Un-spectral dimension and quantum spacetime phases

Piero Nicolini [a], Euro Spallucci [b,*]

[a] *Frankfurt Institute for Advanced Studies (FIAS), Institut für Theoretische Physik, Johann Wolfgang Goethe-Universität, Ruth-Moufang-Strasse 1, 60438 Frankfurt am Main, Germany*
[b] *Dipartimento di Fisica, Università degli Studi di Trieste, I.N.F.N. Sezione di Trieste, Strada Costiera 11, 34014 Trieste, Italy*

ARTICLE INFO

ABSTRACT

In this Letter, we propose a new scenario emerging from the conjectured presence of a minimal length ℓ in the spacetime fabric, on the one side, and the existence of a new scale invariant, continuous mass spectrum, of un-particles on the other side. We introduce the concept of *un-spectral dimension* $\mathbb{D}_U$ of a d-dimensional, euclidean (quantum) spacetime, as the spectral dimension measured by an "un-particle" probe. We find a general expression for the un-spectral dimension $\mathbb{D}_U$ labelling different spacetime phases: a semi-classical phase, where ordinary spectral dimension gets contribution from the scaling dimension d_U of the un-particle probe; a critical "Planckian phase", where four-dimensional spacetime can be effectively considered two-dimensional when $d_U = 1$; a "Trans-Planckian phase", which is accessible to un-particle probes only, where spacetime as we currently understand it looses its physical meaning.

If we look at a fractal, for instance the Cantor set in Fig. 1a), we can grasp the meaning of what could be the spacetime in the presence of strong quantum gravity fluctuations. Indeed fractals capture two of the main features of what we expect to be a quantum spacetime. When extreme energy/small distance regimes are probed, the spacetime itself changes its own nature and exhibits frenzy geometrical and topological fluctuations. The shorter is the spacetime scale probed, the more involved is the fluctuation pattern. Thus, below some fundamental length scale we cannot model spacetime as a smooth manifold any longer, rather it will look like a rough and fragmentated (hyper)surface, e.g. a fractal. Another feature for which a Cantor set, or more generally a fractal, turns out to be quite useful is the *self-similarity*, namely the property of being exactly similar to a part of itself. In other words, fractals are scale invariant because at any magnification there is a smaller piece of the fractal that is similar to the whole. Fig. 1b) is an artistic representation of a fractal spacetime where fractality is represented by a self-similar distribution of holes. From this perspective, quantum gravity seems to be closely connected to both roughness and scale invariance, both features being supported by recent non-perturbative string theory developments like AdS/CFT duality and *M*-theory. A related consideration is given by the (non)renormalizability of gravity following from mass^{-2} dimension (in natural units) of the Newton constant. On the other hand, in a two-dimensional spacetime the gravitational
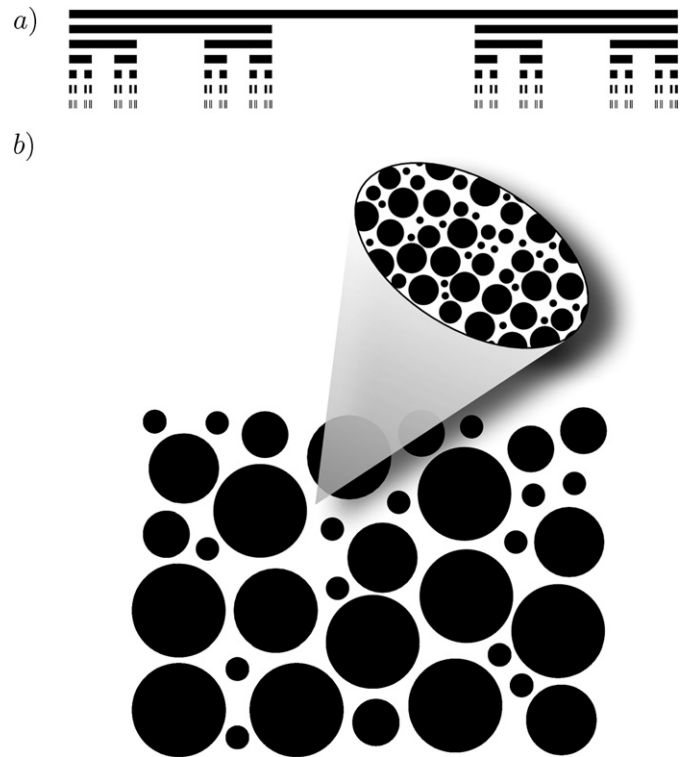

Fig. 1. a) A Cantor set. Increasing energy, one moves from the upper continuous surface to the lower fractal. b) A quantum spacetime, showing fractal self-similarity.

