Phase diagram of the $2 + 1$-dimensional Gross-Neveu model with chiral imbalance

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In this work, the phase diagram of the $2 + 1$-dimensional Gross-Neveu model is investigated with baryon chemical potential as well as chiral chemical potential in the mean-field approximation. We study the theory using two lattice discretizations, which are both based on naive fermions. An inhomogeneous chiral phase is observed only for one of the two discretizations. Our results suggest that this phase disappears in the continuum limit.

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1. Introduction

The Gross-Neveu (GN) model describes N_f fermion flavors with a quartic interaction in the scalar channel. It is frequently used as a toy model for spontaneous breaking of chiral symmetry [1]. In general, four-Fermi theories are of interest in many applications in high energy physics [2–7], condensed matter physics[1] [8–18] but also of theoretical aspects of quantum field theory (e.g., renormalizability in the $1/N$ expansion or in a perturbative approach) [19–21]. In the mean-field approximation, the 1 + 1-dimensional GN model exhibits a symmetric phase, a homogeneous symmetry-broken phase and an inhomogeneous phase, where the chiral condensate is an oscillating function of the spatial coordinate [22–24].

Recently, two independent mean-field studies of the μ-T phase diagram (μ denotes the baryon chemical potential) of the 2 + 1-dimensional GN model were published [25–27]. An important result of these works is that the existence of an inhomogeneous phase depends on the details of the regularization. When the regulator (lattice spacing or Pauli-Villars cutoff) is removed, the inhomogeneous phase disappears.

In this work, we continue our previous lattice field theory investigations [25, 27] by studying the phase diagram of the 2 + 1-dimensional GN model with an additional chiral chemical potential μ_{45}.

2. The 2 + 1-dimensional GN model

The action of the 2 + 1-dimensional GN model with baryon chemical potential μ as well as chiral chemical potential μ_{45} is given by

$$S[\bar{\psi}, \psi] = \int d^3x \left(\sum_{n=1}^{N_f} \bar{\psi}_n \left(\gamma_\nu \partial_\nu + \gamma_0 \mu + \gamma_{45} \gamma_0 \mu_{45} \right) \psi_n - \frac{g^2}{2} \left(\sum_{n=1}^{N_f} \bar{\psi}_n \psi_n \right)^2 \right), \qquad (1)$$

where ψ_n is a four-component massless fermion field, N_f is the number of flavors and g^2 is the coupling of the four-fermion interaction. A reducible representation of the Clifford algebra is chosen, where γ_0, γ_1 and γ_2 are block-diagonal. The upper two components of ψ_n can be interpreted as being "left-handed", the lower two components as "right-handed". There are two linearly independent matrices γ_4 and γ_5, which anti-commute with γ_0, γ_1 and γ_2 and, thus, both fulfill the necessary properties of a suitable γ_5 matrix [7, 31]. Continuous symmetry transformations correspond to the generators $\mathbb{I}_4$ and γ_{45} [2], while discrete chiral symmetry transformations are given by either γ_4 or γ_5 [6, 30]. Since $\gamma_{45} = \mathrm{diag}(+\mathbb{I}_2, -\mathbb{I}_2)$, the chemical potential μ_{45} generates a chiral imbalance. More details can be found in Ref. [27] and will also be discussed in an upcoming publication.

By introducing an auxiliary boson field σ and integrating over the fermion fields, one obtains the effective action

$$S_{\mathrm{eff}}[\sigma] = N_f \left(\frac{1}{2\lambda} \int d^3x\, \sigma^2 - \ln\left(\det(Q) \right) \right) \quad , \quad Z = \int D\sigma\, e^{-S_{\mathrm{eff}}[\sigma]}, \qquad (2)$$

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[1]Especially 2 + 1-dimensional models are often used to describe superconducting electrons in high-temperature superconductors, which are confined to planes.

[2]For N_f fermion fields one needs to consider tensor products with the generators of flavor rotations to get the full set of symmetry transformations.

where

$$Q = \gamma_\nu \partial_\nu + \gamma_0 \mu + \gamma_{45} \gamma_0 \mu_{45} + \sigma \tag{3}$$

is the Dirac operator and $\lambda = N_f g^2$. The effective action is real-valued, if the boson field σ is restricted to be constant in temporal direction (see appendix A of Ref. [27] for a proof). As $S_{\text{eff}} \propto N_f$, the limit $N_f \to \infty$ reduces the relevant configurations in the path integral to the global minima of S_{eff}, i.e., the mean-field approximation becomes exact in that limit. A Ward-Takahashi identity causes σ to be proportional to the chiral condensate $\langle \bar{\psi}_n \psi_n \rangle$. Moreover, the effective action Eq. (2) is invariant under $\sigma \to -\sigma$. Thus, σ is an order parameter for the discrete chiral symmetry.

