

# Uniform Generation of Temporal Graphs with Given Degrees

Daniel Allendorf  

Goethe University Frankfurt, Germany

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## Abstract

Uniform sampling from the set  $\mathcal{G}(\mathbf{d})$  of graphs with a given degree-sequence  $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{N}^n$  is a classical problem in the study of random graphs. We consider an analogue for temporal graphs in which the edges are labeled with integer timestamps. The input to this generation problem is a tuple  $\mathbf{D} = (\mathbf{d}, T) \in \mathbb{N}^n \times \mathbb{N}_{>0}$  and the task is to output a uniform random sample from the set  $\mathcal{G}(\mathbf{D})$  of temporal graphs with degree-sequence  $\mathbf{d}$  and timestamps in the interval  $[1, T]$ . By allowing repeated edges with distinct timestamps,  $\mathcal{G}(\mathbf{D})$  can be non-empty even if  $\mathcal{G}(\mathbf{d})$  is, and as a consequence, existing algorithms are difficult to apply.

We describe an algorithm for this generation problem which runs in expected linear time  $O(M)$  if  $\Delta^{2+\epsilon} = O(M)$  for some constant  $\epsilon > 0$  and  $T - \Delta = \Omega(T)$  where  $M = \sum_i d_i$  and  $\Delta = \max_i d_i$ . Our algorithm applies the switching method of McKay and Wormald [1] to temporal graphs: we first generate a random temporal *multigraph* and then remove self-loops and duplicated edges with switching operations which rewire the edges in a degree-preserving manner.

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## 1 Introduction

Given a sequence of integers  $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{N}^n$ , we can ask if there exists a realization of  $\mathbf{d}$  as a simple graph  $G = (V, E)$  on the vertex set  $V = \{v_1, \dots, v_n\}$  in the sense that the number of incident edges at node  $v_i$  equals  $d_i$  for each  $1 \leq i \leq n$ , and where we call a graph simple if it contains no loops (edges connecting a node to itself) or multi-edges (multiple edges connecting two nodes). Conditions for the realizability as a graph were given by Havel [2] and Hakimi [3], and the provided proofs can be used as an algorithm to construct one specific realization. In general, there can be many graphs matching the same sequence, and given a sequence  $\mathbf{d}$ , we define  $\mathcal{G}(\mathbf{d})$  as the set of *all* realizations. A challenging problem in this context is to provide a sample  $G \in \mathcal{G}(\mathbf{d})$  uniformly at random. Such a sample is useful as it allows us to construct null models for testing the influence of the degrees on properties of interest [4]. In addition, this sampling problem is tightly related to the task of estimating  $|\mathcal{G}(\mathbf{d})|$  [5, 1].

One way to obtain a uniform random sample  $G \in \mathcal{G}(\mathbf{d})$  is to use the *configuration model* of Bollobás [6] to sample random *multigraphs* with sequence  $\mathbf{d}$  until a simple graph is found, but the run time of this scheme is exponential in the maximum degree  $\Delta = \max_i d_i$  [7]. Efficient algorithms have been obtained via the *switching* method of McKay and Wormald [1]. The approach is to again start from a random multigraph but instead of rejecting non-simple graphs outright, loops and multi-edges are removed with switching operations which rewire the edges while preserving the degrees of the nodes. In addition to an algorithm with expected runtime  $O(m)$  for generating graphs with  $m$  edges and bounded degrees  $\Delta^4 = O(m)$  [1, 8], efficient algorithms which use this method have been given for generating  $d$ -regular graphs in expected time  $O(nd + d^4)$  if  $d = o(\sqrt{n})$  [9, 8], and graphs with power-law degrees in

expected time  $O(n)$  if the exponent satisfies  $\gamma > (21 + \sqrt{61})/10$  [10, 8]. Alternatively, there are efficient solutions to various relaxations of the problem. For instance, we may allow the graph to match the sequence only in expectation [11], or use a markov chain to approximate the uniform distribution to an arbitrary degree by increasing the number of transitions performed [12, 13, 14]. See also [15] for a survey of relevant techniques and results.

