On analogs of Cremona automorphisms for matroid fans

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Introduction

Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty and perfection.

— Hermann Weyl, *Symmetry* [Wey52]

In mathematics, a symmetry is a transformation of a mathematical object that preserves its structure or some of its properties. Classical types of symmetries are linear transformations in Euclidean space such as reflection, rotation, translation, or scaling. In the 19th century, mathematicians began to study more abstract types of symmetries, also called *automorphisms*, and group theory was developed as an algebraic tool to describe their properties. For example, Galois discovered that the solvability of a polynomial equation by radicals depends on the group of symmetries of its roots.

In this thesis we study certain symmetries of systems of linear equations, using matroid theory and tropical geometry. For example, consider the set

$$\mathcal{A} := \{x = 0,\ y = 0,\ z = 0,\ x = y,\ x = z,\ y = z\}$$

of equations in the real variables x, y, and z. Geometrically, we can interpret the variables as Cartesian coordinates for the three-dimensional Euclidean space $\mathbb{R}^3$. Then every equation in $\mathcal{A}$ defines a plane through the origin and one can consider the classical symmetries that preserve this plane arrangement. For $\mathcal{A}$, there are the following symmetries and their combinations:

(1) There is a group action of the symmetric group S_3 since $\mathcal{A}$ is completely symmetric in the variables $\{x, y, z\}$. Geometrically, S_3 acts by rotations and reflections through a plane, depending on the sign of the permutation. For example, interchanging x and y corresponds to the reflection through the plane $x = y$.

(2) Since the equations in $\mathcal{A}$ are homogeneous, multiplying all variables by the same non-zero factor preserves every plane. By taking the quotient with respect to scalar transformations, we can also consider $\mathcal{A}$ as line arrangement in the projective plane $\mathbb{P}^2$.

(3) The linear map

$$x \mapsto x, \quad y \mapsto x - y, \quad z \mapsto x - z$$

preserves the set $\mathcal{A}$. For example, the equation $x = y$ is sent to the equation $x = x - y$, which is equivalent to $y = 0$. Geometrically, this map corresponds to a non-orthogonal reflection through the line $x = 2y = 2z$.

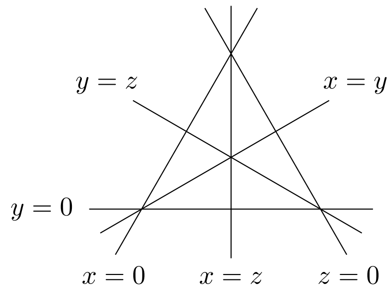

FIGURE 1. A geometric representation of $\mathcal{A}$ as line arrangement in the projective plane.

However, there also exists a non-linear symmetry in this example. Consider the quadratic transformation

$$x \mapsto yz, \quad y \mapsto xz, \quad z \mapsto xy,$$

which is known as *Cremona transformation* in projective geometry, named after Italian mathematician Luigi Cremona (1830–1903), see [Trk08]. The image of $\mathcal{A}$ under the Cremona transformation is given by

$$\mathcal{A}' := \{yz = 0,\ xz = 0,\ xy = 0,\ yz = xz,\ yz = xy,\ xz = xy\}.$$

Each quadratic equation in $\mathcal{A}'$ can be decomposed into two linear equations:

$$\begin{aligned}
yz = 0 &\iff y = 0 \ \text{ or } \ z = 0\\
xz = 0 &\iff x = 0 \ \text{ or } \ z = 0\\
xy = 0 &\iff x = 0 \ \text{ or } \ y = 0\\
yz = xz &\iff y = x \ \text{ or } \ z = 0\\
yz = xy &\iff z = x \ \text{ or } \ y = 0\\
xz = xy &\iff z = y \ \text{ or } \ x = 0
\end{aligned}$$

…and the equations on the right hand side are exactly the equations in $\mathcal{A}$! But some equations appear with multiplicity 3, according to a scheme that is reminiscent of the matrix multiplication

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1\\ 1 & 0 & 1 & 0 & 1 & 0\\ 1 & 1 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1\\1\\1\\1\\1\\1 \end{pmatrix} = \begin{pmatrix} 3\\3\\3\\1\\1\\1 \end{pmatrix}.$$

The goal of this thesis is to gain a better understanding of these Cremona symmetries. When do they exist and which groups do they generate? Are there other types of non-linear symmetries? The search for answers will lead us to matroid theory.

Matroids are combinatorial objects that generalize the notion of dependence from linear algebra. For example, the two equations $x = y$ and $y = 0$ together imply the equation $x = 0$. In the language of matroid theory, $x = 0$ lies in the *closure* $\mathrm{cl}(\{x = y,\ y = 0\})$ of the set $\{x = y,\ y = 0\}$. There is no other equation in $\mathcal{A}$ that follows from $x = y$ and $y = 0$, thus $\{x = y,\ y = 0,\ x = 0\}$ is a *closed* set, also called a *flat* of the matroid M associated to $\mathcal{A}$. Geometrically, the flats of M describe the intersection behavior of the planes in $\mathcal{A}$.

The set $\{x = 0,\ y = z\}$ is also a flat of M, but it only contains two elements. Such a flat is called *disconnected*, in contrast to the *connected* flat $\{x = y,\ y = 0,\ x = 0\}$. There are three other connected flats of "rank 2": $\{y = 0,\ z = 0,\ y = z\}$, $\{x = 0,\ z = 0,\ x = z\}$, and $\{x = y,\ x = z,\ y = z\}$.

The structure of the matroid M can be visualized via its *minimal nested set complex* $\mathcal{N}$. In our example, this is a graph whose vertices are the elements of $\mathcal{A}$ plus the connected flats of rank 2. From every connected flat of rank 2 we draw an edge to each of its elements, and we add edges between each pair of elements that form a disconnected flat.

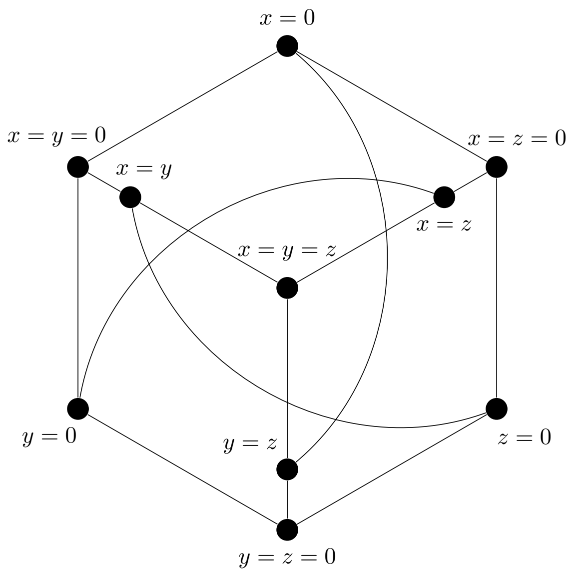

FIGURE 2. The minimal nested set complex of the matroid associated to $\mathcal{A}$.

Every classical symmetry of $\mathcal{A}$ induces an automorphism of $\mathcal{N}$ that maps elements to elements and connected flats to connected flats. However, rotating the outer hexagon by 180 degrees induces an automorphism of $\mathcal{N}$ that does not have this property. This is an example for a *combinatorial Cremona automorphism*, as defined by Shaw and Werner in their recent paper [SW23]. As we will see, geometric and combinatorial Cremona symmetries are closely related, and their connection can be described using tropical geometry.

Background

Recently, exciting new connections between algebraic geometry and combinatorics have been discovered. An important tool is the so-called tropicalization process that translates algebraic varieties into polyhedral objects. The simplest examples are linear spaces, and their tropical analogs are closely related to matroids.

More precisely, given an essential hyperplane arrangement $\mathcal{A}$ in projective space over a field with trivial valuation, its complement $\Omega_{\mathcal{A}}$ can be identified with a linear subvariety of the algebraic torus. The tropicalization of this variety is the support of a polyhedral fan and depends only on the matroid $M(\mathcal{A})$ associated to $\mathcal{A}$. By generalizing this construction, one can associate to every matroid M a *tropical linear space* $\operatorname{trop}(M)$, even if M is not realizable as a hyperplane arrangement, see [MS15, Section 4.2]. This new geometric approach led to great progress in matroid theory, including the development of a combinatorial Hodge theory for matroids, see [Ard18] for a survey. In 2022, June Huh was awarded the Fields Medal for his contributions to these breakthroughs.

There are several natural fan structures on the set $\operatorname{trop}(M)$, and in this thesis we will distinguish between the *coarse Bergman fan* $B_c(M)$, *the minimal nested set fan* $B_m(M)$, and the *fine Bergman fan* $B_f(M)$. These fan structures were studied in [AK06] and [FS05], and Feichtner and Sturmfels gave a criterion for the coarse Bergman fan and the minimal nested set fan to coincide ([FS05, Theorem 5.3]).

Fan structures on $\operatorname{trop}(M(\mathcal{A}))$, where $M(\mathcal{A})$ is the matroid associated to a hyperplane arrangement $\mathcal{A}$ as above, can be used to construct tropical compactifications of the complement $\Omega_{\mathcal{A}}$ by taking the closure in the associated toric varieties ([Tev07, Proposition 2.3]). In this way, the minimal nested set fan induces the minimal wonderful compactification defined by de Concini and Procesi ([DCP95]), and the coarse Bergman fan induces the visible contour compactification defined by Kapranov ([Kap93]).

Assuming that $\mathcal{A}$ is connected, Kurul and Werner proved that every birational automorphism f of $\Omega_{\mathcal{A}}$ can be extended to an automorphism of its visible contour compactification ([KW19, Theorem 5.1]). In the proof, they show that f induces an automorphism of the intrinsic torus whose tropicalization preserves the coarse Bergman fan. By [Kur17, Theorem 7.7], this gives rise to an embedding of the birational automorphism group $\operatorname{Aut}(\Omega_{\mathcal{A}})$ into the automorphism group $\operatorname{Aut}(B_c(M(\mathcal{A})))$ of the coarse Bergman fan. Thus automorphisms of Bergman fans can be viewed as analogs of birational automorphisms for matroids.

In their aforementioned paper [SW23] about the birational geometry of matroids, Shaw and Werner studied automorphism groups of Bergman fans, using tools from the newly developed Hodge theory for matroids such as the Chow ring of a matroid. For a simple matroid M that is not totally disconnected, they prove that every automorphism of the fine Bergman fan $B_f(M)$ is induced by a matroid automorphism of M ([SW23, Theorem 6.3]). However, depending on the matroid M, there may exist automorphisms of the coarse Bergman fan $B_c(M)$ that are not induced by matroid automorphisms.

The arrangement $\mathcal{A}$ that we considered in the introduction, which is also known as the essential braid arrangement of type A_3, and more generally all root system matroids $M(A_n)$ with $n \geq 3$ are examples for this phenomenon. By [AP18], the automorphism group of the moduli space $M_{0,n}^{\mathrm{trop}}$ of stable tropical curves of genus 0 with $n \geq 5$ marked points is isomorphic to the symmetric group S_n. As explained in [SW23, Example 3.1], the coarse Bergman fan $B_c(M(A_n))$ can be identified with $M_{0,n+2}^{\mathrm{trop}}$, hence $\mathrm{Aut}(B_c(M(A_n))) \cong S_{n+2}$ is larger than $\mathrm{Aut}(M(A_n)) \cong S_{n+1}$.

In order to describe these additional automorphisms, Shaw and Werner introduced combinatorial Cremona automorphisms of Bergman fans by proving the following theorem:

Theorem ([SW23, Theorem 8.3])**.** *Let B be a basis of a simple connected matroid M on the ground set E. The Cremona map $\mathrm{crem}_B \colon \mathbb{R}^E \to \mathbb{R}^E$ given by*

$$v_b \mapsto v_{\mathrm{cl}(B\setminus\{b\})} \text{ for all } b \in B \quad \text{and} \quad v_e \mapsto v_e \text{ for all } e \in E \setminus B$$

induces an automorphism of the coarse Bergman fan $B_c(M)$ if and only if the sets $\{\mathrm{cl}(\{b, b'\}) \setminus \{b, b'\}\}_{b,b' \in B}$ form a partition of $E \setminus B$.

This combinatorial condition on a basis of a matroid will be the starting point for our investigations.

Summary

In Chapter 1 we lay the combinatorial groundwork by studying Cremona bases of matroids, which we define as follows:

Definition (1.3.1)**.** Let M be a matroid on the ground set E. A basis B of M is called a *Cremona basis* of M if

$$\bigcup_{b,b' \in B} \mathrm{cl}(\{b, b'\}) = E.$$

For all results in this chapter, we assume that the matroid M is simple, and in this case the definition is equivalent to the condition in [SW23, Theorem 8.3], as we show in Proposition 1.3.2. We will see that the existence of such a basis has strong implications for the structure of the matroid.

With respect to a Cremona basis B, the matroid M can be represented by its *support graph* $G_B(M)$, which is allowed to have parallel edges, see Definition 1.3.4. The vertices of $G_B(M)$ correspond to the basis elements B and the edges of $G_B(M)$ correspond to the non-basis elements $E \setminus B$. More precisely, for an element $e \in E \setminus B$, the assumption that B is a Cremona basis implies that the fundamental circuit $C_B(e)$ of e with respect to B contains exactly two elements of B, and we define these to be the endpoints of e.

In general, for a subset $S \subseteq E$ we define its *support* $\mathrm{supp}_B(S)$ with respect to B as the smallest set of basis elements whose closure contains S, see Proposition 1.2.8. In this way, S can be represented by a subgraph $G_B(S)$ of $G_B(M)$ whose set of vertices is $\mathrm{supp}_B(S)$. If this graph is connected, then we call S *support-connected* with respect to B. By Proposition 1.4.7, this condition is weaker than the usual notion of connectivity for matroids.

In Section 1.5, we apply these tools to describe the flats and the rank function of matroids that have a Cremona basis. We show that the flats can be classified into two types:

Theorem (1.5.2)**.** *Let M be a simple matroid that admits a Cremona basis B and let F be a support-connected flat of M. Then exactly one of the following is true:*

(1) *F is a* coordinate flat*, that is, $F \cap B = \mathrm{supp}_B(F)$.*
(2) *F is a* non-coordinate flat*, that is, $F \cap B = \emptyset \neq F$.*

The rank $\mathrm{rk}(S)$ of every subset $S \subseteq E$ is determined by the cardinality of its support and the type of the closure $\mathrm{cl}(S)$. This follows from Corollary 1.4.8 and the following theorem:

Theorem (1.5.5)**.** *Let M be a simple matroid on the ground set E that admits a Cremona basis B and let $S \subseteq E$ be a subset.*

(1) $\mathrm{cl}(S)$ *is a coordinate flat if and only if* $\mathrm{rk}(S) = |\mathrm{supp}_B(S)|$.
(2) *If* $\mathrm{cl}(S)$ *is a support-connected non-coordinate flat, then* $\mathrm{rk}(S) = |\mathrm{supp}_B(S)| - 1$.

We emphasize that the notions of support and (non-)coordinate flats depend on the choice of a Cremona basis B, and in Sections 1.6–1.8 we consider the case that a matroid has more than one Cremona basis. Every matroid automorphism sends Cremona bases to Cremona bases, thus there is a group action of $\mathrm{Aut}(M)$ on the set $\mathcal{CB}(M)$ of Cremona bases of M. By

studying the shape of the support graph $G_B(B')$ for Cremona bases B and B' of M in Section 1.6, we show that this action is always transitive:

Theorem (1.7.2)**.** *Let M be a simple matroid on the ground set E that admits two Cremona bases B and B'. Then there exists an involutive automorphism $f_{BB'}$ of M with*

$$f_{BB'}(B) = B', \quad f_{BB'}(B') = B, \quad \textit{and} \quad f_{BB'}|_{E\setminus(B\cup B')} = \mathrm{id}.$$

In particular, the matroid automorphism $f_{BB'}$ induces an isomorphism between the support graphs $G_B(M)$ and $G_{B'}(M)$. We call $f_{BB'}$ the *Cremona base change automorphism* with respect to the Cremona bases B and B'.

If we additionally assume that M and the contractions M/e are connected for all $e \in E$, then the Cremona base change automorphisms generate the symmetric group $\mathrm{Sym}(\mathcal{CB}(M))$:

Theorem (1.7.6)**.** *Let M be a simple matroid of rank at least* 3 *on the ground set E. If M and the contractions M/e are connected for all $e \in E$, then the group action* $\mathrm{Aut}(M) \to \mathrm{Sym}(\mathcal{CB}(M))$ *is surjective.*

This leads to a representability criterion for matroids that admit more than one Cremona basis.

Theorem (1.8.2)**.** *Let M be a simple connected matroid of rank at least 3 on the ground set E and assume that the contractions M/e are connected for all $e \in E$. If M admits more than one Cremona basis, then M is representable over any field K with $|K| \geq |E| - \mathrm{rk}(M) + 1$.*

In Section 2.1, we recall the definitions of the *tropical linear space* $\mathrm{trop}(M)$, the *fine Bergman fan* $B_f(M)$, and the *coarse Bergman fan* $B_c(M)$ of a matroid M. Moreover, there is the *minimal nested set fan* $B_m(M)$, which is constructed from the *minimal nested set complex* $\mathcal{N}(M)$ of a matroid M.

Automorphisms of these matroid fans are by definition linear maps that preserve the fan structure and are induced by lattice automorphisms. By Lemma 2.2.3, matroid automorphisms induce automorphisms of these fans, and conversely, by Proposition 2.2.4, every automorphism of the coarse Bergman fan that induces a permutation of the rays corresponding to singletons comes from a matroid automorphism.

In Section 2.3, we study the standard Cremona transformation

$$\mathrm{crem}\colon \mathbb{P}^d_K \dashrightarrow \mathbb{P}^d_K, \quad [x_0 : \ldots : x_d] \mapsto \left[\frac{1}{x_0} : \ldots : \frac{1}{x_d}\right]$$

in the projective space $\mathbb{P}^d_K$ over a field K and determine the hyperplane arrangements for which it induces an automorphism of the complement.

Theorem (2.3.5)**.** *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{P}^d_K$. The standard Cremona transformation* crem *induces an automorphism of the complement $\Omega_{\mathcal{A}}$ if and only if*

$$\mathcal{A} = \{V(x_i) \mid 0 \leq i \leq d\} \cup \{V(x_i + zx_j) \mid 0 \leq i < j \leq d,\ z \in Z_{ij}\}$$

for some collection $Z = \{Z_{ij}\}_{0\leq i<j\leq d}$ of sets $Z_{ij} \subseteq K^\times$ with the property that all sets Z_{ij} are closed under taking multiplicative inverses.

In particular, in this case the coordinate hyperplanes form a Cremona basis of the associated matroid $M(\mathcal{A})$.

In Chapter 3, we study the Cremona automorphisms defined by Shaw and Werner. We give an example where the combinatorial Cremona map is the tropicalization of the Cremona transformation on projective space, using [Kur17, Theorem 7.7]. Moreover, we prove the following criterion for the Cremona map to preserve the minimal nested set structure:

Theorem (3.1.7). *Let M be a simple connected matroid of rank at least 3 and assume that M has a Cremona basis B such that $G_B(M)$ is a complete graph. Then the Cremona map* crem_B *induces an automorphism of the minimal nested set fan $B_m(M)$.*

In Section 3.2, we describe the structure of the *Cremona group* $\mathrm{Cr}(M)$ of a matroid M, which we define as the subgroup of $\mathrm{Aut}(B_c(M))$ generated by matroid automorphisms and Cremona automorphisms.

Theorem (3.2.5). *Let M be a simple connected matroid of rank at least 3. For every Cremona basis B of M, the Cremona group* $\mathrm{Cr}(M)$ *is generated by matroid automorphisms and the Cremona automorphism* crem_B*.*

Under the additional assumption that the contractions M/e are connected for all $e \in E$, we have $\mathrm{Aut}(M)/\mathrm{Aut}_{\mathrm{CB}}(M) \cong S_k$ by Theorem 1.7.6, where $k \in \mathbb{N}_0$ is the number of Cremona bases and $\mathrm{Aut}_{\mathrm{CB}}(M) \subseteq \mathrm{Aut}(M)$ is the normal subgroup of matroid automorphisms that preserve every Cremona basis. In the Cremona group the degree of the symmetric group increases by 1, like in the example $M(A_n)$:

Theorem (3.2.8). *Let M be a simple connected matroid of rank at least 3 and assume that the contractions M/e are connected for all $e \in E$. Then* $\mathrm{Cr}(M)/\mathrm{Aut}_{\mathrm{CB}}(M) \cong S_{k+1}$*.*

For a simple matroid M of rank 3 it has been shown in [SW23, Theorem 9.2] that the automorphism group $\mathrm{Aut}(B_c(M))$ of the coarse Bergman fan coincides with the Cremona group $\mathrm{Cr}(M)$, if M is not a non-trivial parallel connection. In Section 3.3, we show a similar result for the minimal nested set complex $\mathcal{N}(M)$:

Theorem (3.3.2). *Let M be a simple connected matroid of rank 3.*

(1) *If M is the matroid associated to a self-dual non-degenerate projective plane, then* $\mathrm{Aut}(M)$ *is a subgroup of* $\mathrm{Aut}(\mathcal{N}(M))$ *of index 2.*
(2) *Otherwise,* $\mathrm{Aut}(\mathcal{N}(M))$ *is generated by matroid automorphisms and Cremona automorphisms.*

In Chapter 4, we apply our results to root system matroids and obtain a new proof for the isomorphism $\mathrm{Aut}(B_c(M(A_n))) \cong S_{n+2}$ for all $n \geq 3$ by showing that $M(A_n)$ has $n+1$ Cremona bases. For the other root systems, we show the following:

Theorem (4.3.3, 4.3.13). *For all $n \geq 3$, the root system matroid $M(B_n)$ has a unique Cremona basis and* $\mathrm{Aut}(B_c(M(B_n))) \cong \mathrm{Aut}(M(B_n)) \times \mathbb{Z}/2\mathbb{Z}$ *is generated by matroid automorphisms and the unique Cremona map.*

Theorem (4.4.10). *For all $n \geq 4$, the root system matroid $M(D_n)$ has no Cremona bases and* $\mathrm{Aut}(B_c(M(D_n)))$ *is isomorphic to* $\mathrm{Aut}(M(D_n))$*.*

Theorem (4.6.5). *The root system matroid $M(F_4)$ has no Cremona bases and* $\mathrm{Aut}(B_c(M(F_4)))$ *is isomorphic to* $\mathrm{Aut}(M(F_4))$*.*

CHAPTER 1

Cremona bases of matroids

1.1. Introduction

Let E be a finite set of vectors in some vector space V, let's say over the real numbers. For a set of vectors $S \subseteq E$ we define its rank as

$$\mathrm{rk}(S) := \dim(\langle S \rangle),$$

where $\langle S \rangle$ denotes the linear span of S. In this way, we obtain a function $\mathrm{rk}\colon 2^E \to \mathbb{N}_0$ on the power set 2^E that assigns a non-negative integer to every subset of E. Now let us forget the vector space V and just consider E as an abstract set equipped with the function rk. We notice that we can recover some information about the vectors in E from the rank function. For example, a subset $I \subseteq E$ was linearly independent in V if and only if $\mathrm{rk}(I) = |I|$. Moreover, since the rank function is constructed from a vector space, we can deduce that it has the following properties:

(1) The linear span of n vectors cannot have dimension larger than n. Thus $\mathrm{rk}(S) \leq |S|$ for all $S \subseteq E$.
(2) By adding vectors, the dimension of the linear span can only increase. Thus for all subsets $S \subseteq T \subseteq E$ we have $\mathrm{rk}(S) \leq \mathrm{rk}(T)$.
(3) For two subsets $S, T \subseteq E$, the linear span satisfies the relations $\langle S \cup T \rangle = \langle S \rangle + \langle T \rangle$ and $\langle S \cap T \rangle \subseteq \langle S \rangle \cap \langle T \rangle$. Thus the dimension formula for linear subspaces implies

$$\mathrm{rk}(S \cap T) \leq \dim(\langle S \rangle \cap \langle T \rangle) = \mathrm{rk}(S) + \mathrm{rk}(T) - \mathrm{rk}(S \cup T).$$

A finite set E together with a function $\mathrm{rk}\colon 2^E \to \mathbb{N}_0$ satisfying these three conditions is one of many equivalent ways to define a matroid. Conversely, one might ask: given a matroid (E, rk), can we identify E with vectors in a vector space over some field such that the function rk is induced as above from the vector space structure? If this is the case, then the matroid is called representable, but in fact, most matroids are not representable ([Nel18]). There are many other classes of matroids that arise naturally in various contexts, for example in graph theory.

This means that matroids can be used to generalize and compare notions from different areas of mathematics. Motivated by linear independence in vector spaces, we obtain a more general notion of independence by defining a subset $I \subseteq E$ of an arbitrary matroid to be independent if $\mathrm{rk}(I) = |I|$. Other basic concepts for matroids include bases, circuits, closure, flats, and connectivity, and we will summarize these definitions in Sections 1.2 and 1.4. For a detailed introduction to matroid theory we recommend the book by Oxley ([Oxl11]).

The goal of this chapter is to introduce a new class of matroids, or more precisely, a certain type of bases that we call Cremona bases. These bases have interesting properties that will help us to understand the structure of matroids where Cremona bases exist.

The following example illustrates some of the ideas in this chapter. Let $B := \{x_1, \ldots, x_{10}\}$ be a basis of a real vector space V and consider the set

$$\begin{aligned} S := \{&x_1, x_1 - x_2, x_3 + x_4, x_3 - x_4, x_5 - x_6, \\ &x_5 - x_7, x_6 - x_7, x_8 - x_9, x_8 - x_{10}, x_9 + x_{10}\}. \end{aligned}$$

What is the rank of S? We could construct the 10-by-10-matrix representing S and compute its rank, but in this case there exists another approach that we can also apply to non-representable matroids. Note that every vector in S only depends on at most two basis vectors – this is the defining property of a Cremona basis, see Section 1.3. Thus we can visualize S as a graph (with parallel edges allowed) where basis vectors are vertices and linear combinations of two basis vectors are edges:

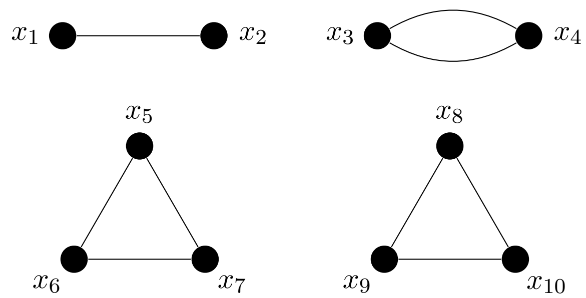

FIGURE 3. The support graph of S with respect to the Cremona basis B.

The resulting graph, which we call the support graph of S, is disconnected and we can consider each of the 4 components separately. By construction, the linear span of all elements of S that belong to the same component is contained in the coordinate subspace spanned by the vertices, so for example $\langle x_3 + x_4, x_3 - x_4 \rangle \subseteq \langle x_3, x_4 \rangle$, and in many cases these subspaces coincide. Indeed, we always have equality in a component where S contains a vertex (such as x_1) or that has parallel edges (such as $x_3 + x_4$ and $x_3 - x_4$). The other two components have identical graphs, but behave differently since the vectors $\{x_5 - x_6, x_5 - x_7, x_6 - x_7\}$ are linearly dependent and generate a non-coordinate subspace of dimension 2. Subspaces corresponding to different components are independent, hence we deduce $\mathrm{rk}(S) = 2 + 2 + 2 + 3 = 9$. We will make this precise in Section 1.5.

Since the support graph depends on the choice of a Cremona basis, we consider in Sections 1.6–1.8 the case that matroid has more than one Cremona basis. Like bases of vector spaces, but unlike bases of matroids in general, Cremona bases have the property that a change of basis is possible. Moreover, assuming a certain connectivity condition, we will prove that every simple matroid with more than one Cremona basis is representable over any sufficiently large field.

1.2. Matroids

In this section, we summarize some basic notions of matroids and define two examples of simple rank 3 matroids that we will refer to repeatedly throughout this thesis. Except for the support, all these definitions can be found in [Oxl11, Chapter 1].

Definition 1.2.1. A *matroid* M is a finite set E together with a *rank function* $\mathrm{rk}\colon 2^E \to \mathbb{N}_0$ on the power set 2^E that satisfies the following axioms:

(1) For all $S \subseteq E$ we have $\mathrm{rk}(S) \leq |S|$.
(2) For all $S, T \subseteq E$ with $S \subseteq T$ we have $\mathrm{rk}(S) \leq \mathrm{rk}(T)$.
(3) For all $S, T \subseteq E$ we have $\mathrm{rk}(S \cup T) + \mathrm{rk}(S \cap T) \leq \mathrm{rk}(S) + \mathrm{rk}(T)$.

We then say that M is a matroid on the *ground set* E. For a subset $S \subseteq E$, the number $\mathrm{rk}(S)$ is called the *rank* of S, and the *rank* of M is defined as $\mathrm{rk}(M) := \mathrm{rk}(E)$.

A subset $I \subseteq E$ is called *independent* if $\mathrm{rk}(I) = |I|$, and *dependent* otherwise. Subsets of independent sets are independent and maximal independent subsets are called *bases*. Conversely, supersets of dependent sets are dependent and minimal dependent subsets are called *circuits*. The collections of independent sets, bases, and circuits each determine the matroid uniquely and provide equivalent definitions of a matroid. If all circuits have at least three elements, then the matroid is called *simple*.

Example 1.2.2. Let V be a vector space over some field K. As in the introduction, if E is a finite set of vectors in V, then the map

$$\mathrm{rk}\colon 2^E \to \mathbb{N}_0, \quad S \mapsto \dim(\langle S \rangle)$$

defines a rank function on E and we call the resulting matroid $M(E)$ the *vector matroid* of E. Important examples are root system matroids, which we will study in Chapter 4. A subset $I \subseteq E$ is independent in $M(E)$ if and only if it is linearly independent in V.

Slightly more generally, let $(v_1, \ldots, v_n)$ be a finite sequence of vectors in V that might contain repetitions of the same vector. Then the *vector matroid* of $(v_1, \ldots, v_n)$ is the matroid on the ground set $\{1, \ldots, n\}$ with rank function

$$\mathrm{rk}\colon 2^{\{1,\ldots,n\}} \to \mathbb{N}_0, \quad S \mapsto \dim(\langle v_i \mid i \in S \rangle).$$

Example 1.2.3. The *cycle matroid* $M(G)$ of a graph G is the matroid on the set of edges of G whose independent subsets are exactly the subsets $I \subseteq E(G)$ such that the induced subgraph $G[I]$ is a forest. The corresponding rank function is given by

$$\mathrm{rk}\colon 2^{E(G)} \to \mathbb{N}_0, \quad S \mapsto |V(G[S])| - \omega(G[S]),$$

where $\omega(G[S])$ denotes the number of components of $G[S]$.

Definition 1.2.4. Let M and M' be matroids with ground sets E and E', respectively. A *matroid isomorphism* $f\colon M \to M'$ is a bijective map $f\colon E \to E'$ with $\mathrm{rk}_{M'}(f(S)) = \mathrm{rk}_M(S)$ for all $S \subseteq E$.

The *automorphisms* of a matroid M, i.e., isomorphisms $M \to M$, form a group $\mathrm{Aut}(M)$ under composition, called the *automorphism group* of M. A matroid is called *representable* if it is isomorphic to some vector matroid.

Proposition 1.2.5. *Let M be a matroid on the ground set E. Then the* closure operator

$$\mathrm{cl}\colon 2^E \to 2^E, \quad S \mapsto \{e \in E \mid \mathrm{rk}(S \cup \{e\}) = \mathrm{rk}(S)\}$$

has the following properties:

(1) *For all $S, T \subseteq E$ with $S \subseteq T$ we have* $\mathrm{cl}(S) \subseteq \mathrm{cl}(T)$.
(2) *For all $S \subseteq E$ we have $S \subseteq \mathrm{cl}(S)$ and* $\mathrm{cl}(S) = \mathrm{cl}(\mathrm{cl}(S))$.
(3) *For all $S \subseteq E$ we have* $\mathrm{rk}(\mathrm{cl}(S)) = \mathrm{rk}(S)$.

A subset $S \subseteq E$ is called *spanning* if $\mathrm{cl}(S) = E$, which is equivalent to $\mathrm{rk}(S) = \mathrm{rk}(E)$. A basic fact about matroids is that all bases are spanning and thus have the same number of elements.

Another important way to describe matroids is by their flats. A *flat* of M is a subset $F \subseteq E$ with $\mathrm{cl}(F) = F$. If F and F' are flats of M, then their *join* $F \vee F' := \mathrm{cl}(F \cup F')$ and their intersection $F \cap F'$ are flats as well since $\mathrm{cl}(F \cap F') \subseteq \mathrm{cl}(F) \cap \mathrm{cl}(F') = F \cap F'$. Together with these operations, the partially ordered set of flats forms a lattice, called the *lattice of flats* $\mathcal{L}(M)$ of M.

If M is the vector matroid of some set of vectors E, then the closure $\mathrm{cl}(S)$ of a set $S \subseteq E$ is equal to the intersection of E with the linear span $\langle S \rangle$, hence the flats of $M(E)$ correspond bijectively to the linear subspaces of V that can be spanned by subsets of E.

Every matroid has the trivial flats $\mathrm{cl}(\emptyset)$, the unique flat of rank 0, and E, the unique flat of rank $\mathrm{rk}(M)$. Flats of rank $\mathrm{rk}(M) - 1$ are also called *hyperplanes*. In a simple matroid, the rank 1 flats are exactly the singletons $\{e\}$ with $e \in E$ and we have $\mathrm{cl}(\emptyset) = \emptyset$. Therefore, a simple matroid of rank 3 is uniquely determined by its hyperplanes, and in this case we will call a hyperplane *connected* if it has more than two elements (see Section 1.4 for the general definition of connectedness).

Example 1.2.6. There exists a unique simple rank 3 matroid M_1 on the ground set $\{1, \ldots, 5\}$ whose connected hyperplanes are $\{1, 2, 4\}$ and $\{1, 3, 5\}$. Indeed, M_1 is isomorphic to the vector matroid of

$$E := \{x_1, x_2, x_3, x_1 - x_2, x_1 - x_3\},$$

where x_1, x_2, x_3 denotes the standard basis of $\mathbb{R}^3$.

We can visualize simple matroids of rank 3 by their *geometric representation*, that is, for each element of the ground set we draw a point in the affine

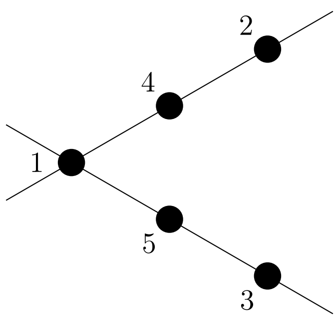

FIGURE 4. The geometric representation of M_1.

plane and for each connected hyperplane we draw a line or a circle through all its points.

Every automorphism of M_1 must fix the element 1, since it is the only element that is contained in two connected hyperplanes. In particular, since some bases contain 1 and some do not, the automorphism group of M_1 does not act transitively on the set of bases. A permutation of $\{2,3,4,5\}$ induces an automorphism of M_1 if and only if it preserves the partition $\{\{2,4\},\{3,5\}\}$, hence the automorphism group of M_1 is isomorphic to the dihedral group D_4 of symmetries of a square (imagine $\{2,4\}$ and $\{3,5\}$ being the diagonals).

Example 1.2.7. There exists a unique simple rank 3 matroid M_2 on the ground set $\{1,\dots,7\}$ whose connected hyperplanes are $\{1,2,4,5\}$, $\{1,3,6\}$, $\{2,3,7\}$, and $\{4,6,7\}$. Indeed, M_2 is isomorphic to the vector matroid of

$$E := \{x_1, x_2, x_3, x_1 - x_2, x_1 + x_2, x_1 - x_3, x_2 - x_3\},$$

where x_1, x_2, x_3 denotes the standard basis of $\mathbb{R}^3$.

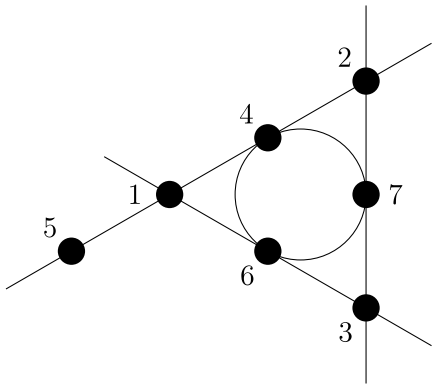

FIGURE 5. The geometric representation of M_2.

Again, the automorphism group of M_2 does not act transitively on the set of bases of M_2 since every automorphism of M_2 fixes the element 5 and thus induces a permutation of $\{1,2,4\}$. Conversely, every permutation of $\{1,2,4\}$ can be extended uniquely to an automorphism of M_2, hence $\mathrm{Aut}(M_2)$ is isomorphic to the symmetric group S_3.

We will continue these examples in Section 1.3 by determining the Cremona bases and the support graph of these matroids. For the rest of this section, let I be an independent set of a matroid M.

Proposition 1.2.8. *For every subset $S \subseteq \mathrm{cl}(I)$, the collection $\mathcal{I} := \{I' \subseteq I \mid S \subseteq \mathrm{cl}(I')\}$ has a unique minimal element $\mathrm{supp}_I(S)$.*

Proof. By assumption, $\mathcal{I}$ is non-empty, so it suffices to show that $\mathcal{I}$ is closed under intersection. Let $I', I'' \subseteq I$ with $S \subseteq \mathrm{cl}(I')$ and $S \subseteq \mathrm{cl}(I'')$. As subsets of the independent set I, the sets I' and I'' as well as their intersection $I' \cap I''$ and their union $I' \cup I''$ are all independent. Since $S \subseteq \mathrm{cl}(I' \cup I'')$, we have

$$\begin{aligned}
\mathrm{rk}((I' \cap I'') \cup S) &= \mathrm{rk}((I' \cup S) \cap (I'' \cup S)) \\
&\leq \mathrm{rk}(I' \cup S) + \mathrm{rk}(I'' \cup S) - \mathrm{rk}(I' \cup I'' \cup S) \\
&= |I'| + |I''| - |I' \cup I''| = |I' \cap I''| = \mathrm{rk}(I' \cap I'')
\end{aligned}$$

and thus $I' \cap I'' \in \mathcal{I}$. $\square$

We call $\mathrm{supp}_I(S)$ the *support* of S with respect to I. For a single element $e \in \mathrm{cl}(I)$, we also write $\mathrm{supp}_I(e)$ instead of $\mathrm{supp}_I(\{e\})$. Clearly, $\mathrm{supp}_I(e) = \{e\}$ if $e \in I$.

