Quantum kinetic theory and dissipative spin hydrodynamics

Dissertation
zur Erlangung des Doktorgrades
der Naturwissenschaften

vorgelegt beim Fachbereich Physik
der Johann Wolfgang Goethe-Universität
in Frankfurt am Main

von

David Wagner

aus Trier

Frankfurt 2023
(D 30)

vom Fachbereich Physik der

Johann Wolfgang Goethe-Universität als Dissertation angenommen.

Dekan: Prof. Dr. Roger Erb

Erstgutachter: Prof. Dr. Dirk H. Rischke

Zweitgutachter: Priv. Doz. Dr. Hendrik van Hees

Datum der Disputation:

Deutschsprachige Zusammenfassung

Die vorliegende Arbeit befasst sich mit der Konstruktion dissipativer relativistischer Hydrodynamik insbesondere für solche Fluide, deren Konstituenten einen nicht verschwindenden Spin aufweisen. In diesem Kapitel soll eine Zusammenfassung der Motivation, der Methoden sowie der Ergebnisse vorgenommen werden.

Einleitung und Motivation

Die an Beschleunigern wie dem Large Hadron Collider (LHC) oder dem Relativistic Heavy Ion Collider (RHIC) durchgeführten Schwerionenkollisionen erlauben einen einzigartigen Einblick in Zustände der Materie, wie sie sonst nur im frühen Universum vorhanden waren. Die dort erreichten hohen Dichten und Temperaturen sind nach aktuellem Kenntnisstand hinreichend, um das sog. „Quark-Gluon Plasma" (QGP) zu erzeugen, einen Zustand, in dem Quarks und Gluonen die relevanten Freiheitsgrade darstellen. Diese Teilchen, welche im Rahmen der Quantenchromodynamik (QCD) stark wechselwirken, sind unter normalen Umständen durch das sog. „confinement" in Hadronen gebunden. Es hat sich in den vergangenen Jahren und Jahrzehnten herausgestellt, dass dieses QGP eine hohe Kollektivität aufweist und sich somit mittels hydrodynamischer Methoden beschreiben lässt [1–3].

Eine interessante Unterklasse der in diesen Schwerionenkollisionen möglichen Messungen besteht in der Betrachtung der Polarisation der emittierten Teilchen. Die Intuition hinter einer solchen Messung liegt darin, dass die Kollisionen nicht immer zentral ablaufen, sodass das entstehende QGP mit einer signifikanten Winkelgeschwindigkeit rotiert. Analog zur Magnetisierung von Materie unter Rotation, dem sog. Barnett-Effekt [4], ist es dann zu erwarten, dass die emittierten Teilchen eine nicht verschwindende Polarisation entwickeln. In der Tat wurde an mehreren Experimenten nachgewiesen, dass die Λ-Hyperonen, welche Baryonen mit dem Spin $1/2$ darstellen, einen Polarisationsgrad aufweisen, der insbesondere bei niedrigen Energien von Null verschieden ist [5–8]. Diese Art der sog. globalen (d.h. über die Impulse der beteiligten Teilchen integrierten) Polarisation kann recht gut mit hydrodynamischen Modellen in Einklang gebracht werden, welche annehmen, dass die Polarisation der Teilchen nur von der Vortizität des Mediums abhängt [9]. Im Gegensatz dazu ist dies für die lokale (impulsabhängige) Polarisation [10] nicht möglich, wobei hier mit der Berücksichtigung der Effekte des sog. „thermalen Schertensors" in den letzten Jahren deutliche Fortschritte erzielt wurden [11–14].

Eine weitere vielversprechende Observable besteht im sog. „alignment" der ϕ- und $K^{\star 0}$-Mesonen, welche einen Spin von 1 aufweisen. Diese Größe ist Teil der Tensorpolarisation von Teilchen und muss somit von der oben beschriebenen (Vektor-) Polarisation unterschieden werden. Während die Vektorpolarisation der Λ-Hyperonen durch den Erwartungswert des Pauli-Lubanski-Vektors gegeben ist und potentiell für alle Teilchen mit nicht verschwindendem Spin vorhanden sein kann, beschreibt die Tensorpolarisation Elemente der Spin-Dichtematrix, die für Teilchen mit Spin kleiner 1 nicht existieren. Als Beispiel für den masselosen Fall sei hier das Photon genannt, welches sowohl zirkular als auch linear polarisiert sein kann: erstere Größe gibt die Vektorpolarisation an, letztere die Tensorpolarisation [15]. Das (globale) alignment der oben genannten ϕ-Mesonen hat sich als signifikant herausgestellt [16–18], was in einer Reihe von theoretischen Erklärungsversuchen resultierte, wobei eine eindeutige Lösung noch aussteht [19–28].

Die Verbindung dieser Arbeit zu den oben genannten theoretischen Erklärungen für Vektor- und Tensorpolarisation von Teilchen besteht darin, dass vielen dieser Ansätze hydrodynamische Modelle zugrunde liegen. Ausgehend von fundamentalen Quantenfeldtheorien, wird über den Weg der kinetischen Theorie eine hydrodynamische Beschreibung von hinreichend stark wechselwirkenden Systemen formuliert, deren fundamentale Freiheitsgrade einen Spin von 0, $^1/_2$ oder 1 aufweisen. Innerhalb dieses Rahmens können dann Ausdrücke für die relevanten Observablen hergeleitet werden, welche sowohl Gleichgewichts- als auch dissipative Effekte mit einbeziehen.

Relativistische Hydrodynamik

Das Anwendungsgebiet der Hydrodynamik, deren Einführung sich Kapitel 2 widmet, sind solche Systeme, die eine hinreichend große Separation zwischen mikro- und makroskopischen Skalen aufweisen und somit durch ihre erhaltenen Ströme beschrieben werden können. In der Thermodynamik, welche die Existenz eines globalen Gleichgewichts voraussetzt und somit als statischer Spezialfall der Hydrodynamik betrachtet werden kann, sind diese Größen beispielsweise durch die (im nicht beschleunigten Fall konstante) Temperatur, den Druck, und das chemische Potential gegeben. In relativistischen Theorien, welche den Fokus dieser Arbeit darstellen, umfassen diese den Energie-Impuls-Tensor $T^{\mu\nu}$, den Gesamtdrehimpulstensor $J^{\lambda\mu\nu}$, sowie weitere erhaltene Ströme wie z.B. den elektrischen Ladungsstrom. Um die Diskussion möglichst einfach zu halten, beschränkt sich diese Arbeit auf ein Fluid, das aus einer Teilchensorte besteht, sodass der erhaltene Strom N^μ als Teilchenstrom aufgefasst werden kann. Die Erhaltung des Gesamtdrehimpulses folgt in der konventionellen Hydrodynamik trivial aus der Energie-Impuls-Erhaltung. Demgegenüber muss diese in der Spin-Hydrodynamik aufgrund der Existenz eines Spin-Beitrags zum Gesamtdrehimpuls explizit mit einbezogen werden. Die zeitliche Entwicklung der oben eingeführten Ströme ist durch die entsprechenden Erhaltungsgleichungen gegeben,

$$\partial_\mu N^\mu = 0 \,, \qquad \partial_\mu T^{\mu\nu} = 0 \,, \qquad \partial_\lambda J^{\lambda\mu\nu} = 0 \,. \tag{1}$$

Hier zeichnet sich bereits das Hauptproblem der dissipativen Hydrodynamik ab: die Anzahl der Erhaltungsgleichungen reicht nicht aus, um die Evolution aller Komponenten der erhaltenen Ströme eindeutig festzulegen! In der idealen Hydrodynamik, welche auf dem Konzept eines lokalen thermodynamischen Gleichgewichtes basiert, kann die Anzahl an unabhängigen Freiheitsgraden jedoch hinreichend stark reduziert werden, sodass die Erhaltungsgleichungen genügen, um die Dynamik des Systems zu bestimmen. Im Speziellen bestimmt die Erhaltungsgleichung für den Energie-Impuls-Tensor die zeitliche Entwicklung der Temperatur $T(x)$ sowie der Vierergeschwindigkeit $u^\mu(x)$, während die Erhaltung des Teilchenstromes das Verhältnis von chemischem Potential zu Temperatur $\alpha(x) \coloneqq \mu(x)/T(x)$ festlegt. Die Erhaltungsgleichung für den Gesamtdrehimpulstensor schließlich führt auf die Zeitentwicklung des sog. Spin-Potentials $\Omega^{\mu\nu}$, welches nur unter Berücksichtigung eines Spintensors auftritt.

Da in der dissipativen Hydrodynamik, welche einen allgemeineren Fall darstellt, die oben angesprochenen Argumente zur Reduktion der unabhängigen Komponenten aufgrund des Fehlens eines Gleichgewichtes nicht verwendet werden können, bleibt hier das Problem der Unbestimmtheit der erhaltenen Ströme bestehen. Es ist somit nötig, für die dissipativen Größen, welche durch die Erhaltungsgleichungen nicht eindeutig bestimmt werden, Bewegungsgleichungen abzuleiten. Eine Methode, dies zu tun, ist die sog. Gradiententenentwicklung, deren Beitrag erster Ordnung die im nichtrelativistischen Fall äußerst erfolgreichen Navier-Stokes-Gleichungen sind. Im relativistischen Fall jedoch stoßen diese Gleichungen an ihre Grenzen, da sie zur Klasse parabolischer Differentialgleichungen gehören und somit die Forderung der speziellen Relativitätstheorie nach Kausalität verletzen. Infolgedessen entwickeln sich im linearen Regime Instabilitäten, welche die relativistische Version der Navier-Stokes-Gleichungen in der Praxis nicht einsetzbar machen [29–32]. Eine weitere Möglichkeit, Bewegungsgleichungen für die dissipativen Größen herzuleiten, besteht darin, eine mikroskopische Theorie zugrunde zu legen, und durch zweckmäßige Näherungen das interessierende makroskopische Verhalten zu extrahieren.

Diese Herangehensweise ist Gegenstand der vorliegenden Arbeit, und die fundamentale mikroskopische Theorie ist durch die Quantenfeldtheorie gegeben.

Feldtheorie, Erhaltungsgrößen, und kinetische Theorie

Die erste Frage, die beantwortet werden muss, um das im letzten Abschnitt beschriebene Programm durchzuführen, besteht darin, wie die makroskopischen erhaltenen Ströme mit den fundamentalen Freiheitsgraden der mikroskopischen Theorie in Verbindung gesetzt werden können, was in Kapitel 3 in Angriff genommen wird.

Teilchen mit Spins 0, $1/2$ und 1 werden durch Quantenfelder repräsentiert, welche sich unter LORENTZ-Transformationen jeweils wie Skalare, Spinoren und Vierervektoren verhalten. Aus den Symmetrien der entsprechenden Wirkungen unter POINCARÉ-Transformationen lassen sich dann die erhaltenen Ströme mittels des NOETHERschen Theorems [33] bestimmen: so folgt aus der Invarianz der Wirkung unter Raumzeit-Translationen die Divergenzfreiheit des Energie-Impuls-Tensors, während die Erhaltung des Gesamtdrehimpulses aus der Invarianz unter LORENTZ-Transformationen resultiert. Die Erhaltung des Ladungs- bzw. Teilchenstroms schließlich folgt aus einer globalen $U(1)$-Symmetrie.

Obwohl sich mit dieser Methode die erhaltenen Ströme als Funktionen der mikroskopischen Felder darstellen lassen, ist die eingangs erwähnte Fragestellung noch nicht vollständig beantwortet: es stellt sich nämlich heraus, dass die Erhaltungsgleichungen invariant unter einer Klasse von sog. „Pseudoeichtransformationen" sind [34]. Im Kontext einer Feldtheorie, welche Teilchen mit nicht verschwindendem Spin beschreibt, folgt aus jeder Wahl einer solchen Pseudoeichung eine bestimmte Aufspaltung des Gesamtdrehimpulses in einen Bahn- und einen Spinanteil [35]. Wenngleich vom Standpunkt der vollen mikroskopischen Theorie alle Pseudoeichungen äquivalent sind, können sie dennoch die Trunkierungen beeinflussen, welche beim Übergang zu einer makroskopischen Theorie zwangsläufig vorgenommen werden müssen. Die Pseudoeichungen, welche in dieser Arbeit Anwendung finden, erfüllen die Anforderung, dass der Spintensor im Fall freier Felder sowie im Gleichgewicht erhalten bleibt und der Energie-Impuls-Tensor somit symmetrisch ist. Die Anschauung, welche hinter diesen Bedingungen steht, besteht darin, dass die Teilchen Bahn- und Spindrehimpuls nur in Kollisionen austauschen können, und dass im Gleichgewicht im Mittel kein Austausch mehr stattfinden sollte.

Eine in Herleitungen der konventionellen Hydrodynamik oft verwendete mikroskopische Formulierung ist durch die kinetische Theorie gegeben, welche Systeme von Teilchen beschreibt, die mittels zeit- und räumlich lokalisierter Stöße wechselwirken. Unter Berücksichtigung von Zweierstößen sowie Verwendung des BOLTZMANNschen Stoßzahlansatzes lässt sich eine Bewegungsgleichung für die Ein-Teilchen-Verteilungsfunktion $f(x,k)$ angeben, welche die Wahrscheinlichkeit beschreibt, ein Teilchen mit Impuls k am Raumzeitpunkt x zu finden [36].

Diese Formulierung ist zunächst der klassischen Physik zuzurechnen, da sie davon ausgeht, Ort und Impuls gleichzeitig beliebig genau bestimmen zu können, was im Rahmen von Quantentheorien aufgrund der HEISENBERGschen Unschärferelation nicht möglich ist. Die analoge Formulierung im Kontext der Quantenmechanik und Quantenfeldtheorie besteht im WIGNER-Funktions-Formalismus, bei dem die Rolle der Verteilungsfunktion von der WIGNER-Funktion $W(x,k)$ übernommen wird, welche eine FOURIER-Transformation der Relativkoordinate der Zweipunktfunktion darstellt und somit im Allgemeinen matrixwertig ist.[1] Im Fall von massiven Spin-$1/2$ Teilchen weist die WIGNER-Funktion vier unabhängige Komponenten auf, welche sich in einem Skalar $\mathcal{F}(x,k)$ und einem Axialvektor $\mathcal{A}^\mu(x,k)$ mit $k \cdot \mathcal{A} = 0$ zusammenfassen lassen. Im Spin-1-Fall dagegen beinhaltet die WIGNER-Funktion neun unabhängige Komponenten, welche neben einem Skalar $f_K(x,k)$ und einem zum Impuls orthogonalen Axialvektor $G^\mu(x,k)$ auch einen symmetrischen, zum Impuls orthogonalen, spurlosen Tensor $F_K^{\mu\nu}$ umfassen. Diese zusätzlichen Freiheitsgrade stehen in direkter Beziehung zum

[1] Eine Ausnahme bildet der Fall von Teilchen mit Spin 0, in dem die WIGNER-Funktion ein Skalar ist und somit nur eine unabhängige Komponente aufweist.

Vorhandensein der Tensorpolarisation bei Teilchen von Spin 1 und höher. In diesem Formalismus geht zwar die Interpretation als Wahrscheinlichkeitsdichte verloren, da die WIGNER-Funktion über Gebiete $\Delta x \Delta k \sim \hbar$ negativ werden kann [37], jedoch lassen sich zumindest im Fall freier Felder alle erhaltenen Ströme mittels dieser Funktion als gewichtete Impulsintegrale ausdrücken. Weiterhin beschreibt die Bewegungsgleichung der WIGNER-Funktion

$$k \cdot \partial W(x,k) = \mathcal{C}(x,k) \tag{2}$$

das Verhalten des Systems zunächst exakt. Die rechte Seite dieses Ausdrucks stellt eine Art Kollisionsintegral dar, welches in einer sinnigen Art und Weise genähert werden muss.

Vor dieser Rechnung, welche Gegenstand der beiden folgenden Kapitel ist, wird ein in Refs. [38–42] angewandtes Hilfsmittel eingeführt: Um eine kompakte Beschreibung aller Freiheitsgrade der WIGNER-Funktion zu erhalten, ist es möglich, eine sog. „Spin“-Variable $\mathfrak{s}^\mu$ einzuführen, und die unabhängigen Komponenten der WIGNER-Funktion als Multipolmomente einer skalaren Funktion $\mathfrak{f}(x,k,\mathfrak{s})$ bezüglich dieser Variable zu definieren. Der Teil dieser Funktion, welcher auf der Massenschale liegt, d.h. für den $k^2 = m^2$ gilt, erfüllt dann gemäß Gl. (2) eine Bewegungsgleichung der Form

$$k \cdot \partial f(x,k,\mathfrak{s}) = C(x,k,\mathfrak{s}) \;, \tag{3}$$

was eine einheitliche Beschreibung von Teilchen mit beliebigem Spin erlaubt. Die physikalisch relevanten Observablen sind immer durch bestimmte Komponenten der WIGNER-Funktion gegeben und beinhalten somit eine Integration über die Variable $\mathfrak{s}$.

Die kinetische Gleichung in der GLW- und KB-Methode

Eine zentrale Frage der kinetischen Theorie besteht darin, den Kollisionsterm in Gl. (3) aufzustellen. Im Zuge dessen wird sich in dieser Arbeit auf die Effekte binärer elastischer Kollisionen beschränkt. Desweiteren wird die Annahme des „molekularen Chaos“ getroffen, welche besagt, dass Teilchen vor einem Stoß unkorreliert sind, und die auch dem BOLTZMANNschen Stoßzahlansatz zugrunde liegt.

Die konkrete Berechnung wird auf zwei Weisen demonstriert: In Kapitel 4 wird die GLW-Methode (nach DE GROOT, VAN LEEUWEN und VAN WEERT) verwendet, welche in Ref. [43] dargelegt ist und in einer direkten Entwicklung der Interaktionsterme nach asymptotischen „in“- und „out“-Zuständen besteht. Kapitel 5 dagegen verwendet die KB-Methode (nach KADANOFF und BAYM), die auf der DYSON-SCHWINGER-Gleichung für die Zweipunktfunktion basiert. Während die GLW-Methode direkt mit den Interaktionstermen arbeitet, drückt die KB-Methode den Kollisionsterm zunächst durch die Selbstenergien der beteiligten Felder aus, welche dann mittels einer diagrammatischen Methode entwickelt werden. Trotz der formalen Unterschiede liefern beide Methoden schließlich die gleichen Ergebnisse, mit dem Unterschied, dass es der KB-Methode auch mit handhabbarem Aufwand gelingt, Effekte der Quantenstatistik zu berücksichtigen.

In beiden Methoden werden neben der oben vorgestellten Annahme des molekularen Chaos, welche die Ursache einer irreversiblen Zeitentwicklung ist, sowie der Restriktion auf binäre Kollisionen zwei essentielle Näherungen vorgenommen. Zunächst werden die Kopplungen als hinreichend schwach angenommen, um die Wechselwirkungsenergien als klein im Vergleich zu den Ruheenergien anzusehen. Dies wird benötigt, um beispielsweise Korrekturen der Ruhemasse der Teilchen perturbativ zu betrachten. Desweiteren wird eine sog. Gradientenentwicklung bis zur ersten Ordnung vorgenommen, welche aus der Matrixwertigkeit der WIGNER-Funktion herrührt und nichtlokale Effekte beschreibt, die darauf zurückzuführen sind, dass die fundamentalen Quantenfelder im Gegensatz zu punktförmigen Teilchen eine endliche Ausdehnung besitzen. Während in der GLW-Methode die Ordnung in beiden Näherungen durch Faktoren von $\hbar$ gezählt werden, ist die KB-Methode hier differenzierter, da sie die Kopplungskonstanten der ersten Näherung aufgrund der anschaulichen Diagrammtechnik klar ersichtlich macht, während die Gradientenentwicklung ebenfalls durch das Auftauchen von $\hbar$ signalisiert wird. Das Hauptresultat der Kapitel 4 und 5 lautet

$$\begin{aligned} k \cdot \partial f(x,k,\mathfrak{s}) = \frac{1}{2} \int & \mathrm{d}\Gamma_1\, \mathrm{d}\Gamma_2\, \mathrm{d}\Gamma'\, \mathrm{d}\bar{S}(k) (2\pi\hbar)^4 \delta^{(4)}(k+k'-k_1-k_2) \mathcal{W}^{(\sigma)} \\ & \times \Big[f(x+\Delta_1-\Delta, k_1, \mathfrak{s}_1) f(x+\Delta_2-\Delta, k_2, \mathfrak{s}_2) \widetilde{f}(x+\Delta'-\Delta, k', \mathfrak{s}') \widetilde{f}(x,k,\bar{\mathfrak{s}}) \\ & - \widetilde{f}(x+\Delta_1-\Delta, k_1, \mathfrak{s}_1) \widetilde{f}(x+\Delta_2-\Delta, k_2, \mathfrak{s}_2) f(x+\Delta'-\Delta, k', \mathfrak{s}') f(x,k,\bar{\mathfrak{s}}) \Big] , \end{aligned} \tag{4}$$

wobei $\sigma \in \{0, 1/2, 1\}$ den Spin der Teilchen darstellt und $\mathrm{d}S(k)$ bzw. $\mathrm{d}\Gamma$ bedeuten, dass über die Phasenraumvariablen $\mathfrak{s}$ bzw. k und $\mathfrak{s}$ integriert wird. Der obige Ausdruck berücksichtigt quantenstatistische Effekte, was durch die Funktionen $\widetilde{f} := 1 \pm f$ ausgedrückt wird, wobei das positive (negative) Vorzeichen für Bosonen (Fermionen) zu verwenden ist. Ein wichtiges Resultat der beiden Methoden, ersichtlich aus Gl. (4), besteht darin, dass in der ersten Ordnung der Gradientenentwicklung sowohl lokale als auch nichtlokale Kollisionen auftreten, wobei letztere Gradientenbeiträge im Kollisionsintegral beschreiben. Die Anschauung hinter diesen Termen besteht darin, dass zwei Teilchen nicht am selben Raumzeitpunkt kollidieren, sondern um eine bestimmte Distanz gegeneinander verschoben sind, welche durch die Vektoren Δ_1, Δ_2, Δ' und Δ bestimmt wird. Auf diese Art besteht zum Zeitpunkt der Kollision ein nicht verschwindender Bahndrehimpuls, welcher unter Beachtung der Gesamtdrehimpulserhaltung in Spin umgewandelt werden kann. Diese nichtlokalen Kollisionen stellen somit einen mikroskopischen Mechanismus zur Polarisation von Teilchen dar, da nur durch sie der Spin geändert werden kann. Wichtig ist zu erwähnen, dass diese Beiträge zwar bereits in Refs. [44, 45] beschrieben, jedoch erst in Refs. [46, 47] und in der vorliegenden Arbeit in LORENTZ-kovarianter Form aufgestellt wurden. Dies ist essentiell, da eine solche Form die Frage, ob es möglich ist, ein Inertialsystem zu finden, in dem die Kollisionen lokal ablaufen, eindeutig negativ beantwortet.

Auch auf das Gleichgewicht haben diese nichtlokalen Kollisionen tiefgreifende Auswirkungen. Üblicherweise wird das lokale Gleichgewicht, welches den Ausgangspunkt für den Übergang zur Hydrodynamik darstellt, durch die Bedingung definiert, dass der Kollisionsterm für die Gleichgewichtsverteilungsfunktion f_{eq} verschwindet. Aufgrund quantenstatistischer Effekte muss diese Funktion (zur ersten Ordnung in der Gradientenentwicklung) die Form

$$f_{\text{eq}}(x,k,\mathfrak{s}) := \left\{ \exp\left[\alpha_0(x) - \beta_0(x)\cdot k - \frac{\sigma\hbar}{2m} \Omega_{0,\mu\nu}(x) \epsilon^{\mu\nu\alpha\beta} k_\alpha \mathfrak{s}_\beta \right] \pm 1 \right\}^{-1} \tag{5}$$

annehmen, wobei die Größen α_0, $\beta_0^\mu \equiv u^\mu / T_0$, und $\Omega_0^{\mu\nu}$ LAGRANGE-Multiplikatoren darstellen, welche im Fall des lokalen Gleichgewichtes mit den bekannten Größen aus der idealen Spin-Hydrodynamik übereinstimmen. Die nichtlokalen Beiträge des Kollisionsterms sorgen nun aber dafür, dass zusätzliche Bedingungen an diese Größen gestellt werden müssen,

$$\partial_{(\mu}\beta_{0,\nu)} = 0 , \quad \partial_\mu \alpha_0 = 0 , \quad \Omega_{0,\mu\nu} = \varpi_{\mu\nu} \equiv -\frac{1}{2} \partial_{[\mu}\beta_{0,\nu]} , \tag{6}$$

die charakteristisch für den Zustand des globalen Gleichgewichts sind. Um nun im folgenden Kapitel auf die Formulierung dissipativer Hydrodynamik eingehen zu können, ist es nötig, die Definition des lokalen Gleichgewichts derart anzupassen, dass nur der lokale Teil des Kollisionsintegrals von f_{eq} zum Verschwinden gebracht wird, sodass die LAGRANGE-Multiplikatoren in Gl. (5) beliebige Funktionen von x sein können.

Dissipative Spin-Hydrodynamik

Ein Verfahren, um konventionelle Hydrodynamik aus kinetischer Theorie herzuleiten, besteht in der sog. Momentenmethode. Hierbei wird die Verteilungsfunktion zunächst als Summe der Gleichgewichtsfunktion f_{eq} sowie einer Abweichung δf dargestellt, und letztere in einer orthogonalen und vollständigen Basis im Impulsraum entwickelt. Die Koeffizienten dieser Entwicklung, die sog. irreduziblen Momente

$$\rho_r^{\mu_1\cdots\mu_\ell} := \int \mathrm{d}\Gamma\, E_k^r k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle} \delta f , \tag{7a}$$

beschreiben immer noch die komplette Dynamik des Systems, ermöglichen aber eine systematische Trunkierung. In Kapitel 6 der vorliegenden Arbeit wird diese Methode in einer verallgemeinerten Form angewandt, welche auch die Abhängigkeit der Verteilungsfunktion von der Spinvariablen $\mathfrak{s}$ berücksichtigt. Konkret werden zusätzlich zu Gl. (7a) die irreduziblen Momente vom Spin-Rang 1 und 2 eingeführt,

$$\tau_r^{\mu,\mu_1\cdots\mu_\ell} := \int \mathrm{d}\Gamma\, E_k^r \mathfrak{s}^\mu k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}\delta f\ , \tag{7b}$$

$$\psi_r^{\mu\nu,\mu_1\cdots\mu_\ell} := \int \mathrm{d}\Gamma\, E_k^r K^{\mu\nu}_{\alpha\beta} \mathfrak{s}^\alpha \mathfrak{s}^\beta k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}\delta f\ , \tag{7c}$$

mit deren Hilfe die Verteilungsfunktion für alle hier betrachteten Teilchen beschrieben werden kann. Es sei angemerkt, dass eine Teilmenge der Momente $\rho_r^{\mu_1\cdots\mu_\ell}$ den dissipativen Anteil des Energie-Impuls-Tensors und des Teilchenstromes beschreibt, während der Spintensor von den Momenten $\tau_r^{\mu,\mu_1\cdots\mu_\ell}$ bestimmt wird. Die Momente $\psi_r^{\mu\nu,\mu_1\cdots\mu_\ell}$ schließlich tauchen in keinem erhaltenen Strom auf, stehen aber mit der Tensorpolarisation in Verbindung.

Für die Zeitentwicklung der in den Gl.en (7) definierten irreduziblen Momente lassen sich unter Verwendung der kinetischen Gleichung exakte Ausdrücke herleiten, welche ein gekoppeltes System (abzählbar) unendlich vieler partieller Differentialgleichungen bilden, was eine Umformulierung der ursprünglichen partiellen Integro-Differentialgleichung (4) darstellt. Die sich nun stellende Frage besteht darin, wie dieses System in einer sinnvollen Weise geschlossen werden kann. Die in dieser Arbeit verwendete Methode basiert auf Refs. [48–50] und verwendet eine Trunkierung in zwei als klein angenommenen Größen: die Knudsen-Zahl $\mathrm{Kn} := \lambda_{\mathrm{mfp}}/L$ beschreibt das Verhältnis aus der mittleren freien Weglänge der Teilchen λ_{mfp} und einer hydrodynamischen Längenskala L, welche invers zu den Gradienten makroskopischer Variablen ist. Diese Zahl gibt an, wie groß die Separation zwischen mikro- und makroskopischen Skalen ist, und bildet somit ein Maß für die Anwendbarkeit der Hydrodynamik. Die zweite Größe, welche eigentlich einen ganzen Satz von Verhältnissen beschreibt, sind die inversen Reynolds-Zahlen $\mathrm{Re}^{-1} \sim \delta f/f_{\mathrm{eq}}$, welche den Betrag von dissipativen Größen im Vergleich zu ihrem Gleichgewichtswert angeben. Diese Zahlen, von denen es so viele wie dissipative Größen gibt, beschreiben die Nähe des Systems zum lokalen Gleichgewicht.

Unter der Annahme, dass die oben genannten Größen klein und von derselben Größenordnung sind, ist es möglich, die Gleichungen für die irreduziblen Momente perturbativ zu behandeln. Dazu wird zunächst die Lösung erster Ordnung gebildet, welche das kinetische Analogon zur Navier-Stokes-Theorie bildet. Um die Gleichungen zur zweiten Ordnung in Kn und Re^{-1} zu trunkieren, werden diese Lösungen dann in die Terme zweiter Ordnung eingesetzt und höhere Beiträge vernachlässigt. Es sollte erwähnt werden, dass sich diese Methode, in Ref. [50] „Inverse-Reynolds Dominance“ (IReD) genannt, von dem in Ref. [51] eingeführten und vielfach verwendeten sog. Denicol-Niemi-Molnár-Rischke (DNMR)-Ansatz unterscheidet, welcher auf der Extraktion der Eigenmoden des linearisierten Kollisionsterms basiert. Zwar resultieren beide Methoden in formal ähnlichen und, wie in Ref. [50] gezeigt, perturbativ äquivalenten Gleichungen, jedoch sind die auftretenden Koeffizienten unterschiedlich. Ein Vorteil des IReD-Ansatzes besteht darin, dass eine Klasse von in der DNMR-Methode auftretenden Termen, welche die Bewegungsgleichungen parabolisch und somit akausal und instabil werden lassen, nicht auftritt, und ihre Effekte in anderen Transportkoeffizienten absorbiert werden. Da diese parabolischen Beiträge in praktischen Anwendungen immer vernachlässigt werden müssen, ist es zu erwarten, dass die IReD-Methode bessere Ergebnisse im Vergleich mit der unterliegenden kinetischen Theorie bringt. Ein weiterer Vorteil, welcher in Ref. [49] demonstriert wurde, besteht in der Anwendbarkeit auf Systeme mit mehreren erhaltenen Ladungsströmen.

Unter Verwendung dieser Methode werden die Momentengleichungen so trunkiert, dass nur noch diejenigen Momente übrig bleiben, deren Navier-Stokes-Beiträge von erster Ordnung in Kn oder Re^{-1} und die somit hydrodynamisch „wichtig“ sind. Dies kommt einer Trunkierung im Tensorrang ℓ der irreduziblen Momente (7) gleich. Der letzte Schritt besteht dann in der Wahl derjenigen Momente, welche als dynamische Freiheitsgrade gewählt werden sollen. Um das Ziel einer hydrodynamischen Theorie zu erfüllen, die erhaltenen Ströme zu beschreiben, ist es hier sinnvoll, diejenigen irreduziblen Momente zu wählen, welche in diesen Größen auftreten. Da die Momente $\psi_r^{\mu\nu,\mu_1\cdots\mu_\ell}$ in keinem erhaltenen

Strom enthalten sind, ist diese Prozedur dort nicht möglich; stattdessen werden diese Momente mit den konventionellen dissipativen Größen aus dem Energie-Impuls-Tensor und dem Teilchenstrom in Verbindung gesetzt.

Hiernach, beschrieben in Abschnitt 6.4, ist das Hauptziel erreicht: Zusätzlich zu den Gleichungen, welche die ideale Spin-Hydrodynamik charakterisieren, existiert ein Satz von Gleichungen zur Beschreibung der dissipativen Komponenten aller erhaltenen Ströme. Diese sind alle vom Typ einer Relaxationsgleichung, wobei die asymptotischen Werte jenen aus der Navier-Stokes-Theorie entsprechen und die charakteristischen Relaxationszeiten von der mikroskopischen Interaktion der Teilchen bestimmt werden. Interessant ist, an welchen Punkten die nichtlokalen Beiträge der Kollisionsterme auftreten: während die oben genannten Relaxationszeiten allein von lokalen Kollisionen bestimmt werden, sind die nichtlokalen Beiträge für die Navier-Stokes-Werte der dissipativen Beiträge des Spintensors verantwortlich. Ebenso bestimmen sie die charakteristische Zeitspanne, welche das Spin-Potential $\Omega_0^{\mu\nu}$ benötigt, um zu seinem Gleichgewichtswert, der thermalen Vortizität $\varpi^{\mu\nu}$, zu relaxieren.

Zum Abschluss kann eine Verbindung zu den eingangs beschriebenen, im Experiment zugänglichen Observablen hergestellt werden. Innerhalb des aufgestellten Rahmens lassen sich diese Größen, d.h. der Pauli-Lubanski-Vektor sowie die Tensorpolarisation, als Funktionen der hydrodynamischen Variablen schreiben, und, was besonders anschaulich ist, im Navier-Stokes-Limes betrachten. Dabei stellt sich heraus, dass die Vektorpolarisation der Teilchen dissipative Korrekturen erhält, welche auf den nichtlokalen Anteilen des Kollisionsterms basieren, und sowohl durch die Vortizität als auch durch den Schertensor des Mediums erzeugt wird. Die Tensorpolarisation hingegen, welche eine rein dissipative Größe ist, wird allein von den lokalen Kollisionen bestimmt. Zum Abschluss der Arbeit wird eine einfache Trunkierung des vollen Modells gewählt, sodass nur ein einziger Koeffizient übrig bleibt, welcher sensitiv für die mikroskopischen Details ist. Auf diese Weise wird ein simpler Zusammenhang zwischen der Tensorpolarisation der Teilchen und dem Scherspannungstensor des Mediums hergestellt.

Ausblick

In dieser Arbeit wurde eine in sich geschlossene Herleitung der dissipativen relativistischen Spin-Hydrodynamik präsentiert. Ausgehend von mikroskopischen Quantenfeldtheorien, welche Teilchen der Spins 0, 1/2 und 1 beschreiben, wurde unter Verwendung des Wigner-Funktions-Formalismus eine kinetische Theorie entwickelt, welche quantenmechanische nichtlokale Effekte in führender Ordnung berücksichtigt. Aus dieser kinetischen Theorie wurden wiederum mittels einer verallgemeinerten Momentenmethode die Gleichungen hergeleitet, welche die Zeitentwicklung der dissipativen Anteile der erhaltenen Ströme beschreiben. Zusammen mit den makroskopischen Erhaltungsgleichungen ist damit die Dynamik des Energie-Impuls-Tensors, des Teilchenstroms und des Gesamtdrehimpulstensors eindeutig festgelegt.

Die so konstruierte Theorie kann auf verschiedene Arten erweitert werden: neben Verbesserungen in der Herleitung, wie z.B. der Berücksichtigung nichtlinearer Beiträge aus den Kollisionsintegralen, sollte im Vordergrund stehen, die hergeleiteten Ausdrücke für Vektor- und Tensorpolarisation mit den entsprechenden experimentellen Daten zu vergleichen. Weiterhin ist geboten, die Gleichungen darauf zu untersuchen, ob sie im linearisierten Fall ein symmetrisch-hyperbolisches System bilden [52]. Eine Wiederholung der hier präsentierten Rechnung für masselose Teilchen, insbesondere für nicht-abelsche Eichfelder, wäre zu begrüßen, um weitere Einsicht in das hydrodynamische Verhalten des QGPs zu gewinnen. Schließlich können die in der Anfangsphase einer Schwerionenkollision entstehenden Magnetfelder sehr stark sein, weswegen es sinnvoll wäre, Effekte elektromagnetischer Felder, welche in Refs. [53–55] für den kollisionsfreien Fall untersucht wurden, auch in die volle Beschreibung mit einzubeziehen, um eine Theorie der Spin-Magnetohydrodynamik zu entwickeln.

Notation

Although most of the notation employed in this thesis will be introduced again before its first usage, here we will provide a comprehensive overview of the basic rules and conventions for reference purposes.

In this work, we choose natural units, i.e., we set $c = \epsilon_0 = \mu_0 = k_B = 1$. The (reduced) PLANCK constant $\hbar$ is not set to unity, since we will use it as a formal indicator for an expansion in gradients as well as coupling constants.

We employ the "mostly minus" convention for the metric tensor in flat spacetime, i.e.,

$$g^{\mu\nu} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}^{\mu\nu} ,$$

The totally antisymmetric LEVI-CIVITA pseudotensor density $\epsilon^{\mu\nu\alpha\beta}$ is defined via $\epsilon^{0123} = -\epsilon_{0123} := 1$. The scalar product of two four-vectors a and b is denoted with a dot, $a \cdot b \equiv a^\mu b_\mu := a^\mu g_{\mu\nu} b^\nu$, and the square of a four-vector is defined as $a^2 := a \cdot a$. Indices on LORENTZ vectors or tensors as well as the components of spinors in DIRAC space are denoted by greek letters. To avoid confusion, the latter are represented predominantly by the letters from the beginning $(\alpha, \beta, \cdots)$ of the greek alphabet. Other lists use latin letters $(i, j, \cdots)$, and three-vectors are printed in bold.
The symmetrization of a rank-two tensor $A^{\mu\nu}$ is given by round brackets, $A^{(\mu\nu)} := A^{\mu\nu} + A^{\nu\mu}$. Similarly, its antisymmetrization is denoted via square brackets, $A^{[\mu\nu]} := A^{\mu\nu} - A^{\nu\mu}$.

An operator $\widehat{O}$ in FOCK space is represented by a wide hat, and its normal-ordered ensemble average is given by the same symbol without a hat, $O := \langle : \widehat{O} : \rangle$. The commutator between two operators $\widehat{O}_1$ and $\widehat{O}_2$ is denoted by square brackets, $[\widehat{O}_1, \widehat{O}_2] := \widehat{O}_1\widehat{O}_2 - \widehat{O}_2\widehat{O}_1$, and the anticommutator is represented by curly brackets $\{\widehat{O}_1, \widehat{O}_2\} := \widehat{O}_1\widehat{O}_2 + \widehat{O}_2\widehat{O}_1$.
A quantum field in a general representation is given the letter $\widehat{\varphi}$. Relativistic scalar fields are represented as $\widehat{\phi}$, while spinor fields get the letter $\widehat{\psi}$. Vector fields are denoted as $\widehat{V}$. The DIRAC matrices $\{\gamma^\mu\ , \mu = 0, 1, 2, 3\}$ are defined via their anticommutator, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. Here, the unit matrix in DIRAC space has been omitted for brevity, as will often be done in the main text. For the contraction of a four-vector a with these matrices, we employ the FEYNMAN slash notation, $\not{a} := a \cdot \gamma = a^\mu \gamma_\mu$. Expressed in arbitrary units of energy, which we choose as MeV, the scalar and vector fields have a dimension of $[\widehat{\phi}] = [\widehat{V}] = \text{MeV}$, whereas the spinor fields fulfill $[\widehat{\psi}] = \text{MeV}^{3/2}$.

The spacetime and momentum coordinates will be denoted most of the time by variants of the letters x and k. If not indicated otherwise, the four-gradient ∂^μ is taken to act with respect to the spacetime coordinates, $\partial^\mu \equiv \partial^\mu_x$. The projectors onto the spaces parallel and orthogonal to the momentum k are given by $E^{\mu\nu} := k^\mu k^\nu / k^2$ and $K^{\mu\nu} := g^{\mu\nu} - E^{\mu\nu}$, respectively, and they manifestly fulfill $E^{\mu\nu} k_\nu = k^\mu$ and $K^{\mu\nu} k_\nu = 0$. The traceless symmetric projector orthogonal to the four-momentum is denoted by $K^{\mu\nu}_{\alpha\beta} := \frac{1}{2} K^{(\mu}_\alpha K^{\nu)}_\beta - \frac{1}{3} K^{\mu\nu} K_{\alpha\beta}$, and it has the property that $K_{\mu\nu} K^{\mu\nu}_{\alpha\beta} = K^{\alpha\beta} K^{\mu\nu}_{\alpha\beta} = 0$.
The measure in momentum space for particles on the mass shell (i.e., where $k^2 = m^2$) is defined as $\mathrm{d}K := \mathrm{d}^3 k / [(2\pi\hbar)^3 k^0]$, and the measure in spin space reads $\mathrm{d}S(k) := [S_0 m / (\varsigma\pi)] \mathrm{d}^4\mathfrak{s}\, \delta(\mathfrak{s}^2 + \varsigma^2) \delta(k \cdot \mathfrak{s})$,

where S_0 and ς depend on the particle spin. The combined measure is denoted as $\mathrm{d}\Gamma := \mathrm{d}K\mathrm{d}S(k)$. Finally, the microscopic dipole tensor is defined as $\Sigma_{\mathfrak{s}}^{\mu\nu} := -(1/m)\epsilon^{\mu\nu\alpha\beta}k_\alpha\mathfrak{s}_\beta$.

The four-velocity is denoted by u^μ and is normalized to unity, $u^2 = 1$. The projector onto the three-space orthogonal to it is defined as $\Delta^{\mu\nu} := g^{\mu\nu} - u^\mu u^\nu$. The comoving derivative will be written as $\frac{\mathrm{d}}{\mathrm{d}\tau} := u \cdot \partial$, and the spacelike gradient is given by $\nabla^\mu := \Delta^{\mu\nu}\partial_\nu$. Considering an ℓ-th rank tensor $A^{\mu_1\cdots\mu_\ell}$, we furthermore define $A^{\langle\mu_1\cdots\mu_\ell\rangle} := \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}A^{\nu_1\cdots\nu_\ell}$, where $\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}$ denotes a projector of tensor-rank 2ℓ that is orthogonal to the four-velocity as well as symmetric and traceless in any pair of upper or lower indices. In the special case $\ell = 2$, this projector reads $\Delta^{\mu\nu}_{\alpha\beta} := \frac{1}{2}\Delta^{(\mu}_{\alpha}\Delta^{\nu)}_{\beta} - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}$. The irreducible components of the derivative of the fluid four-velocity are given by the shear tensor $\sigma^{\mu\nu} := \nabla^{\langle\mu}u^{\nu\rangle}$, the expansion scalar $\theta := \nabla \cdot u$, and the vorticity $\omega^{\mu\nu} := \frac{1}{2}\nabla^{[\mu}u^{\nu]}$. Furthermore, we define the vorticity vector as $\omega^\mu := \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu\omega_{\alpha\beta}$.

Contents

Chapter 1

Introduction

The aim of this thesis is to provide a mostly self-contained derivation of relativistic dissipative spin hydrodynamics. Starting from a microscopic quantum field theory, the tools of quantum kinetic theory are used to construct an effective description of the system in terms of quasiparticles, while keeping the leading-order quantum effects that are related to spin. Then, the macroscopic fluid-dynamical behavior is extracted through a systematic truncation of the kinetic equation, providing a set of equations for the components of the conserved currents present in the system. In this chapter, we will shortly present the status quo of fluid dynamics with spin, and show the connection to the field of ultrarelativistic heavy-ion collisions.

1.1 Fluids with spin: a theoretical challenge

Fluid dynamics has been a tremendously successful field for centuries. Already the nonrelativistic EULER equations [56] are applicable to a variety of problems concerning flows with low viscosities, and their viscous generalization, the set of NAVIER-STOKES equations [57, 58], is widely used in practical applications today. When asking how the universal behavior of fluids emerges from a microscopic theory, one possible route to take is given by kinetic theory, where the fluid constituents are treated as particles whose distribution obeys an evolution equation of BOLTZMANN type. Upon taking the limit of this microscopic theory for systems that are close to equilibrium and feature a sufficient separation of microscopic and macroscopic scales, the equations of dissipative hydrodynamics can be derived. In such a derivation, the fact that an explicit microscopic model is assumed is both a blessing and a curse. On the one hand, it allows for the explicit calculation of all coefficients related to dissipative behavior of the fluid, as they are determined by the microscopic details of the system. On the other hand, certain assumptions have to be made for the fluid to be describable by kinetic theory. These do not necessarily have to overlap with the necessary conditions for fluid dynamics to be valid; there are regimes where kinetic theory does not provide a good description of the system, but fluid dynamics does, and vice versa. Thus, deriving hydrodynamics from a microscopic theory allows for the explicit computation of all terms that appear, but limits the applicability of the resulting equations, at least when using the coefficients as they arise from the microscopic theory.

In the case of nonrelativistic hydrodynamics, a generalization to fluids whose constituents do not behave as pointlike spinless particles has been undertaken and is relevant to the fields of e.g. spintronics [59–62] and micropolar fluids [63]. When deriving these types of theories from a microscopic kinetic approach, the internal degrees of freedom of the particles have to be considered as well. As shown in, e.g., Refs. [64–68], these quantities manifest themselves in the appearance of a microscopic tensor of inertia that determines the rotational energy of the fluid constituent in question. The application to a fluid that consists of particles with nonvanishing spin has been treated extensively in Refs. [69–73], which constitute pioneering works in this direction and introduced many ideas that were further developed in subsequent years.

In the relativistic domain, while the equations of ideal fluid dynamics, the relativistic Euler equations, are uncontroversial, the dissipative case is not as clear cut. The reason for this lies in the fact that the relativistic generalization of the Navier-Stokes equations is acausal and unstable [30], such that an alternative theory has to be provided. One way to derive such a formulation, as in the nonrelativistic case, consists in providing a microscopic foundation in terms of relativistic kinetic theory. Upon considering the near-equilibrium behavior of the system, whose microscopic and macroscopic scales are assumed to be sufficiently well separated, one can derive a set of fluid-dynamical equations which can be causal and stable [74]. It has to be remarked that the issue of finding a viable theory of relativistic dissipative fluid dynamics is not merely an academic issue: at present, this type of theory is (successfully) used in the modeling of relativistic heavy-ion collisions, which we will treat in the next section.

Upon extending these relativistic theories of fluid dynamics to the case where the fluid constituents are particles of nonzero spin, several challenges emerge. First, since spin is fundamentally a quantum-mechanical property, an appropriate version of kinetic theory that includes such effects has to be used. As an even stronger demand, a quantum-field theoretical foundation of kinetic theory is mandatory to incorporate both quantum-mechanical and relativistic effects. These requirements can be fulfilled by considering the covariant Wigner function, which serves as a matrix-valued generalization of the classical phase-space distribution function [75]. Second, in a relativistic setting, the set of conserved quantities on which fluid dynamics is based is not unique. Rather, the conservation laws for energy, momentum, and total angular momentum are invariant under so-called pseudogauge transformations, which essentially redistribute those components of the total angular momentum which are labeled as "orbital" and "spin"-angular momenta, respectively [35, 76, 77].

Besides these difficulties, a number of works in the past years have taken on the challenge of formulating relativistic spin hydrodynamics [38, 78–81]. Regarding the task of deriving relativistic dissipative fluid dynamics with spin from kinetic theory via the method of moments, pioneering work has been done in Refs. [82, 83] for massive particles of spin 1/2. What is missing up to now is a formulation of spin hydrodynamics for particles of higher spin, although investigations into an appropriate kinetic formulation have been undertaken [84–89]. In particular, in light of the experimentally measurable quantities that we will discuss in Subsec. 1.2.1, a hydrodynamic theory for massive vector mesons that have spin 1 is desirable, and will be derived in this work.

1.2 Relativistic heavy-ion collisions and hydrodynamics

One aim of today's collider experiments is to explore the phase diagram of Quantum Chromodynamics (QCD), which constitutes the theory of strong interactions, one of the four fundamental forces of Nature. While the basic equations of QCD are well known, it is notoriously hard to treat analytically due to its negative β-function which lets the energy-dependent coupling of the theory decrease (increase) at higher (lower) energies. This property, called asymptotic freedom [90, 91], is responsible for the fact that the fundamental fields given by quarks and gluons are not observed directly, but rather form hadrons that constitute the effective low-energy degrees of freedom. Thus, in order to construct the phase diagram of QCD, which is depicted in Fig. 1.1, a variety of methods are used (for reviews, see, e.g., Refs. [93–95]). While at zero baryon chemical potential, $\mu_B = 0$, it is possible to study the behavior of QCD from first principles via lattice simulations, this method does not apply at higher μ_B due to the infamous sign problem [96]. In this region, different tools are applied, such as functional methods [97]. One particularly interesting feature of the QCD phase diagram is the crossover from a hadron gas, in which the quarks and gluons are confined, to the so-called quark-gluon plasma (QGP), where they constitute the primary degrees of freedom. At $\mu_B = 0$, this transition happens at a crossover temperature of $T_C \simeq 155$MeV, and might end in a critical point at higher μ_B, cf. Fig. 1.1.

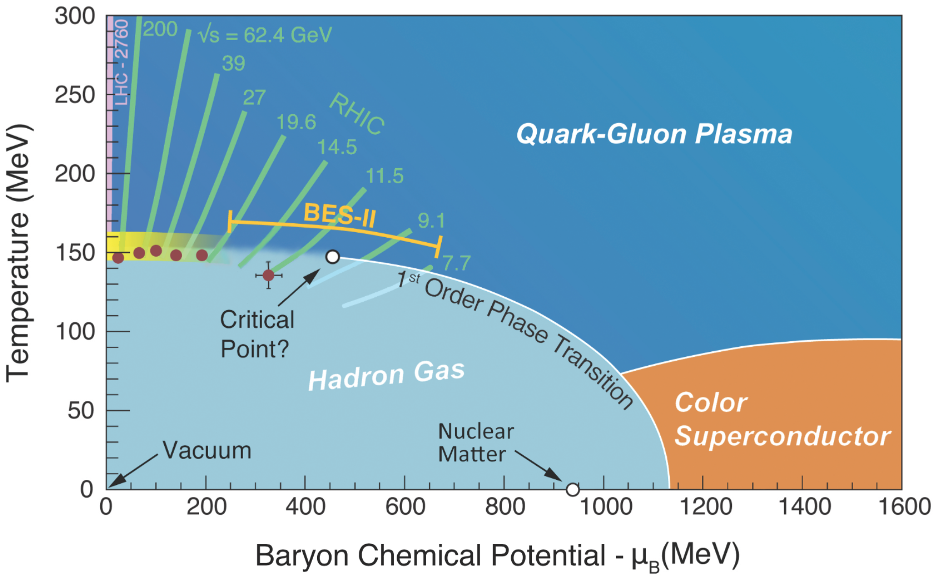

Figure 1.1: The phase diagram of QCD [92].

Since the nuclei in ultrarelativistic heavy-ion collisions are accelerated to almost the speed of light, the QGP can be probed in these experiments. Interestingly, it has been found in the past decades that the QGP exhibits collective traits, implying that it behaves as a fluid rather than a gas of free particles [1, 2]. Using hybrid simulations that combine kinetic and hydrodynamic frameworks to model the different stages of a heavy-ion collision, data such as particle multiplicities and flow coefficients can be reproduced rather well (for a review, see, e.g., Ref. [3]).

1.2.1 Polarization observables

An interesting class of potentially measurable quantities is given by the observables related to the polarization of the particles. These constitute exciting probes of the hydrodynamic nature of the QGP, since the fluid-dynamical gradients of the medium, such as vorticity and shear, can induce a nonvanishing polarization. In the following, we will shortly discuss two observables of this type, namely the polarization of Λ hyperons, and the alignment of ϕ and $K^{\star 0}$ mesons.[1]

Polarization of Λ hyperons

The Λ baryon can decay via the weak interaction into a proton and a pion, $\Lambda \to p + \pi^-$. Since the weak interaction violates parity, the distribution of the decay products over the solid angle depends on the polarization of the Λ baryon. More specifically, denoting all quantities evaluated in the rest frame of the Λ hyperon with a star, we have [9]

$$\frac{\mathrm{d}N}{\mathrm{d}\Omega^\star} = \frac{1}{4\pi}\left(1 + \alpha_\Lambda \boldsymbol{\mathcal{P}}^\star_\Lambda \cdot \hat{\mathbf{k}}^\star_p\right) = \frac{1}{4\pi}\left(1 + \alpha_\Lambda |\boldsymbol{\mathcal{P}}^\star_\Lambda| \cos\xi^\star\right) , \tag{1.1}$$

where $\boldsymbol{\mathcal{P}}^\star_\Lambda$ and $\hat{\mathbf{k}}^\star_p$ are the polarization vector of the Λ particle and the direction of the momentum of the emitted proton, respectively. Furthermore, $\xi^\star$ denotes the angle between the polarization vector and the momentum of the emitted particle, and α_Λ is the so-called decay parameter of the hyperon, which is estimated to be $\alpha_\Lambda \simeq 0.75$ [98]. Equation (1.1) allows to relate the average of the proton

[1] We remark that research is ongoing considering the polarization of other particles, such as, e.g., Ω or Ξ hyperons, cf. Ref. [7].

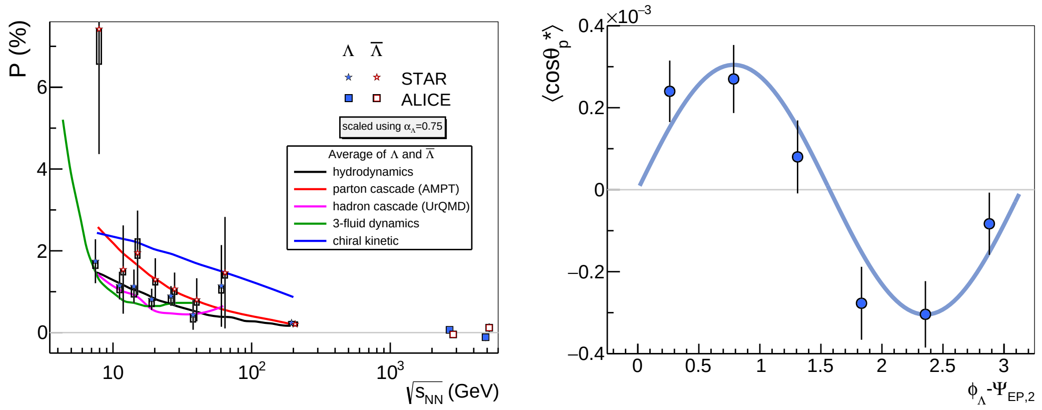

Figure 1.2: Left: The global polarization of Λ hyperons measured by the ALICE and STAR collaborations as a function of center-of-mass energy, compared to various models. Right: Expectation value of $\cos\theta_p^\star$ [9].

momenta along some direction $\mathbf{n}$, an experimentally measurable quantity, to the polarization of the Λ hyperon,

$$\left\langle \hat{\mathbf{k}}_p^\star \cdot \mathbf{n} \right\rangle = \frac{\alpha_\Lambda}{3} \boldsymbol{\mathcal{P}}_\Lambda^\star \cdot \mathbf{n} \,. \tag{1.2}$$

Choosing the vector $\mathbf{n}$ to point in the direction of the total angular momentum of the system, we find the so-called global polarization [9]

$$\overline{\mathcal{P}}_\Lambda^\star = -\frac{8}{\pi\alpha_\Lambda} \left\langle \sin\left(\phi_p^\star - \Psi_{\rm RP}\right)\right\rangle \,, \tag{1.3}$$

where $\phi_p^\star$ denotes the angle between $\hat{\mathbf{k}}_p^\star$ and the impact parameter, and $\Psi_{\rm RP}$ is the reaction-plane angle.[2] The global polarization of Λ hyperons [5–8], shown in Fig. 1.2, can be explained rather well by hydrodynamic models that assume local equilibrium [9]. In these approaches, the particles are polarized through the vorticity of the medium, akin to the famous Barnett effect [4].

On the other hand, one may set $\mathbf{n}$ in Eq. (1.2) to point in the direction of the beam, which we choose to be the z-axis. The longitudinal polarization [10] is then given by

$$\mathcal{P}_\Lambda^z = \frac{1}{\alpha_\Lambda} \frac{\left\langle \cos\theta_p^\star \right\rangle}{\left\langle \cos^2\theta_p^\star \right\rangle} \,. \tag{1.4}$$

This observable is called local polarization, since it is a function of the angle ϕ_Λ, cf. Fig. 1.2. In contrast to the global polarization, there has been a disagreement between theory and experiment, sometimes referred to as the polarization sign puzzle, a name that originates from the fact that the theoretical predictions were of the right magnitude, but opposite sign compared to the data. Recently, models based on local equilibrium have been able to reproduce the data by including previously omitted terms proportional to the so-called thermal shear of the medium [11–14]. One goal of this thesis is to expand on the extensive results obtained in Refs. [82, 83] by obtaining the global and local polarization in a theory of dissipative hydrodynamics featuring transport coefficients that can be systematically improved.

Alignment of ϕ and $K^{\star 0}$ mesons

Another possible observable related to polarization is the so-called alignment of vector mesons, which is given by the difference of the 00-element of their density matrix from the unpolarized value of

[2] In practice, this angle has to be estimated from experiment, and a correction factor has to be introduced [99].

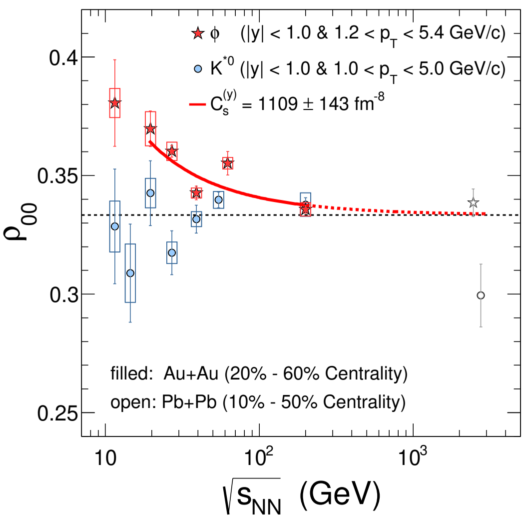

Figure 1.3: The global alignment of ϕ and $K^{\star 0}$ mesons [18].

1/3 [15]. Since both ϕ and $K^{\star 0}$ mesons decay mainly via the parity-conserving strong interaction, their polarization vectors cannot be measured. However, utilizing the decays $\phi \to K^+ + K^-$ and $K^{\star 0} \to K^+ + \pi^-$, one can relate the angular distribution of the decay products to the 00-element of the density matrix via [9]

$$\frac{\mathrm{d}N}{\mathrm{d}\cos\xi^\star} = \frac{3}{4}\left[1 - \varrho_{00} + (3\varrho_{00} - 1)\cos^2\xi^\star\right] . \tag{1.5}$$

While not entering the polarization vector, the 00-element of the spin density matrix is part of the so-called tensor polarization, and constitutes an effect that is only present for particles of spin 1 or higher [15]. The measured (global) alignment of ϕ mesons, displayed in Fig. 1.3, is larger than expected, with the additional complication that the alignment of $K^{\star 0}$ mesons is compatible with zero [16–18]. In recent years, the explanation of these results, which cannot be provided by naively combining the polarizations of the constituent quarks, as the effect would be too weak, has become the subject of intense work [19–27], but an established solution is still missing. In this thesis, while not yet being able to make quantitative predictions, we will, as shown in Ref. [28], arrive at a theoretical description of this type of effect in a hydrodynamic framework, with the alignment of the vector mesons induced by dissipative fluid gradients.

1.3 Overview of this thesis

In order to provide an introduction into the description of macroscopic systems, Chapter 2 starts from a thermodynamic viewpoint and introduces ideal hydrodynamics through the concept of local equilibrium. After presenting different ways to arrive at theories of dissipative fluid dynamics in Sec. 2.3, the inclusion of spin is discussed. Both with and without spin, the conservation equations that govern ideal fluid dynamics are not sufficient to describe all degrees of freedom appearing in the conserved currents. To remedy this, we turn to a formulation of quantum kinetic theory, whose basic building blocks are established in Chapter 3. Introducing the concept of the Wigner function, it is shown how to connect to the macroscopic conserved currents through phase-space integrals. Then, Secs. 3.3, 3.4, and 3.5 discuss the phase-space formulation of the dynamics of massive scalar, spinor, and vector fields, respectively. It is shown how to extend phase space by an additional variable in order to work with a scalar distribution function for fields of any spin. Different pseudogauges are introduced, and kinetic equations for the Wigner function of all fields are derived. On the right-hand sides of these evolution equations, there appear collision terms which specify the scattering between quasiparticles. Chapter 4 is then concerned with computing these collision terms for all of the aforementioned fields. To perform this computation, the so-called GLW method is used, which consists of an expansion in "in"-picture reduced density matrices. Both local and nonlocal contributions are computed, and the spacetime shifts characterizing the nonlocality are given in a manifestly covariant form. In Sec. 4.6, the state of local equilibrium is discussed, and is found to be equal to the one corresponding to global equilibrium, as long as the particles have nonzero spin. In Chapter 5, the computation of the collision terms is repeated, this time using the KB approach, which starts from the DYSON-SCHWINGER equations and assumes a reasonable truncation for the appearing self-energy. The results of the previous chapter are recovered, with the important addition that quantum-statistical effects are retained. This manifests itself also in the discussion of equilibrium in Sec. 5.6, where the known quantum-statistical distributions are recovered. Having laid the groundwork on the underlying quantum kinetic theory, Chapter 6 is concerned with deriving dissipative fluid dynamics. The method of moments is introduced and generalized to the case of particles with nonzero spin, necessitating the introduction of irreducible moments of different ranks in spin. The exact equations of motion of the irreducible moments of all spin-ranks are derived. In order to close the system of moment equations, the IReD approach is employed, which denotes a perturbative scheme where the higher-order terms are approximated by using their NAVIER-STOKES values. This then allows to derive equations of motion for all dissipative degrees of freedom that appear in the conserved currents, thus completing the construction of spin hydrodynamics from kinetic theory. It is found that those spin degrees of vector particles which are responsible for the tensor polarization couple to the standard hydrodynamic fields, leading to corrections in the respective transport coefficients. In Sec. 6.5, the polarization-related observables, i.e., the PAULI-LUBANSKI pseudovector and the alignment, are expressed in terms of hydrodynamic fields, and also shown in the NAVIER-STOKES limit. Finally, Chapter 7 summarizes this work and lists future perspectives.

Chapter 2

Relativistic hydrodynamics

In general, describing the behavior of a macroscopic physical system is a highly nontrivial task. Evaluating the equations of motion for all microscopic degrees of freedom is not feasible, as it would require solving $\mathcal{O}(N_A) \sim \mathcal{O}\left(10^{23}\right)$ coupled differential equations. Instead, it is sensible to take a diametrically opposed approach that does not build on evaluating the evolution of the microscopic degrees of freedom, but rather on a few macroscopic variables that describe the emergent properties of the system. In this chapter we will discuss a few of these approaches, the most well-known being usual thermodynamics, which deals with systems in *thermal equilibrium*. Relaxing the assumption of complete thermal equilibrium yields a theory that deals with spacetime-dependent fields in *local equilibrium*, namely ideal hydrodynamics. Removing this constraint as well, which is necessary in order to be able to describe dissipative processes, leads into the realm of dissipative hydrodynamics. This last and most general approach to describe macroscopic systems, which in contrast to the previous two requires system-specific input beyond an equation of state, will be the main topic of the thesis.

2.1 Thermodynamics

In complete thermal equilibrium, a macroscopic system can be characterized by a comparatively small set of *extensive* and *intensive* quantities, extensive meaning that the quantity scales with the system's size, which does not hold for intensive quantities. Considering a system at rest in the absence of external fields, the extensive thermodynamic quantities are given by the *energy* $\mathcal{E}$, the *entropy* $\mathcal{S}$, the *volume* $\mathcal{V}$, and the *particle number* $\mathcal{N}$. These quantities obey the *first law of thermodynamics*,

$$\mathrm{d}\mathcal{E} = T\mathrm{d}\mathcal{S} - P\mathrm{d}\mathcal{V} + \sum_{i=1}^{N_{\mathrm{spec}}} \mu_i \mathrm{d}\mathcal{N}_i \,, \tag{2.1}$$

where the factors in front of the differentials are the intensive quantities of the system, namely the *temperature* T, the *pressure* P, and the *chemical potentials* $\{\mu_i\}$, where $i = 1, \cdots, N_{\mathrm{spec}}$. In the following, we will always assume that the system consists of a single particle species, i.e., $N_{\mathrm{spec}} = 1$. The intensive quantities can be related to the extensive ones by the relations

$$T := \left.\frac{\partial \mathcal{E}}{\partial \mathcal{S}}\right|_{\mathcal{V},\mathcal{N}} = \left(\left.\frac{\partial \mathcal{S}}{\partial \mathcal{E}}\right|_{\mathcal{V},\mathcal{N}}\right)^{-1} , \tag{2.2a}$$

$$P := -\left.\frac{\partial \mathcal{E}}{\partial \mathcal{V}}\right|_{\mathcal{S},\mathcal{N}} = T\left.\frac{\partial \mathcal{S}}{\partial \mathcal{V}}\right|_{\mathcal{E},\mathcal{N}} , \tag{2.2b}$$

$$\mu := \left.\frac{\partial \mathcal{E}}{\partial \mathcal{N}}\right|_{\mathcal{S},\mathcal{V}} = -T\left.\frac{\partial \mathcal{S}}{\partial \mathcal{N}}\right|_{\mathcal{E},\mathcal{V}} , \tag{2.2c}$$

which follow immediately from Eq. (2.1) by considering either the energy or the entropy as a function of the other extensive variables, i.e., $\mathcal{E} \equiv \mathcal{E}(\mathcal{S}, \mathcal{V}, \mathcal{N})$ or $\mathcal{S} \equiv \mathcal{S}(\mathcal{E}, \mathcal{V}, \mathcal{N})$, respectively.

Given that $\mathcal{E}, \mathcal{S}, \mathcal{V}$ and $\mathcal{N}$ are all extensive quantities, we can deduce that the energy is a homogeneous function of order one, i.e.,

$$\mathcal{E}(\lambda\mathcal{S}, \lambda\mathcal{V}, \lambda\mathcal{N}) = \lambda\mathcal{E}(\mathcal{S}, \mathcal{V}, \mathcal{N}) \;, \tag{2.3}$$

which holds for any $\lambda \in \mathbb{R}^+$. Expanding λ around unity, $\lambda = 1 + \eta$, $\eta \ll 1$, we find

$$\begin{aligned}(1+\eta)\mathcal{E}(\mathcal{S}, \mathcal{V}, \mathcal{N}) &= \mathcal{E}\left[(1+\eta)\mathcal{S}, (1+\eta)\mathcal{V}, (1+\eta)\mathcal{N}\right] \\ &= \mathcal{E}(\mathcal{S}, \mathcal{V}, \mathcal{N}) + \eta\left(\left.\frac{\partial\mathcal{E}}{\partial\mathcal{S}}\right|_{\mathcal{V},\mathcal{N}}\mathcal{S} + \left.\frac{\partial\mathcal{E}}{\partial\mathcal{V}}\right|_{\mathcal{S},\mathcal{N}}\mathcal{V} + \left.\frac{\partial\mathcal{E}}{\partial\mathcal{N}}\right|_{\mathcal{S},\mathcal{V}}\mathcal{N}\right) + \mathcal{O}(\eta^2) \;, \end{aligned} \tag{2.4}$$

from which it follows by equating terms of order $\mathcal{O}(\eta)$ and using Eqs. (2.2) that

$$\mathcal{E} = T\mathcal{S} - P\mathcal{V} + \mu\mathcal{N} \;. \tag{2.5}$$

This relation is called the Euler *equation*. For the following considerations, it is helpful to work not with total quantities, but rather with densities. Introducing the energy density $\varepsilon := \mathcal{E}/\mathcal{V}$, the entropy density $s := \mathcal{S}/\mathcal{V}$ and the particle-number density $n := \mathcal{N}/\mathcal{V}$, we have

$$\varepsilon + P = Ts + \mu n \;. \tag{2.6}$$

Defining the inverse temperature $\beta := 1/T$ and the ratio of chemical potential over temperature $\alpha := \mu/T \equiv \mu\beta$, we can rewrite the first law as

$$\mathrm{d}s = \beta\mathrm{d}\varepsilon - \alpha\mathrm{d}n \;. \tag{2.7}$$

Furthermore, by Legendre-transforming the energy with respect to $\mathcal{S}, \mathcal{V}$, and $\mathcal{N}$ as well as using Eq. (2.5), we find the Gibbs-Duhem relation

$$\mathrm{d}P = s\mathrm{d}T + n\mathrm{d}\mu \;, \tag{2.8}$$

which upon switching from (μ, T) to (α, β) and using the Euler equation takes on the following form,

$$\beta\mathrm{d}P = -(\varepsilon + P)\mathrm{d}\beta + n\mathrm{d}\alpha \;. \tag{2.9}$$

The discussion up to now tacitly assumed a fixed reference frame and did not assess the question of how thermodynamic variables transform upon changing the reference frame, for which a covariant formulation is needed.

2.1.1 Covariant thermodynamics

As we will discuss in more detail in Sec. 3.1, the action of a relativistic theory should be invariant under transformations belonging to the Poincaré group, implying the conservation of the *energy-momentum tensor* $T^{\mu\nu}$ as well as the *total angular momentum tensor* $J^{\lambda\mu\nu}$. Note that the total angular momentum tensor can be decomposed into the sum of the *orbital angular momentum tensor*

$$L^{\lambda\mu\nu} := T^{\lambda[\nu}x^{\mu]} \tag{2.10}$$

and the *spin tensor*

$$\hbar S^{\lambda\mu\nu} := J^{\lambda\mu\nu} - L^{\lambda\mu\nu} \;. \tag{2.11}$$

Note that the square brackets denote antisymmetrization, $A^{[\mu}B^{\nu]} := A^\mu B^\nu - A^\nu B^\mu$. In order to stay consistent with the thermodynamic relations introduced earlier, we further assume that the theory features a conserved *particle four-current* N^μ. Note that these conserved currents are to be identified with *densities* (and not total quantities) when comparing to Eqs. (2.6)–(2.9). This can be seen from the fact that the total charges (i.e., the total particle number $\mathcal{N}$, the total momentum $\mathcal{P}^\mu$, and the total angular momentum $\mathcal{J}^{\mu\nu}$) are given by integrating N^μ, $T^{\mu\nu}$, and $J^{\lambda\mu\nu}$ over a hypersurface Σ. Explicitly, we have

$$\mathcal{N} := \int \mathrm{d}\Sigma_\lambda N^\lambda \ , \tag{2.12a}$$

$$\mathcal{P}^\mu := \int \mathrm{d}\Sigma_\lambda T^{\lambda\mu} \ , \tag{2.12b}$$

$$\mathcal{J}^{\mu\nu} := \int \mathrm{d}\Sigma_\lambda J^{\lambda\mu\nu} \ , \tag{2.12c}$$

where the hypersurface can in particular be chosen to be an equal-time one, $\mathrm{d}\Sigma^\mu \equiv \delta^\mu_0 \mathrm{d}^3x$, such that the total charges are constant in time. The conservation equations for these currents read

$$\partial_\mu N^\mu = 0 \ , \tag{2.13a}$$

$$\partial_\mu T^{\mu\nu} = 0 \ , \tag{2.13b}$$

$$\partial_\lambda J^{\lambda\mu\nu} = \hbar \partial_\lambda S^{\lambda\mu\nu} + T^{[\mu\nu]} = 0 \ . \tag{2.13c}$$

Given the aforementioned general arguments, it is now clear how to connect the thermodynamic variables appearing in the standard relations (2.7) and (2.9) to covariant conserved currents. The energy density is the 00-component of the energy-momentum tensor, $\varepsilon \equiv T^{00}$, while the particle-number density denotes the zeroth component of the particle four-current, i.e., $n \equiv N^0$. Assigning a four-vector S^μ to the entropy density such that $s \equiv S^0$ and expressing the inverse temperature as the zeroth component of a four-vector, $\beta \equiv \beta^0$, we may express the first law (2.7) as

$$\mathrm{d}S^0 = \beta_0 \mathrm{d}T^{00} - \alpha \mathrm{d}N^0 \ . \tag{2.14}$$

At this point it is clear that Eq. (2.14) is the zeroth component of a covariant expression evaluated in some frame, which we can argue to be the rest frame of the medium. First, we note that the four-velocity of the medium u^μ, which in its rest frame (denoted with an index RF) becomes

$$u^\mu_{\mathrm{RF}} = (1,\, 0,\, 0,\, 0)^\mu \ , \tag{2.15}$$

is the only vector at our disposal that is related to the system. In a nonrotating system in thermal equilibrium without external fields, the pressure is isotropic and thus the energy-momentum tensor is diagonal,

$$T^{\mu\nu}_{\mathrm{RF}} = \mathrm{diag}\,(\varepsilon,\, P,\, P,\, P)^{\mu\nu} \equiv \varepsilon u^\mu_{\mathrm{RF}} u^\nu_{\mathrm{RF}} - P\Delta^{\mu\nu}_{\mathrm{RF}} \ , \tag{2.16a}$$

where we defined the projector $\Delta^{\mu\nu} := g^{\mu\nu} - u^\mu u^\nu$. Since u^μ is the only vector at hand, we must have that $N^\mu = nu^\mu$, $S^\mu = su^\mu$, and $\beta^\mu = \beta u^\mu$, i.e.,

$$N^\mu_{\mathrm{RF}} = (n,\, 0,\, 0,\, 0)^\mu \ , \tag{2.16b}$$

$$S^\mu_{\mathrm{RF}} = (s,\, 0,\, 0,\, 0)^\mu \ , \tag{2.16c}$$

$$\beta^\mu_{\mathrm{RF}} = (\beta,\, 0,\, 0,\, 0)^\mu \ . \tag{2.16d}$$

From Eqs. (2.16), we can then deduce the covariant form of the first law,

$$\mathrm{d}S^\mu = \beta_\nu \mathrm{d}T^{\mu\nu} - \alpha \mathrm{d}N^\mu \ , \tag{2.17}$$

which reduces to Eq. (2.7) in the rest frame of the medium. Similarly, we obtain the covariant form of the GIBBS-DUHEM relation (2.9),

$$\mathrm{d}\,(\beta^\mu P) = -T^{\mu\nu}\mathrm{d}\beta_\nu + N^\mu \mathrm{d}\alpha \ . \tag{2.18}$$

The EULER equation (2.6) then reads

$$T^{\mu\nu}\beta_\nu + P\beta^\mu = S^\mu + \alpha N^\mu \ . \tag{2.19}$$

We remark that from Eq. (2.17) we find that, as expected, the entropy four-current has vanishing divergence,

$$\partial_\mu S^\mu = \beta_\nu \partial_\mu T^{\mu\nu} - \alpha \partial_\mu N^\mu = 0 \ . \tag{2.20}$$

Moving systems in thermal equilibrium

It should be stressed that the previous results always assumed complete thermal equilibrium. This restricts the possible values of the system's four-velocity u^μ to the ones that are obtained from the rest-frame values by a constrained class of LORENTZ transformations, namely those that let the four-velocity be a combination of a uniform motion and a rigid rotation.

This condition can be derived along the lines of §10 of Ref. [100], which deals with the nonrelativistic case. First, we may think of the system in question as made up of N smaller systems, which are macroscopic nonetheless. These small systems, which we call cells, are themselves in thermodynamic equilibrium, as well as in equilibrium with each other. Note that, in Sec. 2.2 we will relax the latter assumption. Thermal equilibrium demands the maximization of the total entropy, which is given by

$$\mathcal{S} \coloneqq \int \mathrm{d}\Sigma_\lambda S^\lambda \ . \tag{2.21}$$

The total entropy $\mathcal{S}$ of the system is given by the sum of the entropies of the cells $\mathcal{S}_i$, and the latter have to be functions of the internal energy of the cell, i.e., the difference between the total energy of the cell and its kinetic energy. Thus, we have

$$\mathcal{S} = \sum_{i=1}^{N} \mathcal{S}_i \left(\mathcal{E}_i - M_i(\gamma_i - 1)\right) \ , \tag{2.22}$$

where

$$\gamma_i = \frac{1}{\sqrt{1 - v_i^2}} = \sqrt{1 + \frac{p_i^2}{M_i^2}} \tag{2.23}$$

is the LORENTZ factor, $\mathbf{v}_i \coloneqq \mathbf{u}_i/\gamma_i$ is the three-velocity and $\mathbf{p}_i \coloneqq M_i\gamma_i\mathbf{v}_i$ denotes the momentum of the i-th cell, which has the mass M_i. Furthermore, the conservation of the total four-momentum and the total angular momentum imply

$$\sum_{i=1}^{N} p_i^\mu = \text{const.} \ , \qquad \sum_{i=1}^{N} \epsilon^{\mu\nu\alpha\beta} x_{i,\alpha} p_{i,\beta} = \text{const.} \ , \tag{2.24}$$

where we neglected the possibility of particles having spin for now. Maximizing the entropy (2.22) subject to the constraints (2.24), we find, using the method of LAGRANGE multipliers,

$$\begin{aligned} 0 &= \frac{\partial}{\partial p_j^\mu} \sum_{i=1}^{N} \left[\mathcal{S}_i \left(\mathcal{E}_i - M_i(\gamma_i - 1)\right) + b_\alpha p_i^\alpha + \tilde{\varpi}_{\rho\sigma}\epsilon^{\rho\sigma\alpha\beta} x_{i,\alpha} p_{i,\beta}\right] \\ &= -\frac{u_{j,\mu}}{T_j} + b_\mu + \tilde{\varpi}^{\rho\sigma}\epsilon_{\rho\sigma\alpha\mu} x_j^\alpha \ . \end{aligned} \tag{2.25}$$

Here we employed that

$$\frac{\partial \gamma_i}{\partial p_j^\mu} = \frac{p_{i,\mu}}{\gamma_i M_i^2}\delta_{ij} = \frac{u_{i,\mu}}{\gamma_i M_i}\delta_{ij} \tag{2.26}$$

as well as the definition of the temperature in a moving system

$$\left.\frac{\partial \mathcal{S}_i}{\partial \mathcal{E}_i}\right|_{\mathcal{N}_i} = \frac{\gamma_i}{T_i} \ , \tag{2.27}$$

which reduces to Eq. (2.2a) in the case of $\gamma_i = 1$, as it has to. Rearranging Eq. (2.25) and defining $\varpi^{\alpha\beta} \coloneqq \tilde{\varpi}_{\rho\sigma}\epsilon^{\rho\sigma\alpha\beta}$, we find

$$\frac{u_i^\mu}{T_i} \equiv \beta_i^\mu = b^\mu + \varpi^{\mu\nu} x_{i,\nu} \ , \tag{2.28}$$

where $\varpi^{\mu\nu} = \frac{1}{2}\partial^{[\nu}\beta_i^{\mu]} = \text{const.}$ is the so-called *thermal vorticity*. This condition tells us that (for constant temperature) each cell in the medium may move uniformly with the same magnitude and direction (given by b^μ). Furthermore, there may be a constant rotation or acceleration (characterized by ϖ^{ij} or ϖ^{0i}, respectively). Note that in the case of nonzero acceleration or rotation the temperature is not constant [101].

We will encounter the condition (2.28) again in Chapters 4 and 5, where it will arise from an effective microscopic theory as the condition of *global equilibrium*.

Recovering a known relation

Since Eq. (2.28) implies that the acceleration and rotation of the system is constant, we can write the total four-temperature as $\beta^\mu = b^\mu + \varpi^{\mu\nu}x_\nu$ (where x is continuous now), which can then be inserted into the first law (2.17). Using the antisymmetry of $\varpi^{\mu\nu}$, we find

$$\mathrm{d}S^\mu = b_\nu \mathrm{d}T^{\mu\nu} - \frac{1}{2}\varpi_{\nu\lambda}\mathrm{d}L^{\mu\nu\lambda} - \varpi_{\nu\lambda}T^{\mu\nu}\mathrm{d}x^\lambda - \alpha \mathrm{d}N^\mu \; , \tag{2.29}$$

where we employed the definition (2.10). Note that we still have $\partial_\mu S^\mu = 0$ due to the divergence of the orbital angular momentum tensor, $\partial_\mu L^{\mu\nu\lambda} = T^{[\nu\lambda]}$.

To conclude this section, we consider the case of a rigidly rotating (but not uniformly moving) medium.[1] The four-velocity is

$$u^\mu = \gamma\,(1,\, \mathbf{v})^\mu = \gamma\,(1, \boldsymbol{\omega} \times \mathbf{x})^\mu \; , \tag{2.30}$$

where $\boldsymbol{\omega}$ is the vorticity vector. Via Eq. (2.28), we can then identify

$$b^\mu = \left(\frac{\gamma}{T},\, 0,\, 0,\, 0\right)^\mu \; , \quad \varpi^{0i} = 0 \; , \quad \varpi^{ij} = \frac{\gamma}{T}\epsilon^{ijk}\omega^k \; . \tag{2.31}$$

In order to obtain the nonrelativistic limit, we may approximate the Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - |\boldsymbol{\omega} \times \mathbf{x}|^2}} = 1 + \mathcal{O}\left(|\boldsymbol{\omega}|^2|\mathbf{x}|^2\right) \; , \tag{2.32}$$

such that $T = \text{const.}$ in that regime. Defining the angular momentum vector $L^i := -\frac{1}{2}\epsilon^{ijk}L^{0jk}$, we find for the zeroth component of Eq. (2.29) in the nonrelativistic limit

$$T\mathrm{d}s = \mathrm{d}\varepsilon + \boldsymbol{\omega} \cdot \mathrm{d}\mathbf{L} - \mu \mathrm{d}n \; . \tag{2.33}$$

Here we neglected terms of second order in the velocity, using that $T^{0i} = \rho v^i \sim \mathcal{O}(|\boldsymbol{\omega}||\mathbf{x}|)$, where ρ denotes the density of the medium. As expected, Eq. (2.33) is the first law for nonrelativistic rotating systems known from the literature, cf., e.g., §26 of Ref. [100].

2.2 Ideal hydrodynamics

When speaking of hydrodynamics, the fundamental difference to usual thermodynamics as discussed in Sec. 2.1 lies in the concept of local equilibrium. Thinking of the total system (which we call a *fluid* from now on) as being made up of a large number of small subsystems, which are nonetheless large enough themselves to be considered as macroscopic, we can take each of these cells to be in thermal equilibrium as discussed in the previous section. The crucial difference lies in the relaxation of the assumption of complete thermal equilibrium, i.e., the assumption that all cells are in thermal equilibrium with each other. Without this premise, Eq. (2.28) does not necessarily follow, as the entropy only has to be maximal for a given cell, but not for the whole system; in other words, it has to be maximized only *locally*, but not *globally*. Letting the number of the fluid cells go to infinity

[1] For a discussion of the accelerating case, see Ref. [101].

while shrinking their size to zero (such that the volume of the fluid stays finite), we can move from a set of thermodynamic quantities for each cell $\{T_i,\, P_i,\, u_i^\mu,\, \cdots\}$ to a set of thermodynamic *fields* $\{T(x),\, P(x),\, u^\mu(x),\, \cdots\}$ that depend continuously on spacetime.

The equations of motion for these quantities are given by the conservation equations (2.13). For the purpose of this section, we will ignore the conservation law for the total angular momentum, which will be discussed in Sec. 2.4. As argued in Sec. 2.1, we may decompose the particle four-current and the energy-momentum tensor as

$$N^\mu = n u^\mu \qquad \text{and} \qquad T^{\mu\nu} = \varepsilon u^\mu u^\nu - P\Delta^{\mu\nu}\ . \tag{2.34}$$

Inserting these decompositions into the respective conservation laws, we obtain

$$\partial_\mu N^\mu = \dot{n} + n\theta\ , \tag{2.35a}$$

$$\partial_\mu T^{\mu\nu} = \left[\dot{\varepsilon} + (\varepsilon + P)\,\theta\right] u^\nu + (\varepsilon + P)\,\dot{u}^\nu - \nabla^\nu P\ . \tag{2.35b}$$

Here, we defined the *comoving derivative* $\frac{\mathrm{d}}{\mathrm{d}\tau} \coloneqq u_\mu \partial^\mu$, which is denoted by a dot and reduces to a time derivative in the rest frame of the fluid, where $u^\mu = (1,\, 0,\, 0,\, 0)^\mu$. Similarly, the *spacelike gradient* $\nabla^\mu \coloneqq \Delta^{\mu\nu}\partial_\nu$ was introduced. Furthermore, we defined the so-called *expansion scalar* $\theta \coloneqq \partial_\mu u^\mu$. Projecting Eq. (2.35b) along the fluid four-velocity (by contracting with u_ν) and onto the three-space orthogonal to it (by contracting with $\Delta^\mu{}_\nu$) yields

$$\dot{n} = -n\theta\ , \tag{2.36a}$$

$$\dot{\varepsilon} = -\,(\varepsilon + P)\,\theta\ , \tag{2.36b}$$

$$(\varepsilon + P)\,\dot{u}^\mu = \nabla^\mu P\ , \tag{2.36c}$$

where we used the fact that $u_\mu \dot{u}^\mu = \frac{1}{2}\mathrm{d}u^2/\mathrm{d}\tau = 0$ since the four-velocity is normalized to one. Equations (2.36) are called the relativistic Euler equations and constitute the basis of relativistic ideal fluid dynamics.

Evidently, the ideal fluid is characterized by five variables, namely ε, n, and u^μ, where it has to be noted that u^μ only has three independent components due to its normalization. Recall that the pressure is not an independent quantity, as it is determined in terms of the energy and particle-number density after specifying an *equation of state* of the form $f(P,\varepsilon,n) = 0$ with some function f. Thus, the evolution of an ideal fluid is completely specified by the conservation equations and (covariant) thermodynamics. It has to be stressed that at this point the only information that is specific to the fluid under consideration comes from the equation of state; Eqs. (2.36) are (under the assumption of isotropy in the fluid-rest frame) universal, and the fluid to be studied merely provides the initial conditions.

Lastly, note that the covariant thermodynamic relations (2.17)–(2.19) still hold, and thus the entropy four-current is conserved, cf. Eq. (2.20). Analogous to Eqs. (2.36), its equation of motion reads

$$0 = \partial_\mu S^\mu = \dot{s} + s\theta\ . \tag{2.37}$$

At this point, it should be mentioned that, due to the nonlinearity of the Euler equations, a perfect fluid may develop *shock waves* (discontinuities). At these points, the energy-momentum tensor and the particle four-current are no longer continuously differentiable [102, 103]. The resulting values of $T^{\mu\nu}$ and N^μ on both sides of the discontinuity are then determined by the (relativistic) Rankine-Hugoniot conditions [104]. It can subsequently be shown that shock waves have to increase the entropy [102, 103], i.e., they constitute irreversible processes.

Nonrelativistic limit

We close this section by connecting with the nonrelativistic theory. In order to perform this limit, we have to decompose the fluid four-velocity into its time and space components $u^0 = \gamma$ and $\mathbf{u} = \gamma\mathbf{v}$, which behave as a scalar and a vector under GALILEI-transformations, respectively. In the nonrelativistic case, we have $|\mathbf{v}| =: v \ll 1$,[2] such that the fluid four-velocity reads

$$u^\mu \simeq (1,\, \mathbf{v})^\mu \ , \tag{2.38}$$

where we neglected terms of second order in v. Then, the expansion scalar takes the form

$$\theta \simeq \nabla \cdot \mathbf{v} \ . \tag{2.39}$$

The comoving derivative reads

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = u_\mu \partial^\mu \simeq \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \ , \tag{2.40}$$

while the spacelike gradient becomes

$$\nabla_\mu = \partial_\mu - u_\mu \frac{\mathrm{d}}{\mathrm{d}\tau} \simeq (0,\, \nabla)_\mu \ , \tag{2.41}$$

where we again neglected terms of second order in v. Using that, in the nonrelativistic limit, the particle-number density is directly related to the mass density ρ of the fluid made out of constituents of mass m, $mn = \rho$, and expressing the energy density as $\varepsilon \equiv \rho e$ with the specific internal energy e, we find from Eqs. (2.36)

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) \ , \tag{2.42a}$$

$$\frac{\partial e}{\partial t} + (\mathbf{v} \cdot \nabla)\, e = -\frac{P}{\rho} \nabla \cdot \mathbf{v} \ , \tag{2.42b}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} = -\frac{1}{\rho} \nabla P \ . \tag{2.42c}$$

Note that the zeroth component of Eq. (2.36c) became trivial and was consequently omitted from the system of equations. Manifestly, Eq. (2.42a) denotes a continuity equation for the mass flow and Eq. (2.42b) describes the evolution of the specific internal energy, while Eq. (2.42c) determines the evolution of the fluid velocity.

2.3 Dissipative hydrodynamics

As we have seen in Sec. 2.2, ideal fluids evolve adiabatically (except for shock waves), implying that all processes are reversible. However, any real system features some degree of degree of dissipation, which renders the evolution of the fluid irreversible. Giving up the premise of local thermodynamic equilibrium, we can no longer argue that the energy-momentum tensor is diagonal in the rest frame of the fluid, as there may be nonzero fluxes in energy or momentum. Of course, the conservation equations (2.13) are still valid, and constitute the basis of a theory of dissipative fluids as well. Considering a fluid whose constituents do not have spin, i.e., setting $S^{\lambda\mu\nu} = 0$, the conservation of total angular momentum (2.13c) enforces the energy-momentum tensor to be symmetric.

In order to proceed, we irreducibly decompose N^μ and $T^{\mu\nu}$ with respect to u^μ, which at this point merely is a normalized timelike four-vector that specifies a frame of reference. We obtain

$$N^\mu = n u^\mu + n^\mu \ , \tag{2.43a}$$

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + u^{(\mu} h^{\nu)} - P \Delta^{\mu\nu} + \pi^{\mu\nu} \ , \tag{2.43b}$$

where the round brackets denote symmetrization, $A^{(\mu} B^{\nu)} := A^\mu B^\nu + A^\nu B^\mu$. In addition, $n := u \cdot N$ is the particle-number density in the frame characterized by u^μ, while $n^\mu := \Delta^{\mu\nu} N_\nu$ denotes the

[2] The dimensionless quantity that controls the expansion around the nonrelativistic limit is of course v/c, where c is the speed of light in vacuum.

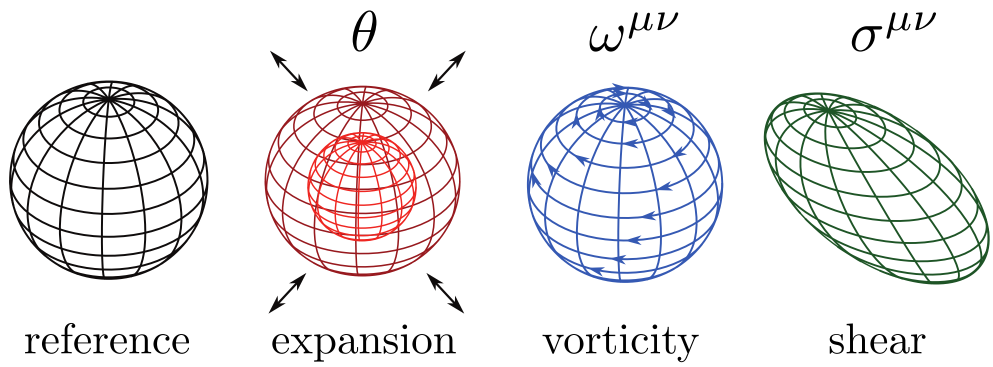

Figure 2.1: Sketch of the intuition behind the irreducible components of the derivative of the four-velocity $\nabla^\mu u^\nu$ (Figure adapted from Ref. [103], p.136).

particle diffusion in that frame. The energy density is still given by $\varepsilon := u_\mu u_\nu T^{\mu\nu}$, but can now be accompanied by an *energy flux* $h^\mu := \Delta^{\mu\nu} u^\alpha T_{\nu\alpha}$. Note that, due to the symmetry of $T^{\mu\nu}$, the energy flux and the momentum density are equal. The isotropic pressure is given by $P := -\frac{1}{3}\Delta_{\mu\nu}T^{\mu\nu}$, while the remaining part is called the *shear-stress tensor* $\pi^{\mu\nu} := \Delta^{\mu\nu}_{\alpha\beta}T^{\alpha\beta}$. Here, the traceless projector $\Delta^{\mu\nu}_{\alpha\beta} := \frac{1}{2}\Delta^{(\mu}_{\alpha}\Delta^{\nu)}_{\beta} - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}$ projects a rank-two tensor onto the subspace orthogonal to u^μ.

The divergences of the particle four-current and the energy-momentum tensor read

$$\partial_\mu N^\mu = \dot{n} + n\theta + \partial_\mu n^\mu \ , \tag{2.44a}$$

$$\begin{aligned}\partial_\mu T^{\mu\nu} = {}& \left[\dot{\varepsilon} + \theta\left(\varepsilon + P\right) + \partial_\mu h^\mu\right] u^\nu + \dot{h}^\nu + \frac{4}{3}\theta h^\nu \\ & + h_\mu(\sigma^{\mu\nu} + \omega^{\mu\nu}) + \left(\varepsilon + P\right)\dot{u}^\nu - \nabla^\nu P + \partial_\mu \pi^{\mu\nu} \ . \end{aligned}\tag{2.44b}$$

Here, we made use of the following irreducible decomposition of the derivative of the four-velocity u^μ,

$$\partial^\mu u^\nu = u^\mu \dot{u}^\nu + \frac{\theta}{3}\Delta^{\mu\nu} + \sigma^{\mu\nu} + \omega^{\mu\nu} \ , \tag{2.45}$$

where we defined the *shear tensor* $\sigma^{\mu\nu} := \Delta^{\mu\nu}_{\alpha\beta}\partial^\alpha u^\beta$ and the *vorticity* $\omega^{\mu\nu} := \frac{1}{2}\nabla^{[\mu}u^{\nu]}$. Intuitively, the expansion scalar θ describes a change of the volume of a fluid cell without any additional motion, while the vorticity describes a rotation at constant volume. The shear tensor can be envisioned as the change of the shape of a fluid cell without changing the volume. These interpretations are visualized in Fig. 2.1.

2.3.1 Hydrodynamic frames

It is important to remark that the variables ε, n, and P a priori do not fulfill the thermodynamic relations (2.6)–(2.9) since they are related to the nonequilibrium fluid. However, one can formally take the particle four-current and the energy-momentum tensor as sums of equilibrium and dissipative parts, and then interpret some of the quantities that appear in Eqs. (2.43) as equilibrium contributions. First, we write

$$N^\mu = N^\mu_0 + \delta N^\mu \ , \tag{2.46a}$$

$$T^{\mu\nu} = T^{\mu\nu}_0 + \delta T^{\mu\nu} \ , \tag{2.46b}$$

where the equilibrium contributions read

$$N_0^\mu := n_0 u^\mu \ , \qquad T_0^{\mu\nu} := \varepsilon_0 u^\mu u^\nu - P_0 \Delta^{\mu\nu} \ . \tag{2.47}$$

Note that the variables n_0, ε_0, and P_0 are the particle-number density, energy density, and pressure characterizing a *fictitious* equilibrium state, and they do fulfill the thermodynamic relations (2.6)–(2.9).[3] In particular, there exists an equation of state, such that the equilibrium pressure $P_0(n_0, \varepsilon_0)$ can be expressed in terms of the equilibrium particle-number density and energy density. Decomposing the deviations from equilibrium as

$$\delta N^\mu = u^\mu \delta n + n^\mu \ , \tag{2.48a}$$
$$\delta T^{\mu\nu} = u^\mu u^\nu \delta\varepsilon + u^{(\mu} h^{\nu)} - \Pi \Delta^{\mu\nu} + \pi^{\mu\nu} \ , \tag{2.48b}$$

we can identify

$$\varepsilon = \varepsilon_0 + \delta\varepsilon \ , \qquad n = n_0 + \delta n \ , \qquad P = P_0 + \Pi \ . \tag{2.49}$$

At this point, it is clear that we have introduced several new variables which are not defined unambiguously, as is evident by simply counting the degrees of freedom: N^μ has four independent components, while $T^{\mu\nu}$, being a symmetric second-rank tensor, has ten. However, we are dealing with five scalar quantities (ε_0, $\delta\varepsilon$, n_0, δn, and Π), three vectors (n^μ, h^μ, and u^μ, which have three independent components each), and one traceless symmetric tensor ($\pi^{\mu\nu}$, which has five independent components). In total, we thus are faced with 19 quantities whose evolution has to be specified, five of which have to be fixed by *defining* the fictitious equilibrium state characterized by n_0, ε_0, and u^μ. These defining relations are known as *matching conditions*, and a specific choice is called a *hydrodynamic frame*. Two hydrodynamic frames are ubiquitous in the literature because of their straightforward physical interpretation, namely the Eckart frame [105] and the Landau frame [102]. In both of these, the deviations of the particle-number and energy density from their equilibrium values are set to zero,

$$\varepsilon_0 := u_\mu u_\nu T^{\mu\nu} \quad \Longrightarrow \quad \delta\varepsilon = 0 \ , \tag{2.50a}$$
$$n_0 := u_\mu N^\mu \quad \Longrightarrow \quad \delta n = 0 \ , \tag{2.50b}$$

implying that n and ε behave as if they were in local thermal equilibrium. The four-velocity is defined in the Eckart frame via

$$N_E^\mu = n_0 u_E^\mu \quad \Longrightarrow \quad n_E^\mu = 0 \ , \tag{2.51}$$

i.e., the rest frame specified by the four-velocity u_E^μ is such that there is no particle diffusion. The price to pay for this definition consists in the energy diffusion not vanishing, $h_E^\mu \neq 0$. In contrast, the Landau frame defines the four-velocity through

$$T^{\mu\nu} u_{L,\nu} = \varepsilon_0 u_L^\mu \quad \Longrightarrow \quad h_L^\mu = 0 \ , \tag{2.52}$$

such that in the rest frame defined through u_L^μ there is no energy diffusion. However, in general there will be a nonvanishing particle diffusion, $n_L^\mu \neq 0$.

In the past years, more general frame choices have gained popularity due to their desirable properties regarding the causality and stability of the resulting system of equations [106–110]. However, a thorough discussion is outside the scope of this thesis, where we will choose the Landau frame.

Inserting Eqs. (2.49) into Eqs. (2.44) and projecting the equation of motion for the energy-momentum tensor with u_ν and $\Delta^\mu{}_\nu$, we find

$$0 = \dot{n}_0 + \delta\dot{n} + (n_0 + \delta n)\theta + \partial_\mu n^\mu \ , \tag{2.53a}$$
$$0 = \dot{\varepsilon}_0 + \delta\dot{\varepsilon} + (\varepsilon_0 + P_0 + \delta\varepsilon + \Pi)\,\theta + (\partial_\mu - \dot{u}_\mu)\,h^\mu - \pi^{\mu\nu}\sigma_{\mu\nu} \tag{2.53b}$$
$$\begin{aligned} 0 = {} & \Delta^{\mu\nu}\dot{h}_\nu + \frac{4}{3}\theta h^\mu + h_\nu(\sigma^{\nu\mu} + \omega^{\nu\mu}) \\ & + (\varepsilon_0 + P_0 + \delta\varepsilon + \Pi)\,\dot{u}^\mu - \nabla^\mu(P_0 + \Pi) + \Delta^{\mu\nu}\partial^\alpha \pi_{\nu\alpha} \ . \end{aligned} \tag{2.53c}$$

[3] We remark that neither the equilibrium nor the dissipative quantities are measurable by themselves, but only their sum, i.e., the components of the physical particle four-current and energy-momentum tensor.

In the LANDAU frame, the conservation equations read

$$\dot{n}_0 = -n_0\theta + \dot{u}_\mu n^\mu - \nabla_\mu n^\mu \ , \tag{2.54a}$$
$$\dot{\varepsilon}_0 = -\left(\varepsilon_0 + P_0 + \Pi\right)\theta + \pi^{\mu\nu}\sigma_{\mu\nu} \tag{2.54b}$$
$$\left(\varepsilon_0 + P_0 + \Pi\right)\dot{u}^\mu = \nabla^\mu(P_0 + \Pi) + \pi^{\mu\nu}\dot{u}_\nu - \Delta^{\mu\nu}\nabla^\alpha\pi_{\nu\alpha} \ . \tag{2.54c}$$

Here we decomposed the partial derivatives $\partial^\mu = u^\mu \frac{\mathrm{d}}{\mathrm{d}\tau} + \nabla^\mu$ and used that the particle diffusion and the shear-stress tensor are orthogonal to the fluid four-velocity.

It is straightforwardly seen that, even after enforcing matching conditions such as Eqs. (2.50)–(2.52), the system of equations (2.54) remains underdetermined. The reason is that we are still dealing with 14 dynamical quantities, while the conservation laws only provide five equations of motion, which determine ε_0, n_0, and u^μ. Intuitively, this underdetermination arises because dissipation results from complicated microscopic processes in the fluid, and thus it is a property of the specific system at hand, as opposed to the general conservation laws that govern ideal fluid dynamics. Thus, we always have to provide some microscopic input that specifies which kind of fluid we aim to describe. This microscopic input consists in additional equations that determine the dissipative quantities, i.e., Π, n^μ, and $\pi^{\mu\nu}$ in the LANDAU frame. The two most prominent approaches to provide these equations are the *gradient expansion*, which relates the dissipative currents to fluid gradients in algebraic equations, and the *MÜLLER-ISRAEL-STEWART (MIS)-type theories*, which keep the dissipative quantities dynamical and provide differential equations for them.

2.3.2 Gradient expansion

The basic idea of the gradient expansion lies in relating the dissipative quantities to derivatives of the fluid-dynamical quantities that characterize the fictitious local equilibrium state, i.e., $\{\varepsilon_0, n_0, u^\mu\}$, or equivalently $\{\alpha_0, \beta_0, u^\mu\}$, where α_0 and β_0 are defined through the equation of state and the thermodynamic relations (2.6)–(2.9). In the LANDAU frame, we find for the relevant dissipative quantities

$$\Pi = -\zeta\theta \ , \tag{2.55a}$$
$$n^\mu = \kappa I^\mu + \lambda J^\mu \ , \tag{2.55b}$$
$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} \ , \tag{2.55c}$$

where we defined $I^\mu := \nabla^\mu\alpha_0$, $J^\mu := \nabla^\mu\beta_0$, and collected terms on the right-hand side that contain only one derivative and fulfill the necessary symmetries. A theory that builds on relations like Eqs. (2.55), which are of first order in derivatives of equilibrium quantities, is commonly referred to as a *first-order theory*. Note that the coefficients ζ, κ, λ, and η are not fixed and have to be calculated from a microscopic approach that describes the fluid under consideration.

First-order entropy

Even though there are no universal values for the coefficients in Eqs. (2.55), we can constrain them to some extent via the second law of thermodynamics. First, note that our fictitious equilibrium state also features an entropy current

$$\begin{aligned} S_0^\mu &= T_0^{\mu\nu}\beta_0 u_\nu + P_0\beta_0 u^\mu - \alpha_0 N_0^\mu \\ &= \big[\underbrace{\beta_0\left(P_0 + \varepsilon_0\right) - \alpha_0 n_0}_{=:s_0}\big]u^\mu \ , \end{aligned} \tag{2.56}$$

cf. Eq. (2.19). Its divergence reads

$$\begin{aligned}\partial_\mu S_0^\mu &= \beta_0 u_\nu \partial_\mu T_0^{\mu\nu} - \alpha_0 \partial_\mu N_0^\mu \\ &= -\beta_0 u_\nu \partial_\mu \delta T^{\mu\nu} + \alpha_0 \partial_\mu \delta N^\mu \\ &= \beta_0 \left(\pi^{\mu\nu}\sigma_{\mu\nu} - \Pi\theta\right) + \alpha_0 \partial_\mu n^\mu \ , \end{aligned} \tag{2.57}$$

where we used the conservation of the total particle four-current and the energy-momentum tensor. Note that, since we are working in the LANDAU frame, it holds that $\delta\varepsilon = \delta n = 0$, $h^\mu = 0$. Defining the first-order correction to the entropy current [111]

$$S_1^\mu \coloneqq -\alpha_0 \delta N^\mu \equiv -\alpha_0 n^\mu \ , \tag{2.58}$$

we find

$$\begin{aligned}\partial_\mu \left(S_0^\mu + S_1^\mu\right) &= \beta_0 \left(\pi^{\mu\nu}\sigma_{\mu\nu} - \Pi\theta\right) - n^\mu I_\mu \\ &\equiv \beta_0 \left(2\eta\sigma^{\mu\nu}\sigma_{\mu\nu} + \zeta\theta^2\right) - \left(\kappa I^\mu + \lambda J^\mu\right) I_\mu \ , \end{aligned} \tag{2.59}$$

where we inserted the first-order gradient expansion (2.55). From the second law of thermodynamics, $\partial_\mu S^\mu \geq 0$, we then find that[4]

$$\lambda \stackrel{!}{=} 0 \ , \quad \zeta \stackrel{!}{\geq} 0 \ , \quad \kappa \stackrel{!}{\geq} 0 \ , \quad \eta \stackrel{!}{\geq} 0 \ . \tag{2.60}$$

Thus, the *bulk viscosity* ζ, the *thermal conductivity* κ, and the *shear viscosity* η should be nonnegative.

Inserting Eq. (2.55) with the constraints (2.60) into Eqs. (2.54), we arrive at the *relativistic NAVIER-STOKES equations*,

$$\dot{n}_0 = -n_0\theta + \kappa \dot{u}_\mu I^\mu - \nabla_\mu \left(\kappa I^\mu\right) \ , \tag{2.61a}$$

$$\dot{\varepsilon}_0 = -\left(\varepsilon_0 + P_0 - \zeta\theta\right)\theta + 2\eta\sigma^{\mu\nu}\sigma_{\mu\nu} \tag{2.61b}$$

$$\left(\varepsilon_0 + P_0 - \zeta\theta\right)\dot{u}^\mu = \nabla^\mu (P_0 - \zeta\theta) + 2\eta\sigma^{\mu\nu}\dot{u}_\nu - 2\Delta^{\mu\nu}\nabla^\alpha \left(\eta\sigma_{\nu\alpha}\right) \ . \tag{2.61c}$$

Unfortunately, Eqs. (2.61) are not usable in practice, as they constitute parabolic equations, which feature an infinite speed of signal propagation, thus being inconsistent with special relativity [103]. This is not merely a conceptual problem, as acausality in the rest frame of the fluid leads to (linear) instabilities in moving frames [29–32]. Going to higher orders in the gradient expansion does not fix this problem, either, as this procedure results in the (relativistic) BURNETT equations [112, 113], which are unstable even in the nonrelativistic regime [114]. At this point, we remark again that, if one chooses nonstandard matching conditions where the deviation from the equilibrium particle-number and energy density are kept, it is possible to formulate first-order theories that are causal and stable [106–110].

Nonrelativistic limit

To conclude this subsection, we show how to obtain the nonrelativistic limit of the conservation equations (2.61). In addition to the nonrelativistic limits introduced in Sec. 2.2, we need to express the shear tensor,

$$\sigma^{0\mu} \simeq 0 \ , \qquad \sigma^{ij} \simeq \Lambda^{ij} - \delta^{ij}\frac{1}{3}\nabla\cdot\mathbf{v} \ , \tag{2.62}$$

where we introduced $\Lambda^{ij} \coloneqq \frac{1}{2}(\partial^i v^j + \partial^j v^i)$. The nonrelativistic limit of the continuity equation (2.61a) reads[5]

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\left(\rho\mathbf{v} - \frac{m}{h}\mathbf{q}\right) \simeq -\nabla\cdot\left(\rho\mathbf{v} - \mathbf{q}\right) \ , \tag{2.63a}$$

[4] Since I^μ is spacelike, we have that $I^\mu I_\mu \leq 0$.

[5] The continuity equation acquires a contribution from the heat current due to the choice of the LANDAU frame [115]. In the ECKART frame, the continuity equation would be unchanged compared to the ideal case, whereas Eqs. (2.63b) and (2.63c) would receive corrections involving the heat flux, which however vanish in the nonrelativistic limit [103].

where we neglected terms of second order in $\mathbf{v}$. Moreover, we identified[6] the heat flux $\mathbf{q} \equiv -h\mathbf{n} = h\kappa\mathbf{I}$, where $h := (\varepsilon_0 + P_0)/n_0$ denotes the enthalpy per particle, which in the nonrelativistic limit is dominated by the rest-mass energy, $h \simeq m$. With these considerations, the energy-conservation equation (2.61b) takes the form

$$\frac{\partial e}{\partial t} + (\mathbf{v}\cdot\nabla)\, e = -\frac{P_0}{\rho}\nabla\cdot\mathbf{v} - \frac{1}{\rho}\nabla\cdot\mathbf{q} + \frac{2\eta}{\rho}\Lambda_{ij}\Lambda^{ij} + \frac{1}{\rho}\left(\zeta - \frac{2}{3}\eta\right)(\nabla\cdot\mathbf{v})^2 \,. \tag{2.63b}$$

Lastly, neglecting terms of second order in v, the evolution equation for the fluid velocity (2.61c) becomes

$$\frac{\partial v^i}{\partial t} + (\mathbf{v}\cdot\nabla)\, v^i = -\frac{1}{\rho}\left\{\partial_i P_0 - \partial_j\left[\eta\left(\partial_i v^j + \partial_j v^i - \frac{2}{3}\delta^i_j \nabla\cdot\mathbf{v}\right) + \zeta\delta^i_j\nabla\cdot\mathbf{v}\right]\right\} \,, \tag{2.63c}$$

which is called the nonrelativistic Navier-Stokes equation. Equations (2.63), together with an equation of state and appropriate choices for the viscosities and conductivities ζ, η, and κ, completely determine the evolution of a nonrelativistic fluid and have wide-ranging practical applications.

2.3.3 MIS-type theories

As mentioned in the previous subsection, the relativistic Navier-Stokes equations are not used in practice, since they feature an infinite speed of signal propagation, which leads to instabilities in the linearized theory when observed in the frame of a (with respect to the system) moving observer [29–32]. A straightforward way to amend this issue is to make the equations hyperbolic by introducing *relaxation-type* equations for the dissipative currents, i.e.,

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta + \cdots \,, \tag{2.64a}$$

$$\tau_n \dot{n}^{\langle\mu\rangle} + n^\mu = \kappa I^\mu + \cdots \,, \tag{2.64b}$$

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \cdots \,, \tag{2.64c}$$

where the dots symbolize terms of higher order in dissipative quantities and fluid gradients, and τ_Π, τ_n, and τ_π denote the characteristic relaxation timescales of Π, n^μ, and $\pi^{\mu\nu}$, respectively. Furthermore, we introduced the notation

$$\dot{n}^{\langle\mu\rangle} := \Delta^{\mu\nu}\dot{n}_\nu \,, \qquad \dot{\pi}^{\langle\mu\nu\rangle} := \Delta^{\mu\nu}_{\alpha\beta}\dot{\pi}^{\alpha\beta} \,, \tag{2.65}$$

where the projector is needed to remove the components of the equations (2.64) which are not independent. The crucial difference between Eqs. (2.64), which are said to be of MIS type, and (2.55), is that in the former the dissipative quantities are kept as independent dynamical degrees of freedom satisfying differential equations, whereas the Navier-Stokes relations (2.55) express them through gradients of fluid-dynamical quantities in an algebraic way. Note that Eqs. (2.64) relax to their respective Navier-Stokes values at asymptotically long times. While such a formulation can render the equations causal and stable [31], it introduces several ambiguities. Besides the values of the relaxation times τ_Π, τ_n, and τ_π, the higher-order terms on the right-hand sides of Eqs. (2.64) have to be specified. In order to find the most general form of these terms, we first have to introduce small parameters that control the expansion. The first dimensionless quantity is the so-called *Knudsen number* Kn, which is defined as

$$\mathrm{Kn} := \frac{\ell_{\mathrm{micro}}}{L_{\mathrm{hydro}}} \,, \tag{2.66}$$

where ℓ_{micro} denotes a microscopic scale, such as, e.g., the mean free path of the particles that constitute the fluid, while L_{hydro} stands for a macroscopic hydrodynamic length scale, which we can associate with the distance over which macroscopic quantities vary, $\partial \sim L^{-1}_{\mathrm{hydro}}$. This parameter quantifies how large the separation between microscopic and macroscopic scales is. The second quantity that

[6] This identification can be seen from an argumentation in §136 of Ref. [102]. Pure heat conduction takes place when the total particle three-current vanishes, i.e., when $\mathbf{n} = -\mathbf{v}/n_0$. The heat flux then is given by $q^i = T^{0i} = (\varepsilon_0 + P_0)u^i = -hn^i$.

controls the quality of the expansion is the *inverse* REYNOLDS *number* Re^{-1}, which is defined as the ratio of a dissipative quantity and its equilibrium value and thus quantifies how close the system is to equilibrium. Since there are several dissipative quantities, each of them has an associated inverse REYNOLDS number,

$$\mathrm{Re}_{\Pi}^{-1} := \frac{|\Pi|}{P_0} , \quad \mathrm{Re}_{n}^{-1} := \frac{\sqrt{-n_\mu n^\mu}}{\beta_0 P_0} , \quad \mathrm{Re}_{\pi}^{-1} := \frac{\sqrt{\pi_{\mu\nu}\pi^{\mu\nu}}}{P_0} . \tag{2.67}$$

Taking into account all dissipative currents and fluid gradients at our disposal, the most general relaxation-type equations up to second order in Kn and Re^{-1} read [51]

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta + \mathcal{J} + \mathcal{K} + \mathcal{R} , \tag{2.68a}$$

$$\tau_n \dot{n}^{\langle\mu\rangle} + n^\mu = \kappa I^\mu + \mathcal{J}^\mu + \mathcal{K}^\mu + \mathcal{R}^\mu , \tag{2.68b}$$

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{K}^{\mu\nu} + \mathcal{R}^{\mu\nu} . \tag{2.68c}$$

Here, the terms

$$\mathcal{J} := -\ell_{\Pi n}\nabla\cdot n - \tau_{\Pi n} n\cdot F - \delta_{\Pi\Pi}\Pi\theta - \lambda_{\Pi n} n\cdot I + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu} , \tag{2.69a}$$

$$\begin{aligned}\mathcal{J}^\mu := &-\tau_n n_\nu \omega^{\nu\mu} - \delta_{nn} n^\mu\theta - \ell_{n\Pi}\nabla^\mu\Pi + \ell_{n\pi}\Delta^{\mu\nu}\nabla_\lambda \pi^\lambda{}_\nu + \tau_{n\Pi}\Pi F^\mu - \tau_{n\pi}\pi^{\mu\nu}F_\nu \\ &- \lambda_{nn} n_\nu \sigma^{\mu\nu} + \lambda_{n\Pi}\Pi I^\mu - \lambda_{n\pi}\pi^{\mu\nu} I_\nu ,\end{aligned} \tag{2.69b}$$

$$\begin{aligned}\mathcal{J}^{\mu\nu} := &\, 2\tau_\pi \pi_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} \\ &- \tau_{\pi n} n^{\langle\mu}F^{\nu\rangle} + \ell_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle} + \lambda_{\pi n} n^{\langle\mu}I^{\nu\rangle} ,\end{aligned} \tag{2.69c}$$

denote contributions of first order in both Kn and Re^{-1}, and we defined $F^\mu := \nabla^\mu P_0$. Similarly, we introduced

$$\mathcal{K} := \tilde{\zeta}_1 \omega_{\mu\nu}\omega^{\mu\nu} + \tilde{\zeta}_2 \sigma_{\mu\nu}\sigma^{\mu\nu} + \tilde{\zeta}_3\theta^2 + \tilde{\zeta}_4 I\cdot I + \tilde{\zeta}_5 F\cdot F + \tilde{\zeta}_6 I\cdot F + \tilde{\zeta}_7 \nabla\cdot I + \tilde{\zeta}_8 \nabla\cdot F , \tag{2.70a}$$

$$\mathcal{K}^\mu := \tilde{\kappa}_1 \sigma^{\mu\nu}I_\nu + \tilde{\kappa}_2\sigma^{\mu\nu}F_\nu + \tilde{\kappa}_3 I^\mu\theta + \tilde{\kappa}_4 F^\mu\theta + \tilde{\kappa}_5\omega^{\mu\nu}I_\nu + \tilde{\kappa}_6\Delta^\mu_\lambda\nabla_\nu\sigma^{\lambda\nu} + \tilde{\kappa}_7\nabla^\mu\theta , \tag{2.70b}$$

$$\begin{aligned}\mathcal{K}^{\mu\nu} := &\,\tilde{\eta}_1\omega^{\lambda\langle\mu}\omega^{\nu\rangle}{}_\lambda + \tilde{\eta}_2\theta\sigma^{\mu\nu} + \tilde{\eta}_3\sigma^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} + \tilde{\eta}_4\sigma_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} + \tilde{\eta}_5 I^{\langle\mu}I^{\nu\rangle} \\ &+ \tilde{\eta}_6 F^{\langle\mu}F^{\nu\rangle} + \tilde{\eta}_7 I^{\langle\mu}F^{\nu\rangle} + \tilde{\eta}_8\nabla^{\langle\mu}I^{\nu\rangle} + \tilde{\eta}_9\nabla^{\langle\mu}F^{\nu\rangle} ,\end{aligned} \tag{2.70c}$$

to collect the terms of second order in the KNUDSEN number.[7] Lastly, the quantities

$$\mathcal{R} := \varphi_1\Pi^2 + \varphi_2 n\cdot n + \varphi_3\pi^{\mu\nu}\pi_{\mu\nu} , \tag{2.71a}$$

$$\mathcal{R}^\mu := \varphi_4\pi^{\mu\nu}n_\nu + \varphi_5\Pi n^\mu , \tag{2.71b}$$

$$\mathcal{R}^{\mu\nu} := \varphi_6\Pi\pi^{\mu\nu} + \varphi_7\pi^{\lambda\langle\mu}\pi^{\nu\rangle}{}_\lambda + \varphi_8 n^{\langle\mu}n^{\nu\rangle} , \tag{2.71c}$$

are of second order in the inverse REYNOLDS number. The prefactors appearing in the equations above are the so-called second-order transport coefficients of the system, which have to be computed from a microscopic theory. The main objective of this thesis is to obtain Eqs. (2.68) as well as their spin-analogues (cf. Sec. 2.4) from a quantum field-theoretical starting point.

To close this section, we remark that another way to approach the task of finding the second-order contributions to Eqs. (2.64) without resorting to a microscopic theory is to extend the analysis done in Subsec. 2.3.2 to second order by considering a higher-order phenomenological entropy current [116, 117]. This type of analysis yields relaxation-type equations as indicated in Eqs. (2.64), but does not include all possible terms that are allowed by symmetry. Furthermore, it does not provide explicit values for the transport coefficients.

2.4 Hydrodynamics with spin

When talking about fluids whose constituent particles have a nonvanishing spin, we have to take a step back and consider the conservation of the total angular momentum (2.13c) in more detail. While for a

[7] These contributions are in principle problematic, as they render the equations of motion parabolic [51]. In Sec. 6 we will show that it is possible to set these terms to zero [50].

fluid consisting of spinless particles the conservation of the total angular momentum simply enforces the symmetry of the energy-momentum tensor, for fluids consisting of fermions or bosons with nonzero spin the spin tensor is in principle an independent quantity that fulfills the equation of motion

$$\hbar\partial_\lambda S^{\lambda\mu\nu} = T^{[\nu\mu]} \; . \tag{2.72}$$

Note that, as we noticed in the case of fluid dynamics without spin, the system is severely underdetermined: The spin tensor features $4 \times 6 = 24$ degrees of freedom, since it is antisymmetric in the last two indices, while the antisymmetric part of the energy-momentum tensor adds six more components. In contrast, Eq. (2.72) determines six components of the spin tensor, which have to characterize the ideal case. Collecting these six independent components in an antisymmetric second-rank tensor $\Omega_0^{\mu\nu}$, we may decompose it with respect to the fluid four-velocity as

$$\Omega_0^{\mu\nu} = u^{[\mu}\kappa_0^{\nu]} + \epsilon^{\mu\nu\alpha\beta}u_\alpha\omega_{0,\beta} \; , \tag{2.73}$$

which can be inverted to yield

$$\kappa_0^\mu = -\Omega_0^{\mu\nu}u_\nu \; , \qquad \omega_0^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu\Omega_{0,\alpha\beta} \; . \tag{2.74}$$

The quantity $\Omega_0^{\mu\nu}$, which will reappear in Sec. 4.6, is called the *spin potential*, with κ_0^μ and ω_0^μ being its electric- and magnetic-like parts, respectively.

Using the vector and tensor structures at our disposal, i.e., u^μ, $g^{\mu\nu}$, and $\Omega_0^{\mu\nu}$, the spin tensor in the ideal case can be decomposed as [38][8]

$$\begin{aligned} S_0^{\lambda\mu\nu} &= A_0u^\lambda\Omega_0^{\mu\nu} + B_0u^\lambda u_\alpha\Omega_0^{\alpha[\mu}u^{\nu]} + C_0u^\lambda\Omega_0^{\alpha[\mu}\Delta^{\nu]}{}_\alpha + D_0u_\alpha\Omega_0^{\alpha[\mu}\Delta^{\nu]\lambda} + E_0\Delta^\lambda{}_\alpha\Omega_0^{\alpha[\mu}u^{\nu]} \\ &= (A_0 - B_0 - C_0)\,u^\lambda u^{[\mu}\kappa_0^{\nu]} + (A_0 - 2C_0)\,u^\lambda\epsilon^{\mu\nu\alpha\beta}u_\alpha\omega_{0,\beta} \\ &\quad + D_0\kappa_0^{[\mu}\Delta^{\nu]\lambda} + E_0u^{[\mu}\epsilon^{\nu]\lambda\alpha\beta}u_\alpha\omega_{0,\beta} \; , \end{aligned} \tag{2.75}$$

where the coefficients $A_0, \cdots, E_0$ are functions of the temperature and the chemical potential, thus not adding additional degrees of freedom. The divergence of the equilibrium part of the spin tensor then reads

$$\begin{aligned} \partial_\lambda S_0^{\lambda\mu\nu} &= \left(\dot{u}^{[\mu}\kappa_0^{\nu]} + u^{[\mu}\dot{\kappa}_0^{\nu]} + \theta u^{[\mu}\kappa_0^{\nu]}\right)(A_0 - B_0 - C_0) + u^{[\mu}\kappa_0^{\nu]}(\dot{A}_0 - \dot{B}_0 - \dot{C}_0) \\ &\quad + \epsilon^{\mu\nu\alpha\beta}\left[(\dot{u}_\alpha\omega_{0,\beta} + u_\alpha\dot{\omega}_{0,\beta} + \theta u_\alpha\omega_{0,\beta})(A_0 - 2C_0) + u_\alpha\omega_{0,\beta}(\dot{A}_0 - 2\dot{C}_0)\right] \\ &\quad + D_0\left(\nabla^{[\nu}\kappa_0^{\mu]} - \kappa_0^{[\mu}u^{\nu]}\theta - \kappa_0^{[\mu}\dot{u}^{\nu]}\right) + \kappa_0^{[\mu}\nabla^{\nu]}D_0 + E_0\left(\nabla_\lambda u^{[\mu}\right)\epsilon^{\nu]\lambda\alpha\beta}u_\alpha\omega_{0,\beta} \\ &\quad + u^{[\mu}\epsilon^{\nu]\lambda\alpha\beta}\left[(u_\lambda\dot{u}_\alpha\omega_{0,\beta} + u_\alpha\nabla_\lambda\omega_{0,\beta})\,E_0 + u_\alpha\omega_{0,\beta}\nabla_\lambda E_0\right] \; . \end{aligned} \tag{2.76}$$

The dissipative parts of the spin tensor can be included through

$$S^{\lambda\mu\nu} = S_0^{\lambda\mu\nu} + \delta S^{\lambda\mu\nu} \; . \tag{2.77}$$

As in the case of the energy-momentum tensor and the particle four-current, equilibrium variables (i.e., $\Omega_0^{\mu\nu}$) require a choice of hydrodynamic frame in the case where dissipation is present. In this thesis, we choose the spin analogue of the LANDAU frame [82], i.e., we demand that

$$u_\lambda\delta S^{\lambda\mu\nu} = 0 \; , \tag{2.78}$$

such that we can rewrite

$$\partial_\lambda\delta S^{\lambda\mu\nu} = -\dot{u}_\lambda\delta S^{\lambda\mu\nu} + \nabla_\lambda S^{\lambda\mu\nu} \; . \tag{2.79}$$

[8] In principle there could also be a term $\sim g^{\lambda[\mu}u^{\nu]}$, cf. Ref. [118]. We did not consider it here because it does not depend on the spin potential and will not emerge from the microscopic theory, as shown in Subsec. 6.2.2.

Projecting Eq. (2.72) with u_μ and $\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}u^\beta$, we obtain the equations of motion for the components of the spin potential,

$$\begin{aligned}
\hbar\dot{\kappa}_0^{\langle\mu\rangle} = &-\frac{\hbar}{A_0 - B_0 - C_0}\Big[\kappa_0^\mu(\dot{A}_0 - \dot{B}_0 - \dot{C}_0) - \epsilon^{\mu\nu\alpha\beta}u_\nu\dot{u}_\alpha\omega_{0,\beta}(A_0 - 2C_0 - E_0) \\
&- D_0\left(\sigma^{\mu\nu} + \omega^{\mu\nu}\right)\kappa_{0,\nu} + \left(A_0 - B_0 - C_0 + \frac{2}{3}D_0\right)\theta\kappa_0^\mu \\
&- \epsilon^{\mu\nu\alpha\beta}u_\nu\left(E_0\nabla_\alpha\omega_{0,\beta} + \omega_{0,\beta}\nabla_\alpha E_0\right) + u_\nu\left(\dot{u}_\lambda - \nabla_\lambda\right)\delta S^{\lambda\mu\nu} - \frac{1}{\hbar}T^{[\mu\nu]}u_\nu\Big] ,
\end{aligned} \tag{2.80a}$$

$$\begin{aligned}
\hbar\dot{\omega}_0^{\langle\mu\rangle} = &-\frac{\hbar}{A_0 - 2C_0}\Big[(\dot{A}_0 - 2\dot{C}_0)\omega_0^\mu + \epsilon^{\mu\nu\alpha\beta}u_\nu\dot{u}_\alpha\kappa_{0,\beta}(A_0 - B_0 - C_0 + D_0) \\
&+ E_0(\sigma^{\mu\nu} + \omega^{\mu\nu})\omega_{0,\nu} + \left(A_0 - 2C_0 - \frac{2}{3}E_0\right)\theta\omega_0^\mu - \epsilon^{\mu\nu\alpha\beta}u_\nu\left(D_0\nabla_\alpha\kappa_{0,\beta} + \kappa_{0,\beta}\nabla_\alpha D_0\right) \\
&+ \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu\left(-\dot{u}^\lambda + \nabla^\lambda\right)\delta S_{\lambda\alpha\beta} + \frac{1}{\hbar}\epsilon^{\mu\nu\alpha\beta}u_\nu T_{\alpha\beta}\Big] ,
\end{aligned} \tag{2.80b}$$

where we used Eq. (2.76). When the dissipative parts are set to zero, Eqs. (2.80a) and (2.80b) determine ideal spin hydrodynamics and have been found to exhibit wavelike behavior in the linear regime around a nonrotating flow [119]. While it is possible to decompose the dissipative contribution to the spin tensor in a general way [120], we will not do so in this chapter. However, employing the formalism presented in Chapter 3, we will be able to determine the equations of motion of the different components of $\delta S^{\lambda\mu\nu}$. We remark that, while still an active area of research, in principle the same points apply that were put forward in Sec. 2.3 concerning the equations of motion for the dissipative degrees of freedom. In particular, when trying to apply Navier-Stokes-type relations to the components of $\delta S^{\lambda\mu\nu}$, Refs. [121, 122] suggest that these theories are plagued with similar instabilities as the standard relativistic Navier-Stokes theories. Thus, a possible way to obtain a causal and stable theory consists in deriving relaxation-type equations for the dissipative components of the spin tensor, which we will do in Chapter 6.

Chapter 3

Field theory and phase space

The thermodynamic and hydrodynamic formulations treated in the last chapter are all built on fundamental conserved quantities. Since we will try to derive the macroscopic behavior of a given system from the bottom up in the following chapters, we have to ask how these conservation laws arise when considering a microscopic field theory. To begin this endeavor, we recapitulate how the basic conserved currents introduced in Subsec. 2.1.1 are connected to fundamental spacetime symmetries.

3.1 Conservation laws

The basic transformations in spacetime are characterized by the POINCARÉ group, which is a semidirect product of the translation group in four-dimensional MINKOWSKI space $\mathbb{R}^{1,3}$ and the LORENTZ group $O(1,3)$. The translation group contains the (finite) translations in space and time, i.e.,

$$x^\mu \overset{\mathbb{R}^{1,3}}{\longrightarrow} x'^\mu = x^\mu + a^\mu \ , \tag{3.1a}$$

with some four-vector a^μ. The LORENTZ group, on the other hand, describes transformations that connect inertial systems which are rotated or moving uniformly relative to each other,

$$x^\mu \overset{O(1,3)}{\longrightarrow} x'^\mu = \Lambda^\mu{}_\nu x^\nu \ , \tag{3.1b}$$

where $\Lambda^\mu{}_\nu$ denotes a general LORENTZ transformation. As a special case, a pure boost $L^\mu{}_\nu$ describes the transformation between systems that are uniformly moving with velocity $\mathbf{v}$ with respect to each other and reads

$$L^\mu{}_\nu \coloneqq \begin{pmatrix} \gamma & -\gamma\mathbf{v}^{\mathrm{T}} \\ -\gamma\mathbf{v} & \mathbb{1} + (\gamma-1)\frac{\mathbf{v}\otimes\mathbf{v}^{\mathrm{T}}}{v^2} \end{pmatrix}^\mu{}_\nu \tag{3.2}$$

where we defined $\gamma \coloneqq 1/\sqrt{1-v^2}$ with the magnitude of the three-velocity $v \coloneqq |\mathbf{v}|$, and $\mathbb{1}$ denotes the three-dimensional unit matrix. On the other hand, a pure rotation $R^\mu{}_\nu$ connects inertial systems that are rotated against each other and takes the form

$$R^\mu{}_\nu \coloneqq \begin{pmatrix} 1 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & \mathbf{R} \end{pmatrix}^\mu{}_\nu \ , \quad \text{with } \mathbf{R} \in SO(3) \ , \tag{3.3}$$

with $\mathbf{0} \coloneqq (0,0,0)^{\mathrm{T}}$. While the set of rotations forms a subgroup of $O(1,3)$ [namely the group $SO(3)$], the boosts do not. Given that the POINCARÉ group is a LIE group, its elements can be expressed via the exponential map, i.e., we have for a general field $\varphi(x)$ that transforms in some representation of the group,

$$\varphi(x) \overset{\mathbb{R}^{1,3}\rtimes O(1,3)}{\longrightarrow} \varphi'(x') = \left\{ \left[\exp\left(ia_\alpha \widehat{P}^\alpha\right) \exp\left(\frac{i}{2}\omega_{\alpha\beta}\widehat{M}^{\alpha\beta}\right)\right] \varphi \right\} (\Lambda^\mu{}_\nu x^\nu + a^\mu) \ , \tag{3.4}$$

where $\widehat{P}$ is the *generator* of translations, while $\widehat{M}$ generates LORENTZ transformations.[1] Note that the ten parameters $\{a_\mu,\, \omega_{\mu\nu}\}$, where $\omega_{\mu\nu} = -\omega_{\nu\mu}$, fully specify the transformation. The commutation relations for the generators read [123]

$$\left[\widehat{P}^\mu, \widehat{P}^\nu\right] = 0\;, \tag{3.5a}$$

$$\left[\widehat{M}^{\mu\nu}, \widehat{P}^\alpha\right] = i\left(g^{\mu\alpha}\widehat{P}^\nu - g^{\nu\alpha}\widehat{P}^\mu\right)\;, \tag{3.5b}$$

$$\left[\widehat{M}^{\mu\nu}, \widehat{M}^{\alpha\beta}\right] = i\left(g^{\mu\alpha}\widehat{M}^{\nu\beta} - g^{\nu\alpha}\widehat{M}^{\mu\beta} - g^{\mu\beta}\widehat{M}^{\nu\alpha} + g^{\nu\beta}\widehat{M}^{\mu\alpha}\right)\;, \tag{3.5c}$$

where square brackets denote the commutator.

NOETHER's (first) theorem tells us that for every continuous symmetry there exists a corresponding conserved current. The symmetry group $\mathscr{G}$ can be characterized by functions Λ and Ω that describe the behavior of coordinates and fields under infinitesimal transformations, respectively, i.e.,

$$x^\mu \xrightarrow{\mathscr{G}} x^\mu + \Lambda^\mu_i(x)\delta\omega^i\;, \tag{3.6a}$$

$$\varphi(x) \xrightarrow{\mathscr{G}} \varphi(x) + \Omega_i(x)\delta\omega^i\;, \tag{3.6b}$$

where $\delta\omega^i$ are the parameters of the respective transformation.

Explicitly, the conserved current $\mathscr{J}$ for a theory of a field φ described by some Lagrangian $\mathcal{L}$ reads [123]

$$\mathscr{J}^\mu_i = \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi\right)}\Omega_i - T^\mu{}_\nu\Lambda^\nu_i\;, \tag{3.7}$$

where we defined the (canonical) energy-momentum tensor

$$T^{\mu\nu} := \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi\right)}\partial^\nu\varphi - g^{\mu\nu}\mathcal{L}\;. \tag{3.8}$$

In Eq. (3.7), the index "i" assumes different forms depending on the generators of the symmetry group. Note also that the field $\varphi(x)$, if it transforms in a nontrivial representation of the LORENTZ group, will have multiple components, which are implicitly summed over in Eqs. (3.7) and (3.8).

Since the POINCARÉ group characterizes the transformations between inertial systems, a sensible relativistic theory should feature it as one of its symmetry groups. In consequence, two conserved currents arise immediately, namely the energy-momentum tensor $T^{\mu\nu}$ as well as the total angular momentum tensor $J^{\lambda\mu\nu}$. The former is a consequence of the invariance under the translation group $\mathbb{R}^{1,3}$, while the latter stems from the invariance under the restricted LORENTZ group $SO^+(1,3)$. Furthermore, if the fields are electrically charged, the action of the theory features a global $U(1)$ symmetry, leading to a conserved electric current, which, given that we are dealing with a single particle species, we may associate with a four-current N^μ which characterizes the difference between particles and antiparticles. In the following, we will call N^μ simply the particle four-current.

In this and the following chapters, since we want to start from a *quantum* field theory, we are going to deal with *operator-valued* fields $\widehat{\varphi}$, which will result in the conserved currents also becoming operators. The quantities that can be related to experiment are then given by the respective (normal-ordered) averages, i.e.,

$$N^\mu \equiv \left\langle :\widehat{N}^\mu:\right\rangle\;,\quad T^{\mu\nu} \equiv \left\langle :\widehat{T}^{\mu\nu}:\right\rangle\;,\quad J^{\lambda\mu\nu} \equiv \left\langle :\widehat{J}^{\lambda\mu\nu}:\right\rangle\;. \tag{3.9}$$

[1]The LORENTZ group also contains the discrete parity and time-reversal transformations, which we will for now ignore, i.e., we consider the proper orthochronous (or restricted) LORENTZ group $SO^+(1,3)$ that is the connected component of $O(1,3)$ containing the identity.