* Corresponding author.
 E-mail addresses: nicolini@th.physik.uni-frankfurt.de (P. Nicolini),
spallucci@trieste.infn.it (E. Spallucci).

coupling constant becomes dimensionless and gravity is expected to be power-counting renormalizable. This special feature is accompanied by the fact that in two dimensions the spacetime is conformally flat and field theories more naturally enjoy properties like conformal invariance. In support of this line of reasoning there is the concept of spectral dimension, one of the most intriguing features of a quantum spacetime. If we expect an increasing degree of fuzziness in the quantum regime, then we must accept the idea that also spacetime dimension should be reviewed. As the classical manifold dissolves into a sort of fractal dust, the very concept of "dimensionality" must change from an assigned property into a dynamical quantity running with the energy scale of the probe. An effective way to measure the actual dimension of a quantum manifold consists in studying the diffusion of a test particle. The dynamics of the process is encoded into the heat kernel equation

$$\Delta K(x, y; s) = \frac{\partial}{\partial s} K(x, y; s) \tag{1}$$

where s is a fictitious diffusion time of dimension of a length squared, Δ is the Laplace operator and $K(x, y; s)$ is the heat kernel, representing the probability density of diffusion from x to y in a "lapse of time s". The initial condition for the diffusion process is that the test particle starts from x at $s = 0$

$$K(x, y; 0) = \frac{\delta^d(x - y)}{\sqrt{\det g_{ab}}} \tag{2}$$

where $\delta^d(x - y)$ is the d-dimensional Dirac delta, d is an integer number representing the topological dimension and g_{ab} is the metric of the manifold. If we consider a closed random path, i.e. $x = y$, we can define the *return probability* by integrating the Kernel over all spacetime and factorizing out the total invariant volume

$$P(s) = \frac{\int d^d x \sqrt{\det g_{ab}} K(x, x; s)}{\int d^d x \sqrt{\det g_{ab}}}. \tag{3}$$

From $P(s)$ we can define the *spectral dimension* as

$$\mathbb{D} = -2 \frac{\partial \ln P_g(s)}{\partial \ln s}. \tag{4}$$

It is easy to show that in flat space, for a "free" diffusion, the return probability is $P(s) = (4\pi s)^{-d/2}$ and the spectral dimension is $\mathbb{D} = d$. In the presence of gravity, the above formula can be yet employed to check an effective dimensional reduction, even if large s limit holds only on local patches of the manifold which approximates the tangent space. The importance of the spectral dimension lies in the fact that it could provide a glimpse about a crucial feature of a quantum manifold: if it turned out that in the quantum gravity regime the actual dimension measured by the diffusion process is two, we could conclude that gravity is a renormalizable theory, overcoming the conventional difficulties about its quantization. As a result there have been many attempts to calculate the spectral dimension [1] and it has been found that $\mathbb{D}$ tends to the value 2 for scales approaching ℓ, an effective minimal length in the manifold [2,3]. However, in all the approaches above it is understood that short distance can be probed only by ultra-relativistic objects with a negligible rest mass. Thus, scale invariance is kinematically realized in a light-cone type limit. Indeed, if we consider the heat equation for a massive particle we find

$$\tilde{\Delta} K(x, y; s) = \frac{\partial}{\partial s} K(x, y; s) \tag{5}$$

where the operator $\tilde{\Delta} = \Delta - m^2$ includes a non-differential term m^2. From the definition (4) we get

$$\mathbb{D} = -2s \frac{\int d^d x \sqrt{\det g_{ab}} \Delta K(x, x; s)}{\int d^d x \sqrt{\det g_{ab}} K(x, x; s)} + 2sm^2. \tag{6}$$

The first term in the r.h.s of (6) leads to a constant, i.e. scale independent, value of the spectral dimension, but the second one is linear in s and diverges for asymptotic diffusion times spoiling a meaningful definition of $\mathbb{D}$. From this viewpoint, one concludes that the spectral dimension can be safely introduced only in a scale invariant framework. In other words, spacetime spectral dimension cannot be probed by massive objects as they break scale invariance.