Example 1.2.9. Let E be a finite set of vectors in a vector space V over some field K and assume that E contains a basis $B := \{x_1, \ldots, x_n\}$ of V. Then B is also a basis of the associated matroid $M(E)$ and for every vector $v \in E$ we have

$$\mathrm{supp}_B(v) = \{x_i \mid a_i \neq 0\},$$

where the scalars $a_1, \ldots, a_n \in K$ are the coefficients in the unique expression $v = \sum_{i=1}^n a_i x_i$ of v as linear combination of the basis vectors.

Corollary 1.2.10. *If $e \in \mathrm{cl}(I) \setminus I$, then $C_I(e) := \mathrm{supp}_I(e) \cup \{e\}$ is the unique circuit of M with $C_I(e) \subseteq I \cup \{e\}$, called the* fundamental circuit *of e with respect to I.*

Proof. If $e \notin I$, then the map $I' \mapsto I' \cup \{e\}$ is an order-preserving bijection between the sets $I' \subseteq I$ with $e \in \mathrm{cl}(I')$ and the dependent subsets of $I \cup \{e\}$. Now the claim follows from Proposition 1.2.8. $\square$

Corollary 1.2.11. *Assume that M is simple.*

(1) *For all $e \in \mathrm{cl}(I) \setminus I$ we have $|\mathrm{supp}_I(e)| \geq 2$.*
(2) *For all $I' \subseteq I$ with $|I| = 2$ we have $\{e \in \mathrm{cl}(I) \mid \mathrm{supp}_I(e) = I'\} = \mathrm{cl}(I') \setminus I'$.*

Proof. Follows from Corollary 1.2.10 since every circuit of a simple matroid has size at least 3. $\square$

The following properties of the support will be used frequently:

Proposition 1.2.12. *Let $S \subseteq \mathrm{cl}(I)$ be a subset.*

(1) $\mathrm{supp}_I(S) = \bigcup_{e \in S} \mathrm{supp}_I(e)$.
(2) $\mathrm{supp}_I(\mathrm{cl}(S)) = \mathrm{supp}_I(S)$.
(3) $\mathrm{rk}(S) \leq |\mathrm{supp}_I(S)|$.

Proof. For all $I' \subseteq I$ we have

$$\begin{aligned} S \subseteq \mathrm{cl}(I') &\iff \forall e \in S : e \in \mathrm{cl}(I') \iff \forall e \in S : \mathrm{supp}_I(e) \subseteq I' \\ &\iff \bigcup_{e \in I} \mathrm{supp}_I(e) \subseteq I', \end{aligned}$$

thus the minimality of the support implies $\mathrm{supp}_I(S) = \bigcup_{e \in I} \mathrm{supp}_I(e)$. In particular, we deduce $\mathrm{supp}_I(S) \subseteq \mathrm{supp}_I(\mathrm{cl}(S))$. Conversely, the inclusion $S \subseteq \mathrm{cl}(\mathrm{supp}_I(S))$ implies $\mathrm{rk}(S) \leq \mathrm{rk}(\mathrm{cl}(\mathrm{supp}_I(S))) = |\mathrm{supp}_I(S)|$ and $\mathrm{cl}(S) \subseteq \mathrm{cl}(\mathrm{supp}_I(S))$, hence $\mathrm{supp}_I(\mathrm{cl}(S)) \subseteq \mathrm{supp}_I(S)$. $\square$

1.3. Cremona bases and support graphs

Let M be a matroid on the ground set E.

Definition 1.3.1. A basis B of M is called a *Cremona basis* of M if

$$\bigcup_{b,b'\in B} \mathrm{cl}(\{b,b'\}) = E.$$

In other words, the rank 2 flats generated by the elements of a Cremona basis cover the ground set.

The following proposition shows that this definition is equivalent to the condition in [SW23, Theorem 8.3] for a simple matroid.

Proposition 1.3.2. *If M is simple, then the following are equivalent for a basis B of M:*

(1) *B is a Cremona basis of M.*
(2) *$|\mathrm{supp}_B(e)| = 2$ for all $e \in E \setminus B$.*
(3) *The sets $\{\mathrm{cl}(\{b,b'\}) \setminus \{b,b'\}\}_{b,b'\in B}$ form a partition of $E \setminus B$.*

We use the convention that the empty set is allowed as an element of a partition.

Proof. Assume that B is a Cremona basis of M and let $e \in E \setminus B$. Then there exist $b, b' \in B$ with $e \in \mathrm{cl}(\{b,b'\})$, thus $\mathrm{supp}_B(e) = \{b,b'\}$ and $b \neq b'$ since M is simple (Corollary 1.2.11).

For any basis B, the equivalence classes with respect to the relation

$$e \sim e' :\Longleftrightarrow \mathrm{supp}_B(e) = \mathrm{supp}_B(e')$$

form a partition of $E \setminus B$. Condition (2) and Corollary 1.2.11 imply that these equivalence classes are exactly the non-empty sets of the form $\mathrm{cl}(\{b,b'\}) \setminus \{b,b'\}$ with $b, b' \in B$. (Note that $\mathrm{cl}(\{b,b'\}) \setminus \{b,b'\} = \emptyset$ if $b = b'$.)

If B is a basis with property (3), then $E \setminus B \subseteq \bigcup_{b,b'\in B} \mathrm{cl}(\{b,b'\})$, hence B is a Cremona basis of M. ☐

In particular, if a simple matroid has a Cremona basis, then knowing the maximal size of the rank 2 flats gives an upper bound on the cardinality of the ground set. This will be an important criterion to show that a given matroid does not admit a Cremona basis.

Corollary 1.3.3. *If M is simple and has a Cremona basis, then*

$$|E| \leq \mathrm{rk}(M) + \binom{\mathrm{rk}(M)}{2}\big(\max_{\mathrm{rk}(F)=2} |F| - 2\big).$$

Proof. We use the notation $\binom{B}{2}$ for the collection of subsets $I \subseteq B$ with $|I| = 2$. If B is a Cremona basis of M, then Proposition 1.3.2 (3) implies

$$\begin{aligned} |E| = |B| + |E \setminus B| &= \mathrm{rk}(M) + \sum_{\{b,b'\}\in\binom{B}{2}} |\mathrm{cl}(\{b,b'\}) \setminus \{b,b'\}| \\ &= \mathrm{rk}(M) + \sum_{\{b,b'\}\in\binom{B}{2}} (|\mathrm{cl}(\{b,b'\})| - 2) \\ &\leq \mathrm{rk}(M) + \binom{\mathrm{rk}(M)}{2}\big(\max_{\mathrm{rk}(F)=2} |F| - 2\big). \end{aligned}$$

☐

Property (2) in Proposition 1.3.2 allows us to visualize simple matroids with a Cremona basis as graphs, where vertices correspond to elements of the Cremona basis and edges correspond to the remaining elements of the ground set.

Definition 1.3.4. Assume that M is simple and has a Cremona basis B. The *support graph* $G_B(S)$ of a subset $S \subseteq E$ with respect to B is defined as the graph with vertices $\mathrm{supp}_B(S)$ and edges $S \setminus B$, where the endpoints of an edge e are the vertices in $\mathrm{supp}_B(e)$.

Note that we allow parallel edges in the support graph. We also write $G_B(M)$ instead of $G_B(E)$ and call it the *support graph* of the matroid M. If $S' \subseteq S \subseteq E$, then $G_B(S')$ is a subgraph of $G_B(S)$. Note that the support graph of a subset $S \subseteq E$ is the smallest subgraph of $G_B(M)$ with $S \subseteq V(G_B(S)) \cup E(G_B(S))$.

Example 1.3.5. Let M_1 be the matroid from Example 1.2.6. Out of the 8 bases of M_1, exactly the 4 bases containing the element 1 are Cremona bases:

- $\{1, 2, 3\}$ is a Cremona basis since $4 \in \mathrm{cl}(\{1, 2\})$ and $5 \in \mathrm{cl}(\{1, 3\})$.
- $\{1, 2, 5\}$ is a Cremona basis since $3 \in \mathrm{cl}(\{1, 5\})$ and $4 \in \mathrm{cl}(\{1, 2\})$.
- $\{1, 3, 4\}$ is a Cremona basis since $2 \in \mathrm{cl}(\{1, 4\})$ and $5 \in \mathrm{cl}(\{1, 3\})$.
- $\{1, 4, 5\}$ is a Cremona basis since $2 \in \mathrm{cl}(\{1, 4\})$ and $3 \in \mathrm{cl}(\{1, 5\})$.
- $\{2, 3, 4\}$ is not a Cremona basis since $|\mathrm{supp}_{\{2,3,4\}}(5)| = 3$.
- $\{2, 3, 5\}$ is not a Cremona basis since $|\mathrm{supp}_{\{2,3,5\}}(4)| = 3$.
- $\{2, 4, 5\}$ is not a Cremona basis since $|\mathrm{supp}_{\{2,4,5\}}(3)| = 3$.
- $\{3, 4, 5\}$ is not a Cremona basis since $|\mathrm{supp}_{\{3,4,5\}}(2)| = 3$.

The support graph $G_B(M_1)$ with respect to the basis $B := \{1, 2, 3\}$ is a path of length 2:

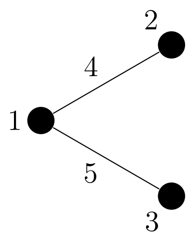

FIGURE 6. The support graph $G_B(M_1)$.

The automorphism group $\mathrm{Aut}(M_1) \cong D_4$ acts transitively on the set of Cremona bases. We will prove in Section 1.7 that this is always the case, which implies that the support graph $G_B(M_1)$ of M_1 is independent of the chosen Cremona basis B up to isomorphism.

Example 1.3.6. Let M_2 be the matroid from Example 1.2.7. As we will check later in Example 1.7.5, M_2 has exactly 3 Cremona bases, $\{1, 2, 3\}$, $\{1, 4, 6\}$, and $\{2, 4, 7\}$. The support graph $G_B(M_1)$ with respect to the basis $B := \{1, 2, 3\}$ is a triangle with one pair of parallel edges.

Note that for every two distinct Cremona bases B and B' of M_2 we have $|B \cap B'| = 1$. This will be a special case of the star property in Section 1.6.

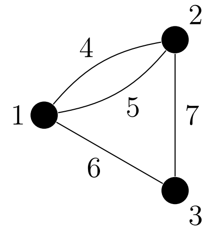

FIGURE 7. The support graph $G_B(M_2)$.

Example 1.3.7 ([Oxl11, Example 1.2.7])**.** Let m and n be non-negative integers with $m \leq n$. The *uniform matroid* $U_{m,n}$ is the rank m matroid on the ground set $\{1, \ldots, n\}$ with rank function $\mathrm{rk}(S) := \min\{|S|, m\}$ for all $S \subseteq \{1, \ldots, n\}$. Every permutation of the ground set induces a matroid automorphism, thus $\mathrm{Aut}(U_{m,n})$ is the symmetric group S_n.

If $m \geq 3$, then $\bigcup_{b,b' \in B} \mathrm{cl}(\{b, b'\}) = B$ for every basis B, thus $U_{m,n}$ has a Cremona basis if and only if $m = n$. Otherwise, if $m \leq 2$, then every basis is a Cremona basis. Thus $U_{3,4}$ is the smallest matroid with no Cremona bases.

Note that every simple matroid of rank 2 or less is isomorphic to a uniform matroid and thus there is no difference between Cremona bases and normal bases. Therefore, we are mainly interested in matroids of rank at least 3.

Example 1.3.8. Let E be a finite set of vectors in a vector space V over some field K and assume that E contains a basis $B := \{x_1, \ldots, x_n\}$ of V. By Example 1.2.9, B is a Cremona basis of the matroid $M(E)$ if and only if every vector $v \in E$ is of the form $v = ax_i + bx_j$ for some $a, b \in K$ and $i, j \in \{1, \ldots, n\}$.

Example 1.3.9. Let G be a finite group and $r \geq 3$ an integer. The *Dowling matroid* $Q_r(G)$ is a certain simple matroid on the ground set

$$\{1, \ldots, r\} \cup \{(g, i, j) \mid g \in G,\ 1 \leq i < j \leq r\}$$

such that $\mathrm{cl}(\{i, j\}) = \{i, j\} \cup \{(g, i, j) \mid g \in G\}$ for all $1 \leq i < j \leq r$. Thus $B := \{1, \ldots, r\}$ is a Cremona basis of $Q_r(G)$ and the support graph $G_B(Q_r(G))$ is a complete graph with r vertices and $|G|$ edges joining any given pair of vertices. There are also rank 2 circuits of non-basis elements that are determined by the group structure as follows: For all $g, h \in G$ and all $1 \leq i < j < k \leq r$ the set $\{(g, i, j), (h, i, k), (gh, i, k)\}$ is a circuit of $Q_r(G)$.

If $|G| \geq 2$, then B is the only Cremona basis of $Q_r(G)$, as will follow from Lemma 1.6.7. For more information about Dowling matroids, see [Oxl11, Section 6.10]. In fact, in order to describe these matroids, Oxley uses a graph that coincides with the support graph $G_B(Q_r(G))$ up to the addition of loops.

There are two cases that we will consider in more detail in Chapter 4: If the group G is trivial, then $Q_r(G)$ is isomorphic to the matroid associated to the root system A_r, and if G is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, then $Q_r(G)$ is isomorphic to the matroid associated to the root system B_r.

By [Oxl11, Theorem 6.10.10], the Dowling matroid $Q_r(G)$ is representable over a field K if and only if G is isomorphic to a subgroup of $K^\times$. The Klein four-group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ cannot be embedded into the group of units of any field, since it contains three elements of order 2, but the polynomial $X^2 - 1$ has at most two roots over any field. Thus the Dowling matroid

$Q_r(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ is an example for a non-representable matroid with a Cremona basis.

Example 1.3.10. Let M be the simple rank 3 matroid associated to a finite non-degenerate projective plane of order $k \geq 2$. Then M has $k^2 + k + 1$ elements and every rank 2 flat has cardinality $k + 1$. If M had a Cremona basis, then Corollary 1.3.3 would imply

$$k^2 + k + 1 \leq 3 + \binom{3}{2}(k - 1) = 3k.$$

But for all $k \geq 2$ we have $k^2 + k + 1 \geq 3k + 1$.

1.4. Connectivity

Motivated by graph theory, another important concept in matroid theory is connectivity, see [Oxl11, Chapter 4]. Usually, the connectedness of a matroid is defined using circuits, but instead we choose an equivalent approach using separators (cf. [Oxl11, Proposition 4.2.1]). This has the advantage that it is easier to compare with the notion of support-connectedness that we introduce in the second part of this section.

Let M be a simple matroid on the ground set E.

Definition 1.4.1. Let $S \subseteq E$ be a subset. A *separator* of S is a subset $T \subseteq S$ such that

$$\operatorname{rk}(S) = \operatorname{rk}(T) + \operatorname{rk}(S \setminus T).$$

The *components* of S are the minimal non-empty separators of S. We call S *connected* if it has exactly one component, and *disconnected* otherwise.

Note that by our definition the empty set is not connected. Separators of E are also called separators of M, and the matroid M is called connected if E is connected. Note that for every subset $T \subseteq S$ we have $\operatorname{rk}(S) \leq \operatorname{rk}(T)+\operatorname{rk}(S\setminus T)$ by the axioms of the rank function. Every set S has the two trivial separators $\emptyset$ and S itself. One can show that the collection of separators of S is closed under union, intersection, and complement, hence the components of S form a partition of S. If T is a separator of S, then every separator of T is also a separator of S, thus the components of S are connected. Moreover, if T is a separator of S, then $\operatorname{cl}(T) \cap \operatorname{cl}(S \setminus T) = \emptyset$ since

$$\begin{aligned}\operatorname{rk}(\operatorname{cl}(T) \cap \operatorname{cl}(S \setminus T)) &\leq \operatorname{rk}(\operatorname{cl}(T)) + \operatorname{rk}(\operatorname{cl}(S \setminus T)) - \operatorname{rk}(\operatorname{cl}(T) \cup \operatorname{cl}(S \setminus T)) \\ &= \operatorname{rk}(T) + \operatorname{rk}(S \setminus T) - \operatorname{rk}(S) = 0.\end{aligned}$$

In particular, we have $\operatorname{cl}(T) \cap S = T$, hence separators of flats are flats.

Lemma 1.4.2. *Let $S' \subseteq S \subseteq E$ be subsets. If T is a separator of S, then $T' := T \cap S'$ is a separator of S'.*

Proof. Using that T is a separator of S, we compute

$$\begin{aligned}\operatorname{rk}(T') + \operatorname{rk}(S' \setminus T') &= \operatorname{rk}(T \cap S') + \operatorname{rk}((S \setminus T) \cap S') \\ &\leq \operatorname{rk}(T) + \operatorname{rk}(S') - \operatorname{rk}(T \cup S') \\ &\quad + \operatorname{rk}(S \setminus T) + \operatorname{rk}(S') - \operatorname{rk}((S \setminus T) \cup S') \\ &= 2\operatorname{rk}(S') + \operatorname{rk}(S) - \operatorname{rk}(T \cup S') - \operatorname{rk}((S \setminus T) \cup S') \\ &\leq \operatorname{rk}(S').\end{aligned}$$

□

This implies that every connected subset of S is contained in a component of S, hence the components of S are exactly the maximal connected subsets of S. In particular, the union of two connected sets S and S' with $S \cap S' \neq \emptyset$ is again connected.

Corollary 1.4.3. *Let $T_1, \ldots, T_k$ be separators of a subset $S \subseteq E$ that form a partition of S. Then $\operatorname{rk}(S) = \sum_{i=1}^{k} \operatorname{rk}(T_i)$.*

Proof. Clear for $k \leq 2$. If $k \geq 3$, then $\operatorname{rk}(S) = \operatorname{rk}(\bigcup_{i=1}^{k-1} T_i) + \operatorname{rk}(T_i)$ and $T_1, \ldots, T_{k-1}$ are separators of $\bigcup_{i=1}^{k-1} T_i$ by Lemma 1.4.2, hence the claim follows by induction on k. □

Lemma 1.4.4. *A basis B of M is a Cremona basis if and only if $B \cap S$ is a Cremona basis of $M|_S$ for every component S of M.*

Proof. First assume B is a Cremona basis of M. Let S be a component of M and let $e \in S \setminus B$. Since B is a Cremona basis of M, there exist $b, b' \in B$ with $e \in \mathrm{cl}(\{b, b'\})$, and it suffices to show that $b, b' \in B \cap S$. If $b, b' \in B \setminus S$, then $e \in \mathrm{cl}(\{b, b'\}) \subseteq E \setminus S$ since $E \setminus S$ is a flat of M, contradiction. Thus we may assume $b \in B \cap S$ and then we deduce $b' \in \mathrm{cl}(\{b, e\}) \subseteq S$ since S is a flat of M.

Conversely, let $e \in E \setminus B$ and assume that $B \cap S$ is a Cremona basis of $M|_S$, where S is the component of M containing e. Then there exist $b, b' \in B \cap S \subseteq B$ with $e \in \mathrm{cl}(\{b, b'\})$. $\square$

Assumption. *For the rest of this section, assume that M has a Cremona basis B in the sense of Definition 1.3.1.*

The following property of subsets of E with respect to the Cremona basis B will be useful.

Definition 1.4.5. Let $S \subseteq E$ be a subset. A *support separator* of S is a subset $T \subseteq S$ such that

$$\mathrm{supp}_B(T) \cap \mathrm{supp}_B(S \setminus T) = \emptyset.$$

The *support components* of S are the minimal non-empty support separators of S, and S is called *support-connected* if it has exactly one support component.

Note that these definitions depend on the choice of the Cremona basis B. Like separators, Proposition 1.2.12 shows that the collection of support separators is closed under union, intersection, and complement, hence the support components of S form a partition of S.

Lemma 1.4.6. *The support components of a subset $S \subseteq E$ correspond bijectively to the components of its support graph $G_B(S)$. In particular, S is support-connected if and only if $G_B(S)$ is connected.*

Proof. If T is a support separator of S, then $G_B(S)$ is the disjoint union of $G_B(T)$ and $G_B(S \setminus T)$, thus $G_B(T)$ is the subgraph induced by the vertices $\mathrm{supp}_B(T)$. Conversely, if vertices $I \subseteq \mathrm{supp}_B(S)$ form a union of components of $G_B(S)$, then

$$S = (S \cap \mathrm{cl}(I)) \cup (S \cap \mathrm{cl}(B \setminus I)),$$

thus $T := S \cap \mathrm{cl}(I)$ is a support separator of S. These constructions are inverse to each other and order-preserving, hence the support components of S correspond to the components of $G_B(S)$. $\square$

This also implies that every support-connected subset of a set $S \subseteq E$ is contained in a support component of S. The next proposition shows that support-connectivity is weaker than connectivity.

Proposition 1.4.7. *Let $S \subseteq E$. Every support separator of S is a separator of S. In particular, if S is connected, then S is support-connected.*

Proof. For every subset $T \subseteq S$ we have

$$S = T \cup (S \setminus T) \subseteq \mathrm{cl}(\mathrm{supp}_B(T)) \cup \mathrm{cl}(\mathrm{supp}_B(S \setminus T)) =: S'.$$

Assume that T is a support separator of S. We claim that $T' := \mathrm{cl}(\mathrm{supp}_B(T))$ is a separator of S'. First note that $\mathrm{cl}(\mathrm{supp}_B(S \setminus T))$ is the complement of T' in S'. Indeed, if $e \in T' \cap \mathrm{cl}(\mathrm{supp}_B(S \setminus T))$, then we have $\mathrm{supp}_B(e) \subseteq \mathrm{supp}_B(T) \cap \mathrm{supp}_B(S \setminus T) = \emptyset$, contradicting the assumption that M is simple. Thus

$$\begin{aligned}\mathrm{rk}(T') + \mathrm{rk}(S' \setminus T') &= |\,\mathrm{supp}_B(T)| + |\,\mathrm{supp}_B(S \setminus T)| \\ &= |\,\mathrm{supp}_B(T) \cup \mathrm{supp}_B(S \setminus T)| = \mathrm{rk}(S').\end{aligned}$$

Now Lemma 1.4.2 implies that $T' \cap S$ is a separator of S. Since $T \subseteq T'$ and $S \setminus T \subseteq S' \setminus T'$, we conclude $T' \cap S = T$. □

Corollary 1.4.8. *Let $S \subseteq E$. Then $\mathrm{rk}(S) = \sum_{i=1}^{k} \mathrm{rk}(S_i)$, where $S_1, \ldots, S_k$ are the support components of S.*

Proof. Follows from Proposition 1.4.7 and Corollary 1.4.3. □

Proposition 1.4.9. *Let $S \subseteq E$. If S is support-connected, then $\mathrm{cl}(S)$ is also support-connected.*

Proof. Assume that S is support-connected and let T be a support separator of $\mathrm{cl}(S)$. We may assume $S \cap T \neq \emptyset$, otherwise replace T by $\mathrm{cl}(S) \setminus T$. Then $S \cap T$ is a support separator of S since $\mathrm{supp}_B(S \cap T) \subseteq \mathrm{supp}_B(T)$ and $\mathrm{supp}_B(S \setminus T) \subseteq \mathrm{supp}_B(\mathrm{cl}(S) \setminus T)$, so we deduce $S \subseteq T \subseteq \mathrm{cl}(S)$ and thus $\mathrm{supp}_B(T) = \mathrm{supp}_B(\mathrm{cl}(S))$ by Proposition 1.2.12. We conclude that $\mathrm{supp}_B(\mathrm{cl}(S) \setminus T) = \emptyset$ and hence $T = \mathrm{cl}(S)$ since M is simple. □

Lemma 1.4.10. *Every separator of M is a support separator of M. In particular, M is connected if and only if $G_B(M)$ is connected.*

Proof. For a separator S of M let $I_1 := S \cap B$ and $I_2 := B \setminus S$. Then $S \subseteq \mathrm{cl}(I_1)$ and $E \setminus S \subseteq \mathrm{cl}(I_2)$ since

$$\mathrm{rk}(E) = |B| = |I_1| + |I_2| = \mathrm{rk}(I_1) + \mathrm{rk}(I_2) \leq \mathrm{rk}(S) + \mathrm{rk}(E \setminus S) = \mathrm{rk}(E).$$

Hence $\mathrm{supp}_B(S) \cap \mathrm{supp}_B(E \setminus S) \subseteq I_1 \cap I_2 = \emptyset$. □

Definition 1.4.11. Let $T \subseteq E$. Then the *contraction* M/T is the matroid on the ground set $E \setminus T$ with rank function

$$\mathrm{rk}_{M/T}(S) := \mathrm{rk}_M(S \cup T) - \mathrm{rk}_M(T)$$

for all $S \subseteq E \setminus T$.

Lemma 1.4.12. *Assume that M is connected.*

(1) *For all $e \in E \setminus B$, the contraction M/e is connected.*
(2) *For all $b \in B$, the contraction M/b is connected if and only if the deletion $G_B(M) - b$ is connected.*

Proof. For the proof of (1), let $e \in E \setminus B$ and let $S \subseteq E \setminus \{e\}$ be a separator of M/e. We may assume $S \cap B \neq \emptyset$, else replace S by its complement in $E \setminus \{e\}$.

Claim: If $b, b' \in B$ are adjacent in $G_B(M)$ and $b \in S$, then also $b' \in S$. Indeed, if $\mathrm{supp}_B(e) = \{b, b'\}$, then $\{b, b'\}$ is a circuit of M/e, hence $b' \in S$ since S is a flat of M/e. Otherwise, by assumption there exists $e' \in E \setminus B$ with $\mathrm{supp}_B(e') = \{b, b'\}$. Then $C := \{b, b', e'\}$ is a circuit of M with $e \notin \mathrm{cl}_M(C)$, thus C is also a circuit of M/e with $C \cap S \neq \emptyset$. Since S is a separator of M/e, we deduce $C \subseteq S$.

Hence $B \subseteq S$ since $G_B(M)$ is connected by Lemma 1.4.10, and we conclude $\mathrm{cl}(S) = E \setminus \{e\}$.

For (2), let $I \subseteq B \setminus \{b\}$ be a set of vertices that form a component of $G_B(M) - b$. Then $E \subseteq \mathrm{cl}(I \cup \{b\}) \cup \mathrm{cl}(B \setminus I)$ and $\mathrm{cl}(I \cup \{b\}) \cap \mathrm{cl}(B \setminus I) = \{b\}$. This implies

$$\begin{aligned}\mathrm{rk}(M/b) &= |I| + 1 + |B| - |I| - 2 \\ &= \mathrm{rk}_M(\mathrm{cl}(I \cup \{b\})) + \mathrm{rk}_M(\mathrm{cl}(B \setminus I)) - 2 \\ &= \mathrm{rk}_{M/b}(\mathrm{cl}(I \cup \{b\}) \setminus \{b\}) + \mathrm{rk}_{M/b}(\mathrm{cl}(B \setminus I)),\end{aligned}$$

hence $\mathrm{cl}(I \cup \{b\}) \setminus \{b\}$ is a separator of M/b.

Conversely, let S be a separator of M/b. Then

$$\begin{aligned}|B| - 1 &= |S \cap B| + |(B \setminus \{b\}) \setminus S| \\ &\leq \mathrm{rk}_{M/b}(S) + \mathrm{rk}_{M/b}((E \setminus \{b\}) \setminus S) = \mathrm{rk}(E \setminus \{b\}) = |B| - 1,\end{aligned}$$

thus $S = \mathrm{cl}_{M/b}(S \cap B)$ and $(E \setminus \{b\}) \setminus S = \mathrm{cl}_{M/b}((B \setminus \{b\}) \setminus S)$. This implies $E \subseteq \mathrm{cl}((S \cap B) \cup \{b\}) \cup \mathrm{cl}(B \setminus S)$, hence the vertices $S \cap B$ form a component of $G_B(M) - b$. □

Corollary 1.4.13. *$G_B(M)$ is 2-connected if and only if M is connected and M/e is connected for all $e \in E$.*

1.5. Coordinate and non-coordinate flats

In this section, we describe the flats and the rank function of matroids that have a Cremona basis. Throughout this section, let M be a simple matroid on the ground set E that admits a Cremona basis B.

Definition 1.5.1. Let F be a flat of M.

(1) F is called a *coordinate flat* if $F \cap B = \operatorname{supp}_B(F)$.
(2) F is called a *non-coordinate flat* if $F \cap B = \emptyset \neq F$.

This definition depends on the chosen Cremona basis, so we will also say that a flat F is a (non-)coordinate flat with respect to B. Note that every non-empty subflat of a non-coordinate flat is a non-coordinate flat.

Theorem 1.5.2. *Every support-connected flat of M is either a coordinate flat or a non-coordinate flat.*

Proof. Let F be a support-connected flat of M. If $e \in F \setminus B$ is an edge joining two vertices $b, b' \in \operatorname{supp}_B(F)$ such that $b \in F$, then $\{b, b', e\}$ is a circuit, hence

$$b' \in \operatorname{cl}(\{b, e\}) \subseteq \operatorname{cl}(F) = F.$$

This means that the vertices $F \cap B$ form a (possibly empty) union of components of $G_B(F)$. But $G_B(F)$ is connected by Lemma 1.4.6, so we have either $F \cap B = \operatorname{supp}_B(F)$ or $F \cap B = \emptyset$. □

Corollary 1.5.3. *Let $S \subseteq E$ be support-connected. If $S \cap B \neq \emptyset$, then* $\operatorname{cl}(S)$ *is a coordinate flat.*

Proof. The closure $\operatorname{cl}(S)$ is also support-connected by Proposition 1.4.9 and cannot be a non-coordinate flat since $\operatorname{cl}(S) \cap B \supseteq S \cap B \neq \emptyset$, hence $\operatorname{cl}(S)$ is a coordinate flat by Theorem 1.5.2. □

Proposition 1.5.4. *The coordinate flats of M correspond bijectively to the subsets of B.*

Proof. If I is a subset of B, then $\operatorname{cl}(I) \cap B = I$ and

$$\operatorname{supp}_B(\operatorname{cl}(I)) = \operatorname{supp}_B(I) = \bigcup_{e \in I} \operatorname{supp}_B(e) = \bigcup_{e \in I} \{e\} = I$$

by Proposition 1.2.12, hence $F := \operatorname{cl}(I)$ is a coordinate flat. Conversely, if F is a coordinate flat, then $F = \operatorname{cl}(I)$ for $I := \operatorname{supp}_B(F)$. Indeed, by Proposition 1.2.12 we have

$$F \subseteq \operatorname{cl}(\operatorname{supp}_B(F)) = \operatorname{cl}(F \cap B) \subseteq \operatorname{cl}(F) = F$$

and hence equality throughout. These constructions are inverse to each other. □

Theorem 1.5.5. *Let $S \subseteq E$ be a subset.*

(1) $\operatorname{cl}(S)$ *is a coordinate flat if and only if* $\operatorname{rk}(S) = |\operatorname{supp}_B(S)|$.
(2) *If* $\operatorname{cl}(S)$ *is a support-connected non-coordinate flat, then* $\operatorname{rk}(S) = |\operatorname{supp}_B(S)| - 1$.

Proof. If $\mathrm{cl}(S)$ is a coordinate flat, then $\mathrm{cl}(S) = \mathrm{cl}(\mathrm{supp}_B(\mathrm{cl}(S)))$ by Proposition 1.5.4 and hence

$$\mathrm{rk}(S) = \mathrm{rk}(\mathrm{cl}(S)) = \mathrm{rk}(\mathrm{cl}(\mathrm{supp}_B(\mathrm{cl}(S)))) = |\,\mathrm{supp}_B(\mathrm{cl}(S))| = |\,\mathrm{supp}_B(S)|.$$

Conversely, if $\mathrm{rk}(\mathrm{cl}(S)) = \mathrm{rk}(S) = |\,\mathrm{supp}_B(S)|$, then the inclusion of flats $\mathrm{cl}(S) \subseteq \mathrm{cl}(\mathrm{supp}_B(S))$ is an equality.

Now assume that $\mathrm{cl}(S)$ is a support-connected non-coordinate flat. We choose a vertex $b \in \mathrm{supp}_B(\mathrm{cl}(S))$ and consider the flat $F := \mathrm{cl}(S \cup \{b\})$. Then F is support-connected with $\mathrm{supp}_B(S) = \mathrm{supp}_B(F)$ and $\mathrm{rk}(F) = \mathrm{rk}(S) + 1$. Since $F \cap B \neq \emptyset$, F is a coordinate flat by Corollary 1.5.3, hence part (1) implies $\mathrm{rk}(S) = \mathrm{rk}(F) - 1 = |\,\mathrm{supp}_B(F)| - 1 = |\,\mathrm{supp}_B(S)| - 1$. □

Lemma 1.5.6. *A coordinate flat F is connected if and only if it is support-connected.*

Proof. Follows from Lemma 1.4.10 applied to the restriction matroid $M|_F$. □

Proposition 1.5.7. *The support graph $G_B(F)$ of a non-coordinate flat F is simple. In particular, $|F| \leq \binom{|\,\mathrm{supp}_B(F)|}{2}$.*

Proof. Let F be a flat whose support graph contains a pair of parallel edges $e, e' \in F \setminus B$ with endpoints $b, b' \in B$. Then the set $\{b, b', e, e'\}$ has rank 2, thus $\{b, b'\} \subseteq \mathrm{cl}(\{e, e'\}) \subseteq \mathrm{cl}(F) = F$. In particular, $F \cap B \neq \emptyset$. By contraposition, if F is a non-coordinate flat, then $G_B(F)$ is simple. The second claim follows from the inequality $|E(G)| \leq \binom{|V(G)|}{2}$ for a simple graph G. □

Corollary 1.5.8. *Let C be a rank 2 circuit of M. Then $|\,\mathrm{supp}_B(C)| \in \{2, 3\}$. If $|\,\mathrm{supp}_B(C)| = 3$, then $G_B(C)$ is a simple triangle.*

Proof. The closure $\mathrm{cl}(C)$ is a connected rank 2 flat and in particular support-connected by Proposition 1.4.7. If $\mathrm{cl}(C)$ is a coordinate flat, then we have $|\,\mathrm{supp}_B(C)| = \mathrm{rk}(C) = 2$ by Theorem 1.5.2. Otherwise, $\mathrm{cl}(C)$ is a non-coordinate flat and $|\,\mathrm{supp}_B(C)| = \mathrm{rk}(C) + 1 = 3$ by Theorems 1.5.2 and 1.5.5. In the latter case, Proposition 1.5.7 implies that $G_B(C)$ is a simple graph with 3 vertices. Since $|C| = |C \setminus B| = 3$, the graph must be a triangle. □

Lemma 1.5.9. *A non-coordinate flat F with $|\,\mathrm{supp}_B(F)| \geq 3$ is connected if and only if $G_B(F)$ is 2-connected.*

Proof. Let S be a subset of F and let $S_1, \ldots, S_k$ be its support components. Then for all $i \in \{1, \ldots, k\}$ the closure $\mathrm{cl}(S_i)$ is contained in F and thus a non-coordinate flat. Thus Corollary 1.4.8 and Theorem 1.5.5 imply

$$\mathrm{rk}(S) = \sum_{i=1}^{k} \mathrm{rk}(S_i) = \sum_{i=1}^{k} |\,\mathrm{supp}_B(S_i)| - 1 = |\,\mathrm{supp}_B(S)| - k.$$

Hence the restriction matroid $M|_F$ is isomorphic to the cycle matroid $M(G_B(F))$, see Example 1.2.3. By [Oxl11, Proposition 4.1.7], the cycle matroid $M(G)$ of a graph G is connected if and only if G is 2-connected. □

Corollary 1.5.10. *Let F be a connected non-coordinate flat and let $b \in$ $\mathrm{supp}_B(F)$. Then*

$$F_{-b} := \{e \in F \mid b \notin \mathrm{supp}_B(e)\} \subseteq F$$

is either empty or a support-connected non-coordinate flat. In both cases, we have $F = F_{-b} \vee \{e\}$ for every $e \in F \setminus F_{-b}$.

Proof. F_{-b} is a flat since it is the intersection of F with the coordinate flat $\mathrm{cl}(E \setminus \{b\})$. Note that $G_B(F_{-b})$ and the deletion graph $G_B(F) - b$ have the same set of edges and for the vertices we have the inclusion $\mathrm{supp}_B(F_{-b}) \subseteq \mathrm{supp}_B(F) \setminus \{b\}$. The graph $G_B(F) - b$ might contain additional vertices of degree 0, however, $G_B(F) - b$ is connected by Lemma 1.5.9, so this only happens if $G_B(F) - b$ consists of a single vertex and F_{-b} is empty. In this case, $F = \{e\}$ is a singleton by Proposition 1.5.7 and the claim is trivial.

