Physics Letters B 692 (2010) 32–35

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Fuzziness at the horizon

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ARTICLE INFO

Article history:
Received 27 March 2010
Received in revised form 6 May 2010
Accepted 3 July 2010
Available online 10 July 2010
Editor: T. Yanagida

Keywords:
Cauchy horizon
Noncommutative black holes

ABSTRACT

We study the stability of the noncommutative Schwarzschild black hole interior by analysing the propagation of a massless scalar field between the two horizons. We show that the spacetime fuzziness triggered by the field higher momenta can cure the classical exponential blue-shift divergence, suppressing the emergence of infinite energy density in a region nearby the Cauchy horizon.

Black holes can exhibit not only an event horizon, namely the outermost surface that physically separates two noncommunicating regions of spacetime, but also inner Cauchy horizons. These internal horizons, null surfaces beyond which spacetime predictability breaks down, have the intriguing properties of showing up a "dual effect" of the conventional red shift, i.e. the blue shift. To understand the physics of this blue shift, it is common procedure to study a radiation represented by a scalar field, propagating in the region between the two horizons. For the sake of clarity we consider a spherically symmetric spacetime region, whose metric can be cast in the form

$$ds^2 = \frac{r^2\,dr^2}{(r_+ - r)(r - r_-)} - \frac{(r_+ - r)(r - r_-)}{r^2}dt^2 - r^2\,d\Omega^2 \qquad (1)$$

where r_- is the Cauchy horizon, r_+ is the event horizon, i.e. $r_- < r < r_+$, r plays the role of a temporal coordinate and t is a spatial one. Introducing tortoise coordinates

$$r^\star = -r - \frac{1}{\kappa_+}\ln(r_+ - r) + \frac{1}{\kappa_-}\ln(r - r_-) \qquad (2)$$

where $\kappa_\pm \equiv (r_+ - r_-)/r_\pm^2$, we can define null coordinates $x_- = -r^* - t$ and $x_+ = -r^* + t$ to study the propagation of a scalar field on this background in terms of the scalar wave equation

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = 0. \qquad (3)$$

The solution of the above equation, let us conclude that the field, in the vicinity of the Cauchy horizon where $x_+ \to \infty$, decays as $\phi \sim x_+^{-2\ell-2}$, with ℓ the multipole order of the field. However this is no longer true for the energy of the field. Indeed, if we consider, the field's rate variation as measured by a free falling observer (FFO) crossing the Cauchy horizon we obtain the infinite result

$$\phi_{,\mu}U^\mu \simeq \phi_{,x_+}\dot{x}_+ \sim x_+^{-2\ell-3}e^{\kappa_- x_+} \qquad (4)$$

where U^μ is the 4-velocity of the observer (the dot denotes differentiation with respect to proper time). Actually the FFO measures a flux of energy given by the square of the above quantity, that is even more divergent. This mechanism of instability due to the infinite blue shift at the Cauchy horizon can be explained in these terms. An external observer would require an infinite time to reach the future null infinity ($x_+ = \infty$) since at the best its velocity is $\dot{x}_+ \simeq 1$. On the other hand, a FFO can reach the Cauchy horizon in a finite proper time, which implies that $\dot{x}_+$ will diverge as $x \to \infty$. From Eq. (4) we see that this divergence overcomes the field decay. As a result the Cauchy horizon is unstable. An extensive study on this subject has basically led to the general conclusion that this infinite amount of energy density at the Cauchy horizon can develop unbounded curvature, disrupting the spacetime geometry [1–18].

Up to now we have mentioned purely classical solutions, since the above analyses have concerned quantum effect at the most in the matter fields propagating on the spacetime manifold. On the other hand, there is a recent class of black hole solutions (QGBHs) obtained by means of quantum gravity arguments, like loop quantum black holes [19–21], asymptotically safe gravity black holes [22,23], generalized uncertainty principle [24,25] and noncommu-

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0370-2693
doi:10.1016/j.physletb.2010.07.007