3.1.1 Pseudogauge transformations

Even though NOETHER's theorem allows to compute conserved currents from the Lagrangian, they are not fixed uniquely. To see this, consider the following transformation,

$$\widehat{S}^{\lambda\mu\nu} \longrightarrow \widehat{S}^{\lambda\mu\nu}_{\text{pgt}} := \widehat{S}^{\lambda\mu\nu} - \widehat{\Phi}^{\lambda\mu\nu} + \hbar\partial_\rho \widehat{Z}^{\mu\nu\lambda\rho} \;, \tag{3.10a}$$

$$\widehat{T}^{\mu\nu} \longrightarrow \widehat{T}^{\mu\nu}_{\text{pgt}} := \widehat{T}^{\mu\nu} + \frac{\hbar}{2}\partial_\lambda \left(\widehat{\Phi}^{\lambda\mu\nu} + \widehat{\Phi}^{\nu\mu\lambda} + \widehat{\Phi}^{\mu\nu\lambda}\right) \;, \tag{3.10b}$$

where $\widehat{\Phi}$ is antisymmetric in the last two indices, $\widehat{\Phi}^{\lambda\mu\nu} = -\widehat{\Phi}^{\lambda\nu\mu}$, and $\widehat{Z}$ is antisymmetric in the first and last pair of indices, $\widehat{Z}^{\mu\nu\lambda\rho} = -\widehat{Z}^{\nu\mu\lambda\rho} = -\widehat{Z}^{\mu\nu\rho\lambda}$. Since we have

$$\partial_\mu\partial_\lambda \left(\widehat{\Phi}^{\lambda\mu\nu} + \widehat{\Phi}^{\nu\mu\lambda} + \widehat{\Phi}^{\mu\nu\lambda}\right) = 0 \;, \tag{3.11}$$

the new energy-momentum tensor is also conserved,

$$\partial_\mu \widehat{T}^{\mu\nu}_{\text{pgt}} = 0 \;. \tag{3.12}$$

Furthermore, due to the relation

$$\partial_\lambda\partial_\rho \widehat{Z}^{\mu\nu\lambda\rho} = 0 \tag{3.13}$$

in conjunction with

$$\widehat{T}^{[\mu\nu]}_{\text{pgt}} = \widehat{T}^{[\mu\nu]} + \hbar\partial_\lambda \widehat{\Phi}^{\lambda\mu\nu} \;, \tag{3.14}$$

the equation of motion for the spin tensor stays unchanged as well,

$$\hbar\partial_\lambda \widehat{S}^{\lambda\mu\nu}_{\text{pgt}} = \widehat{T}^{[\nu\mu]}_{\text{pgt}} \;. \tag{3.15}$$

Finally, the conserved charges are left invariant under the transformation (3.10) as long as boundary terms can be neglected. Denoting the unit vector on the boundary of Σ as $\hat{t}$, this becomes immediately clear for the energy-momentum tensor,

$$\begin{aligned}
\widehat{\mathcal{P}}^\mu_{\text{pgt}} &= \int \mathrm{d}\Sigma_\lambda \widehat{T}^{\lambda\mu}_{\text{pgt}} \\
&= \int_\Sigma \mathrm{d}\Sigma_\lambda \widehat{T}^{\lambda\mu} + \frac{\hbar}{2}\oint_{\partial\Sigma} \mathrm{d}\hat{t}_{\lambda,\nu} \left(\widehat{\Phi}^{\nu\lambda\mu} + \widehat{\Phi}^{\mu\lambda\nu} + \widehat{\Phi}^{\lambda\mu\nu}\right) \\
&\equiv \widehat{\mathcal{P}}^\mu \;.
\end{aligned} \tag{3.16}$$

In the case of the total angular momentum tensor, we first compute

$$\begin{aligned}
\widehat{J}^{\lambda\mu\nu}_{\text{pgt}} &= \widehat{T}^{\lambda[\nu}x^{\mu]} + \frac{\hbar}{2}\left[\partial_\alpha \left(\widehat{\Phi}^{\alpha\lambda\nu} + \widehat{\Phi}^{\lambda\nu\alpha} + \widehat{\Phi}^{\nu\lambda\alpha}\right)x^\mu - \partial_\alpha \left(\widehat{\Phi}^{\alpha\lambda\mu} + \widehat{\Phi}^{\lambda\mu\alpha} + \widehat{\Phi}^{\mu\lambda\alpha}\right)x^\nu\right] \\
&\quad + \hbar\widehat{S}^{\lambda\mu\nu} - \hbar\widehat{\Phi}^{\lambda\mu\nu} \\
&= \widehat{J}^{\lambda\mu\nu} + \frac{\hbar}{2}\partial_\alpha \left(\widehat{\Phi}^{\alpha\lambda[\nu}x^{\mu]} - \widehat{\Phi}^{\lambda\alpha[\nu}x^{\mu]} + x^{[\mu}\widehat{\Phi}^{\nu]\lambda\alpha}\right) \;,
\end{aligned} \tag{3.17}$$

from which we obtain

$$\begin{aligned}
\widehat{\mathcal{J}}^{\mu\nu}_{\text{pgt}} &= \int \mathrm{d}\Sigma_\lambda \widehat{J}^{\lambda\mu\nu}_{\text{pgt}} \\
&= \int_\Sigma \mathrm{d}\Sigma_\lambda \widehat{J}^{\lambda\mu\nu} + \frac{\hbar}{2}\oint_{\partial\Sigma} \mathrm{d}\hat{t}_{\lambda,\alpha} \left(\widehat{\Phi}^{\alpha\lambda[\nu}x^{\mu]} - \widehat{\Phi}^{\lambda\alpha[\nu}x^{\mu]} + x^{[\mu}\widehat{\Phi}^{\nu]\lambda\alpha}\right) \\
&\equiv \widehat{\mathcal{J}}^{\mu\nu} \;.
\end{aligned} \tag{3.18}$$

The transformation (3.10) is called a *pseudogauge transformation* [34] and describes an ambiguity in the definition of the conserved currents. At this point, since the total charges and the equations of motion for the transformed quantities are left invariant, it seems that there is no reason for observables to depend on the pseudogauge. However, as we will see in Chapter 6, a truncation has to be made in order to obtain dissipative hydrodynamics from a microscopic theory. At that point, the chosen pseudogauge might influence the truncation and the resulting theory thus becomes pseudogauge-dependent. This issue will reappear in Subsec. 6.3.4.

In the remainder of this chapter, we will discuss some commonly used pseudogauges for fields of different spin. In order to assign an intuitive meaning to the different pseudogauge choices, however, it is advantageous to express the conserved currents as *phase-space* integrals, the formulation of which will be the subject of the next section.

3.2 Phase-space formulation

3.2.1 Classical systems

In classical mechanics, the dynamics of an ensemble of $N \gg 1$ particles is often conveniently described via a formulation in phase space. Since each particle (labeled by $1 \leq i \leq N$) has a well-defined position $x_i \coloneqq (t_i, \vec{r}_i)$ and a well-defined momentum $k_i \coloneqq (k_i^0, \vec{k}_i)$, there exists a scalar function $F_N(\{x_i\};\{k_i\})$ (depending on all $8N$ particle coordinates) that describes the distribution of positions and momenta. The function F_N is then called an N-particle distribution function, and it describes the probability of finding N particles at the phase-space positions $\{(x_1, k_1), \cdots, (x_N, k_N)\}$. Parametrizing the trajectory of the i-th particle by the parameter τ_i, the N-particle distribution function can be written as [124]

$$F_N(\{x_i\};\{k_i\}) \coloneqq \frac{1}{m^N N!}\Bigg\langle \int \mathrm{d}\tau_1 \cdots \int \mathrm{d}\tau_N \times \sum_{j_1,\cdots,j_N} \prod_{i=1}^{N} \delta^{(4)}\left[x_i - x_{j_i}(\tau_{j_i})\right] \delta^{(4)}\left[k_i - k_{j_i}(\tau_{j_i})\right] \Bigg\rangle_{\mathrm{ens}} , \tag{3.19}$$

where m is the particle mass and the angular brackets denote ensemble averaging. The intuition behind Eq. (3.19) is that all worldlines of the N particles are traced out in phase space via the integrals over $\tau_1, \cdots, \tau_N$, and the sum checks whether the arguments of the function $\{x_i\}, \{k_i\}$ all lie on a worldline. Knowledge of the N-particle distribution function F_N is equivalent to complete knowledge of the system, and thus to solving $\mathcal{O}(N)$ coupled differential equations. What we aim to achieve via a phase-space formulation, however, is a *coarse-grained* description of the system. In the classical picture, this coarse-graining consists in not considering the whole N-particle distribution function, but rather s-particle distributions (with $s < N$), where the information about $N - s$ particles is integrated out,

$$F_s(x_1, \cdots, x_s; k_1, \cdots, k_s) \coloneqq m^{N-s}(N-s)! \int \mathrm{d}\Sigma_{s+1}\mathrm{d}^4k_{s+1} \cdots \int \mathrm{d}\Sigma_N \mathrm{d}^4k_N\, F_N(\{x_i\};\{k_i\}) , \tag{3.20}$$

with Σ_i being three-dimensional spacelike hypersurfaces. The distribution function (3.20) now describes the probability of finding s particles at the phase-space positions $\{(x_1, k_1), \cdots, (x_s, k_s)\}$. Since this function describes less particles than are contained in the system, it does not provide complete information anymore. In particular, the $N - s$ particles that were integrated over act as a source term for the change of the s-particle distribution function, leading to the fact that the evolution of F_s will depend on F_{s+1}, such that the system is not closed and has to be truncated. This dependence on higher-order distribution functions is called the *BBGKY hierarchy*, after Bogoliubov, Born, Green, Kirkwood, and Yvon [125–128].

Taking this procedure to the extreme by setting $s = 1$, we arrive at the notion of the one-particle distribution function $F(x, k) \coloneqq F_1(x, k)$, describing the probability of finding a particle at position x and with momentum k. As discussed above, its evolution depends on the two-particle distribution function F_2, and as such its evolution equation is not closed. One popular way of truncating this system is by introducing the condition of *molecular chaos*, where one assumes the two-particle distribution function to simply be a product of the one-particle distribution functions, i.e.,

$$F_2(x_1, x_2; k_1, k_2) = F(x_1, k_1)F(x_2, k_2) . \tag{3.21}$$

We will encounter this condition again in a slightly different form in Chapters 4 and 5.

Conserved currents

With the concept of the one-particle distribution function, it is possible to reconnect to the conserved quantities introduced earlier in Sec. 3.1. Considering a system made up of the same species of particles in the absence of inelastic processes (i.e., no particles are created or destroyed), the particle four-current N^μ is conserved, with the global charge being given by the total number of particles. It can then be expressed as the average of the four-momentum, i.e.,

$$N^\mu(x) = \int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4} k^\mu F(x,k) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3 k^0} k^\mu f(x,k) \ . \tag{3.22}$$

Here we made use of the fact that for classical particles the momentum is always on shell, $k^2 = m^2$, which allows us to write

$$F(x,k) \equiv 4\pi\hbar\, \delta(k^2 - m^2) f(x,k) \ , \tag{3.23}$$

and subsequently employ the identity

$$\int \mathrm{d}^4k \delta(k^2 - m^2) = \int \mathrm{d}^4k \frac{1}{2k^0} \delta\left(k^0 - \sqrt{\mathbf{k}^2 + m^2}\right) \equiv \int \frac{\mathrm{d}^3\mathbf{k}}{2k^0} \ , \tag{3.24}$$

where in the last $\mathrm{d}^3\mathbf{k}$-integral it is implied that $k^0 = \sqrt{\mathbf{k}^2 + m^2}$. The energy-momentum tensor can be expressed similarly as

$$T^{\mu\nu}(x) = \int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4} k^\mu k^\nu F(x,k) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3 k^0} k^\mu k^\nu f(x,k) \ . \tag{3.25}$$

Equations (3.22) and (3.25) provide relations between the conserved currents (which describe the macroscopic properties of the system) and the one-particle distribution function. This implies that, if the evolution of f is known, the conserved currents follow at once, providing the equations of motion that were missing from the purely macroscopic analysis of Chapter 2. First, however, we have to ask the question of how to provide a phase-space formulation in the case of quantum-mechanical systems.

3.2.2 Quantum systems

In contrast to the classical systems we just discussed, in a quantum system there cannot be a scalar function describing the position and momenta of all particles that looks like Eq. (3.19), since a quantum-mechanical particle does not have well-defined position and momentum at the same time. This can be seen straightforwardly from the fact that the position and momentum operators do not commute. Nevertheless, it is possible to formulate a phase-space description of a quantum theory at the cost of giving up some of the properties of the distribution function that have been taken for granted in Subsec. 3.2.1. We will first illustrate the idea by considering one-dimensional nonrelativistic quantum mechanics before moving on to a relativistic quantum field-theoretical formulation.

One-dimensional nonrelativistic quantum mechanics

Building on Eqs. (3.22) and (3.25), we may reformulate the question of obtaining a phase-space formulation by asking whether there exists a function $W(x,k,t)$ that fulfills

$$\mathrm{Tr}\left[\widehat{\varrho}(\hat{x},\hat{k},t)\widehat{A}(\hat{x},\hat{k})\right] = \int \mathrm{d}x \int \frac{\mathrm{d}k}{2\pi\hbar} A(x,k) W(x,k,t) \ , \tag{3.26}$$

where $\widehat{\varrho}$ is the density matrix and $\widehat{A}$ is an arbitrary operator. Note that, in this example, we returned to nonrelativistic physics, such that the time coordinate takes a special role. Indeed there is an infinite number of functions that satisfy Eq. (3.26), but none of them features all the properties one would expect of a one-particle distribution function, namely

- being real, $W^*(x,k,t) = W(x,k,t)$,
- being nonnegative, $W(x,k,t) \geq 0 \,\forall\, (x,k,t) \in \mathbb{R}^3$, and
- giving the correct marginal distributions, i.e.,

$$\int \frac{\mathrm{d}k}{2\pi\hbar} \, W(x,k,t) = |\Psi(x,t)|^2 \ , \qquad \int \mathrm{d}x \, W(x,k,t) = |\tilde{\Psi}(k,t)|^2 \ , \tag{3.27}$$

  where Ψ is the wave function of the quantum-mechanical particle, and $\tilde{\Psi}$ is its FOURIER transform.

In the literature, one finds several choices for $W(x,k,t)$, depending on the problem at hand; for a review, see Ref. [37].[2] In this thesis, we are going to use the so-called *WIGNER function*

$$\begin{aligned} W(x,k,t) &:= \int \mathrm{d}v \, e^{-\frac{i}{\hbar}kv} \left\langle x + \frac{v}{2} \right| \widehat{\varrho} \left| x - \frac{v}{2} \right\rangle \\ &= \int \mathrm{d}v \, e^{-\frac{i}{\hbar}kv} \Psi^* \left(x + \frac{v}{2} \right) \Psi \left(x - \frac{v}{2} \right) \ , \end{aligned} \tag{3.28}$$

where the second equality holds if the system is in a pure state, i.e., $\widehat{\varrho} = |\Psi\rangle \langle\Psi|$. Note that, while the WIGNER function is real and gives the correct marginal distributions, it is not necessarily nonnegative, such that the strict interpretation as a probability density fails, the reason being that interference effects are incorporated [37]. In order to verify that Eq. (3.26) indeed holds, we have to define $A(x,k)$, which will be the *WIGNER-WEYL* transform of the operator $\widehat{A}$,

$$A(x,k) := \int \mathrm{d}u \, e^{-\frac{i}{\hbar}ku} \left\langle x + \frac{u}{2} \right| \widehat{A} \left| x - \frac{u}{2} \right\rangle \ . \tag{3.29}$$

Inserting the above equation into the left-hand side of Eq. (3.26) and omitting the dependence on $\hat{x}, \hat{k}$ for brevity, we find

$$\begin{aligned} \int \mathrm{d}x \int \frac{\mathrm{d}k}{2\pi\hbar} A(x,k) W(x,k,t) &= \int \mathrm{d}x \int \mathrm{d}v \left\langle x - \frac{v}{2} \right| \widehat{A} \left| x + \frac{v}{2} \right\rangle \left\langle x + \frac{v}{2} \right| \widehat{\varrho} \left| x - \frac{v}{2} \right\rangle \\ &= \int \mathrm{d}y_- \int \mathrm{d}y_+ \, \langle y_- | \, \widehat{A} \, | y_+ \rangle \, \langle y_+ | \, \widehat{\varrho} \, | y_- \rangle \\ &\equiv \mathrm{Tr} \left(\widehat{A} \, \widehat{\varrho} \right) \ . \end{aligned} \tag{3.30}$$

Here we substituted $y_\pm := x \pm v/2$ in the second step. The WIGNER function (3.28) thus fulfills the desired property, as long as the correct transform of the operator in question is used.

Field theory: Covariant WIGNER operator

We now want to generalize the concept of phase space in quantum mechanics to a quantum field theory of some field $\widehat{\varphi}$ that furnishes a certain representation of the inhomogeneous LORENTZ group. In particular, if $\widehat{\varphi}$ has spin j, it transforms in the

$$\left(\tfrac{j}{2}, \tfrac{j}{2}\right) - \text{representation}$$

[2] We remark that one of the possible choices is given by the GLAUBER-SUDARSHAN P representation [129].

of the LORENTZ group if j is integer, and in the

$$\left(\tfrac{2j-1}{4}, \tfrac{2j-1}{4}\right) \otimes \left[\left(\tfrac{1}{2}, 0\right) \oplus \left(0, \tfrac{1}{2}\right)\right] - \text{representation}$$

if j is half-integer. Since the (m, n)-representation of the LORENTZ group is $(2m+1)(2n+1)$-dimensional, we have $(j+1)^2$ $[(2j+1)^2]$ components for integer (half-integer) spin. However, a (massive) field of spin j only has $2j+1$ degrees of freedom, such that the field $\widehat{\varphi}$ fulfills suitable constraint equations that reduce the number of independent components [123]. Building on the definition (3.28), we can generalize the WIGNER function in one dimension to a WIGNER *operator* in 3+1 dimensions,

$$\widehat{W}(x,k) := \kappa \int \mathrm{d}^4 v e^{-\frac{i}{\hbar} k\cdot v} \widehat{\overline{\varphi}}\left(x + \frac{v}{2}\right) \widehat{\varphi}\left(x - \frac{v}{2}\right) , \tag{3.31}$$

where the constant is $|\kappa| = 2/\hbar$ for integer and $|\kappa| = 1$ for half-integer spin, and we defined

$$\widehat{\overline{\varphi}} := \begin{cases} \widehat{\varphi}^\dagger , & j \text{ integer} , \\ \widehat{\varphi}^\dagger \gamma^0 , & j \text{ half-integer} . \end{cases} \tag{3.32}$$

Note that this definition differs from the conventions in Refs. [43, 44] by a factor of $(2\pi\hbar)^4$ in the $\mathrm{d}^4 v$-integration measure, but agrees with the formulation used in Ref. [46].[3] The WIGNER operator (3.31) has $2\lfloor j \rfloor$ LORENTZ indices and an additional 2 DIRAC indices if j is half integer. From the number of degrees of freedom of the field $\widehat{\varphi}$, it follows that the WIGNER operator features $(2j+1)^2$ independent components, which we will verify for the cases $j \in \{0, 1/2, 1\}$ in the following sections. The *WIGNER function* follows from the corresponding operator by taking its normal-ordered expectation value, $W(x,k) := \langle : \widehat{W}(x,k) : \rangle$. We remark that, as soon as the fields $\widehat{\varphi}$ do not transform trivially under LORENTZ transformations, the WIGNER function is no longer real, but fulfills $\widetilde{W} = W^*$, where [43]

$$\widetilde{W} := \begin{cases} W^{\mathrm{T}} , & j \text{ integer} , \\ \gamma^0 W^{\mathrm{T}} \gamma^0 , & j \text{ half-integer} . \end{cases} \tag{3.33}$$

Since the energy-momentum and particle-number operators for free fields are bilinear in derivatives of the fields $\widehat{\overline{\varphi}}$, $\widehat{\varphi}$, we need to formulate expressions of the type

$$\left[\partial^{\mu_1} \cdots \partial^{\mu_n} \widehat{\overline{\varphi}}(x)\right] \left[\partial^{\nu_1} \cdots \partial^{\nu_m} \widehat{\varphi}(x)\right]$$

in terms of integrals over the WIGNER operator. Firstly, note that the inverse of the WIGNER transform is given by

$$\int \frac{\mathrm{d}^4 k}{(2\pi\hbar)^4} \widehat{W}(x,k) = \kappa \widehat{\overline{\varphi}}(x) \widehat{\varphi}(x) . \tag{3.34}$$

Furthermore, we define the so-called BOPP operator [130]

$$D^\mu := k^\mu + \frac{i\hbar}{2}\partial^\mu , \quad D^{*\mu} := k^\mu - \frac{i\hbar}{2}\partial^\mu , \tag{3.35}$$

which fulfills

$$D^\mu \widehat{W}(x,k) = i\hbar\kappa \int \mathrm{d}^4 v e^{-\frac{i}{\hbar} k\cdot v} \widehat{\overline{\varphi}}\left(x + \frac{v}{2}\right) \partial^\mu \widehat{\varphi}\left(x - \frac{v}{2}\right) , \tag{3.36a}$$

$$D^{*\mu} \widehat{W}(x,k) = -i\hbar\kappa \int \mathrm{d}^4 v e^{-\frac{i}{\hbar} k\cdot v} \left[\partial^\mu \widehat{\overline{\varphi}}\left(x + \frac{v}{2}\right)\right] \widehat{\varphi}\left(x - \frac{v}{2}\right) . \tag{3.36b}$$

[3] In this thesis, we adopt the latter convention because it allows the interpretation of the WIGNER function as a particle-number density, as we will see in Chapter 4.

To prove this, we compute, denoting the derivative with respect to v as ∂_v^μ, and abbreviating $\varphi_\pm \coloneqq \varphi(x \pm v/2)$,

$$\begin{aligned}
\partial^\mu \widehat{W}(x,k) &= \kappa \int \mathrm{d}^4 v e^{-\frac{i}{\hbar} k\cdot v} \left[\left(\partial^\mu \widehat{\overline{\varphi}}_+\right) \widehat{\varphi}_- + \widehat{\overline{\varphi}}_+ \left(\partial^\mu \widehat{\varphi}_-\right)\right] \\
&= \kappa \int \mathrm{d}^4 v e^{-\frac{i}{\hbar} k\cdot v} \left[2\left(\partial_v^\mu \widehat{\overline{\varphi}}_+\right) \widehat{\varphi}_- + \widehat{\overline{\varphi}}_+ \left(\partial^\mu \widehat{\varphi}_-\right)\right] \\
&= \kappa \int \mathrm{d}^4 v e^{-\frac{i}{\hbar} k\cdot v} \left[2\frac{i}{\hbar} k^\mu \widehat{\overline{\varphi}}_+ \widehat{\varphi}_- - 2\widehat{\overline{\varphi}}_+ \left(\partial_v^\mu \widehat{\varphi}_-\right) + \widehat{\overline{\varphi}}_+ \left(\partial^\mu \widehat{\varphi}_-\right)\right] \\
&= \kappa \int \mathrm{d}^4 v e^{-\frac{i}{\hbar} k\cdot v} \left[2\frac{i}{\hbar} k^\mu \widehat{\overline{\varphi}}_+ \widehat{\varphi}_- + 2\widehat{\overline{\varphi}}_+ (\partial^\mu \widehat{\varphi}_-)\right] \\
&\equiv 2\frac{i}{\hbar} k^\mu \widehat{W}(x,k) + 2\kappa \int \mathrm{d}^4 v e^{-\frac{i}{\hbar} k\cdot v} \widehat{\overline{\varphi}}_+ \left(\partial^\mu \widehat{\varphi}_-\right) \ .
\end{aligned} \tag{3.37}$$

This proves Eq. (3.36a), while Eq. (3.36b) follows analogously. Combining the identities (3.34) and (3.36), we are able to express any operator bilinear in the fields as

$$\begin{aligned}
&\kappa \left[\partial^{\mu_1} \cdots \partial^{\mu_n} \widehat{\overline{\varphi}}(x)\right] \left[\partial^{\nu_1} \cdots \partial^{\nu_m} \widehat{\varphi}(x)\right] \\
&= (-1)^n \left(\frac{1}{i\hbar}\right)^{n+m} \int \frac{\mathrm{d}^4 k}{(2\pi\hbar)^4} \left(D^{*\mu_1} \cdots D^{*\mu_n}\right) \left(D^{\nu_1} \cdots D^{\nu_m}\right) \widehat{W}(x,k) \ .
\end{aligned} \tag{3.38}$$

Equation (3.38) is the central formula that we will use to represent conserved currents of different fields as momentum-space integrals.

Power-counting in the Planck constant

In the following sections, we will encounter equations of motion for the Wigner functions that involve different powers of $\hbar$. We will then often perform a so-called "$\hbar$-expansion" which consists in writing a quantity $A(x,k)$ as a power series

$$A(x,k) \equiv \sum_{n=0}^{\infty} \hbar^n A^{(n)}(x,k) \tag{3.39}$$

that can then be perturbatively solved in order to determine the coefficients $A^{(n)}(x,k)$.

This kind of expansion, which is well-known in quantum mechanics and thermodynamics [100, 131], is of course independent from the value of the Planck constant, which can in the end be safely set to unity and acts as a book-keeping parameter. As we will see, a power of $\hbar$ that contributes to the power-counting (as opposed to the powers appearing in, e.g., the measure of momentum space) is always accompanied by a derivative, such that the $\hbar$-expansion also becomes an expansion in gradients. The dimensionless quantity that controls the quality of the expansion is given by the ratio of the Compton-wavelength $\lambda_C = \hbar/m$ and a macroscopic length scale L, where we take the gradient to be proportional to its inverse, $\partial \sim L^{-1}$. The classical limit (corresponding to $\hbar \to 0$) is then given by

$$\frac{\hbar/m}{L} \to 0 \ ,$$

which is the well-known eikonal approximation that appears in the transition from wave optics to geometric optics [132] and was already recognized by Schrödinger as the classical limit of quantum mechanics [133]. It should be noted that this classical limit, where the notion of a particle has a well-defined meaning, corresponds also to the limit where kinetic theory, as imagined by Boltzmann, i.e., as a theory of particles colliding in small regions of spacetime, is valid. In the remainder of the thesis, we will mostly contain ourselves to the first order in the $\hbar$-expansion, treating quantum effects (in particular those induced by the particle spin) as small corrections.

3.3 Scalar fields

As a first example, we consider the case of complex scalar fields, which do not feature any spin and transform in the $(0,0)$-representation of the LORENTZ group. Accordingly, all quantities related to the scalar field are denoted with a subscript S.

3.3.1 Dynamics

A complex scalar field $\widehat{\phi}$ is described by the KLEIN-GORDON Lagrangian

$$\widehat{\mathcal{L}}_S = \hbar \left[\left(\partial^\mu \widehat{\phi}^\dagger \right) \left(\partial_\mu \widehat{\phi} \right) - \frac{m^2}{\hbar^2} |\widehat{\phi}|^2 \right] + \widehat{\mathcal{L}}_{S,\text{int}} \,, \tag{3.40}$$

where $\widehat{\mathcal{L}}_{S,\text{int}}$ denotes an interaction Lagrangian, which we assume to be independent of the derivatives of the field. The resulting equations of motion read

$$\left(\Box + \frac{m^2}{\hbar^2} \right) \widehat{\phi} = \widehat{\rho} \,, \tag{3.41a}$$

$$\left(\Box + \frac{m^2}{\hbar^2} \right) \widehat{\phi}^\dagger = \widehat{\rho}^\dagger \,, \tag{3.41b}$$

where we introduced the source terms

$$\widehat{\rho} \coloneqq \frac{1}{\hbar} \frac{\partial \widehat{\mathcal{L}}_{S,\text{int}}}{\partial \widehat{\phi}^\dagger} \,, \qquad \widehat{\rho}^\dagger \coloneqq \frac{1}{\hbar} \frac{\partial \widehat{\mathcal{L}}_{S,\text{int}}}{\partial \widehat{\phi}} \,. \tag{3.42}$$

The GREEN's function $\Delta(x, x')$ of the complex scalar field can be obtained from

$$\left(\Box_x + \frac{m^2}{\hbar^2} \right) \Delta(x, x') = \delta^{(4)}(x - x') \,, \tag{3.43}$$

which in FOURIER space becomes

$$\begin{aligned} &\int \frac{\mathrm{d}^4 k}{(2\pi\hbar)^4} \int \frac{\mathrm{d}^4 k'}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}(k\cdot x + k'\cdot x')} (-k^2 + m^2) \widetilde{\Delta}(k, k') \\ = &\int \frac{\mathrm{d}^4 k}{(2\pi\hbar)^4} \int \frac{\mathrm{d}^4 k'}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}(k\cdot x + k'\cdot x')} (2\pi\hbar)^4 \hbar^2 \delta^{(4)}(k + k') \,. \end{aligned} \tag{3.44}$$

With the definition $\widetilde{\Delta}(k, k') =: (2\pi\hbar)^4 \delta^{(4)}(k + k') \widetilde{\Delta}(k)$, we find the retarded and advanced GREEN's functions

$$\widetilde{\Delta}_R(k) = -\frac{\hbar^2}{k^2 - m^2 + i\eta k^0} \,, \tag{3.45a}$$

$$\widetilde{\Delta}_A(k) = -\frac{\hbar^2}{k^2 - m^2 - i\eta k^0} \,, \tag{3.45b}$$

where the infinitesimal quantity $\eta > 0$ in the denominator indicates that, when evaluating the FOURIER integral via contour integration, the half-circle has to be closed in the lower or upper half-plane, respectively.

3.3.2 WIGNER function

The WIGNER operator is defined [in accordance with Eq. (3.31)] as

$$\widehat{W}(x, k) \coloneqq \frac{2}{\hbar} \int \mathrm{d}^4 v e^{-\frac{i}{\hbar} k\cdot v} \widehat{\phi}^\dagger \left(x + \frac{v}{2} \right) \widehat{\phi} \left(x - \frac{v}{2} \right) \,. \tag{3.46}$$

Acting on it with the operator $D^2 - m^2$ and using Eq. (3.38), we obtain with the help of the Klein-Gordon equation

$$\left(k^2 - m^2 + i\hbar k\cdot\partial - \frac{\hbar^2}{4}\Box\right)\widehat{W}(x,k) = \hbar\widehat{C}(x,k)\ , \tag{3.47}$$

where we defined

$$\widehat{C}(x,k) := -2\int \mathrm{d}^4v e^{-\frac{i}{\hbar}k\cdot v}\widehat{\phi}^\dagger\left(x+\frac{v}{2}\right)\widehat{\rho}\left(x-\frac{v}{2}\right)\ . \tag{3.48}$$

Taking the real and imaginary parts of the expectation value of Eq. (3.47), and defining $\mathcal{D} := \mathrm{Re}\,\langle : \widehat{C} : \rangle$, $\mathcal{C} := \mathrm{Im}\,\langle : \widehat{C} : \rangle$, we find

$$\left(k^2 - m^2 - \frac{\hbar^2}{4}\Box\right)W(x,k) = \hbar\mathcal{D}(x,k)\ , \tag{3.49}$$

$$k\cdot\partial W(x,k) = \mathcal{C}(x,k)\ . \tag{3.50}$$

Here we also made use of the fact that the Wigner function is real. One can see that Eq. (3.49) constitutes a mass-shell equation for the Wigner function, i.e., it will ensure that the momentum k will obey the relativistic energy-momentum relation, $k^2 = m^2$, in the classical limit. Equation (3.50) on the other hand denotes a kinetic equation that will determine the time evolution of $W(x,k)$.

3.3.3 Conserved currents

From the Lagrangian (3.40), we immediately obtain the canonical energy-momentum tensor

$$T^{\mu\nu}_{S,C} + g^{\mu\nu}\mathcal{L}_S = \hbar\left\langle : \left(\partial^{(\mu}\widehat{\phi}^\dagger\right)\left(\partial^{\nu)}\widehat{\phi}\right) : \right\rangle\ . \tag{3.51}$$

The canonical energy-momentum tensor is manifestly symmetric, which is to be expected from the conservation of the total angular momentum, since the scalar field does not carry spin. In order to express the energy-momentum tensor in terms of the Wigner function, we make use of Eq. (3.38) and obtain

$$\begin{aligned} T^{\mu\nu}_{S,C} + g^{\mu\nu}\mathcal{L}_S &= \frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi\hbar)^4}D^{*(\mu}D^{\nu)}W(x,k) \\ &= \int\frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\left(k^\mu k^\nu + \frac{\hbar^2}{4}\partial^\mu\partial^\nu\right)W(x,k)\ . \end{aligned} \tag{3.52}$$

Furthermore, the action of the scalar field is invariant under the global $U(1)$ transformation

$$\widehat{\phi}\to e^{-\frac{i}{\hbar}q\alpha}\widehat{\phi}\ ,\quad \widehat{\phi}^\dagger\to e^{\frac{i}{\hbar}q\alpha}\widehat{\phi}^\dagger\ ,$$

leading to the conserved electric current

$$j^\mu_S = iq\left\langle : \left(\widehat{\phi}^\dagger\partial^\mu\widehat{\phi} - \widehat{\phi}\partial^\mu\widehat{\phi}^\dagger\right) : \right\rangle \equiv iq\left\langle : \widehat{\phi}^\dagger\overleftrightarrow{\partial^\mu}\widehat{\phi} : \right\rangle\ . \tag{3.53}$$

Employing the general prescription (3.38) again, we can express the electric current as

$$\begin{aligned} j^\mu_S &= \frac{q}{2}\int\frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\left(D^{*\mu} + D^\mu\right)W(x,k) \\ &= q\int\frac{\mathrm{d}^4k}{(2\pi\hbar)^4}k^\mu W(x,k)\ , \end{aligned} \tag{3.54}$$

which is a very intuitive result. At this point it becomes clear that there is the general correspondence

$$\text{“ }\overleftrightarrow{\partial^\mu}\text{ between }\widehat{\phi}^\dagger\text{ and }\widehat{\phi}\implies\frac{2}{i\hbar}k^\mu\text{ in the momentum integral ” .}$$

GLW pseudogauge

Even though the scalar field does not have spin, we can still perform a pseudogauge transformation. Considering the vectors and tensors at our disposal, it is clear that the superpotentials $\widehat{\Phi}$, $\widehat{Z}$ can only consist of (at most) one gradient and the metric tensor. Defining

$$\widehat{\Phi}^{\lambda\mu\nu}_{\text{GLW}} \coloneqq \frac{1}{2}\left(\widehat{\phi}^\dagger g^{\lambda[\mu}\partial^{\nu]}\widehat{\phi} + \text{h.c.}\right) , \qquad \widehat{Z}^{\mu\nu\lambda\rho}_{\text{GLW}} \coloneqq -\frac{1}{4\hbar} g^{\xi[\mu}g^{\nu][\lambda}\delta^{\rho]}_\xi |\widehat{\phi}|^2 , \tag{3.55}$$

where "h.c." denotes the hermitian conjugate, we see that

$$\hbar\partial_\rho \widehat{Z}^{\mu\nu\lambda\rho}_{\text{GLW}} = \widehat{\Phi}^{\lambda\mu\nu}_{\text{GLW}} , \tag{3.56}$$

such that [according to Eq. (3.10a)] the spin-tensor does not change in this so-called *GLW pseudogauge* (after DE GROOT, VAN LEEUWEN and VAN WEERT), $\widehat{S}^{\lambda\mu\nu}_{S,\text{KG}} = \widehat{S}^{\lambda\mu\nu}_{S,C} = 0$. Considering that

$$\widehat{\Phi}^{\lambda\mu\nu}_{\text{GLW}} + \widehat{\Phi}^{\mu\nu\lambda}_{\text{GLW}} + \widehat{\Phi}^{\nu\mu\lambda}_{\text{GLW}} = \widehat{\phi}^\dagger g^{\nu[\mu}\partial^{\lambda]}\widehat{\phi} + \text{h.c.} , \tag{3.57}$$

the energy-momentum tensor becomes

$$T^{\mu\nu}_{S,\text{GLW}} = -\frac{\hbar}{2}\left\langle : \widehat{\phi}^\dagger \overleftrightarrow{\partial}^\mu \overleftrightarrow{\partial}^\nu \widehat{\phi} : \right\rangle - g^{\mu\nu}\left\langle : \left\{ \widehat{\mathcal{L}}_S - \frac{\hbar}{2}\left[\widehat{\phi}^\dagger \Box \widehat{\phi} + \left(\partial^\lambda \widehat{\phi}^\dagger\right)\left(\partial_\lambda \widehat{\phi}\right) + \text{h.c.}\right]\right\} : \right\rangle . \tag{3.58}$$

Denoting the last term in the equation above as

$$\begin{aligned} \widehat{\mathcal{L}}_{S,\text{GLW}} &\coloneqq \widehat{\mathcal{L}}_S - \frac{\hbar}{2}\left[\widehat{\phi}^\dagger \Box \widehat{\phi} + \left(\partial^\lambda \widehat{\phi}^\dagger\right)\left(\partial_\lambda \widehat{\phi}\right) + \text{h.c.}\right] \\ &= \widehat{\mathcal{L}}_{S,\text{int}} - \frac{\hbar}{2}\left[\widehat{\phi}^\dagger\left(\Box + \frac{m^2}{\hbar^2}\right)\widehat{\phi} + \text{h.c.}\right] , \end{aligned} \tag{3.59}$$

the energy-momentum tensor can be expressed through the WIGNER function as

$$T^{\mu\nu}_{S,\text{GLW}} + g^{\mu\nu}\mathcal{L}_{S,\text{GLW}} = \int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4} k^\mu k^\nu W(x,k) . \tag{3.60}$$

The right-hand side of the equation above resembles the form we would expect to obtain for the energy-momentum tensor in kinetic theory, with the WIGNER function taking the role of the distribution function.

Before moving on to higher-spin fields, we remark two things about the Lagrangian (3.59): First, if the theory is free, i.e., if $\widehat{\mathcal{L}}_{S,\text{int}} = 0$, it vanishes as soon as the equations of motion are imposed. More precisely, in the case that the interaction term is a polynomial in the fields, we have that

$$\mathcal{L}_{S,\text{int}} \sim \widehat{\phi}^\dagger \widehat{\rho} + \widehat{\rho}^\dagger \widehat{\phi} , \tag{3.61}$$

such that we get from Eq. (3.60) after applying the equations of motion

$$\mathcal{L}_{S,\text{GLW}} \sim \int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\mathcal{D} , \tag{3.62}$$

showing that this term incorporates off-shell effects [43, 134, 135]. Second, we could have started directly from the Lagrangian $\widehat{\mathcal{L}}_{S,\text{GLW}}$, which differs from the canonical Lagrangian $\widehat{\mathcal{L}}_S$ by a total derivative, thus yielding the same action and equations of motion. In that case, since $\widehat{\mathcal{L}}_{S,\text{GLW}}$ depends on second derivatives of the field, we would have to compute the energy-momentum tensor as [43]

$$T^{\mu\nu}_{S,\text{GLW}} + g^{\mu\nu}\mathcal{L}_{S,\text{GLW}} = \left\langle : \left\{ \left[\frac{\partial \widehat{\mathcal{L}}_{S,\text{GLW}}}{\partial(\partial_\mu \widehat{\phi}^\dagger)} + \frac{\partial \widehat{\mathcal{L}}_{S,\text{GLW}}}{\partial(\partial_\mu\partial_\lambda \widehat{\phi}^\dagger)}\overleftrightarrow{\partial_\lambda}\right]\partial^\nu \widehat{\phi} + \text{h.c.}\right\} : \right\rangle , \tag{3.63}$$

yielding precisely the result (3.58). Thus, this pseudogauge transformation is equivalent to changing the Lagrangian by a total divergence.

3.4 Spinor fields

We now move on to DIRAC particles of spin 1/2, which transform in the $(1/2, 0) \oplus (0, 1/2)$-representation of the LORENTZ group. The quantities related to these fields are denoted with a subscript D.

3.4.1 Dynamics

A massive DIRAC field $\widehat{\psi}$ is described by the Lagrangian

$$\widehat{\mathcal{L}}_D := \widehat{\overline{\psi}} \left(\frac{i\hbar}{2} \overleftrightarrow{\not\partial} - m \right) \widehat{\psi} + \widehat{\mathcal{L}}_{D,\mathrm{int}} \ , \tag{3.64}$$

where $\not A := \gamma \cdot A$ for any four-vector A and γ^μ are the DIRAC matrices fulfilling

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \ . \tag{3.65}$$

The Lagrangian (3.64) leads to the following equations of motion,

$$\left(i\hbar \not\partial - m \right) \widehat{\psi} = \hbar \widehat{\rho} \ , \tag{3.66a}$$
$$\widehat{\overline{\psi}} \left(i\hbar \overleftarrow{\not\partial} + m \right) = -\hbar \widehat{\overline{\rho}} \ , \tag{3.66b}$$

where the source terms are given by

$$\widehat{\rho} := -\frac{1}{\hbar} \frac{\partial \widehat{\mathcal{L}}_{D,\mathrm{int}}}{\partial \widehat{\overline{\psi}}} \ , \quad \widehat{\overline{\rho}} := -\frac{1}{\hbar} \frac{\partial \widehat{\mathcal{L}}_{D,\mathrm{int}}}{\partial \widehat{\psi}} \ . \tag{3.67}$$

Note that by acting with the operators $(i\hbar \not\partial + m)$ and $(i\hbar \overleftarrow{\not\partial} - m)$ on Eqs. (3.66a) and (3.66b), respectively, we obtain

$$\left(\Box + \frac{m^2}{\hbar^2} \right) \widehat{\psi} = -\left(i \not\partial + \frac{m}{\hbar} \right) \widehat{\rho} \ , \tag{3.68a}$$
$$\left(\Box + \frac{m^2}{\hbar^2} \right) \widehat{\overline{\psi}} = \widehat{\overline{\rho}} \left(i \overleftarrow{\not\partial} - \frac{m}{\hbar} \right) \ , \tag{3.68b}$$

i.e., all components of the DIRAC fields also fulfill the KLEIN-GORDON equation.

The GREEN's function of the DIRAC field $S(x, x')$ is obtained by solving

$$\left(i\hbar \not\partial - m \right) S(x, x') = \hbar \delta^{(4)}(x - x') \ , \tag{3.69}$$

or in FOURIER space

$$\begin{aligned} &\int \frac{\mathrm{d}^4 k}{(2\pi\hbar)^4} \int \frac{\mathrm{d}^4 k'}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}(k\cdot x + k'\cdot x')} (\not k - m) \widetilde{S}(k, k') \\ = &\int \frac{\mathrm{d}^4 k}{(2\pi\hbar)^4} \int \frac{\mathrm{d}^4 k'}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}(k\cdot x + k'\cdot x')} (2\pi\hbar)^4 \hbar \delta^{(4)}(k + k') \ . \end{aligned} \tag{3.70}$$

Defining $\widetilde{S}(k, k') =: (2\pi\hbar)^4 \delta^{(4)}(k + k') \widetilde{S}(k)$, we find for the retarded and advanced GREEN's functions

$$\widetilde{S}_R(k) = \hbar \frac{\not k + m}{k^2 - m^2 + i\eta k^0} = -\frac{1}{\hbar} \left(\not k + m \right) \widetilde{\Delta}_R(k) \ , \tag{3.71a}$$
$$\widetilde{S}_A(k) = \hbar \frac{\not k + m}{k^2 - m^2 - i\eta k^0} = -\frac{1}{\hbar} \left(\not k + m \right) \widetilde{\Delta}_A(k) \ . \tag{3.71b}$$

3.4.2 Wigner function

The Wigner operator for Dirac fields is defined as

$$\widehat{W}_{\alpha\beta}(x,k) = \int \mathrm{d}^4 v e^{-\frac{i}{\hbar}k\cdot v} \widehat{\overline{\psi}}_\beta \left(x+\frac{v}{2}\right) \widehat{\psi}_\alpha \left(x-\frac{v}{2}\right) , \tag{3.72}$$

where we made the Dirac indices α, β explicit. As we discussed in Subsec. 3.2.2, the Wigner operator formally has 16 components, but only four of them are independent. It is easiest to see which components constitute the independent degrees of freedom by decomposing the Wigner operator in terms of the Clifford algebra, i.e.,

$$\widehat{W} = \frac{1}{4}\left(\widehat{\mathcal{F}} + i\gamma_5 \widehat{\mathcal{P}} + \widehat{\not{\mathcal{V}}} + \gamma_5 \widehat{\not{\mathcal{A}}} + \frac{1}{2}\sigma_{\mu\nu}\widehat{\mathcal{S}}^{\mu\nu}\right) , \tag{3.73}$$

where $\sigma^{\mu\nu} := \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, and the square brackets denote the commutator. Equation (3.73) can be inverted by taking suitably weighted traces over Dirac space,

$$\widehat{\mathcal{F}} = \mathrm{Tr}\, \widehat{W} \ , \ \widehat{\mathcal{P}} = -i\mathrm{Tr}\, \gamma_5 \widehat{W} \ , \ \widehat{\mathcal{V}}^\mu = \mathrm{Tr}\, \gamma^\mu \widehat{W} \ , \ \widehat{\mathcal{A}}^\mu = \mathrm{Tr}\, \gamma^\mu\gamma_5 \widehat{W} \ , \ \widehat{\mathcal{S}}^{\mu\nu} = \mathrm{Tr}\, \sigma^{\mu\nu}\widehat{W} \ . \tag{3.74}$$

The equations of motion for the Wigner operator are found by applying the operator $\not{D} - m$ to Eq. (3.72) and employing Eq. (3.38) in conjunction with the Dirac equation, obtaining

$$\left[\left(\not{k} + \frac{i\hbar}{2}\not{\partial} - m\right)\widehat{W}(x,k)\right]_{\alpha\beta} = \hbar \widehat{C}_{\alpha\beta}(x,k) \ , \tag{3.75}$$

where we defined

$$\widehat{C}_{\alpha\beta}(x,k) := \int \mathrm{d}^4 v e^{-\frac{i}{\hbar}k\cdot v} \widehat{\overline{\psi}}_\beta \left(x+\frac{v}{2}\right) \widehat{\rho}_\alpha \left(x-\frac{v}{2}\right) . \tag{3.76}$$

The equation above, being matrix-valued in Dirac space, can be decomposed w.r.t. the Clifford algebra, and each resulting equation can be separated into real and imaginary parts. We find

$$\mathrm{Re}\,\langle :\widehat{C}:\rangle = \frac{1}{4}\left(\mathcal{D}_\mathcal{F} + i\gamma_5 D_\mathcal{P} + \not{\mathcal{D}}_\mathcal{V} + \gamma_5 \not{\mathcal{D}}_\mathcal{A} + \frac{1}{2}\sigma_{\mu\nu}\mathcal{D}^{\mu\nu}_\mathcal{S}\right) , \tag{3.77a}$$

$$\mathrm{Im}\,\langle :\widehat{C}:\rangle = \frac{1}{4}\left(\mathcal{C}_\mathcal{F} + i\gamma_5 C_\mathcal{P} + \not{\mathcal{C}}_\mathcal{V} + \gamma_5 \not{\mathcal{C}}_\mathcal{A} + \frac{1}{2}\sigma_{\mu\nu}\mathcal{C}^{\mu\nu}_\mathcal{S}\right) , \tag{3.77b}$$

where

$$\begin{aligned} \mathcal{D}_\mathcal{F} = \mathrm{Re}\,\langle :\mathrm{Tr}\,\widehat{C}:\rangle \ , \ \mathcal{D}_\mathcal{P} = -i\mathrm{Re}\,\langle :\mathrm{Tr}\,\gamma_5\widehat{C}:\rangle \ , \ \mathcal{D}^\mu_\mathcal{V} = \mathrm{Re}\,\langle :\mathrm{Tr}\,\gamma^\mu\widehat{C}:\rangle \ , \\ \mathcal{D}^\mu_\mathcal{A} = \mathrm{Re}\,\langle :\mathrm{Tr}\,\gamma^\mu\gamma_5\widehat{C}:\rangle \ , \ \mathcal{D}^{\mu\nu}_\mathcal{S} = \mathrm{Re}\,\langle :\mathrm{Tr}\,\sigma^{\mu\nu}\widehat{C}:\rangle \ , \end{aligned} \tag{3.78a}$$

$$\begin{aligned} \mathcal{C}_\mathcal{F} = \mathrm{Im}\,\langle :\mathrm{Tr}\,\widehat{C}:\rangle \ , \ \mathcal{C}_\mathcal{P} = -i\mathrm{Im}\,\langle :\mathrm{Tr}\,\gamma_5\widehat{C}:\rangle \ , \ \mathcal{C}^\mu_\mathcal{V} = \mathrm{Im}\,\langle :\mathrm{Tr}\,\gamma^\mu\widehat{C}:\rangle \ , \\ \mathcal{C}^\mu_\mathcal{A} = \mathrm{Im}\,\langle :\mathrm{Tr}\,\gamma^\mu\gamma_5\widehat{C}:\rangle \ , \ \mathcal{C}^{\mu\nu}_\mathcal{S} = \mathrm{Im}\,\langle :\mathrm{Tr}\,\sigma^{\mu\nu}\widehat{C}:\rangle \ . \end{aligned} \tag{3.78b}$$

Then, we perform the trace over Eq. (3.76), weighted with the generators of the Clifford algebra $\{1, -i\gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \sigma^{\mu\nu}\}$. Taking the real part of the resulting set of equations yields

$$k\cdot\mathcal{V} - m\mathcal{F} = \hbar\mathcal{D}_\mathcal{F} \ , \tag{3.79a}$$

$$\frac{\hbar}{2}\partial\cdot\mathcal{A} + m\mathcal{P} = -\hbar\mathcal{D}_\mathcal{P} \ , \tag{3.79b}$$

$$k^\mu\mathcal{F} - \frac{\hbar}{2}\partial_\nu\mathcal{S}^{\nu\mu} - m\mathcal{V}^\mu = \hbar\mathcal{D}^\mu_\mathcal{V} \ , \tag{3.79c}$$

$$-\frac{\hbar}{2}\partial^\mu\mathcal{P} + \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}k_\nu\mathcal{S}_{\alpha\beta} + m\mathcal{A}^\mu = -\hbar\mathcal{D}^\mu_\mathcal{A} \ , \tag{3.79d}$$

$$\frac{\hbar}{2}\partial^{[\mu}\mathcal{V}^{\nu]} - \epsilon^{\mu\nu\alpha\beta}k_\alpha\mathcal{A}_\beta - m\mathcal{S}^{\mu\nu} = \hbar\mathcal{D}^{\mu\nu}_\mathcal{S} \ , \tag{3.79e}$$

while the imaginary part gives

$$\partial \cdot \mathcal{V} = 2\mathcal{C}_{\mathcal{F}} \;, \tag{3.80a}$$

$$k \cdot \mathcal{A} = \hbar \mathcal{C}_{\mathcal{P}} \;, \tag{3.80b}$$

$$\frac{\hbar}{2}\partial^\mu \mathcal{F} + k_\nu \mathcal{S}^{\nu\mu} = \hbar \mathcal{C}^\mu_{\mathcal{V}} \;, \tag{3.80c}$$

$$k^\mu \mathcal{P} + \frac{\hbar}{4}\epsilon^{\mu\nu\alpha\beta}\partial_\nu \mathcal{S}_{\alpha\beta} = -\hbar \mathcal{C}^\mu_{\mathcal{A}} \;, \tag{3.80d}$$

$$k^{[\mu}\mathcal{V}^{\nu]} + \frac{\hbar}{2}\epsilon^{\mu\nu\alpha\beta}\partial_\alpha \mathcal{A}_\beta = -\hbar \mathcal{C}^{\mu\nu}_{\mathcal{S}} \;. \tag{3.80e}$$

Equations (3.79) and (3.80) determine the components $\mathcal{P}$, $\mathcal{V}^\mu$, and $\mathcal{S}^{\mu\nu}$ in terms of the independent degrees of freedom $\mathcal{F}$ and $\mathcal{A}^\mu$, where it has to be noted that $\mathcal{A}^\mu$ is subject to the constraint (3.80b) and thus only has three independent components. Thus, as we argued in Subsec. 3.2.2, the Wigner function has $(2\frac{1}{2}+1)^2 = 4$ independent degrees of freedom. The evolution equation for these components can either be found by manipulating Eqs. (3.79), (3.80), or, which is easier, by employing the fact that the Dirac spinor fields fulfill the Klein-Gordon equation component-wise, cf. Eqs. (3.68). Acting with the operators $D^2 + m^2$ and $D^{*2} + m^2$ on the Wigner operator (3.72) and using Eqs. (3.68), we find (taking the average over Fock space)

$$\begin{aligned}\left(D^2 - m^2\right) W_{\alpha\beta}(x,k) &= \hbar \int \mathrm{d}^4 v e^{-\frac{i}{\hbar}k\cdot v} \left\langle : \widehat{\bar{\psi}}_\beta \left(x + \frac{v}{2}\right) \left[\left(i\hbar \not{\partial} + m\right) \widehat{\rho}\left(x - \frac{v}{2}\right)\right]_\alpha : \right\rangle \\ &\equiv \hbar \left[\left(\not{D} + m\right) C(x,k)\right]_{\alpha\beta} \;,\end{aligned} \tag{3.81a}$$

$$\begin{aligned}\left(D^{*2} - m^2\right) W_{\alpha\beta}(x,k) &= -\hbar \int \mathrm{d}^4 v e^{-\frac{i}{\hbar}k\cdot v} \left\langle : \left[\widehat{\bar{\rho}}\left(x + \frac{v}{2}\right)\left(i\hbar \overleftarrow{\not{\partial}} - m\right)\right]_\beta \widehat{\psi}_\alpha \left(x - \frac{v}{2}\right) : \right\rangle \\ &\equiv \hbar \left\{\gamma^0 \left[\left(\not{D} + m\right) C(x,k)\right]^\dagger \gamma^0\right\}_{\alpha\beta} \;,\end{aligned} \tag{3.81b}$$

Taking the sum and difference of these equations, we obtain, employing the notation of Refs. [44, 46],

$$\left(k^2 - m^2 - \frac{\hbar^2}{4}\Box\right) W_{\alpha\beta}(x,k) = \hbar \delta M_{\alpha\beta}(x,k) \;, \tag{3.82}$$

$$k \cdot \partial W_{\alpha\beta}(x,k) = \mathcal{C}_{\alpha\beta}(x,k) \;, \tag{3.83}$$

where we introduced

$$\delta M_{\alpha\beta}(x,k) \coloneqq \frac{1}{2}\left\{\left(\not{D} + m\right) C(x,k) + \gamma^0 \left[\left(\not{D} + m\right) C(x,k)\right]^\dagger \gamma^0\right\}_{\alpha\beta} \;, \tag{3.84a}$$

$$\mathcal{C}_{\alpha\beta}(x,k) \coloneqq -\frac{i}{2}\left\{\left(\not{D} + m\right) C(x,k) - \gamma^0 \left[\left(\not{D} + m\right) C(x,k)\right]^\dagger \gamma^0\right\}_{\alpha\beta} \;. \tag{3.84b}$$

Equations (3.82) and (3.83), in analogy to their scalar-field counterparts (3.49) and (3.50), are to be understood as mass-shell and kinetic equations for the spin-1/2 Wigner function. However, they still have to be supplied with the subsidiary conditions imposed by the Dirac equation, i.e., Eqs. (3.79) and (3.80). Since the Wigner function only has 4 independent components, namely $\mathcal{F}$ and the part of $\mathcal{A}^\mu$ that is orthogonal to the four-momentum, it is sufficient to compute the evolution of these degrees of freedom via taking the trace of Eqs. (3.82) and (3.83), weighted with 1 and $\gamma^\mu\gamma_5$, respectively. The remaining components $\mathcal{P}$, $\mathcal{V}^\mu$, and $\mathcal{S}^{\mu\nu}$ can then be reconstructed via the subsidiary conditions (3.79), (3.80).

Extending phase space

Instead of solving four mass-shell and Boltzmann equations, we may *enlarge the phase space* of the theory from (x, k) to $(x, k, \mathfrak{s})$ to be able to solve only a single equation for a scalar distribution function $\mathfrak{f}(x, k, \mathfrak{s})$ [38–42]. In order to achieve this, we introduce a "spin"-variable $\mathfrak{s}$, that is given by a normalized spacelike four-vector, which is orthogonal to the four-momentum k. Defining the scalar distribution function in extended phase space as

$$\mathfrak{f}(x,k,\mathfrak{s}) \coloneqq \frac{1}{2}\left[\mathcal{F}(x,k) - \mathfrak{s}\cdot\mathcal{A}(x,k)\right] \equiv \frac{1}{2}\mathrm{Tr}\left[\left(\mathbb{1} - \not{\mathfrak{s}}\gamma_5\right)W(x,k)\right] , \tag{3.85}$$

we can express Eqs. (3.82) and (3.83) as

$$\left(k^2 - m^2 - \frac{\hbar^2}{4}\Box\right)\mathfrak{f}(x,k,\mathfrak{s}) = \hbar\mathfrak{M}(x,k,\mathfrak{s}) , \tag{3.86}$$

$$k\cdot\partial\mathfrak{f}(x,k,\mathfrak{s}) = \mathfrak{C}(x,k,\mathfrak{s}) . \tag{3.87}$$

Here we defined

$$\mathfrak{M}(x,k,\mathfrak{s}) \coloneqq \frac{1}{2}\mathrm{Tr}\left[\left(\mathbb{1} - \not{\mathfrak{s}}\gamma_5\right)\delta M(x,k)\right] , \tag{3.88a}$$

$$\mathfrak{C}(x,k,\mathfrak{s}) \coloneqq \frac{1}{2}\mathrm{Tr}\left[\left(\mathbb{1} - \not{\mathfrak{s}}\gamma_5\right)\mathcal{C}(x,k)\right] . \tag{3.88b}$$

Note that Eqs. (3.86) and (3.87) contain the information about all independent components of the Wigner function, as we can obtain $\mathcal{F}$ and the part of $\mathcal{A}^\mu$ that is orthogonal to the four-momentum via suitably weighted integrals over spin space,

$$\mathcal{F}(x,k) = \int \mathrm{d}S(k)\mathfrak{f}(x,k,\mathfrak{s}) , \quad K^{\mu\nu}\mathcal{A}_\nu(x,k) = \int \mathrm{d}S(k)\mathfrak{s}^\mu\mathfrak{f}(x,k,\mathfrak{s}) , \tag{3.89}$$

where $K^{\mu\nu} \coloneqq g^{\mu\nu} - k^\mu k^\nu/k^2$ is the projector onto the subspace orthogonal to k. Here, the integration measure in spin space is defined as

$$\mathrm{d}S(k) \coloneqq \frac{\sqrt{k^2}}{\varsigma\pi}\mathrm{d}^4\mathfrak{s}\,\delta(\mathfrak{s}^2+\varsigma^2)\delta(k\cdot\mathfrak{s}) , \quad \varsigma^2 = 3 , \tag{3.90}$$

and we used the identities

$$\int \mathrm{d}S(k) = 2 , \quad \int \mathrm{d}S(k)\mathfrak{s}^\mu = 0 , \quad \int \mathrm{d}S(k)\mathfrak{s}^\mu\mathfrak{s}^\nu = -2K^{\mu\nu} , \tag{3.91}$$

which are special cases of a general formula derived in Appendix F.1.

Before going on to discuss the conserved currents for Dirac fields, we remark that we will always consider spin effects to be small, i.e., they should not enter in the classical limit $\hbar \to 0$. Thus, we assume that $\mathcal{A}^\mu \sim \mathcal{O}(\hbar)$, from which, taking into account Eqs. (3.79), it follows that $\mathcal{P}, \mathcal{S}^{\mu\nu} \sim \mathcal{O}(\hbar)$ as well, while $m\mathcal{V}^\mu = k^\mu\mathcal{F} + \mathcal{O}(\hbar)$. Note that this structure could have been predicted based solely on the fact that at zeroth order in $\hbar$ (which is equivalent to zeroth order in gradients) there are no four-vectors at our disposal except the four-momentum k. Building on the same argument, since there are no pseudoscalar, pseudovector, or tensor quantities at zeroth order, we may conclude that $\mathcal{D}_\mathcal{P}, \mathcal{C}_\mathcal{P}, \mathcal{D}^\mu_\mathcal{A}, \mathcal{C}^\mu_\mathcal{A}, \mathcal{D}^{\mu\nu}_\mathcal{S}, \mathcal{C}^{\mu\nu}_\mathcal{S} \sim \mathcal{O}(\hbar)$, from which we can conclude that $\mathcal{P} \sim \mathcal{O}(\hbar^2)$. With this power-counting in mind, we have the important relation $k\cdot\mathcal{A} \sim \mathcal{O}(\hbar^2)$, implying that, to second order in $\hbar$, we can replace $K^{\mu\nu}\mathcal{A}_\nu(x,k)$ by $\mathcal{A}^\mu(x,k)$ in Eq. (3.89).

3.4.3 Conserved currents

From Noether's theorem we immediately obtain the canonical energy-momentum tensor as

$$T^{\mu\nu}_{D,C} + g^{\mu\nu}\mathcal{L}_D = \frac{i\hbar}{2}\left\langle :\widehat{\bar{\psi}}\gamma^\mu\overleftrightarrow{\partial^\nu}\widehat{\psi}:\right\rangle , \tag{3.92}$$

while the canonical spin tensor assumes the form

$$S^{\lambda\mu\nu}_{D,C} = \frac{1}{4}\left\langle :\widehat{\bar{\psi}}\left\{\gamma^\lambda,\sigma^{\mu\nu}\right\}\widehat{\psi}:\right\rangle = -\frac{1}{2}\epsilon^{\lambda\mu\nu\alpha}\left\langle :\widehat{\bar{\psi}}\gamma_\alpha\gamma_5\widehat{\psi}:\right\rangle , \tag{3.93}$$

where the curly brackets denote the anticommutator, $\{A, B\} \coloneqq AB + BA$, and we used the identity

$$\gamma^\mu\gamma^\nu\gamma^\lambda = g^{\mu\nu}\gamma^\lambda + g^{\nu\lambda}\gamma^\mu - g^{\mu\lambda}\gamma^\nu + i\epsilon^{\mu\nu\lambda\rho}\gamma_\rho\gamma_5$$

to derive the second equality in Eq. (3.93). Note that the canonical energy-momentum tensor is not symmetric, implying that the canonical spin tensor is not conserved. Furthermore, the canonical spin tensor is totally antisymmetric. In terms of the WIGNER function, we can invoke the general relation (3.38) as well as the definitions (3.74) to obtain

$$T^{\mu\nu}_{D,C} + g^{\mu\nu}\mathcal{L}_D = \int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4} k^\nu \mathcal{V}^\mu(x,k) \,, \tag{3.94}$$

$$S^{\lambda\mu\nu}_{D,C} = -\frac{1}{2}\epsilon^{\lambda\mu\nu\alpha}\int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\mathcal{A}_\alpha(x,k) \,. \tag{3.95}$$

Using Eqs. (3.79c), (3.79e), we can rewrite the energy-momentum tensor in terms of the independent components of the WIGNER function,

$$T^{\mu\nu}_{D,C} + g^{\mu\nu}\mathcal{L}_D = \int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\frac{k^\nu}{m}\left[k^\mu\mathcal{F} + \frac{\hbar}{2m}\partial_\alpha\left(\frac{\hbar}{2m}\partial^{[\mu}k^{\alpha]}\mathcal{F} - \epsilon^{\mu\alpha\beta\gamma}k_\beta\mathcal{A}_\gamma\right) - \hbar\mathcal{D}^\mu_\mathcal{V}\right] + \mathcal{O}(\hbar^3) \,, \tag{3.96}$$

where we used that $\hbar^2\mathcal{D}^{\mu\nu}_\mathcal{S} \sim \mathcal{O}(\hbar^3)$. Using now Eqs. (3.89) and defining

$$\Sigma^{\mu\nu}_\mathfrak{s} \coloneqq -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}k_\alpha\mathfrak{s}_\beta \,, \tag{3.97}$$

we may express the energy-momentum and the spin tensor in extended phase space as

$$\begin{aligned} T^{\mu\nu}_{D,C} + g^{\mu\nu}\mathcal{L}_D &= \int \mathrm{d}\Gamma k^\nu\left(k^\mu + \frac{\hbar}{2}\Sigma^{\mu\alpha}_\mathfrak{s}\partial_\alpha + \frac{\hbar^2}{4m^2}\partial^{[\mu}k^{\alpha]}\partial_\alpha\right)f(x,k,\mathfrak{s}) \\ &\quad - \frac{\hbar}{m}\int\frac{\mathrm{d}^4k}{(2\pi\hbar)^4}k^\nu\mathcal{D}^\mu_\mathcal{V} + \mathcal{O}(\hbar^3) \,, \end{aligned} \tag{3.98}$$

$$S^{\lambda\mu\nu}_{D,C} = \frac{1}{2}\int \mathrm{d}\Gamma\left(k^\lambda\Sigma^{\mu\nu}_\mathfrak{s} + k^\mu\Sigma^{\nu\lambda}_\mathfrak{s} + k^\nu\Sigma^{\lambda\mu}_\mathfrak{s}\right)f(x,k,\mathfrak{s}) + \mathcal{O}(\hbar^2) \,. \tag{3.99}$$

In these expressions, we neglected off-shell terms by substituting

$$\mathfrak{f}(x,k,\mathfrak{s}) \eqqcolon 4m\pi\hbar\delta(k^2-m^2)f(x,k,\mathfrak{s}) \,, \tag{3.100}$$

and we defined the on-shell momentum- and spin-space measure

$$\mathrm{d}\Gamma \coloneqq \mathrm{d}K\,\mathrm{d}S(k) \,, \quad \text{where} \quad \mathrm{d}K \coloneqq \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\hbar)^3k^0} \equiv \frac{\mathrm{d}^4k}{(2\pi\hbar)^4}4\pi\hbar\delta(k^2-m^2) \,. \tag{3.101}$$

Furthermore, in order to reformulate the spin tensor, we employed the SCHOUTEN identity

$$k^\lambda\epsilon^{\mu\nu\alpha\beta} + k^\mu\epsilon^{\nu\alpha\beta\lambda} + k^\nu\epsilon^{\alpha\beta\lambda\mu} + k^\alpha\epsilon^{\beta\lambda\mu\nu} + k^\beta\epsilon^{\lambda\mu\nu\alpha} = 0 \,, \tag{3.102}$$

which holds because at least two of the five free indices have to be equal to each other in four dimensions. When contracted with k_β and $\mathcal{A}_\alpha$, it becomes

$$k^2\epsilon^{\lambda\mu\nu\alpha}\mathcal{A}_\alpha = k^\lambda\epsilon^{\mu\nu\beta\alpha}k_\beta\mathcal{A}_\alpha + k^\mu\epsilon^{\nu\lambda\beta\alpha}k_\beta\mathcal{A}_\alpha + k^\nu\epsilon^{\lambda\mu\beta\alpha}k_\beta\mathcal{A}_\alpha + \mathcal{O}(\hbar^2) \,, \tag{3.103}$$

from which Eq. (3.99) follows.

At this point, we can write down the kinetic representation of the conservation laws (2.13b), (2.13c). They read to first order in $\hbar$

$$0 = \partial_\mu\left(T^{\mu\nu}_{D,C} + g^{\mu\nu}\mathcal{L}_D\right) = \int \mathrm{d}\Gamma k^\nu C(x,k,\mathfrak{s}) \,, \tag{3.104a}$$

$$0 = \hbar\partial_\lambda S^{\lambda\mu\nu}_{D,C} + T^{[\mu\nu]}_{D,C} = \frac{\hbar}{2}\int \mathrm{d}\Gamma\Sigma^{\mu\nu}_\mathfrak{s}C(x,k,\mathfrak{s}) - \frac{\hbar}{m}\int\frac{\mathrm{d}^4k}{(2\pi\hbar)^4}k^{[\nu}\mathcal{D}^{\mu]}_\mathcal{V} \,, \tag{3.104b}$$

where we made use of Eq. (3.87) and defined the on-shell collision kernel

$$\mathfrak{C}(x,k,\mathfrak{s}) =: 4m\pi\hbar C(x,k,\mathfrak{s}) \,. \tag{3.105}$$

Furthermore, in order to derive Eq. (3.104b), we made use of Eq. (3.104a). It is clear that Eq. (3.104a) describes the conservation of linear momentum in a collision, while Eq. (3.104b) specifies how spin is converted into orbital angular momentum. In order to see this, let us assume the form

$$\hbar\mathcal{D}^\mu_\mathcal{V}(x,k) \stackrel{?}{=} -\int \mathrm{d}S\Delta^\mu(x,k,\mathfrak{s})\mathfrak{C}(x,k,\mathfrak{s}) \,, \tag{3.106}$$

such that the conservation of the total angular momentum becomes

$$\int \mathrm{d}\Gamma \left(\frac{\hbar}{2}\Sigma^{\mu\nu}_\mathfrak{s} + k^{[\nu}\Delta^{\mu]}\right) C(x,k,\mathfrak{s}) = 0 \,. \tag{3.107}$$

Here, the second term can be identified as a part of the orbital angular momentum. Thus, a change in the spin of the particle is accompanied by a shift Δ in its position. We will explicitly calculate how this shift looks like in Chapter 4, and we will find that the guess (3.106) is not completely correct.

Even though the conservation law (3.104b) is general, the canonical pseudogauge has some drawbacks in its interpretation. Specifically, consider the divergence of the spin tensor,

$$\begin{aligned}\hbar\partial_\lambda S^{\lambda\mu\nu}_{D,C} &= \frac{\hbar}{2}\int \mathrm{d}\Gamma \left[\Sigma^{\mu\nu}_\mathfrak{s} C(x,k,\mathfrak{s}) + k^{[\mu}\Sigma^{\nu]\lambda}_\mathfrak{s}\partial_\lambda f(x,k,\mathfrak{s})\right] \\ &= \frac{\hbar}{2}\int \mathrm{d}\Gamma k^{[\mu}\Sigma^{\nu]\lambda}_\mathfrak{s}\partial_\lambda f(x,k,\mathfrak{s}) + \frac{\hbar}{m}\int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4} k^{[\nu}\mathcal{D}^{\mu]}_\mathcal{V} \,,\end{aligned} \tag{3.108}$$

where we used Eq. (3.104b) in the second step. Manifestly, the spin tensor is not conserved on its own, not even in the free case where $\mathcal{D}^\mu_\mathcal{V} = 0$. Intuitively, one would expect that the energy-momentum tensor is symmetric in the free case, such that the spin tensor is also conserved on its own.

Belinfante pseudogauge

One way to achieve this symmetrization of the energy-momentum tensor is through the procedure introduced by Belinfante and Rosenfeld [136, 137] by choosing the superpotentials

$$\widehat{\Phi}^{\lambda\mu\nu}_B := \widehat{S}^{\lambda\mu\nu}_{D,C} \,, \quad \widehat{Z}^{\mu\nu\lambda\rho}_B := 0 \,. \tag{3.109}$$

Then, the corresponding conserved currents read

$$T^{\mu\nu}_{D,B} + g^{\mu\nu}\mathcal{L}_D = \frac{i\hbar}{4}\Big\langle :\widehat{\bar{\psi}}\gamma^{(\mu}\overleftrightarrow{\partial}^{\nu)}\widehat{\psi}: \Big\rangle \,, \tag{3.110}$$

$$S^{\lambda\mu\nu}_{D,B} = 0 \,. \tag{3.111}$$

In terms of the Wigner function, the energy-momentum tensor is straightforwardly computed to be

$$\begin{aligned}T^{\mu\nu}_{D,B} + g^{\mu\nu}\mathcal{L}_D &= \frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4} k^{(\nu}\mathcal{V}^{\mu)}(x,k) \\ &= \frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\frac{k^{(\nu}}{m}\left[k^{\mu)}\mathcal{F} + \frac{\hbar}{2m}\partial_\alpha\left(\frac{\hbar}{2m}\partial^{[\mu)}k^{\alpha]}\mathcal{F} - \epsilon^{\mu)\alpha\beta\gamma}k_\beta\mathcal{A}_\gamma\right) - \hbar\mathcal{D}^{\mu)}_\mathcal{V}\right] + \mathcal{O}(\hbar^3) \,,\end{aligned} \tag{3.112}$$

which in extended phase space becomes

$$\begin{aligned}T^{\mu\nu}_{D,B} + g^{\mu\nu}\mathcal{L}_D = &\frac{1}{2}\int \mathrm{d}\Gamma k^{(\nu}\left(k^{\mu)} + \frac{\hbar}{2}\Sigma^{\mu)\alpha}_\mathfrak{s}\partial_\alpha + \frac{\hbar^2}{4m^2}\partial^{[\mu)}k^{\alpha]}\partial_\alpha\right) f(x,k,\mathfrak{s}) \\ &- \frac{\hbar}{2m}\int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4} k^{(\nu}\mathcal{D}^{\mu)}_\mathcal{V} + \mathcal{O}(\hbar^3) \,.\end{aligned} \tag{3.113}$$

In this case, the spin tensor is constantly zero, such that all spin dynamics are absorbed into the energy-momentum tensor.

HW pseudogauge

A less drastic way of ensuring the symmetry of the energy-momentum tensor in the absence of interactions consists in introducing the HW (after Hilgevoord and Wouthuysen) superpotentials [54, 138, 139]

$$\widehat{\Phi}^{\lambda\mu\nu}_{\text{HW}} := \widehat{M}^{[\mu\nu]\lambda} - g^{\lambda[\mu}\widehat{M}_{\alpha}{}^{\nu]\alpha} + \widehat{Q}^{\lambda\mu\nu} , \quad \widehat{Z}^{\mu\nu\lambda\rho}_{\text{HW}} := -\frac{1}{8m}\widehat{\bar{\psi}}\{\sigma^{\mu\nu},\sigma^{\lambda\rho}\}\widehat{\psi} , \tag{3.114}$$

where we defined

$$\widehat{M}^{\lambda\mu\nu} := \frac{i\hbar}{4m}\widehat{\bar{\psi}}\sigma^{\mu\nu}\overleftrightarrow{\partial^{\lambda}}\widehat{\psi} , \qquad \widehat{Q}^{\lambda\mu\nu} := -\frac{\hbar}{4m}\left(\widehat{\bar{\rho}}\gamma^{\lambda}\sigma^{\mu\nu}\widehat{\psi} + \widehat{\bar{\psi}}\sigma^{\mu\nu}\gamma^{\lambda}\widehat{\rho}\right) . \tag{3.115}$$

In this case, it is actually easier to directly evaluate the pseudogauge transformation in terms of the Wigner function. In particular, note that

$$M^{\lambda\mu\nu} = \frac{1}{2m}\int\frac{\mathrm{d}^4k}{(2\pi\hbar)^4}k^{\lambda}\mathcal{S}^{\mu\nu} , \tag{3.116a}$$

$$Q^{\lambda\mu\nu} = \frac{\hbar}{2m}\int\frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\left(-g^{\lambda[\mu}\mathcal{C}^{\nu]}_{\mathcal{V}} + \epsilon^{\lambda\mu\nu\alpha}\mathcal{D}_{\mathcal{A},\alpha}\right) , \tag{3.116b}$$

$$Z^{\mu\nu\lambda\rho} = -\frac{1}{4m}\int\frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\left(g^{\lambda[\mu}g^{\nu]\rho}\mathcal{F} - \epsilon^{\lambda\mu\nu\rho}\mathcal{P}\right) , \tag{3.116c}$$

where we used

$$\gamma^{\lambda}\sigma^{\mu\nu} = ig^{\lambda[\mu}\gamma^{\nu]} - \epsilon^{\lambda\mu\nu\alpha}\gamma_{\alpha}\gamma_5$$

to derive the second equality. Then, employing that for any third-rank tensor A that is antisymmetric in its last two indices it holds that

$$A^{[\mu\nu]\lambda} + A^{[\mu\lambda]\nu} + A^{[\nu\lambda]\mu} = -A^{\nu[\mu\lambda]} \equiv -2A^{\nu\mu\lambda} , \tag{3.117}$$

we find for the energy-momentum tensor

$$\begin{aligned} &T^{\mu\nu}_{D,\text{HW}} + g^{\mu\nu}\mathcal{L}_D \\ &= \frac{1}{m}\int\frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\left[k^{\nu}\left(k^{\mu}\mathcal{F} - \hbar\mathcal{D}^{\mu}_{\mathcal{V}}\right) + \frac{\hbar^2}{4}\left(\partial^{\nu}\partial^{\mu} - g^{\mu\nu}\Box\right)\mathcal{F} + \frac{\hbar^2}{4}\epsilon^{\mu\nu\alpha\beta}\partial_{\alpha}\mathcal{D}_{\mathcal{A},\beta}\right] + \mathcal{O}(\hbar^3) , \end{aligned} \tag{3.118}$$

while the spin tensor becomes [due to Eqs. (3.79d) and (3.80c)]

$$\begin{aligned} S^{\lambda\mu\nu}_{D,\text{HW}} &= \frac{1}{2m}\int\frac{\mathrm{d}^4k}{(2\pi\hbar)^4}k^{\lambda}\mathcal{S}^{\mu\nu} \\ &= \frac{1}{2m}\int\frac{\mathrm{d}^4k}{(2\pi\hbar)^4}k^{\lambda}\left(\frac{\hbar}{2m}\partial^{[\mu}k^{\nu]}\mathcal{F} - \epsilon^{\mu\nu\alpha\beta}k_{\alpha}\mathcal{A}_{\beta}\right) + \mathcal{O}(\hbar^2) . \end{aligned} \tag{3.119}$$

Note that, since the first term in the spin tensor is separately conserved due to momentum conservation, when taking into account Eq. (3.104b) it becomes clear that the HW spin tensor is conserved in the absence of interactions, and consequently the energy-momentum tensor is symmetric in that case. In general, however, the HW energy-momentum tensor is not symmetric.

Lastly, we remark that, in the case of free fields, this set of energy-momentum and spin tensors can also be obtained via the Noether procedure from the Lagrangian [140]

$$\widehat{\mathcal{L}}_{D,\text{HW}} := \frac{\hbar^2}{2m}\left[\left(\partial^{\mu}\widehat{\bar{\psi}}\right)\left(\partial_{\mu}\widehat{\psi}\right) - \frac{m^2}{\hbar^2}\widehat{\bar{\psi}}\widehat{\psi}\right] , \tag{3.120}$$

where the resulting equation of motion is given by the Klein-Gordon equation and the Dirac equation has to be supplied as a subsidiary condition.

GLW pseudogauge

Alternatively, the Lagrangian (3.120) can be changed by a total derivative to give

$$\widehat{\mathcal{L}}_{D,\mathrm{GLW}} \coloneqq -\frac{\hbar^2}{4m}\left[\widehat{\bar{\psi}}\left(\Box + \frac{m^2}{\hbar^2}\right)\widehat{\psi} + \text{h.c.}\right] , \tag{3.121}$$

which yields the GLW currents in the case of free fields. Here, the (Fock-space averages of the) superpotentials read in terms of the Wigner function

$$\Phi^{\lambda\mu\nu}_{\mathrm{GLW}} \coloneqq \frac{1}{2m}\int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4} k^{[\mu}\mathcal{S}^{\nu]\lambda} , \quad Z^{\mu\nu\lambda\rho}_{\mathrm{GLW}} \coloneqq 0 . \tag{3.122}$$

The corresponding energy-momentum tensor is given by

$$T^{\mu\nu}_{D,\mathrm{GLW}} + g^{\mu\nu}\mathcal{L}_D = \frac{1}{m}\int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4} k^\nu \left(k^\mu \mathcal{F} - \hbar \mathcal{D}^\mu_{\mathcal{V}}\right) , \tag{3.123}$$

making it obvious that the hard-to-interpret terms of second order in $\hbar$ in Eq. (3.118) have been removed. Likewise, the spin tensor now reads

$$S^{\lambda\mu\nu}_{D,\mathrm{GLW}} = \frac{1}{2m}\int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\left(k^\lambda \mathcal{S}^{\mu\nu} + \hbar\epsilon^{\lambda\mu\nu\alpha}\mathcal{D}_{\mathcal{A},\alpha} - \frac{\hbar}{2}\epsilon^{\lambda\mu\nu\alpha}\partial_\alpha \mathcal{P}\right) . \tag{3.124}$$

Using Eqs. (3.79d) and (3.79e), this can be rewritten as

$$S^{\lambda\mu\nu}_{D,\mathrm{GLW}} = \frac{1}{2m}\int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4} k^\lambda \left(-\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}k_\alpha \mathcal{A}_\beta - \frac{\hbar}{2m^2}k^{[\mu}\partial^{\nu]}\mathcal{F}\right) + \mathcal{O}(\hbar^2) . \tag{3.125}$$

Given that the last term in the equation above is conserved on its own, we may ask whether it can also be removed by a suitable pseudogauge transformation. Indeed, we may modify the quantities given in Ref. [43] by implementing an additional superpotential

$$Z^{\mu\nu\lambda\rho}_{\mathrm{GLW}'} \coloneqq -\frac{1}{4m^3}\int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4} k^{[\lambda}g^{\rho][\mu}k^{\nu]}\mathcal{F} ,$$

such that (upon using momentum conservation) the spin tensor becomes

$$S^{\lambda\mu\nu}_{D,\mathrm{GLW}'} = -\frac{1}{2m^2}\int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4} k^\lambda \epsilon^{\mu\nu\alpha\beta}k_\alpha \mathcal{A}_\beta + \mathcal{O}(\hbar^2) , \tag{3.126}$$

while the energy-momentum tensor stays unchanged.

Since in this thesis we do not consider the interaction term proportional to the Lagrangian,[4] in extended phase space we have thus the intuitive result

$$T^{\mu\nu}_{D,\mathrm{GLW}'} = \int \mathrm{d}\Gamma k^\mu k^\nu f(x,k,\mathfrak{s}) + \mathcal{O}(\hbar^2) , \tag{3.127}$$

$$S^{\lambda\mu\nu}_{D,\mathrm{GLW}'} = \frac{1}{2}\int \mathrm{d}\Gamma k^\lambda \Sigma^{\mu\nu}_{\mathfrak{s}} f(x,k,\mathfrak{s}) + \mathcal{O}(\hbar^2) , \tag{3.128}$$

which is the formulation we are going to use when constructing spin hydrodynamics in Chapter 6.

3.5 Vector fields

Finally, we consider charged Proca fields $\widehat{V}^\mu$, which transform in the $(1/2, 1/2)$-representation of the Lorentz group. We will mark the quantities referring to these fields with a subscript P.

[4]The omission of this contribution is consistent with the procedure in the next chapters, where mean-field effects will not be considered as well. In the diagrammatic language of Chapter 5, this corresponds to neglecting the tadpole contributions.

3.5.1 Dynamics

The Lagrangian reads

$$\widehat{\mathcal{L}}_P = -\hbar\left(\frac{1}{2}\widehat{V}^{\dagger\mu\nu}\widehat{V}_{\mu\nu} - \frac{m^2}{\hbar^2}\widehat{V}^{\dagger\mu}\widehat{V}_\mu\right) + \widehat{\mathcal{L}}_{P,\mathrm{int}} \;, \tag{3.129}$$

where $\widehat{V}^{\mu\nu} := \partial^{[\mu}\widehat{V}^{\nu]}$ and $\widehat{\mathcal{L}}_{P,\mathrm{int}}$ again denotes a general interaction Lagrangian. With the EULER-LAGRANGE equations, this Lagrangian leads to the following equations of motion,

$$\partial_\mu\widehat{V}^{\mu\nu} + \frac{m^2}{\hbar^2}\widehat{V}^\nu = \widehat{\rho}^\nu \;, \tag{3.130a}$$

$$\partial_\mu\widehat{V}^{\dagger\mu\nu} + \frac{m^2}{\hbar^2}\widehat{V}^{\dagger\nu} = \widehat{\rho}^{\dagger\nu} \;, \tag{3.130b}$$

where the source terms read

$$\widehat{\rho}^\mu := -\frac{1}{\hbar}\frac{\partial\widehat{\mathcal{L}}_{P,\mathrm{int}}}{\partial\widehat{V}^\dagger_\mu} \;, \quad \widehat{\rho}^{\dagger\mu} := -\frac{1}{\hbar}\frac{\partial\widehat{\mathcal{L}}_{P,\mathrm{int}}}{\partial\widehat{V}_\mu} \;. \tag{3.131}$$

Taking the divergence of Eqs. (3.130) yields

$$\partial\cdot\widehat{V} = \frac{\hbar^2}{m^2}\partial\cdot\widehat{\rho}\,, \quad \partial\cdot\widehat{V}^\dagger = \frac{\hbar^2}{m^2}\partial\cdot\widehat{\rho}^\dagger \tag{3.132}$$

which implies that the equations of motion become

$$\left(\Box + \frac{m^2}{\hbar^2}\right)\widehat{V}^\mu = \left(g^{\mu\nu} + \frac{\hbar^2}{m^2}\partial^\mu\partial^\nu\right)\widehat{\rho}_\nu \;, \tag{3.133a}$$

$$\left(\Box + \frac{m^2}{\hbar^2}\right)\widehat{V}^{\dagger\mu} = \left(g^{\mu\nu} + \frac{\hbar^2}{m^2}\partial^\mu\partial^\nu\right)\widehat{\rho}^\dagger_\nu \;. \tag{3.133b}$$

Note that in the free case, i.e., when $\widehat{\rho}^\mu = 0$, each component of the vector field separately fulfills the free KLEIN-GORDON equation. Furthermore, the second terms on the right-hand sides of Eqs. (3.133) can be seen to be potentially problematic in the limit of $m \to 0$; we will shortly come back to this limit later.

We now compute the GREEN's function $G^{\mu\nu}(x,x')$ of the massive vector field, which has to fulfill

$$\left(\Box_x + \frac{m^2}{\hbar^2}\right)G^{\mu\nu}(x,x') - \partial^\nu_x\partial_{x,\alpha}G^{\alpha\mu}(x,x') = g^{\mu\nu}\delta^{(4)}(x-x') \;. \tag{3.134}$$

Transforming Eq. (3.134) into FOURIER space, we obtain

$$\begin{aligned}&\int\frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\int\frac{\mathrm{d}^4k'}{(2\pi\hbar)^4}e^{-\frac{i}{\hbar}(k\cdot x+k'\cdot x')}\left[(-k^2+m^2)\widetilde{G}^{\mu\nu}(k,k') + k^\nu k_\alpha\widetilde{G}^{\alpha\mu}(k,k')\right]\\ &= \int\frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\int\frac{\mathrm{d}^4k'}{(2\pi\hbar)^4}e^{-\frac{i}{\hbar}(k\cdot x+k'\cdot x')}(2\pi\hbar)^4\hbar^2\delta^{(4)}(k+k')g^{\mu\nu} \;.\end{aligned} \tag{3.135}$$

Defining $\widetilde{G}^{\mu\nu}(k,k') =: (2\pi\hbar)^4\delta^{(4)}(k+k')\widetilde{G}^{\mu\nu}(k)$, the equation above implies

$$(-k^2+m^2)\widetilde{G}^{\mu\nu}(k) + k^\nu k_\alpha\widetilde{G}^{\alpha\mu}(k) = \hbar^2 g^{\mu\nu} \;. \tag{3.136}$$

Contracting this equation with $K_{\lambda\mu}k_\nu$ and $k_\mu K_{\nu\lambda}$ yields

$$K_{\lambda\mu}\widetilde{G}^{\mu\nu}(k)k_\nu = 0 \quad \text{and} \quad k_\mu\widetilde{G}^{\mu\nu}(k)K_{\nu\lambda} = 0 \;, \tag{3.137}$$

respectively. Furthermore, from antisymmetrizing Eq. (3.136) it follows that $\widetilde{G}^{\mu\nu}(k)$ is symmetric, implying that we may decompose the GREEN's function as

$$\widetilde{G}^{\mu\nu}(k) = E^{\mu\nu}\widetilde{G}_E(k) + K^{\mu\nu}\widetilde{G}_K(k) \;, \tag{3.138}$$

where $E^{\mu\nu} := k^\mu k^\nu / k^2$. Projecting Eq. (3.136) with $K_{\mu\nu}$ and $E_{\mu\nu}$, we obtain

$$(-k^2 + m^2)\widetilde{G}_K(k) = \hbar^2 \; , \tag{3.139a}$$

$$m^2 \widetilde{G}_E(k) = \hbar^2 \; . \tag{3.139b}$$

Thus, the retarded Green's function reads in Fourier space

$$\widetilde{G}^{\mu\nu}_R(k) = \hbar^2 \left(\frac{E^{\mu\nu}}{m^2} - \frac{K^{\mu\nu}}{k^2 - m^2 + i\eta k^0} \right) \; . \tag{3.140a}$$

Its advanced counterpart is obtained similarly as

$$\widetilde{G}^{\mu\nu}_A(k) = \hbar^2 \left(\frac{E^{\mu\nu}}{m^2} - \frac{K^{\mu\nu}}{k^2 - m^2 - i\eta k^0} \right) \; , \tag{3.140b}$$

where the infinitesimal quantity in the denominator now indicates that the half-circle has to be closed in the upper half-plane. Employing the Green's functions of the scalar field (3.45) and making use of the fact that η is an infinitesimal quantity, we find

$$\widetilde{G}^{\mu\nu}_R(k) = \widetilde{\Delta}_R(k) \left(g^{\mu\nu} - \frac{k^2}{m^2} E^{\mu\nu} \right) \; , \tag{3.141a}$$

$$\widetilde{G}^{\mu\nu}_A(k) = \widetilde{\Delta}_A(k) \left(g^{\mu\nu} - \frac{k^2}{m^2} E^{\mu\nu} \right) \; . \tag{3.141b}$$

The massless case

In the massless case, the equations of motion for the vector field reduce to

$$\partial_\mu \widehat{V}^{\mu\nu} = \widehat{\rho}^\nu \; , \tag{3.142}$$

which yields

$$\partial \cdot \widehat{\rho} = 0 \tag{3.143}$$

upon taking the divergence. This implies that the source term for the massless vector field has to be *conserved*. Furthermore, note that at this point we lose the subsidiary condition (3.132) due to the mass vanishing identically. As expected for a massless vector field, the Lagrangian is now invariant under *gauge transformations*

$$\widehat{V}^\mu \to \widehat{V}^\mu + \partial^\mu \widehat{\Lambda} \; , \tag{3.144}$$

where $\widehat{\Lambda}$ is an arbitrary scalar function. In order to compute physical quantities, the gauge has to be *fixed* in some way, as the equations of motion are ill-posed otherwise. One popular choice of gauge fixing consists in imposing the Lorenz gauge

$$\partial \cdot \widehat{V} = 0 \; , \tag{3.145}$$

which looks identical to Eq. (3.132) for the case of free fields. However, it has to be stressed that the former, although intuitive, is only one peculiar choice of gauge, and there exist many situations in which other gauges may be more suitable.[5] Equation (3.132) on the other hand is enforced by the equations of motion directly.

3.5.2 Wigner function

The Wigner operator for charged vector fields is defined as

$$\widehat{W}^{\mu\nu}(x,k) := -\frac{2}{\hbar} \int \mathrm{d}^4 v e^{-\frac{i}{\hbar} k\cdot v} \widehat{V}^{\dagger\mu} \left(x + \frac{v}{2} \right) \widehat{V}^\nu \left(x - \frac{v}{2} \right) \; , \tag{3.146}$$

[5] As one example, cf. Ref. [141] for a thorough discussion of the Fock-Schwinger gauge.

cf. Eq. (3.31). Acting with the operators D_μ and D^*_μ on the definition above, using the constraint equations (3.132) on the vector fields, and taking the average over Fock space, we find

$$\left(k_\mu + \frac{i\hbar}{2}\partial_\mu\right) W^{\nu\mu}(x,k) = \frac{\hbar}{m^2}\left(k_\mu + \frac{i\hbar}{2}\partial_\mu\right) C^{\nu\mu}(x,k)\ , \tag{3.147a}$$

$$\left(k_\mu - \frac{i\hbar}{2}\partial_\mu\right) W^{\mu\nu}(x,k) = \frac{\hbar}{m^2}\left(k_\mu - \frac{i\hbar}{2}\partial_\mu\right) C^{*\nu\mu}(x,k)\ , \tag{3.147b}$$

where we used the fact that the Wigner function is hermitian and defined

$$C^{\mu\nu}(x,k) := -2\int \mathrm{d}^4v e^{-\frac{i}{\hbar}k\cdot v}\left\langle : \widehat{V}^{\dagger\mu}\left(x+\frac{v}{2}\right)\widehat{\rho}^\nu\left(x-\frac{v}{2}\right) :\right\rangle\ . \tag{3.148}$$

Acting with the operator $D^2 - m^2$ on the Wigner function and using Eqs. (3.133) yields

$$\begin{aligned}\left(k^2 - m^2 - \frac{\hbar^2}{4}\Box + i\hbar k\cdot\partial\right) W^{\mu\nu}(x,k) = &-\hbar C^{\mu\nu}(x,k) \\ &+ \frac{\hbar}{m^2}\left(k^\nu k_\alpha - \frac{\hbar^2}{4}\partial^\nu\partial_\alpha + \frac{i\hbar}{2}k^{(\nu}\partial_{\alpha)}\right) C^{\mu\alpha}(x,k)\ .\end{aligned} \tag{3.149}$$

Equations (3.147), which originate from Eq. (3.132) that removed one dynamical component from the fields $\widehat{V}^\mu$, have the effect of reducing the number of independent degrees of freedom of the Wigner function to nine. It is advantageous to decompose the Wigner function into symmetric and antisymmetric parts,

$$W^{\mu\nu} = W^{\mu\nu}_S + W^{\mu\nu}_A\ ,\quad W^{\mu\nu}_S := \frac{1}{2}W^{(\mu\nu)}\ ,\quad W^{\mu\nu}_A := \frac{1}{2}W^{[\mu\nu]}\ , \tag{3.150}$$

for which we obtain constraint equations by taking the sum and difference of Eqs. (3.147):

$$k_\mu W^{\mu\nu}_S - \frac{i\hbar}{2}\partial_\mu W^{\mu\nu}_A = \frac{\hbar}{m^2}\left[k_\mu\left(i\mathcal{C}^{\mu\nu}_A - \delta M^{\mu\nu}_S\right) + \frac{\hbar}{2}\partial_\mu\left(\mathcal{C}^{\mu\nu}_S + i\delta M^{\mu\nu}_A\right)\right]\ , \tag{3.151a}$$

$$k_\mu W^{\mu\nu}_A - \frac{i\hbar}{2}\partial_\mu W^{\mu\nu}_S = \frac{\hbar}{m^2}\left[k_\mu\left(i\mathcal{C}^{\mu\nu}_S - \delta M^{\mu\nu}_A\right) + \frac{\hbar}{2}\partial_\mu\left(\mathcal{C}^{\mu\nu}_A + i\delta M^{\mu\nu}_S\right)\right]\ . \tag{3.151b}$$

Here, we defined

$$\delta M^{\mu\nu} := -\frac{1}{2}\left(C^{\mu\nu} + C^{*\nu\mu}\right)\ ,\quad \mathcal{C}^{\mu\nu} := \frac{i}{2}\left(C^{\mu\nu} - C^{*\nu\mu}\right)\ , \tag{3.152}$$

and decomposed these objects into symmetric and antisymmetric parts as well. Note that, apart from signs and the operators $\not{D} + m$, Eqs. (3.152) are analogous to Eqs. (3.84). Subtracting Eq. (3.149) from its hermitian conjugate and splitting the result into symmetric and antisymmetric parts, we also obtain kinetic equations for the symmetric and antisymmetric parts of the Wigner function,

$$\begin{aligned}k\cdot\partial W^{\mu\nu}_S = \mathcal{C}^{\mu\nu}_S - \frac{1}{2m^2}\Bigg[&\left(k_\alpha k^{(\mu} - \frac{\hbar^2}{4}\partial_\alpha\partial^{(\mu}\right)\left(\mathcal{C}^{\nu)\alpha}_S - i\delta M^{\nu)\alpha}_A\right) \\ &+\frac{\hbar}{2}\left(k_\alpha\partial^{(\mu} + \partial_\alpha k^{(\mu}\right)\left(i\mathcal{C}^{\nu)\alpha}_A + \delta M^{\nu)\alpha}_S\right)\Bigg]\ ,\end{aligned} \tag{3.153a}$$

$$\begin{aligned}k\cdot\partial W^{\mu\nu}_A = \mathcal{C}^{\mu\nu}_A - \frac{1}{2m^2}\Bigg[&\left(k_\alpha k^{[\mu} - \frac{\hbar^2}{4}\partial_\alpha\partial^{[\mu}\right)\left(i\delta M^{\nu]\alpha}_S - \mathcal{C}^{\nu]\alpha}_A\right) \\ &-\frac{\hbar}{2}\left(k_\alpha\partial^{[\mu} + \partial_\alpha k^{[\mu}\right)\left(i\mathcal{C}^{\nu]\alpha}_S + \delta M^{\nu]\alpha}_A\right)\Bigg]\ .\end{aligned} \tag{3.153b}$$

Since there appear contractions with the momentum k, it appears sensible to further decompose the Wigner function with respect to momentum such that we are able to extract the independent components. This idea is analogous to the Clifford decomposition (3.73) in the spin-1/2 case and yields

$$W_S^{\mu\nu}(x,k) = E^{\mu\nu} f_E + \frac{k^{(\mu}}{2k} F_S^{\nu)} + K^{\mu\nu} f_K + F_K^{\mu\nu} \ , \tag{3.154a}$$

$$W_A^{\mu\nu}(x,k) = i\frac{k^{[\mu}}{2k} F_A^{\nu]} + i\epsilon^{\mu\nu\alpha\beta} \frac{k_\alpha}{m} G_\beta \ , \tag{3.154b}$$

where the components are given by

$$f_E = E_{\mu\nu} W^{\mu\nu} \ , \quad F_S^\mu = K^{\mu(\alpha} \frac{k^{\beta)}}{k} W_{\alpha\beta} \ , \quad f_K = \frac{1}{3} K_{\mu\nu} W^{\mu\nu} \ , \quad F_K^{\mu\nu} = K^{\mu\nu}_{\alpha\beta} W^{\alpha\beta} \ , \tag{3.155}$$

and

$$F_A^\mu = iK^{\mu[\alpha} \frac{k^{\beta]}}{k} W_{\alpha\beta} \ , \quad G^\mu = -\frac{i}{2} \epsilon^{\mu\nu\alpha\beta} \frac{k_\nu}{m} W_{\alpha\beta} \ , \tag{3.156}$$

respectively. Here we introduced the traceless projector onto the subspace orthogonal to the four-momentum,

$$K^{\mu\nu}_{\alpha\beta} := \frac{1}{2}\left(K^\mu_\alpha K^\nu_\beta + K^\nu_\alpha K^\mu_\beta\right) - \frac{1}{3} K^{\mu\nu} K_{\alpha\beta} \ .$$

The vectorial quantities F_A, F_K fulfill $F_S \cdot k = F_A \cdot k = 0$ and thus only have three independent components. The same holds true for the axial vector G. The tensorial quantity F_K on the other hand fulfills $F_K^{\mu\nu} = F_K^{\nu\mu}$, $F_K^{\mu\nu} k_\nu = 0$ and $F^\mu_{K\mu} = 0$, leaving five independent components. Thus, the constraints (3.147) determine F_A, F_S, and f_E in terms of f_K, G, and F_K. Similarly, we decompose the quantities δM and $\mathcal{C}$,

$$\delta M_S^{\mu\nu}(x,k) = E^{\mu\nu} \mathcal{D}_E + \frac{k^{(\mu}}{2k} \mathcal{D}_S^{\nu)} + K^{\mu\nu} \mathcal{D}_K + \mathcal{D}_K^{\mu\nu} \ , \tag{3.157a}$$

$$\delta M_A^{\mu\nu}(x,k) = i\frac{k^{[\mu}}{2k} \mathcal{D}_A^{\nu]} + i\epsilon^{\mu\nu\alpha\beta} \frac{k_\alpha}{m} \mathcal{D}_{G,\beta} \ , \tag{3.157b}$$

$$\mathcal{C}_S^{\mu\nu}(x,k) = E^{\mu\nu} \mathcal{C}_E + \frac{k^{(\mu}}{2k} \mathcal{C}_S^{\nu)} + K^{\mu\nu} \mathcal{C}_K + \mathcal{C}_K^{\mu\nu} \ , \tag{3.157c}$$

$$\mathcal{C}_A^{\mu\nu}(x,k) = i\frac{k^{[\mu}}{2k} \mathcal{C}_A^{\nu]} + i\epsilon^{\mu\nu\alpha\beta} \frac{k_\alpha}{m} \mathcal{C}_{G,\beta} \ , \tag{3.157d}$$

with the components defined analogously to the ones introduced before.

In order to expand the kinetic equations to first order in the Planck constant, we have to clarify which parts of the collision terms enter at leading order. Using the definition of the collision term (3.148) and the constraint (3.132), we obtain

$$\left(k_\mu - \frac{i\hbar}{2}\partial_\mu\right) C^{\mu\nu} \sim \mathcal{O}(\hbar) \ , \qquad \left(k_\mu + \frac{i\hbar}{2}\partial_\mu\right) C^{*\mu\nu} \sim \mathcal{O}(\hbar) \ , \tag{3.158}$$

from which it follows that $\mathcal{C}_E$, $\mathcal{D}_E \sim \mathcal{O}(\hbar)$. As we did in the last section in the spin-1/2 case, we follow Refs. [45, 54] and consider a situation where no initial large (vector- or tensor-) polarization is present. In this case we conclude that G^μ, $F_K^{\mu\nu} \sim \mathcal{O}(\hbar)$ as well as $\mathcal{C}_S^\mu$, $\mathcal{C}_A^\mu$, $\mathcal{D}_S^\mu$, $\mathcal{D}_A^\mu \sim \mathcal{O}(\hbar)$, which follows from the fact that there are no vector or tensor structures at our disposal at order $\mathcal{O}(1)$ which possess the required symmetries of the aforementioned terms.

With these simplifications, we obtain from Eqs. (3.147)

$$f_E = \frac{\hbar^2}{4k^2} K^{\alpha\beta} \partial_\alpha \partial_\beta f_K - \frac{\hbar}{m^2} \mathcal{D}_E + \mathcal{O}(\hbar^3) \ , \tag{3.159a}$$

$$F_S^\nu = \mathcal{O}(\hbar^2) \ , \tag{3.159b}$$

$$kF_A^\nu = \hbar K^{\nu\mu} \partial_\mu f_K + \mathcal{O}(\hbar^2) \ . \tag{3.159c}$$

Furthermore, from Eqs. (3.153) we obtain a simple form of the kinetic equations for the independent components of the WIGNER function,

$$k \cdot \partial f_K = \mathcal{C}_K + \mathcal{O}(\hbar^2) \;, \tag{3.160a}$$
$$k \cdot \partial F_K^{\mu\nu} = \mathcal{C}_K^{\mu\nu} + \mathcal{O}(\hbar^2) \;, \tag{3.160b}$$
$$k \cdot \partial G^\mu = \mathcal{C}_G^\mu + \mathcal{O}(\hbar^2) \;, \tag{3.160c}$$

while the mass-shell equations follow from the real part of Eq. (3.149),

$$(k^2 - m^2) f_K = \hbar \mathcal{D}_K + \mathcal{O}(\hbar^2) \;, \tag{3.161a}$$
$$(k^2 - m^2) F_K^{\mu\nu} = \hbar \mathcal{D}_K^{\mu\nu} + \mathcal{O}(\hbar^2) \;, \tag{3.161b}$$
$$(k^2 - m^2) G^\mu = \hbar \mathcal{D}_G^\mu + \mathcal{O}(\hbar^2) \;. \tag{3.161c}$$

Extending phase space

In order to combine the nine independent components of the WIGNER function into one scalar distribution function, we may, in the same manner as introduced in the last section for spin-1/2 particles, enlarge the phase space by introducing an additional variable $\mathfrak{s}^\mu$, together with a respective measure

$$\mathrm{d}S(k) \coloneqq \frac{3m}{2\varsigma\pi} \mathrm{d}^4\mathfrak{s}\, \delta(\mathfrak{s}^2 + \varsigma^2)\delta(k\cdot\mathfrak{s}) \;, \quad \varsigma^2 \coloneqq 2 \;, \tag{3.162}$$

Note that now the normalization of the spin vector $\mathfrak{s}$ is different compared to the spin-1/2 case, which is dictated by the prefactors that should appear in the conserved quantities.

Defining a distribution function in this enlarged phase space

$$\mathfrak{f}(x,k,\mathfrak{s}) \coloneqq f_K - \mathfrak{s}\cdot G + \frac{5}{8}\mathfrak{s}^\mu\mathfrak{s}^\nu F_{K,\mu\nu} \;, \tag{3.163}$$

Eqs. (3.160) and (3.161) become (up to first order in $\hbar$)

$$(k^2 - m^2)\mathfrak{f}(x,k,\mathfrak{s}) = \hbar\mathfrak{M}(x,k,\mathfrak{s}) \;, \tag{3.164}$$
$$k\cdot\partial\mathfrak{f}(x,k,\mathfrak{s}) = \mathfrak{C}(x,k,\mathfrak{s}) \;, \tag{3.165}$$

where we defined

$$\mathfrak{C}(x,k,\mathfrak{s}) \coloneqq \left(\frac{1}{3}K_{\mu\nu} - \frac{i}{2m}\epsilon_{\mu\nu\alpha\beta}k^\alpha\mathfrak{s}^\beta + \frac{5}{8}\mathfrak{s}_\alpha\mathfrak{s}_\beta K^{\alpha\beta}_{\mu\nu}\right)\mathcal{C}^{\mu\nu}(x,k) \;, \tag{3.166a}$$
$$\mathfrak{M}(x,k,\mathfrak{s}) \coloneqq \left(\frac{1}{3}K_{\mu\nu} - \frac{i}{2m}\epsilon_{\mu\nu\alpha\beta}k^\alpha\mathfrak{s}^\beta + \frac{5}{8}\mathfrak{s}_\alpha\mathfrak{s}_\beta K^{\alpha\beta}_{\mu\nu}\right)\delta M^{\mu\nu}(x,k) \;. \tag{3.166b}$$

Using the identities

$$\int \mathrm{d}S(k) = 3 \;, \quad \int \mathrm{d}S(k)\mathfrak{s}^\mu\mathfrak{s}^\nu = -2K^{\mu\nu} \;, \quad \int \mathrm{d}S(k) K^{\mu\nu}_{\rho\sigma}\mathfrak{s}^\rho\mathfrak{s}^\sigma\mathfrak{s}^\alpha\mathfrak{s}^\beta = \frac{8}{5}K^{\mu\nu,\alpha\beta} \;, \tag{3.167a}$$

and

$$\int \mathrm{d}S(k)\mathfrak{s}^\mu = 0 \;, \qquad \int \mathrm{d}S(k)\mathfrak{s}^\mu\mathfrak{s}^\nu\mathfrak{s}^\alpha = 0 \;, \tag{3.167b}$$

we can obtain the independent components of the WIGNER function via suitably weighted integrations over spin space,

$$f_K(x,k) = \frac{1}{3}\int \mathrm{d}S(k)\mathfrak{f}(x,k,\mathfrak{s}) \;, \tag{3.168a}$$
$$G^\mu(x,k) = \frac{1}{2}\int \mathrm{d}S(k)\mathfrak{s}^\mu\mathfrak{f}(x,k,\mathfrak{s}) \;, \tag{3.168b}$$
$$F_K^{\mu\nu}(x,k) = \int \mathrm{d}S(k) K^{\mu\nu}_{\alpha\beta}\mathfrak{s}^\alpha\mathfrak{s}^\beta\mathfrak{f}(x,k,\mathfrak{s}) \;. \tag{3.168c}$$

3.5.3 Conserved currents

Applying NOETHER's theorem to the Lagrangian (3.129), we find the canonical currents to be

$$T^{\mu\nu}_{P,C} + g^{\mu\nu}\mathcal{L}_P = -\hbar\left\langle :\left(\widehat{V}^{\dagger\mu\alpha}\partial^\nu\widehat{V}_\alpha + \widehat{V}^{\mu\alpha}\partial^\nu\widehat{V}^\dagger_\alpha\right):\right\rangle , \tag{3.169}$$

$$S^{\lambda\mu\nu}_{P,C} = \left\langle :\left(\widehat{V}^{\dagger\lambda[\nu}\widehat{V}^{\mu]} + \widehat{V}^{\lambda[\nu}\widehat{V}^{\dagger\mu]}\right):\right\rangle . \tag{3.170}$$

Using the definition of $\widehat{V}^{\mu\nu}$ and the general formula (3.38), the energy-momentum tensor reads in terms of the WIGNER function

$$\begin{aligned}
&T^{\mu\nu}_{P,C} + g^{\mu\nu}\mathcal{L}_P \\
&= \frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\left(D^\nu D^{*[\mu}W^{\alpha]}{}_\alpha + W_\alpha{}^{[\alpha}\overleftarrow{D^{\mu]}}\overleftarrow{D^{*\nu}}\right) \\
&= \int\frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\left[\left(k^\mu k^\nu + \frac{\hbar^2}{4}\partial^\mu\partial^\nu\right)W^\alpha{}_\alpha - \left(k^\nu k_\alpha + \frac{\hbar^2}{4}\partial^\nu\partial_\alpha\right)W^{\alpha\mu}_S - \frac{i\hbar}{2}k^{[\nu}\partial_{\alpha]}W^{\alpha\mu}_A\right] ,
\end{aligned} \tag{3.171}$$

where we split the WIGNER function into symmetric and antisymmetric parts in the second step. The spin tensor on the other hand can be written as

$$\begin{aligned}
S^{\lambda\mu\nu}_{P,C} &= \frac{i}{2}\int\frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\left[\left(D^{*\lambda} + D^\lambda\right)W^{[\mu\nu]} + W^{\lambda[\mu}\overleftarrow{D^{*\nu]}} - D^{[\nu}W^{\mu]\lambda}\right] \\
&= i\int\frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\left(2k^\lambda W^{\mu\nu}_A + k^{[\mu}W^{\nu]\lambda}_A - \frac{i\hbar}{2}\partial^{[\nu}W^{\mu]\lambda}_S\right) .
\end{aligned} \tag{3.172}$$

Expressing the canonical conserved currents up to first order in $\hbar$ in extended phase space by using Eqs. (3.154) in conjunction with Eqs. (3.159) and (3.168), we have

$$T^{\mu\nu}_{P,C} + g^{\mu\nu}\mathcal{L}_P = \int\mathrm{d}\Gamma k^\mu k^\nu \mathfrak{f}(x,k,\mathfrak{s}) + \mathcal{O}(\hbar^2) , \tag{3.173}$$

$$S^{\lambda\mu\nu}_{P,C} = \int\mathrm{d}\Gamma\left[k^\lambda\left(\Sigma^{\mu\nu}_{\mathfrak{s}} - \frac{\hbar}{3m^2}k^{[\mu}\partial^{\nu]}\right) + \frac{1}{2}k^{[\mu}\Sigma^{\nu]\lambda}_{\mathfrak{s}} + \frac{\hbar}{6}K^{\lambda[\mu}\partial^{\nu]}\right]\mathfrak{f}(x,k,\mathfrak{s}) + \mathcal{O}(\hbar^2) . \tag{3.174}$$

Here, we defined the tensor $\Sigma_{\mathfrak{s}}$ as in Eq. (3.97), and set

$$\mathfrak{f}(x,k,\mathfrak{s}) =: 4\pi\hbar\delta(k^2 - m^2)f(x,k,\mathfrak{s}) , \tag{3.175}$$

which differs from Eq. (3.100) by a factor of mass. This can be understood directly from the dimension of the fields: while a DIRAC spinor (in natural units, where we choose to measure energy in MeV) has dimension $\mathrm{MeV}^{3/2}$, a vector field has dimension MeV. Then, for DIRAC fields the WIGNER function has dimension MeV^{-1}, while the one for vector fields has units of MeV^{-2}. Given that the units of the delta function for the mass shell are MeV^{-2}, it is clear that with the definitions (3.100) and (3.175) the function $f(x,k,\mathfrak{s})$ is dimensionless for any spin.

BELINFANTE pseudogauge

The BELINFANTE pseudogauge is constructed by taking the superpotentials

$$\widehat{\Phi}^{\lambda\mu\nu}_B = \widehat{S}^{\lambda\mu\nu}_{P,C} , \quad \widehat{Z}^{\mu\nu\lambda\rho}_B := 0 , \tag{3.176}$$

which lead to the energy-momentum tensor

$$T^{\mu\nu}_{P,B} + g^{\mu\nu}\mathcal{L}_P = \hbar\left\langle :\left[\widehat{V}^{\dagger\mu\alpha}\widehat{V}_\alpha{}^\nu + \widehat{V}^{\mu\alpha}\widehat{V}^{\dagger\,\nu}_\alpha + \frac{m^2}{\hbar^2}\left(\widehat{V}^{\dagger\mu}\widehat{V}^\nu + \widehat{V}^\mu\widehat{V}^{\dagger\nu}\right)\right]:\right\rangle , \tag{3.177}$$

while the spin tensor vanishes by construction, $S_{P,B}^{\lambda\mu\nu} = 0$. In terms of the WIGNER function we have

$$T_{P,B}^{\mu\nu} + g^{\mu\nu}\mathcal{L}_P = \int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4} \left[\left(k^\mu k^\nu + \frac{\hbar^2}{4}\partial^\mu\partial^\nu \right) W^\alpha{}_\alpha - \left(k_\alpha k^{(\nu} + \frac{\hbar^2}{4}\partial_\alpha\partial^{(\nu} \right) W_S^{\mu)\alpha} \right.$$
$$\left. + i\hbar\partial_\alpha k^{(\nu} W_A^{\mu)\alpha} + \left(k^2 - m^2 + \frac{\hbar^2}{4}\Box \right) W_S^{\mu\nu} \right] . \qquad (3.178)$$

Note that the last term in the expression above constitutes an off-shell effect [cf. Eqs. (3.161)], which we take to be of order $\mathcal{O}(\hbar^2)$ and do not consider further. Thus, in extended phase space we have again

$$T_{P,B}^{\mu\nu} + g^{\mu\nu}\mathcal{L}_P = \int \mathrm{d}\Gamma k^\mu k^\nu f(x,k,\mathfrak{s}) + \mathcal{O}(\hbar^2) . \qquad (3.179)$$

HW pseudogauge

In accordance with the superpotentials introduced in Eq. (3.114) for spin-1/2 particles, we define

$$\widehat{\Phi}_{\mathrm{HW}}^{\lambda\mu\nu} := \widehat{M}^{[\mu\nu]\lambda} - g^{\lambda[\mu}\widehat{M}_\alpha{}^{\nu]\alpha} + \widehat{Q}^{\lambda\mu\nu} , \quad \widehat{Z}_{\mathrm{HW}}^{\mu\nu\lambda\rho} := -\frac{1}{2}\left(\widehat{V}^{\dagger[\mu} g^{\nu][\lambda}\widehat{V}^{\rho]} + \widehat{V}^{[\mu} g^{\nu][\lambda}\widehat{V}^{\dagger\rho]} \right) , \qquad (3.180)$$

where we introduced

$$\widehat{M}^{\lambda\mu\nu} := \frac{1}{2}\left(\widehat{V}^{\dagger\mu}\overleftrightarrow{\partial^\lambda}\widehat{V}^\nu + \widehat{V}^\mu\overleftrightarrow{\partial^\lambda}\widehat{V}^{\dagger\nu} \right) , \quad \widehat{Q}^{\lambda\mu\nu} := \frac{\hbar^2}{m^2} g^{\lambda[\mu}\left(\widehat{V}^{\nu]}\partial\cdot\widehat{\rho}^\dagger + \widehat{V}^{\dagger\nu]}\partial\cdot\widehat{\rho} \right) . \qquad (3.181)$$

As in the case of DIRAC particles, it is easier to evaluate the conserved quantities in terms of the WIGNER function, where the (FOCK-space averages of the) superpotentials read

$$\Phi_{\mathrm{HW}}^{\lambda\mu\nu} = \int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\left(\frac{\hbar}{2}\partial_\rho W_S^{\rho[\mu} g^{\nu]\lambda} + ik^{[\mu}W_A^{\nu]\lambda} \right) , \qquad (3.182a)$$

$$Z_{\mathrm{HW}}^{\mu\nu\lambda\rho} = \frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\left(g^{\nu[\lambda}W_S^{\rho]\mu} - g^{\mu[\lambda}W_S^{\rho]\nu} \right) . \qquad (3.182b)$$

Noting that

$$\Phi_{\mathrm{HW}}^{\lambda\mu\nu} + \Phi_{\mathrm{HW}}^{\mu\nu\lambda} + \Phi_{\mathrm{HW}}^{\nu\mu\lambda} = 2\int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\left(\frac{\hbar}{2}\partial_\rho W_S^{\rho[\mu} g^{\lambda]\nu} + ik^\nu W_A^{\lambda\mu} \right) , \qquad (3.183)$$

we compute the energy-momentum tensor in the HW pseudogauge as

$$T_{P,\mathrm{HW}}^{\mu\nu} + g^{\mu\nu}\mathcal{L}_P = \int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\left[\left(k^\mu k^\nu + \frac{\hbar^2}{4}\partial^\mu\partial^\nu \right) W^\alpha{}_\alpha - \left(k_\rho k^\nu - \frac{\hbar^2}{4}\partial_\rho\partial^\nu \right) W_S^{\rho\mu} \right.$$
$$\left. + \frac{i\hbar}{2}k^{(\nu}\partial_{\rho)}W_A^{\rho\mu} - \frac{g^{\mu\nu}}{2}\left(k^2 - m^2 + \frac{\hbar^2}{4}\Box \right) W^\alpha{}_\alpha \right] . \qquad (3.184)$$

Using the constraint equations (3.151) and employing that $\hbar\mathcal{C}_E$, $\mathcal{D}_E \sim \mathcal{O}(\hbar^2)$, we find

$$T_{P,\mathrm{HW}}^{\mu\nu} + g^{\mu\nu}\mathcal{L}_P = \int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\left[\left(k^\mu k^\nu + \frac{\hbar^2}{4}\partial^\mu\partial^\nu \right) W^\alpha{}_\alpha - \frac{\hbar}{2m^2}k^\nu\left(-k\mathcal{C}_A^\mu - k\mathcal{D}_S^\mu + \hbar K^{\mu\alpha}\partial_\alpha\mathcal{C}_K \right) \right.$$
$$\left. - \frac{g^{\mu\nu}}{2}\left(k^2 - m^2 + \frac{\hbar^2}{4}\Box \right) W^\alpha{}_\alpha \right] + \mathcal{O}(\hbar^3) , \qquad (3.185)$$

such that to first order in the PLANCK constant it holds again that

$$T_{P,\mathrm{HW}}^{\mu\nu} + g^{\mu\nu}\mathcal{L}_P = \int \mathrm{d}\Gamma k^\mu k^\nu f(x,k,\mathfrak{s}) + \mathcal{O}(\hbar^2) . \qquad (3.186)$$

The spin tensor becomes in the HW pseudogauge

$$S^{\lambda\mu\nu}_{P,\text{HW}} = 2i \int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4} k^\lambda W^{\mu\nu}_A$$
$$= \int \mathrm{d}\Gamma k^\lambda \left(\Sigma^{\mu\nu}_{\mathfrak{s}} - \frac{\hbar}{3m^2} k^{[\mu}\partial^{\nu]} \right) f(x,k,\mathfrak{s}) + \mathcal{O}(\hbar^2) \,. \tag{3.187}$$

Note that, as expected, this spin tensor is conserved in the case where the interactions are absent and $k \cdot \partial f(x,k,\mathfrak{s}) = 0$. The divergence of the spin tensor is equal to the antisymmetric part of the energy-momentum tensor, yielding the conservation law for the total angular momentum (valid up to first order in $\hbar$)

$$\hbar\partial_\lambda S^{\lambda\mu\nu}_{P,\text{HW}} = \hbar \int \mathrm{d}\Gamma \Sigma^{\mu\nu}_{\mathfrak{s}} C(x,k,\mathfrak{s}) = -\frac{\hbar}{m} \int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4} \frac{k}{2m} k^{[\nu} \left(\mathcal{C}^{\mu]}_A + \mathcal{D}^{\mu]}_S \right) = T^{[\nu\mu]}_{P,\text{HW}} \,, \tag{3.188}$$

where we also used the conservation of momentum and defined the on-shell part of the collision kernel in extended phase space,

$$\mathfrak{C}(x,k,\mathfrak{s}) =: 4\pi\hbar\delta(k^2-m^2)C(x,k,\mathfrak{s}) \,. \tag{3.189}$$

Note that Eq. (3.188) looks very similar to its spin-1/2 counterpart (3.104b). When identifying the vector parts of the collision terms $\mathcal{D}^\mu_\mathcal{V}$ and $(\mathcal{C}^\mu_A + \mathcal{D}^\mu_S)/2$, the only differences are a factor of k/m, which becomes unity on-shell, and a factor of 2 in front of the spin tensor, which is to be expected due to the higher spin magnitude of vector particles.

Let us note that, as in the cases before, we can derive the HW currents in the free case from the Lagrangian

$$\mathcal{L}_{P,\text{HW}} \coloneqq -\hbar \left[(\partial_\mu \widehat{V}^\dagger_\nu)\partial^\mu \widehat{V}^\nu - (\partial \cdot \widehat{V}^\dagger)\partial \cdot \widehat{V} - \frac{m^2}{\hbar^2} \widehat{V}^\dagger \cdot \widehat{V} \right] \,, \tag{3.190}$$

which differs from the original one by a total divergence.

GLW pseudogauge

Finally, we consider the GLW pseudogauge[6] for spin-1 particles, which in the free case can be derived from the Lagrangian

$$\mathcal{L}_{P,\text{GLW}} \coloneqq \hbar \left[\frac{1}{2} \left(\widehat{V}^{\dagger\mu} \Box \widehat{V}_\mu + \widehat{V}^\mu \Box \widehat{V}^\dagger_\mu \right) + (\partial \cdot \widehat{V}^\dagger)\partial \cdot \widehat{V} + \frac{m^2}{\hbar^2} \widehat{V}^\dagger \cdot \widehat{V} \right] \,, \tag{3.191}$$

which vanishes upon using the equations of motion. Slightly modifying the HW superpotentials (3.180), we define

$$\widehat{\Phi}^{\lambda\mu\nu}_{\text{GLW}} \coloneqq \widehat{\Phi}^{\lambda\mu\nu}_{\text{HW}} - \frac{1}{2} g^{\lambda[\mu}\partial^{\nu]} \widehat{V}^\dagger \cdot \widehat{V} \,, \quad \widehat{Z}^{\mu\nu\lambda\rho}_{\text{GLW}} \coloneqq \widehat{Z}^{\mu\nu\lambda\rho}_{\text{HW}} - \frac{1}{4} \delta^{[\nu}_\alpha g^{\mu][\lambda} g^{\rho]\alpha} \widehat{V}^\dagger \cdot \widehat{V} \,, \tag{3.192}$$

The resulting energy-momentum tensor reads

$$T^{\mu\nu}_{P,\text{GLW}} + g^{\mu\nu}\mathcal{L}_P = \int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4} \left[k^\mu k^\nu W^\alpha{}_\alpha + \frac{\hbar}{2m^2} k^\nu \left(k\mathcal{C}^\mu_A + k\mathcal{C}^\mu_S + \hbar K^{\mu\alpha}\partial_\alpha \mathcal{C}_K \right) \right.$$
$$\left. - \frac{g^{\mu\nu}}{2} \left(k^2 - m^2 - \frac{\hbar^2}{4}\Box \right) W^\alpha{}_\alpha \right] + \mathcal{O}(\hbar^3) \,, \tag{3.193}$$

while the spin tensor stays unchanged.

Analogous to the spin-1/2 case, we can remove the last term in Eq. (3.187) (which is separately conserved due to momentum conservation) by adding a further superpotential to the quantities in the HW pseudogauge,

$$Z^{\mu\nu\lambda\rho}_{\text{GLW}'} \coloneqq -\frac{1}{3m^2} \int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4} k^{[\lambda} g^{\rho][\mu} k^{\nu]} W^\alpha{}_\alpha \,. \tag{3.194}$$

[6] Also referred to as KG (Klein-Gordon) pseudogauge in Ref. [54].

By virtue of momentum conservation, we thus have, analogous to Eq. (3.128),

$$T^{\mu\nu}_{D,\mathrm{GLW}'} = \int \mathrm{d}\Gamma k^\mu k^\nu f(x,k,\mathfrak{s}) + \mathcal{O}(\hbar^2) \;, \tag{3.195}$$

$$S^{\lambda\mu\nu}_{D,\mathrm{GLW}'} = \int \mathrm{d}\Gamma k^\lambda \Sigma^{\mu\nu}_{\mathfrak{s}} f(x,k,\mathfrak{s}) + \mathcal{O}(\hbar^2) \;, \tag{3.196}$$

which is the formulation we will employ for deriving spin hydrodynamics in Chapter 6. Note that we again omitted the contribution proportional to the Lagrangian from the energy-momentum tensor.

To close the discussion of the conserved currents, we remark that, although not treated in this thesis, the formulation of different pseudogauges presented here can also be generalized to the case where nonzero electromagnetic fields are present, which requires a redefinition of the Wigner function to render it gauge-invariant [53, 54, 84, 85, 142, 143].

Chapter 4

The kinetic equation in the GLW approach

In Chapter 3, we showed that the conserved currents can be described as phase-space integrals over the respective WIGNER functions. The time evolution of the latter is given by the kinetic equations for particles of spins 0, 1/2, and 1, which we also computed in the preceding chapter. In extended phase space, they read up to first order in $\hbar$

$$k \cdot \partial \mathfrak{f}(x,k,\mathfrak{s}) = \mathfrak{C}(x,k,\mathfrak{s}) \ , \tag{4.1}$$

where the collision terms are

$$\mathfrak{C}(x,k,\mathfrak{s}) = \begin{cases} \mathcal{C}(x,k) \ , & \text{for spin } 0 \ , \\ \frac{1}{2}\left[\delta_{\beta\alpha} - (\not{\mathfrak{s}}\gamma_5)_{\beta\alpha}\right] \mathcal{C}^{\alpha\beta}(x,k) \ , & \text{for spin } 1/2 \ , \\ \left(\frac{1}{3}K_{\mu\nu} - \frac{i}{2m}\epsilon_{\mu\nu\alpha\beta}k^\alpha \mathfrak{s}^\beta + \frac{5}{8}\mathfrak{s}_\alpha \mathfrak{s}_\beta K^{\alpha\beta}_{\mu\nu}\right) \mathcal{C}^{\mu\nu}(x,k) \ , & \text{for spin } 1 \ , \end{cases} \tag{4.2}$$

cf. Eqs. (3.87) and (3.165). Note that in the case of spin-1/2 particles the indices α, β in the equations above denote components in DIRAC space, while for spin-1 particles they are LORENTZ indices. For scalar particles the spin indices can simply be ignored, such that the $\mathfrak{s}$-dependence in Eq. (4.1) is spurious in that case.

The aim of this chapter is to employ the so-called GLW method [43, 46] (after DE GROOT, VAN LEEUWEN, and VAN WEERT) to express the collision term $\mathfrak{C}$ as a functional of the distribution function $\mathfrak{f}$. In particular, we will impose that the collision kernel should describe *binary elastic scattering*, i.e., we will consider only $2 \to 2$ scatterings, without the possibility of particle creation or annihilation.

In standard kinetic theory, it can be argued that the BOLTZMANN equation for binary elastic scattering takes on the form [43, 115]

$$\begin{aligned} k \cdot \partial f(x,k) &= C_{\text{class}}(x,k) \\ &= \frac{1}{2}\int \mathrm{d}K' \,\mathrm{d}K_1 \,\mathrm{d}K_2 (2\pi\hbar)^4 \delta^{(4)}(k+k'-k_1-k_2) \mathcal{W}_{\mathbf{k}\mathbf{k}'\to\mathbf{k}_1\mathbf{k}_2} \\ &\quad \times \left[f(x,k_1)f(x,k_2)\widetilde{f}(x,k)\widetilde{f}(x,k') - f(x,k)f(x,k')\widetilde{f}(x,k_1)\widetilde{f}(x,k_2)\right] \ , \end{aligned} \tag{4.3}$$

where $\widetilde{f}(x,k) \coloneqq 1 - af(x,k)$, with $a = -1$ for bosons and $a = 1$ for fermions, denote the BOSE-enhancement and PAULI-blocking factors, respectively. For classical particles, one simply has $a = 0$.

The quantity $\mathcal{W}_{\mathbf{k}\mathbf{k}'\to\mathbf{k}_1\mathbf{k}_2}$ is called the transition rate for the two particle-scattering process with momenta $\mathbf{k},\mathbf{k}'$ prior to and momenta $\mathbf{k}_1,\mathbf{k}_2$ after the collision. Note that we have the detailed balance relation $\mathcal{W}_{\mathbf{k}\mathbf{k}'\to\mathbf{k}_1\mathbf{k}_2} = \mathcal{W}_{\mathbf{k}_1\mathbf{k}_2\to\mathbf{k}\mathbf{k}'}$, which can be shown to follow from the unitarity of the scattering matrix [123]. In the following, we will show that the form of the collision term in quantum kinetic theory assumes a similar form as in Eq. (4.3), with a few important modifications owing to the nonzero spin of the particles.

4.1 Basic idea

Recalling the definitions of the collision terms (3.76) and (3.148), it becomes clear that the main task consists in evaluating the expectation value of an operator bilinear in the respective fields and source terms. The basic idea behind the GLW approach lies in employing a complete orthogonal basis of the relevant Fock space in terms of "in"-states. Then, one can express the expectation value of a general operator $\widehat{O}$ as a series of "in"-Wigner functions, which are constructed from the operators creating and annihilating the "in"-states. Truncating this series such that the resulting expression is only bilinear in Wigner functions, one obtains the sought-after expression for the collision term.

4.1.1 Fock space and expectation values

The first step, which consists in deriving a formula for expressing the expectation value of a general operator in terms of "in"-Wigner functions, can be done in a general manner for fields of any spin. In this subsection, we present the general formulation, and specialize to fields of fixed spin in the following sections.

Fock space

We define the "in"-states as

$$|k^n;\sigma^n\rangle_{\text{in}} \coloneqq \widehat{a}^\dagger_{\text{in}}(k^n,\sigma^n)\,|0\rangle \;, \tag{4.4}$$

where we introduced

$$k^n \coloneqq k_1^\mu, k_2^\mu, \cdots, k_n^\mu \;, \quad \sigma^n \coloneqq \sigma_1,\sigma_2,\cdots,\sigma_n \;, \tag{4.5a}$$

as well as

$$\widehat{a}^\dagger_{\text{in}}(k^n,\sigma^n) \coloneqq \widehat{a}^\dagger_{\text{in}}(k_1,\sigma_1)\widehat{a}^\dagger_{\text{in}}(k_2,\sigma_2)\cdots\widehat{a}^\dagger_{\text{in}}(k_n,\sigma_n) \;. \tag{4.5b}$$

Here, the values of the spin-variables σ range from 0 to $2j+1$, where j is the spin of the particle described by a field $\widehat{\varphi}$. These states form a complete and orthogonal basis of the Fock space, i.e., we have the relations

$${}_{\text{in}}\langle k;\sigma|\,k';\sigma'\rangle_{\text{in}} = (2\pi\hbar)^3 2k^0\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{\sigma\sigma'} \;, \tag{4.6a}$$

$$\mathbb{1} = \sum_{n=0}^{\infty}\frac{1}{2^n n!}\sum_{\sigma^n}\int \mathrm{d}K^n\,|k^n;\sigma^n\rangle_{\text{in}\;\text{in}}\langle k^n;\sigma^n| \;. \tag{4.6b}$$

Here we defined

$$\int \mathrm{d}K^n \coloneqq \int \mathrm{d}K_1\,\mathrm{d}K_2\cdots\mathrm{d}K_n \;, \quad \sum_{\sigma^n} \coloneqq \sum_{\sigma_1=1}^{2j+1}\sum_{\sigma_2=1}^{2j+1}\cdots\sum_{\sigma_n=1}^{2j+1} \;. \tag{4.7}$$

The factorial in Eq. (4.6b) is needed to account for double counting, such that the Fock space is spanned by all distinct "in"-states. Furthermore, the factor 2^n arises because of the definition of the measure dK. Note that the same completeness relation also holds for the "out"-states, which describe outgoing free particles. Using these creation and annihilation operators, we define the "in"-fields [123]

$$\begin{aligned}\widehat{\varphi}^a_{\rm in}(x) &\coloneqq \lambda \sum_\sigma \int \frac{\mathrm{d}^4k}{(2\pi\hbar)^3}\Theta(k^0)\delta(k^2-m^2)e^{-\frac{i}{\hbar}k\cdot x}U^a(k,\sigma)\widehat{a}_{\rm in}(k,\sigma)\\ &\equiv \frac{\lambda}{2}\sum_\sigma \int \mathrm{d}K e^{-\frac{i}{\hbar}k\cdot x}U^a(k,\sigma)\widehat{a}_{\rm in}(k,\sigma)\ ,\end{aligned} \tag{4.8}$$

where the prefactor λ is needed to recover the correct dimensions of the respective fields, e.g., $\lambda = 1$ for spin-1/2 particles, whereas $\lambda = \sqrt{\hbar}$ for spin-0 and spin-1 particles. Note that, for simplicity, Eq. (4.8) neglects antiparticle contributions, whose inclusion is demonstrated in Ref. [43]. The quantities $U^a(k,\sigma)$ span the internal space for particles with nonzero spin and correspond to the basis spinors $u^\alpha_r(k)$ and polarization vectors $\epsilon^{(\lambda)\mu}(k)$ in the case of spin-1/2 and spin-1 particles, respectively. The index a collects all internal indices of the field $\widehat{\varphi}$. We require the following orthogonality relation,

$$\overline{U}^a(k,\sigma)U_a(k,\sigma) = \eta\delta_{\sigma\sigma'}\ , \tag{4.9}$$

where the choice of η depends on the spin of the particle, e.g., $\eta = 1$ for spin-0, $\eta = 2m$ for spin-1/2, and $\eta = -1$ for spin-1 particles. Using Eq. (4.8), we define the "in"-Wigner function,[1]

$$W^{ab}_{\rm in}(x,k) \coloneqq \kappa \int \mathrm{d}^4v e^{-\frac{i}{\hbar}k\cdot v}\left\langle \widehat{\overline{\varphi}}^b_{\rm in}\left(x+\frac{v}{2}\right)\widehat{\varphi}^a_{\rm in}\left(x-\frac{v}{2}\right)\right\rangle\ , \tag{4.10}$$

which is consistent with Eq. (3.31).