Otherwise, we have $G_B(F_{-b}) = G_B(F) - b$, so F_{-b} is support-connected. Moreover, Theorem 1.5.5 implies

$$\mathrm{rk}(F_{-b}) = |\,\mathrm{supp}_B(F_{-b})| - 1 = |\,\mathrm{supp}_B(F)| - 2 = \mathrm{rk}(F) - 1.$$

In particular, for every $e \in F \setminus F_{-b}$ we have $\mathrm{rk}(F_{-b} \vee \{e\}) = \mathrm{rk}(F)$ and $F_{-b} \vee \{e\} \subseteq F$, hence $F_{-b} \vee \{e\} = F$. □

Lemma 1.5.11. *Let F and F' be connected flats of $M(A_n)$. Then $F \vee F'$ is disconnected if $\mathrm{supp}_B(F) \cap \mathrm{supp}_B(F') = \emptyset$ and at least one of F and F' is a non-coordinate flat.*

Proof. Assume that $F \vee F'$ is connected. If exactly one of F and F' is a coordinate flat, then $F \vee F'$ is a coordinate flat by Corollary 1.5.3, thus

$$\begin{aligned} |\,\mathrm{supp}_B(F) \cup \mathrm{supp}_B(F')| &= |\,\mathrm{supp}_B(F \vee F')| = \mathrm{rk}(F \vee F') \\ &\leq \mathrm{rk}(F) + \mathrm{rk}(F') = |\,\mathrm{supp}_B(F)| + |\,\mathrm{supp}_B(F')| - 1 \end{aligned}$$

by Proposition 1.2.12 and Theorem 1.5.5. Otherwise, if F and F' are both non-coordinate flats, then

$$\begin{aligned} |\,\mathrm{supp}_B(F) \cup \mathrm{supp}_B(F')| - 1 &= |\,\mathrm{supp}_B(F \vee F')| - 1 \leq \mathrm{rk}(F \vee F') \\ &\leq \mathrm{rk}(F) + \mathrm{rk}(F') \leq |\,\mathrm{supp}_B(F)| + |\,\mathrm{supp}_B(F')| - 2 \end{aligned}$$

by Proposition 1.2.12 and Theorem 1.5.5. In both cases, we deduce that the intersection $\mathrm{supp}_B(F) \cap \mathrm{supp}_B(F')$ is non-empty. □

Lemma 1.5.12. *Let F and F' be support-connected flats of M with*

$$|\,\mathrm{supp}_B(F) \cap \mathrm{supp}_B(F')| = 1.$$

Then $F \vee F'$ is a non-coordinate flat if and only if both F and F' are non-coordinate flats.

Proof. Since F and F' are support-connected flats and the intersection $\mathrm{supp}_B(F) \cap \mathrm{supp}_B(F')$ is non-empty, their join $F \vee F'$ is also support-connected. If at least one of F and F' is a coordinate flat, then $F \vee F'$ is a coordinate flat by Corollary 1.5.3. Otherwise, if F and F' are both non-coordinate flats, then we have

$$\begin{aligned} \mathrm{rk}(F \vee F') &\leq \mathrm{rk}(F) + \mathrm{rk}(F') = |\,\mathrm{supp}_B(F)| + |\,\mathrm{supp}_B(F')| - 2 \\ &= |\,\mathrm{supp}_B(F) \cup \mathrm{supp}_B(F')| + |\,\mathrm{supp}_B(F) \cap \mathrm{supp}_B(F')| - 2 \\ &= |\,\mathrm{supp}_B(F \vee F')| - 1, \end{aligned}$$

hence $F \vee F'$ is a non-coordinate flat by Theorems 1.5.2 and 1.5.5. □

1.6. The star property

We saw in examples that a matroid may have more than one Cremona basis, and in this case the notions of support and (non-)coordinate flats depend on the chosen Cremona basis. How are different Cremona bases related to each other? The goal of this section is to compare two Cremona bases by studying the shape of the support graph of one Cremona basis with respect to the other.

We need the following terminology from graph theory. Every connected graph G satisfies the formula $|E(G)| \geq |V(G)| - 1$ and G is called a *tree* if equality holds. Every tree is simple. A disjoint union of trees is also called a *forest.* A connected graph G is called a *star* if there exists a vertex $v \in V(G)$ (called *central vertex*) such that every edge $e \in E(G)$ is incident to v. In particular, a central vertex is adjacent to every other vertex. Every simple star is a tree. Conversely, if G is a tree, then a vertex adjacent to every other vertex is central. If $|V(G)| \geq 3$, then G has at most one central vertex.

Throughout this section, let M be a simple matroid on the ground set E and assume that M admits two Cremona bases B and B'. We will consider the support graph $G_B(B')$ of B' with respect to B, so all notions related to support are understood to be with respect to B unless stated otherwise. We will refer to the following theorem as the star property of Cremona bases.

Theorem 1.6.1. *Assume that no component of M has rank 2. Then $G_B(B')$ is a spanning forest of stars and $B \cap B'$ is a collection of exactly one central vertex for each star. In particular, $|B \cap B'| \geq 1$ with equality if and only if B' is support-connected with respect to B.*

We will prove this theorem in several steps. First note that $G_B(B')$ is a spanning subgraph of $G_B(M)$. Indeed, for every basis B' we have $\operatorname{supp}_B(B') = \operatorname{supp}_B(\operatorname{cl}(B')) = \operatorname{supp}_B(E)$ by Proposition 1.2.12. Moreover, by Lemma 1.4.6, every component of $G_B(B')$ is of the form $G_B(I')$, where I' is a support component of B' with respect to B.

Lemma 1.6.2. *Let I' be an independent subset of M that is support-connected with respect to B.*

(1) *We have $|I' \cap B| \leq 1$.*
(2) *If $|I' \cap B| = 1$, then $G_B(I')$ is a tree.*
(3) *$G_B(I')$ has at most one pair of parallel edges.*

Proof. Since I' is support-connected, we have

$$|I' \setminus B| = |E(G_B(I'))| \geq |V(G_B(I'))| - 1 = |\operatorname{supp}_B(I')| - 1.$$

On the other hand, $|\operatorname{supp}_B(I')| \geq \operatorname{rk}(I') = |I'|$ by Proposition 1.2.12, hence

$$|I' \cap B| = |I'| - |I' \setminus B| \leq |I'| - |\operatorname{supp}_B(I')| + 1 \leq 1.$$

If $|I' \cap B| = 1$, then $\operatorname{cl}(I')$ is a coordinate flat by Corollary 1.5.3, thus $G_B(I')$ is a tree since the equality $|\operatorname{supp}_B(I')| = \operatorname{rk}(I') = |I'|$ in Theorem 1.5.5 implies

$$|E(G_B(I'))| = |I' \setminus B| = |I'| - 1 = |\operatorname{supp}_B(I')| - 1 = |V(G_B(I'))| - 1.$$

If $G_B(I')$ has at least two pairs of parallel edges, then by removing one edge from each pair we obtain a subset $I'' \subseteq I'$ with $|I'' \setminus B| = |I' \setminus B| - 2$ and

$\operatorname{supp}_B(I'') = \operatorname{supp}_B(I')$. By construction, I'' is still support-connected, thus Proposition 1.2.12 and Theorem 1.5.5 imply

$$\operatorname{rk}(I') \leq |\operatorname{supp}_B(I')| = |\operatorname{supp}_B(I'')| \leq \operatorname{rk}(I'') + 1 = |I''| + 1 = |I'| - 1,$$

contradicting the assumption that I' is independent. □

Using that B' is a Cremona basis, we show that every support component I' of B' that fulfills the condition $|I' \cap B| = 1$ is not only a tree, but also a star.

Proposition 1.6.3. *Let I' be a support component of B' with respect to B. If $|I' \cap B| = 1$, then $G_B(I')$ is a simple star with central vertex $b \in I' \cap B$.*

Proof. Let $I' \cap B = \{b\}$. By Lemma 1.6.2 (2), the support graph $G_B(I')$ of I' is a tree and it suffices to show that b is adjacent to every other vertex. Let $b' \in \operatorname{supp}_B(I') \setminus \{b\}$. Then $b' \notin I'$, thus also $b' \notin B'$ since I' is a support component of B'. By assumption, B' is a Cremona basis of M, so the fundamental circuit $C := C_{B'}(b') \subseteq B' \cup \{b'\}$ has rank 2. Since $C \cap B \neq \emptyset$, Corollary 1.5.3 implies that $\operatorname{cl}(C)$ is a coordinate flat, hence C is contained in the rank 2 flat $\operatorname{cl}(\{b', b''\})$ for some vertex $b'' \in B$. If $b'' \notin B'$, then C would contain two parallel edges, contradicting the fact that $G_B(I')$ is simple. Thus $b'' \in B'$ and $C = \{b', e, b''\}$ for some edge $e \in B'$ with $\operatorname{supp}_B(e) = \{b', b''\}$. Since I' is a support component of B', we have $e, b'' \in I'$, hence $b'' = b$. This shows that b and b' are adjacent in $G_B(I')$. □

It remains to show that, under the conditions of Theorem 1.6.1, we have $|I' \cap B| = 1$ for every support component I' of B'.

Lemma 1.6.4. *If two vertices $b_1, b_2 \in B$ that are adjacent in $G_B(M)$ lie in different components of $G_B(B')$, then $b_1, b_2 \in B'$.*

Proof. Let $e \in E \setminus B$ be an edge with $\operatorname{supp}_B(e) = \{b_1, b_2\}$. By assumption, b_1 and b_2 lie in different components of $G_B(B')$, so we have $e \notin B'$. Since B' is a Cremona basis, the fundamental circuit $C := C_{B'}(e)$ has rank 2, say $C = \{e, e_1, e_2\}$ for some $e_1, e_2 \in B'$. If $\operatorname{cl}(C)$ were a non-coordinate flat, then by Corollary 1.5.8 the support graph $G_B(C)$ would be a triangle, but then e_1 and e_2 would form a path in $G_B(B')$ between b_1 and b_2, contradicting the assumption. Thus Theorem 1.5.2 implies that $\operatorname{cl}(C)$ is a coordinate flat and hence equal to $\operatorname{cl}(\{b_1, b_2\})$. But e_1 and e_2 cannot be edges since b_1 and b_2 are not adjacent in $G_B(B')$, hence $\{b_1, b_2\} = \{e_1, e_2\} \subseteq B'$. □

Proposition 1.6.5. *Assume that M has no component of rank 2. Then $|I' \cap B| = 1$ for every support component I' of B' with respect to B.*

Proof. Let I' be a support component of B' with respect to B. By Lemma 1.6.2 (1), we have $|I' \cap B| \leq 1$ and it remains to show the converse inequality. By Lemma 1.4.10, I' is contained in a component S of M and without loss of generality we may assume that M is connected. Otherwise, we replace M with $M|_S$ and replace the Cremona bases B and B' with $B \cap S$ and $B' \cap S$, respectively, using Lemma 1.4.4.

First assume that I' is a proper support component of B'. Since M is connected, the support graph $G_B(M)$ is connected by Lemma 1.4.10, thus there exists an edge $e \in E \setminus B$ between I' and some other support component of B'. Then the endpoints of e are contained in B' by Lemma 1.6.4, hence $|I' \cap B| \geq 1$.

Otherwise, we have $I' = B'$, i.e., B' is support-connected with respect to B. By Lemma 1.6.2 (3), $G_B(B')$ contains at most one pair of parallel edges. Since $\mathrm{rk}(M) \neq 2$, we can choose a vertex $b_1 \in B$ that is not incident to a pair of parallel edges in $G_B(B')$. We are done if $b_1 \in B \cap B'$, so assume $b_1 \notin B'$. Then the fundamental circuit $C := C_{B'}(b_1)$ has rank 2, say $C = \{b_1, e_1, e_2\}$ for some $e_1, e_2 \in B'$. Since $C \cap B \neq \emptyset$, Corollary 1.5.3 implies that $\mathrm{cl}(C)$ is a coordinate flat, hence C is contained in the rank 2 flat $\mathrm{cl}(\{b_1, b_2\})$ for some vertex $b_2 \in B$. By choice of b_1, only one of e_1 and e_2 can be an edge and the other is equal to b_2, hence $b_2 \in B \cap B' = B \cap I'$. □

This completes the proof of Theorem 1.6.1.

Example 1.6.6. Let M be the uniform matroid of rank 2 on the ground set $\{1, 2, 3, 4\}$ (cf. Example 1.3.7). Then $B := \{1, 2\}$ and $B' := \{3, 4\}$ are Cremona bases of M with $B \cap B' = \emptyset$.

The existence of a second Cremona basis B' of M has consequences for the shape of the support graph $G_B(M)$ with respect to B:

Lemma 1.6.7. *Assume that M has no component of rank 2. Then for all $b_1, b_2 \in B \setminus B'$ we have*

$$|\,\mathrm{cl}(\{b_1, b_2\}) \setminus \{b_1, b_2\}| \leq 1.$$

Proof. Let $b_1, b_2 \in B \setminus B'$ with $\mathrm{cl}(\{b_1, b_2\}) \setminus \{b_1, b_2\} \neq \emptyset$. Then $\mathrm{cl}(\{b_1, b_2\})$ is connected and we have $b_1 \neq b_2$ since M is simple. Proposition 1.4.7 and Theorem 1.5.2 imply that $\mathrm{cl}(\{b_1, b_2\})$ is either a coordinate flat or a non-coordinate flat with respect to B'. By interchanging the roles of B and B' in Theorem 1.6.1, we deduce that $G_{B'}(B)$ is simple, so b_1 and b_2 are edges with different support with respect to B'. In particular, $|\,\mathrm{supp}_{B'}(\mathrm{cl}(\{b_1, b_2\}))| > 2 = \mathrm{rk}(\mathrm{cl}(\{b_1, b_2\}))$, hence $\mathrm{cl}(\{b_1, b_2\})$ is a non-coordinate flat with respect to B' by Theorem 1.5.5. But then we have $|\,\mathrm{cl}(\{b_1, b_2\})| \leq 3$ by Proposition 1.5.7. □

Lemma 1.6.8. *Let I' be a support component of B' with $I' \cap B = \{b_0\}$. Then the vertices $\mathrm{supp}_B(I') \setminus \{b_0\}$ form a union of components of $G_B(M) - b_0$.*

Proof. Assume that there exist vertices $b_1 \in \mathrm{supp}_B(I') \setminus \{b_0\}$ and $b_2 \in B \setminus \mathrm{supp}_B(I')$ that are adjacent in $G_B(M) - b_0$. By assumption, they lie in different components of $G_B(B')$, so Lemma 1.6.4 implies $b_1 \in B'$ and thus $b_1 \in I'$ since I' is a support component of B'. But then we have $b_1 = b_0$ by Lemma 1.6.2, contradiction. □

Proposition 1.6.9. *Assume that M and the contractions M/b are connected for all $b \in B$. If $\mathrm{rk}(M) \neq 2$, then $B = B'$ or $|B \cap B'| = 1$.*

Proof. Assume $B \neq B'$. Then there exists a support component I' of B' with $|I'| \geq 2$. Since M is connected and $\mathrm{rk}(M) \neq 2$, Proposition 1.6.5 implies $|I' \cap B| = 1$, say $I' \cap B = \{b_0\}$. It suffices to show that $I' = B'$. By assumption, M/b_0 is connected, which is equivalent to $G_B(M) - b_0$ being connected by Lemma 1.4.12. Now Lemma 1.6.8 implies that $\mathrm{supp}_B(I') \setminus \{b_0\}$ is empty or equal to $B \setminus \{b_0\}$, but the former is not possible since $|I'| \geq 2$. Hence $\mathrm{supp}_B(I') = B$ and thus $I' = B'$. □

1.7. Cremona base change

Using the star property, we will show in this section that there exists a matroid automorphism interchanging any given pair of Cremona bases.

Lemma 1.7.1. *Let M and M' be matroids with ground sets E and E', respectively. If $f\colon E \to E'$ is a bijective map such that $\mathrm{rk}_{M'}(f(F)) \leq \mathrm{rk}_M(F)$ for all connected flats F of M and $\mathrm{rk}_M(f^{-1}(F')) \leq \mathrm{rk}_{M'}(F')$ for all connected flats F' of M', then f is a matroid isomorphism.*

Proof. Let $S \subseteq E$ and let $S_1, \ldots, S_k$ be the components of $\mathrm{cl}_M(S)$. Then by assumption we have $\mathrm{rk}_{M'}(f(S_i)) \leq \mathrm{rk}_M(S_i)$ for all i, hence

$$\begin{aligned}\mathrm{rk}_{M'}(f(S)) \leq \mathrm{rk}_{M'}(f(\mathrm{cl}_M(S))) &= \mathrm{rk}_{M'}\left(f\big(\bigcup_{i=1}^k S_i\big)\right) = \mathrm{rk}_{M'}\left(\bigcup_{i=1}^k f(S_i)\right)\\ &\leq \sum_{i=1}^k \mathrm{rk}_{M'}(f(S_i)) \leq \sum_{i=1}^k \mathrm{rk}_M(S_i) = \mathrm{rk}_M(\mathrm{cl}(S)) = \mathrm{rk}_M(S)\end{aligned}$$

by Corollary 1.4.3.

Conversely, let $S'_1, \ldots, S'_k$ be the components of $\mathrm{cl}_{M'}(f(S))$. Then by assumption we have $\mathrm{rk}_M(f^{-1}(S'_i)) \leq \mathrm{rk}_{M'}(S'_i)$ for all i, hence

$$\begin{aligned}\mathrm{rk}_M(S) \leq \mathrm{rk}_M(f^{-1}(\mathrm{cl}_{M'}(f(S)))) &= \mathrm{rk}_M\left(f^{-1}\big(\bigcup_{i=1}^k S'_i\big)\right)\\ &= \mathrm{rk}_M\left(\bigcup_{i=1}^k f^{-1}(S'_i)\right) \leq \sum_{i=1}^k \mathrm{rk}_M(f^{-1}(S'_i))\\ &\leq \sum_{i=1}^k \mathrm{rk}_{M'}(S'_i) = \mathrm{rk}_{M'}(\mathrm{cl}_{M'}(f(S))) = \mathrm{rk}_{M'}(f(S))\end{aligned}$$

by Corollary 1.4.3. □

Theorem 1.7.2. *Let M be a simple matroid on the ground set E that admits two Cremona bases B and B'. Then there exists an involutive automorphism $f_{BB'}$ of M with*

$$f_{BB'}(B) = B', \quad f_{BB'}(B') = B, \quad \text{and} \quad f_{BB'}|_{E\setminus(B\cup B')} = \mathrm{id}.$$

The involution $f_{BB'}$ is called the *Cremona base change automorphism* with respect to B and B'.

Proof. By constructing $f_{BB'}$ separately on each component of M, we may assume that M is connected (see Lemma 1.4.4). Setting $f_{BB'}(e) := e$ for all $e \in E \setminus (B \cup B')$ and $f_{BB'}(b) := b$ for all $b \in B \cap B'$, it remains to define $f_{BB'}$ on the symmetric difference of B and B'.

If M has rank 2, then M is a uniform matroid and any permutation of the ground set is a matroid automorphism (cf. Example 1.3.7), thus any bijection between $B \setminus B'$ and $B' \setminus B$ gives rise to an involution with the desired properties.

Otherwise, we know by Theorem 1.6.1 that $G_B(B')$ is a spanning forest of stars and that the intersection $B \cap B'$ contains exactly one central vertex of each star. Thus for any vertex $b \in B \setminus B'$ there exists a unique edge

$e \in B' \setminus B$ joining b to a central vertex $b_0 \in B \cap B'$ of the corresponding star component and we set $f_{BB'}(b) := e$. Note that

$$\operatorname{supp}_B(f_{BB'}(b)) = \{b_0, b\} = \{b_0\} \cup \operatorname{supp}_B(b).$$

Conversely, every edge $e \in B' \setminus B$ connects a central vertex $b_0 \in B \cap B'$ of the corresponding star component with a vertex $b \in B \setminus B'$ and we set $f_{BB'}(e) := b$. Note that

$$\operatorname{supp}_B(e) = \{b_0, f_{BB'}(e)\} = \{b_0\} \cup \operatorname{supp}_B(f_{BB'}(e)).$$

By construction, $f_{BB'}$ is an involution on E with the desired properties. Furthermore, we claim that for every edge $e \in E$ with endpoints $b_1, b_2 \in B \setminus B'$ we have

$$\operatorname{supp}_{B'}(e) = \{f_{BB'}(b_1), f_{BB'}(b_2)\} = f_{BB'}(\operatorname{supp}_B(e)). \tag{$\star$}$$

Indeed, Lemma 1.6.4 implies that b_1 and b_2 lie in the same support component I' of B'. Since B' is a Cremona basis, the fundamental circuit $C := C_{B'}(e)$ has rank 2, say $C = \{e, e_1, e_2\}$ for some $e_1, e_2 \in I'$. By Theorem 1.6.1, we have $|I' \cap B| = 1$, say $I' \cap B = \{b_0\}$, and $G_B(I')$ is a simple star with central vertex b_0. In particular, $b_0 \in \operatorname{supp}_B(b')$ for all $b' \in I'$, hence $b_0 \in \operatorname{supp}_B(C)$. Now Corollary 1.5.8 implies that $G_B(C)$ is a simple triangle with vertices $\{b_0, b_1, b_2\}$. But $f_{BB'}(b_1)$ and $f_{BB'}(b_2)$ are the unique edges in $G_B(I')$ joining b_0 with b_1 and b_2, respectively, hence $C = \{e, f_{BB'}(b_1), f_{BB'}(b_2)\}$.

In order to prove that the involution $f_{BB'}$ is a matroid automorphism, by Lemma 1.7.1 it suffices to show the inequality

$$\operatorname{rk}(f_{BB'}(F)) \leq \operatorname{rk}(F)$$

for all connected flats F of M. Every connected flat of M is either a coordinate flat or a non-coordinate flat with respect to B by Proposition 1.4.7 and Theorem 1.5.2.

First let F be a connected coordinate flat of M. Then Proposition 1.2.12 and Theorem 1.5.5 imply

$$\operatorname{rk}(f_{BB'}(F)) \leq |\operatorname{supp}_B(f_{BB'}(F))| \quad \text{and} \quad |\operatorname{supp}_B(F)| = \operatorname{rk}(F),$$

so we are done if $\operatorname{supp}_B(f_{BB'}(F)) \subseteq \operatorname{supp}_B(F)$. Otherwise, choose a vertex $b_0 \in \operatorname{supp}_B(f_{BB'}(F)) \setminus \operatorname{supp}_B(F)$. By construction of $f_{BB'}$, this means that b_0 is a central vertex of a support component I' of B' and that F contains a vertex $b \in B \setminus B'$ adjacent to b_0 in $G_B(I')$. By Lemma 1.6.8, the vertices $\operatorname{supp}_B(I') \setminus \{b_0\}$ form a union of components of $G_B(M) - b_0$. Since F is connected and $b_0 \notin \operatorname{supp}_B(F)$, we deduce that $F = \operatorname{cl}(I)$ for some subset $I \subseteq \operatorname{supp}_B(I') \setminus \{b_0\}$. In particular, we have $\operatorname{cl}(I) \cap B' = \emptyset$, and for every edge $e \in F \setminus I$ we have

$$f_{BB'}(e) = e \in \operatorname{cl}(\operatorname{supp}_{B'}(e)) \overset{(\star)}{=} \operatorname{cl}(f_{BB'}(\operatorname{supp}_B(e))) \subseteq \operatorname{cl}(f_{BB'}(I)).$$

Hence $f_{BB'}(F) \subseteq \operatorname{cl}(f_{BB'}(I))$ and we conclude

$$\operatorname{rk}(f_{BB'}(F)) \leq \operatorname{rk}(f_{BB'}(I)) \leq |f_{BB'}(I)| = |I| = \operatorname{rk}(I) = \operatorname{rk}(F).$$

Now let F be a connected non-coordinate flat of M with $f_{BB'}(F) \neq F$. Then F contains an edge $e \in B' \setminus B$ and the star property implies that e is incident to a central vertex $b_0 \in B \cap B'$ of the corresponding star component I' of B'. By Lemma 1.5.9, $G_B(F) - b_0$ is a connected subgraph of $G_B(M) - b_0$, so we deduce $\mathrm{supp}_B(F) \subseteq \mathrm{supp}_B(I')$ using Lemma 1.6.8. In particular, for all $b \in \mathrm{supp}_B(F) \setminus \{b_0\}$, the image $f_{BB'}(b)$ is an edge joining b and b_0. Since $f_{BB'}$ is an involution, the vertex $f_{BB'}(e)$ is contained in the set

$$I := \{b \in \mathrm{supp}_B(F) \setminus \{b_0\} \mid f_{BB'}(b) \in F\},$$

which forms a union of components of $G_B(F) - b_0$. Indeed, if $e' \in F$ is an edge joining two vertices $b, b' \in \mathrm{supp}_B(F) \setminus \{b_0\}$ such that $f_{BB'}(b) \in F$, then $C_{B'}(e') = \{e', f_{BB'}(b), f_{BB'}(b')\}$ by ($\star$) and thus $f_{BB'}(b') \in F$. But $G_B(F) - b_0$ is connected, hence $I = \mathrm{supp}_B(F) \setminus \{b_0\}$.

We claim that $f_{BB'}(F) \subseteq \mathrm{cl}(I)$ and thus

$$\mathrm{rk}(f_{BB'}(F)) \leq \mathrm{rk}(I) = |\,\mathrm{supp}_B(F)| - 1 = \mathrm{rk}(F)$$

by Theorem 1.5.5. Let $e \in F$. If $b_0 \notin \mathrm{supp}_B(F)$, then $e \notin B'$, so we have $f_{BB'}(e) = e \in \mathrm{cl}(I)$. Otherwise, we have $\mathrm{supp}_B(e) = \{b_0, b\}$ for some $b \in I$. Then $f_{BB'}(b) \in F$ is an edge parallel to e, but $G_B(F)$ is a simple graph by Proposition 1.5.7, hence $e = f_{BB'}(b)$ and $f_{BB'}(e) = b \in \mathrm{cl}(I)$. This completes the proof. □

Corollary 1.7.3. *For all Cremona bases B and B' of M the support graphs $G_B(M)$ and $G_{B'}(M)$ are isomorphic.*

Proof. The base change automorphism $f_{BB'}$ induces an isomorphism between $G_B(M)$ and $G_{B'}(M)$ since it maps vertices to vertices and for any edge $e \in E \setminus B$ of $G_B(M)$ we have $\mathrm{supp}_{B'}(f_{BB'}(e)) = f_{BB'}(\mathrm{supp}_B(e))$. □

Every matroid automorphism permutes the set $\mathcal{CB}(M)$ of Cremona bases of M and Theorem 1.7.2 shows that the group action $\mathrm{Aut}(M) \to \mathrm{Sym}(\mathcal{CB}(M))$ is transitive, where $\mathrm{Sym}(\mathcal{CB}(M))$ denotes the symmetric group.

Example 1.7.4. Let M_1 be the matroid from Example 1.2.6. We saw in Example 1.3.5 that M_1 has four Cremona bases, $B_{23} := \{1, 2, 3\}$, $B_{25} := \{1, 2, 5\}$, $B_{34} := \{1, 3, 4\}$, and $B_{45} := \{1, 4, 5\}$. We obtain the following Cremona base change automorphisms:

- $f_{B_{23}B_{25}}$ and $f_{B_{34}B_{45}}$ are the transposition (3 5).
- $f_{B_{23}B_{34}}$ and $f_{B_{25}B_{45}}$ are the transposition (2 4).
- $f_{B_{23}B_{45}}$ and $f_{B_{25}B_{34}}$ correspond to the permutation (2 4)(3 5).

The set of Cremona base change automorphisms forms the Klein four group, which is a proper abelian subgroup of $\mathrm{Aut}(M_1) \cong D_4$. The automorphism group $\mathrm{Aut}(M_1)$ is a proper subgroup of $\mathrm{Sym}(\mathcal{CB}(M)) \cong S_4$.

Example 1.7.5. Let M_2 be the matroid from Example 1.2.7. In Example 1.3.6 we determined the support graph $G_{B_0}(M_2)$ with respect to the Cremona basis $B_0 := \{1, 2, 3\}$ and claimed that M_2 has two other Cremona bases, $B_1 := \{1, 4, 6\}$ and $B_2 := \{2, 4, 7\}$. Indeed, for every Cremona basis $B' \neq B_0$ we have $|B_0 \cap B'| = 1$ by Proposition 1.6.9. With respect to B_0, $G_{B_0}(B_1)$ is a spanning star with central vertex 1 and $G_{B_0}(B_2)$ is a spanning star with central vertex 2. Lemma 1.6.7 implies that there is no Cremona basis with central vertex 3. We obtain the following Cremona base change automorphisms:

- $f_{B_0B_1}$ is the permutation (2 4)(3 6).
- $f_{B_0B_2}$ is the permutation (1 4)(3 7).
- $f_{B_1B_2}$ is the permutation (1 2)(6 7).

The Cremona base change automorphisms generate the whole automorphism group $\mathrm{Aut}(M_2) \cong S_3$.

Theorem 1.7.6. *Let M be a simple matroid of rank at least 3 on the ground set E. If M and the contractions M/e are connected for all $e \in E$, then the group action $\mathrm{Aut}(M) \to \mathrm{Sym}(\mathcal{CB}(M))$ is surjective.*

Proof. Let B_0, B_1, and B_2 be pairwise different Cremona bases. By Proposition 1.6.9, we have $|B_i \cap B_j| = 1$ for all $i \neq j$, say $B_0 \cap B_1 = \{b_1\}$ and $B_0 \cap B_2 = \{b_2\}$. The Cremona base change automorphism $f_{B_0 B_1}$ from Theorem 1.7.2 swaps B_0 and B_1 and satisfies $f_{B_0 B_1}|_{E \setminus (B_0 \cup B_1)} = \mathrm{id}$. Since the symmetric group is generated by transpositions, it suffices to show that $f_{B_0 B_1}(B_2) = B_2$. If $b_1 = b_2$, then by construction of $f_{B_0 B_1}$ we have $f_{B_0 B_1}(b_1) = b_1$ and hence $f_{B_0 B_1}|_{B_2} = \mathrm{id}$. Now assume $b_1 \neq b_2$. By Theorem 1.6.1, $G_{B_0}(B_1)$ and $G_{B_0}(B_2)$ are simple spanning stars with central vertices b_1 and b_2, respectively, thus $B_1 \cap B_2 = \{e\}$ for some edge e with $\mathrm{supp}_{B_0}(e) = \{b_1, b_2\}$. By construction of $f_{B_0 B_1}$ we have $f_{B_0 B_1}(e) = b_2$ and $f_{B_0 B_1}(b_2) = e$, hence $f_{B_0 B_1}(B_2) = B_2$. $\square$

1.8. Representability criterion

We conclude this chapter with a representability criterion for matroids that admit more than one Cremona basis.

Theorem 1.8.1. *Let M be a simple connected matroid on the ground set E. If there exist Cremona bases B and $\tilde{B}$ of M with $|B \cap \tilde{B}| = 1$, then M is representable over any field K with $|K| \geq |E| - \mathrm{rk}(M) + 1$.*

Proof. If M has rank 1, then the statement is trivial, and if M has rank 2, then M is the uniform matroid $U_{2,|E|}$ (see Example 1.3.7), which is known to be representable over a field K if and only if $|K| \geq |E| - 1$, see [Oxl11, Proposition 6.5.2].

Let $n := \mathrm{rk}(M) - 1$ and write $B = \{b_0, \ldots, b_n\}$ with $B \cap \tilde{B} = \{b_0\}$. By Theorem 1.6.1, $G_B(B')$ is a simple spanning star with central vertex b_0. We decompose the set of edges $E \setminus B$ into the subset

$$E_0 := \{e \in E \setminus B \mid b_0 \in \mathrm{supp}_B(e)\}$$

of edges incident to b_0 and the subset

$$E_+ := \{e \in E \setminus B \mid b_0 \notin \mathrm{supp}_B(e)\} = \mathrm{cl}(B \setminus \{b_0\}) \setminus B$$

of edges not incident to b_0. Since $\mathrm{cl}(B \setminus \{b_0\}) \cap B' = \emptyset$, we have

$$E_+ = \mathrm{cl}(B \setminus \{b_0\}) \cap f_{BB'}(\mathrm{cl}(B \setminus \{b_0\}))$$

by construction of the Cremona base change automorphism $f_{BB'}$ from Theorem 1.7.2. In particular, E_+ is a flat of M, which is by definition either empty or a non-coordinate flat, thus its support graph is simple by Proposition 1.5.7.

On the set E_0, we consider the equivalence relation

$$e \sim e' \;:\Longleftrightarrow\; E_+ \vee \{e\} = E_+ \vee \{e'\}.$$

Let K be a field with $|K| \geq |E| - \mathrm{rk}(M) + 1$ and let $B' := \{v_0, \ldots, v_n\}$ be a basis of the vector space K^{n+1}. Then $|E_0/{\sim}| \leq |E \setminus B| \leq |K^\times|$, thus we choose an injective map $\iota\colon E_0/{\sim} \to K^\times$ and construct a map $f\colon E \to K^{n+1}$ as follows:

(1) If $b_i \in B$ is a basis vector for any $i \in \{0, \ldots, n\}$, then we set $f(b_i) := v_i$.
(2) If $e \in E_0$ is an edge with $\mathrm{supp}_B(e) = \{b_0, b_i\}$ for some $i \in \{1, \ldots, n\}$, then we set $f(e) := v_0 - \iota([e])v_i$. Note that $\iota([e]) \neq 0$.
(3) If $e \in E_+$ is an edge with $\mathrm{supp}_B(e) = \{b_i, b_j\}$ for some i, j with $1 \leq i < j \leq n$, then we set $f(e) := v_i - v_j$.

Let M' be the matroid associated to the set of vectors $E' := f(E)$. By construction, B' is a Cremona basis of M' (cf. Example 1.3.8) and f is support-preserving with respect to B and B', i.e., $\mathrm{supp}_{B'}(f(e)) = f(\mathrm{supp}_B(e))$ for all $e \in E$. We claim that f induces an isomorphism of matroids between M and M'.

First, we have to check that f is injective. Since $G_B(E_+)$ is simple, for all $1 \leq i < j \leq n$ there exists at most one edge $e \in E \setminus B$ with $\mathrm{supp}_B(e) = \{b_i, b_j\}$. Therefore it suffices to consider part (2) of the construction. Let $e, e' \in E_0$ be edges with $f(e) = f(e')$. Then e and e' are parallel, say $\mathrm{supp}_B(e) = \mathrm{supp}_B(e') = \{b_0, b_i\}$ for some $i \in \{1, \ldots, n\}$, and we have

$E_+ \vee \{e\} = E_+ \vee \{e'\}$. If $b_i \notin \operatorname{supp}_B(E_+)$, then clearly $e = e'$. Otherwise, let F be the support component of E_+ with $b_i \in \operatorname{supp}_B(F)$, which is a non-coordinate flat. Then $F \vee \{e\}$ and $F \vee \{e'\}$ are support components of $E_+ \vee \{e\}$ and $E_+ \vee \{e'\}$, respectively, thus $E_+ \vee \{e\} = E_+ \vee \{e'\}$ implies that we have $F' := F \vee \{e\} = F \vee \{e'\}$. Now Lemma 1.5.12 shows that F' is a non-coordinate flat, hence Proposition 1.5.7 implies $e = e'$.

It remains to verify that f satisfies the condition from Lemma 1.7.1. By Theorem 1.5.2, there are four cases to consider:

(1) Let F be a connected coordinate flat of M. Since f is support-preserving, we have $\operatorname{rk}_{M'}(f(F)) \le |\operatorname{supp}_{B'}(f(F))| = |\operatorname{supp}_B(F)| = \operatorname{rk}_M(F)$ by Proposition 1.2.12 and Theorem 1.5.5.

(2) Let F be a connected non-coordinate flat of M. If $b_0 \notin \operatorname{supp}_B(F)$, then $f(F)$ is contained in the linear subspace

$$U := \langle v_i - v_j \mid 1 \le i < j \le n,\ b_i, b_j \in \operatorname{supp}_B(F)\rangle.$$

Otherwise, if $b_0 \in \operatorname{supp}_B(F)$, then for all $e, e' \in F \cap E_0$ we have $F_{-b_0} \vee \{e\} = F = F_{-b_0} \vee \{e'\}$ by Corollary 1.5.10. Since $F_{-b_0} \subseteq E_+$, we see that all edges $e \in F \cap E_0$ have the same image $a \in K^\times$ under the embedding ι. This means that $f(F)$ is contained in the linear subspace

$$\begin{aligned} U :=\ & \langle v_0 - a v_i \mid 1 \le i \le n,\ b_i \in \operatorname{supp}_B(F)\rangle \\ & + \langle v_i - v_j \mid 1 \le i < j \le n,\ b_i, b_j \in \operatorname{supp}_B(F)\rangle. \end{aligned}$$

In both cases, the subspace U has dimension $|\operatorname{supp}_B(F)| - 1$ and we conclude $\operatorname{rk}(f(F)) \le \dim(U) = |\operatorname{supp}_B(F)| - 1 = \operatorname{rk}(F)$ using Theorem 1.5.5.

(3) Let F' be a connected coordinate flat of M'. Since f is bijective, the inverse f^{-1} is also support-preserving and as in case (1) we have $\operatorname{rk}_M(f^{-1}(F')) \le |\operatorname{supp}_B(f^{-1}(F'))| = |\operatorname{supp}_{B'}(F')| = \operatorname{rk}_{M'}(F')$.

(4) Let F' be a connected non-coordinate flat of M'. If $v_0 \notin \operatorname{supp}_{B'}(F')$, then $f^{-1}(F')$ is contained in the non-coordinate flat E_+. Otherwise, if $v_0 \in \operatorname{supp}_{B'}(F')$, let $e, e' \in F'$ be edges incident to v_0, say $e = v_0 - a v_i$ and $e' = v_0 - b v_j$ for some $a, b \in K^\times$ and $i, j \in \{1, \ldots, n\}$. By Corollary 1.5.10, we have $F'_{-v_0} \vee \{e\} = F'_{-v_0} \vee \{e'\}$. Since $F'_{-v_0} \subseteq \{v_i - v_j \mid 1 \le i < j \le n\}$, this implies $a = b$. Thus $f^{-1}(F')$ is contained in the non-coordinate flat $E_+ \vee \{e\}$.

