_1, \dots, x_n]$ be an ideal. Then there exists a tropical basis.

A proof of this proposition can be found for example in [EKL06]. An independent proof in the context of projections of tropical varieties can be found later in this work.

For linear ideals a tropical basis consisting of the circuits is known. But this basis can be very large, see Section 3.

EXAMPLE 2.19. Adding the following polynomials to $\{f_1, f_2\}$ we get a tropical basis:

$$\begin{aligned} f_3 &:= (3t - t^2)x^2 + (-t^{\frac{1}{2}} + 4t^{\frac{3}{2}} \\ &\quad - 2t^{\frac{5}{2}})x + (t + 2t^2 - t^3); \\ f_4 &:= (3t^3 - t^4)y^2 + (t^{\frac{3}{2}} + 2t^{\frac{5}{2}})y \\ &\quad + (2t + t^2) \end{aligned}$$

The intersection of the corresponding tropical hypersurfaces is the point $(\frac{1}{2}, -\frac{1}{2})$, see Figure 5.

3. Linear spaces

A tropical d -plane in \mathbb{R}^n is a tropical variety $\mathcal{T}(I)$ where I is a linear ideal, i.e. generated by $n - d$ linear polynomials:

$$I = \left\langle \sum_{j=1}^n a_{i,j} x_j : i = 1, \dots, n - d \right\rangle.$$

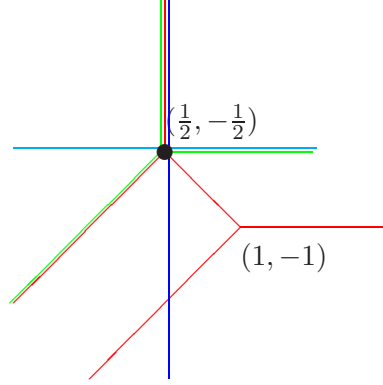


FIGURE 5. A tropical variety

Here $(a_{i,j}) \in K^{(n-d) \times n}$ and $\text{rank}(a_{i,j}) = n - d$.

The *Plücker coordinates* are given by

$$P_{i_1 \dots i_d} := (-1)^{i_1 + \dots + i_d} \det \begin{pmatrix} a_{1,j_1} & \cdots & a_{1,j_{n-d}} \\ \vdots & \ddots & \vdots \\ a_{n-d,j_1} & \cdots & a_{n-d,j_{n-d}} \end{pmatrix} \neq 0$$

Here $1 \leq i_1 < \dots < i_d \leq n$ and $1 \leq j_1 < \dots < j_{n-d} \leq n$ is the complement, $\{i_1, \dots, i_d\} = \{1, \dots, n\} \setminus \{j_1, \dots, j_{n-d}\}$.

Then we can compute a tropical basis for the linear ideal I .

LEMMA 2.20. *A tropical basis for I is given by the circuits*

$$C_{i_0 \dots i_d} = \sum_{r=0}^d (-1)^r P_{i_0 \dots \hat{i}_r \dots i_d} x_{i_r}$$

A proof can be found for example in [Stu02].

Most results about tropical bases were made in the constant coefficient case. That means that we have a trivial valuation of our field K . We can for example take $K = \mathbb{C}$. In this context Bogart, Jensen, Speyer, Sturmfels and Thomas showed in [BJS⁺07] that a linear tropical basis can be very large:

PROPOSITION 2.21. *For any $1 \leq d \leq n$, there is a linear ideal I in $\mathbb{C}[x_1, \dots, x_n]$ such that any tropical basis of linear forms in I has size at least $\frac{1}{d+1} \binom{n}{d}$.*

We revisit the proof.

PROOF. See [BJS⁺07]. Assume that all Plücker coordinates are nonzero. We have then $\binom{n}{d+1}$ circuits, each supported on a different $(d+1)$ -subset of $\{x_1, \dots, x_n\}$.

Let $w \in \mathcal{T}(I)$, i.e. $\text{in}_w(C_{i_0, \dots, i_d})$ is not a monomial for all choices of $\{i_1, \dots, i_d\}$. Because we are in the constant coefficient case that means that the minimum $\min\{w_{i_0}, \dots, w_{i_d}\}$ is achieved twice. But this forces $\min\{w_1, \dots, w_n\}$ to be

achieved at least $n - d + 1$ times.

Let now $w \notin \mathcal{T}(I)$. Then the minimum $\min\{w_1, \dots, w_n\}$ is achieved at most $n - d$ times, w.l.o.g.

$$w_{i_1} = \dots = w_{i_d} < \min\{w_j \mid j \in \{1, \dots, n\} \setminus \{i_1, \dots, i_{n-d}\}\}$$

Because $w \notin \mathcal{T}(I)$ in every tropical basis of linear forms there must be a polynomial f such that $in_w(f)$ is a monomial,

$$in_w(f) = a_i x_i, \quad i \in \{i_1, \dots, i_{n-d}\}$$

So all other x_j with $j \in \{i_1, \dots, i_{n-d}\}$ are eliminated, f is therefore a circuit with support

$$\text{supp}(f) = \{x_i, x_j \mid j \notin \{i_1, \dots, i_{n-d}\}\}$$

Every circuit has a support of $d + 1$ elements, so it has $d + 1$ different subsets with d elements. There are $\binom{n}{d}$ subsets of $\{x_1, \dots, x_n\}$ with d elements. All of these has to be covered by a tropical basis, so there are at least $\frac{1}{d+1} \binom{n}{d}$ elements in a tropical basis of linear forms. \square

But we can show that there are small tropical bases if we drop the assumption that the basis consists of linear polynomials.

CHAPTER 3

Tropical bases via projections

In this chapter we first give some results together with their proofs of Bieri and Groves, see [BG84]. They concern projections of polyhedral complexes, i.e. geometric regular projections. Secondly we study the preimage of a projection of a tropical variety and show that it is a tropical hypersurface. With the results of the first section we get that for every ideal $I \triangleleft K[x_1, \dots, x_n]$ there are $n + 1$ polynomials in I which form a tropical basis. That means that there is always a tropical basis with only few elements. But the degree of these polynomials can be very high (see Chapter 6). In the third section we give an example of the computation of such a tropical basis and in the fourth section we investigate the linear case.

1. Projections

To show that there exist a basis of $n + 1$ polynomials we want to describe the tropical variety via different projections, i.e. surjective linear maps.

If we have an arbitrary polyhedral complex $\Delta \subseteq \mathbb{R}^n$ of dimension m R. Bieri and J. Groves showed (see [BG84]) that there are $n + 1$ projections $\pi_0, \dots, \pi_n : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$ such that

$$\Delta = \bigcap_{i=0}^n \pi_i^{-1}(\pi_i(\Delta))$$

Here we want to revisit the techniques used in this paper. To describe these projections we have to deal with affine subspaces.

DEFINITION 3.1. *Let \mathcal{X} be a finite set of affine linear subspaces $X \subseteq \mathbb{R}^n$. \mathcal{X} is complete if for all $X, Y \in \mathcal{X}$ the intersection $X \cap Y$ is in \mathcal{X} . The dimension of \mathcal{X} is defined as*

$$\dim \mathcal{X} = \max\{\dim X : X \in \mathcal{X}\}$$

and the support, also denoted by \mathcal{X} , is the set

$$\mathcal{X} = \bigcup_{X \in \mathcal{X}} X \subseteq \mathbb{R}^n$$

We say that \mathcal{X} is pure of dimension m if all maximal spaces (with respect to inclusion) are of dimension m .

We will make no difference in the notation between the polyhedral complex and its underlying space to avoid technical notations.

We can assign to each polyhedral complex Δ a finite set \mathcal{X} of affine linear subspaces by taking all underlying affine subspaces of the maximal cells of Δ .

DEFINITION 3.2. When we add all intersections $X_1 \cap \dots \cap X_r$ with $X_1, \dots, X_r \in \mathcal{X}$, r arbitrary, to \mathcal{X} we get a complete set, the completion of \mathcal{X} .

Clearly if the polyhedral complex was of dimension m then $\dim(\mathcal{X}) = m$ and if the polyhedral complex was pure of dimension m then so is \mathcal{X} .

Now we want to project polyhedral complexes $\Delta \subset \mathbb{R}^n$ of dimension m to \mathbb{R}^{m+1} :

DEFINITION 3.3. A rational projection

$$\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$$

is a linear map, described by a matrix $A = (a_{ij})_{i,j} \in M_{m+1 \times n}(\mathbb{Q})$.

Assume now that \mathcal{X} is complete. Then we can define geometrically regular projections.

DEFINITION 3.4. A projection $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$ is geometrically regular with respect to \mathcal{X} if the following hold

- for all $X \in \mathcal{X} : \dim(X) = \dim(\pi(X))$,
- for all $X, Y \in \mathcal{X} : X \subseteq Y \Leftrightarrow \pi(X) \subseteq \pi(Y)$.

For an example of a geometrically regular and a non-regular projection see Figures 1 and 2, respectively.

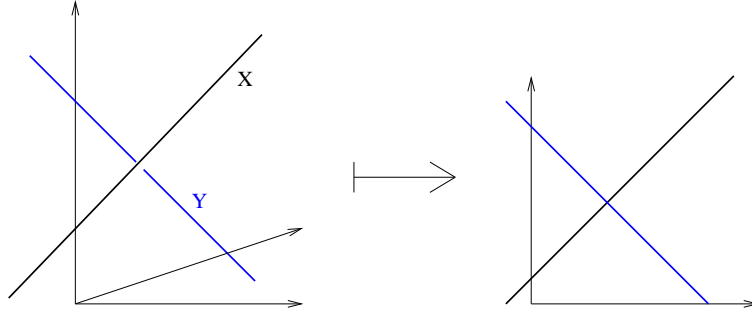


FIGURE 1. Here $\mathcal{X} = \{X, Y\}$ and the projection is regular.

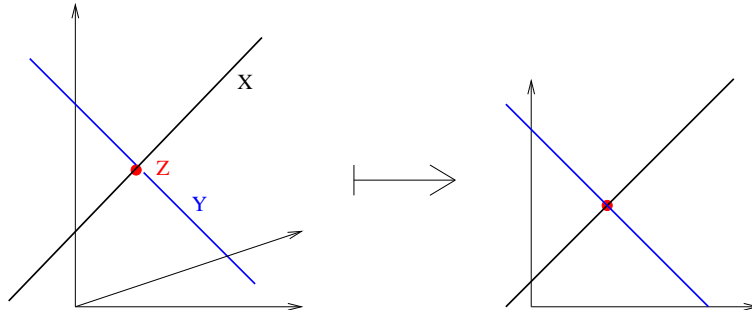


FIGURE 2. Here $\mathcal{X} = \{X, Y, Z\}$ and the projection is nonregular because $\pi(Z) \subseteq \pi(Y)$ and $Z \not\subseteq Y$

In the space of all projections the set of geometrically regular projections with respect to a finite set \mathcal{X} of affine subspaces is generic, i.e. all other projections form a subset of at most dimension $n - 1$.

LEMMA 3.5. *The set of projections which are not geometrically regular is contained in a finite union of hyperplanes within the space of all projections $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$.*

PROOF. Let a projection π be described by the matrix $A = (a_{ij})_{i,j}$. For a geometrically regular projection it is forbidden that for any $X \in \mathcal{X}$ the dimension decreases, i.e. $\dim(X) \neq \dim(\pi(X))$ and that for any two affine subspaces $X \not\subseteq Y \in \mathcal{X}$ holds $\pi(X) \subseteq \pi(Y)$. But in all these cases the entries of A has to fulfill certain equalities. Because \mathcal{X} is finite there are only finite such equalities. \square

LEMMA 3.6 (regular projection lemma). *Let \mathcal{X} be a finite set of affine subspaces of \mathbb{R}^n with $\dim(\mathcal{X}) = m < n$, $\mathcal{Y} \subseteq \mathcal{X}$ a subset with $\dim(\mathcal{Y}) = r \leq m$. Then there are $r + 1$ affine projections $\pi_0, \dots, \pi_r : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$ with the property that for every point $x \in \mathcal{Y}$ there is an index $0 \leq i \leq r$ such that*

$$\pi_i^{-1}(\pi_i(x)) \cap \mathcal{X} = \{x\}.$$

PROOF. We follow the proof of [BG84]. The proof is by induction on the dimension r . If $r = -1$ then $\mathcal{Y} = \emptyset$ and there is nothing to prove. So let $r > -1$. Lemma 3.5 gives us a projection $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$ which is geometrically regular with respect to \mathcal{Y} . Let

$$T := \{y \in \mathcal{Y} \mid \exists x \in \mathcal{X}, x \neq y : \pi(y) = \pi(x)\}$$

Then for all subspaces $Y \in \mathcal{Y}$ we have

$$Y \cap T = \bigcup_{X \in \mathcal{X}, Y \not\subseteq X} Y \cap \pi^{-1}(\pi(X))$$

Because π is regular $\pi(Y) \not\subseteq \pi(X)$ for all $X \not\subseteq Y$. Therefore

$$\dim(Y \cap \pi^{-1}(\pi(X))) < \dim Y.$$

Let \mathcal{Z} be the set of all such intersections:

$$\mathcal{Z} := \{Y \cap \pi^{-1}(\pi(X)) \mid Y \in \mathcal{Y}, X \in \mathcal{X}, Y \not\subseteq X\}$$

\mathcal{Z} is a finite set of affine subspaces of \mathbb{R}^n with $\dim(\mathcal{Z}) < r$. Let \mathcal{Z}' be the smallest complete finite set of subspaces containing \mathcal{X} and \mathcal{Z} . Then the inductive hypothesis applied for \mathcal{Z} and \mathcal{Z}' yields projections π_0, \dots, π_{r-1} such that for every point $z \in \mathcal{Z}$ there is an index $0 \leq i \leq r - 1$ with $\pi_i^{-1}(\pi_i(z)) \cap \mathcal{X} = \{z\}$. This hold especially for all points $z \in T \subseteq \mathcal{Z}'$. But for all other points of $y \in \mathcal{Y}$ the projection $\pi := \pi_r$ suffices $\pi_r^{-1}(\pi_r(y)) \cap \mathcal{X} = \{y\}$. So the assertion follows. \square

So if we assign to each polyhedral complex a finite set of affine subspaces of the same dimension we can detect all points x of it by an appropriate projection. But we want also get the set of affine subspaces as an intersection of preimages of several projections.

LEMMA 3.7. *Let \mathcal{X} be a complete and finite set of affine subspaces pure of dimension m . Then there are $n - m$ projections π_1, \dots, π_{n-m} such that*

$$\bigcap_{i=1}^{n-m} \pi^{-1}(\pi(\mathcal{X}))$$

is pure m -dimensional.

PROOF, SEE [BG84]. We want to find $n - m$ projections

$$\pi_1, \dots, \pi_{n-m} : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$$

with the property, that $\bigcap_{i=1}^{n-m} \pi_i^{-1} \pi_i(\mathcal{X})$ is pure m -dimensional. This is done inductively: Let $\mathcal{X}_0 = \{\mathbb{R}^n\}$. Let π_{n-m} be an arbitrary projection with m -dimensional image $\pi_{n-m}(\mathcal{X})$. Then we define

$$\mathcal{X}_1 := \{\text{affine subspaces in the preimage } \pi_{n-m}^{-1} \pi_{n-m}(\mathcal{X})\}.$$

So all maximal affine subspaces of \mathcal{X}_1 have dimension $n - 1$.

Assume now \mathcal{X}_t is constructed and is $(n - t)$ -dimensional with

$$\mathcal{X}_t = \bigcap_{i=n-m-t+1}^{n-m} \pi_i^{-1} \pi_i(\mathcal{X}).$$

Let \mathcal{B} be the finite set of all $(n - t)$ -dimensional subspaces of \mathbb{R}^n parallel to at least one of the affine subspaces of \mathcal{X}_t .

Then choose $\pi_{n-m-t} : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$ as a projection with

$$\mathbb{R}^n = \ker \pi_{n-m-t} + V \text{ for all } V \in \mathcal{B}.$$

Let \mathcal{Y} be the set of all $(n - 1)$ -dimensional affine subspaces of $\pi_{n-m-t}^{-1} \pi_{n-m-t}(\mathcal{X})$ and let

$$\mathcal{X}_{t+1} = \{Y \cap X \mid Y \in \mathcal{Y}, X \in \mathcal{X}_t\}$$

Now \mathcal{X}_{t+1} is pure of dimension $n - t - 1$.

It follows that π_1, \dots, π_{n-m} are the projections we are searching for. \square

REMARK 3.8. *Clearly the set $\bigcap_{i=1}^{n-m} \pi^{-1}(\pi(\mathcal{X}))$ contains \mathcal{X} .*

COROLLARY 3.9. *Let Δ be a polyhedral complex. Then there are $n+1$ projections π_0, \dots, π_n such that*

$$\Delta = \bigcap_{i=0}^n \pi_i^{-1}(\pi_i(\Delta))$$

PROOF. Combine Lemma 3.6 and Lemma 3.7. \square

2. The preimage of a projection of a tropical variety

Let now

$$\begin{aligned} \pi : \mathbb{R}^n &\rightarrow \mathbb{R}^{m+1} \\ x &\mapsto Ax \end{aligned}$$

be a projection given by a regular matrix $A \in M_{n \times n}(\mathbb{Z})$ with rows denoted by $a^{(1)}, \dots, a^{(m+1)}$.

We want to show that the preimage $\pi^{-1} \pi(\mathcal{T}(I))$ is a tropical hypersurface, generated by a polynomial in I .

Let $u^{(1)}, \dots, u^{(l)} \in \mathbb{Z}^n$ with $l := n - (m + 1)$ be a basis of the orthogonal complement of $\text{span}\{a^{(1)}, \dots, a^{(m+1)}\}$, which is the kernel of π . Set $R = K[x_1, \dots, x_n, \lambda_1^{\pm 1}, \dots, \lambda_l^{\pm 1}]$ and define the ideal $J \triangleleft R$ by

$$J = \langle \tilde{f} \in R : \tilde{f} = f(x_1 \prod_{j=1}^l \lambda_j^{u_1^{(j)}}, \dots, x_n \prod_{j=1}^l \lambda_j^{u_n^{(j)}}) \text{ for some } f \in I \rangle.$$

We can easily describe J by the generators of I :

PROPOSITION 3.10. *Let I be generated by f_1, \dots, f_s . Then J is generated by the polynomials*

$$\tilde{f}_i := f_i(x_1 \prod_{j=1}^l \lambda_j^{u_1^{(j)}}, \dots, x_n \prod_{j=1}^l \lambda_j^{u_n^{(j)}}), \quad i = 1, \dots, s.$$

PROOF. Obviously $\langle \tilde{f}_1, \dots, \tilde{f}_s \rangle \subseteq J$. Let now $f \in J$. Then there exist $g_1, \dots, g_r \in I, k_1, \dots, k_r \in K[x_1, \dots, x_n, \lambda_1, \dots, \lambda_l]$ with $f = \sum_{i=1}^r k_i \tilde{g}_i$. So it suffices to show that each polynomial

$$\tilde{g} = g(x_1 \prod_{j=1}^l \lambda_j^{u_1^{(j)}}, \dots, x_n \prod_{j=1}^l \lambda_j^{u_n^{(j)}})$$

is in the ideal $\langle \tilde{f}_1, \dots, \tilde{f}_s \rangle \subseteq J$.

Because $g \in I$ there exist $h_i \in K[x_1, \dots, x_n]$ with $g = \sum_{i=1}^s h_i \cdot f_i$

$$\begin{aligned} &\Rightarrow g(x_1 \prod_{j=1}^l \lambda_j^{u_1^{(j)}}, \dots, x_n \prod_{j=1}^l \lambda_j^{u_n^{(j)}}) = \\ &\sum h_i(x_1 \prod_{j=1}^l \lambda_j^{u_1^{(j)}}, \dots, x_n \prod_{j=1}^l \lambda_j^{u_n^{(j)}}) f_i(x_1 \prod_{j=1}^l \lambda_j^{u_1^{(j)}}, \dots, x_n \prod_{j=1}^l \lambda_j^{u_n^{(j)}}) \end{aligned}$$

□

THEOREM 3.11. *If I is a prime ideal, then J is also a prime ideal.*

PROOF. Let $U := (u^{(1)}, \dots, u^{(l)}) \in M_{n \times l}(\mathbb{Z})$ and

$$\Phi_U : K[x_1, \dots, x_n] \rightarrow R, \quad x_i \mapsto x_i \cdot \prod_{j=1}^l \lambda_j^{u_i^{(j)}}$$

$$\Psi_U : R \rightarrow R, \quad \lambda_j \mapsto \lambda_j, \quad x_i \mapsto x_i \cdot \prod_{j=1}^l \lambda_j^{u_i^{(j)}}$$

the corresponding K -algebra homomorphisms. Obviously Ψ_U is an isomorphism with inverse Ψ_{-U} .

We have the commutative diagramm

$$\begin{array}{ccc} K[x_1, \dots, x_n] & \xrightarrow{\Phi_0} & R \\ & \searrow \Phi_U & \swarrow \Psi_U \\ & & R \end{array}$$

Because Ψ_U is an isomorphism and

$$\Psi_U(\langle \Phi_0 I \rangle) = \langle \Psi_U(\Phi_0(I)) \rangle = \langle \Phi_U(I) \rangle$$

it suffices to show that $\Phi_0(I)$ is a prime ideal. Lemma 5.21 of [Bro89] states that the extension of a prime ideal of a ring S to an ideal of a polynomial ring $S[\lambda_1, \dots, \lambda_l, \mu_1, \dots, \mu_l]$ remains prime. The projection to the ring with $\lambda_i \cdot \mu_i = 1$ still preserves primality. So with $S = K[x_1, \dots, x_n]$ this means that $\langle \Phi_0(I) \rangle$ is prime and therefore J is prime. \square

The following Theorem shows that the preimage is a tropical variety.

