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Exact formulas for a thin-lens system with an arbitrary number of lenses

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ABSTRACT

We present an angular thin-lens formula giving the angle of refraction β for arbitrary values of the angle of incidence α. With this formula, we find analytical results for the focal length f of a thin-lens system. The number of lenses n and their focal lengths $f_1, f_2, \ldots, f_n$ are abitrary, as are the mutual distances $D_{12}, \ldots, D_{(n-1)n}$ between the lenses. All these results are exact, i.e. not restricted to small or even paraxial angles. In the literature, the 2-lens and 3-lens versions (the last one without proof or derivation) are known [1]. We present the general result for n lenses and for (the positions of) its principal planes.

1. Introduction

Already Gauß noted [2] that the common school textbook form of the lens formula $1/f = 1/i + 1/o$ is incompatible with the use of a decently defined Cartesian coordinate system. This hindrance finally stems from the fact that, for historical reasons, both object distance o and image distance i are counted positive for the constellation of Fig. 1. All formulas in this article are consistent with the form $1/f = 1/i - 1/o$ which was proposed by Gauß to remove the hindrance and its inconsistencies. This means that all quantities (O, o, I, i, α, β, γ, see Fig. 3) except f (and the mutual distances $D_{(i-1)i}$ for a lens system) are *directed*.

2. Single lens

Our starting point is a *non-paraxial* relation between the angles α and β of the incident and the emerging light rays for a single thin lens (see Fig. 1).

$$\tan\beta = \tan\alpha - \tan\gamma \tag{1}$$

Ray tracing experts will recognize Eqs. (1) and (2). However, since ray tracing is usually done by matrix optics at spherical surfaces, the potential of Eqs. (1) and (2) for non-paraxial ray tracing, even for a single lens, seems not to be widely realized.

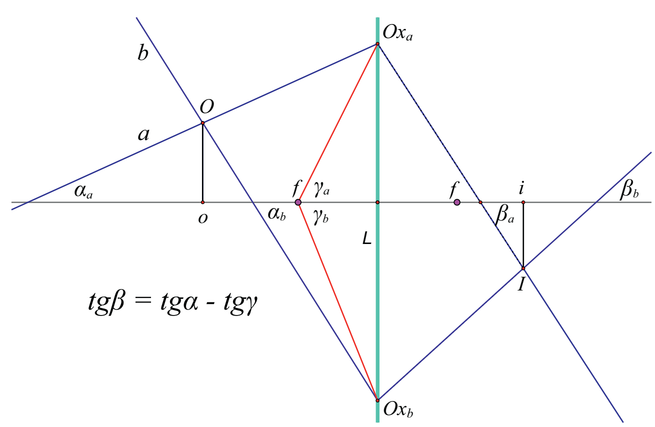

Fig. 1. Relation between the angles α, β and γ for rays a and b refracted by the thin lens L with focal length f. O is the object size, o the object distance, I the image size and i the image distance.

The third angle in the relation is the angle γ between the optical axis and the connecting line from the focal point f at the object side of the lens to the point Ox where the incident ray hits the lens, as shown in Fig. 1 for two rays a and b. Eq. (1) allows the direct determination of $tg\beta$ for arbitrary values of $tg\alpha$. The construction of a second ray immediately produces the image, without recourse to the commonly used special rays. If one takes angle γ at the object side of the lens, Eq. (1) is valid for object positions at both sides of the lens. To formally account for the two orientations, we introduce the left–right parameter $\langle LR\rangle$, which takes the value $\langle LR\rangle = +1$ for light

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rays traveling from left to right. For right-to-left rays, $\langle LR \rangle = -1$. This definition yields a form of Eq. (1), which works for both directions of a light ray:

$$\tan\beta = \tan\alpha - \langle LR \rangle p \cdot Ox \tag{2}$$

where $p = 1/f$ is the optical power of the lens. Proof of Eqs. (1) and (2) follows at the end of the paper. As it turns out, this versatile form enables us to find p and f for an n-lens system.