Re-expressing Fock-space averages

In a subsequent step, we show how to express the expectation value of an arbitrary operator $\widehat{O}$ in terms of the "in"-Wigner function (4.10). Inserting Eq. (4.6b), we can express the expectation value of $\widehat{O}$ as

$$\begin{aligned}\left\langle\widehat{O}\right\rangle &\coloneqq \mathrm{Tr}\,\widehat{\varrho}\,\widehat{O}\\ &= \sum_{n=0}^\infty \frac{1}{2^{2n}(n!)^2}\sum_{\lambda^n,\lambda'^n}\int \mathrm{d}K^n\int \mathrm{d}K'^n\ {}_{\rm in}\langle k^n;\lambda^n|\widehat{O}\,|k'^n;\lambda'^n\rangle_{\rm in}\,{}_{\rm in}\langle k'^n;\lambda'^n|\widehat{\varrho}\,|k^n;\lambda^n\rangle_{\rm in}\ ,\end{aligned} \tag{4.11}$$

where we assume that the initial state has been prepared such that the "in"-particle number operator commutes with the density matrix. The next task is to express the matrix element of the density matrix through expectation values of bilinear products of creation and annihilation operators. For this we first compute, using the cyclicity of the trace,

$$\begin{aligned}\left\langle \widehat{a}^\dagger_{\rm in}(k^n,\lambda^n)\widehat{a}_{\rm in}(k'^n,\lambda'^n)\right\rangle &= \mathrm{Tr}\,\widehat{a}_{\rm in}(k'^n,\lambda'^n)\widehat{\varrho}\,\widehat{a}^\dagger_{\rm in}(k^n,\lambda^n)\\ &= \sum_{k=0}^\infty \frac{1}{2^k k!}\sum_{\sigma^n}\int \mathrm{d}P^k\ {}_{\rm in}\langle p^k,k'^n;\sigma^k,\lambda'^n|\widehat{\varrho}\,|p^k,k^n;\sigma^k,\lambda^n\rangle_{\rm in}\ .\end{aligned} \tag{4.12}$$

The inversion of this relation, proven in Ref. [43], gives

$$\begin{aligned}{}_{\rm in}\langle k'^n;\sigma'^n|\widehat{\varrho}\,|k^n;\sigma^n\rangle_{\rm in} &= \sum_{m=0}^\infty \frac{(-1)^m}{2^{2m}(m!)^2}\sum_{\sigma^m,\sigma'^m}\int \mathrm{d}P^m\int \mathrm{d}P'^m\ {}_{\rm in}\langle p^m;\sigma^m|\,p'^m;\sigma'^m\rangle_{\rm in}\\ &\quad\times\left\langle \widehat{a}^\dagger_{\rm in}(k^n,p^m,\sigma^n,\sigma^m)\widehat{a}_{\rm in}(k'^n,p'^m,\sigma'^n,\sigma'^m)\right\rangle\ ,\end{aligned} \tag{4.13}$$

[1] Because of the exclusion of antiparticles, we omit the normal-ordering prescription.

which we may insert into Eq. (4.11) to obtain

$$\left\langle \widehat{O} \right\rangle = \sum_{n=0}^{\infty} \frac{1}{2^{2n}(n!)^2} \sum_{\rho^n,\rho'^n} \int \mathrm{d}K^n \int \mathrm{d}K'^n \, {}_{\text{in}}\langle k^n;\rho^n|\widehat{O}\,|k'^n;\rho'^n\rangle_{\text{in}}$$
$$\times \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m}(m!)^2} \sum_{\sigma^m,\sigma'^m} \int \mathrm{d}P^m \int \mathrm{d}P'^m \, {}_{\text{in}}\langle p^m;\sigma^m|\, p'^m;\sigma'^m\rangle_{\text{in}}$$
$$\times \left\langle \widehat{a}^\dagger_{\text{in}}(k^n,p^m,\rho^n,\sigma^m)\widehat{a}_{\text{in}}(k'^n,p'^m,\rho'^n,\sigma'^m) \right\rangle \; . \tag{4.14}$$

Introducing a new summation index, $j := n+m$, and using the fact that $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \equiv \sum_{j=0}^{\infty}\sum_{m=0}^{j}$, we arrive at

$$\left\langle \widehat{O} \right\rangle = \sum_{j=0}^{\infty} \frac{1}{2^{2j}(j!)^2} \sum_{\sigma^j,\sigma'^j} \int \mathrm{d}K^j \int \mathrm{d}K'^j \, {}_{\text{in}}\langle\!\langle k^j;\sigma^j|\,\widehat{O}\,|k'^j;\sigma'^j\rangle\!\rangle_{\text{in}} \left\langle \widehat{a}^\dagger_{\text{in}}(k^j,\sigma^j)\widehat{a}_{\text{in}}(k'^j,\sigma'^j) \right\rangle \; , \tag{4.15}$$

where we defined

$${}_{\text{in}}\langle\!\langle k^j;\sigma^j|\,\widehat{O}\,|k'^j;\sigma'^j\rangle\!\rangle_{\text{in}} := \sum_{m=0}^{j} (-1)^m \binom{j}{m}^2 {}_{\text{in}}\langle k^m;\sigma^m|\, k'^m;\sigma'^m\rangle_{\text{in}}$$
$$\times \, {}_{\text{in}}\langle k^{j-m};\sigma^{j-m}|\widehat{O}\,|k'^{j-m};\sigma'^{j-m}\rangle_{\text{in}} \; . \tag{4.16}$$

Note that the expression above is taken to be symmetric (antisymmetric) under the exchange of primed and unprimed momenta if the particles are bosons (fermions). Next we put in the essential assumption of molecular chaos, implying that the expectation value of creation and annihilation operators factorizes pairwise as

$$\left\langle \widehat{a}^\dagger_{\text{in}}(k^n,\sigma^n)\widehat{a}_{\text{in}}(k'^m,\sigma'^m) \right\rangle = \delta_{nm} \sum_{\mathcal{P}} (\pm 1)^{\mathcal{P}} \prod_{j=1}^{n} \left\langle \widehat{a}^\dagger_{\text{in}}(k_j,\sigma_j)\widehat{a}_{\text{in}}(k'_j,\sigma'_j) \right\rangle \; . \tag{4.17}$$

In this context, the symbol $\mathcal{P}$ stands for the summation over all possible permutations of primed and unprimed variables, while the factor $(\pm 1)^{\mathcal{P}}$ gives a sign change for odd permutations if the particles are fermions, and no sign change if they are bosons. In terms of the fields, Eq. (4.17) becomes

$$\left\langle \widehat{\overline{\varphi}}^{b_1}_{\text{in}}(x_1)\cdots\widehat{\overline{\varphi}}^{b_n}_{\text{in}}(x_n)\widehat{\varphi}^{a_1}_{\text{in}}(x'_1)\cdots\widehat{\varphi}^{a_m}_{\text{in}}(x'_m) \right\rangle = \delta_{nm} \sum_{\mathcal{P}} (\pm 1)^{\mathcal{P}} \prod_{j=1}^{n} \left\langle \widehat{\overline{\varphi}}^{b_j}_{\text{in}}(x_j)\widehat{\varphi}^{a_j}_{\text{in}}(x'_j) \right\rangle \; . \tag{4.18}$$

Inverting the definition of the "in"-fields (4.8), we have

$$\frac{1}{2\pi\hbar}\frac{1}{\eta\lambda}\int \mathrm{d}^4x e^{\frac{i}{\hbar}k\cdot x}\overline{U}_a(k,\sigma)\widehat{\varphi}^a_{\text{in}}(x) = \Theta(k^0)\delta(k^2-m^2)\widehat{a}_{\text{in}}(k,\sigma) \; . \tag{4.19}$$

Inserting this expression into Eq. (4.15) and renaming the sum index $j \to n$, we obtain

$$\left\langle \widehat{O} \right\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}^4x^n \int \mathrm{d}^4x'^n \widetilde{O}_{n,a_1b_1\cdots a_nb_n}(x^n;x'^n) \prod_{j=1}^{n} \left\langle \widehat{\overline{\varphi}}^{b_j}_{\text{in}}(x_j)\widehat{\varphi}^{a_j}_{\text{in}}(x'_j) \right\rangle \; , \tag{4.20}$$

where

$$\widetilde{O}_{n,a_1b_1\cdots a_nb_n}(x^n;x'^n) := \frac{1}{(\eta\lambda)^{2n}} \int \frac{\mathrm{d}^4k^n}{(2\pi\hbar)^{4n}} \int \frac{\mathrm{d}^4k'^n}{(2\pi\hbar)^{4n}} \sum_{\sigma^n,\sigma'^n} {}_{\text{in}}\langle\!\langle k^n;\sigma^n|\,\widehat{O}\,|k'^n;\sigma'^n\rangle\!\rangle_{\text{in}}$$
$$\times \left[\prod_{j=1}^{n} e^{-\frac{i}{\hbar}\left(k_jx_j - k'_jx'_j\right)} U_{b_j}(k_j,\sigma_j)\overline{U}_{a_j}(k'_j,\sigma'_j) \right] \; . \tag{4.21}$$

Using the definition of the "in"-Wigner function (4.10), we have

$$\kappa \left\langle \widehat{\varphi}^b_{\rm in} \left(x + \frac{v}{2}\right) \widehat{\varphi}^a_{\rm in} \left(x - \frac{v}{2}\right) \right\rangle = \int \frac{\mathrm{d}^4 k}{(2\pi\hbar)^4} e^{\frac{i}{\hbar} k\cdot v} W^{ab}_{\rm in}(x,k) \; . \tag{4.22}$$

Defining the center and difference variables $\bar{x}_j := (x_j + x'_j)/2$ and $v_j := x_j - x'_j$, Eq. (4.22) in conjunction with Eq. (4.20) then yields

$$\left\langle \widehat{O} \right\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}^4 \bar{x}^n \int \frac{\mathrm{d}^4 \bar{k}^n}{(2\pi\hbar)^{4n}} O_{n,a_1 b_1 \cdots a_n b_n}(\bar{x}^n; \bar{k}^n) \prod_{j=1}^{n} W^{a_j b_j}_{\rm in}(\bar{x}_j, \bar{k}_j) \; , \tag{4.23}$$

where we defined

$$O_{n,a_1 b_1 \cdots a_n b_n}(\bar{x}^n; \bar{k}^n) := \left(\frac{1}{\kappa \lambda^2 \eta^2}\right)^n \int \frac{\mathrm{d}^4 u^n}{(2\pi\hbar)^{4n}} \sum_{\sigma^n, \sigma'^n} {}_{\rm in}\!\left\langle\!\!\left\langle \bar{k}^n - \frac{u^n}{2}; \sigma^n \right| \widehat{O} \left| \bar{k}^n + \frac{u^n}{2}; \sigma'^n \right\rangle\!\!\right\rangle_{\rm in}$$
$$\times \left[\prod_{j=1}^{n} e^{\frac{i}{\hbar} u_j \cdot \bar{x}_j} U_{b_j}\left(\bar{k}_j - \frac{u_j}{2}, \sigma_j\right) \overline{U}_{a_j}\left(\bar{k}_j + \frac{u_j}{2}, \sigma'_j\right) \right] \; . \tag{4.24}$$

Note that in this calculation $\bar{k}$ is the integration variable appearing in Eq. (4.22), we used the emerging delta function $\delta^{(4)}\left(\frac{k_j + k'_j}{2} - \bar{k}_j\right)$ and defined $k_j - k'_j =: u_j$.

Equation (4.23) is the sought-after relation that allows to express the expectation value of any operator in terms of the "in"-Wigner function. In the following sections we will employ this relation to evaluate the collision term for particles of different spins.

Expressing $\boldsymbol{W}$ in terms of $\boldsymbol{W}_{\rm in}$

The first application, however, lies in applying Eq. (4.23) to the Wigner function itself. We recognize that the Wigner function can be expressed as the expectation value

$$W^{ab}(x,k) = \left\langle e^{\frac{i}{\hbar}\widehat{P}\cdot x} \widehat{\Psi}^{ab}(k) e^{-\frac{i}{\hbar}\widehat{P}\cdot x} \right\rangle \; , \tag{4.25}$$

where we introduced[2]

$$\widehat{\Psi}^{ab}(k) := \kappa \int \mathrm{d}^4 v e^{-\frac{i}{\hbar} k\cdot v} \widehat{\varphi}^b \left(\frac{v}{2}\right) \widehat{\varphi}^a \left(-\frac{v}{2}\right) \; , \tag{4.26}$$

and $\widehat{P}$ is the usual momentum operator. Then, applying Eq. (4.23) and relegating details to App. B.1, we obtain the intuitive result

$$W^{ab}(x,k) = W^{ab}_{\rm in}(x,k) + \text{terms of higher order in the density} \; . \tag{4.27}$$

Thus, we may replace every "in"-Wigner function appearing in the expression for the collision term on the right-hand side of the kinetic equation by the full one, up to corrections of higher order in the density, cf. Ref. [43]. Note that this step is indeed important, as the Boltzmann equation has to be closed in terms of the full Wigner function that appears on its left-hand side. We remark that this introduces another approximation into our scheme that is different from the ($\hbar$-)gradient expansion, and essentially corresponds to an expansion in terms of the (dimensionless) coupling constant of the interaction. In the following, to be consistent in the truncation, we will always neglect collisional contributions that arise *inside the collision integral itself.*

The second, and arguably more involved application of Eq. (4.23) lies in expressing the collision term in terms of the Wigner function, which we will demonstrate for particles up to spin 1 in the following.

[2] As noted before, we do not consider the normal ordering due to the formal omission of antiparticle contributions, whose inclusion is demonstrated in Ref. [43], p. 96.

4.2 A helpful theorem

Before proceeding to the actual computation, we state an important theorem that simplifies the calculation significantly. The proof is provided in Appendix A.

Theorem 1. *The kinetic equation for the* WIGNER *function of particles of any spin also holds on the mass shell to any order in* $\hbar$.

Specifically, this means that, when considering the WIGNER function $W(x,k)$ for a general field defined in Eq. (3.31), and expanding it as a series in $\hbar$, we may consider the momentum k to lie on the mass-shell, $k^2 = m^2$. We thus have the following implication,

$$k \cdot \partial W(x,k) = \mathcal{C}(x,k) \quad \Longrightarrow \quad k \cdot \partial W_{\text{on-shell}}(x,k) = \mathcal{C}_{\text{on-shell}}(x,k) \; , \tag{4.28}$$

where

$$W(x,k) =: \delta(k^2 - m^2) W_{\text{on-shell}}(x,k) + W_{\text{off-shell}}(x,k) \; , \tag{4.29a}$$

$$\mathcal{C}(x,k) =: \delta(k^2 - m^2) \mathcal{C}_{\text{on-shell}}(x,k) + \mathcal{C}_{\text{off-shell}}(x,k) \; , \tag{4.29b}$$

with the "off-shell" terms being nonsingular on the mass shell. Note that this does not imply that the WIGNER function does not have off-shell parts. Rather, it provides an evolution equation for the on-shell parts of the WIGNER function. The off-shell parts can then be reconstructed perturbatively from the on-shell ones by employing the mass-shell equations, such as Eqs. (3.49), (3.82), and (3.161).

4.3 Scalar fields

The scalar case has been discussed thoroughly in Ref. [43], but we nonetheless treat it here for completeness. Furthermore, nonlocal collisions have not been taken into account in Ref. [43], which, as we will show, bears no consequences to first order in $\hbar$.

4.3.1 Rewriting expectation values

The FOURIER decomposition of the "in"-fields in the scalar case reads

$$\widehat{\phi}_{\text{in}}(x) := \sqrt{\hbar} \int \frac{\mathrm{d}^4 k}{(2\pi\hbar)^3} \Theta(k^0) \delta(k^2 - m^2) e^{-\frac{i}{\hbar} k \cdot x} \widehat{a}_{\text{in}}(k) \; , \tag{4.30}$$

and the "in"-WIGNER function is defined as

$$W_{\text{in}}(x,k) := \frac{2}{\hbar} \int \mathrm{d}^4 v e^{-\frac{i}{\hbar} k \cdot v} \left\langle \widehat{\phi}^\dagger_{\text{in}} \left(x + \frac{v}{2} \right) \widehat{\phi}_{\text{in}} \left(x - \frac{v}{2} \right) \right\rangle \; . \tag{4.31}$$

Evaluated for spin-0 particles, Eq. (4.23) becomes

$$\left\langle \widehat{O} \right\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}^4 \bar{x}^n \int \frac{\mathrm{d}^4 \bar{k}^n}{(2\pi\hbar)^{4n}} O_n(\bar{x}^n; \bar{k}^n) \prod_{j=1}^{n} W_{\text{in}}(\bar{x}_j, \bar{k}_j) \; , \tag{4.32}$$

where we introduced

$$O_n(\bar{x}^n; \bar{k}^n) := \frac{1}{2^n} \int \frac{\mathrm{d}^4 u^n}{(2\pi\hbar)^{4n}} {}_{\text{in}}\!\left\langle\!\!\left\langle \bar{k}^n - \frac{u^n}{2} \right| \widehat{O} \left| \bar{k}^n + \frac{u^n}{2} \right\rangle\!\!\right\rangle_{\text{in}} \prod_{j=1}^{n} e^{\frac{i}{\hbar} u_j \cdot \bar{x}_j} \; . \tag{4.33}$$

Note that the quantities $U^a(k, \sigma)$ become unity in the spinless case.

Expressing $\mathcal{C}$ in terms of W_{in}

In order to express the collision term as a function of W_{in}, we express it as an average over FOCK space,

$$\mathcal{C}(x,k) = \left\langle e^{\frac{i}{\hbar}\widehat{P}\cdot x}\widehat{\Phi}(k)e^{-\frac{i}{\hbar}\widehat{P}\cdot x}\right\rangle , \tag{4.34}$$

where we introduced

$$\widehat{\Phi}(k) := i\int \mathrm{d}^4 v e^{-\frac{i}{\hbar}k\cdot v}\left[\widehat{\phi}^\dagger\left(\frac{v}{2}\right)\widehat{\rho}\left(-\frac{v}{2}\right) - \widehat{\rho}^\dagger\left(\frac{v}{2}\right)\widehat{\phi}\left(-\frac{v}{2}\right)\right] . \tag{4.35}$$

Then, we can utilize Eq. (4.32) to express the collision term as

$$\mathcal{C}(x,k) = \sum_{n=2}^{\infty}\frac{1}{n!}\int \mathrm{d}^4\bar{x}^n \int \frac{\mathrm{d}^4\bar{k}^n}{(2\pi\hbar)^{4n}}\Phi_n(\bar{x}^n;\bar{k}^n|k)\prod_{j=1}^{n} W_{\text{in}}(x+\bar{x}_j,\bar{k}_j) , \tag{4.36}$$

where we defined the kernel

$$\Phi_n(\bar{x}^n;\bar{k}^n|k) := \frac{1}{2^n}\int \frac{\mathrm{d}^4u^n}{(2\pi\hbar)^{4n}}\,{}_{\text{in}}\!\left\langle\!\!\left\langle \bar{k}^n - \frac{u^n}{2}\right|\widehat{\Phi}(k)\left|\bar{k}^n + \frac{u^n}{2}\right\rangle\!\!\right\rangle_{\text{in}}\prod_{j=1}^{n} e^{\frac{i}{\hbar}u_j\cdot\bar{x}_j} . \tag{4.37}$$

In Eq. (4.36), the n-th term in the sum, being proportional to a product of n WIGNER functions, contributes to the effect of n-particle collisions. In the remainder of this chapter we focus on binary elastic scattering and thus truncate the sum after the $n = 2$ term. Note that, when comparing to the expected result (4.3), we can already anticipate that this truncation will lead to the loss of effects that originate from BOSE-EINSTEIN statistics, since these are characterized by the product of three and four distribution functions [144]. After inserting a complete set of "out"-states and using the fact that the "in"-and "out"-states are momentum eigenstates, we obtain

$$\begin{aligned}
&{}_{\text{in}}\!\left\langle k^2 - \frac{u^2}{2}\right|\widehat{\Phi}(k)\left|k^2 + \frac{u^2}{2}\right\rangle_{\text{in}} \\
&= i\sum_{m=0}^{\infty}\frac{1}{2^m m!}\int \mathrm{d}K'^m\,{}_{\text{in}}\!\left\langle k^2 - \frac{u^2}{2}\right|\left[\widehat{\phi}^\dagger(0)\middle|k'^m\right\rangle_{\text{out}}\,{}_{\text{out}}\!\left\langle k'^m\middle|\widehat{\rho}(0)\right. \\
&\quad \left.-\widehat{\rho}^\dagger(0)\middle|k'^m\right\rangle_{\text{out}}\,{}_{\text{out}}\!\left\langle k'^m\middle|\widehat{\phi}(0)\right]\left|k^2 + \frac{u^2}{2}\right\rangle_{\text{in}}(2\pi\hbar)^4\delta^{(4)}\left(k + \sum_{j=0}^{m}k'_j - k_1 - k_2\right) \\
&= \frac{i}{2}\int \mathrm{d}K'\,{}_{\text{in}}\!\left\langle k^2 - \frac{u^2}{2}\right|\left[\widehat{\phi}^\dagger(0)\middle|k'\right\rangle_{\text{out}}\,{}_{\text{out}}\!\left\langle k'\middle|\widehat{\rho}(0)\right. \\
&\quad \left.-\widehat{\rho}^\dagger(0)\middle|k'\right\rangle_{\text{out}}\,{}_{\text{out}}\!\left\langle k'\middle|\widehat{\phi}(0)\right]\left|k^2 + \frac{u^2}{2}\right\rangle_{\text{in}}(2\pi\hbar)^4\delta^{(4)}\left(k + k' - k_1 - k_2\right) .
\end{aligned} \tag{4.38}$$

Note that in the last equality we assumed some conserved charge to be present (e.g. baryon number or electric charge), and assumed only one species of particles. Under these conditions, the only permissible scattering with two outgoing particles is $2 \to 2$ scattering. In Eq. (4.38), we next have to evaluate the matrix elements of the fields and the source terms. The former can be treated via the YANG-FELDMAN equation [43]

$$\widehat{\phi}(0) = \widehat{\phi}_{\text{in}}(0) + \int \mathrm{d}^4x\Delta_R(-x)\widehat{\rho}(x) , \tag{4.39}$$

leading to

$$\begin{aligned}
{}_{\text{out}}\!\left\langle k'\middle|\widehat{\phi}(0)\middle|k^2 + \frac{u^2}{2}\right\rangle_{\text{in}} &= \sqrt{\hbar}(2\pi\hbar)^3 2k'^0\left[\delta^{(3)}\left(\mathbf{k}' - \mathbf{k}_2 - \frac{\mathbf{u}_2}{2}\right) + (1\leftrightarrow 2)\right] \\
&\quad + \widetilde{\Delta}_R\left(k_1 + k_2 + \frac{u_1+u_2}{2} - k'\right)\,{}_{\text{out}}\!\left\langle k'\middle|\widehat{\rho}(0)\middle|k^2 + \frac{u^2}{2}\right\rangle_{\text{in}} ,
\end{aligned} \tag{4.40}$$

where we introduced the FOURIER-transformed retarded propagator of the scalar field, cf. Eqs. (3.45). The matrix elements of the sources on the other hand can be connected to the transfer matrix $\widehat{t}$ (i.e., the nontrivial part of the scattering matrix [145]) through the relation

$$ {}_{\text{out}}\langle k'|\widehat{\rho}(0)\,|k^2\rangle_{\text{in}} = \frac{1}{\sqrt{\hbar}}\,\langle k,k'|\,\widehat{t}\,|k^2\rangle \ . \tag{4.41} $$

Inserting Eqs. (4.40) and (4.41) into Eq. (4.38), we find

$$ \begin{aligned} & {}_{\text{in}}\left\langle k^2-\frac{u^2}{2}\middle|\widehat{\Phi}(k)\middle|k^2+\frac{u^2}{2}\right\rangle_{\text{in}} \\ &= i(2\pi\hbar)^4\left\{\delta\left(k^0-k_1^0-k_2^0+\sqrt{\left(\mathbf{k}_2-\frac{\mathbf{u}_2}{2}\right)^2+m^2}\right)\delta^{(3)}\left(\mathbf{k}-\mathbf{k}_1-\frac{\mathbf{u}_2}{2}\right)\right. \\ &\quad\times\left\langle k+\frac{u_1+u_2}{2},k_2-\frac{u_2}{2}\middle|\,\widehat{t}\,\middle|k^2+\frac{u^2}{2}\right\rangle+(1\leftrightarrow 2) \\ &\quad-\left[\delta\left(k^0-k_1^0-k_2^0+\sqrt{\left(\mathbf{k}_2+\frac{\mathbf{u}_2}{2}\right)^2+m^2}\right)\delta^{(3)}\left(\mathbf{k}-\mathbf{k}_1+\frac{\mathbf{u}_2}{2}\right)\right. \\ &\quad\times\left.\left\langle k^2-\frac{u^2}{2}\middle|\,\widehat{t}^\dagger\,\middle|k-\frac{u_1+u_2}{2},k_2+\frac{u_2}{2}\right\rangle+(1\leftrightarrow 2)\right] \\ &\quad-\frac{1}{2\hbar}\int \mathrm{d}K'\left\langle k^2-\frac{u^2}{2}\middle|\,\widehat{t}^\dagger\,\middle|k-\frac{u_1+u_2}{2},k'\right\rangle\left\langle k',k+\frac{u_1+u_2}{2}\middle|\,\widehat{t}\,\middle|k^2+\frac{u^2}{2}\right\rangle \\ &\quad\times\left.\left[\widetilde{\Delta}_R\left(k-\frac{u_1+u_2}{2}\right)-\widetilde{\Delta}_R^*\left(k+\frac{u_1+u_2}{2}\right)\right]\delta^{(4)}\left(k+k'-k_1-k_2\right)\right\} . \end{aligned} \tag{4.42} $$

In order to simplify this expression further, we split the transfer matrix $\widehat{t}$ into its real and imaginary parts. The real parts, which were neglected in Ref. [43], correspond to VLASOV-like terms that belong to the left-hand side of the BOLTZMANN equation, cf. e.g. Ref. [146]. In the remainder of this thesis, we will not consider these terms further for simplicity, but including them is straightforward. Thus, we can approximate the transfer matrix by its imaginary part, whose matrix elements can be rewritten through the optical theorem [43]

$$ \frac{i}{2}\,\langle k^2|\,\widehat{t}-\widehat{t}^\dagger\,|p^2\rangle = -\frac{(2\pi\hbar)^4}{16}\int \mathrm{d}Q_1\,\mathrm{d}Q_2\,\delta^{(4)}(q_1+q_2-k_1-k_2)\,\langle k^2|\,\widehat{t}\,|q^2\rangle\,\langle q^2|\,\widehat{t}^\dagger\,|p^2\rangle \ . \tag{4.43} $$

Utilizing this theorem, we are in a position to reinsert Eq. (4.42) into Eq. (4.37) and subsequently into Eq. (4.36). However, in order to obtain a simpler expression it is advantageous to employ Theorem 1 and evaluate everything on the mass shell. The main effect of this usage is that the last line of Eq. (4.42) can be evaluated at the point $u_1=u_2=0$, since otherwise the difference of the propagators would introduce off-shell terms. This can be seen from the identity

$$ \lim_{\eta\to 0}\frac{1}{x\pm i\eta} = \mathscr{P}\frac{1}{x}\mp i\pi\delta(x) \ , \tag{4.44} $$

where $\mathscr{P}$ denotes the CAUCHY principal value. Then, since the retarded propagators are evaluated at the same momentum, we are able to use the important relation

$$ \widetilde{\Delta}_R\,(k)-\widetilde{\Delta}_R^*(k) = 2\pi i\hbar^2\delta(k^2-m^2) \ . \tag{4.45} $$

Defining the on-shell contributions to the WIGNER function and the collision term

$$ W(x,k) =: 4\pi\hbar\delta(k^2-m^2)f(x,k)+W_{\text{off-shell}}(x,k) \ , \tag{4.46a} $$
$$ \mathcal{C}(x,k) =: 4\pi\hbar\delta(k^2-m^2)C(x,k)+\mathcal{C}_{\text{off-shell}}(x,k) \ , \tag{4.46b} $$

where the off-shell parts are taken to be nonsingular at $k^2 = m^2$, we obtain

$$
\begin{aligned}
C(x,k) = \frac{(2\pi\hbar)^4}{32} \int \mathrm{d}^4\bar{x}^2 \int \mathrm{d}K_1\,\mathrm{d}K_2\,\mathrm{d}K' \int \frac{\mathrm{d}^4u^2}{(2\pi\hbar)^8} e^{\frac{i}{\hbar}(u_1\cdot\bar{x}_1 + u_2\cdot\bar{x}_2)} \\
\times \Bigg[f(x+\bar{x}_1,k_1) f(x+\bar{x}_2,k_2)\delta^{(4)}(k+k'-k_1-k_2) \\
\times M\left(k+\frac{u_1+u_2}{2},k',k_1+\frac{u_1}{2},k_2+\frac{u_2}{2}\right) M^*\left(k-\frac{u_1+u_2}{2},k',k_1-\frac{u_1}{2},k_2-\frac{u_2}{2}\right) \\
-\frac{1}{2} f\left(x+\bar{x}_1,k-\frac{u_2}{2}\right) f(x+\bar{x}_2,k')\delta^{(4)}\left(k+k'-k_1-k_2+\frac{u_1}{2}\right) \\
\times M\left(k+\frac{u_1+u_2}{2},k'-\frac{u_2}{2},k_1,k_2\right) M^*\left(k+\frac{u_1-u_2}{2},k'+\frac{u_2}{2},k_1,k_2\right) \\
-\frac{1}{2} f\left(x+\bar{x}_1,k+\frac{u_2}{2}\right) f(x+\bar{x}_2,k')\delta^{(4)}\left(k+k'-k_1-k_2-\frac{u_1}{2}\right) \\
\times M\left(k+\frac{u_2-u_1}{2},k'-\frac{u_2}{2},k_1,k_2\right) M^*\left(k-\frac{u_1+u_2}{2},k'+\frac{u_2}{2},k_1,k_2\right) \Bigg] ,
\end{aligned}
\tag{4.47}
$$

where we introduced the notation

$$
\langle k,k' | \,\widehat{t}\, | k^2 \rangle = \frac{1}{\hbar} \langle k,k' | : \widehat{\mathcal{L}}_{\mathrm{int}}(0) : | k^2 \rangle =: M(k,k',k_1,k_2) \,. \tag{4.48}
$$

Note that we also used the fact that we can neglect the off-shell contributions of the WIGNER functions inside the collision integrals, as they are either of order $\mathcal{O}(\hbar^2)$ or of collisional origin themselves, putting them outside our employed truncation. Finally we expand the on-shell WIGNER function around x as

$$
f(x+\bar{x}_j,k) \simeq f(x,k) + \bar{x}_j \cdot \partial f(x,k) \,, \tag{4.49}
$$

such that we obtain (after performing the $\mathrm{d}\bar{x}^2$-integrations) and making use of the symmetries of M

$$
\begin{aligned}
C(x,k) = \frac{(2\pi\hbar)^4}{32} \int \mathrm{d}K_1\,\mathrm{d}K_2\,\mathrm{d}K' \int \mathrm{d}^4u^2 \\
\times \Bigg(\delta^{(4)}(k+k'-k_1-k_2) \prod_{j=1}^{2} \left\{ \left[\delta^{(4)}(u_j) - i\hbar\partial^\rho_{u_j}\delta^{(4)}(u_j)\partial_\rho \right] f(x,k_j) \right\} \\
\times M\left(k+\frac{u_1+u_2}{2},k',k_1+\frac{u_1}{2},k_2+\frac{u_2}{2}\right) M^*\left(k-\frac{u_1+u_2}{2},k',k_1-\frac{u_1}{2},k_2-\frac{u_2}{2}\right) \\
-\frac{1}{2} \left\{ \left[\delta^{(4)}(u_1) - i\hbar\partial^\rho_{u_1}\delta^{(4)}(u_1)\partial_\rho \right] f\left(x,k-\frac{u_2}{2}\right) \right\} \\
\times \left\{ \left[\delta^{(4)}(u_2) - i\hbar\partial^\rho_{u_2}\delta^{(4)}(u_2)\partial_\rho \right] f(x,k') \right\} \delta^{(4)}\left(k+k'-k_1-k_2+\frac{u_1}{2}\right) \\
\times M\left(k+\frac{u_1+u_2}{2},k'-\frac{u_2}{2},k_1,k_2\right) M^*\left(k+\frac{u_1-u_2}{2},k'+\frac{u_2}{2},k_1,k_2\right) \\
-\frac{1}{2} \left\{ \left[\delta^{(4)}(u_1) - i\hbar\partial^\rho_{u_1}\delta^{(4)}(u_1)\partial_\rho \right] f\left(x,k+\frac{u_2}{2}\right) \right\} \\
\times \left\{ \left[\delta^{(4)}(u_2) - i\hbar\partial^\rho_{u_2}\delta^{(4)}(u_2)\partial_\rho \right] f(x,k') \right\} \delta^{(4)}\left(k+k'-k_1-k_2-\frac{u_1}{2}\right) \\
\times M\left(k+\frac{u_2-u_1}{2},k'-\frac{u_2}{2},k_1,k_2\right) M^*\left(k-\frac{u_1+u_2}{2},k'+\frac{u_2}{2},k_1,k_2\right) \Bigg) .
\end{aligned}
\tag{4.50}
$$

The equation above already features a structure of gain and loss terms, determining the rate of particles scattering into and out of the momentum state k, respectively. However, the loss term, given by the second and third term in the equation above, consists of two contributions, differing in the signs of the momentum-space shifts u_1, u_2. The first terms in curly brackets denote local contributions to the collision term, while the respective second terms characterize nonlocal collisions, since they originate from the fact that the distribution functions in Eq. (4.47) were originally evaluated at different spacetime positions, cf. Eq. (4.49). At this point it is apparent that at zeroth order in $\hbar$ only local collisions take place, while at first order both local and nonlocal contributions enter. Note that, from now on, we will also neglect the momentum dependence of the vertices M.

4.3.2 Local collisions

Evaluating the first terms in curly brackets in Eq. (4.50), i.e., setting $u_1 = u_2 = 0$, we find the local collision term

$$
\begin{aligned}
C_{\text{local}}(x,k) = \frac{1}{2} \int & \mathrm{d}K_1\, \mathrm{d}K_2\, \mathrm{d}K' (2\pi\hbar)^4 \delta^{(4)}(k+k'-k_1-k_2) \frac{|M|^2}{16} \\
& \times \left[f(x,k_1) f(x,k_2) - f(x,k) f(x,k') \right] \;,
\end{aligned} \tag{4.51}
$$

Comparing this expression to Eq. (4.3), we can identify $\mathcal{W}_{\mathbf{k}\mathbf{k}'\to\mathbf{k}_1\mathbf{k}_2} \equiv |M|^2/16$. Furthermore, as expected, we lost the effect of quantum statistics due to our truncation of the collision expansion (4.36), which is valid in the low-density regime [43].

4.3.3 Nonlocal collisions

In order to evaluate the nonlocal terms, we integrate by parts in u_1, u_2 in Eq. (4.50). We find two contributions (which we will label by Roman numbers), one where the u_1, u_2-derivatives act on the distribution functions in the loss terms, and a second one where they act on the momentum-conserving delta functions in the loss terms. In principle, there would be a third contribution where the derivatives act on the vertices M, which vanishes due to our assumption of them being momentum-independent. The first contribution reads

$$
\begin{aligned}
C_{\text{I}}^{\text{nonlocal}}(x,k) &= -i\hbar \frac{|M|^2}{64} \int \mathrm{d}K_1\, \mathrm{d}K_2\, \mathrm{d}K' (2\pi\hbar)^4 \delta^{(4)}(k+k'-k_1-k_2) \\
&\quad \times \int \mathrm{d}^4 u_2 \delta^{(4)}(u_2) \partial_{u_2}^\rho \left[f\left(x, k - \frac{u_2}{2}\right) + f\left(x, k + \frac{u_2}{2}\right) \right] \partial_\rho f(x,k') \\
&= -\frac{i\hbar}{2} \frac{|M|^2}{64} \int \mathrm{d}K_1 \int \mathrm{d}K_2 \int \mathrm{d}K' (2\pi\hbar)^4 \delta^{(4)}(k+k'-k_1-k_2) \\
&\quad \times \partial_k^\rho \left[-f(x,k) + f(x,k) \right] \partial_\rho f(x,k') \\
&= 0 \;.
\end{aligned} \tag{4.52}
$$

The second contribution can be evaluated similarly. Noting that

$$
\partial_{u_1}^\mu \delta^{(4)} \left(k + k' - k_1 - k_2 \pm \frac{u_1}{2} \right) = \pm \frac{1}{2} \partial_{k'}^\mu \delta^{(4)} \left(k + k' - k_1 - k_2 \pm \frac{u_1}{2} \right) \;,
$$

we can rewrite the u_1-derivative acting on the delta function as a k'-derivative and integrate by parts again to obtain

$$
\begin{aligned}
C_{\text{II}}^{\text{nonlocal}}(x,k) &= \frac{i\hbar}{2} \frac{|M|^2}{64} \int \mathrm{d}K_1\, \mathrm{d}K_2\, \mathrm{d}K' (2\pi\hbar)^4 \delta^{(4)}(k+k'-k_1-k_2) \\
&\quad \times \partial_{k'}^\rho \left[f(x,k') - f(x,k') \right] \partial_\rho f(x,k) \\
&= 0 \;.
\end{aligned} \tag{4.53}
$$

Thus, in the case of scalar particles, the nonlocal contributions vanish and the collision term stays local up to first order in $\hbar$. The intuition behind this fact is that nonlocal collisions are needed to exchange orbital and spin angular momenta. More specifically, the exchange of spin will necessitate a nonzero net orbital angular momentum in the collision, which is realized by shifting the position of the particles. However, since the spin of scalar particles is zero, the orbital angular momentum is separately conserved, and such shifts are thus not induced.

As we shall see in the coming sections, this changes for particles with nonzero spin.

4.4 Spinor fields

After having completed the treatment of scalar fields, we move on to DIRAC fermions with spin 1/2. While the general steps are very similar to what has been shown in the last section, there are some subtleties related to the nontrivial internal structure of the fields.

4.4.1 Rewriting expectation values

For a spinor field, the FOURIER decomposition reads

$$\widehat{\psi}_{\text{in}}(x) = \sum_s \int \frac{\mathrm{d}^4 k}{(2\pi\hbar)^3} \Theta(k^0)\delta(k^2 - m^2) e^{-\frac{i}{\hbar}k\cdot x} u_s(k)\widehat{a}_{\text{in}}(k) \,, \tag{4.54}$$

where we introduced the basis spinors u_r, cf. Eq. (4.8). These quantities are constrained by the DIRAC equation to satisfy

$$(\not{k} - m)\, u_r(k) = 0 \,, \tag{4.55}$$

and they are constructed to fulfill the following orthogonality and completeness relations,

$$\overline{u}_{r,\alpha}(k) u_s^{\alpha}(k) = 2m\delta_{rs} \,, \tag{4.56a}$$

$$\sum_r u_r^{\alpha}(k)\overline{u}_{r,\beta}(k) = (\not{k} + m)^{\alpha}{}_{\beta} \equiv 2m\Lambda^{+,\alpha}{}_{\beta}(k) \,, \tag{4.56b}$$

where we made the DIRAC indices explicit[3] and introduced the projector onto positive-energy states

$$\Lambda^+(k) := \frac{\not{k} + m}{2m} \,. \tag{4.57}$$

Accordingly, the "in"-WIGNER function for the DIRAC field is defined as

$$W_{\text{in},\alpha\beta}(x,k) := \int \mathrm{d}^4 v e^{-\frac{i}{\hbar}k\cdot v} \left\langle \widehat{\overline{\psi}}_{\beta}\left(x + \frac{v}{2}\right) \widehat{\psi}_{\alpha}\left(x - \frac{v}{2}\right)\right\rangle \,. \tag{4.58}$$

From Eq. (4.23) we obtain for spin-1/2 particles

$$\left\langle \widehat{O} \right\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}^4 \bar{x}^n \int \frac{\mathrm{d}^4 \bar{k}^n}{(2\pi\hbar)^{4n}} O_{n,\alpha_1\beta_1\cdots\alpha_n\beta_n}(\bar{x}^n;\bar{k}^n) \prod_{j=1}^{n} W_{\text{in}}^{\alpha_j\beta_j}(\bar{x}_j,\bar{k}_j) \,, \tag{4.59}$$

where we defined

$$\begin{aligned} O_{n,\alpha_1\beta_1\cdots\alpha_n\beta_n}(\bar{x}^n;\bar{k}^n) := \frac{1}{(2m)^{2n}} \int \frac{\mathrm{d}^4 u^n}{(2\pi\hbar)^{4n}} \sum_{r^n, r'^n} {}_{\text{in}}\!\left\langle\!\!\left\langle \bar{k}^n - \frac{u^n}{2}; r^n \right| \widehat{O} \left| \bar{k}^n + \frac{u^n}{2}; r'^n \right\rangle\!\!\right\rangle_{\text{in}} \\ \times \left[\prod_{j=1}^{n} e^{\frac{i}{\hbar}u_j\cdot\bar{x}_j} u_{r_j,\beta_j}\left(\bar{k}_j - \frac{u_j}{2}\right) \overline{u}_{r'_j,\alpha_j}\left(\bar{k}_j + \frac{u_j}{2}\right)\right] \,. \end{aligned} \tag{4.60}$$

Expressing $\mathcal{C}_{\alpha\beta}$ in terms of $W_{\text{in},\alpha\beta}$

As demonstrated in the last section for scalar particles, we now express the collision term (3.84b) in terms of the "in"-WIGNER function, which can then be replaced by the full WIGNER function to close the resulting equation. Note that, once the collision term (3.84b) is known, we can proceed to translate it into extended phase space via Eq. (3.88b).

[3] From now on, we will not distinguish between upper and lower DIRAC indices, as they will be traced over in the final result anyway.

Orienting on Eq. (4.34), we express the collision term as the average

$$\mathcal{C}_{\alpha\beta}(x,k) = \left\langle e^{\frac{i}{\hbar}\widehat{P}\cdot x}\widehat{\Phi}_{\alpha\beta}(k)e^{-\frac{i}{\hbar}\widehat{P}\cdot x}\right\rangle \;, \tag{4.61}$$

where we defined the operator

$$\begin{aligned}\widehat{\Phi}_{\alpha\beta}(k) \coloneqq \frac{i}{2}\int \mathrm{d}^4 v e^{-\frac{i}{\hbar}k\cdot v}\bigg\{&\left[\widehat{P}_\mu,\widehat{\bar{\rho}}\left(\frac{v}{2}\right)\gamma^\mu\right]_\beta \widehat{\psi}_\alpha\left(-\frac{v}{2}\right) + m\widehat{\bar{\rho}}_\beta\left(\frac{v}{2}\right)\widehat{\psi}_\alpha\left(-\frac{v}{2}\right)\\ &-\widehat{\bar{\psi}}_\beta\left(\frac{v}{2}\right)\left[\gamma^\mu\widehat{\rho}\left(-\frac{v}{2}\right),\widehat{P}_\mu\right]_\alpha - m\widehat{\bar{\psi}}_\beta\left(\frac{v}{2}\right)\widehat{\rho}_\alpha\left(-\frac{v}{2}\right)\bigg\} \;. \end{aligned}\tag{4.62}$$

Here we used that for an arbitrary operator $\widehat{A}(x)$, the (covariant version of) the HEISENBERG equation of motion states that

$$\left[\widehat{A}(x),\widehat{P}^\mu\right] = i\hbar\partial^\mu\widehat{A}(x) \;. \tag{4.63}$$

Employing Eq. (4.59), the collision term can be written as

$$\mathcal{C}^{\alpha\beta}(x,k) = \sum_{n=2}^{\infty}\frac{1}{n!}\int \mathrm{d}^4\bar{x}^n\int\frac{\mathrm{d}^4\bar{k}^n}{(2\pi\hbar)^{4n}}\Phi^{\alpha\beta}_{n,\alpha_1\beta_1\cdots\alpha_n\beta_n}(\bar{x}^n;\bar{k}^n|k)\prod_{j=1}^{n}W_{\text{in}}^{\alpha_j\beta_j}(x+\bar{x}_j,\bar{k}_j) \;, \tag{4.64}$$

where we defined

$$\begin{aligned}\Phi^{\alpha\beta}_{n,\alpha_1\beta_1\cdots\alpha_n\beta_n}(\bar{x}^n;\bar{k}^n|k) \coloneqq \frac{1}{(2m)^{2n}}\int\frac{\mathrm{d}^4u^n}{(2\pi\hbar)^{4n}}\sum_{r^n,r'^n}{}_{\text{in}}\!\left\langle\!\!\left\langle \bar{k}^n-\frac{u^n}{2};r^n\right|\widehat{\Phi}^{\alpha\beta}(k)\left|\bar{k}^n+\frac{u^n}{2};r'^n\right\rangle\!\!\right\rangle_{\text{in}}\\ \times\left[\prod_{j=1}^{n}e^{\frac{i}{\hbar}u_j\cdot\bar{x}_j}u_{r_j,\beta_j}\left(\bar{k}_j-\frac{u_j}{2}\right)\bar{u}_{r'_j,\alpha_j}\left(\bar{k}_j+\frac{u_j}{2}\right)\right] \;, \end{aligned}\tag{4.65}$$

in analogy to Eq. (4.60). After inserting a complete set of "out"-states and performing the d^4v-integration we obtain

$$\begin{aligned}&{}_{\text{in}}\!\left\langle k^2-\frac{u^2}{2};r^2\right|\widehat{\Phi}_{\alpha\beta}(k)\left|k^2+\frac{u^2}{2};s^2\right\rangle_{\text{in}}\\ &=\frac{i}{4}\sum_{r'}\int\mathrm{d}K'(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\bigg[{}_{\text{out}}\!\left\langle k';r'\right|\widehat{\psi}_\alpha(0)\left|k^2+\frac{u^2}{2};s^2\right\rangle_{\text{in}}\\ &\times{}_{\text{in}}\!\left\langle k^2-\frac{u^2}{2};r^2\right|\widehat{\bar{\rho}}_{\alpha'}(0)\left|k';r'\right\rangle_{\text{out}}\left(\not{k}-\frac{\not{u}_1+\not{u}_2}{2}+m\right)_{\alpha'\beta}-\left(\not{k}+\frac{\not{u}_1+\not{u}_2}{2}+m\right)_{\alpha\alpha'}\\ &\times{}_{\text{out}}\!\left\langle k';r'\right|\widehat{\rho}_{\alpha'}(0)\left|k^2+\frac{u^2}{2};s^2\right\rangle_{\text{in}}\,{}_{\text{in}}\!\left\langle k^2-\frac{u^2}{2};r^2\right|\widehat{\bar{\psi}}_\beta(0)\left|k';r'\right\rangle_{\text{out}}\bigg] \;. \end{aligned}\tag{4.66}$$

As was done in the last section in order to obtain Eq. (4.38), we used that, in the case of a single particle species that features an intrinsic charge, only binary scattering is permitted. The matrix elements of the field operators can be evaluated via the YANG-FELDMAN equation for spin-1/2 particles [43]

$$\widehat{\psi}(0) = \widehat{\psi}_{\text{in}}(0) + \int\mathrm{d}^4x S_R(-x)\widehat{\rho}(x) \;, \tag{4.67}$$

where S_R denotes the retarded fermion propagator introduced in Sec. 3.4. Then, the respective matrix elements can be evaluated to be

$$\begin{aligned}&{}_{\text{out}}\!\left\langle k';r'\right|\widehat{\psi}(0)\left|k^2+\frac{u^2}{2};s^2\right\rangle_{\text{in}}\\ &=(2\pi\hbar)^3 2k'^0\left[u_{s_1}\left(k_1+\frac{u_1}{2}\right)\delta^{(3)}\left(\mathbf{k}'-\mathbf{k}_2-\frac{\mathbf{u}_2}{2}\right)\delta_{r's_2}-(1\leftrightarrow 2)\right]\\ &+\widetilde{S}_R\left(k_1+k_2+\frac{u_1+u_2}{2}-k'\right){}_{\text{out}}\!\left\langle k';r'\right|\widehat{\rho}(0)\left|k^2+\frac{u^2}{2};s^2\right\rangle_{\text{in}} \;. \end{aligned}\tag{4.68}$$

Inserting this relation into Eq. (4.66) and employing the definition of the Fourier-transformed retarded fermion propagator (3.71a), we find

$$\begin{aligned}
&{}_{\text{in}}\left\langle k^2-\frac{u^2}{2};r^2\middle|\widehat{\Phi}_{\alpha\beta}(k)\middle|k^2+\frac{u^2}{2};s^2\right\rangle_{\text{in}} \\
&= im(2\pi\hbar)^4\Bigg(\bigg\{u_{s_1,\alpha}\left(k_1+\frac{u_1}{2}\right)\delta^{(3)}\left(\mathbf{k}-\mathbf{k}_1+\frac{\mathbf{u}_2}{2}\right)\delta\left[k^0+\sqrt{\left(\mathbf{k}_2+\frac{\mathbf{u}_2}{2}\right)^2+m^2}-k_1^0-k_2^0\right] \\
&\quad\times{}_{\text{in}}\left\langle k^2-\frac{u^2}{2};r^2\middle|\widehat{\overline{\rho}}_{\alpha'}(0)\middle|k_2+\frac{u_2}{2};s_2\right\rangle_{\text{out}}\Lambda^+_{\alpha'\beta}\left(k-\frac{u_1+u_2}{2}\right)+(1\leftrightarrow 2)\bigg\} \\
&\quad-\bigg\{\overline{u}_{r_1,\beta}\left(k_1-\frac{u_1}{2}\right)\delta^{(3)}\left(\mathbf{k}-\mathbf{k}_1-\frac{\mathbf{u}_2}{2}\right)\delta\left[k^0+\sqrt{\left(\mathbf{k}_2-\frac{\mathbf{u}_2}{2}\right)^2+m^2}-k_1^0-k_2^0\right] \\
&\quad\times\Lambda^+_{\alpha\alpha'}\left(k+\frac{u_1+u_2}{2}\right){}_{\text{out}}\left\langle k_2-\frac{u_2}{2};r_2\middle|\widehat{\rho}_{\alpha'}(0)\middle|k^2+\frac{u^2}{2};s^2\right\rangle_{\text{in}}+(1\leftrightarrow 2)\bigg\} \\
&\quad-\frac{m}{\hbar}\sum_{r'}\int\mathrm{d}K'\delta^{(4)}(k+k'-k_1-k_2)\left[\widetilde{\Delta}_R\left(k+\frac{u_1+u_2}{2}\right)-\widetilde{\Delta}^*_R\left(k-\frac{u_1+u_2}{2}\right)\right] \\
&\quad\times\Lambda^+_{\alpha\alpha'}\left(k+\frac{u_1+u_2}{2}\right){}_{\text{out}}\left\langle k';r'\middle|\widehat{\rho}_{\alpha'}(0)\middle|k^2+\frac{u^2}{2};s^2\right\rangle_{\text{in}} \\
&\quad\times{}_{\text{in}}\left\langle k^2-\frac{u^2}{2};r^2\middle|\widehat{\overline{\rho}}_{\beta'}(0)\middle|k';r'\right\rangle_{\text{out}}\Lambda^+_{\beta'\beta}\left(k-\frac{u_1+u_2}{2}\right)\Bigg)\,. \qquad (4.69)
\end{aligned}$$

Note that in this expression we made use of the completeness relation (4.56b). Next, we have to employ the relation between the source terms and transfer-matrix elements [i.e., the spin-1/2 equivalent of Eq. (4.41)], which is given by [43]

$$\langle k,k';r,r'|\,\widehat{t}\,\big|k^2;r^2\big\rangle=-\overline{u}_r(k)\,{}_{\text{out}}\langle k';r'|\widehat{\rho}(0)\,\big|k^2;r^2\big\rangle_{\text{in}}\ . \qquad (4.70)$$

Then, we split the transfer matrix into real and imaginary parts, neglect the real parts (as they give rise to Vlasov-like terms, which we do not consider in this work[4]) and make use of the optical theorem, which in the case of spin-1/2 particles reads [43]

$$\begin{aligned}
&\frac{i}{2}\langle k,k';r,r'|\,\widehat{t}-\widehat{t}^\dagger\,\big|k^2;r^2\big\rangle \\
&=-\frac{(2\pi\hbar)^4}{16}\delta^{(4)}(k+k'-k_1-k_2)\sum_{s^2}\int\mathrm{d}Q_1\,\mathrm{d}Q_2\,\langle k,k';r,r'|\,\widehat{t}\,\big|q^2;s^2\big\rangle\big\langle q^2;s^2\big|\,\widehat{t}^\dagger\,\big|k^2;r^2\big\rangle\ . \qquad (4.71)
\end{aligned}$$

Lastly, similar to Eq. (4.48), we define the tree-level vertices of the theory via

$$\begin{aligned}
\langle k,k';r,r'|\,\widehat{t}\,\big|k^2;r^2\big\rangle&=\frac{1}{\hbar}\,\langle k,k';r,r'|:\widehat{\mathcal{L}}_{\text{int}}(0):\big|k^2;r^2\big\rangle \\
&=\overline{u}_{r,\alpha}(k)\overline{u}_{r',\alpha'}(k')u_{r_1,\alpha_1}(k_1)u_{r_2,\alpha_2}(k_2)M^{\alpha\alpha'\alpha_1\alpha_2}(k,k',k_1,k_2)\ . \qquad (4.72)
\end{aligned}$$

Note that, since we are dealing with fermions, we have the symmetry relations

$$M^{\alpha\alpha'\alpha_1\alpha_2}(k,k',k_1,k_2)=-M^{\alpha'\alpha\alpha_1\alpha_2}(k',k,k_1,k_2)=-M^{\alpha\alpha'\alpha_2\alpha_1}(k,k',k_2,k_1)\ .$$

Assuming that the vertex M is symmetric under the exchange $k\leftrightarrow k_1$, $k'\leftrightarrow k_2$, we furthermore have

$$\big\langle k^2;r^2\big|\,\widehat{t}^\dagger\,|k,k';r,r'\rangle=\overline{u}_{r_1,\alpha_1}(k_1)\overline{u}_{r_2,\alpha_2}(k_2)u_{r,\alpha}(k)u_{r',\alpha'}(k')M^{\alpha_1\alpha_2\alpha\alpha'}(k,k',k_1,k_2)\ , \qquad (4.73)$$

where we used that $\widehat{t}=\widehat{t}^\dagger$ at tree level.

[4] These contributions have been called "pure-spin exchange terms" in Ref. [43]. We remark at this point that neglecting these terms is the equivalent of considering the self-energy in the T-matrix approximation in the Kadanoff-Baym approach [46, 147].

These relations can now be inserted into Eq. (4.69) and subsequently into Eq. (4.64), where the sum is truncated at $n = 2$. Performing these steps and making use of Theorem 1, we obtain the following result which only depends on the on-shell parts of the Wigner functions,

$$\begin{aligned}
&C_{\alpha\beta}(x,k)\\
&= (2\pi\hbar)^4\frac{m^4}{2}\int \mathrm{d}K_1\,\mathrm{d}K_2\,\mathrm{d}K'\int \mathrm{d}^4u^2\,\Lambda^+_{\alpha\alpha'}\left(k+\frac{u_1+u_2}{2}\right)\Lambda^+_{\beta'\beta}\left(k-\frac{u_1+u_2}{2}\right)\\
&\quad\times\Bigg(\delta^{(4)}(k+k'-k_1-k_2)\Lambda^+_{\delta_2\alpha_2}(k')\Lambda^+_{\beta_1\gamma_1'}\left(k_1+\frac{u_1}{2}\right)\Lambda^+_{\beta_2\gamma_2'}\left(k_2+\frac{u_2}{2}\right)\Lambda^+_{\delta_1'\gamma_1}\left(k_1-\frac{u_1}{2}\right)\\
&\quad\times\Lambda^+_{\delta_2'\gamma_2}\left(k_2-\frac{u_2}{2}\right)\delta^{\alpha'}_{\alpha_1}\delta^{\beta'}_{\delta_1}\prod_{j=1}^{2}\left\{\left[\delta^{(4)}(u_j)-i\hbar\partial^\rho_{u_j}\delta^{(4)}(u_j)\partial_\rho\right]W^{\gamma_j'\delta_j'}_{\text{on-shell}}(x,k_j)\right\}\\
&\quad\times M^{\alpha_1\alpha_2\beta_1\beta_2}\left(k+\frac{u_1+u_2}{2},k',k_1+\frac{u_1}{2},k_2+\frac{u_2}{2}\right)\\
&\quad\times M^{\gamma_1\gamma_2\delta_1\delta_2}\left(k-\frac{u_1+u_2}{2},k',k_1-\frac{u_1}{2},k_2-\frac{u_2}{2}\right)\\
&\quad-\frac{1}{2}\delta^{(4)}\left(k+k'-k_1-k_2+\frac{u_1}{2}\right)\Lambda^+_{\delta_2\gamma_2'}\left(k'+\frac{u_2}{2}\right)\Lambda^+_{\delta_1\gamma_1'}\left(k+\frac{u_1-u_2}{2}\right)\Lambda^+_{\delta_2'\alpha_2}\left(k'-\frac{u_2}{2}\right)\\
&\quad\times\Lambda^+_{\beta_1\gamma_1}(k_1)\Lambda^+_{\beta_2\gamma_2}(k_2)\delta^{\alpha'}_{\alpha_1}\delta^{\beta'}_{\delta_1'}\left\{\left[\delta^{(4)}(u_1)-i\hbar\partial^\rho_{u_1}\delta^{(4)}(u_1)\partial_\rho\right]W^{\gamma_1'\delta_1'}_{\text{on-shell}}\left(x,k-\frac{u_2}{2}\right)\right\}\\
&\quad\times\left\{\left[\delta^{(4)}(u_2)-i\hbar\partial^\rho_{u_2}\delta^{(4)}(u_2)\partial_\rho\right]W^{\gamma_2'\delta_2'}_{\text{on-shell}}(x,k')\right\}\\
&\quad\times M^{\alpha_1\alpha_2\beta_1\beta_2}\left(k+\frac{u_2+u_1}{2},k'-\frac{u_2}{2},k_1,k_2\right)M^{\gamma_1\gamma_2\delta_1\delta_2}\left(k+\frac{u_1-u_2}{2},k'+\frac{u_2}{2},k_1,k_2\right)\\
&\quad-\frac{1}{2}\delta^{(4)}\left(k+k'-k_1-k_2-\frac{u_1}{2}\right)\Lambda^+_{\delta_2\gamma_2'}\left(k'+\frac{u_2}{2}\right)\Lambda^+_{\delta_1'\alpha_1}\left(k+\frac{u_2-u_1}{2}\right)\Lambda^+_{\delta_2'\alpha_2}\left(k'-\frac{u_2}{2}\right)\\
&\quad\times\Lambda^+_{\beta_1\gamma_1}(k_1)\Lambda^+_{\beta_2\gamma_2}(k_2)\delta^{\alpha'}_{\gamma_1'}\delta^{\beta'}_{\delta_1}\left\{\left[\delta^{(4)}(u_1)-i\hbar\partial^\rho_{u_1}\delta^{(4)}(u_1)\partial_\rho\right]W^{\gamma_1'\delta_1'}_{\text{on-shell}}\left(x,k+\frac{u_2}{2}\right)\right\}\\
&\quad\times\left\{\left[\delta^{(4)}(u_2)-i\hbar\partial^\rho_{u_2}\delta^{(4)}(u_2)\partial_\rho\right]W^{\gamma_2'\delta_2'}_{\text{on-shell}}(x,k')\right\}\\
&\quad\times M^{\alpha_1\alpha_2\beta_1\beta_2}\left(k+\frac{u_2-u_1}{2},k'-\frac{u_2}{2},k_1,k_2\right)M^{\gamma_1\gamma_2\delta_1\delta_2}\left(k-\frac{u_1+u_2}{2},k'+\frac{u_2}{2},k_1,k_2\right)\Bigg)\,. \qquad (4.74)
\end{aligned}$$

In this expression, we expanded the Wigner functions to first order around $\bar{x}_1 = \bar{x}_2 = 0$, cf. Eq. (4.49), and subsequently performed the $\mathrm{d}\bar{x}^2$-integrations. Furthermore, the on-shell contributions of the Wigner function and the collision term are defined via

$$W(x,k) =: 4m\pi\hbar\delta(k^2-m^2)W_{\text{on-shell}}(x,k)+W_{\text{off-shell}}(x,k)\,, \qquad (4.75a)$$

$$\mathcal{C}(x,k) =: 4m\pi\hbar\delta(k^2-m^2)C(x,k)+\mathcal{C}_{\text{off-shell}}(x,k)\,, \qquad (4.75b)$$

where the off-shell terms are defined to be nonsingular on the mass shell. Since we can neglect the off-shell parts of the collision term, we were also able to make use of the relation (4.45) to simplify the term in Eq. (4.69) containing the difference of the retarded scalar propagator and its complex conjugate, which can both be evaluated at the momentum k.
Note that, in accordance with Eq. (3.100), we included a factor of mass when defining the on-shell parts for the fermionic Wigner function. This is to be contrasted with Eqs. (4.46), and has its roots in the different dimension of the fermionic fields as compared to scalar or vector fields.

As in the case of Klein-Gordon fields, the collision term (4.74) contains both local and nonlocal contributions. Due to the nontrivial internal structure of particles with spin, however, there appear a multitude of positive-energy projectors Λ^+, many of which are dependent on u_1, u_2 and thus contribute to the nonlocal collisions. Indeed, as we will see shortly, these terms are responsible for nonvanishing nonlocal contributions to the collision term at first order in $\hbar$. As in the previous section, we will from now on neglect the momentum dependence of the vertices M for simplicity.

4.4.2 Local collisions

In order to obtain the local contributions to the collision term, we set all shifts u_1, u_2 in momentum space to zero, such that we find

$$\begin{aligned} C^{\text{local}}_{\alpha\beta}(x,k) = \frac{m^4}{2}\int & \mathrm{d}K_1\,\mathrm{d}K_2\,\mathrm{d}K'(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}\Lambda^+_{\alpha\alpha'}(k)\Lambda^+_{\beta'\beta}(k) \\ & \times\Bigg\{\Lambda^+_{\delta_2\alpha_2}(k')\Lambda^+_{\beta_1\gamma_1'}(k_1)\Lambda^+_{\beta_2\gamma_2'}(k_2)\Lambda^+_{\delta_1'\gamma_1}(k_1)\Lambda^+_{\delta_2'\gamma_2}(k_2)\delta^{\alpha'}_{\alpha_1}\delta^{\beta'}_{\delta_1}\prod_{j=1}^{2}W^{\gamma_j'\delta_j'}_{\text{on-shell}}(x,k_j) \\ & -\frac{1}{2}\Lambda^+_{\delta_2'\alpha_2}(k')\Lambda^+_{\delta_2\gamma_2'}(k')\Lambda^+_{\beta_1\gamma_1}(k_1)\Lambda^+_{\beta_2\gamma_2}(k_2)\left[\delta^{\alpha'}_{\alpha_1}\delta^{\beta'}_{\delta_1'}\Lambda^+_{\delta_1\gamma_1'}(k)+\delta^{\alpha'}_{\gamma_1'}\delta^{\beta'}_{\delta_1}\Lambda^+_{\delta_1'\alpha_1}(k)\right] \\ & \times W^{\gamma_1'\delta_1'}_{\text{on-shell}}(x,k)W^{\gamma_2'\delta_2'}_{\text{on-shell}}(x,k')\Bigg\}\,. \end{aligned} \tag{4.76}$$

This expression now has to be translated into extended phase space, i.e., we have to use (the on-shell part of) Eq. (3.88b) to compute $C(x,k,\mathfrak{s})$. In order to obtain a result with a straightforward interpretation, we also have to relate the Wigner functions in Eq. (4.76) to the scalar distribution function in phase space $f(x,k,\mathfrak{s})$, cf. Eq. (3.85). However, in principle the Wigner function has more components in its Clifford decomposition than the scalar $\mathcal{F}$ and the axial vector $\mathcal{A}$ that enter the distribution f. As discussed in Subsec. 3.4.2, to first order in $\hbar$ the pseudoscalar component $\mathcal{P}$ vanishes, while the vector and tensor contributions read

$$\mathcal{V}^\mu(x,k)\simeq\frac{k^\mu}{m}\mathcal{F}(x,k)\,,\quad \mathcal{S}^{\mu\nu}(x,k)\simeq\frac{\hbar}{2m^2}\partial^{[\mu}k^{\nu]}\mathcal{F}(x,k)-\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}k_\alpha\mathcal{A}_\beta(x,k)\,, \tag{4.77}$$

where the omitted terms are either of second order in $\hbar$ or of collisional origin. Noting that all Wigner functions in Eq. (4.76) are sandwiched in between two positive-energy projectors, we use the relation

$$\Lambda^+(k)\sigma^{\mu\nu}k_\nu\Lambda^+(k)=0 \tag{4.78}$$

and conclude that

$$\Lambda^+(k)W_{\text{on-shell}}(x,k)\Lambda^+(k)=\int\mathrm{d}S(k)h(k,\mathfrak{s})f(x,k,\mathfrak{s})\,, \tag{4.79}$$

where we defined

$$h(k,\mathfrak{s}):=\frac{1}{2}\left(\mathbb{1}+\gamma_5\not{\mathfrak{s}}\right)\Lambda^+(k)\,. \tag{4.80}$$

Note that due to

$$\left(\mathbb{1}+\gamma_5\not{\mathfrak{s}}\right)\Lambda^+(k)=\Lambda^+(k)\left(\mathbb{1}+\gamma_5\not{\mathfrak{s}}\right)\,,$$

which follows from $k\cdot\mathfrak{s}=0$, it holds that

$$\Lambda^+(k)h(k,\mathfrak{s})=h(k,\mathfrak{s})\Lambda^+(k)=h(k,\mathfrak{s})\,. \tag{4.81}$$

Inserting the representation (4.79) of the on-shell Wigner function into the local collision term (4.76), we obtain

$$\begin{aligned} C^{\text{local}}_{\alpha\beta}(x,k)=\frac{m^4}{4}\int & \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,\mathrm{d}\bar{S}(k)(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2} \\ & \times\Bigg\{\Lambda^+_{\alpha\alpha_1}(k)\Lambda^+_{\delta_1\beta}(k)h_{\delta_2\alpha_2}(k',\mathfrak{s}')h_{\beta_1\gamma_1}(k_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(k_2,\mathfrak{s}_2) \\ & \times f(x,k_1,\mathfrak{s}_1)f(x,k_2,\mathfrak{s}_2)-h_{\delta_2\alpha_2}(k',\mathfrak{s}')h_{\beta_1\gamma_1}(k_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(k_2,\mathfrak{s}_2) \\ & \times\left[\Lambda^+_{\alpha\alpha_1}(k)h_{\delta_1\beta}(k,\bar{\mathfrak{s}})+\Lambda^+_{\delta_1\beta}(k)h_{\alpha\alpha_1}(k,\bar{\mathfrak{s}})\right]f(x,k,\bar{\mathfrak{s}})f(x,k',\mathfrak{s}')\Bigg\}\,, \end{aligned} \tag{4.82}$$

where the momentum- and spin-space measure $\mathrm{d}\Gamma$ has been defined in Eq. (3.101), we made use of the relation

$$\int\mathrm{d}S(k)h(k,\mathfrak{s})=\Lambda^+(k)\,, \tag{4.83}$$

and inserted a spurious $\mathrm{d}\bar{S}(k)$-integration in the first term. Lastly, we write the on-shell part of Eq. (3.88b) as

$$C(x,k,\mathfrak{s}) \coloneqq \frac{1}{2}\left(\mathbb{1}+\gamma_5 \not{\mathfrak{s}}\right)_{\beta\alpha} C_{\alpha\beta}(x,k) \;, \tag{4.84}$$

which then yields

$$\begin{aligned} C_{\text{local}}(x,k,\mathfrak{s}) = \frac{m^4}{4}\int & \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,\mathrm{d}\bar{S}(k)(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2} \\ & \times h_{\delta_2\alpha_2}(k',\mathfrak{s}')h_{\beta_1\gamma_1}(k_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(k_2,\mathfrak{s}_2)\left[h_{\delta_1\beta}(k,\bar{\mathfrak{s}})h_{\beta\alpha_1}(k,\mathfrak{s})+h_{\delta_1\beta}(k,\mathfrak{s})h_{\beta\alpha_1}(k,\bar{\mathfrak{s}})\right] \\ & \times \left[f(x,k_1,\mathfrak{s}_1)f(x,k_2,\mathfrak{s}_2)-f(x,k,\bar{\mathfrak{s}})f(x,k',\mathfrak{s}')\right] \;, \end{aligned} \tag{4.85}$$

where we made use of the identity

$$\int \mathrm{d}\bar{S}(k)\left[h_{\delta_1\beta}(k,\bar{\mathfrak{s}})h_{\beta\alpha_1}(k,\mathfrak{s})+h_{\delta_1\beta}(k,\mathfrak{s})h_{\beta\alpha_1}(k,\bar{\mathfrak{s}})\right] = 2h_{\delta_1\alpha_1}(k,\mathfrak{s}) \;. \tag{4.86}$$

In Eq. (4.85), it is interesting to note that, while the distribution functions in the gain term depend on $(k_1,\mathfrak{s}_1)$ and $(k_2,\mathfrak{s}_2)$, as expected, the arguments of the functions in the loss term are $(k',\mathfrak{s}')$ and $(k,\bar{\mathfrak{s}})$ [and not $(k,\mathfrak{s})$]. As we will see in Sec. 4.5, this is an effect that is present also for particles of spin 1. In the case of spin-1/2 particles, it has been shown in Ref. [44] that it is possible to redefine the collision kernel $C(x,k,\mathfrak{s})$ such that the integration over $\bar{\mathfrak{s}}$ is removed and the loss term takes the expected form.[5] In general, however, it is not possible to perform such a procedure, which is why we refrain from doing it at this point. We will return to this issue in Sec. 4.6.

4.4.3 Nonlocal collisions

Integrating by parts in the variables u_1,u_2 in Eq. (4.74), we split the resulting expression into four terms, which we will label by Roman numbers, as in the last section. Firstly, the u_1,u_2-derivatives act on the first two positive-energy projectors $\Lambda^+(k\pm u_1/2\pm u_2/2)$ on the right-hand side of Eq. (4.74). These derivatives can be evaluated by noting that

$$\partial_u^\mu \Lambda^+\left(k+\frac{u}{2}\right)\Big|_{u=0} = \frac{1}{4m}\gamma^\mu \;, \tag{4.87}$$

leading to

$$\begin{aligned} C^{\text{nonlocal}}_{\text{I},\alpha\beta}(x,k) = \frac{i\hbar}{4m}\frac{m^4}{2}\int & \mathrm{d}K_1\,\mathrm{d}K_2\,\mathrm{d}K'\,(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2) \\ & \times M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}\left[\gamma^\mu_{\alpha\alpha'}\Lambda^+_{\beta'\beta}(k)-\Lambda^+_{\alpha\alpha'}(k)\,\gamma^\mu_{\beta'\beta}\right] \\ & \times \Big\{\delta_{\alpha'\alpha_1}\delta_{\beta'\delta_1}\Lambda^+_{\delta_2\alpha_2}(k')\partial_\mu\left[W^{\beta_1\gamma_1}_{\text{on-shell}}(x,k_1)W^{\beta_2\gamma_2}_{\text{on-shell}}(x,k_2)\right] \\ & \quad -\frac{1}{2}\Lambda^+_{\beta_1\gamma_1}(k_1)\Lambda^+_{\beta_2\gamma_2}(k_2)\left[\delta_{\alpha'\alpha_1}\delta_{\delta_1'\beta'}\delta_{\delta_1\gamma_1'}+\delta_{\alpha'\gamma_1'}\delta_{\delta_1\beta'}\delta_{\delta_1'\alpha_1}\right] \\ & \quad \times \partial_\mu\left[W^{\gamma_1'\delta_1'}_{\text{on-shell}}(x,k)W^{\delta_2\alpha_2}_{\text{on-shell}}(x,k')\right]\Big\} \;. \end{aligned} \tag{4.88}$$

Here we used that

$$W(x,k) = \frac{1}{2}\Lambda^+(k)\mathcal{F}(x,k)+\mathcal{O}(\hbar) \;, \tag{4.89}$$

[5] Such a redefinition is possible since the physical meaning of the spin degree of freedom $\mathfrak{s}$ is limited and all observable quantities are given by integrals over spin space.

implying $\Lambda^+(k)W(x,k)\Lambda^+(k) = W(x,k) + \mathcal{O}(\hbar)$, which we may use since the nonlocal contributions are already of first order in $\hbar$ and the neglected terms are thus of second order. Inserting the relation (4.79) and employing the definition (4.84), we find

$$\begin{aligned} C_{\mathrm{I}}^{\mathrm{nonlocal}}(x,k,\mathfrak{s}) = \frac{i\hbar}{4m}\frac{m^4}{4}\int & \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,\mathrm{d}\bar{S}(k)\,(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2) \\ & \times M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}\,[h(k,\mathfrak{s}),\gamma^\mu]_{\beta'\alpha'} \\ & \times \Big\{\delta_{\alpha'\alpha_1}\delta_{\beta'\delta_1}h_{\delta_2\alpha_2}(k',\mathfrak{s}')h_{\beta_1\gamma_1}(k_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(k_2,\mathfrak{s}_2)\partial_\mu\left[f(x,k_1,\mathfrak{s}_1)f(x,k_2,\mathfrak{s}_2)\right] \\ & \quad - h_{\beta_1\gamma_1}(k_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(k_2,\mathfrak{s}_2)h_{\delta_2\alpha_2}(k',\mathfrak{s}')\left[\delta_{\alpha'\alpha_1}h_{\delta_1\beta'}(k,\bar{\mathfrak{s}})+h_{\alpha'\alpha_1}(k,\bar{\mathfrak{s}})\delta_{\delta_1\beta'}\right] \\ & \quad \times \partial_\mu\left[f(x,k,\bar{\mathfrak{s}})f(x,k',\mathfrak{s}')\right]\Big\}\,. \end{aligned} \tag{4.90}$$

Since the distribution functions only depend on the spin variable at first order in $\hbar$, which follows from the fact that $\mathcal{A}^\mu \sim \mathcal{O}(\hbar)$, we may omit the spin arguments and perform the $\bar{\mathfrak{s}}$-integral trivially to obtain

$$\begin{aligned} C_{\mathrm{I}}^{\mathrm{nonlocal}}(x,k,\mathfrak{s}) = \frac{i\hbar}{4m}\frac{m^4}{2}\int & \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2} \\ & \times h_{\delta_2\alpha_2}(k',\mathfrak{s}')h_{\beta_1\gamma_1}(k_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(k_2,\mathfrak{s}_2)\,[h(k,\mathfrak{s}),\gamma^\mu]_{\delta_1\alpha_1} \\ & \times \partial_\mu\left[f(x,k_1)f(x,k_2)-\frac{1}{2}f(x,k)f(x,k')\right]\,. \end{aligned} \tag{4.91}$$

Here we used that $\{[h(k,\mathfrak{s}),\gamma^\mu],\Lambda^+(k)\} = [h(k,\mathfrak{s}),\gamma^\mu]$ since $h(k,\mathfrak{s})(\not{k}-m) = 0$. In the second contribution to the nonlocal collision term, the u_1, u_2-derivatives act on the remaining projectors in Eq. (4.74). Performing the same steps as outlined above, we compute

$$\begin{aligned} C_{\mathrm{II}}^{\mathrm{nonlocal}}(x,k,\mathfrak{s}) = -\frac{i\hbar}{4m}\frac{m^4}{2}\int & \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2} \\ & \times \Big\{f(x,k_2)\Big[\partial_\mu f(x,k_1)\Big]h_{\delta_2\alpha_2}(k',\mathfrak{s}')h_{\beta_2\gamma_2}(k_2,\mathfrak{s}_2)h_{\delta_1\alpha_1}(k,\mathfrak{s})\,[h(k_1,\mathfrak{s}_1),\gamma^\mu]_{\beta_1\gamma_1} \\ & + f(x,k_1)\Big[\partial_\mu f(x,k_2)\Big]h_{\delta_2\alpha_2}(k',\mathfrak{s}')h_{\beta_1\gamma_1}(k_1,\mathfrak{s}_1)h_{\delta_1\alpha_1}(k,\mathfrak{s})\,[h(k_2,\mathfrak{s}_2),\gamma^\mu]_{\beta_2\gamma_2} \\ & - f(x,k)\left[\partial_\mu f(x,k')\right]h_{\beta_1\gamma_1}(k_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(k_2,\mathfrak{s}_2)h_{\delta_1\alpha_1}(k,\mathfrak{s})\,[h(k',\mathfrak{s}'),\gamma^\mu]_{\delta_2\alpha_2} \\ & - \frac{1}{2}\Big[f(x,k')\partial_\mu f(x,k) - f(x,k)\partial_\mu f(x,k')\Big] \\ & \times h_{\beta_1\gamma_1}(k_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(k_2,\mathfrak{s}_2)h_{\delta_2\alpha_2}(k',\mathfrak{s}')\,[h(k,\mathfrak{s}),\gamma^\mu]_{\delta_1\alpha_1}\Big\}\,. \end{aligned} \tag{4.92}$$

When comparing to the nonlocal collisions for spin-0 particles, cf. Subsec. 4.3.3, Eqs. (4.91) and (4.92) are the essential new contributions that arise from the nontrivial internal structure of the particles. The remaining contributions were already present in the scalar case, where we proved that they vanish. Indeed, as we will show now, this is also the case for Dirac fields, such that the contributions from the positive-energy projectors is the only source of nonlocality at first order in the Planck constant.[6] The third contribution consists in the u_1, u_2-derivatives acting on the arguments of the Wigner functions in the loss terms, leading to

$$\begin{aligned} C_{\mathrm{III}}^{\mathrm{nonlocal}}(x,k,\mathfrak{s}) = -\frac{i\hbar}{4m}\frac{m^4}{4}\int & \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,\mathrm{d}\bar{S}(k)\,(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2) \\ & \times M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}\,[\partial_k^\mu f(x,k)]\,[\partial_\mu f(x,k')] \\ & \times h_{\beta_1\gamma_1}(k_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(k_2,\mathfrak{s}_2)h_{\delta_2\alpha_2}(k',\mathfrak{s}')\,[h(k,\mathfrak{s}),h(k,\bar{\mathfrak{s}})]_{\delta_1\alpha_1}\,. \end{aligned} \tag{4.93}$$

[6] One may wonder why there are no contributions that contain momentum-derivatives from the so-called "Poisson-bracket terms" that arise in the literature [148, 149]. The reason lies in the $\hbar$-gradient expansion, in which these terms, corresponding to the nonlocal contributions III and IV, only give nonvanishing corrections at second and higher orders.

After performing the $\mathrm{d}\bar{S}(k)$-integration trivially, this term vanishes since $h(k,\mathfrak{s})$ and $\Lambda^+(k)$ commute. In the fourth contribution, the derivatives act on the delta functions in the loss terms, yielding

$$\begin{aligned} C_{\mathrm{IV}}^{\mathrm{nonlocal}}(x,k,\mathfrak{s}) = &-\frac{i\hbar}{4m}\frac{m^4}{4}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,\mathrm{d}\bar{S}(k)\,(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2) \\ &\times M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}h_{\beta_1\gamma_1}(k_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(k_2,\mathfrak{s}_2)\,[h(k,\mathfrak{s}),h(k,\bar{\mathfrak{s}})]_{\delta_1\alpha_1} \\ &\times [\partial_\mu f(x,k)]\left[h_{\delta_2\alpha_2}(k',\mathfrak{s}')\partial^\mu_{k'}f(x,k') + \{h(k',\mathfrak{s}'),\gamma^\mu\}_{\delta_2\alpha_2}f(x,k')\right] . \end{aligned} \tag{4.94}$$

As has been the case with $C_{\mathrm{III}}^{\mathrm{nonlocal}}(x,k,\mathfrak{s})$, this term vanishes after performing the $\mathrm{d}\bar{S}(k)$-integration. Then, the total nonlocal collision term is simply given by the sum of Eqs. (4.91) and (4.92), i.e.,

$$\begin{aligned} C_{\mathrm{nonlocal}}(x,k,\mathfrak{s}) = &-\frac{i\hbar}{4m}\frac{m^4}{2}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2} \\ &\times\Big\{ f(x,k_2)\Big[\partial_\mu f(x,k_1)\Big]h_{\delta_2\alpha_2}(k',\mathfrak{s}')h_{\beta_2\gamma_2}(k_2,\mathfrak{s}_2)h_{\delta_1\alpha_1}(k,\mathfrak{s})\,[h(k_1,\mathfrak{s}_1),\gamma^\mu]_{\beta_1\gamma_1} \\ &+ f(x,k_1)\Big[\partial_\mu f(x,k_2)\Big]h_{\delta_2\alpha_2}(k',\mathfrak{s}')h_{\beta_1\gamma_1}(k_1,\mathfrak{s}_1)h_{\delta_1\alpha_1}(k,\mathfrak{s})\,[h(k_2,\mathfrak{s}_2),\gamma^\mu]_{\beta_2\gamma_2} \\ &- f(x,k)\,[\partial_\mu f(x,k')]\,h_{\beta_1\gamma_1}(k_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(k_2,\mathfrak{s}_2)h_{\delta_1\alpha_1}(k,\mathfrak{s})\,[h(k',\mathfrak{s}'),\gamma^\mu]_{\delta_2\alpha_2} \\ &- f(x,k')\,[\partial_\mu f(x,k)]\,h_{\beta_1\gamma_1}(k_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(k_2,\mathfrak{s}_2)h_{\delta_2\alpha_2}(k',\mathfrak{s}')\,[h(k,\mathfrak{s}),\gamma^\mu]_{\delta_1\alpha_1} \\ &- \partial_\mu\Big[f(x,k_1)f(x,k_2) - f(x,k)f(x,k')\Big] \\ &\times h_{\beta_1\gamma_1}(k_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(k_2,\mathfrak{s}_2)h_{\delta_2\alpha_2}(k',\mathfrak{s}')\,[h(k,\mathfrak{s}),\gamma^\mu]_{\delta_1\alpha_1}\Big\} . \end{aligned} \tag{4.95}$$

Interestingly, the terms that are proportional to the derivative of $f(x,k)$ cancel each other, which will get a clear interpretation in the next subsection.

4.4.4 Summary

Collecting the local and nonlocal contributions to the collision kernel given by Eqs. (4.85) and (4.95), respectively, the BOLTZMANN equation for the on-shell distribution function in extended phase space takes on the form

$$\begin{aligned} k\cdot\partial f(x,k,\mathfrak{s}) &= C(x,k,\mathfrak{s}) \\ &= \frac{1}{2}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,\mathrm{d}\bar{S}(k)\,(2\pi\hbar)^4\delta^{(4)}\,(k+k'-k_1-k_2)\,\mathcal{W}^{(1/2)} \\ &\quad\times\Big[f(x+\Delta_1-\Delta,k_1,\mathfrak{s}_1)f(x+\Delta_2-\Delta,k_2,\mathfrak{s}_2) - f(x,k,\bar{\mathfrak{s}})f(x+\Delta'-\Delta,k',\mathfrak{s}')\Big] . \end{aligned} \tag{4.96}$$

Here we defined the (local) transition rate[7]

$$\begin{aligned} \mathcal{W}^{(1/2)} := &\frac{m^4}{2}M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}h_{\beta_1\gamma_1}(k_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(k_2,\mathfrak{s}_2)h_{\delta_2\alpha_2}(k',\mathfrak{s}') \\ &\times[h_{\delta_1\beta}(k,\bar{\mathfrak{s}})h_{\beta\alpha_1}(k,\mathfrak{s}) + h_{\delta_1\beta}(k,\mathfrak{s})h_{\beta\alpha_1}(k,\bar{\mathfrak{s}})] \end{aligned} \tag{4.97}$$

and interpreted the nonlocal terms (which are proportional to gradients of the distribution functions) as the first-order correction in a TAYLOR expansion, e.g.

$$f(x+\Delta,k,\mathfrak{s}) = f(x,k,\mathfrak{s}) + \Delta\cdot\partial f(x,k,\mathfrak{s}) + \mathcal{O}(\hbar^2) , \tag{4.98}$$

[7]Compared to Ref. [46], there is a discrepancy of a factor of two in the kinetic equation, which is compensated by corresponding inverse factors in the transition rate and the nonlocal shifts. The difference merely lies in where the volume factor belonging to the integral over $\bar{\mathfrak{s}}$ is put.

and analogously for the other terms. The spacetime shifts Δ, which are of first order in $\hbar$, are defined as

$$\Delta_1^\mu := -\frac{i\hbar}{8m}\frac{m^4}{\mathcal{W}^{(1/2)}} M^{\alpha_1\alpha_2\beta_1\beta_2} M^{\gamma_1\gamma_2\delta_1\delta_2} h_{\beta_2\gamma_2}(k_2,\mathfrak{s}_2) h_{\delta_2\alpha_2}(k',\mathfrak{s}') h_{\delta_1\alpha_1}(k,\mathfrak{s}) \left[h(k_1,\mathfrak{s}_1),\gamma^\mu\right]_{\beta_1\gamma_1} , \tag{4.99a}$$

$$\Delta_2^\mu := -\frac{i\hbar}{8m}\frac{m^4}{\mathcal{W}^{(1/2)}} M^{\alpha_1\alpha_2\beta_1\beta_2} M^{\gamma_1\gamma_2\delta_1\delta_2} h_{\beta_1\gamma_1}(k_1,\mathfrak{s}_1) h_{\delta_2\alpha_2}(k',\mathfrak{s}') h_{\delta_1\alpha_1}(k,\mathfrak{s}) \left[h(k_2,\mathfrak{s}_2),\gamma^\mu\right]_{\beta_2\gamma_2} , \tag{4.99b}$$

$$\Delta'^\mu := -\frac{i\hbar}{8m}\frac{m^4}{\mathcal{W}^{(1/2)}} M^{\alpha_1\alpha_2\beta_1\beta_2} M^{\gamma_1\gamma_2\delta_1\delta_2} h_{\beta_1\gamma_1}(k_1,\mathfrak{s}_1) h_{\beta_2\gamma_2}(k_2,\mathfrak{s}_2) h_{\delta_1\alpha_1}(k,\mathfrak{s}) \left[h(k',\mathfrak{s}'),\gamma^\mu\right]_{\delta_2\alpha_2} , \tag{4.99c}$$

$$\Delta^\mu := -\frac{i\hbar}{8m}\frac{m^4}{\mathcal{W}^{(1/2)}} M^{\alpha_1\alpha_2\beta_1\beta_2} M^{\gamma_1\gamma_2\delta_1\delta_2} h_{\beta_1\gamma_1}(k_1,\mathfrak{s}_1) h_{\beta_2\gamma_2}(k_2,\mathfrak{s}_2) h_{\delta_2\alpha_2}(k',\mathfrak{s}') \left[h(k,\mathfrak{s}),\gamma^\mu\right]_{\delta_1\alpha_1} , \tag{4.99d}$$

and they are orthogonal to their respective momenta, i.e., they fulfill $\Delta_j \cdot k_j = 0$, where $\Delta_j \in \{\Delta, \Delta', \Delta_1, \Delta_2\}$ and $k_j \in \{k, k', k_1, k_2\}$.

The interpretation of the nonlocal terms now becomes clear: They represent a spacetime shift of the colliding particles. More specifically, each particle j is shifted by a (different) amount Δ_j, in addition to a global shift of $-\Delta$. This also gives a meaning to the fact that there are no nonlocal terms proportional to the gradient of $f(x,k,\mathfrak{s})$, since in that case the individual and the global shift cancels. The fact that the particles do not collide at the same spacetime point is in fact mandatory for particles with spin, since only in that case there is a finite orbital angular momentum at the point of the collision. This, however, is necessary to enable the exchange of orbital angular momentum and spin in a collision, since only the *total* angular momentum $J^{\mu\nu} := (\hbar/2)\Sigma_{\mathfrak{s}}^{\mu\nu} + x^{[\mu}k^{\nu]}$ is conserved [44]. At this point, we remark that similar nonlocalities have also been obtained in a nonrelativistic setting [148, 150, 151]. Furthermore, we stress that the shifts (4.99) are LORENTZ-covariant objects, and thus, apart from situations where the collisions are local and all shifts vanish identically, there is no possibility to choose a so-called "no-jump frame" where they become zero [152].
We can now also see that the form (3.106) which we assumed for the antisymmetric part of the energy-momentum tensor, while conveying the right idea, was a little too simple, since the shift Δ depends on all momenta. We will come back to this quantity in Subsec. 6.2.2, where we will deduce its correct form from the conservation of the total angular momentum, and compute it explicitly in Appendix B.3.

To conclude this section, we seek a connection to the standard BOLTZMANN equation (4.3) and consider the case where the distribution functions do not depend on their respective spin variables. Then, we can perform the integrals over spin space, such that the factors of $h(k_j,\mathfrak{s}_j)$ in $\mathcal{W}^{(1/2)}$ become positive-energy projectors. The unpolarized transition rate is obtained by averaging over the incoming and summing over the outgoing spins, yielding

$$\begin{aligned}\overline{|M(k,k',k_1,k_2)|^2} &:= \frac{1}{4}\sum_{r,r'}\sum_{r^2}\left|\langle k,k';r,r'|\,\widehat{t}\,|k^2;r^2\rangle\right|^2 \\ &= 4m^4 M^{\alpha_1\alpha_2\beta_1\beta_2} M^{\gamma_1\gamma_2\delta_1\delta_2} \Lambda^+_{\beta_1\gamma_1}(k_1)\Lambda^+_{\beta_2\gamma_2}(k_2)\Lambda^+_{\delta_1\alpha_1}(k)\Lambda^+_{\delta_2\alpha_2}(k') .\end{aligned} \tag{4.100}$$

Using the definition above and averaging over the spin variable $\mathfrak{s}$ [i.e., integrating Eq. (4.96) over $\mathfrak{s}$ and dividing by two], we obtain the known form of the spin-averaged BOLTZMANN equation

$$\begin{aligned}k\cdot\partial f(x,k) = \frac{g}{2}\int &\mathrm{d}K_1\,\mathrm{d}K_2\,\mathrm{d}K'(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\frac{\overline{|M(k,k',k_1,k_2)|^2}}{16} \\ &\times\left[f(x,k_1)f(x,k_2) - f(x,k)f(x,k')\right] ,\end{aligned} \tag{4.101}$$

where the spin-degeneracy factor is $g = 2$.

4.5 Vector fields

The methods that have to be applied to determine the kinetic equation for (massive) vector particles with spin 1 are in essence identical to the ones employed for DIRAC spinors, since both types of fields transform in a nontrivial representation of the LORENTZ group. Nevertheless, there are some technical differences due to the different constraint equations that they fulfill.

4.5.1 Rewriting expectation values

In the case of spin 1, the "in"-fields are given by

$$\widehat{V}^{\mu}_{\text{in}}(x) := \sqrt{\hbar}\sum_{\lambda}\int \frac{\mathrm{d}^4k}{(2\pi\hbar)^3}\Theta(k^0)\delta(k^2-m^2)e^{-\frac{i}{\hbar}k\cdot x}\epsilon^{(\lambda)\mu}(k)\widehat{a}_{\text{in}}(k,\sigma)\ , \tag{4.102}$$

where the prefactor of $\sqrt{\hbar}$ is needed to recover the correct units of the vector field and is the same as in the scalar case, cf. Eq. (4.30). Here, the spin index λ runs from -1 to $+1$, and $\epsilon^{(\lambda)\mu}(k)$ are polarization vectors which are required to be orthogonal to their associated momentum k. They are constructed to fulfill the following orthogonality and completeness relations,

$$\epsilon^{*(\lambda)\mu}(k)\epsilon^{(\lambda')}_{\mu}(k) = -\delta_{\lambda\lambda'}\ , \tag{4.103a}$$

$$\sum_{\lambda}\epsilon^{*(\lambda)\mu}(k)\epsilon^{(\lambda)\nu}(k) = -K^{\mu\nu}\ . \tag{4.103b}$$

Note that the negative signs are necessary because of the signature of MINKOWSKI space. Consequently, we define the "in"-WIGNER function for the PROCA field,

$$W^{\mu\nu}_{\text{in}}(x,k) := -\frac{2}{\hbar}\int \mathrm{d}^4v e^{-\frac{i}{\hbar}k\cdot v}\left\langle \widehat{V}^{\dagger\mu}_{\text{in}}\left(x+\frac{v}{2}\right)\widehat{V}^{\nu}_{\text{in}}\left(x-\frac{v}{2}\right)\right\rangle\ , \tag{4.104}$$

where it should be noted that the LORENTZ indices are switched when compared to the general convention (4.10), which however will not influence the calculation significantly. Specializing Eq. (4.23) to the spin-1 case, it reads

$$\left\langle\widehat{O}\right\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\int \mathrm{d}^4\bar{x}^n\int\frac{\mathrm{d}^4\bar{k}^n}{(2\pi\hbar)^{4n}}O_{n,\mu_1\nu_1\cdots\mu_n\nu_n}(\bar{x}^n;\bar{k}^n)\prod_{j=1}^{n}W^{\mu_j\nu_j}_{\text{in}}(\bar{x}_j,\bar{k}_j)\ , \tag{4.105}$$

where we defined

$$\begin{aligned}O_{n,\mu_1\nu_1\cdots\mu_n\nu_n}(\bar{x}^n;\bar{k}^n) := \frac{(-1)^n}{2^n}\int\frac{\mathrm{d}^4u^n}{(2\pi\hbar)^{4n}}\sum_{\lambda^n,\lambda'^n}{}_{\text{in}}\!\left\langle\!\!\left\langle \bar{k}^n-\frac{u^n}{2};\lambda^n\right|\widehat{O}\left|\bar{k}^n+\frac{u^n}{2};\lambda'^n\right\rangle\!\!\right\rangle_{\text{in}}\\ \times\left[\prod_{j=1}^{n}e^{\frac{i}{\hbar}u_j\cdot\bar{x}_j}\epsilon^{(\lambda_j)}_{\mu_j}\left(\bar{k}_j-\frac{u_j}{2}\right)\epsilon^{*(\lambda'_j)}_{\nu_j}\left(\bar{k}_j+\frac{u_j}{2}\right)\right]\ .\end{aligned} \tag{4.106}$$

Note that, as mentioned before, compared to the general case the LORENTZ indices of the WIGNER function are reversed, which is due to its definition (4.104), but plays no role otherwise.

Expressing $\mathcal{C}^{\mu\nu}$ in terms of $W^{\mu\nu}_{\text{in}}$

As we have done in the previous sections, we now turn to expanding the collision term $\mathcal{C}^{\mu\nu}$ in terms of the "in"-WIGNER function (and thus in terms of the full WIGNER function). First we rewrite the collision term as the following FOCK-space average,

$$\mathcal{C}^{\mu\nu}(x,k) = \left\langle e^{\frac{i}{\hbar}\widehat{P}\cdot x}\widehat{\Phi}^{\mu\nu}(k)e^{-\frac{i}{\hbar}\widehat{P}\cdot x}\right\rangle\ , \tag{4.107}$$

with the operator

$$\widehat{\Phi}^{\mu\nu}(k) := -i \int \mathrm{d}^4 v e^{\frac{i}{\hbar} k\cdot v} \left[\widehat{V}^{\dagger\mu}\left(\frac{v}{2}\right) \widehat{\rho}^{\nu}\left(-\frac{v}{2}\right) - \widehat{\rho}^{\dagger\mu}\left(\frac{v}{2}\right) \widehat{V}^{\nu}\left(-\frac{v}{2}\right) \right] . \tag{4.108}$$

Note that in this case, as for spin-0 particles, we do not have to use the HEISENBERG equation of motion (4.63), since the equation of motion for $\widehat{V}^\mu$ is of second order already and there are thus no additional differential operators acting on the source terms, in contrast to the case of DIRAC spinors. Making use of Eq. (4.105), we can express the collision term as

$$\mathcal{C}^{\mu\nu}(x,k) = \sum_{n=2}^{\infty} \frac{1}{n!} \int \mathrm{d}^4 \bar{x}^n \int \frac{\mathrm{d}^4 \bar{k}^n}{(2\pi\hbar)^{4n}} \Phi^{\mu\nu}_{n,\mu_1\nu_1\cdots\mu_n\nu_n}(\bar{x}^n;\bar{k}^n|k) \prod_{j=1}^{n} W_{\text{in}}^{\mu_j\nu_j}(x+\bar{x}_j,\bar{k}_j) , \tag{4.109}$$

where we introduced

$$\begin{aligned}\Phi^{\mu\nu}_{n,\mu_1\nu_1\cdots\mu_n\nu_n}(\bar{x}^n;\bar{k}^n|k) :=& \frac{(-1)^n}{2^n} \int \frac{\mathrm{d}^4 u^n}{(2\pi\hbar)^{4n}} \sum_{\lambda^n,\lambda'^n} {}_{\text{in}}\!\left\langle\!\!\left\langle \bar{k}^n - \frac{u^n}{2};\lambda^n \right| \widehat{\Phi}^{\mu\nu}(k) \left| \bar{k}^n + \frac{u^n}{2};\lambda'^n \right\rangle\!\!\right\rangle_{\text{in}} \\ &\times \left[\prod_{j=1}^{n} e^{\frac{i}{\hbar}u_j\cdot\bar{x}_j} \epsilon^{(\lambda_j)}_{\mu_j}\left(\bar{k}_j - \frac{u_j}{2}\right) \epsilon^{*(\lambda'_j)}_{\nu_j}\left(\bar{k}_j + \frac{u_j}{2}\right) \right] .\end{aligned} \tag{4.110}$$

As discussed before, we will restrict ourselves to the binary-collision approximation and truncate the sum over n after the first term. Furthermore, it should be noted that, considering Eq. (3.165), only the parts of $\Phi^{\mu\nu}$ orthogonal to k^μ will contribute to the kinetic equation.[8]

Inserting a complete set of "out"-states and performing the integration over v yields

$$\begin{aligned}&{}_{\text{in}}\!\left\langle k^2 - \frac{u^2}{2};\lambda^2 \right| \widehat{\Phi}^{\mu\nu}(k) \left| k^2 + \frac{u^2}{2};\lambda'^2 \right\rangle_{\text{in}} \\ =& -\frac{i}{2}\sum_{\sigma'} \int \mathrm{d}K' \, {}_{\text{in}}\!\left\langle k^2 - \frac{u^2}{2};\lambda^2 \right| \left[\widehat{V}^{\dagger\mu}(0) \left| k';\sigma' \right\rangle_{\text{out out}} \left\langle k';\sigma' \right| \widehat{\rho}^{\nu}(0) \right. \\ &\left. - \widehat{\rho}^{\dagger\mu}(0) \left| k';\sigma' \right\rangle_{\text{out out}} \left\langle k';\sigma' \right| \widehat{V}^{\nu}(0) \right] \left| k^2 + \frac{u^2}{2};\lambda'^2 \right\rangle_{\text{in}} (2\pi\hbar)^4 \delta^{(4)}\left(k + k' - k_1 - k_2\right) .\end{aligned} \tag{4.111}$$

In order to evaluate the expectation values involving the full vector fields at the origin, we need to relate them to their "in"-counterparts and the retarded propagator via the YANG-FELDMAN equation for spin-1 particles,

$$\widehat{V}^{\mu}(0) = \widehat{V}^{\mu}_{\text{in}}(0) + \int \mathrm{d}^4 x \, G_R^{\mu\nu}(-x) \widehat{\rho}_\nu(x) . \tag{4.112}$$

Inserting this relation into the expectation values in Eq. (4.111), we obtain

$$\begin{aligned}&{}_{\text{out}}\!\left\langle k';\sigma' \right| \widehat{V}^{\mu}(0) \left| k^2 + \frac{u^2}{2};\lambda'^2 \right\rangle_{\text{in}} \\ =& \sqrt{\hbar}(2\pi\hbar)^3 2k'^0 \left[\epsilon^{(\lambda'_1)\mu}\left(k_1 + \frac{u_1}{2}\right) \delta^{(3)}\left(\mathbf{k}' - \mathbf{k}_2 - \frac{\mathbf{u}_2}{2}\right) \delta_{\sigma'\lambda'_2} + (1\leftrightarrow 2) \right] \\ &+ \widetilde{G}_R^{\mu\nu}\left(k_1 + k_2 + \frac{u_1+u_2}{2} - k'\right) {}_{\text{out}}\!\left\langle k';\sigma' \right| \widehat{\rho}_\nu(0) \left| k^2 + \frac{u^2}{2};\lambda'^2 \right\rangle_{\text{in}} ,\end{aligned} \tag{4.113}$$

[8]This is a consequence of the fact that the distribution function $\mathfrak{f}$ in extended phase space is constructed to contain only the independent components of the WIGNER function.

where we used the FOURIER decomposition of the retarded propagator. Inserting Eq. (4.113) into Eq. (4.111), we find

$$
\begin{aligned}
&{}_{\text{in}}\left\langle k^2-\frac{u^2}{2};\lambda^2\middle|\widehat{\Phi}^{\mu\nu}(k)\middle|k^2+\frac{u^2}{2};\lambda'^2\right\rangle_{\text{in}} \\
&= -\frac{i}{2}\sum_{\sigma'}\int \mathrm{d}K'(2\pi\hbar)^4\delta^{(4)}\left(k+k'-k_1-k_2\right)\Bigg({}_{\text{out}}\left\langle k';\sigma'\middle|\widehat{\rho}^{\nu}(0)\middle|k^2+\frac{u^2}{2};\lambda'^2\right\rangle_{\text{in}} \\
&\times\Bigg\{\sqrt{\hbar}(2\pi\hbar)^3 2k'^0\left[\epsilon^{*(\lambda_1)\mu}\left(k_1-\frac{u_1}{2}\right)\delta^{(3)}\left(\mathbf{k}'-\mathbf{k}_2+\frac{\mathbf{u}_2}{2}\right)\delta_{\sigma'\lambda_2}+(1\leftrightarrow 2)\right] \\
&+\widetilde{G}_R^{*\mu\alpha}\left(k-\frac{u_1+u_2}{2}\right){}_{\text{in}}\left\langle k^2-\frac{u^2}{2};\lambda^2\middle|\widehat{\rho}^{\dagger}_{\alpha}(0)\middle|k';\sigma'\right\rangle_{\text{out}}\Bigg\} \\
&- {}_{\text{in}}\left\langle k^2-\frac{u^2}{2};\lambda^2\middle|\widehat{\rho}^{\dagger\mu}(0)\middle|k';\sigma'\right\rangle_{\text{out}} \\
&\times\Bigg\{\sqrt{\hbar}(2\pi\hbar)^3 2k'^0\left[\epsilon^{(\lambda_1')\nu}\left(k_1+\frac{u_1}{2}\right)\delta^{(3)}\left(\mathbf{k}'-\mathbf{k}_2-\frac{\mathbf{u}_2}{2}\right)\delta_{\sigma'\lambda_2'}+(1\leftrightarrow 2)\right] \\
&+\widetilde{G}_R^{\nu\alpha}\left(k+\frac{u_1+u_2}{2}\right){}_{\text{out}}\left\langle k';\sigma'\middle|\widehat{\rho}_{\alpha}(0)\middle|k^2+\frac{u^2}{2};\lambda'^2\right\rangle_{\text{in}}\Bigg\}\Bigg)\;. \qquad (4.114)
\end{aligned}
$$

In order to evaluate the matrix elements of the source terms in Eq. (4.114), we first rewrite

$$
\begin{aligned}
{}_{\text{out}}\left\langle k';\sigma'\middle|\widehat{\rho}^{\nu}(0)\middle|k^2+\frac{u^2}{2};\lambda'^2\right\rangle_{\text{in}} &= \sum_{\sigma=0}^{3} g^{\sigma\sigma}\epsilon^{(\sigma)\nu}\left(k+\frac{u_1+u_2}{2}\right)\epsilon^{*(\sigma)\alpha}\left(k+\frac{u_1+u_2}{2}\right) \\
&\times {}_{\text{out}}\left\langle k';\sigma'\middle|\widehat{\rho}_{\alpha}(0)\middle|k^2+\frac{u^2}{2};\lambda'^2\right\rangle_{\text{in}}\;, \qquad (4.115)
\end{aligned}
$$

where we introduced a timelike polarization vector $\epsilon^{(0)\mu}(k) \coloneqq k^\mu/k$, such that

$$
\sum_{\sigma'=0}^{3} g^{\sigma'\sigma'}\epsilon^{(\sigma')\mu}(k)\epsilon^{*(\sigma')\nu}(k) = E^{\mu\nu}+K^{\mu\nu} = g^{\mu\nu}\;. \qquad (4.116)
$$

It is important to note that the term containing the timelike polarization vector in the sum in Eq. (4.115) is actually one order higher in $\hbar$ than the others. To see this, consider the action of the respective four-momentum on the source term,

$$
\begin{aligned}
\left(k^\alpha+\frac{u_1^\alpha+u_2^\alpha}{2}\right){}_{\text{out}}\left\langle k';\sigma'\middle|\widehat{\rho}_{\alpha}(0)\middle|k^2+\frac{u^2}{2};\lambda'^2\right\rangle_{\text{in}} &= {}_{\text{out}}\left\langle k';\sigma'\middle|\left[\widehat{\rho}_{\alpha}(0),\widehat{P}^{\alpha}\right]\middle|k^2+\frac{u^2}{2};\lambda'^2\right\rangle_{\text{in}} \\
&= i\hbar\;{}_{\text{out}}\left\langle k';\sigma'\middle|\left(\partial\cdot\widehat{\rho}\right)(0)\middle|k^2+\frac{u^2}{2};\lambda'^2\right\rangle_{\text{in}}\;. \qquad (4.117)
\end{aligned}
$$

If the source was conserved, this term would vanish identically. Even though we do not assume this, we will see shortly that such a term will not contribute to the BOLTZMANN equation to first order in $\hbar$. The contraction of a polarization vector with a source term can be related to the transfer-matrix elements through

$$
\epsilon^{*(\sigma)\alpha}(k)\,{}_{\text{out}}\langle k';\sigma'|\widehat{\rho}_\alpha(0)\,|k^2;\lambda'^2\rangle_{\text{in}} = -\frac{1}{\sqrt{\hbar}}\langle k,k';\sigma,\sigma'|\,\widehat{t}\,|k^2;\lambda'^2\rangle\;. \qquad (4.118)
$$

From this equation and Eq. (4.117) we deduce that all transfer-matrix elements in which one of the polarizations is timelike, i.e., where $\sigma=0$, are one order higher in $\hbar$ than their counterparts where all polarizations are spacelike. Next we recall Eq. (3.141a) and write the retarded vector boson propagator in terms of the retarded GREEN's function of the KLEIN-GORDON field,

$$
\begin{aligned}
\widetilde{G}_R^{\mu\nu}(k) &= \widetilde{\Delta}_R(k)\left(g^{\mu\nu}-\frac{k^2}{m^2}E^{\mu\nu}\right) \\
&= \widetilde{\Delta}_R(k)\sum_{\sigma=0}^{3} g^{\sigma\sigma}\epsilon^{(\sigma)\mu}(k)\epsilon^{*(\sigma)\nu}(k)\left(1-\frac{k^2}{m^2}\delta_{\sigma 0}\right)\;. \qquad (4.119)
\end{aligned}
$$

Inserting Eqs. (4.115), (4.118), and (4.119) into Eq. (4.114), we obtain

$$
\begin{aligned}
&{}_{\text{in}}\left\langle k^2-\frac{u^2}{2};\lambda^2\middle|\widehat{\Phi}^{\mu\nu}(k)\middle|k^2+\frac{u^2}{2};\lambda'^2\right\rangle_{\text{in}} \\
&= -i(2\pi\hbar)^4\sum_{\sigma,\sigma'=0}^{3} g^{\sigma\sigma}g^{\sigma'\sigma'}\left(\delta\left(k^0-k_1^0-k_2^0+\sqrt{\left(\mathbf{k}_2-\frac{\mathbf{u}_2}{2}\right)^2+m^2}\right)\delta^{(3)}\left(\mathbf{k}-\mathbf{k}_1-\frac{\mathbf{u}_2}{2}\right)\delta_{\sigma'\lambda_1}\right. \\
&\times\left\langle k+\frac{u_1+u_2}{2},k_2-\frac{u_2}{2};\sigma,\lambda_2\middle|\widehat{t}\middle|k^2+\frac{u^2}{2};\lambda'^2\right\rangle\epsilon^{(\sigma)\nu}\left(k+\frac{u_1+u_2}{2}\right)\epsilon^{*(\sigma')\mu}\left(k-\frac{u_1+u_2}{2}\right) \\
&+(1\leftrightarrow 2)-\left[\delta\left(k^0-k_1^0-k_2^0+\sqrt{\left(\mathbf{k}_2+\frac{\mathbf{u}_2}{2}\right)^2+m^2}\right)\delta^{(3)}\left(\mathbf{k}-\mathbf{k}_1+\frac{\mathbf{u}_2}{2}\right)\delta_{\sigma\lambda_1'}\right. \\
&\times\left\langle k^2-\frac{u^2}{2};\lambda^2\middle|\widehat{t}^\dagger\middle|k-\frac{u_1+u_2}{2},k_2+\frac{u_2}{2};\sigma',\lambda_2'\right\rangle\epsilon^{*(\sigma')\mu}\left(k-\frac{u_1+u_2}{2}\right)\epsilon^{(\sigma)\nu}\left(k+\frac{u_1+u_2}{2}\right) \\
&\left.+(1\leftrightarrow 2)\right]-\frac{1}{2\hbar}\sum_{\sigma''}\int \mathrm{d}K'\epsilon^{*(\sigma')\mu}\left(k-\frac{u_1+u_2}{2}\right)\epsilon^{(\sigma)\nu}\left(k+\frac{u_1+u_2}{2}\right) \\
&\times\left\langle k^2-\frac{u^2}{2};\lambda^2\middle|\widehat{t}^\dagger\middle|k-\frac{u_1+u_2}{2},k';\sigma',\sigma''\right\rangle\left\langle k',k+\frac{u_1+u_2}{2};\sigma'',\sigma\middle|\widehat{t}\middle|k^2+\frac{u^2}{2};\lambda'^2\right\rangle \\
&\times\left\{\widetilde{\Delta}_R\left(k-\frac{u_1+u_2}{2}\right)\left[1-\frac{\left(k-\frac{u_1+u_2}{2}\right)^2}{m^2}\delta_{\sigma'0}\right]\right. \\
&\left.\left.-\widetilde{\Delta}_R^*\left(k+\frac{u_1+u_2}{2}\right)\left[1-\frac{\left(k+\frac{u_1+u_2}{2}\right)^2}{m^2}\delta_{\sigma0}\right]\right\}\delta^{(4)}\left(k+k'-k_1-k_2\right)\right), \qquad (4.120)
\end{aligned}
$$

where we inserted identities in order to be able to factor out the sums over σ and σ'. As done in the previous sections, in the first four terms on the right-hand side of Eq. (4.120), we separate the real and imaginary parts of $\widehat{t}$ and $\widehat{t}^\dagger$ in order to make use of the optical theorem

$$
\begin{aligned}
\frac{i}{2}\left\langle k^2;\lambda^2\middle|\widehat{t}-\widehat{t}^\dagger\middle|p^2;\lambda'^2\right\rangle = -\frac{(2\pi\hbar)^4}{16}\sum_{\rho^2}\int \mathrm{d}Q_1\,\mathrm{d}Q_2\,\delta^{(4)}(q_1+q_2-k_1-k_2) \\
\times\left\langle k^2;\lambda^2\middle|\widehat{t}\middle|q^2;\rho^2\right\rangle\left\langle q^2;\rho^2\middle|\widehat{t}^\dagger\middle|p^2;\lambda'^2\right\rangle . \qquad (4.121)
\end{aligned}
$$

As mentioned previously, we will not consider the Vlasov-type contributions from the real parts of the transfer matrix further. Lastly, in order to obtain a manifestly covariant expression, we need to relate the transfer-matrix elements to the tree-level vertices of the theory via

$$
\left\langle k,k';\sigma,\sigma'\middle|\widehat{t}\middle|k^2;\lambda'^2\right\rangle = \epsilon_{\gamma_1}^{*(\sigma)}(k)\epsilon_{\gamma_2}^{*(\sigma')}(k')\epsilon_{\delta_1}^{(\lambda_1')}(k_1)\epsilon_{\delta_2}^{(\lambda_2')}(k_2)M^{\gamma_1\gamma_2\delta_1\delta_2}(k,k',k_1,k_2)\,. \qquad (4.122)
$$

Since vector particles constitute bosons, we have the following symmetry relations,

$$
M^{\gamma_1\gamma_2\delta_1\delta_2}(k,k',k_1,k_2) = M^{\gamma_2\gamma_1\delta_1\delta_2}(k',k,k_1,k_2) = M^{\gamma_1\gamma_2\delta_2\delta_1}(k,k',k_2,k_1)\,,
$$

while the assumption that M is symmetric under the exchange of incoming and outgoing momenta (together with the fact that $\widehat{t}^\dagger=\widehat{t}$ at tree level) yields

$$
\left\langle k^2;\lambda'^2\middle|\widehat{t}^\dagger\middle|k,k';\sigma,\sigma'\right\rangle = \epsilon_{\delta_1}^{*(\lambda_1')}(k_1)\epsilon_{\delta_2}^{*(\lambda_2')}(k_2)\epsilon_{\gamma_1}^{(\sigma)}(k)\epsilon_{\gamma_2}^{(\sigma')}(k')M^{\delta_1\delta_2\gamma_1\gamma_2}(k,k',k_1,k_2)\,. \qquad (4.123)
$$

From now on we will only consider the part of the collision term that is orthogonal to the four-momentum, which we will label by an index "⊥", and split it into on- and off-shell contributions, i.e.,

$$
\mathcal{C}_\perp^{\mu\nu}(x,k) := K^\mu{}_{\mu'}K^\nu{}_{\nu'}\mathcal{C}^{\mu'\nu'}(x,k) =: 4\pi\hbar\delta(k^2-m^2)C_\perp^{\mu\nu}(x,k)+\mathcal{C}_{\perp,\text{off-shell}}^{\mu\nu}(x,k)\,. \qquad (4.124a)
$$

As in the previous sections, the off-shell terms are assumed to be nonsingular on the mass shell. Similarly, we define the on-shell components of the Wigner function through

$$
W^{\mu\nu}(x,k) =: 4\pi\hbar\delta(k^2-m^2)W_{\perp,\text{on-shell}}^{\mu\nu}(x,k)+W_{\perp,\text{off-shell}}^{\mu\nu}(x,k)\,, \qquad (4.124b)
$$

and neglect the off-shell contributions inside the collision term on account of Theorem 1.
The following steps are the same as in Secs. 4.3 and 4.4: We use the completeness relation of the polarization vectors (4.103b), expand the WIGNER function to first order around $\bar{x}_1 = \bar{x}_2 = 0$, and perform the $\mathrm{d}\bar{x}^2$-integrations, such that we obtain

$$
\begin{aligned}
&C_{\perp}^{\mu\nu}(x,k) \\
&= \frac{(2\pi\hbar)^4}{32} \int \mathrm{d}K_1\, \mathrm{d}K_2\, \mathrm{d}K' \int \mathrm{d}^4 u^2 K^{\mu}{}_{\mu'} K^{\nu}{}_{\nu'} \left(K - \frac{U_1+U_2}{2}\right)^{\mu'\alpha} \left(K + \frac{U_1+U_2}{2}\right)^{\nu'\beta} \\
&\times \Bigg(\delta^{(4)}(k+k'-k_1-k_2) K'_{\eta_2\gamma_2} \left(K_1 + \frac{U_1}{2}\right)_{\delta_1\beta_1} \left(K_2 + \frac{U_2}{2}\right)_{\delta_2\beta_2} \left(K_1 - \frac{U_1}{2}\right)_{\alpha_1\zeta_1} \\
&\times \left(K_2 - \frac{U_2}{2}\right)_{\alpha_2\zeta_2} g_{\alpha\eta_1} g_{\beta\gamma_1} \prod_{j=1}^{2} \left\{ \left[\delta^{(4)}(u_j) - i\hbar \partial^{\rho}_{u_j} \delta^{(4)}(u_j) \partial_\rho\right] W^{\alpha_j\beta_j}_{\text{on-shell}}(x,k_j) \right\} \\
&\times M^{\gamma_1\gamma_2\delta_1\delta_2}\left(k + \frac{u_1+u_2}{2}, k', k_1 + \frac{u_1}{2}, k_2 + \frac{u_2}{2}\right) \\
&\times M^{\zeta_1\zeta_2\eta_1\eta_2}\left(k - \frac{u_1+u_2}{2}, k', k_1 - \frac{u_1}{2}, k_2 - \frac{u_2}{2}\right) \\
&- \frac{1}{2}\delta^{(4)}\left(k+k'-k_1-k_2+\frac{u_1}{2}\right) \left(K' + \frac{U_2}{2}\right)_{\eta_2\beta_2} \left(K + \frac{U_1-U_2}{2}\right)_{\eta_1\beta_1} \left(K' - \frac{U_2}{2}\right)_{\alpha_2\gamma_2} \\
&\times K_{1,\delta_1\zeta_1} K_{2,\delta_2\zeta_2} g_{\alpha\alpha_1} g_{\beta\gamma_1} \left\{ \left[\delta^{(4)}(u_1) - i\hbar\partial^{\rho}_{u_1}\delta^{(4)}(u_1)\partial_\rho\right] W^{\alpha_1\beta_1}_{\text{on-shell}}\left(x, k - \frac{u_2}{2}\right) \right\} \\
&\times \left\{ \left[\delta^{(4)}(u_2) - i\hbar\partial^{\rho}_{u_2}\delta^{(4)}(u_2)\partial_\rho\right] W^{\alpha_2\beta_2}_{\text{on-shell}}(x,k') \right\} \\
&\times M^{\gamma_1\gamma_2\delta_1\delta_2}\left(k + \frac{u_1+u_2}{2}, k' - \frac{u_2}{2}, k_1, k_2\right) M^{\zeta_1\zeta_2\eta_1\eta_2}\left(k + \frac{u_1-u_2}{2}, k' + \frac{u_2}{2}, k_1, k_2\right) \\
&- \frac{1}{2}\delta^{(4)}\left(k+k'-k_1-k_2-\frac{u_1}{2}\right) \left(K' + \frac{U_2}{2}\right)_{\eta_2\beta_2} \left(K + \frac{U_2-U_1}{2}\right)_{\alpha_1\gamma_1} \left(K' - \frac{U_2}{2}\right)_{\alpha_2\gamma_2} \\
&\times K_{1,\delta_1\zeta_1} K_{2,\delta_2\zeta_2} g_{\alpha\eta_1} g_{\beta\beta_1} \left\{ \left[\delta^{(4)}(u_1) - i\hbar\partial^{\rho}_{u_1}\delta^{(4)}(u_1)\partial_\rho\right] W^{\alpha_1\beta_1}_{\text{on-shell}}\left(x, k + \frac{u_2}{2}\right) \right\} \\
&\times \left\{ \left[\delta^{(4)}(u_2) - i\hbar\partial^{\rho}_{u_2}\delta^{(4)}(u_2)\partial_\rho\right] W^{\alpha_2\beta_2}_{\text{on-shell}}(x,k') \right\} \\
&\times M^{\gamma_1\gamma_2\delta_1\delta_2}\left(k + \frac{u_2-u_1}{2}, k' - \frac{u_2}{2}, k_1, k_2\right) M^{\zeta_1\zeta_2\eta_1\eta_2}\left(k - \frac{u_1+u_2}{2}, k' + \frac{u_2}{2}, k_1, k_2\right) \Bigg) .
\end{aligned}
\tag{4.125}
$$

As in the previous calculations for particles of spins 0 and 1/2, we were able to employ the relation (4.45) since the difference of the retarded GREEN's function and its complex conjugate can be evaluated at the same momentum. Furthermore, we defined the projectors orthogonal to the sum of momenta k and q as

$$
(K+Q)^{\mu\nu} := g^{\mu\nu} - \frac{(k+q)^{\mu}(k+q)^{\nu}}{(k+q)^{\alpha}(k+q)_{\alpha}} .
$$

In Eq. (4.125), it can be seen that to first order the terms in the sum in Eq. (4.120) that include timelike polarizations, i.e., where $\sigma = 0$ or $\sigma' = 0$, do not contribute to the on-shell collision kernel orthogonal to the four-momentum. This is due to the fact that the transfer-matrix elements containing timelike polarizations are one order higher in the $\hbar$-gradient expansion than their counterparts which include only spacelike polarizations, cf. Eq. (4.117).

When comparing Eq. (4.125) to its spin-1/2 analogue (4.74), it becomes clear that the projectors orthogonal to the momentum, which arise from the completeness relation (4.103b), take on the role that the positive-energy projectors Λ^+ played in the case of DIRAC fields. Thus, we may expect that the nonlocality of the collision term will, to first order in $\hbar$, arise solely because of the u_1, u_2-dependence of these projectors. As we will see in Subsec. 4.5.3, this is indeed the case. In the remainder of this section, we will again assume for simplicity that the vertices M do not depend on the momentum.

4.5.2 Local collisions

Taking into account the terms without u_1, u_2-derivatives in Eq. (4.125), we find

$$C^{\mu\nu}_{\perp,\text{local}}(x,k) = \frac{1}{32}\int \mathrm{d}K_1\,\mathrm{d}K_2\,\mathrm{d}K'(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M^{\gamma_1\gamma_2\delta_1\delta_2}M^{\zeta_1\zeta_2\eta_1\eta_2}K^{\mu\alpha}K^{\nu\beta}$$
$$\times\Big[K'_{\eta_2\gamma_2}K_{1,\delta_1\beta_1}K_{2,\delta_2\beta_2}K_{1,\alpha_1\zeta_1}K_{2,\alpha_2\zeta_2}g_{\alpha\eta_1}g_{\beta\gamma_1}W^{\alpha_1\beta_1}_{\text{on-shell}}(x,k_1)W^{\alpha_2\beta_2}_{\text{on-shell}}(x,k_2)$$
$$-\frac{1}{2}K'_{\eta_2\beta_2}K'_{\alpha_2\gamma_2}K_{1,\delta_1\zeta_1}K_{2,\delta_2\zeta_2}\left(K_{\eta_1\beta_1}g_{\alpha\alpha_1}g_{\beta\gamma_1}+K_{\alpha_1\gamma_1}g_{\alpha\eta_1}g_{\beta\beta_1}\right)$$
$$\times W^{\alpha_1\beta_1}_{\text{on-shell}}(x,k)\,W^{\alpha_2\beta_2}_{\text{on-shell}}(x,k')\Big]\,. \tag{4.126}$$

In order to be able to translate this expression into extended phase space, we notice that all Wigner functions in Eq. (4.126) are contracted with projectors that are orthogonal to the respective momentum. Since this contraction removes the components that are parallel to the momentum, we have

$$K^\mu{}_\alpha K^\nu{}_\beta W^{\alpha\beta}_{\text{on-shell}}(x,k) = K^{\mu\nu}f_{K,\text{on-shell}}(x,k) + i\epsilon^{\mu\nu\alpha\beta}\frac{k_\alpha}{m}G_{\text{on-shell},\beta}(x,k) + F^{\mu\nu}_{K,\text{on-shell}}(x,k)$$
$$\equiv \int \mathrm{d}S(k)h^{\mu\nu}(k,\mathfrak{s})f(x,k,\mathfrak{s})\,, \tag{4.127}$$

where we defined

$$h^{\mu\nu}(k,\mathfrak{s}) := \frac{1}{3}K^{\mu\nu} + \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\frac{k_\alpha}{m}\mathfrak{s}_\beta + K^{\mu\nu}_{\alpha\beta}\mathfrak{s}^\alpha\mathfrak{s}^\beta\,, \tag{4.128}$$

cf. Eqs. (3.154), (3.168) and (3.175). Then, Eq. (4.126) becomes

$$C^{\mu\nu}_{\perp,\text{local}}(x,k) = \frac{1}{32}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,\mathrm{d}\bar{S}(k)(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M^{\gamma_1\gamma_2\delta_1\delta_2}M^{\zeta_1\zeta_2\eta_1\eta_2}K^{\mu\alpha}K^{\nu\beta}$$
$$\times\Big[h'_{\gamma_2\eta_2}(k',\mathfrak{s}')h_{\zeta_1\delta_1}(k_1,\mathfrak{s}_1)h_{\zeta_2\delta_2}(k_2,\mathfrak{s}_2)g_{\alpha\eta_1}g_{\beta\gamma_1}f(x,k_1,\mathfrak{s}_1)f(x,k_2,\mathfrak{s}_2)$$
$$-\frac{1}{2}h_{\gamma_2\eta_2}(k',\mathfrak{s}')h_{\zeta_1\delta_1}(k_1,\mathfrak{s}_1)h_{\zeta_2\delta_2}(k_2,\mathfrak{s}_2)\left[h_{\alpha\eta_1}(k,\bar{\mathfrak{s}})g_{\beta\gamma_1}+g_{\alpha\eta_1}h_{\gamma_1\beta}(k,\bar{\mathfrak{s}})\right]$$
$$\times f\left(x,k,\bar{\mathfrak{s}}\right)f\left(x,k',\mathfrak{s}'\right)\Big]\,, \tag{4.129}$$

where we used that

$$K^\mu{}_\alpha h^{\alpha\nu}(k,\mathfrak{s}) = h^{\mu\alpha}(k,\mathfrak{s})K_\alpha{}^\nu = h^{\mu\nu}(k,\mathfrak{s})\,, \tag{4.130}$$

as well as

$$\int \mathrm{d}S(k)h^{\mu\nu}(k,\mathfrak{s}) = K^{\mu\nu}\,. \tag{4.131}$$

Lastly, we use that the on-shell part of the collision kernel in extended phase space can be written as[9]

$$C(x,k,\mathfrak{s}) = H_{\nu\mu}(k,\mathfrak{s})C^{\mu\nu}(x,k)\,, \tag{4.132}$$

where we defined

$$H^{\mu\nu}(k,\mathfrak{s}) := \frac{1}{3}K^{\mu\nu} + \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\frac{k_\alpha}{m}\mathfrak{s}_\beta + \frac{5}{8}K^{\mu\nu}_{\alpha\beta}\mathfrak{s}^\alpha\mathfrak{s}^\beta\,, \tag{4.133}$$

cf. Eq. (3.166a). At this point it is interesting to note that in the case of Dirac fields we did not have to distinguish between h and H, as is evident from Eqs. (4.79) and (4.84). Furthermore, note that Eqs. (4.128) and (4.133) differ only at second order in the spin vector $\mathfrak{s}$. Utilizing Eq. (4.132), we find

$$C_{\text{local}}(x,k,\mathfrak{s}) = \frac{1}{64}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,\mathrm{d}\bar{S}(k)(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M^{\gamma_1\gamma_2\delta_1\delta_2}M^{\zeta_1\zeta_2\eta_1\eta_2}$$
$$\times h'_{\gamma_2\eta_2}(k',\mathfrak{s}')h_{\zeta_1\delta_1}(k_1,\mathfrak{s}_1)h_{\zeta_2\delta_2}(k_2,\mathfrak{s}_2)$$
$$\times\left[h_{\gamma_1}{}^\alpha(k,\bar{\mathfrak{s}})H_{\alpha\eta_1}(k,\mathfrak{s}) + H_{\gamma_1}{}^\alpha(k,\mathfrak{s})h_{\alpha\eta_1}(k,\bar{\mathfrak{s}})\right]$$
$$\times\left[f(x,k_1,\mathfrak{s}_1)f(x,k_2,\mathfrak{s}_2) - f\left(x,k,\bar{\mathfrak{s}}\right)f\left(x,k',\mathfrak{s}'\right)\right]\,, \tag{4.134}$$

[9] We can omit the index "⊥" since H is orthogonal to the four-momentum anyway.

where we used that

$$\int \mathrm{d}\bar{S}(k)\,[h_{\gamma_1}{}^{\alpha}(k,\bar{\mathfrak{s}})H_{\alpha\eta_1}(k,\mathfrak{s}) + H_{\gamma_1}{}^{\alpha}(k,\mathfrak{s})h_{\alpha\eta_1}(k,\bar{\mathfrak{s}})] = 2H_{\gamma_1\eta_1}(k,\mathfrak{s})\,. \tag{4.135}$$

Note the remarkable similarity of Eqs. (4.85) and (4.134), where the difference in prefactors arises because of the different normalization of basis spinors $u_r(k)$ and polarization vectors $\epsilon^{(\lambda)}(k)$, which are $2m$ and 1, respectively. Furthermore, as we found in the spin-1/2 case, the loss term in Eq. (4.134) depends on $(k,\bar{\mathfrak{s}})$, and not on $(k,\mathfrak{s})$. Contrary to the case of Dirac fields, in the case of particles of spin 1 or higher it is not possible to redefine the collision term in extended phase space to remove the integration over $\bar{\mathfrak{s}}$ except in some special cases, such as the equilibrium one discussed in Sec. 4.6.

4.5.3 Nonlocal collisions

The steps to obtain the nonlocal part of the collision term are essentially the same as the ones presented for Dirac fields in Subsec. 4.4.3: Integrating by parts in u_1, u_2 in Eq. (4.125) leaves us with four contributions, which we label by Roman numbers. The first contribution consists of the derivatives acting on the projectors $(K \pm U_1/2 \pm U_2/2)^{\mu\nu}$ in front of everything on the right-hand side, which we evaluate by using that

$$\partial_u^\lambda \left(K + \frac{U}{2}\right)^{\mu\nu}\Bigg|_{u=0} = -\frac{1}{2m^2}g^{\lambda(\mu}k^{\nu)} + \frac{k^\lambda k^\mu k^\nu}{m^4}\,. \tag{4.136}$$

Note that the second term in the equation above will not contribute since the first index of the projectors which the derivative acts on is contracted with a projector orthogonal to k. As a result we obtain

$$\begin{aligned} C^{\mu\nu}_{\perp,\text{nonlocal,I}}(x,k) = {}& \frac{i\hbar}{2m^2}\frac{1}{32}\int \mathrm{d}K_1\,\mathrm{d}K_2\,\mathrm{d}K'(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2) \\ & \times M^{\gamma_1\gamma_2\delta_1\delta_2}M^{\zeta_1\zeta_2\eta_1\eta_2}\left(K^{\rho\mu}k^\alpha K^{\nu\beta} - K^{\mu\alpha}K^{\rho\nu}k^\beta\right) \\ & \times \Big\{ K'_{\eta_2\gamma_2}g_{\alpha\eta_1}g_{\beta\gamma_1}\partial_\rho\left[W_{\text{on-shell},\zeta_1\delta_1}(x,k_1)W_{\text{on-shell},\zeta_2\delta_2}(x,k_2)\right] \\ & \quad - \frac{1}{2}K_{1,\delta_1\zeta_1}K_{2,\delta_2\zeta_2}\left(g_{\alpha\alpha_1}g_{\beta\gamma_1}K_{\eta_1\beta_1} + g_{\alpha\eta_1}g_{\beta\beta_1}K_{\alpha_1\gamma_1}\right) \\ & \quad \times \partial_\rho\left[W_{\text{on-shell},\gamma_2\eta_2}(x,k')W^{\alpha_1\beta_1}_{\text{on-shell}}(x,k)\right]\Big\}\,, \end{aligned} \tag{4.137}$$

where we used that

$$W^{\mu\nu}(x,k) = K^{\mu\nu}f_K(x,k) + \mathcal{O}(\hbar)\,, \tag{4.138}$$

and thus $K^\mu{}_\alpha W^{\alpha\beta}(x,k)K_\beta{}^\nu = W^{\mu\nu}(x,k) + \mathcal{O}(\hbar)$. Translating this equation into extended phase space and using the fact that the distribution functions are independent of the spin variable at zeroth order in $\hbar$ we find

$$\begin{aligned} C_{\text{I}}^{\text{nonlocal}}(x,k,\mathfrak{s}) = {}& -\frac{i\hbar}{2m^2}\frac{1}{32}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2) \\ & \times M^{\gamma_1\gamma_2\delta_1\delta_2}M^{\zeta_1\zeta_2\eta_1\eta_2}h_{\gamma_2\eta_2}(k',\mathfrak{s}')h_{\zeta_1\delta_1}(k_1,\mathfrak{s}_1)h_{\zeta_2\delta_2}(k_2,\mathfrak{s}_2) \\ & \times \left[H^\rho{}_{\eta_1}(k,\mathfrak{s})k_{\gamma_1} - H_{\gamma_1}{}^\rho(k,\mathfrak{s})k_{\eta_1}\right]\partial_\rho\left[f(x,k_1)f(x,k_2) - \frac{1}{2}f(x,k)f(x,k')\right]. \end{aligned} \tag{4.139}$$

In the second contribution to the nonlocal collision term, the u_1, u_2-derivatives act on the remaining projectors, giving

$$
\begin{aligned}
C_{\mathrm{II}}^{\mathrm{nonlocal}}(x,k,\mathfrak{s}) = \frac{i\hbar}{2m^2}\frac{1}{32}\int & \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M^{\gamma_1\gamma_2\delta_1\delta_2}M^{\zeta_1\zeta_2\eta_1\eta_2} \\
\times \Big\{ & f(x,k_2)\,[\partial_\rho f(x,k_1)]\,h_{\gamma_2\eta_2}(k',\mathfrak{s}')h_{\zeta_2\delta_2}(k_2,\mathfrak{s}_2)H_{\gamma_1\eta_1}(k,\mathfrak{s}) \\
& \times [h^\rho{}_{\delta_1}(k_1,\mathfrak{s}_1)k_{1,\zeta_1} - h_{\zeta_1}{}^\rho(k_1,\mathfrak{s}_1)k_{1,\delta_1}] \\
& + f(x,k_1)\,[\partial_\rho f(x,k_2)]\,h_{\gamma_2\eta_2}(k',\mathfrak{s}')h_{\zeta_1\delta_1}(k_1,\mathfrak{s}_1)H_{\gamma_1\eta_1}(k,\mathfrak{s}) \\
& \times [h^\rho{}_{\delta_2}(k_2,\mathfrak{s}_2)k_{2,\zeta_2} - h_{\zeta_2}{}^\rho(k_2,\mathfrak{s}_2)k_{2,\delta_2}] \\
& - f(x,k)\,[\partial_\rho f(x,k')]\,h_{\zeta_1\delta_1}(k_1,\mathfrak{s}_1)h_{\zeta_2\delta_2}(k_2,\mathfrak{s}_2)H_{\gamma_1\eta_1}(k,\mathfrak{s}) \\
& \times \left[h^\rho{}_{\eta_2}(k',\mathfrak{s}')k'_{\gamma_2} - h_{\gamma_2}{}^\rho(k',\mathfrak{s}')k'_{\eta_2}\right] \\
& - \frac{1}{2}\,[f(x,k')\partial_\rho f(x,k) - f(x,k)\partial_\rho f(x,k')]\,h_{\zeta_1\delta_1}(k_1,\mathfrak{s}_1)h_{\zeta_2\delta_2}(k_2,\mathfrak{s}_2) \\
& \times h_{\gamma_2\eta_2}(k',\mathfrak{s}')\,[H^\rho{}_{\eta_1}(k,\mathfrak{s})k_{\gamma_1} - H_{\gamma_1}{}^\rho(k,\mathfrak{s})k_{\eta_1}]\Big\}\,. \qquad (4.140)
\end{aligned}
$$

The third and fourth contributions to the collision term consist of the derivatives acting on the Wigner functions and momentum-conserving delta functions in the loss term, respectively. As we showed in the previous section for spin-1/2 particles, these terms vanish to first order in $\hbar$, which is also true in the spin-1 case. The reason is the same as discussed in Subsec. 4.4.3, namely that the two contributions to the loss term cancel, since the momentum-space shifts u_1, u_2 appear with opposite signs in them and the Wigner functions are proportional to momentum-space projectors to zeroth order in $\hbar$. Thus, we have to first order in the Planck constant[10]

$$C_{\mathrm{III}}^{\mathrm{nonlocal}}(x,k,\mathfrak{s}) = 0\,, \qquad (4.141a)$$
$$C_{\mathrm{IV}}^{\mathrm{nonlocal}}(x,k,\mathfrak{s}) = 0\,, \qquad (4.141b)$$

such that the total nonlocal collision term for spin-1 particles is given by the sum of Eqs. (4.139) and (4.140),

$$
\begin{aligned}
C_{\mathrm{nonlocal}}(x,k,\mathfrak{s}) = \frac{i\hbar}{2m^2}\frac{1}{32}\int & \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M^{\gamma_1\gamma_2\delta_1\delta_2}M^{\zeta_1\zeta_2\eta_1\eta_2} \\
\times \Big\{ & f(x,k_2)\,[\partial_\rho f(x,k_1)]\,h_{\gamma_2\eta_2}(k',\mathfrak{s}')h_{\zeta_2\delta_2}(k_2,\mathfrak{s}_2)H_{\gamma_1\eta_1}(k,\mathfrak{s}) \\
& \times [h^\rho{}_{\delta_1}(k_1,\mathfrak{s}_1)k_{1,\zeta_1} - h_{\zeta_1}{}^\rho(k_1,\mathfrak{s}_1)k_{1,\delta_1}] \\
& + f(x,k_1)\,[\partial_\rho f(x,k_2)]\,h_{\gamma_2\eta_2}(k',\mathfrak{s}')h_{\zeta_1\delta_1}(k_1,\mathfrak{s}_1)H_{\gamma_1\eta_1}(k,\mathfrak{s}) \\
& \times [h^\rho{}_{\delta_2}(k_2,\mathfrak{s}_2)k_{2,\zeta_2} - h_{\zeta_2}{}^\rho(k_2,\mathfrak{s}_2)k_{2,\delta_2}] \\
& - f(x,k)\,[\partial_\rho f(x,k')]\,h_{\zeta_1\delta_1}(k_1,\mathfrak{s}_1)h_{\zeta_2\delta_2}(k_2,\mathfrak{s}_2)H_{\gamma_1\eta_1}(k,\mathfrak{s}) \\
& \times \left[h^\rho{}_{\eta_2}(k',\mathfrak{s}')k'_{\gamma_2} - h_{\gamma_2}{}^\rho(k',\mathfrak{s}')k'_{\eta_2}\right] \\
& - f(x,k')\,[\partial_\rho f(x,k)]\,h_{\zeta_1\delta_1}(k_1,\mathfrak{s}_1)h_{\zeta_2\delta_2}(k_2,\mathfrak{s}_2)h_{\gamma_2\eta_2}(k',\mathfrak{s}') \\
& \times [H^\rho{}_{\eta_1}(k,\mathfrak{s})k_{\gamma_1} - H_{\gamma_1}{}^\rho(k,\mathfrak{s})k_{\eta_1}] \\
& - \partial_\rho\,[f(x,k_1)f(x,k_2) - f(x,k)f(x,k')]\,h_{\zeta_1\delta_1}(k_1,\mathfrak{s}_1)h_{\zeta_2\delta_2}(k_2,\mathfrak{s}_2) \\
& \times h_{\gamma_2\eta_2}(k',\mathfrak{s}')\,[H^\rho{}_{\eta_1}(k,\mathfrak{s})k_{\gamma_1} - H_{\gamma_1}{}^\rho(k,\mathfrak{s})k_{\eta_1}]\Big\}\,. \qquad (4.142)
\end{aligned}
$$

As in the case of Dirac fermions, the terms proportional to the gradient of $f(x,k)$ cancel each other.

[10]As we shall see in Sec. 5.5, this is in accordance with the fact that in the KB approach some contributions belonging to the Poisson-bracket terms vanish.