In this Letter we are going to present a new, scale invariant, procedure to compute $\mathbb{D}$ by means of a massive probe providing a non-trivial modification to the standard definition. It may sound odd to preserve scale invariance in the presence of a massive object, but this problem can be by-passed by using *un-particle probes* borrowed from a recently proposed extension of the elementary particle standard model. The new idea is that there exists a new high-energy sector of the particle standard model where the fundamental objects display a continuous, scale-invariant, mass spectrum in alternative to the discrete mass spectrum of ordinary elementary particles. This new "stuff" is very weakly coupled to ordinary matter below some threshold energy, say some TeV [4]. Beyond these energies the standard model fields interact with a new scale invariant sector described by the so-called Banks–Zaks (BZ) fields. The interaction is mediated by very heavy particles of mass M_U. Below M_U, the interaction leads to non-renormalizable effective couplings of the form $O_{sm} O_{BZ}/M_U^k$ where O_{sm} is an operator with mass dimension d_{sm} built out of the standard model fields and O_{BZ} is the equivalent for BZ fields. The scale invariance properties of the BZ sector emerge below a scale Λ_U, through dimensional transmutation of the BZ fields into un-particle fields. The above interaction term becomes $C_U (\Lambda_U)^{d_{BZ} - d_U} O_{sm} O_U/M_U^k$, where the BZ operators O_{BZ} match un-particle operators O_U; d_U is the un-particle scaling dimension and C_U is a normalization constant. In this scenario the BZ fields decouple from ordinary matter at low energies and therefore the interaction $O_{sm} O_{BZ}/M_U^k$ should not affect the scale invariant properties of un-particles. Even if the scaling dimension can be arbitrarily large, it is customary to assume that $1 < d_U < 2$. Recently it has been argued that un-particle might affect the gravitational interaction too: indeed un-gravity could arise from the un-graviton exchange among massive particles [5]. In addition by exploiting results coming from Cavendish experiments one obtains that eventual un-particle corrections to Newton's law might occur at energy scales higher than TeV, in agreement with the basic hypotheses about un-particle physics. Following this line of reasoning, un-gravity corrections to Schwarzschild metric have been perturbatively derived in [6] and confirmed at the non-perturbative level by solving the field equations derived from an effective action including the un-graviton corrections at all order [7]. As a special result, the Hawking temperature and the Bekenstein entropy of the un-Schwarzschild black hole suggest that the dimension of the horizon is a non-integer number $2d_U$. These examples suggest that un-particle physics provides a tool to implement fractalization of conventional scenarios, by forcing the presence of the scaling dimension d_U. From this vantage point, it is almost compelling to explore the spectral dimension, i.e. the fractal structure of the Planckian spacetime, by means of un-particle probes. For sake of clarity we stress that in this case un-particles do not generate gravity since we are considering a generic manifold either classical or quantum. In the simplest case of a scalar un-particle the Green function turns out to be [8]

$$G_U(x - y) = A_{d_U} \int_0^\infty dm^2 (m^2)^{d_U - 2} G(x - y; m^2)$$

where d_U controls a continuous mass spectrum, while

$$A_{d_U} = \frac{8\pi^{5/2}}{(\Lambda_U^2)^{d_U-1}(2\pi)^{d_U}} \frac{\Gamma(d_U+1/2)}{\Gamma(d_U-1)\Gamma(2d_U)}.$$

