Irreducibility of a Universal Prym–Brill–Noether Locus

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For genus $g = \frac{r(r+1)}{2} + 1$, we prove that via the forgetful map, the universal Prym–Brill–
Noether locus $\mathcal{V}_g^r$ has a unique irreducible component dominating the moduli space $\mathcal{R}_g$
of Prym curves.

1 Introduction

The moduli space $\mathcal{R}_g$ of Prym curves was brought to the attention of algebraic geometers
by Mumford in his influential paper [15], as a way of understanding principally
polarized Abelian varieties. For an element $[C, \eta]$ of $\mathcal{R}_g$, we let $\pi \colon \widetilde{C} \to C$ be the associated
double cover and let $\mathrm{Nm}_\pi \colon \mathrm{Pic}^{2g-2}(\widetilde{C}) \to \mathrm{Pic}^{2g-2}(C)$ be the norm map of this morphism
of curves. In this situation, the preimage of ω_C consists of two disjoint varieties

$$P^+ = \left\{ L \in \mathrm{Pic}^{2g-2}(\widetilde{C}) \mid \mathrm{Nm}(L) = \omega_C \text{ and } h^0(\widetilde{C}, L) \equiv 0 \ (\mathrm{mod}\ 2) \right\}$$

and

$$P^- = \left\{ L \in \mathrm{Pic}^{2g-2}(\widetilde{C}) \mid \mathrm{Nm}(L) = \omega_C \text{ and } h^0(\widetilde{C}, L) \equiv 1 \ (\mathrm{mod}\ 2) \right\}.$$

isomorphic to the Prym variety in $\mathrm{Pic}^0(\widetilde{C})$.

Following this development, Welters emphasized in [17] that Prym–Brill–Noether theory can be employed in order to understand the geometry of subvarieties of Prym varieties. More precisely, he considered the loci

$$V^r(C,\eta) := \left\{ L \in \mathrm{Pic}^{2g-2}(\widetilde{C}) \mid \mathrm{Nm}(L) \cong \omega_C, \; h^0(\widetilde{C}, L) \geq r+1, \text{ and } h^0(\widetilde{C}, L) \equiv r+1 \;(\mathrm{mod}\; 2) \right\}$$

in order to study the singularities of the theta divisor of the associated Prym variety. The relation between Prym–Brill–Noether theory and the study of singularities of theta divisors piqued the interest of other mathematicians. The two papers [17] and [2] showed that when $g \geq \frac{r(r+1)}{2}+1$, the locus $V^r(C,\eta)$ is non-empty of dimension at least $g-1-\frac{r(r+1)}{2}$. In addition, for a generic $[C,\eta] \in \mathcal{R}_g$, the locus $V^r(C,\eta)$ has exactly this dimension when $g \geq \frac{r(r+1)}{2}+1$ and is empty when $g < \frac{r(r+1)}{2}+1$; see [16]. Subsequently in [4], De Concini and Pragacz viewed $V^r(C,\eta)$ as a Lagrangian degeneracy locus (cf. [14]) and computed the class of $V^r(C,\eta)$ in the Prym variety when it has the expected dimension $g-1-\frac{r(r+1)}{2}$.

In recent years, two new perspectives for the study of Prym–Brill–Noether theory emerged. On one hand, it has been studied from the point of view of tropical geometry (see [3] and [13]), thus providing another proof for the dimension estimate of $V^r(C,\eta)$ for a generic $[C,\eta]$ and, on the other hand, from the perspective of moduli theory, with a view to understanding the birational geometry of $\mathcal{R}_g$ for small values of g. It is natural to ask when $g \geq \frac{r(r+1)}{2}+1$ whether the universal Prym–Brill–Noether locus

$$\mathcal{V}_g^r := \left\{ [C,\eta,L] \mid [C,\eta] \in \mathcal{R}_g \text{ and } L \in V^r(C,\eta) \right\}$$