In both cases, we see that $f^{-1}(F')$ is a non-coordinate flat, hence $\operatorname{rk}(f^{-1}(F')) \le |\operatorname{supp}_B(f^{-1}(F'))| - 1 = |\operatorname{supp}_{B'}(F')| - 1 = \operatorname{rk}(F')$ by Theorem 1.5.5. □

Corollary 1.8.2. *Let M be a simple connected matroid of rank at least* 3 *on the ground set E and assume that M/e is connected for all $e \in E$. If M admits more than one Cremona basis, then M is representable over any field K with $|K| \ge |E| - \operatorname{rk}(M) + 1$.*

Proof. Follows from Proposition 1.6.9 and Theorem 1.8.1. □

1.9. Conclusion

In this chapter, we defined Cremona bases of matroids and studied their properties. In particular, the distinction between coordinate and non-coordinate flats will be important for the construction of combinatorial Cremona automorphisms in Section 3.1 and for our computations with root system matroids in Chapter 4. Moreover, the existence of Cremona base change automorphisms will be an important tool to study the structure of the Cremona group of a matroid in Section 3.2.

As the example of the root system D_n in Section 4.4 shows, the results of this section can also be applied to matroids that do not admit a Cremona basis, but can be embedded into a matroid that does. One could try to find a criterion for when such an embedding exists.

If $\mathcal{A}$ is a hyperplane arrangement in projective space such that the standard Cremona transformation induces an automorphism of the complement $\Omega_{\mathcal{A}}$, then the coordinate hyperplanes form a Cremona basis of the matroid $M(\mathcal{A})$, as we will show in Section 2.3. It would be interesting to know if there are other contexts where matroids with Cremona bases arise.

Matroid theory is a rich field and there are many other questions that one could ask about Cremona bases. How do Cremona bases behave with respect to other concepts in matroid theory such as duality? Can we describe the class of matroids that admit a Cremona basis, for example using excluded minors? How are matroids that admit a Cremona basis related to other classes of matroids? One obvious relation is that a non-uniform matroid M of rank at least 4 that admits a Cremona basis cannot be a paving matroid. This leads us to believe that matroids with Cremona bases are very rare.

Conjecture 1.9.1 (cf. [Oxl11, Conjecture 15.5.8])**.** *Asymptotically, almost every matroid does not admit a Cremona basis.*

CHAPTER 2

Matroid fans and hyperplane arrangements

2.1. Matroid fans

In this section, we summarize the construction of fans on the tropical linear space associated to a matroid. For more details see [AK06], [MS15, Section 4.2], and [FS05].

Let M be a simple connected matroid on the ground set E. For an element $e \in E$ we define the function $v_e \colon E \to \mathbb{R}$ by

$$v_e(e') = \begin{cases} 1, & \text{if } e = e' \\ 0, & \text{if } e \neq e' \end{cases}.$$

The functions $\{v_e\}_{e \in E}$ form a basis of the vector space $\mathbb{R}^E$ of real-valued functions on E and for a subset $S \subseteq E$ we write $v_S := \sum_{e \in S} v_e$.

If $w \in \mathbb{R}^E$ is a vector and $S \subseteq E$ a subset, then $w(S) := \sum_{e \in S} w(e)$ is called the *w-weight* of S, and therefore the vectors in $\mathbb{R}^E$ are also called *weight vectors*. Given a weight vector $w \in \mathbb{R}^E$, we obtain a flag

$$\emptyset = S_0 \subsetneq \cdots \subsetneq S_{k+1} = E$$

of subsets of E by sorting the elements of E by w-weight in descending order, i.e., such that

(1) w is constant on $S_{i+1} \setminus S_i$ for all $0 \leq i \leq k$ and
(2) $w|_{S_i \setminus S_{i-1}} > w|_{S_{i+1} \setminus S_i}$ for all $1 \leq i \leq k$.

We denote this flag by $\mathcal{F}(w)$.

Theorem 2.1.1 ([AK06, Proposition 1 and Theorem 1])**.** *The following are equivalent for a weight vector $w \in \mathbb{R}^E$:*

(1) *For every circuit C of M, the minimum of the numbers $(w(i))_{i \in C}$ is attained at least twice.*
(2) *The flag $\mathcal{F}(w)$ is a flag of flats of M.*
(3) *The bases of M with maximal w-weight cover the ground set E.*

Clearly, these conditions are all invariant under translation along the line $\langle v_E \rangle$ and thus well-defined on vectors in the quotient space $\mathbb{R}^E / \langle v_E \rangle$.

Definition 2.1.2. The *tropical linear space* $\operatorname{trop}(M)$ of a matroid M, also called the *matroid fan* or the *Bergman fan*, is the set of all weight vectors $w \in \mathbb{R}^E / \langle v_E \rangle$ that satisfy the equivalent conditions from Theorem 2.1.1.

For a flag $\mathcal{F}$ of flats of M given by $\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E$, we define the cone $\rho_{\mathcal{F}} := \langle v_{F_1}, \ldots, v_{F_k} \rangle_{\mathbb{R}_{\geq 0}}$.

Theorem 2.1.3 ([MS15, Theorem 4.2.6])**.** *The cones $\rho_{\mathcal{F}}$, where $\mathcal{F}$ ranges over the flags of flats of M, form a polyhedral fan with support* $\operatorname{trop}(M)$.

This fan is called the *fine Bergman fan* $B_f(M)$ of M.

Example 2.1.4. Let M_1 be the matroid from Example 1.2.6. This is a matroid on the ground set $E := \{1, \ldots, 5\}$ and we identify $\mathbb{R}^E$ with $\mathbb{R}^5$. The circuits of M_1 are $\{1,2,4\}$, $\{1,3,5\}$, and $\{2,3,4,5\}$. One can check that a weight vector $w = (w_1, \ldots, w_5) \in \mathbb{R}^5$ satisfies condition (1) from Theorem 2.1.1 if and only if at least one of the following conditions holds:

(1) $w_2 = w_4 \leq w_1 = w_3 \leq w_5$	(7) $w_1 = w_2 = w_3 \leq w_4 \leq w_5$
(2) $w_2 = w_4 \leq w_1 = w_5 \leq w_3$	(8) $w_1 = w_2 = w_3 \leq w_5 \leq w_4$
(3) $w_2 = w_4 \leq w_3 = w_5 \leq w_1$	(9) $w_1 = w_2 = w_5 \leq w_3 \leq w_4$
(4) $w_3 = w_5 \leq w_1 = w_2 \leq w_4$	(10) $w_1 = w_2 = w_5 \leq w_4 \leq w_3$
(5) $w_3 = w_5 \leq w_1 = w_4 \leq w_2$	(11) $w_1 = w_4 = w_3 \leq w_2 \leq w_5$
(6) $w_3 = w_5 \leq w_2 = w_4 \leq w_1$	(12) $w_1 = w_4 = w_3 \leq w_5 \leq w_2$
	(13) $w_1 = w_4 = w_5 \leq w_2 \leq w_3$
	(14) $w_1 = w_4 = w_5 \leq w_3 \leq w_2$

The equivalence between conditions (1) and (2) in Theorem 2.1.1 shows that these 14 cases correspond to the 14 chains in the proper part of the lattice of flats of M_1.

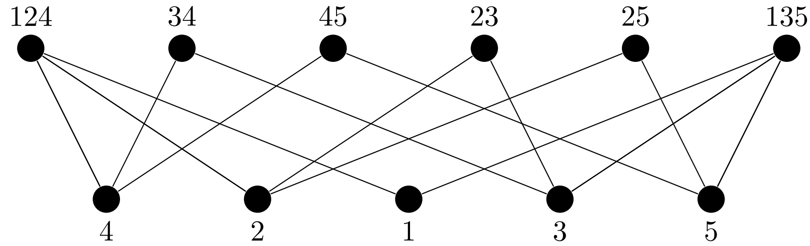

FIGURE 8. The underlying complex of the fine Bergman fan $B_f(M_1)$.

Proposition 2.1.5 ([FS05, Proposition 2.5]). *The tropical linear space* $\operatorname{trop}(M)$ *is a subset of the normal fan of the matroid polytope* P_M*, which is defined as the convex hull of the vectors* $\{v_B \mid B \text{ basis of } M\}$*.*

The induced fan structure on $\operatorname{trop}(M)$ is called the *coarse Bergman fan* $B_c(M)$ of M. By construction, two vectors $w, w' \in \operatorname{trop}(M)$ lie in the interior of the same cone of the coarse Bergman fan if and only if the bases with maximal w-weight coincide with the bases with maximal w'-weight. By [Ham14, Proposition 3.4.1], the coarse Bergman fan is the coarsest fan structure on $\operatorname{trop}(M)$.

Definition 2.1.6. A non-empty set $\mathcal{S}$ of proper connected flats of M is called *nested* if for any antichain $A \subseteq \mathcal{S}$ with $|A| \geq 2$ the join $\bigvee_{F \in A} F$ is disconnected. The nested sets of M form an abstract simplicial complex $\mathcal{N}(M)$, the *minimal nested set complex* of M.

In particular, two proper connected flats F and F' of M are adjacent in the 1-skeleton of $\mathcal{N}(M)$ if and only if F and F' are comparable or $F \vee F'$ is reducible.

For a nested set $\mathcal{S}$ of M we define the cone $\rho_{\mathcal{S}} := \langle v_F \mid F \in \mathcal{S} \rangle_{\mathbb{R}_{\geq 0}}$.

Theorem 2.1.7 ([FS05, Theorem 4.1]). *The cones* $\rho_{\mathcal{S}}$*, where* $\mathcal{S}$ *ranges over the nested sets of* M*, form a fan that refines the coarse Bergman fan* $B_c(M)$*.*

This fan is called the *minimal nested set fan* $B_m(M)$ of M.

Theorem 2.1.8 ([FS05, Theorem 5.3])**.** *Assume that M is connected. Then the minimal nested set fan and the coarse Bergman fan coincide if and only if the matroid $(M/F)|_{G\setminus F}$ is connected for every pair of flats $F \subseteq G$ with G connected.*

Example 2.1.9. We continue Example 2.1.4 with the matroid M_1. The cones containing the weight vectors $w \in \mathbb{R}^5/\langle\mathbb{R}(1,\ldots,1)\rangle$ corresponding to the cases $w_2 = w_4 \leq w_3 = w_5 \leq w_1$ and $w_3 = w_5 \leq w_2 = w_4 \leq w_1$ both induce the same set

$$\{\{1,2,3\},\{1,2,5\},\{1,3,4\},\{1,4,5\}\}$$

of w-maximal bases. In this way, the coarse structure can be obtained from the fine structure by replacing every vertex of degree 2 by an edge.

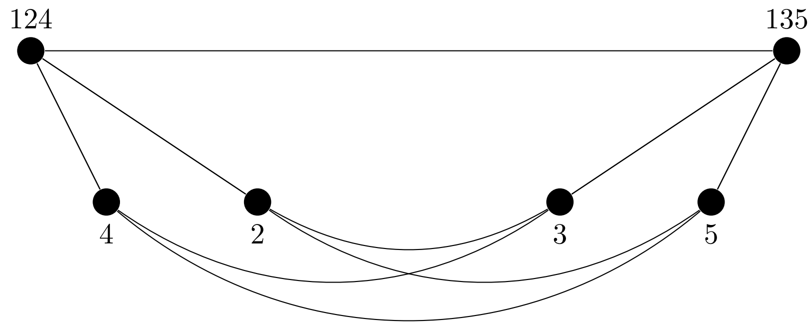

FIGURE 9. The underlying complex of the coarse Bergman fan $B_c(M_1)$.

In the minimal nested set complex $\mathcal{N}(M_1)$, we also remove the rays corresponding to the disconnected flats $\{2,3\}$, $\{2,5\}$, $\{3,4\}$, and $\{4,5\}$, but we keep the ray corresponding to the connected singleton $\{1\}$.

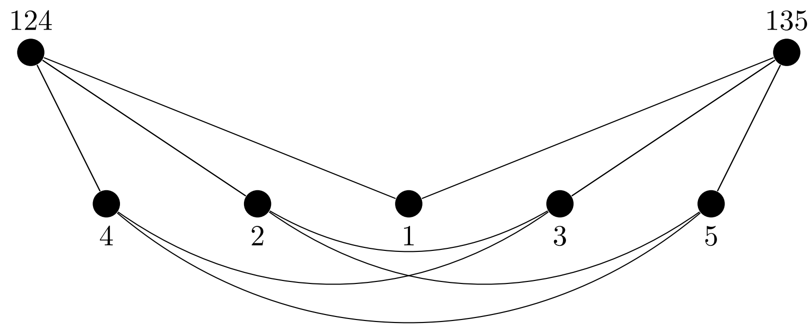

FIGURE 10. The minimal nested set complex $\mathcal{N}(M_1)$.

2.2. Automorphisms of matroid fans

In this section, we define automorphisms of matroid fans and we study how the automorphisms groups with respect to different fan structures are related to each other and to the group of matroid automorphisms.

Let M be a simple connected matroid of rank $r \geq 3$ on the ground set E.

Definition 2.2.1 ([SW23, Definition 2.5])**.** Let Σ be a fan with support $\operatorname{trop}(M)$. An *automorphism* of Σ is the $\mathbb{R}$-linear extension of an automorphism of the lattice $\mathbb{Z}^E/\langle v_E\rangle$ such that the image of every cone of Σ is again a cone of Σ.

We denote by $\operatorname{Aut}(\Sigma)$ the automorphism group of such a fan Σ. Note that $\operatorname{trop}(M)$ generates $\mathbb{R}^E/\langle v_E\rangle$ since M is simple, thus an automorphism φ of Σ is uniquely determined by its restriction $\varphi|_{\operatorname{trop}(M)}$.

Lemma 2.2.2. *Let φ be the $\mathbb{R}$-linear extension of an automorphism of $\mathbb{Z}^E/\langle v_E\rangle$.*

(1) *If φ is an automorphism of a fan Σ with support $\operatorname{trop}(M)$, then $\varphi(\operatorname{trop}(M)) = \operatorname{trop}(M)$.*
(2) *If $\varphi(\operatorname{trop}(M)) = \operatorname{trop}(M)$, then φ is an automorphism of the coarse Bergman fan $B_c(M)$.*

In particular, $\operatorname{Aut}(B_f(M))$ *and* $\operatorname{Aut}(B_m(M))$ *are subgroups of* $\operatorname{Aut}(B_c(M))$.

Proof. Let φ be an automorphism of a fan Σ with support $\operatorname{trop}(M)$. Then by definition φ is an automorphism of the vector space $\mathbb{R}^E/\langle v_E\rangle$, thus φ preserves the dimensions of all cones of Σ. In particular, $\varphi(\operatorname{trop}(M)) = \operatorname{trop}(M)$.

For (2), assume $\varphi(\operatorname{trop}(M)) = \operatorname{trop}(M)$. Then the cones $\varphi(\rho)$, where ρ ranges over the cones of the coarse Bergman fan $B_c(M)$, form a fan Σ' with support $\operatorname{trop}(M)$. By [Ham14, Proposition 3.4.1], the coarse Bergman fan $B_c(M)$ is the coarsest fan on $\operatorname{trop}(M)$, thus Σ' is a triangulation of $B_c(M)$. On the other hand, Σ' has the same number of cones as $B_c(M)$, hence $\Sigma' = B_c(M)$. This shows that φ maps cones of $B_c(M)$ to cones of $B_c(M)$. ☐

For a permutation f of the ground set E let $\varphi_f \colon \mathbb{R}^E/\langle v_E\rangle \to \mathbb{R}^E/\langle v_E\rangle$ be the automorphism induced by the map

$$\mathbb{Z}^E \to \mathbb{Z}^E,\ v_e \mapsto v_{f(e)}.$$

Lemma 2.2.3. *Let f be a matroid automorphism of M. Then φ_f is an automorphism of $B_*(M)$ for every fan structure $* \in \{f, c, m\}$.*

Proof. If f is a matroid automorphism of M, then it maps flags of flats to flags of flats and nested sets to nested sets. Hence the automorphism φ_f is an automorphism of both the fine Bergman fan $B_f(M)$ and the minimal nested set fan $B_m(M)$. Now the claim for the coarse Bergman fan follows from Lemma 2.2.2. ☐

Proposition 2.2.4. *Let φ be an automorphism of the coarse Bergman fan $B_c(M)$ that induces a permutation of the rays corresponding to flats of rank 1. Then φ is induced by a matroid automorphism of M.*

Proof. For any subset $S \subseteq E$ we have the equivalence

$$v_S \in \operatorname{trop}(M) \iff S \text{ is a flat of } M \tag{$\star$}$$

by Theorem 2.1.1. Assume there exists a permutation $f\colon E \to E$ such that $\varphi(\langle v_e \rangle) = \langle v_{f(e)} \rangle$ for all $e \in E$. Since M is connected, for every $e \in E$ we have

$$-v_{f(e)} = v_{E \setminus \{f(e)\}} \notin \operatorname{trop}(M)$$

by $(\star)$ and thus $\langle \varphi(v_e) \rangle_{\geq 0} = \langle v_{f(e)} \rangle_{\geq 0}$. Moreover, by definition φ is derived from a lattice automorphism, hence $\varphi = \varphi_f$.

It remains to show that f is an automorphism of M. If F is a flat of M, then $v_F \in \operatorname{trop}(M)$ by $(\star)$. Since φ is an automorphism of $B_c(M)$, we deduce $\varphi(v_F) = v_{f(F)} \in \operatorname{trop}(M)$, hence $f(F)$ is a flat of M by $(\star)$. $\square$

Depending on the matroid M and on the chosen fan structure, there may or may not exist fan automorphisms that are not induced by matroid automorphisms. For the fine Bergman fan $B_f(M)$, there is the following result:

Theorem 2.2.5 ([SW23, Theorem 6.3])**.** *Let M be a simple matroid that is not totally disconnected. Then every automorphism of the fine Bergman fan $B_f(M)$ is induced by a matroid automorphism.*

For the coarse Bergman fan, however, Shaw and Werner introduced combinatorial Cremona maps as examples for fan automorphisms that are not induced by matroid automorphisms. We will review their construction in Chapter 3.

2.3. Hyperplane arrangements and the Cremona transformation

Let V be a d-dimensional vector space over some field K.

Definition 2.3.1. A *(linear) hyperplane* of V is a linear subspace $H \subseteq V$ with $\dim_K(H) = \dim_K(V) - 1$. A finite set $\mathcal{A} = \{H_1, \ldots, H_k\}$ of hyperplanes of V is also called a *hyperplane arrangement* in V.

Proposition 2.3.2. *Let $\mathcal{A}$ be a hyperplane arrangement in V. Then*

$$2^{\mathcal{A}} \to \mathbb{N}_0,\ S \mapsto \operatorname{codim}_K\big(\bigcap_{H \in S} H\big)$$

defines a rank function on $\mathcal{A}$.

Proof. By choosing normal vectors of the hyperplanes in $\mathcal{A}$, we may write $\mathcal{A} = \{v_1^\perp, \ldots, v_k^\perp\}$ for some $v_1, \ldots, v_k \in V$. Then for any set $I \subseteq \{1, \ldots, k\}$ we have

$$\operatorname{codim}_K\big(\bigcap_{i \in I} v_i^\perp\big) = \operatorname{codim}_K(\langle v_i \mid i \in I\rangle^\perp) = \dim_K\langle v_i \mid i \in I\rangle.$$

Hence $M(\mathcal{A})$ is isomorphic to the vector matroid associated to the vectors $\{v_1, \ldots, v_k\}$. □

In this way, every hyperplane arrangement $\mathcal{A}$ induces a simple matroid $M(\mathcal{A})$. Let $\pi\colon V \to \mathbb{P}(V)$ be the canonical projection to the projective space $\mathbb{P}(V)$. If $\mathcal{A}$ is a hyperplane arrangement in V, then $\mathcal{A}' := \{\pi(H)\}$ is a hyperplane arrangement in $\mathbb{P}(V)$ and we denote by $\Omega_{\mathcal{A}'} := \mathbb{P}(V) \setminus \bigcup \mathcal{A}'$ the *arrangement complement.*

Let $d \geq 3$ and fix coordinates $x_0, \ldots, x_d$ on the projective space $\mathbb{P}^d_K$. Then the *standard Cremona transformation* is the birational map

$$\operatorname{crem}\colon \mathbb{P}^d_K \dashrightarrow \mathbb{P}^d_K, \quad [x_0 : \ldots : x_d] \mapsto \left[\frac{1}{x_0} : \ldots : \frac{1}{x_d}\right],$$

which can also be represented by

$$[x_0 : \ldots : x_d] \mapsto \left[\prod_{\substack{i=0\\ i\neq 0}}^{d} x_d : \prod_{\substack{i=0\\ i\neq 1}}^{d} x_d : \ldots : \prod_{\substack{i=0\\ i\neq d}}^{d} x_d\right].$$

We denote by $\mathbb{T}^d_K \subseteq \mathbb{P}^d_K$ the standard torus.

Lemma 2.3.3. *Let $U \subseteq \mathbb{P}^d_K$ be an open subscheme. Then* crem *induces an automorphism of U if and only if $U \subseteq \mathbb{T}^d_K$ and* $\operatorname{crem}(U) \subseteq U$.

Proof. crem is an involutive automorphism of $\mathbb{T}^d_K$ and contracts all coordinate hyperplanes $V(x_i)$ (intersected with the domain of definition of crem) to a single point. Thus $U \subseteq \mathbb{T}^d_K$ is a necessary condition for $\operatorname{crem}|_U$ to be injective. Clearly, $\operatorname{crem}|_U \in \operatorname{Aut}(U)$ implies $\operatorname{crem}(U) \subseteq U$.

Conversely, let $U \subseteq \mathbb{T}^d_K$ and assume that $\operatorname{crem}(U) \subseteq U$. Since crem is involutive, $\operatorname{crem}|_U \circ \operatorname{crem}|_U = (\operatorname{crem} \circ \operatorname{crem})|_U = \operatorname{id}_U$, hence $\operatorname{crem}|_U$ is an automorphism. □

Note that $\operatorname{crem}(U) \subseteq U$ is equivalent to $\operatorname{crem}(\mathbb{T}^d_K \setminus U) \subseteq \mathbb{T}^d_K \setminus U$: If $\operatorname{crem}(U) \subseteq U$ and $\operatorname{crem}(x) \in U$ for some $x \in \mathbb{T}^d_K \setminus U$, then $\operatorname{crem}(\operatorname{crem}(x)) \in U$ contradicts $\operatorname{crem} \circ \operatorname{crem} = \operatorname{id}$, and vice versa.

Lemma 2.3.4. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{P}^d_K$ and let $V(f) \subseteq \mathbb{P}^d_K$ be an irreducible hypersurface. If a dense subset of $V(f)$ is contained in $\bigcup \mathcal{A}$, then $V(f) \in \mathcal{A}$, so in particular* $\deg(f) = 1$.

Proof. By assumption we have

$$V(f) = \overline{V(f) \cap \bigcup \mathcal{A}} = \overline{\bigcup_{H \in \mathcal{A}} \big(V(f) \cap H\big)} = \bigcup_{H \in \mathcal{A}} \big(V(f) \cap H\big),$$

using that $\mathcal{A}$ is finite. Since $V(f)$ is irreducible, there is a hyperplane $H \in \mathcal{A}$ with $V(f) \subseteq H$. Moreover, H is irreducible and $\dim V(f) = \dim H$, hence $V(f) = H$. □

To a collection $Z = \{Z_{ij}\}_{0 \le i < j \le d}$ of subsets $Z_{ij} \subseteq K^\times$ we associate the hyperplane arrangement

$$\mathcal{A}(Z) := \{V(x_i) \mid 0 \le i \le d\} \cup \{V(x_i + z x_j) \mid 0 \le i < j \le d,\ z \in Z_{ij}\}.$$

Theorem 2.3.5. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{P}^d_K$. The standard Cremona transformation* crem *induces an automorphism of the complement $\Omega_{\mathcal{A}}$ if and only if $\mathcal{A} = \mathcal{A}(Z)$ for some collection Z as above with the property that all sets Z_{ij} are closed under taking multiplicative inverses.*

Proof. Consider the hyperplane arrangement $\mathcal{A}(Z)$ for some Z as above. Then by construction $\Omega_{\mathcal{A}(Z)} \subseteq \mathbb{T}^d_K$. Moreover, for every hyperplane $H := V(x_i + z x_j) \in \mathcal{A}(Z)$ we have

$$\mathrm{crem}(H \cap \mathbb{T}^d_K) = V(x_i + z^{-1} x_j) \cap \mathbb{T}^d_K.$$

Since Z_{ij} is closed under taking multiplicative inverses, we have $V(x_i + z^{-1} x_j) \in \mathcal{A}(Z)$ and hence

$$\mathrm{crem}\big(\bigcup \mathcal{A}(Z) \cap \mathbb{T}^d_K\big) \subseteq \bigcup \mathcal{A}(Z) \cap \mathbb{T}^d_K.$$

By Lemma 2.3.3, the Cremona transformation induces an automorphism of $\Omega_{\mathcal{A}(Z)} = \mathbb{T}^d_K \setminus (\bigcup \mathcal{A}(Z) \cap \mathbb{T}^d_K)$.

Conversely, assume that $\mathrm{crem}|_{\Omega_{\mathcal{A}}}$ is an automorphism. Then

$$\{V(x_i) \mid 0 \le i \le d\} \subseteq \mathcal{A} \quad \text{and} \quad \mathrm{crem}\big(\bigcup \mathcal{A} \cap \mathbb{T}^d_K\big) \subseteq \bigcup \mathcal{A} \cap \mathbb{T}^d_K \qquad (\star)$$

by Lemma 2.3.3. Let $H := V(a_0 x_0 + \cdots + a_d x_d)$ be a hyperplane in $\mathcal{A}$ that is not a coordinate hyperplane, i.e., $|\{k \mid a_k \neq 0\}| \ge 2$.

Claim: $|\{k \mid a_k \neq 0\}| = 2$.

Assuming the claim, we get $H = V(a_i x_i + a_j x_j) = V(x_i + \frac{a_j}{a_i} x_j)$ for some $0 \le i < j \le d$ and some $z := \frac{a_j}{a_i} \in K^\times$. By writing all such hyperplanes H in this form, we obtain a unique collection Z with $\mathcal{A} = \mathcal{A}(Z)$. For every hyperplane $H = V(x_i + z x_j) \in \mathcal{A}$ we have

$$\mathrm{crem}(H \cap \mathbb{T}^d_K) = V(x_i + z^{-1} x_j) \cap \mathbb{T}^d_K.$$

From $(\star)$ we deduce $V(x_i + z^{-1} x_j) \subseteq \bigcup \mathcal{A}$ and hence $V(x_i + z^{-1} x_j) \in \mathcal{A}$ by Lemma 2.3.4. This implies that all sets Z_{ij} are closed under taking multiplicative inverses. □

Proof of the claim. Assume that $|\{k \mid a_k \neq 0\}| > 2$. Let $V(f) \subseteq \mathbb{P}^d_K$ be the hypersurface cut out by the homogeneous polynomial

$$f := \frac{a_0 x_1 \cdots x_d + a_1 x_0 x_2 \cdots x_d + \cdots + a_d x_0 \cdots x_{d-1}}{\prod_{\{k \mid a_k = 0\}} x_k}.$$

Then the equation

$$\begin{aligned}
\operatorname{crem}(H \cap \mathbb{T}^d_K) &= V\Big(\frac{a_0}{x_0} + \cdots + \frac{a_d}{x_d}\Big) \cap \mathbb{T}^d_K \\
&= V(a_0 x_1 \cdots x_d + a_1 x_0 x_2 \cdots x_d + \cdots + a_d x_0 \cdots x_{d-1}) \cap \mathbb{T}^d_K \\
&= V(f) \cap \mathbb{T}^d_K
\end{aligned}$$

and $(\star)$ imply

$$V(f) \subseteq \operatorname{crem}(H \cap \mathbb{T}^d_K) \cup (\mathbb{P}^d_K \setminus \mathbb{T}^d_K) \subseteq \bigcup \mathcal{A}.$$

By construction, f is not divisible by any non-trivial monomial. It suffices to show that f is irreducible, since then Lemma 2.3.4 would imply $V(f) \in \mathcal{A}$, contradicting

$$\deg(f) = d - |\{k \mid a_k = 0\}| = |\{k \mid a_k \neq 0\}| - 1 \geq 2.$$

Let $f = g \cdot h$ be a factorization of f. Then

$$\begin{aligned}
1 &= |\{k \mid a_i \neq 0\}| - (|\{k \mid a_i \neq 0\}| - 1) \\
&= \sum_i \deg_{x_i}(f) - \deg(f) \\
&= \sum_i \big(\deg_{x_i}(g) + \deg_{x_i}(h)\big) - \big(\deg(g) + \deg(h)\big) \\
&= \underbrace{\Big(\sum_i \deg_{x_i}(g) - \deg(g)\Big)}_{\geq 0} + \underbrace{\Big(\sum_i \deg_{x_i}(h) - \deg(h)\Big)}_{\geq 0},
\end{aligned}$$

where deg without index denotes the total degree. Without loss of generality, we may assume that the first summand is 0. This means that g is a monomial, hence g must be constant. $\square$

Corollary 2.3.6. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{P}^d_K$ such that* crem *induces an automorphism of $\Omega_{\mathcal{A}}$. Then the set of coordinate hyperplanes $B := \{V(x_0), \ldots, V(x_d)\}$ is a Cremona basis of the associated matroid $M(\mathcal{A})$.*

Proof. By Theorem 2.3.5, $\mathcal{A}$ is of the form $\mathcal{A}(Z)$ for some Z. If H is a hyperplane of the form $V(x_i + z x_j)$, then $\operatorname{supp}_B(H) = \{V(x_i), V(x_j)\}$, hence the coordinate hyperplanes form a Cremona basis of $M(\mathcal{A})$. $\square$

Example 2.3.7. Consider the hyperplane arrangement

$$\mathcal{A} := \{V(x_0), V(x_1), V(x_2), V(x_0 - x_1), V(x_0 - x_2)\}$$

in $\mathbb{P}^2_K$. Then by Theorem 2.3.5 the Cremona transformation induces an automorphism of $\Omega_{\mathcal{A}}$. The matroid $M(\mathcal{A})$ is isomorphic to the matroid M_1 from Example 1.2.6.

CHAPTER 3

Cremona automorphisms of matroid fans

3.1. Construction

In this section, we review the construction of Cremona automorphisms of the coarse Bergman fan, based on [SW23, Chapter 8].

Let M be a simple connected matroid of rank $r \geq 3$ on the ground set E that admits a Cremona basis B.

Definition 3.1.1 ([SW23, Definition 8.1])**.** The *Cremona map* with respect to B is defined as the $\mathbb{R}$-linear map $\operatorname{crem}_B \colon \mathbb{R}^E \to \mathbb{R}^E$ given by

$$v_b \mapsto v_{\operatorname{cl}(B \setminus \{b\})} \text{ for all } b \in B, \quad v_e \mapsto v_e \text{ for all } e \in E \setminus B.$$

In other words, the vector corresponding to a basis element $b \in B$ is sent to the indicator vector of the coordinate hyperplane $\operatorname{cl}(B \setminus \{b\})$, but the vectors corresponding to non-basis elements are fixed. Since M has rank at least 3, the Cremona map is not induced by a matroid automorphism in the sense of Lemma 2.2.3. The following proposition determines the images of coordinate flats under the Cremona map.

Proposition 3.1.2 ([SW23, Lemma 8.4])**.** *For all $I \subseteq B$ we have*

$$\operatorname{crem}_B(v_{\operatorname{cl}(I)}) = v_{\operatorname{cl}(B \setminus I)} + (|I| - 1)v_E.$$

In particular, the Cremona map crem_B induces an involution on $\mathbb{R}^E / \langle v_E \rangle$.

Proof. Let $I \subseteq B$. Then we compute

$$\begin{aligned}
\operatorname{crem}_B(v_{\operatorname{cl}(I)}) &= \operatorname{crem}_B\Bigg(\sum_{b \in I} v_b + \sum_{\substack{e \in E \setminus B \\ \operatorname{supp}_B(e) \subseteq I}} v_e \Bigg) \\
&= \sum_{b \in I} v_{\operatorname{cl}(B \setminus \{b\})} + \sum_{\substack{e \in E \setminus B \\ \operatorname{supp}_B(e) \subseteq I}} v_e \\
&= \sum_{b \in I} \Bigg(\sum_{b' \in B \setminus \{b\}} v_{b'} + \sum_{\substack{e \in E \setminus B \\ b \notin \operatorname{supp}_B(e)}} v_e \Bigg) + \sum_{\substack{e \in E \setminus B \\ \operatorname{supp}_B(e) \subseteq I}} v_e \\
&= |I| \Bigg(\sum_{b' \in B \setminus I} v_{b'} + \sum_{\substack{e \in E \setminus B \\ \operatorname{supp}_B(e) \cap I = \emptyset}} v_e \Bigg) \\
&\quad + (|I| - 1) \Bigg(\sum_{b' \in I} v_{b'} + \sum_{\substack{e \in E \setminus B \\ |\operatorname{supp}_B(e) \cap I| = 1}} v_e + \sum_{\substack{e \in E \setminus B \\ \operatorname{supp}_B(e) \subseteq I}} v_e \Bigg) \\
&= v_{\operatorname{cl}(B \setminus I)} + (|I| - 1)v_E.
\end{aligned}$$

In particular, we have $\mathrm{crem}_B(v_E) = (r-1)v_E$, thus crem_B descends to a $\mathbb{R}$-linear map $\mathbb{R}^E/\langle v_E\rangle$. Moreover, for all $b \in B$ we have $\mathrm{crem}_B(v_{\mathrm{cl}(B\setminus\{b\})}) = v_b + (r-2)v_E$, hence the induced map is an involution. $\square$

From now on, by the Cremona map crem_B we mean the induced map $\mathbb{R}^E/\langle v_E\rangle \to \mathbb{R}^E/\langle v_E\rangle$ on the quotients.

Theorem 3.1.3 ([SW23, Theorem 8.3])**.** *The Cremona map* crem_B *preserves the tropicalization* $\mathrm{trop}(M) \subseteq \mathbb{R}^E/\langle v_E\rangle$.

To prove this, it suffices to show that every cone of the fine Bergman fan is contained in a cone of the minimal nested set fan. For details, see [SW23, Theorem 8.3].

Example 3.1.4. As in the introduction, let $M(\mathcal{A})$ be the matroid associated to the hyperplane arrangement

$$\mathcal{A} := \{V(x), V(y), V(z), V(x-y), V(x-z), V(y-z)\}.$$

Then the Cremona map $\mathrm{crem}_B \colon \mathbb{R}^6 \to \mathbb{R}^6$ is given by the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

which is the transposition of the matrix from the introduction, and the induced map $\mathrm{crem}_B \colon \mathbb{R}^6/\langle v_E\rangle \to \mathbb{R}^6/\langle v_E\rangle$ on the quotients is given by

$$\begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{pmatrix},$$

with respect to the basis $\{v_1, \dots, v_5\}$.

Example 3.1.5. Let K be a field and consider the hyperplane arrangement

$$\mathcal{A} := \{V(x), V(y), V(z), V(x-y), V(x-z)\}$$

in $\mathbb{P}^2_K$. Then the coordinate hyperplanes $B := \{V(x), V(y), V(z)\}$ form a Cremona basis of the corresponding matroid and by Theorem 2.3.5 the Cremona transformation induces an automorphism of the arrangement complement $\Omega_{\mathcal{A}}$. The map

$$j \colon \mathbb{P}^2_K \to \mathbb{P}^4_K, \quad [x:y:z] \mapsto [x:y:z:x-y:x-z]$$

identifies the complement $\Omega_{\mathcal{A}}$ with the linear variety $V(x_0 - x_1 - x_3, x_0 - x_2 - x_4)$ in the torus $\mathbb{T}^4_K \subseteq \mathbb{P}^4_K$. If we define the monomial map $\varphi \colon \mathbb{T}^4_K \to \mathbb{T}^4_K$ by

$$[x_0 : x_1 : x_2 : x_3 : x_4] \mapsto [x_1x_2 : x_0x_2 : x_0x_1 : -x_2x_3 : -x_1x_4],$$

then the diagram

$$\begin{tikzcd} \Omega_{\mathcal{A}} \arrow[r, hook, "j"] \arrow[d, "\text{crem}"'] & \mathbb{T}^4_K \arrow[d, "\varphi"] \\ \Omega_{\mathcal{A}} \arrow[r, hook, "j"] & \mathbb{T}^4_K \end{tikzcd} \qquad \begin{tikzcd} {[x:y:z]} \arrow[r, mapsto] \arrow[d, mapsto] & {[x:y:z:x-y:x-z]} \arrow[d, mapsto] \\ {[yz:xz:xy]} \arrow[r, mapsto] & {[yz:xz:xy:z(y-x):y(z-x)]} \end{tikzcd}$$

commutes. The embedding $\mathrm{Aut}(\Omega_{\mathcal{A}}) \hookrightarrow \mathrm{Aut}(B_c(M))$ from [Kur17, Theorem 7.7] maps crem to the tropicalization $\mathrm{trop}(\varphi)$ of this map φ. Thus $\mathrm{trop}(\varphi)$ is the linear map given by multiplication with the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

and hence coincides with the definition of the combinatorial Cremona automorphism crem_B.

The following example shows that Cremona maps may not preserve the minimal nested set structure:

Example 3.1.6. Let M_1 be the simple rank 3 matroid on the ground set $\{1, \dots, 5\}$ with rank 2 circuits $\{1, 2, 4\}$ and $\{1, 3, 5\}$. Then $B := \{1, 2, 3\}$ is a Cremona basis of M, but the coordinate flat $\mathrm{cl}(\{2, 3\}) = \{2, 3\}$ is disconnected. This means that the ray $\langle v_1 \rangle$ is a cone of the minimal nested set fan, but the ray $\langle \mathrm{crem}_B(v_1) \rangle = \langle v_{23} \rangle$ is not, hence the Cremona map crem_B does not induce an automorphism of the minimal nested set fan $B_m(M)$.