tative geometry inspired black holes [26–32] (for a review see [33] and the references therein). Independently on their starting point the above solutions, converge on a unique qualitative behavior, namely the absence of any curvature singularity and the presence of more than a horizon. In other words, as far as some sort of smearing effect is concerned due to the fuzziness of spacetime in its quantum gravity regime, the physics of QGBHs has a universal character. This fact has its equivalent on the thermodynamics side: the Hawking temperature admits a maximum, followed by a "SCRAM phase", a thermodynamic stable shut down, characterized by a positive black hole heat capacity. As a consequence, also for the neutral solution, in place of the runaway behavior of the temperature, one finds that the evaporation ends up with a zero temperature extremal black hole, a final configuration entirely governed by a quantum gravity induced minimal length. This new scenario of the evaporation implies a further virtue of QGBHs: a finite temperature prevents any relevant back reaction, namely a self interaction of the radiated energy with its source. Thus we can conclude that these solutions are stable versus back reaction and can describe the entire black hole life until the final configuration. However, having an inner Cauchy horizon is again a source of concern, since we might have the suspect that the interior of these black holes is unstable. As a result the solutions could be no longer singularity free, since the singularity might occur on the inner horizon rather than at the origin, frustrating the efforts that vivified the formulations at the basis of their derivation. In some sense, the instability of QGBHs is even worse with respect to the conventional classical analogs, since it affects the neutral, static case too. To address this problem we need to change our perspective. If we do believe in the tenets of quantum gravity, we have to accept the possibility for a quantum manifold to provide a natural ultraviolet cut-off for any field propagating over it in order to prevent any growth of energy beyond Planckian magnitude. Without loss of generality we will consider the neutral noncommutative spacetime only, even if our analysis holds for the charged solution too and other QGBHs. Indeed noncommutative geometry is just one of the possible effective ways to implement a natural cut-off.

We shall start recalling some properties of the noncommutative geometry inspired black hole, whose line element is given in [27]

$$ds^2 = \left(1 - 2m(r)/r\right) dt^2 - \frac{dr^2}{(1 - 2m(r)/r)} - r^2\, d\Omega^2 \tag{5}$$

with the mass function $m(r) = 4M\gamma(3/2, r^2/4\theta)/\sqrt{\pi}$, where M is the total mass in the spacetime manifold, θ is a parameter encoding noncommutativity and having the dimension of a length squared, while

$$\gamma\left(3/2, r^2/4\theta\right) = \int_0^{r^2/4\theta} dt\, t^{1/2} e^{-t} \tag{6}$$

is the incomplete lower gamma function. The above line element is clearly regular at the origin, where a de Sitter core accounts for the mean value of the quantum fluctuations of the manifold. The metric admits one, two or no horizon depending on the total mass M, respectively equal, larger or smaller than extremal black hole total mass $M_0 \approx 1.9\sqrt{\theta}$. In the following we shall restrict our attention to the case of two horizons. In this scenario the line element (5) in the interior region $r_- < r < r_+$ can be cast in the form

$$ds^2 = \frac{d\tau^2}{(\frac{2m(\tau)}{\tau} - 1)} - \left(\frac{2m(\tau)}{\tau} - 1\right) d\rho^2 - \tau^2\, d\Omega^2 \tag{7}$$

where we introduced the new variables τ and ρ in place of r and t, since in this region they become a temporal and a spatial coordinate, respectively. It is convenient to introduce a temporal tortoise coordinate τ^* defined as $d\tau^* = d\tau/(2m(\tau)/\tau - 1)$ such that $\tau^* \to \pm\infty$ as $\tau \to r_\pm$. In the sequel we shall introduce null coordinates $x_- \equiv -\tau^* - \rho$ and $x_+ \equiv -\tau^* + \rho$ as in [5,6]. Then the metric reads

$$ds^2 = \left(\frac{2m(\tau)}{\tau} - 1\right) dx_-\, dx_+ - \tau^2\, d\Omega^2. \tag{8}$$

The event horizon becomes the null hypersurface $x_- = -\infty$ and the left and right branches of the Cauchy horizon r_- are null hypersurfaces $x_- = \infty$ and $x_+ = \infty$, respectively. In the region between the two horizons (see Fig. 1) we consider the propagation of a massless scalar test field $\phi = \phi(\tau, \rho, \vartheta, \varphi)$ governed by Eq. (3) where the metric is associated to the background geometry (7). Now it is the time to invoke the noncommutative nature of the field. Indeed, up to now, noncommutative effects have been considered to smear the matter generating the background geometry only. We need to extend this procedure to matter propagating over the manifold too. To this purpose, we follow the formulation proposed in [34–40] to get a modified integral measure in the momentum representation of the field, which between the two horizons can be written as

$$\phi\left(\tau^*, \rho, \vartheta, \varphi\right) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_{-\infty}^{\infty} dk\, e^{-k^2\theta/4} e^{-ik\rho} \frac{1}{\tau} \psi_{\ell mk}\left(\tau^*\right) Y_{\ell m}. \tag{9}$$