4.5.4 Summary

We add the nonlocal collision term (4.142) to the local contribution (4.134) and interpret them as the first two contributions in a TAYLOR series, such that the BOLTZMANN equation can be cast in the following form,

$$\begin{aligned} k \cdot \partial f(x,k,\mathfrak{s}) &= C(x,k,\mathfrak{s}) \\ &= \frac{1}{2}\int \mathrm{d}\Gamma_1\, \mathrm{d}\Gamma_2\, \mathrm{d}\Gamma'\, \mathrm{d}\bar{S}(k)\, (2\pi\hbar)^4 \delta^{(4)}(k+k'-k_1-k_2)\mathcal{W}^{(1)} \\ &\quad \times [f(x+\Delta_1-\Delta,k_1,\mathfrak{s}_1)f(x+\Delta_2-\Delta,k_2,\mathfrak{s}_2) - f(x,k,\bar{\mathfrak{s}})f(x+\Delta'-\Delta,k',\mathfrak{s}')]\ , \end{aligned} \tag{4.143}$$

which is formally identical to Eq. (4.96). In the spin-1 case the local transition rate is defined as

$$\begin{aligned} \mathcal{W}^{(1)} := \frac{1}{32} M^{\gamma_1\gamma_2\delta_1\delta_2} M^{\zeta_1\zeta_2\eta_1\eta_2} h_{\zeta_1\delta_1}(k_1,\mathfrak{s}_1) h_{\zeta_2\delta_2}(k_2,\mathfrak{s}_2) h_{\gamma_2\eta_2}(k',\mathfrak{s}') \\ \times [h_{\gamma_1}{}^{\alpha}(k,\bar{\mathfrak{s}}) H_{\alpha\eta_1}(k,\mathfrak{s}) + H_{\gamma_1}{}^{\alpha}(k,\mathfrak{s}) h_{\alpha\eta_1}(k,\bar{\mathfrak{s}})]\ , \end{aligned} \tag{4.144}$$

while the spacetime shifts read[11]

$$\begin{aligned} \Delta_1^\mu &:= \frac{1}{3}\frac{i\hbar}{32m^2}\frac{1}{\mathcal{W}^{(1)}} M^{\gamma_1\gamma_2\delta_1\delta_2} M^{\zeta_1\zeta_2\eta_1\eta_2} h_{\zeta_2\delta_2}(k_2,\mathfrak{s}_2) h_{\gamma_2\eta_2}(k',\mathfrak{s}') H_{\gamma_1\eta_1}(k,\mathfrak{s}) \\ &\quad \times [h^{\mu}{}_{\delta_1}(k_1,\mathfrak{s}_1)k_{1,\zeta_1} - h_{\zeta_1}{}^{\mu}(k_1,\mathfrak{s}_1)k_{1,\delta_1}]\ , \end{aligned} \tag{4.145a}$$

$$\begin{aligned} \Delta_2^\mu &:= \frac{1}{3}\frac{i\hbar}{32m^2}\frac{1}{\mathcal{W}^{(1)}} M^{\gamma_1\gamma_2\delta_1\delta_2} M^{\zeta_1\zeta_2\eta_1\eta_2} h_{\zeta_1\delta_1}(k_1,\mathfrak{s}_1) h_{\gamma_2\eta_2}(k',\mathfrak{s}') H_{\gamma_1\eta_1}(k,\mathfrak{s}) \\ &\quad \times [h^{\mu}{}_{\delta_2}(k_2,\mathfrak{s}_2)k_{2,\zeta_2} - h_{\zeta_2}{}^{\mu}(k_2,\mathfrak{s}_2)k_{2,\delta_2}]\ , \end{aligned} \tag{4.145b}$$

$$\begin{aligned} \Delta'^\mu &:= \frac{1}{3}\frac{i\hbar}{32m^2}\frac{1}{\mathcal{W}^{(1)}} M^{\gamma_1\gamma_2\delta_1\delta_2} M^{\zeta_1\zeta_2\eta_1\eta_2} h_{\zeta_1\delta_1}(k_1,\mathfrak{s}_1) h_{\zeta_2\delta_2}(k_2,\mathfrak{s}_2) H_{\gamma_1\eta_1}(k,\mathfrak{s}) \\ &\quad \times \left[h^{\mu}{}_{\eta_2}(k',\mathfrak{s}')k'_{\gamma_2} - h_{\gamma_2}{}^{\mu}(k',\mathfrak{s}')k'_{\eta_2}\right]\ , \end{aligned} \tag{4.145c}$$

$$\begin{aligned} \Delta^\mu &:= \frac{1}{3}\frac{i\hbar}{32m^2}\frac{1}{\mathcal{W}^{(1)}} M^{\gamma_1\gamma_2\delta_1\delta_2} M^{\zeta_1\zeta_2\eta_1\eta_2} h_{\zeta_1\delta_1}(k_1,\mathfrak{s}_1) h_{\zeta_2\delta_2}(k_2,\mathfrak{s}_2) h_{\gamma_2\eta_2}(k',\mathfrak{s}') \\ &\quad \times [H^{\mu}{}_{\eta_1}(k,\mathfrak{s})k_{\gamma_1} - H_{\gamma_1}{}^{\mu}(k,\mathfrak{s})k_{\eta_1}]\ . \end{aligned} \tag{4.145d}$$

As before, they fulfill $\Delta_j \cdot k_j = 0$. Interestingly, even though spin-1 particles have a richer internal structure than their spin-1/2 counterparts, to first order the BOLTZMANN equations for DIRAC and PROCA fields are formally identical, with minor modifications in the definition of the spacetime shifts and transition rates. In our formalism, the differences between particles of different spin is encoded in their dependence on the spin variable $\mathfrak{s}$, in the sense that the distribution function of a spin-j particle contains contributions of all orders $\leq 2j$ in $\mathfrak{s}$.

Lastly, we seek to establish a connection with the usual BOLTZMANN equation (4.3). Computing the unpolarized transition rate

$$\begin{aligned} \overline{|M(k,k',k_1,k_2)|^2} &:= \frac{1}{9}\sum_{\lambda,\lambda'}\sum_{\lambda^2} \left|\langle k,k';\lambda,\lambda'|\,\hat{t}\,|k^2;\lambda^2\rangle\right|^2 \\ &= \frac{1}{9} M^{\gamma_1\gamma_2\delta_1\delta_2} M^{\zeta_1\zeta_2\eta_1\eta_2} K_{1,\delta_1\zeta_1} K_{2,\delta_2\zeta_2} K_{\eta_1\gamma_1} K'_{\eta_2\gamma_2}\ , \end{aligned} \tag{4.146}$$

integrating Eq. (4.143) over $\mathfrak{s}$, and dividing by three, we find the spin-averaged BOLTZMANN equation

$$\begin{aligned} k \cdot \partial f(x,k) = \frac{g}{2}\int \mathrm{d}K_1\, \mathrm{d}K_2\, \mathrm{d}K' (2\pi\hbar)^4 \delta^{(4)}(k+k'-k_1-k_2) \frac{\overline{|M(k,k',k_1,k_2)|^2}}{16} \\ \times [f(x,k_1)f(x,k_2) - f(x,k)f(x,k')]\ , \end{aligned} \tag{4.147}$$

where, as expected, the spin-degeneracy factor is now $g = 3$.

[11] The factor of 1/3 has to be present to cancel the factor of 3 that arises from trivially computing the $\mathrm{d}\bar{S}(k)$-integral.

4.6 Equilibrium

The main results of this chapter are the kinetic equations for the distribution functions in extended phase space, given by Eqs. (4.51), (4.96), and (4.143) for spin 0, 1/2 and 1, respectively. While in the case of scalar fields, the spin space is trivial and the collisions are purely local, in the cases of nonzero spin the particles do not collide at the same spacetime point x. We will now see what effects this has on the *local-equilibrium* distribution function $f_{\text{eq}}(x, k, \mathfrak{s})$, which is defined by the property that it makes the collision term vanish, i.e., it fulfills $C[f_{\text{eq}}](x, k, \mathfrak{s}) = 0$. Note that this distribution function does not need to satisfy the Boltzmann equation [43, 115], which is what defines *global equilibrium.*

From the form of the kinetic equations, where quantum-statistical effects were neglected, it is apparent that the local-equilibrium distribution function should be of Maxwell-Jüttner form,

$$f_{\text{eq}}(x, k, \mathfrak{s}) =: \exp\left[g(x, k, \mathfrak{s})\right] \ , \tag{4.148}$$

where $g(x, k, \mathfrak{s})$ has to consist of summational invariants, i.e., quantities that are additively conserved in a collision. In our formulation, which assumed a monatomic gas, there are three distinct summational invariants, namely a constant which we take to be unity, the four-momentum k^μ and the total angular momentum $J^{\mu\nu}$. Taking the function $g(x, k, \mathfrak{s})$ to be a linear combination of these quantities, we have [44, 45, 47]

$$\begin{aligned} f_{\text{eq}}(x, k, \mathfrak{s}) &= \exp\left[\alpha_0(x) - b_0(x) \cdot k + \frac{1}{2}\Omega_{0,\mu\nu}(x) J^{\mu\nu}\right] \\ &= \exp\left[\alpha_0(x) - \beta_0(x) \cdot k + \sigma\frac{\hbar}{2}\Omega_{0,\mu\nu}(x)\Sigma_{\mathfrak{s}}^{\mu\nu}\right] \ , \end{aligned} \tag{4.149}$$

where $\sigma \in \{0, 1/2, 1\}$ is the spin of the particles and $\alpha_0(x)$, $\beta_0(x)$, and $\Omega_0(x)$ are arbitrary functions of spacetime, which act as Lagrange multipliers and can be associated to the chemical potential over temperature, the four-temperature and the so-called spin potential, respectively. Note that in the second step we used $J^{\mu\nu} = \sigma\hbar\Sigma_{\mathfrak{s}}^{\mu\nu} + x^{[\mu}k^{\nu]}$, where $\Sigma_{\mathfrak{s}}^{\mu\nu}$ has been defined in Eq. (3.97), and absorbed the second part (constituting the contribution from the orbital angular momentum) into the Lagrange multiplier $\beta_0(x)$, which is defined as

$$\beta_0^\mu(x) \coloneqq b_0^\mu(x) + \Omega_0^{\mu\nu}(x) x_\nu \ . \tag{4.150}$$

Inserting the *Ansatz* (4.149) into the generic form of the collision terms (4.96), (4.143) and expanding to first order in $\hbar$, we find

$$\begin{aligned} C[f_{\text{eq}}](x, k, \mathfrak{s}) = \frac{1}{2}\int & \mathrm{d}\Gamma_1\, \mathrm{d}\Gamma_2\, \mathrm{d}\Gamma'\, \mathrm{d}\bar{S}(k)\, (2\pi\hbar)^4 \delta^{(4)}(k + k' - k_1 - k_2) \mathcal{W}^{(\sigma)} e^{2\alpha_0 - \beta_0\cdot(k+k')} \\ & \times \left[\sigma\frac{\hbar}{2}\Omega_{0,\mu\nu}\left(\Sigma_{\mathfrak{s}_1}^{\mu\nu} + \Sigma_{\mathfrak{s}_2}^{\mu\nu} - \Sigma_{\bar{\mathfrak{s}}}^{\mu\nu} - \Sigma_{\mathfrak{s}'}^{\mu\nu}\right) + (\Delta_1 + \Delta_2 - \Delta - \Delta')\cdot\partial\alpha_0 \right. \\ & \left. - \left(\Delta_1^\mu k_1^\nu + \Delta_2^\mu k_2^\nu - \Delta^\mu k^\nu - \Delta'^\mu k'^\nu\right)\partial_\mu \beta_{0,\nu}\right] \ , \end{aligned} \tag{4.151}$$

where we also used the conservation of momentum.

Weak equivalence principle

At this point, we run into a problem: In order to proceed, we need to use the conservation of the total angular momentum, which reads

$$J^{\mu\nu}(x, k, \mathfrak{s}) + J^{\mu\nu}(x, k', \mathfrak{s}') - J^{\mu\nu}(x, k_1, \mathfrak{s}_1) - J^{\mu\nu}(x, k_2, \mathfrak{s}_2) = 0 \ . \tag{4.152}$$

However, the terms involving the microscopic dipole tensor Σ are not proportional to $\mathfrak{s}$, but to $\bar{\mathfrak{s}}$. In order to remedy this, we have to remember the fact that only quantities which are integrated over the spin variable $\mathfrak{s}$ correspond to observables, which means that, as long as the integrated quantities stay the same, we are free to redefine the objects in extended phase space. Specifically, we may reformulate

$$C(x,k,\mathfrak{s}) \to \widetilde{C}(x,k,\mathfrak{s}) \,, \qquad f(x,k,\mathfrak{s}) \to \widetilde{f}(x,k,\mathfrak{s}) \,, \tag{4.153}$$

as long as [44]

$$\int \mathrm{d}S(k)\mathfrak{s}^{\mu_1}\cdots\mathfrak{s}^{\mu_n}\widetilde{f}(x,k,\mathfrak{s}) = \int \mathrm{d}S(k)\mathfrak{s}^{\mu_1}\cdots\mathfrak{s}^{\mu_n} f(x,k,\mathfrak{s}) \,, \tag{4.154a}$$

$$\int \mathrm{d}S(k)\mathfrak{s}^{\mu_1}\cdots\mathfrak{s}^{\mu_n}\widetilde{C}(x,k,\mathfrak{s}) = \int \mathrm{d}S(k)\mathfrak{s}^{\mu_1}\cdots\mathfrak{s}^{\mu_n} C(x,k,\mathfrak{s}) \,, \tag{4.154b}$$

where $n \in \{0,\cdots,2j\}$ for spin-j particles. From the definition in terms of the components of the Wigner function, we immediately obtain $\widetilde{f}(x,k,\mathfrak{s}) = f(x,k,\mathfrak{s})$, which leaves the collision term to be modified such that the terms which depend on $\bar{\mathfrak{s}}$ switch their argument to $\mathfrak{s}$. In general, this is not possible for particles of spin higher than $1/2$. Since we are interested in equilibrium, however, we may use that the equilibrium distribution function only features a linear dependence on $\mathfrak{s}$ up to first order in $\hbar$, which stems from the fact that quantities of higher order in spin (which may be related to higher-order polarization phenomena, such as alignment [18, 28]) are not connected to conserved quantities directly. Then, it suffices to consider Eq. (4.154b) only up to $n=1$. For parity-conserving interactions it holds that

$$\int \mathrm{d}S_i(k_i)\mathcal{W}^{(\sigma)}\,\mathfrak{s}_i^\mu = 0 \,, \qquad \int \mathrm{d}S_i(k_i)\mathrm{d}S_j(k_j)\mathcal{W}^{(\sigma)}\,\mathfrak{s}_i^\mu K^{\nu\lambda}_{\alpha\beta}\,\mathfrak{s}_j^\alpha\mathfrak{s}_j^\beta = 0 \,, \tag{4.155}$$

where $\mathfrak{s}_i,\mathfrak{s}_j \in \{\mathfrak{s}_1,\mathfrak{s}_2,\mathfrak{s}',\mathfrak{s},\bar{\mathfrak{s}}\}$, since the quantities in Eq. (4.155) transform as pseudotensors under parity, while the only tensor structures at our disposal are given by an odd number of powers of momentum, which transform as tensors. We then make use of the equalities

$$\int \mathrm{d}\bar{S}(k)\left[h^{\alpha\alpha'}(k,\mathfrak{s})h^{\alpha'\beta}(k,\bar{\mathfrak{s}}) + h^{\alpha\alpha'}(k,\bar{\mathfrak{s}})h^{\alpha'\beta}(k,\mathfrak{s})\right] = 2h^{\alpha\beta}(k,\mathfrak{s}) \,, \tag{4.156a}$$

$$\int \mathrm{d}S(k)\mathfrak{s}^\rho \mathrm{d}\bar{S}(k)\bar{\mathfrak{s}}^\sigma\left[h^{\alpha\alpha'}(k,\mathfrak{s})h^{\alpha'\beta}(k,\bar{\mathfrak{s}}) + h^{\alpha\alpha'}(k,\bar{\mathfrak{s}})h^{\alpha'\beta}(k,\mathfrak{s})\right] = 2\int \mathrm{d}S(k)\mathfrak{s}^\rho\mathfrak{s}^\sigma h^{\alpha\beta}(k,\mathfrak{s}) \,, \tag{4.156b}$$

for the terms appearing in the spin-$1/2$ case, and

$$\int \mathrm{d}\bar{S}(k)\left[H^{\mu\alpha}(k,\mathfrak{s})h_\alpha^{\ \nu}(k,\bar{\mathfrak{s}}) + h^{\mu\alpha}(k,\bar{\mathfrak{s}})H_\alpha^{\ \nu}(k,\mathfrak{s})\right] = 2H^{\mu\nu}(k,\mathfrak{s}) \,, \tag{4.157a}$$

$$\int \mathrm{d}S(k)\mathfrak{s}^\rho \mathrm{d}\bar{S}(k)\bar{\mathfrak{s}}^\sigma\left[H^{\mu\alpha}(k,\mathfrak{s})h_\alpha^{\ \nu}(k,\bar{\mathfrak{s}}) + h^{\mu\alpha}(k,\bar{\mathfrak{s}})H_\alpha^{\ \nu}(k,\mathfrak{s})\right] = 2\int \mathrm{d}S(k)\mathfrak{s}^\rho\mathfrak{s}^\sigma H^{\mu\nu}(k,\mathfrak{s}) \,, \tag{4.157b}$$

for the quantities appearing for spin 1. Subsequently, we may replace $\bar{\mathfrak{s}}$ with $\mathfrak{s}$ and remove the $\mathrm{d}\bar{S}$-integral in Eq. (4.151) while redefining the transition rates as

$$\begin{aligned}\widetilde{\mathcal{W}}^{(1/2)} :=\, & m^4 M^{\alpha_1\alpha_2\beta_1\beta_2}(k,k',k_1,k_2)M^{\gamma_1\gamma_2\delta_1\delta_2}(k,k',k_1,k_2)\\ &\times h_{\beta_1\gamma_1}(k_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(k_2,\mathfrak{s}_2)h_{\delta_1\alpha_1}(k,\mathfrak{s})h_{\delta_2\alpha_2}(k',\mathfrak{s}') \,,\end{aligned} \tag{4.158}$$

and

$$\begin{aligned}\widetilde{\mathcal{W}}^{(1)} :=\, & \frac{1}{16}M^{\gamma_1\gamma_2\delta_1\delta_2}(k,k',k_1,k_2)M^{\zeta_1\zeta_2\eta_1\eta_2}(k,k',k_1,k_2)\\ &\times H_{\zeta_1\delta_1}(k_1,\mathfrak{s}_1)H_{\zeta_2\delta_2}(k_2,\mathfrak{s}_2)H_{\gamma_1\eta_1}(k,\mathfrak{s})H_{\gamma_2\eta_2}(k',\mathfrak{s}') \,,\end{aligned} \tag{4.159}$$

respectively.[12] After using the conservation of the total angular momentum, we then find

$$\begin{aligned}\widetilde{C}[f_{\mathrm{eq}}](x,k,\mathfrak{s}) = \frac{1}{2}\int & \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\widetilde{\mathcal{W}}^{(\sigma)}e^{2\alpha_0-\beta_0\cdot(k+k')}\\ &\times\left[\sigma\frac{\hbar}{2}\left(\Omega_{0,\mu\nu}-\varpi_{\mu\nu}\right)\left(\Sigma^{\mu\nu}_{\mathfrak{s}_1}+\Sigma^{\mu\nu}_{\mathfrak{s}_2}-\Sigma^{\mu\nu}_{\mathfrak{s}}-\Sigma^{\mu\nu}_{\mathfrak{s}'}\right)+\left(\Delta_1+\Delta_2-\Delta-\Delta'\right)\cdot\partial\alpha_0\right.\\ &\left.\quad-\frac{1}{2}\left(\Delta_1^\mu k_1^\nu+\Delta_2^\mu k_2^\nu-\Delta^\mu k^\nu-\Delta'^\mu k'^\nu\right)\partial_{(\mu}\beta_{0,\nu)}\right] .\end{aligned} \tag{4.160}$$

[12] In Eq. (4.159), we used that $h^{\mu\nu}(k,\mathfrak{s}) = H^{\mu\nu}(k,\mathfrak{s})$ to linear order in $\mathfrak{s}$ (which are the only relevant terms for this case).

From the equation above, we see that the collision term vanishes exactly only if

$$\partial_{(\mu}\beta_{0,\nu)} = 0 \;, \quad \partial_\mu \alpha_0 = 0 \;, \quad \Omega_{0,\mu\nu} = \varpi_{\mu\nu} \equiv -\frac{1}{2}\partial_{[\mu}\beta_{0,\nu]} \;. \tag{4.161}$$

The conditions above imply that β_0 is at most a linear function of x, i.e.,

$$\beta_0^\mu = b^\mu + \varpi^{\mu\nu} x_\nu \;, \tag{4.162}$$

with b, $\varpi = \text{const}$. Note that this form of the four-temperature β_0 coincides with the one in Eq. (2.28) derived from the global maximization of the entropy. Consequently, the conditions (4.161) constitute *global* equilibrium, which not only makes the collision kernel vanish, but also fulfills the Boltzmann equation. Inserting the equilibrium distribution function subject to the constraints (4.161) into the left-hand side of the kinetic equation, we indeed find

$$k \cdot \partial \left[\exp\left(\alpha_0 - b_\mu k^\mu - k_\mu \varpi^{\mu\nu} x_\nu + \sigma \frac{\hbar}{2} \varpi_{\mu\nu} \Sigma_{\mathfrak{s}}^{\mu\nu} \right) \right] = 0 \;. \tag{4.163}$$

Thus, for particles with spin, when defining the state of local equilibrium by requiring the collision term to vanish exactly, it coincides with the state of global equilibrium. We will inspect this statement more closely when developing dissipative hydrodynamics with spin in Chapter 6.

Chapter 5

The kinetic equation in the KB approach

In this chapter, we show an alternative way of expressing the collision terms (4.2) as functionals of the on-shell distribution functions $f(x, k, \mathfrak{s})$ in extended phase space. This method is originally accredited to Kadanoff and Baym [147, 149] and is based on the Dyson-Schwinger equations. Compared to the GLW method treated in Chapter 4, this approach has the advantage of retaining the effects of quantum statistics, which we expect to appear in the kinetic equation on the grounds of standard kinetic theory, cf. Eq. (4.3).

5.1 Two-point functions

The objects we are going to analyze in the following are the different possible two-point functions of the system in question, which encode correlations between the fields $\widehat{\varphi}$ at different spacetime points. Note that we will always assume the vacuum expectation values of the fields to vanish, i.e., $\langle\widehat{\varphi}\rangle = 0$. Considering the fields at the two spacetime points $x_1 \equiv (t_1, \mathbf{x}_1)$ and $x_2 \equiv (t_2, \mathbf{x}_2)$, there are four basic two-point functions. Firstly, we may not impose any time ordering and define two-point functions where the fields appear in the same (opposite) order compared to the arguments of the Green's function, thus yielding the greater (lesser) propagators[1]

$$G^{>}_{ab}(x_1, x_2) \coloneqq \left\langle \widehat{\varphi}_a(x_1)\widehat{\bar{\varphi}}_b(x_2) \right\rangle \ , \tag{5.1a}$$

$$G^{<}_{ab}(x_1, x_2) \coloneqq \pm \left\langle \widehat{\bar{\varphi}}_b(x_2)\widehat{\varphi}_a(x_1) \right\rangle \ , \tag{5.1b}$$

where the plus and minus signs apply to bosons and fermions, respectively. Alternatively, we may order the field operators according to whether t_1 is larger than t_2 or vice versa. Opting for the standard time ordering, denoted by the operator $\widehat{T}$, gives the Feynman propagator,

$$\begin{aligned} G^{\mathrm{F}}_{ab}(x_1, x_2) &\coloneqq \left\langle \widehat{T}\left[\widehat{\varphi}_a(x_1)\widehat{\bar{\varphi}}_b(x_2)\right] \right\rangle \\ &\equiv \Theta(t_1 - t_2) \left\langle \widehat{\varphi}_a(x_1)\widehat{\bar{\varphi}}_b(x_2) \right\rangle \pm \Theta(t_2 - t_1) \left\langle \widehat{\bar{\varphi}}_b(x_2)\widehat{\varphi}_a(x_1) \right\rangle \\ &\equiv \Theta(t_1 - t_2) G^{>}_{ab}(x_1, x_2) + \Theta(t_2 - t_1) G^{<}_{ab}(x_1, x_2) \ . \end{aligned} \tag{5.1c}$$

[1] We use the term "propagator" somewhat loosely here, since the two-point functions that we are using are differing from the actual propagators by appropriate factors of i, compare, e.g., Refs. [86, 135, 153].

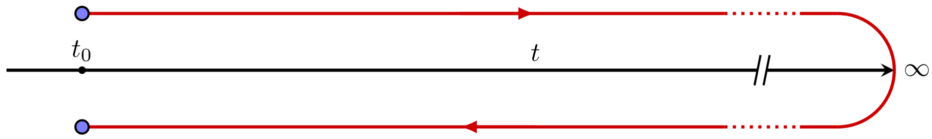

Figure 5.1: The closed-time path $\mathscr{C}$.

Choosing the anti-time ordering, for which we use the operator $\widehat{T}_A$, yields the so-called anti-Feynman or Dyson propagator [154],

$$\begin{aligned} G^{\bar{\mathrm{F}}}_{ab}(x_1,x_2) &:= \left\langle \widehat{T}_A \left[\widehat{\varphi}_a(x_1)\widehat{\overline{\varphi}}_b(x_2)\right]\right\rangle \\ &\equiv \Theta(t_2-t_1)\left\langle \widehat{\varphi}_a(x_1)\widehat{\overline{\varphi}}_b(x_2)\right\rangle \pm \Theta(t_1-t_2)\left\langle \widehat{\overline{\varphi}}_b(x_2)\widehat{\varphi}_a(x_1)\right\rangle \\ &\equiv \Theta(t_2-t_1)G^{>}_{ab}(x_1,x_2)+\Theta(t_1-t_2)G^{<}_{ab}(x_1,x_2)\ . \end{aligned} \tag{5.1d}$$

These two-point functions are not all independent; indeed we manifestly have

$$G^{\bar{\mathrm{F}}}+G^{\mathrm{F}}=G^{>}+G^{<}\ . \tag{5.2}$$

Furthermore, we define the retarded and advanced propagators,

$$G^{R}_{ab}(x_1,x_2) := \Theta(t_1-t_2)\left[G^{>}_{ab}(x_1,x_2)-G^{<}_{ab}(x_1,x_2)\right]\ , \tag{5.3a}$$

$$G^{A}_{ab}(x_1,x_2) := -\Theta(t_2-t_1)\left[G^{>}_{ab}(x_1,x_2)-G^{<}_{ab}(x_1,x_2)\right]\ . \tag{5.3b}$$

Finally, note that the following relations hold,

$$G^{F}=G^{R}+G^{<}=G^{A}+G^{>}\ , \tag{5.4a}$$

$$G^{\bar{F}}=G^{>}-G^{R}=G^{<}-G^{A}\ . \tag{5.4b}$$

5.1.1 The Keldysh contour

The four basic propagators defined in Eqs. (5.1) can be combined by evaluating the two-point function along the so-called Keldysh contour [155]. This contour, which we denote by $\mathscr{C}$, runs from t_0 to ∞, where it turns around and runs from ∞ back to t_0, cf. Fig. 5.1. We then define contour-ordered correlation functions, indicated by the operator $\widehat{T}_{\mathscr{C}}$, such that the fields are ordered according to where their time arguments appear on the Keldysh contour, i.e., the field appearing at the first position has the time argument that is the farthest on the contour. Note that, if fields have to be permuted, appropriate minus signs are added for fermions. Specifically, we consider the contour-ordered two-point function

$$G_{ab}(x_1,x_2) := \left\langle \widehat{T}_{\mathscr{C}}\left[\widehat{\varphi}_a(x_1)\widehat{\overline{\varphi}}_b(x_2)\right]\right\rangle\ . \tag{5.5}$$

Depending on where the arguments t_1 and t_2 lie on the contour $\mathscr{C}$, G will assume different forms. To see this, consider the situation where both t_1 and t_2 lie on the upper half of the contour; in this case the contour-ordering $\widehat{T}_{\mathscr{C}}$ will be equivalent to the standard time-ordering $\widehat{T}$. Similarly, if both t_1 and t_2 lie on the lower half of the contour, $\widehat{T}_{\mathscr{C}}$ acts the same way as the anti-time ordering $\widehat{T}_A$. If t_1 and t_2 lie on the upper (lower) and lower (upper) half of the contour, respectively, t_2 (t_1) will always be further advanced on $\mathscr{C}$ and thus appear in front. Summarizing, we may write

$$G_{ab}(x_1,x_2)=\begin{cases} G^{F}_{ab}(x_1,x_2)\ , & \text{if } t_1,t_2 \text{ on upper half}\ ,\\ G^{<}_{ab}(x_1,x_2)\ , & \text{if } t_1 \text{ on upper half, } t_2 \text{ on lower half}\ ,\\ G^{>}_{ab}(x_1,x_2)\ , & \text{if } t_2 \text{ on upper half, } t_1 \text{ on lower half}\ ,\\ G^{\bar{F}}_{ab}(x_1,x_2)\ , & \text{if } t_1,t_2 \text{ on lower half}\ . \end{cases} \tag{5.6}$$

Denoting these four cases by "++", "+−", "−+", and "−−", where + (−) indicates a position on the upper (lower) half of the contour, we can put the contour-ordered two-point function in the following matrix form,

$$G = \begin{pmatrix} G^{++} & G^{+-} \\ G^{-+} & G^{--} \end{pmatrix} = \begin{pmatrix} G^{F} & G^{<} \\ G^{>} & G^{\bar{F}} \end{pmatrix} . \tag{5.7}$$

The DYSON-SCHWINGER equation for the contour-ordered two-point function reads

$$G^{-1}_{0,ac} G^{AB}_{cb}(x_1,x_2) = -ic^{AB}\delta_{ab}\delta^{(4)}(x_1-x_2) + \frac{i}{\lambda^2}\int \mathrm{d}^4x' \Sigma^{AC}_{ac}(x_1,x')c_{CD}G^{DB}_{cb}(x',x_2) . \tag{5.8a}$$

Here, G_0^{-1} is the inverse free propagator for the respective fields, and Σ is the self-energy of the field. Furthermore, we defined $c \coloneqq \mathrm{diag}(1,-1)$, where the minus sign is needed due to the different orientation of the lower half of the contour. The factor λ, which was introduced in Eq. (4.8) and is equal to $\sqrt{\hbar}$ and 1 for bosons and fermions, respectively, is convenient to factor out due to different powers of $\hbar$ appearing in the inverse free propagators on the left-hand side. Note also that in Eq. (5.8a) the inverse free propagator is taken to act on the argument x_1, and in the case of vector fields the KRONECKER delta has to be replaced by a metric tensor. The adjoint of the DYSON-SCHWINGER equation is given by

$$G^{AB}_{ac}(x_1,x_2)\overleftarrow{G}^{*-1}_{0,cb} = -ic^{AB}\delta_{ab}\delta^{(4)}(x_1-x_2) + \frac{i}{\lambda^2}\int \mathrm{d}^4x' G^{AC}_{ac}(x_1,x')c_{CD}\Sigma^{DB}_{cb}(x',x_2) , \tag{5.8b}$$

where the inverse free propagator now acts on x_2. Explicitly, we have for the lesser propagator

$$\begin{aligned}\left[G^{-1}_{0,ac} - \frac{i}{\lambda^2}\Sigma^{\mathrm{MF}}_{ac}(x_1)\right] G^{<}_{cb}(x_1,x_2) &= \frac{i}{\lambda^2}\int \mathrm{d}^4x' \left[\Sigma^{F}_{ac}(x_1,x')G^{<}_{cb}(x',x_2) - \Sigma^{<}_{ac}(x_1,x')G^{\bar{F}}_{cb}(x',x_2)\right] \\ &= \frac{i}{\lambda^2}\int \mathrm{d}^4x' \left[\Sigma^{R}_{ac}(x_1,x')G^{<}_{cb}(x',x_2) + \Sigma^{<}_{ac}(x_1,x')G^{A}_{cb}(x',x_2)\right] ,\end{aligned} \tag{5.9a}$$

as well as

$$\begin{aligned}G^{<}_{ac}(x_1,x_2)\left[\overleftarrow{G}^{*-1}_{0,cb} - \frac{i}{\lambda^2}\Sigma^{\mathrm{MF}}_{cb}(x_2)\right] &= \frac{i}{\lambda^2}\int \mathrm{d}^4x' \left[G^{F}_{ac}(x_1,x')\Sigma^{<}_{cb}(x',x_2) - G^{<}_{ac}(x_1,x')\Sigma^{\bar{F}}_{cb}(x',x_2)\right] \\ &= \frac{i}{\lambda^2}\int \mathrm{d}^4x' \left[G^{R}_{ac}(x_1,x')\Sigma^{<}_{cb}(x',x_2) + G^{<}_{ac}(x_1,x')\Sigma^{A}_{cb}(x',x_2)\right] ,\end{aligned} \tag{5.9b}$$

where we employed Eqs. (5.4). Here we took into account that the self-energy in principle contains a mean-field part that is responsible for mass- and momentum corrections, i.e.,

$$\Sigma^{++}(x,x') = \Sigma^{\mathrm{MF}}(x)\delta^{(4)}(x-x') + \Sigma^{F}(x,x') , \tag{5.10a}$$

$$\Sigma^{--}(x,x') = -\Sigma^{\mathrm{MF}}(x)\delta^{(4)}(x-x') + \Sigma^{\bar{F}}(x,x') , \tag{5.10b}$$

whereas we have $\Sigma^{\pm\mp}(x,x') = \Sigma^{\lessgtr}(x,x')$.

5.2 The KADANOFF-BAYM equations

In order to arrive at a quantum kinetic theory, we do not need to consider the lesser propagator per se, but rather its WIGNER transform

$$G^{<}_{ab}(x,k) \coloneqq \int \mathrm{d}^4v e^{-\frac{i}{\hbar}k\cdot v} G^{<}_{ab}\left(x-\frac{v}{2}, x+\frac{v}{2}\right) , \tag{5.11a}$$

whose inverse is given by

$$G^{<}_{ab}(x_1,x_2) = \int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4} e^{\frac{i}{\hbar}k\cdot(x_2-x_1)} G^{<}_{ab}\left(\frac{x_1+x_2}{2}, k\right) . \tag{5.11b}$$

This definition differs from (the expectation value of) Eq. (3.31)[2] only by a factor, i.e., $G^<(x,k) \equiv \pm W(x,k)/\kappa$, where κ depends on the spin of the field and has been introduced in Subsec. 3.2.2. In order to conform with the standard notation, we will keep using $G^<(x,k)$ for the WIGNER function in this chapter.

In the following we list some identities of WIGNER transforms [86]. First, as already remarked in Eqs. (3.36), a derivative acting on a function f (which may be matrix-valued) will become a BOPP operator after the WIGNER transform,

$$i\hbar \frac{\partial}{\partial x_1^\mu} f(x_1,x_2) \longrightarrow D_\mu f(x,k) , \qquad -i\hbar \frac{\partial}{\partial x_2^\mu} f(x_1,x_2) \longrightarrow D_\mu^* f(x,k) . \tag{5.12a}$$

Second, the product of two (possibly matrix-valued) functions, where one of them depends on only one argument, becomes in WIGNER space

$$f(x_1)g(x_1,x_2) \longrightarrow f(x)g(x,k) - \frac{i\hbar}{2}\left[\partial_\mu f(x)\right]\left[\partial_k^\mu g(x,k)\right] + \mathcal{O}(\hbar^2) , \tag{5.12b}$$

$$f(x_2)g(x_1,x_2) \longrightarrow f(x)g(x,k) + \frac{i\hbar}{2}\left[\partial_\mu f(x)\right]\left[\partial_k^\mu g(x,k)\right] + \mathcal{O}(\hbar^2) . \tag{5.12c}$$

Finally, the convolution of two functions becomes

$$\int \mathrm{d}^4x' f(x_1,x')g(x',x_2) \longrightarrow f(x,k)g(x,k) - \frac{i\hbar}{2}\left\{f(x,k),g(x,k)\right\}_{\mathrm{PB}} + \mathcal{O}(\hbar^2) , \tag{5.12d}$$

where we defined the POISSON brackets

$$\left\{f(x,k),g(x,k)\right\}_{\mathrm{PB}} \coloneqq \left[\partial_\mu f(x,k)\right]\left[\partial_k^\mu g(x,k)\right] - \left[\partial_k^\mu f(x,k)\right]\left[\partial_\mu g(x,k)\right] . \tag{5.13}$$

Note that these brackets fulfill (for matrix-valued quantities A and B)

$$\{A,B\}_{\mathrm{PB},ab} = -\left\{B^{\mathrm{T}},A^{\mathrm{T}}\right\}_{\mathrm{PB},ba} , \tag{5.14}$$

where T denotes the transpose. Equation (5.12a) was already proved in Sec. 3.2.2 for the case of the WIGNER function, but can be shown to hold for any WIGNER transform by considering

$$\begin{aligned} D^\mu f(x,k) &= \int \mathrm{d}^4y e^{-\frac{i}{\hbar}k\cdot v}\left[k^\mu + \frac{i\hbar}{2}\left(\partial_1^\mu + \partial_2^\mu\right)\right] f\left(x-\frac{v}{2},x+\frac{v}{2}\right) \\ &= \int \mathrm{d}^4v e^{-\frac{i}{\hbar}k\cdot v}\left[i\hbar\overleftarrow{\partial}_v^\mu + \frac{i\hbar}{2}\left(\partial_1^\mu + \partial_2^\mu\right)\right] f\left(x-\frac{v}{2},x+\frac{v}{2}\right) \\ &= i\hbar\int \mathrm{d}^4v e^{-\frac{i}{\hbar}k\cdot v}\partial_1^\mu f\left(x-\frac{v}{2},x+\frac{v}{2}\right) . \end{aligned} \tag{5.15}$$

Here, we defined ∂_1 (∂_2) as a derivative w.r.t. the first (second) argument of the subsequent function. In order to prove Eq. (5.12b), we compute

$$\begin{aligned} f(x_1)g(x_1,x_2) &\longrightarrow \int \mathrm{d}^4v e^{-\frac{i}{\hbar}k\cdot v} f\left(x-\frac{v}{2}\right) g\left(x-\frac{v}{2},x+\frac{v}{2}\right) \\ &= \int \mathrm{d}^4v e^{-\frac{i}{\hbar}k\cdot v}\left[f(x) - \frac{1}{2}v\cdot\partial f(x)\right] g\left(x-\frac{v}{2},x+\frac{v}{2}\right) + \mathcal{O}(\hbar^2) \\ &= \int \mathrm{d}^4v e^{-\frac{i}{\hbar}k\cdot v}\left[f(x) - \frac{1}{2}i\hbar\overleftarrow{\partial}_k^\mu\cdot\partial f(x)\right] g\left(x-\frac{v}{2},x+\frac{v}{2}\right) + \mathcal{O}(\hbar^2) \\ &= f(x)g(x,k) - \frac{i\hbar}{2}\left[\partial_\mu f(x)\right]\left[\partial_k^\mu g(x,k)\right] + \mathcal{O}(\hbar^2) . \end{aligned} \tag{5.16}$$

Here, we assumed the function $g(x_1,x_2)$ to be *sharply peaked* at $x_1 = x_2$ [86], such that we may expand the quantities in the integral around $v = 0$. The idea behind this expansion is that the function g

[2] We repeat that, compared to Refs. [28, 45, 54], we use a different convention in that we assign the factor of $(2\pi\hbar)^4$ to the momentum-space measure, in accordance with Ref. [46].

will be taken to be one of the propagators; if we are in a regime where quantum kinetic theory is applicable, there has to be a sufficient separation of microscopic and macroscopic scales, as we already discussed in previous chapters. While the macroscopic scales will in this case be either related to inverse macroscopic gradients or the mean free path, the microscopic scale is characterized by the range of the interaction, which manifests itself in the correlation length between fields, i.e., the values of the two-point functions at different spacetime points. The proof of Eq. (5.12c) works in the same way as shown in Eq. (5.16), such that we now concentrate on proving Eq. (5.12d),

$$\begin{aligned}
&\int \mathrm{d}^4x' f(x_1,x')g(x',x_2)\\
\longrightarrow &\int \mathrm{d}^4v \int \mathrm{d}^4x' e^{-\frac{i}{\hbar}k\cdot v} f\left(x-\frac{v}{2},x'\right) g\left(x',x+\frac{v}{2}\right)\\
=&\int \mathrm{d}^4y' \int \mathrm{d}^4z' e^{-\frac{i}{\hbar}k\cdot(y'+z')} f\left(x-\frac{y'+z'}{2},x+\frac{y'-z'}{2}\right) g\left(x+\frac{y'-z'}{2},x+\frac{y'+z'}{2}\right)\\
=&\int \mathrm{d}^4y' \int \mathrm{d}^4z' e^{-\frac{i}{\hbar}k\cdot(y'+z')} \left[f\left(x-\frac{y'}{2},x+\frac{y'}{2}\right)-\frac{1}{2}z'\cdot\partial_x f\left(x-\frac{y'}{2},x+\frac{y'}{2}\right)\right]\\
&\times\left[g\left(x-\frac{z'}{2},x+\frac{z'}{2}\right)+\frac{1}{2}y'\cdot\partial_x g\left(x-\frac{z'}{2},x+\frac{z'}{2}\right)\right]+\mathcal{O}(\hbar^2)\\
=&f(x,k)g(x,k)-\frac{i\hbar}{2}\Big\{\left[\partial_\mu f(x,k)\right]\left[\partial_k^\mu g(x,k)\right]-\left[\partial_k^\mu f(x,k)\right]\left[\partial_\mu g(x,k)\right]\Big\}+\mathcal{O}(\hbar^2)\\
\equiv&f(x,k)g(x,k)-\frac{i\hbar}{2}\left\{f(x,k),g(x,k)\right\}_{\mathrm{PB}}+\mathcal{O}(\hbar^2)\,. \qquad (5.17)
\end{aligned}$$

In this calculation, we substituted $v = y' + z'$, $x' = x + (y' - z')/2$ and expanded f (g) around $z' = 0$ $(y' = 0)$. Both of these expansions are permissible since the relevant expansion point is $y' = z' = 0$.

Making use of Eqs. (5.12), we obtain from Eqs. (5.9)

$$\begin{aligned}
&\left\{G^{-1}_{0,ac}-\frac{i}{\lambda^2}\Sigma^{\mathrm{MF}}_{ac}(x)-\frac{\hbar}{2\lambda^2}\left[\partial_\mu\Sigma^{\mathrm{MF}}_{ac}(x)\right]\partial_k^\mu\right\}G^<_{cb}(x,k)\\
=&\frac{i}{\lambda^2}\left[\Sigma^R_{ac}(x,k)G^<_{cb}(x,k)+\Sigma^<_{ac}(x,k)G^A_{cb}(x,k)\right]\\
&+\frac{\hbar}{2\lambda^2}\left[\left\{\Sigma^R(x,k),G^<(x,k)\right\}_{\mathrm{PB},ab}+\left\{\Sigma^<(x,k),G^A(x,k)\right\}_{\mathrm{PB},ab}\right]+\mathcal{O}(\hbar^3)\,, \qquad (5.18a)
\end{aligned}$$

as well as

$$\begin{aligned}
&G^<_{ac}(x,k)\left\{\overleftarrow{G}^{*-1}_{0,cb}-\frac{i}{\lambda^2}\Sigma^{\mathrm{MF}}_{cb}(x)+\frac{\hbar}{2\lambda^2}\overleftarrow{\partial}^\mu_k\left[\partial_\mu\Sigma^{\mathrm{MF}}_{cb}(x)\right]\right\}\\
=&\frac{i}{\lambda^2}\left[G^R_{ac}(x,k)\Sigma^<_{cb}(x,k)+G^<_{ac}(x,k)\Sigma^A_{cb}(x,k)\right]\\
&+\frac{\hbar}{2\lambda^2}\left[\left\{G^R(x,k),\Sigma^<(x,k)\right\}_{\mathrm{PB},ab}+\left\{G^<(x,k),\Sigma^A(x,k)\right\}_{\mathrm{PB},ab}\right]+\mathcal{O}(\hbar^3)\,, \qquad (5.18b)
\end{aligned}$$

where G_0^{-1} now denotes the appropriate Wigner transform of the inverse propagator. Equations (5.18) are the *Kadanoff-Baym (KB) equations* in general form, which we will analyze in the following for fields of different spin. Note that they are expanded to second order in the Planck constant. This is necessary since we will have to cancel a factor of $\hbar$ on both sides when deriving the kinetic equations, cf. the discussions that led to Eqs. (3.50), (3.83), and (3.153).

Quasiparticle approximation

In order to simplify the KB equations, we will furthermore need to introduce the so-called *quasiparticle approximation*, which builds on the fact that, as long as we are in a regime where quantum kinetic theory is valid, the transformed lesser propagator should behave like a distribution function for particles

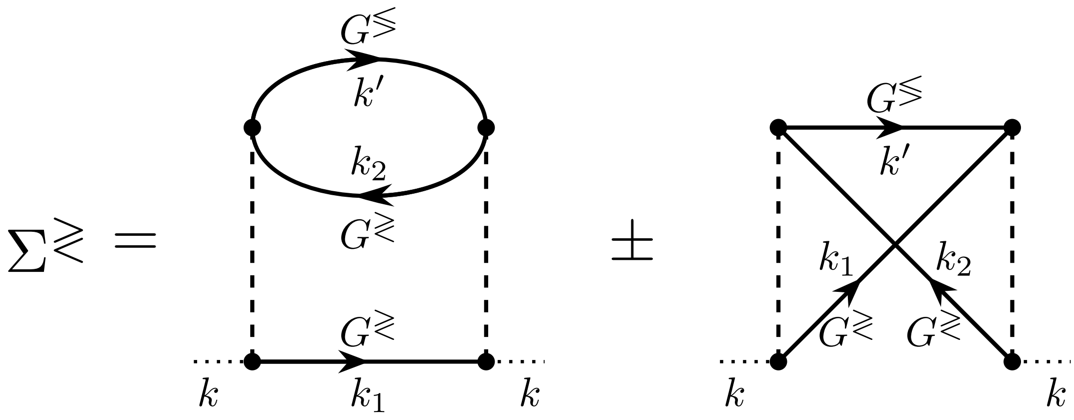

Figure 5.2: Greater and lesser self-energies (in phase space) in the T-matrix approximation.

on the mass shell. Explicitly, we consider the Wigner transform of a retarded or advanced quantity X, which can be both the Green's function and the self-energy [156]. Making use of Eqs. (5.3) as well as the Fourier transform of the Heaviside function,

$$\widetilde{\Theta}(\omega) = \frac{1}{2}\delta(\omega) + \frac{1}{2\pi i}\mathscr{P}\frac{1}{\omega} \equiv \lim_{\eta\to 0}\frac{1}{2\pi i}\frac{1}{\omega - i\eta} \ , \tag{5.19}$$

where $\mathscr{P}$ denotes a principal-value integration [cf. Eq. (4.44)], we find

$$\begin{aligned} X^R(x,k) &= \frac{1}{2\pi i}\lim_{\eta\to 0}\int \mathrm{d}p_0 \frac{1}{p_0 - k_0 - i\eta}\left[X^>(x,p_0,\mathbf{k}) - X^<(x,p_0,\mathbf{k})\right] \\ &= \frac{1}{2}\left[X^>(x,k) - X^<(x,k)\right] + \frac{1}{2\pi i}\mathscr{P}\int \mathrm{d}p_0 \frac{X^>(x,p_0,\mathbf{k}) - X^<(x,p_0,\mathbf{k})}{p_0 - k_0} \end{aligned} \tag{5.20a}$$

as well as

$$\begin{aligned} X^A(x,k) &= \frac{1}{2\pi i}\lim_{\eta\to 0}\int \mathrm{d}p_0 \frac{1}{p_0 - k_0 + i\eta}\left[X^>(x,p_0,\mathbf{k}) - X^<(x,p_0,\mathbf{k})\right] \\ &= -\frac{1}{2}\left[X^>(x,k) - X^<(x,k)\right] + \frac{1}{2\pi i}\mathscr{P}\int \mathrm{d}p_0 \frac{X^>(x,p_0,\mathbf{k}) - X^<(x,p_0,\mathbf{k})}{p_0 - k_0} \ . \end{aligned} \tag{5.20b}$$

With these relations, we are able to express the KB equations solely in terms of greater and lesser quantities. The quasiparticle approximation then consists in inserting Eqs. (5.20) and neglecting the principal-value integrations, which depend only on the off-shell parts of the quantity X. These parts are, according to our analysis from Chapter 3, which of course remains valid, at least of first order in $\hbar$, such that their contributions in the Poisson bracket terms is of third order. In fact, we will verify in the following sections that these off-shell components are of second order in $\hbar$, such that the principal-value integrals enter at third order only. Thus, neglecting these terms does not introduce a further approximation besides the $\hbar$-expansion.

Self-energies

Another approximation, however, has to enter when expressing the self-energies. In this thesis, similar to Ref. [46], we consider the self-energies in the *T-matrix approximation*. In this formulation, the greater and lesser self-energies are given by the FEYNMAN diagrams displayed in Fig. 5.2. After a computation shown in Appendix B.2.1, we obtain them as

$$\Sigma_{ab}^{\gtrless}(x,k) = \frac{1}{2\lambda^6}\int \frac{d^4k_1}{(2\pi\hbar)^4}\frac{d^4k_2}{(2\pi\hbar)^4}\frac{d^4k'}{(2\pi\hbar)^4}(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2) \\ \times G_{a_1b_1}^{\gtrless}(x,k_1)G_{a_2b_2}^{\gtrless}(x,k_2)G_{b'a'}^{\lessgtr}(x,k')M_{aa'a_1a_2}M_{b_1b_2bb'} . \tag{5.21}$$

Here, M, as introduced in Chapter 4, denotes the tree-level vertex of the theory, which we assume to be independent of momentum. In the case of spin-0 particles, it is a scalar quantity, while for higher-spin particles it is given by a rank-four tensor in the respective internal space. For example, if we were to consider DIRAC fermions in the NAMBU-JONA-LASINIO (NJL) model, the interaction Lagrangian reads [157, 158]

$$\widehat{\mathcal{L}}_{\text{int,NJL}} := \sum_c G_c \left[\widehat{\overline{\psi}}(x)\Gamma^{(c)}\widehat{\psi}(x)\right]\left[\widehat{\overline{\psi}}(x)\Gamma^{(c)}\widehat{\psi}(x)\right] . \tag{5.22}$$

Here, the index c runs over the possible channels [i.e., scalar (S), pseudoscalar (P), vector (V), axial vector (A), and antisymmetric tensor (T), weighted with couplings G_c], while the matrices Γ denote the corresponding element of the CLIFFORD algebra, i.e., $\Gamma^{(S)} := \mathbb{1}$, $\Gamma^{(P)} := -i\gamma_5$, $\Gamma^{(V)} := \gamma^\mu$, $\Gamma^{(A)} := \gamma_5\gamma^\mu$, and $\Gamma^{(T)} := \sigma^{\mu\nu}$, where a sum over the possible LORENTZ indices of Γ is understood in Eq. (5.22). In this setup, the vertex is [46]

$$M^{\alpha_1\alpha_2\beta_1\beta_2} \equiv \frac{2G_c}{\hbar}\left(\Gamma^{(c)\alpha_1\beta_1}\Gamma^{(c)\alpha_2\beta_2} - \Gamma^{(c)\alpha_1\beta_2}\Gamma^{(c)\alpha_2\beta_1}\right) , \tag{5.23}$$

where the minus sign is necessary to preserve the antisymmetry under fermion exchange. Similarly, if we were to consider massive vector bosons interacting via a scalar four-point interaction of strength G,

$$\widehat{\mathcal{L}}_{\text{int,V}} := \hbar G\left(\widehat{V}^\dagger \cdot \widehat{V}\right)^2 , \tag{5.24}$$

the vertex would read [28]

$$M^{\mu\nu\alpha\beta} \equiv 2\hbar^2 G(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha}) , \tag{5.25}$$

where the positive sign takes care of the symmetry under boson exchange.

Note that we did not consider the mean-field part of the self-energies in this approximation, which corresponds to tadpole-type diagrams [86, 146]. These contributions, as can be deduced from the KB equations (5.18), are responsible for corrections to the momentum and the energy of the particles as well as VLASOV-type terms, which are usually written on the left-hand side of the BOLTZMANN equation. The fact that we set $\Sigma^{\text{MF}} = 0$ is equivalent to neglecting the real parts of the transfer matrix in the GLW approach in Chapter 4, cf. the remarks after Eqs. (4.42), (4.70), and (4.121).

Taking these considerations into account and inserting Eqs. (5.20) into the KB equations, we arrive at

$$G_{0,ac}^{-1}G_{cb}^{<}(x,k) = \frac{i}{2\lambda^2}\left[\Sigma_{ac}^{>}(x,k)G_{cb}^{<}(x,k) - \Sigma_{ac}^{<}(x,k)G_{cb}^{>}(x,k)\right] \\ + \frac{\hbar}{4\lambda^2}\left[\left\{\Sigma^{>}(x,k), G^{<}(x,k)\right\}_{\text{PB},ab} - \left\{\Sigma^{<}(x,k), G^{>}(x,k)\right\}_{\text{PB},ab}\right] , \tag{5.26a}$$

$$G_{ac}^{<}(x,k)\overleftarrow{G}_{0,cb}^{*-1} = \frac{i}{2\lambda^2}\left[G_{ac}^{>}(x,k)\Sigma_{cb}^{<}(x,k) - G_{ac}^{<}(x,k)\Sigma_{cb}^{>}(x,k)\right] \\ + \frac{\hbar}{4\lambda^2}\left[\left\{G^{>}(x,k), \Sigma^{<}(x,k)\right\}_{\text{PB},ab} - \left\{G^{<}(x,k), \Sigma^{>}(x,k)\right\}_{\text{PB},ab}\right] , \tag{5.26b}$$

where terms of third and higher orders in $\hbar$ have been omitted. In the following sections, we will use this form of the KB equations to derive kinetic equations for particles of spins 0, $1/2$, and 1.

5.3 Scalar fields

Let us consider Eqs. (5.26) for charged scalar fields; in that case the inverse free propagator in real space is simply given by $G_0^{-1}(x) = \Box + m^2/\hbar^2$, such that the respective WIGNER transforms read

$$\hbar^2 G_0^{-1} = m^2 - D^2 = -k^2 + m^2 - i\hbar k\cdot\partial + \frac{\hbar^2}{4}\Box \,, \tag{5.27a}$$

$$\hbar^2 \overleftarrow{G}_0^{*-1} = m^2 - \overleftarrow{D}^{*2} = -k^2 + m^2 + i\hbar k\cdot\overleftarrow{\partial} + \frac{\hbar^2}{4}\overleftarrow{\Box} \,. \tag{5.27b}$$

Multiplying Eqs. (5.26) by $\hbar^2$ and remembering that $\lambda = \sqrt{\hbar}$, we thus find

$$\begin{aligned}&\left(-k^2 + m^2 - i\hbar k\cdot\partial + \frac{\hbar^2}{4}\Box\right) G^<(x,k)\\ &= \frac{i\hbar}{2}\left[\Sigma^>(x,k)G^<(x,k) - \Sigma^<(x,k)G^>(x,k)\right]\\ &\quad + \frac{\hbar^2}{4}\left[\left\{\Sigma^>(x,k),G^<(x,k)\right\}_{\mathrm{PB}} - \left\{\Sigma^<(x,k),G^>(x,k)\right\}_{\mathrm{PB}}\right] \,,\end{aligned} \tag{5.28a}$$

$$\begin{aligned}&\left(-k^2 + m^2 + i\hbar k\cdot\partial + \frac{\hbar^2}{4}\Box\right) G^<(x,k)\\ &= \frac{i\hbar}{2}\left[G^>(x,k)\Sigma^<(x,k) - G^<(x,k)\Sigma^>(x,k)\right]\\ &\quad + \frac{\hbar^2}{4}\left[\left\{G^>(x,k),\Sigma^<(x,k)\right\}_{\mathrm{PB}} - \left\{G^<(x,k),\Sigma^>(x,k)\right\}_{\mathrm{PB}}\right] \,.\end{aligned} \tag{5.28b}$$

Taking the sum and difference of Eqs. (5.28) and truncating at first order in $\hbar$, we obtain

$$\left(k^2 - m^2\right) G^<(x,k) = 0 \,, \tag{5.29a}$$

$$k\cdot\partial G^<(x,k) = \frac{1}{2}\left[G^>(x,k)\Sigma^<(x,k) - G^<(x,k)\Sigma^>(x,k)\right] \,. \tag{5.29b}$$

Here we canceled a global factor of $i\hbar$ in Eq. (5.29b) and employed Eq. (5.14), which for scalar particles simply reads

$$\left\{G^{\lessgtr},\Sigma^{\gtrless}\right\}_{\mathrm{PB}} = -\left\{\Sigma^{\gtrless},G^{\lessgtr}\right\}_{\mathrm{PB}} \,. \tag{5.30}$$

Equation (5.29a) reveals that the WIGNER function $G^<$ is on shell up to first order in $\hbar$. Thus, we do not need to invoke Theorem 1 to evaluate the BOLTZMANN equation (5.29b) on the mass shell, since up to first order in $\hbar$ no off-shell contributions arise. Comparing Eq. (5.29b) to Eq. (3.50), we can identify the collision terms,

$$\frac{1}{2}\left[G^>(x,k)\Sigma^<(x,k) - G^<(x,k)\Sigma^>(x,k)\right] = -\hbar\,\mathrm{Im}\int \mathrm{d}^4v\, e^{-\frac{i}{\hbar}k\cdot v}\left\langle \widehat{\phi}^\dagger\left(x+\frac{v}{2}\right)\widehat{\rho}\left(x-\frac{v}{2}\right)\right\rangle \,. \tag{5.31}$$

Writing the lesser and greater GREEN's functions as

$$G^<(x,k) = 2\pi\hbar^2\delta(k^2-m^2)f(x,k) \,, \qquad G^>(x,k) = 2\pi\hbar^2\delta(k^2-m^2)\widetilde{f}(x,k) \,, \tag{5.32}$$

where we used that $\kappa \equiv 2/\hbar$ for scalar particles, and inserting the self-energies in the T-matrix approximation (5.21), we arrive at

$$\begin{aligned}k\cdot\partial f(x,k) = {}& \frac{1}{2}\int \mathrm{d}K'\,\mathrm{d}K_1\,\mathrm{d}K_2 (2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\mathcal{W}^{(0)}\\ &\times\left[f(x,k_1)f(x,k_2)\widetilde{f}(x,k)\widetilde{f}(x,k') - f(x,k)f(x,k')\widetilde{f}(x,k_1)\widetilde{f}(x,k_2)\right] \,,\end{aligned} \tag{5.33}$$

where the restriction to the mass shell is understood and we defined $\mathcal{W}^{(0)} := M^2/16$. The function $\widetilde{f} \equiv 1+f$ gives the BOSE enhancement factors, which is shown in Appendix B.2.2. As expected, our result agrees with the expectations from standard kinetic theory (4.3). When comparing to the result from the GLW approach (4.51), it becomes apparent that in the KB approach the effect of quantum statistics is retained rather easily.[3]

[3] These effects are also obtainable in the GLW approach, cf., e.g., Ref. [144]. However, the required amount of work is higher, since the expansion of the collision kernel in terms of WIGNER functions has to be extended to fourth order.

5.4 Spinor fields

Next we turn to the description of DIRAC fields, which is more complicated due to the internal structure of the fields and thus of the WIGNER functions. Considering our results from Sec. 4.4, we can already expect that the collision term will acquire nonlocal parts.

5.4.1 Structure of the equations

The first difference to the case of scalar fields lies in the inverse free propagator, which now takes the form $G_0^{-1}(x) = i\partial\!\!\!/ - m/\hbar$ in real space. Thus, the needed WIGNER transforms are given by

$$\hbar G_0^{-1} = D\!\!\!\!/ - m = k\!\!\!/ - m + \frac{i\hbar}{2}\partial\!\!\!/ \;, \tag{5.34a}$$

$$\hbar \overleftarrow{G}_0^{*-1} = \overleftarrow{D\!\!\!\!/}^* - m = k\!\!\!/ - m - \frac{i\hbar}{2}\overleftarrow{\partial\!\!\!/} \;. \tag{5.34b}$$

Omitting the spinor indices, the KB equations then read (using that $\lambda = 1$)

$$\left(D\!\!\!\!/ - m\right) G^<(x,k) = I_{\text{coll}} \;, \tag{5.35a}$$

$$G^<(x,k)\left(\overleftarrow{D\!\!\!\!/}^* - m\right) = \gamma^0 I_{\text{coll}}^\dagger \gamma^0 \;, \tag{5.35b}$$

where it is understood that $\overleftarrow{D\!\!\!\!/}^* := \gamma \cdot \overleftarrow{D}^*$ and we defined

$$\begin{aligned} I_{\text{coll}} := \frac{i\hbar}{2}&\left[\Sigma^>(x,k)G^<(x,k) - \Sigma^<(x,k)G^>(x,k)\right] \\ &+ \frac{\hbar^2}{4}\left[\left\{\Sigma^>(x,k), G^<(x,k)\right\}_{\text{PB}} - \left\{\Sigma^<(x,k), G^>(x,k)\right\}_{\text{PB}}\right] \;. \end{aligned} \tag{5.36}$$

Here we employed that $\gamma^0 (G^<)^\dagger \gamma^0 = G^<$ and similarly for the self-energy. Furthermore, we used the relation (5.14), which for DIRAC fermions takes the form

$$\left\{G^{\lessgtr}, \Sigma^{\gtrless}\right\}_{\text{PB}} = -\gamma^0 \left\{\Sigma^{\gtrless}, G^{\lessgtr}\right\}_{\text{PB}}^\dagger \gamma^0 \;. \tag{5.37}$$

Comparing Eq. (5.35a) to Eq. (3.75), we can connect the collision terms,

$$I_{\text{coll}} = -\hbar \int \mathrm{d}^4 v\, e^{-\frac{i}{\hbar}k\cdot v} \left\langle \widehat{\bar{\psi}}\left(x + \frac{v}{2}\right) \widehat{\rho}\left(x - \frac{v}{2}\right)\right\rangle \;. \tag{5.38}$$

Acting with $D\!\!\!\!/ + m$ on Eq. (5.35a) and with $\overleftarrow{D\!\!\!\!/}^* + m$ on Eq. (5.35b), we find

$$\left(D^2 - m^2\right) G^<(x,k) = (D\!\!\!\!/ + m) I_{\text{coll}} \;, \tag{5.39a}$$

$$\left(D^{*2} - m^2\right) G^<(x,k) = \gamma^0 \left[(D\!\!\!\!/ + m) I_{\text{coll}}\right]^\dagger \gamma^0 \;, \tag{5.39b}$$

Adding and subtracting these equations yields

$$\left(k^2 - m^2 - \frac{\hbar^2}{4}\Box\right) G^<(x,k) = \frac{1}{2}\left\{(D\!\!\!\!/ + m) I_{\text{coll}} + \gamma^0 \left[(D\!\!\!\!/ + m) I_{\text{coll}}\right]^\dagger \gamma^0\right\} \;, \tag{5.40a}$$

$$i\hbar k \cdot \partial G^<(x,k) = \frac{1}{2}\left\{(D\!\!\!\!/ + m) I_{\text{coll}} - \gamma^0 \left[(D\!\!\!\!/ + m) I_{\text{coll}}\right]^\dagger \gamma^0\right\} \;. \tag{5.40b}$$

Since we want to compute the distribution function in extended phase space, we make use of the definition (3.85) (together with the fact that $\kappa = 1$ for DIRAC fields, i.e., $G^< \equiv -W$) to obtain

$$\left(k^2 - m^2 - \frac{\hbar^2}{4}\Box\right) \mathfrak{f}(x,k,\mathfrak{s}) = -\frac{1}{2}\text{Re}\,\text{Tr}\left[(\mathbb{1} + \gamma_5 \mathfrak{s}\!\!\!/)(D\!\!\!\!/ + m) I_{\text{coll}}\right] \;, \tag{5.41a}$$

$$\hbar k \cdot \partial \mathfrak{f}(x,k,\mathfrak{s}) = -\frac{1}{2}\text{Im}\,\text{Tr}\left[(\mathbb{1} + \gamma_5 \mathfrak{s}\!\!\!/)(D\!\!\!\!/ + m) I_{\text{coll}}\right] \;. \tag{5.41b}$$

Structure of the Wigner function

Before being able to evaluate these equations up to first order in the Planck constant, we need to clarify the structure of the Wigner functions appearing inside the collision integral I_{coll}. At this point we introduce another approximation (that was also employed in the GLW approach), namely neglecting collisional contributions to the Wigner functions that appear inside the collision terms themselves. This is justified since the self-energy (5.21) is of second order in the coupling, such that collisional contributions to the Wigner functions inside $\Sigma^{\gtrless}$ would be at least of fourth order. Practically, this means that the structure of the Wigner function is determined by (a subset of) Eqs. (3.79) and (3.80) with the right-hand sides set to zero. As before, we decompose the Wigner transforms of the greater and lesser Green's functions in terms of the Clifford algebra,

$$G^{<} = \frac{1}{4}\left(\mathcal{F} + i\gamma_5\mathcal{P} + \not{\mathcal{V}} + \gamma_5\not{\mathcal{A}} + \frac{1}{2}\sigma_{\mu\nu}\mathcal{S}^{\mu\nu}\right) , \tag{5.42a}$$

$$G^{>} = \frac{1}{4}\left(\widetilde{\mathcal{F}} + i\gamma_5\widetilde{\mathcal{P}} + \widetilde{\not{\mathcal{V}}} + \gamma_5\widetilde{\not{\mathcal{A}}} + \frac{1}{2}\sigma_{\mu\nu}\widetilde{\mathcal{S}}^{\mu\nu}\right) . \tag{5.42b}$$

Then, up to first order in $\hbar$, the components of the Wigner function which are not independent read

$$\mathcal{P}(x,k) \simeq 0 , \quad \mathcal{V}^\mu(x,k) \simeq \frac{k^\mu}{m}\mathcal{F}(x,k) , \quad \mathcal{S}^{\mu\nu}(x,k) \simeq \frac{\hbar}{2m^2}\partial^{[\mu}k^{\nu]}\mathcal{F}(x,k) - \frac{1}{m}\epsilon^{\mu\nu\alpha\beta}k_\alpha\mathcal{A}_\beta(x,k) , \tag{5.43a}$$

$$\widetilde{\mathcal{P}}(x,k) \simeq 0 , \quad \widetilde{\mathcal{V}}^\mu(x,k) \simeq \frac{k^\mu}{m}\widetilde{\mathcal{F}}(x,k) , \quad \widetilde{\mathcal{S}}^{\mu\nu}(x,k) \simeq \frac{\hbar}{2m^2}\partial^{[\mu}k^{\nu]}\widetilde{\mathcal{F}}(x,k) - \frac{1}{m}\epsilon^{\mu\nu\alpha\beta}k_\alpha\widetilde{\mathcal{A}}_\beta(x,k) . \tag{5.43b}$$

We may thus write the greater and lesser Green's functions as

$$G^{\gtrless}(x,k) \simeq G^{\gtrless}_{\text{qc}}(x,k) + G^{\gtrless}_{\nabla}(x,k) , \tag{5.44}$$

where we introduced the "quasiclassical" contributions

$$G^{<}_{\text{qc}}(x,k) \coloneqq \frac{1}{2}\Lambda^+(k)\left[\mathcal{F}(x,k) + \gamma_5\not{\mathcal{A}}(x,k)\right] , \tag{5.45a}$$

$$G^{>}_{\text{qc}}(x,k) \coloneqq \frac{1}{2}\Lambda^+(k)\left[\widetilde{\mathcal{F}}(x,k) + \gamma_5\widetilde{\not{\mathcal{A}}}(x,k)\right] , \tag{5.45b}$$

as well as the "gradient" terms

$$G^{<}_{\nabla}(x,k) \coloneqq \frac{\hbar}{8m^2}\sigma^{\mu\nu}k_\nu\partial_\mu\mathcal{F}(x,k) , \tag{5.46a}$$

$$G^{>}_{\nabla}(x,k) \coloneqq \frac{\hbar}{8m^2}\sigma^{\mu\nu}k_\nu\partial_\mu\widetilde{\mathcal{F}}(x,k) . \tag{5.46b}$$

The quantities $\Lambda^+(k) = (\not{k} + m)/(2m)$ denote positive-energy projectors and have been introduced in Eq. (4.57). Furthermore, it is important to note that the quasiclassical terms have contributions both at zeroth and first order in $\hbar$, while the gradient terms are of first order only. From Eq. (5.40a) we see that off-shell effects are either of second order in $\hbar$ or of collisional origin, such that we may approximate $k_1^2 \simeq k_2^2 \simeq k'^2 \simeq m^2$ inside the collision integrals. Translating the components of the Wigner function into extended phase space via Eq. (3.89), we can express the quasiclassical contributions as

$$G^{<}_{\text{qc}}(x,k) = -4m\pi\hbar\delta(k^2 - m^2)\int \mathrm{d}S(k)h(k,\mathfrak{s})f(x,k,\mathfrak{s}) , \tag{5.47a}$$

$$G^{>}_{\text{qc}}(x,k) = 4m\pi\hbar\delta(k^2 - m^2)\int \mathrm{d}S(k)h(k,\mathfrak{s})\widetilde{f}(x,k,\mathfrak{s}) , \tag{5.47b}$$

where the quantity $h(k,\mathfrak{s}) = \frac{1}{2}(\mathbb{1} + \gamma_5 \not{\mathfrak{s}})\Lambda^+(k)$ is known from Eq. (4.80) and the prefactors are in accordance with earlier definitions, cf. Eq. (3.100). Note that the minus sign in $G^<_{\text{qc}}$ is needed because of the definition (5.1b). The gradient contributions on the other hand read in extended phase space

$$G^<_\nabla(x,k) = -4m\pi\hbar\delta(k^2 - m^2)\frac{\hbar}{8m^2}\sigma^{\mu\nu}k_\nu\partial_\mu \int \mathrm{d}S(k)f(x,k,\mathfrak{s}) \;, \tag{5.48a}$$

$$G^>_\nabla(x,k) = 4m\pi\hbar\delta(k^2 - m^2)\frac{\hbar}{8m^2}\sigma^{\mu\nu}k_\nu\partial_\mu \int \mathrm{d}S(k)\widetilde{f}(x,k,\mathfrak{s}) \;, \tag{5.48b}$$

where we again took care of the minus sign from fermionic statistics.

Structure of the self-energies

Since the self-energies (5.21) contain Green's functions, we can also split them in the same way as shown before,

$$\Sigma^{\gtrless}(x,k) \simeq \Sigma^{\gtrless}_{\text{qc}}(x,k) + \Sigma^{\gtrless}_\nabla(x,k) \;. \tag{5.49}$$

Here we defined the quasiclassical contribution as

$$\begin{aligned}\Sigma^{\gtrless}_{\text{qc},\alpha\beta}(x,k) \coloneqq \frac{1}{2}\int &\frac{\mathrm{d}^4k_1}{(2\pi\hbar)^4}\frac{\mathrm{d}^4k_2}{(2\pi\hbar)^4}\frac{\mathrm{d}^4k'}{(2\pi\hbar)^4}(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}\\ &\times G^{\gtrless}_{\text{qc},\alpha_1\beta_1}(x,k_1)G^{\gtrless}_{\text{qc},\alpha_2\beta_2}(x,k_2)G^{\lessgtr}_{\text{qc},\beta'\alpha'}(x,k') \;,\end{aligned} \tag{5.50}$$

and the gradient part reads

$$\begin{aligned}\Sigma^{\gtrless}_{\nabla,\alpha\beta}(x,k) \coloneqq \frac{1}{2}\int &\frac{\mathrm{d}^4k_1}{(2\pi\hbar)^4}\frac{\mathrm{d}^4k_2}{(2\pi\hbar)^4}\frac{\mathrm{d}^4k'}{(2\pi\hbar)^4}(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}\\ &\times \Big[G^{\gtrless}_{\nabla,\alpha_1\beta_1}(x,k_1)G^{\gtrless}_{\text{qc},\alpha_2\beta_2}(x,k_2)G^{\lessgtr}_{\text{qc},\beta'\alpha'}(x,k')\\ &\quad + G^{\gtrless}_{\text{qc},\alpha_1\beta_1}(x,k_1)G^{\gtrless}_{\nabla,\alpha_2\beta_2}(x,k_2)G^{\lessgtr}_{\text{qc},\beta'\alpha'}(x,k')\\ &\quad + G^{\gtrless}_{\text{qc},\alpha_1\beta_1}(x,k_1)G^{\gtrless}_{\text{qc},\alpha_2\beta_2}(x,k_2)G^{\lessgtr}_{\nabla,\beta'\alpha'}(x,k')\Big] \;.\end{aligned} \tag{5.51}$$

Note that we could restrict ourselves to terms that are linear in the gradient contributions to the Green's functions, since they are already of first order in $\hbar$. Employing Eqs. (5.47) and (5.48), we can express the quasiclassical parts of the lesser and greater self-energies as

$$\begin{aligned}\Sigma^<_{\text{qc},\alpha\beta}(x,k) = \frac{m^3}{2}\int &\mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}\\ &\times h_{\alpha_1\beta_1}(k_1,\mathfrak{s}_1)h_{\alpha_2\beta_2}(k_2,\mathfrak{s}_2)h_{\beta'\alpha'}(k',\mathfrak{s}')f(x,k_1,\mathfrak{s}_1)f(x,k_2,\mathfrak{s}_2)\widetilde{f}(x,k',\mathfrak{s}')\end{aligned} \tag{5.52a}$$

and

$$\begin{aligned}\Sigma^>_{\text{qc},\alpha\beta}(x,k) = -\frac{m^3}{2}\int &\mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}\\ &\times h_{\alpha_1\beta_1}(k_1,\mathfrak{s}_1)h_{\alpha_2\beta_2}(k_2,\mathfrak{s}_2)h_{\beta'\alpha'}(k',\mathfrak{s}')\widetilde{f}(x,k_1,\mathfrak{s}_1)\widetilde{f}(x,k_2,\mathfrak{s}_2)f(x,k',\mathfrak{s}') \;,\end{aligned} \tag{5.52b}$$

respectively. The gradient parts on the other hand take the forms

$$\begin{aligned}\Sigma^<_{\nabla,\alpha\beta}(x,k) = \frac{\hbar}{8m^2}\frac{m^3}{2}\int &\mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}\\ \times\Big\{&\sigma^{\mu\nu}_{\alpha_1\beta_1}k_{1,\nu}h_{\alpha_2\beta_2}(k_2,\mathfrak{s}_2)h_{\beta'\alpha'}(k',\mathfrak{s}')\Big[\partial_\mu f(x,k_1,\mathfrak{s}_1)\Big]f(x,k_2,\mathfrak{s}_2)\widetilde{f}(x,k',\mathfrak{s}')\\ &+ h_{\alpha_1\beta_1}(k_1,\mathfrak{s}_1)\sigma^{\mu\nu}_{\alpha_2\beta_2}k_{2,\nu}h_{\beta'\alpha'}(k',\mathfrak{s}')f(x,k_1,\mathfrak{s}_1)\Big[\partial_\mu f(x,k_2,\mathfrak{s}_2)\Big]\widetilde{f}(x,k',\mathfrak{s}')\\ &+ h_{\alpha_1\beta_1}(k_1,\mathfrak{s}_1)h_{\alpha_2\beta_2}(k_2,\mathfrak{s}_2)\sigma^{\mu\nu}_{\beta'\alpha'}k'_\nu f(x,k_1,\mathfrak{s}_1)f(x,k_2,\mathfrak{s}_2)\Big[\partial_\mu\widetilde{f}(x,k',\mathfrak{s}')\Big]\Big\} \;,\end{aligned} \tag{5.53a}$$

as well as

$$\Sigma^{>}_{\nabla,\alpha\beta}(x,k) = -\frac{\hbar}{8m^2}\frac{m^3}{2}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}$$
$$\times\Big\{\sigma^{\mu\nu}_{\alpha_1\beta_1}k_{1,\nu}h_{\alpha_2\beta_2}(k_2,\mathfrak{s}_2)h_{\beta'\alpha'}(k',\mathfrak{s}')\left[\partial_\mu\widetilde{f}(x,k_1,\mathfrak{s}_1)\right]\widetilde{f}(x,k_2,\mathfrak{s}_2)f(x,k',\mathfrak{s}')$$
$$+h_{\alpha_1\beta_1}(k_1,\mathfrak{s}_1)\sigma^{\mu\nu}_{\alpha_2\beta_2}k_{2,\nu}h_{\beta'\alpha'}(k',\mathfrak{s}')\widetilde{f}(x,k_1,\mathfrak{s}_1)\left[\partial_\mu\widetilde{f}(x,k_2,\mathfrak{s}_2)\right]f(x,k',\mathfrak{s}')$$
$$+h_{\alpha_1\beta_1}(k_1,\mathfrak{s}_1)h_{\alpha_2\beta_2}(k_2,\mathfrak{s}_2)\sigma^{\mu\nu}_{\beta'\alpha'}k'_\nu\widetilde{f}(x,k_1,\mathfrak{s}_1)\widetilde{f}(x,k_2,\mathfrak{s}_2)\left[\partial_\mu f(x,k',\mathfrak{s}')\right]\Big\}\ . \tag{5.53b}$$

With these expressions, we are now able to take on the Boltzmann equation in a structured manner.

5.4.2 Evaluating the kinetic equation

Considering Eq. (5.41b) in conjunction with the decompositions (5.44) and (5.49), we can split its right-hand side into four contributions,

$$4m\pi\hbar\delta(k^2-m^2)k\cdot\partial f(x,k,\mathfrak{s}) = \mathcal{I}_{\mathrm{qc}}(x,k,\mathfrak{s})+\mathcal{I}_{\partial}(x,k,\mathfrak{s})+\mathcal{I}_{\nabla}(x,k,\mathfrak{s})+\mathcal{I}_{\mathrm{PB}}(x,k,\mathfrak{s})\ , \tag{5.54}$$

where we already discarded the off-shell contributions on the left-hand side due to Theorem 1.[4] Here, the first term contains all quasiclassical contributions,

$$\mathcal{I}_{\mathrm{qc}}(x,k,\mathfrak{s}) := \frac{1}{4}\mathrm{Re}\,\mathrm{Tr}\left\{(\mathbb{1}+\gamma_5\not{\mathfrak{s}})(\not{k}+m)\left[\Sigma^{<}_{\mathrm{qc}}(x,k)G^{>}_{\mathrm{qc}}(x,k)-\Sigma^{>}_{\mathrm{qc}}(x,k)G^{<}_{\mathrm{qc}}(x,k)\right]\right\}$$
$$= m\mathrm{Re}\,\mathrm{Tr}\left\{h(k,\mathfrak{s})\left[\Sigma^{<}_{\mathrm{qc}}(x,k)G^{>}_{\mathrm{qc}}(x,k)-\Sigma^{>}_{\mathrm{qc}}(x,k)G^{<}_{\mathrm{qc}}(x,k)\right]\right\}\ , \tag{5.55}$$

where the real part arises because of the imaginary unit in the collision term (5.36) and we employed the definition of $h(k,\mathfrak{s})$. Note that $\mathcal{I}_{\mathrm{qc}}$ contains parts of zeroth as well as first order in $\hbar$. Comparing $\mathcal{I}_{\mathrm{qc}}$ to the collision term (5.36), there is a global sign change due to identifying $G^{<}$ with the negative of the distribution function, cf. Eqs. (5.41). The second term on the right-hand side of Eq. (5.54), which is of first order in $\hbar$, denotes the contribution from the derivative contained in the operator $\not{D}$,

$$\mathcal{I}_{\partial}(x,k,\mathfrak{s}) := -\frac{\hbar}{8}\mathrm{Im}\,\mathrm{Tr}\left\{(\mathbb{1}+\gamma_5\not{\mathfrak{s}})\not{\partial}\left[\Sigma^{<}_{\mathrm{qc}}(x,k)G^{>}_{\mathrm{qc}}(x,k)-\Sigma^{>}_{\mathrm{qc}}(x,k)G^{<}_{\mathrm{qc}}(x,k)\right]\right\}\ , \tag{5.56}$$

while the third term (which is also of first order in $\hbar$) collects the gradient contributions from the propagators inside the collision terms [cf. Eq. (5.44)],

$$\mathcal{I}_{\nabla}(x,k,\mathfrak{s}) := m\mathrm{Re}\,\mathrm{Tr}\big\{h(k,\mathfrak{s})\big[\Sigma^{<}_{\mathrm{qc}}(x,k)G^{>}_{\nabla}(x,k)-\Sigma^{>}_{\mathrm{qc}}(x,k)G^{<}_{\nabla}(x,k)$$
$$+\Sigma^{<}_{\nabla}(x,k)G^{>}_{\mathrm{qc}}(x,k)-\Sigma^{>}_{\nabla}(x,k)G^{<}_{\mathrm{qc}}(x,k)\big]\big\}\ . \tag{5.57}$$

The fourth term finally contains the contributions of the Poisson-brackets,

$$\mathcal{I}_{\mathrm{PB}}(x,k,\mathfrak{s}) := \frac{m\hbar}{2}\mathrm{Im}\,\mathrm{Tr}\Big\{h(k,\mathfrak{s})\Big[\big\{\Sigma^{<}_{\mathrm{qc}}(x,k),G^{>}_{\mathrm{qc}}(x,k)\big\}_{\mathrm{PB}}-\big\{\Sigma^{>}_{\mathrm{qc}}(x,k),G^{<}_{\mathrm{qc}}(x,k)\big\}_{\mathrm{PB}}\Big]\Big\}$$
$$= \frac{m\hbar}{2}\mathrm{Im}\,\mathrm{Tr}\Big(h(k,\mathfrak{s})\Big\{\left[\partial_\mu\Sigma^{<}_{\mathrm{qc}}(x,k)\right]\left[\partial^\mu_k G^{>}_{\mathrm{qc}}(x,k)\right]-\left[\partial^\mu_k\Sigma^{<}_{\mathrm{qc}}(x,k)\right]\left[\partial_\mu G^{>}_{\mathrm{qc}}(x,k)\right]$$
$$-\left[\partial_\mu\Sigma^{>}_{\mathrm{qc}}(x,k)\right]\left[\partial^\mu_k G^{<}_{\mathrm{qc}}(x,k)\right]+\left[\partial^\mu_k\Sigma^{>}_{\mathrm{qc}}(x,k)\right]\left[\partial_\mu G^{<}_{\mathrm{qc}}(x,k)\right]\Big\}\Big)\ , \tag{5.58}$$

where we could neglect all gradient terms, as they would be at least of second order in $\hbar$. In the following, we will compute these four contributions, with the result that the quasiclassical one $\mathcal{I}_{\mathrm{qc}}$ gives the local collisions, while the terms $\mathcal{I}_{\partial}$, $\mathcal{I}_{\nabla}$, and $\mathcal{I}_{\mathrm{PB}}$ are responsible for the nonlocal parts.

[4] Actually, when considering the mass-shell equation, one can show that the off-shell terms are zero up to first order in $\hbar$, as was done in Ref. [46].

Quasiclassical terms

The quasiclassical terms are obtained straightforwardly by inserting Eqs. (5.47) and (5.52) into Eq. (5.55), giving the known form of the local on-shell collision term,

$$\mathcal{I}_{\text{qc}}(x,k,\mathfrak{s}) = 4m\pi\hbar\delta(k^2-m^2)\frac{1}{2}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,\mathrm{d}\bar{S}(k)(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\mathcal{W}^{(1/2)} \times\left(f_1 f_2 \widetilde{f}'\widetilde{f} - \widetilde{f}_1\widetilde{f}_2 f' f\right) , \tag{5.59}$$

where we abbreviated

$$f_1 \coloneqq f(x,k_1,\mathfrak{s}_1)\,, \qquad f_2 \coloneqq f(x,k_2,\mathfrak{s}_2)\,, \qquad f' \coloneqq f(x,k',\mathfrak{s}')\,, \qquad f \coloneqq f(x,k,\bar{\mathfrak{s}})\,. \tag{5.60}$$

Note that, due to the WIGNER functions always including one spin-space integral, the distribution function f does not depend on $\mathfrak{s}$, but rather $\bar{\mathfrak{s}}$. The transition rate reads

$$\begin{aligned}\mathcal{W}^{(1/2)} &= m^4\mathrm{Re}\left[M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}h_{\alpha_1\beta_1}(k_1,\mathfrak{s}_1)h_{\alpha_2\beta_2}(k_2,\mathfrak{s}_2)h_{\beta'\alpha'}(k',\mathfrak{s}')h_{\beta\gamma}(k,\bar{\mathfrak{s}})h_{\gamma\alpha}(k,\mathfrak{s})\right]\\ &= \frac{m^4}{2}M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}h_{\alpha_1\beta_1}(k_1,\mathfrak{s}_1)h_{\alpha_2\beta_2}(k_2,\mathfrak{s}_2)h_{\beta'\alpha'}(k',\mathfrak{s}')\\ &\quad\times\left[h_{\beta\gamma}(k,\bar{\mathfrak{s}})h_{\gamma\alpha}(k,\mathfrak{s})+h_{\beta\gamma}(k,\mathfrak{s})h_{\gamma\alpha}(k,\bar{\mathfrak{s}})\right] ,\end{aligned} \tag{5.61}$$

which agrees with our result from the GLW approach (4.97). Here we used that $h^\dagger = \gamma^0 h\gamma^0$, together with the symmetries of M. In particular, besides the antisymmetries upon changes in the first and second pair of indices due to fermionic statistics, we assume that

$$\gamma^0_{\alpha\beta}\gamma^0_{\alpha'\beta'}M^*_{\beta\beta'\beta_1\beta_2}\gamma^0_{\beta_1\alpha_1}\gamma^0_{\beta_2\alpha_2} = M_{\alpha_1\alpha_2\alpha\alpha'}\,, \tag{5.62}$$

cf. Eq. (4.73).

Gradient terms

Next we take on the gradient contributions $\mathcal{I}_\partial$ and $\mathcal{I}_\nabla$ to the collision kernel. Upon inserting the expressions for $\Sigma^\gtrless$ and $G^\gtrless$, the former becomes

$$\mathcal{I}_\partial(x,k,\mathfrak{s}) = 4m\pi\hbar\delta(k^2-m^2)\frac{1}{2}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,\mathrm{d}\bar{S}(k)(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2) \times\mathcal{T}^{(a)}_\mu\partial^\mu\left(f_1 f_2 \widetilde{f}'\widetilde{f} - \widetilde{f}_1\widetilde{f}_2 f' f\right) , \tag{5.63}$$

where we defined

$$\begin{aligned}\mathcal{T}^{(a)}_\mu &\coloneqq -\frac{\hbar m^3}{8}\mathrm{Im}\left[(\mathbb{1}+\gamma_5\bar{\mathfrak{s}}\!\!\!/)_{\gamma\delta}(\gamma_\mu)_{\delta\alpha}M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}\right.\\ &\qquad\left.\times h_{\alpha_1\beta_1}(k_1,\mathfrak{s}_1)h_{\alpha_2\beta_2}(k_2,\mathfrak{s}_2)h_{\beta'\alpha'}(k',\mathfrak{s}')h_{\beta\gamma}(k,\bar{\mathfrak{s}})\right]\\ &= \frac{i\hbar m^3}{16}M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}h_{\alpha_1\beta_1}(k_1,\mathfrak{s}_1)h_{\alpha_2\beta_2}(k_2,\mathfrak{s}_2)h_{\beta'\alpha'}(k',\mathfrak{s}')\left[h(k,\mathfrak{s}),\gamma_\mu\right]_{\beta\alpha}\,.\end{aligned} \tag{5.64}$$

Here we used the symmetries of M in the second step, together with the fact that we can approximate $h(k,\bar{\mathfrak{s}}) \simeq \frac{1}{2}\Lambda^+(k)$ since the $\bar{\mathfrak{s}}$-dependent part of $f(x,k,\bar{\mathfrak{s}})$ is at least of first order in $\hbar$, such that the neglected terms are of order $\mathcal{O}(\hbar^2)$. Furthermore, we employed that $\gamma_5\bar{\mathfrak{s}}\!\!\!/$ and Λ^+ commute. Note that, comparing to Eq. (4.99d), we have $\mathcal{T}^{(a)}_\mu = -\frac{1}{2}\mathcal{W}^{(1/2)}\Delta_\mu$.

The second gradient contribution $\mathcal{I}_\nabla$ becomes

$$\mathcal{I}_\nabla(x,k,\mathfrak{s}) = 4m\pi\hbar\delta(k^2-m^2)\frac{1}{2}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,\mathrm{d}\bar{S}(k)(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2) \\ \times\Big\{\mathcal{T}_1^{\mu\nu}k_{1,\nu}\left[(\partial_\mu f_1)f_2\widetilde{f}'\widetilde{f}-(\partial_\mu\widetilde{f}_1)\widetilde{f}_2 f' f\right]+\mathcal{T}_2^{\mu\nu}k_{2,\nu}\left[f_1(\partial_\mu f_2)\widetilde{f}'\widetilde{f}-\widetilde{f}_1(\partial_\mu\widetilde{f}_2)f'f\right] \\ +\mathcal{T}'^{\mu\nu}k'_\nu\left[f_1f_2(\partial_\mu\widetilde{f}')\widetilde{f}-\widetilde{f}_1\widetilde{f}_2(\partial_\mu f')f\right]+\mathcal{T}^{\mu\nu}k_\nu\left[f_1f_2\widetilde{f}'(\partial_\mu\widetilde{f})-\widetilde{f}_1\widetilde{f}_2f'(\partial_\mu f)\right]\Big\}\,, \tag{5.65}$$

with the quantities

$$\mathcal{T}_1^{\mu\nu} := \frac{\hbar m^2}{8}\mathrm{Re}\left[h_{\gamma\alpha}(k,\mathfrak{s})M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}\sigma^{\mu\nu}_{\alpha_1\beta_1}h_{\alpha_2\beta_2}(k_2,\mathfrak{s}_2)h_{\beta'\alpha'}(k',\mathfrak{s}')h_{\beta\gamma}(k,\bar{\mathfrak{s}})\right]\,, \tag{5.66a}$$

$$\mathcal{T}_2^{\mu\nu} := \frac{\hbar m^2}{8}\mathrm{Re}\left[h_{\gamma\alpha}(k,\mathfrak{s})M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}h_{\alpha_1\beta_1}(k_1,\mathfrak{s}_1)\sigma^{\mu\nu}_{\alpha_2\beta_2}h_{\beta'\alpha'}(k',\mathfrak{s}')h_{\beta\gamma}(k,\bar{\mathfrak{s}})\right]\,, \tag{5.66b}$$

$$\mathcal{T}'^{\mu\nu} := \frac{\hbar m^2}{8}\mathrm{Re}\left[h_{\gamma\alpha}(k,\mathfrak{s})M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}h_{\alpha_1\beta_1}(k_1,\mathfrak{s}_1)h_{\alpha_2\beta_2}(k_2,\mathfrak{s}_2)\sigma^{\mu\nu}_{\beta'\alpha'}h_{\beta\gamma}(k,\bar{\mathfrak{s}})\right]\,, \tag{5.66c}$$

$$\mathcal{T}^{\mu\nu} := \frac{\hbar m^2}{8}\mathrm{Re}\left[h_{\gamma\alpha}(k,\mathfrak{s})M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}h_{\alpha_1\beta_1}(k_1,\mathfrak{s}_1)h_{\alpha_2\beta_2}(k_2,\mathfrak{s}_2)h_{\beta'\alpha'}(k',\mathfrak{s}')\sigma^{\mu\nu}_{\beta\gamma}\right]\,. \tag{5.66d}$$

In order to simplify these expressions, we note that

$$\sigma^{\mu\nu}k_{1,\nu} = im\left[\gamma^\mu,\Lambda^+(k_1)\right]\,,\quad \sigma^{\mu\nu}k_{2,\nu} = im\left[\gamma^\mu,\Lambda^+(k_2)\right]\,,\quad \sigma^{\mu\nu}k'_\nu = im\left[\gamma^\mu,\Lambda^+(k')\right]\,. \tag{5.67}$$

Since the contributions of the distribution functions that are proportional to $\mathfrak{s}_1$, $\mathfrak{s}_2$, and $\mathfrak{s}'$ are of first order in $\hbar$, we may replace

$$\sigma^{\mu\nu}k_{1,\nu} \simeq im\left[\gamma^\mu,h(k_1,\mathfrak{s}_1)\right]\,,\quad \sigma^{\mu\nu}k_{2,\nu} \simeq im\left[\gamma^\mu,h(k_2,\mathfrak{s}_2)\right]\,,\quad \sigma^{\mu\nu}k'_\nu \simeq im\left[\gamma^\mu,h(k',\mathfrak{s}')\right] \tag{5.68}$$

inside the collision terms. To simplify Eq. (5.66d), we compute

$$\sigma^{\mu\nu}k_\nu h(k,\mathfrak{s}) = i\left(\gamma^\mu \not{k} - k^\mu\right)h(k,\mathfrak{s}) = im\gamma^\mu h(k,\mathfrak{s}) - ik^\mu h(k,\mathfrak{s})\,. \tag{5.69}$$

Inserting these results into Eqs. (5.66) and using the symmetries of M, we obtain

$$\mathcal{T}_1^{\mu\nu}k_{1,\nu} = -\frac{i\hbar m^3}{8}M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}\left[h(k_1,\mathfrak{s}_1),\gamma^\mu\right]_{\alpha_1\beta_1}h_{\alpha_2\beta_2}(k_2,\mathfrak{s}_2)h_{\beta'\alpha'}(k',\mathfrak{s}')h_{\beta\alpha}(k,\mathfrak{s})\,, \tag{5.70a}$$

$$\mathcal{T}_2^{\mu\nu}k_{2,\nu} = -\frac{i\hbar m^3}{8}M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}h_{\alpha_1\beta_1}(k_1,\mathfrak{s}_1)\left[h(k_2,\mathfrak{s}_2),\gamma^\mu\right]_{\alpha_2\beta_2}h_{\beta'\alpha'}(k',\mathfrak{s}')h_{\beta\alpha}(k,\mathfrak{s})\,, \tag{5.70b}$$

$$\mathcal{T}'^{\mu\nu}k'_\nu = -\frac{i\hbar m^3}{8}M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}h_{\alpha_1\beta_1}(k_1,\mathfrak{s}_1)h_{\alpha_2\beta_2}(k_2,\mathfrak{s}_2)\left[h(k',\mathfrak{s}'),\gamma^\mu\right]_{\beta'\alpha'}h_{\beta\alpha}(k,\mathfrak{s})\,, \tag{5.70c}$$

$$\mathcal{T}^{\mu\nu}k_\nu = -\frac{i\hbar m^3}{16}M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}h_{\alpha_1\beta_1}(k_1,\mathfrak{s}_1)h_{\alpha_2\beta_2}(k_2,\mathfrak{s}_2)h_{\beta'\alpha'}(k',\mathfrak{s}')\left[h(k,\mathfrak{s}),\gamma^\mu\right]_{\beta\alpha}\,. \tag{5.70d}$$

Comparing these expressions to the results (4.99) from the GLW approach, we find

$$\mathcal{T}_1^{\mu\nu}k_{1,\nu} = \mathcal{W}^{(1/2)}\Delta_1^\mu\,,\quad \mathcal{T}_2^{\mu\nu}k_{2,\nu} = \mathcal{W}^{(1/2)}\Delta_2^\mu\,,\quad \mathcal{T}'^{\mu\nu}k'_\nu = \mathcal{W}^{(1/2)}\Delta'^\mu\,,\quad \mathcal{T}^{\mu\nu}k_\nu = \frac{1}{2}\mathcal{W}^{(1/2)}\Delta^\mu\,. \tag{5.71}$$

Poisson brackets

The final contribution to the collision term is given by the Poisson-bracket terms, where we have to evaluate spacetime and momentum derivatives acting on the Green's functions and the self-energies, cf. Eq. (5.13). In order to compute the momentum derivatives acting on the (quasiclassical) Green's functions, we note that

$$\partial_k^\mu h(k,\mathfrak{s}) = \frac{1}{4m}(\mathbb{1}+\gamma_5\not{\mathfrak{s}})\gamma^\mu\,, \tag{5.72}$$

which yields

$$\partial_k^\mu G^<_{\rm qc}(x,k) \simeq -4m\pi\hbar\delta(k^2-m^2)\int \mathrm{d}S(k)\frac{1}{2}\left[\frac{\gamma^\mu}{2m}+\Lambda^+(k)\partial_k^\mu\right] f(x,k,\mathfrak{s})\;, \tag{5.73a}$$

$$\partial_k^\mu G^>_{\rm qc}(x,k) \simeq 4m\pi\hbar\delta(k^2-m^2)\int \mathrm{d}S(k)\frac{1}{2}\left[\frac{\gamma^\mu}{2m}+\Lambda^+(k)\partial_k^\mu\right] \widetilde{f}(x,k,\mathfrak{s})\;, \tag{5.73b}$$

where we used again that $f(x,k,\mathfrak{s})$ is spin-independent and on shell up to terms of order $\mathcal{O}(\hbar)$. The momentum derivatives applied to $\Sigma^{\gtrless}_{\rm qc}$ only act on the momentum-conserving delta functions. As we already did in the GLW approach, we rewrite them as derivatives with respect to k' and perform integration by parts. Then, we find

$$\begin{aligned}\partial_k^\mu \Sigma^<_{{\rm qc},\alpha\beta}(x,k) = &-\frac{m^3}{2}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}\\ &\times h_{\alpha_1\beta_1}(k_1,\mathfrak{s}_1)h_{\alpha_2\beta_2}(k_2,\mathfrak{s}_2)f_1f_2\frac{1}{2}\left[\frac{\gamma^\mu}{2m}+\Lambda^+(k')\partial_{k'}^\mu\right]_{\beta'\alpha'}\widetilde{f}'\;,\end{aligned} \tag{5.74a}$$

$$\begin{aligned}\partial_k^\mu \Sigma^>_{{\rm qc},\alpha\beta}(x,k) = &-\frac{m^3}{2}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}\\ &\times h_{\alpha_1\beta_1}(k_1,\mathfrak{s}_1)h_{\alpha_2\beta_2}(k_2,\mathfrak{s}_2)\widetilde{f}_1\widetilde{f}_2\frac{1}{2}\left[\frac{\gamma^\mu}{2m}+\Lambda^+(k')\partial_{k'}^\mu\right]_{\beta'\alpha'}f'\;.\end{aligned} \tag{5.74b}$$

Inserting these expressions into the Poisson-bracket contribution to the collision term (5.58), we obtain

$$\begin{aligned}\mathcal{I}_{\rm PB}(x,k,\mathfrak{s}) = &\,4m\pi\hbar\delta(k^2-m^2)\frac{1}{2}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,\mathrm{d}\bar{S}(k)(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\\ &\times\Big\{\mathcal{T}^{(b)}_\mu\left[(\partial^\mu f_1f_2\widetilde{f}')\widetilde{f}-(\partial^\mu\widetilde{f}_1\widetilde{f}_2f')f\right]+\mathcal{T}^{(c)}_\mu\left[f_1f_2\widetilde{f}'(\partial^\mu\widetilde{f})-\widetilde{f}_1\widetilde{f}_2f'(\partial^\mu f)\right]\\ &+\overline{\mathcal{T}}\left[(\partial_\mu f_1f_2\widetilde{f}')\partial_k^\mu\widetilde{f}-(\partial_\mu\widetilde{f}_1\widetilde{f}_2f')\partial_k^\mu f+f_1f_2(\partial_{k'}^\mu\widetilde{f}')\partial_\mu\widetilde{f}-\widetilde{f}_1\widetilde{f}_2(\partial_{k'}^\mu f')\partial_\mu f\right]\Big\}\;,\end{aligned} \tag{5.75}$$

where we introduced the quantities

$$\begin{aligned}\mathcal{T}^{(b)}_\mu &:= \frac{\hbar m^3}{8}\mathrm{Im}\left[h_{\gamma\alpha}(k,\mathfrak{s})M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}h_{\alpha_1\beta_1}(k_1,\mathfrak{s}_1)h_{\alpha_2\beta_2}(k_2,\mathfrak{s}_2)h_{\beta'\alpha'}(k',\mathfrak{s}')(\gamma_\mu)_{\beta\gamma}\right]\\ &= \frac{i\hbar m^3}{16}M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}h_{\alpha_1\beta_1}(k_1,\mathfrak{s}_1)h_{\alpha_2\beta_2}(k_2,\mathfrak{s}_2)h_{\beta'\alpha'}(k',\mathfrak{s}')\left[h(k,\mathfrak{s}),\gamma_\mu\right]_{\beta\alpha}\;,\\ &= \mathcal{T}^{(a)}_\mu \equiv -\frac{1}{2}\mathcal{W}^{(1/2)}\Delta_\mu\;,\end{aligned} \tag{5.76a}$$

$$\begin{aligned}\mathcal{T}^{(c)}_\mu &:= \frac{\hbar m^3}{8}\mathrm{Im}\left[h_{\beta\alpha}(k,\mathfrak{s})M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}h_{\alpha_1\beta_1}(k_1,\mathfrak{s}_1)h_{\alpha_2\beta_2}(k_2,\mathfrak{s}_2)(\gamma_\mu)_{\beta'\alpha'}\right]\\ &= 0\;,\end{aligned} \tag{5.76b}$$

$$\begin{aligned}\overline{\mathcal{T}} &:= \frac{\hbar m^4}{4}\mathrm{Im}\left[h_{\beta\alpha}(k,\mathfrak{s})M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}h_{\alpha_1\beta_1}(k_1,\mathfrak{s}_1)h_{\alpha_2\beta_2}(k_2,\mathfrak{s}_2)h_{\beta'\alpha'}(k',\mathfrak{s}')\right]\\ &= 0\;.\end{aligned} \tag{5.76c}$$

The fact that both $\mathcal{T}^{(c)}_\mu$ and $\overline{\mathcal{T}}$ vanish follows after using the symmetries of the vertex M to show that the expressions in square brackets are purely real.[5]

[5]We remark that the statement that those terms are zero is related to the vanishing of the third and fourth contribution to the nonlocal collision term in the GLW approach, cf. Eqs. (4.93) and (4.94), and generalizes that result to quantum statistics.

5.4.3 Summary

After collecting the results from the previous subsection, we find from Eq. (5.54)

$$
\begin{aligned}
k\cdot\partial f(x,k,\mathfrak{s}) = \frac{1}{2}\int & \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,\mathrm{d}\bar{S}(k)(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\mathcal{W}^{(1/2)} \\
& \times\Big\{ f_1 f_2 \widetilde{f}'\widetilde{f} - \widetilde{f}_1\widetilde{f}_2 f' f + (\Delta_1^\mu - \Delta^\mu)\Big[(\partial_\mu f_1) f_2 \widetilde{f}'\widetilde{f} - (\partial_\mu \widetilde{f}_1)\widetilde{f}_2 f' f\Big] \\
& + (\Delta_2^\mu - \Delta^\mu)\Big[f_1(\partial_\mu f_2)\widetilde{f}'\widetilde{f} - \widetilde{f}_1(\partial_\mu \widetilde{f}_2) f' f\Big] \\
& + (\Delta'^\mu - \Delta^\mu)\Big[f_1 f_2(\partial_\mu \widetilde{f}')\widetilde{f} - \widetilde{f}_1\widetilde{f}_2(\partial_\mu f') f\Big]\Big\} . \qquad (5.77)
\end{aligned}
$$

When interpreting the quantities of first order in $\hbar$ as the leading terms in a TAYLOR series, we obtain

$$
\begin{aligned}
k\cdot\partial f(x,k,\mathfrak{s}) = \frac{1}{2}\int & \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,\mathrm{d}\bar{S}(k)(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\mathcal{W}^{(1/2)} \\
& \times\Big[f(x+\Delta_1-\Delta,k_1,\mathfrak{s}_1)f(x+\Delta_2-\Delta,k_2,\mathfrak{s}_2)\widetilde{f}(x+\Delta'-\Delta,k',\mathfrak{s}')\widetilde{f}(x,k,\bar{\mathfrak{s}}) \\
& - \widetilde{f}(x+\Delta_1-\Delta,k_1,\mathfrak{s}_1)\widetilde{f}(x+\Delta_2-\Delta,k_2,\mathfrak{s}_2)f(x+\Delta'-\Delta,k',\mathfrak{s}')f(x,k,\bar{\mathfrak{s}})\Big] .
\end{aligned} \qquad (5.78)
$$

Provided that $\widetilde{f} = 1 - f$ (cf. Appendix B.2.2), Eq. (5.78) provides the generalization of Eq. (4.96) to quantum statistics, and reduces to that equation in the limit of classical statistics, where $\widetilde{f} \to 1$. All conclusions that were put forward in the preceding chapter, such as the connection to the case where there is no dependence on the spin variables, cf. Eq. (4.101), of course remain valid. An important novel effect that arises when including quantum statistics consists in an altered equilibrium state, as we will see in Sec. 5.6.

5.5 Vector fields

Finally, we will consider the KB equations for charged massive vector fields, akin to the discussion in Sec. 4.5. Note that, compared to the earlier definition of the WIGNER function for spin-1 fields (3.146), the GREEN's function $G^{<\mu\nu}$ is defined with the indices switched, i.e., $G^{<\mu\nu} \equiv -(\hbar/2)W^{\nu\mu}$.

5.5.1 Structure of the equations

The inverse free propagator for PROCA fields reads in real space[6]

$$
-G_0^{-1,\mu\nu}(x) = \left(\Box + \frac{m^2}{\hbar^2}\right) g^{\mu\nu} - \partial^\mu\partial^\nu , \qquad (5.79)
$$

such that its WIGNER transform is

$$
-\hbar^2 G_0^{-1,\mu\nu} = \left(-D^2 + m^2\right) g^{\mu\nu} + D^\mu D^\nu , \qquad (5.80a)
$$

$$
-\hbar^2 \overleftarrow{G}_0^{*-1,\mu\nu} = \big(-\overleftarrow{D}^{*2} + m^2\big) g^{\mu\nu} + \overleftarrow{D}^{*\mu}\overleftarrow{D}^{*\nu} . \qquad (5.80b)
$$

Then, the KB equations (5.26) (with $\lambda = \sqrt{\hbar}$) read

$$
\left(-D^2 + m^2\right) G^{<\mu\nu}(x,k) + D^\mu D_\alpha G^{<\alpha\nu} = -I_{\text{coll}}^{\mu\nu} , \qquad (5.81a)
$$

$$
\left(-D^{*2} + m^2\right) G^{<\mu\nu}(x,k) + D^{*\nu} D^*_\alpha G^{<\mu\alpha} = -I_{\text{coll}}^{*\nu\mu} , \qquad (5.81b)
$$

[6] Compared to the discussion in Sec. 3.5.1, there is a sign difference. Alternatively, we could have defined the self-energy in a different way, cf. Ref. [86].

where we defined

$$I^{\mu\nu}_{\text{coll}} := \frac{i\hbar}{2}\left[\Sigma^{>\mu\alpha}(x,k)G^{<\nu}_{\alpha}(x,k) - \Sigma^{<\mu\alpha}(x,k)G^{>\nu}_{\alpha}(x,k)\right] + \frac{\hbar^2}{4}\left[\left\{\Sigma^{>}(x,k),G^{<}(x,k)\right\}^{\mu\nu}_{\text{PB}} - \left\{\Sigma^{<}(x,k),G^{>}(x,k)\right\}^{\mu\nu}_{\text{PB}}\right] . \tag{5.82}$$

Note that $(G^{<})^\dagger = G^{<}$ is hermitian, and the identity (5.14) for the Poisson brackets reads

$$\left\{G^{\lessgtr}(x,k),\Sigma^{\gtrless}(x,k)\right\}^{\mu\nu}_{\text{PB}} = -\left\{G^{\lessgtr}(x,k),\Sigma^{\gtrless}(x,k)\right\}^{*\nu\mu}_{\text{PB}} . \tag{5.83}$$

Acting with D_μ (D^*_ν) on the first (second) equation of (5.81), we find the subsidiary conditions

$$D_\mu G^{<\mu\nu}(x,k) = -\frac{1}{m^2}D_\mu I^{\mu\nu}_{\text{coll}} , \qquad D^*_\nu G^{<\mu\nu}(x,k) = -\frac{1}{m^2}D_\nu I^{*\nu\mu}_{\text{coll}} , \tag{5.84}$$

which are complex conjugates of each other. Putting them to use, the KB equations become

$$\left(-D^2+m^2\right)G^{<\mu\nu}(x,k) = -I^{\mu\nu}_{\text{coll}} + \frac{1}{m^2}D^\mu D_\alpha I^{\alpha\nu}_{\text{coll}} , \tag{5.85a}$$

$$\left(-D^{*2}+m^2\right)G^{<\mu\nu}(x,k) = -I^{*\nu\mu}_{\text{coll}} + \frac{1}{m^2}D^{*\nu} D^*_\alpha I^{*\alpha\mu}_{\text{coll}} . \tag{5.85b}$$

Comparing these to Eq. (3.149), we can identify

$$I^{\mu\nu}_{\text{coll}} = -\hbar^2\int \mathrm{d}^4v e^{-\frac{i}{\hbar}k\cdot v}\left\langle \widehat{V}^{\dagger\nu}\left(x+\frac{v}{2}\right)\widehat{\rho}^{\mu}\left(x-\frac{v}{2}\right)\right\rangle , \tag{5.86}$$

where the flipped Lorentz indices and the prefactors arise due to the definition of $G^{<\mu\nu}$. From Eqs. (5.85) we find the mass-shell and Boltzmann equations,

$$\left(k^2-m^2-\frac{\hbar^2}{4}\Box\right)G^{<\mu\nu}(x,k) = \frac{1}{2}\left(I^{\mu\nu}_{\text{coll}} - \frac{1}{m^2}D^\mu D_\alpha I^{\alpha\nu}_{\text{coll}} + \text{h.c.}\right) , \tag{5.87a}$$

$$k\cdot\partial G^{<\mu\nu}(x,k) = -\frac{i}{2\hbar}\left(I^{\mu\nu}_{\text{coll}} - \frac{1}{m^2}D^\mu D_\alpha I^{\alpha\nu}_{\text{coll}} - \text{h.c.}\right) . \tag{5.87b}$$

In order to translate these expressions into extended phase space, we have to contract them with

$$H^{\mu\nu}(k,\mathfrak{s}) = \frac{1}{3}K^{\mu\nu} + \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\frac{k_\alpha}{m}\mathfrak{s}_\beta + \frac{5}{8}K^{\mu\nu}_{\alpha\beta}\mathfrak{s}^\alpha\mathfrak{s}^\beta , \tag{5.88}$$

cf. Eq. (4.132). Based on the discussion in Subsec. 3.5.2, we can argue that parts of the terms on the right-hand sides of Eqs. (5.87) involving Bopp operators acting on the collision integrals do not contribute in our truncation of the $\hbar$-gradient expansion. The reason is that the operator $H^{\mu\nu}$ is orthogonal to the four-momentum, thus projecting out contributions where D^μ is given by its zeroth-order term k^μ. Then, to first order in $\hbar$ we can write the mass-shell and kinetic equations as[7]

$$\left(k^2-m^2\right)\mathfrak{f}(x,k,\mathfrak{s}) = -\frac{2}{\hbar}\text{Re}\left[H_{\mu\nu}(k,\mathfrak{s})\left(I^{\mu\nu}_{\text{coll}} - \frac{i\hbar}{2m^2}\partial^\mu k_\alpha I^{\alpha\nu}_{\text{coll}}\right)\right] , \tag{5.89a}$$

$$k\cdot\partial\mathfrak{f}(x,k,\mathfrak{s}) = -\frac{2}{\hbar^2}\text{Im}\left[H_{\mu\nu}(k,\mathfrak{s})\left(I^{\mu\nu}_{\text{coll}} - \frac{i\hbar}{2m^2}\partial^\mu k_\alpha I^{\alpha\nu}_{\text{coll}}\right)\right] , \tag{5.89b}$$

where we used the fact that H is hermitian, $H^{*\mu\nu} = H^{\nu\mu}$. From this point on, we can proceed as in the previous section.

[7] As remarked before, we used that $\mathfrak{f} = H_{\nu\mu}W^{\mu\nu} = -(2/\hbar)H_{\mu\nu}G^{<\mu\nu}$, where we also employed that $\kappa = -2/\hbar$.

Structure of the Wigner function

Similar to the case of Dirac fields, we have to discuss the structure of the Green's functions that appear inside the collision integrals. As before, we can neglect collisional contributions to the Green's functions since the resulting terms would be of fourth order in the coupling. Then, their structure is determined by Eqs. (3.151) with the right-hand sides set to zero. We remind the reader that these constraints result from the fact that free Proca fields have a vanishing divergence, thus reducing the number of independent degrees of freedom. When decomposing the Wigner functions as

$$G^{<\mu\nu}(x,k) = E^{\mu\nu} f_E + K^{\mu\nu} f_K + \frac{k^{(\mu}}{2k} F_S^{\nu)} + i\frac{k^{[\mu}}{2k} F_A^{\nu]} + i\epsilon^{\mu\nu\alpha\beta}\frac{k_\alpha}{m} G_\beta + F_K^{\mu\nu} \;, \tag{5.90a}$$

$$G^{>\mu\nu}(x,k) = E^{\mu\nu} \widetilde{f}_E + K^{\mu\nu} \widetilde{f}_K + \frac{k^{(\mu}}{2k} \widetilde{F}_S^{\nu)} + i\frac{k^{[\mu}}{2k} \widetilde{F}_A^{\nu]} + i\epsilon^{\mu\nu\alpha\beta}\frac{k_\alpha}{m} \widetilde{G}_\beta + \widetilde{F}_K^{\mu\nu} \;, \tag{5.90b}$$

we can, according to Eqs. (3.159), express the dependent parts up to order $\mathcal{O}(\hbar)$ as

$$f_E \simeq 0 \;, \quad F_S^\mu \simeq 0 \;, \quad F_A^\mu \simeq \frac{\hbar}{m} K^{\mu\nu}\partial_\nu f_K \;, \tag{5.91a}$$

$$\widetilde{f}_E \simeq 0 \;, \quad \widetilde{F}_S^\mu \simeq 0 \;, \quad \widetilde{F}_A^\mu \simeq \frac{\hbar}{m} K^{\mu\nu}\partial_\nu \widetilde{f}_K \;. \tag{5.91b}$$

With these results, we can decompose the Green's functions into quasiclassical and gradient contributions,

$$G^{\gtrless\mu\nu}(x,k) \simeq G^{\gtrless\mu\nu}_{\text{qc}}(x,k) + G^{\gtrless\mu\nu}_{\nabla}(x,k) \;, \tag{5.92}$$

where we defined

$$G^{<\mu\nu}_{\text{qc}}(x,k) \coloneqq K^{\mu\nu} f_K(x,k) + i\epsilon^{\mu\nu\alpha\beta}\frac{k_\alpha}{m} G_\beta(x,k) + F_K^{\mu\nu}(x,k) \;, \tag{5.93a}$$

$$G^{>\mu\nu}_{\text{qc}}(x,k) \coloneqq K^{\mu\nu} \widetilde{f}_K(x,k) + i\epsilon^{\mu\nu\alpha\beta}\frac{k_\alpha}{m} \widetilde{G}_\beta(x,k) + \widetilde{F}_K^{\mu\nu}(x,k) \;, \tag{5.93b}$$

as well as

$$G^{<\mu\nu}_{\nabla}(x,k) \coloneqq \frac{i\hbar}{2m^2} k^{[\mu}\partial^{\nu]} f_K(x,k) \;, \tag{5.94a}$$

$$G^{>\mu\nu}_{\nabla}(x,k) \coloneqq \frac{i\hbar}{2m^2} k^{[\mu}\partial^{\nu]} \widetilde{f}_K(x,k) \;. \tag{5.94b}$$

Here we already employed that we may take all momenta in the kinetic equation to be on shell. Remembering the definition of $h^{\mu\nu}$ from Eq. (4.128),

$$h^{\mu\nu}(k,\mathfrak{s}) = \frac{1}{3}K^{\mu\nu} + \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\frac{k_\alpha}{m}\mathfrak{s}_\beta + K^{\mu\nu}_{\alpha\beta}\mathfrak{s}^\alpha\mathfrak{s}^\beta \;, \tag{5.95}$$

we can express the quasiclassical contributions in extended phase space,

$$G^{<\mu\nu}_{\text{qc}}(x,k) \coloneqq -2\pi\hbar^2\delta(k^2-m^2)\int \mathrm{d}S(k) h^{\nu\mu}(k,\mathfrak{s}) f(x,k,\mathfrak{s}) \;, \tag{5.96a}$$

$$G^{>\mu\nu}_{\text{qc}}(x,k) \coloneqq -2\pi\hbar^2\delta(k^2-m^2)\int \mathrm{d}S(k) h^{\nu\mu}(k,\mathfrak{s}) \widetilde{f}(x,k,\mathfrak{s}) \;, \tag{5.96b}$$

where we again used that $G^{<\mu\nu} = -(\hbar/2)W^{\nu\mu}$. The gradient contributions on the other hand read

$$G^{<\mu\nu}_{\nabla}(x,k) \coloneqq -2\pi\hbar^2\delta(k^2-m^2)\frac{i\hbar}{2m^2}k^{[\nu}\partial^{\mu]}\frac{1}{3}\int \mathrm{d}S(k) f(x,k,\mathfrak{s}) \;, \tag{5.97a}$$

$$G^{>\mu\nu}_{\nabla}(x,k) \coloneqq -2\pi\hbar^2\delta(k^2-m^2)\frac{i\hbar}{2m^2}k^{[\nu}\partial^{\mu]}\frac{1}{3}\int \mathrm{d}S(k) \widetilde{f}(x,k,\mathfrak{s}) \;. \tag{5.97b}$$

Structure of the self-energies

The results for the structure of the WIGNER function can now be inserted into the self-energies to obtain

$$\Sigma^{\gtrless\mu\nu}(x,k) \simeq \Sigma^{\gtrless\mu\nu}_{\mathrm{qc}}(x,k) + \Sigma^{\gtrless\mu\nu}_{\nabla}(x,k) \;, \tag{5.98}$$

where we defined (remembering that $\lambda = \sqrt{\hbar}$)

$$\begin{aligned}\Sigma^{\gtrless\mu\nu}_{\mathrm{qc}}(x,k) &\coloneqq \frac{1}{2\hbar^3}\int \frac{\mathrm{d}^4k_1}{(2\pi\hbar)^4}\frac{\mathrm{d}^4k_2}{(2\pi\hbar)^4}\frac{\mathrm{d}^4k'}{(2\pi\hbar)^4}(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}\\ &\quad\times G^{\gtrless}_{\mathrm{qc},\mu_1\nu_1}(x,k_1)G^{\gtrless}_{\mathrm{qc},\mu_2\nu_2}(x,k_2)G^{\lessgtr}_{\mathrm{qc},\nu'\mu'}(x,k')\;, \end{aligned}\tag{5.99}$$

$$\begin{aligned}\Sigma^{\gtrless\mu\nu}_{\nabla}(x,k) &\coloneqq \frac{1}{2\hbar^3}\int \frac{\mathrm{d}^4k_1}{(2\pi\hbar)^4}\frac{\mathrm{d}^4k_2}{(2\pi\hbar)^4}\frac{\mathrm{d}^4k'}{(2\pi\hbar)^4}(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}\\ &\quad\times\Big[G^{\gtrless}_{\nabla,\mu_1\nu_1}(x,k_1)G^{\gtrless}_{\mathrm{qc},\mu_2\nu_2}(x,k_2)G^{\lessgtr}_{\mathrm{qc},\nu'\mu'}(x,k')\\ &\qquad+G^{\gtrless}_{\mathrm{qc},\mu_1\nu_1}(x,k_1)G^{\gtrless}_{\nabla,\mu_2\nu_2}(x,k_2)G^{\lessgtr}_{\mathrm{qc},\nu'\mu'}(x,k')\\ &\qquad+G^{\gtrless}_{\mathrm{qc},\mu_1\nu_1}(x,k_1)G^{\gtrless}_{\mathrm{qc},\mu_2\nu_2}(x,k_2)G^{\lessgtr}_{\nabla,\nu'\mu'}(x,k')\Big]\;. \end{aligned}\tag{5.100}$$

These terms can then be translated into extended phase space, with the results

$$\begin{aligned}\Sigma^{<\mu\nu}_{\mathrm{qc}}(x,k) &= -\frac{1}{16}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}\\ &\quad\times h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)h_{\mu'\nu'}(k',\mathfrak{s}')f(x,k_1,\mathfrak{s}_1)f(x,k_2,\mathfrak{s}_2)\widetilde{f}(x,k',\mathfrak{s}')\;, \end{aligned}\tag{5.101a}$$

$$\begin{aligned}\Sigma^{>\mu\nu}_{\mathrm{qc}}(x,k) &= -\frac{1}{16}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}\\ &\quad\times h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)h_{\mu'\nu'}(k',\mathfrak{s}')\widetilde{f}(x,k_1,\mathfrak{s}_1)\widetilde{f}(x,k_2,\mathfrak{s}_2)f(x,k',\mathfrak{s}')\;, \end{aligned}\tag{5.101b}$$

and

$$\begin{aligned}\Sigma^{<\mu\nu}_{\nabla}(x,k) &= -\frac{i\hbar}{2m^2}\frac{1}{16}\frac{1}{3}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}\\ &\quad\times\Big\{h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)h_{\mu'\nu'}(k',\mathfrak{s}')\Big[k_{1,[\nu_1}\partial_{\mu_1]}f(x,k_1,\mathfrak{s}_1)\Big]f(x,k_2,\mathfrak{s}_2)\widetilde{f}(x,k',\mathfrak{s}')\\ &\qquad+h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)h_{\mu'\nu'}(k',\mathfrak{s}')f(x,k_1,\mathfrak{s}_1)\Big[k_{2,[\nu_2}\partial_{\mu_2]}f(x,k_2,\mathfrak{s}_2)\Big]\widetilde{f}(x,k',\mathfrak{s}')\\ &\qquad+h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)f(x,k_1,\mathfrak{s}_1)f(x,k_2,\mathfrak{s}_2)\Big[k'_{[\mu'}\partial_{\nu']}\widetilde{f}(x,k',\mathfrak{s}')\Big]\Big\}\;, \end{aligned}\tag{5.102a}$$

$$\begin{aligned}\Sigma^{>\mu\nu}_{\nabla}(x,k) &= -\frac{i\hbar}{2m^2}\frac{1}{16}\frac{1}{3}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}\\ &\quad\times\Big\{h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)h_{\mu'\nu'}(k',\mathfrak{s}')\Big[k_{1,[\nu_1}\partial_{\mu_1]}\widetilde{f}(x,k_1,\mathfrak{s}_1)\Big]\widetilde{f}(x,k_2,\mathfrak{s}_2)f(x,k',\mathfrak{s}')\\ &\qquad+h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)h_{\mu'\nu'}(k',\mathfrak{s}')\widetilde{f}(x,k_1,\mathfrak{s}_1)\Big[k_{2,[\nu_2}\partial_{\mu_2]}\widetilde{f}(x,k_2,\mathfrak{s}_2)\Big]f(x,k',\mathfrak{s}')\\ &\qquad+h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)\widetilde{f}(x,k_1,\mathfrak{s}_1)\widetilde{f}(x,k_2,\mathfrak{s}_2)\Big[k'_{[\mu'}\partial_{\nu']}f(x,k',\mathfrak{s}')\Big]\Big\}\;. \end{aligned}\tag{5.102b}$$

With the self-energies and the WIGNER functions expressed in this way, we are in a position to compute the kinetic equation.

5.5.2 Evaluating the kinetic equation

According to Theorem 1, we only need to evaluate the on-shell part of the BOLTZMANN equation (5.89b). With the aid of the results of the previous subsection, it can be expressed as

$$4\pi\hbar\delta(k^2-m^2)k\cdot\partial f(x,k,\mathfrak{s}) = \mathcal{I}_{\mathrm{qc}}(x,k,\mathfrak{s}) + \mathcal{I}_{\partial}(x,k,\mathfrak{s}) + \mathcal{I}_{\nabla}(x,k,\mathfrak{s}) + \mathcal{I}_{\mathrm{PB}}(x,k,\mathfrak{s})\;, \tag{5.103}$$

which looks very similar to Eq. (5.54). As before, the term $\mathcal{I}_{\text{qc}}$ collects the quasiclassical parts of the collision terms,

$$\mathcal{I}_{\text{qc}}(x,k,\mathfrak{s}) \coloneqq \frac{1}{\hbar}\text{Re}\left\{H_{\mu\nu}(k,\mathfrak{s})\left[\Sigma^{<\mu\alpha}_{\text{qc}}(x,k)G^{>}_{\text{qc},\alpha}{}^{\nu}(x,k) - \Sigma^{>\mu\alpha}_{\text{qc}}(x,k)G^{<}_{\text{qc},\alpha}{}^{\nu}(x,k)\right]\right\}\ . \tag{5.104}$$

The action of the derivative on the collision term is captured by $\mathcal{I}_\partial$,

$$\mathcal{I}_\partial(x,k,\mathfrak{s}) \coloneqq \frac{1}{2m^2}\text{Im}\left\{H_{\mu\nu}(k,\mathfrak{s})\partial^\mu k_\alpha\left[\Sigma^{<\alpha\beta}_{\text{qc}}(x,k)G^{>}_{\text{qc},\beta}{}^{\nu}(x,k) - \Sigma^{>\alpha\beta}_{\text{qc}}(x,k)G^{<}_{\text{qc},\beta}{}^{\nu}(x,k)\right]\right\}\ , \tag{5.105}$$

while $\mathcal{I}_\nabla$ contains the gradient terms,

$$\begin{aligned}\mathcal{I}_\nabla(x,k,\mathfrak{s}) \coloneqq \frac{1}{\hbar}\text{Re}\Big\{H_{\mu\nu}(k,\mathfrak{s})\Big[&\Sigma^{<\mu\alpha}_{\text{qc}}(x,k)G^{>}_{\nabla,\alpha}{}^{\nu}(x,k) - \Sigma^{>\mu\alpha}_{\nabla}(x,k)G^{<}_{\text{qc},\alpha}{}^{\nu}(x,k)\\ &+\Sigma^{<\mu\alpha}_{\nabla}(x,k)G^{>}_{\text{qc},\alpha}{}^{\nu}(x,k) - \Sigma^{>\mu\alpha}_{\text{qc}}(x,k)G^{<}_{\nabla,\alpha}{}^{\nu}(x,k)\Big]\Big\}\ .\end{aligned} \tag{5.106}$$

The POISSON-bracket terms are described by $\mathcal{I}_{\text{PB}}$,

$$\begin{aligned}\mathcal{I}_{\text{PB}}(x,k,\mathfrak{s}) &\coloneqq \frac{1}{2}\text{Im}\left\{H_{\mu\nu}(k,\mathfrak{s})\left[\left\{\Sigma^<_{\text{qc}}(x,k),G^>_{\text{qc}}(x,k)\right\}^{\mu\nu}_{\text{PB}} - \left\{\Sigma^>_{\text{qc}}(x,k),G^<_{\text{qc}}(x,k)\right\}^{\mu\nu}_{\text{PB}}\right]\right\}\\ &= \frac{1}{2}\text{Im}\Big(H_{\mu\nu}(k,\mathfrak{s})\Big\{\left[\partial_\rho\Sigma^{<\mu\alpha}_{\text{qc}}\right]\left[\partial^\rho_k G^{>}_{\text{qc},\alpha}{}^{\nu}\right] - \left[\partial^\rho_k\Sigma^{<\mu\alpha}_{\text{qc}}\right]\left[\partial_\rho G^{>}_{\text{qc},\alpha}{}^{\nu}\right]\\ &\quad - \left[\partial_\rho\Sigma^{>\mu\alpha}_{\text{qc}}\right]\left[\partial^\rho_k G^{<}_{\text{qc},\alpha}{}^{\nu}\right] - \left[\partial^\rho_k\Sigma^{>\mu\alpha}_{\text{qc}}\right]\left[\partial_\rho G^{<}_{\text{qc},\alpha}{}^{\nu}\right]\Big\}\Big)\ .\end{aligned} \tag{5.107}$$

In the following, we compute these contributions, showing that $\mathcal{I}_\nabla$ and $\mathcal{I}_{\text{PB}}$ are responsible for the nonlocal collisions.

Quasiclassical terms

Inserting the quasiclassical GREEN's functions and self-energies (5.96) and (5.101), we readily find

$$\begin{aligned}\mathcal{I}_{\text{qc}}(x,k,\mathfrak{s}) = 4\pi\hbar\delta(k^2-m^2)\frac{1}{2}\int &\text{d}\Gamma_1\,\text{d}\Gamma_2\,\text{d}\Gamma'\,\text{d}\bar{S}(k)(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\mathcal{W}^{(1)}\\ &\times\left(f_1f_2\widetilde{f}'\widetilde{f} - \widetilde{f}_1\widetilde{f}_2f'f\right)\ ,\end{aligned} \tag{5.108}$$

where we used the abbreviations (5.60) and defined

$$\begin{aligned}\mathcal{W}^{(1)} &\coloneqq \frac{1}{16}\text{Re}\left[M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)h_{\mu'\nu'}(k',\mathfrak{s}')H_\mu{}^\alpha(k,\mathfrak{s})h_{\alpha\nu}(k,\bar{\mathfrak{s}})\right]\\ &= \frac{1}{32}M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)h_{\mu'\nu'}(k',\mathfrak{s}')\\ &\quad\times\left[H_\mu{}^\alpha(k,\mathfrak{s})h_{\alpha\nu}(k,\bar{\mathfrak{s}}) + h_\mu{}^\alpha(k,\bar{\mathfrak{s}})H_{\alpha\nu}(k,\mathfrak{s})\right]\ .\end{aligned} \tag{5.109}$$

In order to arrive at this expression, we used that h and H are hermitian, as well as the assumed symmetry of the vertex $M^{*\mu\mu'\mu_1\mu_2} = M^{\mu_1\mu_2\mu\mu'}$. As expected, it agrees with the local result from the GLW approach (4.144), with the difference that quantum statistics are included.

Gradient terms

We compute the term $\mathcal{I}_\partial$ first, obtaining

$$\begin{aligned}\mathcal{I}_{\text{qc}}(x,k,\mathfrak{s}) = 4\pi\hbar\delta(k^2-m^2)\frac{1}{2}\int &\text{d}\Gamma_1\,\text{d}\Gamma_2\,\text{d}\Gamma'\,\text{d}\bar{S}(k)(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\mathcal{W}^{(1)}\\ &\times\mathscr{T}^{(a)}_\rho\partial^\rho\left(f_1f_2\widetilde{f}'\widetilde{f} - \widetilde{f}_1\widetilde{f}_2f'f\right)\ ,\end{aligned} \tag{5.110}$$

where we defined

$$\begin{aligned}\mathscr{T}_\rho^{(a)} &:= \frac{\hbar}{2m^2}\frac{1}{16}\text{Im}\left[M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)h_{\mu'\nu'}(k',\mathfrak{s}')k_\mu H_\rho{}^\alpha(k,\mathfrak{s})h_{\alpha\nu}(k,\bar{\mathfrak{s}})\right]\\ &\simeq \frac{\hbar}{2m^2}\frac{1}{16}\frac{1}{3}\text{Im}\left[M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)h_{\mu'\nu'}(k',\mathfrak{s}')k_\mu H_{\rho\nu}(k,\mathfrak{s})\right]\\ &= -\frac{i\hbar}{64m^2}\frac{1}{3}M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)h_{\mu'\nu'}(k',\mathfrak{s}')\\ &\quad\times\left[H_{\rho\nu}(k,\mathfrak{s})k_\mu - H_{\mu\rho}(k,\mathfrak{s})k_\nu\right]\ . \end{aligned} \tag{5.111}$$

Here we used the fact that $h^{\mu\nu}(k,\bar{\mathfrak{s}})$ can be replaced by $\frac{1}{3}K^{\mu\nu}$, since the distribution function $f(x,k,\bar{\mathfrak{s}})$ is spin-independent at zeroth order in $\hbar$. Comparing to Eqs. (4.145), we see that $\mathscr{T}_\rho^{(a)} = -\frac{1}{2}\mathcal{W}^{(1)}\Delta_\rho$.

The contribution $\mathcal{I}_\nabla$ can be evaluated to

$$\begin{aligned}\mathcal{I}_\nabla(x,k,\mathfrak{s}) &= 4\pi\hbar\delta(k^2-m^2)\frac{1}{2}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,\mathrm{d}\bar{S}(k)(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\\ &\times\left\{\mathscr{T}_1^\rho\left[(\partial_\rho f_1)f_2\widetilde{f}'\widetilde{f}-(\partial_\rho\widetilde{f}_1)\widetilde{f}_2f'f\right]+\mathscr{T}_2^\rho\left[f_1(\partial_\rho f_2)\widetilde{f}'\widetilde{f}-\widetilde{f}_1(\partial_\rho\widetilde{f}_2)f'f\right]\right.\\ &\left.+\mathscr{T}'^\rho\left[f_1f_2(\partial_\rho\widetilde{f}')\widetilde{f}-\widetilde{f}_1\widetilde{f}_2(\partial_\rho f')f\right]+\mathscr{T}^\rho\left[f_1f_2\widetilde{f}'(\partial_\rho\widetilde{f})-\widetilde{f}_1\widetilde{f}_2f'(\partial_\rho f)\right]\right\}\ , \end{aligned} \tag{5.112}$$

with the tensors

$$\mathscr{T}_1^\rho := -\frac{\hbar}{32m^2}\frac{1}{3}\text{Im}\left[M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}k_{1,[\nu_1}\delta^\rho_{\mu_1]}h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)h_{\mu'\nu'}(k',\mathfrak{s}')H_\mu{}^\alpha(k,\mathfrak{s})h_{\alpha\nu}(k,\bar{\mathfrak{s}})\right]\ , \tag{5.113a}$$

$$\mathscr{T}_2^\rho := -\frac{\hbar}{32m^2}\frac{1}{3}\text{Im}\left[M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)k_{2,[\nu_2}\delta^\rho_{\mu_2]}h_{\mu'\nu'}(k',\mathfrak{s}')H_\mu{}^\alpha(k,\mathfrak{s})h_{\alpha\nu}(k,\bar{\mathfrak{s}})\right]\ , \tag{5.113b}$$

$$\mathscr{T}'^\rho := -\frac{\hbar}{32m^2}\frac{1}{3}\text{Im}\left[M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)k'_{[\mu'}\delta^\rho_{\nu']}H_\mu{}^\alpha(k,\mathfrak{s})h_{\alpha\nu}(k,\bar{\mathfrak{s}})\right]\ , \tag{5.113c}$$

$$\mathscr{T}^\rho := -\frac{\hbar}{32m^2}\frac{1}{3}\text{Im}\left[M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)h_{\mu'\nu'}(k',\mathfrak{s}')H_\mu{}^\alpha(k,\mathfrak{s})k_{[\alpha}\delta^\rho_{\nu]}\right]\ . \tag{5.113d}$$

In order to simplify these expressions, we use the fact that the components of f_1, f_2, f', and f that depend on the spin variables are at least of order $\mathcal{O}(\hbar)$. Then we may replace

$$\frac{1}{3}k_{1,[\nu_1}\delta^\rho_{\mu_1]} = \frac{1}{3}k_{1,[\nu_1}K^\rho_{1,\mu_1]} \simeq k_{1,[\nu_1}h^\rho_{1,\mu_1]}(k_1,\mathfrak{s}_1)\ , \tag{5.114}$$

and similar for the cases dependent on k_2 and k'. Furthermore, for the same reason we can approximate $h^{\mu\nu}(k,\bar{\mathfrak{s}}) \simeq \frac{1}{3}K^{\mu\nu}$ in Eqs. (5.113a)–(5.113c). Lastly, in Eq. (5.113d) we utilize that H is orthogonal to the four-momentum. With these results and the symmetries of M, we find

$$\begin{aligned}\mathscr{T}_1^\rho = \frac{i\hbar}{32m^2}\frac{1}{3}&M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)h_{\mu'\nu'}(k',\mathfrak{s}')H_{\mu\nu}(k,\mathfrak{s})\\ &\times\left[h^\rho{}_{\mu_1}(k_1,\mathfrak{s}_1)k_{1,\nu_1}-k_{1,\mu_1}h_{\nu_1}{}^\rho(k_1,\mathfrak{s}_1)\right]\ , \end{aligned} \tag{5.115a}$$

$$\begin{aligned}\mathscr{T}_2^\rho = \frac{i\hbar}{32m^2}\frac{1}{3}&M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)h_{\mu'\nu'}(k',\mathfrak{s}')H_{\mu\nu}(k,\mathfrak{s})\\ &\times\left[h^\rho{}_{\mu_2}(k_2,\mathfrak{s}_2)k_{2,\nu_2}-k_{2,\mu_2}h_{\nu_2}{}^\rho(k_2,\mathfrak{s}_2)\right]\ , \end{aligned} \tag{5.115b}$$

$$\begin{aligned}\mathscr{T}'^\rho = \frac{i\hbar}{32m^2}\frac{1}{3}&M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)H_{\mu\nu}(k,\mathfrak{s})\\ &\times\left[h^\rho{}_{\nu'}(k',\mathfrak{s}')k'_{\mu'}-k'_{\nu'}h_{\mu'}{}^\rho(k',\mathfrak{s}')\right]\ , \end{aligned} \tag{5.115c}$$

$$\begin{aligned}\mathscr{T}^\rho = \frac{i\hbar}{64m^2}\frac{1}{3}&M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)h_{\mu'\nu'}(k',\mathfrak{s}')\\ &\times\left[H^\rho{}_\nu(k,\mathfrak{s})k_\mu-k_\nu H_\mu{}^\rho(k,\mathfrak{s})\right]\ . \end{aligned} \tag{5.115d}$$

Comparing to the shifts (4.145), we find

$$\mathscr{T}_1^\rho = \mathcal{W}^{(1)}\Delta_1^\rho\,, \quad \mathscr{T}_2^\rho = \mathcal{W}^{(1)}\Delta_2^\rho\,, \quad \mathscr{T}'^\rho = \mathcal{W}^{(1)}\Delta'^\rho\,, \quad \mathscr{T}^\rho = \frac{1}{2}\mathcal{W}^{(1)}\Delta^\rho\,. \tag{5.116}$$

Poisson brackets

In order to evaluate $\mathcal{I}_{\text{PB}}$, we first need to compute the momentum derivatives acting on the Green's functions and self-energies. For this we note that, since we may approximate the Green's functions on the right-hand sides of the Boltzmann equation as being spin-independent, we can evaluate the momentum-derivative acting on the quantities h approximately as

$$\partial_k^\rho h^{\mu\nu}(k,\mathfrak{s}) \simeq \frac{1}{3}\partial_k^\rho K^{\mu\nu} = -\frac{1}{3m^2}k^{(\mu}g^{\nu)\rho}\,. \tag{5.117}$$

We then obtain from Eqs. (5.96)

$$\partial_k^\rho G_{\text{qc}}^{<\mu\nu}(x,k) \simeq -2\pi\hbar^2\delta(k^2-m^2)\int \mathrm{d}S(k)\frac{1}{3}\left(-\frac{k^{(\nu}}{m^2}g^{\mu)\rho} + K^{\nu\mu}\partial_k^\rho\right)f(x,k,\mathfrak{s})\,, \tag{5.118a}$$

$$\partial_k^\rho G_{\text{qc}}^{>\mu\nu}(x,k) \simeq -2\pi\hbar^2\delta(k^2-m^2)\int \mathrm{d}S(k)\frac{1}{3}\left(-\frac{k^{(\nu}}{m^2}g^{\mu)\rho} + K^{\nu\mu}\partial_k^\rho\right)\widetilde{f}(x,k,\mathfrak{s})\,, \tag{5.118b}$$

where we could neglect the off-shell contributions since they are of higher order in either $\hbar$ or the coupling constant. The momentum derivatives on the self-energies, which act only on the momentum-conserving delta function, are again rewritten as k'-derivatives, giving

$$\begin{aligned}\partial_k^\rho \Sigma_{\text{qc}}^{<\mu\nu}(x,k) \simeq{}& \frac{1}{16}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}\\ &\times h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)f_1f_2\frac{1}{3}\left(-\frac{k'^{(\nu}}{m^2}g^{\mu)\rho} + K'^{\nu\mu}\partial_{k'}^\rho\right)\widetilde{f}'\,,\end{aligned} \tag{5.119a}$$

$$\begin{aligned}\partial_k^\rho \Sigma_{\text{qc}}^{>\mu\nu}(x,k) \simeq{}& \frac{1}{16}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}\\ &\times h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)\widetilde{f}_1\widetilde{f}_2\frac{1}{3}\left(-\frac{k'^{(\nu}}{m^2}g^{\mu)\rho} + K'^{\nu\mu}\partial_{k'}^\rho\right)f'\,.\end{aligned} \tag{5.119b}$$

We compute the contribution as

$$\begin{aligned}\mathcal{I}_{\text{PB}}(x,k,\mathfrak{s}) ={}& 4\pi\hbar\delta(k^2-m^2)\frac{1}{2}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,\mathrm{d}\bar{S}(k)(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\\ &\times\Big\{\mathscr{T}_\rho^{(b)}\left[(\partial^\rho f_1f_2\widetilde{f}')\widetilde{f} - (\partial^\rho\widetilde{f}_1\widetilde{f}_2f')f\right] + \mathscr{T}_\rho^{(c)}\left[f_1f_2\widetilde{f}'(\partial^\rho\widetilde{f}) - \widetilde{f}_1\widetilde{f}_2f'(\partial^\rho f)\right]\\ &+\overline{\mathscr{T}}\left[(\partial_\rho f_1f_2\widetilde{f}')\partial_k^\rho\widetilde{f} - (\partial_\rho\widetilde{f}_1\widetilde{f}_2f')\partial_k^\rho f + f_1f_2(\partial_{k'}^\rho\widetilde{f}')\partial_\rho\widetilde{f} - \widetilde{f}_1\widetilde{f}_2(\partial_{k'}^\rho f')\partial_\rho f\right]\Big\}\,,\end{aligned} \tag{5.120}$$

where we defined

$$\begin{aligned}\mathscr{T}_\rho^{(b)} \coloneqq{}& -\frac{\hbar}{32m^2}\frac{1}{3}\text{Im}\left[M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)h_{\mu'\nu'}(k',\mathfrak{s}')H_\mu{}^\alpha(k,\mathfrak{s})k_{(\alpha}g_{\nu)\rho}\right]\\ ={}& -\frac{i\hbar}{64m^2}\frac{1}{3}M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)h_{\mu'\nu'}(k',\mathfrak{s}')\\ &\times\left[H_{\rho\nu}(k,\mathfrak{s})k_\mu - k_\nu H_{\mu\rho}(k,\mathfrak{s})\right]\,,\end{aligned} \tag{5.121a}$$

$$\begin{aligned}\mathscr{T}_\rho^{(c)} \coloneqq{}& -\frac{\hbar}{32m^2}\frac{1}{3}\text{Im}\left[M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)k'_{(\nu'}g_{\mu')\rho}H_{\mu\nu}(k,\mathfrak{s})\right]\\ ={}& 0\,,\end{aligned} \tag{5.121b}$$

$$\begin{aligned}\overline{\mathscr{T}} \coloneqq{}& \frac{\hbar}{32m^2}\frac{1}{3}\text{Im}\left[M^{\mu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\nu\nu'}h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)h_{\mu'\nu'}(k',\mathfrak{s}')H_{\mu\nu}(k,\mathfrak{s})\right]\\ ={}& 0\,.\end{aligned} \tag{5.121c}$$

As expected, we find $\mathscr{T}_\rho^{(b)} = -\frac{1}{2}\mathcal{W}^{(1)}\Delta_\rho$.