Theorem 3.1.7. *Assume that M has a Cremona basis B such that $G_B(M)$ is a complete graph. Then the Cremona map* crem_B *induces an automorphism of the minimal nested set fan $B_m(M)$.*

Proof. Since $G_B(M)$ is a complete graph, all coordinate flats of M are connected by Lemma 1.5.6 and thus the Cremona map crem_B preserves the rays of the minimal nested set fan.

It remains to show that crem_B maps nested sets to nested sets. Let $\mathcal{S} \subseteq \mathcal{L}_{\mathrm{conn}}(M) \setminus \{\emptyset, E\}$ be a nested set of proper connected flats of M and let $\mathcal{A}$ be an antichain in $\mathcal{S}' := \{\mathrm{crem}_B(F) \mid F \in \mathcal{S}\}$. If $\mathcal{A} \subseteq \mathcal{S}$, then $\bigvee_{F \in \mathcal{A}} F$ is disconnected by assumption, so we may assume that $\mathcal{A}$ contains a coordinate flat.

On the other hand, we claim that every antichain $\mathcal{A} \subseteq \mathcal{S}'$ contains at most one coordinate flat. Indeed, if $F_1, F_2 \in \mathcal{S}'$ are coordinate flats, then $\mathrm{crem}_B(F_1)$ and $\mathrm{crem}_B(F_2)$ are coordinate flats in $\mathcal{S}$ and their join is connected since $G_B(M)$ is a complete graph. Since $\mathcal{S}$ is nested, $\mathrm{crem}_B(F_1)$ and $\mathrm{crem}_B(F_2)$ are comparable, hence the same is true for F_1 and F_2.

Now let $I \subseteq B$ such that $\mathrm{cl}(I)$ is the unique coordinate flat in $\mathcal{A}$. We claim that $\mathrm{supp}_B(F) \subseteq B \setminus I$ for all flats $F \in \mathcal{A} \setminus \{\mathrm{cl}(I)\}$. Otherwise, if $\mathrm{supp}_B(F) \cap I \neq \emptyset$ for some flat F, then $\{F, \mathrm{cl}(B \setminus I)\}$ is an antichain in $\mathcal{S}$ since $\mathrm{crem}_B(\mathrm{cl}(I)) = \mathrm{cl}(B \setminus I)$, thus $F \vee \mathrm{cl}(B \setminus I)$ is disconnected. Then $F \vee \mathrm{cl}(B \setminus I)$ is not a coordinate flat and not support-connected by Corollary 1.5.3, hence $F \subseteq \mathrm{cl}(I)$, contradicting the antichain property of $\mathcal{A}$.

We deduce that $\bigvee_{F\in\mathcal{A}\setminus\{\mathrm{cl}(I)\}} F$ is a flat with support contained in $B \setminus I$. By assumption it is disconnected and thus not a coordinate flat. Hence $\bigvee_{F\in\mathcal{A}} F = \mathrm{cl}(I) \vee (\bigvee_{F\in\mathcal{A}\setminus\{\mathrm{cl}(I)\}} F)$ is disconnected by Lemma 1.5.11. $\square$

3.2. The Cremona group of a matroid

Let M be a simple connected matroid of rank $r \geq 3$.

Definition 3.2.1. The *Cremona group* $\mathrm{Cr}(M)$ of M is the subgroup of $\mathrm{Aut}(B_c(M))$ generated by matroid automorphisms and Cremona automorphisms.

By definition, $\mathrm{Cr}(M) = \mathrm{Aut}(M)$ if M does not admit a Cremona basis.

Corollary 3.2.2. *Assume that M has a Cremona basis B such that $G_B(M)$ is a complete graph. Then* $\mathrm{Cr}(M) \subseteq \mathrm{Aut}(B_m(M))$.

Proof. If M has a Cremona basis B such that $G_B(M)$ is a complete graph, then by Corollary 1.7.3 this is true for every Cremona basis of M. Thus the claim follows from Theorem 3.1.7. □

Proposition 3.2.3. *If M has exactly one Cremona basis, then* $\mathrm{Cr}(M) \cong \mathrm{Aut}(M) \times \mathbb{Z}/2\mathbb{Z}$.

Proof. Let B be the unique Cremona basis of M and let f be a matroid automorphism of M. Since matroid automorphisms preserve Cremona bases, we have $f(B) = B$. Thus for all $b \in B$ we have

$$\begin{aligned}(\mathrm{crem}_B \circ f)(v_b) &= \mathrm{crem}_B(v_{f(b)}) = v_{\mathrm{cl}(B \setminus \{f(b)\})} \\ &= f(v_{\mathrm{cl}(B\setminus\{b\})}) = (f \circ \mathrm{crem}_B)(v_b)\end{aligned}$$

and for all $e \in E \setminus B$ we have

$$(\mathrm{crem}_B \circ f)(v_e) = \mathrm{crem}_B(v_{f(e)}) = v_{f(e)} = f(v_e) = (f \circ \mathrm{crem}_B)(v_e),$$

hence crem_B commutes with f. Now the claim follows since crem_B is an involution by Proposition 3.1.2. □

We now show that Cremona automorphisms with respect to different Cremona bases are conjugate by matroid automorphisms.

Proposition 3.2.4. *Assume that M has two Cremona bases B and B'. Then the Cremona automorphisms crem_B and $\mathrm{crem}_{B'}$ satisfy the relation*

$$f_{BB'} \circ \mathrm{crem}_B = \mathrm{crem}_{B'} \circ f_{BB'},$$

where $f_{BB'}$ is the Cremona base change automorphism from Theorem 1.7.2.

Proof. We verify that $f_{BB'} \circ \mathrm{crem}_B$ and $\mathrm{crem}_{B'} \circ f_{BB'}$ agree on the spanning set $\{v_e \mid e \in E\}$ of $\mathbb{R}^E/\langle v_E \rangle$. First let $b \in B \cap B'$. Then $f_{BB'}(b) = b$ and thus

$$\begin{aligned}(f_{BB'} \circ \mathrm{crem}_B)(v_b) &= f_{BB'}(v_{\mathrm{cl}(B\setminus\{b\})}) = v_{\mathrm{cl}(B'\setminus\{b\})} \\ &= \mathrm{crem}_{B'}(v_b) = (\mathrm{crem}_{B'} \circ f_{BB'})(v_b).\end{aligned}$$

Now let $b \in B \setminus B'$ and $e \in B' \setminus B$ with $f_{BB'}(b) = e$. Then

$$\begin{aligned}(f_{BB'} \circ \mathrm{crem}_B)(v_b) &= f_{BB'}(v_{\mathrm{cl}(B\setminus\{b\})}) = v_{\mathrm{cl}(B'\setminus\{e\})} \\ &= \mathrm{crem}_{B'}(v_e) = (\mathrm{crem}_{B'} \circ f_{BB'})(v_b)\end{aligned}$$

and

$$(f_{BB'} \circ \mathrm{crem}_B)(v_e) = f_{BB'}(v_e) = v_b = \mathrm{crem}_{B'}(v_b) = (\mathrm{crem}_{B'} \circ f_{BB'})(v_e).$$

The last case where $e \in E \setminus (B \cup B')$ is trivial:

$$(f_{BB'} \circ \mathrm{crem}_B)(v_e) = f_{BB'}(v_e) = v_e = \mathrm{crem}_{B'}(v_e) = (\mathrm{crem}_{B'} \circ f_{BB'})(v_e).$$

This completes the proof. $\square$

In particular, we deduce the following theorem.

Theorem 3.2.5. *For every Cremona basis B of M, the Cremona group $\mathrm{Cr}(M)$ is generated by matroid automorphisms and the Cremona automorphism crem_B.*

Proof. Follows from Proposition 3.2.4. $\square$

Example 3.2.6. Let M_1 be the matroid from Example 1.2.6 on the ground set $\{1, \ldots, 5\}$ with automorphism group $\mathrm{Aut}(M_1) \cong D_4$. We saw in Example 1.3.5 that M_1 has four Cremona bases, $B_{23} := \{1, 2, 3\}$, $B_{25} := \{1, 2, 5\}$, $B_{43} := \{1, 4, 3\}$, and $B_{45} := \{1, 4, 5\}$, and we computed the Cremona base change automorphisms of M_1 in Example 1.7.4. In Example 2.1.9 we computed the coarse Bergman fan $B_c(M_1)$:

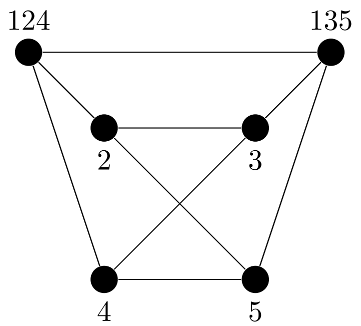

FIGURE 11. The underlying complex of the coarse Bergman fan $B_c(M_1)$.

We deduce that

$$\mathrm{Aut}(B_c(M_1)) \subseteq (S_3 \times S_3) \rtimes \mathbb{Z}/2\mathbb{Z}$$

since the complement graph of the underlying complex of $B_c(M_1)$ is the disjoint union of two triangles.

The matroid automorphisms induce automorphisms of the coarse Bergman fan that can be represented by permutations of the set of vertices

$$\{2, 3, 4, 5, 124, 135\}.$$

For example, the transpositions (2 4) and (3 5) are realized by matroid automorphisms as well as the permutation (2 3)(4 5)(124 135).

We compute the Cremona automorphisms:

$$\begin{aligned}
\mathrm{crem}_{B_{23}} &: v_1 \mapsto v_{\{2,3\}}, \quad v_2 \mapsto v_{\{1,3,5\}}, \quad v_3 \mapsto v_{\{1,2,4\}}, \quad v_4 \mapsto v_4, \quad v_5 \mapsto v_5 \\
\mathrm{crem}_{B_{25}} &: v_1 \mapsto v_{\{2,5\}}, \quad v_2 \mapsto v_{\{1,3,5\}}, \quad v_3 \mapsto v_3, \quad v_4 \mapsto v_4, \quad v_5 \mapsto v_{\{1,2,4\}} \\
\mathrm{crem}_{B_{43}} &: v_1 \mapsto v_{\{4,3\}}, \quad v_2 \mapsto v_2, \quad v_3 \mapsto v_{\{1,2,4\}}, \quad v_4 \mapsto v_{\{1,3,5\}}, \quad v_5 \mapsto v_5 \\
\mathrm{crem}_{B_{45}} &: v_1 \mapsto v_{\{4,5\}}, \quad v_2 \mapsto v_2, \quad v_3 \mapsto v_3, \quad v_4 \mapsto v_{\{1,3,5\}}, \quad v_5 \mapsto v_{\{1,2,4\}}
\end{aligned}$$

The Cremona group contains for example the map

$$(3\ 5) \circ \mathrm{crem}_{B_{23}} \circ \mathrm{crem}_{B_{25}},$$

which is the transposition (3 124). The matroid automorphisms together with this transposition generate $(S_3 \times S_3) \rtimes \mathbb{Z}/2\mathbb{Z}$, hence

$$\mathrm{Aut}(B_c(M_1)) = \mathrm{Cr}(M_1) = (S_3 \times S_3) \rtimes \mathbb{Z}/2\mathbb{Z}.$$

Proposition 3.2.7. *Assume that M has two Cremona bases B and B' with $|B \cap B'| = 1$. Then the Cremona automorphisms crem_B and $\mathrm{crem}_{B'}$ satisfy the relation*

$$\mathrm{crem}_B \circ \mathrm{crem}_{B'} = f_{BB'} \circ \mathrm{crem}_B,$$

where $f_{BB'}$ is the Cremona base change automorphism from Theorem 1.7.2.

Proof. As in the proof of Proposition 3.2.4, we verify that $\mathrm{crem}_B \circ \mathrm{crem}_{B'}$ and $f_{BB'} \circ \mathrm{crem}_B$ agree on the spanning set $\{v_e \mid e \in E\}$ of $\mathbb{R}^E/\langle v_E\rangle$. First let $b \in B \cap B'$. Then by assumption $\mathrm{cl}(B' \setminus \{b\})$ is a non-coordinate flat with respect to B and thus

$$\begin{aligned}(\mathrm{crem}_B \circ \mathrm{crem}_{B'})(v_b) &= \mathrm{crem}_B(v_{\mathrm{cl}(B'\setminus\{b\})}) = v_{\mathrm{cl}(B'\setminus\{b\})} \\ &= f_{BB'}(v_{\mathrm{cl}(B\setminus\{b\})}) = (f_{BB'} \circ \mathrm{crem}_B)(v_b).\end{aligned}$$

Now let $b \in B \setminus B'$ and $e \in B' \setminus B$ with $f_{BB'}(b) = e$. Then $\mathrm{cl}(B \setminus \{b\}) = \mathrm{cl}(B' \setminus \{e\})$, thus we compute

$$\begin{aligned}(\mathrm{crem}_B \circ \mathrm{crem}_{B'})(v_b) &= \mathrm{crem}_B(v_b) = v_{\mathrm{cl}(B\setminus\{b\})} = v_{\mathrm{cl}(B'\setminus\{e\})} \\ &= f_{BB'}(v_{\mathrm{cl}(B\setminus\{b\})}) = (f_{BB'} \circ \mathrm{crem}_B)(v_b)\end{aligned}$$

and

$$\begin{aligned}(\mathrm{crem}_B \circ \mathrm{crem}_{B'})(v_e) &= \mathrm{crem}_B(v_{\mathrm{cl}(B'\setminus\{e\})}) = \mathrm{crem}_B(v_{\mathrm{cl}(B\setminus\{b\})}) \\ &= v_b = f_{BB'}(v_e) = (f_{BB'} \circ \mathrm{crem}_B)(v_e).\end{aligned}$$

The last case where $e \in E \setminus (B \cup B')$ is trivial:

$$(\mathrm{crem}_B \circ \mathrm{crem}_{B'})(v_e) = \mathrm{crem}_B(v_e) = v_e = f_{BB'}(v_e) = (f_{BB'} \circ \mathrm{crem}_B)(v_e).$$

This completes the proof. □

Let $\mathrm{Aut}_{\mathrm{CB}}(M)$ be the subgroup of $\mathrm{Aut}(M)$ defined by

$$\mathrm{Aut}_{\mathrm{CB}}(M) := \{f \in \mathrm{Aut}(M) \mid f(B) = B \text{ for every Cremona basis } B \text{ of } M\}.$$

Theorem 3.2.8. *Assume that the contraction M/e is connected for all $e \in E$. Then $\mathrm{Cr}(M)/\mathrm{Aut}_{\mathrm{CB}}(M) \cong S_{k+1}$, where $k \in \mathbb{N}_0$ is the number of Cremona bases of M.*

Proof. By Theorem 1.7.6, the action of $\mathrm{Aut}(M)$ on the set of Cremona bases of M is surjective. By construction, $\mathrm{Aut}_{\mathrm{CB}}(M)$ is the kernel of this action, thus $\mathrm{Aut}(M)/\mathrm{Aut}_{\mathrm{CB}}(M) \cong S_k$. The same argument as in the proof of Proposition 3.2.3 shows that Cremona automorphisms commute with all elements in $\mathrm{Aut}_{\mathrm{CB}}(M)$. Let $\{B_1, \dots, B_k\}$ be the set of Cremona bases of M. Then for every $i \in \{1, \dots, k\}$ the Cremona automorphism crem_{B_i} can be identified with the transposition swapping i and $k+1$. Indeed, by Propositions 3.2.4 and 3.2.7 we have the relations

$$\mathrm{crem}_{B_i} \circ \mathrm{crem}_{B_j} = \mathrm{crem}_{B_j} \circ f_{B_iB_j} = f_{B_iB_j} \circ \mathrm{crem}_{B_i}$$

and these elements all correspond to the 3-cycle $(j\ i\ n+1)$. □

Corollary 3.2.9. *Assume that the contraction M/e is connected for all $e \in E$. Then the index of* $\mathrm{Aut}(M)$ *in* $\mathrm{Cr}(M)$ *is equal to the number of Cremona bases plus 1.*

Example 3.2.10. Let M_2 be the matroid from Example 1.2.7. Since $G_B(M_2)$ is a complete graph, Corollary 1.4.13 shows that the condition in Theorem 3.2.8 is fulfilled. We saw in Example 1.7.5 that M_2 has three Cremona bases and that $\mathrm{Aut}(M_2)$ is isomorphic to the symmetric group S_3. Hence $\mathrm{Aut}_{\mathrm{CB}}(M_2)$ is trivial and $\mathrm{Cr}(M_2) \cong S_4$.

3.3. Automorphisms of the minimal nested set complex in rank 3

For matroids of rank 3, the following is known:

Theorem 3.3.1 ([SW23, Corollary 9.5])**.** *Let M be a simple matroid of rank 3 which is not a non-trivial parallel connection. Then* $\mathrm{Aut}(B_c(M)) = \mathrm{Cr}(M)$.

In this section, we will show a similar result for the automorphism group $\mathrm{Aut}(\mathcal{N}(M))$ of the minimal nested set complex $\mathcal{N} := \mathcal{N}(M)$. The difference between $\mathrm{Aut}(\mathcal{N}(M))$ and $\mathrm{Aut}(B_m(M))$ is that there might be automorphisms of $\mathcal{N}$ that are not induced by lattice automorphisms.

Theorem 3.3.2. *Let M be a simple connected matroid of rank 3.*

(1) *If M is the matroid associated to a self-dual non-degenerate projective plane, then* $\mathrm{Aut}(M)$ *is a subgroup of* $\mathrm{Aut}(\mathcal{N}(M))$ *of index 2.*
(2) *Otherwise,* $\mathrm{Aut}(\mathcal{N}(M))$ *is generated by matroid automorphisms and Cremona automorphisms.*

We will prove this theorem in several steps. Since M has rank 3, the minimal nested set complex $\mathcal{N}(M)$ is a graph with vertices $\mathcal{E} \sqcup \mathcal{H}$, where $\mathcal{E}$ denotes the set of singletons (lines) of M and $\mathcal{H}$ the set of connected hyperplanes of M. For an automorphism f of $\mathcal{N}(M)$, we denote by

$$\mathcal{E}_{\mathrm{inv}}(f) := \{\ell \in \mathcal{E} \mid f(\ell) \in \mathcal{H}\} \quad \text{and} \quad \mathcal{H}_{\mathrm{inv}}(f) := \{H \in \mathcal{H} \mid f(H) \in \mathcal{E}\}$$

the set of lines and hyperplanes, respectively, whose rank is inverted by f. Clearly, $|\mathcal{E}_{\mathrm{inv}}(f)| = |\mathcal{H}_{\mathrm{inv}}(f)|$ and we call this number the *inversion index* $\mathrm{inv}(f)$ of f. We have $0 \leq \mathrm{inv}(f) \leq \min\{|\mathcal{E}|, |\mathcal{H}|\}$ and f is called *rank-preserving* if and only if $\mathrm{inv}(f) = 0$, which is equivalent to being a matroid automorphism, see for example [BdM06, Corollary 2.9].

Lemma 3.3.3. *Let f be an automorphism of the minimal nested set complex $\mathcal{N}(M)$ of M.*

(1) *If $H \in \mathcal{H}$ is a hyperplane with $H \cap \mathcal{E}_{inv}(f) \neq \emptyset$, then $H \in \mathcal{H}_{inv}(f)$.*
(2) *Let $\ell, \ell' \in \mathcal{E}_{inv}(f)$ with $\ell \neq \ell'$. Then ℓ and ℓ' are not adjacent in $\mathcal{N}(M)$ and have exactly one common neighbor in $\mathcal{N}(M)$, namely the unique hyperplane $\ell \vee \ell'$ containing both.*
(3) *Let $H, H' \in \mathcal{H}_{inv}(f)$ with $H \neq H'$. Then H and H' have exactly one common neighbor ℓ in $\mathcal{N}(M)$, namely $\ell = H \cap H'$. Moreover, $\ell \in \mathcal{E}_{inv}(f)$.*

Proof.

(1) Let $\ell \in H \cap \mathcal{E}_{\mathrm{inv}}(f)$. If $f(H)$ is a hyperplane, then $f(\ell)$ and $f(H)$ would be hyperplanes adjacent in $\mathcal{N}(M)$, contradiction.
(2) If ℓ and ℓ' were adjacent in $\mathcal{N}$, then $f(\ell)$ and $f(\ell')$ would be hyperplanes adjacent in $\mathcal{N}$, contradiction. This means that their closure $\mathrm{cl}(\{\ell, \ell'\})$ is an irreducible hyperplane of M and thus a common neighbor of ℓ and ℓ'. On the other hand, since two distinct hyperplanes of a simple rank 3 matroid have at most one common element, $f(\ell)$ and $f(\ell')$ have at most one common neighbor. The claim follows by applying f^{-1}.
(3) By assumption, $f(H)$ and $f(H')$ are lines and lie in $\mathcal{E}_{\mathrm{inv}}(f^{-1})$. Now (2) implies that $f(H)$ and $f(H')$ have a unique common neighbor in

$\mathcal{N}$, which is a hyperplane H''. Applying f^{-1} shows that $f^{-1}(H'')$ is the unique common neighbor of H and H', hence $\ell := f^{-1}(H'') \in \mathcal{E}_{\mathrm{inv}}(f)$. $\square$

Theorem 3.3.4. *If $\mathcal{N}$ has an automorphism f with inversion index $|\mathcal{E}|$, then M is the matroid associated to a self-dual non-degenerate projective plane. In this case,* $\mathrm{Aut}(M)$ *is a subgroup of* $\mathrm{Aut}(\mathcal{N})$ *of index 2.*

Proof. Let f be an automorphism of $\mathcal{N}$ with $\mathrm{inv}(f) = |\mathcal{E}|$. Then f gives a bijection between $\mathcal{E}$ and $\mathcal{H}$. Indeed, if $H \in \mathcal{H}$ is any hyperplane and $\ell \in H$, then $f(H)$ is adjacent to the hyperplane $f(\ell)$ and must be a line, hence $\mathcal{H}_{\mathrm{inv}}(f) = \mathcal{H}$. By Lemma 3.3.3 (2), $\mathcal{N}$ is a bipartite graph, so we can regard $\mathcal{N}$ as incidence structure with points $\mathcal{E}$ and lines $\mathcal{H}$. Moreover, Lemma 3.3.3 (2) and (3) translate exactly to the axioms of a projective plane. The projective plane is non-degenerate since $|\mathcal{H}| = |\mathcal{E}|$ implies that M has more than one irreducible hyperplane, and f corresponds to a self-duality of the plane.

Since $\mathcal{N}$ is a connected bipartite graph, any $f' \in \mathrm{Aut}(\mathcal{N})$ has inversion index 0 or $|\mathcal{E}|$. If f' is another automorphism of $\mathcal{N}$ with inversion index $|\mathcal{E}|$, then $f' \circ f$ is rank-preserving, thus $\mathrm{Aut}(\mathcal{N})$ is generated by $\mathrm{Aut}(M)$ and f. Since f^2 is rank-preserving, $\mathrm{Aut}(M)$ is a subgroup of $\mathrm{Aut}(\mathcal{N})$ of index 2. $\square$

Conversely, if M is a self-dual non-degenerate projective plane with points P, then the self-duality clearly corresponds to an automorphism f of the minimal nested set complex with $\mathrm{inv}(f) = |P|$.

Example 3.3.5. The *Fano matroid* is the simple rank 3 matroid F_7 on the ground set $E := \{1, \dots, 7\}$ with rank 2 flats $\{1, 2, 4\}$, $\{1, 3, 6\}$, $\{1, 5, 7\}$, $\{2, 3, 5\}$, $\{2, 6, 7\}$, $\{3, 4, 7\}$, and $\{4, 5, 6\}$. The map

$$\begin{aligned}&v_1 \mapsto v_{\{2,3,5\}},\ v_2 \mapsto v_{\{1,3,6\}},\ v_3 \mapsto v_{\{1,2,4\}},\ v_4 \mapsto v_{\{3,4,7\}},\\ &v_5 \mapsto v_{\{1,5,7\}},\ v_6 \mapsto v_{\{2,6,7\}},\ v_7 \mapsto v_{\{4,5,6\}}\end{aligned}$$

corresponding to the self-duality of F_7 induces an involution $\varphi\colon \mathbb{R}^E/\langle v_E\rangle \to \mathbb{R}^E/\langle v_E\rangle$ that preserves the tropicalization $\mathrm{trop}(M)$ and gives rise to an automorphism of the minimal nested set complex $\mathcal{N}(M)$.

However, φ is not an automorphism of the coarse Bergman fan since is not induced by a lattice automorphism: With respect to the basis $\{v_1, \dots, v_6\}$, we can represent φ by the matrix

$$\begin{pmatrix} 0 & 1 & 1 & -1 & 0 & -1 \\ 1 & 0 & 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 \end{pmatrix}$$

whose determinant is -8.

Proposition 3.3.6. *Let $f \in \mathrm{Aut}(\mathcal{N})$. Then* $\mathrm{inv}(f) \neq 2$.

Proof. Let $\mathcal{E}_{\mathrm{inv}}(f) = \{e_1, e_2\}$. Then $H := \mathrm{cl}(\{e_1, e_2\}) \in \mathcal{H}_{\mathrm{inv}}(f)$ by Lemma 3.3.3 (1) and (2). Let H' be the other hyperplane in $\mathcal{H}_{\mathrm{inv}}(f)$. By Lemma 3.3.3 (3), we may assume $H \cap H' = \{e_1\}$.

Claim: $\deg(e_1) = 2$.

By construction, H and H' are neighbors of e_1 and by Lemma 3.3.3 (1) e_1 is not contained in any other hyperplane. Assume there exists a line ℓ adjacent to e_1. Then ℓ is not adjacent to e_2, since H is the unique common neighbor of e_1 and e_2 by Lemma 3.3.3 (2). Consider the irreducible hyperplane $H'' := \mathrm{cl}(\{\ell, e_2\})$. By Lemma 3.3.3 (1), $H'' \in \mathcal{H}_{\mathrm{inv}}(f)$. If $H'' = H$, then $e_1 \in H''$, contradicting the choice of ℓ, but if $H'' = H'$, then $e_2 \in H \cap H'$, contradiction.

But $\deg(e_1) = \deg(f(e_1))$ and the hyperplane $f(e_1)$ must have degree ≥ 3, contradiction. □

Theorem 3.3.7. *Let $f \in \mathrm{Aut}(\mathcal{N})$ with inversion index $1 < \mathrm{inv}(f) < |\mathcal{E}|$. Then $B := \mathcal{E}_{inv}(f)$ is a Cremona basis of M such that $G_B(M)$ is a complete graph. In particular, $\mathrm{inv}(f) = \mathrm{rk}(M) = 3$.*

Proof. Let $n := \mathrm{inv}(f)$. We call the elements $e_1, \ldots, e_n$ of $\mathcal{E}_{\mathrm{inv}}(f)$ *coordinates.* By Lemma 3.3.3 (1) and (2), the *coordinate hyperplanes* $H_{ij} := \mathrm{cl}(\{e_i, e_j\})$ lie in $\mathcal{H}_{\mathrm{inv}}(f)$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$.

Claim 1: Let $\ell \in \mathcal{E} \setminus \mathcal{E}_{\mathrm{inv}}(f)$. Then there exists a unique hyperplane $H \in \mathcal{H}_{\mathrm{inv}}(f)$ with $\ell \in H$. Moreover, H is a coordinate hyperplane with $|H \cap \mathcal{E}_{\mathrm{inv}}(f)| \geq n - 1$.

By Lemma 3.3.3 (2), ℓ is adjacent to at most one coordinate, say e_1. Using $n > 1$, we consider the irreducible hyperplanes $H_i := \mathrm{cl}(\{\ell, e_i\})$ for $i \in \{2, \ldots, n\}$. By Lemma 3.3.3 (1), $H_i \in \mathcal{H}_{\mathrm{inv}}(f)$ for all i. On the other hand, since $\ell \notin \mathcal{E}_{\mathrm{inv}}(f)$, Lemma 3.3.3 (3) implies that ℓ is contained in at most one hyperplane $H \in \mathcal{H}_{\mathrm{inv}}(f)$, hence $H := H_i = H_j$ for all $i, j \in \{2, \ldots, n\}$. In particular, $\{e_2, \ldots, e_n\} \subseteq H \cap \mathcal{E}_{\mathrm{inv}}(f)$. Since $n > 2$ by Proposition 3.3.6, this means that H is a coordinate hyperplane.

Since $\mathrm{inv}(f) < |\mathcal{E}|$, such a line $\ell \in \mathcal{E} \setminus \mathcal{E}_{\mathrm{inv}}(f)$ exists and by the claim we may assume that $H := \mathrm{cl}(\{e_2, \ldots, e_n\})$ is a hyperplane.

Claim 2: $\mathrm{rk}(\{e_1, \ldots, e_n\}) = 3$ and $\mathcal{H}_{\mathrm{inv}}(f) = \{H_{12}, \ldots, H_{1n}, H\}$.

If $e_1 \in H$, then H is the only coordinate hyperplane and Claim 1 implies $\mathcal{E} \subseteq H$, contradiction to $\mathrm{rk}(M) = 3$. Hence $e_1 \notin H$ and $\mathrm{rk}(\{e_1, \ldots, e_n\}) = 3$. In particular, $H \neq H_{1i}$ for all $i \in \{2, \ldots, n\}$. Moreover, if $H_{1i} = H_{1j}$ for some $i, j \in \{2, \ldots, n\}$, then $\{e_i, e_j\} \subseteq H_{1i} \cap H$, hence $i = j$. Now the claim follows from $|\mathcal{H}_{\mathrm{inv}}(f)| = n$.

Since $e_1 \in H_{12} \setminus H$, we have $|H_{12} \cap H| \leq 1$ and thus $H_{12} \cap \mathcal{E}_{\mathrm{inv}}(f) = \{e_1, e_2\}$. Let $\ell \in H_{12} \setminus \mathcal{E}_{\mathrm{inv}}(f)$. Then ℓ is adjacent to e_i for all $i \in \{3, \ldots, n\}$. Indeed, if ℓ and e_i lie in a common hyperplane H', then $H' \in \mathcal{H}_{\mathrm{inv}}(f)$ by Lemma 3.3.3 (1), so $H' = H_{12}$ by Claim 1, contradiction since $e_i \notin H_{12}$. But Lemma 3.3.3 (2) shows that ℓ is adjacent to at most one coordinate, hence $n = 3$.

Together with Claim 2, this means that $\mathcal{E}_{\mathrm{inv}}(f)$ is a basis of M and Claim 1 shows that $\mathcal{E}_{\mathrm{inv}}$ is a Cremona basis. □

Proposition 3.3.8. *Let $f \in \mathrm{Aut}(\mathcal{N})$ with $\mathcal{E}_{inv}(f) = \{\ell\}$ and $\mathcal{H}_{inv}(f) = \{H\}$. Then $\ell \notin H$.*

Proof. Assume $\ell \in H$ and let $k, m \in \mathbb{N}$ be minimal with $f^k(\ell) = H$ and $f^m(H) = \ell$. We show that all cases lead to contradictions:

Case 1: $k \geq 2$.

Claim 1: ℓ is not adjacent to $f^i(\ell)$ for all $0 < i < k$.

Since $i + 1 \leq k$, $f(\ell)$ and $f^{i+1}(\ell)$ are both hyperplanes, thus not adjacent. The claim follows by applying f^{-1}.

Claim 2: We have $k \leq m$.

Applying f^m to the edge $\{H, \ell\}$ implies that $\ell = f^m(H)$ and $f^m(\ell)$ are adjacent. If $m < k$, then this contradicts Claim 1.

Claim 3: ℓ is adjacent to $f^{k+i}(\ell)$ for all $0 \leq i < k$.

Clear for $i = 0$, so assume $i > 0$. Since $i < m$ by Claim 2, $\ell' := f^{k+i}(\ell)$ is a line with $\ell' \neq \ell$. Moreover, $\ell' \notin H$, else applying f^{-k} to the edge $\{\ell', H\}$ would contradict Claim 1. By Lemma 3.3.3 (1), there is no hyperplane containing both ℓ and ℓ', hence ℓ and ℓ' are adjacent.

Now applying f^{-1} to Claim 3 shows that $f^{-1}(\ell) \in H \cap f^{-1}(H)$. Let $\ell' \in f^{-1}(H) \setminus \{f^{-1}(\ell), f^{-2}(\ell)\}$. Since $\ell' \neq f^{-1}(\ell)$, we have $\ell' \notin H$, so by Lemma 3.3.3 (1) ℓ' and ℓ are adjacent. Applying f shows that $f(\ell')$ is adjacent to both H and $f(\ell) = f^{-1}(H)$, hence $f(\ell') = f^{-1}(\ell)$, contradiction to $\ell' \neq f^{-2}(\ell)$.

Case 2: $k = 1$, $m \geq 2$.

Then $\mathcal{H}_{\text{inv}}(f^2) = \mathcal{H}_{\text{inv}}(f)$ and Lemma 3.3.3 (1) implies $f^{-1}(\ell) \notin H'$ for all hyperplanes $H' \neq H$. On the other hand, $f^{-1}(\ell) \notin H$. Indeed, applying f to the edge $\{\ell, H\}$ shows that $f(H) \in H$. Since $m \geq 2$, $f(H) \neq \ell$, so $f(H)$ and ℓ are not adjacent. Applying f^{-1} shows that $f^{-1}(\ell) \notin H$.

Hence $f^{-1}(\ell)$ is not contained in any irreducible hyperplane and we have

$$|\mathcal{E}| - 1 = \deg(f^{-1}(\ell)) = \deg(\ell) \leq |\mathcal{E}| + 1 - |H| \leq |\mathcal{E}| - 2,$$

contradiction.

Case 3: $k = 1$, $m = 1$.

Let $\ell_1, \ell_2 \in H \setminus \{\ell\}$ with $\ell_1 \neq \ell_2$ and consider the lines $f(\ell_1)$ and $f(\ell_2)$. Since ℓ_1 and ℓ_2 are adjacent to H, $f(\ell_1)$ and $f(\ell_2)$ are adjacent to $f(H) = \ell$. In particular, since $\ell \in H$, $f(\ell_1)$ and $f(\ell_2)$ are not elements of H. Moreover, since ℓ_1 and ℓ_2 are not adjacent to each other, neither are $f(\ell_1)$ and $f(\ell_2)$. Thus $f(\ell_1)$ and $f(\ell_2)$ are both adjacent to $H' := \mathrm{cl}(\{f(\ell_1), f(\ell_2)\}) \in \mathcal{H}$. Clearly $H' \neq \ell$, thus $f^{-1}(H') \neq f^{-1}(\ell) = H$.

If $f^{-1}(H')$ and H are distinct hyperplanes with two common neighbors, ℓ_1 and ℓ_2, then we have a contradiction. Thus $f^{-1}(H')$ must be a line, hence $H' = H$. But $H' \neq H$ since $f(\ell_1) \in H' \setminus H$, contradiction. □

Corollary 3.3.9. *There is no automorphism f of $\mathcal{N}(M)$ with inversion index 1.*

Proof. Assume there exists an automorphism f of $\mathcal{N}(M)$ with inversion index 1. Let $\mathcal{E}_{\text{inv}}(f) = \{\ell\}$ and $\mathcal{H}_{\text{inv}}(f) = \{H\}$. By Lemma 3.3.3 (1), $\ell \notin H'$ for all $H' \in \mathcal{H} \setminus \{H\}$, and by Proposition 3.3.8, $\ell \notin H$. Thus $\deg(\ell) = |\mathcal{E}| - 1$, since ℓ is not contained in any irreducible hyperplane, i.e., totally disconnected to any other line. But $f^k(\ell) = H$ for some $k \in \mathbb{N}$, hence

$$|H| = \deg(H) = \deg(\ell) = |\mathcal{E}| - 1.$$

We deduce $H = \mathcal{E} \setminus \{\ell\}$, which means that H is a proper separator of M, contradicting the assumption that M is connected. $\square$

This completes the proof of Theorem 3.3.2.

CHAPTER 4

Root system matroids

4.1. Introduction

A root system Φ is a finite set of non-zero vectors (called roots) in Euclidean space with special geometric properties, one of them being that the reflection across the linear hyperplane $\alpha^\perp$ perpendicular to any root $\alpha \in \Phi$ maps the root system to itself. These reflections generate a finite group of isometries, called the Weyl group $W(\Phi)$ of the root system. Every root system can be decomposed into irreducible root systems, which have been completely classified: There are four infinite families A_n, B_n, C_n, D_n and five exceptional cases E_8, E_7, E_6, F_4, G_2. Root systems are closely connected to Coxeter groups and play an important role in the theory of Lie groups and Lie algebras, for more information see for example [Bou02, Chapter 6] or [Hum90].

The goal of this chapter is to investigate the existence of Cremona bases and Cremona automorphisms for the vector matroids that arise from irreducible root systems. More precisely, since root systems contain for every root α also its negative $-\alpha$, we are interested in the associated simple matroids.

Definition 4.1.1. Let Φ be a root system. Then the following matroids are isomorphic:

(1) The simplification of the vector matroid of Φ.
(2) The vector matroid associated to any subset $\Phi^+ \subseteq \Phi$ with the property $|\Phi^+ \cap \{\alpha, -\alpha\}| = 1$ for all $\alpha \in \Phi$.
(3) The matroid of the hyperplane arrangement $\{\alpha^\perp \mid \alpha \in \Phi\}$.

We call this matroid $M(\Phi)$ the *root system matroid* associated to the root system Φ.

Proof. Follows from Proposition 2.3.2 and from the fact that a root system has the property $\langle\alpha\rangle \cap \Phi = \{\alpha, -\alpha\}$ for all α. □

While the roots of a root system are required to satisfy the so-called integrality condition with respect to the inner product, the root system matroid does not remember the lengths of the roots nor the inner product on the vector space. First, this implies that the irreducible root systems B_n and C_n give rise to the same matroid, since they differ only by the lengths of their roots. Second, this means that root system matroids can be also represented over other fields than the real numbers.