The presence of an exponential damping factor encodes the effect of the noncommutative UV regularization. Here τ has to be thought as an implicit function of the variable τ^*, while $Y_{\ell m} = Y_{\ell m}(\vartheta, \varphi)$ denotes the spherical harmonics. Analogous modifications have been already efficiently employed in a variety of contexts, namely to describe a traversable wormhole sustained by quantum geometry fluctuations [41], to remove the initial cosmological singularity and drive the inflation without an inflaton field [42] and to get corrections to the Unruh thermal bath by means of a nonlocal deformation of conventional field theories [43–46]. Further contributions concern the modification of the Newton potential in the presence of noncommutative spacetime coordinates [47], the evaporation of the noncommutative black holes in terms of gravitational amplitudes for boson and fermion fields [48–50] and for up to ten spatial dimensions in particle detectors at the LHC [51] and the calculation of the spectral dimension of a quantum spacetime [52]. As a result we obtain the "radial" function $\psi_{\ell mk}$ obeys the equation

$$\frac{d^2\psi_{\ell mk}}{d\tau^{*2}} + \left[k^2 - V\left(\tau^*\right)\right]\psi = 0 \tag{10}$$

where the potential is given by

$$V\left(\tau^*\right) = \frac{g_{\rho\rho}}{\tau}\left[\frac{\ell(\ell+1)}{\tau} + 2\frac{g_{\rho\rho}}{\tau} + \partial_\tau g_{\rho\rho}\right]. \tag{11}$$

For our purposes we are interested in the asymptotic solutions of (10) as $\tau^* \to -\infty$. The potential $V(\tau^*)$ decays exponentially in time as

$$V\left(\tau^*\right) \approx e^{\mp\alpha_\pm \tau^*}, \quad \tau^* \to \pm\infty \tag{12}$$

where $2\alpha_\pm \equiv \pm(dg_{00}/dr)_{r=r_\pm}$. Finally, near the Cauchy horizon the asymptotic solutions of (10) are

$$e^{-ik\rho}\psi_{\ell mk}\left(\tau^*\right) \approx e^{\pm ikx_\pm}\left[1 + O\left(e^{\alpha_- \tau^*}\right)\right]. \tag{13}$$

Up to exponentially vanishing corrections, the solution are plane waves approaching the left and right branch of the horizon r_-, respectively. As in [7] the energy density in the scalar field ϕ as measured by a freely falling observer near a horizon with four velocity U^{μ} will be proportional to

$$\mathcal{E} = \left(\phi_{,\alpha} U^{\alpha}\right)\left(\phi_{,\beta} U^{\beta}\right) + \frac{1}{2}\phi_{,\alpha}\phi^{*,\alpha}. \tag{14}$$

Since $-\tau^* \pm \rho = \text{const}$ are null surfaces and taking into account for the form of ϕ nearby the horizon, we have that the energy density is dominated by the term $|\phi_{,\alpha} U^{\alpha}|^2$. Therefore we can restrict our analysis to the term $\phi_{,\alpha} U^{\alpha}$ only. To this purpose, we need the form of the velocity vector field associated to the FFO which can be written as

$$U = U^{x_-}\frac{\partial}{\partial x_-} + U^{x_+}\frac{\partial}{\partial x_+} \tag{15}$$

where

$$U^{x_\pm} = -\frac{\tau}{2m(\tau)-\tau}\left(h \mp \sqrt{h^2 + \frac{2m(\tau)-\tau}{\tau}}\right), \tag{16}$$

with h a dimensionless parameter. If $h > 0$ the FFO worldline enters region *III* from region *I* and exits region *III* through the left-hand branch ($x_- = \infty$) of the inner horizon. If $h < 0$ the worldline enters region *III* from region *II* and exits through the right-hand ($x_+ = \infty$) branch of r_-. If $h = 0$ the worldline will move through the region *III* passing through the bifurcation points of the horizon r_-. In the vicinity of the Cauchy horizon, we have for $h > 0$

$$U^{x_+} \approx \frac{1}{2}, \qquad U^{x_-} \approx e^{-\alpha_- \tau^*} = e^{\alpha_-(x_- + x_+)/2} \tag{17}$$

whereas for $h < 0$

$$U^{x_+} \approx e^{\alpha_-(x_- + x_+)/2}, \qquad U^{x_-} \approx \frac{1}{2}. \tag{18}$$

Hence, for $h > 0$ and asymptotically for $x_- \to \infty$ we have

$$\phi_{,\alpha} U^{\alpha} \approx e^{\alpha_-(x_- + x_+)/2}\frac{\partial \phi}{\partial x_-} + \frac{1}{2}\frac{\partial \phi}{\partial x_+} \tag{19}$$

whereas for $h < 0$ and $x_+ \to \infty$

$$\phi_{,\alpha} U^{\alpha} \approx \frac{1}{2}\frac{\partial \phi}{\partial x_-} + e^{\alpha_-(x_- + x_+)/2}\frac{\partial \phi}{\partial x_+}. \tag{20}$$