THEOREM 3.12. *Let $I \triangleleft K[x_1, \dots, x_n]$ be an m -dimensional prime ideal and $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$ be a rational projection. Then $\pi^{-1}(\pi(\mathcal{T}(I)))$ is a tropical variety with*

$$(6) \quad \pi^{-1}(\pi(\mathcal{T}(I))) = \mathcal{T}(J \cap K[x_1, \dots, x_n]).$$

If I is a prime ideal, then J is prime and therefore $J \cap K[x_1, \dots, x_n]$ too. If additionally the dimension of the preimage $\pi^{-1}\pi(\mathcal{T}(I))$ is $(n-1)$ -dimensional (and therefore $J \cap K[x_1, \dots, x_n]$ has Krull dimension $n-1$) and Krull's Hauptidealsatz says that $J \cap K[x_1, \dots, x_n]$ is a principal ideal.

To show the theorem we have to prove some small lemmata:

LEMMA 3.13. *For any $w \in \mathcal{T}(J \cap K[x_1, \dots, x_n])$ and $u \in \text{span}\{u^{(1)}, \dots, u^{(l)}\}$ we have $w + u \in \mathcal{T}(J \cap K[x_1, \dots, x_n])$.*

PROOF. Let $u = \sum_{j=1}^l \mu_j u^{(j)}$ with $\mu_1, \dots, \mu_l \in \mathbb{Q}$. The case of real μ_i then follows by taking the closure.

Let $w \in \mathcal{T}(J \cap K[x_1, \dots, x_n])$. Since $\mathcal{T}(J \cap K[x_1, \dots, x_n])$ is closed, we can assume without loss of generality that there exists $z \in \mathcal{V}(J \cap K[x_1, \dots, x_n])$ with $\text{ord } z = w$. Define $y = (y', y'') \in (\bar{K}^*)^{n+l}$ by

$$y = (y', y'') = \left(z_1 t^{\sum_{j=1}^l \mu_j u_1^{(j)}}, \dots, z_n t^{\sum_{j=1}^l \mu_j u_n^{(j)}}, t^{-\mu_1}, \dots, t^{-\mu_l} \right).$$

For any $f \in I$, the point y is a zero of the polynomial

$$f(x_1 \prod_{j=1}^l \lambda_j^{u_1^{(j)}}, \dots, x_n \prod_{j=1}^l \lambda_j^{u_n^{(j)}}) \in R,$$

and thus $y \in \mathcal{V}(J)$. Hence, $y' \in \mathcal{V}(J \cap K[x_1, \dots, x_n])$. Moreover,

$$\text{ord } y' = (w_1 + \sum_{j=1}^l \mu_j u_1^{(j)}, \dots, w_n + \sum_{j=1}^l \mu_j u_n^{(j)}) = w + \sum_{j=1}^l \mu_j u^{(j)} = w + u,$$

which proves the claim. \square

Because we are interested in the ideal I , it is important to prove that the elimination ideal of J is a subset of I :

LEMMA 3.14. *Let $I \triangleleft K[x_1, \dots, x_n]$ be an ideal. Then $J \cap K[x_1, \dots, x_n] \subseteq I$.*

PROOF. Let $p = \sum_i h_i g_i$ be a polynomial in $J \cap K[x_1, \dots, x_n]$ with

$$g_i = f_i(x_1 \prod_{j=1}^l \lambda_j^{u_1^{(j)}}, \dots, x_n \prod_{j=1}^l \lambda_j^{u_n^{(j)}}) \in R \text{ and } f_i \in I.$$

Since p is independent of $\lambda_1, \dots, \lambda_l$ we have

$$p = p|_{\lambda_1=1, \dots, \lambda_l=1} = \sum_i h_i|_{\lambda_1=1, \dots, \lambda_l=1} f_i \in I.$$

\square

To prove the theorem we first concentrate on special projections, so called algebraically regular projections. The general case holds also but we need another proposition for this case. Besides, most of the projections are algebraically regular (with respect to an ideal).

DEFINITION 3.15. *We call a projection algebraically regular for I if for each $i \in \{1, \dots, l\}$ the elimination ideal $J \cap K[x_1, \dots, x_n, \lambda_1, \dots, \lambda_i]$ has a finite basis \mathcal{F}_i such that in every polynomial $f \in \mathcal{F}_i$ the coefficients of the powers of λ_i (when considering f as a polynomial in λ_i) are monomials in $x_1, \dots, x_n, \lambda_1, \dots, \lambda_{i-1}$.*

For the definition of elimination ideals see for instance [CLO05].

LEMMA 3.16. *The set of projections which are not algebraically regular is contained in a finite union of hyperplanes within the space of all projections $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$.*

PROOF. It suffices to show that for the choice of $u^{(l)}$, we just have to avoid a lower-dimensional subset of $\mathbb{R}^n \setminus \{0\}$. For $u^{(1)}, \dots, u^{(l-1)}$ we can then argue inductively (however, an explicit description then becomes more technical). Assume that I is generated by f_1, \dots, f_s . As we have seen in proposition 3.10 J is generated by the polynomials

$$\tilde{f}_i := f_i(x_1 \prod_{j=1}^l \lambda_j^{u_1^{(j)}}, \dots, x_n \prod_{j=1}^l \lambda_j^{u_n^{(j)}}), \quad i = 1, \dots, s.$$

\tilde{f}_i is of the form

$$\tilde{f}_i = \sum_{\alpha \in \mathcal{A}_i} c_\alpha x^\alpha \lambda_1^{\sum \alpha_j u_j^{(1)}} \dots \lambda_l^{\sum \alpha_j u_j^{(l)}}$$

with $\mathcal{A}_i \subset \mathbb{Z}^n$ finite. It is sufficient for being an algebraically regular projection that all λ_j^k have monomial coefficients. That is the case if

$$\sum \alpha_j u_j^{(l)} \neq \sum \beta_j u_j^{(l)}$$

for all $\alpha, \beta \in \mathcal{A}_i$ with $\alpha \neq \beta$. So we have to choose $u^{(l)}$ from the subset

$$\bigcap_j \{u \in \mathbb{R}^n : \sum \alpha_i u_i^{(l)} \neq \sum \beta_i u_i^{(l)} \text{ for all } \alpha, \beta \in \mathcal{A}_j \text{ with } \alpha \neq \beta\}.$$

Hence, the algebraically non-regular projections are contained in a finite number of hyperplanes. This is an underdimensional subset of \mathbb{R}^n . \square

Now we can prove the theorem for this special case:

THEOREM 3.17. *Let $I \triangleleft K[x_1, \dots, x_n]$ be a prime ideal and $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$ be an algebraically regular projection. Then $\pi^{-1}\pi(\mathcal{T}(I))$ is a tropical variety with*

$$(7) \quad \pi^{-1}\pi(\mathcal{T}(I)) = \mathcal{T}(J \cap K[x_1, \dots, x_n]).$$

PROOF. Let $w \in \pi^{-1}\pi(\mathcal{T}(I))$. Since the right hand set of (7) is closed, we can assume without loss of generality that there exists $z' \in \mathcal{V}(I)$ and $u \in \text{span}\{u^{(1)}, \dots, u^{(l)}\}$ with $\text{ord } z' = w + u$. For any $f \in I$, the point

$$z := (z', 1)$$

is a zero of the polynomial

$$f(x_1 \prod_{j=1}^l \lambda_j^{u_1^{(j)}}, \dots, x_n \prod_{j=1}^l \lambda_j^{u_n^{(j)}}) \in R,$$

and thus $z \in \mathcal{V}(J)$. Hence, $z' \in \mathcal{V}(J \cap K[x_1, \dots, x_n])$. By Lemma 3.13, $w \in \mathcal{T}(J \cap K[x_1, \dots, x_n])$ as well.

Let now $w \in \mathcal{T}(J \cap K[x_1, \dots, x_n])$. Again we can assume that there is a $z \in \mathcal{V}(J \cap K[x_1, \dots, x_n]) \subseteq (\bar{K}^*)^n$ with $w = \text{ord}(z)$. The projection is algebraically regular which means that the generators of the elimination ideals $J \cap K[x_1, \dots, x_n, \lambda_1, \dots, \lambda_l]$ have only monomials as coefficients with respect to λ_i . By the Extension Theorem (see, e.g., [CLO05]), we can extend the root z inductively to a root $\tilde{z} \in \mathcal{V}(J)$ with the same first n entries. The definition of J says that

$$z' := (z_1 \tilde{z}_{n+1}^{u_1^{(1)}} \cdots \tilde{z}_{n+l}^{u_1^{(l)}}, \dots, z_n \tilde{z}_{n+1}^{u_n^{(1)}} \cdots \tilde{z}_{n+l}^{u_n^{(l)}})$$

is a root of I . Then

$$\text{ord}(z') = \text{ord}(z) + \sum_{i=1}^l \text{ord}(\tilde{z}_{n+i}) u^{(i)}$$

which means that $\text{ord}(z) = w \in \pi^{-1}\pi(\mathcal{T}(I))$. \square

So we have now proved the theorem for the special case of an algebraically regular projection. We needed this assumption to lift the root of a polynomial in the elimination ideal to a root of the ideal itself. But in general we can not lift the root but the point in the tropical variety. This gives us a Tropical Extension Theorem. With that the general case and therefore the Theorem 3.12 is proved.

THEOREM 3.18 (Tropical Extension Theorem). *Let $I \triangleleft K[x_0, \dots, x_n]$ be an ideal and $I_1 = I \cap K[x_1, \dots, x_n]$ be its first elimination ideal. For any $w \in \mathcal{T}(I_1)$ there exists a point $\tilde{w} = (w_0, \dots, w_n) \in \mathbb{R}^{n+1}$ with $w_i = \tilde{w}_i$ for $1 \leq i \leq n$ and $\tilde{w} \in \mathcal{T}(I)$.*

PROOF. First let $w \in \text{ord}(\mathcal{V}(I_1))$, so that there exists $z \in \mathcal{V}(I_1)$ with $\text{ord}(z) = w$. Let $\mathcal{G} = \{g_1, \dots, g_s\}$ be a reduced Gröbner basis of I with respect to a lexicographical term order with $x_0 > x_i$, $1 \leq i \leq n$. I.e.,

$$g_i = h_i(x_1, \dots, x_n)x_0^{\deg_{x_0} g_i} + \text{terms of lower degree in } x_0.$$

There are two cases to consider:

Case 1: $z \notin \mathcal{V}(h_1, \dots, h_s)$. Then by the classical Extension Theorem there is a root \tilde{z} of I which extends z , so $\text{ord}(\tilde{z}) =: \tilde{w}$ extends w .

Case 2: $z \in \mathcal{V}(h_1, \dots, h_s)$. Then $w = \text{ord}(z) \in \mathcal{T}(h_1, \dots, h_s)$. Let $\mathcal{P} = \{p_1, \dots, p_t\}$ be a tropical basis of I .

Let p_j be any of these polynomials. p_j has the form

$$p_j = q_j(x_1, \dots, x_n)x_0^{\deg_{x_0} p_j} + \text{terms of lower degree in } x_0.$$

Since \mathcal{G} is a lexicographic Gröbner basis, we have $q_j(x_1, \dots, x_n) =: \sum k_\alpha x^\alpha \in \langle h_1, \dots, h_s \rangle$. Hence, the minimum

$$\min_{\alpha} \{\text{ord}(k_\alpha) + \alpha_1 x_1 + \dots + \alpha_n x_n\}$$

is attained twice at w . We can pick a sufficiently small value $w_0^{(j)} \in \mathbb{R}$ so that all terms $x_1^{m_1} \dots x_n^{m_n} x_0^{m_0}$ of p_j with $m_0 < \deg_{x_0} p_j$ have a larger value $m_1 w_1 + \dots + m_n w_n + m_0 w_0^{(j)}$. But then the minimum of all values of all terms of p_j is attained at least twice; it is

$$\min_{\alpha} \{\text{ord}(k_\alpha) + \alpha_1 x_1 + \dots + \alpha_n x_n\} + \deg_{x_0} p_j \cdot w_0^{(j)}.$$

So $(w_0^{(j)}, w_1, \dots, w_n) \in \mathcal{T}(h_j)$.

By setting $w_0 = \min_j \{w_0^{(j)}\}$ and $\tilde{w} := (w_0, \dots, w_n) \in \mathcal{T}(I)$, we obtain the desired extension of w .

Let now $w = \lim_{i \rightarrow \infty} w^{(i)}$ be in the closure of $\text{ord}(\mathcal{V}(I_1))$. Then there exist $\tilde{w}^{(i)} \in \mathcal{T}(I)$ with $\tilde{w}_j^{(i)} = w_j^{(i)}$ for $1 \leq j \leq n$. Let $\mathcal{P} = \{p_1, \dots, p_t\}$ be again a tropical basis of I . Then we can assume w.l.o.g. that the minimum of $\text{trop}(p_k)$, $1 \leq k \leq t$ for $\tilde{w}^{(i)}$ is attained at the same terms. This gives us conditions for the $\tilde{w}_0^{(i)}$:

$$k^{(i)} \leq \tilde{w}_0^{(i)} \leq l^{(i)} \quad (\text{one of them can be } \pm\infty).$$

These bounds vary continuously with $w^{(i)}$. So we can choose \tilde{w}_0 arbitrarily in $[\lim k^{(i)}, \lim l^{(i)}]$ (only one of the limites can be $\pm\infty$). \square

Now we can construct the tropical variety $\mathcal{T}(I)$ via projections.

THEOREM 3.19. *Let $I \triangleleft K[x_1, \dots, x_n]$ be a prime ideal generated by the polynomials f_1, \dots, f_r . Then there exist $g_0, \dots, g_n \in I$ such that*

$$(8) \quad \mathcal{T}(I) = \bigcap_{i=0}^n \mathcal{T}(g_i)$$

and thus $\mathcal{G} := \{f_1, \dots, f_r, g_0, \dots, g_n\}$ is a tropical basis for I of cardinality $r + n + 1$.

PROOF. Let \mathcal{X} be the complete set generated by all underlying affine subspaces of the cells of $\mathcal{T}(I)$. Then combine Theorem 3.12, Corollary 3.9 and Lemma 3.14 to get the desired result. \square

Moreover we can say:

COROLLARY 3.20. *Let $I \triangleleft K[x_1, \dots, x_n]$ be a prime ideal of dimension $\dim(I) = m$ generated by the polynomials f_1, \dots, f_r . Then there exist $g_0, \dots, g_{n-m} \in I$ such that the cells of maximal dimension of $\bigcap_{i=0}^{n-m} \mathcal{T}(g_i)$ gives the tropical variety $\mathcal{T}(I)$, i.e.*

$$\mathcal{T}(I) = \bigcap_{i=0}^{n-m} \overline{\mathcal{T}(g_i)^m \setminus \mathcal{T}(g_i)^{m-1}}$$

PROOF. The proof of Lemma 3.6 gives us a projection $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$ such that the set of points T with

$$T := \{y \in \mathcal{T}(I) \mid \exists x \in \mathcal{T}(I), x \neq y : \pi(y) = \pi(x)\}$$

has dimension less than m . Combining this with Lemma 3.7 and Lemma 3.14 gives us the assertion. \square

For an arbitrary ideal I we also have the following theorem.

THEOREM 3.21. *Let $I \triangleleft K[x_1, \dots, x_n]$ be an arbitrary ideal. Then there are $n + 1$ polynomials g_0, \dots, g_n with*

$$(9) \quad \mathcal{T}(I) = \bigcap_{j=0}^n \mathcal{T}(g_j)$$

PROOF. First let $I = \sqrt{I}$. Then $I = P_1 \cap \dots \cap P_r$ with prime ideals P_i (see for example [AM69]). For each P_i there are polynomials $g_{i,0}, \dots, g_{i,n}$ such that

$$\mathcal{T}(P_i) = \bigcap_{j=0}^n \mathcal{T}(g_{i,j})$$

But then

$$\begin{aligned} \mathcal{T}(I) &= \bigcup_{i=1}^r \mathcal{T}(P_i) = \bigcup_{i=1}^r \bigcap_{j=0}^n \mathcal{T}(g_{i,j}) = \\ &= \bigcap_{j=0}^n \bigcup_{i=1}^r \mathcal{T}(g_{i,j}) = \bigcap_{j=0}^n \mathcal{T}(g_{1,j} \cdots g_{r,j}) \end{aligned}$$

So if we choose $g_j := g_{1,j} \cdots g_{r,j}$ we have the desired assertion for radical ideals.

If I is not a radical then there are $g_0, \dots, g_n \in \sqrt{I}$ with

$$\mathcal{T}(I) = \mathcal{T}(\sqrt{I}) = \bigcap_{j=0}^n \mathcal{T}(g_j).$$

Going over to appropriate powers of g_j we get the desired result. \square

3. Example

Theorem 3.19 (and its proof) gives not only the existence of such projections, it provides also an algorithm for computing a tropical variety. Here we will give an example.

Let $I \triangleleft \mathbb{Q}[x, y, z]$ be generated by

$$f_1 := 2 + y - 4x^2y + x^2y^2 + 2xy^2$$

$$f_2 := xyz - 2z + 4xyz^2 - 2 + z^2$$

and let ord be the 2-adic valuation.

For the first projection π_3 we take the one with kernel $(0, 0, 1)$, so it is the projection on the plane $z = 0$. Then $J \cap K[x, y, z]$ is generated by f_1 and the tropical variety is

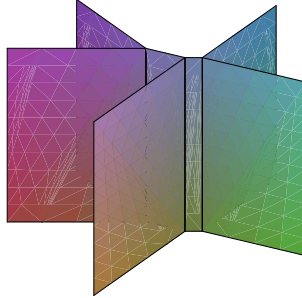


FIGURE 3. The tropical hypersurface $\mathcal{T}(f_1)$

$$V_1 := \{(x, y, z) \in \mathbb{R}^3 \mid x = 1/2 - y, 1 \leq y \leq 2, z \in \mathbb{R}\},$$

$$V_2 := \{(x, y, z) \in \mathbb{R}^3 \mid y = 1, -1/2 \leq x, z \in \mathbb{R}\},$$

$$V_3 := \{(x, y, z) \in \mathbb{R}^3 \mid x = 1, y \leq -2, z \in \mathbb{R}\},$$

$$V_4 := \{(x, y, z) \in \mathbb{R}^3 \mid y = -1 - 2x, x \leq -3/2, z \in \mathbb{R}\},$$

$$V_5 := \{(x, y, z) \in \mathbb{R}^3 \mid x = -1 - y, y \leq -2, z \in \mathbb{R}\},$$

$$V_6 := \{(x, y, z) \in \mathbb{R}^3 \mid y = -2x, -1/2 \leq x \leq 1, z \in \mathbb{R}\},$$

$$V_7 := \{(x, y, z) \in \mathbb{R}^3 \mid y = 2, x \leq -3/2, z \in \mathbb{R}\}.$$

The kernel of the second projection π_2 should not lie in the subspaces parallel to the supporting affine subspaces of the above sets V_i , $i = 1, \dots, 7$. We can choose for example $\langle (1, 1, 0) \rangle$.

$$\begin{aligned}
J \cap K[x, y, z] = & \\
\langle & 192xyz + 1008xyz^2 + 16xy + 2176xyz^3 + 1996xyz^4 + 448z^5xy \\
& - 260z^6xy + 1153xyz^8 + 712xyz^7 - 128x^2z - 32y^2z - 1728x^2z^3 \\
& - 896x^2z^2 - 512x^2z^4 - 594y^2z^3 - 240y^2z^2 - 666y^2z^4 - 368x^2z^7 \\
& + 288x^2z^6 + 1120x^2z^5 + 64x^2z^8 + 52y^2z^7 - 16y^2z^6 - 335y^2z^5 + 16y^2z^8 \rangle
\end{aligned}$$

The elimination ideal is generated by our third polynomial f_3 . This gives us a 1-dimensional set \mathcal{X} , which consists of the supporting affine subspaces of the tropical prevariety $\mathcal{T}(f_1) \cap \mathcal{T}(f_3)$.

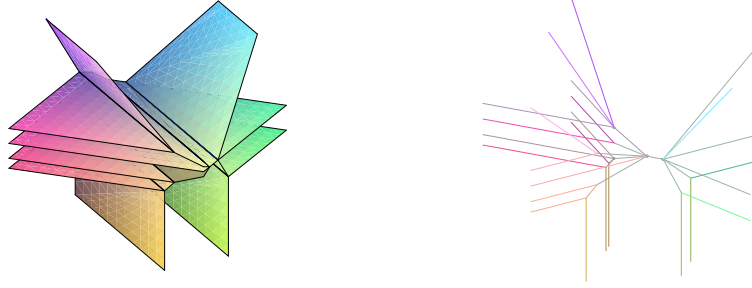


FIGURE 4. The tropical hypersurface $\mathcal{T}(f_3)$ and the intersection $\mathcal{T}(f_1) \cap \mathcal{T}(f_3)$

To choose a geometrically regular projection with respect to $\mathcal{T}(f_1) \cap \mathcal{T}(f_3)$ we can for example take $(2, 4, 1)$ as a kernel for π_1 . Computing the polynomial f_4 in the elimination ideal (it has 63 terms) and the intersection of the tropical variety of all three polynomials we get the intersection seen on the left in Figure 5.

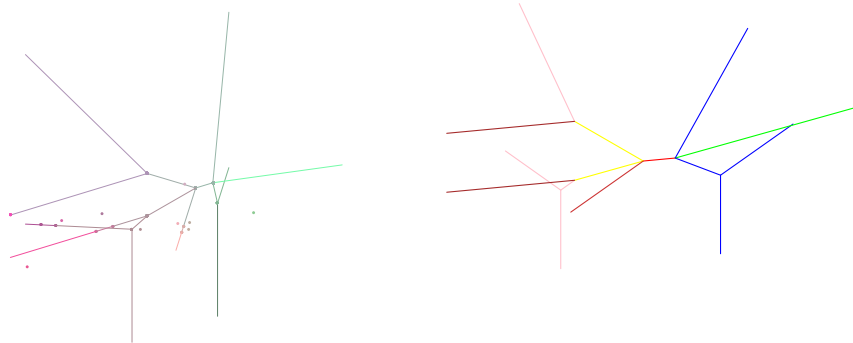


FIGURE 5. The intersection $\mathcal{T}(f_1) \cap \mathcal{T}(f_3) \cap \mathcal{T}(f_4)$ and the tropical variety $\mathcal{T}(I)$

We can see that there are many 0-cells not belonging to the tropical variety $T(I)$. So we need one more polynomial to cancel them. We can take for example the polynomial corresponding to the projection π_0 with kernel $\langle(1, 3, 1)\rangle$. It has 47 terms and the resulting intersection of all four tropical hypersurfaces is the tropical variety $T(I)$, see Figure 5.