3. Properties of lens systems

All properties of a thin-lens system are solely determined by the focal lengths of its individual lenses $f_1, f_2, \ldots, f_n$ (or their corresponding powers $p_1, \ldots, p_n$) and their mutual pair distances $D_{12}, \ldots, D_{(n-1)n}$, which we take positive by definition, see **Directedness and consistency** below. Object distance o, image distance i and focal length f all are measured relative to the so-called principal planes P_R and P_L, which by themselves are functions of f and D. The exact formulas for P_R and P_L are presented below. For the moment we use the well known graphical construction [3] by which the principal plane P_R lies at the intersection of left-side parallel ray and associated right-side focus ray, see Fig. 2. The principal plane P_L is defined the other way around, see Fig. 4.

3.1. 2-lens system

We consider an initially parallel ray traversing two lenses from left to right, finally crossing the optical axis at f_R. Since parallel means ($\tan\alpha_1 = 0$ and $Ox1 = O$), Eq. (2) tells us:

$$\begin{aligned}
&\tan\beta_1 = -\langle LR \rangle p_1 O \quad \text{Using } \langle LR \rangle^2 = 1 : \\
&\tan\beta_2 = \tan\alpha_2 - \langle LR \rangle p_2 Ox2 \\
&= \tan\beta_1 - \langle LR \rangle p_2 (O + \langle LR \rangle D_{12} \tan\beta_1) \\
&= -\langle LR \rangle (p_1 + p_2 - p_1 p_2 D_{12}) O \\
&= -\langle LR \rangle (p_1 + p_2 (1 - p_1 D_{12})) O
\end{aligned} \tag{3}$$

In the second-to-last line of Eq. (3) we recognize the well known expression for the power of a 2-lens system with powers p_1, p_2 and mutual distance D_{12}. This formula yields the typifying focal length $f_{12} = 1/p_{12}$ and not the coordinate version f_L or f_R, which is consistent with the fact that we only used angles and vertical quantities (O, Ox) to get the result. The 2-lens f_{12} resulting from Eq. (3) is shown in Fig. 2. For angle β we get:

$$\tan\beta_2 = \tan\beta_f = -\langle LR \rangle p_{12} O \tag{4}$$

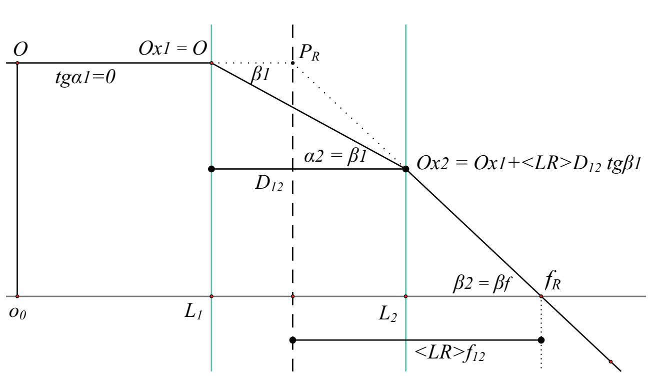

Fig. 2. Parallel ray traversing two lenses and hitting the focal point f_R at an angle $\beta_2 = \beta_f$.

where β_f is the angle under which an initially horizontal ray hits the optical axis at the corresponding focal point, in this case f_R.

On the way to a general n-lens formula, the last line of Eq. (3) is useful. It starts to show the recursive properties of the sought after formulas. We will see this clearer in the 3-lens case below.