has a unique irreducible component dominating the moduli space $\mathcal{R}_g$. This is true for $g > \frac{r(r+1)}{2}+1$ because the fibre above a general $[C,\eta] \in \mathcal{R}_g$ is irreducible; see [5, Examples 6.2]. However, as pointed out in [12], this was not known for $g = \frac{r(r+1)}{2}+1$. The present paper aims at showing that when $g = \frac{r(r+1)}{2}+1$, the moduli space $\mathcal{V}_g^r$ has a unique irreducible component dominating $\mathcal{R}_g$. In the interest of proving this result, we will consider the compactification $\overline{\mathcal{R}}_g$ of the moduli space of Prym curves $\mathcal{R}_g$; see [1] and [10]. Ultimately, we degenerate to the boundary locus of $\overline{\mathcal{R}}_g$ and employ the theory of limit linear series, adapted to our situation.

2   Prym Linear Series

Our goal in this section is to provide a suitable definition of Prym linear series and then use it to prove our main result. We will start by recalling some definitions regarding limit linear series, while referring the reader to [6] for a thorough study.

First, recall that a g^r_d on a smooth curve Y is defined to be a pair (V, L) of a degree d line bundle L together with an $(r+1)$-dimensional vector subspace $V \subseteq H^0(Y, L)$. This definition extends naturally to curves of compact type.

Let Y be a genus g curve of compact type. A crude limit g^r_d consists of a g^r_d (V_i, L_i) for every irreducible component Y_i of Y, further satisfying the following property:

• Let q be a node of Y connecting two irreducible components, Y_j and Y_k, and let $0 \le a_0 < \cdots < a_r \le d$ and $0 \le b_0 < \cdots < b_r \le d$ be the vanishing orders at q of the sections in V_j and V_k, respectively. Then, for any $0 \le i \le r$ we have

$$a_i + b_{r-i} \ge d.$$

If for any node q all the inequalities are in fact equalities, the limit linear system is called refined.

Lastly, we define the Brill–Noether number associated to a smooth pointed curve and a g^r_d on it. Let $(Y, p_1, \ldots, p_n)$ be a smooth genus g curve together with n points on it, and let $l = (V, L)$ be a g^r_d on Y. Let $0 \le a^i_0 < \cdots < a^i_r \le d$ be the vanishing orders at p_i of the sections in V. The Brill–Noether number of the g^r_d with respect to the points p_i is defined as

$$\rho(l, p_1, \ldots, p_n) := g - (r+1)(g - d + r) - \sum_{i=1}^{n} \sum_{j=0}^{r} a^i_j + n \cdot \frac{r(r+1)}{2}.$$

Having these definitions, we are ready to particularize to our situation.

Let $[C, \eta] \in \mathcal{R}_g$ be a generic Prym curve. Then, we know from [17, Lemma 3.2] that a generic element $L \in V^r(C, \eta)$ satisfies $h^0(\widetilde{C}, L) = r + 1$. Moreover, when $g = \frac{r(r+1)}{2} + 1$ we know from [16, Theorem 1.1] that all $L \in V^r(C, \eta)$ satisfy $h^0(\widetilde{C}, L) = r + 1$. In particular, the line bundle L can be viewed as a g^r_{2g-2} on the curve $\widetilde{C}$. Furthermore, up to restricting to an open subset, we can view all irreducible components of $\mathcal{V}^r_g$ dominating $\mathcal{R}_g$ as contained in the moduli space $\mathcal{G}^r_{2g-2}(\mathcal{R}_g)$ parametrizing limit g^r_{2g-2} over double covers $[\pi \colon \widetilde{C} \to C]$ where $\widetilde{C}$ is of compact type. We ask what points can appear in the compactification of $\mathcal{V}^r_g$ inside this space.