4. The Newton polytopes for the linear case

As mentioned earlier, an ideal generated by linear forms may not have a small tropical basis if we restrict the basis to consist of linear forms. Using our results from Section 2, we can provide a short basis at the price of increased degrees. A natural question is to provide a good characterization for the Newton polytopes of the resulting basis polynomials. Here, we briefly discuss the special case of a prime ideal I generated by two linear polynomials

$$F = \sum_{i=1}^n a_i x_i + a_{n+1}, \quad G = \sum_{i=1}^n b_i x_i + b_{n+1} \in K[x_1, \dots, x_n].$$

In order to characterize the Newton polytope of the additional polynomials in the tropical basis, we consider the resultant of the polynomials f, g

$$\begin{aligned} f &= a_1 x_1 \lambda^{v_1} + \dots + a_n x_n \lambda^{v_n} + a_{n+1} \\ g &= b_1 x_1 \lambda^{v_1} + \dots + b_n x_n \lambda^{v_n} + b_{n+1} \end{aligned}$$

in $K[x_1, \dots, x_n, \lambda]$. For a general introduction to the theory of resultants see [CLO05]. Assume that the components v_i are distinct. Then w.l.o.g. we can assume $v_1 > v_2 > \dots > v_n > v_{n+1} := 0$.

In order to apply the results of Gelfand, Kapranov and Zelevinsky [GKZ90] regarding the Newton polytope of the resultant, we consider the representation

$$\text{Res}_\lambda(f, g) = \sum_{p, q} c_{p, q} a^p b^q x^{p+q}$$

with $p = (p_1, \dots, p_{n+1}), q = (q_1, \dots, q_{n+1}) \in \mathbb{Z}_+^{n+1}$ and understand the coefficients $a_i, b_j, i = 1, \dots, n+1, j = 1, \dots, n+1$ as variables. Then the following hold.

PROPOSITION 3.22. *The Newton polytope is contained in the set $\mathcal{Q}_n \subset \mathbb{Z}^{2n+2}$ of nonnegative integer points (p, q) with*

$$\begin{aligned} (1) \quad & \sum_{i=1}^{n+1} p_i = \sum_{j=1}^{n+1} q_j = v_1, \\ (2) \quad & \sum_{i=1}^{n+1} v_i p_i + \sum_{j=1}^{n+1} v_j q_j = v_1^2, \\ (3) \quad & \sum_{\substack{1 \leq k \leq n \\ 0 \leq v_1 - v_k \leq i}} (i - v_1 + v_k) p_k + \sum_{\substack{1 \leq l \leq n \\ 0 \leq v_1 - v_l \leq j}} (j - v_1 + v_l) q_l \geq ij \quad (0 \leq i, j \leq v_1). \end{aligned}$$

PROOF. In [GKZ90, Proposition 1] and [GKZ90, Theorem 4] the resultant of two arbitrary polynomials in the variable λ is considered. So the resultant is a polynomial in the $v_1 + 1$ coefficients of the first and the $v_1 + 1$ coefficient of the second polynomial. This gives points $(p, q) \in \mathbb{Z}^{2v_1+2}$. But in our case many of these coefficients are 0. So we have to consider only points (p, q) where

the corresponding entries are 0. So we can reduce to the case of $(p, q) \in \mathbb{Z}^{2n+2}$. This proves the Proposition. \square

COROLLARY 3.23. *The set of integer points in the Newton polytope of the resultant, $\text{New}(\text{Res}_\lambda(f, g)) \subset \mathbb{Z}^n$, is contained in the image of \mathcal{Q}_n under the mapping*

$$(p_1, \dots, p_{n+1}, q_1, \dots, q_{n+1}) \mapsto (p_1 + q_1, \dots, p_n + q_n)$$

PROOF. $p_i + q_i$ is just the exponent of x_i in the representation $\text{Res}_\lambda(f, g) = \sum_{p, q} c_{p, q} a^p b^q x^{p+q}$ of the resultant. But after inserting the coefficients of the polynomials f and g some terms may cancel. Nevertheless the image of \mathcal{Q}_n under the mapping above is an upper bound for the Newton polytope. \square

[GKZ90] tells us that the vertices (p, q) of \mathcal{Q}_n are of the following form:

$$(1) \quad q_j = \sum_{\substack{1 \leq i \leq n \\ \sum_{k=1}^i p_k = v_1 - v_j}} (v_i - v_{i+1}) \text{ and}$$

$$(2) \quad p_i = \sum_{\substack{1 \leq j \leq n \\ \sum_{l=1}^j p_l = v_1 - v_i}} (v_j - v_{j+1}).$$

THEOREM 3.24. \mathcal{Q}_n has

$$N(n) := \sum_{i_1=1}^{n+1} \sum_{i_2=1}^{i_1} \dots \sum_{i_n=1}^{i_{n-1}} 1$$

vertices.

PROOF. The entries of p have to be of the following form:

- $p_1 \in \{v_1 - v_j \mid 1 \leq j \leq n+1\}$
- $p_i \in \{v_1 - v_j - \sum_{k=1}^{i-1} p_k \mid v_j \leq v_1 - \sum_{k=1}^{i-1} p_k\}$ for $2 \leq i \leq n$
- $p_{n+1} = v_1 - \sum_{k=1}^{n-1} p_k$

This gives us $\sum_{i_1=1}^{n+1} \sum_{i_2=1}^{i_1} \dots \sum_{i_n=1}^{i_{n-1}} 1$ possibilities. \square

n	$N(n)$
1	2
2	6
3	20
4	70
5	252
6	924
7	3432

For small n this gives the following number of vertices:

COROLLARY 3.25. *The number of vertices of \mathcal{Q}_n equals the number of points (p_1, \dots, p_{n+1}) with $\sum p_i = n$.*

So if we look at the images of the vertices of \mathcal{Q}_n we can decide how many vertices $\text{New}(\text{Res}_\lambda(f, g))$ has.

COROLLARY 3.26. *The set of vertices of $\text{New}(\text{Res}_\lambda(f, g))$ is contained in the image of the vertices of \mathcal{Q}_n under the mapping above.*

If not all v_i are pairwise disjoint, then we have to look at the subset \mathcal{Q}_m with $m+1 = |\{v_1, \dots, v_n, 0\}|$, $\{v_1, \dots, v_n, 0\} = \{w_1, \dots, w_m, w_{m+1} = 0\}$. Then the integer points of $\text{New}(\text{Res}_\lambda(f, g))$ are the exponents of the terms in all polynomials

$$(10) \quad \prod_{j=1}^m \left(\sum_{\substack{i \in \{1, \dots, n\} \\ v_i = w_j}} a_i x_i \right)^{p_j} \cdot (a_{n+1} + \sum_{\substack{i \in \{1, \dots, n\} \\ v_i = 0}} a_i x_i)^{p_{m+1}} \\ \cdot \prod_{j=1}^m \left(\sum_{\substack{i \in \{1, \dots, n\} \\ v_i = w_j}} b_i x_i \right)^{q_j} \cdot (b_{n+1} + \sum_{\substack{i \in \{1, \dots, n\} \\ v_i = 0}} b_i x_i)^{q_{m+1}}$$

where $(p, q) \in \mathbb{Z}^{2m+2}$ ranges over all points of \mathcal{Q}_m

EXAMPLE 3.27. Let $I = \langle 2x + y - 4, x + 2y + z - 1 \rangle$ and $\text{ord}(\cdot)$ be the 2-adic valuation (see Figure 6 for a figure of $\mathcal{T}(I)$).

Actually, the first projection can be chosen arbitrarily. We choose a projec-

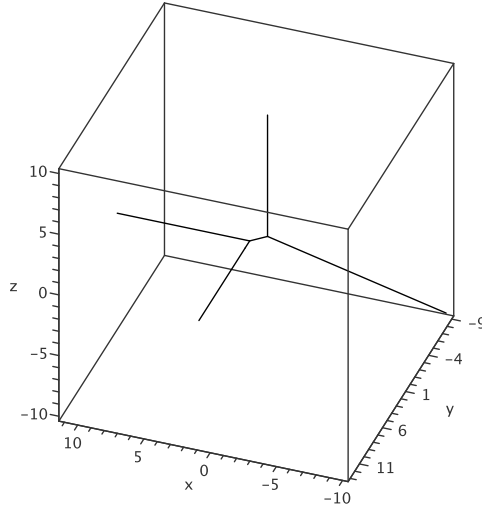


FIGURE 6. Tropical line $\mathcal{T}(I)$ in 3-space

tion π_1 whose kernel is generated by $(0, 0, 1)$. Then the tropical hypersurface $\pi_1^{-1}\pi_1(\mathcal{T}(I))$ satisfies

$$\pi_1^{-1}\pi_1(\mathcal{T}(I)) = \mathcal{T}(2x + y - 4),$$

and the Newton polytope of that polynomial is a triangle.

Now we choose π_2 with kernel $(1, 2, 0)$. Then the resultant is

$$\text{Res}_\lambda(2x\lambda + y\lambda^2 - 4, x\lambda + 2y\lambda^2 + z - 1) = y \cdot (6x^2 + 6x^2z + 49y + 14yz + yz^2)$$

The exponents of the resultant are the exponents of the polynomial (10) for the set \mathcal{Q}_m with $m = 2$ ($m + 1 = |\{2, 1, 0\}|$),

$$\mathcal{Q}_2 = \{(2, 0, 0, 0, 0, 2), (1, 1, 0, 0, 1, 1), (1, 0, 1, 1, 0, 1), (1, 0, 1, 0, 2, 0), (0, 2, 0, 1, 0, 1), (0, 1, 1, 1, 1, 0), (0, 0, 2, 2, 0, 0)\}.$$

We get the exponents

$$(2, 1, 0), (2, 1, 1), (0, 2, 0), (0, 2, 1), (0, 2, 2).$$

The convex hull is a quadrangle.

For a picture of the intersection of $\mathcal{T}(2x + y - 4)$ and $\mathcal{T}(6x^2 + 6x^2z + 49y + 14yz + yz^2)$, see Figure 7.

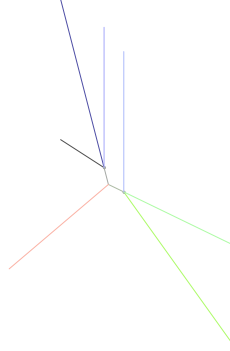


FIGURE 7. The intersection $\mathcal{T}(2x + y - 4) \cap \mathcal{T}(6x^2 + 6x^2z + 49y + 14yz + yz^2)$

By choosing π_3 with kernel generated by $(1, 0, 1)$, we obtain the polynomial $3xy + 2x - yz + 4z$. Its Newton polytope is again a quadrangle.

The intersection of all three tropical hypersurfaces gives us the tropical variety $\mathcal{T}(I)$. So we do not need another projection. Adding these three nonlinear polynomials to the basis of I yields a tropical basis.

5. Bounds on the degree of the polynomials

In this section we try to determine the effect of the entries of the $u^{(i)}$, $i = 1, \dots, l$ on the degree of the generating polynomial.

First we examine the case of a projection $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ and a 1-dimensional tropical variety $\mathcal{T}(I)$, where I is generated by two linear polynomials f_1, f_2 :

$$\begin{aligned} f_1 &:= a_1x_1 + a_2x_2 + a_3x_3 + a_4 \\ f_2 &:= b_1x_1 + b_2x_2 + b_3x_3 + b_4 \end{aligned}$$

The kernel of π is 1-dimensional, generated by a vector v . Let for example $v = (1, 2, 3)$. Then the resultant looks like:

$$\det \begin{vmatrix} a_3x_3 & a_2x_2 & a_1x_1 & a_4 & 0 \\ 0 & a_3x_3 & a_2x_2 & a_1x_1 & a_4 \\ b_2x_2 & b_1x_1 & b_4 & 0 & 0 \\ 0 & b_2x_2 & b_1x_1 & b_4 & 0 \\ 0 & 0 & b_2x_2 & b_1x_1 & b_4 \end{vmatrix}$$

So the resultant has total degree at most 4, and therefore the generating polynomial of the elimination ideal, too.

EXAMPLE 3.28. If you take the following polynomials you get an example, where it has indeed total degree 4:

$$\begin{aligned} f_1 &:= x_1 + 2x_2 + 3x_3 + 1 \\ f_2 &:= 2x_1 + 5x_2 + 7 \end{aligned}$$

Let $v = (1, 2, 3)$. Then we get the modified polynomials

$$\begin{aligned} \tilde{f}_1 &:= x_1\lambda + 2x_2\lambda^2 + 3x_3\lambda^3 + 1 \\ \tilde{f}_2 &:= 2x_1\lambda + 5x_2\lambda^2 + 7 \end{aligned}$$

Elimination of λ gives the polynomial

$$60x_1^3x_3 + 25x_1^2x_2^2 - 1428x_1x_2x_3 + 405x_2^3 + 3087x_3^2$$

Let now be $v = (v_1, v_2, v_3)$ be arbitrary. Then the total degree of the generating polynomial can again be bounded by the degree of the resultant, which is at most $2 \cdot \max\{v_1, v_2, v_3\}$.

EXAMPLE 3.29. Let $v = (0, 1, 1)$ and

$$\begin{aligned} f_1 &:= x_1 + 2x_2 + 3x_3 + 1 \\ f_2 &:= 2x_1 + 5x_2 + 2x_3 + 7 \end{aligned}$$

Then modifying with λ and eliminating λ gives

$$-4x_1x_3 + x_1x_2 - 19x_3 - 9x$$

So the total degree is $2 = 2v_2$.

THEOREM 3.30. Let now f_1, f_2 be arbitrary, not necessary linear. Then the resultant has total degree at most $2 \cdot \deg f_1 \cdot \deg f_2 \cdot \max\{v_1, v_2, v_3\}$.

PROOF. If there is a term $x_i^{\deg f_i}$ in f_i with $v_i = \max\{v_1, v_2, v_3\}$ then $\deg f_i \cdot \max\{v_1, v_2, v_3\}$ is the λ -degree of the modified polynomial \tilde{f}_i . So it is an upper bound for the λ -degree of the modified polynomial. Then the Sylvester matrix has at most $\deg f_2 \cdot \max\{v_1, v_2, v_3\}$ rows with entries of total degree at most $\deg f_1$ and $\deg f_1 \cdot \max\{v_1, v_2, v_3\}$ rows with entries of total degree at most $\deg f_2$. This gives the upper bound. \square

EXAMPLE 3.31. Let $v = (0, 1, 3)$ and

$$f_1 = x_3^2 + 3x_2x_3 - 5$$

$$f_2 = x_2^3 + 3x_1^3 + 2x_1x_2 - 5x_2^2x_1 - 1$$

Then the modified polynomials have λ -degree 6 and 9 and the Sylvester matrix has the following structure:

$$\begin{pmatrix} * & 0 & * & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & * & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & * & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & * & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 & * & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & * & 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * & 0 & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & * & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & * & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & * & * & * & * \end{pmatrix}$$

So if you multiply the entries in the diagonal you get a term of degree $2 \cdot \deg f_1 \cdot \deg f_2 \cdot \max\{v_1, v_2, v_3\} = 2 \cdot 2 \cdot 3 \cdot 3 = 36$.

But if we have more than two defining polynomials of our ideal I we cannot calculate a classical resultant to find an upper bound for the degree of the polynomial.

Let f_1, \dots, f_{n-m} be polynomials in $K[x_1, \dots, x_n]$ defining an m -dimensional tropical variety $\mathcal{T}(f_1) \cap \dots \cap \mathcal{T}(f_{n-m}) = \mathcal{T}(\langle f_1, \dots, f_{n-m} \rangle)$. Let

$$\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-m+1}$$

an arbitrary projection with kernel generated by $u^{(1)}, \dots, u^{(n-m-1)} \in \mathbb{Z}^n$. Let $U = (u_i^{(j)})_{i,j}$ and abbreviate $r = n - m$.

Then we modify our polynomials to get polynomials

$$\tilde{f}_1, \dots, \tilde{f}_r \in K[x_1, \dots, x_n][\lambda_1, \dots, \lambda_{r-1}].$$

Let A_i be the support of f'_i as a polynomial in the λ_i . So A_i lies in the sublattice of \mathbb{Z}^n generated by the rows of U .

To these subsets A_i there is a polynomial $R_{A_1, \dots, A_r}(\tilde{f}_1, \dots, \tilde{f}_r)$, the mixed- (A_1, \dots, A_r) -resultant, see [GKZ08], with the following properties:

- It is a polynomial in $K[x_1, \dots, x_n]$.
- It is a linear combination with integer coefficients of the coefficients of the \tilde{f}_i regarded as polynomials in the λ_i .
- If the \tilde{f}_i have a common root then $R_{A_1, \dots, A_r}(\tilde{f}_1, \dots, \tilde{f}_r) = 0$
- It is homogeneous with respect to each \tilde{f}_i .
- It is a polynomial function of the \tilde{f}_i , i.e. it lies in the ideal generated by $\tilde{f}_1, \dots, \tilde{f}_r$ and therefore in I .

To state the degree of the generating polynomial explicitly we need the degree of homogeneity of the (A_1, \dots, A_r) -resultant. Therefore we use mixed volumes:

DEFINITION 3.32. *Let P_1, \dots, P_n be polytopes in \mathbb{R}^n . Then the mixed volume $MV_n(P_1, \dots, P_n)$ is defined by the coefficient of $\lambda_1 \cdot \dots \cdot \lambda_n$ of the polynomial*

$$\text{vol}(\lambda_1 \cdot P_1 + \dots + \lambda_n \cdot P_n)$$

Let $Q_i = \text{New}(\tilde{f}_i) = \text{conv } A_i$, $i = 1, \dots, r$. Then the following holds:

PROPOSITION 3.33 ([GKZ08] Chapter 8, Prop. 1.6). *The degree of homogeneity of $R_{A_1, \dots, A_r}(\tilde{f}_1, \dots, \tilde{f}_r)$ with respect to \tilde{f}_i is equal to the mixed volume of all the polytopes Q_j with $j \neq i$.*

If $r = 2$ then the (A_1, A_2) -resultant is the normal resultant $\text{Res}_\lambda(f'_1, f'_2)$ and the mixed volume of $\text{New}(f'_i)$ is $\deg_\lambda(f'_j)$, $j \neq i$, if the f'_i are irreducible. So the total degree of the resultant is at most

$$2 \cdot \deg \tilde{f}_1 \cdot \deg \tilde{f}_2 = \deg \tilde{f}_1 \cdot MV_1(Q_2) + \deg \tilde{f}_2 \cdot MV_1(Q_1).$$

This leads to the following theorem:

THEOREM 3.34. *Let $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$ be a projection and $I = \langle f_1, \dots, f_{n-m} \rangle$ an m -dimensional ideal in $K[x_1, \dots, x_n]$. The total degree of the generating polynomial of the preimage of the projection $\pi^{-1}\pi(\mathcal{T}(I))$ is at most*

$$\sum_{i=1}^r \deg(\tilde{f}_i) \cdot MV_{n-m-1}(Q_1, \dots, \hat{Q}_i, \dots, Q_{n-m})$$

where Q_i are the Newton polytopes of the modified polynomials \tilde{f}_i with respect to the new variables $\lambda_1, \dots, \lambda_{n-m-1}$.

Mixed fiber polytopes

Fiber polytopes were first introduced by Billera and Sturmfels in [BS92]. They arise when you look at projections $Q = \pi(P)$ of a convex polytope P . A fiber polytope is a convex polytope which is the average of all fibers $\pi^{-1}(x), x \in Q$. It turned out that fiber polytopes generalize the concept of secondary polytopes which were first introduced by Gel'fand, Kapranov and Zelevinsky, see [GKZ08]. With the concept of fiber polytopes one can define mixed fiber polytopes (in the same way as one can define mixed volumes with volumes). It turned out that mixed fiber polytopes describe the Newton polytope of a certain polynomial, the generating polynomial of a tropical hypersurface which is a projection of a tropical variety. This was shown by Esterov and Khovanskii, see [EK08], or Sturmfels and Yu [SY08].

In this chapter we give a basic definition of fiber polytopes and describe their connections to certain polyhedral subdivisions. After that we specialize to the case of secondary polytopes (see [BS92, Zie98]) and introduce for further use in the next chapter the notion of mixed fiber polytopes. At last we give a short excursus on mixed volumes which we will need in Chapter 6. The results of this chapter can be found for example in [Zie98]. We need them as a preparation for Chapter 5.

1. Fiber polytopes

Let $Q \subset \mathbb{R}^q$ be a polytope. A *polytope bundle* over Q is a function which assigns to each $x \in Q$ a polytope $\mathcal{B}(x) \in \mathbb{R}^p$ in a nicely way (for further details see [BS92]). In our case we take the fibers $\mathcal{B}(x) = \pi^{-1}(x) \cap P$ of a projection $\pi : P \rightarrow Q$. The Minkowski integral is defined as the subset

$$\int_Q B(x)dx := \left\{ \int_Q \gamma(x)dx \mid \gamma \text{ is a section of } \pi \right\}$$

where a section γ is a (continuous) map $\mathbb{R}^q \rightarrow \mathbb{R}^p$ with $\pi \circ \gamma = id$ and $\gamma(Q) = \gamma(\pi(P)) \subseteq P$. Now we define:

DEFINITION 4.1. *Let $\pi : \mathbb{R}^p \rightarrow \mathbb{R}^q$ be a linear map, $P \subset \mathbb{R}^p$ a p -polytope and $Q := \pi(P) \subset \mathbb{R}^q$ a q -polytope. Then the fiber polytope is defined as the Minkowski integral*

$$\Sigma_\pi(P) = \int_Q (\pi^{-1}(x) \cap P)dx.$$

Each fiber $\pi^{-1}(x) \cap P$ has dimension $p - q$ and so the fiber polytope is a polytope of dimension $p - q$ in \mathbb{R}^p (see [BS92]). It lies in an affine subspace which is parallel to the kernel of π .