*Temporal graphs* are capable of modeling not only the topology but also the time structure of networks [16]. Possibly the most common type of temporal graph augments each edge of a classical graph with an integer timestamp. Here, we work by the following definition, which is well suited towards uniform generation.

► **Definition 1 (Temporal Graph).** *A temporal (multi-)graph  $G = (V, E)$  consists of a set of nodes  $V = \{v_1, \dots, v_n\}$  and a (multi-)set of temporal edges  $E = \{e_1, \dots, e_m\}$  where each temporal edge is a tuple  $(\{u, v\}, t) \in \{\{u, v\} : u, v \in V\} \times \mathbb{N}_{>0}$ .*

In terms of semantics, the presence of an edge  $(\{u, v\}, t)$  in the edge set  $E$  indicates that the nodes  $u$  and  $v$  are connected at time  $t$ . This allows us to express new relationships between the nodes. For instance, we may consider a node  $v$  *reachable* from a node  $u$  if and only if there exists a path from  $u$  to  $v$  which traverses its edges in ascending order of time. Note that temporal graphs of this type can be seen as an example of edge-labeled graphs (see [17]). While our findings here extend to general edge-labeled graphs, our labelings arise simply due to choosing a graph uniformly at random, whereas in the study of edge-labeled graphs one is more concerned with special kinds of labelings (for instance, see [18]).

For the purpose of modeling networks, it makes sense to restrict ourselves to *simple* temporal graphs which exclude certain types of "unnatural" edges. Here, we call a temporal multigraph  $G = (V, E)$  *simple* if the edge set  $E$  contains no loops and no multi-edges between the same nodes and with the same timestamp, i.e. iff  $u \neq v$  for all  $(\{u, v\}, t) \in E$  and  $e \neq e'$  for all  $e, e' \in E$ . We remark that there are alternative definitions. A more restrictive possibility is to consider a temporal graph simple only if each pair of nodes is connected at most once via a temporal edge (for instance, see [19] for connectivity thresholds of such graphs). However, there are examples of networks in which repeated connections do indeed carry semantic value, and it is rather straightforward to generate a graph without repeated connections by assigning timestamps to the edges of a classical simple graph (see [20] for a model of this kind). The definition used here also suffices to ensure that a process which generates a random temporal multigraph outputs each simple temporal graph with the same probability (see section 2), which is crucial towards uniform generation.

Equipped with these definitions we are now able to state the uniform generation problem for temporal graphs. Given a tuple  $\mathbf{D} = (\mathbf{d}, T) \in \mathbb{N}^n \times \mathbb{N}_{>0}$ , say that a simple temporal graph  $G$  over the node set  $V = \{v_1, \dots, v_n\}$  realizes  $\mathbf{D}$  if the timestamps of all edges lie in the interval  $[1, T]$  and the sum of the numbers of incident edges at node  $v_i$  over all  $T$  timestamps equals  $d_i$  for each  $1 \leq i \leq n$ . In addition, define the set  $\mathcal{G}(\mathbf{D})$  of realizations of  $\mathbf{D}$  as a simple temporal graph and consider the problem of sampling a realization  $G \in \mathcal{G}(\mathbf{D})$  uniformly at random. Observe that as a consequence of allowing repeated connections, a given tuple  $\mathbf{D} = (\mathbf{d}, T)$  can be realizable as a simple temporal graph even if the sequence  $\mathbf{d}$  is not realizable as a classical simple graph. In fact, the classical generation problem corresponds to the case  $T = 1$ . This also implies that existing switching algorithms are difficult to apply. Even if we attempt to distribute the degrees among individual sequences  $\mathbf{d}_t, 1 \leq t \leq T$  such that  $\mathbf{d} = \sum_t \mathbf{d}_t$  and generate a classical graph for each sequence, it is not trivial to obtain an overall temporal graph with the correct distribution as different choices of the sequences can correspond to vastly different numbers of realizations. In fact, it is not difficult to see that there are choices of the sequences  $\mathbf{d}_t, 1 \leq t \leq T$  which are not realizable at

all. Thus, one would expect that this generation problem requires switchings which operate on the temporal graph as a whole. Indeed, this is the key feature of the method which we investigate here. The resulting algorithm, which we call T-GEN, generates simple temporal graphs with bounded degrees. Our main result is as follows (see section 4 for the proof).