Let $[\pi \colon \widetilde{C} \to C] \in \overline{\mathcal{R}}_g$ such that C is of compact type and admits a unique irreducible component X satisfying $\eta_X \not\cong \mathcal{O}_X$. For this component X, we denote by $p^X_1, \ldots, p^X_{s_X}$ its nodes and by $g^X_1, \ldots, g^X_{s_X}$ the genera of the connected components of $C \setminus X$ glued to X at these points. For an irreducible component Y of C, different from X, we denote by q^Y the node glueing Y to the connected component of $C \setminus Y$ containing X,

and by $p_1^Y, \ldots, p_{s_Y}^Y$ the other nodes of Y. We denote by $g_0^Y, g_1^Y, \ldots, g_{s_Y}^Y$ the genera of the connected components of $C \setminus Y$ glued to Y at these points.

Using the above notations, we can define the concept of a Prym limit g_{2g-2}^r:

Definition 2.1.   A Prym limit g_{2g-2}^r, denoted L, is a crude limit g_{2g-2}^r on $\widetilde{C}$ satisfying the following two conditions:

1. For the unique component $\widetilde{X}$ of $\widetilde{C}$ above X, the $\widetilde{X}$-aspect $L_{\widetilde{X}}$ of L satisfies

$$Nm_{\pi_{|\widetilde{X}}} L_{\widetilde{X}} \cong \omega_X \left(\sum_{i=1}^{s} 2g_i^X p_i^X \right).$$

2. For a component Y of C different from X, we denote by Y_1 and Y_2 the two irreducible components of $\widetilde{C}$ above it. We identify these two components with Y via the map π. With this identification the Y_1 and Y_2 aspects of L satisfy the following:

$$L_{Y_1} \otimes L_{Y_2} \cong \omega_Y \left((2g - 2 + 2g_0^Y) q^Y + \sum_{i=1}^{s} g_i^Y p_i^Y \right).$$

Because the points in the boundary need to respect the norm condition, we immediately obtain that:

Lemma 2.1.   Let $[\pi : \widetilde{C} \to C] \in \overline{\mathcal{R}}_g$ with $\widetilde{C}$ of compact type, and let $\overline{\mathcal{V}}_g^r$ be the closure of $\mathcal{V}_g^r$ inside $\mathcal{G}_{2g-2}^r(\mathcal{R}_g)$. Then the fibre of the map $\overline{\mathcal{V}}_g^r \to \overline{\mathcal{R}}_g$ over the point $[\pi : \widetilde{C} \to C]$ is contained in the locus of Prym limit g_{2g-2}^r on $[\pi : \widetilde{C} \to C]$.

We are now ready to use a degeneration argument in order to prove our main result.

Theorem 2.2.   When $g = \frac{r(r+1)}{2} + 1$, the space $\mathcal{V}_g^r$ has a unique irreducible component dominating $\mathcal{R}_g$.

Proof.   We consider the boundary divisor $\Delta_1 \subseteq \overline{\mathcal{R}}_g$ whose generic point is of the form $[Y \cup_x E, \mathcal{O}_Y, \eta_E \neq \mathcal{O}_E]$, where Y and E are components of genus $g-1$ and 1, respectively. Let $[Y_1 \cup_{x_1} \widetilde{E} \cup_{x_2} Y_2 \to Y \cup_x E]$ be the double cover associated to a generic element of Δ_1. We want to describe the locus of Prym limit g_{2g-2}^r's on such a double cover.

Let L be a Prym limit g^r_{2g-2} on $[Y_1 \cup_{x_1} \widetilde{E} \cup_{x_2} Y_2 \to Y \cup_x E]$. The additivity of the Brill–Noether numbers implies the following:

$$\rho(2g-1, r, 2g-2) = -r \geq \rho(L_{Y_1}, x_1) + \rho(L_{\widetilde{E}}, x_1, x_2) + \rho(L_{Y_2}, x_2).$$

But we know from [7, Theorem 1.1] and [9, Proposition 1.4.1] that $\rho(L_{Y_1}, x_1) \geq 0$, $\rho(L_{Y_2}, x_2) \geq 0$ and $\rho(L_{\widetilde{E}}, x_1, x_2) \geq -r$. It is clear that these are in fact equalities and L is a refined limit g^r_{2g-2}.