Because P and Q are polyhedral complexes we need only to consider sections γ which are piecewise linear. Integration of the linear components uses then classical Riemann integrals:

If $R \subseteq Q$ is a polytope on which γ is linear then

$$\int_R \gamma(x) dx = \text{vol}(R) \cdot \gamma(r_0)$$

where r_0 is the barycenter of R .

So we can integrate the section γ componentwise.

EXAMPLE 4.2.

Let $P \subset \mathbb{R}^3$ be the polytope with vertices $\{(0, 0, 0), (2, 0, 0), (0, 2, 0), (0, 0, 2)\}$ and π the map

$$\pi : \mathbb{R}^3 \rightarrow \mathbb{R}, x \mapsto (1, 1, 0) \cdot x$$

Then the image of P under π is the interval $[0, 2]$.

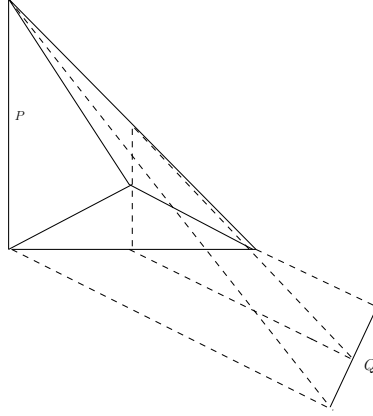


FIGURE 1. The polytope P and its projection Q

For $x \in [0, 2]$ the fiber $\pi^{-1}(x)$ is a quadrangle with a fixed normal fan. Therefore we can express the Minkowski integral by the sum.

$$\begin{aligned} \Sigma_\pi(P) &= \sum_{i=0}^1 (\pi^{-1}(i + \frac{1}{2}) \cap P) = 2 \cdot (\pi^{-1}(1) \cap P) = \\ &= \text{conv}\{(2, 0, 0), (0, 2, 0), (2, 0, 2), (0, 2, 2)\}, \text{ see figure 2.} \end{aligned}$$

Each piecewise linear section is a monotonous path from a point p_0 with $\pi(p_0) = 0$ to a point p_1 with $\pi(p_1) = 2$.

Define the following linear sections:

$$\begin{aligned} \gamma_1 : t &\mapsto t \cdot (1, 0, 0), t \in \mathbb{R} \\ \gamma_2 : t &\mapsto t \cdot (0, 1, 0), t \in \mathbb{R} \\ \gamma_3 : t &\mapsto (0, 0, 2) + t \cdot (1, 0, -1), t \in \mathbb{R} \\ \gamma_4 : t &\mapsto (0, 0, 2) + t \cdot (0, 1, -1), t \in \mathbb{R} \end{aligned}$$

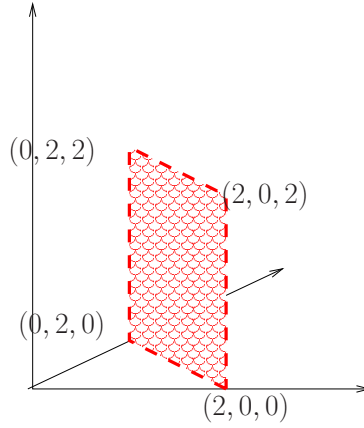


FIGURE 2. The fiber polytope

Then the corresponding points in the fiber polytope are

$$\int_Q \gamma_1(x) dx = \text{vol}(Q) \cdot \gamma_1(1) = (2, 0, 0), \int_Q \gamma_2(x) dx = \text{vol}(Q) \cdot \gamma_2(1) = (0, 2, 0),$$

$$\int_Q \gamma_3(x) dx = \text{vol}(Q) \cdot \gamma_3(1) = (2, 0, 2), \int_Q \gamma_4(x) dx = \text{vol}(Q) \cdot \gamma_4(1) = (0, 2, 2).$$

These are exactly the vertices of the fiber polytope as computed above. Theorem 4.7 shows that this is not an accident.

We can also compute fiber polytopes if P or Q are not full dimensional. Then we have to restrict the projection π to the corresponding affine subspaces. In the full dimensional case we can compute the fiber polytopes using the program TrIm (see [SY08]).

2. Polyhedral subdivisions

Let $\pi : \mathbb{R}^p \rightarrow \mathbb{R}^q$ be again a projection and $\pi(P) = Q$ for polytopes P, Q .

DEFINITION 4.3. *A polyhedral subdivision of Q is π -induced if it is of the form $\{\pi(F) : F \in \mathcal{F}\}$ where \mathcal{F} is a collection of faces of P , and if $\pi(F) \subseteq \pi(F')$ implies $F = F' \cap \pi^{-1}\pi(F)$. We will denote this subdivision by \mathcal{F} .*

EXAMPLE 4.4. The subdivisions of Chapter 1 where we project the lower faces of the extended newton polytope on a subdivision of the ordinary newton polytope are π -induced where π is the projection forgetting the last coordinate. All subdivisions which arise in this way as the projection of lower faces are called *regular*.

DEFINITION 4.5. *A π -induced subdivision \mathcal{F} is tight if $\dim(F) = \dim(\pi(F))$ for all q -dimensional $F \in \mathcal{F}$.*

We can order the set of all π -induced subdivisions partially by

$$\mathcal{F}_1 \leq \mathcal{F}_2 \Leftrightarrow \bigcup_{F \in \mathcal{F}_1} F \subseteq \bigcup_{F \in \mathcal{F}_2} F$$

We can get regular π -induced subdivisions in the following way:

DEFINITION 4.6. *Let $\pi : \mathbb{R}^p \rightarrow \mathbb{R}^q$ be a projection with $\pi(P) = Q$ and let $c \in \mathbb{R}^p$. Define a linear map*

$$\begin{aligned} \pi^c &: \mathbb{R}^p \rightarrow \mathbb{R}^{q+1} \\ x &\mapsto \begin{pmatrix} \pi(x) \\ cx \end{pmatrix} \end{aligned}$$

Then $Q^c := \pi^c(P)$ is a polytope which projects to Q under the map which forgets the last coordinate.

The lower faces of Q^c induce a subdivision of P , called π -coherent.

A π -coherent subdivision is obviously regular and by choosing

$$\mathcal{F}^c := \{P \cap (\pi^c)^{-1}(F) \mid F \text{ a lower face of } Q^c\}$$

it is π -induced.

Note that not all regular π -induced subdivisions are π -coherent. For an example see [Zie98].

Sturmfels and Billera showed that there is an important correspondence between fiber polytopes and the set of all π -coherent subdivisions:

THEOREM 4.7 (Billera-Sturmfels [BS92]). *The face lattice of $\Sigma_\pi(P)$ is isomorphic to the poset of all π -coherent subdivisions of $\pi(P)$. Here the vertices of $\Sigma_\pi(P)$ correspond to the tight π -coherent subdivisions of $\pi(P)$*

EXAMPLE 4.8. In example 4.2 the given sections are the mappings giving the tight π -coherent subdivisions of $Q = \pi(P)$, so it was not accidental that we computed the vertices of Q .

3. Secondary polytopes

A secondary polytope is a special case of a fiber polytope. It arises as a fiber polytope of a projection of an n -simplex. Let $\Delta_n := \text{conv}\{e_i : 0 \leq i \leq n\}$ where $e_0 := 0$ and e_i the i -th standard unit vector.

DEFINITION 4.9. *Let Q be a q -polytope with $n+1$ vertices v_0, \dots, v_n and π the projection $\mathbb{R}^n \rightarrow \mathbb{R}^q$ which sends e_i to v_i . Then the secondary polytope of Q is defined as*

$$\Sigma(Q) := (q+1)\Sigma_\pi(\Delta_n)$$

In the case of the projection of an n -simplex every regular subdivision is π -coherent. Together with Theorem 4.7 Billera and Sturmfels showed

COROLLARY 4.10 ([BS92], see also [Zie98]). *The secondary polytope $\Sigma(Q)$ is the convex hull in \mathbb{R}^n of the vectors*

$$\Phi_\Delta := \frac{1}{\text{vol}(Q)} \cdot \sum_{\tau=[v_{i_0}, \dots, v_{i_q}] \in \Delta} \text{vol}(\tau)(e_{i_0} + \dots + e_{i_q})$$

as Δ ranges over all triangulations of Q . (The triangulations are the tight π -coherent subdivisions of Q .)

EXAMPLE 4.11. Let Q be the convex hull of the vertices $v_0 = (0, 0, 0)$, $v_1 = (0, 1, 0)$, $v_2 = (1, 0, 0)$, $v_3 = (1, 1, 0)$, $v_4 = (0, 0, 1)$ and $v_5 = (0, 1, 1)$, see figure 3.

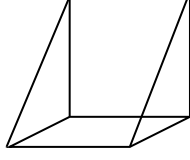


FIGURE 3. The polytope Q

So Q has 6 vertices and π is the projection

$$\pi : \mathbb{R}^5 \rightarrow \mathbb{R}^3, e_i \mapsto v_i$$

With Corollary 4.10 we can compute the vertices of $\Sigma(Q)$: Q has 6 triangulations, see Figure 4.

Corollary 4.10 gives us 6 vertices of the secondary polytope:

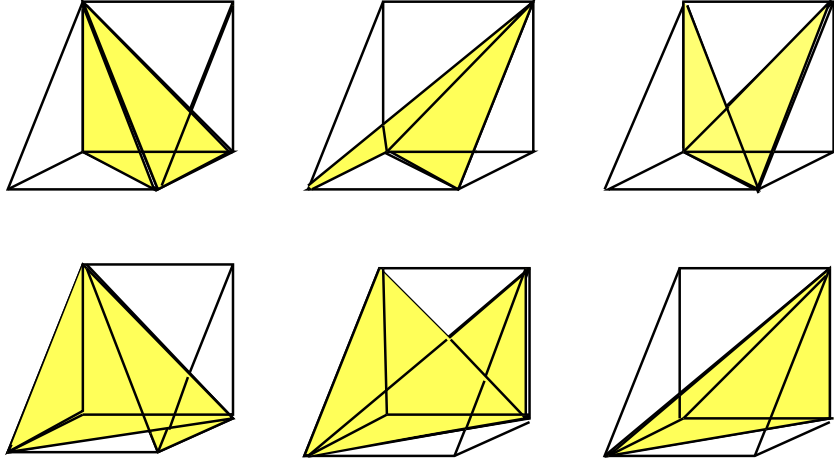


FIGURE 4. The triangulations of Q

$$\begin{aligned} 2 \cdot \frac{1}{6} \cdot ((0, 1, 0, 1, 1) + (1, 1, 0, 0, 1) + (1, 1, 1, 0, 1)) &= \frac{1}{3} \cdot (2, 3, 1, 1, 3) \\ 2 \cdot \frac{1}{6} \cdot ((0, 1, 1, 1, 0) + (0, 0, 1, 1, 1) + (1, 0, 1, 0, 1)) &= \frac{1}{3} \cdot (1, 1, 3, 2, 2) \\ 2 \cdot \frac{1}{6} \cdot ((0, 1, 0, 1, 1) + (0, 1, 1, 0, 1) + (1, 0, 1, 0, 1)) &= \frac{1}{3} \cdot (1, 2, 2, 1, 3) \\ 2 \cdot \frac{1}{6} \cdot ((1, 1, 0, 1, 0) + (1, 1, 1, 1, 0) + (1, 0, 1, 1, 1)) &= \frac{1}{3} \cdot (3, 2, 2, 3, 1) \\ 2 \cdot \frac{1}{6} \cdot ((1, 1, 0, 1, 0) + (1, 1, 0, 1, 1) + (1, 1, 1, 0, 1)) &= \frac{1}{3} \cdot (3, 3, 1, 2, 2) \\ 2 \cdot \frac{1}{6} \cdot ((0, 1, 1, 1, 0) + (1, 0, 1, 1, 0) + (1, 0, 1, 1, 1)) &= \frac{1}{3} \cdot (2, 1, 3, 3, 1) \end{aligned}$$

If we enter these points in `polymake` we get that the convex hull is a 2- dimensional hexagon.

```
application polytope
version 2.3
type RationalPolytope
```

```
POINTS
1 1 1 3 2 2
1 1 2 2 1 3
1 3 2 2 3 1
1 3 3 1 2 2
1 2 1 3 3 1
1 2 3 1 1 3
```

```
VERTICES
1 1 1 3 2 2
1 1 2 2 1 3
1 3 2 2 3 1
1 3 3 1 2 2
1 2 1 3 3 1
1 2 3 1 1 3
```

```
DIM
2
```

4. Mixed fiber polytopes

Mixed fiber polytopes were first invented by McMullen [McM04]. Esterov and Khovanskii discovered that they were useful to compute the Newton polytope of a polynomial which describes the projection of a complete intersection, see [EK08]. Sturmfels and Yu give a self-contained introduction to this topic, see [SY08], and present their program `TrIm` which was developed for computing mixed fiber polytopes. The details will be explained in the next chapter, here we review the definition of a mixed fiber polytope and describe it as a formal sum of fiber polytopes analogously to a statement about mixed volumes.

Let P_1, \dots, P_c be polytopes in \mathbb{R}^p and $\lambda_1, \dots, \lambda_c \geq 0$. Then the fiber polytope of the Minkowski sum

$$P_\lambda := \lambda_1 \cdot P_1 + \dots + \lambda_c \cdot P_c$$

depends polynomially on the parameters $\lambda_1, \dots, \lambda_c$. This polynomial is homogeneous of degree $q + 1$:

$$\Sigma_\pi(\lambda_1 \cdot P_1 + \dots + \lambda_c \cdot P_c) = \sum_{i_1 + \dots + i_c = q+1} \lambda_1^{i_1} \dots \lambda_c^{i_c} M_{i_1 \dots i_c}$$

The polytopes $M_{i_1 \dots i_c}$ are uniquely determined.

DEFINITION 4.12. For $c = q + 1$ the mixed fiber polytope is defined as the coefficient of $\lambda_1 \dots \lambda_c$:

$$\Sigma_\pi(P_1, \dots, P_c) := M_{1 \dots 1}$$

In the special case $P_1 = \dots = P_{q+1} = P$ holds:

$$\Sigma_\pi(P_\lambda) = \Sigma_\pi(\lambda_1 + \dots + \lambda_{q+1}) \cdot P = (\lambda_1 + \dots + \lambda_{q+1})^{q+1} \Sigma_\pi(P)$$

Expanding the sum yields that the coefficient of $\lambda_1 \cdots \lambda_{q+1}$ is $(q+1)! \cdot \Sigma_\pi(P)$. Therefore

$$\Sigma_\pi(P) = \frac{1}{(q+1)!} \Sigma_\pi(P, \dots, P)$$

This reduces the case of a fiber polytope to a special case of the mixed fiber polytope.

Mixed fiber polytopes can be expressed by Minkowski sums and formal differences of conventional fiber polytopes (see [EK08]). For this, consider a formal subtraction on the semigroup of polytopes with the Minkowski summation by

$$P - Q = R \iff P = Q + R.$$

The formal subtraction is a well-defined operation because if $R + Q = R' + Q$ then $R = R'$. (A vertex of the sum has to be a sum of vertices $r + q = r' + q$ for the same vertex $q \in Q$ where all vertices r, r', q maximize the same linear function. So R and R' must have the same vertices.)

PROPOSITION 4.13. *The extension of the Minkowski sum with formal differences turns the semigroup of convex polytopes into a group, the group of virtual polytopes.*

PROOF. Association passes on to the virtual polytopes, $\{0\}$ is the neutral element and $-P$ is the inverse of P because

$$P - P = 0 \iff P = \{0\} + P.$$

□

With this definition we can state:

THEOREM 4.14. *For any polytopes $P_1, \dots, P_r \subseteq \mathbb{R}^n$ we have*

$$(11) \quad \Sigma_\psi(P_1, \dots, P_r) = \sum_{k=1}^n (-1)^{n+k} \sum_{i_1 < \dots < i_k} \Sigma_\psi(P_{i_1} + \dots + P_{i_k}).$$

PROOF. The proof is analogous to similar statements on the mixed volume (see for example [Ewa96, Thm. 2]). Denoting the right hand side of (11) by $g(P_1, \dots, P_r)$, we observe that for $\lambda_1, \dots, \lambda_r > 0$ the expression $g(\lambda_1 P_1, \dots, \lambda_r P_r)$ is a polynomial in $\lambda_1, \dots, \lambda_r$. For $P_1 = \{0\}$, the definition of g implies

$$\begin{aligned} & (-1)^{r+1} \cdot g(\{0\}, P_2, \dots, P_r) \\ &= \sum_{2 \leq i \leq r} \Sigma_\psi(P_i) - \left(\sum_{2 \leq j \leq r} \Sigma_\psi(\{0\} + P_j) + \sum_{2 \leq i < j \leq r} \Sigma_\psi(P_i + P_j) \right) \\ &+ \left(\sum_{2 \leq j < k \leq r} \Sigma_\psi(\{0\} + P_j + P_k) + \sum_{2 \leq i < j < k \leq r} \Sigma_\psi(P_i + P_j + P_k) \right) \pm \dots \\ &= 0, \end{aligned}$$

where the last equality follows from re-arranging the parenthesis and $\Sigma_\psi(P) = \Sigma_\psi(\{0\} + P)$. As a consequence, the polynomial $g(0 \cdot P_1, \lambda_2 \cdot P_2, \dots, \lambda_r \cdot P_r) = 0$ evaluates to zero for all $\lambda_2, \dots, \lambda_r$; i.e., it is the zero polynomial. Thus, in the polynomial $g(\lambda_1 \cdot P_1, \lambda_2 \cdot P_2, \dots, \lambda_r \cdot P_r)$, the coefficients of all monomials $\lambda_{i_1} \cdots \lambda_{i_r}$ with $1 \notin \{i_1, \dots, i_r\}$ vanish. By symmetry, this statement also holds for all terms in which not all indices from $\{1, \dots, r\}$ occur. Hence, there is only one monomial with nonzero coefficient, namely $\lambda_1 \cdots \lambda_r$. So this coefficient has to be the mixed fiber polytope $\Sigma_\psi(P_1, \dots, P_r)$. \square

For $q = 1$ and for $\lambda_1 = \lambda_2 = 1$ it follows that the equation

$$\Sigma_\pi(P_1 + P_2) = \Sigma_\pi(P_1, P_2) + \Sigma_\pi(P_1) + \Sigma_\pi(P_2)$$

holds. It is very useful when we want to compute the mixed fiber polytope.

EXAMPLE 4.15. Let P and Q be the following polytopes.

```
> P := convhull([1, 2, 3], [3, 2, 3], [3, 1, 0], [0, 3, 2]);
> Q := convhull([2, 0, 1], [1, 0, 0], [0, 2, 1], [2, 2, 1]);
```

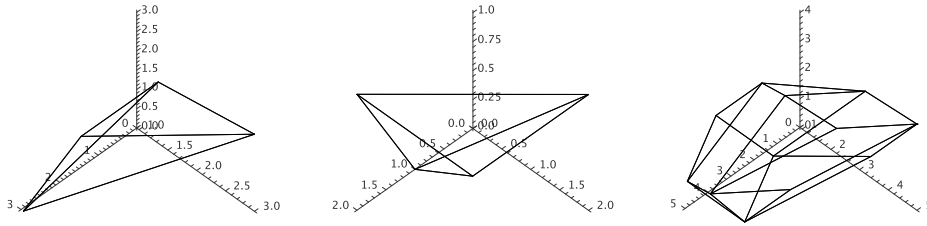


FIGURE 5. P , Q and the Minkowski sum $P+Q$

Then Maple computes the following polytopes, where π is the projection with kernel generated by \mathbf{p} .

```
> p := [0, 1, 1];
> M := mixedfiber(P, Q, p);
> F[1] := Faserpolytop(P, p);
      F[1] := POLYTOPE(3, 2, 3, 3)
> F[2] := Faserpolytop(Q, p);
      F[2] := POLYTOPE(3, 2, 4, 4)
> F[3] := Faserpolytop(minkowskisum(P, Q), p);
      F[3] := POLYTOPE(3, 2, 9, 9)
```

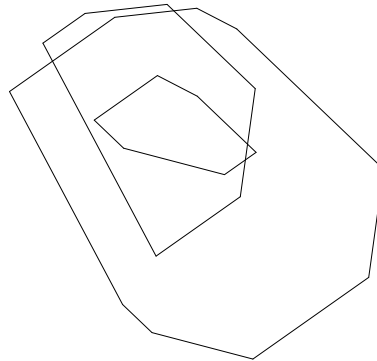


FIGURE 6. The polytopes $F[1]+F[2]=\Sigma_{\pi}(P) + \Sigma_{\pi}(Q)$, $F[3]=\Sigma_{\pi}(P+Q)$ and $M=\Sigma_{\pi}(P,Q)$

They have the following vertices.

```
> vertices(minkowskisum(F[1], F[2]));
[23/2, 8, 17/2], [19/2, 9, 15/2], [15/2, 11, 11/2], [35/2, 8, 17/2],
[33/2, 9, 15/2], [21/2, 11, 11/2]

> vertices(F[3]);
[49/2, 13, 37/2], [37/2, 14, 35/2], [33/2, 15, 33/2], [23/2, 20, 23/2],
[33/2, 22, 19/2], [67/2, 20, 23/2], [69/2, 19, 25/2], [69/2, 13, 37/2],
[55/2, 22, 19/2]

> vertices(M);
[[4, 9, 6], [7, 6, 9], [9, 11, 4], [13, 5, 10], [17, 5, 10],
[17, 11, 4]]
```

One can see that the mixed fiber polytope (the small polytope) is the difference between the fiber polytope of $P+Q$ (the big polytope) and the sum of the fiber polytopes of P and Q (the midsize polytope).