5.5.3 Summary

Collecting our results, the kinetic equation becomes

$$\begin{aligned} k\cdot\partial f(x,k,\mathfrak{s}) = \frac{1}{2}\int & \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,\mathrm{d}\bar{S}(k)(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\mathcal{W}^{(1)} \\ & \times\Big\{ f_1 f_2 \widetilde{f}'\widetilde{f} - \widetilde{f}_1\widetilde{f}_2 f' f + (\Delta_1^\mu-\Delta^\mu)\Big[(\partial_\mu f_1) f_2\widetilde{f}'\widetilde{f} - (\partial_\mu\widetilde{f}_1)\widetilde{f}_2 f' f\Big] \\ & + (\Delta_2^\mu-\Delta^\mu)\Big[f_1(\partial_\mu f_2)\widetilde{f}'\widetilde{f} - \widetilde{f}_1(\partial_\mu\widetilde{f}_2) f' f\Big] \\ & + (\Delta'^\mu-\Delta^\mu)\Big[f_1 f_2(\partial_\mu\widetilde{f}')\widetilde{f} - \widetilde{f}_1\widetilde{f}_2(\partial_\mu f') f\Big]\Big\}\,, \end{aligned} \tag{5.122}$$

or, in a more compact form,

$$\begin{aligned} k\cdot\partial f(x,k,\mathfrak{s}) = \frac{1}{2}\int & \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,\mathrm{d}\bar{S}(k)(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\mathcal{W}^{(1)} \\ \times\Big[& f(x+\Delta_1-\Delta,k_1,\mathfrak{s}_1)f(x+\Delta_2-\Delta,k_2,\mathfrak{s}_2)\widetilde{f}(x+\Delta'-\Delta,k',\mathfrak{s}')\widetilde{f}(x,k,\bar{\mathfrak{s}}) \\ & -\widetilde{f}(x+\Delta_1-\Delta,k_1,\mathfrak{s}_1)\widetilde{f}(x+\Delta_2-\Delta,k_2,\mathfrak{s}_2)f(x+\Delta'-\Delta,k',\mathfrak{s}')f(x,k,\bar{\mathfrak{s}})\Big]\,. \end{aligned} \tag{5.123}$$

As in the scalar case, we have $\widetilde{f} = 1+f$, which tells us that this equation generalizes the result (4.143) obtained in the GLW approach to quantum statistics, and reduces to it if we take the limit $\widetilde{f}\to 1$.

5.6 Equilibrium

With the collision terms for particles of spins 0, 1/2, and 1 at hand, we are in a position to compute equilibrium. In principle, the discussion of Sec. 4.6, in particular the concept of using a weak equivalence principle to redefine the transition rates in order to be able to use the conservation of the total angular momentum, remains valid, so we do not repeat it here. The most important modification to the case of classical statistics lies in the form of the equilibrium distribution function: in order to be able to use the summational invariance of charge, four-momentum, and total angular momentum, we have to demand that the distribution function takes on the following form,

$$f_{\mathrm{eq}}(x,k,\mathfrak{s}) := \{\exp[g(x,k,\mathfrak{s})]+a\}^{-1}\,, \tag{5.124}$$

where $g(x,k,\mathfrak{s})$ consists of summational invariants. In the case of bosonic fields, such as the Klein-Gordon or Proca ones, we have to set $a=-1$, leading to a distribution function of Bose-Einstein type. On the other hand, for Dirac fields which follow fermionic statistics, it holds that $a=1$, which yields a Fermi-Dirac distribution function. In order to recover the form of the distribution function for classical particles that was introduced in Sec. 4.6, we simply have to set $a=0$.

Since the conclusions of that section are still valid, we find that, in order for the collision term to vanish, it has to hold that

$$\partial_{(\mu}\beta_{0,\nu)}=0\,,\quad \partial_\mu\alpha_0=0\,,\quad \Omega_{0,\mu\nu}=\varpi_{\mu\nu}\equiv-\frac{1}{2}\partial_{[\mu}\beta_{0,\nu]}\,, \tag{5.125}$$

showing that, when defined in the usual way, local implies global equilibrium as soon as the particles have a nonvanishing spin. When moving on now to derive hydrodynamics from quantum kinetic theory, we will argue that this statement can be somewhat relaxed, provided that suitable conditions are fulfilled.

Chapter 6

Dissipative spin hydrodynamics

After having computed the kinetic equations for the on-shell single-particle distribution function in extended phase space, $f(x, k, \mathfrak{s})$, we will now return to the problem of developing dissipative hydrodynamics with spin that was initially posed in Chapter 2.

6.1 Relevant scales and local equilibrium

At the end of Chapters 4 and 5, we found that, for particles with nonzero spin, local and global equilibrium coincide, a fact that hinged on the spacetime shifts Δ_1, Δ_2, Δ', and Δ appearing in the collision integral. This statement is rather contrary to the usual way of deriving hydrodynamics from kinetic theory, where, in local equilibrium, the LAGRANGE multipliers $\alpha_0(x)$ and $\beta_0(x)$ are arbitrary functions of spacetime [43, 115, 153], which is consistent with the way we introduced hydrodynamics in Chapter 2. As we shall discuss now, it is possible to transfer the concept of this kind of local equilibrium to spin hydrodynamics as well, namely by considering the different scales that are involved in the problem. When constructing hydrodynamics from kinetic theory, there are three characteristic length scales one has to consider: the effective range of the interaction ℓ_{int}, which should be a lot smaller than the mean free path λ_{mfp} of the particles. Furthermore, the hydrodynamic length scale L_{hydro}, i.e., the scale over which macroscopic quantities vary considerably, should be much larger than the mean free path in order for the hydrodynamic limit (where collisions take place so frequently that local equilibrium in the usual sense is reached quickly) to be valid. In the case of kinetic theory with spin, a fourth scale enters through the spacetime shifts, which are of the order of the COMPTON wavelength λ_C [as evidenced by the factor $\hbar/m$ in Eqs. (4.99) and (4.145)] and should therefore be at most of the order of the interaction length scale in order for the quasiparticle picture (and thus kinetic theory) to work. In total we then have the following ordering of scales,

$$\Delta \lesssim \ell_{\text{int}} \ll \lambda_{\text{mfp}} \ll L_{\text{hydro}} \,. \tag{6.1}$$

From these scales, we may construct two dimensionless quantities: first, the usual KNUDSEN number [cf. Eq. (2.66)]

$$\text{Kn} \coloneqq \frac{\lambda_{\text{mfp}}}{L_{\text{hydro}}} \,, \tag{6.2}$$

which controls the applicability of the hydrodynamic limit. Second, we can define a "quantum KNUDSEN number"

$$\varkappa \coloneqq \frac{\Delta}{L_{\text{hydro}}} \ll \text{Kn} \,, \tag{6.3}$$

where the inequality follows from the ordering of scales (6.1). Note that, in principle, one could assign another scale $\ell_{\text{vort}} \sim |\omega|^{-1}$ to the vorticity [82, 83, 159], which is different from the other hydrodynamic gradients since it can be present even in global equilibrium, but for simplicity we will not do so here, i.e., we will assume that $\ell_{\text{vort}} \sim L_{\text{hydro}}$.

Taking into account these dimensionless quantities, it becomes clear that the terms in the second and third lines of Eq. (4.160), which are responsible for the equivalence of local and global equilibrium, are much smaller than the usual hydrodynamic corrections. Thus, we may *redefine* the necessary condition on the local-equilibrium distribution function f_{eq} to be

$$C[f_{\text{eq}}](x,k,\mathfrak{s}) \sim \mathcal{O}(\varkappa) \;, \tag{6.4}$$

such that the collision term does not have to vanish exactly, but rather be sufficiently small. This definition also ensures that the local-equilibrium distribution function only needs to make the local part of the collision term vanish. Then, the local-equilibrium distribution function is given by

$$f_{\text{eq}}(x,k,\mathfrak{s}) = \left\{ \exp\left[-\alpha_0(x) + \beta_0^\mu(x) k_\mu - \sigma \frac{\hbar}{2} \Omega_{0,\mu\nu}(x) \Sigma_{\mathfrak{s}}^{\mu\nu} \right] + a \right\}^{-1} \;, \tag{6.5}$$

where the LAGRANGE multipliers α_0, β_0, and Ω_0 are now arbitrary functions of spacetime, in line with the usual formulation of hydrodynamics. At this point we reiterate that Eq. (6.5) is to be understood in a perturbative way, i.e., in practical calculations we have to use the form

$$f_{\text{eq}}(x,k,\mathfrak{s}) = f_{0\mathbf{k}}(x) \left[1 + \widetilde{f}_{0\mathbf{k}}(x) \sigma \frac{\hbar}{2} \Omega_{0,\mu\nu}(x) \Sigma_{\mathfrak{s}}^{\mu\nu} \right] + \mathcal{O}(\hbar^2) \;, \tag{6.6a}$$

$$f_{0\mathbf{k}}(x) \coloneqq \left\{ \exp\left[-\alpha_0(x) + \beta_0^\mu(x) k_\mu \right] + a \right\}^{-1} \;. \tag{6.6b}$$

Without loss of generality we may split the full one-particle distribution function into equilibrium and dissipative contributions,

$$f(x,k,\mathfrak{s}) = f_{\text{eq}}(x,k,\mathfrak{s}) + \delta f_{\mathbf{k}\mathfrak{s}} \;. \tag{6.7}$$

In order for dissipative hydrodynamics to be a viable approximation, we will require that the inverse REYNOLDS numbers, which have been introduced in Sec. 2.3, are small, $\text{Re}^{-1} \ll 1$. In terms of the distribution function, we have

$$\text{Re}^{-1} \sim \frac{\delta f}{f_{\text{eq}}} \;. \tag{6.8}$$

However, we have to distinguish between different inverse REYNOLDS numbers. Since we may expect the effects of spin to be small, the quantities that originate from the parts of the distribution function which are linear or bilinear in the spin vector $\mathfrak{s}$ should be smaller than the usual dissipative corrections. In the following sections, we will derive hydrodynamic equations up to second order in both KNUDSEN and inverse REYNOLDS numbers.

6.2 The method of moments

While the equilibrium distribution function (6.5) is known, we do not know the functional form of the deviation $\delta f_{\mathbf{k}\mathfrak{s}}$. Following Refs. [51, 83], we will employ the so-called *method of moments* (which is essentially a multipole expansion in momentum space) to isolate the parts of $\delta f_{\mathbf{k}\mathfrak{s}}$ that are important for hydrodynamics.

6.2.1 Expansion of the single-particle distribution function

First, without loss of generality, we may write the deviation from equilibrium as

$$\delta f_{\mathbf{k}\mathfrak{s}} = f_{0\mathbf{k}} \widetilde{f}_{0\mathbf{k}} \left(\phi_{\mathbf{k}} - \mathfrak{s}_\mu \zeta_{\mathbf{k}}^\mu + \mathfrak{s}_\alpha \mathfrak{s}_\beta K_{\mu\nu}^{\alpha\beta} \xi_{\mathbf{k}}^{\mu\nu} \right) \;, \tag{6.9}$$

where $\phi_{\mathbf{k}}$, $\zeta_{\mathbf{k}}$, and $\xi_{\mathbf{k}}$ are functions of the momentum only. Note that this does not constitute an approximation since, for particles of spin $j \leq 1$, the distribution function in extended phase space depends on at most two powers of the spin vector.[1] These functions may then be further expanded as

$$\phi_{\mathbf{k}} = \sum_{\ell=0}^{\infty} \lambda^{\mu_1 \cdots \mu_\ell} k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle} \ , \tag{6.10a}$$

$$\zeta^{\mu}_{\mathbf{k}} = \sum_{\ell=0}^{\infty} \eta^{\mu,\mu_1 \cdots \mu_\ell} k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle} \ , \tag{6.10b}$$

$$\xi^{\mu\nu}_{\mathbf{k}} = \sum_{\ell=0}^{\infty} \vartheta^{\mu\nu,\mu_1 \cdots \mu_\ell} k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle} \ , \tag{6.10c}$$

where the tensors λ, η, and ϑ are functions of the energy in the fluid-rest frame, $E_{\mathbf{k}}$, only. Equations (6.10) constitute expansions in terms of the set of irreducible tensors[2]

$$1,\ k^{\langle\mu\rangle},\ k^{\langle\mu}k^{\nu\rangle},\ k^{\langle\mu}k^{\nu}k^{\lambda\rangle},\ \cdots \ , \tag{6.11}$$

which form a complete and orthogonal basis. Specifically, for any function F depending on $E_{\mathbf{k}}$ only, we have the orthogonality relation

$$\int \mathrm{d}K k^{\langle\mu_1} \cdots k^{\mu_n\rangle} k_{\langle\nu_1} \cdots k_{\nu_m\rangle} F(E_{\mathbf{k}}) = \frac{m!\delta_{mn}}{(2m+1)!!} \Delta^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_m} \int \mathrm{d}K \left(m^2 - E^2_{\mathbf{k}}\right)^m F(E_{\mathbf{k}}) \ . \tag{6.12}$$

The tensors introduced in Eqs. (6.10) can be further expressed as

$$\lambda^{\mu_1\cdots\mu_\ell} = \sum_{n=0}^{\infty} c_n^{\mu_1\cdots\mu_\ell} P^{(0,\ell)}_{\mathbf{k}n} \ , \tag{6.13a}$$

$$\eta^{\mu,\mu_1\cdots\mu_\ell} = \sum_{n=0}^{\infty} d_n^{\mu,\mu_1\cdots\mu_\ell} P^{(1,\ell)}_{\mathbf{k}n} \ , \tag{6.13b}$$

$$\vartheta^{\mu\nu,\mu_1\cdots\mu_\ell} = \sum_{n=0}^{\infty} e_n^{\mu\nu,\mu_1\cdots\mu_\ell} P^{(2,\ell)}_{\mathbf{k}n} \ , \tag{6.13c}$$

where the quantities $P^{(j,\ell)}_{\mathbf{k}n}$ are polynomials in energy fulfilling

$$\int \mathrm{d}K \omega^{(\ell)} P^{(j,\ell)}_{\mathbf{k}m} P^{(j,\ell)}_{\mathbf{k}n} = \delta_{mn} \ , \qquad \omega^{(\ell)} \coloneqq g \frac{W^{(\ell)}}{(2\ell+1)!!} (m^2 - E^2_{\mathbf{k}})^\ell f_{0\mathbf{k}} \widetilde{f}_{0\mathbf{k}} \ . \tag{6.14}$$

In the massless limit, the $P^{(j,\ell)}_{\mathbf{k}n}$ are given by associated LAGUERRE polynomials. From demanding that $P^{(j,\ell)}_{\mathbf{k}0} = 1$, we find the normalization to be

$$W^{(\ell)} = \frac{(-1)^\ell}{J_{2\ell,\ell}} \ , \quad \text{where} \ \ J_{nq} \coloneqq \frac{\partial I_{nq}}{\partial \alpha_0} \ , \tag{6.15}$$

and

$$I_{nq}(\alpha_0, \beta_0) \coloneqq \frac{(-1)^q}{(2q+1)!!} \int \mathrm{d}\Gamma\, E^{n-2q}_{\mathbf{k}} (m^2 - E^2_{\mathbf{k}})^q f_{0\mathbf{k}} \ . \tag{6.16}$$

The thermodynamic integrals I_{nq} defined above are ubiquitous in the following calculations, and we show how to evaluate them analytically for BOLTZMANN statistics in Appendix F.2. Employing the

[1] If one would want to analyze higher-spin particles, higher powers of $\mathfrak{s}$ have to be included. Analogously, for spin-1/2 particles, we have simply $\xi_{\mathbf{k}} = 0$, while for spin-0 particles only $\phi_{\mathbf{k}}$ is nonzero.

[2] Irreducibility is to be understood with respect to the little group of the four-velocity u^μ, i.e., the subgroup of LORENTZ transformations that leave the four-velocity invariant. For massive particles, this group is isomorphic to the rotation group $SO(3)$; the decomposition into irreducible tensors is thus equivalent to an expansion in terms of spherical harmonics [43].

orthogonality relation (6.12) as well as the spin-space integrals (3.167), we can express the tensors introduced in Eqs. (6.13) as integrals over $\delta f_{\mathbf{k}\mathfrak{s}}$,

$$c_n^{\mu_1\cdots\mu_\ell} = \frac{W^{(\ell)}}{\ell!}\int \mathrm{d}\Gamma P_{\mathbf{k}n}^{(0,\ell)} k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}\delta f_{\mathbf{k}\mathfrak{s}} \ , \tag{6.17a}$$

$$d_n^{\mu,\mu_1\cdots\mu_\ell} = \frac{g}{2}\frac{W^{(\ell)}}{\ell!}\int \mathrm{d}\Gamma P_{\mathbf{k}n}^{(1,\ell)} \mathfrak{s}^\mu k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}\delta f_{\mathbf{k}\mathfrak{s}} \ , \tag{6.17b}$$

$$e_n^{\mu\nu,\mu_1\cdots\mu_\ell} = \frac{5g}{8}\frac{W^{(\ell)}}{\ell!}\int \mathrm{d}\Gamma K^{\mu\nu}_{\alpha\beta}\mathfrak{s}^\alpha\mathfrak{s}^\beta P_{\mathbf{k}n}^{(2,\ell)} k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}\delta f_{\mathbf{k}\mathfrak{s}} \ . \tag{6.17c}$$

Subsequently, we define the set of so-called *irreducible moments* of $\delta f_{\mathbf{k}\mathfrak{s}}$,

$$\rho_r^{\mu_1\cdots\mu_\ell} := \int \mathrm{d}\Gamma E_{\mathbf{k}}^r k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}\delta f_{\mathbf{k}\mathfrak{s}} \ , \tag{6.18a}$$

$$\tau_r^{\mu,\mu_1\cdots\mu_\ell} := \int \mathrm{d}\Gamma E_{\mathbf{k}}^r \mathfrak{s}^\mu k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}\delta f_{\mathbf{k}\mathfrak{s}} \ , \tag{6.18b}$$

$$\psi_r^{\mu\nu,\mu_1\cdots\mu_\ell} := \int \mathrm{d}\Gamma E_{\mathbf{k}}^r K^{\mu\nu}_{\alpha\beta}\mathfrak{s}^\alpha\mathfrak{s}^\beta k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}\delta f_{\mathbf{k}\mathfrak{s}} \ , \tag{6.18c}$$

which will constitute the dynamical objects of our theory. The polynomials $P_{\mathbf{k}n}^{(j,\ell)}$ can be expanded in powers of energy as

$$P_{\mathbf{k}n}^{(j,\ell)} = \sum_{r\in\mathbb{S}_\ell^{(j)}}^{n} a_{nr}^{(j,\ell)} E_{\mathbf{k}}^r \ , \tag{6.19}$$

where the coefficients $a_{nr}^{(j,\ell)}$ are determined by Gram-Schmidt orthonormalization [51]. Explicitly, we show how to construct them in Appendix F.3.
By inserting Eq. (6.19) into Eqs. (6.17), we are then able to express the functions $\phi_{\mathbf{k}}$, $\zeta_{\mathbf{k}}$, and $\xi_{\mathbf{k}}$ through the irreducible moments,

$$\phi_{\mathbf{k}} = \sum_{\ell=0}^{\infty}\sum_{n\in\mathbb{S}_\ell^{(0)}} \mathcal{H}_{\mathbf{k}n}^{(0,\ell)}\rho_n^{\mu_1\cdots\mu_\ell} k_{\langle\mu_1}\cdots k_{\mu_\ell\rangle} \ , \tag{6.20a}$$

$$\zeta_{\mathbf{k}}^\mu = \sum_{\ell=0}^{\infty}\sum_{n\in\mathbb{S}_\ell^{(1)}} \mathcal{H}_{\mathbf{k}n}^{(1,\ell)}\tau_n^{\mu,\mu_1\cdots\mu_\ell} k_{\langle\mu_1}\cdots k_{\mu_\ell\rangle} \ , \tag{6.20b}$$

$$\xi_{\mathbf{k}}^{\mu\nu} = \sum_{\ell=0}^{\infty}\sum_{n\in\mathbb{S}_\ell^{(2)}} \mathcal{H}_{\mathbf{k}n}^{(2,\ell)}\psi_n^{\mu\nu,\mu_1\cdots\mu_\ell} k_{\langle\mu_1}\cdots k_{\mu_\ell\rangle} \ , \tag{6.20c}$$

where we introduced

$$\mathcal{H}_{\mathbf{k}n}^{(j,\ell)} := \widetilde{g}^{(j)}\frac{W^{(\ell)}}{\ell!}\sum_{m\in\mathbb{S}_\ell^{(j)}} P_{\mathbf{k}m}^{(j,\ell)} a_{mn}^{(j,\ell)} \ , \qquad \widetilde{g}^{(j)} := \begin{cases} 1 & , \ j=0 \ , \\ g/2 & , \ j=1 \ , \\ 5g/8 & , \ j=2 \ . \end{cases} \tag{6.21}$$

The quantity $\mathbb{S}_\ell^{(j)}$ denotes the set of moments of spin-rank j and tensor-rank ℓ that are included in the employed basis,[3] and thus quantifies the *truncation* that is used for practical calculations. To obtain exact results, the size of the basis should be infinite and thus $\mathbb{S}_\ell^{(j)} \to \mathbb{N}_0$.

[3] For comparison with, e.g., Ref. [51] it should be noted that sums running from 0 to N_ℓ appearing in this reference correspond to $\mathbb{S}_\ell^{(0)} = \{0, 1, \cdots, N_\ell\}$.

Note that, when combining Eqs. (6.9), (6.18), and (6.20), we can express any moment in terms of all others with the same tensor-rank in momentum and spin,

$$\rho_r^{\mu_1\cdots\mu_\ell} = \sum_{n\in\mathbb{S}_\ell^{(0)}} \mathcal{F}_{-r,n}^{(0,\ell)} \rho_n^{\mu_1\cdots\mu_\ell} \,, \tag{6.22a}$$

$$\tau_r^{\mu,\mu_1\cdots\mu_\ell} = \sum_{n\in\mathbb{S}_\ell^{(1)}} \mathcal{F}_{-r,n}^{(1,\ell)} \tau_n^{\mu,\mu_1\cdots\mu_\ell} \,, \tag{6.22b}$$

$$\psi_r^{\mu\nu,\mu_1\cdots\mu_\ell} = \sum_{n\in\mathbb{S}_\ell^{(2)}} \mathcal{F}_{-r,n}^{(2,\ell)} \psi_n^{\mu\nu,\mu_1\cdots\mu_\ell} \,. \tag{6.22c}$$

Here we introduced the thermodynamic integrals

$$\mathcal{F}_{rn}^{(j,\ell)} := \frac{1}{\widetilde{g}^{(j)}} \frac{\ell!}{(2\ell+1)!!} \int \mathrm{d}\Gamma f_{0\mathbf{k}} \widetilde{f}_{0\mathbf{k}} E_{\mathbf{k}}^{-r} \mathcal{H}_{\mathbf{k}n}^{(j,\ell)} \left(m^2 - E_{\mathbf{k}}^2\right)^\ell \,. \tag{6.23}$$

The expressions (6.22) are exact if the irreducible moment on the left-hand side is included in the basis, and an approximation in the case that it is not. In particular, Eqs. (6.22) can be used to approximate moments with $r < 0$.

Lastly, we remark that not all components of the irreducible moments of nonzero spin-rank [cf. Eqs. (6.18b) and (6.18c)] are independent due to the spin vector fulfilling $k \cdot \mathfrak{s} = 0$. To see this, consider the projection of the moments of spin-rank one onto the four-velocity,

$$u_\mu \tau_r^{\mu,\mu_1\cdots\mu_\ell} = -\int \mathrm{d}\Gamma E_{\mathbf{k}}^{r-1} \mathfrak{s}_\mu k^{\langle\mu\rangle} k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle} \delta f_{\mathbf{k}\mathfrak{s}} \,. \tag{6.24}$$

The expression above shows that the component of $\tau_r^{\mu,\mu_1\cdots\mu_\ell}$ that is parallel to the fluid four-velocity in its first index is not an independent quantity; a similar reasoning also holds for the components of $\psi_r^{\mu\nu,\mu_1\cdots\mu_\ell}$ parallel to u^μ in any of the first two indices. In order to explicitly remove these dependent components, we make use of the fact that $\mathfrak{s}$ and $\zeta_{\mathbf{k}}$ are orthogonal to the four-momentum and write

$$\mathfrak{s}_\mu \zeta_{\mathbf{k}}^\mu = \mathfrak{s}_\mu \left(\Delta^\mu{}_\nu + u^\mu u_\nu\right) \zeta_{\mathbf{k}}^\nu = \mathfrak{s}_\mu \left(\Delta^\mu{}_\nu + \frac{k^{\langle\mu\rangle} k_{\langle\nu\rangle}}{E_{\mathbf{k}}^2}\right) \zeta_{\mathbf{k}}^{\langle\nu\rangle} \equiv \mathfrak{s}_\mu \Xi^\mu{}_\nu \zeta_{\mathbf{k}}^{\langle\nu\rangle} \,, \tag{6.25}$$

where we defined

$$\Xi^{\mu\nu} := \Delta^{\mu\nu} + \frac{k^{\langle\mu\rangle} k^{\langle\nu\rangle}}{E_{\mathbf{k}}^2} \,. \tag{6.26}$$

Similarly, we can rewrite

$$\mathfrak{s}_\alpha \mathfrak{s}_\beta K_{\mu\nu}^{\alpha\beta} \xi_{\mathbf{k}}^{\mu\nu} = \mathfrak{s}_\alpha \mathfrak{s}_\beta K_{\gamma\delta}^{\alpha\beta} \Xi_{\mu\nu}^{\gamma\delta} \xi_{\mathbf{k}}^{\langle\mu\nu\rangle} \,, \tag{6.27}$$

where we introduced accordingly

$$\Xi_{\alpha\beta}^{\mu\nu} := \frac{1}{2}\left(\Xi^\mu{}_\alpha \Xi^\nu{}_\beta + \Xi^\nu{}_\alpha \Xi^\mu{}_\beta\right) - \frac{1}{\Xi^2} \Xi^{\mu\gamma} \Xi^\nu{}_\gamma \Xi^\delta{}_\alpha \Xi_{\delta\beta} \,, \tag{6.28}$$

with $\Xi^2 := \Xi^{\mu\nu}\Xi_{\mu\nu} = 2 + m^4/E_{\mathbf{k}}^4$. Using this method to only retain the independent components of the irreducible moments, the expansion of the deviation of the single-particle distribution function from local equilibrium assumes the following form,

$$\begin{aligned} \delta f_{\mathbf{k}\mathfrak{s}} = f_{0\mathbf{k}} \widetilde{f}_{0\mathbf{k}} \sum_{\ell=0}^{\infty} k_{\langle\mu_1} \cdots k_{\mu_\ell\rangle} \Bigg(& \sum_{n\in\mathbb{S}_\ell^{(0)}} \mathcal{H}_{\mathbf{k}n}^{(0,\ell)} \rho_n^{\mu_1\cdots\mu_\ell} - \mathfrak{s}^\mu \Xi_{\mu\nu} \sum_{n\in\mathbb{S}_\ell^{(1)}} \mathcal{H}_{\mathbf{k}n}^{(1,\ell)} \tau_n^{\langle\nu\rangle,\mu_1\cdots\mu_\ell} \\ & + \mathfrak{s}_\alpha \mathfrak{s}_\beta K_{\gamma\delta}^{\alpha\beta} \Xi_{\mu\nu}^{\gamma\delta} \sum_{n\in\mathbb{S}_\ell^{(2)}} \mathcal{H}_{\mathbf{k}n}^{(2,\ell)} \psi_n^{\langle\mu\nu\rangle,\mu_1\cdots\mu_\ell} \Bigg) \,. \end{aligned} \tag{6.29}$$

Up to this point, as long as all irreducible moments are included, no approximation has been made, i.e., knowing the evolution of all irreducible moments is equivalent to solving the full Boltzmann equation. However, in order to arrive at a finite set of equations of motion describing the hydrodynamic degrees of freedom, we need to establish a connection between the kinetic expressions of the conserved currents and the irreducible moments.

6.2.2 Conserved currents and ideal spin hydrodynamics

In this thesis, we consider the modified GLW pseudogauge, i.e., Eq. (3.60) for spin 0, Eqs. (3.127) and (3.128) for spin 1/2, and Eqs. (3.195) and (3.196) for spin 1. Supplementing the energy-momentum and spin tensors listed in those equations by an expression for the particle-number current, cf. Eq. (3.54), we have up to order $\mathcal{O}(\hbar)$

$$N^\mu(x) = \int \mathrm{d}\Gamma k^\mu f(x,k,\mathfrak{s}) \; , \tag{6.30}$$

$$T^{\mu\nu}(x) = \int \mathrm{d}\Gamma k^\mu k^\nu f(x,k,\mathfrak{s}) \; , \tag{6.31}$$

$$S^{\lambda\mu\nu}(x) = \sigma \int \mathrm{d}\Gamma k^\lambda \Sigma_{\mathfrak{s}}^{\mu\nu} f(x,k,\mathfrak{s}) \; . \tag{6.32}$$

Note that the spin-0 case, where the spin tensor vanishes, is included in this formulation. Considering the definitions of the irreducible moments (6.18), we can express the particle-number density and the diffusion current as

$$n_0 = u_\mu N_0^\mu \quad = I_{10} \; , \tag{6.33a}$$
$$n^\mu = \Delta^{\mu\nu}\delta N_\nu = \rho_0^\mu \; . \tag{6.33b}$$

The matching condition (2.50b) lets the dissipative part of the particle-number density vanish,

$$\delta n = u_\mu \delta N^\mu = \rho_1 = 0 \; . \tag{6.34}$$

The components of the energy-momentum tensor read

$$\varepsilon_0 = u_\mu u_\nu T_0^{\mu\nu} \quad = I_{20} \; , \tag{6.35a}$$
$$P_0 = -\frac{1}{3}\Delta_{\mu\nu}T_0^{\mu\nu} \quad = I_{21} \; , \tag{6.35b}$$
$$\Pi = -\frac{1}{3}\Delta_{\mu\nu}\delta T^{\mu\nu} = -\frac{m^2}{3}\rho_0 \; , \tag{6.35c}$$
$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\delta T^{\alpha\beta} \quad = \rho_0^{\mu\nu} \; . \tag{6.35d}$$

Here, we already used the matching conditions (2.50a) and (2.52), which imply that

$$\delta\varepsilon = u_\mu u_\nu \delta T^{\mu\nu} \quad = \rho_2 = 0 \; , \tag{6.36}$$
$$h^\mu = \Delta^{\mu\alpha}u^\beta \delta T_{\alpha\beta} = \rho_1^\mu = 0 \; . \tag{6.37}$$

Since the variables characterizing local equilibrium in kinetic theory are α_0 and β_0, it is advisable to rewrite the equations of motion for ε_0 and n_0 (2.54) in terms of these quantities. By employing the relations

$$\dot{\varepsilon}_0 = J_{20}\dot{\alpha}_0 - J_{30}\dot{\beta}_0 \; , \qquad \dot{n}_0 = J_{10}\dot{\alpha}_0 - J_{20}\dot{\beta}_0 \; , \tag{6.38}$$

we find

$$\dot{\alpha}_0 = \frac{1}{D_{20}}\left\{-J_{30}\left(n_0\theta + \partial_\mu n^\mu\right) + J_{20}\left[\left(\varepsilon_0 + P_0 + \Pi\right)\theta - \pi^{\mu\nu}\sigma_{\mu\nu}\right]\right\} \; , \tag{6.39a}$$

$$\dot{\beta}_0 = \frac{1}{D_{20}}\left\{-J_{20}\left(n_0\theta + \partial_\mu n^\mu\right) + J_{10}\left[\left(\varepsilon_0 + P_0 + \Pi\right)\theta - \pi^{\mu\nu}\sigma_{\mu\nu}\right]\right\} \; , \tag{6.39b}$$

$$\dot{u}^\mu = \frac{1}{\varepsilon_0 + P_0}\left(F^\mu + \nabla^\mu \Pi - \Pi \dot{u}^\mu + \pi^{\mu\nu}\dot{u}_\nu - \Delta^{\mu\nu}\nabla^\alpha \pi_{\nu\alpha}\right) \; , \tag{6.39c}$$

where we defined

$$D_{nq} \coloneqq J_{n+1,q}J_{n-1,q} - J_{nq}^2 \; . \tag{6.40}$$

At this point, we remark that the relation[4]

$$F^\mu = \frac{n_0}{\beta_0}I^\mu - \frac{\varepsilon_0 + P_0}{\beta_0}\nabla^\mu \beta_0 \tag{6.41}$$

[4]We remind the reader that $F^\mu \coloneqq \nabla^\mu P_0$, $I^\mu \coloneqq \nabla^\mu \alpha_0$.

holds.

In the more involved case of the spin tensor, we first insert the local-equilibrium distribution function (6.6a) into Eq. (6.32) to obtain

$$\begin{aligned} S_0^{\lambda\mu\nu} &= \frac{2\sigma^2\hbar}{gm^2}\Big[u^\lambda u^{[\mu}\kappa_0^{\nu]}(m^2 J_{10}-J_{30}+J_{31}) + u^\lambda\epsilon^{\mu\nu\alpha\beta}u_\alpha\omega_{0,\beta}(m^2J_{10}+2J_{31}) \\ &\qquad -u^{[\mu}\epsilon^{\nu]\lambda\alpha\beta}u_\alpha\omega_{0,\beta}J_{31} + \Delta^{\lambda[\mu}\kappa_0^{\nu]}J_{31}\Big] \\ &= \frac{2\sigma^2\hbar}{gm^2}\Big[-2u^\lambda u^{[\mu}\kappa_0^{\nu]}J_{31} + u^\lambda\epsilon^{\mu\nu\alpha\beta}u_\alpha\omega_{0,\beta}(J_{30}-J_{31}) \\ &\qquad -u^{[\mu}\epsilon^{\nu]\lambda\alpha\beta}u_\alpha\omega_{0,\beta}J_{31} + \Delta^{\lambda[\mu}\kappa_0^{\nu]}J_{31}\Big] , \end{aligned} \tag{6.42}$$

where we used the fact that integrals of an odd number of spin vectors over spin space vanish, and decomposed the spin potential according to Eq. (2.73). Furthermore, we employed that $m^2J_{10} = J_{30} - 3J_{31}$. Comparing this expression to the more general one (2.75), we find the following explicit expressions for the quantities $A_0, \cdots, E_0$ in terms of thermodynamic integrals,

$$A_0 \equiv \frac{2\sigma^2\hbar}{g}J_{10} , \qquad B_0 \equiv \frac{2\sigma^2\hbar}{gm^2}J_{30} , \qquad C_0 = D_0 = E_0 \equiv -\frac{2\sigma^2\hbar}{gm^2}J_{31} . \tag{6.43}$$

Making use of the definitions of the irreducible moments of spin-rank one [cf. Eq. (6.18b)], the matching condition (2.78) yields

$$u_\lambda \delta S^{\lambda\mu\nu} = -\frac{\sigma}{m}\epsilon^{\mu\nu\alpha\beta}\left(u_\alpha\tau_{2,\beta} + \tau_{1,\beta,\alpha}\right) = 0 , \tag{6.44}$$

from which it follows that

$$\tau_1^{[\mu,\nu]} = u^{[\mu}\tau_2^{\nu]} . \tag{6.45}$$

Note that only the antisymmetric part of the moment of spin- and momentum-rank one is determined by τ_2^μ, while the symmetric part remains unconstrained. Then, the dissipative part of the spin tensor takes the form

$$\delta S^{\lambda\mu\nu} = -\frac{\sigma}{m}\epsilon^{\mu\nu\alpha\beta}\left[\frac{1}{2}u_\alpha\tau_{1,(\langle\beta\rangle,}{}^{\langle\lambda\rangle)} + \frac{1}{3}\Delta^\lambda{}_\alpha\left(m^2\tau_{0,\beta}-\tau_{2,\beta}\right) + \tau_{0,\beta,\alpha}{}^\lambda\right] . \tag{6.46}$$

In the following, we will rename the moments that appear in the equation above as[5]

$$\mathfrak{p}^\mu := \tau_0^{\langle\mu\rangle} , \qquad \mathfrak{n}^\mu := \tau_2^\mu , \qquad \mathfrak{z}^{\mu\nu} := \tau_1^{(\langle\mu\rangle,\langle\nu\rangle)} , \qquad \mathfrak{q}^{\lambda\mu\nu} := \tau_0^{\lambda,\mu\nu} . \tag{6.47}$$

It should be noted that these types of moments are connected to certain higher-order ones, as we can see by employing Eq. (6.24):

$$u_\mu\tau_r^\mu = -\tau_{r-1\mu}^{\mu,} , \tag{6.48a}$$

$$u_\mu\tau_r^{\mu,\nu} = -\tau_{r-1\mu}^{\mu,\nu} - \frac{1}{3}\left(m^2\tau_{r-1}^{\langle\nu\rangle} - \tau_{r+1}^{\langle\nu\rangle}\right) , \tag{6.48b}$$

$$u_\mu\tau_r^{\mu,\nu\lambda} = -\tau_{r-1\ \mu}^{\mu,\nu\lambda} - \frac{2}{5}\left(m^2\tau_{r-1}^{\langle\nu,\lambda\rangle} - \tau_{r+1}^{\langle\nu,\lambda\rangle}\right) . \tag{6.48c}$$

Inserting Eqs. (6.43) and (6.46) into the equations of motion for the components of Ω_0 (2.80), we find the equation of motion for κ_0 to be

$$\begin{aligned} \frac{4\sigma^2\hbar}{gm}J_{31}\dot{\kappa}_0^{\langle\mu\rangle} &= \frac{4\sigma^2\hbar}{gm}\Big\{-\kappa_0^\mu\left(K_{31}\dot{\alpha}_0 - K_{41}\dot{\beta}_0 + \frac{4}{3}J_{31}\theta\right) - \frac{1}{2}J_{30}\epsilon^{\mu\nu\alpha\beta}u_\nu\dot{u}_\alpha\omega_{0,\beta} \\ &\quad + \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu\left[J_{31}\nabla_\alpha\omega_{0,\beta} + \omega_{0,\beta}\left(K_{31}I_\alpha - K_{41}\nabla_\alpha\beta_0\right)\right] \\ &\quad + \frac{1}{2}J_{31}\left(\sigma^{\mu\nu}+\omega^{\mu\nu}\right)\kappa_{0,\nu}\Big\} + \sigma\epsilon^{\mu\nu\alpha\beta}u_\nu\Big[\frac{1}{2}(\sigma_{\lambda\alpha}+\omega_{\lambda\alpha})\mathfrak{z}_\beta{}^\lambda \\ &\quad + \frac{1}{3}(\nabla_\alpha - \dot{u}_\alpha)(m^2\mathfrak{p}_\beta - \mathfrak{n}_\beta) + (\nabla^\lambda - \dot{u}^\lambda)\mathfrak{q}_{\beta\alpha\lambda}\Big] - \frac{m}{\hbar}T^{[\mu\nu]}u_\nu , \end{aligned} \tag{6.49a}$$

[5] In contrast to Ref. [82], we define $\mathfrak{p}^\mu$ to be orthogonal to the fluid four-velocity.

whereas the evolution of ω_0 is determined by

$$\begin{aligned}\frac{2\sigma^2\hbar}{gm}(J_{30}-J_{31})\dot{\omega}_0^{\langle\mu\rangle} = &-\frac{2\sigma^2\hbar}{gm}\bigg\{\left[(K_{30}-K_{31})\dot{\alpha}_0-(K_{40}-K_{41})\dot{\beta}_0+\left(J_{30}-\frac{1}{3}J_{31}\right)\theta\right]\omega_0^\mu \\ &+\epsilon^{\mu\nu\alpha\beta}u_\nu\left[J_{31}\nabla_\alpha\kappa_{0,\beta}+\kappa_{0,\beta}(K_{31}I_\alpha-K_{41}\nabla_\alpha\beta_0)-3J_{31}\dot{u}_\alpha\kappa_{0,\beta}\right] \\ &-J_{31}(\sigma^{\mu\nu}+\omega^{\mu\nu})\omega_{0,\nu}\bigg\}-\sigma\left[\frac{1}{2}(\dot{u}_\lambda-\nabla_\lambda)\mathfrak{z}^{\lambda\mu}+u_\nu\Delta^\mu{}_\alpha(\nabla_\lambda-\dot{u}_\lambda)\mathfrak{q}^{[\nu\alpha]\lambda}\right. \\ &\left.-\frac{1}{3}\left(\sigma^{\mu\nu}+\omega^{\mu\nu}-\frac{2}{3}\theta\Delta^{\mu\nu}\right)(m^2\mathfrak{p}_\nu-\mathfrak{n}_\nu)\right]-\frac{m}{\hbar}\epsilon^{\mu\nu\alpha\beta}u_\nu T_{\alpha\beta}\;. \end{aligned} \tag{6.49b}$$

Here we introduced the thermodynamic integrals

$$K_{nq} \coloneqq \frac{\partial J_{nq}}{\partial\alpha_0}\equiv\frac{\partial^2 I_{nq}}{\partial\alpha_0^2}\;. \tag{6.50}$$

Note that the last terms in the equations above are not problematic in the limit of $\hbar\to 0$, since the antisymmetric part of the energy-momentum tensor is at least of second order in $\hbar$.

The system of eleven equations which is given by Eqs. (6.39) and (6.49) specifies the time evolution of all components of the Lagrange multipliers α_0, β_0^μ, and $\Omega_0^{\mu\nu}$, and thus provides the foundation of ideal spin hydrodynamics. In order to close the system of equations, we need to determine the evolution of the dissipative quantities Π, n^μ, $\pi^{\mu\nu}$, $\mathfrak{p}^\mu$, $\mathfrak{z}^{\mu\nu}$, and $\mathfrak{q}^{\lambda\mu\nu}$. Note that we do not need to consider the evolution of $\mathfrak{n}^\mu$, since its independent components are determined by the matching condition (6.45). In particular, using Eq. (6.24), we have

$$\frac{2}{3}\mathfrak{n}^{\langle\mu\rangle}=-\mathfrak{q}^{\nu\mu}{}_\nu-\frac{m^2}{3}\mathfrak{p}^\mu\;. \tag{6.51}$$

The antisymmetric part of $T^{\mu\nu}$

We still need to express the antisymmetric part of the energy-momentum tensor which appears in Eqs. (6.49) in terms of kinetic quantities. When taking the divergence of the spin tensor, we find

$$\begin{aligned}\hbar\partial_\lambda S^{\lambda\mu\nu}=&\,\hbar\sigma\frac{1}{2}\int[\mathrm{d}\Gamma](2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\widetilde{\mathcal{W}}^{(\sigma)}\Sigma_{\mathfrak{s}}^{\mu\nu} \\ &\times\Big[f(x+\Delta_1,k_1,\mathfrak{s}_1)f(x+\Delta_2,k_2,\mathfrak{s}_2)\widetilde{f}(x+\Delta',k',\mathfrak{s}')\widetilde{f}(x+\Delta,k,\mathfrak{s}) \\ &-\widetilde{f}(x+\Delta_1,k_1,\mathfrak{s}_1)\widetilde{f}(x+\Delta_2,k_2,\mathfrak{s}_2)f(x+\Delta',k',\mathfrak{s}')f(x+\Delta,k,\mathfrak{s})\Big]\;,\end{aligned} \tag{6.52}$$

where we introduced $[\mathrm{d}\Gamma]\coloneqq\mathrm{d}\Gamma\,\mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'$. Note that we neglected a term $\sim\Delta\cdot\partial(f_1f_2\widetilde{f}'\widetilde{f}-\widetilde{f}_1\widetilde{f}_2f'f)$, since it is at least of order $\mathcal{O}(\hbar\varkappa\mathrm{Re}^{-1})$. Furthermore, we already employed the weak equivalence principle, which is applicable due to $\Sigma_{\mathfrak{s}}^{\mu\nu}$ being linear in the spin vector and the assumption (4.155) on the transition rate. Subsequently, we may use the symmetries of the collision term as well as the conservation of the total angular momentum (4.152) to write

$$\begin{aligned}\hbar\partial_\lambda S^{\lambda\mu\nu}=&-\frac{1}{8}\int[\mathrm{d}\Gamma](2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\widetilde{\mathcal{W}}^{(\sigma)}\left(k^{[\nu}\Delta^{\mu]}+k'^{[\nu}\Delta'^{\mu]}-k_1^{[\nu}\Delta_1^{\mu]}-k_2^{[\nu}\Delta_2^{\mu]}\right) \\ &\times\Big[f(x+\Delta_1,k_1,\mathfrak{s}_1)f(x+\Delta_2,k_2,\mathfrak{s}_2)\widetilde{f}(x+\Delta',k',\mathfrak{s}')\widetilde{f}(x+\Delta,k,\mathfrak{s}) \\ &-\widetilde{f}(x+\Delta_1,k_1,\mathfrak{s}_1)\widetilde{f}(x+\Delta_2,k_2,\mathfrak{s}_2)f(x+\Delta',k',\mathfrak{s}')f(x+\Delta,k,\mathfrak{s})\Big]\;.\end{aligned} \tag{6.53}$$

Then, by employing the symmetries of the collision term again, we find a familiar form for the antisymmetric part of the energy-momentum tensor,

$$\begin{aligned}T^{[\mu\nu]}=&\,\frac{1}{2}\int[\mathrm{d}\Gamma](2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\widetilde{\mathcal{W}}^{(\sigma)}\Delta^{[\mu}k^{\nu]} \\ &\times\Big[f(x,k_1,\mathfrak{s}_1)f(x,k_2,\mathfrak{s}_2)\widetilde{f}(x,k',\mathfrak{s}')\widetilde{f}(x,k,\mathfrak{s})-\widetilde{f}(x,k_1,\mathfrak{s}_1)\widetilde{f}(x,k_2,\mathfrak{s}_2)f(x,k',\mathfrak{s}')f(x,k,\mathfrak{s})\Big]\;.\end{aligned} \tag{6.54}$$

Here we did not consider the nonlocal contributions to the distribution functions since expressions which are quadratic in those shifts go beyond our truncation. Note that, while we derived it here via the conservation of the total angular momentum, we can also compute $T^{[\mu\nu]}$ explicitly, as is demonstrated in Appendix B.3. After splitting the distribution function into local-equilibrium and dissipative parts and only considering terms linear in gradients or dissipative quantities, we find

$$\begin{aligned}\frac{m}{\hbar}T^{[\mu\nu]} = \frac{1}{2}\int [\mathrm{d}\Gamma](2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\widetilde{\mathcal{W}}^{(\sigma)}\frac{m}{\hbar}\Delta^{[\mu}k^{\nu]}f_{0\mathbf{k}}f_{0\mathbf{k}'}\widetilde{f}_{0\mathbf{k}_1}\widetilde{f}_{0\mathbf{k}_2} \\ \times\left[-\frac{\sigma\hbar}{2m}\left(\widetilde{\Omega}_{0,\alpha\beta}-\widetilde{\varpi}_{\alpha\beta}\right)\left(k_1^\alpha\mathfrak{s}_1^\beta+k_2^\alpha\mathfrak{s}_2^\beta-k^\alpha\mathfrak{s}^\beta-k'^\alpha\mathfrak{s}'^\beta\right)-\mathfrak{s}_\alpha\left(\zeta^\alpha_{\mathbf{k}_1}+\zeta^\alpha_{\mathbf{k}_2}-\zeta^\alpha_{\mathbf{k}}-\zeta^\alpha_{\mathbf{k}'}\right)\right],\end{aligned} \tag{6.55}$$

where we defined $\widetilde{\Omega}^{\mu\nu} := \epsilon^{\mu\nu\alpha\beta}\Omega_{\alpha\beta}$. Here, we already multiplied by a factor of $m/\hbar\sim\lambda_C^{-1}$ as it appears in the conservation equations (6.49). Furthermore, since $\Sigma^{\mu\nu}_{\mathfrak{s}}$ is linear in the spin vector, only the moments $\tau_r^{\mu,\mu_1\cdots\mu_\ell}$ can contribute.

In order to get a clearer interpretation of Eq. (6.55), we have to ask on which quantities it can depend, which are constrained by the requirement that $T^{[\mu\nu]}$ is an antisymmetric second-rank tensor. We can see that the only terms dependent on the local-equilibrium quantities that fulfill the required symmetries are the components of the difference between the spin potential and the thermal vorticity, i.e.,

$$u^{[\mu}\left(\Omega_0^{\nu]\alpha}-\varpi^{\nu]\alpha}\right)u_\alpha \qquad \text{and} \qquad \left(\Omega_0^{\langle\mu\rangle\langle\nu\rangle}-\varpi^{\langle\mu\rangle\langle\nu\rangle}\right). \tag{6.56}$$

Note that these contributions do not have to be present with the same coefficient. In the dissipative sector, we have to remember that the moments $\tau_r^{\mu,\mu_1\cdots\mu_\ell}$ transform as axial vectors in the first Lorentz index. Then, the only types of tensors that we can build which transform appropriately are the duals of $u_\alpha\tau_{r,\beta}$, $\tau_{n,\alpha,\beta}$, and $u_\alpha t_{r,\beta}$, where $t_r^\mu := \tau_r^{\alpha,\mu}{}_\alpha$. Thus, we can write the antisymmetric part of the energy-momentum tensor as

$$\begin{aligned}\frac{m}{\hbar}T^{[\mu\nu]} = -\Gamma^{(\kappa)}u^{[\mu}\left(\kappa_0^{\nu]}+\varpi^{\nu]\alpha}u_\alpha\right)+\Gamma^{(\omega)}\left(\epsilon^{\mu\nu\alpha\beta}u_\alpha\omega_{0,\beta}-\varpi^{\langle\mu\rangle\langle\nu\rangle}\right) \\ +\epsilon^{\mu\nu\alpha\beta}\left(u_\alpha\sum_{n\in\mathbb{S}_0^{(1)}}\gamma_n^{(0)}\tau_{n,\beta}+\sum_{n\in\mathbb{S}_1^{(1)}}\gamma_n^{(1)}\tau_{n,\langle\alpha\rangle,\beta}+u_\alpha\sum_{n\in\mathbb{S}_2^{(1)}}\gamma_n^{(2)}t_{n,\beta}\right),\end{aligned} \tag{6.57}$$

where we defined the coefficients

$$\begin{aligned}\Gamma^{(\kappa)} := \frac{\sigma}{12}\int[\mathrm{d}\Gamma](2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\widetilde{\mathcal{W}}^{(\sigma)}\Delta^{[\mu}k^{\nu]}f_{0\mathbf{k}}f_{0\mathbf{k}'}\widetilde{f}_{0\mathbf{k}_1}\widetilde{f}_{0\mathbf{k}_2} \\ \times u_\mu\Delta^\rho_\nu u^\sigma\epsilon_{\rho\sigma\alpha\beta}\left(k_1^\alpha\mathfrak{s}_1^\beta+k_2^\alpha\mathfrak{s}_2^\beta-k^\alpha\mathfrak{s}^\beta-k'^\alpha\mathfrak{s}'^\beta\right),\end{aligned} \tag{6.58a}$$

$$\begin{aligned}\Gamma^{(\omega)} := -\frac{\sigma}{24}\int[\mathrm{d}\Gamma](2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\widetilde{\mathcal{W}}^{(\sigma)}\Delta^{[\mu}k^{\nu]}f_{0\mathbf{k}}f_{0\mathbf{k}'}\widetilde{f}_{0\mathbf{k}_1}\widetilde{f}_{0\mathbf{k}_2} \\ \times\Delta^\rho_\mu\Delta^\sigma_\nu\epsilon_{\rho\sigma\alpha\beta}\left(k_1^\alpha\mathfrak{s}_1^\beta+k_2^\alpha\mathfrak{s}_2^\beta-k^\alpha\mathfrak{s}^\beta-k'^\alpha\mathfrak{s}'^\beta\right),\end{aligned} \tag{6.58b}$$

which are related to the components of the spin potential, as well as the quantities

$$\begin{aligned}\gamma_n^{(0)} := \frac{1}{6}\int[\mathrm{d}\Gamma](2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\widetilde{\mathcal{W}}^{(\sigma)}\frac{m}{\hbar}\Delta^\mu k^\nu f_{0\mathbf{k}}f_{0\mathbf{k}'}\widetilde{f}_{0\mathbf{k}_1}\widetilde{f}_{0\mathbf{k}_2} \\ \times\epsilon_{\mu\nu\alpha\beta}u^\alpha\Xi^{\beta\gamma}\mathfrak{s}_\gamma\left(\mathcal{H}^{(1,0)}_{\mathbf{k}_1n}+\mathcal{H}^{(1,0)}_{\mathbf{k}_2n}-\mathcal{H}^{(1,0)}_{\mathbf{k}n}-\mathcal{H}^{(1,0)}_{\mathbf{k}'n}\right),\end{aligned} \tag{6.58c}$$

$$\begin{aligned}\gamma_n^{(1)} := \frac{1}{24}\int[\mathrm{d}\Gamma](2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\widetilde{\mathcal{W}}^{(\sigma)}\frac{m}{\hbar}\Delta^\mu k^\nu f_{0\mathbf{k}}f_{0\mathbf{k}'}\widetilde{f}_{0\mathbf{k}_1}\widetilde{f}_{0\mathbf{k}_2} \\ \times\epsilon_{\mu\nu\alpha\beta}\Xi^{\alpha\gamma}\mathfrak{s}_\gamma\left(\mathcal{H}^{(1,1)}_{\mathbf{k}_1n}k_1^{\langle\beta\rangle}+\mathcal{H}^{(1,1)}_{\mathbf{k}_2n}k_2^{\langle\beta\rangle}-\mathcal{H}^{(1,1)}_{\mathbf{k}n}k^{\langle\beta\rangle}-\mathcal{H}^{(1,1)}_{\mathbf{k}'n}k'^{\langle\beta\rangle}\right),\end{aligned} \tag{6.58d}$$

$$\begin{aligned}\gamma_n^{(2)} := \frac{1}{10}\int[\mathrm{d}\Gamma](2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\widetilde{\mathcal{W}}^{(\sigma)}\frac{m}{\hbar}\Delta^\mu k^\nu f_{0\mathbf{k}}f_{0\mathbf{k}'}\widetilde{f}_{0\mathbf{k}_1}\widetilde{f}_{0\mathbf{k}_2} \\ \times\epsilon_{\mu\nu\alpha\beta}u^\alpha\Xi_{\delta\gamma}\mathfrak{s}^\gamma\left(\mathcal{H}^{(1,2)}_{\mathbf{k}_1n}k_1^{\langle\beta}k_1^{\delta\rangle}+\mathcal{H}^{(1,2)}_{\mathbf{k}_2n}k_2^{\langle\beta}k_2^{\delta\rangle}-\mathcal{H}^{(1,2)}_{\mathbf{k}n}k^{\langle\beta}k^{\delta\rangle}-\mathcal{H}^{(1,2)}_{\mathbf{k}'n}k'^{\langle\beta}k'^{\delta\rangle}\right),\end{aligned} \tag{6.58e}$$

which multiply the respective irreducible moments.
Then, using Eq. (6.41) to replace the gradients of β_0 and introducing the vorticity vector $\omega^\mu := \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu\omega_{\alpha\beta}$, the quantities that appear in the evolution equations for κ_0^μ and ω_0^μ read

$$-\frac{m}{\hbar}T^{[\mu\nu]}u_\nu = -\Gamma^{(\kappa)}\left[\kappa_0^\mu + \frac{\beta_0}{2}\left(\frac{F^\mu}{\varepsilon_0+P_0} + \dot{u}^\mu\right)\right] + \frac{\Gamma^{(\kappa)}}{2h}I^\mu - \sum_{n\in\mathbb{S}_1^{(1)}}\gamma_n^{(1)}w_n^\mu \,, \tag{6.59a}$$

$$-\frac{m}{\hbar}\epsilon^{\mu\nu\alpha\beta}u_\nu T_{\alpha\beta} = -\Gamma^{(\omega)}\left(\omega_0^\mu + \beta_0\omega^\mu\right) - \sum_{n\in\mathbb{S}_0^{(1)}}\gamma_n^{(0)}\tau_n^\mu - \sum_{n\in\mathbb{S}_2^{(1)}}\gamma_n^{(2)}t_n^\mu \,, \tag{6.59b}$$

where we also defined the dual of the irreducible moments of spin-rank one and momentum rank two,

$$w_r^\mu := \epsilon^{\mu\nu\alpha\beta}u_\nu\tau_{r,\alpha,\beta} \,. \tag{6.60}$$

Note that here we explicitly computed the components of the thermal vorticity,

$$\varpi^{\mu\nu}u_\nu = \frac{1}{2}\left(\beta_0\dot{u}^\mu - \nabla^\mu\beta_0\right) = \frac{\beta_0}{2}\left(\frac{F^\mu}{\varepsilon_0+P_0} + \dot{u}^\mu\right) - \frac{I^\mu}{2h} \,, \tag{6.61a}$$

$$\epsilon^{\mu\nu\alpha\beta}u_\nu\varpi_{\alpha\beta} = -2\beta_0\omega^\mu \,, \tag{6.61b}$$

where we made use of Eq. (6.41). In the following we will define for brevity $\Gamma^{(I)} := (2h)^{-1}$.

Summarizing the equations for the spin potential

Putting the considerations on the antisymmetric part of the energy-momentum tensor to use, we may reformulate Eqs. (6.49) as

$$\begin{aligned}
&\frac{4\sigma^2\hbar}{gm}J_{31}\dot{\kappa}_0^{\langle\mu\rangle} + \Gamma^{(\kappa)}\kappa_0^\mu + \sum_{n\in\mathbb{S}_1^{(1)}}\gamma_n^{(1)}w_n^\mu \\
&= -\frac{\beta_0\Gamma^{(\kappa)}}{\varepsilon_0+P_0}F^\mu + \Gamma^{(\kappa)}\Gamma^{(I)}I^\mu + \frac{4\sigma^2\hbar}{gm}\bigg\{ -\kappa_0^\mu\left(K_{31}\dot{\alpha}_0 - K_{41}\dot{\beta}_0 + \frac{4}{3}J_{31}\theta\right) \\
&\quad - \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu\left[J_{30}\dot{u}_\alpha\omega_{0,\beta} - J_{31}\nabla_\alpha\omega_{0,\beta} - \omega_{0,\beta}\left(K_{31}I_\alpha - K_{41}\nabla_\alpha\beta_0\right)\right] + \frac{1}{2}J_{31}\left(\sigma^{\mu\nu}+\omega^{\mu\nu}\right)\kappa_{0,\nu}\bigg\} \\
&\quad + \sigma\epsilon^{\mu\nu\alpha\beta}u_\nu\left[\frac{1}{2}(\sigma_{\lambda\alpha}+\omega_{\lambda\alpha})\mathfrak{z}_\beta{}^\lambda + \frac{1}{3}(\nabla_\alpha - \dot{u}_\alpha)(m^2\mathfrak{p}_\beta - \mathfrak{n}_\beta) + (\nabla^\lambda - \dot{u}^\lambda)\mathfrak{q}_{\beta\alpha\lambda}\right] \,,
\end{aligned} \tag{6.62a}$$

$$\begin{aligned}
&\frac{2\sigma^2\hbar}{gm}(J_{30}-J_{31})\dot{\omega}_0^{\langle\mu\rangle} + \Gamma^{(\omega)}\omega_0^\mu + \sum_{n\in\mathbb{S}_0^{(1)}}\gamma_n^{(0)}\tau_n^\mu + \sum_{n\in\mathbb{S}_2^{(1)}}\gamma_n^{(2)}t_n^\mu \\
&= -\beta_0\Gamma^{(\omega)}\omega^\mu - \frac{2\sigma^2\hbar}{gm}\bigg\{\left[(K_{30}-K_{31})\dot{\alpha}_0 - (K_{40}-K_{41})\dot{\beta}_0 + \left(J_{30} - \frac{1}{3}J_{31}\right)\theta\right]\omega_0^\mu \\
&\quad + \epsilon^{\mu\nu\alpha\beta}u_\nu\left[J_{31}\nabla_\alpha\kappa_{0,\beta} + \kappa_{0,\beta}(K_{31}I_\alpha - K_{41}\nabla_\alpha\beta_0) - 3J_{31}\dot{u}_\alpha\kappa_{0,\beta}\right] - J_{31}(\sigma^{\mu\nu}+\omega^{\mu\nu})\omega_{0,\nu}\bigg\} \\
&\quad - \sigma\left[\frac{1}{2}(\dot{u}_\lambda - \nabla_\lambda)\mathfrak{z}^{\lambda\mu} + u_\nu\Delta^\mu{}_\alpha(\nabla_\lambda - \dot{u}_\lambda)\mathfrak{q}^{[\nu\alpha]\lambda} - \frac{1}{3}\left(\sigma^{\mu\nu}+\omega^{\mu\nu} - \frac{2}{3}\theta\Delta^{\mu\nu}\right)(m^2\mathfrak{p}_\nu - \mathfrak{n}_\nu)\right] \,,
\end{aligned} \tag{6.62b}$$

where we used the equation of motion for the four-velocity (6.39c) and neglected nonlinear terms.[6] This form of the evolution equations for the components of the spin potential makes it clear that, provided that $\Gamma^{(\kappa)}$ and $\Gamma^{(\omega)}$ are positive, they are of relaxation type. The respective relaxation times

[6] In principle, since we aim to provide a theory that is accurate to second order in Knudsen and inverse Reynolds numbers, we should keep these terms. We neglect them for consistency since they constitute nonlinear contributions emerging from the collision term, which will be omitted also in the following section. However, considering the results of Ref. [160], they should be included in the future.

are controlled by the nonlocal part of the collision terms, as is manifest from the coefficients (6.58). More specific, the relaxation times will be determined by the inverses of $\Gamma^{(\kappa)}$ and $\Gamma^{(\omega)}$, implying that, in line with our expectations, larger nonlocal contributions lead to faster relaxation towards the respective components of the thermal vorticity. In the limit of local collisions, where $\Delta \to 0$ and thus $\Gamma^{(\kappa)}, \Gamma^{(\omega)} \to 0$, the relaxation times become infinite and the components of the spin tensor will follow wave-type equations [119]. Moreover, since the coefficients $\Gamma^{(\kappa)}$ and $\Gamma^{(\omega)}$ originate from the nonlocal part of the collision term, we expect them to be small, such that the damping of the spin waves is low, necessitating a dynamical treatment of the spin potential, as opposed to taking it to be equal to the thermal vorticity. Further research on this point will be undertaken in the future.

Furthermore, some of the dissipative terms in the equations above are of first order in Knudsen and inverse Reynolds numbers, which will make it necessary to include the equations of motion for ω_0^μ and κ_0^μ in the truncation procedure which we will establish in Sec. 6.3. This is in contrast to the equations of motion for α_0, β_0, and u^μ, which do not couple to dissipative quantities at first order in Knudsen and inverse Reynolds numbers. Note that, when setting all terms of second order to zero, we obtain algebraic relations for the components of the spin potential, which, if dissipative quantities are set to zero as well, reduce to the appropriate projections of the thermal vorticity.

6.2.3 Tensor polarization

Evidently, in the preceding discussion of the conserved currents, the moments of spin-rank two (6.18c) did not appear at all, which raises the question of their significance. In order to get a clearer understanding of what is relevant, we have to remember that the quantities which are measured in experiment are certain entries of the spin-density matrix $\widehat{\varrho}(k)$ of the particles [5, 10, 16, 18]. The *vector polarization* of a particle is encoded in the expectation value of the Pauli-Lubanski operator [15]

$$S^\mu(k) := \mathrm{Tr}\left[\widehat{S}^\mu \widehat{\varrho}(k)\right] , \tag{6.63}$$

where

$$\widehat{S}^\mu := -\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta}\widehat{J}_{\nu\alpha}\widehat{P}_\beta , \tag{6.64}$$

with $\widehat{J}$ and $\widehat{P}$ being the total-angular momentum and momentum operators in the relevant representation, respectively. However, the spin-density matrix of a particle of spin $j \geq 1$ has a richer internal structure, featuring higher-order polarization observables that involve more powers of the Pauli-Lubanski operator. In particular, the *tensor polarization*, which constitutes a traceless symmetric tensor,[7] is defined as [15]

$$\Theta^{\mu\nu}(k) := \frac{1}{2}\sqrt{\frac{3}{2}}\mathrm{Tr}\left\{\left[\widehat{S}^{(\mu}\widehat{S}^{\nu)} + \frac{2\sigma(\sigma+1)}{3}K^{\mu\nu}\right]\widehat{\varrho}(k)\right\} . \tag{6.65}$$

In Appendix C, it is shown that these quantities can be connected to the single-particle distribution function in the following way,

$$S^\mu(k) = \frac{\sigma}{N(k)}\int \mathrm{d}\Sigma_\lambda k^\lambda \int \mathrm{d}S(k)\mathfrak{s}^\mu f(x,k,\mathfrak{s}) , \tag{6.66}$$

$$\Theta^{\mu\nu}(k) = \frac{1}{2}\sqrt{\frac{3}{2}}\frac{1}{N(k)}\int \mathrm{d}\Sigma_\lambda k^\lambda \int \mathrm{d}S(k)K^{\mu\nu}_{\alpha\beta}\mathfrak{s}^\alpha\mathfrak{s}^\beta f(x,k,\mathfrak{s}) , \tag{6.67}$$

where we defined

$$N(k) := \int \mathrm{d}\Sigma_\gamma k^\gamma \int \mathrm{d}S(k) f(x,k,\mathfrak{s}) . \tag{6.68}$$

Thus, the components of the distribution function that are bilinear in the spin vector (i.e., the moments of spin-rank two) determine the tensor polarization of particles.

[7] We remark that this type of polarization becomes the linear polarization of light in the massless case.

When integrating the expressions (6.66) and (6.67), which determine the so-called *local* polarization, over momentum space as well, we find the expressions for the *global* vector and tensor polarization,

$$\overline{S}^\mu := \frac{1}{\overline{N}} \int \mathrm{d}K N(k) S^\mu(k) = \frac{\sigma}{\overline{N}} \int \mathrm{d}\Sigma_\lambda \left(-\frac{\sigma\hbar}{m} \int \mathrm{d}K k^\lambda \widetilde{\Omega}^{\mu\alpha} k_\alpha f_{0\mathbf{k}} \widetilde{f}_{0\mathbf{k}} + u^\lambda \tau_1^\mu + \tau_0^{\mu,\lambda} \right) , \tag{6.69}$$

$$\overline{\Theta}^{\mu\nu} := \frac{1}{\overline{N}} \int \mathrm{d}K N(k) \Theta^{\mu\nu}(k) = \frac{1}{2}\sqrt{\frac{3}{2}} \frac{1}{\overline{N}} \int \mathrm{d}\Sigma_\lambda \left(u^\lambda \psi_1^{\mu\nu} + \psi_0^{\mu\nu,\lambda} \right) , \tag{6.70}$$

where we introduced

$$\overline{N} := \int \mathrm{d}K N(k) . \tag{6.71}$$

Note that, just as the moments of spin-rank one, the irreducible moments of spin-rank two that appear here are connected to the traces of higher-order ones,

$$u_\mu \psi_r^{\mu\nu} = -\psi_{r-1\,\mu}^{\mu\nu,} , \tag{6.72a}$$

$$u_\mu u_\nu \psi_r^{\mu\nu} = \psi_{r-2\,\mu\nu}^{\mu\nu,} - \frac{1}{3} u_\mu u_\nu \left(m^2 \psi_{r-2}^{\mu\nu} - \psi_r^{\mu\nu} \right) , \tag{6.72b}$$

$$u_\mu \psi_r^{\mu\nu,\lambda} = -\psi_{r-1\,\mu}^{\mu\nu,\lambda} - \frac{1}{3} \left(m^2 \psi_{r-1}^{\nu\langle\lambda\rangle} - \psi_{r+1}^{\nu\langle\lambda\rangle} \right) , \tag{6.72c}$$

$$u_\mu u_\nu \psi_r^{\mu\nu,\lambda} = \psi_{r-2\,\mu\nu}^{\mu\nu,\lambda} + \frac{2}{5} \left(m^2 \psi_{r-2}^{\mu\langle\lambda\rangle,}{}_{\mu} - \psi_r^{\mu\langle\lambda\rangle,}{}_{\mu} \right) - \frac{1}{5} u_\mu u_\nu \left(m^2 \psi_{r-2}^{\mu\nu,\lambda} - \psi_r^{\mu\nu,\lambda} \right) . \tag{6.72d}$$

6.2.4 General equations of motion

It is clear that, in order to arrive at hydrodynamic equations, we need to know the evolution of the irreducible moments (6.18). In order to obtain these, we rewrite the Boltzmann equation in extended phase space as

$$\delta\dot{f}_{\mathbf{k}\mathfrak{s}} = E_{\mathbf{k}}^{-1} C(x,k,\mathfrak{s}) - \dot{f}_{\mathrm{eq}}(x,k,\mathfrak{s}) - E_{\mathbf{k}}^{-1} k^{\langle\mu\rangle} \nabla_\mu f_{\mathrm{eq}}(x,k,\mathfrak{s}) - E_{\mathbf{k}}^{-1} k^{\langle\mu\rangle} \nabla_\mu \delta f_{\mathbf{k}\mathfrak{s}} . \tag{6.73}$$

Then, by acting with a comoving derivative on the definition of the irreducible moments and using Eq. (6.73), we are able to derive exact evolution equations for them. The explicit calculations are shown in Appendix D.

Spin-rank zero

As is evident from Eqs. (6.33) and (6.35), the dissipative components of the particle four-current and the energy-momentum tensor are determined by the moments of spin-rank zero and momentum-rank zero, one and two. Thus, we compute the equations of motion for these type of moments. Defining the thermodynamic integrals

$$G_{nm} := J_{n0} J_{m0} - J_{n-1,0} J_{m+1,0} , \tag{6.74}$$

we find for the moment of zeroth rank in momentum after a longer, but straightforward computation

$$\begin{aligned} \dot{\rho}_r - C_{r-1} = {} & \alpha_r^{(0)} \theta - \frac{G_{2r}}{D_{20}} \Pi\theta + \frac{G_{2r}}{D_{20}} \pi^{\mu\nu}\sigma_{\mu\nu} + \frac{G_{3r}}{D_{20}} \partial_\mu n^\mu + (r-1) \rho_{r-2}^{\mu\nu} \sigma_{\mu\nu} \\ & + r \rho_{r-1}^\mu \dot{u}_\mu - \nabla_\mu \rho_{r-1}^\mu - \frac{1}{3} \left[(r+2)\rho_r - (r-1) m^2 \rho_{r-2} \right] \theta , \end{aligned} \tag{6.75a}$$

where we made use of the equations of motion for α_0 and β_0 [cf. Eqs. (6.39a) and (6.39b)] and defined

$$\alpha_r^{(0)} := (1-r) I_{r1} - I_{r0} - \frac{1}{D_{20}} \left[G_{2r} (\varepsilon_0 + P_0) - G_{3r} n_0 \right] . \tag{6.76a}$$

The equation of motion for the moments of momentum-rank one is

$$\begin{aligned}\dot{\rho}_r^{\langle\mu\rangle} - C_{r-1}^{\langle\mu\rangle} = {}& \alpha_r^{(1)} I^\mu + \rho_r^\nu \omega^\mu{}_\nu + \frac{1}{3}[(r-1)m^2\rho_{r-2}^\mu - (r+3)\rho_r^\mu]\theta - \Delta_\lambda^\mu \nabla_\nu \rho_{r-1}^{\lambda\nu} + r\rho_{r-1}^{\mu\nu}\dot{u}_\nu \\ &+ \frac{1}{5}\left[(2r-2)m^2\rho_{r-2}^\nu - (2r+3)\rho_r^\nu\right]\sigma_\nu^\mu + \frac{1}{3}\left[m^2 r\rho_{r-1} - (r+3)\rho_{r+1}\right]\dot{u}^\mu \\ &+ \frac{\beta_0 J_{r+2,1}}{\varepsilon_0 + P_0}(\Pi\dot{u}^\mu - \nabla^\mu\Pi + \Delta_\nu^\mu \partial_\lambda \pi^{\lambda\nu}) - \frac{1}{3}\nabla^\mu(m^2\rho_{r-1} - \rho_{r+1}) + (r-1)\rho_{r-2}^{\mu\nu\lambda}\sigma_{\lambda\nu} ,\end{aligned} \tag{6.75b}$$

where we employed Eq. (6.39c) and introduced

$$\alpha_r^{(1)} := J_{r+1,1} - \frac{n_0}{\varepsilon_0 + P_0} J_{r+2,1} . \tag{6.76b}$$

Note that the equation of motion has been projected orthogonal to the four-velocity, i.e., we have $\dot{\rho}_r^{\langle\mu\rangle} := \Delta^\mu{}_\nu u \cdot \partial \rho_r^\nu$. Lastly, the moments of rank two in momentum follow the evolution equation

$$\begin{aligned}\dot{\rho}_r^{\langle\mu\nu\rangle} - C_{r-1}^{\langle\mu\nu\rangle} = {}& 2\alpha_r^{(2)}\sigma^{\mu\nu} - \frac{2}{7}\left[(2r+5)\rho_r^{\lambda\langle\mu} - 2m^2(r-1)\rho_{r-2}^{\lambda\langle\mu}\right]\sigma_\lambda^{\nu\rangle} + 2\rho_r^{\lambda\langle\mu}\omega^{\nu\rangle}{}_\lambda \\ &+ \frac{2}{15}[(r+4)\rho_{r+2} - (2r+3)m^2\rho_r + (r-1)m^4\rho_{r-2}]\sigma^{\mu\nu} + \frac{2}{5}\nabla^{\langle\mu}\left(\rho_{r+1}^{\nu\rangle} - m^2\rho_{r-1}^{\nu\rangle}\right) \\ &- \frac{2}{5}\left[(r+5)\rho_{r+1}^{\langle\mu} - rm^2\rho_{r-1}^{\langle\mu}\right]\dot{u}^{\nu\rangle} - \frac{1}{3}\left[(r+4)\rho_r^{\mu\nu} - m^2(r-1)\rho_{r-2}^{\mu\nu}\right]\theta \\ &+ (r-1)\rho_{r-2}^{\mu\nu\lambda\rho}\sigma_{\lambda\rho} - \Delta_{\alpha\beta}^{\mu\nu}\nabla_\lambda\rho_{r-1}^{\alpha\beta\lambda} + r\rho_{r-1}^{\mu\nu\lambda}\dot{u}_\lambda ,\end{aligned} \tag{6.75c}$$

where we defined

$$\alpha_r^{(2)} := I_{r+2,1} + (r-1)I_{r+2,2} . \tag{6.76c}$$

The second terms on the left-hand sides of Eqs. (6.75) are called generalized irreducible collision terms and are defined as

$$C_r^{\langle\mu_1\cdots\mu_\ell\rangle} := \int \mathrm{d}\Gamma E_{\mathbf{k}}^r k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle} C(x,k,\mathfrak{s}) . \tag{6.77}$$

Note that in Eqs. (6.75) it appears that no moments of spin rank higher than zero enter; however, we will see in Subsec 6.3.2 that the generalized collision terms will introduce a coupling to the moments of spin-rank two.

Spin-rank one

The calculation to obtain the equations of motion for the moments of spin-rank one [cf. Eq. (6.18b)] is similar to the one for the moments of spin-rank zero. The main difference consists in the fact that the equilibrium contributions are now given by the terms of first order in $\hbar$ in Eq. (6.5). We find for the moments of rank one in spin and rank zero in momentum

$$\begin{aligned}\dot{\tau}_r^{\langle\mu\rangle} - \mathfrak{C}_{r-1}^{\langle\mu\rangle} = {}& \frac{\sigma\hbar}{gm}\Big\{2\omega_0^\mu\left[K_{r+1,0}\dot{\alpha}_0 - K_{r+2,0}\dot{\beta}_0 + (J_{r+1,0} + rJ_{r+1,1})\theta\right] + J_{r+1,0}\left(2\dot{\omega}_0^{\langle\mu\rangle} - \widetilde{\Omega}_0^{\langle\mu\rangle\nu}\dot{u}_\nu\right) \\ &- J_{r+1,1}\Delta_\lambda^\mu\nabla_\nu\widetilde{\Omega}_0^{\lambda\nu} - \widetilde{\Omega}_0^{\langle\mu\rangle\nu}\left[K_{r+1,1}I_\nu - K_{r+2,1}(\nabla_\nu\beta_0 + \beta_0\dot{u}_\nu)\right]\Big\} + r\dot{u}_\nu\tau_{r-1}^{\langle\mu\rangle,\nu} \\ &+ (r-1)\sigma_{\alpha\beta}\tau_{r-2}^{\langle\mu\rangle,\alpha\beta} - \Delta_\lambda^\mu\nabla_\nu\tau_{r-1}^{\lambda,\nu} - \frac{1}{3}\left[(r+2)\tau_r^{\langle\mu\rangle} - (r-1)m^2\tau_{r-2}^{\langle\mu\rangle}\right]\theta .\end{aligned} \tag{6.78a}$$

Comparing the right-hand sides of Eqs. (6.75a) and (6.78a), we see that the only difference in structure comes from the equilibrium terms, since the contributions that involve the dissipative quantities arise solely from the rank of the projected momenta that appear. Furthermore, note that we, in contrast to Eq. (6.75a), did not insert the evolution equation for α_0, β_0, and u^μ yet for brevity. For the spin moment of tensor-rank one in momentum we find the equation of motion

$$\begin{aligned}\dot{\tau}_r^{\langle\mu\rangle,\langle\nu\rangle} - \mathfrak{C}_{r-1}^{\langle\mu\rangle,\langle\nu\rangle} = &-\frac{\sigma\hbar}{gm}\Big\{ J_{r+2,1}\dot{\widetilde{\Omega}}_0^{\langle\mu\rangle\langle\nu\rangle} + J_{r+2,1}\Delta^\mu{}_\rho u_\lambda \nabla^\nu \widetilde{\Omega}_0^{\rho\lambda} + 2\beta_0 K_{r+3,2}\widetilde{\Omega}_0^{\langle\mu\rangle\lambda}\sigma^\nu{}_\lambda \\ &+ 2\omega_0^\mu \left[K_{r+2,1}I^\nu - K_{r+3,1}\left(\nabla^\nu\beta_0 + \beta_0\dot{u}^\nu\right)\right] + \widetilde{\Omega}_0^{\langle\mu\rangle\langle\nu\rangle}\Big(K_{r+2,1}\dot{\alpha}_0 - K_{r+3,1}\dot{\beta}_0 \\ &+ \frac{5}{3}\beta_0 K_{r+3,2}\theta\Big)\Big\} + \omega^\nu{}_\rho \tau_r^{\langle\mu\rangle,\rho} + \frac{1}{3}\left[(r-1)m^2\tau_{r-2}^{\langle\mu\rangle,\nu} - (r+3)\tau_r^{\langle\mu\rangle,\nu}\right]\theta \\ &+ \frac{1}{5}\left[(2r-2)m^2\tau_{r-2}^{\langle\mu\rangle,\lambda} - (2r+3)\tau_r^{\langle\mu\rangle,\lambda}\right]\sigma^\nu{}_\lambda + \frac{1}{3}\dot{u}^\nu\left[m^2 r\tau_{r-1}^{\langle\mu\rangle} - (r+3)\tau_{r+1}^{\langle\mu\rangle}\right] \\ &- \frac{1}{3}\Delta^\mu_\lambda\nabla^\nu\left(m^2\tau_{r-1}^\lambda - \tau_{r+1}^\lambda\right) + r\dot{u}_\rho\tau_{r-1}^{\langle\mu\rangle,\nu\rho} - \Delta^\nu_\lambda\Delta^\mu_\alpha\nabla_\rho\tau_{r-1}^{\alpha,\lambda\rho} + (r-1)\sigma_{\lambda\rho}\tau_{r-2}^{\langle\mu\rangle,\nu\lambda\rho} ,\end{aligned} \tag{6.78b}$$

where we see the same similarities to Eq. (6.75b) that were mentioned in Eq. (6.78a). Finally, for the spin moment of tensor-rank two in momentum the equation of motion reads

$$\begin{aligned}\dot{\tau}_r^{\langle\mu\rangle,\langle\nu\lambda\rangle} - \mathfrak{C}_{r-1}^{\langle\mu\rangle,\langle\nu\lambda\rangle} = &\frac{2\sigma\hbar}{gm}\Big\{\widetilde{\Omega}^{\langle\mu\rangle\langle\nu}\left[K_{r+3,2}I^{\lambda\rangle} - K_{r+4,2}\left(\nabla^{\lambda\rangle}\beta_0 + \beta_0\dot{u}^{\lambda\rangle}\right)\right] + K_{r+3,2}\Delta^\mu_\gamma\Delta^{\nu\lambda}_{\alpha\beta}\nabla^\alpha\widetilde{\Omega}^{\gamma\beta} \\ &- 2\beta_0 K_{r+4,2}\omega_0^\mu\sigma^{\nu\lambda}\Big\} + r\dot{u}_\rho\tau_{r-1}^{\langle\mu\rangle,\nu\lambda\rho} + \frac{2}{5}\left[rm^2\tau_{r-1}^{\langle\mu\rangle,\langle\nu} - (r+5)\tau_{r+1}^{\langle\mu\rangle,\langle\nu}\right]\dot{u}^{\lambda\rangle} \\ &- \Delta^\mu_\gamma\Delta^{\nu\lambda}_{\alpha\beta}\nabla_\rho\tau_{r-1}^{\gamma,\alpha\beta\rho} + \Delta^\mu_\rho\frac{2}{5}\Delta^{\nu\lambda}_{\alpha\beta}\nabla^\beta\left(\tau_{r+1}^{\rho,\alpha} - m^2\tau_{r-1}^{\rho,\alpha}\right) \\ &+ \frac{1}{3}\left[(r-1)m^2\tau_{r-2}^{\langle\mu\rangle,\nu\lambda} - (r+4)\tau_r^{\langle\mu\rangle,\nu\lambda}\right]\theta + (r-1)\sigma_{\rho\tau}\tau_{r-2}^{\langle\mu\rangle,\nu\lambda\rho\tau} \\ &+ \frac{2}{7}\left[2(r-1)m^2\tau_{r-2}^{\langle\mu\rangle,\rho\langle\nu} - (2r+5)\tau_r^{\langle\mu\rangle,\rho\langle\nu}\right]\sigma^{\lambda\rangle}_\rho + 2\tau_r^{\langle\mu\rangle,\rho\langle\nu}\omega^{\lambda\rangle}_\rho \\ &+ \frac{2}{15}\left[(r-1)m^4\tau_{r-2}^{\langle\mu\rangle} - (2r+3)m^2\tau_r^{\langle\mu\rangle} + (r+4)\tau_{r+2}^{\langle\mu\rangle}\right]\sigma^{\nu\lambda} .\end{aligned} \tag{6.78c}$$

The generalized irreducible collision terms appearing on the left-hand sides of Eqs. (6.78) (which we denoted with a different font to preclude ambiguities) are defined as

$$\mathfrak{C}_r^{\langle\mu\rangle,\langle\mu_1\cdots\mu_\ell\rangle} := \int \mathrm{d}\Gamma E^r_{\mathbf{k}}\mathfrak{s}^{\langle\mu\rangle}k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}C(x,k,\mathfrak{s}) . \tag{6.79}$$

Spin-rank two

The equations of motion for the irreducible moments of spin-rank two are again obtained by acting with a comoving derivative on Eq. (6.18c) and using the Boltzmann equation (6.73). For the moments of tensor-rank zero in momentum we find

$$\begin{aligned}\dot{\psi}_r^{\langle\mu\nu\rangle} - \mathfrak{C}_{r-1}^{\langle\mu\nu\rangle} = &\frac{1}{3}\left[(r-1)m^2\psi_{r-2}^{\langle\mu\nu\rangle} - (r+2)\psi_r^{\langle\mu\nu\rangle}\right]\theta + r\dot{u}_\gamma\psi_{r-1}^{\langle\mu\nu\rangle,\gamma} \\ &- \Delta^{\mu\nu}_{\alpha\beta}\nabla_\gamma\psi_{r-1}^{\alpha\beta,\gamma} + (r-1)\psi_{r-2}^{\langle\mu\nu\rangle,\alpha\beta}\sigma_{\alpha\beta} .\end{aligned} \tag{6.80a}$$

Again, we notice the same structures appearing on the right-hand side as in Eqs. (6.75a) and (6.78a). However, in contrast to the moments of spin-ranks zero and one, there are no terms present that emerge from local equilibrium, since there is no conserved quantity associated with the tensor polarization. The equation of motion for the moment of tensor-rank one in momentum reads

$$\begin{aligned}\dot{\psi}_r^{\langle\mu\nu\rangle,\langle\lambda\rangle} - \mathfrak{C}_{r-1}^{\langle\mu\nu\rangle,\langle\lambda\rangle} = &\frac{1}{3}\left[m^2 r\psi_{r-1}^{\langle\mu\nu\rangle} - (r+3)\psi_{r+1}^{\langle\mu\nu\rangle}\right]\dot{u}^\lambda - \frac{1}{3}\Delta^{\mu\nu}_{\alpha\beta}\nabla^\lambda\left(m^2\psi_{r-1}^{\alpha\beta} - \psi_{r+1}^{\alpha\beta}\right) \\ &+ \psi_r^{\langle\mu\nu\rangle,\alpha}\omega^\lambda{}_\alpha + \frac{1}{3}\left[(r-1)m^2\psi_{r-2}^{\langle\mu\nu\rangle,\lambda} - (r+3)\psi_r^{\langle\mu\nu\rangle,\lambda}\right]\theta \\ &+ \frac{1}{5}\left[(2r-2)m^2\psi_{r-2}^{\langle\mu\nu\rangle,\alpha} - (2r+3)\psi_r^{\langle\mu\nu\rangle,\alpha}\right]\sigma_\alpha{}^\lambda \\ &+ r\dot{u}_\alpha\psi_{r-1}^{\langle\mu\nu\rangle,\lambda\alpha} - \Delta^{\mu\nu}_{\alpha\beta}\Delta^\lambda_\gamma\nabla_\delta\psi_{r-1}^{\alpha\beta,\gamma\delta} + (r-1)\psi_{r-2}^{\langle\mu\nu\rangle,\lambda\alpha\beta}\sigma_{\alpha\beta} ,\end{aligned} \tag{6.80b}$$

while the one for momentum-rank two is given by

$$\begin{aligned}\dot{\psi}_r^{\langle\mu\nu\rangle,\langle\lambda\alpha\rangle} - \mathfrak{C}_{r-1}^{\langle\mu\nu\rangle,\langle\lambda\alpha\rangle} &= \frac{2}{5}\left[m^2 r\psi_{r-1}^{\langle\mu\nu\rangle,\langle\lambda} - (r+5)\psi_{r+1}^{\langle\mu\nu\rangle,\langle\lambda}\right]\dot{u}^{\alpha\rangle} - \frac{2}{5}\Delta^{\mu\nu}_{\rho\sigma}\Delta^{\lambda\alpha}_{\beta\gamma}\nabla^\gamma\left[m^2\psi_{r-1}^{\rho\sigma,\beta} - \psi_{r+1}^{\rho\sigma,\beta}\right] \\ &+ 2\psi_r^{\langle\mu\nu\rangle,\beta\langle\lambda}\omega^{\alpha\rangle}{}_\beta + \frac{1}{3}\left[m^2(r-1)\psi_{r-2}^{\langle\mu\nu\rangle,\lambda\alpha} - (r+4)\psi_r^{\langle\mu\nu\rangle,\lambda\alpha}\right]\theta \\ &+ \frac{2}{7}\left[2m^2(r-1)\psi_{r-2}^{\langle\mu\nu\rangle,\beta\langle\lambda} - (2r+5)\psi_r^{\langle\mu\nu\rangle,\beta\langle\lambda}\right]\sigma^{\alpha\rangle}{}_\beta \\ &+ \frac{2}{15}\sigma^{\lambda\alpha}\left[m^4(r-1)\psi_{r-2}^{\langle\mu\nu\rangle} - m^2(2r+3)\psi_r^{\langle\mu\nu\rangle} + (r+4)\psi_{r+2}^{\langle\mu\nu\rangle}\right] \\ &+ r\psi_{r-1}^{\langle\mu\nu\rangle,\lambda\alpha\beta}\dot{u}_\beta - \Delta^{\mu\nu}_{\rho\sigma}\Delta^{\lambda\alpha}_{\beta\gamma}\nabla_\delta\psi_{r-1}^{\rho\sigma,\beta\gamma\delta} + (r-1)\psi_{r-2}^{\langle\mu\nu\rangle,\lambda\alpha\beta\gamma}\sigma_{\beta\gamma}\,. \end{aligned} \tag{6.80c}$$

For reasons that will become clear later, we furthermore need the equations of motion for the moments of tensor-ranks three and four in momentum, which read

$$\begin{aligned}\dot{\psi}_r^{\langle\mu\nu\rangle,\langle\lambda\alpha\beta\rangle} - \mathfrak{C}_{r-1}^{\langle\mu\nu\rangle,\langle\lambda\alpha\beta\rangle} &= \frac{1}{3}\left[(r-1)m^2\psi_{r-2}^{\langle\mu\nu\rangle,\lambda\alpha\beta} - (r+5)\psi_r^{\langle\mu\nu\rangle,\lambda\alpha\beta}\right]\theta + 3\psi_r^{\langle\mu\nu\rangle,\kappa\langle\lambda\alpha}\omega^{\beta\rangle}{}_\kappa \\ &+ \frac{6}{35}\left[(r-1)m^4\psi_{r-2}^{\langle\mu\nu\rangle,\langle\lambda} - (2r+5)m^2\psi_r^{\langle\mu\nu\rangle,\langle\lambda} + (r+6)\psi_{r+2}^{\langle\mu\nu\rangle,\langle\lambda}\right]\sigma^{\alpha\beta\rangle} \\ &+ \frac{1}{3}\left[m^2(2r-2)\psi_{r-2}^{\langle\mu\nu\rangle,\kappa\langle\lambda\alpha} - (2r+7)\psi_r^{\langle\mu\nu\rangle,\kappa\langle\lambda\alpha}\right]\sigma^{\beta\rangle}{}_\kappa + r\dot{u}_\kappa\psi_{r-1}^{\langle\mu\nu\rangle,\kappa\lambda\alpha\beta} \\ &- \frac{3}{7}\Delta^{\mu\nu}_{\gamma\delta}\Delta^{\lambda\alpha\beta}_{\kappa\zeta\eta}\nabla^\kappa\left(m^2\psi_{r-1}^{\gamma\delta,\zeta\eta} - \psi_{r+1}^{\gamma\delta,\zeta\eta}\right) - \Delta^{\mu\nu}_{\gamma\delta}\Delta^{\lambda\alpha\beta}_{\kappa\zeta\eta}\nabla_\rho\psi_{r-1}^{\gamma\delta,\kappa\zeta\eta\rho} \\ &+ \frac{3}{7}\left[m^2 r\psi_{r-1}^{\langle\mu\nu\rangle,\langle\lambda\alpha} - (r+7)\psi_{r+1}^{\langle\mu\nu\rangle,\langle\lambda\alpha}\right]\dot{u}^{\beta\rangle} + (r-1)\sigma_{\gamma\delta}\psi_{r-2}^{\langle\mu\nu\rangle,\lambda\alpha\beta\gamma\delta}\,, \end{aligned} \tag{6.80d}$$

and

$$\begin{aligned}\dot{\psi}_r^{\langle\mu\nu\rangle,\langle\lambda\alpha\beta\gamma\rangle} - \mathfrak{C}_{r-1}^{\langle\mu\nu\rangle,\langle\lambda\alpha\beta\gamma\rangle} &= \frac{1}{3}\left[(r-1)m^2\psi_{r-2}^{\langle\mu\nu\rangle,\lambda\alpha\beta\gamma} - (r+6)\psi_r^{\langle\mu\nu\rangle,\lambda\alpha\beta\gamma}\right]\theta + 4\psi_r^{\langle\mu\nu\rangle,\kappa\langle\lambda\alpha\beta}\omega^{\gamma\rangle}{}_\kappa \\ &+ \frac{4}{21}\left[(r-1)m^4\psi_{r-2}^{\langle\mu\nu\rangle,\langle\lambda\alpha} - (2r+7)m^2\psi_r^{\langle\mu\nu\rangle,\langle\lambda\alpha} + (r+8)\psi_{r+2}^{\langle\mu\nu\rangle,\langle\lambda\alpha}\right]\sigma^{\beta\gamma\rangle} \\ &+ \frac{4}{11}\left[m^2(2r-2)\psi_{r-2}^{\langle\mu\nu\rangle,\kappa\langle\lambda\alpha\beta} - (2r+9)\psi_r^{\langle\mu\nu\rangle,\kappa\langle\lambda\alpha\beta}\right]\sigma^{\gamma\rangle}{}_\kappa \\ &+ r\dot{u}_\kappa\psi_{r-1}^{\langle\mu\nu\rangle,\kappa\lambda\alpha\beta\gamma} - \frac{4}{9}\Delta^{\mu\nu}_{\xi\delta}\Delta^{\lambda\alpha\beta\gamma}_{\kappa\zeta\eta\rho}\nabla^\kappa\left(m^2\psi_{r-1}^{\xi\delta,\zeta\eta\rho} - \psi_{r+1}^{\xi\delta,\zeta\eta\rho}\right) \\ &- \Delta^{\mu\nu}_{\xi\delta}\Delta^{\lambda\alpha\beta\gamma}_{\kappa\zeta\eta\rho}\nabla_\sigma\psi_{r-1}^{\xi\delta,\kappa\zeta\eta\rho\sigma} + \frac{4}{9}\left[m^2 r\psi_{r-1}^{\langle\mu\nu\rangle,\langle\lambda\alpha\beta} - (r+9)\psi_{r+1}^{\langle\mu\nu\rangle,\langle\lambda\alpha\beta}\right]\dot{u}^{\gamma\rangle} \\ &+ (r-1)\sigma_{\delta\rho}\psi_{r-2}^{\langle\mu\nu\rangle,\lambda\alpha\beta\gamma\delta\rho}\,, \end{aligned} \tag{6.80e}$$

respectively. Note that these equations are equivalent to the ones presented in Ref. [161] for moments of spin-rank zero. In Eqs. (6.80), the generalized irreducible collision terms are given by

$$\mathfrak{C}_r^{\langle\mu\nu\rangle,\langle\mu_1\cdots\mu_\ell\rangle} := \int \mathrm{d}\Gamma E_{\mathbf{k}}^r K_{\alpha\beta}^{\langle\mu\nu\rangle}\mathfrak{s}^\alpha\mathfrak{s}^\beta k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}C(x,k,\mathfrak{s})\,. \tag{6.81}$$

6.3 Closing the system of equations

The equations of motion (6.75), (6.78), and (6.80) are exact (besides the approximations made in deriving the Boltzmann equation), but do not form a closed system of equations. This can be straightforwardly seen by the fact that the evolution of the moment of tensor-rank ℓ in momentum depends on the moments of tensor-rank $\ell+1$ and $\ell+2$. Furthermore, a moment of rank r in energy couples to moments of energy-rank $r-1$ and $r-2$ as well. Thus, one has to find a way to sensibly truncate and close this system of equations, which we will do by keeping terms up to second order in Knudsen and inverse Reynolds numbers.

6.3.1 IReD: Basic idea

We explain the scheme that we will use by first considering the example of a fluid made of spin-0 particles, where the distribution function does not depend on the spin variable and the moments $\tau_r^{\mu,\mu_1\cdots\mu_\ell}$ and $\psi_r^{\mu\nu,\mu_1\cdots\mu_\ell}$ vanish. Then, one can express the generalized irreducible collision terms (6.77) in terms of the irreducible moments as [51]

$$C_{r-1}^{\langle\mu_1\cdots\mu_\ell\rangle} = -\sum_{n\in\mathbb{S}_\ell^{(0)}} \mathcal{A}_{rn}^{(\ell)} \rho_n^{\mu_1\cdots\mu_\ell} + \mathcal{O}(\mathrm{Re}^{-2}) , \tag{6.82}$$

where $\mathcal{A}^{(\ell)}$ is a matrix whose dimension is equal to the number of elements in the basis $\mathbb{S}_\ell^{(0)}$. Note that here the second terms on the right-hand side denote nonlinear contributions to the collision term that are at least quadratic in inverse Reynolds numbers. Then, the moment equations (6.75) take the form of a system of coupled relaxation-type equations. One possible way to truncate the system, which we denote "DNMR approach" and is detailed in Ref. [51], consists of diagonalizing the matrices $\mathcal{A}^{(\ell)}$ in order to find the slowest eigenmodes, i.e., the eigenvectors of $\mathcal{A}^{(\ell)}$ that correspond to the smallest eigenvalues.[8] Subsequently, one can re-express these eigenvectors in terms of irreducible moments and, neglecting terms of third order in Knudsen and inverse Reynolds numbers, close the moment equations in terms of the fluid-dynamical ones, i.e., ρ_0, ρ_0^μ, and $\rho_0^{\mu\nu}$.

In this thesis, however, we will use a different approach, which has its nonrelativistic origins in works by Struchtrup [48] and has recently been employed for deriving second-order dissipative hydrodynamics with multiple conserved charges [49]. In this approach, one first multiplies the moment equations (6.75) with the inverse of the linearized collision matrix, $\tau^{(\ell)} := (\mathcal{A}^{(\ell)})^{-1}$, such that they take the form

$$\sum_{n\in\mathbb{S}_0^{(0)}} \tau_{rn}^{(0)} \dot{\rho}_n + \rho_r = \frac{3}{m^2}\zeta_r\theta + \mathcal{O}(\mathrm{Kn}\,\mathrm{Re}^{-1}, \mathrm{Re}^{-2}) , \tag{6.83a}$$

$$\sum_{n\in\mathbb{S}_1^{(0)}} \tau_{rn}^{(1)} \dot{\rho}_n^{\langle\mu\rangle} + \rho_r^\mu = \kappa_r I^\mu + \mathcal{O}(\mathrm{Kn}\,\mathrm{Re}^{-1}, \mathrm{Re}^{-2}) , \tag{6.83b}$$

$$\sum_{n\in\mathbb{S}_2^{(0)}} \tau_{rn}^{(2)} \dot{\rho}_n^{\langle\mu\nu\rangle} + \rho_r^{\mu\nu} = 2\eta_r\sigma^{\mu\nu} + \mathcal{O}(\mathrm{Kn}\,\mathrm{Re}^{-1}, \mathrm{Re}^{-2}) , \tag{6.83c}$$

where we defined the Navier-Stokes values

$$\zeta_r := \frac{m^2}{3}\sum_{n\in\mathbb{S}_0^{(0)}} \tau_{rn}^{(0)}\alpha_n^{(0)} , \qquad \kappa_r := \sum_{n\in\mathbb{S}_1^{(0)}} \tau_{rn}^{(1)}\alpha_n^{(1)} , \qquad \eta_r := \sum_{n\in\mathbb{S}_2^{(0)}} \tau_{rn}^{(2)}\alpha_n^{(2)} . \tag{6.84}$$

Noting that the first terms on the left-hand sides of Eqs. (6.83) are of order $\mathcal{O}(\mathrm{Kn}\,\mathrm{Re}^{-1})$, we find the asymptotic matching conditions for the moments,

$$\rho_r = \mathcal{R}_{rn}^{(0)}\rho_n + \text{h.o.t.} , \tag{6.85a}$$

$$\rho_r^\mu = \mathcal{R}_{rn}^{(1)}\rho_n^\mu + \text{h.o.t.} , \tag{6.85b}$$

$$\rho_r^{\mu\nu} = \mathcal{R}_{rn}^{(2)}\rho_n^{\mu\nu} + \text{h.o.t.} , \tag{6.85c}$$

where we defined

$$\mathcal{R}_{rn}^{(0)} := \frac{\zeta_r}{\zeta_n} , \qquad \mathcal{R}_{rn}^{(1)} := \frac{\kappa_r}{\kappa_n} , \qquad \mathcal{R}_{rn}^{(2)} := \frac{\eta_r}{\eta_n} \tag{6.86}$$

[8] One has to pick the smallest eigenvalue of $\mathcal{A}^{(\ell)}$ since the relaxation times τ that appear in an equation of the type $\tau\dot{\rho}+\rho = \cdots$ are related to the inverse of $\mathcal{A}^{(\ell)}$, such that the smallest eigenvalue gives rise to the longest relaxation time.

and abbreviated the terms of higher orders in Kn and Re $^{-1}$ as "h.o.t." (higher-order terms). Choosing $n = 0$ in the equations above, we are able to express any moment in terms of the hydrodynamic quantities,

$$\rho_r = -\frac{3}{m^2}\mathcal{R}^{(0)}_{r0}\Pi + \text{h.o.t.} , \tag{6.87a}$$

$$\rho^\mu_r = \mathcal{R}^{(1)}_{r0} n^\mu + \text{h.o.t.} , \tag{6.87b}$$

$$\rho^{\mu\nu}_r = \mathcal{R}^{(2)}_{r0}\pi^{\mu\nu} + \text{h.o.t.} . \tag{6.87c}$$

These relations can then be used to replace all moments in Eqs. (6.83) evaluated at $r = 0$ to obtain hydrodynamic equations of the form

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta_0\theta + \mathcal{J} + \mathcal{R} , \tag{6.88a}$$

$$\tau_n \dot{n}^{\langle\mu\rangle} + n^\mu = \kappa_0 I^\mu + \mathcal{J}^\mu + \mathcal{R}^\mu , \tag{6.88b}$$

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta_0\sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{R}^{\mu\nu} , \tag{6.88c}$$

where the relaxation times are given by

$$\tau_\Pi \coloneqq \sum_{n\in\mathbb{S}^{(0)}_0} \tau^{(0)}_{0n}\mathcal{R}^{(0)}_{n0} , \qquad \tau_n \coloneqq \sum_{n\in\mathbb{S}^{(0)}_1} \tau^{(1)}_{0n}\mathcal{R}^{(1)}_{n0} , \qquad \tau_\pi \coloneqq \sum_{n\in\mathbb{S}^{(0)}_2} \tau^{(2)}_{0n}\mathcal{R}^{(2)}_{n0} \tag{6.89}$$

and the second-order terms on the right-hand side have been introduced in Eqs. (2.69) and (2.71).

We note that this method, which has been termed Inverse-Reynolds Dominance (IReD) approach [50] or order-of-magnitude approximation [49], is equivalent to the DNMR approach up to second order in Kn and Re $^{-1}$, as has been shown in Ref. [50]. However, a major advantage compared to the DNMR method is that no terms of second order in the Knudsen number appear in the equations of motion. These terms, which are written in the most general form in Eqs. (2.70), can potentially render the equations parabolic (and thus acausal), and are consequently omitted in practical applications. However, in the IReD approach, the effect of these terms, which is potentially large [160], is resummed into the (hyperbolic) terms of first order in both Knudsen and inverse Reynolds numbers [cf. Eqs. (2.69)], manifesting itself in different expressions for the transport coefficients [50]. Furthermore, we note that, in contrast to the results of the DNMR method, the values of the transport coefficients in the IReD approach are compatible with the constraints imposed from a phenomenological expression for the entropy to second order such that the second law of thermodynamics is fulfilled [52].

Thus, assuming that Knudsen and inverse Reynolds numbers are of the same order of magnitude, in order to stay accurate to second order also in practical applications this method is preferable over the usual DNMR prescription. Indeed, in Ref. [162], the DNMR and IReD approximations were considered in the context of an exactly solvable system of coupled relaxation equations akin to the ones arising from kinetic theory, and it was found that IReD is able to stay accurate in a wide range of situations. In comparison, the accuracy of DNMR depends on how much the slowest microscopic timescale dominates the macroscopic dynamics. Furthermore, the results of that reference confirm that the performance of DNMR is impeded by neglecting the terms of second order in the Knudsen number.[9]

6.3.2 Generalized irreducible collision terms

In order to develop spin hydrodynamics for particles up to spin 1, we will now apply the truncation prescription outlined for spin-0 systems in the previous subsection to the moment equations (6.75),

[9] We remark that a similar picture emerges when considering the results of Ref. [163], where the heat-flow problem was solved with the DNMR approach. In particular, the "21/37" approach introduced there, which works rather well, is similar in spirit to IReD, although more moments are made dynamical.

(6.78), and (6.80). In order to see which moments couple to each other, we first have to evaluate Eqs. (6.77), (6.79), and (6.81) for the relevant ranks in momentum.

Inserting Eq. (6.7) into the generic form of the collision integral developed in Chapter 5 and linearizing the resulting expression in the deviation from equilibrium $\delta f_{\mathbf{k}\mathfrak{s}}$, we find

$$C(x,k,\mathfrak{s}) = C_0(x,k,\mathfrak{s}) + \bar{C}(x,k,\mathfrak{s}) \ . \tag{6.90}$$

Here, the first term collects the nonzero parts of the collision term that depend only on the local-equilibrium distribution function,

$$\begin{aligned} C_0(x,k,\mathfrak{s}) &:= \frac{1}{2}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\widetilde{\mathcal{W}}^{(\sigma)} f_{0\mathbf{k}}f_{0\mathbf{k}'}\widetilde{f}_{0\mathbf{k}_1}\widetilde{f}_{0\mathbf{k}_2} \\ &\times\Bigg[-\frac{\sigma\hbar}{2m}\left(\widetilde{\Omega}_{0,\mu\nu}-\widetilde{\varpi}_{\mu\nu}\right)\left(k_1^\mu\mathfrak{s}_1^\nu + k_2^\mu\mathfrak{s}_2^\nu - k^\mu\mathfrak{s}^\nu - k'^\mu\mathfrak{s}'^\nu\right) \\ &+\left(\partial_\mu\alpha_0\right)\left(\Delta_1^\mu+\Delta_2^\mu-\Delta^\mu-\Delta'^\mu\right) - \frac{1}{2}\left(\Delta_1^\mu k_1^\nu + \Delta_2^\mu k_2^\nu - \Delta^\mu k^\nu - \Delta'^\mu k'^\nu\right)\partial_{(\mu}\beta_{0,\nu)}\Bigg] \ . \end{aligned} \tag{6.91}$$

Note that this term is merely the part of the collision term that we neglected while defining local equilibrium at the beginning of this section, cf. Eq. (6.4).
The second term in Eq. (6.90) on the other hand describes the contributions from the deviations from equilibrium,

$$\begin{aligned} \bar{C}(x,k,\mathfrak{s}) &:= \frac{1}{2}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,\mathrm{d}\bar{S}(k)(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\mathcal{W}^{(\sigma)} \\ &\times f_{0\mathbf{k}}f_{0\mathbf{k}'}\widetilde{f}_{0\mathbf{k}_1}\widetilde{f}_{0\mathbf{k}_2}\Big[\phi_{\mathbf{k}_1}+\phi_{\mathbf{k}_2}-\phi_{\mathbf{k}}-\phi_{\mathbf{k}'} - \left(\mathfrak{s}_{1,\mu}\zeta^\mu_{\mathbf{k}_1} + \mathfrak{s}_{2,\mu}\zeta^\mu_{\mathbf{k}_2} - \bar{\mathfrak{s}}_\mu\zeta^\mu_{\mathbf{k}} - \mathfrak{s}'_\mu\zeta^\mu_{\mathbf{k}'}\right) \\ &+\mathfrak{s}_{1,\alpha}\mathfrak{s}_{1,\beta}K^{\alpha\beta}_{1,\mu\nu}\xi^{\mu\nu}_{\mathbf{k}_1} + \mathfrak{s}_{2,\alpha}\mathfrak{s}_{2,\beta}K^{\alpha\beta}_{2,\mu\nu}\xi^{\mu\nu}_{\mathbf{k}_2} - \bar{\mathfrak{s}}_\alpha\bar{\mathfrak{s}}_\beta K^{\alpha\beta}_{\mu\nu}\xi^{\mu\nu}_{\mathbf{k}} - \mathfrak{s}'_\alpha\mathfrak{s}'_\beta K'^{\alpha\beta}_{\mu\nu}\xi^{\mu\nu}_{\mathbf{k}'}\Big] \ . \end{aligned} \tag{6.92}$$

Note that in both equations above we again neglected terms of order $\mathcal{O}(\varkappa\,\mathrm{Re}^{-1})$ as well as terms of second order in inverse Reynolds numbers. While the former contribution will give corrections to the contributions (2.69), the latter one leads to terms in the hydrodynamic equations that are nonlinear in the dissipative currents, cf. Eq. (2.71). The coefficients in front of these terms do not necessarily need to be small, as shown in Ref. [160]. However, due to the complexity of computing the nonlinear contributions of the collision integrals, we postpone their analysis to future work.

In the following, we will evaluate the generalized irreducible collision integrals, which always feature integrations over both momentum and spin space. To simplify the notation, we introduce the nonvanishing integrals over spin space involving the transition rate,

$$\mathcal{M} := \frac{1}{2}(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\int[\mathrm{d}S]\,\mathrm{d}\bar{S}(k)\,\mathcal{W}^{(\sigma)} \ , \tag{6.93a}$$

$$\mathcal{N}^{\mu\nu}_{(\mathfrak{s}_i\mathfrak{s}_j)} := \frac{1}{2}(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\int[\mathrm{d}S]\,\mathrm{d}\bar{S}(k)\,\mathcal{W}^{(\sigma)}\mathfrak{s}_i^\mu\mathfrak{s}_j^\nu \ , \tag{6.93b}$$

$$\mathcal{N}^{\mu\nu}_{(\mathfrak{s})} := \frac{1}{2}(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\int[\mathrm{d}S]\,\mathrm{d}\bar{S}(k)\,\mathcal{W}^{(\sigma)}K^{\mu\nu}_{\alpha\beta}\mathfrak{s}^\alpha\mathfrak{s}^\beta \ , \tag{6.93c}$$

$$\mathcal{N}^{\mu\nu,\alpha\beta}_{(\mathfrak{s}_i\mathfrak{s}_j)} := \frac{1}{2}(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\int[\mathrm{d}S]\,\mathrm{d}\bar{S}(k)\,\mathcal{W}^{(\sigma)}K^{\mu\nu}_{i,\rho\sigma}\mathfrak{s}_i^\rho\mathfrak{s}_i^\sigma\,K^{\alpha\beta}_{j,\zeta\eta}\mathfrak{s}_j^\zeta\mathfrak{s}_j^\eta \ , \tag{6.93d}$$

$$\mathcal{M}^{\mu\nu}_{(\mathfrak{s}_i\mathfrak{s}_j)} := \Xi^{\nu}_{j\ \alpha}\mathcal{N}^{\mu\alpha}_{(\mathfrak{s}_i\mathfrak{s}_j)} \ , \tag{6.93e}$$

$$\mathcal{M}^{\mu\nu}_{(\mathfrak{s})} := \Xi^{\mu\nu}_{j,\gamma\delta}\mathcal{N}^{\gamma\delta}_{(\mathfrak{s})} \ , \tag{6.93f}$$

$$\mathcal{M}^{\mu\nu,\alpha\beta}_{(\mathfrak{s}_i\mathfrak{s}_j)} := \Xi^{\alpha\beta}_{j,\gamma\delta}\mathcal{N}^{\mu\nu,\gamma\delta}_{(\mathfrak{s}_i\mathfrak{s}_j)} \ . \tag{6.93g}$$

In these expressions, $\mathfrak{s}_i \in \{\mathfrak{s}_1,\mathfrak{s}_2,\mathfrak{s}',\bar{\mathfrak{s}},\mathfrak{s}\}$, and the momentum k_i is determined by the associated spin vector, i.e., k_1 by $\mathfrak{s}_1$, k_2 by $\mathfrak{s}_2$, k' by $\mathfrak{s}'$, and k by $\mathfrak{s}$ and $\bar{\mathfrak{s}}$. Similarly, Ξ_j is the tensor Ξ introduced in Eq. (6.28) with the momentum being k_j.

Spin-rank zero

Inserting the moment expansion (6.29), Eq. (6.77) becomes

$$C_{r-1}^{\langle\mu_1\cdots\mu_\ell\rangle} = \int [\mathrm{d}K]\; E_{\mathbf{k}}^{r-1} k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle} f_{0\mathbf{k}} f_{0\mathbf{k}'} \widetilde{f}_{0\mathbf{k}_1}\widetilde{f}_{0\mathbf{k}_2} \sum_{\ell'=0}^{\infty}\Bigg[\sum_{n\in\mathbb{S}_{\ell'}^{(0)}} \mathcal{M}\Big(\mathcal{H}_{\mathbf{k}_1 n}^{(0,\ell')} k_{\langle 1,\nu_1}\cdots k_{1,\nu_{\ell'}\rangle}$$
$$+ \mathcal{H}_{\mathbf{k}_2 n}^{(0,\ell')} k_{\langle 2,\nu_1}\cdots k_{2,\nu_{\ell'}\rangle} - \mathcal{H}_{\mathbf{k}' n}^{(0,\ell')} k'_{\langle\nu_1}\cdots k'_{\nu'_\ell\rangle} - \mathcal{H}_{\mathbf{k} n}^{(0,\ell')} k_{\langle\nu_1}\cdots k_{\nu_{\ell'}\rangle}\Big)\rho_n^{\nu_1\cdots\nu_{\ell'}}$$
$$+ \sum_{n\in\mathbb{S}_{\ell'}^{(2)}} \Big(\mathcal{M}_{(\mathfrak{s}_1)\rho\sigma}\mathcal{H}_{\mathbf{k}_1 n}^{(2,\ell')} k_{\langle 1,\nu_1}\cdots k_{1,\nu_{\ell'}\rangle} + \mathcal{M}_{(\mathfrak{s}_2)\rho\sigma}\mathcal{H}_{\mathbf{k}_2 n}^{(2,\ell')} k_{\langle 2,\nu_1}\cdots k_{2,\nu_{\ell'}\rangle}$$
$$- \mathcal{M}_{(\mathfrak{s}')\rho\sigma}\mathcal{H}_{\mathbf{k}' n}^{(2,\ell')} k'_{\langle\nu_1}\cdots k'_{\nu_{\ell'}\rangle} - \mathcal{M}_{(\bar{\mathfrak{s}})\rho\sigma}\mathcal{H}_{\mathbf{k} n}^{(2,\ell')} k_{\langle\nu_1}\cdots k_{\nu_{\ell'}\rangle}\Big)\psi_n^{\langle\rho\sigma\rangle,\nu_1\cdots\nu_{\ell'}}\Bigg]$$
$$\equiv -\sum_{\ell'=0}^{\infty}\Bigg[\sum_{n\in\mathbb{S}_{\ell'}^{(0)}} (\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_{\ell'}}\,\rho_n^{\nu_1\cdots\nu_{\ell'}} + \sum_{n\in\mathbb{S}_{\ell'}^{(2)}} (A_{rn})^{\mu_1\cdots\mu_\ell}_{\rho\sigma,\nu_1\cdots\nu_{\ell'}}\,\psi_n^{\langle\rho\sigma\rangle,\nu_1\cdots\nu_{\ell'}}\Bigg]\,. \tag{6.94}$$

Here we used that, when considering interactions that do not violate parity, all integrals over the transition rate weighted with an odd number of spin vectors vanish, cf. Eq. (4.155). Furthermore, we defined

$$(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_{\ell'}} := -\int [\mathrm{d}K]\; E_{\mathbf{k}}^{r-1} k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle} f_{0\mathbf{k}} f_{0\mathbf{k}'} \widetilde{f}_{0\mathbf{k}_1}\widetilde{f}_{0\mathbf{k}_2}\mathcal{M}\Big(\mathcal{H}_{\mathbf{k}_1 n}^{(0,\ell')} k_{\langle 1,\nu_1}\cdots k_{1,\nu_{\ell'}\rangle}$$
$$+ \mathcal{H}_{\mathbf{k}_2 n}^{(0,\ell')} k_{\langle 2,\nu_1}\cdots k_{2,\nu_{\ell'}\rangle} - \mathcal{H}_{\mathbf{k}' n}^{(0,\ell')} k'_{\langle\nu_1}\cdots k'_{\nu_{\ell'}\rangle} - \mathcal{H}_{\mathbf{k} n}^{(0,\ell')} k_{\langle\nu_1}\cdots k_{\nu_{\ell'}\rangle}\Big)\,, \tag{6.95a}$$
$$(A_{rn})^{\mu_1\cdots\mu_\ell}_{\rho\sigma,\nu_1\cdots\nu_{\ell'}} := -\int [\mathrm{d}K]\; E_{\mathbf{k}}^{r-1} k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle} f_{0\mathbf{k}} f_{0\mathbf{k}'} \widetilde{f}_{0\mathbf{k}_1}\widetilde{f}_{0\mathbf{k}_2}$$
$$\times\Big(\mathcal{M}_{(\mathfrak{s}_1)\rho\sigma}\mathcal{H}_{\mathbf{k}_1 n}^{(2,\ell')} k_{\langle 1,\nu_1}\cdots k_{1,\nu_{\ell'}\rangle} + \mathcal{M}_{(\mathfrak{s}_2)\rho\sigma}\mathcal{H}_{\mathbf{k}_2 n}^{(2,\ell')} k_{\langle 2,\nu_1}\cdots k_{2,\nu_{\ell'}\rangle}$$
$$- \mathcal{M}_{(\mathfrak{s}')\rho\sigma}\mathcal{H}_{\mathbf{k}' n}^{(2,\ell')} k'_{\langle\nu_1}\cdots k'_{\nu_{\ell'}\rangle} - \mathcal{M}_{(\bar{\mathfrak{s}})\rho\sigma}\mathcal{H}_{\mathbf{k} n}^{(2,\ell')} k_{\langle\nu_1}\cdots k_{\nu_{\ell'}\rangle}\Big)\,. \tag{6.95b}$$

Considering the irreducible moments fulfilling the required symmetries, we find the irreducible collision terms of spin-rank zero to be

$$C_{r-1} = -\sum_{n\in\mathbb{S}_0^{(0)}} \mathcal{A}_{rn}^{(0)}\rho_n - \sum_{n\in\mathbb{S}_2^{(2)}} A_{rn}^{(0,2)} p_n\,, \tag{6.96a}$$
$$C_{r-1}^{\langle\mu\rangle} = -\sum_{n\in\mathbb{S}_1^{(0)}} \mathcal{A}_{rn}^{(1)}\rho_n^\mu - \sum_{n\in\mathbb{S}_1^{(2)}} A_{rn}^{(1,1)} p_n^\mu - \sum_{n\in\mathbb{S}_3^{(2)}} A_{rn}^{(1,3)} q_n^\mu\,, \tag{6.96b}$$
$$C_{r-1}^{\langle\mu\nu\rangle} = -\sum_{n\in\mathbb{S}_2^{(0)}} \mathcal{A}_{rn}^{(2)}\rho_n^{\mu\nu} - \sum_{n\in\mathbb{S}_0^{(2)}} A_{rn}^{(2,0)}\psi_n^{\langle\mu\nu\rangle} - \sum_{n\in\mathbb{S}_2^{(2)}} A_{rn}^{(2,2)} p_n^{\mu\nu} - \sum_{n\in\mathbb{S}_4^{(2)}} A_{rn}^{(2,4)} q_n^{\mu\nu}\,. \tag{6.96c}$$

Here, we introduced for brevity the partial traces of moments of spin-rank two that transform as scalars, vectors, and traceless symmetric tensors, respectively:[10]

$$p_n := \psi_n^{\langle\mu\nu\rangle,}{}_{\mu\nu}\,,\quad p_n^\mu := \psi_n^{\langle\mu\alpha\rangle,}{}_\alpha\,,\quad q_n^\mu := \psi_n^{\langle\alpha\beta\rangle,\mu}{}_{\alpha\beta}\,,\quad p_n^{\mu\nu} := \psi_n^{\langle\alpha\beta\rangle,\gamma}{}_\alpha\Delta^{\mu\nu}_{\beta\gamma}\,,\quad q_n^{\mu\nu} := \psi_n^{\langle\alpha\beta\rangle,\mu\nu}{}_{\alpha\beta}\,. \tag{6.97}$$

In Eqs. (6.96), the coefficients $\mathcal{A}_{rn}$ determining the coupling of the moments of spin-rank zero to themselves read

$$\mathcal{A}_{rn}^{(\ell)} := \frac{1}{2\ell+1}\Delta^{\nu_1\cdots\nu_\ell}_{\mu_1\cdots\mu_\ell}(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\,, \tag{6.98}$$

[10] We remark at this point that the equations of motion for these moments can be obtained by taking the appropriate traces of the moments of spin-rank two and momentum-rank one to four, cf. Eqs. (6.80).

while the quantities A_{rn} that control the coupling of the moments of spin-rank two to the ones of spin-rank zero are given by

$$\begin{aligned}
A_{rn}^{(0,2)} &:= \frac{1}{5}\Delta^{\rho\sigma,\nu_1\nu_2}(A_{rn})_{\rho\sigma,\nu_1\nu_2} , & A_{rn}^{(1,1)} &:= \frac{1}{5}\Delta^{\rho}_{\mu}\Delta^{\sigma\nu_1}(A_{rn})^{\mu}_{\rho\sigma,\nu_1} , \\
A_{rn}^{(1,3)} &:= \frac{1}{7}\Delta^{\nu_1}_{\mu}\Delta^{\rho\sigma,\nu_2\nu_3}(A_{rn})^{\mu}_{\rho\sigma,\nu_1\nu_2\nu_3} , & A_{rn}^{(2,0)} &:= \frac{1}{5}\Delta^{\rho\sigma}_{\mu\nu}(A_{rn})^{\mu\nu}_{\rho\sigma} , \\
A_{rn}^{(2,2)} &:= \frac{12}{35}\Delta^{\rho\alpha}_{\mu\nu}\Delta_{\alpha}{}^{\sigma,\nu_1\nu_2}(A_{rn})^{\mu\nu}_{\rho\sigma,\nu_1\nu_2} , & A_{rn}^{(2,4)} &:= \frac{1}{9}\Delta^{\nu_1\nu_2}_{\mu\nu}\Delta^{\rho\sigma,\nu_3\nu_4}(A_{rn})^{\mu\nu}_{\rho\sigma,\nu_1\nu_2\nu_3\nu_4} .
\end{aligned} \tag{6.99}$$

Evidently, there is a coupling between the moments of spin-ranks zero and two through the collision terms.[11] The moments of spin-rank one on the other hand do not appear because of their transformation properties under parity. Furthermore, note that the terms C_0 do not contribute to the equations above, since there always appears an odd number of spin vectors in these integrals.

Spin-rank one

In the case of the irreducible moments of spin-rank one, there are two contributions, one from the equilibrium terms (6.91), and one from the terms explicitly involving $\delta f_{\mathbf{k}\mathfrak{s}}$ (6.92). We can immediately write down the generalized irreducible collision integrals as

$$\mathfrak{C}_{r-1}^{\langle\mu\rangle,\langle\mu_1\cdots\mu_\ell\rangle} = -\sum_{\ell'=0}^{\infty}\sum_{n\in\mathbb{S}_{\ell'}^{(1)}}(\mathcal{B}_{rn})^{\mu,\mu_1\cdots\mu_\ell}_{\nu,\nu_1\cdots\nu_{\ell'}}\tau_n^{\langle\nu\rangle,\nu_1\cdots\nu_{\ell'}} + \mathfrak{C}_{0,r-1}^{\langle\mu\rangle,\langle\mu_1\cdots\mu_\ell\rangle} , \tag{6.100}$$

where we defined

$$\begin{aligned}
(\mathcal{B}_{rn})^{\mu,\mu_1\cdots\mu_\ell}_{\nu,\nu_1\cdots\nu_{\ell'}} := \int [\mathrm{d}K]\; & E_{\mathbf{k}}^{r-1}k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}f_{0\mathbf{k}}f_{0\mathbf{k}'}\widetilde{f}_{0\mathbf{k}_1}\widetilde{f}_{0\mathbf{k}_2} \\
&\times\Big(\mathcal{M}^{\mu}_{(\mathfrak{s}\mathfrak{s}_1)^\nu}\mathcal{H}^{(1,\ell')}_{\mathbf{k}_1 n}k_{\langle 1,\nu_1}\cdots k_{1,\nu_{\ell'}\rangle} + \mathcal{M}^{\mu}_{(\mathfrak{s}\mathfrak{s}_2)^\nu}\mathcal{H}^{(1,\ell')}_{\mathbf{k}_2 n}k_{\langle 2,\nu_1}\cdots k_{2,\nu_{\ell'}\rangle} \\
&-\mathcal{M}^{\mu}_{(\mathfrak{s}\mathfrak{s}')^\nu}\mathcal{H}^{(1,\ell')}_{\mathbf{k}' n}k'_{\langle\nu_1}\cdots k'_{\nu_{\ell'}\rangle} - \mathcal{M}^{\mu}_{(\mathfrak{s}\bar{\mathfrak{s}})^\nu}\mathcal{H}^{(1,\ell')}_{\mathbf{k} n}k_{\langle\nu_1}\cdots k_{\nu_{\ell'}\rangle}\Big) ,
\end{aligned} \tag{6.101}$$

and collected the terms coming from the local-equilibrium distribution function in

$$\mathfrak{C}_{0,r-1}^{\langle\mu\rangle,\langle\mu_1\cdots\mu_\ell\rangle} := \int \mathrm{d}\Gamma\, E_{\mathbf{k}}^{r-1}\mathfrak{s}^{\langle\mu\rangle}k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}C_0(x,k,\mathfrak{s}) . \tag{6.102}$$

Note that, in our power-counting, there are no first-order terms on the right-hand sides of Eqs. (6.78), such that the Navier-Stokes values of the moments of spin-rank one will be provided by the generalized irreducible collision terms (6.102). Since we are interested in keeping only the irreducible moments which are hydrodynamically important, i.e., whose Navier-Stokes values are of first order in the Knudsen number, we only evaluate a subset of the collision integrals corresponding to these moments. Given the vectors and tensors at our disposal, there is only a limited number of quantities of first order in Kn which are orthogonal to the four-velocity and transform as an axial vector in one index, namely an axial vector, two antisymmetric tensors, and a rank-three tensor, i.e.,

$$\left(\widetilde{\Omega}_0^{\mu\nu} - \widetilde{\varpi}^{\mu\nu}\right)u_\nu \equiv 2\omega_0^\mu + 2\beta_0\omega^\mu , \quad \widetilde{\Omega}_0^{\langle\mu\rangle\langle\nu\rangle} - \widetilde{\varpi}^{\langle\mu\rangle\langle\nu\rangle} , \quad \epsilon^{\mu\nu\alpha\beta}u_\alpha I_\beta , \quad \text{and} \quad \sigma_\rho{}^{\langle\nu}\epsilon^{\lambda\rangle\mu\alpha\rho}u_\alpha .$$

Orienting on the equations of motion for the spin potential (6.62), we reformulate the antisymmetric tensors that may appear as Navier-Stokes values equivalently as vectors, namely

$$I^\mu \qquad \text{and} \qquad \epsilon^{\mu\nu\alpha\beta}u_\nu(\widetilde{\Omega}_{\alpha\beta} - \widetilde{\varpi}_{\alpha\beta}) \equiv 2\kappa_0^\mu + \beta_0\left(\frac{2F^\mu}{\varepsilon_0+P_0} - \frac{I^\mu}{\beta_0 h}\right) ,$$

where we used the equation of motion for the fluid four-velocity and neglected nonlinear terms, in accordance with our earlier approximation to the collision terms. Accordingly, we can anticipate that

[11] We remark that a coupling of this form has also been obtained in Ref. [73].

only the antisymmetric components of the moments $\tau_r^{\mu,\nu}$ will have nonvanishing NAVIER-STOKES values, such that we may equivalently consider the moments w_r^μ defined in Eq. (6.60). Furthermore, the symmetries of the NAVIER-STOKES values suggest to decompose the moments of tensor-rank two in momentum according to

$$\tau_r^{\langle\mu\rangle,\nu\lambda} = \frac{3}{5}\Delta^{\mu\langle\nu} t_r^{\lambda\rangle} - \frac{2}{3} t_{r,\rho}{}^{\langle\lambda}\epsilon^{\nu\rangle\mu\alpha\rho} u_\alpha + \tau_r^{\langle\mu,\nu\lambda\rangle} , \tag{6.103}$$

with

$$t_r^\mu = \tau_r^{\alpha,\mu}{}_\alpha , \qquad t_r^{\mu\nu} := \tau_{r,\alpha,\beta}{}^{\langle\mu}\epsilon^{\nu\rangle\alpha\beta\rho} u_\rho . \tag{6.104}$$

Then, after defining

$$\mathfrak{C}^\mu_{w,r-1} := \epsilon^{\mu\nu\alpha\beta} u_\nu \mathfrak{C}_{r-1,\alpha,\beta} , \qquad \mathfrak{C}^{\mu\nu}_{t,r-1} := \mathfrak{C}_{r-1,\alpha,\beta}{}^{\langle\mu}\epsilon^{\nu\rangle\alpha\beta\rho} u_\rho , \tag{6.105}$$

the relevant collision integrals are

$$\mathfrak{C}^{\langle\mu\rangle}_{r-1} = 2g_r^{(0)}\left(\omega_0^\mu + \beta_0\omega^\mu\right) - \sum_{n\in\mathbb{S}_0^{(1)}} \mathcal{B}^{(0)}_{rn}\tau_n^{\langle\mu\rangle} - \sum_{n\in\mathbb{S}_2^{(1)}} \mathcal{B}^{(0,2)}_{rn} t_n^\mu , \tag{6.106a}$$

$$\mathfrak{C}^{\alpha,\mu}_{r-1\alpha} = 2g_r^{(2)}\left(\omega_0^\mu + \beta_0\omega^\mu\right) - \sum_{n\in\mathbb{S}_0^{(1)}} \mathcal{B}^{(2,0)}_{rn}\tau_n^{\langle\mu\rangle} - \sum_{n\in\mathbb{S}_2^{(1)}} \mathcal{B}^{(2)}_{rn} t_n^\mu , \tag{6.106b}$$

$$\mathfrak{C}^\mu_{w,r-1} = 2g_r^{(\kappa)}\left(\kappa_0^\mu + \frac{\beta_0 F^\mu}{\varepsilon_0 + P_0}\right) + g_r^{(I)} I^\mu - \sum_{n\in\mathbb{S}_1^{(1)}} \mathcal{B}^{(1)}_{rn} w_n^\mu , \tag{6.106c}$$

$$\mathfrak{C}^{\mu\nu}_{t,r-1} = h_r^{(2)}\beta_0\sigma^{\mu\nu} - \sum_{n\in\mathbb{S}_2^{(1)}} \overline{\mathcal{B}}^{(2)}_{rn} t_n^{\mu\nu} , \tag{6.106d}$$

with the terms $\mathcal{B}_{rn}$ defined as

$$\begin{aligned}
\mathcal{B}^{(0)}_{rn} &:= \frac{1}{3}\Delta^\nu_\mu (\mathcal{B}_{rn})^\mu_\nu , & \mathcal{B}^{(0,2)}_{rn} &:= \frac{1}{5}\Delta^{\nu_1}_\mu \Delta^{\nu_2\nu}(B_{rn})^\mu_{\nu,\nu_1\nu_2} , \\
\mathcal{B}^{(2,0)}_{rn} &:= \frac{1}{3}\Delta^\nu_\mu \Delta_{\alpha\beta}(\mathcal{B}_{rn})^{\alpha,\mu\beta}_\nu , & \mathcal{B}^{(2)}_{rn} &:= \frac{1}{5}\Delta^{\nu_1}_\mu \Delta^{\nu_2\nu}\Delta_{\alpha\beta}(\mathcal{B}_{rn})^{\alpha,\mu\beta}_{\nu,\nu_1\nu_2} , \\
\mathcal{B}^{(1)}_{rn} &:= \frac{1}{6}\Delta^{[\nu}_\mu \Delta^{\nu_1]}_{\mu_1}(\mathcal{B}_{rn})^{\mu,\mu_1}_{\nu,\nu_1} , & \overline{\mathcal{B}}^{(2)}_{rn} &:= \frac{1}{15}\Delta^\nu_\alpha \Delta^{\nu_1\nu_2}_{\beta\gamma}(\mathcal{B}_{rn})^{\alpha,\beta\gamma}_{\nu,\nu_1\nu_2} .
\end{aligned} \tag{6.107}$$

Furthermore, we defined $g_r^{(I)} := g_r^{(\alpha)} - g_r^{(\kappa)}/h$ and introduced the coefficients

$$g_r^{(0)} := \frac{\sigma\hbar}{12m}\int [\mathrm{d}K] E_{\mathbf{k}}^{r-1} f_{0\mathbf{k}} f_{0\mathbf{k}'} \widetilde{f}_{0\mathbf{k}_1}\widetilde{f}_{0\mathbf{k}_2} g_{\alpha\beta}\left(E_{\mathbf{k}_1}\mathcal{N}^{\alpha\beta}_{(\mathfrak{s}\mathfrak{s}_1)} + E_{\mathbf{k}_2}\mathcal{N}^{\alpha\beta}_{(\mathfrak{s}\mathfrak{s}_2)} - E_{\mathbf{k}}\mathcal{N}^{\alpha\beta}_{(\mathfrak{s}\mathfrak{s})} - E_{\mathbf{k}'}\mathcal{N}^{\alpha\beta}_{(\mathfrak{s}\mathfrak{s}')}\right) , \tag{6.108a}$$

$$g_r^{(2)} := \frac{\sigma\hbar}{12m}\int [\mathrm{d}K] E_{\mathbf{k}}^{r-1} f_{0\mathbf{k}} f_{0\mathbf{k}'} \widetilde{f}_{0\mathbf{k}_1}\widetilde{f}_{0\mathbf{k}_2} k_{\langle\alpha}k_{\beta\rangle}\left(E_{\mathbf{k}_1}\mathcal{N}^{\alpha\beta}_{(\mathfrak{s}\mathfrak{s}_1)} + E_{\mathbf{k}_2}\mathcal{N}^{\alpha\beta}_{(\mathfrak{s}\mathfrak{s}_2)} - E_{\mathbf{k}}\mathcal{N}^{\alpha\beta}_{(\mathfrak{s}\mathfrak{s})} - E_{\mathbf{k}'}\mathcal{N}^{\alpha\beta}_{(\mathfrak{s}\mathfrak{s}')}\right) , \tag{6.108b}$$

$$g_r^{(\kappa)} := \frac{\sigma\hbar}{12m}\int [\mathrm{d}K] E_{\mathbf{k}}^{r-1} f_{0\mathbf{k}} f_{0\mathbf{k}'} \widetilde{f}_{0\mathbf{k}_1}\widetilde{f}_{0\mathbf{k}_2} k_{[\langle\alpha\rangle}\left(k_{1,\beta}\mathcal{N}^{\langle\beta\rangle]\alpha}_{(\mathfrak{s}\mathfrak{s}_1)} + k_{2,\beta}\mathcal{N}^{\langle\beta\rangle]\alpha}_{(\mathfrak{s}\mathfrak{s}_2)} - k_\beta\mathcal{N}^{\langle\beta\rangle]\alpha}_{(\mathfrak{s}\mathfrak{s})} - k'_\beta\mathcal{N}^{\langle\beta\rangle]\alpha}_{(\mathfrak{s}\mathfrak{s}')}\right) , \tag{6.108c}$$

$$\begin{aligned}
g_r^{(\alpha)} := \frac{1}{6}\int [\mathrm{d}\Gamma] E_{\mathbf{k}}^{r-1} (2\pi\hbar)^4\delta^{(4)}(k + k' - k_1 - k_2)\widetilde{W}^{(\sigma)} f_{0\mathbf{k}} f_{0\mathbf{k}'} \widetilde{f}_{0\mathbf{k}_1}\widetilde{f}_{0\mathbf{k}_2} \\
\times \epsilon^{\mu\nu\alpha\beta} u_\nu \mathfrak{s}_\alpha k_\beta \left(\Delta_{1,\mu} + \Delta_{2,\mu} - \Delta_\mu - \Delta'_\mu\right) ,
\end{aligned} \tag{6.108d}$$

$$\begin{aligned}
h_r^{(2)} := -\frac{1}{10}\int [\mathrm{d}\Gamma] E_{\mathbf{k}}^{r-1} (2\pi\hbar)^4\delta^{(4)}(k_1 + k_2 - k - k')\widetilde{\mathcal{W}}^{(\sigma)} f_{0\mathbf{k}} f_{0\mathbf{k}'} \widetilde{f}_{0\mathbf{k}_1}\widetilde{f}_{0\mathbf{k}_2} k_{\langle\beta}k^{\zeta\rangle}\epsilon^{\eta\alpha\beta\gamma}\mathfrak{s}_\alpha u_\gamma \\
\times \left(\Delta_{1,\langle\zeta}k_{1,\eta\rangle} + \Delta_{2,\langle\zeta}k_{2,\eta\rangle} - \Delta_{\langle\zeta}k_{\eta\rangle} - \Delta'_{\langle\zeta}k'_{\eta\rangle}\right) ,
\end{aligned} \tag{6.108e}$$

which will determine the NAVIER-STOKES values of the components of the spin tensor. From these equations, we can see that the irreducible moments of spin-rank one do not feature a coupling to moments of spin-ranks zero or two, but couple among themselves. Furthermore, it is apparent that the symmetric moments of tensor-rank one in momentum are, in our power-counting, not among the quantities which are hydrodynamically important in the sense that they do not have Navier-Stokes values which are of first order in the Knudsen number. The same holds true for the completely traceless component of the irreducible moment of spin-rank one and momentum-rank three, cf. Eq. (6.103).