3.2. 3-lens system

Although the 2-lens result Eq. (4) is geometrically obvious, see Fig. 2, it becomes productive however only in cooperation with Eq. (2). The same is true for the 3-lens system. Even without a figure of the 3-lens case, we can cope with it thanks to the emerging recursive properties, since:

$$\begin{aligned}
&\tan\beta_1 = -\langle LR \rangle p_1 O \\
&\tan\beta_2 = -\langle LR \rangle p_{12} O \\
&\tan\beta_3 = \tan\beta_2 - \langle LR \rangle p_3 (O + \langle LR \rangle D_{12} \tan\beta_1 + \langle LR \rangle D_{23} \tan\beta_2) \\
&= -\langle LR \rangle p_{12} O - \langle LR \rangle p_3 (O + \langle LR \rangle p_1 D_{12} + \langle LR \rangle p_{12} D_{23}) \\
&= -\langle LR \rangle (p_{12} + p_3 (1 - p_1 D_{12} - p_{12} D_{23})) O \\
&= -\langle LR \rangle p_{123} O
\end{aligned} \tag{5}$$

The calculation produces a recursive version of p_{123}. Of course, with the known 2-lens results, one can write down the low-level result for p_{123} or f_{123}, which is identical to the result in [1]:

$$\begin{aligned}
p_{123} = {} & (p_1 + p_2 + p_3) - (p_1 p_2 + p_1 p_3) D_{12} + \\
& - (p_1 p_3 + p_2 p_3) D_{23} + p_1 p_2 p_3 D_{12} D_{23}
\end{aligned} \tag{6}$$

Ref. [1] seems to be relatively unknown, in the internet one only finds an educated guess for the 3-lens EFL (Effective Focal Length) which we compare with our recursive f-version:

$$\begin{aligned}
f_{123} &= \frac{f_1 f_{12} f_3}{f_1 f_{12} + f_1 f_3 - D_{23} f_1 - D_{12} f_{12}} \\
EFL_{123} &= \frac{f_{12} f_3}{f_{12} + f_3 - D_{23}}
\end{aligned} \tag{7}$$

Comparison shows that the term $D_{12} f_{12}$ is missing in the denominator of the educated guess, see for instance [4].

3.3. n-lens system

Continuation of the process with more lenses can be brought into the following form:

$$\begin{aligned}
&p_1 = p_1 \\
&p_{12} = p_1 + p_2 (1 - p_1 D_{12}) \\
&p_{123} = p_{12} + p_3 (1 - p_1 D_{12} - p_{12} D_{23}) \\
&p_{1234} = p_{123} + p_4 (1 - p_1 D_{12} - p_{12} D_{23} - p_{123} D_{34}) \\
&p_{12345} = p_{1234} + p_5 (1 - p_1 D_{12} - p_{12} D_{23} - p_{123} D_{34} - p_{1234} D_{45}) \\
&p_{1 \cdots n} = p_{1 \cdots (n-1)} + p_n \left(1 - \sum_{i=1}^{(n-1)} p_{1 \ldots i} D_{i(i+1)} \right)
\end{aligned} \tag{8}$$

The pattern is so obvious, that the result for arbitrary n is immediately clear. However, for many lenses, even the recursive version becomes longer and longer: the sum version in the last line of Eq. (8) contains $n + 1$ terms. Fortunately, there exist two ways to generate 2-term versions. The first way is to realize that geometrically, the part in brackets is the vertical coordinate (in units of O) of the point where an initially parallel ray hits lens number n.

Therefore:

$$p_{1\cdots n} = p_{1\cdots(n-1)} + p_n\left(\frac{Oxn(\alpha_1 = 0)}{O}\right) \tag{9}$$

There is a further way: With the help of Eq. (8), it is possible (for $n > 2$) to generate formulas without explicit occurrence of the pair distances $D_{k(k+1)}$. The result is:

$$p_{1\cdots n} = \left(\frac{p_{1\cdots(n-k)}p_{(n-k)\cdots n} - p_{1\cdots(n-k-1)}p_{(n-k+1)\cdots n}}{p_{(n-k)}}\right) \quad \text{for } 1 \le k \le n \tag{10}$$