The rank of a root system is defined as the dimension of the vector space spanned by the roots, and thus coincides with the rank of the corresponding root system matroid. As we are only interested in matroids of rank at least 3, we will not consider the exceptional root system G_2 of rank 2.

Since reflections preserving the root system induce automorphisms of the corresponding matroid, root system matroids have a large automorphism group that is similar to the Weyl group of the root system. The Weyl groups and the matroid automorphism groups are known for all irreducible root systems.

Theorem 4.1.2 ([DSFT11, Theorem 1.2])**.** *For the matroid $M(\Phi)$ associated to an irreducible root system Φ, the number of elements and the automorphism group are given in the following table:*

Φ	$\lvert M(\Phi)\rvert$	$\mathrm{Aut}(M(\Phi))$
A_n	$\frac{n(n+1)}{2}$	$W(A_n) \cong S_{n+1}$
B_n	n^2	$W(B_n)/\{\pm\mathrm{id}\} \cong (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n$
D_n	$n(n-1)$	$W(F_4)/\{\pm\mathrm{id}\}$ *if* $n=4$ $W(B_n)/\{\pm\mathrm{id}\}$ *if* $n \geq 5$
E_6	36	$W(E_6)$
E_7	63	$W(E_7)/\{\pm\mathrm{id}\}$
E_8	120	$W(E_8)/\{\pm\mathrm{id}\}$
F_4	24	$W^\sigma(F_4)/\{\pm\mathrm{id}\}$

Here, $W^\sigma(F_4)$ denotes the group generated by the Weyl group $W(F_4)$ and a certain isometry σ of $\mathbb{R}^4$.

As we mentioned in the introduction, the automorphism group of the coarse Bergman fan is known for the root system A_n:

Theorem 4.1.3 ([SW23, Section 3], [AP18, Theorem 3.13])**.** *For all $n \geq 3$, the automorphism group of $B_c(M(A_n))$ is isomorphic to the symmetric group S_{n+2}. It is generated by matroid automorphisms and a Cremona map.*

By applying our results from Chapters 1 and 3, we are able to give a new proof of this result and to compute the automorphisms of the coarse Bergman fan for other root system matroids. The following theorem summarizes our results in this chapter:

Theorem 4.1.4. *We show the following results on the number of Cremona bases and the automorphism groups of the coarse Bergman fan and the minimal nested set complex for root system matroids:*

Φ	$\lvert\mathcal{CB}(M(\Phi))\rvert$	$\mathrm{Aut}(B_c(M(\Phi))) \cong \mathrm{Cr}(M(\Phi))$
A_n	$n+1$	S_{n+2}
B_n	1	$\mathrm{Aut}(M(B_n)) \times \mathbb{Z}/2\mathbb{Z}$
D_n	0	$\mathrm{Aut}(M(D_n))$
E_6, E_7, E_8	0	*open*
F_4	0	$\mathrm{Aut}(M(F_4))$

To prove these results, we consider each root system matroid in a separate section. We will make repeated use of the following fact:

Theorem 4.1.5 ([ARW07, Theorem 1.2])**.** *For root system matroids, the coarse Bergman fan and the minimal nested set fan coincide.*

4.2. A_n

Let $x_1, \dots, x_n$ be a basis of a vector space V over some field K and assume $n \geq 3$. We define the root system matroid $M(A_n)$ corresponding to the irreducible root system A_n (divided by the lineality space) as the vector matroid associated to the set of roots

$$\Phi^+(A_n) := \{x_i \mid 1 \leq i \leq n\} \cup \{x_i - x_j \mid 1 \leq i < j \leq n\}.$$

(We will sometimes use $x_j - x_i$ as alternative notation for the matroid element $x_i - x_j$ in order to avoid case distinctions.)

$M(A_n)$ is a simple matroid of rank n and isomorphic to the Dowling matroid $Q_n(G)$ for the trivial group $G = \{e\}$, see Example 1.3.9.

In this section we give a new proof of Theorem 4.1.3 that does not rely on the interpretation of $B_c(M(A_n))$ as a moduli space. We start by determining the Cremona bases of $M(A_n)$.

Theorem 4.2.1. *$M(A_n)$ has exactly the following $n + 1$ Cremona bases:*

(1) *$B := B_0 := \{x_i \mid 1 \leq i \leq n\}$ is a Cremona basis of $M(A_n)$.*
(2) *$B_i := \{x_1 - x_i, \dots, x_{i-1} - x_i, x_i, x_i - x_{i+1}, \dots, x_i - x_n\}$ is a Cremona basis of $M(A_n)$ for all $1 \leq i \leq n$.*

The support graph $G_B(M(A_n)))$ is the complete simple graph on n vertices.

Proof. B is clearly a basis of $M(A_n)$ and we have $\operatorname{supp}_B(x_i - x_j) = \{x_i, x_j\}$ for all $1 \leq i < j \leq n$, thus B is a Cremona basis. Since $|\operatorname{cl}(\{x_i, x_j\}) \setminus \{x_i, x_j\}| = 1$ for all $1 \leq i < j \leq n$, the support graph $G_B(M(A_n))$ of $M(A_n)$ with respect to B is the complete simple graph on the vertices $x_1, \dots, x_n$. In particular, $M(A_n)$ is connected by Lemma 1.4.10 and for every root $v \in \Phi^+(A_n)$ the contraction $M(A_n)/v$ is connected by Lemma 1.4.12.

Moreover, for any $1 \leq i \leq n$, the basis B_i is a Cremona basis of $M(A_n)$:

- We have $\operatorname{supp}_{B_i}(x_j) = \{x_i, x_i - x_j\}$ for all $1 \leq j \leq n$ with $j \neq i$.
- We have $\operatorname{supp}_{B_i}(x_j - x_k) = \{x_i - x_j, x_i - x_k\}$ for all $1 \leq j < k \leq n$ with $j, k \neq i$.

With respect to B, the support graph $G_B(B_i)$ is a spanning star with central vertex x_i. It follows from Theorem 1.6.1 and Proposition 1.6.9 that there are no other Cremona bases. $\square$

Example 4.2.2. For $n = 3$, the support graph $G_B(M(A_3))$ of $M(A_3)$ with respect to the Cremona basis B is a triangle:

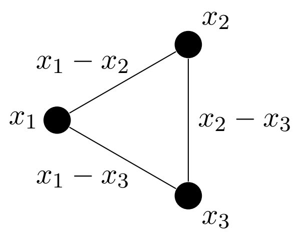

FIGURE 12. The support graph $G_B(M(A_3))$.

By Theorem 4.1.2, the automorphism group of $M(A_n)$ coincides with the Weyl group of A_n, which is isomorphic to the symmetric group S_{n+1}. We obtain a new interpretation of this isomorphism.

Corollary 4.2.3. *The group action* $\mathrm{Aut}(M(A_n)) \to \mathrm{Sym}(\mathcal{CB}(M(A_n))) \cong S_{n+1}$ *of matroid automorphisms on the set of Cremona bases* $\mathcal{CB}(M)$ *of* M *is an isomorphism.*

Proof. The isomorphism $\mathrm{Sym}(\mathcal{CB}(M)) \cong S_{n+1}$ follows from Theorem 4.2.1. The automorphisms of $M(A_n)$ permute the set of Cremona bases, so for every matroid automorphism $f \in \mathrm{Aut}(M(A_n))$ there is a unique permutation $\sigma\colon \{0, \dots, n\} \to \{0, \dots, n\}$ with $f(B_i) = B_{\sigma(i)}$ for all $0 \le i \le n$. We claim that the induced group homomorphism from $\mathrm{Aut}(M(A_n))$ to S_{n+1} is bijective.

Indeed, if f is an automorphism of $M(A_n)$ with $f(B_i) = B_i$ for all i, then f is the identity since it preserves the intersections $B_0 \cap B_i = \{x_i\}$ for all $1 \le i \le n$ and $B_i \cap B_j = \{x_i - x_j\}$ for all $1 \le i < j \le n$. Surjectivity has been shown in Theorem 1.7.6. □

Theorem 4.2.4 (cf. Theorem 4.1.3)**.** *The Cremona group of* $M(A_n)$ *is isomorphic to* S_{n+2}*. It is generated by matroid automorphisms and a single Cremona automorphism.*

Proof. This follows from Theorem 4.2.1 and Corollary 4.2.3 by applying Theorem 3.2.8. □

It remains to prove that there are no other automorphisms of the coarse Bergman fan $B_c(M(A_n))$, which coincides with the minimal nested set fan $B_m(M(A_n))$ by Theorem 4.1.5. We will show this by computing the 1-skeleton of the minimal nested set complex $\mathcal{N}(M(A_n))$.

Proposition 4.2.5. *Let* $J \subseteq \{1, \dots, n\}$ *be non-empty. Then*

$$F_J := \{x_i \mid i \in J\} \cup \{x_i - x_j \mid i, j \in J,\ i < j\}$$

is a connected coordinate flat of $M(A_n)$ *of rank* $|J|$ *with support* $\{x_i \mid i \in J\}$.

Proof. We have

$$F_J = \bigcup_{i,j \in J} \mathrm{cl}(\{x_i, x_j\}) = \mathrm{cl}(\{x_i \mid i \in J\}),$$

thus F_J is a coordinate flat of rank $|J|$ with support $\{x_i \mid i \in J\}$. The support graph $G_B(F_J)$ is a complete graph, hence F_J is connected by Lemma 1.5.6. □

Proposition 4.2.6. *Let* $J \subseteq \{1, \dots, n\}$ *with* $|J| \ge 2$*. Then*

$$F_J^= := \{x_i - x_j \mid i, j \in J,\ i < j\}$$

is a connected non-coordinate flat of $M(A_n)$ *of rank* $|J| - 1$ *with support* $\{x_i \mid i \in J\}$.

Proof. Consider the linear subspace U of V that is obtained by intersecting the subspace $\langle x_i \mid i \in J\rangle$ with the hyperplane $(\sum_{i \in J} x_i)^\perp$. Then $F_J^= = \Phi^+(A_n) \cap U$, thus $F_J^=$ is a non-coordinate flat of $M(A_n)$. The support graph $G_B(F_J^=)$ is the complete graph with vertices $\{x_i \mid i \in J\}$, so Theorem 1.5.5 implies that $F_J^=$ has rank $|J| - 1$. If $|J| = 2$, then clearly $F_J^=$ is connected, and for $|J| \ge 3$ this follows from Lemma 1.5.9. □

Theorem 4.2.7. *Every support-connected flat of* $M(A_n)$ *is connected. The connected flats of* $M(A_n)$ *are exactly the flats of the form* F_J *and* $F_J^=$ *as defined in Propositions 4.2.5 and 4.2.6, respectively.*

Proof. Let F be a support-connected flat of $M(A_n)$ with support $\{x_i \mid i \in J\}$ for some $J \subseteq I$. Then F and J are non-empty. If F is a coordinate flat, then $F = F_J$ by Proposition 1.5.4, hence F is connected by Proposition 4.2.5. Otherwise, F is a non-coordinate flat by Theorem 1.5.2 and its support graph $G_B(F)$ is connected by Lemma 1.4.6. Whenever $x_i - x_j$ and $x_j - x_k$ are edges of $G_B(F)$ for pairwise distinct $i, j, k \in \{1, \dots, n\}$, we have

$$x_i - x_k \in \operatorname{cl}(\{x_i - x_j, x_j - x_k\}) \subseteq F.$$

This shows that $G_B(F)$ is a complete graph, hence $F = F_J^{=}$ and F is connected by Proposition 4.2.6. □

Excluding the trivial coordinate flat $F_{\{1,\dots,n\}} = \Phi^+(A_n)$, these are exactly the vertices of the minimal nested set complex $\mathcal{N}(M(A_n))$. Next, we compute the degrees of these vertices in the 1-skeleton of $\mathcal{N}(M(A_n))$.

Lemma 4.2.8. *Let F and F' be connected flats of $M(A_n)$. Then $F \vee F'$ is disconnected if and only if $\operatorname{supp}_B(F) \cap \operatorname{supp}_B(F') = \emptyset$ and at least one of F and F' is a non-coordinate flat.*

Proof. Assume that $F \vee F'$ is disconnected. Then $F \vee F'$ is not support-connected by Theorem 4.2.7, thus $\operatorname{supp}_B(F) \cap \operatorname{supp}_B(F') = \emptyset$. Moreover, if F and F' were both coordinate flats, then $F \vee F'$ would also be a coordinate flat and thus connected by Proposition 4.2.5, contradiction. The other direction was shown in Lemma 1.5.11. □

Proposition 4.2.9. *Let F be a proper connected flat of $M(A_n)$ of rank r. Then F has degree*

$$d(r, n) := 2^{r+1} + 2^{n-r+1} - n - 6$$

in the 1-skeleton of $\mathcal{N}(M(A_n))$.

Proof. By Theorem 4.2.7, F is either a coordinate flat of the form F_J or a non-coordinate flat of the form $F_J^{=}$. First assume $F = F_J$ for some $\emptyset \subsetneq J \subsetneq \{1, \dots, n\}$. Then $\operatorname{rk}(F) = |J|$ by Proposition 4.2.5. Using Theorem 4.2.7, we count the neighbors of F_J in the 1-skeleton of $\mathcal{N}(M(A_n))$:

(1) F_J is adjacent to all smaller connected coordinate flats, i.e., to all flats $F_{J'}$ with $\emptyset \subsetneq J' \subsetneq J$. There are $2^{|J|} - 2$ such flats.
(2) F_J is adjacent to all larger proper connected coordinate flats, i.e., to all flats $F_{J'}$ with $J \subsetneq J' \subsetneq \{1, \dots, n\}$. There are $2^{n-|J|} - 2$ such flats.
(3) F_J is adjacent to all smaller connected non-coordinate flats, i.e., to all flats $F_{J'}^{=}$ with $J' \subseteq J$ and $|J'| \geq 2$. There are $2^{|J|} - |J| - 1$ such flats.
(4) There are no larger connected non-coordinate flats.
(5) F_J is adjacent to all proper connected coordinate flats $F_{J'}$ such that $F_J \vee F_{J'}$ is disconnected, but by Lemma 4.2.8 there are no such flats.
(6) F_J is adjacent to all connected non-coordinate flats $F_{J'}^{=}$ with $|J'| \geq 2$ such that $F_J \vee F_{J'}^{=}$ is disconnected. By Lemma 4.2.8, this is the case if and only if $J' \subseteq \{1, \dots, n\} \setminus J$. There are $2^{n-|J|} - (n - |J|) - 1$ such flats.

Writing $r := |J|$ for the rank of F_J, we see that in total F_J has degree

$$\begin{aligned}&2^r - 2 + 2^{n-r} - 2 + 2^r - r - 1 + 2^{n-r} - (n-r) - 1\\ =\ &2^{r+1} + 2^{n-r+1} - n - 6.\end{aligned}$$

Now assume $F = F_J^{=}$ for some $J \subseteq \{1, \ldots, n\}$ with $|J| \geq 2$. Then $\mathrm{rk}(F) = |J| - 1$ by Proposition 4.2.6. Using Theorem 4.2.7, we count the neighbors of $F_J^{=}$ in the 1-skeleton of $\mathcal{N}(M(A_n))$:

(1) There are no smaller connected coordinate flats.
(2) $F_J^{=}$ is adjacent to all larger proper connected coordinate flats, i.e., to all flats $F_{J'}$ with $J \subseteq J' \subsetneq \{1, \ldots, n\}$. There are $2^{n-|J|} - 1$ such flats.
(3) $F_J^{=}$ is adjacent to all smaller connected non-coordinate flats, i.e., to all flats $F_{J'}^{=}$ with $J' \subsetneq J$ and $|J'| \geq 2$. There are $2^{|J|} - |J| - 2$ such flats.
(4) $F_J^{=}$ is adjacent to all larger connected non-coordinate flats, i.e., to all flats $F_{J'}^{=}$ with $J \subsetneq J' \subseteq \{1, \ldots, n\}$. There are $2^{n-|J|} - 1$ such flats.
(5) $F_J^{=}$ is adjacent to all proper connected coordinate flats $F_{J'}$ such that $F_J^{=} \vee F_{J'}$ is disconnected. By Lemma 4.2.8, this is the case if and only if $\emptyset \subsetneq J' \subseteq \{1, \ldots, n\} \setminus J$. There are $2^{n-|J|} - 1$ such flats.
(6) $F_J^{=}$ is adjacent to all connected non-coordinate flats $F_{J'}^{=}$ with $|J'| \geq 2$ such that $F_J^{=} \vee F_{J'}^{=}$ is disconnected. By Lemma 4.2.8, this is the case if and only if $J' \subseteq \{1, \ldots, n\} \setminus J$. There are $2^{n-|J|} - (n - |J|) - 1$ such flats.

Writing $r := |J| - 1$ for the rank of $F_J^{=}$, we see that in total $F_J^{=}$ has degree

$$\begin{aligned}&2^{n-r-1} - 1 + 2^{r+1} - r - 3 + 2^{n-r-1} - 1\\ +\ &2^{n-r-1} - 1 + 2^{n-r-1} - (n-r-1) - 1\\ =\ &2^{r+1} + 2^{n-r+1} - n - 6.\end{aligned}$$ ☐

Example 4.2.10. For $n = 3$, we get $d(1,3) = d(2,3) = 3$, see Figure 13.

Corollary 4.2.11. *The vertices of $\mathcal{N}(M(A_n))$ with maximal degree in the 1-skeleton are exactly the singletons and the connected hyperplanes.*

Proof. In Proposition 4.2.9 we saw that the degrees in the 1-skeleton are given by the numbers $d(r,n)$. We note the following properties of these numbers:

(1) For all $1 \leq r < n$ we have $d(r,n) = d(n-r,n)$.
(2) The first difference of $d(r,n)$ in r is a strictly increasing function in r:
$$d(r+1,n) - d(r,n) = 2^{r+1} - 2^{n-r}$$

In particular, $d(1,n) = d(n-1,n)$ is the maximal degree in the 1-skeleton of $\mathcal{N}(M(A_n))$. The corresponding flats are exactly the singletons and the connected hyperplanes. ☐

Example 4.2.12. For the root system A_{10}, we obtain the following degrees in the 1-skeleton of $\mathcal{N}(M(A_{10}))$:

r	1	2	3	4	5	6	7	8	9
$d(r,10)$	1012	504	256	144	112	144	256	504	1012

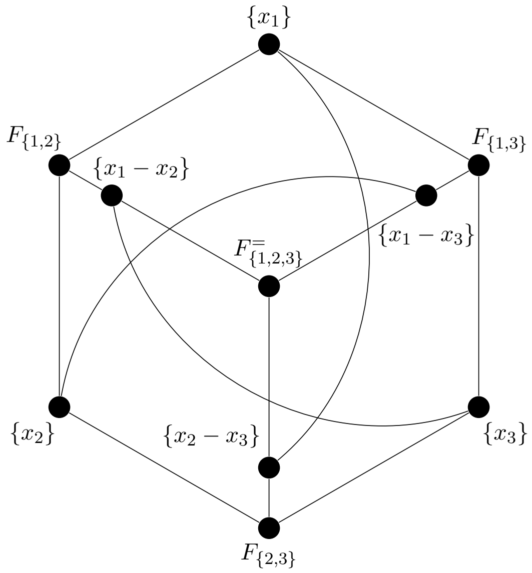

FIGURE 13. The minimal nested set complex $\mathcal{N}(M(A_3))$.

Theorem 4.2.13. *The automorphism group of the coarse Bergman fan $B_c(M(A_n))$ is generated by matroid automorphisms and Cremona automorphisms, hence it is isomorphic to the Cremona group* $\mathrm{Cr}(M(A_n))$.

Proof. The case $n = 3$ follows from Section 3.3, thus we may assume $n \geq 4$. Let f be an automorphism of $B_c(M(A_n))$. By Theorem 4.1.5, the coarse Bergman fan $B_c(M(A_n))$ and the minimal nested set fan $B_m(M(A_n))$ coincide, thus f induces an automorphism of the minimal nested set complex $\mathcal{N}(M(A_n))$. In particular, f preserves the subgraph G of the 1-skeleton of $\mathcal{N}(M(A_n))$ induced by the vertices of maximal degree.

By Corollary 4.2.11, the vertices of G are exactly the singletons and the connected hyperplanes. Thus f induces a bijection between the sets

$$\mathcal{E}_{\mathrm{inv}}(f) := \{v \in \Phi^+(A_n) \mid \mathrm{rk}\, f(\{v\}) = n - 1\}$$

and

$$\mathcal{H}_{\mathrm{inv}}(f) := \{H \subseteq \Phi^+(A_n) \text{ connected hyperplane} \mid \mathrm{rk}\, f(H) = 1\}.$$

In particular, since the vertices $\mathcal{H}_{\mathrm{inv}}(f)$ form an independent set in G, the same is true for $\mathcal{E}_{\mathrm{inv}}(f)$. Note that for every connected hyperplane H of $M(A_n)$ we have $H \in \mathcal{H}_{\mathrm{inv}}(f)$ if $H \cap \mathcal{E}_{\mathrm{inv}}(f) \neq \emptyset$.

If $\mathcal{E}_{\mathrm{inv}}(f)$ is empty, then Proposition 2.2.4 shows that f is induced by a matroid automorphism, so we may assume $\mathcal{E}_{\mathrm{inv}}(f) \neq \emptyset$. Moreover, since the group of matroid automorphisms $\mathrm{Aut}(M(A_n))$ acts transitively on the ground set E, we may assume $\mathcal{E}_{\mathrm{inv}}(f) \cap B \neq \emptyset$, otherwise replace f by $f \circ g$ for a suitable automorphism g of $M(A_n)$. We claim that $\mathcal{E}_{\mathrm{inv}}(f)$ is a Cremona basis of $M(A_n)$.

Case 1: $|\mathcal{E}_{\mathrm{inv}}(f) \cap B| \geq 3$. Then every element $v \in \Phi^+(A_n) \setminus B$ has at least one neighbor in $\mathcal{E}_{\mathrm{inv}}(f)$, but $\mathcal{E}_{\mathrm{inv}}(f)$ is an independent set in G, thus $\mathcal{E}_{\mathrm{inv}}(f) \subseteq B$. On the other hand, $H \cap \mathcal{E}_{\mathrm{inv}}(f) \neq \emptyset$ for every coordinate hyperplane of M, thus $|\mathcal{H}_{\mathrm{inv}}(f)| \geq n$ and we conclude $\mathcal{E}_{\mathrm{inv}}(f) = B$.

Case 2: $|\mathcal{E}_{\mathrm{inv}}(f) \cap B| = 2$. Let $v \in \mathcal{E}_{\mathrm{inv}}(f) \setminus B$. Then $\mathrm{supp}_B(v) = \mathcal{E}_{\mathrm{inv}}(f) \cap B$ since $\mathcal{E}_{\mathrm{inv}}(f)$ is an independent set in G, but $G_B(M(A_n))$ is a simple graph, hence $|\mathcal{E}_{\mathrm{inv}}(f)| \leq 3$. On the other hand, as in Case 1 we have $H \cap \mathcal{E}_{\mathrm{inv}}(f) \neq \emptyset$ for every coordinate hyperplane of M, thus $|\mathcal{H}_{\mathrm{inv}}(f)| \geq n$, contradiction.

Case 3: $|\mathcal{E}_{\mathrm{inv}}(f) \cap B| = 1$, say $\mathcal{E}_{\mathrm{inv}}(f) \cap B = \{x_i\}$ for some $i \in \{1, \ldots, n\}$. There are $n-1$ coordinate hyperplanes H of M with $x_i \in H$, thus $|\mathcal{E}_{\mathrm{inv}}(f)| \geq n-1$, but then $\mathcal{E}_{\mathrm{inv}}(f)$ also contains some non-basis element $v \in \Phi^+(A_n) \setminus B$ and we have $F^{=}_{\{1,\ldots,n\}} \in \mathcal{H}_{\mathrm{inv}}(f)$, hence $|\mathcal{E}_{\mathrm{inv}}(f)| \geq n$. On the other hand, we have $b \in \mathrm{supp}_B(v)$ for all $v \in \mathcal{E}_{\mathrm{inv}}(f) \setminus B$ since $\mathcal{E}_{\mathrm{inv}}(f)$ is an independent set in G, which implies $|\mathcal{E}_{\mathrm{inv}}(f)| \leq n$ using that $G_B(M(A_n))$ is a simple graph.

We conclude that $|\mathcal{E}_{\mathrm{inv}}(f)| = n$ and $G_B(\mathcal{E}_{\mathrm{inv}}(f))$ is a simple spanning star of $G_B(M(A_n))$ with central vertex x_i, hence $\mathcal{E}_{\mathrm{inv}}(f)$ is the Cremona basis B_i.

Since $\mathcal{E}_{\mathrm{inv}}(f)$ is a Cremona basis, the Cremona map $\mathrm{crem}_{\mathcal{E}_{\mathrm{inv}}(f)}$ induces an automorphism of $B_c(M(A_n))$ by Theorem 3.1.3. It suffices to show that $\mathrm{crem}_{\mathcal{E}_{\mathrm{inv}}(f)} \circ f$ is induced by a matroid automorphism. There are n coordinate hyperplanes of M with respect to the Cremona basis $\mathcal{E}_{\mathrm{inv}}(f)$ and all of them lie in $\mathcal{H}_{\mathrm{inv}}(f)$. On the other hand, $|\mathcal{H}_{\mathrm{inv}}(f)| = |\mathcal{E}_{\mathrm{inv}}(f)| = n$. This means that f induces a bijection between the singletons corresponding to elements of the Cremona basis $\mathcal{E}_{\mathrm{inv}}(f)$ and the coordinate hyperplanes with respect to $\mathcal{E}_{\mathrm{inv}}(f)$. Hence the composition $\mathrm{crem}_{\mathcal{E}_{\mathrm{inv}}(f)} \circ f$ maps singletons to singletons and applying Proposition 2.2.4 completes the proof. $\square$

4.3. B_n

Let $x_1, \ldots, x_n$ be a basis of a vector space V over some field K with $\mathrm{char}(K) \neq 2$ and assume $n \geq 3$. We define the root system matroid $M(B_n)$ corresponding to the irreducible root system B_n as the vector matroid associated to the set of roots

$$\Phi^+(B_n) := \{x_i \mid 1 \leq i \leq n\} \cup \{x_i \pm x_j \mid 1 \leq i < j \leq n\}.$$

(We will sometimes use $x_j - x_i$ as alternative notation for the matroid element $x_i - x_j$ in order to avoid case distinctions.)

$M(B_n)$ is a simple matroid of rank n and isomorphic to the Dowling matroid $Q_n(\mathbb{Z}/2\mathbb{Z})$, see Example 1.3.9.

Theorem 4.3.1. *$B := \{x_i \mid 1 \leq i \leq n\}$ is the only Cremona basis of $M(B_n)$. The support graph $G_B(M(B_n))$ is a complete graph.*

Proof. B is clearly a basis of $M(B_n)$ and for all $1 \leq i < j \leq n$ we have

$$\mathrm{cl}(\{x_i, x_j\}) = \{x_i, x_j, x_i + x_j, x_i - x_j\},$$

thus B is a Cremona basis. The support graph $G_B(M(B_n))$ of $M(B_n)$ with respect to B is a complete graph with two parallel edges $x_i + x_j$ and $x_i - x_j$ joining any two vertices x_i and x_j with $i < j$. In particular, $M(B_n)$ is connected by Lemma 1.4.10 and for every root $v \in \Phi^+(B_n)$ the contraction $M(B_n)/v$ is connected by Lemma 1.4.12. It follows from Proposition 1.6.9 and Lemma 1.6.7 that there are no other Cremona bases. ☐

Example 4.3.2. For $n = 3$, the support graph of $M(B_3)$ looks like this:

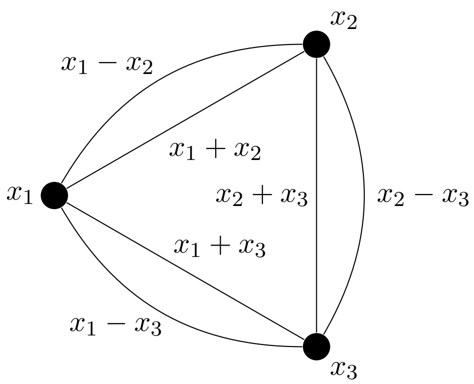

FIGURE 14. The support graph $G_B(M(B_3))$.

Theorem 4.3.3. *The Cremona group of $M(B_n)$ is isomorphic to*

$$\mathrm{Aut}(M(B_n)) \times \mathbb{Z}/2\mathbb{Z} \cong ((\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n) \times \mathbb{Z}/2\mathbb{Z}.$$

It is generated by matroid automorphisms and the unique Cremona automorphism.

Proof. Follows from Proposition 3.2.3 and Theorem 4.1.2. ☐

Proposition 4.3.4. *Let $J \subseteq \{1, \ldots, n\}$ be non-empty. Then*

$$F_J := \{x_i \mid i \in J\} \cup \{x_i \pm x_j \mid i, j \in J,\ i < j\}$$

is a connected coordinate flat of $M(B_n)$ of rank $|J|$ with support $\{x_i \mid i \in J\}$.

Proof. We have

$$F_J = \bigcup_{i,j\in J} \mathrm{cl}(\{x_i, x_j\}) = \mathrm{cl}(\{x_i \mid i \in J\}),$$

thus F_J is a coordinate flat of rank $|J|$ with support $\{x_i \mid i \in J\}$. The support graph $G_B(F_J)$ is a complete graph, hence F_J is connected by Lemma 1.5.6. $\square$

Proposition 4.3.5. *Let $J \subseteq \{1, \dots, n\}$ with $|J| \geq 2$ and choose a sign function $s\colon J \to \{-1, 1\}$. Then*

$$\begin{aligned} F_J^s := \{&x_i - x_j \mid i, j \in J,\ s(i) = s(j),\ i < j\} \\ \cup \{&x_i + x_j \mid i, j \in J,\ s(i) \neq s(j),\ i < j\} \end{aligned}$$

is a connected non-coordinate flat of $M(B_n)$ of rank $|J| - 1$ with support $\{x_i \mid i \in J\}$. Note that $F_J^s = F_J^{-s}$.

Proof. Consider the linear subspace U of V that is obtained by intersecting the subspace $\langle x_i \mid i \in J\rangle$ with the hyperplane $(\sum_{i\in J} s(i)x_i)^{\perp}$. Then $F_J^s = \Phi^+(B_n) \cap U$, thus F_J^s is a non-coordinate flat of $M(B_n)$. The support graph $G_B(F_J^s)$ is a complete simple graph with vertices $\{x_i \mid i \in J\}$, so Theorem 1.5.5 implies that F_J^s has rank $|J| - 1$. If $|J| = 2$, then clearly F_J^s is connected, and for $|J| \geq 3$ this follows from Lemma 1.5.9. $\square$

Theorem 4.3.6. *Every support-connected flat of $M(B_n)$ is connected. The connected flats of $M(B_n)$ are exactly the flats of the form F_J and F_J^s as defined in Propositions 4.3.4 and 4.3.5, respectively.*

Proof. Let F be a support-connected flat of $M(B_n)$ with support $\{x_i \mid i \in J\}$ for some $J \subseteq I$. Then F and J are non-empty. If F is a coordinate flat, then $F = F_J$ by Proposition 1.5.4, hence F is connected by Proposition 4.3.4. Otherwise, F is a non-coordinate flat by Theorem 1.5.2 and its support graph $G_B(F)$ is connected by Lemma 1.4.6. Whenever $x_i - \epsilon_1 x_j$ and $x_j - \epsilon_2 x_k$ are edges of $G_B(F)$ for pairwise distinct $i, j, k \in \{1, \dots, n\}$ and signs $\epsilon_1, \epsilon_2 \in \{-1, 1\}$, we have

$$x_i - \epsilon_1\epsilon_2 x_k \in \mathrm{cl}(\{x_i - \epsilon_1 x_j, x_j - \epsilon_2 x_k\}) \subseteq F.$$

This shows that $G_B(F)$ is a complete simple graph. Choose any vertex $x_i \in \mathrm{supp}_B(F)$ and define the function

$$s\colon J \to \{-1, 1\},\ j \mapsto \begin{cases} 1 & \text{if } i = j \text{ or } x_i - x_j \in F \\ -1 & \text{if } x_i + x_j \in F \end{cases}.$$

Then by construction $F = F_J^s$ and F is connected by Proposition 4.3.5. $\square$

Excluding the trivial coordinate flat $F_{\{1,\dots,n\}} = \Phi^+(B_n)$, these are exactly the vertices of the minimal nested set complex $\mathcal{N}(M(B_n))$. Next, we compute the degrees of these vertices in the 1-skeleton of the minimal nested set complex.

Lemma 4.3.7. *Let F and F' be connected flats of $M(B_n)$. Then $F \vee F'$ is disconnected if and only if $\mathrm{supp}_B(F) \cap \mathrm{supp}_B(F') = \emptyset$ and at least one of F and F' is a non-coordinate flat.*

Proof. Assume that $F \vee F'$ is disconnected. Then $F \vee F'$ is not support-connected by Theorem 4.3.6, thus $\mathrm{supp}_B(F) \cap \mathrm{supp}_B(F') = \emptyset$. Moreover, if F and F' were both coordinate flats, then $F \vee F'$ would also be a coordinate flat and thus connected by Proposition 4.3.4, contradiction. The other direction was shown in Lemma 1.5.11. $\square$

Lemma 4.3.8. *Let $J \subseteq \{1, \dots, n\}$. Then there are exactly*

$$\frac{3^{|J|} - 1}{2} - |J|$$

connected non-coordinate flats F of $M(B_n)$ with $\mathrm{supp}_B(F) \subseteq \{x_i \mid i \in J\}$.

Proof. By Theorem 4.3.6, the connected non-coordinate flats of $M(B_n)$ with $\mathrm{supp}_B(F) \subseteq \{x_i \mid i \in J\}$ are exactly the flats of the form $F^s_{J'}$ as in Proposition 4.3.5 with $J' \subseteq J$. For a fixed $J' \subseteq J$ with $|J'| \geq 2$, there are $2^{|J'|-1}$ flats of type $F^s_{J'}$, since there are $2^{|J'|}$ possible choices for s, but s and $-s$ yield the same flat. Thus we compute:

$$\begin{aligned}
|\{F^s_{J'}\}_{J' \subseteq J,\ |J'| \geq 2,\ s\colon J' \to \{-1,1\}}| &= \sum_{J' \subseteq J, |J'| \geq 2} 2^{|J'|-1} \\
&= \sum_{k=2}^{|J|} \binom{|J|}{k} 2^{k-1} \\
&= \frac{1}{2}\left(\sum_{k=0}^{|J|} \binom{|J|}{k} 2^k - 2|J| - 1 \right) \\
&= \frac{1}{2}\left(3^{|J|} - 2|J| - 1 \right) \\
&= \frac{3^{|J|} - 1}{2} - |J|
\end{aligned}$$

$\square$

Proposition 4.3.9. *Let F be a proper connected coordinate flat of $M(B_n)$ of rank r. Then F has degree*

$$d_1(r, n) := \frac{3^r + 3^{n-r}}{2} + 2^r + 2^{n-r} - n - 5$$

in the 1-skeleton of $\mathcal{N}(M(B_n))$.

Proof. By Theorem 4.3.6, $F = F_J$ for some J with $\emptyset \subsetneq J \subsetneq \{1, \dots, n\}$. Then $\mathrm{rk}(F) = |J|$ by Proposition 4.3.4. Using Theorem 4.3.6, we count the neighbors of F_J in the 1-skeleton of $\mathcal{N}(M(B_n))$:

(1) F_J is adjacent to all smaller connected coordinate flats, i.e., to all flats $F_{J'}$ with $\emptyset \subsetneq J' \subsetneq J$. There are $2^{|J|} - 2$ such flats.
(2) F_J is adjacent to all larger proper connected coordinate flats, i.e., to all flats $F_{J'}$ with $J \subsetneq J' \subsetneq \{1, \dots, n\}$. There are $2^{n-|J|} - 2$ such flats.
(3) F_J is adjacent to all smaller connected non-coordinate flats, i.e., to all flats $F^s_{J'}$ with $J' \subseteq J$ and $|J'| \geq 2$. By Lemma 4.3.8, there are $\frac{3^{|J|}-1}{2} - |J|$ such flats.
(4) There are no larger connected non-coordinate flats.

(5) F_J is adjacent to all proper connected coordinate flats $F_{J'}$ such that $F_J \vee F_{J'}$ is disconnected, but by Lemma 4.3.7 there are no such flats.
(6) F_J is adjacent to all connected non-coordinate flats $F^s_{J'}$ such that $F_J \vee F^s_{J'}$ is disconnected. By Lemma 4.3.7, this is the case if and only if $J' \subseteq \{1, \dots, n\} \setminus J$. By Lemma 4.3.8, there are $\frac{3^{n-|J|}-1}{2} - (n - |J|)$ such flats.

Writing $r := |J|$ for the rank of F_J, we see that in total F_J has degree

$$\begin{aligned} &2^r - 2 + 2^{n-r} - 2 + \frac{3^r - 1}{2} - r + \frac{3^{n-r} - 1}{2} - (n - r) \\ =\ &\frac{3^r + 3^{n-r}}{2} + 2^r + 2^{n-r} - n - 5. \end{aligned}$$

□

Proposition 4.3.10. *Let F be a connected non-coordinate flat of $M(B_n)$ of rank r. Then F has degree*

$$d_2(r, n) := \frac{3^{n-r} - 1}{2} + 2^{r+1} + 2^{n-r} - n - 5$$

in the 1-skeleton of $\mathcal{N}(M(B_n))$.