Spin-rank two

In the case of the generalized irreducible collision terms of spin-rank two, Eq. (6.81), we can already anticipate that the equilibrium-type terms (6.91) will vanish. Then, we can write

$$\mathfrak{C}_{r-1}^{\langle\mu\nu\rangle,\langle\mu_1\cdots\mu_\ell\rangle} = -\sum_{\ell'=0}^{\infty}\left[\sum_{n\in\mathbb{S}_{\ell'}^{(2)}}(\mathcal{D}_{rn})^{\mu\nu,\mu_1\cdots\mu_\ell}_{\rho\sigma,\nu_1\cdots\nu_{\ell'}}\,\psi_n^{\langle\rho\sigma\rangle,\nu_1\cdots\nu_{\ell'}} + \sum_{n\in\mathbb{S}_{\ell'}^{(0)}}(D_{rn})^{\mu\nu,\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_{\ell'}}\,\rho_n^{\nu_1\cdots\nu_{\ell'}}\right] , \tag{6.109}$$

where we introduced

$$\begin{aligned}(\mathcal{D}_{rn})^{\mu\nu,\mu_1\cdots\mu_\ell}_{\rho\sigma,\nu_1\cdots\nu_{\ell'}} :=& -\int [\mathrm{d}K]\; E_{\mathbf{k}}^{r-1}k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}f_{0\mathbf{k}}f_{0\mathbf{k}'}\widetilde{f}_{0\mathbf{k}_1}\widetilde{f}_{0\mathbf{k}_2}\\ &\times\Big(\mathcal{M}^{\mu\nu}_{(\mathfrak{s}\mathfrak{s}_1)\rho\sigma}\mathcal{H}^{(2,\ell')}_{\mathbf{k}_1 n}k_{\langle 1,\nu_1}\cdots k_{1,\nu_{\ell'}\rangle} + \mathcal{M}^{\mu\nu}_{(\mathfrak{s}\mathfrak{s}_2)\rho\sigma}\mathcal{H}^{(2,\ell')}_{\mathbf{k}_2 n}k_{\langle 2,\nu_1}\cdots k_{2,\nu_{\ell'}\rangle}\\ &- \mathcal{M}^{\mu\nu}_{(\mathfrak{s}\mathfrak{s}')\rho\sigma}\mathcal{H}^{(2,\ell')}_{\mathbf{k}' n}k'_{\langle\nu_1}\cdots k'_{\nu_{\ell'}\rangle} - \mathcal{M}^{\mu\nu}_{(\mathfrak{s}\bar{\mathfrak{s}})\rho\sigma}\mathcal{H}^{(2,\ell')}_{\mathbf{k} n}k_{\langle\nu_1}\cdots k_{\nu_{\ell'}\rangle}\Big)\,, \end{aligned} \tag{6.110a}$$

$$\begin{aligned}(D_{rn})^{\mu\nu,\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_{\ell'}} :=& -\int [\mathrm{d}K]\; E_{\mathbf{k}}^{r-1}k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}f_{0\mathbf{k}}f_{0\mathbf{k}'}\widetilde{f}_{0\mathbf{k}_1}\widetilde{f}_{0\mathbf{k}_2}\mathcal{N}^{\mu\nu}_{(\mathfrak{s})}\Big(\mathcal{H}^{(0,\ell')}_{\mathbf{k}_1 n}k_{\langle 1,\nu_1}\cdots k_{1,\nu_{\ell'}\rangle}\\ &+\mathcal{H}^{(0,\ell')}_{\mathbf{k}_2 n}k_{\langle 2,\nu_1}\cdots k_{2,\nu_{\ell'}\rangle} - \mathcal{H}^{(0,\ell')}_{\mathbf{k}' n}k'_{\langle\nu_1}\cdots k'_{\nu_{\ell'}\rangle} - \mathcal{H}^{(0,\ell')}_{\mathbf{k} n}k_{\langle\nu_1}\cdots k_{\nu_{\ell'}\rangle}\Big)\,. \end{aligned} \tag{6.110b}$$

Since the equations of motion for the irreducible moments of spin-rank two (6.80) do not feature terms of first order in Knudsen or inverse Reynolds numbers on the right-hand side, contributions of this order can only arise from the generalized irreducible collision integrals (6.81). The previous discussion of the possible tensor structures appearing in these terms makes it clear that the terms responsible for these first-order contributions are the ones involving the moments of spin-rank zero. However, there one only needs to take into account the moments up to tensor-rank two in momentum, since the Navier-Stokes values of higher-order moments are also not of first order, cf. Eqs. (6.75). Thus, the generalized irreducible collision integrals of spin-rank two which contain first-order contributions read

$$\mathfrak{C}_{r-1}^{\langle\mu\nu\rangle} = -\sum_{n\in\mathbb{S}_0^{(2)}}\mathcal{D}^{(0)}_{rn}\psi_n^{\langle\mu\nu\rangle} - \sum_{n\in\mathbb{S}_2^{(2)}}\mathcal{D}^{(0,2)}_{rn}p_n^{\mu\nu} - \sum_{n\in\mathbb{S}_4^{(2)}}\mathcal{D}^{(0,4)}_{rn}q_n^{\mu\nu} - \sum_{n\in\mathbb{S}_2^{(0)}}D^{(0,2)}_{rn}\rho_n^{\mu\nu}\,, \tag{6.111a}$$

$$\mathfrak{C}_{p,r-1}^{\mu\nu} = -\sum_{n\in\mathbb{S}_0^{(2)}}\mathcal{D}^{(2,0)}_{rn}\psi_n^{\langle\mu\nu\rangle} - \sum_{n\in\mathbb{S}_2^{(2)}}\mathcal{D}^{(2)}_{rn}p_n^{\mu\nu} - \sum_{n\in\mathbb{S}_4^{(2)}}\mathcal{D}^{(2,4)}_{rn}q_n^{\mu\nu} - \sum_{n\in\mathbb{S}_2^{(0)}}D^{(2,2)}_{rn}\rho_n^{\mu\nu}\,, \tag{6.111b}$$

$$\mathfrak{C}_{q,r-1}^{\mu\nu} = -\sum_{n\in\mathbb{S}_0^{(2)}}\mathcal{D}^{(4,0)}_{rn}\psi_n^{\langle\mu\nu\rangle} - \sum_{n\in\mathbb{S}_2^{(2)}}\mathcal{D}^{(4,2)}_{rn}p_n^{\mu\nu} - \sum_{n\in\mathbb{S}_4^{(2)}}\mathcal{D}^{(4)}_{rn}q_n^{\mu\nu} - \sum_{n\in\mathbb{S}_2^{(0)}}D^{(4,2)}_{rn}\rho_n^{\mu\nu}\,, \tag{6.111c}$$

$$\mathfrak{C}_{p,r-1}^{\mu} = -\sum_{n\in\mathbb{S}_1^{(2)}}\mathcal{D}^{(1)}_{rn}p_n^{\mu} - \sum_{n\in\mathbb{S}_3^{(2)}}\mathcal{D}^{(1,3)}_{rn}q_n^{\mu} - \sum_{n\in\mathbb{S}_1^{(0)}}D^{(1,1)}_{rn}\rho_n^{\mu}\,, \tag{6.111d}$$

$$\mathfrak{C}_{q,r-1}^{\mu} = -\sum_{n\in\mathbb{S}_1^{(2)}}\mathcal{D}^{(3,1)}_{rn}p_n^{\mu} - \sum_{n\in\mathbb{S}_3^{(2)}}\mathcal{D}^{(3)}_{rn}q_n^{\mu} - \sum_{n\in\mathbb{S}_1^{(0)}}D^{(3,1)}_{rn}\rho_n^{\mu}\,, \tag{6.111e}$$

$$\mathfrak{C}_{p,r-1} = -\sum_{n\in\mathbb{S}_2^{(2)}}\overline{\mathcal{D}}^{(2)}_{rn}p_n - \sum_{n\in\mathbb{S}_0^{(0)}}D^{(2,0)}_{rn}\rho_n\,, \tag{6.111f}$$

where the quantities p_n and q_n have been introduced in Eq. (6.97), and we defined the corresponding collision terms as

$$\begin{aligned}
&\mathfrak{C}_{p,r-1} := \mathfrak{C}^{\alpha\beta,}_{r-1\alpha\beta} , \quad \mathfrak{C}^{\mu}_{p,r-1} := \mathfrak{C}^{\langle\mu\alpha\rangle,}_{r-1\quad\alpha} , \quad \mathfrak{C}^{\mu}_{q,r-1} := \mathfrak{C}^{\alpha\beta,\mu}_{r-1\ \alpha\beta} , \\
&\mathfrak{C}^{\mu\nu}_{p,r-1} := \mathfrak{C}^{\langle\alpha\beta\rangle,\gamma}_{r-1\quad\alpha}\Delta^{\mu\nu}_{\beta\gamma} , \quad \mathfrak{C}^{\mu\nu}_{q,r-1} := \mathfrak{C}^{\alpha\beta,\mu\nu}_{r-1\quad\alpha\beta} .
\end{aligned} \tag{6.112}$$

The coefficients $\mathcal{D}_{rn}$ appearing in Eqs. (6.111) that determine the coupling between the moments of spin-rank two read

$$\begin{aligned}
\mathcal{D}^{(0)}_{rn} &:= \frac{1}{5}\Delta^{\rho\sigma}_{\mu\nu}(\mathcal{D}_{rn})^{\mu\nu}_{\rho\sigma} , & \mathcal{D}^{(0,2)}_{rn} &:= \frac{12}{35}\Delta^{\rho\alpha}_{\mu\nu}\Delta_{\alpha}^{\ \sigma,\nu_1\nu_2}(\mathcal{D}_{rn})^{\mu\nu}_{\rho\sigma,\nu_1\nu_2} , \\
\mathcal{D}^{(0,4)}_{rn} &:= \frac{1}{9}\Delta^{\nu_1\nu_2}_{\mu\nu}\Delta^{\rho\sigma,\nu_3\nu_4}(\mathcal{D}_{rn})^{\mu\nu}_{\rho\sigma,\nu_1\nu_2\nu_3\nu_4} , & \mathcal{D}^{(2,0)}_{rn} &:= \frac{1}{5}\Delta^{\rho\sigma}_{\beta\gamma}\Delta_{\alpha\delta}(\mathcal{D}_{rn})^{\alpha\beta,\gamma\delta}_{\rho\sigma} , \\
\mathcal{D}^{(2)}_{rn} &:= \frac{12}{35}\Delta^{\rho\zeta}_{\beta\gamma}\Delta_{\zeta}^{\ \sigma,\nu_1\nu_2}\Delta_{\alpha\delta}(\mathcal{D}_{rn})^{\alpha\beta,\gamma\delta}_{\rho\sigma,\nu_1\nu_2} , & \mathcal{D}^{(2,4)}_{rn} &:= \frac{1}{9}\Delta^{\nu_1\nu_2}_{\beta\gamma}\Delta^{\rho\sigma,\nu_3\nu_4}\Delta_{\alpha\delta}(\mathcal{D}_{rn})^{\alpha\beta,\gamma\delta}_{\rho\sigma,\nu_1\nu_2\nu_3\nu_4} , \\
\mathcal{D}^{(4,0)}_{rn} &:= \frac{1}{5}\Delta^{\rho\sigma}_{\gamma\delta}\Delta_{\alpha\beta,\mu\nu}(\mathcal{D}_{rn})^{\alpha\beta,\gamma\delta\mu\nu}_{\rho\sigma} , & \mathcal{D}^{(4,2)}_{rn} &:= \frac{12}{35}\Delta^{\rho\zeta}_{\gamma\delta}\Delta_{\zeta}^{\ \sigma,\nu_1\nu_2}\Delta_{\alpha\beta,\mu\nu}(\mathcal{D}_{rn})^{\alpha\beta,\gamma\delta\mu\nu}_{\rho\sigma,\nu_1\nu_2} , \\
\mathcal{D}^{(4)}_{rn} &:= \frac{1}{9}\Delta^{\nu_1\nu_2}_{\gamma\delta}\Delta^{\rho\sigma,\nu_3\nu_4}\Delta_{\alpha\beta,\mu\nu}(\mathcal{D}_{rn})^{\alpha\beta,\gamma\delta\mu\nu}_{\rho\sigma,\nu_1\nu_2\nu_3\nu_4} , & \mathcal{D}^{(1)}_{rn} &:= \frac{1}{5}\Delta^{\rho}_{\mu}\Delta^{\sigma\nu_1}\Delta_{\alpha\beta}(\mathcal{D}_{rn})^{\alpha,\mu\beta}_{\rho\sigma,\nu_1} , \\
\mathcal{D}^{(1,3)}_{rn} &:= \frac{1}{7}\Delta^{\nu_1}_{\mu}\Delta^{\sigma\rho,\nu_2\nu_3}\Delta_{\alpha\beta}(\mathcal{D}_{rn})^{\alpha,\mu\beta}_{\rho\sigma,\nu_1\nu_2\nu_3} , & \mathcal{D}^{(3,1)}_{rn} &:= \frac{1}{5}\Delta^{\rho}_{\mu}\Delta^{\sigma\nu_1}\Delta_{\alpha\beta,\gamma\delta}(\mathcal{D}_{rn})^{\alpha\beta,\mu\gamma\delta}_{\rho\sigma,\nu_1} , \\
\mathcal{D}^{(3)}_{rn} &:= \frac{1}{7}\Delta^{\nu_1}_{\mu}\Delta^{\sigma\rho,\nu_2\nu_3}\Delta_{\alpha\beta,\gamma\delta}(\mathcal{D}_{rn})^{\alpha\beta,\mu\gamma\delta}_{\rho\sigma,\nu_1\nu_2\nu_3} , & \overline{\mathcal{D}}^{(2)}_{rn} &:= \frac{1}{5}\Delta^{\rho\sigma,\nu_1\nu_2}\Delta_{\alpha\beta,\mu\nu}(\mathcal{D}_{rn})^{\alpha\beta,\mu\nu}_{\rho\sigma,\nu_1\nu_2} ,
\end{aligned} \tag{6.113}$$

while the quantities D_{rn} which characterize the coupling of the moments of spin-rank zero to the ones of second rank in spin are given by

$$\begin{aligned}
D^{(0,2)}_{rn} &:= \frac{1}{5}\Delta^{\nu_1\nu_2}_{\mu\nu}(D_{rn})^{\mu\nu}_{\nu_1\nu_2} , & D^{(2,2)}_{rn} &:= \frac{1}{5}\Delta^{\nu_1\nu_2}_{\beta\gamma}(D_{rn})^{\alpha\beta,\gamma\delta}_{\nu_1\nu_2}\Delta_{\alpha\delta} , \\
D^{(4,2)}_{rn} &:= \frac{1}{5}\Delta^{\nu_1\nu_2}_{\gamma\delta}\Delta_{\alpha\beta,\mu\nu}(D_{rn})^{\alpha\beta,\gamma\delta\mu\nu}_{\nu_1\nu_2} , & D^{(1,1)}_{rn} &:= \frac{1}{3}\Delta^{\nu_1}_{\mu}\Delta_{\alpha\beta}(D_{rn})^{\alpha\mu,\beta}_{\nu_1} , \\
D^{(3,1)}_{rn} &:= \frac{1}{3}\Delta^{\nu_1}_{\mu}\Delta_{\alpha\beta,\gamma\delta}(D_{rn})^{\alpha\beta,\mu\gamma\delta}_{\nu_1} , & D^{(2,0)}_{rn} &:= \Delta_{\alpha\beta,\gamma\delta}(D_{rn})^{\alpha\beta,\gamma\delta} .
\end{aligned} \tag{6.114}$$

6.3.3 Obtaining the Navier-Stokes relations

Spin-ranks zero and two

From the discussion in the previous subsection, it has become clear that the equations of motion for the irreducible moments of spin-ranks zero and two are coupled and must thus be solved together. When considering the relevant scalar moments (i.e., ρ_r and p_r), we can write the system of Eqs. (6.75a) and (the contraction of) (6.80c) as

$$\sum_n \begin{pmatrix} \mathcal{A}^{(0)} & A^{(0,2)} \\ D^{(2,0)} & \overline{\mathcal{D}}^{(2)} \end{pmatrix}_{rn} \begin{pmatrix} \rho \\ p \end{pmatrix}_n \simeq \begin{pmatrix} \alpha^{(0)} \\ 0 \end{pmatrix}_r \theta , \tag{6.115}$$

where the omitted terms are at least of second order in Kn and Re^{-1} and we used the form of the collision terms C_{r-1} and $\mathfrak{C}_{p,r-1}$, cf. Eqs. (6.96a) and (6.111f). From this point on, we omit the sets that characterize the truncation for notational simplicity (they are implicitly contained in the size of the linearized collision matrices). After inverting the matrix on the left-hand side of the equation above [excluding the rows and columns corresponding to ρ_1 and ρ_2, which vanish due to the matching conditions (6.34) and (6.36)],

$$\begin{pmatrix} \tau^{(\rho)}_S & \tau^{(\rho p)}_S \\ \tau^{(p\rho)}_S & \tau^{(p)}_S \end{pmatrix}_{rn} := \begin{pmatrix} \mathcal{A}^{(0)} & A^{(0,2)} \\ D^{(2,0)} & \overline{\mathcal{D}}^{(2)} \end{pmatrix}^{-1}_{rn} , \tag{6.116}$$

we find the NAVIER-STOKES solutions

$$\rho_r \simeq \frac{3}{m^2}\zeta_r\theta \;, \qquad p_r \simeq \frac{3}{m^2}\chi_r\theta \;, \tag{6.117}$$

where we introduced

$$\zeta_r := \frac{m^2}{3}\sum_n \tau^{(\rho)}_{S,rn}\alpha^{(0)}_n \;, \qquad \chi_r := \frac{m^2}{3}\sum_n \tau^{(p\rho)}_{S,rn}\alpha^{(0)}_n \;. \tag{6.118}$$

Similarly, Eqs. (6.75b) and the respective traces of Eqs. (6.80b) and (6.80d) yield

$$\sum_n \begin{pmatrix} \mathcal{A}^{(1)} & A^{(1,1)} & A^{(1,3)} \\ D^{(1,1)} & \mathcal{D}^{(1)} & \mathcal{D}^{(1,3)} \\ D^{(3,1)} & \mathcal{D}^{(3,1)} & \mathcal{D}^{(3)} \end{pmatrix}_{rn} \begin{pmatrix} \rho^\mu \\ p^\mu \\ q^\mu \end{pmatrix}_n \simeq \begin{pmatrix} \alpha^{(1)} \\ 0 \\ 0 \end{pmatrix}_r I^\mu \;, \tag{6.119}$$

which, after defining the inverse

$$\begin{pmatrix} \tau^{(\rho)}_V & \tau^{(\rho p)}_V & \tau^{(\rho q)}_V \\ \tau^{(p\rho)}_V & \tau^{(p)}_V & \tau^{(pq)}_V \\ \tau^{(q\rho)}_V & \tau^{(qp)}_V & \tau^{(q)}_V \end{pmatrix}_{rn} := \begin{pmatrix} \mathcal{A}^{(1)} & A^{(1,1)} & A^{(1,3)} \\ D^{(1,1)} & \mathcal{D}^{(1)} & \mathcal{D}^{(1,3)} \\ D^{(3,1)} & \mathcal{D}^{(3,1)} & \mathcal{D}^{(3)} \end{pmatrix}^{-1}_{rn} \;, \tag{6.120}$$

where the row and column corresponding to ρ^μ_1 is excluded due to the matching condition (6.37), gives the NAVIER-STOKES values

$$\rho^\mu_r \simeq \kappa_r I^\mu \;, \qquad p^\mu_r \simeq \varphi^{(1)}_r I^\mu \;, \qquad q^\mu_r \simeq \varphi^{(3)}_r I^\mu \;, \tag{6.121}$$

where

$$\kappa_r := \sum_n \tau^{(\rho)}_{V,rn}\alpha^{(1)}_n \;, \qquad \varphi^{(1)}_r := \sum_n \tau^{(p\rho)}_{V,rn}\alpha^{(1)}_n \;, \qquad \varphi^{(3)}_r := \sum_n \tau^{(q\rho)}_{V,rn}\alpha^{(1)}_n \;. \tag{6.122}$$

Finally, the equations for the tensor-valued quantities, i.e., Eqs. (6.75c) and (6.80a) together with appropriate traces of Eqs. (6.80c) and (6.80e) read to first order

$$\sum_n \begin{pmatrix} \mathcal{A}^{(2)} & A^{(2,0)} & A^{(2,2)} & A^{(2,4)} \\ D^{(0,2)} & \mathcal{D}^{(0)} & \mathcal{D}^{(0,2)} & \mathcal{D}^{(0,4)} \\ D^{(2,2)} & \mathcal{D}^{(2,0)} & \mathcal{D}^{(2)} & \mathcal{D}^{(2,4)} \\ D^{(4,2)} & \mathcal{D}^{(4,0)} & \mathcal{D}^{(4,2)} & \mathcal{D}^{(4)} \end{pmatrix}_{rn} \begin{pmatrix} \rho^{\mu\nu} \\ \psi^{\langle\mu\nu\rangle} \\ p^{\mu\nu} \\ q^{\mu\nu} \end{pmatrix}_n \simeq \begin{pmatrix} 2\alpha^{(2)} \\ 0 \\ 0 \\ 0 \end{pmatrix}_r \sigma^{\mu\nu} \;, \tag{6.123}$$

We define the inverse matrix

$$\begin{pmatrix} \tau^{(\rho)}_T & \tau^{(\rho\psi)}_T & \tau^{(\rho p)}_T & \tau^{(\rho q)}_T \\ \tau^{(\psi\rho)}_T & \tau^{(\psi)}_T & \tau^{(\psi p)}_T & \tau^{(\psi q)}_T \\ \tau^{(p\rho)}_T & \tau^{(p\psi)}_T & \tau^{(p)}_T & \tau^{(pq)}_T \\ \tau^{(q\rho)}_T & \tau^{(q\psi)}_T & \tau^{(qp)}_T & \tau^{(q)}_T \end{pmatrix}_{rn} := \begin{pmatrix} \mathcal{A}^{(2)} & A^{(2,0)} & A^{(2,2)} & A^{(2,4)} \\ D^{(0,2)} & \mathcal{D}^{(0)} & \mathcal{D}^{(0,2)} & \mathcal{D}^{(0,4)} \\ D^{(2,2)} & \mathcal{D}^{(2,0)} & \mathcal{D}^{(2)} & \mathcal{D}^{(2,4)} \\ D^{(4,2)} & \mathcal{D}^{(4,0)} & \mathcal{D}^{(4,2)} & \mathcal{D}^{(4)} \end{pmatrix}^{-1}_{rn} \tag{6.124}$$

and obtain the NAVIER-STOKES values of the tensorial moments

$$\rho^{\mu\nu}_r \simeq 2\eta_r\sigma^{\mu\nu} \;, \qquad \psi^{\langle\mu\nu\rangle}_r \simeq 2\xi^{(0)}_r\sigma^{\mu\nu} \;, \qquad p^{\mu\nu}_r \simeq 2\xi^{(2)}_r\sigma^{\mu\nu} \;, \qquad q^{\mu\nu}_r \simeq 2\xi^{(4)}_r\sigma^{\mu\nu} \;, \tag{6.125}$$

where the coefficients are defined as

$$\eta_r := \sum_n \tau^{(\rho)}_{T,rn}\alpha^{(2)}_n \;, \qquad \xi^{(0)}_r := \sum_n \tau^{(\psi\rho)}_{T,rn}\alpha^{(2)}_n \;, \qquad \xi^{(2)}_r := \sum_n \tau^{(p\rho)}_{T,rn}\alpha^{(2)}_n \;, \qquad \xi^{(4)}_r := \sum_n \tau^{(q\rho)}_{T,rn}\alpha^{(2)}_n \;. \tag{6.126}$$

Spin-rank one

In the case of the relevant irreducible moments of spin-rank one, we have to consider the moment equations (6.78) in conjunction with the equations of motion for the components of the spin potential (6.62), since the latter are of relaxation type as well. Then, in the NAVIER-STOKES limit we have the following system of equations for the axial-vector valued quantities,

$$\sum_n \begin{pmatrix} \delta_{n0} & \gamma_n^{(0)}/\Gamma^{(\omega)} & \gamma_n^{(2)}/\Gamma^{(\omega)} \\ -2g_r^{(0)}\delta_{n0} & \mathcal{B}_{rn}^{(0)} & \mathcal{B}_{rn}^{(0,2)} \\ -2g_r^{(2)}\delta_{n0} & \mathcal{B}_{rn}^{(2,0)} & \mathcal{B}_{rn}^{(2)} \end{pmatrix} \begin{pmatrix} \omega_0^\mu \\ \tau_n^{\langle\mu\rangle} \\ t_n^\mu \end{pmatrix} \simeq \begin{pmatrix} -1 \\ 2g_r^{(0)} \\ 2g_r^{(2)} \end{pmatrix} \beta_0 \omega^\mu , \tag{6.127}$$

where we used the expressions (6.106). Inverting the matrix

$$\begin{pmatrix} \mathfrak{T}_A^{(\omega)} & \vec{\mathfrak{T}}_A^{(\omega\tau)} & \vec{\mathfrak{T}}_A^{(\omega t)} \\ \vec{\mathfrak{T}}_A^{(\tau\omega)} & \mathfrak{T}_A^{(\tau)} & \mathfrak{T}_A^{(\tau t)} \\ \vec{\mathfrak{T}}_A^{(t\omega)} & \mathfrak{T}_A^{(t\tau)} & \mathfrak{T}_A^{(t)} \end{pmatrix}_{rn} := \begin{pmatrix} 1 & \vec{\gamma}^{(0)}/\Gamma^{(\omega)} & \vec{\gamma}^{(2)}/\Gamma^{(\omega)} \\ -2\vec{g}^{(0)} & \mathcal{B}^{(0)} & \mathcal{B}^{(0,2)} \\ -2\vec{g}^{(2)} & \mathcal{B}^{(2,0)} & \mathcal{B}^{(2)} \end{pmatrix}_{rn}^{-1} , \tag{6.128}$$

where $\vec{g}^{(j)} := (g_0^{(j)}, \cdots, g_r^{(j)})$, and analogous for $\vec{\gamma}^{(j)}$ and $\vec{\mathfrak{T}}_A^{(j)}$, we find the NAVIER-STOKES relations

$$\omega_0^\mu \simeq -\mathfrak{e}^{(\omega)}\omega^\mu , \qquad \tau_r^{\langle\mu\rangle} \simeq \mathfrak{e}_r^{(0)}\omega^\mu , \qquad t_r^\mu \simeq \mathfrak{e}_r^{(2)}\omega^\mu , \tag{6.129}$$

where we introduced

$$\mathfrak{e}^{(\omega)} := \beta_0 \left[\mathfrak{T}_A^{(\omega)} - 2\sum_n \left(\mathfrak{T}_{A,n}^{(\omega\tau)} g_n^{(0)} + \mathfrak{T}_{A,n}^{(\omega t)} g_n^{(2)} \right) \right] , \tag{6.130a}$$

$$\mathfrak{e}_r^{(0)} := \beta_0 \left[-\mathfrak{T}_{A,r}^{(\tau\omega)} + 2\sum_n \left(\mathfrak{T}_{A,rn}^{(\tau)} g_n^{(0)} + \mathfrak{T}_{A,rn}^{(\tau t)} g_n^{(2)} \right) \right] , \tag{6.130b}$$

$$\mathfrak{e}_r^{(2)} := \beta_0 \left[-\mathfrak{T}_{A,r}^{(t\omega)} + 2\sum_n \left(\mathfrak{T}_{A,rn}^{(t\tau)} g_n^{(0)} + \mathfrak{T}_{A,rn}^{(t)} g_n^{(2)} \right) \right] . \tag{6.130c}$$

The vector-valued moments w_r^μ have to be combined with the component κ_0^μ of the spin potential, yielding

$$\sum_n \begin{pmatrix} \delta_{n0} & \gamma_n^{(1)}/\Gamma^{(\kappa)} \\ -2g_r^{(\kappa)}\delta_{n0} & \mathcal{B}_{rn}^{(1)} \end{pmatrix} \begin{pmatrix} \kappa^\mu \\ w_n^\mu \end{pmatrix} \simeq \begin{pmatrix} -1 \\ 2g_r^{(\kappa)} \end{pmatrix} \frac{\beta_0 F^\mu}{\varepsilon_0 + P_0} + \begin{pmatrix} \Gamma^{(I)} \\ g_r^{(I)} \end{pmatrix} I^\mu . \tag{6.131}$$

We then invert the collision matrix

$$\begin{pmatrix} \mathfrak{T}_V^{(\kappa)} & \vec{\mathfrak{T}}_V^{(\kappa w)} \\ \vec{\mathfrak{T}}_V^{(w\kappa)} & \mathfrak{T}_V^{(w)} \end{pmatrix}_{rn} := \begin{pmatrix} 1 & \vec{\gamma}^{(1)}/\Gamma^{(\kappa)} \\ -2\vec{g}^{(\kappa)} & \mathcal{B}^{(1)} \end{pmatrix}_{rn}^{-1} , \tag{6.132}$$

and find the NAVIER-STOKES values

$$\kappa_0^\mu \simeq -\mathfrak{a}^{(\kappa)}F^\mu + \mathfrak{b}^{(\kappa)}I^\mu , \qquad w_r^\mu \simeq \mathfrak{a}_r^{(1)}F^\mu + \mathfrak{b}_r^{(1)}I^\mu . \tag{6.133}$$

Here we defined

$$\mathfrak{a}^{(\kappa)} := \frac{\beta_0}{\varepsilon_0 + P_0} \left(\mathfrak{T}_V^{(\kappa)} - 2\sum_n \mathfrak{T}_{V,n}^{(\kappa w)} g_n^{(\kappa)} \right) , \qquad \mathfrak{b}^{(\kappa)} := \mathfrak{T}_V^{(\kappa)}\Gamma^{(I)} + \sum_n \mathfrak{T}_{V,n}^{(\kappa w)} g_n^{(\kappa)} , \tag{6.134a}$$

$$\mathfrak{a}_r^{(1)} := \frac{\beta_0}{\varepsilon_0 + P_0} \left(-\mathfrak{T}_{V,r}^{(w\kappa)} + 2\sum_n \mathfrak{T}_{V,rn}^{(w)} g_n^{(\kappa)} \right) , \qquad \mathfrak{b}_r^{(1)} := \mathfrak{T}_{V,r}^{(w\kappa)}\Gamma^{(I)} + 2\sum_n \mathfrak{T}_{V,rn}^{(w)} g_n^{(I)} . \tag{6.134b}$$

The equations determining the NAVIER-STOKES values of the tensor-valued moments, which are not connected with any component of the spin potential, read

$$\sum_n \overline{\mathcal{B}}_{rn}^{(2)} t_n^{\mu\nu} \simeq h_r^{(2)} \beta_0 \sigma^{\mu\nu} , \tag{6.135}$$

With the inverse

$$\mathfrak{T}^{(t)}_{T,rn} := \left(\overline{\mathcal{B}}^{(2)}\right)^{-1}_{rn} , \tag{6.136}$$

we find the Navier-Stokes values

$$t^{\mu\nu}_r \simeq \mathfrak{d}^{(2)}_r \sigma^{\mu\nu} , \tag{6.137}$$

where we introduced

$$\mathfrak{d}^{(2)}_r := \beta_0 \sum_n \mathfrak{T}^{(t)}_{T,rn} h^{(2)}_n . \tag{6.138}$$

It is important to note that, as in the case of the moments of spin-rank zero, when inverting the matrix in Eq. (6.127), one has to exclude the rows and columns corresponding to the moment $\tau_2^{\langle\mu\rangle} \equiv \mathfrak{n}^{\langle\mu\rangle}$, since it is fixed by the matching condition (6.51).

6.3.4 The choice of closure

The Navier-Stokes relations (6.117), (6.121), (6.125), (6.129), (6.133), and (6.137) are crucial, as they can now be employed as shown in Subsec. 6.3.1 to close the equations of motion of a given tensor-rank in spin and momentum in terms of any moment that fulfills the required symmetries. The final question to answer consists in the choice of moments that are used to describe the system.

Spin-rank zero

The choice of moments of spin-rank zero is obvious, since we are aiming to describe the dynamics of the energy-momentum tensor and particle four-current. Thus, we choose Π, n^μ, and $\pi^{\mu\nu}$ as the dynamical quantities, and find the asymptotic matching conditions

$$\rho_r \simeq -\frac{3}{m^2}\mathcal{R}^{(0)}_{r0}\Pi , \qquad \rho^\mu_r \simeq \mathcal{R}^{(1)}_{r0} n^\mu , \qquad \rho^{\mu\nu}_r \simeq \mathcal{R}^{(2)}_{r0}\pi^{\mu\nu} , \tag{6.139}$$

where

$$\mathcal{R}^{(0)}_{r0} := \frac{\zeta_r}{\zeta_0} , \qquad \mathcal{R}^{(1)}_{r0} := \frac{\kappa_r}{\kappa_0} , \qquad \mathcal{R}^{(2)}_{r0} := \frac{\eta_r}{\eta_0} . \tag{6.140}$$

Note that we can extend these relations to moments with $r < 0$ by defining

$$\mathcal{R}^{(\ell)}_{-r,0} := \sum_n \mathcal{F}^{(0,\ell)}_{rn}\mathcal{R}^{(\ell)}_{n0} , \tag{6.141}$$

cf. Eqs. (6.22).

Spin-rank one

Similarly, in the case of the moments of spin-rank one we may choose the moments that appear in the spin tensor, i.e., Eq. (6.46). These moments are expected to yield the most important dynamics since the spin tensor in our chosen pseudogauge is conserved in the absence of interactions or in global equilibrium, whereas, e.g., the canonical spin tensor is not conserved even in these cases. This consideration fixes the choice of closure for the moments of momentum-ranks zero and two, but not for those of momentum-rank one, since just their antisymmetric part is hydrodynamically important, whereas only the symmetric part of $\tau_1^{\mu,\nu}$ appears in the spin tensor. We then choose to close the system for these moments in terms of $\tau_0^{\mu,\nu}$, since this quantity appears in the global vector polarization (6.69). Restricting ourselves to the hydrodynamically important moments that have Navier-Stokes values of first order and defining $\mathfrak{q}^\mu := t^\mu_0$, $\mathfrak{w}^\mu := w^\mu_0$, as well as $\mathfrak{t}^{\mu\nu} := t^{\mu\nu}_0$, we find the asymptotic matching conditions to be

$$\tau^{\langle\mu\rangle}_r \simeq \mathcal{Q}^{(10)}_{r0}\mathfrak{p}^\mu , \qquad t^\mu_r \simeq \mathcal{Q}^{(12)}_{r0}\mathfrak{q}^\mu , \qquad w^\mu_r \simeq \mathcal{Q}^{(11)}_{r0}\mathfrak{w}^\mu + \mathcal{Q}^{(\kappa)}_r \kappa^\mu_0 , \qquad t^{\mu\nu}_r \simeq \mathcal{Q}^{(22)}_{r0}\mathfrak{t}^{\mu\nu} , \tag{6.142}$$

where we defined

$$\mathcal{Q}_{r0}^{(10)} := \frac{1}{\mathfrak{e}_0^{(0)}}\left[\mathfrak{e}_r^{(0)}\left(1-\delta_{r2}\right) - \frac{3}{2}\mathfrak{e}_0^{(2)}\delta_{r2}\right] - \frac{m^2}{2}\delta_{r2}\,, \qquad \mathcal{Q}_{r0}^{(12)} := \frac{\mathfrak{e}_r^{(2)}}{\mathfrak{e}_0^{(2)}}\,, \qquad \mathcal{Q}_{r0}^{(22)} := \frac{\mathfrak{d}_r^{(2)}}{\mathfrak{d}_0^{(2)}}\,, \tag{6.143a}$$

as well as

$$\mathcal{Q}_{r0}^{(11)} := \frac{\mathfrak{a}^{(\kappa)}\mathfrak{b}_r^{(1)} + \mathfrak{a}_r^{(1)}\mathfrak{b}^{(\kappa)}}{\mathfrak{a}^{(\kappa)}\mathfrak{b}_0^{(1)} + \mathfrak{a}_0^{(1)}\mathfrak{b}^{(\kappa)}}\,, \qquad \mathcal{Q}_r^{(\kappa)} := \frac{\mathfrak{a}_0^{(1)}\mathfrak{b}_r^{(1)} - \mathfrak{a}_r^{(1)}\mathfrak{b}_0^{(1)}}{\mathfrak{a}^{(\kappa)}\mathfrak{b}_0^{(1)} + \mathfrak{a}_0^{(1)}\mathfrak{b}^{(\kappa)}}\,. \tag{6.143b}$$

Note that the moments w_r^μ cannot be expressed only in terms of $\mathfrak{w}^\mu$, which is a consequence of two Navier-Stokes values appearing, namely F^μ and I^μ.[12] In order to make Eq. (6.142) hold for any r, we had to include the case $r=2$, where the matching condition (6.51) has to be obeyed, which is the reason for the peculiar form of $\mathcal{Q}_{r0}^{(10)}$. These relations are again extended to include $r<0$ by introducing

$$\mathcal{Q}_{-r,0}^{(1\ell)} := \sum_n \mathcal{F}_{rn}^{(1,\ell)}\mathcal{Q}_{n0}^{(1\ell)}\,, \qquad \mathcal{Q}_{-r,0}^{(2\ell)} := \sum_n \mathcal{F}_{rn}^{(1,\ell)}\mathcal{Q}_{n0}^{(2\ell)}\,, \qquad \mathcal{Q}_{-r,0}^{(\kappa)} := \sum_n \mathcal{F}_{rn}^{(1,1)}\mathcal{Q}_{n0}^{(\kappa)}\,. \tag{6.144}$$

Using Eqs. (6.48), this asymptotic matching allows us to express the components of the moments $\tau_r^{\mu,\mu_1\cdots\mu_\ell}$ that are parallel to the fluid four-velocity in the first index as

$$u_\mu \tau_r^{\mu,\nu} \simeq \mathcal{X}_r^{(10)}\mathfrak{p}^\mu + \mathcal{X}_r^{(12)}\mathfrak{q}^\mu\,, \tag{6.145a}$$

$$u_\mu \tau_r^{\mu} \simeq u_\mu \tau_r^{\mu,\mu_1\cdots\mu_\ell} \simeq 0 \qquad \text{for} \quad \ell \geq 2\,, \tag{6.145b}$$

where we defined

$$\mathcal{X}_r^{(10)} := -\frac{1}{3}\left(m^2\mathcal{Q}_{r-1,0}^{(10)} - \mathcal{Q}_{r+1,0}^{(10)}\right)\,, \qquad \mathcal{X}_r^{(12)} := -\mathcal{Q}_{r-1,0}^{(12)}\,. \tag{6.146}$$

Note that we already made use of the fact that the symmetrized components of the moments of spin- and momentum-rank one that are orthogonal to u^μ are to be neglected since they do not have Navier-Stokes values of first order.

Spin-rank two

Lastly, the moments of spin-rank two are different, since they do not appear in any conserved quantity. One way to close the system of equations there would be to choose the moments that appear in the global tensor polarization (6.70), i.e., $\psi_1^{\mu\nu}$ and $\psi_0^{\mu\nu,\lambda}$. However, in this work we go in a different direction by returning to the original problem that second-order hydrodynamics was introduced to solve, namely the acausality (and thus instability) of the relativistic Navier-Stokes theory. This acausality is rooted in the fact that inserting the Navier-Stokes solutions for dissipative quantities into the respective conservation laws leads to parabolic equations. However, since the moments of second rank in spin do not appear in any conservation law, this problem never arises there, which is why we simply stick with the Navier-Stokes solutions [i.e., Eqs. (6.115), (6.119), and (6.123)] for these moments. Notice that, up to second order in Kn and Re^{-1}, we can relate the moments of spin-rank two to the bulk viscous pressure, the particle diffusion current, and the shear-stress tensor via

$$\begin{aligned} p_r &\simeq -\frac{3}{m^2}\mathcal{T}_{r0}^{(00)}\Pi\,, \qquad & p_r^\mu &\simeq \mathcal{T}_{r0}^{(11)}n^\mu\,, \qquad & q_r^\mu &\simeq \mathcal{T}_{r0}^{(13)}n^\mu\,,\\ \psi_r^{\langle\mu\nu\rangle} &\simeq \mathcal{T}_{r0}^{(20)}\pi^{\mu\nu}\,, \qquad & p_r^{\mu\nu} &\simeq \mathcal{T}_{r0}^{(22)}\pi^{\mu\nu}\,, \qquad & q_r^{\mu\nu} &\simeq \mathcal{T}_{r0}^{(24)}\pi^{\mu\nu}\,, \end{aligned} \tag{6.147}$$

where

$$\begin{aligned} \mathcal{T}_{r0}^{(00)} &:= \frac{\chi_r}{\zeta_0}\,, \qquad & \mathcal{T}_{r0}^{(11)} &:= \frac{\varphi_r^{(1)}}{\kappa_0}\,, \qquad & \mathcal{T}_{r0}^{(13)} &:= \frac{\varphi_r^{(3)}}{\kappa_0}\,,\\ \mathcal{T}_{r0}^{(20)} &:= \frac{\xi_r^{(0)}}{\eta_0}\,, \qquad & \mathcal{T}_{r0}^{(22)} &:= \frac{\xi_r^{(2)}}{\eta_0}\,, \qquad & \mathcal{T}_{r0}^{(24)} &:= \frac{\xi_r^{(4)}}{\eta_0}\,. \end{aligned} \tag{6.148}$$

[12] In order to relate (κ_0^μ, w_r^μ) to $(\kappa_0^\mu, \mathfrak{w}^\mu)$, one has to solve a matrix equation, leading to both κ_0^μ and $\mathfrak{w}^\mu$ appearing. This is very similar to the case when multiple conserved charges are present, cf. Ref. [49].

Equations (6.147) can be extended to $r<0$ via

$$\mathcal{T}^{(m\ell)}_{-r,0} \coloneqq \sum_n \mathcal{F}^{(2,\ell)}_{rn} \mathcal{T}^{(m\ell)}_{n0} \; . \tag{6.149}$$

Evidently, there is a large number of possibilities to construct these kind of relations; one could also, e.g., relate the moments p_r^μ and q_r^μ (or their tensorial analogues), which would introduce an ambiguity in the second-order terms of their respective evolution equations. By considering only the Navier-Stokes limit of the moments of spin-rank two, we circumvent these ambiguities, and may simply use Eqs. (6.147) to replace the moments of spin-rank two in the evolution equations for Π, n^μ, and $\pi^{\mu\nu}$, thus closing the system in terms of quantities appearing in the energy-momentum tensor and the particle four-current. Nonetheless, as we shall see in the following section, due to the evolution equations of the moments of spin-rank zero and two being coupled, we do need the second-order equations (6.80), as they will be responsible for corrections to the second-order transport coefficients appearing in the equations of motion for Π, n^μ, and $\pi^{\mu\nu}$.

Finally, we remark that this asymptotic matching permits us to write down explicit expressions for the components of the irreducible moments of spin-rank two that are parallel to the fluid four-velocity, which read

$$u_\mu \psi_r^{\mu\langle\nu\rangle} \simeq \Upsilon_r^{(01)} n^\nu \; , \tag{6.150a}$$

$$u_\mu u_\nu \psi_r^{\mu\nu} \simeq -\frac{3}{m^2}\Upsilon_r^{(00)}\Pi \; , \tag{6.150b}$$

$$u_\mu \psi_r^{\mu\langle\nu\rangle,\lambda} \simeq \Upsilon_r^{(12)}\pi^{\nu\lambda} - \frac{1}{m^2}\Delta^{\nu\lambda}\Upsilon_r^{(10)}\Pi \; , \tag{6.150c}$$

$$u_\mu u_\nu \psi_r^{\mu\nu,\lambda} \simeq \Upsilon_r^{(11)} n^\lambda \; , \tag{6.150d}$$

$$u_\mu \psi_r^{\mu\langle\nu\rangle,\lambda\alpha} \simeq \frac{3}{5}\Upsilon_r^{(21)}\Delta^{\nu\langle\lambda} n^{\alpha\rangle} \; , \tag{6.150e}$$

$$u_\mu u_\nu \psi_r^{\mu\nu,\lambda\alpha} \simeq \Upsilon_r^{(22)}\pi^{\lambda\alpha} \; , \tag{6.150f}$$

$$u_\mu \psi_r^{\mu\langle\nu\rangle,\lambda\alpha\beta} \simeq \frac{5}{7}\Upsilon_r^{(32)}\Delta^{\nu\langle\lambda}\pi^{\alpha\beta\rangle} \; , \tag{6.150g}$$

$$u_\mu u_\nu \psi_r^{\mu\nu,\lambda\alpha\beta} \simeq u_\mu \psi_r^{\mu\nu,\lambda\alpha\beta\gamma} \simeq 0 \; . \tag{6.150h}$$

Here we defined the coefficients

$$\Upsilon_r^{(01)} \coloneqq -\mathcal{T}^{(11)}_{r-1,0} \; , \tag{6.151a}$$

$$\Upsilon_r^{(00)} \coloneqq \sum_n (\mathcal{E}^{(00)})^{-1}_{rn}\mathcal{T}^{(00)}_{n-2,0} \equiv -\Upsilon^{(10)}_{r-1} \; , \tag{6.151b}$$

$$\Upsilon_r^{(12)} \coloneqq \frac{1}{3}\left(\mathcal{T}^{(20)}_{r+1,0} - m^2\mathcal{T}^{(20)}_{r-1,0}\right) - \mathcal{T}^{(22)}_{r-1,0} \; , \tag{6.151c}$$

$$\Upsilon_r^{(11)} \coloneqq \sum_n (\mathcal{E}^{(11)})^{-1}_{rn}\left[\mathcal{T}^{(13)}_{n-2,0} + \frac{2}{5}\left(m^2\mathcal{T}^{(11)}_{n-2,0} - \mathcal{T}^{(11)}_{n0}\right)\right] \; , \tag{6.151d}$$

$$\Upsilon_r^{(21)} \coloneqq \frac{2}{5}\left(\mathcal{T}^{(11)}_{r+1,0} - m^2\mathcal{T}^{(11)}_{r-1,0}\right) - \mathcal{T}^{(13)}_{r-1,0} \; , \tag{6.151e}$$

$$\Upsilon_r^{(22)} \coloneqq \sum_n (\mathcal{E}^{(22)})^{-1}_{rn}\left[\mathcal{T}^{(24)}_{n-2,0} + \frac{4}{7}\left(m^2\mathcal{T}^{(22)}_{n-2,0} - \mathcal{T}^{(22)}_{n0}\right) + \frac{2}{15}\left(m^4\mathcal{T}^{(20)}_{n-2,0} - 2m^2\mathcal{T}^{(20)}_{n,0} + \mathcal{T}^{(20)}_{n+2,0}\right)\right] \; , \tag{6.151f}$$

$$\Upsilon_r^{(32)} \coloneqq \frac{3}{7}\left(\mathcal{T}^{(22)}_{r+1,0} - m^2\mathcal{T}^{(22)}_{r-1,0}\right) - \mathcal{T}^{(24)}_{r-1,0} \; , \tag{6.151g}$$

which contain the matrices

$$\mathcal{E}^{(00)}_{rn} \coloneqq \frac{1}{3}\left(2\delta_{rn} + m^2\mathcal{F}^{(2,0)}_{2-r,n}\right) \; , \tag{6.152a}$$

$$\mathcal{E}^{(11)}_{rn} \coloneqq \frac{1}{5}\left(4\delta_{rn} + m^2\mathcal{F}^{(2,1)}_{2-r,n}\right) \; , \tag{6.152b}$$

$$\mathcal{E}^{(22)}_{rn} \coloneqq \frac{1}{7}\left(6\delta_{rn} + m^2\mathcal{F}^{(2,2)}_{2-r,n}\right) \; . \tag{6.152c}$$

These expressions follow from Eqs. (6.72) and the asymptotic matching conditions (6.147). Furthermore, we used the relation between the original moments $\psi_r^{\langle\mu\nu\rangle,\mu_1\cdots\mu_\ell}$ and their scalar-,vector-, and tensor-valued traces,

$$\psi_r^{\langle\mu\nu\rangle,\lambda} \simeq \frac{3}{5}\Delta^{\lambda\langle\mu}p_r^{\nu\rangle} \,, \tag{6.153a}$$

$$\psi_r^{\langle\mu\nu\rangle,\lambda\alpha} \simeq \frac{1}{5}\Delta^{\mu\nu,\lambda\alpha}p_r + \frac{12}{7}\Delta_{\gamma\delta}\Delta^{\lambda\alpha,\gamma\langle\mu}p_r^{\nu\rangle\delta} \,, \tag{6.153b}$$

$$\psi_r^{\langle\mu\nu\rangle,\lambda\alpha\beta} \simeq \frac{3}{7}\Delta^{\mu\nu,\gamma\delta}\Delta^{\lambda\alpha\beta}_{\gamma\delta\rho}q_r^\rho \,, \tag{6.153c}$$

$$\psi_r^{\langle\mu\nu\rangle,\lambda\alpha\beta\gamma} \simeq \frac{5}{9}\Delta^{\mu\nu,\rho\sigma}\Delta^{\lambda\alpha\beta\gamma}_{\rho\sigma\zeta\eta}q_r^{\zeta\eta} \,, \tag{6.153d}$$

where terms of second order in Kn and Re^{-1} have been omitted.

A note on pseudogauge dependence

Before we go on to write down the final hydrodynamic equations, let us come back to the issue of pseudogauge dependence raised in Chapter 3. A priori, kinetic theory does not know about pseudogauges, since the conservation laws, which are built on collisional invariants, are independent of any such choice. Nevertheless, the pseudogauge choice *does* enter in the hydrodynamic theory that we are constructing, and this happens through the choice of closure which we discussed above. Even though we may separate the moments according to whether their Navier-Stokes values are of first order in Kn, the question in terms of which moments to close the system remains open. The way we answered it simply consisted of keeping the moments that appear in the conserved currents, which is a pseudogauge-dependent statement. However, since to second order in Kn and Re^{-1} the asymptotic matching, i.e., Eqs. (6.139), (6.142), and (6.147), can be done to relate *any* two moments that transform in the same way, all possible closures of the system of equations should be equivalent. This statement is valid provided that we are in the appropriate hydrodynamic regime and that the transport coefficients are completely resummed, i.e., that the size of all bases is taken to infinity, $\mathbb{S}_\ell^{(j)} \to \mathbb{N}_0 \;\forall\, \ell, j$.

6.4 Hydrodynamic equations

Now that we have obtained the asymptotic matching conditions (6.139), (6.142), and (6.147), we can follow the procedure outlined in Subsec. 6.3.1 to close the system of equations.

6.4.1 Energy-momentum tensor and particle four-current

We start by outlining the procedure on how to obtain the hydrodynamic equation for the bulk viscous pressure, with details relegated to Appendix E. The full system of equations for ρ_r and p_r reads

$$\begin{pmatrix}\dot{\rho}_r\\ \dot{p}_r\end{pmatrix} + \sum_n \begin{pmatrix}\mathcal{A}^{(0)} & A^{(0,2)}\\ D^{(2,0)} & \mathcal{D}^{(2)}\end{pmatrix}_{rn}\begin{pmatrix}\rho\\ p\end{pmatrix}_n = \begin{pmatrix}\alpha^{(0)}\\ 0\end{pmatrix}_r \theta + \cdots \,, \tag{6.154}$$

where the dots denote terms of second order in Kn and Re^{-1} which we do not write explicitly here, but are shown in Appendix E. We then invert the collision matrix as shown in the previous section, which lets the equation for the bulk viscous pressure (given by the first row of the resulting vector equation) become

$$\sum_n \left(\tau^{(\rho)}_{S,0n}\dot{\rho}_n + \tau^{(\rho p)}_{S,0n}\dot{p}_n\right) - \frac{3}{m^2}\Pi = \frac{3}{m^2}\zeta_0\theta + \cdots \,. \tag{6.155}$$

Employing now the asymptotic matching conditions (6.139) and (6.147), we find a relaxation-type equation

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta_0 \theta - \ell_{\Pi n} \nabla \cdot n - \tau_{\Pi n} n \cdot F - \delta_{\Pi\Pi} \Pi \theta - \lambda_{\Pi n} n \cdot I + \lambda_{\Pi \pi} \pi^{\mu\nu} \sigma_{\mu\nu} \,, \tag{6.156}$$

where the transport coefficients are listed in Appendix E. Note that no potentially problematic terms of second order in the Knudsen number appear (denoted by $\mathcal{K}$ in Ref. [51]), which is an effect of performing the asymptotic matching in the IReD way, i.e., by relating only terms of first order in the inverse Reynolds number. Furthermore, due to the replacements (6.147) we are able to absorb the effect of the moments of spin-rank two into a modification of the transport coefficients without altering the form of the relaxation equations. At this point, we remark that the modification of the shear viscosity was already noted in Ref. [73], where it was referred to as a deviation from the "isotropic viscosity".

The same logic is applied to the system of equations for the moments ρ_r^μ, p_r^μ, and q_r^μ, yielding the following equation of motion for the particle diffusion current,

$$\begin{aligned}\tau_n \dot{n}^{\langle\mu\rangle} + n^\mu = \kappa_0 I^\mu - \lambda_{n\omega} n_\nu \omega^{\nu\mu} - \delta_{nn} n^\mu \theta - \ell_{n\Pi} \nabla^\mu \Pi + \ell_{n\pi} \Delta^{\mu\nu} \nabla_\lambda \pi^\lambda{}_\nu + \tau_{n\Pi} \Pi F^\mu \\ - \tau_{n\pi} \pi^{\mu\nu} F_\nu - \lambda_{nn} n_\nu \sigma^{\nu\mu} + \lambda_{n\Pi} \Pi I^\mu - \lambda_{n\pi} \pi^{\mu\nu} I_\nu \,,\end{aligned} \tag{6.157}$$

with the coefficients listed again in Appendix E. Lastly, we apply the procedure outlined above to the system of equations for $\rho_r^{\mu\nu}$, $\psi_r^{\langle\mu\nu\rangle}$, $p_r^{\mu\nu}$, and $q_r^{\mu\nu}$, obtaining

$$\begin{aligned}\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta_0 \sigma^{\mu\nu} + \lambda_{\pi\omega} \pi_\lambda{}^{\langle\mu} \omega^{\nu\rangle\lambda} - \delta_{\pi\pi} \pi^{\mu\nu} \theta - \tau_{\pi\pi} \pi^{\lambda\langle\mu} \sigma^{\nu\rangle}{}_\lambda + \lambda_{\pi\Pi} \Pi \sigma^{\mu\nu} \\ - \tau_{\pi n} n^{\langle\mu} F^{\nu\rangle} + \ell_{\pi n} \nabla^{\langle\mu} n^{\nu\rangle} + \lambda_{\pi n} n^{\langle\mu} I^{\nu\rangle} \,.\end{aligned} \tag{6.158}$$

The system of equations (6.156)–(6.158), together with the evolution equations for the equilibrium quantities α_0, β_0, and u^μ, cf. Eqs. (6.39), governs the time evolution of all components of the energy-momentum tensor and the particle four-current of the system up to first order in $\hbar$ and second order in Kn and Re^{-1}. Remarkably, these equations are formally identical to their spin-0 counterparts given in Ref. [50], with the feedback effects from the dynamically generated tensor polarization in the system encoded in a modification of the transport coefficients. Furthermore, it is apparent that the degrees of freedom contained in the spin tensor do not influence the standard hydrodynamic quantities, which is due to our assumption that the interaction conserves parity in combination with the restriction to first order in the Planck constant. Lastly, we remark that in principle there should appear terms of second order in inverse Reynolds numbers in Eqs. (6.156)–(6.158), which we neglected when linearizing the collision terms. These contributions can in principle be calculated in the same way as the coefficients of order $\mathcal{O}(\mathrm{Kn}\,\mathrm{Re}^{-1})$, although they are rather difficult to compute explicitly [160] and are thus left for future work.

6.4.2 Spin tensor

As is evident by our analysis of their Navier-Stokes limit (6.127), the equations of motion for the axial vectors ω_0^μ, $\mathfrak{p}^{\langle\mu\rangle}$, and $\mathfrak{t}^\mu$ are coupled and have to be treated in the same way as illustrated in the subsection above. The resulting set of equations is then given by

$$\begin{aligned}&\tau_\omega \dot{\omega}_0^{\langle\mu\rangle} + \tau_{\omega\mathfrak{p}} \dot{\mathfrak{p}}^{\langle\mu\rangle} + \tau_{\omega\mathfrak{q}} \dot{\mathfrak{q}}^{\langle\mu\rangle} + \omega_0^\mu \\ &= -\mathfrak{e}^{(\omega)} \omega^\mu + \mathfrak{K}_{\omega\theta} \theta \omega_0^\mu + \mathfrak{K}_{\omega\theta\mathfrak{p}} \theta \mathfrak{p}^\mu + \mathfrak{K}_{\omega\theta\mathfrak{q}} \theta \mathfrak{q}^\mu + \mathfrak{K}_{\omega\sigma} \sigma^{\mu\nu} \omega_{0,\nu} + \mathfrak{K}_{\omega\sigma\mathfrak{p}} \sigma^{\mu\nu} \mathfrak{p}_\nu + \mathfrak{K}_{\omega\sigma\mathfrak{q}} \sigma^{\mu\nu} \mathfrak{q}_\nu + \mathfrak{K}_{\omega\mathfrak{t}} \mathfrak{t}^{\mu\nu} \omega_\nu \\ &\quad + \epsilon^{\mu\nu\alpha\beta} u_\nu \Big(\mathfrak{h}_{\omega\mathfrak{w}} \nabla_\alpha \mathfrak{w}_\beta + \mathfrak{h}_{\omega\kappa} \nabla_\alpha \kappa_{0,\beta} + \mathfrak{K}_{\omega I\mathfrak{w}} I_\alpha \mathfrak{w}_\beta + \mathfrak{K}_{\omega F\mathfrak{w}} F_\alpha \mathfrak{w}_\beta + \mathfrak{K}_{\omega I\kappa} I_\alpha \kappa_{0,\beta} + \mathfrak{K}_{\omega F\kappa} F_\alpha \kappa_{0,\beta} \Big) \,,\end{aligned} \tag{6.159a}$$

$$
\begin{aligned}
&\tau_{\mathfrak{p}}\dot{\mathfrak{p}}^{\langle\mu\rangle} + \tau_{\mathfrak{p}\mathfrak{q}}\dot{\mathfrak{q}}^{\langle\mu\rangle} + \tau_{\mathfrak{p}\omega}\dot{\omega}_0^{\langle\mu\rangle} + \mathfrak{p}^\mu \\
&= \mathfrak{e}_0^{(0)}\omega^\mu + \mathfrak{K}_{\mathfrak{p}\theta}\theta\mathfrak{p}^\mu + \mathfrak{K}_{\mathfrak{p}\theta\mathfrak{q}}\theta\mathfrak{q}^\mu + \mathfrak{K}_{\mathfrak{p}\theta\omega}\theta\omega_0^\mu + \mathfrak{K}_{\mathfrak{p}\sigma}\sigma^{\mu\nu}\mathfrak{p}_\nu + \mathfrak{K}_{\mathfrak{p}\sigma\mathfrak{q}}\sigma^{\mu\nu}\mathfrak{q}_\nu + \mathfrak{K}_{\mathfrak{p}\sigma\omega}\sigma^{\mu\nu}\omega_{0,\nu} + \mathfrak{K}_{\mathfrak{p}\mathfrak{t}}\mathfrak{t}^{\mu\nu}\omega_\nu \\
&\quad + \epsilon^{\mu\nu\alpha\beta}u_\nu\Big(\mathfrak{h}_{\mathfrak{p}\mathfrak{w}}\nabla_\alpha\mathfrak{w}_\beta + \mathfrak{h}_{\mathfrak{p}\kappa}\nabla_\alpha\kappa_{0,\beta} + \mathfrak{K}_{\mathfrak{p}I\mathfrak{w}}I_\alpha\mathfrak{w}_\beta + \mathfrak{K}_{\mathfrak{p}F\mathfrak{w}}F_\alpha\mathfrak{w}_\beta + \mathfrak{K}_{\mathfrak{p}I\kappa}I_\alpha\kappa_{0,\beta} + \mathfrak{K}_{\mathfrak{p}F\kappa}F_\alpha\kappa_{0,\beta}\Big) ,
\end{aligned} \tag{6.159b}
$$

and

$$
\begin{aligned}
&\tau_{\mathfrak{q}}\dot{\mathfrak{q}}^{\langle\mu\rangle} + \tau_{\mathfrak{q}\omega}\dot{\omega}_0^{\langle\mu\rangle} + \tau_{\mathfrak{q}\mathfrak{p}}\dot{\mathfrak{p}}^{\langle\mu\rangle} + \mathfrak{q}^\mu \\
&= \mathfrak{e}_0^{(2)}\omega^\mu + \mathfrak{K}_{\mathfrak{q}\theta}\theta\mathfrak{q}^\mu + \mathfrak{K}_{\mathfrak{q}\theta\omega}\theta\omega_0^\mu + \mathfrak{K}_{\mathfrak{q}\theta\mathfrak{p}}\theta\mathfrak{p}^\mu + \mathfrak{K}_{\mathfrak{q}\sigma}\sigma^{\mu\nu}\mathfrak{q}_\nu + \mathfrak{K}_{\mathfrak{q}\sigma\omega}\sigma^{\mu\nu}\omega_{0,\nu} + \mathfrak{K}_{\mathfrak{q}\sigma\mathfrak{p}}\sigma^{\mu\nu}\mathfrak{p}_\nu + \mathfrak{K}_{\mathfrak{q}\mathfrak{t}}\mathfrak{t}^{\mu\nu}\omega_\nu \\
&\quad + \epsilon^{\mu\nu\alpha\beta}u_\nu\Big(\mathfrak{h}_{\mathfrak{q}\mathfrak{w}}\nabla_\alpha\mathfrak{w}_\beta + \mathfrak{h}_{\mathfrak{q}\kappa}\nabla_\alpha\kappa_{0,\beta} + \mathfrak{K}_{\mathfrak{q}I\mathfrak{w}}I_\alpha\mathfrak{w}_\beta + \mathfrak{K}_{\mathfrak{q}F\mathfrak{w}}F_\alpha\mathfrak{w}_\beta + \mathfrak{K}_{\mathfrak{q}I\kappa}I_\alpha\kappa_{0,\beta} + \mathfrak{K}_{\mathfrak{q}F\kappa}F_\alpha\kappa_{0,\beta}\Big) ,
\end{aligned} \tag{6.159c}
$$

with the explicit expressions for the coefficients relegated to Appendix E. The vectorial quantities κ_0^μ and $\mathfrak{w}^\mu$ follow the equations

$$
\begin{aligned}
&\tau_\kappa\dot{\kappa}_0^{\langle\mu\rangle} + \tau_{\kappa\mathfrak{w}}\dot{\mathfrak{w}}^{\langle\mu\rangle} + \kappa_0^\mu \\
&= -\mathfrak{a}^{(\kappa)}F^\mu + \mathfrak{b}^{(\kappa)}I^\mu + \mathfrak{K}_{\kappa\theta}\theta\kappa_0^\mu + \mathfrak{K}_{\kappa\theta\mathfrak{w}}\theta\mathfrak{w}^\mu + \mathfrak{K}_{\kappa\sigma}\sigma^{\mu\nu}\kappa_{0,\nu} + \mathfrak{K}_{\kappa\sigma\mathfrak{w}}\sigma^{\mu\nu}\mathfrak{w}_\nu + \mathfrak{K}_{\kappa\omega}\omega^{\mu\nu}\kappa_{0,\nu} + \mathfrak{K}_{\kappa\omega\mathfrak{w}}\omega^{\mu\nu}\mathfrak{w}_\nu \\
&\quad + \mathfrak{h}_{\kappa\mathfrak{t}}\Delta^\mu{}_\lambda\nabla_\nu\mathfrak{t}^{\nu\lambda} + \mathfrak{K}_{\kappa I\mathfrak{t}}\mathfrak{t}^{\mu\nu}I_\nu + \mathfrak{K}_{\kappa F\mathfrak{t}}\mathfrak{t}^{\mu\nu}F_\nu + \epsilon^{\mu\nu\alpha\beta}u_\nu\Big(\mathfrak{h}_{\kappa\omega}\nabla_\alpha\omega_{0,\beta} + \mathfrak{h}_{\kappa\mathfrak{p}}\nabla_\alpha\mathfrak{p}_\beta + \mathfrak{h}_{\kappa\mathfrak{q}}\nabla_\alpha\mathfrak{q}_\beta \\
&\quad + \mathfrak{K}_{\kappa I\omega}I_\alpha\omega_{0,\beta} + \mathfrak{K}_{\kappa I\mathfrak{p}}I_\alpha\mathfrak{p}_\beta + \mathfrak{K}_{\kappa I\mathfrak{q}}I_\alpha\mathfrak{q}_\beta + \mathfrak{K}_{\kappa F\omega}F_\alpha\omega_{0,\beta} + \mathfrak{K}_{\kappa F\mathfrak{p}}F_\alpha\mathfrak{p}_\beta + \mathfrak{K}_{\kappa F\mathfrak{q}}F_\alpha\mathfrak{q}_\beta\Big) ,
\end{aligned} \tag{6.160a}
$$

and

$$
\begin{aligned}
&\tau_{\mathfrak{w}}\dot{\mathfrak{w}}^{\langle\mu\rangle} + \tau_{\mathfrak{w}\kappa}\dot{\kappa}_0^{\langle\mu\rangle} + \mathfrak{w}^\mu \\
&= \mathfrak{a}_0^{(1)}F^\mu + \mathfrak{b}_0^{(1)}I^\mu + \mathfrak{K}_{\mathfrak{w}\theta}\theta\mathfrak{w}^\mu + \mathfrak{K}_{\mathfrak{w}\theta\kappa}\theta\kappa_0^\mu + \mathfrak{K}_{\mathfrak{w}\sigma}\sigma^{\mu\nu}\mathfrak{w}_\nu + \mathfrak{K}_{\mathfrak{w}\sigma\kappa}\sigma^{\mu\nu}\kappa_{0,\nu} + \mathfrak{K}_{\mathfrak{w}\omega}\omega^{\mu\nu}\mathfrak{w}_\nu + \mathfrak{K}_{\mathfrak{w}\omega\kappa}\omega^{\mu\nu}\kappa_{0,\nu} \\
&\quad + \mathfrak{h}_{\mathfrak{w}\mathfrak{t}}\Delta^\mu{}_\lambda\nabla_\nu\mathfrak{t}^{\nu\lambda} + \mathfrak{K}_{\mathfrak{w}I\mathfrak{t}}\mathfrak{t}^{\mu\nu}I_\nu + \mathfrak{K}_{\mathfrak{w}F\mathfrak{t}}\mathfrak{t}^{\mu\nu}F_\nu + \epsilon^{\mu\nu\alpha\beta}u_\nu\Big(\mathfrak{h}_{\mathfrak{w}\omega}\nabla_\alpha\omega_{0,\beta} + \mathfrak{h}_{\mathfrak{w}\mathfrak{p}}\nabla_\alpha\mathfrak{p}_\beta + \mathfrak{h}_{\mathfrak{w}\mathfrak{q}}\nabla_\alpha\mathfrak{q}_\beta \\
&\quad + \mathfrak{K}_{\mathfrak{w}I\omega}I_\alpha\omega_{0,\beta} + \mathfrak{K}_{\mathfrak{w}I\mathfrak{p}}I_\alpha\mathfrak{p}_\beta + \mathfrak{K}_{\mathfrak{w}I\mathfrak{q}}I_\alpha\mathfrak{q}_\beta + \mathfrak{K}_{\mathfrak{w}F\omega}F_\alpha\omega_{0,\beta} + \mathfrak{K}_{\mathfrak{w}F\mathfrak{p}}F_\alpha\mathfrak{p}_\beta + \mathfrak{K}_{\mathfrak{w}F\mathfrak{q}}F_\alpha\mathfrak{q}_\beta\Big) ,
\end{aligned} \tag{6.160b}
$$

respectively. The equation for the traceless symmetric tensor $\mathfrak{t}^{\mu\nu}$ on the other hand is simpler to obtain and reads

$$
\begin{aligned}
\tau_{\mathfrak{t}}\dot{\mathfrak{t}}^{\langle\mu\nu\rangle} + \mathfrak{t}^{\mu\nu} &= \mathfrak{d}_0^{(2)}\sigma^{\mu\nu} + \mathfrak{K}_{\mathfrak{t}\theta}\theta\mathfrak{t}^{\mu\nu} + \mathfrak{h}_{\mathfrak{t}\mathfrak{w}}\nabla^{\langle\mu}\mathfrak{w}^{\nu\rangle} + \mathfrak{h}_{\mathfrak{t}\kappa}\nabla^{\langle\mu}\kappa_0^{\nu\rangle} + \mathfrak{K}_{\mathfrak{t}I\mathfrak{w}}I^{\langle\mu}\mathfrak{w}^{\nu\rangle} + \mathfrak{K}_{\mathfrak{t}F\mathfrak{w}}F^{\langle\mu}\mathfrak{w}^{\nu\rangle} \\
&\quad + \mathfrak{K}_{\mathfrak{t}I\kappa}I^{\langle\mu}\kappa_0^{\nu\rangle} + \mathfrak{K}_{\mathfrak{t}F\kappa}F^{\langle\mu}\kappa_0^{\nu\rangle} + \mathfrak{K}_{\mathfrak{t}\omega\omega}\omega_0^{\langle\mu}\omega^{\nu\rangle} + \mathfrak{K}_{\mathfrak{t}\omega\mathfrak{p}}\mathfrak{p}^{\langle\mu}\omega^{\nu\rangle} + \mathfrak{K}_{\mathfrak{t}\omega\mathfrak{q}}\mathfrak{q}^{\langle\mu}\omega^{\nu\rangle} \\
&\quad + \sigma_\alpha{}^{\langle\mu}\epsilon^{\nu\rangle\alpha\beta\gamma}u_\beta\left(\mathfrak{K}_{\mathfrak{t}\sigma\omega}\omega_{0,\gamma} + \mathfrak{K}_{\mathfrak{t}\sigma\mathfrak{p}}\mathfrak{p}_\gamma + \mathfrak{K}_{\mathfrak{t}\sigma\mathfrak{q}}\mathfrak{q}_\gamma\right) .
\end{aligned} \tag{6.161}
$$

The equations of motion (6.159)–(6.161) provide the time evolution of all degrees of freedom of the spin tensor, both equilibrium $(\omega_0^\mu, \kappa_0^\mu)$ and dissipative $(\mathfrak{p}^\mu, \mathfrak{q}^\mu, \mathfrak{w}^\mu, \mathfrak{t}^{\mu\nu})$ ones.[13] Also here there should in principle appear both terms that are nonlinear in the moments $\mathfrak{p}^\mu, \mathfrak{q}^\mu, \mathfrak{w}^\mu$, and $\mathfrak{t}^{\mu\nu}$, as well as terms of second order containing the usual dissipative quantities Π, n^μ, and $\pi^{\mu\nu}$, and fluid-dynamical gradients. These contributions, as the nonlinear ones in the previous subsection, were neglected due to linearizing the collision integrals.

6.5 Polarization observables

By now we have achieved our main goal of determining the equations of motion of all components of N^μ, $T^{\mu\nu}$, and $S^{\lambda\mu\nu}$ which are hydrodynamically important in the sense that they have Navier-Stokes solutions of order $\mathcal{O}(\mathrm{Kn})$. What is left to do is to relate the observables that are measured in experiment to these quantities.

[13] When counting the number of independent degrees of freedom, we find that the spin tensor consists of 3+3=6 ideal and 3+3+3+5=14 dissipative quantities, yielding a total of 20 components. This is lower than the theoretically possible 24 independent degrees of freedom, and has its roots in our kinetic description of the spin tensor.

6.5.1 Pauli-Lubanski pseudovector

In order to obtain the local polarization in terms of hydrodynamic quantities, we need to evaluate Eq. (6.66). Inserting the local-equilibrium distribution function (6.6a) as well as the expansion of the deviation from equilibrium in terms of irreducible moments (6.29) and truncating the expansion such that only quantities with first-order Navier-Stokes values are retained, we find

$$S^\mu(k) = \frac{\sigma}{N(k)} \int \mathrm{d}\Sigma_\lambda k^\lambda f_{0\mathbf{k}} \widetilde{f}_{0\mathbf{k}} \Bigg\{ -\frac{\hbar\sigma}{m}\widetilde{\Omega}_0^{\mu\alpha} k_\alpha + 2K^{\mu\gamma}\Xi_{\gamma\nu}\Big[\mathfrak{x}_{\mathfrak{p}}^{(0)}\mathfrak{p}^\nu + \mathfrak{x}_{\mathfrak{q}}^{(0)}\mathfrak{q}^\nu + \epsilon^{\nu\alpha\beta\delta}k_\alpha u_\beta \left(\mathfrak{x}_{\mathfrak{w}}^{(1)}\mathfrak{w}_\delta + \mathfrak{x}_\kappa^{(1)}\kappa_{0,\delta}\right) + \mathfrak{x}_{\mathfrak{q}}^{(2)} k^{\langle\nu}k^{\alpha\rangle}\mathfrak{q}_\alpha + \mathfrak{x}_{\mathfrak{t}}^{(2)} k_{\langle\alpha}k_{\beta\rangle}\mathfrak{t}_\rho{}^\alpha\epsilon^{\beta\nu\delta\rho}u_\delta\Big]\Bigg\} , \tag{6.162}$$

where we defined the coefficients

$$\mathfrak{x}_{\mathfrak{p}}^{(0)} := \sum_{n\neq 2}\mathcal{H}_{\mathbf{k}n}^{(1,0)}\mathcal{Q}_{n0}^{(10)} - \frac{m^2}{2}\mathcal{H}_{\mathbf{k}2}^{(1,0)} , \tag{6.163a}$$

$$\mathfrak{x}_{\mathfrak{q}}^{(0)} := -\frac{3}{2}\mathcal{H}_{\mathbf{k}2}^{(1,0)} , \tag{6.163b}$$

$$\mathfrak{x}_w^{(1)} := \frac{1}{2}\sum_n \mathcal{H}_{\mathbf{k}n}^{(1,1)}\mathcal{Q}_{n0}^{(11)} , \tag{6.163c}$$

$$\mathfrak{x}_\kappa^{(1)} := \frac{1}{2}\sum_n \mathcal{H}_{\mathbf{k}n}^{(1,1)}\mathcal{Q}_{n}^{(\kappa)} , \tag{6.163d}$$

$$\mathfrak{x}_{\mathfrak{q}}^{(2)} := \frac{3}{5}\sum_n \mathcal{H}_{\mathbf{k}n}^{(1,2)}\mathcal{Q}_{n0}^{(12)} , \tag{6.163e}$$

$$\mathfrak{x}_{\mathfrak{t}}^{(2)} := -\frac{2}{3}\sum_n \mathcal{H}_{\mathbf{k}n}^{(1,2)}\mathcal{Q}_{n0}^{(22)} \tag{6.163f}$$

and also used the asymptotic matching (6.142). Note that this expression is only accurate up to first order in Knudsen and inverse Reynolds numbers due to the usage of the asymptotic matching. The expression (6.162) determines the local polarization, i.e., it retains the dependence on momentum space. If instead we are interested in the global polarization, we have to consider the integrated expression (6.69), which yields

$$\overline{S}^\mu = \frac{\sigma}{\overline{N}} \int \mathrm{d}\Sigma_\lambda \Bigg\{ -\frac{2\hbar\sigma}{gm}\left(u^\lambda\omega_0^\mu J_{20} + u^\mu\omega_0^\lambda J_{21} - \epsilon^{\mu\lambda\alpha\beta}u_\alpha\kappa_{0,\beta}J_{21}\right) + \mathcal{Q}_{10}^{(10)}u^\lambda\mathfrak{p}^\mu - u^\mu\left[\frac{1}{3}\left(m^2\mathcal{Q}_{-1,0}^{(10)} - \mathcal{Q}_{10}^{(10)}\right)\mathfrak{p}^\lambda + \mathcal{Q}_{-1,0}^{(12)}\mathfrak{q}^\lambda\right] + \frac{1}{2}\epsilon^{\mu\lambda\alpha\beta}u_\alpha\mathfrak{w}_\beta\Bigg\} . \tag{6.164}$$

where we employed that $\widetilde{\Omega}_0^{\mu\nu} = -2u^{[\mu}\omega_0^{\nu]} + 2\epsilon^{\mu\nu\alpha\beta}u_\alpha\kappa_{0,\beta}$. It is instructive to compute these expressions in the Navier-Stokes limit as well to be able to relate the polarization to gradients of the standard hydrodyamical variables. We find for the local polarization

$$S_{\mathrm{NS}}^\mu(k) = \frac{\sigma}{N(k)} \int \mathrm{d}\Sigma_\lambda k^\lambda f_{0\mathbf{k}}\widetilde{f}_{0\mathbf{k}} \Bigg\{ -\frac{2\hbar\sigma}{m}\left[\mathfrak{e}^{(\omega)}u^{[\mu}\omega^{\nu]} + \epsilon^{\mu\nu\alpha\beta}u_\alpha\left(\mathfrak{b}^{(\kappa)}I_\beta - \mathfrak{a}^{(\kappa)}F_\beta\right)\right]k_\nu + 2K^{\mu\gamma}\Xi_{\gamma\nu}\Bigg[\left(\mathfrak{x}_{\mathfrak{p}}^{(0)}\mathfrak{e}_0^{(0)} + \mathfrak{x}_{\mathfrak{q}}^{(0)}\mathfrak{e}_0^{(2)}\right)\omega^\nu + \left(\mathfrak{x}_{\mathfrak{w}}^{(1)}\mathfrak{b}_0^{(1)} + \mathfrak{x}_\kappa^{(1)}\mathfrak{b}^{(\kappa)}\right)\epsilon^{\nu\alpha\beta\delta}k_\alpha u_\beta I_\delta + \left(\mathfrak{x}_{\mathfrak{w}}^{(1)}\mathfrak{a}_0^{(1)} - \mathfrak{x}_\kappa^{(1)}\mathfrak{a}^{(\kappa)}\right)\epsilon^{\nu\alpha\beta\delta}k_\alpha u_\beta F_\delta + \mathfrak{x}_{\mathfrak{q}}^{(2)}\mathfrak{e}_0^{(2)}k^{\langle\nu}k^{\alpha\rangle}\omega_\alpha + \mathfrak{x}_{\mathfrak{t}}^{(2)}\mathfrak{d}_0^{(2)}k_{\langle\alpha}k_{\beta\rangle}\sigma_\rho{}^\alpha\epsilon^{\beta\nu\delta\rho}u_\delta\Bigg]\Bigg\} . \tag{6.165}$$

Note that there appears a term proportional to the shear tensor, which is similar, although not identical, to the one found in Ref. [11]. In the aforementioned reference, this type of term was found to be vital to match the experimental data on the local polarization. The answer to the question whether the shear contribution in Eq. (6.165) is able to do the same is left for future work. The global polarization on the other hand reads in the NAVIER-STOKES limit

$$\overline{S}^{\mu}_{\rm NS} = \frac{\sigma}{\overline{N}} \int {\rm d}\Sigma_\lambda \bigg\{ \frac{2\hbar\sigma}{gm} \Big[\mathfrak{e}^{(\omega)} \left(J_{20} u^\lambda \omega^\mu + J_{21} u^\mu \omega^\lambda \right) - J_{21} \epsilon^{\mu\lambda\alpha\beta} u_\alpha \left(\mathfrak{a}^{(\kappa)} F_\beta - \mathfrak{b}^{(\kappa)} I_\beta \right) \Big] + \mathcal{Q}^{(10)}_{10} \mathfrak{e}^{(0)}_0 u^\lambda \omega^\mu \\ - u^\mu \left[\frac{1}{3} \left(m^2 \mathcal{Q}^{(10)}_{-1,0} - \mathcal{Q}^{(10)}_{10} \right) \mathfrak{e}^{(0)}_0 + \mathcal{Q}^{(12)}_{-1,0} \mathfrak{e}^{(2)}_0 \right] \omega^\lambda + \frac{1}{2} \epsilon^{\mu\lambda\alpha\beta} u_\alpha \left(\mathfrak{a}^{(1)}_0 F_\beta + \mathfrak{b}^{(1)}_0 I_\beta \right) \bigg\} . \tag{6.166}$$

Here, the shear-dependent term from the local polarization has vanished upon performing the integration and thus only contributes to the local, but not the global polarization.

6.5.2 Alignment

In order to compute the tensor polarization, we now have to evaluate Eq. (6.67). Notably, there will be no contributions from local equilibrium in this expression, since $f_{\rm eq}(x, k, \mathfrak{s})$ is at most of first order in the spin vector. Inserting the expansion of the deviation from local equilibrium (6.29) and truncating the sums such that only quantities with NAVIER-STOKES values of first order are kept, we arrive at the following expression:

$$\Theta^{\mu\nu}(k) = \frac{4}{5}\sqrt{\frac{3}{2}} \frac{1}{N(k)} \int {\rm d}\Sigma_\lambda k^\lambda f_{0\mathbf{k}} \widetilde{f}_{0\mathbf{k}} K^{\mu\nu}_{\rho\sigma} \Xi^{\rho\sigma}_{\alpha\beta} \Big[\mathfrak{y}^{(1)}_\pi \pi^{\alpha\beta} + \mathfrak{y}^{(2)}_\pi \Delta^{\alpha\beta}_{\gamma\delta} k^{\langle\gamma} k_{\rho\rangle} \pi^{\rho\delta} + \mathfrak{y}^{(3)}_\pi k^{\langle\alpha} k^\beta k^\gamma k^{\delta\rangle} \pi_{\gamma\delta} \\ + \mathfrak{y}^{(1)}_n k^{\langle\alpha} n^{\beta\rangle} + \mathfrak{y}^{(2)}_n k^{\langle\alpha} k^\beta k^{\gamma\rangle} n_\gamma + \mathfrak{y}_\Pi k^{\langle\alpha} k^{\beta\rangle} \Pi \Big] . \tag{6.167}$$

Here, the asymptotic matching (6.147) was used in conjunction with Eqs. (6.153) and we defined the coefficients

$$\mathfrak{y}^{(1)}_\pi := \sum_n \mathcal{H}^{(2,0)}_{\mathbf{k}n} \mathcal{T}^{(20)}_{n0} , \tag{6.168a}$$

$$\mathfrak{y}^{(2)}_\pi := \frac{12}{7} \sum_n \mathcal{H}^{(2,2)}_{\mathbf{k}n} \mathcal{T}^{(22)}_{n0} , \tag{6.168b}$$

$$\mathfrak{y}^{(3)}_\pi := \frac{5}{9} \sum_n \mathcal{H}^{(2,4)}_{\mathbf{k}n} \mathcal{T}^{(24)}_{n0} , \tag{6.168c}$$

$$\mathfrak{y}^{(1)}_n := \frac{3}{5} \sum_n \mathcal{H}^{(2,1)}_{\mathbf{k}n} \mathcal{T}^{(11)}_{n0} , \tag{6.168d}$$

$$\mathfrak{y}^{(2)}_n := \frac{3}{7} \sum_n \mathcal{H}^{(2,3)}_{\mathbf{k}n} \mathcal{T}^{(13)}_{n0} , \tag{6.168e}$$

$$\mathfrak{y}_\Pi := -\frac{3}{5m^2} \sum_n \mathcal{H}^{(2,2)}_{\mathbf{k}n} \mathcal{T}^{(00)}_{n0} . \tag{6.168f}$$

Equation (6.167) clarifies that all dissipative currents that are present in the energy-momentum tensor and the particle four-current contribute to the tensor polarization of the system. In order to relate this to the quantity that is measured in experiment (which is called *alignment*), we have to consider the 00-element of the spin-density matrix. As is shown in Appendix C, the alignment is given by

$$\varrho_{00}(k) = \frac{1}{3} - \sqrt{\frac{2}{3}} \epsilon^{(0)}_\mu(k) \epsilon^{(0)}_\nu(k) \Theta^{\mu\nu}(k) , \tag{6.169}$$

where $\epsilon^{(\lambda)\mu}(k)$ denotes the polarization vector of a spin-1 particle with momentum k and spin-projection λ onto a given quantization axis, such that, in the particle-rest frame where $k^\star = (m, 0, 0, 0)$, we have $\epsilon^{(0)\mu}(k^\star) \equiv (0, 0, 0, 1)^\mu$ when choosing the z-axis. The value of $\varrho_{00} = 1/3$ corresponds to the unpolarized case, whereas the second term $\sim \Theta^{zz}(k^\star)$ determines the degree of tensor polarization. Note that, in our expression (6.67), the fluid-dynamical gradients enter at first order, even though they will be multiplied by (potentially small) viscosities ζ_0, η_0, and conductivity κ_0. This is in contrast to quark-coalescence models (see, e.g., Refs. [24, 26, 27]), where the tensor polarization of a vector meson is built from the (vector) polarizations of its constituent quarks. In that case, the resulting polarization has to be of second order, since the quark polarization is mainly induced by vorticity.[14] Whether the expression (6.169) is able to reproduce the large values of the alignment observed in experiment [18] will be the subject of future work.

Lastly, the global tensor polarization is given by Eq. (6.70). Making use of the expressions (6.150), we can express it to first order in Kn and Re^{-1} as

$$\overline{\Theta}^{\mu\nu} = \frac{1}{2}\sqrt{\frac{3}{2}}\frac{1}{\overline{N}}\int \mathrm{d}\Sigma_\lambda \bigg[\left(u^\mu u^\nu - \frac{1}{3}\Delta^{\mu\nu}\right)\left(-\frac{3}{m^2}\Upsilon_1^{(00)} u^\lambda \Pi + \Upsilon_0^{(11)} n^\lambda\right) + \frac{3}{5}\mathcal{T}_{00}^{(11)} n^{\langle\mu}\Delta^{\nu\rangle\lambda} + u^{(\mu}\left(\Upsilon_1^{(01)} n^{\nu)} u^\lambda - \frac{1}{m^2}\Delta^{\nu)\lambda}\Upsilon_0^{(10)}\Pi + \Upsilon_0^{(12)}\pi^{\nu)\lambda}\right) + \mathcal{T}_{10}^{(20)}\pi^{\mu\nu}u^\lambda \bigg] . \tag{6.170}$$

A simple application

To get a clearer understanding of what our theory tells about the alignment of vector mesons, let us consider the basic case treated in Ref. [28], where the effects of the bulk viscous pressure and the particle diffusion are neglected to study the effect of the shear stress on the tensor polarization. Choosing the simplest possible truncation, we set

$$\mathbb{S}_0^{(0)} = \mathbb{S}_1^{(0)} = \mathbb{S}_1^{(2)} = \mathbb{S}_2^{(2)} = \mathbb{S}_3^{(2)} = \mathbb{S}_4^{(2)} = \emptyset\,, \quad \mathbb{S}_2^{(0)} = \{0\}\,, \quad \mathbb{S}_0^{(2)} = \{1\}\,. \tag{6.171}$$

Then, Eq. (6.169) becomes

$$\varrho_{00}(k) = \frac{1}{3} - \frac{4}{15}\frac{\int \mathrm{d}\Sigma_\alpha k^\alpha \xi\, \beta_0 f_{0\mathbf{k}} \mathcal{H}_{\mathbf{k}1}^{(2,0)} \epsilon_\beta^{(0)} \epsilon_\gamma^{(0)} K_{\mu\nu}^{\beta\gamma} \Xi_{\rho\sigma}^{\mu\nu}\pi^{\rho\sigma}}{\int \mathrm{d}\Sigma_\alpha k^\alpha f_{0\mathbf{k}}\left(1 - 3\mathcal{H}_{\mathbf{k}0}^{(0,0)}\Pi/m^2 + \mathcal{H}_{\mathbf{k}0}^{(0,2)}\pi^{\mu\nu}k_{\langle\mu}k_{\nu\rangle}\right)}\,, \tag{6.172}$$

where we defined

$$\xi \coloneqq -\frac{1}{\beta_0}\frac{\mathcal{D}_{10}^{(0,2)}}{\mathcal{D}_{11}^{(0)}} \equiv \frac{1}{\beta_0}\mathcal{T}_{10}^{(20)}\,. \tag{6.173}$$

Note that this coefficient is the only quantity in Eq. (6.172) that depends on the microscopic interactions of the particles, and can be computed via a method summarized in Appendix F.4. In the case of a four-point interaction, this coefficient is of the order of a percent, cf. Fig. 6.1. However, one should keep in mind that this result assumes a fluid that is made solely out of a single species of spin-1 particles, and thus the numerical value may change when developing a theory that incorporates the coupled dynamics of a more realistic system consisting of both spin-1/2 and spin-1 particles.

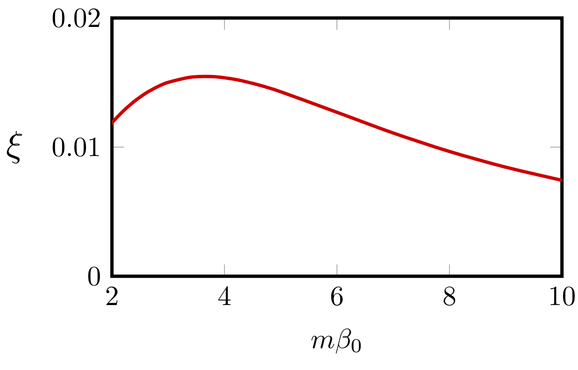

Figure 6.1: The coefficient ξ for the case of a four-point interaction, $\widehat{\mathcal{L}}_{\text{int}} \sim (\widehat{V}^\dagger \cdot \widehat{V})^2$.

[14] It is easy to see this from a symmetry perspective: $\Theta^{\mu\nu}$ is a symmetric traceless second-rank tensor, and the only way to build such an object from vorticities is by combining two of them, e.g., $\omega^{\langle\mu}\omega^{\nu\rangle}$.

Chapter 7

Conclusion and outlook

After finishing the program of constructing dissipative spin hydrodynamics from quantum field theory, let us briefly review the steps needed to achieve this goal and comment on tasks that can be undertaken in the future.

7.1 Summary

In Chapter 1 of this thesis, we initially posed the question how a fluid comprised of particles with spin behaves. As was elaborated in Chapter 2, fluids are characterized by the fact that their microscopic and macroscopic scales are sufficiently separated, such that it is expected that the dynamics of the system can be sufficiently well described by the quantities appearing in the conserved currents. In the case of fluids consisting of particles with nonzero spin, the conservation law for the total angular momentum has to be included explicitly since it does not follow immediately from the conservation of the energy-momentum tensor, which can acquire an antisymmetric part. This, in turn, necessitates including the components of the spin tensor as additional hydrodynamic degrees of freedom, both in the ideal and the dissipative case.

Since the goal was to derive a theory of relativistic dissipative spin hydrodynamics, it proved necessary to provide a description of the dissipative degrees of freedom that went beyond usual NAVIER-STOKES theory in order to avoid acausal and thus unstable behavior. To provide this description, we employed the formalism of quantum kinetic theory, which describes the behavior of quantum fields in the limit that the wavepackets are localized enough to treat them as quasiparticles. In Chapter 3 we computed the conserved currents given by the underlying quantum field theories for particles of spins 0, $1/2$, and 1, and connected them to phase-space integrals over the WIGNER function, which constitutes a quantum field-theoretical generalization of the classical single-particle distribution function. However, the conserved currents are not uniquely defined, but only up to pseudogauge transformations, which leave the conservation laws and the total charges invariant. The standard canonical currents have the drawback that the energy-momentum tensor is in general not symmetric even in the case of free fields, which implies that the spin tensor is also not conserved in that case. For our purposes, we chose the modified GLW pseudogauge, which features the property that the energy-momentum tensor is symmetric in the case of free fields and in global equilibrium, and thus provides an intuitive representation of the currents in phase space.

In Chapter 4, we computed the collision terms appearing in the kinetic equations for the WIGNER functions. Employing the GLW method outlined in Ref. [43], we computed both local and nonlocal collision terms. It turned out that, to first order in $\hbar$, KLEIN-GORDON fields do not feature nonlocalities, in contrast to DIRAC and PROCA fields. This aligns with the intuitive picture of orbital angular

momentum and spin being exchanged in a collision which does not take place in a single spacetime point. The shifts that specify how far the particles are displaced from the center of momentum have been calculated in a covariant form, showing that there does not exist a “no-jump frame” where the collisions are local, except in the case where the shifts were identically zero to begin with.
Chapter 5 repeated this calculation, but in the approach pioneered by KADANOFF and BAYM, which is based on the DYSON-SCHWINGER equations on the KELDYSH contour. While the method seems different at first glance, we were able to demonstrate that, as long as the same approximations as in the GLW approach are made, the KB approach leads to the same collision terms, with the important modification that quantum-statistical effects are retained. Taking these into account enabled us to show that, in equilibrium, the distribution function has to be of BOSE-EINSTEIN or FERMI-DIRAC form, depending on whether the particles are bosons or fermions, respectively.
In both approaches, in order for the collision term to vanish exactly, we found that the conditions have to be those of global equilibrium, i.e., the four-temperature has to be a KILLING vector and the chemical potential over temperature has to be constant. This is at odds with the concept of local equilibrium known from standard fluid dynamics, and constitutes an effect of including nonlocal collisions.

In order to formulate a theory of spin hydrodynamics that reduces to the familiar hydrodynamic form if spin effects are excluded, we modified the definition of local equilibrium in Chapter 6, allowing the collision term to be nonzero, provided that the corrections originate only from nonlocal terms, which are comparatively small. With this definition of local equilibrium, we separated the dissipative parts of the distribution function, and applied a generalization of the method of moments that included both its dependence on momentum as well as on spin.
After having derived the equations of motion for all irreducible moments, we had to close the system with an appropriate truncation to extract the dynamics of the hydrodynamic degrees of freedom. This closure was then achieved via a suitable application of the IReD method, which expresses all moments solely in terms of the dissipative quantities of interest. In this way, we obtained equations of motion for all components of the energy-momentum tensor and the particle four-current. Interestingly, the degrees of freedom connected to the tensor polarization of the particles couple to these quantities, modifying the resulting transport coefficients.
We furthermore showed that, in addition to the six degrees of freedom characterizing the ideal part of the spin tensor, 14 dissipative quantities (two axial vectors, one vector, and one traceless symmetric tensor) feature NAVIER-STOKES values of first order and are thus hydrodynamically important. Because the spin tensor is not conserved, all of these quantities follow relaxation-type dynamics. While the relaxation of the dissipative quantities to their NAVIER-STOKES values is determined by local collisions and can be attributed to particles moving with the fluid that tends to isotropize, the spin potential relaxes to the thermal vorticity on a timescale determined by the nonlocal contributions to the collision term.
Lastly, we employed these results to derive formulae for the observables related to polarization, namely the PAULI-LUBANSKI vector and the alignment, expressed solely in terms of hydrodynamic fields. With these expressions, it is then possible to relate the microscopic properties of polarization to the macroscopic conserved currents, which are governed by hydrodynamics. The result for the local polarization shows a dependence on the shear tensor, which has already been observed in earlier studies, albeit in slightly different form, and has proven crucial to explain the measurements. The formula for the alignment on the other hand shows that, in this framework based on quantum kinetic theory that takes the vector mesons as the fundamental degrees of freedom, the tensor polarization of spin-1 particles is a purely dissipative effect that depends on the magnitude of the shear-stress tensor, the bulk viscous pressure, and the diffusion current.

7.2 Future perspectives

There are various ways to build on the work done in this thesis. First, the formulae which determine the polarization and alignment of particles should be tested in hydrodynamic simulations of heavy-ion collisions to see if they are able to describe the data. In particular, whether or not the expression for the alignment is able to explain the large measured signal for ϕ mesons lets one assess the viability of a hydrodynamic treatment for vector mesons inside the QGP.

Since we have provided explicit expressions for all resummed transport coefficients, one can compute them to arbitrary precision. This is especially important when considering the linearized theory, since in order for it to be symmetric hyperbolic, certain relations between transport coefficients have to be fulfilled. In the case of ultrarelativistic (spinless) fluid dynamics, these conditions have been found not to be valid in the DNMR truncation, whereas the IReD approach preserves them, given that the size of the basis is sent to infinity [52]. Thus, it is important to check whether or not the IReD approach taken in this work also features this desirable property.

In order to improve our understanding of the QGP with quarks and gluons as the fundamental degrees of freedom, the work done here for spin-1 particles can be extended to cover the case of originally massless gluons. This will introduce complications related to gauge symmetry, as well as the non-abelian nature of the particles. Nevertheless, most of the methods presented here should be applicable with suitable modifications. Following this line, a coupled theory including both fermions and bosons would be of interest. Steps towards this goal have already been undertaken in, e.g., Ref. [86], and our results on the dynamics of the spin degrees of freedom in the presence of nonlocal collisions can be used to extend it.

Finally, the effects of electromagnetic fields were not treated in this work, even though some studies on their inclusion have been done, cf. Refs. [28, 54, 55]. In order to treat them properly, the effects of the (in GLW language) "pure-spin exchange" terms, or equivalently (in KB language) "mean-field self-energy" have to be included, since they lead to the emergence of Vlasov-type terms in the kinetic equation. The formulation of such a theory of spin-magnetohydrodynamics would be desirable as well, since, especially in the early stages of heavy-ion collisions, the magnetic fields can be very strong.

Appendix A

Cancellation of off-shell terms

This appendix contains the proof of Theorem 1, which is valid for general fields as we introduced them in Subsec. 3.2.2. In addition to possible constraint equations, any field fulfills the KLEIN-GORDON equation

$$\left(\Box + \frac{m^2}{\hbar^2}\right)\widehat{\varphi} = \widehat{\rho} \,, \tag{A.1}$$

where $\widehat{\rho}$ is a general source term depending on the interaction Lagrangian.

Stating the ingredients

For a more compact presentation, let us recapitulate the basic objects introduced in the main text. The WIGNER function is defined in accordance with Eq. (3.31)

$$W(x,k) = \kappa \int \mathrm{d}^4 v e^{-\frac{i}{\hbar}k\cdot v}\langle :\widehat{\overline{\varphi}}_+ \widehat{\varphi}_- :\rangle \,. \tag{A.2}$$

We remind the reader of the notation

$$\widehat{\overline{\varphi}} := \begin{cases} \widehat{\varphi}^\dagger \,, & j \text{ integer} \,, \\ \widehat{\varphi}^\dagger \gamma^0 \,, & j \text{ half-integer} \,, \end{cases} \tag{A.3}$$

from which it follows that the WIGNER function fulfills $\widetilde{W} = W^*$, where

$$\widetilde{W} := \begin{cases} W^{\mathrm{T}} \,, & j \text{ integer} \,, \\ \gamma^0 W^{\mathrm{T}} \gamma^0 \,, & j \text{ half-integer} \,. \end{cases} \tag{A.4}$$

Acting with the BOPP operator $D^\mu := k^\mu + \frac{i\hbar}{2}\partial^\mu$ on the WIGNER function gives

$$D^\mu W(x,k) = i\hbar\kappa \int \mathrm{d}^4 v e^{-\frac{i}{\hbar}k\cdot v}\langle \widehat{\overline{\varphi}}_+ \partial^\mu \widehat{\varphi}_- \rangle \,, \tag{A.5}$$

$$D^{*\mu} W(x,k) = -i\hbar\kappa \int \mathrm{d}^4 v e^{-\frac{i}{\hbar}k\cdot v/\hbar}\langle (\partial^\mu \widehat{\overline{\varphi}}_+) \widehat{\varphi}_- \rangle \,. \tag{A.6}$$

General proof

Applying a combination of BOPP operators such that the left-hand side of Eq. (A.1) is reproduced under the integral, we find

$$(D^2 - m^2) W(x,k) = -\hbar^2 \kappa \int \mathrm{d}^4 v e^{-\frac{i}{\hbar}k\cdot v}\langle \widehat{\overline{\varphi}}_+ \widehat{\rho}_- \rangle =: \hbar C(x,k) \,. \tag{A.7a}$$

We can repeat the same procedure now with the complex conjugated BOPP operators to obtain

$$(D^{*2} - m^2)C(x,k) = \hbar^3\kappa \int \mathrm{d}^4v e^{-\frac{i}{\hbar}k\cdot v}\langle\widehat{\rho}_+\widehat{\rho}_-\rangle =: \hbar Z(x,k)\ . \tag{A.7b}$$

Note that, as for the WIGNER function, we again have $\widetilde{Z} = Z^*$. Applying the "tilde"-operator to Eqs. (A.7) and using the relations for Z and W, we can isolate the real and imaginary parts, i.e.,

$$(D^2 - m^2)\mathrm{Re}\,W = \frac{\hbar}{2}(C + \widetilde{C})\ , \tag{A.8a}$$

$$(D^2 - m^2)i\mathrm{Im}\,W = \frac{\hbar}{2}(C - \widetilde{C})\ , \tag{A.8b}$$

$$(D^{*2} - m^2)\frac{1}{2}(C + \widetilde{C}) = \hbar\mathrm{Re}\,Z\ , \tag{A.8c}$$

$$(D^{*2} - m^2)\frac{1}{2}(C - \widetilde{C}) = i\hbar\mathrm{Im}\,Z\ . \tag{A.8d}$$

Here we suppressed the arguments (x,k) for brevity and used that

$$\mathrm{Re}\,W = \frac{1}{2}(W + \widetilde{W})\ , \qquad i\mathrm{Im}\,W = \frac{1}{2}(W - \widetilde{W})\ , \tag{A.9}$$

and similarly for Z. At this point we remark that

$$D^2 = k^2 + i\hbar k\cdot\partial - \frac{\hbar^2}{4}\Box\ , \qquad D^{*2} = k^2 - i\hbar k\cdot\partial - \frac{\hbar^2}{4}\Box\ . \tag{A.10}$$

Abbreviating

$$C_S := \frac{1}{2}(C + \widetilde{C})\ , \qquad C_A := \frac{1}{2}(C - \widetilde{C})\ , \tag{A.11}$$

we obtain the real and imaginary parts of Eqs. (A.8) as

$$\left(k^2 - m^2 - \frac{\hbar^2}{4}\Box\right)\mathrm{Re}\,W = \hbar\mathrm{Re}\,C_S\ , \tag{A.12a}$$

$$k\cdot\partial\mathrm{Re}\,W = \mathrm{Im}\,C_S\ , \tag{A.12b}$$

$$\left(k^2 - m^2 - \frac{\hbar^2}{4}\Box\right)\mathrm{Im}\,W = \hbar\mathrm{Im}\,C_A\ , \tag{A.12c}$$

$$-k\cdot\partial\mathrm{Im}\,W = \mathrm{Re}\,C_A\ , \tag{A.12d}$$

$$\left(k^2 - m^2 - \frac{\hbar^2}{4}\Box\right)\mathrm{Re}\,C_S + \hbar k\cdot\partial\mathrm{Im}\,C_S = \hbar\mathrm{Re}\,Z\ , \tag{A.12e}$$

$$\left(k^2 - m^2 - \frac{\hbar^2}{4}\Box\right)\mathrm{Im}\,C_S - \hbar k\cdot\partial\mathrm{Re}\,C_S = 0\ , \tag{A.12f}$$

$$\left(k^2 - m^2 - \frac{\hbar^2}{4}\Box\right)\mathrm{Re}\,C_A + \hbar k\cdot\partial\mathrm{Im}\,C_A = 0\ , \tag{A.12g}$$

$$\left(k^2 - m^2 - \frac{\hbar^2}{4}\Box\right)\mathrm{Im}\,C_A - \hbar k\cdot\partial\mathrm{Re}\,C_A = \hbar\mathrm{Im}\,Z\ . \tag{A.12h}$$

We now expand all quantities as power series in $\hbar$, e.g.,

$$W = \sum_{j=0}^{\infty}\hbar^j W^{(j)}\ . \tag{A.13}$$

Then, Eqs. (A.12a), (A.12c), (A.12f) and (A.12g) become mass-shell like equations,

$$\left(k^2-m^2\right)\mathrm{Re}\,W^{(j)} = \mathrm{Re}\,C_S^{(j-1)} + \frac{\Box}{4}\mathrm{Re}\,W^{(j-2)}\;, \tag{A.14a}$$

$$\left(k^2-m^2\right)\mathrm{Im}\,W^{(j)} = \mathrm{Im}\,C_A^{(j-1)} + \frac{\Box}{4}\mathrm{Im}\,W^{(j-2)}\;, \tag{A.14b}$$

$$\left(k^2-m^2\right)\mathrm{Im}\,C_S^{(j)} = k\cdot\partial\mathrm{Re}\,C_S^{(j-1)} + \frac{\Box}{4}\mathrm{Im}\,C_S^{(j-2)}\;, \tag{A.14c}$$

$$\left(k^2-m^2\right)\mathrm{Re}\,C_A^{(j)} = -k\cdot\partial\mathrm{Im}\,C_A^{(j-1)} + \frac{\Box}{4}\mathrm{Re}\,C_A^{(j-2)}\;, \tag{A.14d}$$

while Eqs. (A.12b) and (A.12d) become kinetic equations,

$$k\cdot\partial\mathrm{Re}\,W^{(j)} = \mathrm{Im}\,C_S^{(j)}\;, \tag{A.15a}$$

$$k\cdot\partial\mathrm{Im}\,W^{(j)} = -\mathrm{Re}\,C_A^{(j)}\;. \tag{A.15b}$$

In these expressions it is understood that quantities formally of negative order in $\hbar$ are set to zero. Comparing this equation to the formulation in Eq. (4.28), we can identify

$$\mathcal{C} \equiv \mathrm{Im}\,C_S - i\mathrm{Re}\,C_A\;. \tag{A.16}$$

Splitting the Wigner function and the collision kernel into on- and off-shell parts,

$$W^{(j)} = \delta(k^2-m^2)W^{(j)}_{\text{on-shell}} + W^{(j)}_{\text{off-shell}}\;, \tag{A.17}$$

$$C_S^{(j)} = \delta(k^2-m^2)C^{(j)}_{S,\text{on-shell}} + C^{(j)}_{S,\text{off-shell}}\;, \tag{A.18}$$

$$C_A^{(j)} = \delta(k^2-m^2)C^{(j)}_{A,\text{on-shell}} + C^{(j)}_{A,\text{off-shell}}\;, \tag{A.19}$$

we obtain the off-shell components from Eqs. (A.14) as

$$\mathrm{Re}\,W^{(j)}_{\text{off-shell}} = (k^2-m^2)^{-1}\left[\mathrm{Re}\,C_S^{(j-1)} + \frac{\Box}{4}\mathrm{Re}\,W^{(j-2)}\right]\;, \tag{A.20a}$$

$$\mathrm{Im}\,W^{(j)}_{\text{off-shell}} = (k^2-m^2)^{-1}\left[\mathrm{Im}\,C_A^{(j-1)} + \frac{\Box}{4}\mathrm{Im}\,W^{(j-2)}\right]\;, \tag{A.20b}$$

$$\mathrm{Im}\,C^{(j)}_{S,\text{off-shell}} = (k^2-m^2)^{-1}\left[k\cdot\partial\mathrm{Re}\,C_S^{(j-1)} + \frac{\Box}{4}\mathrm{Im}\,C_S^{(j-2)}\right]\;, \tag{A.20c}$$

$$\mathrm{Re}\,C^{(j)}_{A,\text{off-shell}} = (k^2-m^2)^{-1}\left[-k\cdot\partial\mathrm{Im}\,C_A^{(j-1)} + \frac{\Box}{4}\mathrm{Re}\,C_A^{(j-2)}\right]\;. \tag{A.20d}$$

Then, the off-shell parts on the left-hand sides of Eqs. (A.15) at order j read

$$k\cdot\partial\mathrm{Re}\,W^{(j)}_{\text{off-shell}} = (k^2-m^2)^{-1}\left[k\cdot\partial\mathrm{Re}\,C_S^{(j-1)} + \frac{\Box}{4}\mathrm{Im}\,C_S^{(j-2)}\right]\;, \tag{A.21a}$$

$$k\cdot\partial\mathrm{Im}\,W^{(j)}_{\text{off-shell}} = (k^2-m^2)^{-1}\left[k\cdot\partial\mathrm{Im}\,C_A^{(j-1)} - \frac{\Box}{4}\mathrm{Re}\,C_A^{(j-2)}\right]\;, \tag{A.21b}$$

where we used Eqs. (A.15) at order $j-2$. Comparing to Eqs. (A.20c) and (A.20d), we find

$$k\cdot\partial\mathrm{Re}\,W^{(j)}_{\text{off-shell}} = \mathrm{Im}\,C^{(j)}_{S,\text{off-shell}}\;, \tag{A.22a}$$

$$k\cdot\partial\mathrm{Im}\,W^{(j)}_{\text{off-shell}} = -\mathrm{Re}\,C^{(j)}_{A,\text{off-shell}}\;, \tag{A.22b}$$

implying that the off-shell terms cancel in the Boltzmann equation to any order in $\hbar$. Thus, we can write

$$k\cdot\partial\mathrm{Re}\,W^{(j)}_{\text{on-shell}} = \mathrm{Im}\,C^{(j)}_{S,\text{on-shell}}\;, \tag{A.23a}$$

$$k\cdot\partial\mathrm{Im}\,W^{(j)}_{\text{on-shell}} = -\mathrm{Re}\,C^{(j)}_{A,\text{on-shell}}\;, \tag{A.23b}$$

where the restriction to the mass shell is understood. This proves our theorem.

Appendix B

Notes on the collision integrals

In this appendix, we present calculations on the GLW and the KB approaches that concern the computation of the collision integrals. In the GLW approach, we will relate the "in"-Wigner function to the full one, and show that both can be identified up to first order in $\hbar$ and to lowest order in the density, neglecting contributions of collisional origin inside the collision integrals themselves. In the KB approach, we will compute the self-energy in the T-matrix approximation, which has been shown to fall in the class of approximations to the self-energy that preserve the macroscopic conservation laws. Furthermore, we will show that the functions $\widetilde{f}$ in the KB approach incorporate quantum statistics. Lastly, we will explicitly compute the antisymmetric part of the energy-momentum tensor in the GLW pseudogauge.

B.1 GLW: The Wigner function and its "in"-counterpart

Here we will show that, to lowest order in the density, the Wigner function is equal to its "in"-counterpart.

We start by restating the observation from Eq. (4.25), namely that the Wigner function can be written as the following Fock-space average,

$$W^{ab}(x,k) = \left\langle e^{\frac{i}{\hbar}\widehat{P}\cdot x}\widehat{\Psi}^{ab}(k)e^{-\frac{i}{\hbar}\widehat{P}\cdot x}\right\rangle \,, \tag{B.1}$$

where

$$\widehat{\Psi}^{ab}(k) := \kappa\int \mathrm{d}^4 v e^{-\frac{i}{\hbar}k\cdot v}\widehat{\overline{\varphi}}^b\left(\frac{v}{2}\right)\widehat{\varphi}^a\left(-\frac{v}{2}\right) \,. \tag{B.2}$$

Since the "in"-states are eigenstates of the total momentum, we may replace $e^{-\frac{i}{\hbar}\hat{P}\cdot x}\left|k^n;\sigma^n\right\rangle_{\mathrm{in}} = \prod_{j=1}^n e^{-\frac{i}{\hbar}k_j\cdot x}\left|k^n;\sigma^n\right\rangle_{\mathrm{in}}$. Rethinking the steps that led to Eq. (4.23), we find

$$W^{ab}(x,k) = \sum_{n=0}^{\infty}\frac{1}{n!}\int \mathrm{d}^4\bar{x}^n\int\frac{\mathrm{d}^4\bar{k}^n}{(2\pi\hbar)^{4n}}\Psi^{ab}_{n,a_1b_1\cdots a_nb_n}(\bar{x}^n;\bar{k}^n|k)\prod_{j=1}^n W_{\mathrm{in}}^{a_jb_j}(x+\bar{x}_j,\bar{k}_j) \,, \tag{B.3}$$

where

$$\Psi^{ab}_{n,a_1b_1\cdots a_nb_n}(\bar{x}^n;\bar{k}^n|k) := \left(\frac{1}{\kappa\lambda^2\eta^2}\right)^n\int\frac{\mathrm{d}^4u^n}{(2\pi\hbar)^{4n}}\sum_{\sigma^n,\sigma'^n}{}_{\mathrm{in}}\!\left\langle\!\!\left\langle \bar{k}^n-\frac{u^n}{2};\sigma^n\right|\widehat{\Psi}^{ab}(k)\left|\bar{k}^n+\frac{u^n}{2};\sigma'^n\right\rangle\!\!\right\rangle_{\mathrm{in}}$$

$$\times\left[\prod_{j=1}^n e^{\frac{i}{\hbar}u_j\cdot\bar{x}_j}U_{b_j}\left(\bar{k}_j-\frac{u_j}{2},\sigma_j\right)\overline{U}_{a_j}\left(\bar{k}_j+\frac{u_j}{2},\sigma'_j\right)\right] \,. \tag{B.4}$$

Following Ref. [43], we compute Ψ_n^{ab} for $n = 0, 1$.

In the case $n = 0$, inserting Eq. (4.16) and making use of the completeness of the "out"-states, we obtain

$$\Psi_0^{ab}(0|k) = \kappa \sum_{m=0}^{\infty} \frac{1}{2^m m!} \sum_{\sigma^m} \int \mathrm{d}K^m \, \langle 0| \widehat{\overline{\varphi}}^b(0) \, |k^m; \sigma^m\rangle_{\text{out}} \times {}_{\text{out}}\langle k^m; \sigma^m | \widehat{\varphi}^a(0) \, |0\rangle \, (2\pi\hbar)^4 \delta^{(4)}\left(k + \sum_{\ell=1}^{m} k_\ell\right) , \tag{B.5}$$

where we employed the fact that the "out"-states are eigenstates of the momentum as well. Since the zeroth component of the momentum is always positive, the delta function vanishes, such that

$$\Psi_0^{ab}(0|k) = 0 \, . \tag{B.6}$$

Similarly, the $n = 1$ case yields

$$\Psi_{1,cd}^{ab}(x; p|k) = \frac{1}{\kappa\lambda^2\eta^2} \int \frac{\mathrm{d}^4u}{(2\pi\hbar)^4} \sum_{\sigma,\sigma'} e^{\frac{i}{\hbar}u\cdot x} U_d\left(p - \frac{u}{2}, \sigma\right) \overline{U}_c\left(p + \frac{u}{2}, \sigma'\right) \times \left[{}_{\text{in}}\left\langle p - \frac{u}{2}; \sigma \middle| \widehat{\Psi}^{ab} \middle| p + \frac{u}{2}; \sigma' \right\rangle_{\text{in}} - {}_{\text{in}}\left\langle p - \frac{u}{2}; \sigma \middle| \, p + \frac{u}{2}; \sigma' \right\rangle_{\text{in}} \left\langle 0 \middle| \widehat{\Psi}^{ab} \middle| 0 \right\rangle \right] .$$

Note that the second term vanishes for the same reasons as Ψ_0^{ab}. Inserting a complete set of "out"-states again, we have

$$\Psi_{1,cd}^{ab}(x; p|k) = \frac{1}{\lambda^2\eta^2} \int \frac{\mathrm{d}^4u}{(2\pi\hbar)^4} \sum_{\sigma,\sigma'} e^{\frac{i}{\hbar}u\cdot x} U_d\left(p - \frac{u}{2}, \sigma\right) \overline{U}_c\left(p + \frac{u}{2}, \sigma'\right) \times \sum_{m=0}^{\infty} \frac{1}{2^m m!} \sum_{\sigma^m} \int \mathrm{d}P'^m \, {}_{\text{in}}\left\langle p - \frac{u}{2}; \sigma \middle| \widehat{\overline{\varphi}}^b(0) \middle| p'^m; \sigma^m \right\rangle_{\text{out}} \times {}_{\text{out}}\left\langle p'^m; \sigma^m \middle| \widehat{\varphi}^a(0) \middle| p + \frac{u}{2}; \sigma' \right\rangle_{\text{in}} (2\pi\hbar)^4 \delta^{(4)}\left(k + \sum_{\ell=1}^{m} p'_\ell - p\right) . \tag{B.7}$$

Using the fact that the one-particle "in"- or "out"-states are orthogonal, we find that [using Eq. (4.8)]

$$\left\langle 0 \middle| \widehat{\varphi}^a(0) \middle| p + \frac{u}{2}; \sigma' \right\rangle_{\text{in}} = \lambda U^a\left(p + \frac{u}{2}, \sigma'\right) , \tag{B.8}$$

which may be used to obtain an explicit expression for the $m = 0$-term in Eq. (B.7). The other terms in the respective sum require that

$$p^2 = (k + p')^2 = 2m^2 + 2k \cdot p' \geq 4m^2 ,$$

which implies including the possibility of creating particles with masses larger than twice the mass of the original one. This possibility we will neglect, such that only the $m = 0$ term in Eq. (B.7) contributes, yielding

$$\Psi_{1,cd}^{ab}(x; p|k) = \frac{1}{\eta^2} \int \frac{\mathrm{d}^4u}{(2\pi\hbar)^4} \sum_{\sigma,\sigma'} e^{\frac{i}{\hbar}u\cdot x} U_d\left(p - \frac{u}{2}, \sigma\right) \overline{U}_c\left(p + \frac{u}{2}, \sigma'\right) \times \overline{U}^b\left(p - \frac{u}{2}, \sigma\right) U^a\left(p + \frac{u}{2}, \sigma'\right) (2\pi\hbar)^4 \delta^{(4)}(k - p) . \tag{B.9}$$

Truncating the sum in Eq. (B.3) after the first term (higher orders would lead to nonlinear dependencies of W^{ab} on W_{in}^{ab}, which are of higher order in the density) and expanding the "in"-WIGNER function around x, we obtain

$$W^{ab}(x, k) = \frac{1}{\eta^2} \int \mathrm{d}^4u \sum_{\sigma,\sigma'} U_d\left(k - \frac{u}{2}, \sigma\right) \overline{U}_c\left(k + \frac{u}{2}, \sigma'\right) \overline{U}^b\left(k - \frac{u}{2}, \sigma\right) U^a\left(k + \frac{u}{2}, \sigma'\right) \times \left\{ W_{\text{in}}^{cd}(x, k)\delta^{(4)}(u) - i\hbar \left[\partial_u^\rho \delta^{(4)}(u)\right] \partial_\rho W_{\text{in}}^{cd}(x, k) \right\} . \tag{B.10}$$

At this point, we differentiate between particles of different spin.

Spin 0

In the case of scalar particles, we have $\eta \equiv 1$ and $U \equiv 1$, such that the second contribution in Eq. (B.10) becomes a vanishing boundary term. Then we simply find

$$W(x,k) = W_{\text{in}}(x,k) \; , \tag{B.11}$$

allowing us to identify the WIGNER function and its "in"-counterpart inside the collision integral.

Spin 1/2

For DIRAC particles, $\eta \equiv 2m$ and $U \equiv u$, where u is the usual basis spinor fulfilling the orthogonality and completeness relations (4.56). Then, the WIGNER function becomes

$$\begin{aligned} W_{\alpha\beta}(x,k) &= \Lambda^+_{\alpha\gamma}(k) W_{\text{in},\gamma\delta}(x,k)\Lambda^+_{\delta\beta}(k) \\ &- i\hbar \int \mathrm{d}^4 u \Lambda^+_{\alpha\gamma}\left(k+\frac{u}{2}\right)\Lambda^+_{\delta\beta}\left(k-\frac{u}{2}\right)\left[\partial^\rho_u \delta^{(4)}(u)\right]\partial_\rho W^{\gamma\delta}_{\text{in}}(x,k) \; . \end{aligned} \tag{B.12}$$

We then integrate by parts in the second term and employ

$$\partial^\rho_u \Lambda^+\left(k+\frac{u}{2}\right) = \frac{1}{4m}\gamma^\rho \; , \tag{B.13}$$

such that

$$W_{\alpha\beta}(x,k) = \Lambda^+_{\alpha\gamma}(k) W_{\text{in},\gamma\delta}(x,k)\Lambda^+_{\delta\beta}(k) + \frac{i\hbar}{4m}\left[\gamma^\rho_{\alpha\gamma}\Lambda^+_{\delta\beta}(k) - \Lambda^+_{\alpha\gamma}(k)\gamma^\rho_{\delta\beta}\right]\partial_\rho W_{\text{in},\gamma\delta}(x,k) \; . \tag{B.14}$$

Subsequently, we use that, by virtue of Eqs. (3.79) and (3.80), the structure of the "in"-WIGNER function in DIRAC space is given by

$$W_{\text{in}}(x,k) = \frac{1}{2}\Lambda^+(k)\left[\mathcal{F}_{\text{in}}(x,k) + \gamma_5\gamma\cdot\mathcal{A}_{\text{in}}(x,k)\right] + \frac{\hbar}{8m^2}\sigma^{\mu\nu}k_\nu\partial_\mu\mathcal{F}_{\text{in}}(x,k) \; . \tag{B.15}$$

Employing that, because $\mathcal{A}^\mu_{\text{in}}$ is orthogonal to the four-momentum, $\Lambda^+(k)$ and $\gamma_5\gamma\cdot\mathcal{A}_{\text{in}}$ commute, we find (neglecting terms of second order in $\hbar$)

$$\begin{aligned} W_{\alpha\beta}(x,k) &= \frac{1}{2}\Lambda^+(k)\left[\mathcal{F}_{\text{in}}(x,k) + \gamma_5\gamma\cdot\mathcal{A}_{\text{in}}(x,k)\right] + \frac{i\hbar}{8m}\left[\gamma^\mu_{\alpha\gamma}\Lambda^+_{\gamma\beta}(k) - \Lambda^+_{\alpha\delta}(k)\gamma^\mu_{\delta\beta}\right]\partial_\mu\mathcal{F}_{\text{in}}(x,k) \\ &= \frac{1}{2}\Lambda^+(k)\left[\mathcal{F}_{\text{in}}(x,k) + \gamma_5\gamma\cdot\mathcal{A}_{\text{in}}(x,k)\right] + \frac{\hbar}{8m^2}\sigma^{\mu\nu}k_\nu\partial_\mu\mathcal{F}_{\text{in}}(x,k) \\ &\equiv W_{\text{in},\alpha\beta}(x,k) \; , \end{aligned} \tag{B.16}$$

where we used Eq. (4.78). As expected, W and W_{in} are equivalent up to terms of higher order in the density and $\hbar$.

Spin 1

In the case of PROCA fields, we have $\eta \equiv -1$ and $U \equiv \epsilon$, where the polarization vectors ϵ fulfill the orthogonality and completeness relations (4.103). Using these, Eq. (B.10) becomes

$$W^{\mu\nu} = K^\mu_\alpha K^\nu_\beta W^{\alpha\beta}_{\text{in}}(x,k) - i\hbar\int \mathrm{d}^4 u \left(K-\frac{U}{2}\right)^\mu_\alpha \left(K+\frac{U}{2}\right)^\nu_\beta \left[\partial^\rho_u\delta^{(4)}(u)\right]\partial_\rho W^{\alpha\beta}_{\text{in}}(x,k) \; , \tag{B.17}$$

where we defined

$$\left(K\pm\frac{U}{2}\right)^{\mu\nu} := g^{\mu\nu} - (k\pm u)^{-2}\left(k\pm\frac{u}{2}\right)^\mu\left(k\pm\frac{u}{2}\right)^\nu \; . \tag{B.18}$$

Integrating by parts and using

$$\partial_u^\rho \left(K \pm \frac{U}{2}\right)^{\mu\nu} \bigg|_{u=0} = \mp k^{-2} \left(\frac{1}{2} g^{\rho(\mu} k^{\nu)} - \frac{k^\rho k^\mu k^\nu}{k^2}\right) \tag{B.19}$$

as well as the fact that $k \cdot \partial W_{\text{in}}^{\mu\nu}(x,k) = 0$, we can evaluate the second term in Eq. (B.17) to get

$$\begin{aligned} W^{\mu\nu}(x,k) &= K_\alpha^\mu K_\beta^\nu W_{\text{in}}^{\alpha\beta}(x,k) + i\hbar \partial_u \left[\left(K - \frac{U}{2}\right)_\alpha^\mu \left(K + \frac{U}{2}\right)_\beta^\nu\right]_{u=0} \partial_\rho W_{\text{in}}^{\alpha\beta}(x,k) \\ &= K_\alpha^\mu K_\beta^\nu W_{\text{in}}^{\alpha\beta}(x,k) + \frac{i\hbar}{2} k^{-2} \left(\partial^{(\mu} k_{\alpha)} K_\beta^\nu - \partial^{(\nu} k_{\beta)} K_\alpha^\mu\right) W_{\text{in}}^{\alpha\beta}(x,k) \, . \end{aligned}$$

Remembering that $k_\mu W_{S,\text{in}}^{\mu\nu} \sim \mathcal{O}(\hbar^2)$ and $k_\mu W_{A,\text{in}}^{\mu\nu} = (i\hbar/2)\partial^\nu f_{K,\text{in}} + \mathcal{O}(\hbar^2)$, cf. Eqs. (3.159), we may rewrite this expression up to first order in $\hbar$ as

$$W^{\mu\nu}(x,k) = K_\alpha^\mu K_\beta^\nu W_{\text{in}}^{\alpha\beta}(x,k) + \frac{i\hbar}{2} \frac{k^{[\mu}}{k^2} \partial^{\nu]} f_{K,\text{in}}(x,k) \equiv W_{\text{in}}^{\mu\nu}(x,k) \, . \tag{B.20}$$

Manifestly, we may again identify the Wigner function with its "in"-counterpart to this order.

B.2 KB: Self-energy and quantum statistics

B.2.1 Approximating the self-energy

In this part of the appendix, we motivate the form of the diagrams considered in Fig. 5.2, and show how to compute them.

Notes on deriving the self-energy

The self-energy appearing in the KB equations has to be approximated in some way, as computing it exactly is usually impossible. As has been shown by Baym in Ref. [164], an approximation preserves the macroscopic conservation laws if it is "Φ-derivable", i.e., if there exists a functional Φ such that[1]

$$\Sigma_{ab}^{AB}(x_1,x_2) = \frac{\delta \Phi[G]}{\delta G^{AB,ab}(x_1,x_2)} \, . \tag{B.21}$$

The Φ-functional that we consider is given by the set of closed two-particle irreducible (2PI) diagrams in Fig. B.1, where the plus and minus signs are for fermions and bosons, respectively. The combinatorial prefactors become unity when taking the derivative with respect to the Green's function. Note that, as mentioned before, we do not consider the Hartree-Fock contributions, which renormalize the energy and momentum of the particles and provide a self-consistent Vlasov-type term in the kinetic equation. In order to include these contributions, we would have to consider an additional "double-bubble" like diagram in the Φ-functional (together with the corresponding exchange term due to boson or fermion exchange symmetry) [135].

[1] In Ref. [165] it has been shown that the conservation laws are even obeyed exactly, i.e., not only to the order in the coupling that defines the truncation of the self-energy. This also holds when the ($\hbar$-)gradient approximation is implemented consistently.

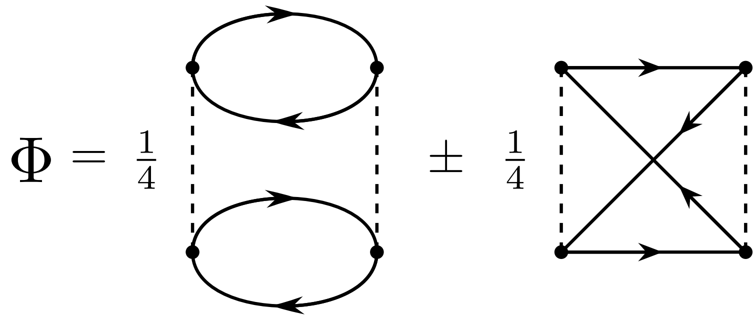

Figure B.1: The Φ-functional consisting of closed 2PI diagrams. Note that the HARTREE-FOCK contributions are omitted.

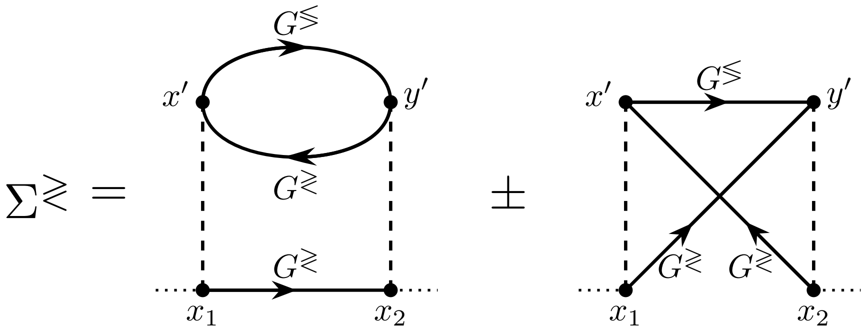

Figure B.2: Greater and lesser self-energies in position space. The dots connected by dashed lines symbolize the tensors V.

Computing the diagrams

The self-energy diagrams shown in Fig. B.2 are obtained from taking the variational derivative of the Φ-functional. Their diagrammatic form is easily constructed by removing one internal line from the closed diagrams in Fig. B.1. Then, they read

$$\Sigma^{\gtrless}_{ab}(x_1,x_2) = \int \mathrm{d}^4x'\,\mathrm{d}^4y' V_{aa'a_1a_2}(x_1,x')V_{b_2b_1b'b}(y',x_2)\left[G^{\gtrless}_{a_1b_1}(x_1,x_2)G^{\gtrless}_{a_2b_2}(x',y')G^{\lessgtr}_{a'b'}(y',x')\right.$$
$$\left.\pm G^{\gtrless}_{a_1b_2}(x_1,y')G^{\gtrless}_{a_2b_1}(x',x_2)G^{\lessgtr}_{b'a'}(y',x')\right] \,. \tag{B.22}$$

Here, the quantities V, which are fourth-rank tensors in the internal space of the fields, are symbolized by the dashed lines. Furthermore, the plus and minus signs belong to bosons and fermions, respectively. For simplicity, we take the quantities V to be local, i.e., $V_{aa'a_1a_2}(x,y) = V_{aa'a_1a_2}\delta^{(4)}(x-y)$, where $V_{aa'a_1a_2}$ is a constant. Then, the self-energies are given by

$$\Sigma^{\gtrless}_{ab}(x_1,x_2) = V_{aa'a_1a_2}V_{b_2b_1b'b}\left[G^{\gtrless}_{a_1b_1}(x_1,x_2)G^{\gtrless}_{a_2b_2}(x_1,x_2)G^{\lessgtr}_{b'a'}(x_2,x_1)\right.$$
$$\left.\pm G^{\gtrless}_{a_1b_2}(x_1,x_2)G^{\gtrless}_{a_2b_1}(x_1,x_2)G^{\lessgtr}_{b'a'}(x_2,x_1)\right] \,. \tag{B.23}$$

After inserting the WIGNER transform

$$G^{\gtrless}_{ab}(x_1,x_2) = \int \frac{\mathrm{d}^4p}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}p\cdot(x_1-x_2)} G^{\gtrless}_{ab}\left(\frac{x_1+x_2}{2},p\right) \,, \tag{B.24}$$

the self-energies become

$$\Sigma^{\gtrless}_{ab}(x_1,x_2) = \int \frac{\mathrm{d}^4k_1}{(2\pi\hbar)^4}\frac{\mathrm{d}^4k_2}{(2\pi\hbar)^4}\frac{\mathrm{d}^4k'}{(2\pi\hbar)^4} V_{aa'a_1a_2}V_{b_2b_1b'b}e^{-\frac{i}{\hbar}(k_1+k_2-k')\cdot(x_1-x_2)}$$
$$\times\Bigg[G^{\gtrless}_{a_1b_1}\left(\frac{x_1+x_2}{2},k_1\right)G^{\gtrless}_{a_2b_2}\left(\frac{x_1+x_2}{2},k_2\right)G^{\lessgtr}_{b'a'}\left(\frac{x_1+x_2}{2},k'\right)$$
$$\pm G^{\gtrless}_{a_1b_2}\left(\frac{x_1+x_2}{2},k_1\right)G^{\gtrless}_{a_2b_1}\left(\frac{x_1+x_2}{2},k_2\right)G^{\lessgtr}_{b'a'}\left(\frac{x_1+x_2}{2},k'\right)\Bigg] . \tag{B.25}$$

The Wigner transform of the self-energy then reads

$$\begin{aligned}\Sigma^{\gtrless}_{ab}(x,k) &= \int \mathrm{d}^4v e^{-\frac{i}{\hbar}k\cdot v}\Sigma^{\gtrless}_{ab}\left(x-\frac{v}{2},x+\frac{v}{2}\right)\\
&= \int \frac{\mathrm{d}^4k_1}{(2\pi\hbar)^4}\frac{\mathrm{d}^4k_2}{(2\pi\hbar)^4}\frac{\mathrm{d}^4k'}{(2\pi\hbar)^4}(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)V_{aa'a_1a_2}V_{b_2b_1b'b}\\
&\quad\times\left[G^{\gtrless}_{a_1b_1}(x,k_1)G^{\gtrless}_{a_2b_2}(x,k_2)G^{\lessgtr}_{b'a'}(x,k')\pm G^{\gtrless}_{a_1b_2}(x,k_1)G^{\gtrless}_{a_2b_1}(x,k_2)G^{\lessgtr}_{b'a'}(x,k')\right]\\
&= \frac{1}{2}\int \frac{\mathrm{d}^4k_1}{(2\pi\hbar)^4}\frac{\mathrm{d}^4k_2}{(2\pi\hbar)^4}\frac{\mathrm{d}^4k'}{(2\pi\hbar)^4}(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\\
&\quad\times G^{\gtrless}_{a_1b_1}(x,k_1)G^{\gtrless}_{a_2b_2}(x,k_2)G^{\lessgtr}_{b'a'}(x,k')\left(V_{aa'a_1a_2}\pm V_{aa'a_2a_1}\right)\left(V_{b_2b_1b'b}\pm V_{b_1b_2b'b}\right) .\end{aligned} \tag{B.26}$$

Next we define

$$V_{aa'a_1a_2}\pm V_{aa'a_2a_1} =: \frac{1}{\lambda^3}M_{aa'a_1a_2} , \tag{B.27}$$

where the factors of λ are necessary to be consistent with the earlier definitions.[2] Note that we have by construction the symmetries $M_{aa'a_1a_2} = \pm M_{a'aa_1a_2} = \pm M_{aa'a_2a_1}$. Then , we find

$$\begin{aligned}\Sigma^{\gtrless}_{ab}(x,k) = \frac{1}{2\lambda^6}\int &\frac{\mathrm{d}^4k_1}{(2\pi\hbar)^4}\frac{\mathrm{d}^4k_2}{(2\pi\hbar)^4}\frac{\mathrm{d}^4k'}{(2\pi\hbar)^4}(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\\
&\times G^{\gtrless}_{a_1b_1}(x,k_1)G^{\gtrless}_{a_2b_2}(x,k_2)G^{\lessgtr}_{b'a'}(x,k')M_{aa'a_1a_2}M_{b_1b_2bb'} .\end{aligned} \tag{B.28}$$

Thus, we have obtained Eq. (5.21), which represents the diagrams in Fig. 5.2 in phase space.