Permutations of the formula (given by the index k) stem from the fundamental left–right symmetry of all the compound powers. Permuting up to n or down to 1 is possible but trivial, giving $p_{1\cdots n} = p_{n\cdots 1}$. For $n = 4$ via $n = 3$ down to $n = 2$ for example, one gets the following (omitting $k = 1$ and $k = n$):

$$\begin{aligned}
&p_{1234} = \frac{p_{12}p_{234} - p_1p_{34}}{p_2} = \frac{p_{123}p_{34} - p_{12}p_4}{p_3}\\
&p_{123} = \frac{p_{12}p_{23} - p_1p_3}{p_2} \quad p_{234} = \frac{p_{23}p_{34} - p_2p_4}{p_3}\\
&p_{12} = p_1 + p_2(1 - p_1D_{12})\\
&p_{23} = p_1 + p_2(1 - p_1D_{12})\\
&p_{34} = p_3 + p_4(1 - p_3D_{34})
\end{aligned} \tag{11}$$

One clearly sees that the information about the mutual distances is explicit only at the 2-lens level. All calculations above level 2 can be made with the implicit versions, which make them much simpler. Actually, the formulas presented in this paper enable the analysis and design of complex systems like camera zoom objectives. Details will be given in a separate paper.

4. Directedness and consistency

As said before in Section 1, the conventions in connection with the school textbook lens equation lead to inconsistencies. More specifically, they corrupt the definition of the magnification as $M = I/O = i/o$: If one takes both i and o positive in Fig. 1 or Fig. 3, one is forced to throw away the information about the inversion of the image I with respect to the object O. Moreover, when describing the use of a collecting lens as magnifying glass, the conventions lead to absurdities like: “The image I is on the same side of the lens as the object O, so I have to count the image distance i negative, while at the same time leaving the object distance o positive even when the object is on the negative-coordinate side of the lens”. As Gauß already pointed out 200 years ago, all this would be mended by the adoption of his directed version of the lens formula $1/f = 1/i - 1/o$. Following his advice, we use directed varieties not only of O, Ox, o, I and i, but also of the angles α,β and γ. According to Cartesian rules, angles are counted positive when rotating counter clockwise from the optical axis and negative when rotating clockwise, as indicated by the plus and minus signs in Fig. 3.

Since f and p, due to their typifying definition as positive for a collecting lens and negative for a diverging lens, are no directed quantities, they need special treatment. The same is true for the mutual pair distances $D_{k\,(k+1)}$, since we take them always to be positive, see Fig. 2 and Eq. (3). As it turns out, all that is needed is to attach the already introduced left–right parameter $\langle LR\rangle$ to f, p and $D_{k\,(k+1)}$ whenever they occur. As a consequence, inconsistencies regarding for instance the magnification M disappear, which for this reason is dubbed $M_{\langle LR\rangle}$:

$$M_{\langle LR\rangle} = \frac{I}{O} = \frac{i}{o} = \frac{\langle LR\rangle f}{\langle LR\rangle f + o} = \frac{1}{1 + \langle LR\rangle p \cdot o} \tag{12}$$

Eq. (12) retains the information about the inversion of the image with the added bonus that it works for both directions of a light ray, all owing to the simple prescription given above. In fact, by applying the prescription to the Gaussian lens formula $1/f = 1/i - 1/o$, we get a lens formula which is directed, consistent and not confined to the Cartesian Convention which allows only light rays coming from the left:

$$\frac{1}{\langle LR\rangle f} = \frac{1}{i} - \frac{1}{o} \tag{13}$$

The subsequent discovery of the n-lens formulas presented in this paper hopefully provides further arguments in favor of the abolition of $1/f = 1/i + 1/o$.

5. Proof of the single-lens equation

We consider a lens with two of the usual special rays, namely those crossing the focal point on one side and running parallel to the optical axis at the other side. The third ray is an arbitrary one running at angles α and β with respect to the optical axis, see Fig. 3.