► **Theorem 2.** *Given a realizable tuple  $\mathbf{D} = (\mathbf{d}, T)$  which satisfies  $\Delta^{2+\epsilon} = O(M)$  for a constant  $\epsilon > 0$  and  $T - \Delta = \Omega(T)$ , T-GEN outputs a uniform random sample  $G \in \mathcal{G}(\mathbf{D})$  in expected time  $O(M)$ .*

As is customary, we assume an underlying sequence of tuples  $(\mathbf{D})_M$  and give the asymptotic runtime as  $M \rightarrow \infty$ . In particular, the conditions on the input tuple can be understood as being imposed on functions  $\Delta(M), T(M)$ .

The general idea of T-GEN is to apply the switching method of [1] to temporal graphs. To this end, we define a *temporal configuration model* (see section 2) which samples a random temporal multigraph with the property that the probability of a given graph only depends on the contained loops and temporal multi-edges, i.e. multiple edges between the same nodes and with the same timestamp. In particular, this implies that a simple temporal graph output by this model has the uniform distribution. Usually, the obtained graph is not simple, but if the input tuple  $\mathbf{D}$  satisfies the conditions imposed in Theorem 2, then the number of non-simple edges is sufficiently small to allow for efficient removal. A property of the random model with interesting consequences is that the expected number of loops is not affected by the temporal component, whereas the expected number of temporal double-edges scales with  $O(1/T)$  (see Lemma 4). This shifted balance between the types of non-simple edges allows for a looser bound on the degrees in terms of the number of edges (e.g.  $\Delta^{2+\epsilon} = O(M)$  as opposed to  $\Delta^4 = O(m)$ ).

A challenging aspect of generating simple temporal graphs is that the degree-sequences  $\mathbf{d}_t, 1 \leq t \leq T$  implied by a random temporal multigraph may not be realizable. This necessitates the use of switching operations which rewire edges across different time slices of the graph and assign fresh timestamps to newly created edges (see Definition 7 and Figure 1 for an example). However, this in turn implies that the number of available timestamps for an edge affects the distribution of graphs after each switching, and to ensure uniformity, it becomes necessary to account for the number of timestamps when correcting the distribution. To discuss this matter, we briefly describe the technique used by [1] to correct the distribution after each switching.

Generally speaking, when defining a switching operation  $\theta$  we fix two subsets  $\mathcal{S}, \mathcal{S}' \subseteq \mathcal{M}(\mathbf{d})$  of the set  $\mathcal{M}(\mathbf{d})$  of all multigraphs matching the sequence  $\mathbf{d}$ . Specifying the edges rewired by  $\theta$  then associates each graph  $G \in \mathcal{S}$  with a subset  $\mathcal{F}(G) \subseteq \mathcal{S}'$  of graphs in  $\mathcal{S}'$  which can be produced by performing a type  $\theta$  switching on  $G$ , and each graph  $G' \in \mathcal{S}'$  with a subset  $\mathcal{B}(G') \subseteq \mathcal{S}$  of graphs on which we can perform a type  $\theta$  switching which produces  $G'$ . Given this setup, the goal is to start from a uniform random graph  $G \in \mathcal{S}$  and perform a type  $\theta$  switching to obtain a uniform random graph  $G' \in \mathcal{S}'$ . To this end, let  $f(G) = |\mathcal{F}(G)|$ ,  $b(G') = |\mathcal{B}(G')|$ , and assume that  $\mathcal{F}(G) \neq \emptyset$  for every  $G \in \mathcal{S}$  and  $\mathcal{B}(G') \neq \emptyset$  for every  $G' \in \mathcal{S}'$ . Then, if we start from a graph  $G$  uniformly distributed in  $\mathcal{S}$ , and perform a uniform random type  $\theta$  switching on  $G$ , the probability of producing a given graph  $G' \in \mathcal{S}'$  is