CHAPTER 5

The Newton polytope of a projection

In this chapter we analyse the subdivision of the Newton polytope of a polynomial defining the projection of a tropical variety, i.e. the subdivision of $\text{New}(g) \in K[x_1, \dots, x_n]$ such that $\mathcal{T}(g) = \pi^{-1}\pi(\mathcal{T}(I))$ for an m -dimensional ideal $I \triangleleft K[x_1, \dots, x_n]$ and a projection $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$. To get some information on this subdivision we have to make some assumptions on the tropical variety $\mathcal{T}(I)$.

In the first section some properties of tropical varieties are defined. In the second section these properties are carried over to the tropical variety of the ideal J (see Section 2). In the third and fourth section we describe the Newton polytope of g and its subdivision by the use of mixed fiber polytopes. This is an extension of the results of Sturmfels, Tevelev and Yu (see [ST08, STY07, SY08]) and Esterov and Khovanskii (see [Est08, EK08]). At last we give an example where such a subdivision is computed.

1. Transversal intersections

An intersection $X = X_1 \cap \dots \cap X_k$ of tropical hypersurfaces X_i can have several properties. For the following the most important property is transversality. In this section we give the definition of transversality and of some related properties together with some examples. Some of the definitions, especially the definition of transversality, can be found in [Vig07].

Let $k \leq n$ and $f_1, \dots, f_k \in K[x_1, \dots, x_n]$. We set $X_i := \mathcal{T}(f_i)$; X_i is a tropical hypersurface in \mathbb{R}^n . Let X be the intersection $X = X_1 \cap \dots \cap X_k$.

DEFINITION 5.1. *The intersection is called proper if $\dim X = n - k$.*

In order to study the intersection X , it is useful to consider the union

$$U := \bigcup_{i=1}^k X_k$$

as well, since U is a tropical hypersurface and thus comes with a natural subdivision. (It is the tropical hypersurface of $f_1 \cdots f_k$.)

Let C be a non-empty cell of X . Then C can be written as $C = \bigcap_{i=1}^k C_i$, where C_i is a cell of X_i , minimal with $C \subset C_i$.

Consider C as a cell of the union U . Then the dual cell C^\vee of C with regard to U is given by

$$C^\vee = C_1^\vee + \dots + C_k^\vee.$$

DEFINITION 5.2. *The intersection $X = X_1 \cap \cdots \cap X_k$ is called transversal along C if*

$$(12) \quad \dim(C^\vee) = \dim(C_1^\vee) + \cdots + \dim(C_k^\vee).$$

The intersection $X_1 \cap \cdots \cap X_k$ is transversal if for each subset $J \subseteq \{1, \dots, k\}$ of cardinality at least 2 the intersection $\bigcap_{j \in J} X_j$ is proper and transversal along each cell.

EXAMPLE 5.3. Let $f_1 = x + 2y + z - 4$, $f_2 = 3x - y + 2z + 1$ and the valuation $\text{ord} : \mathbb{Q} \mapsto \mathbb{R}_\infty$ be the 2-adic valuation. Then $X = \mathcal{T}(f_1) \cap \mathcal{T}(f_2)$ is a proper intersection because the dimension of X is $3 - 2 = 1$, see Figure 1.

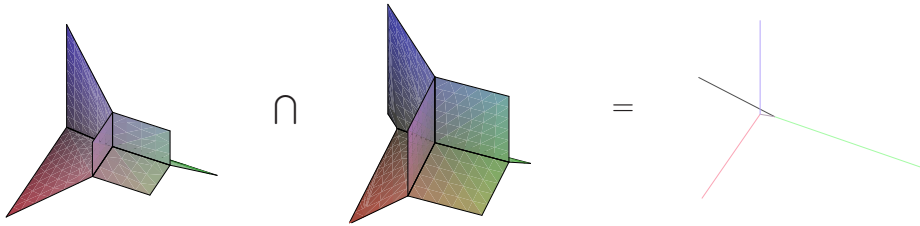


FIGURE 1. A proper intersection of two tropical hypersurfaces

Figure 2 shows the union of the two hypersurfaces and the corresponding subdivision of $\text{New}(f_1, f_2)$.

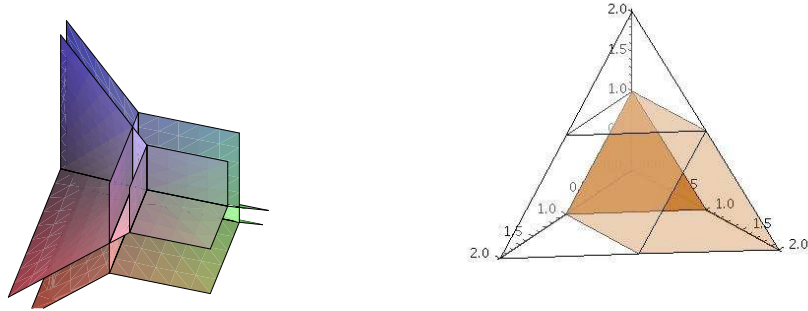


FIGURE 2. The union $\mathcal{T}(f_1) \cup \mathcal{T}(f_2)$ and the corresponding subdivision of the Newton polytope

DEFINITION 5.4. *A proper intersection $\mathcal{T}(f_1) \cap \cdots \cap \mathcal{T}(f_k)$ is called a complete intersection if*

$$\mathcal{T}(\langle f_1, \dots, f_k \rangle) = \mathcal{T}(f_1) \cap \cdots \cap \mathcal{T}(f_k).$$

To count the number of solutions of a system of polynomial equations the term of Newton-nondegeneracy is very important, see [Ber75].

DEFINITION 5.5. *The polynomials $f_1, \dots, f_k \in K[x_1, \dots, x_n] \setminus \{0\}$ are called Newton-nondegenerate if for any collection of faces*

$$A_1 \subset \text{New}(f_1), \dots, A_k \subset \text{New}(f_k),$$

such that the sum $A_1 + \dots + A_k$ is at most a $(k-1)$ -dimensional face of the subdivision of the sum $\text{New}(f_1) + \dots + \text{New}(f_k)$, the restrictions $f_1|_{A_1}, \dots, f_k|_{A_k}$ have no common zeros in $(\overline{K}^)^n$. Here $f_i|_{A_i}$ is the sum of terms in f_i with support in A_i .*

Otherwise we call the polynomials $f_1, \dots, f_k \in K[x_1, \dots, x_n]$ Newton - degenerate.

The solutions of a system of polynomial equations $f_1 = \dots = f_k = 0$ define the algebraic variety of $\langle f_1, \dots, f_k \rangle$. So it is natural that we need the same assumption to determine the Newton polytope of the polynomial defining our tropical hypersurface, see for example [EK08]. We will revisit their results in Section 3

EXAMPLE 5.6. Let $f_1, f_2 \in \mathbb{Q}[x, y, z]$ be again the polynomials given by

$$f_1 = x + 2y + z - 4, \quad f_2 = 3x - y + 2z + 1$$

Here $k = 2$ and we want to find faces of $\text{New}(f_1)$ and of $\text{New}(f_2)$ such that the sum is 1-dimensional. Because the restriction of f_i to a 0-dimensional face has never solutions in $(\mathbb{C}^*)^3$ we need parallel 1-dimensional faces of $\text{New}(f_1)$ and $\text{New}(f_2)$. We denote with $[a, b] \leq P$ the face of a polytope P with vertices a, b .

1. $[(1, 0, 0), (0, 1, 0)] \leq \text{New}(f_1), \text{New}(f_2)$
 $\{(x, y, z) \in (\mathbb{C}^*)^3 \mid x + 2y = 3x - y = 0\} = \emptyset$
2. $[(1, 0, 0), (0, 0, 1)] \leq \text{New}(f_1), \text{New}(f_2)$
 $\{(x, y, z) \in (\mathbb{C}^*)^3 \mid x + z = 3x + 2z = 0\} = \emptyset$
3. $[(0, 0, 1), (0, 1, 0)] \leq \text{New}(f_1), \text{New}(f_2)$
 $\{(x, y, z) \in (\mathbb{C}^*)^3 \mid 2y + z = -y + 2z = 0\} = \emptyset$
4. $[(1, 0, 0), (0, 0, 0)] \leq \text{New}(f_1), \text{New}(f_2)$
 $\{(x, y, z) \in (\mathbb{C}^*)^3 \mid x - 4 = 3x + 1 = 0\} = \emptyset$
5. $[(0, 0, 0), (0, 1, 0)] \leq \text{New}(f_1), \text{New}(f_2)$
 $\{(x, y, z) \in (\mathbb{C}^*)^3 \mid 2y - 4 = -y + 1 = 0\} = \emptyset$
6. $[(0, 0, 1), (0, 0, 0)] \leq \text{New}(f_1), \text{New}(f_2)$
 $\{(x, y, z) \in (\mathbb{C}^*)^3 \mid z - 4 = 2z + 1 = 0\} = \emptyset$

So f_1, f_2 are Newton-nondegenerate.

2. Lifting of the properties

Let now I be again an m -dimensional ideal in $K[x_1, \dots, x_n]$ and $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$ a projection. Using the algebraic characterization of $\pi^{-1}\pi\mathcal{T}(I)$ derived in Chapter 3, we will deduce an alternative characterization of the Newton polytope.

Remember the definition of J : For $f \in I$ let

$$\tilde{f} = f(x_1 \prod_{j=1}^l \lambda_j^{u_j^{(1)}}, \dots, x_n \prod_{j=1}^l \lambda_j^{u_j^{(l)}})$$

where $u^{(1)}, \dots, u^{(l)}$ are vectors generating the kernel of the projection π we are interested in. Then J was defined by

$$J := \langle \tilde{f} : f \in I \rangle \triangleleft K[x_1, \dots, x_n, \lambda_1, \dots, \lambda_l].$$

Lemma 3.10 states that J is generated by $\tilde{f}_1, \dots, \tilde{f}_s$.

Let $\rho : \mathbb{R}^{n+l} \rightarrow \mathbb{R}^n$ be the projection forgetting the last l coordinates, and let $\rho^\circ : \mathbb{R}^{n+l} \rightarrow \mathbb{R}^l$ be the projection onto the last l coordinates (i.e., the map forgetting the first n coordinates). Then we can show that all interesting properties lift from I to J . Remember that we have already seen in Theorem 3.11 that primality is preserved.

LEMMA 5.7. *Given $f_1, \dots, f_{n-m} \in K[x_1, \dots, x_n]$, let $X = \bigcap_{i=1}^{n-m} \mathcal{T}(f_i)$.*

- (1) *If the intersection is a proper intersection then the intersection $\tilde{X} = \bigcap_{i=1}^{n-m} \mathcal{T}(\tilde{f}_i)$ is a proper intersection.*
- (2) *If the intersection is transversal then the intersection $\tilde{X} = \bigcap_{i=1}^{n-m} \mathcal{T}(\tilde{f}_i)$ is transversal.*
- (3) *If f_1, \dots, f_{n-m} are Newton-nondegenerate then $\tilde{f}_1, \dots, \tilde{f}_{n-m}$ are Newton-nondegenerate.*
- (4) *If the intersection is complete, i.e. $\mathcal{T}(\langle f_1, \dots, f_{n-m} \rangle) = \bigcap_{i=1}^{n-m} \mathcal{T}(f_i)$ then the intersection $\mathcal{T}(\langle \tilde{f}_1, \dots, \tilde{f}_{n-m} \rangle) = \bigcap_{i=1}^{n-m} \mathcal{T}(\tilde{f}_i)$ is complete.*

PROOF. The set of vectors α in the support of f_i is in 1-1-correspondence with the set of vectors in the support of \tilde{f}_i by the linear mapping

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ v_1^{(1)} & \dots & \dots & v_n^{(1)} \\ \vdots & & & \vdots \\ v_1^{(l)} & \dots & \dots & v_n^{(l)} \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = M \cdot \alpha.$$

Since this mapping is one-to-one, the Newton polytopes $\text{New}(f_i)$ and $\text{New}(\tilde{f}_i)$ have the same dimension. Furthermore all polynomials \tilde{f}_i , $i = 1, \dots, n-m$, are

homogeneous with respect to the vectors orthogonal to the columns of M , i.e.

$$(-v_1^{(j)}, \dots, -v_n^{(j)}, 0, \dots, 0, 1, 0, \dots, 0)$$

for $j = 1, \dots, l$ where the 1 is at the $(n + j)$ -th position.

(1) Let \tilde{C} be a cell in $\mathcal{T}(\tilde{f}_1) \cap \dots \cap \mathcal{T}(\tilde{f}_{n-m})$. Then $\tilde{C} = \tilde{C}_1 \cap \dots \cap \tilde{C}_{n-m}$ for cells $\tilde{C}_i \in \mathcal{T}(\tilde{f}_i)$. Each \tilde{C}_i is dual to a cell \tilde{C}_i^\vee in the subdivided Newton polytope $\text{New}(\tilde{f}_i)$. It's image under ρ is a cell C_i^\vee in $\text{New}(f_i)$ and $\rho(\tilde{C}_i) = C_i$. Obviously $\rho(\tilde{C}) \subseteq C_1 \cap \dots \cap C_{n-m}$. So $\rho(\tilde{C})$ is at most m -dimensional and therefore its preimage is at most $(l+m)$ -dimensional. That means that $\mathcal{T}(\tilde{f}_1) \cap \dots \cap \mathcal{T}(\tilde{f}_{n-m})$ is a proper intersection.

(2) The equation (12) holds also for the lifted cells, because the mapping does not change the dimensions of the cells. Part (1) shows that $\mathcal{T}(\tilde{f}_1) \cap \dots \cap \mathcal{T}(\tilde{f}_{n-m})$ is a proper intersection and thus it is a transversal intersection.

(3) Let f_1, \dots, f_{n-m} be Newton-nondegenerate. Let now $\tilde{A}_i \subset \text{New}(\tilde{f}_i)$, $i = 1, \dots, n - m$, be a collection of faces such that the sum $\tilde{A}_1 + \dots + \tilde{A}_{n-m}$ is at most an $(n - m - 1)$ -dimensional face of the subdivision of $\text{New}(\tilde{f}_1) + \dots + \text{New}(\tilde{f}_{n-m})$. Then any common zero $(c_1, \dots, c_{n+l}) \in (\bar{K}^*)^{n+l}$ induces a common zero $(c_1 \prod_{j=1}^l c_{n+j}^{v_1^{(j)}}, \dots, c_n \prod_{j=1}^l c_{n+j}^{v_n^{(j)}}) \in (\bar{K}^*)^n$ of $\rho(\tilde{A}_1) = A_1, \dots, \rho(\tilde{A}_{n-m}) = A_{n-m}$ which are faces of $\text{New}(f_i)$ such that $\sum_{i=1}^{n-m} A_i$ is at most an $(n - m - 1)$ -dimensional face of $\sum_{i=1}^{n-m} \text{New}(f_i)$ (ρ does not change the dimension of the cells). But f_1, \dots, f_{n-m} are Newton-nondegenerate, so this is a contradiction and the polynomials $\tilde{f}_1, \dots, \tilde{f}_{n-m}$ are Newton-nondegenerate.

(4) Let $\tilde{f} = \sum_{i=1}^k \bar{g}_i \in \langle \tilde{f}_1, \dots, \tilde{f}_{n-m} \rangle$ with \bar{g}_i homogeneous of different degrees with respect to the vectors $(-v_1^{(j)}, \dots, -v_n^{(j)}, 0, \dots, 0, 1, 0, \dots, 0)$ for $j = 1, \dots, l$ and let $w \in \bigcap_{i=1}^{n-m} \mathcal{T}(\tilde{f}_i)$. If the minimum in $\text{trop}(\tilde{f})$ is attained only once at w then there is a \bar{g}_i such that the minimum of $\text{trop}(\bar{g}_i)$ is attained only once at w and these two minima are equal. Since \bar{g}_i is homogeneous there exist $\lambda^\beta = \lambda_1^{\beta_1} \dots \lambda_l^{\beta_l}$, $\beta_j \in \mathbb{Z}$, such that $\bar{g}_i = \lambda^\beta \cdot \tilde{g}_i$ with $\tilde{g}_i \in \langle f_1, \dots, f_{n-m} \rangle$. But the minimum of $\text{trop}(\bar{g}_i)$ is $\beta_1 w_{n+1} + \dots + \beta_l w_{n+l}$ plus the minimum of $\text{trop}(\tilde{g}_i)$. So the minimum of $\text{trop}(\bar{g}_i)$ is attained only once at the point $(w_1 + \sum_{j=1}^l v_1^{(j)} w_{n+j}, \dots, w_n + \sum_{j=1}^l v_n^{(j)} w_{n+j}) \in \bigcap_{i=1}^{n-m} \mathcal{T}(f_i)$. But this is a contradiction and so $w \in \mathcal{T}(\langle \tilde{f}_1, \dots, \tilde{f}_{n-m} \rangle)$. \square

3. The Newton polytope of a projection

Let $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$ be again a projection represented by the matrix A and $I \triangleleft K[x_1, \dots, x_n]$ an m -dimensional ideal. To describe the Newton polytope of a polynomial f with $\pi(\mathcal{T}(I)) = \mathcal{T}(f)$ we need the complementary linear map

$$\pi^\circ : \mathbb{R}^n \rightarrow \mathbb{R}^{n-m-1} = \mathbb{R}^l$$

whose kernel is the row space of A . Then Sturmfels and Yu state the following theorem.

THEOREM 5.8 ([EK08], [SY08], Thm. 4.1). *If the coefficients of the polynomials f_1, \dots, f_{n-m} are generic then the Newton polytope of $\pi(\langle f_1, \dots, f_{n-m} \rangle)$ is affinely isomorphic to the mixed fiber polytope $\Sigma_{\pi^\circ}(\text{New}(f_1), \dots, \text{New}(f_{n-m}))$.*

A proof using tropical geometry can be found in [ST08]. Originally the theorem was stated and proved by Esterov and Khovanskii in [EK08]. In their work we can see what it means that the coefficients are generic: the polynomials have to be Newton-nondegenerate.

In our situation we have not only the projections π and π° but also the projections $\rho : \mathbb{R}^{n+l} \rightarrow \mathbb{R}^n$ which forgets the last l coordinates and $\rho : \mathbb{R}^{n+l} \rightarrow \mathbb{R}^l$ which forgets the first n coordinates. (They are orthogonal to each other.) So in our case we get the theorem

THEOREM 5.9. *Let f_1, \dots, f_{n-m} be Newton-nondegenerate, and let the intersection*

$$Y = \bigcap_{i=1}^{n-m} Y_i = \bigcap_{i=1}^{n-m} \mathcal{T}(f_i) = \mathcal{T}(I)$$

be complete with $I := \langle f_1, \dots, f_{n-m} \rangle$. The Newton polytope of the tropical hypersurface $\pi^{-1}\pi(\mathcal{T}(I))$ is affinely isomorphic to

$$\rho(\Sigma_{\rho^\circ}(\text{New}(\tilde{f}_1), \dots, \text{New}(\tilde{f}_{n-m}))).$$

PROOF. By Chapter 3, we have

$$\pi^{-1}\pi(\mathcal{T}(I)) = \mathcal{T}(J \cap K[x_1, \dots, x_n]),$$

where $J = \langle \tilde{f}_1, \dots, \tilde{f}_{n-m} \rangle$. By Theorem 5.8, the Newton polytope of the defining polynomial of the right hand side is (up to an affine isomorphism) given by

$$\Sigma_{\rho^\circ}(\text{New}(\tilde{f}_1), \dots, \text{New}(\tilde{f}_{n-m}))$$

in the $(n+l)$ -dimensional space \mathbb{R}^{n+l} . Applying the canonical projection ρ which maps the mixed fiber polytope isomorphic onto its image proves the claim. \square

The following corollary of Theorem 5.8 and Theorem 5.9 combines the algebraic and the geometric viewpoint.

COROLLARY 5.10. *In the setup of Theorem 5.9, up to an affine isomorphism, the following mixed fiber polytopes coincide:*

$$\Sigma_{\pi^\circ}(\text{New}(f_1), \dots, \text{New}(f_{n-m})) = \rho(\Sigma_{\rho^\circ}(\text{New}(\tilde{f}_1), \dots, \text{New}(\tilde{f}_{n-m})))$$

4. The subdivision of the Newton polytope

In this section we study the subdivision of the Newton polytope of the defining polynomial of $\pi^{-1}\pi(\mathcal{T}(I))$. Each cell of that subdivision gives a description of $\pi^{-1}\pi(\mathcal{T}(I))$ locally. These cells are described by fiber polytopes. In the next section we will also show how to patchwork these local fiber polytopes.

We concentrate on the case of a transversal intersection, and we will always assume that this genericity assumption also holds for the local cells.

Let $k := n - m$ and

$$X = \mathcal{T}(f_1) \cap \cdots \cap \mathcal{T}(f_k) \subset \mathbb{R}^n.$$

By Theorem 5.9 the dual subdivision of X lives in $\pi(\mathbb{R}^n) \cap \mathbb{Z}^n$, that means all the vertices have integer coordinates.

In the local version of Theorem 5.9 we have to assume that the preimage of a cell $\pi(C)$ is unique in X .

