**Online Appendix 1: Proof of Equations (2) and (3)**

The CRRA lifetime utility function describing the annuitant´s preferences is given by:

|  |  |  |
| --- | --- | --- |
|  | $$U\_{0}=E\_{0}^{π}\left[β∙\frac{C\_{1}^{1-γ}}{1-γ}+β^{2}∙p\frac{C\_{2}^{1-γ}}{1-γ}\right].$$ | (OA1) |

where the coefficient of relative risk aversion is $γ$, the rate of time preference is $β$, the probability of survival to $t=2 \left(t=1\right)$ of $p \left(1\right)$, and the expectation $E\_{0}^{π}$at time $t=0$ under the subjective probability measure $π$. The realization and the probability of consumption $C\_{1} \left(C\_{2}\right)$ at time $t=1$ $\left(t=2\right)$ are as follows:

|  |  |  |
| --- | --- | --- |
| $$C\_{1}=\left\{\begin{matrix}\left(1-y\right)⋅uS\_{0}&with probability&π\_{u}\\\left(1+y\right) ⋅dS\_{0}&with probability&π\_{d}=1-π\_{u}\end{matrix}\right.$$ | and | (OA2) |
| $$C\_{2}=\left\{\begin{matrix}\left(1-y\right)⋅u^{2}S\_{0}&with probability&π\_{u}^{2}\\\left(1+y\right)⋅udS\_{0}&with probability&π\_{ud}\\\left(1-y\right)⋅duS\_{0}&with probability&π\_{du}\\\left(1+y\right)⋅d^{2}S\_{0}&with probability&π\_{d}^{2}\end{matrix}\right.$$ |  | (OA3) |

where $π\_{u} (π\_{d}=1-π\_{u})$ measures the subjective probability for an increase (decrease). Next we factor out $β/(1-γ)$ and replace the expected consumption by its realization and probability:

$$U\_{0}=\frac{β}{1-γ} \{ \left[\left(1-y\right) uS\_{0}\right]^{1-γ}⋅π\_{u}+\left[\left(1+y\right) dS\_{0}\right]^{1-γ}⋅π\_{d}+βp ( \left[ \left(1-y\right) u^{2}S\_{0}\right]^{1-γ}⋅π\_{u}^{2} +\left[\left(1+y\right) udS\_{0}\right]^{1-γ}⋅π\_{ud}+\left[\left(1-y\right) duS\_{0}\right]^{1-γ}⋅π\_{du}+\left[\left(1+y\right) d^{2}S\_{0}\right]^{1-γ}⋅π\_{d}^{2} )\}$$

Rearranging terms we get:

|  |  |  |
| --- | --- | --- |
|  | $$U\_{0}=\frac{β}{1-γ}⋅S\_{0}^{1-γ}\left\{\left(1-y\right)^{1-γ}⋅B+\left(1+y\right)^{1-γ}⋅A\right\}$$ | (OA4) |

with

|  |  |  |
| --- | --- | --- |
|  | $$A=d^{1-γ}⋅π\_{d}+βp\left(ud^{1-γ}⋅π\_{ud}+d^{2\left(1-γ\right)}⋅π\_{d}^{2}\right)$$ | (OA5a) |
|  | $$B=u^{1-γ}⋅π\_{u}+βp\left(u^{2\left(1-γ\right)}⋅π\_{u}^{2}+du^{1-γ}⋅π\_{du}\right).$$ | (OA5b) |

Calculating the derivative of $U\_{0}$ with respect to $y$ and setting it equal to zero gives us:

|  |  |  |
| --- | --- | --- |
|  | $$\frac{dU\_{0}}{dy}=\frac{β⋅S\_{0}^{1-γ}}{1-γ}\left\{-\left(1-γ\right)⋅\left(1-y\right)^{-γ}⋅B+\left(1-γ\right)⋅\left(1+y\right)^{-γ}⋅A\right\}=0$$ | (OA6) |

Solving for the smoothing factor $y$ yields:

|  |  |  |
| --- | --- | --- |
|  | $$y=\frac{A^{\frac{1}{γ}}-B^{\frac{1}{γ}}}{A^{\frac{1}{γ}}+ B^{\frac{1}{γ}}}$$ | (OA7) |