The heat kernel $K_U(x, y; s)$ can be obtained from

$$G_U(x-y) = \int_0^\infty ds\, K_U(x, y; s)$$

$$= A_{d_U} \int_0^\infty ds \int_0^\infty dm^2\, (m^2)^{d_U-2} K(x, y; s) \quad (7)$$

where $K(x, y; s)$ is the solution of Eq. (5). We can study the diffusion of an un-field

$$\Delta_U K_U(x, y; s) = \frac{\partial}{\partial s} K_U(x, y; s) \quad (8)$$

where the un-Laplacian acquires an extra term depending on the un-particle sector $\Delta_U = \Delta - (d_U - 1)/s$. Eq. (8) can be classified as an inhomogeneous heat equation, whose initial conditions are like in (2). Employing a one-dimensional heat conduction analogy, we could say that our problem is equivalent to that of a bar which is subjected to a time dependent "heat source" $(d_U - 1)/s$. In other words, the heat released by the un-particle term is spatially uniform along the length of the bar and the scale invariance is preserved. We notice that for $d_U = 1$ the above equation becomes the homogeneous heat equation, in agreement with the fact that un-particle corrections vanish for $d_U = 1$ as in Ref. [5]. Therefore, from Eq. (4), we can define the un-spectral dimension as

$$\mathbb{D}_U = -2s \frac{\int d^dx\sqrt{\det g_{ab}}\,\Delta K_U(x, x; s)}{\int d^dx\sqrt{\det g_{ab}}\,K_U(x, x; s)} + \frac{2\Gamma(d_U)}{\Gamma(d_U-1)}. \quad (9)$$

The above formula can be manipulated to obtain

$$\mathbb{D}_U = \mathbb{D} + 2d_U - 2, \quad (10)$$

Eq. (10) is the main result of this work. In analogy with the Hausdorff dimension (see Ref. [9]), we see that in the chosen range for the scale dimension $\mathbb{D}_U \geqslant \mathbb{D}$, while $\mathbb{D}_U = \mathbb{D}$ for $d_U = 1$ only. This increase of the dimension measured by the diffusion process can be explained in terms of the presence of an additional sector, i.e. the un-particles, with respect to the conventional standard model fields calculation. On the other hand, for the specific case $\mathbb{D} = 2$, one finds that the un-spectral dimension depends uniquely on d_U and is $\mathbb{D}_U = 2d_U$. As a result for a diffusion process in a flat plane, $d = 2$ and

$$K_p(x, x; s) = A_{d_U} \int_0^\infty dm^2\, (m^2)^{d_U-2}\, \frac{e^{-m^2s}}{(4\pi s)^{d/2}},$$

we obtain $\mathbb{D} = d = 2$ and $\mathbb{D}_U = 2d_U$. As a first application of Eq. (10) we are going to investigate the nature of the un-Schwarzschild black hole horizon, but we need to do a little step forward. The horizon is a curved surface and therefore the diffusion must take into account this effect through non-trivial Seeley–deWitt coefficients in the heat kernel representation. A further modification occurs in the Laplace operator which acquires a non-minimal coupling to the Ricci scalar to preserve scale invariance $\Delta \longrightarrow \Delta_g \equiv \Delta - \xi_d R$ with $\xi_d \equiv (1/4)(d-2)/(d-1)$. As a result the heat kernel in the presence of gravity reads

$$K_g(x, x; s)$$

$$= A_{d_U} \int_0^\infty dm^2\, (m^2)^{d_U-2}\, \frac{e^{-m^2s}}{(4\pi s)^{d/2}} \left[a_0 + \sum_{n=1}^\infty s^n a_n(x, x) \right]. \quad (11)$$

Since the un-particle "heat source" preserves scale invariance and does not affect the manifold coordinate x, its contribution $2\Gamma(d_U)/\Gamma(d_U - 1)$ will be unchanged in the presence of gravity. Here, for the sake of clarity, we provide only the gravity primary corrections

$$\mathbb{D}_U = d + 2d_U - 2 - 2s \frac{\int d^dx\sqrt{\det g_{ab}}[a_1 + 2a_2s + \cdots]}{\int d^dx\sqrt{\det g_{ab}}[a_0 + a_1s + \cdots]}.$$

We remind that the above formula holds for small diffusion times only. Indeed for a generic topological dimension gravity introduces a scale in the conventional term for the spectral dimension. This is the reason why there is a breaking of the scale invariance analogous to the introduction of a mass as in (5). This is not the case for $d = 2$. Indeed, when one considers the un-Schwarzschild black hole horizon, we have a conformal invariant diffusion, propagating on a conformally flat manifold. Thus, the Green function (7) reduces to the flat space un-particle Green function and we can conclude that $\mathbb{D}_U = 2d_U$, indicating "fractalization" of the surface. This would confirm the argument in [7] according to which the un-Schwarzschild horizon is exactly a $2d_U$-dimensional fractal surface built up by un-gravitons trapped at the Schwarzschild radius.