B.2.2 Quantum statistics

Here we want to prove that the distribution functions $\widetilde{f}$ introduced in Eqs. (5.32), (5.47) and (5.96) indeed represent the expected Pauli-blocking and Bose-enhancement factors.

Starting from Eqs. (5.1), we find that

$$G^{>}_{ab}(x_1,x_2) - G^{<}_{ab}(x_1,x_2) = \begin{cases}\left\langle\left[\widehat{\varphi}_a(x_1),\widehat{\overline{\varphi}}_b(x_2)\right]\right\rangle & \text{for bosons ,}\\ \left\langle\left\{\widehat{\varphi}_a(x_1),\widehat{\overline{\varphi}}_b(x_2)\right\}\right\rangle & \text{for fermions .}\end{cases} \tag{B.29}$$

Then, we remember the expression for the (anti)commutators of massive fields of different spins [166],

$$\left[\widehat{\phi}(x_1),\widehat{\phi}^\dagger(x_2)\right] = i\hbar\Delta(x_1-x_2) , \tag{B.30a}$$
$$\left\{\widehat{\psi}_\alpha(x_1),\widehat{\overline{\psi}}_\beta(x_2)\right\} = (i\hbar\partial\!\!\!/ + m)_{\alpha\beta}i\Delta(x_1-x_2) , \tag{B.30b}$$
$$\left[\widehat{V}^\mu(x_1),\widehat{V}^{\dagger\nu}(x_2)\right] = -i\hbar\left(g^{\mu\nu}+\frac{\hbar^2}{m^2}\partial^\mu\partial^\nu\right)\Delta(x_1-x_2) , \tag{B.30c}$$

[2] Actually there should be a factor of $1/\lambda^4$ since the vertices M are contracted with four fields, but we factored out a contribution of $1/\lambda^2$ in the Kadanoff-Baym equations for convenience.

where

$$i\Delta(x_1 - x_2) \coloneqq \int \frac{\mathrm{d}^4 p}{(2\pi\hbar)^3} \mathrm{sgn}(p_0)\delta(p^2 - m^2) e^{-\frac{i}{\hbar} p\cdot(x_1 - x_2)} \tag{B.31}$$

denotes the invariant Pauli-Jordan function. Computing the Wigner transform of Eq. (B.29) yields for the different fields

$$G^{>}(x,k) - G^{<}(x,k) = 2\pi\hbar^2 \delta(k^2 - m^2) \;, \tag{B.32a}$$

$$G^{>}_{\alpha\beta}(x,k) - G^{<}_{\alpha\beta}(x,k) = 4m\pi\hbar\Lambda^{+}(k)\delta(k^2 - m^2) \;, \tag{B.32b}$$

$$G^{>\mu\nu}(x,k) - G^{<\mu\nu}(x,k) = -2\pi\hbar^2 K^{\mu\nu}\delta(k^2 - m^2) \;, \tag{B.32c}$$

where we assumed that $k_0 > 0$. Then, we express the Wigner functions as

$$G^{<}(x,k) = 2\pi\hbar^2\delta(k^2 - m^2) f(x,k) \;, \tag{B.33a}$$

$$G^{>}(x,k) = 2\pi\hbar^2\delta(k^2 - m^2) \widetilde{f}(x,k) \;, \tag{B.33b}$$

$$G^{<}_{\alpha\beta}(x,k) = -4m\pi\hbar\delta(k^2 - m^2) \int \mathrm{d}S(k) h_{\alpha\beta}(k,\mathfrak{s}) f(x,k,\mathfrak{s}) \;, \tag{B.33c}$$

$$G^{>}_{\alpha\beta}(x,k) = 4m\pi\hbar\delta(k^2 - m^2) \int \mathrm{d}S(k) h_{\alpha\beta}(k,\mathfrak{s}) \widetilde{f}(x,k,\mathfrak{s}) \;, \tag{B.33d}$$

$$G^{<\mu\nu}(x,k) = -2m\pi\hbar^2\delta(k^2 - m^2) \int \mathrm{d}S(k) h^{\nu\mu}(k,\mathfrak{s}) f(x,k,\mathfrak{s}) \;, \tag{B.33e}$$

$$G^{>\mu\nu}(x,k) = -2m\pi\hbar^2\delta(k^2 - m^2) \int \mathrm{d}S(k) h^{\nu\mu}(k,\mathfrak{s}) \widetilde{f}(x,k,\mathfrak{s}) \;, \tag{B.33f}$$

where we could neglect the gradient and off-shell contributions, since they fall outside our employed truncation as all Wigner functions treated here appear inside the collision integrals. Finally, we insert spurious spin-space integrals on the right-hand sides of Eqs. (B.32b) and (B.32c), obtaining

$$2\pi\hbar^2\delta(k^2 - m^2) \left[\widetilde{f}(x,k) - f(x,k) - 1\right] = 0 \;, \tag{B.34a}$$

$$4m\pi\hbar\delta(k^2 - m^2) \int \mathrm{d}S(k) h_{\alpha\beta}(k,\mathfrak{s}) \left[\widetilde{f}(x,k,s) + f(x,k,\mathfrak{s}) - 1\right] = 0 \;, \tag{B.34b}$$

$$2\pi\hbar^2\delta(k^2 - m^2) \int \mathrm{d}S(k) h^{\mu\nu}(k,\mathfrak{s}) \left[\widetilde{f}(x,k,s) - f(x,k,\mathfrak{s}) - 1\right] = 0 \;. \tag{B.34c}$$

Thus, as expected, we can conclude that, to first order in $\hbar$, we may identify

$$\widetilde{f}(x,k) = 1 + f(x,k) \qquad \text{for Klein-Gordon fields} \;, \tag{B.35a}$$

$$\widetilde{f}(x,k,\mathfrak{s}) = 1 - f(x,k,\mathfrak{s}) \qquad \text{for Dirac fields} \;, \tag{B.35b}$$

$$\widetilde{f}(x,k,\mathfrak{s}) = 1 + f(x,k,\mathfrak{s}) \qquad \text{for Proca fields} \;. \tag{B.35c}$$

B.3 The antisymmetric part of the energy-momentum tensor

In this section, we will explicitly compute the antisymmetric part of the GLW energy-momentum tensors for spin-1/2 and spin-1 particles, in order to confirm that they indeed take the form (6.54) which was obtained by postulating the conservation of the total angular momentum $J^{\mu\nu} = \sigma\hbar\Sigma^{\mu\nu}_{\mathfrak{s}} + \Delta^{[\mu}k^{\nu]}$. We will carry out this computation in the KB formalism, since it is more straightforward to implement and retains quantum-statistical effects.

Dirac fields

From Eq. (3.123), we have

$$T^{[\mu\nu]}_{D} = -\frac{\hbar}{m} \int \frac{\mathrm{d}^4 k}{(2\pi\hbar)^4} \mathcal{D}^{[\mu}_{\mathcal{V}} k^{\nu]} \;. \tag{B.36}$$

Using the definition of $\mathcal{D}_\mathcal{V}$ (3.78a) and connecting it with the collision integral in the KB approach (5.36), we obtain

$$\mathcal{D}^\mu_\mathcal{V}(x,k) = \frac{1}{2}\text{Im Tr}\left[\gamma^\mu\left(\Sigma^> G^< - \Sigma^< G^>\right)\right] , \tag{B.37}$$

where we omitted the Poisson-bracket terms. Taking only the quasiclassical contributions (5.52) to the self-energies and writing the Wigner functions in extended phase space according to Eqs. (5.47), we find

$$\mathcal{D}^\mu_\mathcal{V}(x,k) = -4m\pi\hbar\delta(k^2-m^2)\frac{1}{2}\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,\mathrm{d}S(k)\,\mathrm{d}\bar{S}(k)(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2) \\ \times\,\mathcal{U}^\mu\left(f_1 f_2 \widetilde{f}'\widetilde{f} - \widetilde{f}_1\widetilde{f}_2 f' f\right) , \tag{B.38}$$

where we defined[3]

$$\mathcal{U}^\mu := \frac{m^3}{4}\text{Im}\left[M_{\alpha\alpha'\alpha_1\alpha_2}M_{\beta_1\beta_2\beta\beta'}h_{\alpha_1\beta_1}(k_1,\mathfrak{s}_1)h_{\alpha_2\beta_2}(k_2,\mathfrak{s}_2)h_{\beta'\alpha'}(k',\mathfrak{s}')h_{\beta\delta}(k,\mathfrak{s})\gamma^\mu_{\delta\alpha}\right] \\ = \frac{1}{\hbar}\mathcal{W}^{(1/2)}\Delta^\mu . \tag{B.39}$$

Then, after using the weak equivalence principle, the antisymmetric part of the energy-momentum tensor becomes

$$T^{[\mu\nu]}_D = \frac{1}{2}\int[\mathrm{d}\Gamma](2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\widetilde{\mathcal{W}}^{(1/2)}\Delta^{[\mu}k^{\nu]}\left(f_1 f_2\widetilde{f}'\widetilde{f} - \widetilde{f}_1\widetilde{f}_2 f' f\right) , \tag{B.40}$$

in agreement with Eq. (6.54).

Proca fields

In the case of vector fields, we start from Eq. (3.188),

$$T^{[\mu\nu]}_P = \frac{\hbar}{m}\int\frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\frac{k}{2m}\left(\mathcal{C}^{[\mu}_A + \mathcal{D}^{[\mu}_S\right)k^{\nu]} . \tag{B.41}$$

Using the definitions (3.152) and (3.157), we can express the relevant collision terms as

$$\frac{k}{2}\left(\mathcal{C}^\mu_A + \mathcal{D}^\mu_S\right) = -\text{Re}\,C^{\mu\nu}k_\nu = -\frac{1}{\hbar}\text{Im}\left[k_\nu\left(\Sigma^{<\nu\alpha}G^{>\mu}_\alpha - \Sigma^{>\nu\alpha}G^{<\mu}_\alpha\right)\right] , \tag{B.42}$$

where we employed Eq. (5.82) in the last equality and neglected the Poisson-bracket terms again. With the quasiclassical contributions (5.101), we find Eq. (B.42) to be

$$\frac{k}{2}\left(\mathcal{C}^\mu_A + \mathcal{D}^\mu_S\right)(x,k) = 4\pi\hbar\delta(k^2-m^2)\frac{1}{2}\int\mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma'\,\mathrm{d}S(k)\,\mathrm{d}\bar{S}(k)(2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2) \\ \times\,\mathscr{U}^\mu\left(f_1 f_2\widetilde{f}'\widetilde{f} - \widetilde{f}_1\widetilde{f}_2 f' f\right) , \tag{B.43}$$

where we introduced[4]

$$\mathscr{U}^\mu := -\frac{1}{16}\frac{1}{3}\text{Im}\left[M^{\nu\mu'\mu_1\mu_2}M^{\nu_1\nu_2\alpha\nu'}h_{\nu_1\mu_1}(k_1,\mathfrak{s}_1)h_{\nu_2\mu_2}(k_2,\mathfrak{s}_2)h_{\mu'\nu'}(k',\mathfrak{s}')h^\mu{}_\alpha(k,\mathfrak{s})k_\nu\right] \\ = \frac{m^2}{\hbar}\mathcal{W}^{(1)}\Delta^\mu . \tag{B.44}$$

Using the weak equivalence principle, we thus find

$$T^{[\mu\nu]}_P = \frac{1}{2}\int[\mathrm{d}\Gamma](2\pi\hbar)^4\delta^{(4)}(k+k'-k_1-k_2)\widetilde{\mathcal{W}}^{(1)}\Delta^{[\mu}k^{\nu]}\left(f_1 f_2\widetilde{f}'\widetilde{f} - \widetilde{f}_1\widetilde{f}_2 f' f\right) , \tag{B.45}$$

as expected from Eq. (6.54).

[3] Note that there is an additional factor of 1/2 due to the spurious $\mathrm{d}\bar{S}(k)$-integral, which is necessary since $\mathcal{W}^{(1/2)}$ depends on $\bar{\mathfrak{s}}$.

[4] We replaced H by h in the definition of Δ since the components of spin-rank two will not contribute in the final expression.

Appendix C

Polarization observables in kinetic theory

To give a clearer meaning to the different components of the WIGNER functions introduced in the main text, in this appendix we will connect them to the observables related to polarization that are measured in experiment [5, 10, 16, 18]. This discussion largely follows the one presented in Ref. [167] for spin-1/2 particles.

The spin-density matrix of a particle is defined as

$$\varrho_{\sigma\sigma'}(k) \coloneqq \frac{\langle \widehat{a}^\dagger_{\sigma'}(k)\widehat{a}_\sigma(k)\rangle}{\sum_{\sigma''}\langle \widehat{a}^\dagger_{\sigma''}(k)\widehat{a}_{\sigma''}(k)\rangle} \; . \tag{C.1}$$

The goal is to relate the WIGNER function [i.e., the normal-ordered expectation value of Eq. (3.31)] to the averages over creation and annihilation operators appearing in Eq. (C.1). Expressing the fields in terms of creation and annihilation operators

$$\widehat{\varphi}^a(x) \coloneqq \frac{\lambda}{2}\sum_\sigma \int \mathrm{d}K \left[e^{-\frac{i}{\hbar}k\cdot x} U^a(k,\sigma)\widehat{a}_\sigma(k) + e^{\frac{i}{\hbar}k\cdot x} V^a(k,\sigma)\widehat{b}^\dagger_\sigma(k)\right] \tag{C.2}$$

and inserting them into the WIGNER function, we obtain $W = W_+ + W_- + W_S$, where $W_\pm$ denote the particle and antiparticle contributions, respectively (i.e., their associated momenta are timelike with $k^0 > 0$ or $k^0 < 0$), while W_S denotes the WIGNER function whose momentum is spacelike. These three quantities read explicitly

$$\begin{aligned} W_+^{ab}(x,k) &= \frac{\kappa\lambda^2}{4}\sum_{\sigma,\sigma'}\int \mathrm{d}P \int \mathrm{d}P' (2\pi\hbar)^4 \delta^{(4)}\left(k - \frac{p+p'}{2}\right) \\ &\quad \times e^{\frac{i}{\hbar}(p-p')\cdot x}\overline{U}^b(p,\sigma)U^a(p',\sigma')\langle \widehat{a}^\dagger_\sigma(p)\widehat{a}_{\sigma'}(p')\rangle \; , \end{aligned} \tag{C.3a}$$

$$\begin{aligned} W_-^{ab}(x,k) &= \pm\frac{\kappa\lambda^2}{4}\sum_{\sigma,\sigma'}\int \mathrm{d}P \int \mathrm{d}P' (2\pi\hbar)^4 \delta^{(4)}\left(k + \frac{p+p'}{2}\right) \\ &\quad \times e^{\frac{i}{\hbar}(p-p')\cdot x}\overline{V}^b(p,\sigma)V^a(p',\sigma')\langle \widehat{b}^\dagger_{\sigma'}(p')\widehat{b}_\sigma(p)\rangle \; , \end{aligned} \tag{C.3b}$$

$$\begin{aligned} W_S^{ab}(x,k) &= \frac{\kappa\lambda^2}{4}\sum_{\sigma,\sigma'}\int \mathrm{d}P \int \mathrm{d}P' (2\pi\hbar)^4 \delta^{(4)}\left(k - \frac{p-p'}{2}\right) \\ &\quad \times \left[e^{\frac{i}{\hbar}(p+p')\cdot x}\overline{U}^b(p,\sigma)V^a(p',\sigma')\langle \widehat{a}^\dagger_\sigma(p)\widehat{b}^\dagger_{\sigma'}(p')\rangle \right. \\ &\quad \left. + e^{-\frac{i}{\hbar}(p+p')\cdot x}\overline{V}^b(p,\sigma)U^a(p',\sigma')\langle \widehat{b}_\sigma(p)\widehat{a}_{\sigma'}(p')\rangle \right] \; , \end{aligned} \tag{C.3c}$$

where the positive and negative signs in W_- correspond to bosons and fermions, respectively. The integral of the positive-energy WIGNER function over a hypersurface Σ reads

$$\int \mathrm{d}\Sigma_\alpha k^\alpha W_+^{ab}(x,k) \equiv k^0 \int \mathrm{d}^3x\, W_+^{ab}(x,k)$$
$$= \frac{\kappa\lambda^2}{2} \sum_{\sigma,\sigma'} (2\pi\hbar)\delta(k^2-m^2)\Theta(k^0)\overline{U}^b(k,\sigma)U^a(k,\sigma')\langle \widehat{a}_\sigma^\dagger(k)\widehat{a}_{\sigma'}(k)\rangle \ . \tag{C.4}$$

Making use of the orthogonality relations of the polarization vectors (4.9), we find the sought-after relation

$$(2\pi\hbar)\delta(k^2-m^2)\Theta(k^0)\langle \widehat{a}_\sigma^\dagger(k)\widehat{a}_{\sigma'}(k)\rangle = \frac{2}{\kappa\lambda^2\eta^2}\int \mathrm{d}\Sigma_\alpha k^\alpha \overline{U}_a(k,\sigma')W_+^{ab}(x,k)U_b(k,\sigma) \ , \tag{C.5}$$

which lets us express the spin-density matrix of the particles as

$$\varrho_{\sigma\sigma'}(k) = \frac{\int \mathrm{d}\Sigma_\alpha k^\alpha \overline{U}_a(k,\sigma)W_+^{ab}(x,k)U_b(k,\sigma')}{\sum_{\sigma''}\int \mathrm{d}\Sigma_\alpha k^\alpha \overline{U}_a(k,\sigma'')W_+^{ab}(x,k)U_b(k,\sigma'')} \ . \tag{C.6}$$

Note that a similar relation holds also for the antiparticles.

In the next step we will derive expressions for the vector and tensor polarization, which are defined as (for a particle of spin S) [15]

$$S^\mu(k) \coloneqq \mathrm{Tr}\left[\widehat{S}^\mu \widehat{\varrho}(k)\right] \ , \tag{C.7}$$

$$\Theta^{\mu\nu}(k) \coloneqq \frac{1}{2}\sqrt{\frac{3}{2}}\mathrm{Tr}\left\{\left[\widehat{S}^{(\mu}\widehat{S}^{\nu)} + \frac{2S(S+1)}{3}K^{\mu\nu}\right]\widehat{\varrho}(k)\right\} \ , \tag{C.8}$$

where

$$\widehat{S}^\mu \coloneqq -\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta}\widehat{J}_{\nu\alpha}\widehat{P}_\beta \tag{C.9}$$

denotes the PAULI-LUBANSKI operator divided by the particle mass [140, 167]. Here, $\widehat{J}^{\mu\nu}$ is the generator of LORENTZ transformations, while $\widehat{P}^\mu$ generates spacetime translations. We can represent the matrix elements of the operator $\widehat{S}^\mu$ as

$$\langle k,\sigma|\,\widehat{S}^\mu\,|k,\sigma'\rangle = -\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta}k_\nu D^S([k])^{-1}D^S(J_{\alpha\beta})D^S([k]) \ , \tag{C.10}$$

where $D^S(J^{\mu\nu})$ and $D^S([k])$ are the spin-S representations of the total angular-momentum operator and the standard LORENTZ boost to the four-momentum k^μ, respectively. From this relation we can infer

$$S^\mu(k) = -\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta}k_\nu \mathrm{Tr}\left[D^S([k])^{-1}D^S(J_{\alpha\beta})D^S([k])\varrho(k)\right] \ , \tag{C.11}$$

$$\Theta^{\mu\nu}(k) = \frac{1}{2}\sqrt{\frac{3}{2}}\Bigg\{-\frac{1}{4}\epsilon^{\alpha\beta\gamma(\mu}\epsilon^{\nu)\rho\sigma\lambda}\frac{k_\alpha k_\lambda}{m^2}\mathrm{Tr}\left[D^S([k])^{-1}D^S(J_{\beta\gamma})D^S(J_{\rho\sigma})D^S([k])\varrho(k)\right]$$
$$+\frac{2S(S+1)}{3}K^{\mu\nu}\Bigg\} \ . \tag{C.12}$$

At this point, we treat the cases of different nontrivial representations of the LORENTZ group[1] separately.

DIRAC fields

In the case of DIRAC fermions, we have to consider the $(1/2,0)\oplus(0,1/2)$-representation, where (making the DIRAC indices $i,j,\cdots$ and the spin indices $r,s,\cdots$ explicit)

$$D^S(J_{\alpha\beta})_{ij} = \frac{i}{4}[\gamma_\alpha,\gamma_\beta]_{ij} \ , \qquad D^S([k])_{r,i} = \frac{1}{\sqrt{2m}}u_{r,i}(k) \ , \tag{C.13}$$

[1] For scalar particles, which transform in the $(0,0)$-representation, both the vector and tensor polarization vanish.

with the basis spinor u. Then, upon using the completeness relation of the Dirac spinors (4.56b), we find for the vector polarization

$$S^\mu(k) = -\frac{1}{4m^2}\epsilon^{\mu\nu\alpha\beta}k_\nu \frac{i}{4}\frac{\int \mathrm{d}\Sigma_\lambda k^\lambda \mathrm{Tr}\left\{(\not{k}+m)[\gamma_\alpha,\gamma_\beta](\not{k}+m)W(x,k)\right\}}{\int \mathrm{d}\Sigma_\lambda k^\lambda \mathrm{Tr}\left[(\not{k}+m)W(x,k)\right]} \;. \tag{C.14}$$

Decomposing the Wigner function according to the Clifford algebra (3.73), evaluating the traces and making use of Eqs. (3.79) (with vanishing right-hand sides) yields

$$S^\mu(k) = \frac{1}{2}\frac{\int \mathrm{d}\Sigma_\lambda k^\lambda \mathcal{A}^\mu(x,k)}{\int \mathrm{d}\Sigma_\lambda k^\lambda \mathcal{F}(x,k)} \;. \tag{C.15}$$

When translating this expression into extended phase space, we find

$$S^\mu(k) = \frac{1}{2}\frac{1}{N(k)}\int \mathrm{d}\Sigma_\gamma k^\gamma \int \mathrm{d}S(k)\mathfrak{s}^\mu f(x,k,\mathfrak{s}) \;, \tag{C.16}$$

where we defined

$$N(k) \coloneqq \int \mathrm{d}\Sigma_\gamma k^\gamma \mathcal{F}(x,k) \equiv \int \mathrm{d}\Sigma_\gamma k^\gamma \int \mathrm{d}S(k) f(x,k,\mathfrak{s}) \;. \tag{C.17}$$

The tensor polarization on the other hand is given by

$$\Theta^{\mu\nu}(k) = \frac{1}{4}\sqrt{\frac{3}{2}}\left\{\epsilon^{\alpha\beta\gamma(\mu}\epsilon^{\nu)\rho\sigma\lambda}\frac{k_\alpha k_\lambda}{64m^3}\frac{\int \mathrm{d}\Sigma_\xi k^\xi \mathrm{Tr}\left\{(\not{k}+m)[\gamma_\beta,\gamma_\gamma][\gamma_\rho,\gamma_\sigma](\not{k}+m)W(x,k)\right\}}{\int \mathrm{d}\Sigma_\xi k^\xi \mathrm{Tr}\left[(\not{k}+m)W(x,k)\right]} + K^{\mu\nu}\right\} \;, \tag{C.18}$$

and vanishes upon performing the traces, as expected.

Proca fields

For massive spin-1 particles, we work in the $(1/2, 1/2)$ representation of the Lorentz group, where

$$D^S(J_{\beta\gamma})^{\mu\nu} = i(g^\mu_\beta g^\nu_\gamma - g^\mu_\gamma g^\nu_\beta) \;, \quad (D^S(J_{\beta\gamma})D^S(J_{\rho\sigma}))^{\mu\nu} = g^\mu_\beta g^\nu_\rho g_{\gamma\sigma} + g^\mu_\gamma g^\nu_\sigma g_{\beta\rho} - g^\mu_\beta g^\nu_\sigma g_{\gamma\rho} - g^\mu_\gamma g^\nu_\rho g_{\beta\sigma} \;. \tag{C.19}$$

In a basis where the polarization vectors in the particle rest frame [i.e., the frame where $k^{\star\mu} = (m,0,0,0)^\mu$] coincide with the Cartesian axes, $\epsilon^{(\lambda)\mu}(k^\star) = -g^{\lambda\mu}$, we can furthermore express the standard Lorentz transformation as

$$D^S([k])^{\mu\lambda} = \epsilon^{(\lambda)\mu}(k) \;. \tag{C.20}$$

Upon inserting this representation into the vector polarization (C.11), we find

$$S^\mu(k) = -i\epsilon^{\mu\nu\alpha\beta}\frac{k_\nu}{m}\frac{\int \mathrm{d}\Sigma_\gamma k^\gamma W_{\alpha\beta}(x,k)}{\int \mathrm{d}\Sigma_\gamma k^\gamma K_{\rho\sigma}W^{\rho\sigma}(x,k)} \;, \tag{C.21}$$

which in extended phase space becomes

$$S^\mu(k) = \frac{1}{N(k)}\int \mathrm{d}\Sigma_\gamma k^\gamma \int \mathrm{d}S(k)\mathfrak{s}^\mu f(x,k,\mathfrak{s}) \;, \tag{C.22}$$

where we redefined

$$N(k) \coloneqq \int \mathrm{d}\Sigma_\gamma k^\gamma K_{\rho\sigma}W^{\rho\sigma}(x,k) = \int \mathrm{d}\Sigma_\gamma k^\gamma \int \mathrm{d}S(k) f(x,k,\mathfrak{s}) \tag{C.23}$$

and used Eqs. (3.168). Note that, although the quantity $N(k)$ is now defined in terms of the spin-1 Wigner function, the expressions for the vector polarization (C.16) and (C.22) (up to an expected factor of two) formally coincide when expressed in extended phase space.[2] The tensor polarization becomes

$$\begin{aligned}\Theta^{\mu\nu}(k) &= \frac{1}{2}\sqrt{\frac{3}{2}}\left[2\epsilon^{\mu\alpha\beta\gamma}\epsilon^{\nu\rho\sigma\lambda}\frac{k_\alpha k_\lambda}{m^2}g_{\gamma\sigma}K_{\beta\eta}K_{\rho\zeta}\frac{\int \mathrm{d}\Sigma_\epsilon k^\epsilon W^{\eta\zeta}(x,k)}{\int \mathrm{d}\Sigma_\epsilon k^\epsilon K_{\phi\psi}W^{\phi\psi}(x,k)} + \frac{4}{3}K^{\mu\nu}\right] \\ &= \sqrt{\frac{3}{2}}\left[(K^\mu_\alpha K^\nu_\beta - K^{\mu\nu}K_{\alpha\beta})\frac{\int \mathrm{d}\Sigma_\gamma k^\gamma W^{\alpha\beta}(x,k)}{\int \mathrm{d}\Sigma_\gamma k^\gamma K_{\rho\sigma}W^{\rho\sigma}(x,k)} + \frac{2}{3}K^{\mu\nu}\right] \\ &= \sqrt{\frac{3}{2}}K^{\mu\nu}_{\alpha\beta}\frac{\int \mathrm{d}\Sigma_\gamma k^\gamma W^{\alpha\beta}(x,k)}{\int \mathrm{d}\Sigma_\gamma k^\gamma K_{\rho\sigma}W^{\rho\sigma}(x,k)}\ , \end{aligned} \tag{C.24}$$

where we employed the completeness relation of the polarization vectors (4.103b). Translating this expression into integrals over spin space, we finally have

$$\Theta^{\mu\nu}(k) = \frac{1}{2}\sqrt{\frac{3}{2}}\frac{1}{N(k)}\int \mathrm{d}\Sigma_\gamma k^\gamma \int \mathrm{d}S(k) K^{\mu\nu}_{\alpha\beta}\mathfrak{s}^\alpha \mathfrak{s}^\beta f(x,k,\mathfrak{s})\ . \tag{C.25}$$

[2] This is of course also an effect of suitable choices of the measure in spin space.

Appendix D

The evolution equations of the irreducible moments

The purpose of this appendix is to show the steps to arrive at the equations of motion for the irreducible moments, i.e., Eqs. (6.75), (6.78), and (6.80) in the main text. Considering the definitions of the moments (6.18) in conjunction with the fact that the spin vector is not a function of spacetime, it becomes clear that the only difference in the equations of motion for moments of different rank in spin can come from the equilibrium terms in the BOLTZMANN equation (6.73). Thus, we will consider a general moment

$$\chi_r^{\mu_1\cdots\mu_\ell} := \int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^r k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle} \delta f_{\mathbf{k}\mathfrak{s}} \,, \tag{D.1}$$

where the function $F(\mathfrak{s})$ can be 1, $\mathfrak{s}^\mu$, or $K^{\mu\nu}_{\alpha\beta}\mathfrak{s}^\alpha\mathfrak{s}^\beta$, depending on the spin-rank of the moment one wants to consider.

In the following we will use the following identities for the irreducible tensors,

$$k^{\langle\mu\rangle}k^{\langle\nu\rangle} = k^{\langle\mu}k^{\nu\rangle} + \frac{1}{3}\Delta^{\mu\nu}\left(k^{\langle\xi\rangle}k_\xi\right) \,, \tag{D.2a}$$

$$k^{\langle\mu\rangle}k^{\langle\nu\rangle}k^{\langle\lambda\rangle} = k^{\langle\mu}k^{\nu}k^{\lambda\rangle} + \frac{1}{5}\left(\Delta^{\mu\nu}k^{\langle\lambda\rangle} + 2 \text{ perm.}\right)\left(k^{\langle\xi\rangle}k_\xi\right) \,, \tag{D.2b}$$

$$\begin{aligned} k^{\langle\mu\rangle}k^{\langle\nu\rangle}k^{\langle\lambda\rangle}k^{\langle\rho\rangle} &= k^{\langle\mu}k^{\nu}k^{\lambda}k^{\rho\rangle} + \frac{1}{7}\left(\Delta^{\mu\nu}k^{\langle\lambda}k^{\rho\rangle} + 5 \text{ perm.}\right)\left(k^{\langle\xi\rangle}k_\xi\right) \\ &\quad + \frac{1}{15}\left(\Delta^{\mu\nu}\Delta^{\lambda\rho} + 2 \text{ perm.}\right)\left(k^{\langle\xi\rangle}k_\xi\right)^2 \,, \end{aligned} \tag{D.2c}$$

$$\begin{aligned} k^{\langle\mu\rangle}k^{\langle\nu\rangle}k^{\langle\lambda\rangle}k^{\langle\rho\rangle}k^{\langle\alpha\rangle} &= k^{\langle\mu}k^{\nu}k^{\lambda}k^{\rho}k^{\alpha\rangle} + \frac{1}{9}\left(\Delta^{\mu\nu}k^{\langle\lambda}k^{\rho}k^{\alpha\rangle} + 9 \text{ perm.}\right)\left(k^{\langle\xi\rangle}k_\xi\right) \\ &\quad + \frac{1}{35}\left(\Delta^{\mu\nu}\Delta^{\lambda\rho}k^{\langle\alpha\rangle} + 14 \text{ perm.}\right)\left(k^{\langle\xi\rangle}k_\xi\right)^2 \,, \end{aligned} \tag{D.2d}$$

$$\begin{aligned} k^{\langle\mu\rangle}k^{\langle\nu\rangle}k^{\langle\lambda\rangle}k^{\langle\rho\rangle}k^{\langle\alpha\rangle}k^{\langle\beta\rangle} &= k^{\langle\mu}k^{\nu}k^{\lambda}k^{\rho}k^{\alpha}k^{\beta\rangle} + \frac{1}{11}\left(\Delta^{\mu\nu}k^{\langle\lambda}k^{\rho}k^{\alpha}k^{\beta\rangle} + 14 \text{ perm.}\right)\left(k^{\langle\xi\rangle}k_\xi\right) \\ &\quad + \frac{1}{63}\left(\Delta^{\mu\nu}\Delta^{\lambda\rho}k^{\langle\alpha}k^{\beta\rangle} + 44 \text{ perm.}\right)\left(k^{\langle\xi\rangle}k_\xi\right)^2 \\ &\quad + \frac{1}{105}\left(\Delta^{\mu\nu}\Delta^{\lambda\rho}\Delta^{\alpha\beta} + 14 \text{ perm.}\right)\left(k^{\langle\xi\rangle}k_\xi\right)^3 \,, \end{aligned} \tag{D.2e}$$

which, in conjunction with

$$k^{\langle\alpha\rangle}k_\alpha = m^2 - E_{\mathbf{k}}^2 \,,$$

will allow us to use the orthogonality relation (6.12). In the expressions above, "+ x perm." stands for x other distinct permutations of LORENTZ indices.

Momentum-rank zero

We apply the comoving derivative on the definition (D.1) and obtain

$$\dot{\chi}_r = r\dot{u}_\mu \chi^\mu_{r-1} + \int \mathrm{d}\Gamma F(\mathfrak{s}) E^r_{\mathbf{k}} \delta \dot{f}_{\mathbf{k}\mathfrak{s}} \; . \tag{D.3}$$

Making use of the Boltzmann equation (6.73), we find

$$\dot{\chi}_r - \mathfrak{C}_{r-1} = r\dot{u}_\mu \chi^\mu_{r-1} - \int \mathrm{d}\Gamma F(\mathfrak{s}) E^{r-1}_{\mathbf{k}} \left(E_{\mathbf{k}} \dot{f}_{\mathrm{eq}} + k \cdot \nabla f_{\mathrm{eq}} \right) - \int \mathrm{d}\Gamma F(\mathfrak{s}) E^{r-1}_{\mathbf{k}} k_\nu \nabla^\nu \delta f_{\mathbf{k}\mathfrak{s}} \; , \tag{D.4}$$

where we defined the generalized irreducible collision terms

$$\mathfrak{C}^{\langle \mu_1 \cdots \mu_\ell \rangle}_r := \int \mathrm{d}\Gamma F(\mathfrak{s}) E^r_{\mathbf{k}} k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle} C(x,k,\mathfrak{s}) \; . \tag{D.5}$$

We start with the last term in Eq. (D.4) and compute

$$\begin{aligned} - \int \mathrm{d}\Gamma F(\mathfrak{s}) E^{r-1}_{\mathbf{k}} k_\nu \nabla^\nu \delta f_{\mathbf{k}\mathfrak{s}} &= -\theta \chi_r - \nabla_\nu \chi^\nu_{r-1} + (r-1)(\nabla^\nu u^\mu) \int \mathrm{d}\Gamma F(\mathfrak{s}) E^{r-2}_{\mathbf{k}} k_{\langle \nu \rangle} k_{\langle \mu \rangle} \delta f_{\mathbf{k}\mathfrak{s}} \\ &= -\nabla_\nu \chi^\nu_{r-1} + (r-1)\sigma_{\mu\nu} \chi^{\mu\nu}_{r-2} - \frac{\theta}{3} \left[(r+2)\chi_r - (r-1) m^2 \chi_{r-2} \right] \; , \end{aligned} \tag{D.6}$$

such that we find

$$\begin{aligned} \dot{\chi}_r - \mathfrak{C}_{r-1} = &- \int \mathrm{d}\Gamma F(\mathfrak{s}) E^{r-1}_{\mathbf{k}} \left(E_{\mathbf{k}} \dot{f}_{\mathrm{eq}} + k \cdot \nabla f_{\mathrm{eq}} \right) - \frac{\theta}{3} \left[(r+2)\chi_r - (r-1) m^2 \chi_{r-2} \right] \\ &+ r\dot{u}_\mu \chi^\mu_{r-1} - \nabla_\nu \chi^\nu_{r-1} + (r-1)\sigma_{\mu\nu} \chi^{\mu\nu}_{r-2} \; . \end{aligned} \tag{D.7}$$

The term that distinguishes the cases of different ranks in spin is the first one on the right-hand side of the equation above, which is nonzero only for spin-ranks zero and one. In the case of the irreducible moments of spin-rank two, this term vanishes, such that we find Eq. (6.80a) in the main text upon setting $\chi \equiv \psi$. To evaluate the nonvanishing equilibrium contributions, we note that

$$\int \mathrm{d}S(k) f_{\mathrm{eq}} = g f_{0\mathbf{k}} \; , \tag{D.8a}$$

$$\int \mathrm{d}S(k) \mathfrak{s}^\mu f_{\mathrm{eq}} = -\frac{\sigma\hbar}{m} f_{0\mathbf{k}} \widetilde{f}_{0\mathbf{k}} \widetilde{\Omega}^{\mu\nu}_0 k_\nu \; . \tag{D.8b}$$

Furthermore, a derivative acting on $f_{0\mathbf{k}}$ can be written as

$$\partial^\mu f_{0\mathbf{k}} = \frac{\partial f_{0\mathbf{k}}}{\partial \alpha_0} \left(\partial^\mu \alpha_0 - E_{\mathbf{k}} \partial^\mu \beta_0 - \beta_0 k_{\langle \alpha \rangle} \partial^\mu u^\alpha \right) \; . \tag{D.9}$$

In the case of the irreducible moments of zeroth rank in spin [i.e., $F(\mathfrak{s}) = 1$], we compute

$$\begin{aligned} - \int \mathrm{d}\Gamma E^{r-1}_{\mathbf{k}} \left(E_{\mathbf{k}} \dot{f}_{\mathrm{eq}} + k \cdot \nabla f_{\mathrm{eq}} \right) &= -g \int \mathrm{d}K E^{r-1}_{\mathbf{k}} \left(E_{\mathbf{k}} \dot{f}_{0\mathbf{k}} + k \cdot \nabla f_{0\mathbf{k}} \right) \\ &= -J_{r0} \dot{\alpha}_0 + J_{r+1,0} \dot{\beta}_0 + \theta \left[(1-r) I_{r1} - I_{r0} \right] \; . \end{aligned} \tag{D.10}$$

Note that, in order to obtain the term proportional to the expansion scalar, we used the product rule instead of directly evaluating the derivative acting on the local-equilibrium distribution function.

Inserting Eq. (D.10) into the general expression (D.7) and making use of the evolution equations for α_0 and β_0, we obtain Eq. (6.75a) in the main text. In the case where $F(\mathfrak{s}) = \mathfrak{s}^\mu$, we evaluate

$$\begin{aligned}
&- \int \mathrm{d}\Gamma \mathfrak{s}^\mu E_{\mathbf{k}}^{r-1} \left(E_{\mathbf{k}} \dot{f}_{\mathrm{eq}} + k \cdot \nabla f_{\mathrm{eq}} \right) \\
&= \frac{\sigma\hbar}{m} \int \mathrm{d}K E_{\mathbf{k}}^{r-1} \left(E_{\mathbf{k}} u \cdot \partial + k \cdot \nabla \right) f_{0\mathbf{k}} \widetilde{f}_{0\mathbf{k}} \widetilde{\Omega}_0^{\mu\nu} k_\nu \\
&= \frac{\sigma\hbar}{gm} \left(J_{r+1,0} \dot{\widetilde{\Omega}}_0^{\mu\nu} u_\nu - J_{r+1,1} \nabla_\nu \widetilde{\Omega}_0^{\mu\nu} \right) + \frac{\sigma\hbar}{m} \widetilde{\Omega}_0^{\mu\nu} \int \mathrm{d}K E_{\mathbf{k}}^{r-1} k_\nu \left(E_{\mathbf{k}} u \cdot \partial + k \cdot \nabla \right) f_{0\mathbf{k}} \widetilde{f}_{0\mathbf{k}} \\
&= \frac{\sigma\hbar}{gm} \Big\{ J_{r+1,0} \dot{\widetilde{\Omega}}_0^{\mu\nu} u_\nu - J_{r+1,1} \nabla_\nu \widetilde{\Omega}_0^{\mu\nu} + 2\omega_0^\mu \left[K_{r+1,0} \dot{\alpha}_0 - K_{r+2,0} \dot{\beta}_0 + \theta (J_{r+1,0} + r J_{r+1,1}) \right] \\
&\quad + \beta_0 K_{r+2,1} \widetilde{\Omega}_0^{\mu\nu} \dot{u}_\nu - \widetilde{\Omega}_0^{\mu\nu} \left(K_{r+1,1} I_\nu - K_{r+2,1} \nabla_\nu \beta_0 \right) \Big\} . \qquad \text{(D.11)}
\end{aligned}$$

Here, we used that $\widetilde{\Omega}_0^{\mu\nu} u_\nu = 2\omega_0^\mu$. With these equilibrium terms, we obtain Eq. (6.78a).

Momentum-rank one

We act with the comoving derivative on the moment of tensor-rank one in momentum,

$$\begin{aligned}
\dot{\chi}_r^{\langle\mu\rangle} &= \Delta^\mu_\nu \frac{\mathrm{d}}{\mathrm{d}\tau} \int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^r k^{\langle\mu\rangle} \delta f_{\mathbf{k}\mathfrak{s}} \\
&= r \dot{u}_\alpha \chi_{r-1}^{\mu\alpha} + \frac{1}{3} \dot{u}^\mu \left[m^2 r \chi_{r-1} - (r+3) \chi_{r+1} \right] + \Delta^\mu_\nu \int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^r k^{\langle\nu\rangle} \delta \dot{f}_{\mathbf{k}\mathfrak{s}} , \qquad \text{(D.12)}
\end{aligned}$$

and subsequently evaluate the second term by employing the BOLTZMANN equation:

$$\Delta^\mu_\nu \int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^r k^{\langle\nu\rangle} \delta \dot{f}_{\mathbf{k}\mathfrak{s}} = \mathfrak{C}_{r-1}^{\langle\nu\rangle} - \Delta^\mu_\nu \int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^{r-1} k^{\langle\nu\rangle} \left(E_{\mathbf{k}} \dot{f}_{\mathrm{eq}} + k_\rho \nabla^\rho f_{\mathrm{eq}} + k_\rho \nabla^\rho \delta f_{\mathbf{k}\mathfrak{s}} \right) . \qquad \text{(D.13)}$$

First, we analyze the last term in the equation above:

$$\begin{aligned}
&- \Delta^\mu_\nu \int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^{r-1} k^{\langle\nu\rangle} k_\rho \nabla^\rho \delta f_{\mathbf{k}\mathfrak{s}} \\
&= -\Delta^\mu_\nu \nabla_\rho \left[\int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^{r-1} k^{\langle\nu\rangle} \left(E_{\mathbf{k}} u^\rho + k^{\langle\rho\rangle} \right) \delta f_{\mathbf{k}\mathfrak{s}} \right] \\
&\quad + (r-1) \int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^{r-2} k^{\langle\mu\rangle} \left(k_\rho \nabla^\rho E_{\mathbf{k}} \right) \delta f_{\mathbf{k}\mathfrak{s}} + \Delta^\mu_\nu \int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^{r-1} k^{\langle\rho\rangle} \left(\nabla_\rho k^{\langle\nu\rangle} \right) \delta f_{\mathbf{k}\mathfrak{s}} \\
&= -\frac{4}{3} \theta \chi_r^\mu - \Delta^\mu_\nu \nabla_\rho \chi_{r-1}^{\nu\rho} - \frac{1}{3} \nabla^\mu \left(m^2 \chi_{r-1} - \chi_{r+1} \right) - (\sigma^{\mu\alpha} - \omega^{\mu\alpha}) \chi_{r,\alpha} \\
&\quad + (r-1) \left[\frac{\theta}{3} \left(m^2 \chi_{r-2}^\mu - \chi_r^\mu \right) + \chi_{r-2}^{\mu\rho\lambda} \sigma_{\rho\lambda} + \frac{2}{5} \left(m^2 \chi_{r-2}^\rho - \chi_r^\rho \right) \sigma^\mu{}_\rho \right] . \qquad \text{(D.14)}
\end{aligned}$$

Then, we can combine our results so far to yield

$$\begin{aligned}
\dot{\chi}_r^{\langle\mu\rangle} - \mathfrak{C}_{r-1}^{\langle\mu\rangle} = &- \int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^{r-1} k^{\langle\mu\rangle} \left(E_{\mathbf{k}} \dot{f}_{\mathrm{eq}} + k_\rho \nabla^\rho f_{\mathrm{eq}} \right) + \frac{1}{3} \dot{u}^\mu \left[m^2 r \chi_{r-1} - (r+3) \chi_{r+1} \right] \\
&+ \omega^\mu{}_\alpha \chi_r^\alpha - \Delta^\mu_\beta \nabla_\alpha \chi_{r-1}^{\beta\alpha} - \frac{1}{3} \nabla^\mu \left(m^2 \chi_{r-1} - \chi_{r+1} \right) + \frac{\theta}{3} \left[(r-1) m^2 \chi_{r-2}^\mu - (r+3) \chi_r^\mu \right] \\
&+ r \dot{u}_\alpha \chi_{r-1}^{\mu\alpha} + \frac{1}{5} \left[(2r-2) m^2 \chi_{r-2}^\alpha - (2r+3) \chi_r^\alpha \right] \sigma^\mu{}_\alpha + (r-1) \chi_{r-2}^{\mu\alpha\beta} \sigma_{\alpha\beta} . \qquad \text{(D.15)}
\end{aligned}$$

In the case of the moments of spin-rank two, $\chi \equiv \psi$, the equilibrium terms vanish and we find Eq. (6.80b). For spin-ranks zero and one, we need to evaluate these terms separately again. For $\chi \equiv \rho$, we compute

$$- \int \mathrm{d}\Gamma E_{\mathbf{k}}^{r-1} k^{\langle\mu\rangle} \left(E_{\mathbf{k}} \dot{f}_{\mathrm{eq}} + k_\rho \nabla^\rho f_{\mathrm{eq}} \right) = J_{r+1,1} I^\mu - J_{r+2,1} \left(\nabla^\mu \beta_0 + \beta_0 \dot{u}^\mu \right) , \qquad \text{(D.16)}$$

which, after inserting the result into Eq. (D.15) and making use of the equation of motion for the four-velocity, yields Eq. (6.75b). Analogous to the previous case, for $\chi \equiv \tau$ we have

$$\begin{aligned}
&- \int \mathrm{d}\Gamma \mathfrak{s}^\lambda E_{\mathbf{k}}^{r-1} k^{\langle\mu\rangle} \left(E_{\mathbf{k}} \dot{f}_{\text{eq}} + k_\rho \nabla^\rho f_{\text{eq}} \right) \\
&= \frac{\sigma\hbar}{m} \int \mathrm{d}K E_{\mathbf{k}}^{r-1} k^{\langle\mu\rangle} \left(E_{\mathbf{k}} u \cdot \partial + k \cdot \nabla \right) f_{0\mathbf{k}} \widetilde{f}_{0\mathbf{k}} \widetilde{\Omega}_0^{\lambda\nu} k_\nu \\
&= -\frac{\sigma\hbar}{gm} \left(J_{r+2,1} \dot{\widetilde{\Omega}}_0^{\lambda\langle\mu\rangle} + J_{r+2,1} u_\nu \nabla^\mu \widetilde{\Omega}_0^{\lambda\nu} \right) + \widetilde{\Omega}_0^{\lambda\nu} \frac{\sigma\hbar}{m} \int \mathrm{d}K E_{\mathbf{k}}^{r-1} k_\nu k^{\langle\mu\rangle} \left(E_{\mathbf{k}} u \cdot \partial + k \cdot \nabla \right) f_{0\mathbf{k}} \widetilde{f}_{0\mathbf{k}} \\
&= \frac{\sigma\hbar}{gm} \Bigg\{ - J_{r+2,1} \dot{\widetilde{\Omega}}_0^{\lambda\langle\mu\rangle} - J_{r+2,1} u_\nu \nabla^\mu \widetilde{\Omega}_0^{\lambda\nu} + \widetilde{\Omega}_0^{\lambda\langle\mu\rangle} \left(-K_{r+2,1} \dot{\alpha}_0 + K_{r+3,1} \dot{\beta}_0 - \frac{5}{3} \beta_0 K_{r+3,2} \theta \right) \\
&\quad - 2\beta_0 K_{r+3,2} \widetilde{\Omega}_0^{\lambda\nu} \sigma^\mu{}_\nu + 2\omega_0^\lambda \left[-K_{r+2,1} I^\mu + K_{r+3,1} \left(\nabla^\mu \beta_0 + \beta_0 \dot{u}^\mu \right) \right] \Bigg\} . \qquad \text{(D.17)}
\end{aligned}$$

Inserting this result into Eq. (D.15), we find Eq. (6.78b) in the main text.

Momentum-rank two

Differentiating the moment of second rank in momentum and projecting the result onto the subspace of tensors that are orthogonal to u^μ, symmetric and traceless, yields

$$\begin{aligned}
\dot{\chi}_r^{\langle\mu\nu\rangle} &= \Delta^{\mu\nu}_{\alpha\beta} \frac{\mathrm{d}}{\mathrm{d}\tau} \int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^r k^{\langle\alpha} k^{\beta\rangle} \delta f_{\mathbf{k}\mathfrak{s}} \\
&= \Delta^{\mu\nu}_{\alpha\beta} \left[r \dot{u}_\lambda \int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^{r-1} k^{\langle\alpha} k^{\beta\rangle} k^{\langle\lambda\rangle} \delta f_{\mathbf{k}\mathfrak{s}} + \int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^r \left(\frac{\mathrm{d}}{\mathrm{d}\tau} k^{\langle\alpha} k^{\beta\rangle} \right) \delta f_{\mathbf{k}\mathfrak{s}} \right. \\
&\quad \left. + \int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^r k^{\langle\alpha} k^{\beta\rangle} \delta \dot{f}_{\mathbf{k}\mathfrak{s}} \right] . \qquad \text{(D.18)}
\end{aligned}$$

We start by computing the first term in Eq. (D.18),

$$\Delta^{\mu\nu}_{\alpha\beta} r \dot{u}_\lambda \int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^{r-1} k^{\langle\alpha} k^{\beta\rangle} k^{\langle\lambda\rangle} \delta f_{\mathbf{k}\mathfrak{s}} = r \dot{u}_\lambda \chi_{r-1}^{\lambda\mu\nu} + \frac{2}{5} r \dot{u}^{\langle\mu} \left(m^2 \chi_{r-1}^{\nu\rangle} - \chi_{r+1}^{\nu\rangle} \right) . \qquad \text{(D.19)}$$

Using the projected derivative of the traceless symmetric projector of rank two orthogonal to the four-velocity,

$$\Delta^{\mu\nu}_{\alpha\beta} \frac{\mathrm{d}}{\mathrm{d}\tau} \Delta^{\alpha\beta}_{\gamma\delta} = -\dot{u}^{\langle\mu} \Delta^{\nu\rangle}_{(\gamma} u_{\delta)} , \qquad \text{(D.20)}$$

the second contribution in Eq. (D.18) gives

$$\Delta^{\mu\nu}_{\alpha\beta} \int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^r \left(\frac{\mathrm{d}}{\mathrm{d}\tau} k^{\langle\alpha} k^{\beta\rangle} \right) \delta f_{\mathbf{k}\mathfrak{s}} = -2 \dot{u}^{\langle\mu} \chi_{r+1}^{\nu\rangle} . \qquad \text{(D.21)}$$

Putting the Boltzmann equation to use, the third term in Eq. (D.18) reads

$$\begin{aligned}
\int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^r k^{\langle\mu} k^{\nu\rangle} \delta \dot{f}_{\mathbf{k}} &= \mathfrak{C}_{r-1}^{\mu\nu} - \int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^r k^{\langle\mu} k^{\nu\rangle} \dot{f}_{\text{eq}} - \int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^{r-1} k^{\langle\mu} k^{\nu\rangle} k^\lambda \nabla_\lambda f_{\text{eq}} \\
&\quad - \int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^{r-1} k^{\langle\mu} k^{\nu\rangle} k^\lambda \nabla_\lambda \delta f_{\mathbf{k}\mathfrak{s}} . \qquad \text{(D.22)}
\end{aligned}$$

As in the preceding computations, we first concentrate on evaluating the contributions that contain the deviation of the distribution function from equilibrium. Considering the last term in Eq. (D.22), we find

$$
\begin{aligned}
&-\int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^{r-1} k^{\langle\mu} k^{\nu\rangle} k^\lambda \nabla_\lambda \delta f_{\mathbf{k}\mathfrak{s}} \\
&= -\Delta^{\mu\nu}_{\alpha\beta}\nabla_\lambda \left[\int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^{r-1} k^{\langle\alpha} k^{\beta\rangle} k^{\langle\lambda\rangle}\delta f_{\mathbf{k}\mathfrak{s}}\right] + \Delta^{\mu\nu}_{\alpha\beta}\int \mathrm{d}\Gamma F(\mathfrak{s}) \nabla_\lambda \left(E_{\mathbf{k}}^{r-1} k^{\langle\alpha} k^{\beta\rangle} k^{\langle\lambda\rangle}\right)\delta f_{\mathbf{k}\mathfrak{s}} \\
&= -\theta\chi_r^{\mu\nu} - \Delta^{\mu\nu}_{\alpha\beta}\nabla_\lambda \chi_{r-1}^{\alpha\beta\lambda} - \frac{2}{5}\nabla^{\langle\mu}\left(m^2\chi_{r-1}^{\nu\rangle} - \chi_{r+1}^{\nu\rangle}\right) \\
&\quad + \int \mathrm{d}\Gamma F(\mathfrak{s}) k^{\langle\lambda\rangle}\left[E_{\mathbf{k}}^{r-1}\Delta^{\mu\nu}_{\alpha\beta}\nabla_\lambda\left(k^{\langle\alpha\rangle}k^{\langle\beta\rangle} - \frac{1}{3}\Delta^{\alpha\beta}k^{\langle\sigma\rangle}k_\sigma\right) + (r-1)E_{\mathbf{k}}^{r-2}k^{\langle\mu}k^{\nu\rangle}k^{\langle\rho\rangle}\nabla_\lambda u_\rho\right]\delta f_{\mathbf{k}\mathfrak{s}} \\
&= -\theta\chi_r^{\mu\nu} - \Delta^{\mu\nu}_{\alpha\beta}\nabla_\lambda \chi_{r-1}^{\alpha\beta\lambda} - \frac{2}{5}\nabla^{\langle\mu}\left(m^2\chi_{r-1}^{\nu\rangle} - \chi_{r+1}^{\nu\rangle}\right) \\
&\quad + \int \mathrm{d}\Gamma F(\mathfrak{s}) k^{\langle\lambda\rangle}\left[-2E_{\mathbf{k}}^{r}\Delta^{\mu\nu}_{\alpha\beta}k^\alpha\nabla_\lambda u^\beta + (r-1)E_{\mathbf{k}}^{r-2}\left(k^{\langle\mu\rangle}k^{\langle\nu\rangle} - \frac{1}{3}\Delta^{\mu\nu}k^{\langle\sigma\rangle}k_\sigma\right)k^{\langle\rho\rangle}\nabla_\lambda u_\rho\right]\delta f_{\mathbf{k}\mathfrak{s}} \\
&= -\theta\chi_r^{\mu\nu} - \Delta^{\mu\nu}_{\alpha\beta}\nabla_\lambda \chi_{r-1}^{\alpha\beta\lambda} - \frac{2}{5}\nabla^{\langle\mu}\left(m^2\chi_{r-1}^{\nu\rangle} - \chi_{r+1}^{\nu\rangle}\right) - 2\chi_r^{\lambda\langle\mu}\left(\sigma^{\nu\rangle}{}_\lambda - \omega^{\nu\rangle}{}_\lambda + \frac{1}{3}\theta\Delta^{\nu\rangle}{}_\lambda\right) \\
&\quad + (r-1)\nabla_\lambda u_\rho \chi_{r-2}^{\mu\nu\lambda\rho} + \frac{2}{7}(r-1)\left(m^2\chi_{r-2}^{\lambda\langle\mu} - \chi_r^{\lambda\langle\mu}\right)\left(2\sigma^{\nu\rangle}{}_\lambda + \frac{2}{3}\theta\Delta^{\nu\rangle}{}_\lambda\right) \\
&\quad + \frac{1}{7}(r-1)\theta\left(m^2\chi_{r-2}^{\mu\nu} - \chi_r^{\mu\nu}\right) + \frac{2}{15}\left[m^4(r-1)\chi_{r-2} - m^2(2r+3)\chi_r + (r+4)\chi_{r+2}\right]\sigma^{\mu\nu} \,. \qquad \text{(D.23)}
\end{aligned}
$$

Combining these results, we find

$$
\begin{aligned}
\dot{\chi}_r^{\langle\mu\nu\rangle} - \mathfrak{C}_{r-1}^{\langle\mu\nu\rangle} = &-\int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^{r} k^{\langle\mu} k^{\nu\rangle}\dot{f}_{\mathrm{eq}} - \int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^{r-1} k^{\langle\mu} k^{\nu\rangle} k^\lambda \nabla_\lambda f_{\mathrm{eq}} \\
&+ \frac{2}{15}\Big[m^4(r-1)\chi_{r-2} - m^2(2r+3)\chi_r + (r+4)\chi_{r+2}\Big]\sigma^{\mu\nu} + r\dot{u}_\lambda\chi_{r-1}^{\lambda\mu\nu} \\
&+ \frac{2}{5}\dot{u}^{\langle\mu}\Big[rm^2\chi_{r-1}^{\nu\rangle} - (r+5)\chi_{r+1}^{\nu\rangle}\Big] - \frac{2}{5}\nabla^{\langle\mu}\left(m^2\chi_{r-1}^{\nu\rangle} - \chi_{r+1}^{\nu\rangle}\right) \\
&+ \frac{\theta}{3}\Big[m^2(r-1)\chi_{r-2}^{\mu\nu} - (r+4)\chi_r^{\mu\nu}\Big] + (r-1)\chi_{r-2}^{\mu\nu\lambda\rho}\sigma_{\lambda\rho} - \Delta^{\mu\nu}_{\alpha\beta}\nabla_\lambda\chi_{r-1}^{\alpha\beta\lambda} + 2\chi_r^{\lambda\langle\mu}\omega^{\nu\rangle}{}_\lambda \\
&+ \frac{2}{7}\Big[2m^2(r-1)\chi_{r-2}^{\lambda\langle\mu} - (2r+5)\chi_r^{\lambda\langle\mu}\Big]\sigma^{\nu\rangle}{}_\lambda \,. \qquad \text{(D.24)}
\end{aligned}
$$

Setting $F(\mathfrak{s}) = K^{\mu\nu}_{\alpha\beta}\mathfrak{s}^\alpha\mathfrak{s}^\beta$ (and thus $\chi \equiv \psi$), the equilibrium terms vanish and we find Eq. (6.80c). For the moments of spin-rank zero, $\chi \equiv \rho$, the equilibrium term becomes

$$
-\int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^{r} k^{\langle\alpha} k^{\beta\rangle}\dot{f}_{\mathrm{eq}} - \int \mathrm{d}\Gamma F(\mathfrak{s}) E_{\mathbf{k}}^{r-1} k^{\langle\mu} k^{\nu\rangle} k^\lambda \nabla_\lambda f_{\mathrm{eq}} = 2\beta_0 J_{r+3,2}\sigma^{\mu\nu} \,, \qquad \text{(D.25)}
$$

which then yields Eq. (6.75c) by virtue of the identity

$$
\beta_0 J_{nq} = I_{n-1,q-1} + (n-2q) I_{n-1,q} \,. \qquad \text{(D.26)}
$$

For $\chi \equiv \tau$, the equilibrium terms are computed as

$$
\begin{aligned}
&-\int \mathrm{d}\Gamma \mathfrak{s}^\lambda E_{\mathbf{k}}^{r} k^{\langle\alpha} k^{\beta\rangle}\dot{f}_{\mathrm{eq}} - \int \mathrm{d}\Gamma \mathfrak{s}^\lambda E_{\mathbf{k}}^{r-1} k^{\langle\mu} k^{\nu\rangle} k^\beta \nabla_\beta f_{\mathrm{eq}} \\
&= \frac{\sigma\hbar}{m}\int \mathrm{d}K E_{\mathbf{k}}^{r-1} k^{\langle\mu} k^{\nu\rangle} k_\alpha \left(E_{\mathbf{k}} u\cdot\partial + k\cdot\nabla\right) f_{0\mathbf{k}}\widetilde{f}_{0\mathbf{k}}\widetilde{\Omega}_0^{\lambda\alpha} \\
&= \frac{2\sigma\hbar}{gm}\Big\{K_{r+3,2}\Delta^{\mu\nu}_{\alpha\beta}\nabla^\alpha\widetilde{\Omega}^{\lambda\beta} + \widetilde{\Omega}^{\lambda\langle\mu}\left[K_{r+3,2}I^{\nu\rangle} - K_{r+4,2}\left(\nabla^{\nu\rangle}\beta_0 + \beta_0\dot{u}^{\nu\rangle}\right)\right] \\
&\qquad - 2\beta_0 K_{r+4,2}\omega_0^\lambda\sigma^{\mu\nu}\Big\} \,, \qquad \text{(D.27)}
\end{aligned}
$$

leading to Eq. (6.78c).

Momentum-rank three

The moment of tensor-rank three in momentum works as the ones before. Applying the comoving derivative gives

$$\begin{aligned}\dot{\chi}_r^{\langle\mu\nu\lambda\rangle} &= \Delta^{\mu\nu\lambda}_{\alpha\beta\gamma}\frac{\mathrm{d}}{\mathrm{d}\tau}\int \mathrm{d}\Gamma F(\mathfrak{s}) E^r_{\mathbf{k}} k^{\langle\alpha}k^{\beta}k^{\gamma\rangle}\delta f_{\mathbf{k}\mathfrak{s}} \\ &= \Delta^{\mu\nu\lambda}_{\alpha\beta\gamma}\left[r\dot{u}_\kappa\int \mathrm{d}\Gamma F(\mathfrak{s}) E^{r-1}_{\mathbf{k}} k^{\langle\alpha}k^{\beta}k^{\gamma\rangle}k^{\langle\kappa\rangle}\delta f_{\mathbf{k}\mathfrak{s}} + \int \mathrm{d}\Gamma F(\mathfrak{s}) E^r_{\mathbf{k}}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}k^{\langle\alpha}k^{\beta}k^{\gamma\rangle}\right)\delta f_{\mathbf{k}\mathfrak{s}}\right. \\ &\quad \left. + \int \mathrm{d}\Gamma F(\mathfrak{s}) E^r_{\mathbf{k}} k^{\langle\alpha}k^{\beta}k^{\gamma\rangle}\delta \dot{f}_{\mathbf{k}\mathfrak{s}}\right] . \end{aligned} \tag{D.28}$$

The first term in the equation above yields

$$\Delta^{\mu\nu\lambda}_{\alpha\beta\gamma} r\dot{u}_\kappa \int \mathrm{d}\Gamma F(\mathfrak{s}) E^{r-1}_{\mathbf{k}} k^{\langle\alpha}k^{\beta}k^{\gamma\rangle}k^{\langle\kappa\rangle}\delta f_{\mathbf{k}\mathfrak{s}} = r\dot{u}_\kappa \chi^{\kappa\mu\nu\lambda}_{r-1} + \frac{3}{7}r\dot{u}^{\langle\mu}\left(m^2\chi^{\nu\lambda\rangle}_{r-1} - \chi^{\nu\lambda\rangle}_{r+1}\right) , \tag{D.29}$$

whereas the second term becomes

$$\Delta^{\mu\nu\lambda}_{\alpha\beta\gamma}\int \mathrm{d}\Gamma F(\mathfrak{s}) E^r_{\mathbf{k}}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}k^{\langle\alpha}k^{\beta}k^{\gamma\rangle}\right)\delta f_{\mathbf{k}\mathfrak{s}} = -3\dot{u}^{\langle\mu}\chi^{\nu\lambda\rangle}_{r+1} . \tag{D.30}$$

After employing the Boltzmann equation, we obtain for the third term in Eq. (D.28)

$$\int \mathrm{d}\Gamma F(\mathfrak{s}) E^r_{\mathbf{k}} k^{\langle\mu}k^{\nu}k^{\lambda\rangle}\delta \dot{f}_{\mathbf{k}} = \mathfrak{C}^{\mu\nu\lambda}_{r-1} - \int \mathrm{d}\Gamma F(\mathfrak{s}) E^{r-1}_{\mathbf{k}} k^{\langle\mu}k^{\nu}k^{\lambda\rangle}k^{\kappa}\nabla_\kappa\delta f_{\mathbf{k}\mathfrak{s}} . \tag{D.31}$$

Note that we could set the terms involving the equilibrium distribution function to zero already, since no first-order structure involving only fluid gradients with the appropriate symmetries exists. Considering the second term in the equation above, we compute

$$\begin{aligned} &-\int \mathrm{d}\Gamma F(\mathfrak{s}) E^{r-1}_{\mathbf{k}} k^{\langle\mu}k^{\nu}k^{\lambda\rangle}k^{\langle\kappa\rangle}\nabla_\kappa\delta f_{\mathbf{k}\mathfrak{s}} \\ &= -\Delta^{\mu\nu\lambda}_{\alpha\beta\gamma}\nabla_\kappa\left[\int \mathrm{d}\Gamma F(\mathfrak{s}) E^{r-1}_{\mathbf{k}} k^{\langle\alpha}k^{\beta}k^{\gamma\rangle}k^{\langle\kappa\rangle}\delta f_{\mathbf{k}\mathfrak{s}}\right] + \Delta^{\mu\nu\lambda}_{\alpha\beta\gamma}\int \mathrm{d}\Gamma F(\mathfrak{s}) k^{\langle\kappa\rangle}\nabla_\kappa\left(E^{r-1}_{\mathbf{k}} k^{\langle\alpha}k^{\beta}k^{\gamma\rangle}\right)\delta f_{\mathbf{k}\mathfrak{s}} \\ &= -\theta\chi^{\mu\nu\lambda}_r - \Delta^{\mu\nu\lambda}_{\alpha\beta\gamma}\nabla_\kappa\chi^{\kappa\alpha\beta\gamma}_{r-1} - \frac{3}{7}\nabla^{\langle\mu}\left(m^2\chi^{\nu\lambda\rangle}_{r-1} - \chi^{\nu\lambda\rangle}_{r+1}\right) \\ &\quad + \int \mathrm{d}\Gamma F(\mathfrak{s}) k^{\langle\kappa\rangle}\left[E^{r-1}_{\mathbf{k}}\Delta^{\mu\nu\lambda}_{\alpha\beta\gamma}\nabla_\kappa k^{\langle\alpha}k^{\beta}k^{\gamma\rangle} + (r-1)E^{r-2}_{\mathbf{k}} k^{\langle\mu}k^{\nu}k^{\lambda\rangle}k^{\langle\rho\rangle}\nabla_\lambda u_\rho\right]\delta f_{\mathbf{k}\mathfrak{s}} \\ &= -2\theta\chi^{\mu\nu\lambda}_r - \Delta^{\mu\nu\lambda}_{\alpha\beta\gamma}\nabla_\kappa\chi^{\kappa\alpha\beta\gamma}_{r-1} - \frac{3}{7}\nabla^{\langle\mu}\left(m^2\chi^{\nu\lambda\rangle}_{r-1} - \chi^{\nu\lambda\rangle}_{r+1}\right) - 3\chi^{\kappa\langle\mu\nu}_r\sigma^{\lambda\rangle}{}_\kappa + 3\chi^{\kappa\langle\mu\nu}_r\omega^{\lambda\rangle}{}_\kappa \\ &\quad - \frac{6}{5}\sigma^{\langle\mu\nu}\left(m^2\chi^{\lambda\rangle}_r - \chi^{\lambda\rangle}_{r+2}\right) + (r-1)\left[\sigma_{\kappa\rho}\chi^{\kappa\rho\mu\nu\lambda}_{r-2} + \frac{\theta}{3}\left(m^2\chi^{\mu\nu\lambda}_{r-2} - \chi^{\mu\nu\lambda}_r\right)\right. \\ &\quad \left. + \frac{2}{3}\left(m^2\chi^{\kappa\langle\mu\nu}_{r-2} - \chi^{\kappa\langle\mu\nu}_r\right)\sigma^{\lambda\rangle}{}_\kappa + \frac{6}{35}\sigma^{\langle\mu\nu}\left(m^4\chi^{\lambda\rangle}_{r-2} - 2m^2\chi^{\lambda\rangle}_r + \chi^{\lambda\rangle}_{r+2}\right)\right] . \end{aligned} \tag{D.32}$$

Combining the expressions, we find

$$\begin{aligned}\dot{\chi}^{\langle\mu\nu\lambda\rangle}_r - \mathfrak{C}^{\langle\mu\nu\lambda\rangle}_{r-1} &= \frac{6}{35}\sigma^{\langle\mu\nu}\left[m^4(r-1)\chi^{\lambda\rangle}_{r-2} - m^2(2r+5)\chi^{\lambda\rangle}_r + (r+6)\chi^{\lambda\rangle}_{r+2}\right] + r\dot{u}_\kappa\chi^{\kappa\mu\nu\lambda}_{r-1} \\ &\quad + \frac{3}{7}\dot{u}^{\langle\mu}\left[rm^2\chi^{\nu\lambda\rangle}_{r-1} - (r+7)\chi^{\nu\lambda\rangle}_{r+1}\right] - \frac{3}{7}\nabla^{\langle\mu}\left(m^2\chi^{\nu\lambda\rangle}_{r-1} - \chi^{\nu\lambda\rangle}_{r+1}\right) \\ &\quad + \frac{\theta}{3}\left[m^2(r-1)\chi^{\mu\nu\lambda}_{r-2} - (r+5)\chi^{\mu\nu\lambda}_r\right] + (r-1)\chi^{\mu\nu\lambda\rho\kappa}_{r-2}\sigma_{\rho\kappa} - \Delta^{\mu\nu\lambda}_{\alpha\beta\gamma}\nabla_\kappa\chi^{\alpha\beta\gamma\kappa}_{r-1} \\ &\quad + 3\chi^{\kappa\langle\mu\nu}_r\omega^{\lambda\rangle}{}_\kappa + \frac{1}{3}\left[2m^2(r-1)\chi^{\kappa\langle\mu\nu}_{r-2} - (2r+7)\chi^{\kappa\langle\mu\nu}_r\right]\sigma^{\lambda\rangle}{}_\kappa . \end{aligned} \tag{D.33}$$

Upon specifying $\chi \equiv \psi$, we obtain Eq. (6.80d).

Momentum-rank four

Letting the comoving derivative act on the irreducible moment of tensor-rank four gives

$$\begin{aligned}\dot{\chi}_r^{\langle\mu\nu\lambda\rho\rangle} &= \Delta^{\mu\nu\lambda\rho}_{\alpha\beta\gamma\delta}\frac{\mathrm{d}}{\mathrm{d}\tau}\int \mathrm{d}\Gamma F(\mathfrak{s})E^r_{\mathbf{k}}k^{\langle\alpha}k^{\beta}k^{\gamma}k^{\delta\rangle}\delta f_{\mathbf{k}\mathfrak{s}} \\ &= \Delta^{\mu\nu\lambda\rho}_{\alpha\beta\gamma\delta}\left[r\dot{u}_\kappa\int \mathrm{d}\Gamma F(\mathfrak{s})E^{r-1}_{\mathbf{k}}k^{\langle\alpha}k^{\beta}k^{\gamma}k^{\delta\rangle}k^{\langle\kappa\rangle}\delta f_{\mathbf{k}\mathfrak{s}} + \int \mathrm{d}\Gamma F(\mathfrak{s})E^r_{\mathbf{k}}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}k^{\langle\alpha}k^{\beta}k^{\gamma}k^{\delta\rangle}\right)\delta f_{\mathbf{k}\mathfrak{s}}\right. \\ &\quad \left. + \int \mathrm{d}\Gamma F(\mathfrak{s})E^r_{\mathbf{k}}k^{\langle\alpha}k^{\beta}k^{\gamma}k^{\delta\rangle}\delta\dot{f}_{\mathbf{k}\mathfrak{s}}\right] . \end{aligned} \tag{D.34}$$

The first two terms give

$$\Delta^{\mu\nu\lambda\rho}_{\alpha\beta\gamma\delta}r\dot{u}_\kappa\int \mathrm{d}\Gamma F(\mathfrak{s})E^{r-1}_{\mathbf{k}}k^{\langle\alpha}k^{\beta}k^{\gamma}k^{\delta\rangle}k^{\langle\kappa\rangle}\delta f_{\mathbf{k}\mathfrak{s}} = r\dot{u}_\kappa\chi^{\kappa\mu\nu\lambda\rho}_{r-1} + \frac{4}{9}r\dot{u}^{\langle\mu}\left(m^2\chi^{\nu\lambda\rho\rangle}_{r-1} - \chi^{\nu\lambda\rho\rangle}_{r+1}\right) \tag{D.35}$$

and

$$\Delta^{\mu\nu\lambda\rho}_{\alpha\beta\gamma\delta}\int \mathrm{d}\Gamma F(\mathfrak{s})E^r_{\mathbf{k}}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}k^{\langle\alpha}k^{\beta}k^{\gamma}k^{\delta\rangle}\right)\delta f_{\mathbf{k}\mathfrak{s}} = -3\dot{u}^{\langle\mu}\chi^{\nu\lambda\rho\rangle}_{r+1} , \tag{D.36}$$

respectively, while the Boltzmann equation lets the third term take the form

$$\int \mathrm{d}\Gamma F(\mathfrak{s})E^r_{\mathbf{k}}k^{\langle\mu}k^{\nu}k^{\lambda}k^{\rho\rangle}\delta\dot{f}_{\mathbf{k}} = \mathfrak{C}^{\mu\nu\lambda\rho}_{r-1} - \int \mathrm{d}\Gamma F(\mathfrak{s})E^{r-1}_{\mathbf{k}}k^{\langle\mu}k^{\nu}k^{\lambda}k^{\rho\rangle}k^{\kappa}\nabla_\kappa\delta f_{\mathbf{k}\mathfrak{s}} , \tag{D.37}$$

where we employed again that the equilibrium terms vanish. With the help of the identities (D.2) we find

$$\begin{aligned} &-\int \mathrm{d}\Gamma F(\mathfrak{s})E^{r-1}_{\mathbf{k}}k^{\langle\mu}k^{\nu}k^{\lambda}k^{\rho\rangle}k^{\langle\kappa\rangle}\nabla_\kappa\delta f_{\mathbf{k}\mathfrak{s}} \\ &= -\Delta^{\mu\nu\lambda\rho}_{\alpha\beta\gamma\delta}\nabla_\kappa\left[\int \mathrm{d}\Gamma F(\mathfrak{s})E^{r-1}_{\mathbf{k}}k^{\langle\alpha}k^{\beta}k^{\gamma}k^{\delta\rangle}k^{\langle\kappa\rangle}\delta f_{\mathbf{k}\mathfrak{s}}\right] \\ &\quad + \Delta^{\mu\nu\lambda\rho}_{\alpha\beta\gamma\delta}\int \mathrm{d}\Gamma F(\mathfrak{s})k^{\langle\kappa\rangle}\nabla_\kappa\left(E^{r-1}_{\mathbf{k}}k^{\langle\alpha}k^{\beta}k^{\gamma}k^{\delta\rangle}\right)\delta f_{\mathbf{k}\mathfrak{s}} \\ &= -\theta\chi^{\mu\nu\lambda\rho}_r - \Delta^{\mu\nu\lambda\rho}_{\alpha\beta\gamma\delta}\nabla_\kappa\chi^{\kappa\alpha\beta\gamma\delta}_{r-1} - \frac{4}{9}\nabla^{\langle\mu}\left(m^2\chi^{\nu\lambda\rho\rangle}_{r-1} - \chi^{\nu\lambda\rho\rangle}_{r+1}\right) \\ &\quad + \int \mathrm{d}\Gamma F(\mathfrak{s})k^{\langle\kappa\rangle}\left[E^{r-1}_{\mathbf{k}}\Delta^{\mu\nu\lambda\rho}_{\alpha\beta\gamma\delta}\nabla_\kappa k^{\langle\alpha}k^{\beta}k^{\gamma}k^{\delta\rangle} + (r-1)E^{r-2}_{\mathbf{k}}k^{\langle\mu}k^{\nu}k^{\lambda}k^{\rho\rangle}k^{\langle\xi\rangle}\nabla_\lambda u_\xi\right]\delta f_{\mathbf{k}\mathfrak{s}} \\ &= -\frac{7}{3}\theta\chi^{\mu\nu\lambda\rho}_r - \Delta^{\mu\nu\lambda\rho}_{\alpha\beta\gamma\delta}\nabla_\kappa\chi^{\kappa\alpha\beta\gamma\delta}_{r-1} - \frac{4}{9}\nabla^{\langle\mu}\left(m^2\chi^{\nu\lambda\rho\rangle}_{r-1} - \chi^{\nu\lambda\rho\rangle}_{r+1}\right) - 4\chi^{\kappa\langle\mu\nu\lambda}_r\sigma^{\rho\rangle}{}_\kappa + 4\chi^{\kappa\langle\mu\nu\lambda}_r\omega^{\rho\rangle}{}_\kappa \\ &\quad - \frac{12}{7}\sigma^{\langle\mu\nu}\left(m^2\chi^{\lambda\rho\rangle}_r - \chi^{\lambda\rho\rangle}_{r+2}\right) + (r-1)\Big[\sigma_{\kappa\xi}\chi^{\kappa\xi\mu\nu\lambda\rho}_{r-2} + \frac{\theta}{3}\left(m^2\chi^{\mu\nu\lambda\rho}_{r-2} - \chi^{\mu\nu\lambda\rho}_r\right) \\ &\quad + \frac{8}{11}\left(m^2\chi^{\kappa\langle\mu\nu\lambda}_{r-2} - \chi^{\kappa\langle\mu\nu\lambda}_r\right)\sigma^{\rho\rangle}{}_\kappa + \frac{4}{21}\sigma^{\langle\mu\nu}\left(m^4\chi^{\lambda\rho\rangle}_{r-2} - 2m^2\chi^{\lambda\rho\rangle}_r + \chi^{\lambda\rho\rangle}_{r+2}\right)\Big] . \end{aligned} \tag{D.38}$$

When putting these expressions together, we arrive at

$$\begin{aligned}\dot{\chi}_r^{\langle\mu\nu\lambda\rho\rangle} - \mathfrak{C}^{\langle\mu\nu\lambda\rho\rangle}_{r-1} &= \frac{4}{21}\sigma^{\langle\mu\nu}\Big[m^4(r-1)\chi^{\lambda\rho\rangle}_{r-2} - m^2(2r+7)\chi^{\lambda\rho\rangle}_r + (r+8)\chi^{\lambda\rho\rangle}_{r+2}\Big] + r\dot{u}_\kappa\chi^{\kappa\mu\nu\lambda\rho}_{r-1} \\ &\quad + \frac{4}{9}\dot{u}^{\langle\mu}\Big[rm^2\chi^{\nu\lambda\rho\rangle}_{r-1} - (r+9)\chi^{\nu\lambda\rho\rangle}_{r+1}\Big] - \frac{4}{9}\nabla^{\langle\mu}\left(m^2\chi^{\nu\lambda\rho\rangle}_{r-1} - \chi^{\nu\lambda\rho\rangle}_{r+1}\right) \\ &\quad + \frac{\theta}{3}\Big[m^2(r-1)\chi^{\mu\nu\lambda\rho}_{r-2} - (r+6)\chi^{\mu\nu\lambda\rho}_r\Big] + (r-1)\chi^{\mu\nu\lambda\rho\xi\kappa}_{r-2}\sigma_{\xi\kappa} - \Delta^{\mu\nu\lambda\rho}_{\alpha\beta\gamma\delta}\nabla_\kappa\chi^{\alpha\beta\gamma\delta\kappa}_{r-1} \\ &\quad + 4\chi^{\kappa\langle\mu\nu\lambda}_r\omega^{\rho\rangle}{}_\kappa + \frac{4}{11}\Big[2m^2(r-1)\chi^{\kappa\langle\mu\nu\lambda}_{r-2} - (2r+9)\chi^{\kappa\langle\mu\nu\lambda}_r\Big]\sigma^{\rho\rangle}{}_\kappa , \end{aligned} \tag{D.39}$$

which yields Eq. (6.80e) when setting $\chi \equiv \psi$.

Appendix E

Calculations for second-order hydrodynamics

In this appendix we will show how to arrive at the equations of dissipative relativistic second-order hydrodynamics (6.156)–(6.161), and list the transport coefficients appearing therein.

E.1 Derivation of the hydrodynamic equations

In order to derive the hydrodynamic equations, we start from the equations of motion for the irreducible moments and use the asymptotic matching conditions to rewrite the terms of second order as functions of the hydrodynamic variables. Then, it is straightforward to obtain the final form of the second-order equations through the inversion of the linearized collision matrix.

E.1.1 Energy-momentum tensor and particle four-current

The quantities appearing in the energy-momentum tensor and the particle four-current, i.e., the bulk viscous pressure Π, the particle-diffusion current n^μ, and the shear-stress tensor $\pi^{\mu\nu}$, all couple with certain contractions of moments of spin-rank two, as was discussed in the main text. Thus, we have to evaluate the respective equations together, which we group by their transformation properties. Explicitly, we have two types of scalar moments (ρ_r and p_r) that can be matched to Π, three types of vectorial moments (ρ_r^μ, p_r^μ, and q_r^μ) that will play into the equation of motion for n^μ, and four tensorial moments ($\rho_r^{\mu\nu}$, $\psi_r^{\langle\mu\nu\rangle}$, $p_r^{\mu\nu}$, and $q_r^{\mu\nu}$) contributing to $\pi^{\mu\nu}$.

Note that, beside the asymptotic matchings (6.139), (6.142), and (6.147), we need to express the comoving derivatives of α_0, β_0, and u^μ that appear in terms of second order in the moment equations. For them, we employ the evolution equations (6.39) to first order in Kn and Re^{-1}, giving

$$\dot{\alpha}_0 \simeq \mathcal{H}\theta \,, \qquad \dot{\beta}_0 \simeq \overline{\mathcal{H}}\theta \,, \qquad \dot{u}^\mu \simeq \frac{F^\mu}{\varepsilon_0 + P_0} \,, \tag{E.1}$$

where we introduced

$$\mathcal{H} \coloneqq \frac{J_{20}(\varepsilon_0 + P_0) - J_{30} n_0}{D_{20}} \,, \qquad \overline{\mathcal{H}} \coloneqq \frac{J_{10}(\varepsilon_0 + P_0) - J_{20} n_0}{D_{20}} \,. \tag{E.2}$$

These relations can then be used to express the derivatives of any function $\mathscr{F}$ which depends on the equilibrium distribution function as

$$\partial^\mu \mathscr{F} \simeq u^\mu \theta \left(\frac{\partial \mathscr{F}}{\partial \alpha_0} \mathcal{H} + \frac{\partial \mathscr{F}}{\partial \beta_0} \overline{\mathcal{H}} \right) + I^\mu \left(\frac{\partial \mathscr{F}}{\partial \alpha_0} + \frac{1}{h} \frac{\partial \mathscr{F}}{\partial \beta_0} \right) - \frac{\beta_0}{\varepsilon_0 + P_0} \frac{\partial \mathscr{F}}{\partial \beta_0} F^\mu \,. \tag{E.3}$$

Here we defined the enthalpy per particle $h \coloneqq (\varepsilon_0 + P_0)/n_0$ and replaced the gradients of β_0 via the relation (6.41).

Scalars

First, we rewrite the moment equation (6.75a) by employing the asymptotic matching (6.139), yielding

$$\tilde{\tau}_{\Pi,r} \dot{\Pi} + \frac{m^2}{3} C_{r-1} = -\frac{m^2}{3} \alpha_r^{(0)} \theta - \tilde{\ell}_{\Pi n,r} \nabla_\mu n^\mu - \tilde{\tau}_{\Pi n,r} n_\mu F^\mu - \tilde{\delta}_{\Pi\Pi,r} \Pi \theta - \tilde{\lambda}_{\Pi n,r} n_\mu I^\mu + \tilde{\lambda}_{\Pi\pi,r} \pi^{\mu\nu} \sigma_{\mu\nu} \,. \tag{E.4}$$

Here and in the following, the r-dependent coefficients are listed in Subsec. E.2.1. The second scalar equation is given by the contraction of the moment equation (6.80c). Here, we have to employ the asymptotic matching (6.147) for the moments of spin-rank two, as well as the expressions (6.150) for their components parallel to u^μ. Taking these together with Eqs. (6.153), all moments of second rank in spin have to be replaced via the following rules:

$$\psi_r^{\mu\nu} \simeq -\frac{3}{m^2} \Upsilon_r^{(00)} \Pi \left(u^\mu u^\nu - \frac{1}{3} \Delta^{\mu\nu} \right) + \Upsilon_r^{(01)} u^{(\mu} n^{\nu)} + \mathcal{T}_{r0}^{(20)} \pi^{\mu\nu} \,, \tag{E.5a}$$

$$\psi_r^{\mu\nu,\lambda} \simeq \Upsilon_r^{(11)} n^\lambda \left(u^\mu u^\nu - \frac{1}{3} \Delta^{\mu\nu} \right) + \Upsilon_r^{(12)} u^{(\mu} \pi^{\nu)\lambda} - \frac{1}{m^2} \Upsilon_r^{(10)} \Pi u^{(\mu} \Delta^{\nu)\lambda} + \frac{3}{5} \mathcal{T}_{r0}^{(11)} n^{\langle\mu} \Delta^{\nu\rangle\lambda} \,, \tag{E.5b}$$

$$\begin{aligned} \psi_r^{\mu\nu,\lambda\alpha} \simeq{}& \Upsilon_r^{(22)} \pi^{\lambda\alpha} \left(u^\mu u^\nu - \frac{1}{3} \Delta^{\mu\nu} \right) + \frac{3}{5} \Upsilon_r^{(21)} u^{(\mu} \Delta^{\nu)\langle\lambda} n^{\alpha\rangle} \\ &- \frac{3}{5m^2} \mathcal{T}_{r0}^{(00)} \Pi \Delta^{\mu\nu,\lambda\alpha} + \frac{12}{7} \mathcal{T}_{r0}^{(22)} \Delta_{\gamma\delta} \Delta^{\lambda\alpha,\gamma\langle\mu} \pi^{\nu\rangle\delta} \,, \end{aligned} \tag{E.5c}$$

$$\psi_r^{\mu\nu,\lambda\alpha\beta} \simeq \frac{5}{7} \Upsilon_r^{(32)} u^{(\mu} \Delta^{\nu)\langle\lambda} \pi^{\alpha\beta\rangle} + \frac{3}{7} \mathcal{T}_{r0}^{(13)} \Delta^{\mu\nu,\gamma\delta} \Delta^{\lambda\alpha\beta}_{\gamma\delta\rho} n^\rho \,, \tag{E.5d}$$

$$\psi_r^{\mu\nu,\lambda\alpha\beta\gamma} \simeq \frac{5}{9} \mathcal{T}_{r0}^{(24)} \Delta^{\mu\nu,\rho\sigma} \Delta^{\lambda\alpha\beta\gamma}_{\rho\sigma\zeta\eta} \pi^{\zeta\eta} \,. \tag{E.5e}$$

All moments of spin-rank two and momentum-rank higher than four do not feature contributions of first order in KNUDSEN and inverse REYNOLDS numbers. Then, the contraction of Eq. (6.80c) reads

$$\tilde{\tau}_{p\Pi,r} \dot{\Pi} + \frac{m^2}{3} \mathfrak{C}_{p,r-1} = -\tilde{\ell}_{p\Pi n,r} \nabla_\mu n^\mu - \tilde{\tau}_{p\Pi n,r} n_\mu F^\mu - \tilde{\delta}_{p\Pi\Pi,r} \Pi \theta - \tilde{\lambda}_{p\Pi n,r} n_\mu I^\mu + \tilde{\lambda}_{p\Pi\pi,r} \pi^{\mu\nu} \sigma_{\mu\nu} \,, \tag{E.6}$$

which (as expected) looks very similar to Eq. (E.4), with the difference that the first-order term $\sim \theta$ is absent. Upon inserting the form of the linearized collision terms C_{r-1} and $\mathfrak{C}_{p,r-1}$ and inverting the corresponding matrix [cf. Eq. (6.115)], we obtain Eq. (6.156) in the main text. The coefficients appearing therein are listed in Subsec. E.2.2.

Vectors

Rewriting the moment equation (6.75b) with the asymptotic matching (6.139) gives

$$\begin{aligned} \tilde{\tau}_{n,r} \dot{n}^{\langle\mu\rangle} - C_{r-1}^{\langle\mu\rangle} ={}& \alpha_r^{(1)} I^\mu - \tilde{\tau}_{n,r} n_\nu \omega^{\nu\mu} - \tilde{\delta}_{nn,r} n^\mu \theta - \tilde{\ell}_{n\Pi,r} \nabla^\mu \Pi + \tilde{\ell}_{n\pi,r} \Delta^{\mu\nu} \nabla_\lambda \pi^\lambda{}_\nu + \tilde{\tau}_{n\Pi,r} \Pi F^\mu \\ &- \tilde{\tau}_{n\pi,r} \pi^{\mu\nu} F_\nu - \tilde{\lambda}_{nn,r} \sigma^{\mu\nu} n_\nu + \tilde{\lambda}_{n\Pi,r} \Pi I^\mu - \tilde{\lambda}_{n\pi,r} \pi^{\mu\nu} I_\nu \,, \end{aligned} \tag{E.7}$$

where the coefficients are listed again in Subsec. E.2.1. The equations of motion for the moments of spin-rank two that are relevant (namely p_r^μ and q_r^μ) are obtained by contracting Eqs. (6.80b) and (6.80d). They yield

$$\tilde{\tau}_{pn,r}\dot{n}^{\langle\mu\rangle} - \mathfrak{C}^\mu_{p,r-1} = -\tilde{\lambda}_{pn\omega,r}n_\nu\omega^{\nu\mu} - \tilde{\delta}_{pnn,r}n^\mu\theta - \tilde{\ell}_{pn\Pi,r}\nabla^\mu\Pi + \tilde{\ell}_{pn\pi,r}\Delta^{\mu\nu}\nabla_\lambda\pi^\lambda{}_\nu + \tilde{\tau}_{pn\Pi,r}\Pi F^\mu - \tilde{\tau}_{pn\pi,r}\pi^{\mu\nu}F_\nu - \tilde{\lambda}_{pnn,r}\sigma^{\mu\nu}n_\nu + \tilde{\lambda}_{pn\Pi,r}\Pi I^\mu - \tilde{\lambda}_{pn\pi,r}\pi^{\mu\nu}I_\nu \tag{E.8}$$

and

$$\tilde{\tau}_{qn,r}\dot{n}^{\langle\mu\rangle} - \mathfrak{C}^\mu_{q,r-1} = -\tilde{\lambda}_{qn\omega,r}n_\nu\omega^{\nu\mu} - \tilde{\delta}_{qnn,r}n^\mu\theta - \tilde{\ell}_{qn\Pi,r}\nabla^\mu\Pi + \tilde{\ell}_{qn\pi,r}\Delta^{\mu\nu}\nabla_\lambda\pi^\lambda{}_\nu + \tilde{\tau}_{qn\Pi,r}\Pi F^\mu - \tilde{\tau}_{qn\pi,r}\pi^{\mu\nu}F_\nu - \tilde{\lambda}_{qnn,r}\sigma^{\mu\nu}n_\nu + \tilde{\lambda}_{qn\Pi,r}\Pi I^\mu - \tilde{\lambda}_{qn\pi,r}\pi^{\mu\nu}I_\nu\ , \tag{E.9}$$

respectively. Writing the three equations above as one vector equation and inverting the collision matrix (6.119) then gives Eq. (6.157), with the coefficients defined in Subsec. E.2.2.

Tensors

Applying the asymptotic matching (6.139) to the moment equation (6.75c) of energy-rank r gives

$$\tilde{\tau}_{\pi,r}\dot{\pi}^{\langle\mu\nu\rangle} - C^{\langle\mu\nu\rangle}_{r-1} = 2\alpha_r^{(2)}\sigma^{\mu\nu} + 2\tilde{\tau}_{\pi,r}\pi_\lambda{}^{\langle\mu}\omega^{\nu\rangle\lambda} - \tilde{\delta}_{\pi\pi,r}\pi^{\mu\nu}\theta - \tilde{\tau}_{\pi\pi,r}\pi^{\lambda\langle\mu}\sigma^{\nu\rangle}{}_\lambda + \tilde{\lambda}_{\pi\Pi,r}\Pi\sigma^{\mu\nu} - \tilde{\tau}_{\pi n,r}n^{\langle\mu}F^{\nu\rangle} + \tilde{\ell}_{\pi n,r}\nabla^{\langle\mu}n^{\nu\rangle} + \tilde{\lambda}_{\pi n,r}n^{\langle\mu}I^{\nu\rangle}\ . \tag{E.10}$$

After applying the asymptotic matching (6.147), Eq. (6.80a) becomes

$$\tilde{\tau}_{\psi\pi,r}\dot{\pi}^{\langle\mu\nu\rangle} - \mathfrak{C}^{\langle\mu\nu\rangle}_{r-1} = \tilde{\lambda}_{\psi\pi\omega,r}\pi_\lambda{}^{\langle\mu}\omega^{\nu\rangle\lambda} - \tilde{\delta}_{\psi\pi\pi,r}\pi^{\mu\nu}\theta - \tilde{\tau}_{\psi\pi\pi,r}\pi^{\lambda\langle\mu}\sigma^{\nu\rangle}{}_\lambda + \tilde{\lambda}_{\psi\pi\Pi,r}\Pi\sigma^{\mu\nu} - \tilde{\tau}_{\psi\pi n,r}n^{\langle\mu}F^{\nu\rangle} + \tilde{\ell}_{\psi\pi n,r}\nabla^{\langle\mu}n^{\nu\rangle} + \tilde{\lambda}_{\psi\pi n,r}n^{\langle\mu}I^{\nu\rangle}\ . \tag{E.11}$$

In order to obtain the remaining two tensor equations, i.e., the equations of motion for $p_r^{\mu\nu}$ and $q_r^{\mu\nu}$, we have to contract Eqs. (6.80c) and (6.80e) appropriately, which yield

$$\tilde{\tau}_{p\pi,r}\dot{\pi}^{\langle\mu\nu\rangle} - \mathfrak{C}^{\mu\nu}_{p,r-1} = \tilde{\lambda}_{p\pi\omega,r}\pi_\lambda{}^{\langle\mu}\omega^{\nu\rangle\lambda} - \tilde{\delta}_{p\pi\pi,r}\pi^{\mu\nu}\theta - \tilde{\tau}_{p\pi\pi,r}\pi^{\lambda\langle\mu}\sigma^{\nu\rangle}{}_\lambda + \tilde{\lambda}_{p\pi\Pi,r}\Pi\sigma^{\mu\nu} - \tilde{\tau}_{p\pi n,r}n^{\langle\mu}F^{\nu\rangle} + \tilde{\ell}_{p\pi n,r}\nabla^{\langle\mu}n^{\nu\rangle} + \tilde{\lambda}_{p\pi n,r}n^{\langle\mu}I^{\nu\rangle} \tag{E.12}$$

and

$$\tilde{\tau}_{q\pi,r}\dot{\pi}^{\langle\mu\nu\rangle} - \mathfrak{C}^{\mu\nu}_{q,r-1} = \tilde{\lambda}_{q\pi\omega,r}\pi_\lambda{}^{\langle\mu}\omega^{\nu\rangle\lambda} - \tilde{\delta}_{q\pi\pi,r}\pi^{\mu\nu}\theta - \tilde{\tau}_{q\pi\pi,r}\pi^{\lambda\langle\mu}\sigma^{\nu\rangle}{}_\lambda + \tilde{\lambda}_{q\pi\Pi,r}\Pi\sigma^{\mu\nu} - \tilde{\tau}_{q\pi n,r}n^{\langle\mu}F^{\nu\rangle} + \tilde{\ell}_{q\pi n,r}\nabla^{\langle\mu}n^{\nu\rangle} + \tilde{\lambda}_{q\pi n,r}n^{\langle\mu}I^{\nu\rangle}\ . \tag{E.13}$$

Then, after inverting the collision matrix (6.123), we obtain Eq. (6.158) in the main text.

E.1.2 Spin tensor

In the case of the spin tensor, there are two moments of axial-vector type ($\tau_r^{\langle\mu\rangle}$ and t_r^μ), which have to be treated together with the respective component ω_0^μ of the spin potential. Similarly, one vectorial moment w_r^μ is determined together with κ_0^μ, whereas there is only one moment of tensor type $t_r^{\mu\nu}$ that does not couple to the spin potential. First, we have to express the moments that appear in the second-order terms of Eqs. (6.78), i.e., we have to write down relations analogous to the ones in Eqs. (E.5). Taking into account Eqs. (6.145) as well as the asymptotic matching (6.142), we find

$$\tau_r^\mu \simeq \mathcal{Q}_{r0}^{(10)}\mathfrak{p}^\mu\ , \tag{E.14a}$$

$$\tau_r^{\mu,\nu} \simeq \mathcal{X}_r^{(10)}u^\mu\mathfrak{p}^\nu + \mathcal{X}_r^{(12)}u^\mu\mathfrak{q}^\nu + \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\alpha\left(\mathcal{Q}_{r0}^{(11)}\mathfrak{w}_\beta + \mathcal{Q}_r^{(\kappa)}\kappa_{0,\beta}\right)\ , \tag{E.14b}$$

$$\tau_r^{\mu,\nu\lambda} \simeq \frac{3}{5}\mathcal{Q}_{r0}^{(12)}\Delta^{\mu\langle\nu}\mathfrak{q}^{\lambda\rangle} - \frac{2}{3}\mathcal{Q}_{r0}^{(22)}\mathfrak{t}_\rho{}^{\langle\lambda}\epsilon^{\nu\rangle\mu\alpha\rho}u_\alpha\ , \tag{E.14c}$$

while the higher moments can be approximated as zero. We can then proceed in the same way as shown before.

Axial vectors

We have to start with the evolution equation for the magnetic component of the spin potential, ω_0^μ, which reads

$$\tilde{\tau}_\omega \dot{\omega}_0^{\langle\mu\rangle} - \mathfrak{C}_\omega^\mu = \tilde{\mathfrak{K}}_{\omega\theta}\theta\omega_0^\mu + \tilde{\mathfrak{K}}_{\omega\theta\mathfrak{p}}\theta\mathfrak{p}^\mu + \tilde{\mathfrak{K}}_{\omega\sigma}\sigma^{\mu\nu}\omega_{0,\nu} + \tilde{\mathfrak{K}}_{\omega\sigma\mathfrak{p}}\sigma^{\mu\nu}\mathfrak{p}_\nu + \tilde{\mathfrak{K}}_{\omega\mathfrak{t}}\mathfrak{t}^{\mu\nu}\omega_\nu \\ + \epsilon^{\mu\nu\alpha\beta}u_\nu\left(\tilde{\mathfrak{h}}_{\omega\kappa}\nabla_\alpha\kappa_{0,\beta} + \tilde{\mathfrak{K}}_{\omega I\kappa,r}I_\alpha\kappa_{0,\beta} + \tilde{\mathfrak{K}}_{\omega F\kappa,r}F_\alpha\kappa_{0,\beta}\right) , \tag{E.15}$$

where we defined

$$\mathfrak{C}_\omega^\mu := -\omega_0^\mu - \sum_n \frac{\gamma_n^{(0)}}{\Gamma^{(\omega)}}\tau_n^\mu - \sum_n \frac{\gamma_n^{(2)}}{\Gamma^{(\omega)}}t_n^\mu - \beta_0\omega^\mu , \tag{E.16}$$

cf. Eq. (6.62b). After using the relations (E.14), we find from the moment equation (6.78a):

$$\tilde{\tau}_{\mathfrak{p},r}\dot{\mathfrak{p}}^{\langle\mu\rangle} + \tilde{\tau}_{\mathfrak{p}\omega,r}\dot{\omega}_0^{\langle\mu\rangle} - \mathfrak{C}_{r-1}^{\langle\mu\rangle} = \tilde{\mathfrak{K}}_{\mathfrak{p}\theta,r}\theta\mathfrak{p}^\mu + \tilde{\mathfrak{K}}_{\mathfrak{p}\theta\mathfrak{q},r}\theta\mathfrak{q}^\mu + \tilde{\mathfrak{K}}_{\mathfrak{p}\theta\omega,r}\theta\omega_0^\mu + \tilde{\mathfrak{K}}_{\mathfrak{p}\sigma,r}\sigma^{\mu\nu}\mathfrak{p}_\nu + \tilde{\mathfrak{K}}_{\mathfrak{p}\sigma\mathfrak{q},r}\sigma^{\mu\nu}\mathfrak{q}_\nu \\ + \tilde{\mathfrak{K}}_{\mathfrak{p}\sigma\omega,r}\sigma^{\mu\nu}\omega_{0,\nu} + \epsilon^{\mu\nu\alpha\beta}u_\nu\Big(\tilde{\mathfrak{h}}_{\mathfrak{p}\mathfrak{w},r}\nabla_\alpha\mathfrak{w}_\beta + \tilde{\mathfrak{h}}_{\mathfrak{p}\kappa,r}\nabla_\alpha\kappa_{0,\beta} + \tilde{\mathfrak{K}}_{\mathfrak{p}I\mathfrak{w},r}I_\alpha\mathfrak{w}_\beta \\ + \tilde{\mathfrak{K}}_{\mathfrak{p}F\mathfrak{w},r}F_\alpha\mathfrak{w}_\beta + \tilde{\mathfrak{K}}_{\mathfrak{p}I\kappa,r}I_\alpha\kappa_{0,\beta} + \tilde{\mathfrak{K}}_{\mathfrak{p}F\kappa,r}F_\alpha\kappa_{0,\beta}\Big) . \tag{E.17}$$

The moment equation (6.78c), after being contracted with $\Delta_{\mu\lambda}$, becomes

$$\tilde{\tau}_{\mathfrak{q},r}\dot{\mathfrak{q}}^{\langle\mu\rangle} - \mathfrak{C}_{r-1\alpha}^{\alpha,\mu} = \tilde{\mathfrak{K}}_{\mathfrak{q}\theta,r}\theta\mathfrak{q}^\mu + \tilde{\mathfrak{K}}_{\mathfrak{q}\theta\mathfrak{p},r}\theta\mathfrak{p}^\mu + \tilde{\mathfrak{K}}_{\mathfrak{q}\sigma,r}\sigma^{\mu\nu}\mathfrak{q}_\nu + \tilde{\mathfrak{K}}_{\mathfrak{q}\sigma\omega,r}\sigma^{\mu\nu}\omega_{0,\nu} \\ + \tilde{\mathfrak{K}}_{\mathfrak{q}\sigma\mathfrak{p},r}\sigma^{\mu\nu}\mathfrak{p}_\nu + \tilde{\mathfrak{K}}_{\mathfrak{q}\mathfrak{t},r}\mathfrak{t}^{\mu\nu}\omega_\nu + \epsilon^{\mu\nu\alpha\beta}u_\nu\Big(\tilde{\mathfrak{h}}_{\mathfrak{q}\mathfrak{w},r}\nabla_\alpha\mathfrak{w}_\beta + \tilde{\mathfrak{h}}_{\mathfrak{q}\kappa,r}\nabla_\alpha\kappa_{0,\beta} + \tilde{\mathfrak{K}}_{\mathfrak{q}I\mathfrak{w},r}I_\alpha\mathfrak{w}_\beta \\ + \tilde{\mathfrak{K}}_{\mathfrak{q}F\mathfrak{w},r}F_\alpha\mathfrak{w}_\beta + \tilde{\mathfrak{K}}_{\mathfrak{q}I\kappa,r}I_\alpha\kappa_{0,\beta} + \tilde{\mathfrak{K}}_{\mathfrak{q}F\kappa,r}F_\alpha\kappa_{0,\beta}\Big) . \tag{E.18}$$

The r-dependent coefficients can be found as before in Subsec. E.2.1.

Note that in these equations there are no terms $\sim \omega^{\mu\nu}\mathfrak{p}_\nu, \omega^{\mu\nu}\mathfrak{q}_\nu, \omega^{\mu\nu}\omega_{0,\nu}$. The reason for this lies in the fact that up to second order we may replace $\mathfrak{p}^\mu, \mathfrak{q}^\mu$, and ω_0^μ by their NAVIER-STOKES values, which are proportional to ω^μ, and we have $\omega^{\mu\nu}\omega_\nu = 0$. Similarly, there are no terms $\sim \epsilon^{\langle\mu\rangle\nu\alpha\beta}\omega_{\nu\alpha}\kappa_\beta$, since we cannot have four linearly independent vectors orthogonal to u^μ. Lastly, terms $\sim \sigma_{\mu\rho}\mathfrak{t}_\gamma{}^\rho\epsilon^{\nu\mu\delta\gamma}u_\delta$ vanish to second order since $\mathfrak{t}^{\mu\nu} \sim \sigma^{\mu\nu}$.

The inversion of the collision matrix (6.127) then produces Eqs. (6.159), with the transport coefficients again given in Subsec. E.2.2.

Vectors

The first vector-valued quantity appearing in the spin tensor is the electric part of the spin potential κ_0^μ, whose evolution equation reads

$$\tilde{\tau}_\kappa\dot{\kappa}_0^{\langle\mu\rangle} - \mathfrak{C}_\kappa^\mu = \tilde{\mathfrak{K}}_{\kappa\theta}\theta\kappa_0^\mu + \tilde{\mathfrak{K}}_{\kappa\sigma}\sigma^{\mu\nu}\kappa_{0,\nu} + \tilde{\mathfrak{K}}_{\kappa\omega}\omega^{\mu\nu}\kappa_{0,\nu} + \tilde{\mathfrak{h}}_{\kappa\mathfrak{t}}\Delta^\mu{}_\lambda\nabla_\nu\mathfrak{t}^{\nu\lambda} + \tilde{\mathfrak{K}}_{\kappa F\mathfrak{t}}\mathfrak{t}^{\mu\nu}F_\nu \\ + \epsilon^{\mu\nu\alpha\beta}u_\nu\left(\tilde{\mathfrak{h}}_{\kappa\omega}\nabla_\alpha\omega_{0,\beta} + \tilde{\mathfrak{h}}_{\kappa\mathfrak{p}}\nabla_\alpha\mathfrak{p}_\beta + \tilde{\mathfrak{K}}_{\kappa I\omega}I_\alpha\omega_{0,\beta} + \tilde{\mathfrak{K}}_{\kappa F\omega}F_\alpha\omega_{0,\beta} + \tilde{\mathfrak{K}}_{\kappa F\mathfrak{p}}F_\alpha\mathfrak{p}_\beta\right) , \tag{E.19}$$

where we defined

$$\mathfrak{C}_\kappa^\mu := -\kappa_0^\mu - \sum_n \frac{\gamma_n^{(1)}}{\Gamma^{(\kappa)}}w_n^\mu - \frac{\beta_0 F^\mu}{\varepsilon_0 + P_0} + \Gamma^{(I)}I^\mu , \tag{E.20}$$

cf. Eq. (6.62a). On the other hand, contracting Eq. (6.78b) with $\epsilon^{\mu\nu\alpha\beta}u_\nu$ gives

$$\begin{aligned}\tilde{\tau}_{\mathfrak{w},r}\dot{\mathfrak{w}}^{\langle\mu\rangle} + \tilde{\tau}_{\mathfrak{w}\kappa,r}\dot{\kappa}_0^{\langle\mu\rangle} - \mathfrak{C}^{\mu}_{\mathfrak{w},r-1} = {}& \tilde{\mathfrak{K}}_{\mathfrak{w}\theta,r}\theta\mathfrak{w}^\mu + \tilde{\mathfrak{K}}_{\mathfrak{w}\theta\kappa,r}\theta\kappa_0^\mu + \tilde{\mathfrak{K}}_{\mathfrak{w}\sigma,r}\sigma^{\mu\nu}\mathfrak{w}_\nu + \tilde{\mathfrak{K}}_{\mathfrak{w}\sigma\kappa,r}\sigma^{\mu\nu}\kappa_{0,\nu}\\ &+ \tilde{\mathfrak{K}}_{\mathfrak{w}\omega,r}\omega^{\mu\nu}\mathfrak{w}_\nu + \tilde{\mathfrak{K}}_{\mathfrak{w}\omega\kappa,r}\omega^{\mu\nu}\kappa_{0,\nu} + \tilde{\mathfrak{h}}_{\mathfrak{w}\mathfrak{t},r}\Delta^\mu{}_\lambda\nabla_\nu\mathfrak{t}^{\nu\lambda} + \tilde{\mathfrak{K}}_{\mathfrak{w}I\mathfrak{t},r}\mathfrak{t}^{\mu\nu}I_\nu\\ &+ \tilde{\mathfrak{K}}_{\mathfrak{w}F\mathfrak{t},r}\mathfrak{t}^{\mu\nu}F_\nu + \epsilon^{\mu\nu\alpha\beta}u_\nu\Big(\tilde{\mathfrak{h}}_{\mathfrak{w}\mathfrak{p},r}\nabla_\alpha\mathfrak{p}_\beta + \tilde{\mathfrak{h}}_{\mathfrak{w}\mathfrak{q},r}\nabla_\alpha\mathfrak{q}_\beta + \tilde{\mathfrak{K}}_{\mathfrak{w}I\omega,r}I_\alpha\omega_{0,\beta}\\ &+ \tilde{\mathfrak{K}}_{\mathfrak{w}I\mathfrak{p},r}I_\alpha\mathfrak{p}_\beta + \tilde{\mathfrak{K}}_{\mathfrak{w}I\mathfrak{q},r}I_\alpha\mathfrak{q}_\beta + \tilde{\mathfrak{K}}_{\mathfrak{w}F\mathfrak{p},r}F_\alpha\mathfrak{p}_\beta + \tilde{\mathfrak{K}}_{\mathfrak{w}F\mathfrak{q},r}F_\alpha\mathfrak{q}_\beta\Big)\,. \end{aligned} \tag{E.21}$$

Inverting the collision matrix (6.131), we find Eqs. (6.160).

Tensor

Contracting the moment equation (6.78c) with $\Delta^{\mu\nu}_{\gamma\delta}\epsilon^{\delta\alpha\beta\rho}u_\rho$ yields

$$\begin{aligned}\tilde{\tau}_{\mathfrak{t},r}\dot{\mathfrak{t}}^{\langle\mu\nu\rangle} - \mathfrak{C}^{\mu\nu}_{\mathfrak{t},r-1} = {}& \tilde{\mathfrak{K}}_{\mathfrak{t}\theta,r}\theta\mathfrak{t}^{\mu\nu} + \tilde{\mathfrak{h}}_{\mathfrak{t}\mathfrak{w},r}\nabla^{\langle\mu}\mathfrak{w}^{\nu\rangle} + \tilde{\mathfrak{h}}_{\mathfrak{t}\kappa,r}\nabla^{\langle\mu}\kappa_0^{\nu\rangle} + \tilde{\mathfrak{K}}_{\mathfrak{t}I\mathfrak{w},r}I^{\langle\mu}\mathfrak{w}^{\nu\rangle} + \tilde{\mathfrak{K}}_{\mathfrak{t}F\mathfrak{w},r}F^{\langle\mu}\mathfrak{w}^{\nu\rangle}\\ &+ \tilde{\mathfrak{K}}_{\mathfrak{t}I\kappa,r}I^{\langle\mu}\kappa_0^{\nu\rangle} + \tilde{\mathfrak{K}}_{\mathfrak{t}F\kappa,r}F^{\langle\mu}\kappa_0^{\nu\rangle} + \tilde{\mathfrak{K}}_{\mathfrak{t}\omega\omega,r}\omega_0^{\langle\mu}\omega^{\nu\rangle} + \tilde{\mathfrak{K}}_{\mathfrak{t}\omega\mathfrak{p},r}\mathfrak{p}^{\langle\mu}\omega^{\nu\rangle} + \tilde{\mathfrak{K}}_{\mathfrak{t}\omega\mathfrak{q},r}\mathfrak{q}^{\langle\mu}\omega^{\nu\rangle}\\ &+ \sigma_\alpha{}^{\langle\mu}\epsilon^{\nu\rangle\alpha\beta\gamma}u_\beta\left(\tilde{\mathfrak{K}}_{\mathfrak{t}\sigma\omega,r}\omega_{0,\gamma} + \tilde{\mathfrak{K}}_{\mathfrak{t}\sigma\mathfrak{p},r}\mathfrak{p}_\gamma + \tilde{\mathfrak{K}}_{\mathfrak{t}\sigma\mathfrak{q},r}\mathfrak{q}_\gamma\right)\,, \end{aligned} \tag{E.22}$$

which, after the inversion of the collision term, gives Eq. (6.161) in the main text.

E.2 Transport coefficients

In this section, we list the transport coefficients appearing in the main text as well as in the first part of this appendix. To obtain them, one has to perform the contractions of the various moment equations as described in the previous section. The following contractions of irreducible tensors prove helpful in performing these computations,

$$\Delta^{\mu_1\cdots\mu_{\ell-1}\lambda}_{\nu_1\cdots\nu_{\ell-1}\lambda} = \frac{2\ell+1}{2\ell-1}\Delta^{\mu_1\cdots\mu_{\ell-1}}_{\nu_1\cdots\nu_{\ell-1}}\,, \qquad \Delta^{\mu\nu}_{\gamma\delta}\Delta^{\zeta\gamma}_{\alpha\xi}\Delta^{\delta}{}_{\zeta}{}^{\xi}{}_{\beta} = \frac{7}{12}\Delta^{\mu\nu}_{\alpha\beta}\,, \tag{E.23}$$

cf., e.g., Ref. [43] for the first identity.

E.2.1 Primary coefficients

To keep the presentation clear, we first list the r-dependent coefficients appearing in the moment equations after performing the asymptotic matching, but before inverting the linearized collision terms.

$T^{\mu\nu}$: Scalars

The coefficients appearing in Eq. (E.4) read

$$\tilde{\tau}_{\Pi,r} \coloneqq \mathcal{R}^{(0)}_{r0}\,, \tag{E.24a}$$

$$\tilde{\ell}_{\Pi n,r} \coloneqq \frac{m^2}{3}\left(\frac{G_{3r}}{D_{20}} - \mathcal{R}^{(1)}_{r-1,0}\right)\,, \tag{E.24b}$$

$$\tilde{\tau}_{\Pi n,r} \coloneqq \frac{m^2}{3(\varepsilon_0+P_0)}\left(r\mathcal{R}^{(1)}_{r-1,0} - \frac{G_{3r}}{D_{20}} + \frac{\partial\mathcal{R}^{(1)}_{r-1,0}}{\partial\ln\beta_0}\right)\,, \tag{E.24c}$$

$$\tilde{\delta}_{\Pi\Pi,r} \coloneqq \frac{\partial \mathcal{R}^{(0)}_{r0}}{\partial \alpha_0}\mathcal{H} + \frac{\partial \mathcal{R}^{(0)}_{r0}}{\partial \beta_0}\overline{\mathcal{H}} + \frac{r+2}{3}\mathcal{R}^{(0)}_{r0} - \frac{m^2}{3}\left[\frac{G_{2r}}{D_{20}} + (r-1)\mathcal{R}^{(0)}_{r-2,0}\right] , \tag{E.24d}$$

$$\tilde{\lambda}_{\Pi n,r} \coloneqq -\frac{m^2}{3}\left(\frac{\partial \mathcal{R}^{(1)}_{r-1,0}}{\partial \alpha_0} + \frac{1}{h}\frac{\partial \mathcal{R}^{(1)}_{r-1,0}}{\partial \beta_0}\right) , \tag{E.24e}$$

$$\tilde{\lambda}_{\Pi\pi,r} \coloneqq \frac{m^2}{3}\left[(1-r)\mathcal{R}^{(2)}_{r-2,0} - \frac{G_{2r}}{D_{20}}\right] . \tag{E.24f}$$

The ones in Eq. (E.6) on the other hand are defined as

$$\tilde{\tau}_{p\Pi,r} \coloneqq \mathcal{T}^{(00)}_{r0} , \tag{E.25a}$$

$$\tilde{\ell}_{p\Pi n,r} \coloneqq -\frac{m^2}{3}\left[\mathcal{T}^{(13)}_{r-1,0} + \frac{2}{5}\left(m^2\mathcal{T}^{(11)}_{r-1,0} - \mathcal{T}^{(11)}_{r+1,0}\right)\right] , \tag{E.25b}$$

$$\begin{aligned}\tilde{\tau}_{p\Pi n,r} \coloneqq \frac{m^2}{3(\varepsilon_0+P_0)}\bigg\{&\frac{2}{5}\left[m^2 r\mathcal{T}^{(11)}_{r-1,0} - (r+5)\mathcal{T}^{(11)}_{r+1,0}\right] - 2\Upsilon^{(21)}_r + r\mathcal{T}^{(13)}_{r-1,0} \\ &+ \frac{\partial}{\partial \ln\beta_0}\left[\mathcal{T}^{(13)}_{r-1,0} + \frac{2}{5}\left(m^2\mathcal{T}^{(11)}_{r-1,0} - \mathcal{T}^{(11)}_{r+1,0}\right)\right]\bigg\} ,\end{aligned} \tag{E.25c}$$

$$\tilde{\delta}_{p\Pi\Pi,r} \coloneqq \frac{\partial \mathcal{T}^{(00)}_{r0}}{\partial \alpha_0}\mathcal{H} + \frac{\partial \mathcal{T}^{(00)}_{r0}}{\partial \beta_0}\overline{\mathcal{H}} + \frac{4}{9}\left(m^2\Upsilon^{(10)}_{r-1} - \Upsilon^{(10)}_{r+1}\right) - \frac{1}{3}\left[m^2(r-1)\mathcal{T}^{(00)}_{r-2,0} - (r+4)\mathcal{T}^{(00)}_{r0}\right] , \tag{E.25d}$$

$$\tilde{\lambda}_{p\Pi n,r} \coloneqq -\frac{m^2}{3}\left(\frac{\partial}{\partial \alpha_0} + \frac{1}{h}\frac{\partial}{\partial \beta_0}\right)\left[\frac{2}{5}\left(m^2\mathcal{T}^{(11)}_{r-1,0} - \mathcal{T}^{(11)}_{r+1,0}\right) + \mathcal{T}^{(13)}_{r-1,0}\right] , \tag{E.25e}$$

$$\begin{aligned}\tilde{\lambda}_{p\Pi\pi,r} \coloneqq \frac{m^2}{3}\bigg\{&\frac{4}{5}\left(m^2\Upsilon^{(12)}_{r-1} - \Upsilon^{(12)}_{r+1}\right) + 2\Upsilon^{(32)}_{r-1} - \frac{2}{7}\left[2m^2(r-1)\mathcal{T}^{(22)}_{r-2,0} - (2r+5)\mathcal{T}^{(22)}_{r0}\right] \\ &- \frac{2}{15}\left[m^4(r-1)\mathcal{T}^{(20)}_{r-2,0} - m^2(2r+3)\mathcal{T}^{(20)}_{r0} + (r+4)\mathcal{T}^{(20)}_{r+2,0}\right] - (r-1)\mathcal{T}^{(24)}_{r-2,0}\bigg\}\end{aligned} \tag{E.25f}$$

N^μ : Vectors

The quantities appearing in Eq. (E.7) read

$$\tilde{\tau}_{n,r} \coloneqq \mathcal{R}^{(1)}_{r0} , \tag{E.26a}$$

$$\tilde{\delta}_{nn,r} \coloneqq \frac{r+3}{3}\mathcal{R}^{(1)}_{r0} - \frac{r-1}{3}m^2\mathcal{R}^{(1)}_{r-2,0} + \frac{\partial \mathcal{R}^{(1)}_{r0}}{\partial \alpha_0}\mathcal{H} + \frac{\partial \mathcal{R}^{(1)}_{r0}}{\partial \beta_0}\overline{\mathcal{H}} , \tag{E.26b}$$

$$\tilde{\ell}_{n\Pi,r} \coloneqq \frac{\beta_0 J_{r+2,1}}{\varepsilon_0+P_0} + \frac{1}{m^2}\mathcal{R}^{(0)}_{r+1,0} - \mathcal{R}^{(0)}_{r-1,0} , \tag{E.26c}$$

$$\tilde{\ell}_{n\pi,r} \coloneqq \frac{\beta_0 J_{r+2,1}}{\varepsilon_0+P_0} - \mathcal{R}^{(2)}_{r-1,0} , \tag{E.26d}$$

$$\tilde{\tau}_{n\Pi,r} \coloneqq \frac{1}{\varepsilon_0+P_0}\left(\frac{\beta_0 J_{r+2,1}}{\varepsilon_0+P_0} - r\mathcal{R}^{(0)}_{r-1,0} + \frac{r+3}{m^2}\mathcal{R}^{(0)}_{r+1,0} - \frac{\partial \mathcal{R}^{(0)}_{r-1,0}}{\partial \ln\beta_0} + \frac{1}{m^2}\frac{\partial \mathcal{R}^{(0)}_{r+1,0}}{\partial \ln\beta_0}\right) , \tag{E.26e}$$

$$\tilde{\tau}_{n\pi,r} \coloneqq \frac{1}{\varepsilon_0+P_0}\left(\frac{\beta_0 J_{r+2,1}}{\varepsilon_0+P_0} - r\mathcal{R}^{(2)}_{r-1,0} - \frac{\partial \mathcal{R}^{(2)}_{r-1,0}}{\partial \ln\beta_0}\right) , \tag{E.26f}$$

$$\tilde{\lambda}_{nn,r} \coloneqq \frac{1}{5}\left[(2r+3)\mathcal{R}^{(1)}_{r0} - 2(r-1)m^2\mathcal{R}^{(1)}_{r-2,0}\right] , \tag{E.26g}$$

$$\tilde{\lambda}_{n\Pi,r} \coloneqq \frac{\partial \mathcal{R}^{(0)}_{r-1,0}}{\partial \alpha_0} + \frac{1}{h}\frac{\partial \mathcal{R}^{(0)}_{r-1,0}}{\partial \beta_0} - \frac{1}{m^2}\left(\frac{\partial \mathcal{R}^{(0)}_{r+1,0}}{\partial \alpha_0} + \frac{1}{h}\frac{\partial \mathcal{R}^{(0)}_{r+1,0}}{\partial \beta_0}\right) , \tag{E.26h}$$

$$\tilde{\lambda}_{n\pi,r} \coloneqq \frac{\partial \mathcal{R}^{(2)}_{r-1,0}}{\partial \alpha_0} + \frac{1}{h}\frac{\partial \mathcal{R}^{(2)}_{r-1,0}}{\partial \beta_0} , \tag{E.26i}$$

whereas the ones in Eq. (E.8) are given by

$$\tilde{\tau}_{pn,r} \coloneqq \mathcal{T}^{(11)}_{r0} \,, \tag{E.27a}$$

$$\tilde{\lambda}_{pn\omega,r} \coloneqq -\frac{1}{2}\mathcal{T}^{(11)}_{r0} + \frac{2}{3}\Upsilon^{(21)}_{r-1} + \frac{5}{9}\left(m^2\Upsilon^{(01)}_{r-1} - \Upsilon^{(01)}_{r+1}\right) \,, \tag{E.27b}$$

$$\begin{aligned}\tilde{\delta}_{pnn,r} \coloneqq{}& \left(\mathcal{H}\frac{\partial}{\partial\alpha_0} + \overline{\mathcal{H}}\frac{\partial}{\partial\beta_0}\right)\mathcal{T}^{(11)}_{r0} \\ &- \frac{1}{3}\left[(r-1)m^2\mathcal{T}^{(11)}_{r-2,0} - (r+3)\mathcal{T}^{(11)}_{r0} - \frac{1}{3}\Upsilon^{(21)}_{r-1} - \frac{10}{9}\left(m^2\Upsilon^{(01)}_{r-1} - \Upsilon^{(01)}_{r+1}\right)\right] \,,\end{aligned} \tag{E.27c}$$

$$\tilde{\ell}_{pn\Pi,r} \coloneqq -\frac{1}{m^2}\mathcal{T}^{(00)}_{r-1,0} \,, \tag{E.27d}$$

$$\tilde{\ell}_{pn\pi,r} \coloneqq \frac{1}{3}\left(\mathcal{T}^{(20)}_{r+1,0} - m^2\mathcal{T}^{(20)}_{r-1,0}\right) - \mathcal{T}^{(22)}_{r-1,0} \,, \tag{E.27e}$$

$$\tilde{\tau}_{pn\Pi,r} \coloneqq \frac{1}{m^2(\varepsilon_0 + P_0)}\left(4\Upsilon^{(10)}_{r} - r\mathcal{T}^{(00)}_{r-1,0} - \frac{\partial\mathcal{T}^{(00)}_{r-1,0}}{\partial\ln\beta_0}\right) \,, \tag{E.27f}$$

$$\begin{aligned}\tilde{\tau}_{pn\pi,r} \coloneqq{}& \frac{1}{\varepsilon_0 + P_0}\Bigg\{\Upsilon^{(12)}_{r} + \frac{1}{3}\left[(r+3)\mathcal{T}^{(20)}_{r+1,0} - m^2 r\mathcal{T}^{(20)}_{r-1,0}\right] \\ &+ \frac{1}{3}\frac{\partial}{\partial\ln\beta_0}\left(\mathcal{T}^{(20)}_{r+1,0} - m^2\mathcal{T}^{(20)}_{r-1,0} - 3\mathcal{T}^{(22)}_{r-1,0}\right) - r\mathcal{T}^{(22)}_{r-1,0}\Bigg\} \,,\end{aligned} \tag{E.27g}$$

$$\begin{aligned}\tilde{\lambda}_{pnn,r} \coloneqq{}& \frac{1}{9}\left(m^2\Upsilon^{(01)}_{r-1} - \Upsilon^{(01)}_{r+1}\right) + \frac{23}{15}\Upsilon^{(21)}_{r-1} - \frac{3}{5}(r-1)\mathcal{T}^{(13)}_{r-2,0} \\ &- \frac{1}{50}\left[(2r-2)m^2\mathcal{T}^{(11)}_{r-2,0} - (2r+3)\mathcal{T}^{(11)}_{r0}\right] \,,\end{aligned} \tag{E.27h}$$

$$\tilde{\lambda}_{pn\Pi,r} \coloneqq \frac{1}{m^2}\left(\frac{\partial}{\partial\alpha_0} + \frac{1}{h}\frac{\partial}{\partial\beta_0}\right)\mathcal{T}^{(00)}_{r-1,0} \,, \tag{E.27i}$$

$$\tilde{\lambda}_{pn\pi,r} \coloneqq \frac{1}{3}\left(\frac{\partial}{\partial\alpha_0} + \frac{1}{h}\frac{\partial}{\partial\beta_0}\right)\left(m^2\mathcal{T}^{(20)}_{r-1,0} - \mathcal{T}^{(20)}_{r+1,0} + 3\mathcal{T}^{(22)}_{r-1,0}\right) \,. \tag{E.27j}$$

Interestingly, due to the contraction there now appears a nontrivial transport coefficient coupling n^μ and $\omega^{\mu\nu}$. The coefficients in Eq. (E.9) are

$$\tilde{\tau}_{qn,r} \coloneqq \mathcal{T}^{(13)}_{r0} \,, \tag{E.28a}$$

$$\tilde{\lambda}_{qn\omega,r} \coloneqq \frac{3}{5}\mathcal{T}^{(13)}_{r0} \,, \tag{E.28b}$$

$$\tilde{\delta}_{qnn,r} \coloneqq \left(\mathcal{H}\frac{\partial}{\partial\alpha_0} + \overline{\mathcal{H}}\frac{\partial}{\partial\beta_0}\right)\mathcal{T}^{(13)}_{r0} + \frac{1}{3}\left[(r+5)\mathcal{T}^{(13)}_{r0} - m^2(r-1)\mathcal{T}^{(13)}_{r-2,0}\right] \,, \tag{E.28c}$$

$$\tilde{\ell}_{qn\Pi,r} \coloneqq \frac{3}{5m^2}\left(\mathcal{T}^{(00)}_{r+1,0} - m^2\mathcal{T}^{(00)}_{r-1,0}\right) \,, \tag{E.28d}$$

$$\tilde{\ell}_{qn\pi,r} \coloneqq \frac{6}{35}\left(\mathcal{T}^{(22)}_{r+1,0} - m^2\mathcal{T}^{(22)}_{r-1,0}\right) - \mathcal{T}^{(24)}_{r-1,0} \,, \tag{E.28e}$$

$$\tilde{\tau}_{qn\Pi,r} \coloneqq \frac{3}{5m^2(\varepsilon_0 + P_0)}\left[(r+7)\mathcal{T}^{(00)}_{r+1,0} - m^2 r\mathcal{T}^{(00)}_{r-1,0} - \frac{\partial}{\partial\ln\beta_0}\left(m^2\mathcal{T}^{(00)}_{r-1,0} - \mathcal{T}^{(00)}_{r+1,0}\right)\right] \,, \tag{E.28f}$$

$$\begin{aligned}\tilde{\tau}_{qn\pi,r} \coloneqq{}& \frac{1}{\varepsilon_0 + P_0}\Bigg\{\frac{6}{35}\left[(r+7)\mathcal{T}^{(22)}_{r+1,0} - m^2 r\mathcal{T}^{(22)}_{r-1,0}\right] - \frac{6}{35}\frac{\partial}{\partial\ln\beta_0}\left(m^2\mathcal{T}^{(22)}_{r-1,0} - \mathcal{T}^{(22)}_{r+1,0}\right) \\ &- r\mathcal{T}^{(24)}_{r-1,0} - \frac{\partial\mathcal{T}^{(24)}_{r-1,0}}{\partial\ln\beta_0}\Bigg\} \,,\end{aligned} \tag{E.28g}$$

$$\begin{aligned}\tilde{\lambda}_{qnn,r} \coloneqq{}& \frac{53}{225}\left[(2r+7)\mathcal{T}^{(13)}_{r0} - m^2(2r-2)\mathcal{T}^{(13)}_{r-2,0}\right] \\ &- \frac{18}{125}\left[m^4(r-1)\mathcal{T}^{(11)}_{r-2,0} - m^2(2r+5)\mathcal{T}^{(11)}_{r0} + (r+6)\mathcal{T}^{(11)}_{r+2,0}\right] \,,\end{aligned} \tag{E.28h}$$

$$\tilde{\lambda}_{qn\Pi,r} \coloneqq \frac{3}{5m^2}\left(\frac{\partial}{\partial\alpha_0} + \frac{1}{h}\frac{\partial}{\partial\beta_0}\right)\left(m^2\mathcal{T}^{(00)}_{r-1,0} - \mathcal{T}^{(00)}_{r+1,0}\right) \,, \tag{E.28i}$$

$$\tilde{\lambda}_{qn\pi,r} := \left(\frac{\partial}{\partial\alpha_0} + \frac{1}{h}\frac{\partial}{\partial\beta_0}\right)\left[\frac{6}{35}\left(m^2\mathcal{T}^{(22)}_{r-1,0} - \mathcal{T}^{(22)}_{r+1,0}\right) + \mathcal{T}^{(24)}_{r-1,0}\right] . \tag{E.28j}$$

$T^{\mu\nu}$: Tensors

The coefficients in Eq. (E.10) are defined as

$$\tilde{\tau}_{\pi,r} := \mathcal{R}^{(2)}_{r0} , \tag{E.29a}$$

$$\tilde{\delta}_{\pi\pi,r} := \frac{1}{3}\left[(r+4)\mathcal{R}^{(2)}_{r0} - m^2(r-1)\mathcal{R}^{(2)}_{r-2,0}\right] + \frac{\partial\mathcal{R}^{(2)}_{r0}}{\partial\alpha_0}\mathcal{H} + \frac{\partial\mathcal{R}^{(2)}_{r0}}{\partial\beta_0}\overline{\mathcal{H}} , \tag{E.29b}$$

$$\tilde{\tau}_{\pi\pi,r} := \frac{2}{7}\left[(2r+5)\mathcal{R}^{(2)}_{r0} - 2m^2(r-1)\mathcal{R}^{(2)}_{r-2,0}\right] , \tag{E.29c}$$

$$\tilde{\lambda}_{\pi\Pi,r} := -\frac{2}{5m^2}\left[(r+4)\mathcal{R}^{(0)}_{r+2,0} - (2r+3)m^2\mathcal{R}^{(0)}_{r0} + (r-1)m^4\mathcal{R}^{(0)}_{r-2,0}\right] , \tag{E.29d}$$

$$\tilde{\tau}_{\pi n,r} := \frac{2}{5(\varepsilon_0+P_0)}\left[(r+5)\mathcal{R}^{(1)}_{r+1,0} - rm^2\mathcal{R}^{(1)}_{r-1,0} + \frac{\partial\mathcal{R}^{(1)}_{r+1,0}}{\partial\ln\beta_0} - m^2\frac{\partial\mathcal{R}^{(1)}_{r-1,0}}{\partial\ln\beta_0}\right] , \tag{E.29e}$$

$$\tilde{\ell}_{\pi n,r} := \frac{2}{5}\left(\mathcal{R}^{(1)}_{r+1,0} - m^2\mathcal{R}^{(1)}_{r-1,0}\right) , \tag{E.29f}$$

$$\tilde{\lambda}_{\pi n,r} := \frac{2}{5}\left[\frac{\partial\mathcal{R}^{(1)}_{r+1,0}}{\partial\alpha_0} + \frac{1}{h}\frac{\partial\mathcal{R}^{(1)}_{r+1,0}}{\partial\beta_0} - m^2\left(\frac{\partial\mathcal{R}^{(1)}_{r-1,0}}{\partial\alpha_0} + \frac{1}{h}\frac{\partial\mathcal{R}^{(1)}_{r-1,0}}{\partial\beta_0}\right)\right] . \tag{E.29g}$$

The terms appearing in Eq. (E.11) on the other hand read

$$\tilde{\tau}_{\psi\pi,r} := \mathcal{T}^{(20)}_{r0} , \tag{E.30a}$$

$$\tilde{\lambda}_{\psi\pi\omega,r} := 2\Upsilon^{(12)}_{r-1} , \tag{E.30b}$$

$$\tilde{\delta}_{\psi\pi\pi,r} := \frac{1}{3}\left[(r+2)\mathcal{T}^{(20)}_{r0} - m^2(r-1)\mathcal{T}^{(20)}_{r-2,0} + 2\Upsilon^{(12)}_{r-1}\right] + \mathcal{H}\frac{\partial\mathcal{T}^{(20)}_{r0}}{\partial\alpha_0} + \overline{\mathcal{H}}\frac{\partial\mathcal{T}^{(20)}_{r0}}{\partial\beta_0} , \tag{E.30c}$$

$$\tilde{\tau}_{\psi\pi\pi,r} := 2\Upsilon^{(12)}_{r-1} - \frac{12}{7}(r-1)\mathcal{T}^{(22)}_{r-2,0} , \tag{E.30d}$$

$$\tilde{\lambda}_{\psi\pi\Pi,r} := \frac{2}{m}\Upsilon^{(10)}_{r-1} - \frac{3}{5m^2}(r-1)\mathcal{T}^{(00)}_{r-2,0} , \tag{E.30e}$$

$$\tilde{\tau}_{\psi\pi n,r} := -\frac{3}{5(\varepsilon_0+P_0)}\left(\frac{\partial\mathcal{T}^{(11)}_{r-1,0}}{\partial\ln\beta_0} + r\mathcal{T}^{(11)}_{r-1,0}\right) + \frac{2}{\varepsilon_0+P_0}\Upsilon^{(01)}_{r} , \tag{E.30f}$$

$$\tilde{\ell}_{\psi\pi n,r} := -\frac{3}{5}\mathcal{T}^{(11)}_{r-1,0} , \tag{E.30g}$$

$$\tilde{\lambda}_{\psi\pi n,r} := -\frac{3}{5}\frac{\partial\mathcal{T}^{(11)}_{r-1,0}}{\partial\alpha_0} - \frac{1}{h}\frac{3}{5}\frac{\partial\mathcal{T}^{(11)}_{r-1,0}}{\partial\beta_0} , \tag{E.30h}$$

whereas the ones in Eq. (E.12) are given by

$$\tilde{\tau}_{p\pi,r} := \mathcal{T}^{(22)}_{r0} , \tag{E.31a}$$

$$\tilde{\lambda}_{p\pi\omega,r} := \mathcal{T}^{(22)}_{r0} , \tag{E.31b}$$

$$\tilde{\delta}_{p\pi\pi,r} := \left(\mathcal{H}\frac{\partial}{\partial\alpha_0} + \overline{\mathcal{H}}\frac{\partial}{\partial\beta_0}\right)\mathcal{T}^{(22)}_{r0} - \frac{1}{3}\left[m^2(r-1)\mathcal{T}^{(22)}_{r-2,0} - (r+4)\mathcal{T}^{(22)}_{r0}\right] , \tag{E.31c}$$

$$\begin{aligned}\tilde{\tau}_{p\pi\pi,r} :=\ & \frac{3}{49}\left[2m^2(r-1)\mathcal{T}^{(22)}_{r-2,0} - (2r+5)\mathcal{T}^{(22)}_{r0}\right] - (r-1)\mathcal{T}^{(24)}_{r-2,0} \\ & - \frac{2}{15}\left[m^4(r-1)\mathcal{T}^{(20)}_{r-2,0} - m^2(2r+3)\mathcal{T}^{(20)}_{r0} + (r+4)\mathcal{T}^{(20)}_{r+2,0}\right] ,\end{aligned} \tag{E.31d}$$

$$\tilde{\lambda}_{p\pi\Pi,r} := \frac{7}{15m^2}\left(m^2\Upsilon^{(10)}_{r-1} - \Upsilon^{(10)}_{r+1}\right) , \tag{E.31e}$$

$$\tilde{\tau}_{p\pi n,r} := \frac{1}{\varepsilon_0 + P_0}\left\{\frac{7}{50}\left[(r+5)\mathcal{T}^{(11)}_{r+1,0} - m^2 r\mathcal{T}^{(11)}_{r-1,0}\right] - \frac{1}{10} r\mathcal{T}^{(13)}_{r-1,0} - \frac{\partial}{\partial \ln\beta_0}\left[\frac{7}{50}\left(m^2\mathcal{T}^{(11)}_{r-1,0} - \mathcal{T}^{(11)}_{r+1,0}\right) + \frac{1}{10}\mathcal{T}^{(13)}_{r-1,0}\right]\right\} , \tag{E.31f}$$

$$\tilde{\ell}_{p\pi n,r} := \frac{7}{50}\left(\mathcal{T}^{(11)}_{r+1,0} - m^2\mathcal{T}^{(11)}_{r-1,0}\right) - \frac{1}{10}\mathcal{T}^{(13)}_{r-1,0} , \tag{E.31g}$$

$$\tilde{\lambda}_{p\pi n,r} := -\left(\frac{\partial}{\partial\alpha_0} + \frac{1}{h}\frac{\partial}{\partial\beta_0}\right)\left[\frac{7}{50}\left(m^2\mathcal{T}^{(11)}_{r-1,0} - \mathcal{T}^{(11)}_{r+1,0}\right) + \frac{1}{10}\mathcal{T}^{(13)}_{r-1,0}\right] . \tag{E.31h}$$

Lastly, the coefficients in Eq. (E.13) are defined as

$$\tilde{\tau}_{q\pi,r} := \mathcal{T}^{(24)}_{r0} , \tag{E.32a}$$

$$\tilde{\lambda}_{q\pi\omega,r} := -\frac{10}{3}\mathcal{T}^{(24)}_{r0} , \tag{E.32b}$$

$$\tilde{\delta}_{q\pi\pi,r} := \mathcal{H}\frac{\partial\mathcal{T}^{(24)}_{r0}}{\partial\alpha_0} + \overline{\mathcal{H}}\frac{\partial\mathcal{T}^{(24)}_{r0}}{\partial\beta_0} - \frac{1}{3}\left[m^2(r-1)\mathcal{T}^{(24)}_{r-2,0} - (r+6)\mathcal{T}^{(24)}_{r0}\right] , \tag{E.32c}$$

$$\tilde{\tau}_{q\pi\pi,r} := \frac{10}{49}\left[(2r+9)\mathcal{T}^{(24)}_{r0} - m^2(2r-2)\mathcal{T}^{(24)}_{r-2,0}\right] - \frac{288}{1715}\left[m^4(r-1)\mathcal{T}^{(22)}_{r-2,0} - m^2(2r+7)\mathcal{T}^{(22)}_{r0} + (r+8)\mathcal{T}^{(22)}_{r+2,0}\right] , \tag{E.32d}$$

$$\tilde{\lambda}_{q\pi\Pi,r} := -\frac{36}{175}\left[m^4(r-1)\mathcal{T}^{(00)}_{r-2,0} - m^2(2r+7)\mathcal{T}^{(00)}_{r0} + (r+8)\mathcal{T}^{(00)}_{r+2,0}\right] , \tag{E.32e}$$

$$\tilde{\tau}_{q\pi n,r} := \frac{12}{35(\varepsilon_0 + P_0)}\left[(r+9)\mathcal{T}^{(13)}_{r+1,0} - m^2 r\mathcal{T}^{(13)}_{r-1,0} - \frac{\partial}{\partial\ln\beta_0}\left(m^2\mathcal{T}^{(13)}_{r-1,0} - \mathcal{T}^{(13)}_{r+1,0}\right)\right] , \tag{E.32f}$$

$$\tilde{\ell}_{q\pi n,r} := -\frac{12}{35}\left(m^2\mathcal{T}^{(13)}_{r-1,0} - \mathcal{T}^{(13)}_{r+1,0}\right) , \tag{E.32g}$$

$$\tilde{\lambda}_{q\pi n,r} := -\left(\frac{\partial}{\partial\alpha_0} + \frac{1}{h}\frac{\partial}{\partial\beta_0}\right)\left(m^2\mathcal{T}^{(13)}_{r-1,0} - \mathcal{T}^{(13)}_{r+1,0}\right) . \tag{E.32h}$$

This concludes the list of coefficients appearing in the equations for the components of the energy-momentum tensor and particle four-current.