The value $VI\_{0}$ of the PLA for the insurer is given by:

|  |  |  |
| --- | --- | --- |
|  | $$VI\_{0}= \frac{N\left(1+p\right)S\_{0}⋅[q^{2}⋅u^{2}- \left( 1-q\right)^{2}⋅d^{2}]}{\left(1+i\right)^{2}}⋅y,$$ | (OA8) |

with a risk-neutral probability of an upward jump $q$ and riskless interest rate$ i$. $S\_{0}$ and $(1+p)$ are positive variables. Consequently, the value of the insurer is a linear function of the smoothing parameter $y$ if:

|  |  |  |
| --- | --- | --- |
|  | $$q^{2}⋅u^{2}>\left(1-q\right)^{2}⋅d^{2} or q>\frac{d}{u+d} .$$ | (OA9) |

Replacing $q$ with its definition $((1+i)-d)/(u-d)$ gives us the following result:

|  |  |  |
| --- | --- | --- |
|  | $$\frac{\left(1+i\right)-d}{u-d}>\frac{d}{u+d}$$ | (OA10) |

Rearranging terms and assuming $u>d>0$, we get:

|  |  |  |
| --- | --- | --- |
|  | $$\frac{u}{d}>\frac{2u-\left(1+i\right)}{1+i}$$ | (OA11) |

Since $u > 1+i$ by definition, finally we can solve for $d$:

|  |  |  |
| --- | --- | --- |
|  | $$\frac{\left(1+i\right)u}{2u-\left(1+i\right)}>d$$ | (OA12) |

**Online Appendix 2: Illustration of Smoothing Outcomes**

This Appendix illustrates the implications of the mechanics described in Equations (15a-f). The surplus payout policy $PS\_{t}$ depends on the actual vis-à-vis targeted level of contingency reserves $CR\_{t}$.as well as the distributed surplus $DS\_{t}$. This is demonstrated in Figure OA2.1 for three levels of distributed surplus $DS\_{t}$, low (40), medium (100), and high (160), a contingency reserve target of 250, and a surplus payment in the previous period of 100. For example, if the distributed surplus is low (solid gray line) and the actual contingency reserve is above the target level, e.g. 300, the annuitant will receive a payout 80. Hence, surplus payouts are higher than the distributed surplus in this period (80 vs. 40), but lower than in the previous period (80 vs. 100). By contrast, if the contingency reserve is below the target, e.g. 200, and the distributed surplus is high (dashed gray line), the annuitant only receives 125 and the remaining surplus is used to replenish the contingency reserve.

**Figure OA2.1: Surplus Payout Policies for Alternative Contingency Reserve Levels and Distributed Surpluses**

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Notes: Apportionment of distributed surpluses $DS\_{t}$ to surplus payments $PS\_{t}$ for varying contingency reserve levels $CR\_{t}$. Other parameters: Surplus payment in previous year $PS\_{t-1}$: 100; Target contingency reserve: 250; Boundary b: 1.25; Constant population ($I\_{t}=I\_{t-1}$). Source: Authors’ calculations.

Figure OA2 demonstrates how this algorithm smooths surplus payments $PS\_{t}$ over time for a random path of distributed surpluses $DS\_{t}$ and for three different levels of the initial contingency reserve $CR\_{0}$, below target level (150), at target level (250), and above target level (350). We see that the smoothing algorithm reduces the volatility of surplus payouts to the annuitants (solid black line) versus the distributed surplus $DS\_{t}$ (dotted black line). While the randomly generated distributed surplus has a volatility of 59, the smoothing algorithm is able to reduce the volatility in the surplus payments $PS\_{t}$ by about one half. For an initial contingency reserve at (below/above) target level, the volatility of surplus payments is 27 (34/26).

**Figure OA2.2: Time Paths of Distributed Surplus, Contingency Reserve, and Surplus Payment**

 $CR\_{0}$ below Target $CR\_{0}$ at Target $CR\_{0}$ above Target

Notes: Selected simulated paths of distributed surplus $DS\_{t}$, surplus payment $PS\_{t}$, and contingency reserve $CR\_{t}$ for three different initial levels of contingency reserve $CR\_{0}$: below target = 150, at target = 250, and above target = 350. Other parameters: Year zero surplus $PS\_{0}$: 100; Boundary b: 1.25; Constant population ($I\_{t}=I\_{t-1}$). Source: Authors’ calculations.