LEMMA 5.11. *Let C be a cell of X and $\pi^{-1}\pi(C) \cap X = \{C\}$. If C^\vee denotes the corresponding cell of C in the dual subdivision of the union $\mathcal{T}(f_1) \cup \cdots \cup \mathcal{T}(f_k)$ then $C^\vee = C_1^\vee + \cdots + C_k^\vee$ with $C_i^\vee \in \text{New}(f_i)$ and the corresponding dual cell of $\pi(C) \subseteq \pi(\mathcal{T}(I))$ in the subdivision of the Newton polytope of the defining polynomial of $\pi(\mathcal{T}(I))$ is affinely isomorphic to*

$$\Sigma_\pi^\circ(C_1^\vee, \dots, C_k^\vee).$$

PROOF. For $1 \leq i \leq k$ let g_i be the polynomial with support C_i whose coefficients are induced by f_i . Then the local cone of $\mathcal{T}(I)$ at $p \in C$ is defined by these polynomials, i.e. $\text{LC}_p(\mathcal{T}(I)) = \mathcal{T}(\langle g_1, \dots, g_k \rangle)$. If C is the only preimage of $\pi(C)$, then

$$\text{LC}_{\pi(p)} \pi(\mathcal{T}(I)) = \pi(\text{LC}_p \mathcal{T}(I)).$$

By Theorem 5.9 the image $\pi(\mathcal{T}(\langle g_1, \dots, g_k \rangle))$ is dual to $\Sigma_\pi^\circ(C_1^\vee, \dots, C_k^\vee)$. \square

Note that each C_i^\vee is the dual of a cell of dimension at most $n - 1$, so it has dimension at least 1. Thus the sum $C_1^\vee + \cdots + C_k^\vee$ is a mixed cell (cf. [Vig07]). As explained above, we can consider $\pi(\mathcal{T}(I))$ as an m -dimensional complex in an $(m + 1)$ -dimensional space. Every j -dimensional face F of $\pi(\mathcal{T}(I))$ is either the projection of a unique j -dimensional face of $\mathcal{T}(I)$ (see Lemma 5.11), or the intersection of the images of faces of $\mathcal{T}(I)$. Since every cell in the tropical hypersurface $\pi^{-1}\pi(\mathcal{T}(I))$ arises in this way, we obtain:

THEOREM 5.12. *Let $I \triangleleft K[x_1, \dots, x_n]$ an m -dimensional ideal, generated by generic polynomials f_1, \dots, f_{n-m} and $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$ a projection. Then up to affine isomorphisms of the cells, the cells of the dual subdivision of $\pi^{-1}\pi \mathcal{T}(I)$ are of the form*

$$\sum_{i=1}^p \Sigma_{\pi^\circ}(C_{i1}^\vee, \dots, C_{ik}^\vee) \text{ for some } p \in \mathbb{N}.$$

Here $k = n - m$ and F_1, \dots, F_p are faces of $\mathcal{T}(f_1) \cap \cdots \cap \mathcal{T}(f_k)$ and the dual cell of $F_i \subseteq U = \mathcal{T}(f_1) \cup \cdots \cup \mathcal{T}(f_k)$ is given by $F_i^\vee = C_{i1}^\vee + \cdots + C_{ik}^\vee$ with faces C_{i1}, \dots, C_{ik} of $\mathcal{T}(f_1), \dots, \mathcal{T}(f_k)$.

Specifically, for $p = 1$ the full-dimensional cells are of the form $\Sigma_{\pi^\circ}(C_1^\vee, \dots, C_k^\vee)$, where $C_1^\vee + \cdots + C_k^\vee$ is a mixed cell in $\text{New}(f_1) + \cdots + \text{New}(f_k)$.

PROOF. For any cell $D \in \pi(\mathcal{T}(I))$ let F_1, \dots, F_p be the cells in $\mathcal{T}(I)$ minimal with $D \in \pi(F_i)$ and $C_{i1}^\vee + \cdots + C_{ik}^\vee$ the dual cells in the subdivision of $\text{New}(f_1 \cdots f_k)$. Then

$$\text{LC}_D \pi(\mathcal{T}(I)) = \bigcup_{i=1}^p \pi(\text{LC}_{x^{(i)}} \mathcal{T}(I)),$$

where d is a point in D and $x^{(i)}$ is the preimage of d in F_i . As above the image $\pi(\text{LC}_{x^{(i)}} \mathcal{T}(I))$ is dual to $\Sigma_{\pi^\circ}(C_{i1}^\vee, \dots, C_{ik}^\vee)$. Because the normal fan of a sum of two polytopes is the union of the normal fans of the two polytopes it holds:

$$\bigcup_{i=1}^p \pi(\text{LC}_{x^{(i)}} \mathcal{T}(I)) \text{ is dual to } \sum_{i=1}^p \Sigma_{\pi^\circ}(C_{i1}^\vee, \dots, C_{ik}^\vee)$$

This proves the claim. \square

Hence, every dual cell of the tropical hypersurface $\pi^{-1}(\pi(I))$ is indexed by some p -tuple of ‘‘formal’’ mixed fiber polytopes

$$\Sigma_{\pi^\circ}(C_{i1}^\vee, \dots, C_{ik}^\vee).$$

5. Constructing the dual subdivision

In this section, we will explain the patchworking of the local mixed fiber polytopes. Theorem 5.15 describes the offset of the fiber polytope of a facet of a simplex to the face of the fiber polytope of that simplex. Theorem 5.16 is the generalization to faces of mixed fiber polytopes of mixed cells and corollary 5.17 gives us the desired patchworking of the cells in the dual subdivision of $\pi(\mathcal{T}(I))$.

For simplicity, we assume that we know a vertex v of $\pi(\mathcal{T}(I))$ and the corresponding m -dimensional cell C of the dual subdivision of $S := \pi(\mathcal{T}(I))$. We explain how to pass over to a neighbouring cell.

Locally around v , the tropical variety $\mathcal{T}(I)$ defines the m -dimensional fan $\Gamma := \text{LC}_v(\mathcal{T}(I))$. In order to determine the neighbouring cell of C , we consider the 1-skeleton of that fan. Γ is geometrically dual to $\pi(C)$.

Consider a fixed direction vector w of one of the rays of Γ . Let v' be the neighbouring vertex of v on S with regard to this ray, and let Γ' be the corresponding local fan. Further let D be the dual cell corresponding to v' .

Up to affine isomorphisms, we can express C and D as

$$\begin{aligned} C &= \sum_{i=1}^{p_1} \Sigma_{\pi^\circ}(C_{i1}^\vee + \dots + C_{ik}^\vee), \\ D &= \sum_{i=1}^{p_2} \Sigma_{\pi^\circ}(D_{i1}^\vee + \dots + D_{ik}^\vee). \end{aligned}$$

as described in Corollary 5.12.

However, it turns out that due to the affine isomorphisms, these polytopes C and D do not necessarily share a common facet.

In order to characterize the translation involved it suffices to characterize the offset from C to the ‘‘common face’’ (up to a translation) of C and D . This common face is given by $\text{face}_w(C)$ and by $\text{face}_{-w}(D)$ (up to translation).

We denote by Z this face and use the notation

$$Z = \sum_{i=1}^{p_1} \Sigma_{\pi^\circ}(Z_{i1}^\vee + \cdots + Z_{ik}^\vee).$$

In the simplest case, C consists of only one summand. Then in the representation of Z one of the terms Z_{ij} is a face of the summand of C_{ij} and the other terms coincide.

\overline{C} comes from a fiber polytope $\Sigma_{\pi^\circ}(F_1, \dots, F_k)$ and \overline{D} comes from a fiber polytope $\Sigma_{\pi^\circ}(F'_1, \dots, F'_k)$.

The reason for the need of translations is that mixed fiber polytopes are not “inclusion-preserving” with respect to the face structure. In fact, the following example shows that this effect already happens in the unmixed situation.

EXAMPLE 5.13. Let two convex polytopes $P_1 = \text{conv}\{(0, 0), (1, 0), (0, 1)\}$ and $P_2 = \text{conv}\{(1, 1), (2, 1), (1, 2)\}$ and kernel direction $\pi^\circ = (2, 3)$ be given. Then

$$\begin{aligned} \Sigma_{\pi^\circ}(P_1, P_2) &= \Sigma_{\pi^\circ}(P_1 + P_2) - \Sigma_{\pi^\circ}(P_1) - \Sigma_{\pi^\circ}(P_2) \\ &= \text{conv}\{(6, 12), (12, 8)\} - \text{conv}\{(0, \frac{3}{2}), (\frac{3}{2}, \frac{1}{2})\} - \text{conv}\{(3, \frac{9}{2}), (\frac{9}{2}, \frac{7}{2})\} \\ &= \text{conv}\{(3, 6), (6, 4)\}, \end{aligned}$$

but for the face $F := \text{conv}\{(1, 0), (0, 1)\}$ of P_1 we have

$$\begin{aligned} \Sigma_{\pi^\circ}(F, P_2) &= \Sigma_{\pi^\circ}(F + P_2) - \Sigma_{\pi^\circ}(F) - \Sigma_{\pi^\circ}(P_2) \\ &= \text{conv}\{(\frac{9}{2}, 9), (9, 6)\} - \text{conv}\{(\frac{1}{2}, \frac{1}{2})\} - \text{conv}\{(3, \frac{9}{2}), (\frac{9}{2}, \frac{7}{2})\} \\ &= \text{conv}\{(1, 4), (4, 2)\}, \end{aligned}$$

and thus $\Sigma_{\pi^\circ}(F, P_2) \not\subseteq \Sigma_{\pi^\circ}(P_1, P_2)$.

Mappings π° to \mathbb{R}^1 . For the case that the mapping π° maps to the one-dimensional space, we will give explicit descriptions of the offsets of the mixed fiber polytopes of the mixed cells. We can compute the fiber polytope as follows.

Let $\pi^{\circ P} = \min_{x \in P} \pi^\circ(x)$ and $\pi^{\circ P} = \max_{x \in P} \pi^\circ(x)$.

Then

$$\Sigma_{\pi^\circ}(P) = \sum_{i=\pi^{\circ P}}^{\pi^{\circ P}-1} \int_i^{i+1} (\pi^{\circ-1}(x) \cap P) dx = \sum_{i=\pi^{\circ P}}^{\pi^{\circ P}-1} (\pi^{\circ-1}(i + \frac{1}{2}) \cap P).$$

Note that in general for a face F of a polytope P we do not have that set-theoretically $\Sigma_{\pi^\circ}(F)$ is a face of $\Sigma_{\pi^\circ}(P)$. As a consequence, in general for two polytopes P_1 and P_2 with a common face the polytopes $\Sigma_{\pi^\circ}(P_1)$ and $\Sigma_{\pi^\circ}(P_2)$ do not have a common face.

EXAMPLE 5.14. Let $\pi^\circ : \mathbb{R}^3 \rightarrow \mathbb{R}, x \mapsto (1, 1, 1) \cdot x$, P be the standard cube and F the face

$$\text{face}_{(0, -1, 0)} P = \text{conv}\{(0, 0, 0), (1, 0, 1), (1, 0, 0), (0, 0, 1)\}.$$

So $\pi^\circ(P) = [0, 3]$, $\pi^\circ(F) = [0, 2]$ and the fiber polytopes are given by Figure 3. We see that $\Sigma_{\pi^\circ}(F) + (1, \frac{1}{2}, 1)$ is a face of $\Sigma_{\pi^\circ}(P)$.

$$\begin{aligned}
\Sigma_{\pi}^{\circ}(P) &= (\pi^{\circ-1}(1/2) \cap P) & \Sigma_{\pi^{\circ}}(F) &= (\pi^{\circ-1}(1/2) \cap F) \\
&+ (\pi^{\circ-1}(3/2) \cap P) & &+ (\pi^{\circ-1}(3/2) \cap F) \\
&+ (\pi^{\circ-1}(5/2) \cap P) & &= \\
&= & &
\end{aligned}$$

FIGURE 3. The fiber polytopes of P and F

In the following we characterize the affine isomorphism between the fiber polytope of a face of a polytope and the face of a fiber polytope in a simple case. Define for a polytope P and $i \in \mathbb{N}$

$$A(i, P) := \arg \max_{x \in P \cap \pi^{\circ-1}(i + \frac{1}{2})} w^T x$$

THEOREM 5.15. *Let F be an $(n-1)$ -polytope in \mathbb{R}^n , $\pi^{\circ} : \mathbb{R}^n \rightarrow \mathbb{R}$, $v \in \mathbb{R}^n \setminus \text{aff } F$ and $P = \text{conv}\{F \cup \{v\}\}$. Let w be an outer normal vector of the face F of P , i.e. $\text{face}_w(P) = F$.*

If $\pi^{\circ}(v) \in \pi^{\circ}(F)$ then $\Sigma_{\pi^{\circ}}(F)$ is a face of $\Sigma_{\pi^{\circ}}(P)$.

If $\pi^{\circ}(v) > \max_{x \in F} \pi^{\circ}(x)$ then

$$(13) \quad \Sigma_{\pi^{\circ}}(F) + \sum_{\substack{\pi^{\circ}(v)-1 \\ \max_{x \in F} \pi^{\circ}(x)}}^{\pi^{\circ}(v)-1} \arg \max_{x \in P \cap \pi^{\circ-1}(i + \frac{1}{2})} w^T x = \text{face}_w(\Sigma_{\pi^{\circ}}(P))$$

If $\pi^{\circ}(v) < \min_{x \in F} \pi^{\circ}(x)$ then

$$(14) \quad \Sigma_{\pi^{\circ}}(F) + \sum_{\pi^{\circ}(v)}^{\min_{x \in F} \pi^{\circ}(x)-1} \arg \max_{x \in P \cap \pi^{\circ-1}(i + \frac{1}{2})} w^T x = \text{face}_w(\Sigma_{\pi^{\circ}}(P))$$

Here, we assumed that $\arg \max$ is unique.

PROOF. The points in (13) are exactly the points in $\Sigma_{\pi^{\circ}}(P)$ which maximize the objective function $x \mapsto w^T x$. \square

The next theorem describes the relation between the face of a mixed fiber polytope of C and D and the mixed fiber polytope of two faces of the two polytopes C and D , where all faces maximize the same linear map.

THEOREM 5.16. *Let $\pi^\circ : \mathbb{R}^n \rightarrow \mathbb{R}$ be a projection and C, D polytopes in \mathbb{R}^n . Let $w \in \mathbb{R}^n$. Then*

$$\begin{aligned} \Sigma_{\pi^\circ}(\text{face}_w(C), \text{face}_w(D)) + \sum_{i=\pi^\circ_{C+D}}^{\pi^\circ_{\text{face}_w(C+D)}-1} A(i, C+D) + \sum_{i=\pi^\circ_{\text{face}_w(C+D)}}^{\pi^\circ_{C+D}-1} A(i, C+D) = \text{face}_w \Sigma_{\pi^\circ}(C, D) \\ + \sum_{i=\pi^\circ_C}^{\pi^\circ_{\text{face}_w(C)}-1} A(i, C) + \sum_{i=\pi^\circ_{\text{face}_w(C)}}^{\pi^\circ_C-1} A(i, C) + \sum_{i=\pi^\circ_D}^{\pi^\circ_{\text{face}_w(D)}-1} A(i, D) + \sum_{i=\pi^\circ_{\text{face}_w(D)}}^{\pi^\circ_D-1} A(i, D) \end{aligned}$$

PROOF. The image of π° is contained in \mathbb{R} so with theorem 4.14 the following equation holds

$$\Sigma_{\pi^\circ}(C + D) = \Sigma_{\pi^\circ}(C, D) + \Sigma_{\pi^\circ}(C) + \Sigma_{\pi^\circ}(D)$$

But this carries forward to the faces of the polytopes

$$\text{face}_w \Sigma_{\pi^\circ}(C + D) = \text{face}_w \Sigma_{\pi^\circ}(C, D) + \text{face}_w \Sigma_{\pi^\circ}(C) + \text{face}_w \Sigma_{\pi^\circ}(D)$$

For a face $F = \text{face}_w(P)$ of an n -polytope P in \mathbb{R}^n holds

$$\Sigma_{\pi^\circ}(F) + \text{face}_w \int_{\pi^\circ(P) \setminus \pi^\circ(F)} \pi^{\circ-1}(x) \cap P \, dx = \text{face}_w(\Sigma_{\pi^\circ}(P))$$

Including this in the equation above yields

$$\begin{aligned} \Sigma_{\pi^\circ}(\text{face}_w(C + D)) + \text{face}_w \int_{\pi^\circ(C+D) \setminus \pi^\circ(\text{face}_w(C+D))} \pi^{\circ-1}(x) \cap (C + D) \, dx \\ = \text{face}_w \Sigma_{\pi^\circ}(C, D) + \Sigma_{\pi^\circ}(\text{face}_w(C)) + \text{face}_w \int_{\pi^\circ(C) \setminus \pi^\circ(\text{face}_w(C))} \pi^{\circ-1}(x) \cap C \, dx + \\ \Sigma_{\pi^\circ}(\text{face}_w(D)) + \text{face}_w \int_{\pi^\circ(D) \setminus \pi^\circ(\text{face}_w(D))} \pi^{\circ-1}(x) \cap D \, dx \end{aligned}$$

Using $\text{face}_w(C + D) = \text{face}_w(C) + \text{face}_w(D)$ gives us

$$\begin{aligned} \Sigma_{\pi^\circ}(\text{face}_w(C), \text{face}_w(D)) + \text{face}_w \int_{\pi^\circ(C+D) \setminus \pi^\circ(\text{face}_w(C+D))} \pi^{\circ-1}(x) \cap (C + D) \, dx \\ = \text{face}_w \Sigma_{\pi^\circ}(C, D) + \text{face}_w \int_{\pi^\circ(C) \setminus \pi^\circ(\text{face}_w(C))} \pi^{\circ-1}(x) \cap C \, dx + \text{face}_w \int_{\pi^\circ(D) \setminus \pi^\circ(\text{face}_w(D))} \pi^{\circ-1}(x) \cap D \, dx \end{aligned}$$

By changing the integrals by sums we get the assertion. \square

COROLLARY 5.17. *In the case of a mixed cell $C + D$ and its facet $\text{face}_w(C) + D$ ($\text{face}_w(D) = D$), the difference*

$$v(C, D, w) := \Sigma_{\pi^\circ}(\text{face}_w(C), D) - \text{face}_w(\Sigma_{\pi^\circ}(C, D))$$

is a vector.

PROOF. π° is a projection to \mathbb{R} , so it holds that

$$\min \pi^\circ(\text{face}_w C + D) - \min \pi^\circ(C + D) = \min \pi^\circ(\text{face}_w C) - \min \pi^\circ(C)$$

and

$$\max \pi^\circ(C + D) - \max \pi^\circ(\text{face}_w C + D) = \max \pi^\circ(C) - \max \pi^\circ(\text{face}_w C)$$

w is orthogonal to D , so face_w of the Minkowski sums of theorem 5.16 corresponding to the ranges above differ only by a vector. □

This vector is needed to describe the translation between two cells in the subdivision: If $C_1 + D_1$ and $C_2 + D_2$ are two neighbouring cells with respect to a vector w in the Minkowski sum of the two Newton polytopes, then the vector to patchwork the mixed fiber polytopes $\Sigma_{\pi^\circ}(C_1, D_1)$ and $\Sigma_{\pi^\circ}(C_2, D_2)$ is $v(C_1, D_1, w) - v(C_2, D_2, -w)$.

EXAMPLE 5.18. Let $f_1 = x + 2y + z - 4$, $f_2 = 3x - y + 2z + 1$, the valuation $\text{ord} : \mathbb{Q} \mapsto \mathbb{R}_\infty$ be the 2-adic valuation and let

$$\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2, x \mapsto \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot x$$

be the projection with kernel $\langle (2, -1, 1) \rangle$. Then the defining polynomial of $\pi^{-1}\pi \mathcal{T}(\langle f_1, f_2 \rangle)$ is

$$g := -338x - 18z^2 + 483xyz + 25yz^3 + 343y^2x^2$$

After applying the monomial map

$$B : \mathbb{Q}[x, y, z] \rightarrow \mathbb{Q}[x, y],$$

$$x \mapsto x, y \mapsto x^2y, z \mapsto y^2$$

induced by the projection matrix, we get

$$B(g) := -338x - 18y^2 + 483x^3y^2 + 25x^2y^4 + 343x^6y^2$$

For the subdivided newton polytope and the tropical variety see Figure 4.

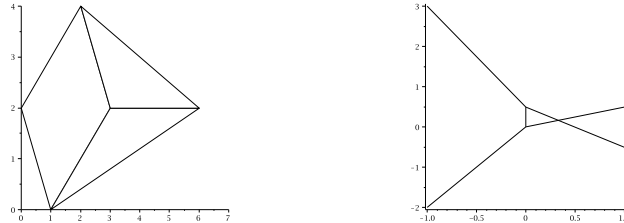


FIGURE 4. $\text{New}(B(g))$ and $\mathcal{T}(B(g))$

In the dual subdivision of $\text{New}(f_1 \cdots f_k)$ are two mixed 3-cells which corresponds to the two points of the tropical line $\mathcal{T}(\langle f_1, f_2 \rangle)$. If the newton polytopes of f_1 and f_2 are given by Figure 5 then the topdimensional mixed cells of the subdivision of $\text{New}(f_1) + \text{New}(f_2)$ are

$$F_3 + [v_0, v_2], G_4 + [w_1, w_3],$$

see Figure 6.

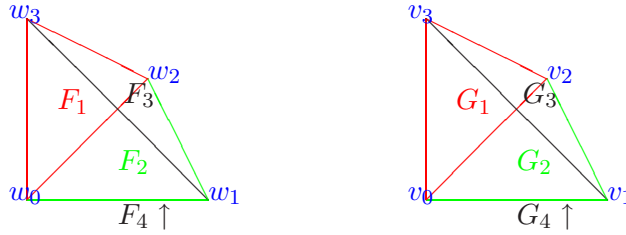


FIGURE 5. The Newton polytopes of f_1 and f_2

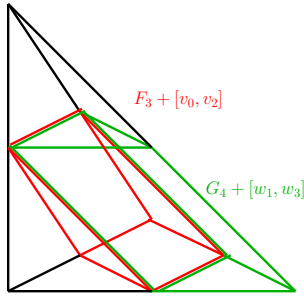


FIGURE 6. The sum of the Newton polytopes and the mixed cells

So we want to compute as in Theorem 5.12 the polytopes $\Sigma_{\pi^\circ}(F_3, [v_0, v_2])$ and $\Sigma_{\pi^\circ}(G_4, [w_1, w_3])$ where

$$\pi^\circ : \mathbb{R}^3 \rightarrow \mathbb{R}, x \mapsto (2, -1, 1) \cdot x$$

The mixed fiber polytope can be computed using the equality

$$\Sigma_{\pi^\circ}(P_1 + P_2) = \Sigma_{\pi^\circ}(P_1, P_2) + \Sigma_{\pi^\circ}(P_1) + \Sigma_{\pi^\circ}(P_2),$$

see Theorem 4.14. After projecting with π we get the polytopes on the right side of Figure 7 and after translation with $(-1, -1)$ and $(1, 1)$, respectively. We get the left side, see Corollary 5.17.