$S^{\lambda\mu\nu}$: Axial vectors

First, we list the coefficients appearing in Eq. (E.15),

$$\tilde{\tau}_\omega := \frac{2\sigma^2\hbar}{mg\Gamma^{(\omega)}}(J_{30} - J_{31}) , \tag{E.33a}$$

$$\tilde{\mathfrak{K}}_{\omega\theta} := -\frac{2\sigma^2\hbar}{mg\Gamma^{(\omega)}}\left[(K_{30} - K_{31})\mathcal{H} - (K_{40} - K_{41})\overline{\mathcal{H}} + \left(J_{30} - \frac{1}{3}J_{31}\right)\right] , \tag{E.33b}$$

$$\tilde{\mathfrak{K}}_{\omega\theta\mathfrak{p}} := -\frac{\sigma m^2}{3\Gamma^{(\omega)}} , \tag{E.33c}$$

$$\tilde{\mathfrak{K}}_{\omega\sigma} := \frac{2\sigma^2\hbar}{mg\Gamma^{(\omega)}}J_{31} , \tag{E.33d}$$

$$\tilde{\mathfrak{K}}_{\omega\sigma\mathfrak{p}} := \frac{\sigma m^2}{2\Gamma^{(\omega)}} , \tag{E.33e}$$

$$\tilde{\mathfrak{K}}_{\omega\mathfrak{t}} := \frac{\sigma}{\Gamma^{(\omega)}} , \tag{E.33f}$$

$$\tilde{\mathfrak{h}}_{\omega\kappa} := -\frac{2\sigma^2\hbar}{mg\Gamma^{(\omega)}}J_{31} , \tag{E.33g}$$

$$\tilde{\mathfrak{K}}_{\omega I\kappa,r} := -\frac{2\sigma^2\hbar}{mg\Gamma^{(\omega)}}\left(K_{31} - \frac{1}{h}K_{41}\right) , \tag{E.33h}$$

$$\tilde{\mathfrak{K}}_{\omega F\kappa,r} := -\frac{2\sigma^2\hbar}{mg\Gamma^{(\omega)}}\frac{1}{\varepsilon_0+P_0}\left(\beta_0 K_{41}-3J_{31}\right) . \tag{E.33i}$$

The quantities appearing in Eq. (E.17) read

$$\tilde{\tau}_{\mathfrak{p},r} := \mathcal{Q}^{(10)}_{r0} , \tag{E.34a}$$

$$\tilde{\tau}_{\mathfrak{p}\omega,r} := -\frac{2\sigma\hbar}{gm}J_{r+1,0} , \tag{E.34b}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{p}\theta,r} := -\frac{1}{3}\left[(r+2)\mathcal{Q}^{(10)}_{r0}-m^2(r-1)\mathcal{Q}^{(10)}_{r-2,0}+\mathcal{X}^{(10)}_{r-1}\right]-\left(\mathcal{H}\frac{\partial}{\partial\alpha_0}+\overline{\mathcal{H}}\frac{\partial}{\partial\beta_0}\right)\mathcal{Q}^{(10)}_{r0} , \tag{E.34c}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{p}\theta\mathfrak{q},r} := -\frac{1}{3}\mathcal{X}^{(12)}_{r-1} , \tag{E.34d}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{p}\theta\omega,r} := \frac{2\sigma\hbar}{gm}\left[K_{r+1,0}\mathcal{H}-K_{r+2,0}\overline{\mathcal{H}}+J_{r+1,0}+\left(r-\frac{2}{3}\right)J_{r+1,1}\right] , \tag{E.34e}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{p}\sigma,r} := -\mathcal{X}^{(10)}_{r-1} , \tag{E.34f}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{p}\sigma\mathfrak{q},r} := -\mathcal{X}^{(12)}_{r-1}+(r-1)\frac{3}{5}\mathcal{Q}^{(12)}_{r-2,0} , \tag{E.34g}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{p}\sigma\omega,r} := \frac{2\sigma\hbar}{gm}J_{r+1,1} , \tag{E.34h}$$

$$\tilde{\mathfrak{h}}_{\mathfrak{p}\mathfrak{w},r} := \frac{1}{2}\mathcal{Q}^{(11)}_{r-1,0} , \tag{E.34i}$$

$$\tilde{\mathfrak{h}}_{\mathfrak{p}\kappa,r} := \frac{1}{2}\mathcal{Q}^{(\kappa)}_{r-1,0}+\frac{2\sigma\hbar}{gm}J_{r+1,1} , \tag{E.34j}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{p}I\mathfrak{w},r} := \frac{1}{2}\left(\frac{\partial}{\partial\alpha_0}+\frac{1}{h}\frac{\partial}{\partial\beta_0}\right)\mathcal{Q}^{(11)}_{r-1,0} , \tag{E.34k}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{p}F\mathfrak{w},r} := -\frac{1}{\varepsilon_0+P_0}\left(\frac{1}{2}\frac{\partial\mathcal{Q}^{(11)}_{r-1,0}}{\partial\ln\beta_0}+\frac{r}{2}\mathcal{Q}^{(11)}_{r0}\right) , \tag{E.34l}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{p}I\kappa,r} := \frac{1}{2}\left(\frac{\partial}{\partial\alpha_0}+\frac{1}{h}\frac{\partial}{\partial\beta_0}\right)\mathcal{Q}^{(\kappa)}_{r-1,0}+\frac{2\sigma\hbar}{gm}\left(K_{r+1,1}-\frac{1}{h}K_{r+2,1}\right) , \tag{E.34m}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{p}F\kappa,r} := \frac{1}{\varepsilon_0+P_0}\left(-\frac{1}{2}\frac{\partial\mathcal{Q}^{(\kappa)}_{r-1,0}}{\partial\ln\beta_0}-\frac{r}{2}\mathcal{Q}^{(\kappa)}_{r0}+\frac{2\sigma\hbar}{gm}J_{r+1,0}\right) . \tag{E.34n}$$

Similarly, the coefficients in Eq. (E.18) are defined as

$$\tilde{\tau}_{\mathfrak{q},r} := \mathcal{Q}^{(12)}_{r0} , \tag{E.35a}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{q}\theta,r} := \frac{2}{9}\left(\mathcal{X}^{(12)}_{r+1}-m^2\mathcal{X}^{(12)}_{r-1}\right)+\frac{1}{3}\left[(r-1)m^2\mathcal{Q}^{(12)}_{r-2,0}-(r+4)\mathcal{Q}^{(12)}_{r0}\right]-\left(\mathcal{H}\frac{\partial}{\partial\alpha_0}+\overline{\mathcal{H}}\frac{\partial}{\partial\beta_0}\right)\mathcal{Q}^{(12)}_{r0} , \tag{E.35b}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{q}\theta\mathfrak{p},r} := \frac{2}{9}\left(\mathcal{X}^{(10)}_{r+1}-m^2\mathcal{X}^{(10)}_{r-1}\right) , \tag{E.35c}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{q}\sigma,r} := \frac{1}{10}\left[2(r-1)m^2\mathcal{Q}^{(12)}_{r-2,0}-(2r+5)\mathcal{Q}^{(12)}_{r0}\right]+\frac{1}{15}\left(\mathcal{X}^{(12)}_{r+1}-m^2\mathcal{X}^{(12)}_{r-1}\right) , \tag{E.35d}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{q}\sigma\omega,r} := -\frac{4\sigma\hbar}{gm}\beta_0 K_{r+4,2} , \tag{E.35e}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{q}\sigma\mathfrak{p},r} := \frac{2}{15}\left[(r-1)m^4\mathcal{Q}^{(10)}_{r-2,0}-(2r+3)m^2\mathcal{Q}^{(10)}_{r0}+(r+4)\mathcal{Q}^{(10)}_{r+2,0}\right]+\frac{1}{15}\left(\mathcal{X}^{(10)}_{r+1}-m^2\mathcal{X}^{(10)}_{r-1}\right) , \tag{E.35f}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{q}\mathfrak{t},r} := -\frac{10}{9}\mathcal{Q}^{(22)}_{r0} , \tag{E.35g}$$

$$\tilde{\mathfrak{h}}_{\mathfrak{q}\mathfrak{w},r} := \frac{1}{6}\left(\mathcal{Q}^{(11)}_{r+1,0}-m^2\mathcal{Q}^{(11)}_{r-1,0}\right) , \tag{E.35h}$$

$$\tilde{\mathfrak{h}}_{\mathfrak{q}\kappa,r} := \frac{1}{6}\left(\mathcal{Q}^{(\kappa)}_{r+1}-m^2\mathcal{Q}^{(\kappa)}_{r-1}\right)+\frac{10}{3}\frac{\sigma\hbar}{gm}K_{r+3,2} , \tag{E.35i}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{q}I\mathfrak{w},r} := \frac{1}{6}\left(\frac{\partial}{\partial\alpha_0}+\frac{1}{h}\frac{\partial}{\partial\beta_0}\right)\left(\mathcal{Q}^{(11)}_{r+1,0}-m^2\mathcal{Q}^{(11)}_{r-1,0}\right) , \tag{E.35j}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{q}F\mathfrak{w},r} := \frac{1}{\varepsilon_0+P_0}\frac{1}{6}\left[rm^2\mathcal{Q}^{(11)}_{r-1,0}-(r+5)\mathcal{Q}^{(11)}_{r+1,0}-\frac{\partial}{\partial\ln\beta_0}\left(\mathcal{Q}^{(11)}_{r+1,0}-m^2\mathcal{Q}^{(11)}_{r-1,0}\right)\right] , \tag{E.35k}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{q}I\kappa,r} := \frac{1}{6}\left(\frac{\partial}{\partial\alpha_0}+\frac{1}{h}\frac{\partial}{\partial\beta_0}\right)\left(\mathcal{Q}^{(\kappa)}_{r+1}-m^2\mathcal{Q}^{(\kappa)}_{r-1}\right)+\frac{10}{3}\frac{\sigma\hbar}{gm}\left(K_{r+3,2}-\frac{1}{h}K_{r+4,2}\right) , \tag{E.35l}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{q}F\kappa,r} := \frac{1}{\varepsilon_0+P_0}\frac{1}{6}\left[rm^2\mathcal{Q}^{(\kappa)}_{r-1}-(r+5)\mathcal{Q}^{(\kappa)}_{r+1}-\frac{\partial}{\partial\ln\beta_0}\left(\mathcal{Q}^{(\kappa)}_{r+1}-m^2\mathcal{Q}^{(\kappa)}_{r-1}\right)\right] . \tag{E.35m}$$

$S^{\lambda\mu\nu}$: Vectors

The coefficients that appear in Eq. (E.19) are given by

$$\tilde{\tau}_\kappa := \frac{4\sigma^2\hbar}{gm\Gamma^{(\kappa)}}J_{31} , \tag{E.36a}$$

$$\tilde{\mathfrak{K}}_{\kappa\theta} := -\frac{4\sigma^2\hbar}{gm\Gamma^{(\kappa)}}\left(K_{31}\mathcal{H}-K_{41}\overline{\mathcal{H}}+\frac{4}{3}J_{31}\right) , \tag{E.36b}$$

$$\tilde{\mathfrak{K}}_{\kappa\sigma} := \frac{2\sigma^2\hbar}{gm\Gamma^{(\kappa)}}J_{31} , \tag{E.36c}$$

$$\tilde{\mathfrak{K}}_{\kappa\omega} := \frac{2\sigma^2\hbar}{gm\Gamma^{(\kappa)}}J_{31} , \tag{E.36d}$$

$$\tilde{\mathfrak{h}}_{\kappa\mathfrak{t}} := -\frac{\sigma}{\Gamma^{(\kappa)}} , \tag{E.36e}$$

$$\tilde{\mathfrak{K}}_{\kappa F\mathfrak{t}} := \frac{\sigma}{\Gamma^{(\kappa)}(\varepsilon_0+P_0)} , \tag{E.36f}$$

$$\tilde{\mathfrak{h}}_{\kappa\omega} := \frac{2\sigma^2\hbar}{gm\Gamma^{(\kappa)}}J_{31} , \tag{E.36g}$$

$$\tilde{\mathfrak{h}}_{\kappa\mathfrak{p}} := \frac{\sigma m^2}{2\Gamma^{(\kappa)}} , \tag{E.36h}$$

$$\tilde{\mathfrak{K}}_{\kappa I\omega} := \frac{2\sigma^2\hbar}{gm\Gamma^{(\kappa)}}\left(K_{31}-\frac{1}{h}K_{41}\right) , \tag{E.36i}$$

$$\tilde{\mathfrak{K}}_{\kappa F\omega} := \frac{2\sigma^2\hbar}{gm\Gamma^{(\kappa)}}\frac{1}{\varepsilon_0+P_0}\left(\beta_0K_{41}-J_{30}\right) , \tag{E.36j}$$

$$\tilde{\mathfrak{K}}_{\kappa F\mathfrak{p}} := -\frac{\sigma m^2}{2\Gamma^{(\kappa)}(\varepsilon_0+P_0)} . \tag{E.36k}$$

On the other hand, the coefficients in Eq. (E.21) are

$$\tilde{\tau}_{\mathfrak{w},r} := \mathcal{Q}^{(11)}_{r0} , \tag{E.37a}$$

$$\tilde{\tau}_{\mathfrak{w}\kappa,r} := \mathcal{Q}^{(\kappa)}_{r}+\frac{4\sigma\hbar}{gm}J_{r+2,1} , \tag{E.37b}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{w}\theta,r} := \frac{1}{3}\left[m^2(r-1)\mathcal{Q}^{(11)}_{r-2,0}-(r+3)\mathcal{Q}^{(11)}_{r0}\right]-\left(\mathcal{H}\frac{\partial}{\partial\alpha_0}+\overline{\mathcal{H}}\frac{\partial}{\partial\beta_0}\right)\mathcal{Q}^{(11)}_{r0} , \tag{E.37c}$$

$$\begin{aligned}\tilde{\mathfrak{K}}_{\mathfrak{w}\theta\kappa,r} := &-\frac{4\sigma\hbar}{gm}\left(K_{r+2,1}\mathcal{H}-K_{r+3,1}\overline{\mathcal{H}}+\frac{5}{3}\beta_0K_{r+3,2}-\frac{1}{3}J_{r+2,1}\right)-\left(\mathcal{H}\frac{\partial}{\partial\alpha_0}+\overline{\mathcal{H}}\frac{\partial}{\partial\beta_0}\right)\mathcal{Q}^{(\kappa)}_{r}\\ &+\frac{1}{3}\left[m^2(r-1)\mathcal{Q}^{(\kappa)}_{r-2}-(r+3)\mathcal{Q}^{(\kappa)}_{r}\right] ,\end{aligned} \tag{E.37d}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{w}\sigma,r} := -\frac{1}{10}\left[2m^2(r-1)\mathcal{Q}^{(11)}_{r-2,0}-(2r+3)\mathcal{Q}^{(11)}_{r0}\right] , \tag{E.37e}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{w}\sigma\kappa,r} := \frac{2\sigma\hbar}{gm}\left(2\beta_0K_{r+3,2}-J_{r+2,1}\right)-\frac{1}{10}\left[2m^2(r-1)\mathcal{Q}^{(\kappa)}_{r-2}-(2r+3)\mathcal{Q}^{(\kappa)}_{r}\right] , \tag{E.37f}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{w}\omega,r} \coloneqq \frac{1}{2}\mathcal{Q}^{(11)}_{r0} \,, \tag{E.37g}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{w}\omega\kappa,r} \coloneqq \frac{1}{2}\mathcal{Q}^{(\kappa)}_{r} + \frac{2\sigma\hbar}{gm}J_{r+2,1} \,, \tag{E.37h}$$

$$\tilde{\mathfrak{h}}_{\mathfrak{w}\mathfrak{t},r} \coloneqq -\mathcal{Q}^{(22)}_{r-1,0} \,, \tag{E.37i}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{w}I\mathfrak{t},r} \coloneqq -\left(\frac{\partial}{\partial\alpha_0} + \frac{1}{h}\frac{\partial}{\partial\beta_0}\right)\mathcal{Q}^{(22)}_{r-1,0} \,, \tag{E.37j}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{w}F\mathfrak{t},r} \coloneqq \frac{1}{\varepsilon_0 + P_0}\left(r\mathcal{Q}^{(22)}_{r-1,0} + \frac{\partial}{\partial\ln\beta_0}\mathcal{Q}^{(22)}_{r-1,0}\right) \,, \tag{E.37k}$$

$$\tilde{\mathfrak{h}}_{\mathfrak{w}\mathfrak{p},r} \coloneqq \frac{1}{3}\left(m^2\mathcal{Q}^{(10)}_{r-1,0} - \mathcal{Q}^{(10)}_{r+1,0}\right) \,, \tag{E.37l}$$

$$\tilde{\mathfrak{h}}_{\mathfrak{w}\mathfrak{q},r} \coloneqq -\frac{1}{2}\mathcal{Q}^{(12)}_{r-1,0} \,, \tag{E.37m}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{w}I\omega,r} \coloneqq \frac{2\sigma\hbar}{gm}\left(K_{r+2,1} - \frac{1}{h}K_{r+3,1}\right) \,, \tag{E.37n}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{w}I\mathfrak{p},r} \coloneqq \frac{1}{3}\left(\frac{\partial}{\partial\alpha_0} + \frac{1}{h}\frac{\partial}{\partial\beta_0}\right)\left(m^2\mathcal{Q}^{(10)}_{r-1,0} - \mathcal{Q}^{(10)}_{r+1,0}\right) \,, \tag{E.37o}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{w}I\mathfrak{q},r} \coloneqq -\frac{1}{2}\left(\frac{\partial}{\partial\alpha_0} + \frac{1}{h}\frac{\partial}{\partial\beta_0}\right)\mathcal{Q}^{(12)}_{r-1,0} \,, \tag{E.37p}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{w}F\mathfrak{p},r} \coloneqq -\frac{1}{3(\varepsilon_0 + P_0)}\left[m^2 r\mathcal{Q}^{(10)}_{r-1,0} - (r+3)\mathcal{Q}^{(10)}_{r+1,0} + \frac{\partial}{\partial\ln\beta_0}\left(m^2\mathcal{Q}^{(10)}_{r-1,0} - \mathcal{Q}^{(10)}_{r+1,0}\right)\right] \,, \tag{E.37q}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{w}F\mathfrak{q},r} \coloneqq \frac{1}{2(\varepsilon_0 + P_0)}\left(r\mathcal{Q}^{(12)}_{r-1,0} + \frac{\partial}{\partial\ln\beta_0}\mathcal{Q}^{(12)}_{r-1,0}\right) \,. \tag{E.37r}$$

$S^{\lambda\mu\nu}$: Tensor

In Eq. (E.22), the following coefficients were introduced,

$$\tilde{\tau}_{\mathfrak{t},r} \coloneqq \mathcal{Q}^{(22)}_{r0} \,, \tag{E.38a}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{t}\theta,r} \coloneqq \frac{1}{3}\left[m^2(r-1)\mathcal{Q}^{(22)}_{r-2,0} - (r+4)\mathcal{Q}^{(22)}_{r0}\right] - \left(\mathcal{H}\frac{\partial}{\partial\alpha_0} + \overline{\mathcal{H}}\frac{\partial}{\partial\beta_0}\right)\mathcal{Q}^{(22)}_{r0} \,, \tag{E.38b}$$

$$\tilde{\mathfrak{h}}_{\mathfrak{t}\mathfrak{w},r} \coloneqq \frac{3}{10}\left(\mathcal{Q}^{(11)}_{r+1,0} - m^2\mathcal{Q}^{(11)}_{r-1,0}\right) \,, \tag{E.38c}$$

$$\tilde{\mathfrak{h}}_{\mathfrak{t}\kappa,r} \coloneqq \frac{3}{10}\left(\mathcal{Q}^{(\kappa)}_{r+1} - m^2\mathcal{Q}^{(\kappa)}_{r-1}\right) \,, \tag{E.38d}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{t}I\mathfrak{w},r} \coloneqq \frac{3}{10}\left(\frac{\partial}{\partial\alpha_0} + \frac{1}{h}\frac{\partial}{\partial\beta_0}\right)\left(\mathcal{Q}^{(11)}_{r+1,0} - m^2\mathcal{Q}^{(11)}_{r-1,0}\right) \,, \tag{E.38e}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{t}F\mathfrak{w},r} \coloneqq \frac{3}{10(\varepsilon_0 + P_0)}\left[rm^2\mathcal{Q}^{(11)}_{r-1,0} - (r+5)\mathcal{Q}^{(11)}_{r+1,0} - \frac{\partial}{\partial\ln\beta_0}\left(\mathcal{Q}^{(11)}_{r+1,0} - m^2\mathcal{Q}^{(11)}_{r-1,0}\right)\right] \,, \tag{E.38f}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{t}I\kappa,r} \coloneqq \frac{3}{10}\left(\frac{\partial}{\partial\alpha_0} + \frac{1}{h}\frac{\partial}{\partial\beta_0}\right)\left(\mathcal{Q}^{(11)}_{r+1,0} - m^2\mathcal{Q}^{(11)}_{r-1,0}\right) + \frac{6\sigma\hbar}{gm}\left(K_{r+3,2} - \frac{1}{h}K_{r+4,2}\right) \,, \tag{E.38g}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{t}F\kappa,r} \coloneqq \frac{3}{10(\varepsilon_0 + P_0)}\left[rm^2\mathcal{Q}^{(\kappa)}_{r-1} - (r+5)\mathcal{Q}^{(\kappa)}_{r+1} - \frac{\partial}{\partial\ln\beta_0}\left(\mathcal{Q}^{(\kappa)}_{r+1} - m^2\mathcal{Q}^{(\kappa)}_{r-1}\right)\right] \,, \tag{E.38h}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{t}\omega\omega,r} \coloneqq \frac{6\sigma\hbar}{gm}K_{r+3,2} \,, \tag{E.38i}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{t}\omega\mathfrak{p},r} \coloneqq \frac{3}{5}\left(\mathcal{X}^{(10)}_{r+1} - m^2\mathcal{X}^{(10)}_{r-1}\right) \,, \tag{E.38j}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{t}\omega\mathfrak{q},r} \coloneqq \frac{3}{5}\left(\mathcal{X}^{(12)}_{r+1} - m^2\mathcal{X}^{(12)}_{r-1}\right) + \frac{9}{10}\mathcal{Q}^{(12)}_{r0} \,, \tag{E.38k}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{t}\sigma\omega,r} := -\frac{4\sigma\hbar}{gm}\beta_0 K_{r+4,2} + \frac{6\sigma\hbar}{gm}K_{r+3,2} \,, \tag{E.38l}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{t}\sigma\mathfrak{p},r} := -\frac{1}{5}\left(\mathcal{X}^{(10)}_{r+1} - m^2\mathcal{X}^{(10)}_{r-1}\right) + \frac{2}{15}\left[m^4(r-1)\mathcal{Q}^{(10)}_{r-2,0} - (2r+3)m^2\mathcal{Q}^{(10)}_{r0} + (r+4)\mathcal{Q}^{(10)}_{r+2,0}\right] \,, \tag{E.38m}$$

$$\tilde{\mathfrak{K}}_{\mathfrak{t}\sigma\mathfrak{q},r} := -\frac{1}{5}\left(\mathcal{X}^{(12)}_{r+1} - m^2\mathcal{X}^{(12)}_{r-1}\right) - \frac{1}{10}\left[2(r-1)m^2\mathcal{Q}^{(12)}_{r-2,0} - (2r+5)\mathcal{Q}^{(12)}_{r0}\right] \,. \tag{E.38n}$$

E.2.2 Total coefficients

After inverting the linearized collision terms, the transport coefficients appearing in Eqs. (6.156)–(6.161) are simply given by weighted sums of the r-dependent coefficients appearing in the previous subsection.

Components of $T^{\mu\nu}$ and N^{μ}

The second-order coefficients in Eq. (6.156) are given by

$$\tau_\Pi := \sum_r \left(\tau^{(\rho)}_{S,0r}\tilde{\tau}_{\Pi,r} + \tau^{(\rho p)}_{S,0r}\tilde{\tau}_{p\Pi,r}\right) \,, \tag{E.39a}$$

$$\ell_{\Pi n} := \sum_r \left(\tau^{(\rho)}_{S,0r}\tilde{\ell}_{\Pi n,r} + \tau^{(\rho p)}_{S,0r}\tilde{\ell}_{p\Pi n,r}\right) \,, \tag{E.39b}$$

$$\tau_{\Pi n} := \sum_r \left(\tau^{(\rho)}_{S,0r}\tilde{\tau}_{\Pi n,r} + \tau^{(\rho p)}_{S,0r}\tilde{\tau}_{p\Pi n,r}\right) \,, \tag{E.39c}$$

$$\delta_{\Pi\Pi} := \sum_r \left(\tau^{(\rho)}_{S,0r}\tilde{\delta}_{\Pi\Pi,r} + \tau^{(\rho p)}_{S,0r}\tilde{\delta}_{p\Pi\Pi,r}\right) \,, \tag{E.39d}$$

$$\lambda_{\Pi n} := \sum_r \left(\tau^{(\rho)}_{S,0r}\tilde{\lambda}_{\Pi n,r} + \tau^{(\rho p)}_{S,0r}\tilde{\lambda}_{p\Pi n,r}\right) \,, \tag{E.39e}$$

$$\lambda_{\Pi\pi} := \sum_r \left(\tau^{(\rho)}_{S,0r}\tilde{\lambda}_{\Pi\pi,r} + \tau^{(\rho p)}_{S,0r}\tilde{\lambda}_{p\Pi\pi,r}\right) \,. \tag{E.39f}$$

The first contribution denotes the case where there is no tensor polarization, while the second terms give (presumably small) corrections. Similarly, the second-order transport coefficients in Eq. (6.157) read

$$\tau_n := \sum_r \left(\tau^{(\rho)}_{V,0r}\tilde{\tau}_{n,r} + \tau^{(\rho p)}_{V,0r}\tilde{\tau}_{pn,r} + \tau^{(\rho q)}_{V,0r}\tilde{\tau}_{qn,r}\right) \,, \tag{E.40a}$$

$$\lambda_{n\omega} := \sum_r \left(\tau^{(\rho)}_{V,0r} + \tau^{(\rho p)}_{V,0r}\tilde{\lambda}_{pn\omega,r} + \tau^{(\rho q)}_{V,0r}\tilde{\lambda}_{qn\omega,r}\right) \,, \tag{E.40b}$$

$$\delta_{nn} := \sum_r \left(\tau^{(\rho)}_{V,0r}\tilde{\delta}_{nn,r} + \tau^{(\rho p)}_{V,0r}\tilde{\delta}_{pnn,r} + \tau^{(\rho q)}_{V,0r}\tilde{\delta}_{qnn,r}\right) \,, \tag{E.40c}$$

$$\ell_{n\Pi} := \sum_r \left(\tau^{(\rho)}_{V,0r}\tilde{\ell}_{n\Pi,r} + \tau^{(\rho p)}_{V,0r}\tilde{\ell}_{pn\Pi,r} + \tau^{(\rho q)}_{V,0r}\tilde{\ell}_{qn\Pi,r}\right) \,, \tag{E.40d}$$

$$\ell_{n\pi} := \sum_r \left(\tau^{(\rho)}_{V,0r}\tilde{\ell}_{n\pi,r} + \tau^{(\rho p)}_{V,0r}\tilde{\ell}_{pn\pi,r} + \tau^{(\rho q)}_{V,0r}\tilde{\ell}_{qn\pi,r}\right) \,, \tag{E.40e}$$

$$\tau_{n\Pi} := \sum_r \left(\tau^{(\rho)}_{V,0r}\tilde{\tau}_{n\Pi,r} + \tau^{(\rho p)}_{V,0r}\tilde{\tau}_{pn\Pi,r} + \tau^{(\rho q)}_{V,0r}\tilde{\tau}_{qn\Pi,r}\right) \,, \tag{E.40f}$$

$$\tau_{n\pi} := \sum_r \left(\tau^{(\rho)}_{V,0r}\tilde{\tau}_{n\pi,r} + \tau^{(\rho p)}_{V,0r}\tilde{\tau}_{pn\pi,r} + \tau^{(\rho q)}_{V,0r}\tilde{\tau}_{qn\pi,r}\right) \,, \tag{E.40g}$$

$$\lambda_{nn} := \sum_r \left(\tau^{(\rho)}_{V,0r}\tilde{\lambda}_{nn,r} + \tau^{(\rho p)}_{V,0r}\tilde{\lambda}_{pnn,r} + \tau^{(\rho q)}_{V,0r}\tilde{\lambda}_{qnn,r}\right) \,, \tag{E.40h}$$

$$\lambda_{n\Pi} := \sum_r \left(\tau^{(\rho)}_{V,0r} \tilde{\lambda}_{n\Pi,r} + \tau^{(\rho p)}_{V,0r} \tilde{\lambda}_{pn\Pi,r} + \tau^{(\rho q)}_{V,0r} \tilde{\lambda}_{qn\Pi,r} \right) \ , \tag{E.40i}$$

$$\lambda_{n\pi} := \sum_r \left(\tau^{(\rho)}_{V,0r} \tilde{\lambda}_{n\pi,r} + \tau^{(\rho p)}_{V,0r} \tilde{\lambda}_{pn\pi,r} + \tau^{(\rho q)}_{V,0r} \tilde{\lambda}_{qn\pi,r} \right) \ . \tag{E.40j}$$

Finally, the second-order transport coefficients appearing in the equation of motion for the shear-stress tensor (6.158) are defined as

$$\tau_{\pi} := \sum_r \left(\tau^{(\rho)}_{T,0r} \tilde{\tau}_{\pi,r} + \tau^{(\rho\psi)}_{T,0r} \tilde{\tau}_{\psi\pi,r} + \tau^{(\rho p)}_{T,0r} \tilde{\tau}_{p\pi,r} + \tau^{(\rho q)}_{T,0r} \tilde{\tau}_{q\pi,r} \right) \ , \tag{E.41a}$$

$$\lambda_{\pi\omega} := \sum_r \left(2\tau^{(\rho)}_{T,0r} + \tau^{(\rho\psi)}_{T,0r} \tilde{\lambda}_{\psi\pi\omega,r} + \tau^{(\rho p)}_{T,0r} \tilde{\lambda}_{p\pi\omega,r} + \tau^{(\rho q)}_{T,0r} \tilde{\lambda}_{q\pi\omega,r} \right) \ , \tag{E.41b}$$

$$\delta_{\pi\pi} := \sum_r \left(\tau^{(\rho)}_{T,0r} \tilde{\delta}_{\pi\pi,r} + \tau^{(\rho\psi)}_{T,0r} \tilde{\delta}_{\psi\pi\pi,r} + \tau^{(\rho p)}_{T,0r} \tilde{\delta}_{p\pi\pi,r} + \tau^{(\rho q)}_{T,0r} \tilde{\delta}_{q\pi\pi,r} \right) \ , \tag{E.41c}$$

$$\tau_{\pi\pi} := \sum_r \left(\tau^{(\rho)}_{T,0r} \tilde{\tau}_{\pi\pi,r} + \tau^{(\rho\psi)}_{T,0r} \tilde{\tau}_{\psi\pi\pi,r} + \tau^{(\rho p)}_{T,0r} \tilde{\tau}_{p\pi\pi,r} + \tau^{(\rho q)}_{T,0r} \tilde{\tau}_{q\pi\pi,r} \right) \ , \tag{E.41d}$$

$$\lambda_{\pi\Pi} := \sum_r \left(\tau^{(\rho)}_{T,0r} \tilde{\lambda}_{\pi\Pi,r} + \tau^{(\rho\psi)}_{T,0r} \tilde{\lambda}_{\psi\pi\Pi,r} + \tau^{(\rho p)}_{T,0r} \tilde{\lambda}_{p\pi\Pi,r} + \tau^{(\rho q)}_{T,0r} \tilde{\lambda}_{q\pi\Pi,r} \right) \ , \tag{E.41e}$$

$$\tau_{\pi n} := \sum_r \left(\tau^{(\rho)}_{T,0r} \tilde{\tau}_{\pi n,r} + \tau^{(\rho\psi)}_{T,0r} \tilde{\tau}_{\psi\pi n,r} + \tau^{(\rho p)}_{T,0r} \tilde{\tau}_{p\pi n,r} + \tau^{(\rho q)}_{T,0r} \tilde{\tau}_{q\pi n,r} \right) \ , \tag{E.41f}$$

$$\ell_{\pi n} := \sum_r \left(\tau^{(\rho)}_{T,0r} \tilde{\ell}_{\pi n,r} + \tau^{(\rho\psi)}_{T,0r} \tilde{\ell}_{\psi\pi n,r} + \tau^{(\rho p)}_{T,0r} \tilde{\ell}_{p\pi n,r} + \tau^{(\rho q)}_{T,0r} \tilde{\ell}_{q\pi n,r} \right) \ , \tag{E.41g}$$

$$\lambda_{\pi n} := \sum_r \left(\tau^{(\rho)}_{T,0r} \tilde{\lambda}_{\pi n,r} + \tau^{(\rho\psi)}_{T,0r} \tilde{\lambda}_{\psi\pi n,r} + \tau^{(\rho p)}_{T,0r} \tilde{\lambda}_{p\pi n,r} + \tau^{(\rho q)}_{T,0r} \tilde{\lambda}_{q\pi n,r} \right) \ . \tag{E.41h}$$

Components of $S^{\lambda\mu\nu}$

We now turn to the transport coefficients that are present in the equations of motion for the components of the spin tensor.

The coefficients in Eq. (6.159a) read

$$\tau_{\omega} := \mathfrak{T}^{(\omega)}_{A} \tilde{\tau}_{\omega} + \sum_r \mathfrak{T}^{(\omega\tau)}_{A,r} \tilde{\tau}_{\mathfrak{p}\omega,r} \ , \tag{E.42a}$$

$$\tau_{\omega\mathfrak{p}} := \sum_r \mathfrak{T}^{(\omega\tau)}_{A,r} \tilde{\tau}_{\mathfrak{p},r} \ , \tag{E.42b}$$

$$\tau_{\omega\mathfrak{q}} := \sum_r \mathfrak{T}^{(\omega t)}_{A,r} \tilde{\tau}_{\mathfrak{q},r} \ , \tag{E.42c}$$

$$\mathfrak{K}_{\omega\theta} := \mathfrak{T}^{(\omega)}_{A} \tilde{\mathfrak{K}}_{\omega\theta} + \sum_r \mathfrak{T}^{(\omega\tau)}_{A,r} \tilde{\mathfrak{K}}_{\mathfrak{p}\theta\omega,r} \ , \tag{E.42d}$$

$$\mathfrak{K}_{\omega\theta\mathfrak{p}} := \mathfrak{T}^{(\omega)}_{A} \tilde{\mathfrak{K}}_{\omega\theta\mathfrak{p}} + \sum_r \left(\mathfrak{T}^{(\omega\tau)}_{A,r} \tilde{\mathfrak{K}}_{\mathfrak{p}\theta,r} + \mathfrak{T}^{(\omega t)}_{A,r} \tilde{\mathfrak{K}}_{\mathfrak{q}\theta\mathfrak{p},r} \right) \ , \tag{E.42e}$$

$$\mathfrak{K}_{\omega\theta\mathfrak{q}} := \sum_r \left(\mathfrak{T}^{(\omega\tau)}_{A,r} \tilde{\mathfrak{K}}_{\mathfrak{p}\theta\mathfrak{q},r} + \mathfrak{T}^{(\omega t)}_{A,r} \tilde{\mathfrak{K}}_{\mathfrak{q}\theta,r} \right) \ , \tag{E.42f}$$

$$\mathfrak{K}_{\omega\sigma} := \mathfrak{T}^{(\omega)}_{A} \tilde{\mathfrak{K}}_{\omega\sigma} + \sum_r \left(\mathfrak{T}^{(\omega\tau)}_{A,r} \tilde{\mathfrak{K}}_{\mathfrak{p}\sigma\omega,r} + \mathfrak{T}^{(\omega t)}_{A,r} \tilde{\mathfrak{K}}_{\mathfrak{q}\sigma\omega,r} \right) \ , \tag{E.42g}$$

$$\mathfrak{K}_{\omega\sigma\mathfrak{p}} := \mathfrak{T}^{(\omega)}_{A} \tilde{\mathfrak{K}}_{\omega\sigma\mathfrak{p}} + \sum_r \left(\mathfrak{T}^{(\omega\tau)}_{A,r} \tilde{\mathfrak{K}}_{\mathfrak{p}\sigma,r} + \mathfrak{T}^{(\omega t)}_{A,r} \tilde{\mathfrak{K}}_{\mathfrak{q}\sigma\mathfrak{p},r} \right) \ , \tag{E.42h}$$

$$\mathfrak{K}_{\omega\sigma\mathfrak{q}} := \sum_r \left(\mathfrak{T}^{(\omega\tau)}_{A,r} \tilde{\mathfrak{K}}_{\mathfrak{p}\sigma\mathfrak{q},r} + \mathfrak{T}^{(\omega t)}_{A,r} \tilde{\mathfrak{K}}_{\mathfrak{q}\sigma,r} \right) \ , \tag{E.42i}$$

$$\mathfrak{K}_{\omega\mathfrak{t}} := \mathfrak{T}^{(\omega)}_{A} \tilde{\mathfrak{K}}_{\omega\mathfrak{t}} + \sum_r \mathfrak{T}^{(\omega t)}_{A,r} \tilde{\mathfrak{K}}_{\mathfrak{q}\mathfrak{t},r} \;, \tag{E.42j}$$

$$\mathfrak{h}_{\omega\mathfrak{w}} := \sum_r \left(\mathfrak{T}^{(\omega\tau)}_{A,r} \tilde{\mathfrak{h}}_{\mathfrak{p}\mathfrak{w},r} + \mathfrak{T}^{(\omega t)}_{A,r} \tilde{\mathfrak{h}}_{\mathfrak{q}\mathfrak{w},r} \right) \;, \tag{E.42k}$$

$$\mathfrak{h}_{\omega\kappa} := \mathfrak{T}^{(\omega)}_{A} \tilde{\mathfrak{h}}_{\omega\kappa} + \sum_r \left(\mathfrak{T}^{(\omega\tau)}_{A,r} \tilde{\mathfrak{h}}_{\mathfrak{p}\kappa,r} + \mathfrak{T}^{(\omega t)}_{A,r} \tilde{\mathfrak{h}}_{\mathfrak{q}\kappa,r} \right) \;, \tag{E.42l}$$

$$\mathfrak{K}_{\omega I\mathfrak{w}} := \sum_r \left(\mathfrak{T}^{(\omega\tau)}_{A,r} \tilde{\mathfrak{K}}_{\mathfrak{p}I\mathfrak{w},r} + \mathfrak{T}^{(\omega t)}_{A,r} \tilde{\mathfrak{K}}_{\mathfrak{q}I\mathfrak{w},r} \right) \;, \tag{E.42m}$$

$$\mathfrak{K}_{\omega F\mathfrak{w}} := \sum_r \left(\mathfrak{T}^{(\omega\tau)}_{A,r} \tilde{\mathfrak{K}}_{\mathfrak{p}F\mathfrak{w},r} + \mathfrak{T}^{(\omega t)}_{A,r} \tilde{\mathfrak{K}}_{\mathfrak{q}F\mathfrak{w},r} \right) \;, \tag{E.42n}$$

$$\mathfrak{K}_{\omega I\kappa} := \mathfrak{T}^{(\omega)}_{A} \tilde{\mathfrak{K}}_{\omega I\kappa} + \sum_r \left(\mathfrak{T}^{(\omega\tau)}_{A,r} \tilde{\mathfrak{K}}_{\mathfrak{p}I\kappa,r} + \mathfrak{T}^{(\omega t)}_{A,r} \tilde{\mathfrak{K}}_{\mathfrak{q}I\kappa,r} \right) \;, \tag{E.42o}$$

$$\mathfrak{K}_{\omega F\kappa} := \mathfrak{T}^{(\omega)}_{A} \tilde{\mathfrak{K}}_{\omega F\kappa} + \sum_r \left(\mathfrak{T}^{(\omega\tau)}_{A,r} \tilde{\mathfrak{K}}_{\mathfrak{p}F\kappa,r} + \mathfrak{T}^{(\omega t)}_{A,r} \tilde{\mathfrak{K}}_{\mathfrak{q}F\kappa,r} \right) \;. \tag{E.42p}$$

Here, there are no contributions from moments of other spin ranks. However, the quantities transforming as axial vectors (ω^μ, $\mathfrak{p}^\mu$, and $\mathfrak{q}^\mu$) couple to each other, as can be seen in the transport coefficients. Similarly, the ones in Eq. (6.159b) are given by

$$\tau_{\mathfrak{p}} := \sum_r \mathfrak{T}^{(\tau)}_{A,0r} \tilde{\tau}_{\mathfrak{p},r} \;, \tag{E.43a}$$

$$\tau_{\mathfrak{p}\mathfrak{q}} := \sum_r \mathfrak{T}^{(\tau t)}_{A,0r} \tilde{\tau}_{\mathfrak{q},r} \;, \tag{E.43b}$$

$$\tau_{\mathfrak{p}\omega} := \mathfrak{T}^{(\tau\omega)}_{A,0} \tilde{\tau}_{\omega} + \sum_r \mathfrak{T}^{(\tau)}_{A,0r} \tilde{\tau}_{\mathfrak{p}\omega,r} \;, \tag{E.43c}$$

$$\mathfrak{K}_{\mathfrak{p}\theta} := \mathfrak{T}^{(\tau\omega)}_{A,0} \tilde{\mathfrak{K}}_{\omega\theta\mathfrak{p}} + \sum_r \left(\mathfrak{T}^{(\tau)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{p}\theta,r} + \mathfrak{T}^{(\tau t)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{q}\theta\mathfrak{p},r} \right) \;, \tag{E.43d}$$

$$\mathfrak{K}_{\mathfrak{p}\theta\mathfrak{q}} := \sum_r \left(\mathfrak{T}^{(\tau)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{p}\theta\mathfrak{q},r} + \mathfrak{T}^{(\tau t)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{q}\theta,r} \right) \;, \tag{E.43e}$$

$$\mathfrak{K}_{\mathfrak{p}\theta\omega} := \mathfrak{T}^{(\tau\omega)}_{A,0} \tilde{\mathfrak{K}}_{\omega\theta} + \sum_r \mathfrak{T}^{(\tau)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{p}\theta\omega,r} \;, \tag{E.43f}$$

$$\mathfrak{K}_{\mathfrak{p}\sigma} := \mathfrak{T}^{(\tau\omega)}_{A,0} \tilde{\mathfrak{K}}_{\omega\sigma\mathfrak{p}} + \sum_r \left(\mathfrak{T}^{(\tau)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{p}\sigma,r} + \mathfrak{T}^{(\tau t)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{q}\sigma\mathfrak{p},r} \right) \;, \tag{E.43g}$$

$$\mathfrak{K}_{\mathfrak{p}\sigma\mathfrak{q}} := \sum_r \left(\mathfrak{T}^{(\tau)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{p}\sigma\mathfrak{q},r} + \mathfrak{T}^{(\tau t)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{q}\sigma,r} \right) \;, \tag{E.43h}$$

$$\mathfrak{K}_{\mathfrak{p}\sigma\omega} := \mathfrak{T}^{(\tau\omega)}_{A,0} \tilde{\mathfrak{K}}_{\omega\sigma} + \sum_r \left(\mathfrak{T}^{(\tau)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{p}\sigma\omega,r} + \mathfrak{T}^{(\tau t)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{q}\sigma\omega,r} \right) \;, \tag{E.43i}$$

$$\mathfrak{K}_{\mathfrak{p}\mathfrak{t}} := \mathfrak{T}^{(\tau\omega)}_{A,0} \tilde{\mathfrak{K}}_{\omega\mathfrak{t}} + \sum_r \mathfrak{T}^{(\tau t)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{q}\mathfrak{t},r} \;, \tag{E.43j}$$

$$\mathfrak{h}_{\mathfrak{p}\mathfrak{w}} := \sum_r \left(\mathfrak{T}^{(\tau)}_{A,0r} \tilde{\mathfrak{h}}_{\mathfrak{p}\mathfrak{w},r} + \mathfrak{T}^{(\tau t)}_{A,0r} \tilde{\mathfrak{h}}_{\mathfrak{q}\mathfrak{w},r} \right) \;, \tag{E.43k}$$

$$\mathfrak{h}_{\mathfrak{p}\kappa} := \mathfrak{T}^{(\tau\omega)}_{A,0} \tilde{\mathfrak{h}}_{\omega\kappa} + \sum_r \left(\mathfrak{T}^{(\tau)}_{A,0r} \tilde{\mathfrak{h}}_{\mathfrak{p}\kappa,r} + \mathfrak{T}^{(\tau t)}_{A,0r} \tilde{\mathfrak{h}}_{\mathfrak{q}\kappa,r} \right) \;, \tag{E.43l}$$

$$\mathfrak{K}_{\mathfrak{p}I\mathfrak{w}} := \sum_r \left(\mathfrak{T}^{(\tau)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{p}I\mathfrak{w},r} + \mathfrak{T}^{(\tau t)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{q}I\mathfrak{w},r} \right) \;, \tag{E.43m}$$

$$\mathfrak{K}_{\mathfrak{p}F\mathfrak{w}} := \sum_r \left(\mathfrak{T}^{(\tau)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{p}F\mathfrak{w},r} + \mathfrak{T}^{(\tau t)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{q}F\mathfrak{w},r} \right) \;, \tag{E.43n}$$

$$\mathfrak{K}_{\mathfrak{p}I\kappa} := \mathfrak{T}^{(\tau\omega)}_{A,0} \tilde{\mathfrak{K}}_{\omega I\kappa} + \sum_r \left(\mathfrak{T}^{(\tau)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{p}I\kappa,r} + \mathfrak{T}^{(\tau t)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{q}I\kappa,r} \right) \;, \tag{E.43o}$$

$$\mathfrak{K}_{\mathfrak{p}F\kappa} := \mathfrak{T}^{(\tau\omega)}_{A,0} \tilde{\mathfrak{K}}_{\omega F\kappa} + \sum_r \left(\mathfrak{T}^{(\tau)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{p}F\kappa,r} + \mathfrak{T}^{(\tau t)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{q}F\kappa,r} \right) \;. \tag{E.43p}$$

The coefficients in the equation of motion for $\mathfrak{q}^\mu$ (6.159c) are defined as

$$\tau_{\mathfrak{q}} := \sum_r \mathfrak{T}^{(t)}_{A,0r} \tilde{\tau}_{\mathfrak{q},r} \ , \tag{E.44a}$$

$$\tau_{\mathfrak{q}\omega} := \mathfrak{T}^{(t\omega)}_{A,0} \tilde{\tau}_{\omega} + \sum_r \mathfrak{T}^{(t\tau)}_{A,0r} \tilde{\tau}_{\mathfrak{p}\omega,r} \ , \tag{E.44b}$$

$$\tau_{\mathfrak{q}\mathfrak{p}} := \sum_r \mathfrak{T}^{(t\tau)}_{A,0r} \tilde{\tau}_{\mathfrak{p},r} \ , \tag{E.44c}$$

$$\mathfrak{K}_{\mathfrak{q}\theta} := \sum_r \left(\mathfrak{T}^{(t\tau)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{p}\theta\mathfrak{q},r} + \mathfrak{T}^{(t)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{q}\theta,r} \right) \ , \tag{E.44d}$$

$$\mathfrak{K}_{\mathfrak{q}\theta\omega} := \mathfrak{T}^{(t\omega)}_{A,0} \tilde{\mathfrak{K}}_{\omega\theta} + \sum_r \mathfrak{T}^{(t\tau)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{p}\theta\omega,r} \ , \tag{E.44e}$$

$$\mathfrak{K}_{\mathfrak{q}\theta\mathfrak{p}} := \mathfrak{T}^{(t\omega)}_{A,0} \tilde{\mathfrak{K}}_{\omega\theta\mathfrak{p}} + \sum_r \left(\mathfrak{T}^{(t\tau)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{p}\theta,r} + \mathfrak{T}^{(t)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{q}\theta\mathfrak{p},r} \right) \ , \tag{E.44f}$$

$$\mathfrak{K}_{\mathfrak{q}\sigma} := \sum_r \left(\mathfrak{T}^{(t\tau)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{p}\sigma\mathfrak{q},r} + \mathfrak{T}^{(t)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{q}\sigma,r} \right) \ , \tag{E.44g}$$

$$\mathfrak{K}_{\mathfrak{q}\sigma\omega} := \mathfrak{T}^{(t\omega)}_{A,0} \tilde{\mathfrak{K}}_{\omega\sigma} + \sum_r \left(\mathfrak{T}^{(t\tau)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{p}\sigma\omega,r} + \mathfrak{T}^{(t)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{q}\sigma\omega,r} \right) \ , \tag{E.44h}$$

$$\mathfrak{K}_{\mathfrak{q}\sigma\mathfrak{p}} := \mathfrak{T}^{(t\omega)}_{A,0} \tilde{\mathfrak{K}}_{\omega\sigma\mathfrak{p}} + \sum_r \left(\mathfrak{T}^{(t\tau)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{p}\sigma,r} + \mathfrak{T}^{(t)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{q}\sigma\mathfrak{p},r} \right) \ , \tag{E.44i}$$

$$\mathfrak{K}_{\mathfrak{q}\mathfrak{t}} := \mathfrak{T}^{(t\omega)}_{A,0} \tilde{\mathfrak{K}}_{\omega\mathfrak{t}} + \sum_r \mathfrak{T}^{(t)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{q}\mathfrak{t},r} \ , \tag{E.44j}$$

$$\mathfrak{h}_{\mathfrak{q}\mathfrak{w}} := \sum_r \left(\mathfrak{T}^{(t\tau)}_{A,0r} \tilde{\mathfrak{h}}_{\mathfrak{p}\mathfrak{w},r} + \mathfrak{T}^{(t)}_{A,0r} \tilde{\mathfrak{h}}_{\mathfrak{q}\mathfrak{w},r} \right) \ , \tag{E.44k}$$

$$\mathfrak{h}_{\mathfrak{q}\kappa} := \mathfrak{T}^{(t\omega)}_{A,0} \tilde{\mathfrak{h}}_{\omega\kappa} + \sum_r \left(\mathfrak{T}^{(t\tau)}_{A,0r} \tilde{\mathfrak{h}}_{\mathfrak{p}\kappa,r} + \mathfrak{T}^{(t)}_{A,0r} \tilde{\mathfrak{h}}_{\mathfrak{q}\kappa,r} \right) \ , \tag{E.44l}$$

$$\mathfrak{K}_{\mathfrak{q}I\mathfrak{w}} := \sum_r \left(\mathfrak{T}^{(t\tau)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{p}I\mathfrak{w},r} + \mathfrak{T}^{(t)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{q}I\mathfrak{w},r} \right) \ , \tag{E.44m}$$

$$\mathfrak{K}_{\mathfrak{q}F\mathfrak{w}} := \sum_r \left(\mathfrak{T}^{(t\tau)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{p}F\mathfrak{w},r} + \mathfrak{T}^{(t)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{q}F\mathfrak{w},r} \right) \ , \tag{E.44n}$$

$$\mathfrak{K}_{\mathfrak{q}I\kappa} := \mathfrak{T}^{(t\omega)}_{A,0} \tilde{\mathfrak{K}}_{\omega I\kappa} + \sum_r \left(\mathfrak{T}^{(t\tau)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{p}I\kappa,r} + \mathfrak{T}^{(t)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{q}I\kappa,r} \right) \ , \tag{E.44o}$$

$$\mathfrak{K}_{\mathfrak{q}F\kappa} := \mathfrak{T}^{(t\omega)}_{A,0} \tilde{\mathfrak{K}}_{\omega F\kappa} + \sum_r \left(\mathfrak{T}^{(t\tau)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{p}F\kappa,r} + \mathfrak{T}^{(t)}_{A,0r} \tilde{\mathfrak{K}}_{\mathfrak{q}F\kappa,r} \right) \ . \tag{E.44p}$$

The quantities appearing in the equation of motion for κ_0^μ (6.160a) are given by

$$\tau_{\kappa} := \mathfrak{T}^{(\kappa)}_{V} \tilde{\tau}_{\kappa} + \sum_r \mathfrak{T}^{(\kappa w)}_{V,r} \tilde{\tau}_{\mathfrak{w}\kappa,r} \ , \tag{E.45a}$$

$$\tau_{\kappa\mathfrak{w}} := \sum_r \mathfrak{T}^{(\kappa w)}_{V,r} \tilde{\tau}_{\mathfrak{w},r} \ , \tag{E.45b}$$

$$\mathfrak{K}_{\kappa\theta} := \mathfrak{T}^{(\kappa)}_{V} \tilde{\mathfrak{K}}_{\kappa\theta} + \sum_r \mathfrak{T}^{(\kappa w)}_{V,r} \tilde{\mathfrak{K}}_{\mathfrak{w}\theta\kappa,r} \ , \tag{E.45c}$$

$$\mathfrak{K}_{\kappa\theta\mathfrak{w}} := \sum_r \mathfrak{T}^{(\kappa w)}_{V,r} \tilde{\mathfrak{K}}_{\mathfrak{w}\theta,r} \ , \tag{E.45d}$$

$$\mathfrak{K}_{\kappa\sigma} := \mathfrak{T}^{(\kappa)}_{V} \tilde{\mathfrak{K}}_{\kappa\sigma} + \sum_r \mathfrak{T}^{(\kappa w)}_{V,r} \tilde{\mathfrak{K}}_{\mathfrak{w}\sigma\kappa,r} \ , \tag{E.45e}$$

$$\mathfrak{K}_{\kappa\sigma\mathfrak{w}} := \sum_r \mathfrak{T}^{(\kappa w)}_{V,r} \tilde{\mathfrak{K}}_{\mathfrak{w}\sigma,r} \ , \tag{E.45f}$$

$$\mathfrak{K}_{\kappa\omega} := \mathfrak{T}^{(\kappa)}_{V} \tilde{\mathfrak{K}}_{\kappa\omega} + \sum_r \mathfrak{T}^{(\kappa w)}_{V,r} \tilde{\mathfrak{K}}_{\mathfrak{w}\omega\kappa,r} \ , \tag{E.45g}$$

$$\mathfrak{K}_{\kappa\omega\mathfrak{w}} := \sum_r \mathfrak{T}^{(\kappa w)}_{V,r} \tilde{\mathfrak{K}}_{\mathfrak{w}\omega,r} \;, \tag{E.45h}$$

$$\mathfrak{h}_{\kappa\mathfrak{t}} := \mathfrak{T}^{(\kappa)}_{V} \tilde{\mathfrak{h}}_{\kappa\mathfrak{t}} + \sum_r \mathfrak{T}^{(\kappa w)}_{V,r} \tilde{\mathfrak{K}}_{\mathfrak{w}\mathfrak{t},r} \;, \tag{E.45i}$$

$$\mathfrak{K}_{\kappa I\mathfrak{t}} := \sum_r \mathfrak{T}^{(\kappa w)}_{V,r} \tilde{\mathfrak{K}}_{\mathfrak{w}I\mathfrak{t},r} \;, \tag{E.45j}$$

$$\mathfrak{K}_{\kappa F\mathfrak{t}} := \mathfrak{T}^{(\kappa)}_{V} \tilde{\mathfrak{K}}_{\kappa F\mathfrak{t}} + \sum_r \mathfrak{T}^{(\kappa w)}_{V,r} \tilde{\mathfrak{K}}_{\mathfrak{w}F\mathfrak{t},r} \;, \tag{E.45k}$$

$$\mathfrak{h}_{\kappa\omega} := \mathfrak{T}^{(\kappa)}_{V} \tilde{\mathfrak{h}}_{\kappa\omega} \;, \tag{E.45l}$$

$$\mathfrak{h}_{\kappa\mathfrak{p}} := \mathfrak{T}^{(\kappa)}_{V} \tilde{\mathfrak{h}}_{\kappa\mathfrak{p}} + \sum_r \mathfrak{T}^{(\kappa w)}_{V,r} \tilde{\mathfrak{h}}_{\mathfrak{w}\mathfrak{p},r} \;, \tag{E.45m}$$

$$\mathfrak{h}_{\kappa\mathfrak{q}} := \sum_r \mathfrak{T}^{(\kappa w)}_{V,r} \tilde{\mathfrak{h}}_{\mathfrak{w}\mathfrak{q},r} \;, \tag{E.45n}$$

$$\mathfrak{K}_{\kappa I\omega} := \mathfrak{T}^{(\kappa)}_{V} \tilde{\mathfrak{K}}_{\kappa I\omega} + \sum_r \mathfrak{T}^{(\kappa w)}_{V,r} \tilde{\mathfrak{K}}_{\mathfrak{w}I\omega,r} \;, \tag{E.45o}$$

$$\mathfrak{K}_{\kappa I\mathfrak{p}} := \sum_r \mathfrak{T}^{(\kappa w)}_{V,r} \tilde{\mathfrak{K}}_{\mathfrak{w}I\mathfrak{p},r} \;, \tag{E.45p}$$

$$\mathfrak{K}_{\kappa I\mathfrak{q}} := \sum_r \mathfrak{T}^{(\kappa w)}_{V,r} \tilde{\mathfrak{K}}_{\mathfrak{w}I\mathfrak{q},r} \;, \tag{E.45q}$$

$$\mathfrak{K}_{\kappa F\omega} := \mathfrak{T}^{(\kappa)}_{V} \tilde{\mathfrak{K}}_{\kappa F\omega} \;, \tag{E.45r}$$

$$\mathfrak{K}_{\kappa F\mathfrak{p}} := \mathfrak{T}^{(\kappa)}_{V} \tilde{\mathfrak{K}}_{\kappa F\mathfrak{p}} + \sum_r \mathfrak{T}^{(\kappa w)}_{V,r} \tilde{\mathfrak{K}}_{\mathfrak{w}F\mathfrak{p},r} \;, \tag{E.45s}$$

$$\mathfrak{K}_{\kappa F\mathfrak{q}} := \sum_r \mathfrak{T}^{(\kappa w)}_{V,r} \tilde{\mathfrak{K}}_{\mathfrak{w}F\mathfrak{q},r} \;. \tag{E.45t}$$

The coefficients in Eq. (6.160b) read

$$\tau_{\mathfrak{w}} := \sum_r \mathfrak{T}^{(w)}_{V,0r} \tilde{\tau}_{\mathfrak{w},r} \;, \tag{E.46a}$$

$$\tau_{\mathfrak{w}\kappa} := \mathfrak{T}^{(w\kappa)}_{V,0} \tilde{\tau}_{\kappa} + \sum_r \mathfrak{T}^{(w)}_{V,0r} \tilde{\tau}_{\mathfrak{w}\kappa,r} \;, \tag{E.46b}$$

$$\mathfrak{K}_{\mathfrak{w}\theta} := \sum_r \mathfrak{T}^{(w)}_{V,0r} \tilde{\mathfrak{K}}_{\mathfrak{w}\theta,r} \;, \tag{E.46c}$$

$$\mathfrak{K}_{\mathfrak{w}\theta\kappa} := \mathfrak{T}^{(w\kappa)}_{V,0} \tilde{\mathfrak{K}}_{\kappa\theta} + \sum_r \mathfrak{T}^{(w)}_{V,0r} \tilde{\mathfrak{K}}_{\mathfrak{w}\theta\kappa,r} \;, \tag{E.46d}$$

$$\mathfrak{K}_{\mathfrak{w}\sigma} := \sum_r \mathfrak{T}^{(w)}_{V,0r} \tilde{\mathfrak{K}}_{\mathfrak{w}\sigma,r} \;, \tag{E.46e}$$

$$\mathfrak{K}_{\mathfrak{w}\sigma\kappa} := \mathfrak{T}^{(w\kappa)}_{V,0} \tilde{\mathfrak{K}}_{\kappa\sigma} + \sum_r \mathfrak{T}^{(w)}_{V,0r} \tilde{\mathfrak{K}}_{\mathfrak{w}\sigma\kappa,r} \;, \tag{E.46f}$$

$$\mathfrak{K}_{\mathfrak{w}\omega} := \sum_r \mathfrak{T}^{(w)}_{V,0r} \tilde{\mathfrak{K}}_{\mathfrak{w}\omega,r} \;, \tag{E.46g}$$

$$\mathfrak{K}_{\mathfrak{w}\omega\kappa} := \mathfrak{T}^{(w\kappa)}_{V,0} \tilde{\mathfrak{K}}_{\kappa\omega} + \sum_r \mathfrak{T}^{(w)}_{V,0r} \tilde{\mathfrak{K}}_{\mathfrak{w}\omega\kappa,r} \;, \tag{E.46h}$$

$$\mathfrak{h}_{\mathfrak{w}\mathfrak{t}} := \mathfrak{T}^{(w\kappa)}_{V,0} \tilde{\mathfrak{h}}_{\kappa\mathfrak{t}} + \sum_r \mathfrak{T}^{(w)}_{V,0r} \tilde{\mathfrak{K}}_{\mathfrak{w}\mathfrak{t},r} \;, \tag{E.46i}$$

$$\mathfrak{K}_{\mathfrak{w}I\mathfrak{t}} := \sum_r \mathfrak{T}^{(w)}_{V,0r} \tilde{\mathfrak{K}}_{\mathfrak{w}I\mathfrak{t},r} \;, \tag{E.46j}$$

$$\mathfrak{K}_{\mathfrak{w}F\mathfrak{t}} := \mathfrak{T}^{(w\kappa)}_{V,0} \tilde{\mathfrak{K}}_{\kappa F\mathfrak{t}} + \sum_r \mathfrak{T}^{(w)}_{V,0r} \tilde{\mathfrak{K}}_{\mathfrak{w}F\mathfrak{t},r} \;, \tag{E.46k}$$

$$\mathfrak{h}_{\mathfrak{w}\omega} := \mathfrak{T}^{(w\kappa)}_{V,0} \tilde{\mathfrak{h}}_{\kappa\omega} \;, \tag{E.46l}$$

$$\mathfrak{h}_{\mathfrak{w}\mathfrak{p}} := \mathfrak{T}^{(w\kappa)}_{V,0}\tilde{\mathfrak{h}}_{\kappa\mathfrak{p}} + \sum_r \mathfrak{T}^{(w)}_{V,0r}\tilde{\mathfrak{h}}_{\mathfrak{w}\mathfrak{p},r} \;, \tag{E.46m}$$

$$\mathfrak{h}_{\mathfrak{w}\mathfrak{q}} := \sum_r \mathfrak{T}^{(w)}_{V,0r}\tilde{\mathfrak{h}}_{\mathfrak{w}\mathfrak{q},r} \;, \tag{E.46n}$$

$$\mathfrak{K}_{\mathfrak{w}I\omega} := \mathfrak{T}^{(w\kappa)}_{V,0}\tilde{\mathfrak{K}}_{\kappa I\omega} + \sum_r \mathfrak{T}^{(w)}_{V,0r}\tilde{\mathfrak{K}}_{\mathfrak{w}I\omega,r} \;, \tag{E.46o}$$

$$\mathfrak{K}_{\mathfrak{w}I\mathfrak{p}} := \sum_r \mathfrak{T}^{(w)}_{V,0r}\tilde{\mathfrak{K}}_{\mathfrak{w}I\mathfrak{p},r} \;, \tag{E.46p}$$

$$\mathfrak{K}_{\mathfrak{w}I\mathfrak{q}} := \sum_r \mathfrak{T}^{(w)}_{V,0r}\tilde{\mathfrak{K}}_{\mathfrak{w}I\mathfrak{q},r} \;, \tag{E.46q}$$

$$\mathfrak{K}_{\mathfrak{w}F\omega} := \mathfrak{T}^{(w\kappa)}_{V,0}\tilde{\mathfrak{K}}_{\kappa F\omega} \;, \tag{E.46r}$$

$$\mathfrak{K}_{\mathfrak{w}F\mathfrak{p}} := \mathfrak{T}^{(w\kappa)}_{V,0}\tilde{\mathfrak{K}}_{\kappa F\mathfrak{p}} + \sum_r \mathfrak{T}^{(w)}_{V,0r}\tilde{\mathfrak{K}}_{\mathfrak{w}F\mathfrak{p},r} \;, \tag{E.46s}$$

$$\mathfrak{K}_{\mathfrak{w}F\mathfrak{q}} := \sum_r \mathfrak{T}^{(w)}_{V,0r}\tilde{\mathfrak{K}}_{\mathfrak{w}F\mathfrak{q},r} \;. \tag{E.46t}$$

Lastly, the coefficients appearing in the equation of motion for the tensor $\mathfrak{t}^{\mu\nu}$ (6.161) are given by

$$\tau_{\mathfrak{t}} := \sum_r \mathfrak{T}^{(t)}_{T,0r}\tilde{\tau}_{\mathfrak{t},r} \;, \tag{E.47a}$$

$$\mathfrak{K}_{\mathfrak{t}\theta} := \sum_r \mathfrak{T}^{(t)}_{T,0r}\tilde{\mathfrak{K}}_{\mathfrak{t}\theta,r} \;, \tag{E.47b}$$

$$\mathfrak{h}_{\mathfrak{t}\mathfrak{w}} := \sum_r \mathfrak{T}^{(t)}_{T,0r}\tilde{\mathfrak{h}}_{\mathfrak{t}\mathfrak{w},r} \;, \tag{E.47c}$$

$$\mathfrak{h}_{\mathfrak{t}\kappa} := \sum_r \mathfrak{T}^{(t)}_{T,0r}\tilde{\mathfrak{h}}_{\mathfrak{t}\kappa,r} \;, \tag{E.47d}$$

$$\mathfrak{K}_{\mathfrak{t}I\mathfrak{w}} := \sum_r \mathfrak{T}^{(t)}_{T,0r}\tilde{\mathfrak{K}}_{\mathfrak{t}I\mathfrak{w},r} \;, \tag{E.47e}$$

$$\mathfrak{K}_{\mathfrak{t}F\mathfrak{w}} := \sum_r \mathfrak{T}^{(t)}_{T,0r}\tilde{\mathfrak{K}}_{\mathfrak{t}F\mathfrak{w},r} \;, \tag{E.47f}$$

$$\mathfrak{K}_{\mathfrak{t}I\kappa} := \sum_r \mathfrak{T}^{(t)}_{T,0r}\tilde{\mathfrak{K}}_{\mathfrak{t}I\kappa,r} \;, \tag{E.47g}$$

$$\mathfrak{K}_{\mathfrak{t}F\kappa} := \sum_r \mathfrak{T}^{(t)}_{T,0r}\tilde{\mathfrak{K}}_{\mathfrak{t}F\kappa,r} \;, \tag{E.47h}$$

$$\mathfrak{K}_{\mathfrak{t}\omega\omega} := \sum_r \mathfrak{T}^{(t)}_{T,0r}\tilde{\mathfrak{K}}_{\mathfrak{t}\omega\omega,r} \;, \tag{E.47i}$$

$$\mathfrak{K}_{\mathfrak{t}\omega\mathfrak{p}} := \sum_r \mathfrak{T}^{(t)}_{T,0r}\tilde{\mathfrak{K}}_{\mathfrak{t}\omega\mathfrak{p},r} \;, \tag{E.47j}$$

$$\mathfrak{K}_{\mathfrak{t}\omega\mathfrak{q}} := \sum_r \mathfrak{T}^{(t)}_{T,0r}\tilde{\mathfrak{K}}_{\mathfrak{t}\omega\mathfrak{q},r} \;, \tag{E.47k}$$

$$\mathfrak{K}_{\mathfrak{t}\sigma\omega} := \sum_r \mathfrak{T}^{(t)}_{T,0r}\tilde{\mathfrak{K}}_{\mathfrak{t}\sigma\omega,r} \;, \tag{E.47l}$$

$$\mathfrak{K}_{\mathfrak{t}\sigma\mathfrak{p}} := \sum_r \mathfrak{T}^{(t)}_{T,0r}\tilde{\mathfrak{K}}_{\mathfrak{t}\sigma\mathfrak{p},r} \;, \tag{E.47m}$$

$$\mathfrak{K}_{\mathfrak{t}\sigma\mathfrak{q}} := \sum_r \mathfrak{T}^{(t)}_{T,0r}\tilde{\mathfrak{K}}_{\mathfrak{t}\sigma\mathfrak{q},r} \;. \tag{E.47n}$$

Appendix F

Useful formulae

This appendix is concerned with deriving several expressions that are used in the main text. Specifically, we show how to compute general spin-space and thermodynamic integrals in Secs. F.1 and F.2, whereas Sec. F.3 demonstrates how to construct the orthogonal polynomials used in the expansion of the distribution function. Lastly, the purpose of Sec. F.4 is to show a method to treat collision integrals one may need to compute.

F.1 General spin-space integrals

We want to consider a general integral over spin space of the form

$$I^{\mu_1\cdots\mu_n} := \int \mathrm{d}S(k)\, \mathfrak{s}^{\mu_1}\cdots\mathfrak{s}^{\mu_n} \ , \tag{F.1}$$

where the measure is in general (for on-shell momenta where $k^2 = m^2$)

$$\mathrm{d}S(k) := S_0 \frac{m}{\varsigma\pi} \mathrm{d}^4\mathfrak{s}\,\delta(\mathfrak{s}^2 + \varsigma^2)\delta(k\cdot\mathfrak{s}) \ . \tag{F.2}$$

Comparing this to Eqs. (3.90) and (3.162), we can see that $\varsigma^2 = 3$ and $S_0 = 1$ for Dirac fermions, whereas $\varsigma^2 = 2$ and $S_0 = 3/2$ for Proca particles. Given that the integral (F.1) only depends on the momentum k and has to be orthogonal to it in all indices, it can only depend on combinations of the projector $K^{\mu\nu}$. Considering its symmetry, we find that it has to be of the form

$$I^{\mu_1\cdots\mu_n} = \begin{cases} I^{(n)} K^{(\mu_1\mu_2}\cdots K^{\mu_{n-1}\mu_n)} & , \quad \text{if } n \text{ is even} \ , \\ 0 & , \quad \text{if } n \text{ is odd} \ . \end{cases} \tag{F.3}$$

Here, the round brackets denote the symmetrization in all indices,

$$K^{(\mu_1\mu_2}\cdots K^{\mu_{n-1}\mu_n)} := \frac{1}{(n-1)!!}\sum_{\mathcal{P}} K^{\mu_1\mu_2}\cdots K^{\mu_{n-1}\mu_n} \ , \tag{F.4}$$

where the factor $(n-1)!!$ counts the terms in the sum over permutations, which is denoted by $\mathcal{P}$. At this point, we only need to compute $I^{(n)}$. In order to do this, we first notice what happens when we contract the tensor (F.4) with $K_{\mu_1\mu_2}$. We have

$$K_{\mu_1\mu_2}K^{(\mu_1\mu_2}\cdots K^{\mu_{n-1}\mu_n)} = \frac{n+1}{n-1} K^{(\mu_3\mu_4}\cdots K^{\mu_{n-1}\mu_n)} \ , \tag{F.5}$$

which becomes clear when considering which types of contractions can happen inside the symmetrized tensor of rank n. Then, we have upon complete contraction

$$K_{\mu_1\mu_2}\cdots K_{\mu_{n-1}\mu_n}K^{(\mu_1\mu_2}\cdots K^{\mu_{n-1}\mu_n)} = n+1\ . \tag{F.6}$$

Now we are able to compute $I^{(n)}$ as

$$I^{(n)} = \frac{1}{n+1}K_{\mu_1\mu_2}\cdots K_{\mu_{n-1}\mu_n}I^{\mu_1\cdots\mu_n} = \frac{(-\varsigma^2)^{n/2}}{n+1}\int \mathrm{d}S(k)\ . \tag{F.7}$$

The last integral is a Lorentz scalar and easily calculated in the particle rest frame, where $\delta(k\cdot\mathfrak{s}) = \delta(\mathfrak{s}^0)/m$,

$$\int \mathrm{d}S(k) = \frac{S_0}{\varsigma\pi}\int \mathrm{d}^3\mathfrak{s}\,\delta(-\mathfrak{s}^2+\varsigma^2) = \frac{S_0}{2\varsigma^2\pi}\int \mathrm{d}^3\mathfrak{s}\,\delta(-|\mathfrak{s}|+\varsigma) = 2S_0\ . \tag{F.8}$$

In total we thus have

$$\int \mathrm{d}S(k)\mathfrak{s}^{\mu_1}\cdots\mathfrak{s}^{\mu_n} = \begin{cases} S_0\dfrac{2(-\varsigma^2)^{n/2}}{n+1}K^{(\mu_1\mu_2}\cdots K^{\mu_{n-1}\mu_n)} & ,\quad \text{if } n \text{ is even}\ ,\\ 0 & ,\quad \text{if } n \text{ is odd}\ .\end{cases} \tag{F.9}$$

In particular, the first few nonzero integrals read

$$\int \mathrm{d}S(k) = 2S_0\ , \tag{F.10a}$$

$$\int \mathrm{d}S(k)\mathfrak{s}^\mu\mathfrak{s}^\nu = -S_0\frac{2\varsigma^2}{3}K^{\mu\nu}\ , \tag{F.10b}$$

$$\int \mathrm{d}S(k)\mathfrak{s}^\mu\mathfrak{s}^\nu\mathfrak{s}^\alpha\mathfrak{s}^\beta = S_0\frac{2\varsigma^4}{15}\left(K^{\mu\nu}K^{\alpha\beta}+K^{\mu\alpha}K^{\nu\beta}+K^{\mu\beta}K^{\nu\alpha}\right)\ . \tag{F.10c}$$

F.2 Thermodynamic integrals

The basic thermodynamic integral we have to evaluate is given by [cf. Eq. (6.16)]

$$\begin{aligned} I_{nq} &:= \frac{1}{(2q+1)!!}\int \mathrm{d}\Gamma E_{\mathbf{k}}^{n-2q}\left(-k^{\langle\alpha\rangle}k_\alpha\right)^q f_{0\mathbf{k}} \\ &= \frac{1}{(2q+1)!!}\int \mathrm{d}\Gamma E_{\mathbf{k}}^{n-2q}\left(E_{\mathbf{k}}^2-m^2\right)^q f_{0\mathbf{k}}\ , \end{aligned} \tag{F.11}$$

where we used that $k^{\langle\alpha\rangle}k_\alpha = m^2 - E_{\mathbf{k}}^2$. Since this integral is a scalar, we can evaluate it in any frame, which most conveniently is chosen to be the fluid rest frame, where $u^\mu \equiv (1,\mathbf{0})$ and $E_{\mathbf{k}} \equiv k^0 = \sqrt{k^2+m^2}$. Switching to spherical coordinates and performing the angular integrations, we find

$$\begin{aligned} I_{nq} &= \frac{g}{(2q+1)!!}\frac{1}{2\pi^2}\int_0^\infty \mathrm{d}k\, k^{2q+2}(k^2+m^2)^{(n-1)/2-q}\left[e^{\beta_0\sqrt{k^2+m^2}-\alpha_0}+a\right]^{-1} \\ &= \frac{g}{(2q+1)!!}\frac{\beta_0^{-n-2}}{2\pi^2}\int_z^\infty \mathrm{d}y\,\left(y^2-z^2\right)^{q+1/2}y^{n-2q}\left[e^{y-\alpha_0}+a\right]^{-1}\ , \end{aligned} \tag{F.12}$$

where $g = 2s+1$ is the degeneracy factor for spin-s particles and we substituted $y := \beta_0\sqrt{k^2+m^2}$ in the second equality. Furthermore, $z := m\beta_0$ is the ratio of mass over temperature that quantifies how far we are in the relativistic regime. In the following we consider classical statistics, i.e., $a = 0$. The remaining task then consists of evaluating an integral of the form

$$\mathcal{I}_{ab}(z) := \int_z^\infty \mathrm{d}y\,\left(y^2-z^2\right)^{b-1/2}y^a e^{-y}\ , \tag{F.13}$$

with $a = n - 2q$ and $b = q + 1$. In the ultrarelativistic limit $z \to 0$, the integral is evaluated straightforwardly as

$$\mathcal{I}_{ab}(0) = \int_0^\infty \mathrm{d}y\, y^{2b+a-1} e^{-y} = \Gamma(a + 2b) \ . \tag{F.14}$$

In the case where $z > 0$, it is advantageous to first define $x := y/z$, such that

$$\mathcal{I}_{ab}(z) = z^{a+2b} \int_1^\infty \mathrm{d}x \ \left(x^2 - 1\right)^{b-1/2} x^a e^{-zx} \ . \tag{F.15}$$

Then, we note that the following recursion holds,

$$\mathcal{I}_{ab}(z) = \mathcal{I}_{a+2,b-1}(z) - \mathcal{I}_{a,b-1}(z) \ , \tag{F.16}$$

which can be applied iteratively to obtain

$$\mathcal{I}_{ab}(z) = \sum_{j=0}^{b} (-1)^j \binom{b}{j} z^{2j} \mathcal{I}_{a+2b-2j,0}(z) \ . \tag{F.17}$$

However, the integrals $\mathcal{I}_{a+2b-2j,0}(z)$ can be evaluated by remembering the definition of the Bickley-Naylor function [168]

$$\mathrm{Ki}_r(z) := \int_0^\infty \mathrm{d}\tau \cosh^{-r}\tau e^{-z\cosh\tau} = \int_1^\infty \mathrm{d}x \, \frac{x^{-r}}{\sqrt{x^2 - 1}} e^{-zx} \equiv z^r \mathcal{I}_{-r,0}(z) \ , \tag{F.18}$$

which then yields

$$\mathcal{I}_{ab}(z) = z^{a+2b} \sum_{j=0}^{b} (-1)^j \binom{b}{j} \mathrm{Ki}_{2j-2b-a}(z) \ . \tag{F.19}$$

Summarizing our results, we have

$$I_{nq} = \frac{g e^{\alpha_0}}{(2q+1)!!} \frac{\beta_0^{-n-2}}{2\pi^2} \times \begin{cases} z^{n+2} \sum_{j=0}^{q+1} (-1)^j \binom{q+1}{j} \mathrm{Ki}_{2j-2-n}(z) \ , & z > 0 \\ \Gamma(n+2) \ , & z = 0 \end{cases} . \tag{F.20}$$

In order to implement Eq. (F.20) efficiently in the case $z > 0$, we note that the Bickley-Naylor function fulfills the following recursion relation for $r \geq 2$,

$$r\mathrm{Ki}_{r+1}(z) = (r - 1)\mathrm{Ki}_{r-1}(z) - z\mathrm{Ki}_r(z) + z\mathrm{Ki}_{r-2}(z) \ , \tag{F.21}$$

with the starting values given by [169]

$$\mathrm{Ki}_0(z) = K_0(z) \ , \tag{F.22a}$$

$$\mathrm{Ki}_1(z) = \frac{\pi}{2} \left\{1 - z\left[\mathbf{L}_{-1}(z)K_0(z) + \mathbf{L}_0(z)K_1(z)\right]\right\} \ , \tag{F.22b}$$

where $K_r(z)$ is the modified Bessel function of the second kind and $\mathbf{L}_r(z)$ denotes the modified Struve function. In the cases where $r < 0$, the Bickley-Naylor function can be expressed as

$$\mathrm{Ki}_r(z) = (-1)^r \frac{\mathrm{d}^r}{\mathrm{d}z^r} K_0(z) \ . \tag{F.23}$$

The result (F.20) is the general solution that will hold for any value of n and q, as long as the integral converges, which is ensured by demanding that $q > -3/2$ for $z > 0$ and $n > -2$ for $z = 0$. Nevertheless, for the cases where $n - 2q \geq 0$ we may establish a simpler formula that circumvents the use of Bickley-Naylor functions. Orienting on the method presented in Chapter XIII of Ref. [43], we first note that we can express the integral (F.13) as

$$\mathcal{I}_{ab}(z) = (-1)^a (2b - 1)!! z^{a+2b} \frac{\mathrm{d}^a}{\mathrm{d}z^a} \frac{K_b(z)}{z^b} \ , \tag{F.24}$$

which is easily proved by employing the integral representation of the modified Bessel function

$$K_r(z) \equiv \frac{z^{-r}}{(2r-1)!!} \int_z^\infty \mathrm{d}\tau \left(\tau^2 - z^2\right)^{r-1/2} e^{-\tau} \tag{F.25}$$

and substituting $\chi := \tau/z$ in the integral. Note that the relation (F.24) holds for all $a \in \mathbb{Z}$, where derivatives of negative order are to be interpreted as integrals of the respective positive order. On the other hand, as long as $a \geq 0$, it is also possible to establish

$$\frac{\mathrm{d}^a}{\mathrm{d}z^a} \frac{K_b(z)}{z^b} = \sum_{j=0}^{\lfloor a/2 \rfloor} (-1)^{a-j} (2j-1)!! \binom{a}{2j} \frac{K_{b+a-j}(z)}{z^{b+j}} , \tag{F.26}$$

which is based on the identity

$$\frac{\mathrm{d}}{\mathrm{d}z} \frac{K_b(z)}{z^b} = -\frac{K_{b+1}(z)}{z^b} \tag{F.27}$$

and can be proved by induction. Combining Eqs. (F.24) and (F.26), we obtain

$$\mathcal{I}_{ab}(z) = (2b-1)!! \sum_{j=0}^{\lfloor a/2 \rfloor} (-1)^j (2j-1)!! \binom{a}{2j} z^{a+b-j} K_{b+a-j}(z) , \tag{F.28}$$

which in turn allows us to express the thermodynamic integral I_{nq} for $n - 2q \geq 0$ as

$$I_{nq} = g e^{\alpha_0} \frac{\beta_0^{-n-2}}{2\pi^2} \sum_{j=0}^{\lfloor (n-2q)/2 \rfloor} (-1)^j (2j-1)!! \binom{n-2q}{2j} z^{n+1-q-j} K_{n+1-q-j}(z) . \tag{F.29}$$

Here we assumed that $z > 0$, since the $z = 0$ case does not change compared to Eq. (F.20).

F.3 Orthogonal polynomials

In this section, we show how to evaluate the coefficients $a_{nr}^{(j,\ell)}$ introduced in Eq. (6.19), orienting on Ref. [51]. Requiring that the polynomials $P_{\mathbf{k}n}^{(j,\ell)}$ are orthonormal, cf. Eq. (6.14), we have

$$\begin{aligned}
\delta_{mn} &= \int \mathrm{d}K \omega^{(\ell)} \sum_{r \in \mathbb{S}_\ell^{(j)}}^{n} \sum_{s \in \mathbb{S}_\ell^{(j)}}^{m} a_{nr}^{(j,\ell)} a_{ms}^{(j,\ell)} E_{\mathbf{k}}^{r+s} \\
&= \sum_{r \in \mathbb{S}_\ell^{(j)}}^{n} \sum_{s \in \mathbb{S}_\ell^{(j)}}^{m} a_{nr}^{(j,\ell)} a_{ms}^{(j,\ell)} \frac{J_{r+s+2\ell,\ell}}{J_{2\ell,\ell}} .
\end{aligned} \tag{F.30}$$

Next, we define the matrix $\mathcal{D}^{(j,\ell n)}$ whose elements are $\mathcal{D}_{rs}^{(j,\ell n)} := J_{r+s+2\ell,\ell}$ [51]. Note that the dimension of this matrix is as large as the number of elements included in the basis $\mathbb{S}_\ell^{(j)}$ that are smaller or equal to n. Then, the orthonormality requirement reads

$$\sum_{r \in \mathbb{S}_\ell^{(j)}}^{n} \sum_{s \in \mathbb{S}_\ell^{(j)}}^{m} a_{nr}^{(j,\ell)} a_{ms}^{(j,\ell)} \mathcal{D}_{rs}^{(j,\ell n)} = \delta_{mn} J_{2\ell,\ell} \quad \forall\, m, n \in \mathbb{S}_\ell^{(j)} . \tag{F.31}$$

To see what this equation tells us, consider it for $m = s_0$, where s_0 denotes the smallest element of the set $\mathbb{S}_j^{(\ell)}$:

$$\sum_{r \in \mathbb{S}_\ell^{(j)}}^{n} a_{nr}^{(j,\ell)} \mathcal{D}_{rs_0}^{(j,\ell n)} = 0 . \tag{F.32}$$

Moving on to $m = s_1$, where s_1 is the next-smallest element, we find after using Eq. (F.32)

$$\sum_{r\in\mathbb{S}_\ell^{(j)}}^{n} a_{nr}^{(j,\ell)} \mathcal{D}_{rs_1}^{(j,\ell n)} = 0 \; . \tag{F.33}$$

Iterating this procedure, we find

$$\sum_{s\in\mathbb{S}_\ell^{(j)}}^{n} a_{nr}^{(j,\ell)} \mathcal{D}_{rs}^{(j,\ell n)} = 0 \quad \forall\, s \neq n \; . \tag{F.34}$$

In the case $s = n$, we obtain

$$a_{nn}^{(j,\ell)} \sum_{r\in\mathbb{S}_\ell^{(j)}}^{n} a_{nr}^{(j,\ell)} \mathcal{D}_{rn}^{(j,\ell n)} = J_{2\ell,\ell} \; , \tag{F.35}$$

where we used Eq. (F.34). In total, we thus have

$$\sum_{r\in\mathbb{S}_\ell^{(j)}}^{n} \frac{a_{nr}^{(j,\ell)}}{a_{nn}^{(j,\ell)}} \mathcal{D}_{rs}^{(j,\ell n)} = \frac{J_{2\ell,\ell}}{\left(a_{nn}^{(j,\ell)}\right)^2} \delta_{ns} \; . \tag{F.36}$$

The solution of this equation is

$$\left(a_{nn}^{(j,\ell)}\right)^2 = \left(\mathcal{D}^{-1}\right)_{nn}^{(j,\ell n)} J_{2\ell,\ell} \; , \qquad a_{nm}^{(j,\ell)} = \frac{\left(\mathcal{D}^{-1}\right)_{mn}^{(j,\ell n)}}{\left(\mathcal{D}^{-1}\right)_{nn}^{(j,\ell n)}} a_{nn}^{(j,\ell)} \; , \tag{F.37}$$

which is the explicit relation were looking to obtain.

F.4 Collision integrals

As has become clear in the main text, almost always collision integrals have to be evaluated to compute transport coefficients. While in rare cases these integrals can be done analytically [170, 171], in most cases they have to be performed numerically. Since the expressions we deal with often involve a large number of terms, in this section we present a method introduced in Chapter XIII of Ref. [43] that can be used to automatize this computation. The basic idea consists in separating the integrals into a sum of elementary collision integrals

$$J^{(a,b,d,e,f)} := \int [\mathrm{d}K] e^{-\beta P_T\cdot u} (P_T^2)^a (P_T\cdot u)^b (Q\cdot u)^d (Q'\cdot u)^e (-Q\cdot Q')^f \delta^{(4)}(k + k' - k_1 - k_2) \; , \tag{F.38}$$

where the momenta k, k', k_1, and k_2 can be expressed in terms of the total momentum P_T and the relative momenta Q, Q' via

$$k^\mu = \frac{1}{2}\left(P_T^\mu + Q^\mu\right) \; , \tag{F.39a}$$

$$k'^\mu = \frac{1}{2}\left(P_T^\mu - Q^\mu\right) \; , \tag{F.39b}$$

$$k_1^\mu = \frac{1}{2}\left(P_T^\mu + Q'^\mu\right) \; , \tag{F.39c}$$

$$k_2^\mu = \frac{1}{2}\left(P_T^\mu - Q'^\mu\right) \; . \tag{F.39d}$$

Next we follow the steps in Ref. [43] and make use of the identity

$$\int_z^\infty \mathrm{d}y \, \left(y^2 - z^2\right)^{b-1/2} y^a e^{-y} = z^{a+2b} \sum_{j=0}^{b} (-1)^j \binom{b}{j} \mathrm{Ki}_{2j-2b-a}(z) \; , \tag{F.40}$$

which we proved in Sec. F.2. The result for the basic integral (F.38) then reads

$$J^{(a,b,d,e,f)} = \beta^{-4-2a-b-d-e-2f} \frac{16\pi^3}{(2\pi\hbar)^{12}} \sum_{g=0}^{\min(d,e)} K(d,e,g)\sigma^{(f,g)} \sum_{h=0}^{\frac{d+e}{2}+1} \binom{\frac{d+e}{2}+1}{h} (-1)^h \\ \times \int_{2z}^{\infty} \mathrm{d}v \left[v^2 - (2z)^2\right]^{(d+e)/2+f+1} v^{2(a-1)+b+3} \mathrm{Ki}_{-b-d-e-2+2h}(v) \;, \tag{F.41}$$

where we introduced the following factors,

$$K(d,e,g) := \begin{cases} \dfrac{d!e!}{(d-g)!!(d+g+1)!!(e-g)!!(e+g+1)!!} \;, & \text{if } (d-g), (e-g) \text{ even} \;, \\ 0 \;, & \text{otherwise} \;, \end{cases} \tag{F.42a}$$

$$\sigma^{(f,g)} := \begin{cases} (2g+1)\dfrac{f!\,2^g}{(f+g+1)!} \dfrac{\left(\frac{f+g}{2}\right)!}{\left(\frac{f-g}{2}\right)!} \;, & \text{if } (f-g) \text{ even} \;, \\ 0 \;, & \text{otherwise} \;. \end{cases} \tag{F.42b}$$

The remaining task then consists of expanding the collision integrals in question as sums of the basic integrals (F.41). Note that the tensors $\Xi^{\mu\nu}$, $\Xi^{\mu\nu}_{\alpha\beta}$, which may appear in the integrals related to irreducible moments of spin-rank higher than zero, do not allow for a straightforward expression in terms of polynomials of P_T, Q, and Q'. This is the case because of the factors of energy appearing in the denominator, leading to

$$\Xi^{\mu\nu} = \Delta^{\mu\nu} + \frac{(P_T^{\langle\mu\rangle} + Q^{\langle\mu\rangle})(P_T^{\langle\nu\rangle} + Q^{\langle\nu\rangle})}{(P_T \cdot u + Q \cdot u)^2} \;, \tag{F.43}$$

and similarly for $\Xi^{\mu\nu}_{\alpha\beta}$. In order to bring these terms into the form required by Eq. (F.38) as well, we expand them around the nonrelativistic limit [formally equivalent to taking the limit $k^\mu \simeq (m, \mathbf{0})^\mu$], leading to

$$\Xi^{\mu\nu} \simeq \Delta^{\mu\nu} \;, \quad \Xi^{\mu\nu}_{\alpha\beta} \simeq \Delta^{\mu\nu}_{\alpha\beta} \;. \tag{F.44}$$

The plot 6.1 is generated with this leading-order approximation.

Bibliography

[1] J.-Y. Ollitrault. “Anisotropy as a signature of transverse collective flow”. *Phys. Rev. D* 46 (1992), pp. 229–245. doi: 10.1103/PhysRevD.46.229.

[2] K. Dusling, W. Li, and B. Schenke. “Novel collective phenomena in high-energy proton–proton and proton–nucleus collisions”. *Int. J. Mod. Phys. E* 25.01 (2016), p. 1630002. doi: 10.1142/S0218301316300022. arXiv: 1509.07939 [nucl-ex].

[3] H. Elfner and B. Müller. “The exploration of hot and dense nuclear matter: Introduction to relativistic heavy-ion physics” (Oct. 2022). arXiv: 2210.12056 [nucl-th].

[4] S. J. Barnett. “Gyromagnetic and Electron-Inertia Effects”. *Rev. Mod. Phys.* 7 (2 Apr. 1935), pp. 129–166. doi: 10.1103/RevModPhys.7.129.

[5] L. Adamczyk et al. “Global Λ hyperon polarization in nuclear collisions: evidence for the most vortical fluid”. *Nature* 548 (2017), pp. 62–65. doi: 10.1038/nature23004. arXiv: 1701.06657 [nucl-ex].

[6] S. Acharya et al. “Global polarization of $\Lambda\bar{\Lambda}$ hyperons in Pb-Pb collisions at $\sqrt{s_{NN}} = 2.76$ and 5.02 TeV”. *Phys. Rev. C* 101.4 (2020). [Erratum: Phys.Rev.C 105, 029902 (2022)], p. 044611. doi: 10.1103/PhysRevC.101.044611. arXiv: 1909.01281 [nucl-ex].

[7] J. Adam et al. “Global Polarization of Ξ and Ω Hyperons in Au+Au Collisions at $\sqrt{s_{NN}} =$ 200 GeV”. *Phys. Rev. Lett.* 126.16 (2021), p. 162301. doi: 10.1103/PhysRevLett.126.162301. arXiv: 2012.13601 [nucl-ex].

[8] R. Abou Yassine et al. “Measurement of global polarization of Λ hyperons in few-GeV heavy-ion collisions”. *Phys. Lett. B* 835 (2022), p. 137506. doi: 10.1016/j.physletb.2022.137506. arXiv: 2207.05160 [nucl-ex].

[9] F. Becattini and M. A. Lisa. “Polarization and Vorticity in the Quark–Gluon Plasma”. *Ann. Rev. Nucl. Part. Sci.* 70 (2020), pp. 395–423. doi: 10.1146/annurev-nucl-021920-095245. arXiv: 2003.03640 [nucl-ex].

[10] J. Adam et al. “Polarization of Λ ($\bar{\Lambda}$) hyperons along the beam direction in Au+Au collisions at $\sqrt{s_{NN}} = 200$ GeV”. *Phys. Rev. Lett.* 123.13 (2019), p. 132301. doi: 10.1103/PhysRevLett.123.132301. arXiv: 1905.11917 [nucl-ex].

[11] F. Becattini, M. Buzzegoli, and A. Palermo. “Spin-thermal shear coupling in a relativistic fluid”. *Phys. Lett. B* 820 (2021), p. 136519. doi: 10.1016/j.physletb.2021.136519. arXiv: 2103.10917 [nucl-th].

[12] F. Becattini, M. Buzzegoli, G. Inghirami, I. Karpenko, and A. Palermo. “Local Polarization and Isothermal Local Equilibrium in Relativistic Heavy Ion Collisions”. *Phys. Rev. Lett.* 127.27 (2021), p. 272302. doi: 10.1103/PhysRevLett.127.272302. arXiv: 2103.14621 [nucl-th].

[13] B. Fu, S. Y. F. Liu, L. Pang, H. Song, and Y. Yin. “Shear-Induced Spin Polarization in Heavy-Ion Collisions”. *Phys. Rev. Lett.* 127.14 (2021), p. 142301. doi: 10.1103/PhysRevLett.127.142301. arXiv: 2103.10403 [hep-ph].

[14] S. Y. F. Liu and Y. Yin. “Spin polarization induced by the hydrodynamic gradients”. *JHEP* 07 (2021), p. 188. doi: 10.1007/JHEP07(2021)188. arXiv: 2103.09200 [hep-ph].

[15] E. Leader. *Spin in Particle Physics.* Cambridge University Press, 2001.

[16] S. Acharya et al. "Evidence of Spin-Orbital Angular Momentum Interactions in Relativistic Heavy-Ion Collisions". *Phys. Rev. Lett.* 125.1 (2020), p. 012301. doi: 10.1103/PhysRevLett.125.012301. arXiv: 1910.14408 [nucl-ex].

[17] B. Mohanty, S. Kundu, S. Singha, and R. Singh. "Spin alignment measurement of vector mesons produced in high energy collisions". *Mod. Phys. Lett. A* 36.39 (2021), p. 2130026. doi: 10.1142/S0217732321300263. arXiv: 2112.04816 [nucl-ex].

[18] M. S. Abdallah et al. "Pattern of global spin alignment of ϕ and K^{*0} mesons in heavy-ion collisions". *Nature* 614.7947 (2023), pp. 244–248. doi: 10.1038/s41586-022-05557-5. arXiv: 2204.02302 [hep-ph].

[19] Z.-T. Liang and X.-N. Wang. "Spin alignment of vector mesons in non-central A+A collisions". *Phys. Lett. B* 629 (2005), pp. 20–26. doi: 10.1016/j.physletb.2005.09.060. arXiv: nucl-th/0411101.

[20] Y.-G. Yang, R.-H. Fang, Q. Wang, and X.-N. Wang. "Quark coalescence model for polarized vector mesons and baryons". *Phys. Rev. C* 97.3 (2018), p. 034917. doi: 10.1103/PhysRevC.97.034917. arXiv: 1711.06008 [nucl-th].

[21] X.-L. Sheng, L. Oliva, and Q. Wang. "What can we learn from the global spin alignment of ϕ mesons in heavy-ion collisions?" *Phys. Rev. D* 101.9 (2020), p. 096005. doi: 10.1103/PhysRevD.101.096005. arXiv: 1910.13684 [nucl-th].

[22] X.-L. Sheng, Q. Wang, and X.-N. Wang. "Improved quark coalescence model for spin alignment and polarization of hadrons". *Phys. Rev. D* 102.5 (2020), p. 056013. doi: 10.1103/PhysRevD.102.056013. arXiv: 2007.05106 [nucl-th].

[23] X.-L. Xia, H. Li, X.-G. Huang, and H. Zhong Huang. "Local spin alignment of vector mesons in relativistic heavy-ion collisions". *Phys. Lett. B* 817 (2021), p. 136325. doi: 10.1016/j.physletb.2021.136325. arXiv: 2010.01474 [nucl-th].

[24] K. J. Gonçalves and G. Torrieri. "Spin alignment of vector mesons as a probe of spin hydrodynamics and freeze-out". *Phys. Rev. C* 105.3 (2022), p. 034913. doi: 10.1103/PhysRevC.105.034913. arXiv: 2104.12941 [nucl-th].

[25] B. Müller and D.-L. Yang. "Anomalous spin polarization from turbulent color fields". *Phys. Rev. D* 105.1 (2022), p. L011901. doi: 10.1103/PhysRevD.105.L011901. arXiv: 2110.15630 [nucl-th].

[26] X.-L. Sheng, L. Oliva, Z.-T. Liang, Q. Wang, and X.-N. Wang. "Spin alignment of vector mesons in heavy-ion collisions" (May 2022). arXiv: 2205.15689 [nucl-th].

[27] X.-L. Sheng, L. Oliva, Z.-T. Liang, Q. Wang, and X.-N. Wang. "Relativistic spin dynamics for vector mesons" (June 2022). arXiv: 2206.05868 [hep-ph].

[28] D. Wagner, N. Weickgenannt, and E. Speranza. "Generating tensor polarization from shear stress". *Phys. Rev. Res.* 5.1 (2023), p. 013187. doi: 10.1103/PhysRevResearch.5.013187. arXiv: 2207.01111 [nucl-th].

[29] W. A. Hiscock and L. Lindblom. "Stability and causality in dissipative relativistic fluids". *Annals Phys.* 151 (1983), pp. 466–496. doi: 10.1016/0003-4916(83)90288-9.

[30] W. A. Hiscock and L. Lindblom. "Generic instabilities in first-order dissipative relativistic fluid theories". *Phys. Rev. D* 31 (1985), pp. 725–733. doi: 10.1103/PhysRevD.31.725.

[31] S. Pu, T. Koide, and D. H. Rischke. "Does stability of relativistic dissipative fluid dynamics imply causality?" *Phys. Rev. D* 81 (2010), p. 114039. doi: 10.1103/PhysRevD.81.114039. arXiv: 0907.3906 [hep-ph].

[32] L. Gavassino, M. Antonelli, and B. Haskell. "Thermodynamic Stability Implies Causality". *Phys. Rev. Lett.* 128.1 (2022), p. 010606. doi: 10.1103/PhysRevLett.128.010606. arXiv: 2105.14621 [gr-qc].

[33] E. Noether. "Invariant Variation Problems". *Gott. Nachr.* 1918 (1918), pp. 235–257. doi: 10.1080/00411457108231446. arXiv: physics/0503066.

[34] F. W. Hehl. “On the Energy Tensor of Spinning Massive Matter in Classical Field Theory and General Relativity”. *Rept. Math. Phys.* 9 (1976), pp. 55–82. doi: 10.1016/0034-4877(76)90016-1.

[35] E. Leader and C. Lorcé. “The angular momentum controversy: What’s it all about and does it matter?” *Phys. Rept.* 541.3 (2014), pp. 163–248. doi: 10.1016/j.physrep.2014.02.010. arXiv: 1309.4235 [hep-ph].

[36] L. Boltzmann. *Vorlesungen über Gastheorie.* 1896.

[37] H.-W. Lee. “Theory and application of the quantum phase-space distribution functions”. *Physics Reports* 259 (1995), pp. 147–211.

[38] W. Florkowski, R. Ryblewski, and A. Kumar. “Relativistic hydrodynamics for spin-polarized fluids”. *Prog. Part. Nucl. Phys.* 108 (2019), p. 103709. doi: 10.1016/j.ppnp.2019.07.001. arXiv: 1811.04409 [nucl-th].

[39] S. Bhadury, W. Florkowski, A. Jaiswal, A. Kumar, and R. Ryblewski. “Relativistic dissipative spin dynamics in the relaxation time approximation”. *Phys. Lett. B* 814 (2021), p. 136096. doi: 10.1016/j.physletb.2021.136096. arXiv: 2002.03937 [hep-ph].

[40] J. Zamanian, M. Marklund, and G. Brodin. “Scalar quantum kinetic theory for spin-1/2 particles: Mean field theory”. *New J. Phys.* 12 (2010), p. 043019. doi: 10.1088/1367-2630/12/4/043019.

[41] R. Ekman, F. A. Asenjo, and J. Zamanian. “Fully relativistic kinetic equation for spin-1/2 particles in the long scale-length approximation”. *Phys. Rev.* E96.2 (2017), p. 023207. doi: 10.1103/PhysRevE.96.023207. arXiv: 1702.00722 [physics.plasm-ph].

[42] R. Ekman, H. Al-Naseri, J. Zamanian, and G. Brodin. “Relativistic kinetic theory for spin-1/2 particles: Conservation laws, thermodynamics, and linear waves”. *Phys. Rev.* E100.2 (2019), p. 023201. doi: 10.1103/PhysRevE.100.023201. arXiv: 1904.08727 [physics.plasm-ph].

[43] S. R. de Groot, W. A. van Leeuwen, and C. G. van Weert. *Relativistic Kinetic Theory. Principles and Applications.* North-Holland, 1980.

[44] N. Weickgenannt, E. Speranza, X.-l. Sheng, Q. Wang, and D. H. Rischke. “Derivation of the nonlocal collision term in the relativistic Boltzmann equation for massive spin-1/2 particles from quantum field theory”. *Phys. Rev. D* 104.1 (2021), p. 016022. doi: 10.1103/PhysRevD.104.016022. arXiv: 2103.04896 [nucl-th].

[45] N. Weickgenannt, E. Speranza, X.-l. Sheng, Q. Wang, and D. H. Rischke. “Generating Spin Polarization from Vorticity through Nonlocal Collisions”. *Phys. Rev. Lett.* 127.5 (2021), p. 052301. doi: 10.1103/PhysRevLett.127.052301. arXiv: 2005.01506 [hep-ph].

[46] D. Wagner, N. Weickgenannt, and D. H. Rischke. “Lorentz-covariant nonlocal collision term for spin-1/2 particles”. *Phys. Rev. D* 106.11 (2022), p. 116021. doi: 10.1103/PhysRevD.106.116021. arXiv: 2210.06187 [nucl-th].

[47] D. Wagner, N. Weickgenannt, and E. Speranza. “Quantum kinetic theory with interactions for massive vector bosons” (June 2023). arXiv: 2306.05936 [nucl-th].

[48] H. Struchtrup. “Stable transport equations for rarefied gases at high orders in the Knudsen number”. *Phys. Fluids* 16 (2004), pp. 3921–3934. doi: https://doi.org/10.1063/1.1782751.

[49] J. A. Fotakis, E. Molnár, H. Niemi, C. Greiner, and D. H. Rischke. “Multicomponent relativistic dissipative fluid dynamics from the Boltzmann equation”. *Phys. Rev. D* 106.3 (2022), p. 036009. doi: 10.1103/PhysRevD.106.036009. arXiv: 2203.11549 [nucl-th].

[50] D. Wagner, A. Palermo, and V. E. Ambruş. “Inverse-Reynolds-dominance approach to transient fluid dynamics”. *Phys. Rev. D* 106.1 (2022), p. 016013. doi: 10.1103/PhysRevD.106.016013. arXiv: 2203.12608 [nucl-th].

[51] G. S. Denicol, H. Niemi, E. Molnar, and D. Rischke. “Derivation of transient relativistic fluid dynamics from the Boltzmann equation”. *Phys. Rev. D* 85 (2012). [Erratum: Phys.Rev.D 91, 039902 (2015)], p. 114047. doi: 10.1103/PhysRevD.85.114047. arXiv: 1202.4551 [nucl-th].

[52] L. Gavassino. "Is relativistic hydrodynamics always symmetric-hyperbolic in the linear regime?" *Phys. Rev. D* 107.6 (2023), p. 065013. doi: 10.1103/PhysRevD.107.065013. arXiv: 2210.05067 [nucl-th].

[53] N. Weickgenannt, X.-L. Sheng, E. Speranza, Q. Wang, and D. H. Rischke. "Kinetic theory for massive spin-1/2 particles from the Wigner-function formalism". *Phys. Rev.* D100.5 (2019), p. 056018. doi: 10.1103/PhysRevD.100.056018. arXiv: 1902.06513 [hep-ph].

[54] N. Weickgenannt, D. Wagner, and E. Speranza. "Pseudogauges and relativistic spin hydrodynamics for interacting Dirac and Proca fields". *Phys. Rev. D* 105.11 (2022), p. 116026. doi: 10.1103/PhysRevD.105.116026. arXiv: 2204.01797 [nucl-th].

[55] D. Wagner, N. Weickgenannt, and E. Speranza. "Kinetic Theory for Massive Spin-1 Particles". *Acta Phys. Polon. Supp.* 16.1 (2023), p. 42. doi: 10.5506/APhysPolBSupp.16.1-A42. arXiv: 2207.14060 [nucl-th].

[56] L. Euler. "Principes généraux du mouvement des fluides". *Mémoires de l'académie des sciences de Berlin* (1757), pp. 274–315.

[57] C. Navier. "On the laws of motion of fluids taking into consideration the adhesion of the molecules". *Ann. Chim. Phys* 19 (1822), pp. 234–245.

[58] G. G. Stokes et al. "On the effect of the internal friction of fluids on the motion of pendulums" (1851).

[59] M. Matsuo, J. Ieda, E. Saitoh, and S. Maekawa. "Effects of mechanical rotation on spin currents". *Physical review letters* 106.7 (2011), p. 076601.

[60] M. Matsuo, J. Ieda, E. Saitoh, and S. Maekawa. "Spin-dependent inertial force and spin current in accelerating systems". *Physical Review B* 84.10 (2011), p. 104410.

[61] M. Matsuo, Y. Ohnuma, and S. Maekawa. "Theory of spin hydrodynamic generation". *Physical Review B* 96.2 (2017), p. 020401.

[62] R. Takahashi et al. "Spin hydrodynamic generation". *Nature Physics* 12.1 (2016), pp. 52–56.

[63] G. Lukaszewicz. *Micropolar Fluids, Theory and Applications*. Birkhäuser Boston, 1999.

[64] J. S. Dahler and N. F. Sather. "Kinetic theory of loaded spheres. i". *The Journal of Chemical Physics* 38.10 (1963), pp. 2363–2382.

[65] S. I. Sandler and J. S. Dahler. "Kinetic theory of loaded spheres. ii". *The Journal of Chemical Physics* 43.5 (1965), pp. 1750–1759.

[66] S. I. Sandler and J. S. Dahler. "Transport properties of polyatomic fluids. II. A dilute gas of spherocylinders". *The Journal of Chemical Physics* 44.3 (1966), pp. 1229–1237.

[67] I. L. McLaughlin and J. S. Dahler. "Transport Properties of Polyatomic Fluids. III. The Transport—Relaxation Equations for a Dilute Gas of Rough Spheres". *The Journal of Chemical Physics* 44.12 (1966), pp. 4453–4459.

[68] B. J. McCoy, S. I. Sandler, and J. S. Dahler. "Transport properties of polyatomic fluids. IV. The kinetic theory of a dense gas of perfectly rough spheres". *The Journal of Chemical Physics* 45.10 (1966), pp. 3485–3512.

[69] L. Waldmann. "Die Boltzmann-Gleichung für gase mit rotierenden molekülen". *Zeitschrift für Naturforschung A* 12.8 (1957), pp. 660–662.

[70] R. F. Snider. "Quantum-Mechanical Modified Boltzmann Equation for Degenerate Internal States". *The Journal of Chemical Physics* 32.4 (1960), pp. 1051–1060.

[71] S. Hess and L. Waldmann. "Kinetic theory for a dilute gas of particles with spin". *Zeitschrift für Naturforschung A* 21.10 (1966), pp. 1529–1546.

[72] S. Hess and L. Waldmann. "Kinetic Theory for a Dilute Gas of Particles with Spin. II. Relaxation Coefficients". *Zeitschrift für Naturforschung A* 23.12 (1968), pp. 1893–1902.

[73] S. Hess and L. Waldmann. "Kinetic Theory for a Dilute Gas of Particles with" Spin"". *Zeitschrift für Naturforschung A* 26.6 (1971), pp. 1057–1071.

[74] T. S. Olson. "Stability and causality in the Israel-Stewart Energy Frame Theory". *Annals Phys.* 199 (1990), p. 18. DOI: 10.1016/0003-4916(90)90366-V.

[75] E. P. Wigner. "On the quantum correction for thermodynamic equilibrium". *Phys. Rev.* 40 (1932), pp. 749–760. DOI: 10.1103/PhysRev.40.749.

[76] M. H. L. Pryce. "The Mass center in the restricted theory of relativity and its connection with the quantum theory of elementary particles". *Proc. Roy. Soc. Lond.* A195 (1948), pp. 62–81. DOI: 10.1098/rspa.1948.0103.

[77] T. D. Newton and E. P. Wigner. "Localized states for elementary systems". *Reviews of Modern Physics* 21.3 (1949), p. 400.

[78] W. Florkowski, B. Friman, A. Jaiswal, and E. Speranza. "Relativistic fluid dynamics with spin". *Phys. Rev.* C97.4 (2018), p. 041901. DOI: 10.1103/PhysRevC.97.041901. arXiv: 1705.00587 [nucl-th].

[79] S. Bhadury, W. Florkowski, A. Jaiswal, A. Kumar, and R. Ryblewski. "Dissipative Spin Dynamics in Relativistic Matter". *Phys. Rev. D* 103.1 (2021), p. 014030. DOI: 10.1103/PhysRevD.103.014030. arXiv: 2008.10976 [nucl-th].

[80] A. Daher, A. Das, W. Florkowski, and R. Ryblewski. "Equivalence of canonical and phenomenological formulations of spin hydrodynamics" (Feb. 2022). arXiv: 2202.12609 [nucl-th].

[81] S. Bhadury, W. Florkowski, A. Jaiswal, A. Kumar, and R. Ryblewski. "Relativistic spin-magnetohydrodynamics" (Apr. 2022). arXiv: 2204.01357 [nucl-th].

[82] N. Weickgenannt, D. Wagner, E. Speranza, and D. H. Rischke. "Relativistic second-order dissipative spin hydrodynamics from the method of moments". *Phys. Rev. D* 106.9 (2022), p. 096014. DOI: 10.1103/PhysRevD.106.096014. arXiv: 2203.04766 [nucl-th].

[83] N. Weickgenannt, D. Wagner, E. Speranza, and D. H. Rischke. "Relativistic dissipative spin hydrodynamics from kinetic theory with a nonlocal collision term". *Phys. Rev. D* 106.9 (2022), p. L091901. DOI: 10.1103/PhysRevD.106.L091901. arXiv: 2208.01955 [nucl-th].

[84] H. T. Elze, M. Gyulassy, and D. Vasak. "Transport Equations for the QCD Gluon Wigner Operator". *Phys. Lett. B* 177 (1986), pp. 402–408. DOI: 10.1016/0370-2693(86)90778-1.

[85] D. Vasak, M. Gyulassy, and H. T. Elze. "Quantum Transport Theory for Abelian Plasmas". *Annals Phys.* 173 (1987), pp. 462–492. DOI: 10.1016/0003-4916(87)90169-2.

[86] S. Mrowczynski and U. W. Heinz. "Towards a relativistic transport theory of nuclear matter". *Annals Phys.* 229 (1994), pp. 1–54. DOI: 10.1006/aphy.1994.1001.

[87] Y. A. Markov and M. A. Markova. "Semiclassical kinetic equations for gluons allowing for spin". *Theor. Math. Phys.* 111 (1997), pp. 601–612. DOI: 10.1007/BF02634271.

[88] X.-G. Huang, P. Mitkin, A. V. Sadofyev, and E. Speranza. "Zilch Vortical Effect, Berry Phase, and Kinetic Theory". *JHEP* 10 (2020), p. 117. DOI: 10.1007/JHEP10(2020)117. arXiv: 2006.03591 [hep-th].

[89] K. Hattori, Y. Hidaka, N. Yamamoto, and D.-L. Yang. "Wigner functions and quantum kinetic theory of polarized photons". *JHEP* 02 (2021), p. 001. DOI: 10.1007/JHEP02(2021)001. arXiv: 2010.13368 [hep-ph].

[90] D. J. Gross and F. Wilczek. "Ultraviolet Behavior of Nonabelian Gauge Theories". *Phys. Rev. Lett.* 30 (1973). Ed. by J. C. Taylor, pp. 1343–1346. DOI: 10.1103/PhysRevLett.30.1343.

[91] H. D. Politzer. "Reliable Perturbative Results for Strong Interactions?" *Phys. Rev. Lett.* 30 (1973). Ed. by J. C. Taylor, pp. 1346–1349. DOI: 10.1103/PhysRevLett.30.1346.

[92] A. Aprahamian et al. "Reaching for the horizon: The 2015 long range plan for nuclear science" (Oct. 2015).

[93] D. H. Rischke. "The Quark gluon plasma in equilibrium". *Prog. Part. Nucl. Phys.* 52 (2004), pp. 197–296. DOI: 10.1016/j.ppnp.2003.09.002. arXiv: nucl-th/0305030.

[94] M. A. Stephanov. "QCD phase diagram: An Overview". *PoS* LAT2006 (2006). Ed. by T. Blum et al., p. 024. DOI: 10.22323/1.032.0024. arXiv: hep-lat/0701002.

[95] K. Fukushima and T. Hatsuda. “The phase diagram of dense QCD”. *Rept. Prog. Phys.* 74 (2011), p. 014001. doi: 10.1088/0034-4885/74/1/014001. arXiv: 1005.4814 [hep-ph].

[96] P. de Forcrand. “Simulating QCD at finite density”. *PoS* LAT2009 (2009). Ed. by C. Liu and Y. Zhu, p. 010. doi: 10.22323/1.091.0010. arXiv: 1005.0539 [hep-lat].

[97] N. Dupuis et al. “The nonperturbative functional renormalization group and its applications”. *Phys. Rept.* 910 (2021), pp. 1–114. doi: 10.1016/j.physrep.2021.01.001. arXiv: 2006.04853 [cond-mat.stat-mech].

[98] M. Ablikim et al. “Polarization and Entanglement in Baryon-Antibaryon Pair Production in Electron-Positron Annihilation”. *Nature Phys.* 15 (2019), pp. 631–634. doi: 10.1038/s41567-019-0494-8. arXiv: 1808.08917 [hep-ex].

[99] A. M. Poskanzer and S. A. Voloshin. “Methods for analyzing anisotropic flow in relativistic nuclear collisions”. *Phys. Rev. C* 58 (1998), pp. 1671–1678. doi: 10.1103/PhysRevC.58.1671. arXiv: nucl-ex/9805001.

[100] L. D. Landau and E. M. Lifschitz. *Statistische Physik Teil 1*. Haan-Gruiten: Verlag Europa-Lehrmittel Nourney Vollmer GmbH & Co. KG, 1991. isbn: 978-3808556542.

[101] F. Becattini. “Thermodynamic equilibrium with acceleration and the Unruh effect”. *Phys. Rev. D* 97.8 (2018), p. 085013. doi: 10.1103/PhysRevD.97.085013. arXiv: 1712.08031 [gr-qc].

[102] L. D. Landau and E. M. Lifschitz. *Hydrodynamik*. Haan-Gruiten: Verlag Europa-Lehrmittel Nourney Vollmer GmbH & Co. KG, 1991. isbn: 9783808555545.

[103] L. Rezzolla and O. Zanotti. *Relativistic Hydrodynamics*. Oxford University Press, Sept. 2013. isbn: 9780198528906. doi: 10.1093/acprof:oso/9780198528906.001.0001.

[104] A. H. Taub. “Relativistic Rankine-Hugoniot Equations”. *Phys. Rev.* 74 (1948), pp. 328–334. doi: 10.1103/PhysRev.74.328.

[105] C. Eckart. “The Thermodynamics of irreversible processes. 3.. Relativistic theory of the simple fluid”. *Phys. Rev.* 58 (1940), pp. 919–924. doi: 10.1103/PhysRev.58.919.

[106] F. S. Bemfica, M. M. Disconzi, and J. Noronha. “Nonlinear Causality of General First-Order Relativistic Viscous Hydrodynamics”. *Phys. Rev. D* 100.10 (2019), p. 104020. doi: 10.1103/PhysRevD.100.104020. arXiv: 1907.12695 [gr-qc].

[107] P. Kovtun. “First-order relativistic hydrodynamics is stable”. *JHEP* 10 (2019), p. 034. doi: 10.1007/JHEP10(2019)034. arXiv: 1907.08191 [hep-th].

[108] R. E. Hoult and P. Kovtun. “Stable and causal relativistic Navier-Stokes equations”. *JHEP* 06 (2020), p. 067. doi: 10.1007/JHEP06(2020)067. arXiv: 2004.04102 [hep-th].

[109] F. S. Bemfica, M. M. Disconzi, and J. Noronha. “General-Relativistic Viscous Fluid Dynamics” (Sept. 2020). arXiv: 2009.11388 [gr-qc].

[110] J. Noronha, M. Spaliński, and E. Speranza. “Transient Relativistic Fluid Dynamics in a General Hydrodynamic Frame” (May 2021). arXiv: 2105.01034 [nucl-th].

[111] G. S. Denicol and D. H. Rischke. *Microscopic Foundations of Relativistic Fluid Dynamics*. Springer International Publishing, 2021. doi: 10.1007/978-3-030-82077-0.

[112] D. Burnett. “The Distribution of Velocities in a Slightly Non-Uniform Gas”. *Proceedings of the London Mathematical Society* s2-39.1 (1935), pp. 385–430. doi: https://doi.org/10.1112/plms/s2-39.1.385.

[113] D. Burnett. “The Distribution of Molecular Velocities and the Mean Motion in a Non-Uniform Gas”. *Proceedings of the London Mathematical Society* s2-40.1 (Jan. 1936), pp. 382–435. issn: 0024-6115. doi: 10.1112/plms/s2-40.1.382.

[114] A. V. Bobylev. “The Chapman-Enskog and Grad methods for solving the Boltzmann equation”. In: *Akademiia Nauk SSSR Doklady*. Vol. 262. 1. 1982, pp. 71–75.

[115] C. Cercignani and G. M. Kremer. *The Relativistic Boltzmann Equation: Theory and Applications*. Birkhäuser, 2002.

[116] W. Israel. “Nonstationary irreversible thermodynamics: A causal relativistic theory”. *Annals of Physics* 100.1 (1976), pp. 310–331. ISSN: 0003-4916. DOI: https://doi.org/10.1016/0003-4916(76)90064-6.

[117] W. Israel and J. M. Stewart. “Transient relativistic thermodynamics and kinetic theory”. *Annals Phys.* 118 (1979), pp. 341–372. DOI: 10.1016/0003-4916(79)90130-1.

[118] A. D. Gallegos, U. Gursoy, and A. Yarom. “Hydrodynamics, spin currents and torsion” (Mar. 2022). arXiv: 2203.05044 [hep-th].

[119] V. E. Ambrus, R. Ryblewski, and R. Singh. “Spin waves in spin hydrodynamics”. *Phys. Rev. D* 106.1 (2022), p. 014018. DOI: 10.1103/PhysRevD.106.014018. arXiv: 2202.03952 [hep-ph].

[120] F. Becattini and L. Tinti. “Thermodynamical inequivalence of quantum stress-energy and spin tensors”. *Phys. Rev.* D84 (2011), p. 025013. DOI: 10.1103/PhysRevD.84.025013. arXiv: 1101.5251 [hep-th].

[121] A. Daher, A. Das, and R. Ryblewski. “Stability studies of first order spin-hydrodynamic frameworks” (Sept. 2022). arXiv: 2209.10460 [nucl-th].

[122] G. Sarwar, M. Hasanujjaman, J. R. Bhatt, H. Mishra, and J.-E. Alam. “The causality and stability of relativistic spin-hydrodynamics” (Sept. 2022). arXiv: 2209.08652 [nucl-th].

[123] S. Weinberg. *The Quantum theory of fields. Vol. 1: Foundations.* Cambridge University Press, June 2005. ISBN: 978-0-521-67053-1, 978-0-511-25204-4. DOI: 10.1017/CBO9781139644167.

[124] G. V. Vereshchagin and A. G. Aksenov. *Relativistic Kinetic Theory: With Applications in Astrophysics and Cosmology.* Cambridge University Press, 2017. DOI: 10.1017/9781107261365.

[125] N. N. Bogoliubov. “Kinetic Equations”. *Journal of Physics-USSR* 10.256 (1946), p. 265.

[126] M. Born and H. S. Green. “A general kinetic theory of liquids I. The molecular distribution functions”. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* 188.1012 (1946), pp. 10–18. DOI: 10.1098/rspa.1946.0093.

[127] J. G. Kirkwood. “The Statistical Mechanical Theory of Transport Processes I. General Theory”. *The Journal of Chemical Physics* 14.3 (1946), pp. 180–201. DOI: 10.1063/1.1724117. eprint: https://doi.org/10.1063/1.1724117.

[128] J. Yvon. “La théorie statistique des fluides et l’équation d’état”. *Paris,Hermann & cie* (1935).

[129] R. J. Glauber. “Coherent and incoherent states of the radiation field”. *Phys. Rev.* 131 (1963), pp. 2766–2788. DOI: 10.1103/PhysRev.131.2766.

[130] F. Bopp. “La mécanique quantique est-elle une mécanique statistique classique particulière?” *Annales de l’institut Henri Poincaré* 15.2 (1956), pp. 81–112. URL: http://eudml.org/doc/79057.

[131] L. D. Landau and E. M. Lifschitz. *Quantenmechanik.* Haan-Gruiten: Verlag Europa-Lehrmittel Nourney Vollmer GmbH & Co. KG, 1986. ISBN: 978-3808556368.

[132] L. D. Landau and E. M. Lifschitz. *Klassische Feldtheorie.* Haan-Gruiten: Verlag Europa-Lehrmittel Nourney Vollmer GmbH & Co. KG, 1992. ISBN: 978-3808555620.

[133] E. Schrödinger. “Quantisierung als Eigenwertproblem”. *Annalen Phys.* 384.6 (1926), pp. 489–527. DOI: 10.1002/andp.19263840602.

[134] S. Jeon and L. G. Yaffe. “From quantum field theory to hydrodynamics: Transport coefficients and effective kinetic theory”. *Phys. Rev. D* 53 (1996), pp. 5799–5809. DOI: 10.1103/PhysRevD.53.5799. arXiv: hep-ph/9512263.

[135] Y. B. Ivanov, J. Knoll, and D. N. Voskresensky. “Selfconsistent approach to off-shell transport”. *Phys. Atom. Nucl.* 66 (2003), pp. 1902–1920. DOI: 10.1134/1.1619502. arXiv: nucl-th/0303006.

[136] F. Belinfante. “On the spin angular momentum of mesons”. *Physica* 6.7-12 (1939), pp. 887–898.

[137] L. Rosenfeld. “On the energy-momentum tensor”. *Mém. Acad. Roy. Belg.* 18 (1940), pp. 1–30.

[138] J. Hilgevoord and S. Wouthuysen. "On the spin angular momentum of the Dirac particle". *Nuclear Physics* 40 (1963), pp. 1–12. issn: 0029-5582. doi: https://doi.org/10.1016/0029-5582(63)90246-3.

[139] J. Hilgevoord and E. de Kerf. "The covariant definition of spin in relativistic quantum field theory". *Physica* 31.7 (1965), pp. 1002–1016.

[140] E. Speranza and N. Weickgenannt. "Spin tensor and pseudo-gauges: from nuclear collisions to gravitational physics". *Eur. Phys. J. A* 57.5 (2021), p. 155. doi: 10.1140/epja/s10050-021-00455-2. arXiv: 2007.00138 [nucl-th].

[141] Triyanta. *The Fock-Schwinger Gauge*. University of Tasmania, 1991.

[142] H. T. Elze, M. Gyulassy, and D. Vasak. "Transport Equations for the QCD Quark Wigner Operator". *Nucl. Phys.* B276 (1986), pp. 706–728. doi: 10.1016/0550-3213(86)90072-6.

[143] H.-T. Elze and U. W. Heinz. "Quark - Gluon Transport Theory". *Phys. Rept.* 183 (1989). Ed. by R. C. Hwa and X.-N. Wang, pp. 81–135. doi: 10.1016/0370-1573(89)90059-8.

[144] W. de Boer and C. van Weert. "Relativistic kinetic theory of quantum systems: III. Transport equation for a neutrino system". *Physica A: Statistical Mechanics and its Applications* 86.1 (1977), pp. 67–79. issn: 0378-4371. doi: https://doi.org/10.1016/0378-4371(77)90062-0.

[145] M. D. Schwartz. *Quantum Field Theory and the Standard Model*. Cambridge University Press, Mar. 2014. isbn: 978-1-107-03473-0, 978-1-107-03473-0.

[146] E. Calzetta and B. L. Hu. "Nonequilibrium quantum fields: Closed-time-path effective action, Wigner function, and Boltzmann equation". *Phys. Rev. D* 37 (10 1988), pp. 2878–2900. doi: 10.1103/PhysRevD.37.2878.

[147] G. Baym and L. P. Kadanoff. "Conservation Laws and Correlation Functions". *Phys. Rev.* 124 (1961), pp. 287–299. doi: 10.1103/PhysRev.124.287.

[148] K. Morawetz, V. Spicka, and P. Lipavsky. "Retarded versus time nonlocal quantum kinetic equations". *Annals Phys.* 294 (2001), p. 135. doi: 10.1006/aphy.2001.6197. arXiv: cond-mat/0005287.

[149] L. P. Kadanoff and G. Baym. *Quantum statistical mechanics : Green's function methods in equilibrium and nonequilibrium problems*. 2018.

[150] K. Morawetz. "Nonequilibrium thermodynamics with binary quantum correlations". *Phys. Rev. E* 96.3 (2017), p. 032106. doi: 10.1103/PhysRevE.96.032106. arXiv: 1708.05602 [cond-mat.stat-mech].

[151] K. Morawetz. "Correlational latent heat by nonlocal quantum kinetic theory". *Phys. Rev. B* 97.19 (2018), p. 195142. doi: 10.1103/PhysRevB.97.195142. arXiv: 1805.09683 [cond-mat.str-el].

[152] X.-L. Sheng, Q. Wang, and D. H. Rischke. "Lorentz-covariant kinetic theory for massive spin-1/2 particles" (Feb. 2022). arXiv: 2202.10160 [nucl-th].

[153] L. D. Landau and E. M. Lifschitz. *Physikalische Kinetik*. Haan-Gruiten: Verlag Europa-Lehrmittel Nourney Vollmer GmbH & Co. KG, 1990. isbn: 9783808556245.

[154] E. A. Calzetta and B.-L. B. Hu. *Nonequilibrium Quantum Field Theory*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2008. isbn: 978-1-00-929003-6, 978-1-00-928998-6, 978-1-00-929002-9, 978-0-511-42147-1, 978-0-521-64168-5. doi: 10.1017/9781009290036.

[155] L. V. Keldysh. "Diagram technique for nonequilibrium processes". *Zh. Eksp. Teor. Fiz.* 47 (1964), pp. 1515–1527.

[156] X.-L. Sheng, N. Weickgenannt, E. Speranza, D. H. Rischke, and Q. Wang. "From Kadanoff-Baym to Boltzmann equations for massive spin-1/2 fermions". *Phys. Rev. D* 104.1 (2021), p. 016029. doi: 10.1103/PhysRevD.104.016029. arXiv: 2103.10636 [nucl-th].

[157] Y. Nambu and G. Jona-Lasinio. "Dynamical Model of Elementary Particles Based on an Analogy with Superconductivity. 1." *Phys. Rev.* 122 (1961). Ed. by T. Eguchi, pp. 345–358. doi: 10.1103/PhysRev.122.345.

[158] Y. Nambu and G. Jona-Lasinio. "Dynamical model of elementary particles based on an analogy with superconductivity. II." *Phys. Rev.* 124 (1961). Ed. by T. Eguchi, pp. 246–254. doi: 10.1103/PhysRev.124.246.

[159] S. Li, M. A. Stephanov, and H.-U. Yee. "Nondissipative Second-Order Transport, Spin, and Pseudogauge Transformations in Hydrodynamics". *Phys. Rev. Lett.* 127.8 (2021), p. 082302. doi: 10.1103/PhysRevLett.127.082302. arXiv: 2011.12318 [hep-th].

[160] E. Molnár, H. Niemi, G. S. Denicol, and D. H. Rischke. "Relative importance of second-order terms in relativistic dissipative fluid dynamics". *Phys. Rev. D* 89.7 (2014), p. 074010. doi: 10.1103/PhysRevD.89.074010. arXiv: 1308.0785 [nucl-th].

[161] C. V. P. de Brito and G. S. Denicol. "Third-order relativistic dissipative fluid dynamics from the method of moments" (Feb. 2023). arXiv: 2302.09097 [nucl-th].

[162] D. Wagner and L. Gavassino. "The regime of applicability of Israel-Stewart hydrodynamics" (Sept. 2023). arXiv: 2309.14828 [nucl-th].

[163] G. S. Denicol et al. "Solving the heat-flow problem with transient relativistic fluid dynamics". *Phys. Rev. D* 89.7 (2014), p. 074005. doi: 10.1103/PhysRevD.89.074005. arXiv: 1207.6811 [nucl-th].

[164] G. Baym. "Selfconsistent approximation in many body systems". *Phys. Rev.* 127 (1962), pp. 1391–1401. doi: 10.1103/PhysRev.127.1391.

[165] J. Knoll, Y. B. Ivanov, and D. N. Voskresensky. "Exact conservation laws of the gradient expanded Kadanoff-Baym equations". *Annals Phys.* 293 (2001), pp. 126–146. doi: 10.1006/aphy.2001.6185. arXiv: nucl-th/0102044.

[166] W. Greiner and J. Reinhardt. *Field quantization.* 1996.

[167] F. Becattini. "Polarization in relativistic fluids: a quantum field theoretical derivation". *Lect. Notes Phys.* 987 (2021), pp. 15–52. doi: 10.1007/978-3-030-71427-7_2. arXiv: 2004.04050 [hep-th].

[168] J. J. Florentin, M. Abramowitz, and I. A. Stegun. "Handbook of Mathematical Functions." *American Mathematical Monthly* 73 (1966), p. 1143.

[169] W. Florkowski, E. Maksymiuk, R. Ryblewski, and M. Strickland. "Exact solution of the (0+1)-dimensional Boltzmann equation for a massive gas". *Phys. Rev. C* 89.5 (2014), p. 054908. doi: 10.1103/PhysRevC.89.054908. arXiv: 1402.7348 [hep-ph].

[170] G. S. Denicol and J. Noronha. "Exact results for the Boltzmann collision operator in $\lambda\phi^4$ theory" (Sept. 2022). arXiv: 2209.10370 [nucl-th].

[171] D. Wagner, V. E. Ambrus, and E. Molnar. "Analytical structure of the binary collision integral and the ultrarelativistic limit of transport coefficients of an ideal gas" (Sept. 2023). arXiv: 2309.09335 [physics.flu-dyn].