The Dirac operator can be expressed in terms of irreducible representations of the Clifford algebra [28],

$$Q[\mu, \mu_{45}, \sigma] = \text{diag}\left(Q^{(2)}[\mu + \mu_{45}, \sigma], \tilde{Q}^{(2)}[\mu - \mu_{45}, \sigma]\right), \tag{4}$$

where $Q^2[\mu, \sigma]$ and $\tilde{Q}^{(2)}[\mu, \sigma]$ are defined as

$$Q^{(2)}[\mu, \sigma] = +\tau_2(\partial_0 + \mu) + \tau_3 \partial_1 + \tau_1 \partial_2 + \sigma, \tag{5}$$

$$\tilde{Q}^{(2)}[\mu, \sigma] = -\tau_2(\partial_0 + \mu) - \tau_3 \partial_1 - \tau_1 \partial_2 + \sigma. \tag{6}$$

The Dirac operators $Q^{(2)}$ and $\tilde{Q}^{(2)}$ contain two inequivalent 2×2 representations, $(\gamma_0, \gamma_1, \gamma_2) = (\tau_2, \tau_3, \tau_1)$ and $(\gamma_0, \gamma_1, \gamma_2) = -(\tau_2, \tau_3, \tau_1)$, with τ_j denoting the Pauli matrices. They act on the "left-handed" and "right-handed" components introduced above. For $\mu_{45} = 0$ their chemical potentials are the same, but for $\mu_{45} \neq 0$ they differ, $\mu + \mu_{45}$ and $\mu - \mu_{45}$, indicating chiral imbalance. The determinants of $Q^{(2)}$ and $\tilde{Q}^{(2)}$ are invariant under $\sigma \to -\sigma$ as well as under $\mu \to -\mu$. A consequence of the latter is that $\det Q$ as well as effective action (2) are invariant under the exchange of μ and μ_{45}. This property is used to check our numerical results presented in section 4.

One can show that

$$\det Q^{(4)}[\mu, \sigma] = \left(\det Q^{(2)}[\mu, \sigma]\right)^2 = \left(\det \tilde{Q}^{(2)}[\mu, \sigma]\right)^2 \tag{7}$$

with $Q^{(4)}[\mu, \sigma] = Q[\mu, 0, \sigma]$ (see Ref. [27] for the proof). In consequence, the same phase diagram would be obtained if a Dirac operator

$$Q'[\mu, \mu_{45}, \sigma] = \text{diag}\left(Q^{(4)}[\mu + \mu_{45}, \sigma], Q^{(4)}[\mu - \mu_{45}, \sigma]\right) \tag{8}$$

would be used in the action. Then, the spinor fields would obtain an additional degree of freedom, independent of chirality, that one can interpret as "isospin" and μ_{45} would serve as an isospin chemical potential. We will explore this concept further in an upcoming publication.

3. Lattice discretizations

We use lattice discretizations of Eq. (2) similar to that discussed in section 4 of Ref. [27]. The main difference is that we use a lattice also in temporal direction instead of a superposition of plane waves. The cubic space-time volume is $\beta \times V = \beta \times L \times L$ and the corresponding lattice has spacing a and $N_t \times N_s \times N_s$ sites, i.e., $\beta = aN_t$ and $L = aN_s$. The boundary conditions are antiperiodic in temporal direction and periodic in the two spatial directions.