In order that the FFO can measure a nondivergent amount of field energy density near the r_- horizon, the appropriate derivative of the field times the exponential blue-shift factor must be finite. In the classical picture [5,7] from the last two relations above one concludes that the e^{-ikx_+} waves are singular along the left branch of r^- and the e^{ikx_-} waves become singular along the right branch of r_-. We shall see that due to the noncommutativity of the field, (19) and (20) stay bounded at the Cauchy horizon. Indeed, we find that for the left-going component

$$\phi_{,\alpha} U^{\alpha} \sim \frac{x_-}{\theta^{3/2}} e^{\alpha_-(x_- + x_+)/2} e^{-x_-^2/\theta} \tag{21}$$

which vanishes as $x_- \to \infty$, keeping x_+ constant. Analogously, we find for the right-going component

$$\phi_{,\alpha} U^{\alpha} \sim \frac{x_+}{\theta^{3/2}} e^{\alpha_-(x_- + x_+)/2} e^{-x_+^2/\theta} \tag{22}$$

which vanishes as $x_+ \to \infty$, for x_- fixed. The above result confirms that, probing higher momenta the field basically triggers the noncommutative nature of the manifold, which shows graininess and prevents any spacetime resolution beyond the value $\sqrt{\theta}$. This

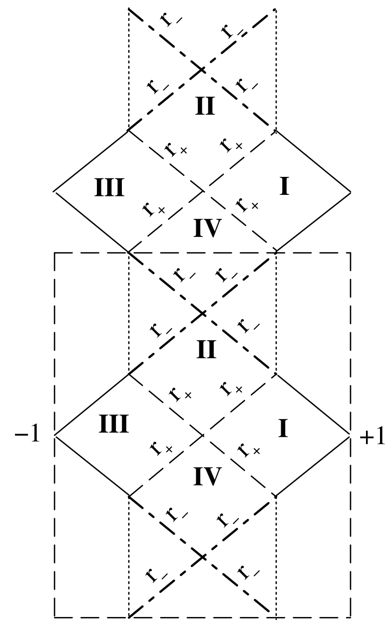

Fig. 1. The Carter–Penrose diagram of the manifold. The conformal diagram of the maximally extended noncommutative inspired Schwarzschild spacetime. The radii $r_\pm$ represent the outer and inner horizons, respectively. The central singularity appearing in the Reissner–Nordström metric is now replaced by a regular de Sitter core, dotted line. The upper and bottom part of the box indicated by the dashed line can be identified to make the manifold cyclic in the time coordinate.

let us also conclude that in this framework no mass inflation can occur. To this purpose, we define

$$T_{ab} = \mathcal{E}_{in} l_a l_b + \mathcal{E}_{out} n_a n_b \tag{23}$$

as the two-dimensional section of the stress tensor, which describes the cross flowing stream of infalling and outgoing of light like particles following null geodesics. Here l_a is the radial null vector pointing inwards, n_a is the radial null vector pointing outwards, while $\mathcal{E}_{in}$ and $\mathcal{E}_{out}$ represent the energy density of the fluxes. The mass inflation is a huge boom of the black hole internal mass parameter, which becomes classically unbounded at the Cauchy horizon. Contrary to the intuition, the inflation is due to both the outflux and the blueshifted influx of a collapsing star as shown in [9]. On the other hand, in the present framework, energy densities cannot diverge even at the Cauchy horizon. Therefore, the mass inflation which is in general proportional to the product $T^{ab}T_{ab}$ will not take place. The above analysis concerns the leading contribution to the energy density of the field in the vicinity of the Cauchy horizon along the lines of [5,6]. Since higher order terms fall off faster, we argue that the stability of QGBH interiors can be shown in general. However, according to the theory of the stability in [3,4], an exponential decay is a mere necessary condition only: in other words even if each term of the expansion is vanishing, their global contribution could yet destabilize the solution. Furthermore, we have addressed here the most simple case of classical perturbation of the manifold, while in general the field could be quantized. In such a case, according to previous contributions [15], the stress tensor $\langle T_{ab}\rangle$ is expected to have an even worse UV behavior. For the above reasons, we think that, after the present analysis, further investigations will be necessary, also to include the other spacetimes within the class of QGBHs.

Acknowledgements

P.N. would like to thank the Universidad de los Andes, Bogotá, Colombia for the kind hospitality during the period of work on this project. P.N. is supported by the Helmholtz International Center for FAIR within the framework of the LOEWE program (Landesoffensive zur EntwicklungWissenschaftlich-Ökonomischer Exzellenz) launched by the State of Hesse.

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