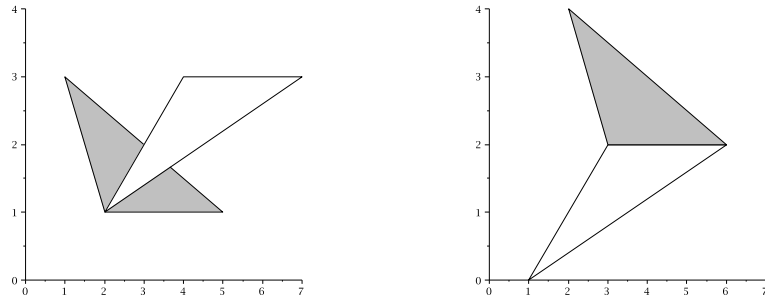


FIGURE 7. The fiber polytopes of the mixed cells

So we have to add (up to a translation) the sum of two mixed fiber polytopes, namely $\Sigma_\alpha([w_1, w_3], [v_0, v_1]) + \Sigma_\alpha([w_2, w_3], [v_0, v_2])$ to get the full subdivision as in Figure 4.

CHAPTER 6

Selfintersections

In this chapter we analyse the projections of tropical curves and derive some bounds on the complexity of the image. Tropical curves were studied by Vigeland for instance, see [Vig07]. He derived some bounds on the number of vertices and the number of edges of a tropical curve which is a transversal intersection of tropical hypersurfaces depending on the degrees of this tropical hypersurfaces. In contrast to this we give bounds for the number of vertices of the image of such a tropical curve. In the last chapter we had to analyse the exact structure of the image using the Newton polytope. For tropical lines $L \subseteq \mathbb{R}^n$ we give now bounds on the number of vertices of the image depending only on the dimension n .

1. Tropical curves in \mathbb{R}^n

Let now \mathcal{C} be a tropical curve in \mathbb{R}^n , i.e. \mathcal{C} is a balanced graph embedded in \mathbb{R}^n . The most simple curves are tropical lines, i.e. tropical varieties of 1-dimensional ideals which are generated by linear forms.

If $\mathcal{C} = L$ is a tropical line in \mathbb{R}^n it has the following combinatorics: It has $n + 1$ rays emanating in the directions e_1, \dots, e_n and $-e_1 - \dots - e_n$ to infinity. There are several types of lines in \mathbb{R}^n depending on where the rays to infinity emanate. The nondegenerate ones are by definition the lines which are 3-valenced graphs, i.e. at each vertex emanate exactly 3 edges.

EXAMPLE 6.1. In \mathbb{R}^3 there are 3 nondegenerate types of lines (see [RGST05]):

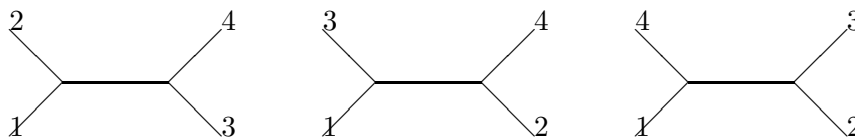
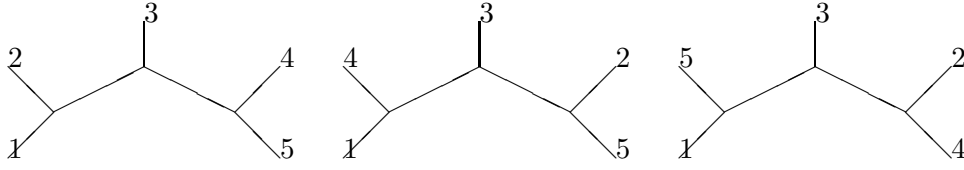


FIGURE 1. Here i for $1 \leq n$ denotes the rays emanating in direction e_i and $n + 1$ the ray emanating in direction $-e_1 - \dots - e_n$.

EXAMPLE 6.2. In \mathbb{R}^4 there are 15 different nondegenerate types, Figure 2 shows for example the ones with the 3 in the middle.

In general in \mathbb{R}^n there are many different types. To define the selfintersection points let $n > 2$ and $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^2$ be an arbitrary rational projection. One can describe this projection by a matrix A .

FIGURE 2. Some nondegenerate tropical lines in \mathbb{R}^4

DEFINITION 6.3. Let \mathcal{C} be a 1-dimensional polyhedral complex (for example a tropical curve). If the projections $\pi(c_1), \pi(c_2)$ of two non-adjacent 1-cells intersect in a point p , then p is called a selfintersection point of \mathcal{C} .

For an example of a selfintersection point, see Figure 3.

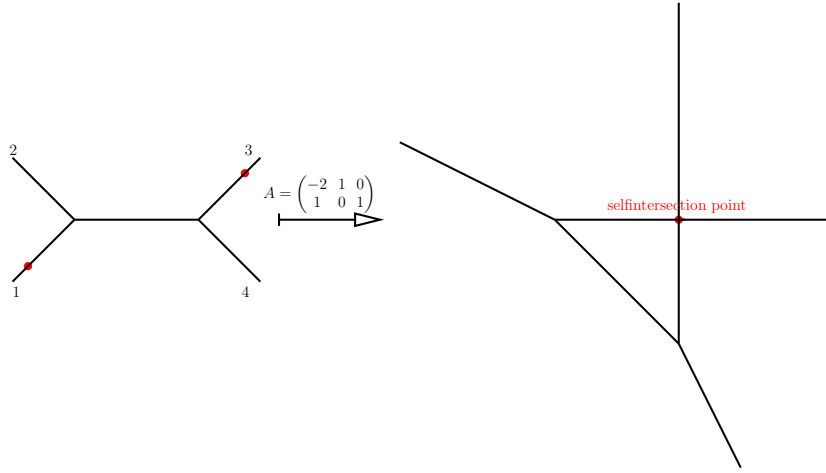


FIGURE 3. A selfintersection point of a line

As said before we want to specify the number of selfintersection points of a tropical curve \mathcal{C} . We state here the theorems, the definition of a caterpillar and the proofs can be found in the next sections.

THEOREM 6.4. As a lower bound for the number of selfintersection points of tropical curves in $\mathbb{R}^n, n \geq 3$, we get:

- (1) There exist a tropical line $L_n \subset \mathbb{R}^n$ and a projection $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^2$ such that L_n has

$$\sum_{i=1}^{n-2} i = \binom{n-1}{2}$$

selfintersection points.

- (2) There exist a tropical curve $\mathcal{C} \subset \mathbb{R}^n$ which is a transversal intersection of $n-1$ tropical hypersurfaces of degrees d_1, \dots, d_{n-1} and a projection $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^2$ such that \mathcal{C} has at least

$$(d_1 \cdot \dots \cdot d_{n-1})^2 \cdot \binom{n-1}{2}$$

selfintersection points.

THEOREM 6.5. *As an upper bound we get:
The image of a tropical line L_n in \mathbb{R}^n which is a caterpillar can have at most*

$$\sum_{i=1}^{n-2} i = \binom{n-1}{2}$$

selfintersection points.

Although we prove Theorem 6.5 only in the special case of a caterpillar we conjecture that it holds also in the general case.

2. Lower bounds for the number of selfintersection points

In this section we proof Theorem 6.4. This will be done in several steps. We start with the first part in the special case $n = 3$. Then we give an inductive proof of the general assertion. The second part will also be proved in two steps, first for the case $n = 3, 4$ and then for the general case.

2.1. A tropical line in \mathbb{R}^3 . Additionally to the fact that there is a tropical line in \mathbb{R}^3 and a corresponding projection with one selfintersection point we want to analyse which projections to a given tropical line fulfill this condition.

Let for this purpose $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a rational projection. One can describe this projection by the kernel $\langle (v_1, v_2, v_3) \rangle$ or by a matrix with rows orthogonal to the kernel. If $v_1 = 0$ we can see by an easy computation, that a projection with kernel generated by $(0, v_2, v_3)$ is either geometrically non-regular with respect to a tropical line or there are no selfintersections in the image of this line. If $v_1 \neq 0$ then π is can be described by the matrix

$$M := \begin{pmatrix} x & 1 & 0 \\ y & 0 & 1 \end{pmatrix}$$

with $x, y \in \mathbb{Q}$.

So we can concentrate on the case, where π is described by the matrix M .

Let L be a tropical line in \mathbb{R}^3 of type [12, 34], i.e. a line with vertices

$$(p_1, p_2, p_3), (p_1 + a, p_2 + a, p_3)$$

and rays emanating from the two vertices in directions $(1, 0, 0), (0, 1, 0)$ and $(-1, -1, -1), (0, 0, 1)$, respectively.

There are four possibilities for an intersection in the image of π , depending on the intersecting rays. Here we can assume that $a = 1$ and $p_i = 0, i = 1, 2, 3$, because all lines of the same type are isomorphic.

If we want to find for example the region where the images of the two rays $\lambda \cdot (0, 0, 1), \lambda > 0$ and $(1, 1, 0) + \mu \cdot (0, 1, 0), \mu > 0$ intersect, we see that $x = -\mu - 1$ and $y = \lambda$. So for $\lambda, \mu > 0$ there is an intersection, if $x < -1$ and $y > 0$.

The intersection of all other possible pairs of rays lead to three other regions, see Figure 4 for the union of all four regions.

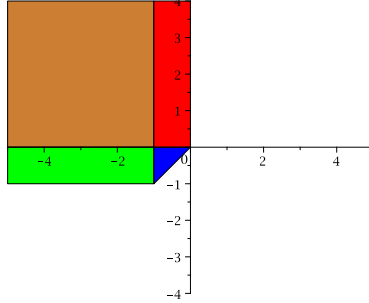


FIGURE 4. The region where a line of type $[12, 34]$ is selfintersecting

In particular we have seen that there is a tropical line and a projection with one selfintersection point.

2.2. A tropical line in \mathbb{R}^n . Now we can prove the general case of the first part of Theorem 6.4.

THEOREM 6.6. *For $n \geq 3$, there exist a line $L_n \subset \mathbb{R}^n$ and a projection $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^2$ such that $\pi(L_n)$ has*

$$\sum_{i=1}^{n-2} i = \binom{n-1}{2}$$

selfintersection points.

PROOF. Let L_n be the line in Figure 5. We want to prove a stronger result:

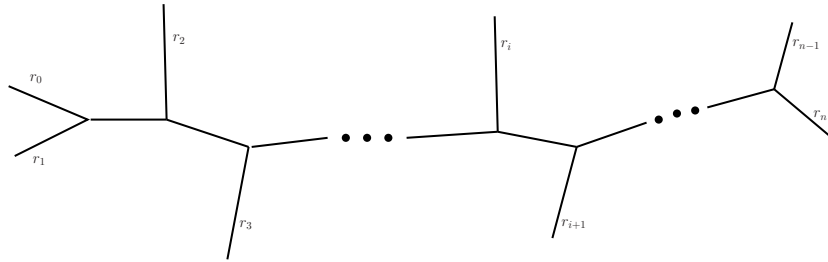


FIGURE 5. The line L_n

(*) *There is a projection $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^2$ such that each ray r_i emanating in direction e_{i+1} , $n-1 \geq i \geq 2$ intersects in the image with the ray r_1 emanating in direction e_2 , $\pi(r_i) \cap \pi(r_1) := p_i$, such that p_i lies between p_{i-1} and p_{i+1} for $i > 2$ and all images $\pi(r_i)$, $n-1 \geq i \geq 2$, do not intersect with the images of the bounded edges.*

The proof is by induction so we begin with $n = 3$. We have already seen that the tropical line L_3 of type $[12][34]$ has one selfintersection point.

Let now $n + 1$ be arbitrary. Map the line with the projection σ defined by

$$\sigma : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n, x \mapsto \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \cdot x$$

Then L_{n+1} is mapped to the nondegenerate line L_n . So by assumption there is a projection $\pi' : \mathbb{R}^n \rightarrow \mathbb{R}^2$ satisfying (*). Then $\pi'\sigma : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^2$ maps r_{n+1} to a point on $\pi'(L_n)$. The corresponding matrix of $\pi'\sigma$ has the form

$$A_{\pi'\sigma} = \begin{pmatrix} a'_{11} & \dots & a'_{1n+1} & 0 \\ a'_{21} & \dots & a'_{2n+1} & 0 \end{pmatrix}.$$

Figure 6 shows an example for $\pi'(L_n)$ for $n = 4$.

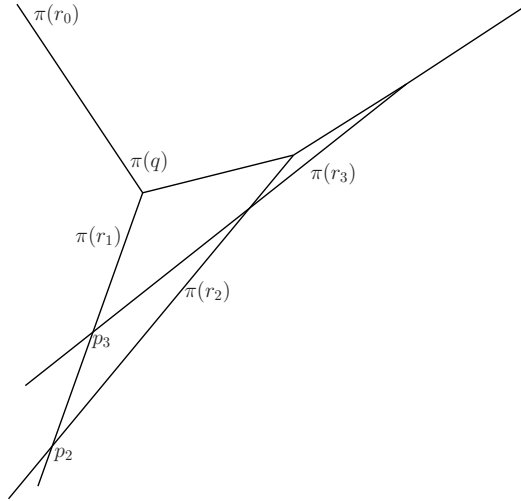


FIGURE 6. $\pi'(L_4)$

Let $\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^2$ be defined by a matrix with the same columns as $A_{\pi'\sigma}$ except the last. Because of the balancing condition we can choose the image of e_{n+1} such that the ray r_n is mapped to a ray lying under the images of the bounded edges and has an intersection point p_n with $\pi(r_1) = \pi'\sigma(r_1)$ lying above p_{n-1} . Because of the induction assumption $\pi(r_n)$ intersects with all $\pi(r_i), 1 \leq i \leq n - 1$. So $\pi(r_n)$ has $n - 1$ selfintersection points. So there are

$$\sum_{i=1}^{n-2} i + (n - 1) = \sum_{i=1}^{n-1} i$$

intersection points under π . □

EXAMPLE 6.7. Let L_4 be of the type [12][3][45], see Figure 7. Then σ is defined by

$$\sigma : \mathbb{R}^4 \rightarrow \mathbb{R}^3, x \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot x$$

and $\sigma(L_4) =: L_3$ is of type [12][34], see Figure 7.

Choose π' as



FIGURE 7. A line of type [12][3][45] and its image under σ , a line of type [12][34].

$$\pi' : \mathbb{R}^3 \rightarrow \mathbb{R}^2, x \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \cdot x$$

Then $\pi'\sigma(L_4)$ is the tropical hypersurface seen in Figure 8 and has one selfintersection point.

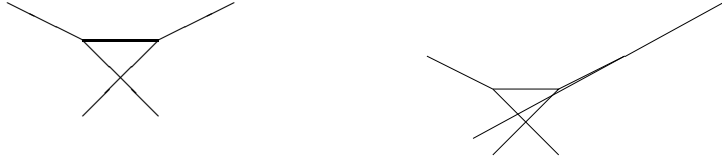


FIGURE 8. One selfintersection point of $\pi'\sigma(L_4)$ and the image of L_4 .

So we can choose as an image of e_4 for example $(-7, -4)$. Then the image of L_4 under

$$\pi : \mathbb{R}^4 \rightarrow \mathbb{R}^2, x \mapsto \begin{pmatrix} 1 & -2 & -1 & -3 \\ -1 & 1 & -1 & -2 \end{pmatrix} \cdot x$$

is the hypersurface seen in Figure 8 which has 3 selfintersection points.

Now we want to prove the second part of Theorem 6.4. Here we first concentrate again on the case of a tropical curve in \mathbb{R}^3 .

THEOREM 6.8. *There exist a tropical curve $\mathcal{C} \subset \mathbb{R}^3$ which is a transversal intersection of two tropical hypersurfaces of degrees d_1 and d_2 and a projection $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ such that \mathcal{C} has at least*

$$(d_1 \cdot \dots \cdot d_{n-1})^2$$

selfintersection points.

PROOF. Let f_1 and f_2 be linear polynomials such that $L := \mathcal{T}(f_1) \cap \mathcal{T}(f_2)$ is a nondegenerate line in \mathbb{R}^3 .

Choose a projection π such that L has one self-intersection point under π . This point is the intersection of the image of two rays r_1, r_2 , each emanating in one of the directions $e_1, e_2, e_3, -e_1 - e_2 - e_3$.

Now perturb the tropical hypersurfaces $\mathcal{T}(f_1)$ and $\mathcal{T}(f_2)$ a little bit. That means the coefficients of f_i and the coefficients of the correspondend perturbed polynomial differ by a small value (with respect to the valuation). Do this generically $d_1 - 1$ and $d_2 - 1$ times to get linear polynomials $f_1^{\epsilon_1}, \dots, f_1^{\epsilon_{d_1-1}}$ and $f_2^{\delta_1}, \dots, f_2^{\delta_{d_2-1}}$, respectively, such that

$$\mathcal{T}(f_1 \cdot f_1^{\epsilon_1} \cdot \dots \cdot f_1^{\epsilon_{d_1-1}}) \cap \mathcal{T}(f_2 \cdot f_2^{\delta_1} \cdot \dots \cdot f_2^{\delta_{d_2-1}})$$

is a 1-dimensional intersection of two tropical hypersurfaces of degrees d_1 and d_2 . It has $d_1 \cdot d_2$ rays emanating in each of the four directions $e_1, e_2, e_3, -e_1 - e_2 - e_3$.

If we project now with π and if the difference of the valuation of the coefficients is small enough then each of the rays emanating in the same directions as r_1 intersect with all the rays emanating in the same direction as r_2 . This gives us $(d_1 \cdot d_2)^2$ self-intersection points. \square

EXAMPLE 6.9. Let $K = \mathbb{C}\{\{t\}\}$ the field of puiseux series and let

$$\begin{aligned} f_1 &:= t^{\epsilon_1}x^2 + (t^{\epsilon_1} + t^{\epsilon_2})xy + (t^{1+\epsilon_1} + t^{1+\epsilon_3})xz + (t^{3+\epsilon_1} + t^{3+\epsilon_4})x + t^{\epsilon_2}y^2 + \\ &\quad (t^{1+\epsilon_2} + t^{1+\epsilon_3})yz + (t^{3+\epsilon_2} + t^{3+\epsilon_4})y + t^{2+\epsilon_3}z^2 + (t^{4+\epsilon_3} + t^{4+\epsilon_4})z + t^{6+\epsilon_4} \\ f_2 &:= t^{2+\delta_1}x^2 + (t^{1+\delta_1} + t^{1+\delta_2})xy + (t^{1+\delta_1} + t^{1+\delta_3})xz + (t^{1+\delta_1} + t^{1+\delta_4})x + \\ &\quad t^{\delta_2}y^2 + (t^{\delta_2} + t^{\delta_3})yz + (t^{\delta_2} + t^{\delta_4})y + t^{\delta_3}z^2 + (t^{\delta_3} + t^{\delta_4})z + t^{\delta_4} \end{aligned}$$

with $\epsilon_1 := \frac{1}{1000}$, $\epsilon_2 := \frac{3}{1000}$, $\epsilon_3 := \frac{5}{1000}$, $\epsilon_4 := \frac{7}{1000}$, $\delta_1 := \frac{11}{1000}$, $\delta_2 := \frac{13}{1000}$, $\delta_3 := \frac{17}{1000}$, $\delta_4 := \frac{1}{1000}$.

So each tropical variety $\mathcal{T}(f_i)$ is the union of two tropical hyperplanes and therefore $d_1 = d_2 = 2$. The intersection $\mathcal{T}(f_1) \cap \mathcal{T}(f_2)$ is a tropical curve with 4 unbounding rays in each of the directions $e_1, e_2, e_3, -e_1 - e_2 - e_3$. Take as projection

$$\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2, x \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

This is a direction where the image of the tropical line defined by setting each $\epsilon_i = \delta_i = 0$ has one selfintersection point. So we can calculate the resultant of the two polynomials p_1, p_2 :

```
> p[1]:=simplify(1^4*subs(x=x/1,subs(y=y/1^2,subs(z=z*1,f[1]))) );
> p[2]:=simplify(1^4*subs(x=x/1,subs(y=y/1^2,subs(z=z*1,f[2]))) );
> R:=resultant(p[1], p[2], 1);
```

The resultant has 21039 terms and describes the image $\pi(\mathcal{T}(f_1) \cap \mathcal{T}(f_2))$ which has 16 selfintersection points coming from the intersection of 4 rays with 4 rays, which gives us the bound of the theorem. But there are other selfintersection

points, in total there are 28 selfintersection points. You can see this by the induced subdivision of the Newton polytope of the resultant, see Figure 9.

```
> subdiv(S, t, [a, b], false);
```

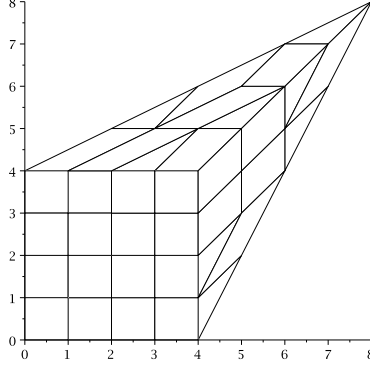


FIGURE 9. The subdivided Newton polytope of the polynomial generating the hypersurface $\pi^{-1}\pi(I)$

Here S is the resultant where the variables x, y, z are expressed in the new variables $a = xz, b = yz^2$.

3. Upper bounds for the number of selfintersection points

In this section we proof the Theorem 6.5. For this pupose we first analyse the case of a tropical line in \mathbb{R}^4 to illustrate some ideas.

Let L be a tropical line in \mathbb{R}^4 . We divide L in 3 pieces, see Figure 10.

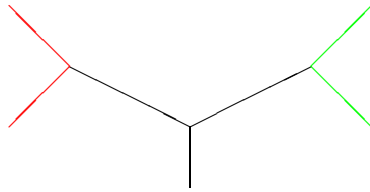


FIGURE 10. The line L

Let B be the image of the black ray in the middle together with the bounded edges of L , R the image of the two red rays on the left and G the image of the green rays on the right. We distinguish now in 4 cases:

Case 1: $B \cap R = \emptyset$ and $B \cap G = \emptyset$

Then R and G can intersect in at most 3 points. If they would intersect in 4 points then because of the convexity we could find two parallel lines $H1$ and $H2$ such that one black edge is in the halfspace $H1^+$ and another black edge is

in the halfspace H_2^- , see Figure 11. But these two edges have to intersect in one point, so this is a contradiction.