$$\sum_{G \in \mathcal{B}(G')} \frac{1}{|\mathcal{S}|f(G)}.$$

which depends on  $G'$  if  $\mathcal{F}(G)$  and  $\mathcal{B}(G')$  vary over different choices of  $G$  and  $G'$ . To correct this, *rejection* steps can be used which restart the algorithm with a certain probability. Before

performing the switching, we *f-reject* (forward reject) with probability  $1 - f(G)/\bar{f}(\mathcal{S})$  where  $\bar{f}(\mathcal{S})$  is an upper bound on  $f(G)$  over all graphs  $G \in \mathcal{S}$ , and after performing the switching, we *b-reject* (backward reject) with probability  $1 - \underline{b}(\mathcal{S}')/b(G')$  where  $\underline{b}(\mathcal{S}')$  is a lower bound on  $b(G')$  over all graphs  $G' \in \mathcal{S}'$ . The probability of producing  $G'$  is now

$$\sum_{G' \in \mathcal{B}(G')} \frac{1}{|\mathcal{S}|} \frac{f(G)}{f(G)} \frac{\underline{b}(\mathcal{S}')}{\bar{f}(\mathcal{S}) b(G')} = \frac{b(G')}{|\mathcal{S}|} \frac{\underline{b}(\mathcal{S}')}{\bar{f}(\mathcal{S}) b(G')} = \frac{\underline{b}(\mathcal{S}')}{|\mathcal{S}| \bar{f}(\mathcal{S})}$$

which only depends on  $\mathcal{S}$  and  $\mathcal{S}'$ , implying that  $G'$  has the uniform distribution if  $G$  does.

Of course, this way of correcting the distribution is efficient only if the typical values of  $f(G)$  and  $b(G')$  do not deviate too much from  $\bar{f}(\mathcal{S})$  and  $\underline{b}(\mathcal{S}')$ . To avoid a high probability of restarting in cases where this does not hold, Gao and Wormald [9] introduced the idea of using additional switchings which partially equalize the probabilities. This is done by defining each phase of a switching algorithm as a markov process which in each step either performs a main kind of switching to remove a non-simple edge, or an additional switching to equalize the probabilities.

T-GEN similarly uses additional switchings but without the use of a markov chain. Instead, we always perform the main kind of switching first and then an additional switching which targets specific edges involved in the main switching performed. Concretely, the issue due to timestamps is that the typical number of available timestamps for an edge is  $\Omega(T)$ , whereas the corresponding lower bound is  $T - (\Delta - 1)$  due to the possibility of a graph in which the edge in question has a multiplicity of  $\Delta$ . Fortunately, the conditions imposed in Theorem 2 suffice to ensure that the highest multiplicity of any edge in the initial graph is bounded by a constant  $\eta = O(1)$  with high probability (see Lemma 5, section 2), and we can also show that any edges created by the switching algorithm itself only increase the multiplicity up to a constant  $\mu = O(1)$  (Lemma 6, section 3). Now, after performing a main kind of switching, we partition the subset  $\mathcal{S}'$  which contains the obtained graph into the subsets  $\mathcal{S}'_{\mathbf{m} < \mu}$  and  $\mathcal{S}' \setminus \mathcal{S}'_{\mathbf{m} < \mu}$  by the multiplicities  $\mathbf{m}$  of specific edges involved in the switching. We then equalize the probabilities of the graphs in  $\mathcal{S}'_{\mathbf{m} < \mu}$  with the standard rejection step (which is now efficient due to  $\mu = O(1)$ ), and reset the probability of graphs in  $\mathcal{S}' \setminus \mathcal{S}'_{\mathbf{m} < \mu}$  to zero by rejecting these graphs. To equalize the probabilities between graphs in  $\mathcal{S}'_{\mathbf{m} < \mu}$  and  $\mathcal{S}' \setminus \mathcal{S}'_{\mathbf{m} < \mu}$ , we define auxiliary switching operations which map the graphs in  $\mathcal{S}'_{\mathbf{m} < \mu}$  to graphs in  $\mathcal{S}' \setminus \mathcal{S}'_{\mathbf{m} < \mu}$  and an identity switching which maps any graph in  $\mathcal{S}'_{\mathbf{m} < \mu}$  to itself, and specify a probability distribution over these two kinds of switchings which ensures that all graphs in  $\mathcal{S}'$  are produced with the same probability via switchings which involve the specific edges.