We denote by $0 \leq a_0 < a_1 < \cdots < a_r \leq 2g-2$ and $0 \leq b_0 < b_1 < \cdots < b_r \leq 2g-2$ the vanishing orders for the Y_1 and Y_2 aspects, respectively. The equality $\rho(L_{\widetilde{E}}, x_1, x_2) = -r$ implies that $a_i + b_{r-i} = 2g-2$ for all $0 \leq i \leq r$.

The genericity of $[Y_2, x_2] \in \mathcal{M}_{g-1,1}$ together with $\rho(L_{Y_2}, x_2) = 0$ imply that $h^0(Y_2, L_{Y_2}(-b_i x_2)) = r + 1 - i$ for all $0 \leq i \leq r$. Using that $L_{Y_1} \otimes L_{Y_2} \cong \omega_Y(2g \cdot x)$ and the Riemann–Roch theorem, we obtain

$$h^0(Y_1, L_{Y_1}(-(2 + a_{r-i})q)) = g + r - 1 - a_{r-i} - i.$$

Choosing $i = 0$ we get $a_r = g + r - 1$. Inverting the roles of the a_i's and b_i's we obtain that $a_0 = g - r - 1$. Because we have the divisorial equivalences

$$a_i x_1 + b_{r-i} x_2 \equiv a_j x_1 + b_{r-j} x_2$$

on the elliptic curve E for every $0 \leq i, j \leq r$, we obtain that $a_i - a_{i-1} \geq 2$ for every $1 \leq i \leq r$. This implies that $a_i = g - r + 2i - 1$ for every $0 \leq i \leq r$.

We now view the moduli space $\mathcal{M}_{g-1,1}$ as embedded in $\overline{\mathcal{R}}_g$ via the map $\pi \colon \mathcal{M}_{g-1,1} \to \overline{\mathcal{R}}_g$ sending a pointed curve $[Y, x] \in \mathcal{M}_{g-1,1}$ to $[Y \cup_x E, \mathcal{O}_Y, \eta_E]$ where $[E, x]$ is a generic elliptic curve and η_E is a 2-torsion line bundle on E. For the ramification sequence $\alpha = (g - r - 1, \ldots, g - 1)$ associated to the vanishing orders $a = (a_0, \ldots, a_r) = (g - r - 1, \ldots, g + r - 1)$, we consider the locus $\mathcal{G}^r_{2g-2}(\alpha)$ parametrizing pairs $[C, p, L]$ where $[C, p] \in \mathcal{M}_{g-1,1}$ and L is a g^r_{2g-2} having vanishing orders greater or equal to a at the point p. Then the locus of Prym limit g^r_{2g-2} over $\mathrm{Im}(\pi)$ is birationally isomorphic to $\mathcal{G}^r_{2g-2}(\alpha)$.

We know from [8, Lemma 3.6] that $\mathcal{G}^r_{2g-2}(\alpha)$ has a unique irreducible component dominating $\mathcal{M}_{g-1,1}$. Moreover,

$$\deg(\mathcal{G}^r_{2g-2}(\alpha) \to \mathcal{M}_{g-1,1}) = 2^{\frac{r(r-1)}{2}} \cdot (g-1)! \cdot \prod_{i=1}^{r} \frac{(i-1)!}{(2i-1)!}$$

as stated on the second page of [11]. On the other hand, we have from [4, Theorem 9] that

$$\deg(\mathcal{V}_g^r \to \mathcal{R}_g) = 2^{\frac{r(r-1)}{2}} \cdot (g-1)! \cdot \prod_{i=1}^{r} \frac{(i-1)!}{(2i-1)!}.$$

We conclude that all dominant irreducible components of $\mathcal{V}_g^r$ contain $\mathcal{G}_{2g-2}^r(\alpha)$ in their closure. From this we get that each such component map to $\mathcal{R}_g$ with degree at least $2^{\frac{r(r-1)}{2}} \cdot (g-1)! \cdot \prod_{i=1}^{r} \frac{(i-1)!}{(2i-1)!}$, implying unicity. ∎

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