For numerical computations it is convenient to work in momentum space. When using naive fermions, a weight function $\tilde{W}_2$ with specific properties has to be included in the interaction term of the GN model, to ensure the correct continuum limit (see Refs. [19, 27, 32], in particular appendix 2 of Ref. [19]). The Dirac operator is then

$$Q(p, q) = i\delta_{p,q} \sum_{\nu=0}^{2} \gamma_\nu \sin\left(p_\nu - \delta_{\nu,0}\, i(\mu + \gamma_{45}\mu_{45})\right) + \delta_{p_0,q_0}\tilde{W}_2(\mathbf{p} - \mathbf{q})\tilde{\sigma}(\mathbf{p} - \mathbf{q}), \qquad (9)$$

where all quantities are expressed in units of a, i.e., dimensionless. In this work as well as in our previous work [27] we use and compare two weight functions corresponding to two different lattice discretizations,

$$\tilde{W}_2'(\mathbf{p}) = \prod_{\nu=1,2} \tilde{W}_1'(p_\nu) \quad , \quad \tilde{W}_1'(p_\nu) = \frac{\cos(p_\nu) + 1}{2} \qquad (10)$$

and

$$\tilde{W}_2''(\mathbf{p}) = \prod_{\nu=1,2} \tilde{W}_1''(p_\nu) \quad , \quad \tilde{W}_1''(p_\nu) = \Theta(\pi/2 - |p_\nu|), \qquad (11)$$

where $\Theta(p)$ is the Heaviside function.

4. Numerical results

The phase diagram at $\mu_{45} = 0$ is known from Refs. [26, 27]. There is a symmetric phase ($\sigma = 0$) at large μ and/or T and a homogeneous symmetry-broken phase with a spatially constant field $\sigma = \bar{\sigma} \neq 0$ at small μ and small T. For $W_2 = W_2''$ and finite lattice spacing there is also an inhomogeneous phase, where σ is a varying function of the spatial coordinates ($\sigma = \sigma(\mathbf{x})$), at small T between the symmetric and the homogeneous symmetry-broken phase. This phase, however, vanishes in the continuum limit. There is no inhomogeneous phase for $W_2 = W_2'$, neither at finite lattice spacing nor in the continuum limit.

In this section we discuss numerical results for the two lattice discretizations (10) and (11) with the aim to extend the phase diagram to $\mu_{45} \neq 0$. Technical aspects like scale setting and tuning of the coupling constant are explained in Ref. [27]. As usual, all dimensionful quantities are expressed in units of $\sigma_0 = \sigma|_{\mu=0,\mu_{45}=0,T\approx0}$, the vacuum expectation value of σ.

4.1 The homogeneous phase diagram

When only allowing a homogeneous condensate, i.e., $\tilde{\sigma}(\mathbf{p}) = \bar{\sigma}\delta_{\mathbf{p},0}$, the lattice discretizations with W_2' and W_2'' are identical (cf. Eq. (10) and Eq. (11) for $\mathbf{p} = 0$). The effective action at given μ, μ_{45} and T is a function of just one variable and can be minimized numerically in a straightforward way to determine the physically preferred value of $\bar{\sigma}$.

Fig. 1 shows the phase diagram in 3-dimensional μ-μ_{45}-T space at finite lattice spacing and space-time volume. We note that these results reflect the previously discussed $\mu \leftrightarrow \mu_{45}$ symmetry of the effective action. The observed phase boundary for $\mu_{45} = 0$ is close to the continuum result from Ref. [33], where small deviations are due to discretization errors and finite volume
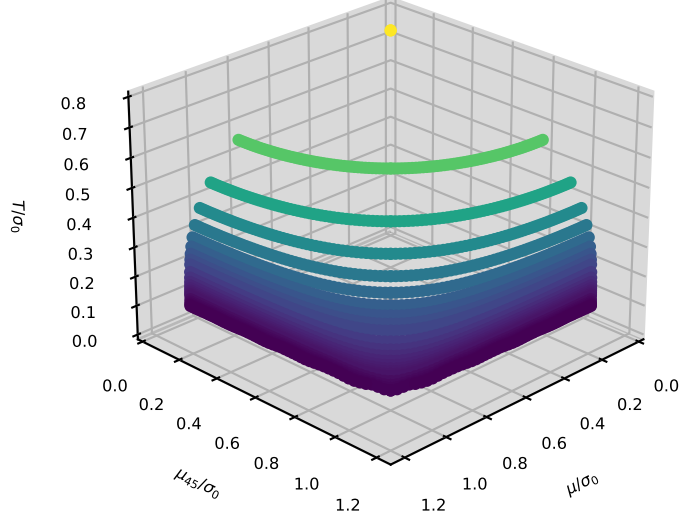
Figure 1: Phase diagram of the chirally imbalanced 2 + 1-dimensional GN model with restriction to a homogeneous condensate in 3-dimensional μ-μ_{45}-T space for $a\sigma_0 = 0.2327$ and $L\sigma_0 = 27.92$.
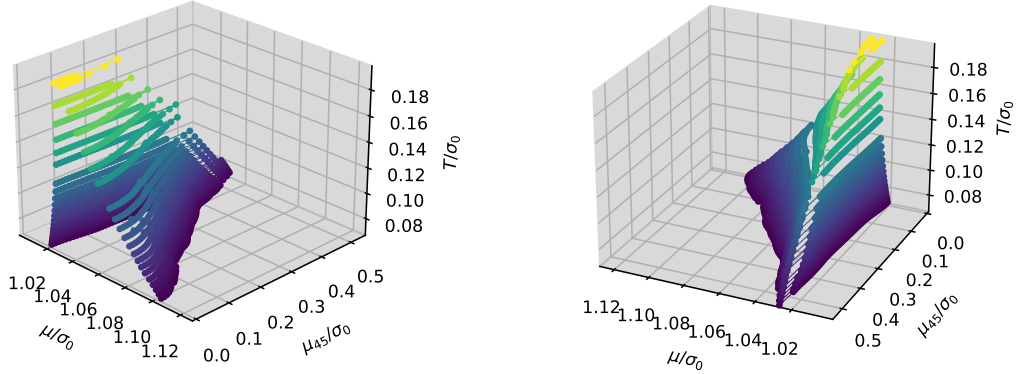