## 2 Temporal Configuration Model

The temporal configuration model samples a random temporal multigraph matching a given tuple  $\mathbf{D} = (\mathbf{d}, T) \in \mathbb{N}^n \times \mathbb{N}_{>0}$  (provided that  $M = \sum_i d_i$  is even). It can be implemented as follows. First, for each node index  $i \in \{1, \dots, n\}$ , put  $d_i$  marbles labeled  $i$  into an urn. Then, starting from the empty graph  $G = (V, \emptyset)$  on the node set  $V = \{v_1, \dots, v_n\}$ , add edges by iteratively performing the following steps until the urn is empty:

1. Draw two marbles from the urn uniformly at random (without replacement), and let  $i, j$  denote the labels of those marbles.
2. Draw a timestamp  $t$  uniformly at random from the set of timestamps  $[1, T]$ .
3. Add the temporal edge  $(\{v_i, v_j\}, t)$  to the graph  $G$ .

In the following, we analyze the output distribution of this random model. We first give some definitions to characterize edges in a temporal multigraph. Given two nodes  $v_i, v_j \in V$  and a timestamp  $t \in [1, T]$ , define  $w_{i,j,t}$  as the number of edges between  $v_i$  and  $v_j$  with timestamp  $t$  in the graph, and call  $w_{i,j,t}$  the *temporal multiplicity* of the edge  $(\{v_i, v_j\}, t)$ . Then, if  $w_{i,j,t} \geq 2$ , say that the edge is contained in a *temporal multi-edge*, and in the special cases  $w_{i,j,t} = 2$  and  $w_{i,j,t} = 3$ , refer to the multi-edge as a *double-edge* and *triple-edge*, respectively. In addition, define  $m_{i,j} = \sum_t w_{i,j,t}$  as the total number of edges between  $v_i$  and  $v_j$  over all timestamps, call  $m_{i,j}$  the *multiplicity* of the node set  $\{v_i, v_j\}$ , and if  $m_{i,j} \geq 2$ , say that  $\{v_i, v_j\}$  is contained in a *ordinary multi-edge*. Finally, call an edge  $(\{v_i\}, t)$  which connects a node  $v_i$  to itself a *loop* at  $v_i$ , and in the cases where  $w_{i,t} = 1$  and  $w_{i,t} = 2$ , refer to the edge as a *temporal single-loop* and *temporal double-loop*, respectively.

Now, let  $\mathcal{M}(\mathbf{D})$  denote the set of temporal multigraphs matching a realizable tuple  $\mathbf{D}$ ,  $\mathbf{W}(G)$  the  $n \times n \times T$  matrix such that the entries  $\mathbf{W}_{i,j,t}$  where  $i \neq j$  contain the temporal multiplicities of the temporal multi-edges in a graph  $G$  and the entries  $\mathbf{W}_{i,i,t}$  the temporal multiplicities of the loops, and let  $\mathcal{S}(\mathbf{W}) \subseteq \mathcal{M}(\mathbf{D})$  denote the subset of multigraphs with temporal multi-edge and loop multiplicities  $\mathbf{W}$ . Then, we obtain the following result.