Case 2: $B \cap R = \emptyset$ and $B \cap G \neq \emptyset$

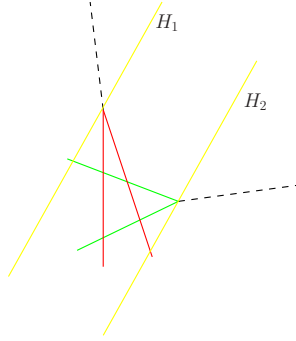


FIGURE 11. The intersection of R and G and the 2 dividing hypersurfaces.

Then G can intersect B in only one point, see Figure 12. R can not be in the cone with apex P spanned by a parallel vector to the unbounded black ray and a vector through the point Q , otherwise it would intersect B . But then R can intersect G in at most 2 points because each red ray can only intersect one green ray, see Figure 12. So we have at most 3 self-intersection points.

Case 3: $B \cap R \neq \emptyset$ and $B \cap G = \emptyset$

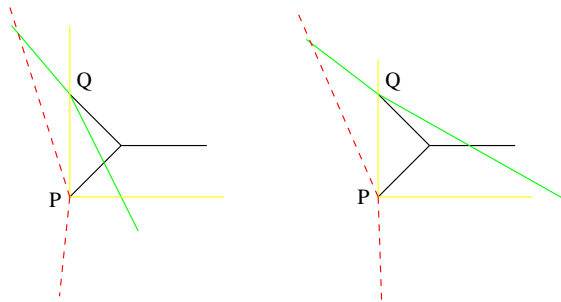


FIGURE 12. The intersection of R and G and the restricting cone

This case is analog to Case 2.

Case 4: $B \cap R \neq \emptyset$ and $B \cap G \neq \emptyset$

Let H_1 be the line through Q and parallel to the image of the unbounding black ray, H_2 the line through P and parallel to the same ray. R and G can both only have one intersection point with B , because of the convexity one red ray has to be in H_1^- and one green ray in H_2^+ , see Figure 13. Let H be the line through P and Q . Because G and R intersect B one red and green ray has

to be in H^+ and one in H^- , respectively. But then at most one red ray and one green ray can intersect each other and we have at most 3 self-intersection points.

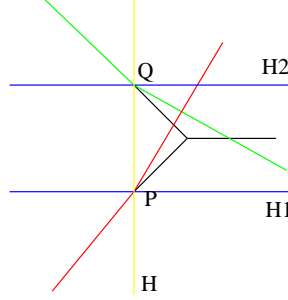


FIGURE 13. The intersection of R and G and the restricting hypersurfaces

We have seen that the image of a tropical line $L_3 \subset \mathbb{R}^3$ can have at most 1 and the image of a tropical line $L_4 \subset \mathbb{R}^4$ can have at most 3 selfintersection points. Now we treat the special case of caterpillar line, i.e. a tropical line $L_n \subset \mathbb{R}^n$ of the trivial form seen in Figure 14.

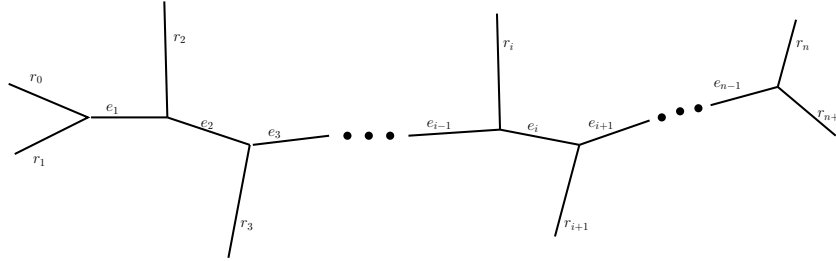


FIGURE 14. A caterpillar line

Note that for $n = 3, 4$ every tropical line is a caterpillar and so the theorem holds in these dimensions in the general situation. We conjecture that also for $n > 4$ the assertion holds but we can only proof it in this special case.

THEOREM 6.10. *The image of a tropical line L_n in \mathbb{R}^n , $n \geq 3$ which is a caterpillar can have at most*

$$\sum_{i=1}^{n-2} i = \binom{n-1}{2}$$

selfintersection points.

PROOF. The proof is by induction. For $n = 3, 4$ we have seen the assertion. So assume the assertion is true for $n \geq 3$. We want to show it for $n + 1$. Let L_{n+1} be a nondegenerate line and π a projection, described by the matrix A . Construct a line L_n as follows: Choose a vertex v of L_{n+1} where two rays emanate. Let w.l.o.g. one ray emanates in direction e_{n+1} and one in direction

$-e_1 - \dots - e_{n+1}$ (otherwise permute the coordinates before). Then map the line L_{n+1} with the projection σ defined by

$$\sigma : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n, x \mapsto \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \cdot x$$

Then L_{n+1} is mapped to the nondegenerate line L_n . Define $\pi' : \mathbb{R}^n \rightarrow \mathbb{R}^2$ by the matrix

$$A' := \begin{pmatrix} a_{1,1} & \dots & a_{1,n} + a_{1,n+1} \\ a_{2,1} & \dots & a_{2,n} + a_{2,n+1} \end{pmatrix}$$

The image $\pi'(L_n)$ equals $\pi(L_{n+1})$ except at the vertex $\pi(v)$, see Figure 15. Let r_0 and r_1 be the rays emanating in directions $-e_1 - \dots - e_{n+1}$ and e_{n+1} , respectively. Their images in $\pi(L_{n+1})$ are represented by solid blue rays. The dashed blue ray $\pi'(\tilde{r})$ is the image of the substitute \tilde{r} for them in $\sigma(L_{n+1})$.

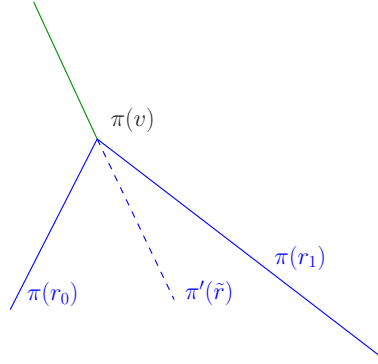


FIGURE 15. The reduction $n + 1 \rightarrow n$

So we have to compare the selfintersection points lying on the two solid rays with the points on the dashed ray. For a 1-dimensional subcomplex \mathcal{Y} of a polyhedral complex \mathcal{X} define

$$SIP(\mathcal{Y}) := \{z \in \mathbb{R}^2 \mid \exists y \in \mathcal{Y} \exists x \in \mathcal{X}, x \neq y : z = \pi(x) = \pi(y)\}$$

Then we want to show

Claim: $\#SIP(r_0) + \#SIP(r_1) - \#SIP(\tilde{r}) \leq n - 1$.

Each line L_{n+1} has $n - 1$ bounded edges. One of them emanate from v , so because of the concavity its image cannot intersect with $\pi(r_0)$ or $\pi(r_1)$. Each other bounded edge contribute to the above sum with at most 1 because its image cannot intersect with $\pi(r_0)$ and $\pi(r_1)$ and not with $\pi'(\tilde{r})$.

This leads to the definition of the contribution of an edge $c = e_i$ or $c = r_i$.

$$\text{contr}(c) := \delta_{\pi(c) \cap \pi(r_0)} - \delta_{\pi(c) \cap \pi'(\tilde{r})} + \delta_{\pi(c) \cap \pi(r_1)}.$$

where $\delta_{a \cap b} = \begin{cases} 1 & \text{if } a \cap b \neq \emptyset \\ 0 & \text{if } a \cap b = \emptyset \end{cases}$. Then

$$\sum_{\substack{c \text{ an edge of } L_{n+1} \\ c \neq r_0, r_1}} \text{contr}(c) = \#SIP(r_0) + \#SIP(r_1) - \#SIP(\tilde{r})$$

We want to find for each ray r_i with $\text{contr}(r_i) = 1$ a bounded edge $e(r_i)$ such that

$$\text{contr}(e(r_i)) \leq 0$$

and if there are r_i, r_k with $\text{contr}(r_i) = \text{contr}(r_k) = 1$ and $e(r_i) = e(r_k)$ then

$$\text{contr}(e(r_i)) = -1 \text{ and } e(r_j) \neq e(r_i) \text{ for all } j \neq i, k \text{ and } \text{contr}(r_j) = 1.$$

If no bounded edge of L_{n+1} has contribution 1 then there are at most $n-1$ rays with contribution 1 because of the concavity condition, see Figure 16. Otherwise let $e_{i_1}, \dots, e_{i_r}, i_1 < i_2 < \dots < i_r$, be the bounded edges with $\text{contr}(e_{i_j}) \neq 0$, i.e. $\text{contr}(e_s) = 0$ for all $s \notin \{i_1, \dots, i_r\}$.

Now we assign to each r_i with $\text{contr}(r_i) = 1$ an edge $e(r_i)$ as said above according to i and the contributions of the edges e_{i_j} .

Step 1: $2 \leq i \leq i_1$ (Note: $i_1 \geq 2$)

We assign each r_i with positive contribution to the adjacent edge lying before r_i :

$$e(r_i) = e_{i-1} \text{ for } \text{contr}(r_i) = 1.$$

But not all the rays r_i with $2 \leq i \leq i_1$ can have $\text{contr}(r_i) = 1$, see Figure 16.

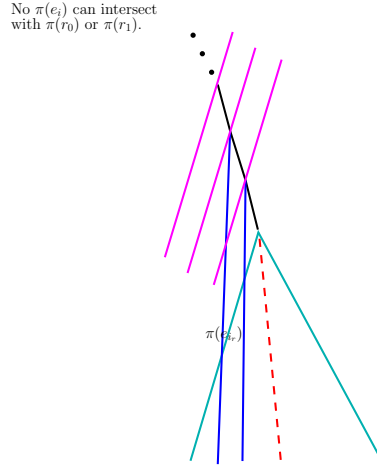


FIGURE 16. Not all rays have contribution 1

So there is a bounded edge e_l which can be used later.

Step 2: $i_j + 1 \leq i \leq i_{j+1}, j \geq 1$

Here we have to differentiate further between the following cases:

Case 2A: $\text{contr}(e_{i_j}) = 1 \wedge \text{contr}(e_{i_{j+1}}) = 1$

Then we have one of the situations of Figure 17 (up to a change of the roles of r_0 and r_1).

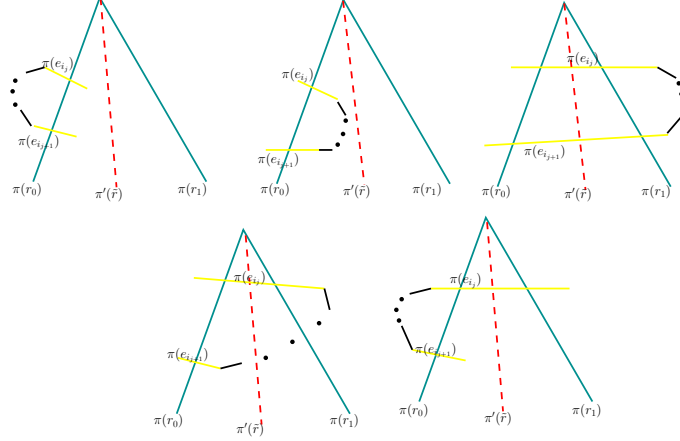


FIGURE 17. The dots indicate how the edges were connected

In each of the cases there has to be a ray r_k , $i_j + 1 \leq k \leq i_{j+1}$, with $\text{contr}(r_k) \leq 0$. The arguments are analog to the one used in step 1. So we assign r_i to the adjacent edge e_i lying after r_i .

$$e(r_i) = e_i \text{ for } i_j + 1 \leq i \leq i_{j+1} - 1, \text{ contr}(r_i) = 1$$

If $\text{contr}(r_{i_{j+1}}) = 1$ we cannot do the same. But then we have seen that there is the edge e_k which is not used yet. So we can assign $e(r_{i_{j+1}}) = e_k$.

Case 2B: $\text{contr}(e_{i_j}) = 1 \wedge \text{contr}(e_{i_{j+1}}) = -1$

Then we have one of the situations of Figure 19.

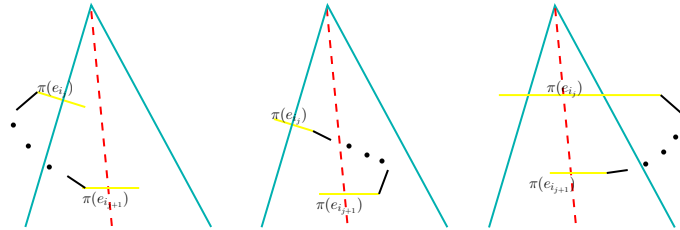


FIGURE 18. These 3 situations can not occur because all e_i with $i_j < i < i_{j+1}$ have $\text{contr}(e_i) = 0$.

In all possible 3 situations we can assign again to the edge lying after r_i .

$$e(r_i) = e_i \text{ for } i_j + 1 \leq i \leq i_{j+1}, \text{ contr}(r_i) = 1$$

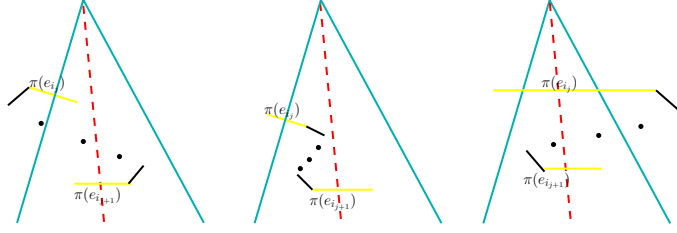


FIGURE 19. These 3 situations can occur.

Case 2C: $\text{contr}(e_{i_j}) = -1 \wedge \text{contr}(e_{i_{j+1}}) = 1$

Then we can assign

$$e(r_i) = e_{i-1} \text{ for } i_j + 1 \leq i \leq i_{j+1}, \text{contr}(r_i) = 1$$

Note that e_{i_j} is used at most twice.

Case 2D: $\text{contr}(e_{i_j}) = -1 \wedge \text{contr}(e_{i_{j+1}}) = -1$

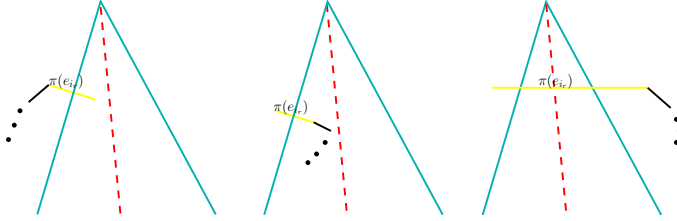
Then we have again no problem and assign

$$e(r_i) = e_{i-1} \text{ for } i_j + 1 \leq i \leq i_{j+1}, \text{contr}(r_i) = 1$$

Step 3: $i_r + 1 \leq i \leq n + 1$

Case 3A: $\text{contr}(e_{i_r}) = 1$

Then we have one of the situations of Figure 20.

FIGURE 20. These 3 situations can occur. The dots indicate where e_{i_r} is connected with the last rays r_n, r_{n+1}

In each case not all r_i , $i_r + 1 \leq i \leq n + 1$ can have $\text{contr}(r_i) = 1$. this holds because of the concavity condition, see Figure 21.

Now we can assign

$$e(r_i) = e_i \text{ for } i_r + 1 \leq i \leq n - 1, \text{contr}(r_i) = 1.$$

Because not all rays have contribution 1 there is a ray r_k , $i_r + 1 \leq k \leq n + 1$ with $\text{contr}(r_k) \neq 1$. So it remains to assign the last rays r_n and r_{n+1} to an appropriate edge if necessary.

If $\text{contr}(r_n) = 1$ and $\text{contr}(r_{n+1}) = 0$ we assign $e(r_n) = e_l$, where e_l is defined in step 1. If $\text{contr}(r_n) = 0$ and $\text{contr}(r_{n+1}) = 1$ we assign $e(r_{n+1}) = e_l$. If both $\text{contr}(r_n) = 1$ and $\text{contr}(r_{n+1}) = 1$ we assign $e(r_n) = e_l$ and $e(r_{n+1}) = e_k$.

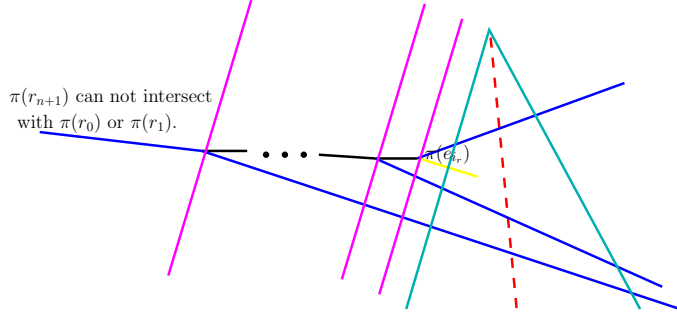


FIGURE 21. Not all rays can have contribution 1.

Case 3B: $\text{contr}(e_{i_r}) = -1$

Then we can assign

$$e(r_i) = e_{i-1} \text{ for } i_r + 1 \leq i \leq n, \text{ contr}(r_i) = 1$$

and

$$e(r_{n+1}) = e_l \text{ if } \text{contr}(r_{n+1}) = 1, l \text{ defined in Step 1.}$$

Now we have assigned for each ray r_i with $\text{contr}(r_i) = 1$ a bounded edge $e(r_i)$ with the conditions described above.

That means that by going over from n to $n + 1$ at most

$$\#\{\text{bounded edges of } L_{n+1}\} = n - 1$$

selfintersection points can arise.

It follows that

$$\#\text{SIP}(L_{n+1}) \leq \#\text{SIP}(L_n) + n - 1 \leq \sum_{i=1}^{n-1} i.$$

□

So we have derived an upper bound for the number of selfintersection points of a tropical line. But we want to have a bound for tropical curves of higher degree, too. Every selfintersection point is a singular point of the projection, i.e. a vertex which is not 3-valent. For such points hold:

LEMMA 6.11. *Let f be a polynomial in $K[x, y]$. If p is a singular point of $\mathcal{T}(f)$, dual to an n -gon for $n \geq 4$, then $p \in \mathcal{T}(f_x) \cap \mathcal{T}(f_y)$.*

PROOF. After derivation there is at least one face of the n -gon p^\vee which is a translate of a cell C in $\text{New}(f_x)$. So $p \in C^\vee \subset \mathcal{T}(f_x)$. Analogously $p \in \mathcal{T}(f_y)$. So the assertion holds. □

Therefore, under the conditions on the curve described, any selfintersection point is not only contained in $\pi(\mathcal{T}(I))$ but in the intersection $\mathcal{T}_1 \cap \mathcal{T}_2$ where $\mathcal{T}_i := \mathcal{T}(\frac{\partial f}{\partial x_i})$ and f is the polynomial describing the projection. This intersection is mostly not a finite intersection.

But because selfintersection points are vertices of f , they are also contained in the stable intersection

$$\mathcal{T}_1 \cap_{st} \mathcal{T}_2 = \lim_{\epsilon \rightarrow 0} (\mathcal{T}_1)_\epsilon \cap (\mathcal{T}_2)_\epsilon$$

where $(\mathcal{T}_i)_\epsilon$ are curves which are near \mathcal{T}_i but intersect each other only in finitely many points.

The theorem of Bernstein deals with the stable intersection of tropical hypersurfaces. It uses intersection multiplicities. For a vertex v in an intersection $X_1 \cap \dots \cap X_n$ the multiplicity is defined by $m_v := MV(F_1, \dots, F_n)$ where $F_1 + \dots + F_n = v^\vee$ is the dual cell in the subdivided polytope $P_1 + \dots + P_n$, where P_i is the Newton polytope of X_i .

THEOREM 6.12 (Tropical Bernstein, see [BB07, RGST05, ST09]). *Suppose the tropical hypersurfaces $X_1, \dots, X_n \subset \mathbb{R}^n$ with Newton polytopes P_1, \dots, P_n intersect in finitely many points. Then the number of intersection points counted with multiplicity is $MV_n(P_1, \dots, P_n)$.*

Furthermore the stable intersection of n tropical hypersurfaces X_1, \dots, X_n always consists of $MV_n(P_1, \dots, P_n)$ points counted with multiplicities.

So a simple bound on the number of selfintersection points is given by the mixed volume of the Newton polynomials of $\frac{\partial f}{\partial x_1}$ and $\frac{\partial f}{\partial x_2}$.

THEOREM 6.13. *Let $\pi(\mathcal{T}(I)) = \mathcal{T}(f)$ be the image of a tropical curve. Then the number of selfintersection points of $\pi(\mathcal{T}(I))$ is bounded above by*

$$\min \left(\text{vol}(\text{New}(f)), MV_\Lambda \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right) \right).$$

PROOF. Since the selfintersection points are singular points in the tropical hypersurface $\mathcal{T}(f)$, another bound on their number is given by the volume of $\text{New}(f)$. \square

EXAMPLE 6.14. Let $Y = \mathcal{T}(f_1) \cap \mathcal{T}(f_2)$ be again the intersection of example 6.9 and f the resultant corresponding to π . The volume of $\text{New}(f)$ is 32 and the mixed volume of the Newton polytopes of the derivatives is 56 and so we get together with Theorem 6.13 the inequalities

$$\begin{aligned} (d_1 + d_2)^2 \leq \#SIP(Y) &\leq \min \left(\text{vol}(\text{New}(f)), MV \left(\text{New} \left(\frac{\partial f}{\partial x_1} \right), \text{New} \left(\frac{\partial f}{\partial x_2} \right) \right) \right) \\ &\Leftrightarrow 16 \leq \#SIP(Y) \leq 32 \end{aligned}$$

We see that the upper bound given by the volume of the Newton polytope is much better than the one given by the mixed volume. The reason is that most of the selfintersection points have multiplicity 2 and we count them twice.

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