Proof. By Theorem 4.3.6, $F = F^s_J$ for some $J \subseteq \{1, \dots, n\}$ with $|J| \geq 2$ and some $s\colon J \to \{-1, 1\}$. Then $\operatorname{rk}(F) = |J| - 1$ by Proposition 4.3.5. Using Theorem 4.3.6, we count the neighbors of F^s_J in the 1-skeleton of $\mathcal{N}(M(B_n))$:

(1) There are no smaller connected coordinate flats.
(2) F^s_J is adjacent to all larger proper connected coordinate flats, i.e., to all flats $F_{J'}$ with $J \subseteq J' \subsetneq \{1, \dots, n\}$. There are $2^{n-|J|} - 1$ such flats.
(3) F^s_J is adjacent to all smaller connected non-coordinate flats, i.e., to all flats $F^{s'}_{J'}$ with $J' \subsetneq J$, $|J'| \geq 2$, and $s' = \pm s|_{J'}$. Since s' is uniquely determined up to sign, there are $2^{|J|} - |J| - 2$ such flats.
(4) F^s_J is adjacent to all larger connected non-coordinate flats, i.e., to all flats $F^{s'}_{J'}$ with $J \subsetneq J' \subseteq \{1, \dots, n\}$ and $s'|_J = \pm s$. For fixed J', there are $2^{|J'|-|J|}$ possible choices for s' up to sign, hence there are

$$\begin{aligned} \sum_{J \subsetneq J' \subseteq \{1,\dots,n\}} 2^{|J'|-|J|} &= \sum_{k=|J|+1}^{n} \big|\{J' \mid J \subseteq J' \subseteq \{1, \dots, n\},\ |J'| = k\}\big| \cdot 2^{k-|J|} \\ &= \sum_{k=|J|+1}^{n} \binom{n - |J|}{k - |J|} 2^{k-|J|} = \sum_{k=1}^{n-|J|} \binom{n - |J|}{k} 2^k \\ &= 3^{n-|J|} - 1 \end{aligned}$$

such flats.
(5) F^s_J is adjacent to all proper connected coordinate flats $F_{J'}$ such that $F^s_J \vee F_{J'}$ is disconnected. By Lemma 4.3.7, this is the case if and only if $\emptyset \subsetneq J' \subseteq \{1, \dots, n\} \setminus J$. There are $2^{n-|J|} - 1$ such flats.
(6) F^s_J is adjacent to all connected non-coordinate flats $F^{s'}_{J'}$ such that $F^s_J \vee F^{s'}_{J'}$ is disconnected. By Lemma 4.3.7, this is the case if and only

if $J' \subseteq \{1, \dots, n\} \setminus J$. By Lemma 4.3.8, there are $\frac{3^{n-|J|}-1}{2} - (n - |J|)$ such flats.

Writing $r := |J| - 1$ for the rank of F_J^s, we see that in total F_J^s has degree

$$\begin{aligned}
&2^{n-r-1} - 1 + 2^{r+1} - r - 3 + 3^{n-r-1} - 1 \\
&+ 2^{n-r-1} - 1 + \frac{3^{n-r-1} - 1}{2} - n + r + 1 \\
&= \frac{3^{n-r} - 1}{2} + 2^{r+1} + 2^{n-r} - n - 5.
\end{aligned}$$

□

Corollary 4.3.11. *The vertices of $\mathcal{N}(M(B_n))$ with maximal degree in the 1-skeleton are exactly the singletons and the coordinate hyperplanes.*

Proof. In Propositions 4.3.9 and 4.3.10 we saw that the degrees in the 1-skeleton are given by the numbers $d_1(r, n)$ and $d_2(r, n)$. We note the following properties of these numbers:

(1) For all $1 \leq r < n$ we have $d_1(r, n) = d_1(n - r, n)$.

(2) For all $1 \leq r < n$ we have $d_1(r, n) - d_2(r, n) = \frac{3^r+1}{2} - 2^r \geq 0$, hence $d_1(r, n) \geq d_2(r, n)$ with equality if and only if $r = 1$.

(3) The first differences of $d_1(r, n)$ and $d_2(r, n)$ in r are strictly increasing functions in r:

$$\begin{aligned}
d_1(r + 1, n) - d_1(r, n) &= 3^r - 3^{n-r-1} + 2^r - 2^{n-r-1} \\
d_2(r + 1, n) - d_2(r, n) &= -3^{n-r-1} + 2^{r+1} - 2^{n-r-1}
\end{aligned}$$

In particular, $d_1(1, n) = d_2(1, n) = d_1(n - 1, n)$ is the maximal degree in the 1-skeleton of $\mathcal{N}(M(B_n))$. The corresponding flats are exactly the singletons and the coordinate hyperplanes. □

Example 4.3.12. For the root system B_{10}, we obtain the following degrees in the 1-skeleton of the minimal nested set complex:

r	1	2	3	4	5	6	7	8	9
$d_1(r, 10)$	10342	3530	1228	470	292	470	1228	3530	10342
$d_2(r, 10)$	10342	3529	1222	445	202	169	262	505	1012

Theorem 4.3.13 (cf. [SW23, Proposition 10.3])**.** *The automorphism group of the coarse Bergman fan $B_c(M(B_n))$ is generated by matroid automorphisms and the unique Cremona automorphism, hence it is isomorphic to the Cremona group* $\mathrm{Cr}(M(B_n))$.

Proof. Let f be an automorphism of $B_c(M(B_n))$. By Theorem 4.1.5, the coarse Bergman fan $B_c(M(B_n))$ and the minimal nested set fan $B_m(M(B_n))$ coincide, thus f induces an automorphism of the minimal nested set complex $\mathcal{N}(M(B_n))$. In particular, f preserves the subgraph G of the 1-skeleton of $\mathcal{N}(M(B_n))$ induced by the vertices of maximal degree. By Corollary 4.3.11, the vertices of G are exactly the singletons and the coordinate hyperplanes. We compute the degrees in G:

(1) Let $x_i \in \Phi^+(B_n)$ for some $i \in \{1, \dots, n\}$. Then the neighbors in G of the singleton $\{x_i\}$ are the $n - 1$ coordinate hyperplanes containing x_i and the $2\binom{n-1}{2}$ singletons corresponding to non-basis elements $v \in \Phi^+(B_n)$ with $x_i \notin \mathrm{supp}_B(v)$, hence $\{x_i\}$ has degree $(n - 1) + 2\binom{n-1}{2} = (n - 1)^2$.

(2) Let $v \in \Phi^+(B_n) \setminus B$ be a non-basis element. Then the neighbors in G of the singleton $\{v\}$ are the $n-2$ coordinate hyperplanes containing v, the $n-2$ singletons corresponding to basis elements $x_i \in B$ with $x_i \notin \operatorname{supp}_B(v)$, and the $2\binom{n-2}{2}$ singletons corresponding to non-basis elements $v' \in \Phi^+(B_n)$ with $\operatorname{supp}_B(v) \cap \operatorname{supp}_B(v') = \emptyset$. Hence v has degree $2(n-2) + 2\binom{n-2}{2} = (n-1)(n-2)$.

(3) Let $H \subseteq \Phi^+(B_n)$ be a coordinate hyperplane of B. Then the neighbors in G of H are exactly the singletons contained in H. Hence H has degree $|H| = (n-1) + 2\binom{n-1}{2} = (n-1)^2$.

We see that the singletons corresponding to non-basis elements have smaller degree than the other vertices, thus the set

$$\mathcal{E}_{\mathrm{inv}}(f) := \{v \in \Phi^+(A_n) \mid \operatorname{rk} f(\{v\}) = n-1\}$$

is contained in B. Moreover, f restricts to an automorphism of the subgraph G' induced by the coordinate singletons and coordinate hyperplanes. This is a connected bipartite graph and thus $\mathcal{E}_{\mathrm{inv}}(f)$ is either empty or equal to the Cremona basis B. By Theorem 3.1.3, the Cremona map crem_B induces an automorphism of $B_c(M(A_n))$, thus we may assume $\mathcal{E}_{\mathrm{inv}}(f) = \emptyset$, otherwise replace f by $\mathrm{crem}_B \circ f$. Now Proposition 2.2.4 shows that f is induced by a matroid automorphism. $\square$

4.4. D_n

Let $x_1, \dots, x_n$ be a basis of a vector space V over some field K with $\mathrm{char}(K) \neq 2$ and assume $n \geq 4$. We define the root system matroid $M(D_n)$ corresponding to the irreducible root system D_n as the vector matroid associated to the set of roots

$$\Phi^+(D_n) := \{x_i \pm x_j \mid 1 \leq i < j \leq n\}.$$

(We will sometimes use $x_j - x_i$ as alternative notation for the matroid element $x_i - x_j$ in order to avoid case distinctions.)

$M(D_n)$ is a simple matroid of rank n. Since $\Phi^+(D_n)$ is a subset of $\Phi^+(B_n)$, the root system matroid $M(D_n)$ is isomorphic to the restriction $M(B_n)|_{\Phi^+(D_n)}$ of the root system matroid corresponding to B_n.

Proposition 4.4.1. *Let $J \subseteq \{1, \dots, n\}$ with $|J| \geq 3$. Then*

$$F_J^{\pm} := \{x_i \pm x_j \mid i, j \in J,\ i < j\}$$

is a connected flat of $M(D_n)$ of rank $|J|$.

Proof. $F_J^{\pm}$ is equal to the intersection of $\Phi^+(D_n)$ with the linear subspace $\langle x_i \mid i \in J\rangle$ and thus a flat of $M(D_n)$. Let B be the unique Cremona basis of $M(B_n)$ from Theorem 4.3.1 and let $v, v' \in F_J^{\pm}$. It suffices to show that v and v' lie in the same component of $F_J^{\pm}$.

If $|\operatorname{supp}_B(v) \cap \operatorname{supp}_B(v')| = 1$, say $v = x_i - \epsilon x_j$ and $v' = x_j - \epsilon' x_k$ with $\epsilon, \epsilon' \in \{-1, 1\}$ and $i, j, k \in J$, then $x_i - \epsilon\epsilon' x_k \in \mathrm{cl}(\{v, v'\}) \cap F_J^{\pm}$, thus v and v' lie in the same component of $F_J^{\pm}$.

Otherwise, there exists an element $v'' \in F_J^{\pm}$ with

$$|\operatorname{supp}_B(v) \cap \operatorname{supp}_B(v'')| = |\operatorname{supp}_B(v'') \cap \operatorname{supp}_B(v')| = 1,$$

using the assumption $|J| \geq 3$ in the case that $\operatorname{supp}_B(v) = \operatorname{supp}_B(v')$. Then v and v' both lie in the component of $F_J^{\pm}$ that contains v''. □

Proposition 4.4.2. *Let $J \subseteq \{1, \dots, n\}$ with $|J| \geq 2$ and choose a sign function $s\colon J \to \{-1, 1\}$. Then*

$$\begin{aligned} F_J^s := &\{x_i - x_j \mid i, j \in J,\ s(i) = s(j),\ i < j\} \\ &\cup \{x_i + x_j \mid i, j \in J,\ s(i) \neq s(j),\ i < j\} \end{aligned}$$

is a connected flat of $M(D_n)$ of rank $|J| - 1$. Note that $F_J^s = F_J^{-s}$.

Proof. Follows from Proposition 4.3.5 since F_J^s coincides with the definition of the flat F_J^s of $M(B_n)$, see ibid. □

Theorem 4.4.3. *The connected flats of $M(D_n)$ are exactly the flats of the form $F_J^{\pm}$ and F_J^s as defined in Propositions 4.4.1 and 4.4.2, respectively.*

Proof. Let F be a connected flat of $M(D_n)$. Then the closure $\overline{F}$ of F in $M(B_n)$ is a connected flat of $M(B_n)$ with $\overline{F} \cap \Phi^+(D_n) = F$. Let $J \subseteq I$ such that $\{x_i \mid i \in J\}$ is the support of $\overline{F}$ with respect to the Cremona basis of $M(B_n)$. By Theorem 4.3.6, we have either $\overline{F} = F_J$ as defined in Proposition 4.3.4 or $\overline{F} = F_J^s$ as defined in Proposition 4.3.5 for some sign function $s\colon J \to \{-1, 1\}$. In the latter case, we have $F = F_J^s \cap \Phi^+(D_n) = F_J^s$ by Proposition 4.4.1, so we may assume $\overline{F} = F_J$. We consider the following cases:

(1) If $|J| = 1$, say $J = \{i\}$ for some $i \in \{1, \dots, n\}$, then $F = \{x_i\} \cap \Phi^+(D_n) = \emptyset$, contradicting the assumption that F is connected.
(2) If $|J| = 2$, say $J = \{i, j\}$ for some $i, j \in \{1, \dots, n\}$ with $i < j$, then $F = \{x_i, x_j, x_i \pm x_j\} \cap \Phi^+(D_n) = \{x_i \pm x_j\}$. This is a flat of rank 2 with two elements and thus not connected, contradiction.
(3) If $|J| \geq 3$, then $F = F_J \cap \Phi^+(D_n) = F_J^{\pm}$ and we are done. $\square$

Corollary 4.4.4. *$M(D_n)$ has no Cremona bases.*

Proof. By Theorem 4.4.3, we have $|F| \leq 3$ for all flats F of rank 2. If $M(D_n)$ had a Cremona basis, then Corollary 1.3.3 would imply

$$2\binom{n}{2} = |\Phi^+(D_n)| \leq n + \binom{n}{2}\big(\max_{x,y\in\Phi^+(D_n)} |\operatorname{cl}(\{x, y\})| - 2\big) \leq n + \binom{n}{2},$$

but for all $n \geq 4$ we have $\binom{n}{2} > n$. $\square$

Although $M(D_n)$ has no Cremona bases, we will call flats of the form $F_J^{\pm}$ coordinate flats and flats of the form F_J^s non-coordinate flats, in analogy to their corresponding flats in $M(B_n)$. Let B be the unique Cremona basis of $M(B_n)$ from Theorem 4.3.1.

Lemma 4.4.5. *Let F and F' be connected flats of $M(D_n)$. Then $F \vee F'$ is disconnected if and only if one of the following is true:*

(a) $\operatorname{supp}_B(F) \cap \operatorname{supp}_B(F') = \emptyset$ *and at least one of F and F' is a non-coordinate flat.*
(b) $\operatorname{rk}(F) = 1 = \operatorname{rk}(F')$ *and* $\operatorname{supp}_B(F) = \operatorname{supp}_B(F')$.

Proof. Let F and F' be connected flats of $M(D_n)$ such that $F \vee F'$ is disconnected. Then the closures $\overline{F}$ and $\overline{F'}$ of F and F', respectively, in $M(B_n)$ are connected flats of $M(B_n)$ with $\overline{F} \cap \Phi^+(D_n) = F$ and $\overline{F'} \cap \Phi^+(D_n) = F'$. Moreover, we have $(\overline{F} \vee \overline{F'}) \cap \Phi^+(D_n) = F \vee F'$.

Assume first that $\overline{F} \vee \overline{F'}$ is disconnected. Then Lemma 4.3.7 implies $\operatorname{supp}_B(\overline{F}) \cap \operatorname{supp}_B(\overline{F'}) = \emptyset$ and that at least one of $\overline{F}$ and $\overline{F'}$ is a non-coordinate flat, thus the same is true for F and F'.

Now assume that $\overline{F} \vee \overline{F'}$ is connected and let $J := \operatorname{supp}_B(\overline{F} \vee \overline{F'})$. Then by Theorem 4.3.6 we have either $\overline{F} \vee \overline{F'} = F_J$ or $\overline{F} \vee \overline{F'} = F_J^s$ for some sign function $s\colon J \to \{-1, 1\}$. In the latter case we have $F \vee F' = F_J^s$, which is connected by Proposition 4.4.2, contradicting our assumption. Thus $\overline{F} \vee \overline{F'} = F_J$. If $|J| \geq 3$, then we have $F \vee F' = F_J^{\pm}$, which is connected by Proposition 4.4.1, and again we have a contradiction. Thus $|J| = 2$ and $F \vee F' = F_J \cap \Phi^+(D_n) = \{x_i + x_j, x_i - x_j\}$ for some $i, j \in \{1, \dots, n\}$ with $i < j$, hence we conclude $\operatorname{rk}(F) = 1 = \operatorname{rk}(F')$ and $\operatorname{supp}_B(F) = \operatorname{supp}_B(F')$.

Conversely, if condition (a) holds, then $F \vee F'$ is disconnected by Lemma 1.5.11. In the case of condition (b) we have $F \vee F' = \{x_i + x_j, x_i - x_j\}$ for some $i, j \in \{1, \dots, n\}$ with $i < j$, hence $F \vee F'$ is disconnected. $\square$

Proposition 4.4.6. *Let F be a proper connected coordinate flat of $M(D_n)$ of rank r. Then F has degree*

$$d_1(r, n) := \frac{3^r + 3^{n-r}}{2} + 2^r + 2^{n-r} - \binom{r}{2} - n - r - 5$$

in the 1-skeleton of $\mathcal{N}(M(D_n))$.

Proof. By Theorem 4.4.3, $F = F_J^{\pm}$ for some $J \subseteq \{1, \dots, n\}$ with $|J| \geq 3$. Then $\operatorname{rk}(F) = |J|$ by Proposition 4.4.1. Using Theorem 4.4.3, we count the neighbors of $F_J^{\pm}$ in the 1-skeleton of $\mathcal{N}(M(D_n))$:

(1) $F_J^{\pm}$ is adjacent to all smaller connected coordinate flats, i.e., to all flats $F_{J'}^{\pm}$ with $J' \subsetneq J$ and $|J'| \geq 3$. There are $2^{|J|} - \binom{|J|}{2} - |J| - 2$ such flats.
(2) $F_J^{\pm}$ is adjacent to all larger proper connected coordinate flats, i.e., to all flats $F_{J'}^{\pm}$ with $J \subsetneq J' \subsetneq \{1, \dots, n\}$. There are $2^{n-|J|} - 2$ such flats.
(3) $F_J^{\pm}$ is adjacent to all smaller connected non-coordinate flats, i.e., to all flats $F_{J'}^{s}$ with $J' \subseteq J$ and $|J'| \geq 2$. By Lemma 4.3.8, there are $\frac{3^{|J|}-1}{2} - |J|$ such flats.
(4) There are no larger connected non-coordinate flats.
(5) $F_J^{\pm}$ is adjacent to all proper connected coordinate flats $F_{J'}^{\pm}$ such that $F_J^{\pm} \vee F_{J'}^{\pm}$ is disconnected, but by Lemma 4.4.5 there are no such flats.
(6) $F_J^{\pm}$ is adjacent to all connected non-coordinate flats $F_{J'}^{s}$ such that $F_J^{\pm} \vee F_{J'}^{s}$ is disconnected. By Lemma 4.4.5, this is the case if and only if $J' \subseteq \{1, \dots, n\} \setminus J$. By Lemma 4.3.8, there are $\frac{3^{n-|J|}-1}{2} - (n - |J|)$ such flats.

Writing $r := |J|$ for the rank of $F_J^{\pm}$, we see that in total $F_J^{\pm}$ has degree

$$\begin{aligned} & 2^r - \binom{r}{2} - r - 2 + 2^{n-r} - 2 + \frac{3^r - 1}{2} - r + \frac{3^{n-r} - 1}{2} - (n - r) \\ = & \frac{3^r + 3^{n-r}}{2} + 2^r + 2^{n-r} - \binom{r}{2} - n - r - 5. \end{aligned}$$ ◻

Proposition 4.4.7. *Let F be a connected non-coordinate flat of $M(D_n)$ of rank r. Then F has degree*

$$d_2(r, n) := \frac{3^{n-r} - 1}{2} + 2^{r+1} + 2^{n-r} - \binom{n - r - 1}{2} - 2n + r - 4$$

in the 1-skeleton of $\mathcal{N}(M(D_n))$.

Proof. By Theorem 4.4.3, $F = F_J^{s}$ for some $J \subseteq \{1, \dots, n\}$ with $|J| \geq 2$ and some $s \colon J \to \{-1, 1\}$. Then $\operatorname{rk}(F) = |J| - 1$ by Proposition 4.4.2. Using Theorem 4.4.3, we count the neighbors of F_J^{s} in the 1-skeleton of $\mathcal{N}(M(D_n))$:

(1) There are no smaller connected coordinate flats.
(2) F_J^{s} is adjacent to all larger proper connected coordinate flats, i.e., to all flats $F_{J'}^{\pm}$ with $J \subseteq J' \subsetneq \{1, \dots, n\}$ and $|J'| \geq 3$. If $|J| = 2$, then there are $2^{n-|J|} - 2$ such flats, else there are $2^{n-|J|} - 1$ such flats.
(3) F_J^{s} is adjacent to all smaller connected non-coordinate flats, i.e., to all flats $F_{J'}^{s'}$ with $J' \subsetneq J$, $|J'| \geq 2$, and $s' = \pm s|_{J'}$. Since s' is uniquely determined up to sign, there are $2^{|J|} - |J| - 2$ such flats.
(4) F_J^{s} is adjacent to all larger connected non-coordinate flats, i.e., to all flats $F_{J'}^{s'}$ with $J \subsetneq J' \subseteq \{1, \dots, n\}$ and $s'|_J = \pm s$. For fixed J',

there are $2^{|J'|-|J|}$ possible choices for s up to sign, hence there are

$$
\begin{aligned}
\sum_{J\subsetneq J'\subseteq\{1,\dots,n\}} 2^{|J'|-|J|} &= \sum_{k=|J|+1}^{n} |\{J' \mid J\subsetneq J'\subseteq\{1,\dots,n\},\ |J'|=k\}|\cdot 2^{k-|J|}\\
&= \sum_{k=|J|+1}^{n} \binom{n-|J|}{k-|J|} 2^{k-|J|} = \sum_{k=1}^{n-|J|}\binom{n-|J|}{k}2^k\\
&= 3^{n-|J|}-1
\end{aligned}
$$

such flats.

(5) F_J^s is adjacent to all proper connected coordinate flats $F_{J'}^{\pm}$ such that $F_J^s \vee F_{J'}^{\pm}$ is disconnected. By Lemma 4.4.5, this is the case if and only if $J'\subseteq\{1,\dots,n\}\setminus J$ with $|J'|\geq 3$. There are $2^{n-|J|}-\binom{n-|J|}{2}-(n-|J|)-1$ such flats.

(6) F_J^s is adjacent to all connected non-coordinate flats $F_{J'}^{s'}$ such that $F_J^s\vee F_{J'}^{s'}$ is disconnected. By condition (a) in Lemma 4.4.5, this is the case if $J'\subseteq\{1,\dots,n\}\setminus J$. By Lemma 4.3.8, there are $\frac{3^{n-|J|}-1}{2}-(n-|J|)$ such flats. Moreover, if $|J|=2$, then by condition (b) in Lemma 4.4.5 there is one additional neighbor, which compensates for the missing neighbor in case (2).

Writing $r:=|J|-1$ for the rank of F_J^s, we see that in total F_J^s has degree

$$
\begin{aligned}
&2^{n-r-1}-1+2^{r+1}-r-3+3^{n-r-1}-1\\
&+2^{n-r-1}-\binom{n-r-1}{2}-(n-r-1)-1+\frac{3^{n-r-1}-1}{2}-(n-r-1)\\
&=\frac{3^{n-r}-1}{2}+2^{r+1}+2^{n-r}-\binom{n-r-1}{2}-2n+r-4. \qquad\square
\end{aligned}
$$

Corollary 4.4.8. *The vertices of $\mathcal{N}(M(D_n))$ with maximal degree in the 1-skeleton are exactly the singletons.*

Proof. In Propositions 4.4.6 and 4.4.7 we saw that the degrees in the 1-skeleton are given by the numbers $d_1(r,n)$ and $d_2(r,n)$. We note the following properties of these numbers:

(1) The first differences of $d_1(r,n)$ and $d_2(r,n)$ in r are strictly increasing functions in r:

$$
\begin{aligned}
d_1(r+1,n)-d_1(r,n) &= 3^r-3^{n-r-1}+2^r-2^{n-r-1}-r-1\\
d_2(r+1,n)-d_2(r,n) &= -3^{n-r-1}+2^{r+1}-2^{n-r-1}+n-r-1
\end{aligned}
$$

(2) We have $d_2(1,n)>d_1(n-1,n)$ since $d_2(1,n)-d_1(n-1,n)=n-1$.

(3) We have $d_2(1,n)>d_2(n-1,n)$ since

$$
d_2(1,n)-d_2(n-1,n)=\frac{3^{n-1}-1}{2}-2^{n-1}-\frac{n(n-3)}{2}>0.
$$

(4) We have $d_2(1,n)>d_1(2,n)$ since

$$
d_2(1,n)-d_1(2,n)=3^{n-2}+2^{n-2}-\binom{n-2}{2}-n>0.
$$

Since $M(D_n)$ has no connected coordinate flat of rank 1, this shows that $d_2(1, n)$ is the maximal degree in the 1-skeleton of $\mathcal{N}(M(D_n))$. The corresponding flats are exactly the singletons. $\square$

Example 4.4.9. For the root system D_{10}, we obtain the following degrees in the 1-skeleton of the minimal nested set complex:

r	1	2	3	4	5	6	7	8	9
$d_1(r, 10)$	(10341)	(3527)	1222	460	277	449	1200	3494	10297
$d_2(r, 10)$	10306	3501	1201	430	192	163	259	504	1012

Theorem 4.4.10. *The automorphism group* $\operatorname{Aut}(B_c(M(D_n)))$ *of the coarse Bergman fan coincides with the group* $\operatorname{Aut}(M(D_n))$ *of matroid automorphisms.*

Proof. Let f be an automorphism of $B_c(M(D_n))$. By Theorem 4.1.5, the coarse Bergman fan $B_c(M(D_n))$ and the minimal nested set fan $B_m(M(D_n))$ coincide, thus f induces an automorphism of the minimal nested set complex $\mathcal{N}(M(D_n))$. In particular, f preserves the subgraph G of the 1-skeleton of $\mathcal{N}(M(D_n))$ induced by the vertices of maximal degree. By Corollary 4.4.8, the vertices of G are exactly the singletons. Thus the claim follows from Proposition 2.2.4. $\square$

4.5. E_6, E_7, and E_8

Let $x_1, \ldots, x_8$ be a basis of an 8-dimensional vector space V over some field K with $\mathrm{char}(K) \neq 2$. We define the root system matroid $M(E_8)$ corresponding to the irreducible root system E_8 as the vector matroid associated to the set of roots

$$\begin{aligned}\Phi^+(E_8) &:= \{x_i \pm x_j \mid 1 \leq i < j \leq n\} \\ &\quad \cup \{x_1 + \epsilon_2 x_2 + \cdots + \epsilon_8 x_8 \mid \epsilon_2, \ldots, \epsilon_8 \in \{-1, 1\},\ \epsilon_2 \cdots \epsilon_8 = 1\}.\end{aligned}$$

$M(E_8)$ is a simple matroid of rank 8. Let $x_1^\vee, \ldots, x_8^\vee$ be the dual basis to $x_1, \ldots, x_8$. The subset

$$\Phi^+(E_7) := \Phi^+(E_8) \cap V(x_7^\vee = x_8^\vee)$$

is a flat of $M(E_8)$ of rank 7 and we define $M(E_7) := M(E_8)|_{\Phi^+(E_7)}$. Likewise, the subset

$$\Phi^+(E_6) := \Phi^+(E_7) \cap V(x_6^\vee = x_7^\vee)$$

is a flat of $M(E_8)$ of rank 6 and we define $M(E_6) := M(E_8)|_{\Phi^+(E_6)}$.

By calculating all closures $\mathrm{cl}(\{v, v'\})$ with $v, v' \in \Phi^+(E_8)$, we determine the connected flats of rank 2:

Proposition 4.5.1. *$M(E_8)$ has the following connected flats of rank 2:*

(1) $\langle x_1 = x_2 = x_3 \rangle := \{x_1 - x_2, x_1 - x_3, x_2 - x_3\}$.

(2) $\langle x_1 + x_2, x_3 + \cdots + x_8 \rangle := \{x_1 + x_2, x_1 + x_2 \pm (x_3 + \cdots + x_8)\}$.

Up to permutations and sign changes, every connected flat of rank 2 is of exactly one of these types.

Corollary 4.5.2. *$M(E_6)$, $M(E_7)$, and $M(E_8)$ have no Cremona bases.*

Proof. By Proposition 4.5.1 we have $|F| \leq 3$ for all connected rank 2 flats F of $M(E_8)$. Since $M(E_7)$ and $M(E_6)$ are restrictions of $M(E_8)$, they also have this property. If $M(E_n)$ had a Cremona basis for some $n \in \{6, 7, 8\}$, then Corollary 1.3.3 would imply

$$|M(E_n)| \leq n + \binom{n}{2} = \frac{n(n+1)}{2}.$$

However, we have $|M(E_8)| = 120 > 36 = \frac{8 \cdot 9}{2}$, $|M(E_7)| = 63 > 28 = \frac{7 \cdot 8}{2}$, and $|M(E_6)| = 36 > 21 = \frac{6 \cdot 7}{2}$. □

It does not seem possible to embed these matroids into a matroid that has a Cremona basis, as we did for the root system matroids of type D_n. Thus classifying all connected flats seems to be difficult, in particular for E_8.

Conjecture 4.5.3. *For all $n \in \{6, 7, 8\}$ we have*

$$\mathrm{Aut}(B_c(M(E_n))) = \mathrm{Aut}(M(E_n)).$$

We believe that this can be shown by computing the degrees of all connected flats in the 1-skeleton of the respective minimal nested set complexes, and we would expect that these numbers are asymmetric like in Example 4.4.9 for D_{10}.

4.6. F_4

Let $x_1, \ldots, x_4$ be a basis of a 4-dimensional vector space V over some field K with $\mathrm{char}(K) \neq 2$. We define the root system matroid $M(F_4)$ corresponding to the irreducible root system F_4 as the vector matroid associated to the set of roots

$$\begin{aligned}\Phi^+(F_4) := \{&x_i \mid 1 \leq i \leq 4\} \\ &\cup \{x_i \pm x_j \mid 1 \leq i < j \leq 4\} \\ &\cup \{x_1 \pm x_2 \pm x_3 \pm x_4\}.\end{aligned}$$

$M(F_4)$ is a simple matroid of rank 4.

Proposition 4.6.1. *$M(F_4)$ has the following connected flats of rank 2:*

(1) $\langle x_1, x_2 \rangle := \{x_1, x_2, x_1 \pm x_2\}$.
(2) $\langle x_1, x_2 + x_3 + x_4 \rangle := \{x_1, x_1 \pm (x_2 + x_3 + x_4)\}$.
(3) $\langle x_1 = x_2 = x_3 \rangle := \{x_1 - x_2, x_1 - x_3, x_2 - x_3\}$.
(4) $\langle x_1 = x_2, x_3 = x_4 \rangle := \{x_1 - x_2, x_3 - x_4, x_1 - x_2 \pm (x_3 - x_4)\}$.

Up to permutations and sign changes, every connected flat of rank 2 is of exactly one of these types.

Proof. See Appendix 4.7. □

Corollary 4.6.2. *$M(F_4)$ has no Cremona bases.*

Proof. Proposition 4.6.1 shows that $\max_{v,v' \in \Phi^+(F_4)} |\operatorname{cl}(\{v, v'\})| = 4$. If $M(F_4)$ had a Cremona basis, then Corollary 1.3.3 would imply

$$24 \leq 4 + 6 \cdot \big(\max_{v,v' \in \Phi^+(F_4)} |\operatorname{cl}(\{v, v'\})| - 2 \big) = 16,$$

contradiction. □

Proposition 4.6.3. *$M(F_4)$ has the following connected flats of rank 3:*

(1) $\langle x_1, x_2, x_3 \rangle := \{x_1, x_2, x_3, x_1 \pm x_2, x_1 \pm x_3, x_2 \pm x_3\}$.
(2) $\langle x_1, x_2, x_3 = x_4 \rangle := \{x_1, x_2, x_1 \pm x_2, x_3 - x_4, x_1 \pm x_2 \pm (x_3 - x_4)\}$.
(3) $\langle x_1 = x_2 = x_3 = x_4 \rangle := \{x_i - x_j \mid 1 \leq i < j \leq 4\} \cup \{x_1 + x_2 - x_3 - x_4, x_1 - x_2 + x_3 - x_4, x_1 - x_2 - x_3 + x_4\}$.

Up to permutations and sign changes, every connected flat of rank 3 is of exactly one of these types.

Proof. See Appendix 4.7. □

Let $\mathcal{N}(F_4)$ be the minimal nested set complex of $M(F_4)$.

Proposition 4.6.4. *Let F be a connected flat of $M(F_4)$.*

(1) *If $\operatorname{rk}(F) = 1$, then F has degree 26 in the 1-skeleton of $\mathcal{N}(F_4)$.*
(2) *If $\operatorname{rk}(F) = 2$, then F has degree 8 or 9 in the 1-skeleton of $\mathcal{N}(F_4)$.*
(3) *If $\operatorname{rk}(F) = 3$, then F has degree 16 in the 1-skeleton of $\mathcal{N}(F_4)$.*

Proof. See Appendix 4.7. □

Theorem 4.6.5. *The automorphism group $\operatorname{Aut}(B_c(M(F_4)))$ of the coarse Bergman fan coincides with the group $\operatorname{Aut}(M(F_4))$ of matroid automorphisms.*

Proof. Follows from Proposition 4.6.4 and Proposition 2.2.4. □

4.7. Appendix: Computations for F_4

We determine the flats of F_4 by calculating the parallel classes in the contraction $M(F_4)/F$ for every flat F. Behind each parallel class, we write two numbers (x, y), where x is the number of elements in the parallel class and y is the number of parallel classes of this type up to permutations and sign changes.