► **Theorem 3.** *Let  $G$  be a temporal multigraph output by the temporal configuration model on an input tuple  $\mathbf{D}$ . Then,  $G$  is uniformly distributed in the set  $\mathcal{S}(\mathbf{W}(G)) \subseteq \mathcal{M}(\mathbf{D})$ .*

**Proof of Theorem 3.** For a given tuple  $\mathbf{D} = (\mathbf{d}, T)$ , let  $P_i = \{i_1, \dots, i_{d_i}\}$  where  $1 \leq i \leq n$ ,  $P = \bigcup_{1 \leq i \leq n} P_i$ , and define a *temporal configuration* of  $\mathbf{D}$  as a partition of the set  $P$  into  $M/2$  subsets of size two in which each subset is assigned an integer  $t \in [1, T]$ . Then, the temporal configuration model samples a temporal configuration uniformly at random and outputs the corresponding temporal multigraph (by identifying the nodes corresponding to the labels and replacing the subsets together with the assigned integers by temporal edges). Thus, the probability of a given graph  $G$  is proportional to the number of temporal configurations corresponding to  $G$ , which equals the number of ways to label the edges in  $G$  with the labels in  $P$  which give a distinct temporal configuration. Denote this number by  $C_P(G)$ , and observe that if  $H_1 = (V, E_1)$ ,  $H_2 = (V, E_2)$  are the subgraphs of the non-simple edges and simple edges in  $G = (V, E)$ , respectively, then as  $E_1 \cap E_2 = \emptyset$  and  $E_1 \cup E_2 = E$ , we have  $C_P(G) = C_{P_1}(H_1)C_{P_2}(H_2) \prod_i \binom{d_i}{k_i}$  where  $\mathbf{k}$  denotes the degree-sequence of  $H_2$  (arbitrarily),  $P_1 = \bigcup_i \{i_1, \dots, i_{d_i - k_i}\}$  and  $P_2 = \bigcup_i \{i_1, \dots, i_{k_i}\}$ . In addition, we have  $C_{P_2}(H_2) = \prod_i k_i!$  as all edges incident at the nodes in the simple temporal graph  $H_2$  are distinct, which implies that all possible labelings result in a distinct temporal configuration. Finally, observe that  $\mathbf{W}(G)$  determines both  $H_1$  and  $\mathbf{k}$ , and thus  $C_P(G)$  only depends on  $\mathbf{W}(G)$ . ◀

Note that the set of *simple* temporal graphs matching  $\mathbf{D}$  corresponds to the special case  $\mathcal{G}(\mathbf{D}) = \mathcal{S}(\mathbf{0}^{n \times n \times T}) \subseteq \mathcal{M}(\mathbf{D})$ . Thus, Theorem 3 implies that a simple temporal graph output by the random model is uniformly distributed in the set  $\mathcal{G}(\mathbf{D})$ . In general, the probability of obtaining a simple temporal graph is small (see Lemma 4 below). Still, there are conditions under which the numbers and multiplicities of non-simple edges are manageable. We state these conditions in terms of the following properties of a degree sequence  $\mathbf{d} \in \mathbb{N}_{>0}^n$ :

$$\Delta = \max_{1 \leq i \leq n} d_i, \quad M = \sum_{1 \leq i \leq n} d_i, \quad M_2 = \sum_{1 \leq i \leq n} d_i(d_i - 1).$$

i.e.  $\Delta$  is the maximum degree,  $M$  the first moment and  $M_2$  the second moment.

We start by giving a condition under which the numbers and multiplicities of temporal multi-edges and temporal multi-loops is not too large.

► **Lemma 4.** *Let  $\mathbf{D} = (\mathbf{d}, T)$  be a tuple which satisfies  $\Delta^2 = o(M)$  and  $\Delta = O(T)$ , and  $G$  a graph output by the temporal configuration model when given  $\mathbf{D}$  as input. Then, the expected number of temporal double-edges in  $G$  is at most  $O(M_2^2/M^2T)$ , the expected number of temporal single-loops is at most  $O(M_2/M)$ , and with high probability there are no temporal double-loops or temporal triple-edges.*