Up to now, we have considered the case of "classical" background manifold in the sense that two points (events) can be arbitrarily closed. In other words, we have not considered the intrinsic uncertainty in the localization of a single point when it is left free to fluctuate quantum mechanically. Since our model of "quantum manifold" would like to account for both a short-distance increasing loss of resolution and self similarity, it is compelling to understand how the un-spectral dimension behaves in regards of both properties. To this purpose, we implement the graininess in spacetime along the lines of Ref. [3] by studying a diffusion process governed by the same heat equation as in (8), but with a modified initial condition $K_\ell(x, y; 0) = \frac{\rho_\ell(x,y)}{\sqrt{\det g_{ab}}}$. $\rho_\ell(x, y)$ is a Gaussian distribution replacing the former Dirac-delta. The width ℓ is the minimal uncertainty in the distance between two fluctuating points, or the best resolution which is compatible with the quantum nature of the background manifold. This loss of resolution primarily affects the early, short-distance, stages of the diffusion process, while at distance large with respect to ℓ, the diffusion is insensitive to the graininess of the manifold. From a thermal point of view, the manifold behaves like a "boiling surface", whose thermal instability sustains the Gaussian profile preventing it from collapsing into a Dirac delta. In what follows, the role of fluctuations in the Riemannian curvature, which is a geometrical attribute of a classical, smooth, manifold becomes less and less relevant with respect to the graininess of the manifold itself. Thus, for our next purpose it is enough to consider the flat, but accounting for quantum uncertainty, heat kernel

$$K_\ell(x, y; s) = A_{d_U} \int_0^\infty dm^2\, (m^2)^{d_U-2}\, \frac{e^{-m^2s} e^{-\frac{(x-y)^2}{4(s+\ell^2)}}}{[4\pi(s+\ell^2)]^{d/2}}.$$

The resulting un-spectral dimension turns out to be

$$\mathbb{D}_U = \frac{s}{s+\ell^2} d - 2 + 2d_U. \quad (12)$$

Eq. (12) is the second main result of this work. It provides a new physical interpretation of the fundamental constant ℓ as the *transition scale* between different phases of the background spacetime.

Long random walks, where $s \gg \ell^2$, test a semi-classical manifold characterized by $\mathbb{D}_U = d - 2 + 2d_U$. For the special case $d = 2$ the manifold un-spectral dimension is totally determined by the scaling parameter d_U, as in the case of the un-Schwarzschild black hole. Conversely, for $d_U = 1$ the un-matter effects decouple and the un-spectral dimension matches the topological dimension d. In the *critical*, say Planckian, regime we have $s \approx \ell^2$ and $\mathbb{D}_U = 2d_U - 2 + d/2$. For $d = 4$, we see that at Planck scale spacetime dimension is totally determined by the scaling dimension, as it is $\mathbb{D}_U = 2d_U$, just like in the case of the un-Schwarzschild black hole. In the particular case $d_U = 1$, we obtain the dynamical reduction to $\mathbb{D}_U = 2$ necessary to get a power counting renormalizable quantum theory of gravity. However, this is not the end of story. Un-matter allows us to access a new "trans-Planckian" phase which cannot be probed by any sort of ordinary matter. Short paths, where $s \ll \ell^2$, measure $\mathbb{D}_U = 2d_U - 2 + O(s/l^2)$ which is non-negative only in virtue of the scaling dimension $d_U \geqslant 1$. We see that for $d_U < 2$ the un-particle probe scatters across something which we can dub "spacetime vapor" to be consistent with the thermal interpretation of the diffusion process. Moreover, as $d_U \to 1$ then $\mathbb{D}_U \to 0$ leading to the ultimate disintegration of space and time as we understand them. This new picture follows from the introduction of un-spectral dimension, as a dimension measured by a scale invariant continuous mass spectrum probe. It may be worth to remark that even in the "worst-case-scenario", where un-particles were not found at LHC as physical objects, the definition (9) would provide an alternative realization of a scale invariant diffusion process, not advocating a light-cone limit. The "extra-bonus" of this approach is to bring into the definition of un-spectral dimension the real parameter d_U leading to a clear fractalization of the background space(time) and the appearance of a new phase which is forbidden to standard matter probes. As both spectral and Hausdorff dimensions are employed in a variety of diffusion problems, we expect the un-spectral dimension to have an equivalent impact in frameworks different from the one considered in the present Letter.