(1) rank 1:
- $\{x_1\}$ (1, 4)
- $\{x_1 + x_2\}$ (1, 12)
- $\{x_1 + x_2 + x_3 + x_4\}$ (1, 8)

(2) rank 2 over $\{x_1\}$:
- $\langle x_1, x_2\rangle = \{x_1, x_2, x_1 \pm x_2\}$ (3, 3)
- $\{x_1, x_2 + x_3\}$ (1, 6, reducible)
- $\langle x_1, x_1 + x_2 + x_3 + x_4\rangle = \{x_1, x_1 \pm (x_2 + x_3 + x_4)\}$ (2, 4)

(3) rank 2 over $\{x_1 + x_2\}$:
- $\langle x_1, x_2\rangle$ (3, 1)
- $\{x_1 + x_2, x_3\}$ (1, 2, reducible)
- $\langle x_1 + x_2, x_1 + x_3\rangle = \{x_1 + x_2, x_1 + x_3, x_2 - x_3\}$ (2, 4)
- $\langle x_1 + x_2, x_3 + x_4\rangle = \{x_1 + x_2, x_3 + x_4, x_1 + x_2 \pm (x_3 + x_4)\}$ (3, 2)
- $\{x_1 + x_2, x_1 - x_2 + x_3 + x_4\}$ (1, 4, reducible)

(4) rank 2 over $\{x_1 + x_2 + x_3 + x_4\}$:
- $\langle x_1, x_1 + x_2 + x_3 + x_4\rangle$ (2, 4)
- $\langle x_1 + x_2, x_3 + x_4\rangle$ (3, 3)
- $\{x_1 + x_2 + x_3 + x_4, x_1 - x_2\}$ (1, 6, reducible)

(5) rank 3 over $\langle x_1, x_2\rangle$:
- $\langle x_1, x_2, x_3\rangle = \langle x_1, x_2\rangle \cup \{x_3, x_1 \pm x_3, x_2 \pm x_3\}$ (5, 2)
- $\langle x_1, x_2, x_3 + x_4\rangle = \langle x_1, x_2\rangle \cup \{x_3 + x_4, x_1 \pm x_2 \pm (x_3 + x_4)\}$ (5, 2)

(6) rank 3 over $\{x_1, x_2 + x_3\}$:
- $\langle x_1, x_2, x_3\rangle$ (7, 1)
- $\langle x_1, x_2 + x_3, x_4\rangle$ (7, 1)
- $\langle x_1, x_2 + x_3, x_2 + x_4\rangle = \{x_1\} \cup \langle x_2 + x_3, x_2 + x_4\rangle$ (2, 2, reducible)
- $\langle x_1, x_2 + x_3, x_1 + x_2 - x_3 + x_4\rangle = \langle x_1, x_1 + x_2 - x_3 + x_4\rangle \cup \{x_2 + x_3\}$ (2, 2, reducible)

(7) rank 3 over $\langle x_1, x_1 + x_2 + x_3 + x_4\rangle$:
- $\langle x_1, x_2, x_3 + x_4\rangle$ (6, 3)
- $\langle x_1, x_1 + x_2 + x_3 + x_4, x_2 - x_3\rangle = \langle x_1, x_1 + x_2 + x_3 + x_4\rangle \cup \{x_2 - x_3\}$ (1, 3, reducible)

(8) rank 3 over $\langle x_1 + x_2, x_1 + x_3\rangle$:
- $\langle x_1, x_2, x_3\rangle$ (6, 1)
- $\langle x_1 + x_2, x_1 + x_3, x_4\rangle = \langle x_1 + x_2, x_1 + x_3\rangle \cup \{x_4\}$ (1, 1, reducible)
- $\langle x_1 + x_2, x_1 + x_3, x_1 + x_4\rangle = \{x_1 + x_2, x_1 + x_3, x_1 + x_4, x_2 - x_3, x_2 - x_4, x_3 - x_4, x_1 + x_2 + x_3 - x_4, x_1 - x_2 + x_3 + x_4, x_1 + x_2 - x_3 + x_4\}$ (6, 2)
- $\langle x_1 + x_2, x_1 + x_3, x_1 - x_2 - x_3 + x_4\rangle = \langle x_1 + x_2, x_1 + x_3\rangle \cup \{x_1 - x_2 - x_3 + x_4\}$ (1, 2, reducible)

(9) rank 3 over $\langle x_1 + x_2, x_3 + x_4\rangle$:

- $\langle x_1, x_2, x_3 + x_4\rangle$ (5, 2)
- $\langle x_1+x_2, x_1+x_3, x_1-x_4\rangle = \{x_1+x_2, x_1+x_3, x_1-x_4, x_2-x_3, x_2+x_4, x_3+x_4, x_1+x_2+x_3+x_4, x_1-x_2+x_3-x_4, x_1+x_2-x_3-x_4\}$ (5, 2)

(10) rank 3 over $\{x_1 + x_2, x_1 - x_2 + x_3 + x_4\}$:
- $\langle x_1, x_2, x_3 + x_4\rangle$ (7, 1)
- $\langle x_1+x_2, x_1-x_2+x_3+x_4, x_3\rangle = \langle x_3, x_1-x_2+x_3+x_4\rangle \cup \{x_1+x_2\}$ (2, 2, reducible)
- $\langle x_1 + x_2, x_1 + x_3, x_1 + x_4\rangle$ (7, 1)
- $\langle x_1+x_2, x_1-x_2+x_3+x_4, x_1-x_3\rangle = \langle x_1+x_2, x_1-x_3\rangle \cup \{x_1-x_2+x_3+x_4\}$ (2, 2, reducible)

Thus we have the following irreducible flats:
(1) 4 rank 1 flats of type $\{x_1\}$
(2) 12 rank 1 flats of type $\{x_1 + x_2\}$
(3) 8 rank 1 flats of type $\{x_1 + x_2 + x_3 + x_4\}$
(4) 6 rank 2 flats of type $\langle x_1, x_2\rangle$
(5) 16 rank 2 flats of type $\langle x_1, x_1 + x_2 + x_3 + x_4\rangle$
(6) 16 rank 2 flats of type $\langle x_1 + x_2, x_1 + x_3\rangle$
(7) 12 rank 2 flats of type $\langle x_1 + x_2, x_3 + x_4\rangle$
(8) 4 rank 3 flats of type $\langle x_1, x_2, x_3\rangle$
(9) 12 rank 3 flats of type $\langle x_1, x_2, x_3 + x_4\rangle$
(10) 8 rank 3 flats of type $\langle x_1 + x_2, x_1 + x_3, x_1 + x_4\rangle$

We compute the degrees in the minimal nested set complex:
(1) The 4 rank 1 flats of type $\{x_1\}$ have degree 26:
- 3: contained in the rank 2 flats $\langle x_1, x_2\rangle$
- 4: contained in the rank 2 flats $\langle x_1, x_1 + x_2 + x_3 + x_4\rangle$
- 6: disconnected to the rank 1 flats $\{x_2 + x_3\}$
- 3: contained in the rank 3 flats $\langle x_1, x_2, x_3\rangle$
- 6: contained in the rank 3 flats $\langle x_1, x_2, x_3 + x_4\rangle$
- 4: disconnected to the rank 2 flats $\langle x_2 + x_3, x_2 + x_4\rangle$

(2) The 12 rank 1 flats of type $\{x_1 + x_2\}$ have degree 26:
- 1: contained in the rank 2 flat $\langle x_1, x_2\rangle$
- 4: contained in the rank 2 flats $\langle x_1 + x_2, x_1 + x_3\rangle$
- 2: contained in the rank 2 flats $\langle x_1 + x_2, x_3 + x_4\rangle$
- 2: disconnected to the rank 1 flats $\{x_3\}$
- 4: disconnected to the rank 1 flats $\{x_1 - x_2 + x_3 + x_4\}$
- 2: contained in the rank 3 flats $\langle x_1, x_2, x_3\rangle$
- 3: contained in the rank 3 flats $\langle x_1+x_2, x_3, x_4\rangle$, $\langle x_1, x_2, x_3+x_4\rangle$, and $\langle x_1, x_2, x_3 - x_4\rangle$
- 4: contained in the rank 3 flats $\langle x_1 + x_2, x_1 + x_3, x_1 + x_4\rangle$
- 4: disconnected to the rank 2 flats $\langle x_3, x_1 - x_2 + x_3 + x_4\rangle$

(3) The 8 rank 1 flats of type $\{x_1 + x_2 + x_3 + x_4\}$ have degree 26:
- 4: contained in the rank 2 flats $\langle x_1, x_1 + x_2 + x_3 + x_4\rangle$
- 3: contained in the rank 2 flats $\langle x_1 + x_2, x_3 + x_4\rangle$
- 6: disconnected to the rank 1 flats $\{x_1 - x_2\}$
- 6: contained in the rank 3 flats $\langle x_1, x_2, x_3 + x_4\rangle$
- 3: contained in the rank 3 flats $\langle x_1 + x_2, x_1 + x_3, x_1 - x_4\rangle$, $\langle x_1 + x_2, x_1 - x_3, x_1 + x_4\rangle$, and $\langle x_1 - x_2, x_1 + x_3, x_1 + x_4\rangle$

- 4: disconnected to the rank 2 flats $\langle x_1 - x_2, x_1 - x_3 \rangle$

(4) The 6 rank 2 flats of type $\langle x_1, x_2 \rangle$ have degree 8:
- 4: contains 4 rank 1 flats
- 2: contained in the rank 3 flats $\langle x_1, x_2, x_3 \rangle$
- 2: contained in the rank 3 flats $\langle x_1, x_2, x_3 + x_4 \rangle$

(5) The 16 rank 2 flats of type $\langle x_1, x_1 + x_2 + x_3 + x_4 \rangle$ have degree 9:
- 3: contains 3 rank 1 flats
- 3: contained in the rank 3 flats $\langle x_1, x_2, x_3 + x_4 \rangle$
- 3: disconnected to the rank 1 flats $\{x_2 - x_3\}$

(6) The 16 rank 2 flats of type $\langle x_1 + x_2, x_1 + x_3 \rangle$ have degree 9:
- 3: contains 3 rank 1 flats
- 1: contained in the rank 3 flat $\langle x_1, x_2, x_3 \rangle$
- 2: contained in the rank 3 flats $\langle x_1 + x_2, x_1 + x_3, x_1 + x_4 \rangle$
- 1: disconnected to the rank 1 flat $\{x_4\}$
- 2: disconnected to the rank 1 flats $\{x_1 - x_2 - x_3 + x_4\}$

(7) The 12 rank 2 flats of type $\langle x_1 + x_2, x_3 + x_4 \rangle$ have degree 8:
- 4: contains 4 rank 1 flats
- 2: contained in the rank 3 flats $\langle x_1, x_2, x_3 + x_4 \rangle$
- 2: contained in the rank 3 flats $\langle x_1 + x_2, x_3 + x_4, x_1 + x_3 \rangle$

(8) The 4 rank 3 flats of type $\langle x_1, x_2, x_3 \rangle$ have degree 16:
- 9: contains 9 rank 1 flats
- 3: contains the rank 3 flats $\langle x_1, x_2 \rangle$
- 4: contains the rank 3 flats $\langle x_1 + x_2, x_1 + x_3 \rangle$

(9) The 12 rank 3 flats of type $\langle x_1, x_2, x_3 + x_4 \rangle$ have degree 16:
- 9: contains 9 rank 1 flats
- 1: contains the rank 2 flat $\langle x_1, x_2 \rangle$
- 4: contains the rank 2 flats $\langle x_1, x_1 + x_2 + x_3 + x_4 \rangle$
- 2: contains the rank 2 flats $\langle x_1 + x_2, x_3 + x_4 \rangle$ and $\langle x_1 - x_2, x_3 + x_4 \rangle$

(10) The 8 rank 3 flats of type $\langle x_1 + x_2, x_1 + x_3, x_1 + x_4 \rangle$ have degree 16:
- 9: contains 9 rank 1 flats
- 4: contains the rank 2 flats $\langle x_1 + x_2, x_1 + x_3 \rangle$
- 3: contains the rank 2 flats $\langle x_1 + x_2, x_3 - x_4 \rangle$

4.8. Conclusion

In this chapter, we applied our results from Chapters 1 and 3 to root system matroids. We saw that the matroids associated to the root systems A_n and B_n have Cremona bases and we determined their Cremona groups as well as the automorphisms of their coarse Bergman fans, using computations in the 1-skeleton minimal nested set structure. In terms of their structure with respect to Cremona bases, these matroids are opposite extremes: A_n has many Cremona bases, while B_n has only one Cremona basis. For the root systems D_n and F_4 we saw that there are no Cremona bases and that all automorphisms of the coarse Bergman fans are induced by matroid automorphisms.

It is a challenge to determine the class of matroids M such that the automorphism group $\mathrm{Aut}(B_c(M))$ of the coarse Bergman fan is generated by matroid automorphisms and Cremona maps. We believe that our proofs for A_n and B_n can be simultaneously generalized to prove the following:

Conjecture 4.8.1. *Let M be a simple connected matroid that admits a Cremona basis B and assume that the coarse Bergman fan $B_c(M)$ and the minimal nested set fan $B_m(M)$ coincide. Then* $\mathrm{Aut}(B_c(M))$ *is generated by matroid automorphisms and Cremona maps.*

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Zusammenfassung

Über Analoga von Cremonaautomorphismen für Matroidfächer

Symmetrie, ob man ihre Bedeutung weit oder eng faßt, ist eine Idee, vermöge derer der Mensch durch die Jahrtausende seiner Geschichte versucht hat, Ordnung, Schönheit und Vollkommenheit zu begreifen und zu schaffen.

— Hermann Weyl, *Symmetrie* [Wey52]

In der Mathematik ist eine Symmetrie eine Transformation eines mathematischen Objekts, die seine Struktur oder manche seiner Eigenschaften erhält. Klassische Typen von Symmetrien sind lineare Transformationen im euklidischen Raum wie Spiegelungen, Drehungen, Parallelverschiebungen und Skalierungen. Im 19. Jahrhundert begannen Mathematiker, abstraktere Typen von Symmetrien zu untersuchen, auch *Automorphismen* genannt, und die Gruppentheorie wurde entwickelt als algebraisches Werkzeug zur Beschreibung ihrer Eigenschaften. Zum Beispiel entdeckte Galois, dass die Auflösbarkeit einer polynomialen Gleichung durch Radikale von der Symmetriegruppe ihrer Nullstellen abhängt.

In dieser Arbeit benutzen wir Matroidtheorie und tropische Geometrie, um bestimmte Symmetrien von linearen Gleichungssystemen zu untersuchen. Betrachten wir zum Beispiel das lineare Gleichungssystem

$$\mathcal{A} := \{x = 0,\ y = 0,\ z = 0,\ x = y,\ x = z,\ y = z\}$$

in den reellen Variablen x, y und z. Geometrisch können wir die Variablen als kartesische Koordinaten für den dreidimensionalen euklidischen Raum $\mathbb{R}^3$ interpretieren. Dann beschreibt jede Gleichung in $\mathcal{A}$ eine Ebene durch den Nullpunkt und wir können die klassischen Symmetrien betrachten, die dieses Ebenenarrangement erhalten. Für $\mathcal{A}$ gibt es die folgenden Symmetrien und ihre Kombinationen:

(1) Es gibt eine Gruppenoperation der symmetrischen Gruppe S_3, da $\mathcal{A}$ komplett symmetrisch in den Variablen $\{x, y, z\}$ ist. Geometrisch operiert S_3 durch Drehungen und Ebenenspiegelungen, je nach Vorzeichen der Permutation. Zum Beispiel entspricht das Vertauschen der Variablen x and y der Spiegelung an der Ebene $x = y$.

(2) Da die Gleichungen in $\mathcal{A}$ homogen sind, wird jede Ebene erhalten, wenn alle Variablen um denselben Faktor ungleich null skaliert werden. Indem wir zu dem Quotientenraum bezüglich Skalierungen übergehen, können wir $\mathcal{A}$ auch als Geradenarrangement in der projektiven Ebene $\mathbb{P}^2$ betrachten.

(3) Die lineare Abbildung

$$x \mapsto x, \quad y \mapsto x - y, \quad z \mapsto x - z$$

erhält die Menge $\mathcal{A}$. Zum Beispiel wird die Gleichung $x = y$ abgebildet auf die Gleichung $x = x - y$, welche äquivalent ist zu $y = 0$. Geometrisch entspricht diese Abbildung einer nicht-senkrechten Spiegelung an der Gerade $x = 2y = 2z$.

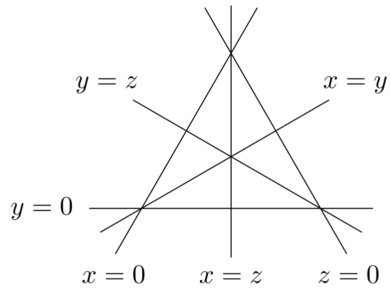

Abb. 15. Eine geometrische Darstellung von $\mathcal{A}$ als Geraden-arrangement in der projektiven Ebene.

Es gibt jedoch auch eine nichtlineare Symmetrie in diesem Beispiel. Betrachten wir die quadratische Transformation

$$x \mapsto yz, \quad y \mapsto xz, \quad z \mapsto xy,$$

welche in der projektiven Geometrie bekannt ist als *Cremonatransformation*, benannt nach dem italienischen Mathematiker Luigi Cremona (1830–1903). Das Bild von $\mathcal{A}$ unter der Cremonatransformation ist die Gleichungsmenge

$$\mathcal{A}' := \{yz = 0,\ xz = 0,\ xy = 0,\ yz = xz,\ yz = xy,\ xz = xy\}.$$

Jede quadratische Gleichung in $\mathcal{A}'$ zerfällt in zwei lineare Gleichungen:

$$\begin{aligned}
yz = 0 &\iff y = 0 \text{ or } z = 0\\
xz = 0 &\iff x = 0 \text{ or } z = 0\\
xy = 0 &\iff x = 0 \text{ or } y = 0\\
yz = xz &\iff y = x \text{ or } z = 0\\
yz = xy &\iff z = x \text{ or } y = 0\\
xz = xy &\iff z = y \text{ or } x = 0
\end{aligned}$$

…und die Gleichungen, die wir erhalten, sind genau die Gleichungen in $\mathcal{A}$! Manche Gleichungen kommen jedoch dreimal vor, das Schema erinnert an die Matrixmultiplikation

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1\\ 1 & 0 & 1 & 0 & 1 & 0\\ 1 & 1 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1\\1\\1\\1\\1\\1 \end{pmatrix} = \begin{pmatrix} 3\\3\\3\\1\\1\\1 \end{pmatrix}.$$

Ziel dieser Arbeit ist es, diese Cremonasymmetrien besser zu verstehen. Wann existieren sie und welche Gruppen erzeugen sie? Gibt es andere Arten von nichtlinearen Symmetrien? Die Suche nach Antworten führt uns zur Theorie der Matroide.

Matroide sind kombinatorische Objekte, die den Begriff der Abhängigkeit aus der linearen Algebra verallgemeinern. Zum Beispiel lässt sich aus den beiden Gleichungen $x = y$ und $y = 0$ die Gleichung $x = 0$ folgern. In der Sprache der Matroide sagt man, dass $x = 0$ im *Abschluss* $\mathrm{cl}(\{x = y,\ y = 0\})$ der Menge $\{x = y,\ y = 0\}$ liegt. Es gibt keine andere Gleichung aus $\mathcal{A}$, die man aus $x = y$ und $y = 0$ herleiten kann, daher bildet $\{x = y,\ y = 0,\ x = 0\}$ eine *abgeschlossene Menge* des zu $\mathcal{A}$ gehörigen Matroids M. Geometrisch beschreiben die abgeschlossenen Mengen von M das Schnittverhalten der Ebenen in $\mathcal{A}$.

Die Menge $\{x = 0,\ y = z\}$ ist auch eine abgeschlossene Menge von M, aber sie enthält nur zwei Elemente und wird daher *unzusammenhängend* genannt, im Gegensatz zur *zusammenhängenden* abgeschlossenen Menge $\{x = y,\ y = 0,\ x = 0\}$. Es gibt drei andere abgeschlossene Mengen vom Rang 2: $\{y = 0,\ z = 0,\ y = z\}$, $\{x = 0,\ z = 0,\ x = z\}$ und $\{x = y,\ x = z,\ y = z\}$.

Die Struktur des Matroids M kann mithilfe des minimalen Nested-set-Fächers $\mathcal{N}$ visualisiert werden. In unserem Beispiel ist dies ein Graph, dessen Knoten die Elemente aus $\mathcal{A}$ plus die zusammenhängenden abgeschlossenen Mengen vom Rang 2 sind. Von jeder zusammenhängenden abgeschlossenen Menge vom Rang 2 aus zeichnen wir je eine Kante zu allen ihren Elementen, außerdem fügen wir Kanten ein zwischen Paaren von Elementen, die eine unzusammenhängende abgeschlossene Menge bilden.

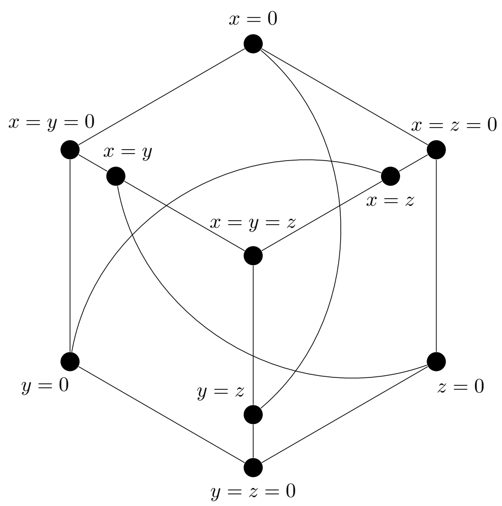

Abb. 16. Der zu $\mathcal{A}$ gehörige minimale Nested-set-Fächer.

Jede klassische Symmetrie von $\mathcal{A}$ induziert einen Automorphismus von $\mathcal{N}$, der Elemente auf Elemente und abgeschlossene Mengen auf abgeschlossene Mengen abbildet. Es gibt allerdings auch einen Automorphismus von $\mathcal{N}$, der diese Eigenschaft nicht hat, nämlich die Rotation des äußeren Sechsecks um 180 Grad. Dieser Automorphismus ist ein Beispiel für einen kombinatorischen Cremonaautomorphismus, wie er von Shaw und Werner in dem kürzlich erschienenen Paper [SW23] eingeführt wurde. Wir werden sehen, dass geometrische und kombinatorische Cremonasymmetrien eng verwandt sind und dass sich ihr Zusammenhang mit tropischer Geometrie beschreiben lässt.

In letzter Zeit wurden aufregende neue Zusammenhänge zwischen algebraischer Geometrie und Kombinatorik entdeckt. Ein wichtiges Werkzeug ist die sogenannte Tropikalisierung, die algebraische Varietäten in polyedrische Objekte übersetzt. Die einfachsten Beispiele sind lineare Räume, deren tropische Analoga eng mit Matroiden zusammenhängen.

Genauer gesagt kann das Komplement eines essentiellen Hyperebenenarrangements $\mathcal{A}$ im projektiven Raum über einem Körper mit trivialer Bewertung mit einer linearen Untervarietät des algebraischen Torus identifiziert werden. Die Tropikalisierung dieser Varietät ist der Träger eines polyedrischen Fächers und hängt nur von dem zu $\mathcal{A}$ gehörigen Matroid $M(\mathcal{A})$ ab. Allgemeiner kann man zu jedem Matroid M einen *tropischen linearen Raum* $\mathrm{trop}(M)$ konstruieren, auch wenn sich M nicht als Hyperebenenarrangement darstellen lässt, siehe [MS15, Abschnitt 4.2]. Dieser neue geometrische Ansatz hat zu großem Fortschritt in der Matroidtheorie geführt, unter anderem zur Entwicklung einer kombinatorischen Hodgetheorie für Matroide, siehe [Ard18] für einen Überblick. Für seine Beiträge zu diesen Entwicklungen wurde June Huh im Jahr 2022 mit der Fields-Medaille geehrt.

Es gibt mehrere natürliche Fächerstrukturen auf der Menge $\mathrm{trop}(M)$ und in dieser Arbeit werden wir zwischen dem groben Bergmanfächer $B_c(M)$, dem minimalen Nested-set-Fächer $B_m(M)$ und dem feinen Bergmanfächer $B_f(M)$ unterscheiden. Diese Fächerstrukturen wurden in [AK06] und [FS05] beschrieben, und Feichtner und Sturmfels gaben ein Kriterium dafür an, dass der grobe Bergmanfächer und der minimale Nested-set-Fächer übereinstimmen [FS05, Satz 5.3].

Fächerstrukturen auf $\mathrm{trop}(M(\mathcal{A}))$, wobei $M(\mathcal{A})$ wie oben das zu einem Hyperebenenarrangement $\mathcal{A}$ gehörige Matroid ist, können benutzt werden, um tropische Kompaktifizierungen des Komplements $\Omega_{\mathcal{A}}$ zu konstruieren, indem man den Abschluss in den dazugehörigen torischen Varietäten nimmt ([Tev07, Proposition 2.3]). Auf diese Weise induziert der minimale Nested-set-Fächer die minimale wundervolle Kompaktifizierung von de Concini und Procesi, während der grobe Bergmanfächer die Visible-contour-Kompaktifizierung induziert, die von Kapranov definiert wurde ([Kap93]).

Unter der Annahme, dass $\mathcal{A}$ zusammenhängend ist, haben Kurul und Werner gezeigt, dass sich jeder birationale Automorphismus f von $\Omega_{\mathcal{A}}$ auf einen Automorphismus der Visible-contour-Kompaktifizierung erweitern lässt ([KW19, Satz 5.1]). In dem Beweis zeigen sie, dass f einen Automorphismus des intrinsischen Torus induziert, dessen Tropikalisierung den groben Bergmanfächer erhält. Nach [Kur17, Satz 7.7] erhält man so eine Einbettung der birationalen Automorphismengruppe $\mathrm{Aut}(\Omega_{\mathcal{A}})$ in die Automorphismengruppe $\mathrm{Aut}(B_c(M(\mathcal{A})))$ des groben Bergmanfächers. Automorphismen von Bergmanfächern können also als Analoga von birationalen Automorphismen für Matroide betrachtet werden.

In dem oben erwähnten Paper [SW23] über die birationale Geometrie von Matroiden haben Shaw und Werner Automorphismengruppen von Bergmanfächern untersucht mithilfe der neu entwickelten Hodgetheorie für Matroide, unter anderem mithilfe des Chowrings eines Matroids. Für ein einfaches Matroid M, das nicht total unzusammenhängend ist, zeigen sie, dass jeder

Automorphismus des feinen Bergmanfächers $B_f(M)$ von einem Matroidautomorphismus von M induziert wird ([SW23, Satz 6.3]). Abhängig vom Matroid M kann es jedoch Automorphismen des groben Bergmanfächers geben, die nicht von Matroidautomorphismen induziert sind.

Das Arrangement $\mathcal{A}$, das wir am Anfang betrachtet haben und welches auch als essentielles Zopfarrangement A_3 bekannt ist, und allgemeiner alle Wurzelsystemmatroide $M(A_n)$ mit $n \geq 3$ sind Beispiele für dieses Phänomen. Nach [AP18] ist die Automorphismengruppe des Modulraums $M_{0,n}^{\mathrm{trop}}$ der stabilen tropischen Kurven mit Geschlecht 0 und $n \geq 5$ markierten Punkten isomorph zur symmetrischen Gruppe S_n. Wie in [SW23, Beispiel 3.1] erklärt wird, kann der grobe Bergmanfächer $B_c(M(A_n))$ mit $M_{0,n+2}^{\mathrm{trop}}$ identifiziert werden, daher ist $\mathrm{Aut}(B_c(M(A_n))) \cong S_{n+2}$ größer als $\mathrm{Aut}(M(A_n)) \cong S_{n+1}$.

Um diese zusätzlichen Automorphismen zu beschreiben, haben Shaw und Werner kombinatorische Cremonaautomorphismen von Bergmanfächern eingeführt und den folgenden Satz gezeigt:

Satz ([SW23, Satz 8.3])**.** *Sei B eine Basis eines einfachen zusammenhängenden Matroids M auf der Grundmenge E. Die Cremonaabbildung $\mathrm{crem}_B\colon \mathbb{R}^E \to \mathbb{R}^E$, die durch*

$$v_b \mapsto v_{\mathrm{cl}(B\setminus\{b\})} \text{ für alle } b \in B \quad \text{und} \quad v_e \mapsto v_e \text{ für alle } e \in E \setminus B$$

gegeben ist, induziert genau dann einen Automorphismus des groben Bergmanfächers $B_c(M)$, wenn die Mengen $\{\mathrm{cl}(\{b, b'\}) \setminus \{b, b'\}\}_{b,b' \in B}$ eine Partition von $E \setminus B$ bilden.

Diese kombinatorische Bedingung an eine Basis eines Matroids bildet den Ausgangspunkt für unsere Untersuchungen.

In Kapitel 1 legen wir das kombinatorische Fundament und untersuchen Cremonabasen von Matroiden, die wie folgt definiert sind:

Definition (1.3.1)**.** Sei M ein Matroid auf der Grundmenge E. Eine Basis B von M heißt *Cremonabasis* von M, wenn gilt:

$$\bigcup_{b,b' \in B} \mathrm{cl}(\{b, b'\}) = E.$$

Für alle Resultate in diesem Kapitel nehmen wir an, dass das Matroid M einfach ist, und in diesem Fall stimmt die Definition überein mit der Bedingung in [SW23, Satz 8.3], wie wir in Proposition 1.3.2 zeigen werden. Wir werden sehen, dass die Existenz einer solchen Basis weitreichende Folgen für die Struktur des Matroids hat.

Bezüglich einer Cremonabasis B kann das Matroid M durch seinen *Trägergraphen* $G_B(M)$ dargestellt werden, wobei wir parallele Kanten erlauben, siehe Definition 1.3.4. Die Knoten von $G_B(M)$ entsprechen den Basiselementen B und die Kanten von $G_B(M)$ entsprechen den restlichen Elementen $E \setminus B$. Genauer gesagt folgt für ein Element $e \in E \setminus B$ aus der Annahme, dass B eine Cremonabasis ist, dass der Fundamentalkreis $C_B(e)$ von e bezüglich B genau zwei Elemente von B enthält, und genau diese wählen wir als Endpunkte von e.

Allgemeiner definieren wir für eine Teilmenge $S \subseteq E$ den *Träger* $\mathrm{supp}_B(S)$ bezüglich B als kleinste Menge von Basiselementen, deren Abschluss S enthält, siehe Proposition 1.2.8. Auf diese Weise kann S als ein Untergraph $G_B(S)$ von $G_B(M)$ dargestellt werden, dessen Knotenmenge $\operatorname{supp}_B(S)$ ist. Wenn dieser Graph zusammenhängend ist, dann nennen wir S *trägerzusammenhängend* bezüglich B. Nach Proposition 1.4.7 ist diese Bedingung schwächer als der übliche Zusammenhangsbegriff für Matroide.

In Abschnitt 1.5 wenden wir diese Begriffe an, um die abgeschlossenen Mengen und die Rangfunktion von Matroiden, die eine Cremonabasis haben, zu beschreiben. Wir zeigen, dass die abgeschlossenen Mengen in zwei Klassen eingeteilt werden können:

Theorem (1.5.2)**.** *Sei M ein einfaches Matroid, das eine Cremonabasis B hat, und sei F eine trägerzusammenhängende abgeschlossene Menge von M. Dann ist genau eine der folgenden Aussagen wahr:*

(1) *F ist eine abgeschlossene* Koordinatenmenge*, das heißt,* $F \cap B = \operatorname{supp}_B(F)$.
(2) *F ist eine abgeschlossene* Nichtkoordinatenmenge*, das heißt,* $F \cap B = \emptyset \neq F$.

Für jede Teilmenge $S \subseteq E$ ist der Rang $\operatorname{rk}(S)$ festgelegt durch die Mächtigkeit ihres Trägers und der Klasse des Abschlusses $\operatorname{cl}(S)$. Das folgt aus Korollar 1.4.8 und dem folgenden Satz:

Satz (1.5.5)**.** *Sei M ein einfaches Matroid auf der Grundmenge E, das eine Cremonabasis B hat, und sei $S \subseteq E$ eine Teilmenge.*

(1) $\operatorname{cl}(S)$ *ist genau dann eine abgeschlossene Koordinatenmenge, wenn* $\operatorname{rk}(S) = |\operatorname{supp}_B(S)|$.
(2) *Wenn* $\operatorname{cl}(S)$ *eine trägerzusammenhängende Nichtkoordinatenmenge ist, dann gilt* $\operatorname{rk}(S) = |\operatorname{supp}_B(S)| - 1$.

Wir betonen, dass die Begriffe Träger und abgeschlossene (Nicht-)Koordinatenmengen von der Wahl einer Cremonabasis B abhängen, und in den Abschnitten 1.6–1.8 betrachten wir den Fall, dass ein Matroid mehr als eine Cremonabasis hat. Jeder Matroidautomorphismus bildet Cremonabasen auf Cremonabasen ab, daher gibt es eine Gruppenoperation von $\operatorname{Aut}(M)$ auf der Menge $\mathcal{CB}(M)$ der Cremonabasen von M. Indem wir in Abschnitt 1.6 die Form des Trägergraphen $G_B(B')$ für Cremonabasen B und B' von M untersuchen, zeigen wir, dass diese Gruppenoperation immer transitiv ist:

Satz (1.7.2)**.** *Sei M ein einfaches Matroid auf der Grundmenge E, das zwei Cremonabasen B und B' hat. Dann gibt es einen selbstinversen Automorphismus $f_{BB'}$ von M mit*

$$f_{BB'}(B) = B', \quad f_{BB'}(B') = B, \quad \textit{und} \quad f_{BB'}|_{E\setminus(B\cup B')} = \mathrm{id}.$$

Insbesondere induziert der Matroidautomorphismus $f_{BB'}$ einen Isomorphismus zwischen den Trägergraphen $G_B(M)$ und $G_{B'}(M)$. Wir nennen $f_{BB'}$ den *Cremonabasiswechselautomorphismus* bezüglich der Cremonabasen B und B'.

Wenn wir zusätzlich annehmen, dass M und die Zusammenziehungen M/e für alle $e \in E$ zusammenhängend sind, dann erzeugen die Cremonabasiswechselautomorphismen die symmetrische Gruppe $\operatorname{Sym}(\mathcal{CB}(M))$:

Satz (1.7.6)**.** *Sei M ein einfaches Matroid vom Rang mindestens 3 auf der Grundmenge E. Wenn M und die Zusammenziehungen M/e für alle $e \in E$ zusammenhängend sind, dann ist die Gruppenwirkung* $\mathrm{Aut}(M) \to \mathrm{Sym}(\mathcal{CB}(M))$ *surjektiv.*

Dies führt zu einem Darstellbarkeitskriterium für Matroide, die mehr als eine Cremonabasis haben.

Satz (1.8.2)**.** *Sei M ein einfaches zusammenhängendes Matroid vom Rang mindestens 3 auf der Grundmenge E und nehmen wir an, dass die Zusammenziehungen M/e für alle $e \in E$ zusammenhängend sind. Wenn M mehr als eine Cremonabasis hat, dann ist M darstellbar über jedem Körper K mit* $|K| \geq |E| - \mathrm{rk}(M) + 1$.

In Abschnitt 2.1 wiederholen wir die Definitionen des *tropischen linearen Raums* $\mathrm{trop}(M)$, des *feinen Bergmanfächers* $B_f(M)$ und des *groben Bergmanfächers* $B_c(M)$ eines Matroids M. Es gibt außerdem den *minimalen Nested-set-Fächer* $B_m(M)$, der aus dem *minimalen Nested-set-Komplex* $\mathcal{N}(M)$ konstruiert wird.

Automorphismen dieser Matroidfächer sind definiert als lineare Abbildungen, die die Fächerstruktur erhalten und von einem Gitterautomorphismus induziert sind. Nach Lemma 2.2.3 induzieren Matroidautomorphismen Fächerautomorphismen, und umgekehrt kommt laut Proposition 2.2.4 jeder Automorphismus des groben Bergmanfächers, der eine Permutation der zu den Einermengen gehörenden Strahlen induziert, von einem Matroidautomorphismus.

In Abschnitt 2.3 untersuchen wir die Standardcremonatransformation

$$\mathrm{crem}\colon \mathbb{P}^d_K \dashrightarrow \mathbb{P}^d_K, \quad [x_0 : \ldots : x_d] \mapsto \left[\frac{1}{x_0} : \ldots : \frac{1}{x_d}\right]$$

im projektiven Raum $\mathbb{P}^d_K$ über einem Körper K und bestimmen die Hyperebenenarrangements, für die sie einen Automorphismus des Komplements induziert.

Satz (2.3.5)**.** *Sei $\mathcal{A}$ ein Hyperebenenarrangement in $\mathbb{P}^d_K$. Die Standardcremonatransformation* crem *induziert genau dann einen Automorphismus des Komplements $\Omega_{\mathcal{A}}$, wenn*

$$\mathcal{A} = \{V(x_i) \mid 0 \leq i \leq d\} \cup \{V(x_i + zx_j) \mid 0 \leq i < j \leq d,\ z \in Z_{ij}\}$$

gilt für eine Familie $Z = \{Z_{ij}\}_{0 \leq i < j \leq d}$ von Mengen $Z_{ij} \subseteq K^\times$ mit der Eigenschaft, dass alle Mengen Z_{ij} abgeschlossen sind unter Kehrwertbildung.

Insbesondere bilden die Koordinatenhyperebenen eine Cremonabasis des dazugehörigen Matroids $M(\mathcal{A})$.

In Kapitel 3 betrachten wir die von Shaw und Werner eingeführten Cremonaautomorphismen. Wir geben ein Beispiel an, in dem die kombinatorische Cremonaabbildung mit der Tropikalisierung der Cremonatransformation im projektiven Raum übereinstimmt, unter Verwendung von [Kur17, Satz 7.7]. Außerdem beweisen wir folgendes Kriterium dafür, wann die Cremonaabbildung die minimale Nested-set-Struktur erhält:

Satz (3.1.7)**.** *Sei M ein einfaches zusammenhängendes Matroid vom Rang mindestens 3 und nehmen wir an, dass M eine Cremonabasis B hat, sodass*

$G_B(M)$ ein vollständiger Graph ist. Dann induziert die Cremonaabbildung crem_B *einen Automorphismus des minimalen Nested-set-Fächers $B_m(M)$.*

In Abschnitt 3.2 beschreiben wir die Struktur der *Cremonagruppe* $\mathrm{Cr}(M)$ eines Matroids M, die wir definieren als die von den Matroidautomorphismen und den Cremonaautomorphismen erzeugte Untergruppe von $\mathrm{Aut}(B_c(M))$.

Satz (3.2.5)**.** *Sei M ein einfaches zusammenhängendes Matroid vom Rang mindestens 3. Für jede Cremonabasis B von M ist die Cremonagruppe* $\mathrm{Cr}(M)$ *von Matroidautomorphismen und dem Cremonaautomorphismus* crem_B *erzeugt.*

Unter der zusätzlichen Voraussetzung, dass die Zusammenziehungen M/e für alle $e \in E$ zusammenhängend sind, haben wir nach Satz 1.7.6 $\mathrm{Aut}(M)/\mathrm{Aut}_{\mathrm{CB}}(M) \cong S_k$, wobei $k \in \mathbb{N}_0$ die Anzahl der Cremonabasen und $\mathrm{Aut}_{\mathrm{CB}}(M) \subseteq \mathrm{Aut}(M)$ der Normalteiler der Matroidautomorphismen ist, die jede Cremonabasis erhalten. In der Cremonagruppe erhöht sich der Grad der symmetrischen Gruppe um 1, ähnlich wie im Beispiel $M(A_n)$:

Satz (3.2.8)**.** *Sei M ein einfaches zusammenhängendes Matroid vom Rang mindestens 3 und nehmen wir an, dass die Zusammenziehungen M/e für alle $e \in E$ zusammenhängend sind. Dann ist* $\mathrm{Cr}(M)/\mathrm{Aut}_{\mathrm{CB}}(M) \cong S_{k+1}$.

Für ein einfaches Matroid M vom Rang 3 ist nach [SW23, Satz 9.2] bekannt, dass die Automorphismengruppe $\mathrm{Aut}(B_c(M))$ des groben Bergmanfächers mit der Cremonagruppe $\mathrm{Cr}(M)$ übereinstimmt, wenn M keine nichttriviale Parallelverbindung ist. In Abschnitt 3.3 zeigen wir ein ähnliches Resultat für den minimalen Nested-set-Komplex:

Satz (3.3.2)**.** *Sei M ein einfaches zusammenhängendes Matroid vom Rang 3.*

(1) *Wenn M das Matroid einer selbstdualen, nicht ausgearteten projektiven Ebene ist, dann ist* $\mathrm{Aut}(M)$ *eine Untergruppe von* $\mathrm{Aut}(\mathcal{N}(M))$ *vom Index 2.*
(2) *Andernfalls wird* $\mathrm{Aut}(\mathcal{N}(M))$ *erzeugt von Matroidautomorphismen und Cremonaautomorphismen.*

In Kapitel 4 wenden wir unsere Resultate auf Wurzelsystemmatroide an und erhalten einen neuen Beweis für den Isomorphismus $\mathrm{Aut}(B_c(M(A_n))) \cong S_{n+2}$, indem wir zeigen, dass $M(A_n)$ $n+1$ Cremonabasen besitzt. Für die anderen Wurzelsysteme zeigen wir folgende Aussagen:

Satz (4.3.3, 4.3.13)**.** *Für alle $n \geq 3$ hat das Wurzelsystemmatroid $M(B_n)$ genau eine Cremonabasis und* $\mathrm{Aut}(B_c(M(B_n))) \cong \mathrm{Aut}(M(B_n)) \times \mathbb{Z}/2\mathbb{Z}$ *wird erzeugt von Matroidautomorphismen und der einzigen Cremonaabbildung.*

Satz (4.4.10)**.** *Für alle $n \geq 4$ hat das Wurzelsystemmatroid $M(D_n)$ keine Cremonabasen und* $\mathrm{Aut}(B_c(M(D_n)))$ *ist isomorph zu* $\mathrm{Aut}(M(D_n))$.

Satz (4.6.5)**.** *Das Wurzelsystemmatroid $M(F_4)$ hat keine Cremonabasen und* $\mathrm{Aut}(B_c(M(F_4)))$ *ist isomorph zu* $\mathrm{Aut}(M(F_4))$.