The point where the light ray hits the lens is designated Ox, where x is the coordinate of the hitting point relative to the object size O. x can take any arbitrary value.

Following the left–right prescription,

$$\tan\gamma = \frac{Ox}{\langle LR\rangle f} = \langle LR\rangle p \cdot Ox \tag{14}$$

For Eq. (1), as seen in Fig. 3 and using Eqs. (12) and (14) this means:

$$\begin{aligned}
&\tan\beta = \tan\alpha - \tan\gamma\\
&\frac{Ox - I}{-i} = \frac{O(1-x)}{o} - \frac{Ox}{\langle LR\rangle f}\\
&\frac{Ox}{O}\left(\frac{1}{\langle LR\rangle f} - \frac{1}{i} + \frac{1}{o}\right) = \frac{1}{o} - \frac{I}{O \cdot i} = \frac{1}{o} - \frac{i}{o \cdot i} = 0
\end{aligned} \tag{15}$$

Comparison with Eq. (13) completes the proof of Eqs. (1) and (2) while attesting their compatibility with the same.

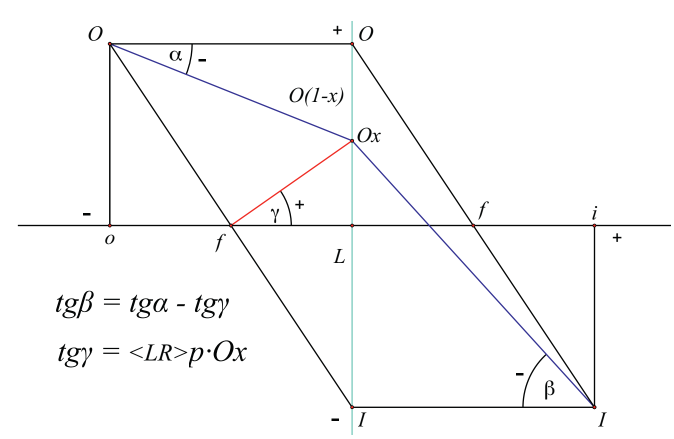

Fig. 3. Arbitrary ray with angle of incidence α clamped between two focus-to-parallel rays. All quantities (O, Ox, o, I, i, α, β, γ) except f are *directed*. f can be positive or negative.

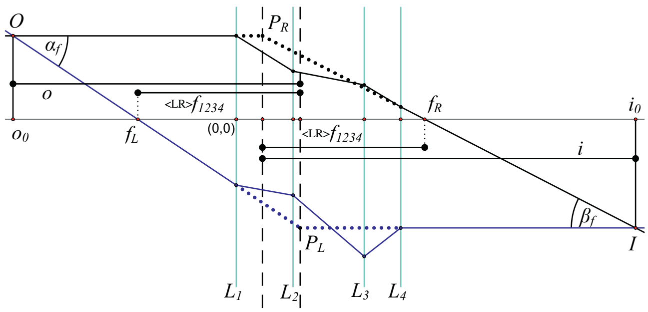

Fig. 4. Ray tracing for a 4-lens system. Included are o, i and f. o_0 and $\boldsymbol{i}_0$ are the coordinates of object and image respectively.

Table 1
Numerical values underlying Fig.4. The precision is restricted for lack of space.

f_1	f_2	f_3	f_4	D_{12}	D_{23}	D_{34}
4.80	−4.00	3.00	−5.00	2.037	2.661	1.281

p_{12}	p_{23}	p_{34}	p_{123}	p_{234}	p_{1234}	f_{1234}
0.064	0.296	0.221	0.202	0.046	0.172	5.815

P_L	P_R	f_L	f_R	$\tan\alpha_f$	$\tan\beta_f$	$M_{\measuredangle}$
2.291	0.939	−3.524	6.754	−0.668	−0.514	−0.769

O	o_0	o	I	i_0	i	$M_{\langle LR\rangle}$
2.990	−7.998	−10.28	−3.886	14.311	13.372	−1.300

6. Ray tracing and principal planes

Solving the consistency and direction issues enables to use Eq. (2) for *non-paraxial* ray tracing, see Fig. 4 for a 4-lens example.