Acknowledgements

References

[1] S. Carlip, arXiv:0909.3329 [gr-qc];
   J. Ambjorn, J. Jurkiewicz, R. Loll, Phys. Rev. Lett. 95 (2005) 171301;
   L. Modesto, arXiv:0812.2214 [gr-qc];
   L. Modesto, arXiv:0905.1665 [gr-qc];
   F. Caravelli, L. Modesto, arXiv:0905.2170 [gr-qc];
   E. Magliaro, C. Perini, L. Modesto, arXiv:0911.0437 [gr-qc];
   O. Lauscher, M. Reuter, JHEP 0510 (2005) 050;
   D. Benedetti, Phys. Rev. Lett. 102 (2009) 111303.
[2] A. Smailagic, E. Spallucci, J. Phys. A 36 (2003) L467;
   A. Smailagic, E. Spallucci, J. Phys. A 36 (2003) L517;
   A. Smailagic, E. Spallucci, J. Phys. A 37 (2004) 1;
   A. Smailagic, E. Spallucci, J. Phys. A 37 (2004) 7169 (Erratum);
   E. Spallucci, A. Smailagic, P. Nicolini, Phys. Rev. D 73 (2006) 084004;
   P. Nicolini, A. Smailagic, E. Spallucci, ESA Spec. Publ. 637 (2006) 11.1, arXiv:hep-th/0507226;
   P. Nicolini, J. Phys. A 38 (2005) L631;
   P. Nicolini, A. Smailagic, E. Spallucci, Phys. Lett. B 632 (2006) 547;
   T.G. Rizzo, JHEP 0609 (2006) 021;
   S. Ansoldi, P. Nicolini, A. Smailagic, E. Spallucci, Phys. Lett. B 645 (2007) 261;
   E. Spallucci, A. Smailagic, P. Nicolini, Phys. Lett. B 670 (2009) 449;
   P. Nicolini, E. Spallucci, Class. Quant. Grav. 27 (2010) 015010;
   R. Casadio, P. Nicolini, JHEP 0811 (2008) 072;
   P. Nicolini, Int. J. Mod. Phys. A 24 (2009) 1229;
   M. Rinaldi, arXiv:0908.1949 [gr-qc];
   P. Nicolini, M. Rinaldi, arXiv:0910.2860 [hep-th];
   D. Batic, P. Nicolini, Phys. Lett. B 692 (2010) 32, arXiv:1001.1158 [gr-qc];
   M. Bleicher, P. Nicolini, J. Phys. Conf. Ser. 237 (2010) 012008, arXiv:1001.2211 [hep-ph];
   A. Smailagic, E. Spallucci, Phys. Lett. B 688 (2010) 82.
[3] L. Modesto, P. Nicolini, Phys. Ref. D 81 (2010) 104040, arXiv:0912.0220 [hep-th].
[4] H. Georgi, Phys. Rev. Lett. 98 (2007) 221601.
[5] H. Goldberg, P. Nath, Phys. Rev. Lett. 100 (2008) 031803.
[6] J.R. Mureika, Phys. Lett. B 660 (2008) 561;
   J.R. Mureika, Phys. Rev. D 79 (2009) 056003.
[7] P. Gaete, J.A. Helayel-Neto, E. Spallucci, Phys. Lett. B 693 (2010) 155, arXiv:1005.0234 [hep-ph].
[8] P. Gaete, E. Spallucci, Phys. Lett. B 661 (2008) 319.
[9] S. Ansoldi, A. Aurilia, E. Spallucci, Phys. Rev. D 56 (1997) 2352;
   S. Ansoldi, A. Aurilia, E. Spallucci, Chaos Solitons Fractals 10 (1999) 197;
   A. Aurilia, S. Ansoldi, E. Spallucci, Class. Quant. Grav. 19 (2002) 3207.