Figure 2: "Phase diagram" of the chirally imbalanced 2 + 1-dimensional GN model obtained via stability analyses in 3-dimensional μ-μ_{45}-T space for $a\sigma_0 = 0.2327$ and $L\sigma_0 = 23.27$ and the discretization with $W_2 = W_2''$. The two plots show the same data from different angles.

corrections. There are also slight differences to our lattice results from Ref. [27], because of different discretizations in temporal direction.

For large temperature, $T/\sigma_0 \gtrsim 0.4$, the phase boundary exhibits an approximate rotational symmetry in the μ-μ_{45} plane, i.e., the phase boundary is crudely described by $\mu^2 + \mu_{45}^2 = \text{const.}$ The behavior is completely different at low temperatures, $T/\sigma_0 \lesssim 0.1$, where the phase boundary resembles a square in the μ-μ_{45} plane.

4.2 Instabilities with respect to spatially inhomogeneous perturbations

To study possibly existing inhomogeneous phases, one has to allow for arbitrary modulations, i.e., non-vanishing $\sigma(\mathbf{p})$, and to find the global minima of the effective action. In principle, such a minimization is possible within our lattice field theory approach, but it is a computationally extremely challenging task. For 1-dimensional modulations and $\mu_{45} = 0$ we were successfully using a local minimization algorithm with several randomly chosen initial field configurations [27]. In the near future we plan to use similar techniques to search for 2-dimensional modulations that are global minima, both at $\mu_{45} = 0$ and $\mu_{45} \neq 0$.

In this work, a simpler strategy is applied and the existence of inhomogeneous phases is studied via stability analyses. For given μ, μ_{45} and T we check the stability of the homogeneous condensate $\bar{\sigma}$, as determined in the previous subsection, with respect to spatially inhomogeneous perturbations $\delta\tilde{\sigma}(\mathbf{p})$ (for details we refer to Ref. [27]). If there is a perturbation $\delta\tilde{\sigma}(\mathbf{p})$, which lowers the effective action compared to the homogeneous condensate, an inhomogeneous phase is indicated. In this way regions in the phase diagram can be mapped, which correspond to or are part of an inhomogeneous phase.

Since there is no inhomogeneous phase in the continuum limit for chirally balanced matter [26, 27], we now explore, whether $\mu_{45} \neq 0$ might favor the existence of such a phase. We start with the discretization with $W_2 = W_2''$ and show in Fig. 2 the boundaries of the instability region in 3-dimensional μ-μ_{45}-T space at finite lattice spacing and space-time volume. The instability region resembles a tetrahedron and is adjacent to the homogeneously broken phase depicted in Fig. 1. Increasing $|\mu_{45}|$ seems to disfavor an inhomogeneous phase, because the instability region shrinks.

Computations at several values of the lattice spacing suggest that the instability region vanishes in the continuum limit also at $\mu_{45} \neq 0$. This is supported by computations with the discretization with $W_2 = W_2'$, where no instability region is observed, neither at $\mu_{45} = 0$ (as discussed in Ref. [27]) nor at $\mu_{45} \neq 0$. These results will be shown and discussed in more detail in an upcoming publication.

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