Inspecting two triangles featuring angle β_f directly at the optical axis in Fig. 2 or Fig. 4, namely a large one bounded by P_R and a small one bounded by the righter most lens of the system, using Eqs. (8) and (9) and generalizing for n lenses it can be seen that:

$$\tan\beta_{f_{1\cdots n}} = \frac{O}{-\langle LR\rangle f_{1\cdots n}} = \frac{O\left(1-\sum_{i=1}^{(n-1)} p_{1\cdots i}D_{i(i+1)}\right)}{-(P_R+\langle LR\rangle f_{1\cdots n}-L_n)} \tag{16}$$

Cross multiplying in Eq. (16) and dividing out O gives:

$$P_R-L_n = -\langle LR\rangle f_{1\cdots n}\sum_{i=1}^{(n-1)} p_{1\cdots i}D_{i(i+1)} \tag{17a}$$

For P_L one has to consider a light ray starting out parallel from the right, which gives:

$$P_L-L_1 = -\langle LR\rangle f_{1\cdots n}\sum_{i=1}^{(n-1)} p_{(i+1)\cdots n}D_{i(i+1)} \tag{17b}$$

Note that the p–D sums for P_L and P_R are complementary.

With the help of Eqs. (8)–(10), one can generate formulas for P_L and P_R (relative to L_1 and L_n, respectively) which do not show explicitly the $D_{k(k+1)}$'s:

$$\begin{aligned} P_L-L_1 &= -\langle LR\rangle[f_{1\cdots n}(1+f_1\cdot p_{2\cdots n})-f_1] \\ P_R-L_n &= -\langle LR\rangle[f_{1\cdots n}(1+p_{1\cdots(n-1)}\cdot f_n)-f_n] \end{aligned} \tag{18}$$

where $p_{1\ldots n}=1/f_{1\ldots n}$, $\langle LR\rangle=+1$ for P_R, $\langle LR\rangle=-1$ for P_L.

Combining all, we can write down the n-lens version of Eq. (2).

$$\begin{aligned} \tan\beta_n &= \tan\alpha_1\,[1+\langle LR\rangle p_{1\cdots n}o]-\langle LR\rangle p_{1\cdots n}O \\ &= \frac{\tan\alpha_1}{M_{\langle LR\rangle}}-\langle LR\rangle p_{1\cdots n}O \end{aligned} \tag{19}$$

Setting object size O to 0 yields the angular magnification $M_{\measuredangle}$:

$$M_{\measuredangle} = \frac{\tan\beta_n}{\tan\alpha_1} = \frac{1}{M_{\langle LR\rangle}} \tag{20}$$

Note that it is not necessary to shift object and/or image to infinity.

7. Test of the formulas

To allow testing and control, we include Table 1 with numerical values underlying Fig. 4. Units are scaled, for instance f and $D_{i\,(i+1)}$ in cm yield p in cm^{-1}. Lens L_1 is placed at the origin (0, 0) of the coordinate system.

It should be noted that Fig. 4 is completely calculated, the graphics however yield perfect support to the numerical results. Especially the confirmation of the permutability of the focal length results (see upper part of Eq. (11)) and the fact that using directed quantities works so well and simple is very satisfying.

8. Conclusion

Respecting the directedness of object and image distance yield exact formulas for the focal length $f_{1\ldots n}$, the angle of refraction β_n, the magnification M and (the positions of) the principal planes P_R and P_L of a lens system.

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[4] http://www-ferp.ucsd.edu/LASERLAB/TUTOR/lensdesign.shtml