**Proof of Lemma 4.** It is straightforward to check that the probability that  $G$  contains  $m$  given temporal edges is  $O(M^{-m}T^{-m})$ . Thus, the expected number of temporal double-edges in a graph output by the temporal configuration model is

$$O\left(\sum_{1 \leq i, j \leq n} \sum_{1 \leq t \leq T} \frac{4 \binom{d_i}{2} \binom{d_j}{2}}{M^2 T^2}\right) = O\left(\frac{M_2^2}{M^2 T}\right),$$

the expected number of temporal single-loops is

$$O\left(\sum_{1 \leq i \leq n} \sum_{1 \leq t \leq T} \frac{\binom{d_i}{2}}{M T}\right) = O\left(\frac{M_2}{M}\right),$$

the expected number of temporal triple-edges is

$$O\left(\sum_{1 \leq i, j \leq n} \sum_{1 \leq t \leq T} \frac{6 \binom{d_i}{3} \binom{d_j}{3}}{M^3 T^3}\right) = O\left(\frac{\Delta^2 M_2^2}{M^3 T^2}\right) = O\left(\frac{\Delta^2}{M}\right) = o(1),$$

and the expected number of temporal double-loops is

$$O\left(\sum_{1 \leq i \leq n} \sum_{1 \leq t \leq T} \frac{3 \binom{d_i}{4}}{M^2 T^2}\right) = O\left(\frac{\Delta^2 M_2}{M^2 T}\right) = O\left(\frac{\Delta^2}{M}\right) = o(1). \quad \blacktriangleleft$$

In addition, the following condition ensures that there is only a small probability that a node is incident with many non-simple edges, or that the graph contains an ordinary multi-edge of high multiplicity.

► **Lemma 5.** *Let  $\mathbf{D} = (\mathbf{d}, T)$  be a tuple which satisfies  $\Delta^{2+\epsilon} = O(M)$  for a constant  $\epsilon > 0$  and  $\Delta = O(T)$ , and  $G$  a graph output by the temporal configuration model when given  $\mathbf{D}$  as input. Then, with high probability, the largest number of incident temporal double-edges at any node in  $G$  is at most  $\kappa = \lfloor 1 + 1/\epsilon \rfloor$ , the largest number of incident temporal single-loops at any node in  $G$  is at most  $\lambda = \lfloor 1 + 1/\epsilon \rfloor$ , and the highest multiplicity of any edge in  $G$  is at most  $\eta = \lfloor 2 + 2/\epsilon \rfloor$ .*

**Proof of Lemma 5.** Define  $\delta = \frac{\epsilon}{2+\epsilon}$  and note that  $\Delta^{2+\epsilon} = O(M) \implies \Delta = O(M^{\delta/\epsilon})$ , and if  $\epsilon > 0$ , then  $\Delta^2/M = O(M^{-\delta}) = o(1)$ . Let  $K_m$  denote the number of nodes incident with  $m$  temporal double-edges in a graph output by the temporal configuration model. Then

$$\begin{aligned} \mathbb{E}[K_m] &= O\left(\sum_{1 \leq i \leq n} \sum_{1 \leq j_1, \dots, j_m \leq n} \sum_{1 \leq t_1, \dots, t_m \leq T} \frac{\binom{2m}{2, \dots, 2} \binom{d_i}{2m} \prod_{k=1}^m \binom{d_{j_k}}{2}}{M^{2m} T^{2m}}\right) \\ &= O\left(\frac{M_{2m} M_2^m}{M^{2m} T^m}\right) = O\left(\frac{\Delta^{2m-1}}{M^{m-1}}\right). \end{aligned}$$

where  $M_k = \sum_{1 \leq i \leq n} \prod_{1 \leq j \leq k} (d_i - j + 1)$  and the last equality follows by  $M_{k+1} < \Delta M_k$ . Thus, if  $\Delta^{2+\epsilon} = O(M)$  for some constant  $\epsilon > 0$ , the probability of at least one node incident with more than  $\kappa = \lfloor 1 + 1/\epsilon \rfloor$  temporal double-edges is at most

$$\begin{aligned}
\sum_{m=\kappa+1}^{\lfloor \Delta/2 \rfloor} \mathbb{E}[K_m] &= O\left(\sum_{m=\kappa+1}^{\lfloor \Delta/2 \rfloor} \frac{\Delta^{2m-1}}{M^{m-1}}\right) \\
&= O\left(\sum_{m=\kappa+1}^{\lfloor \Delta/2 \rfloor} M^{-\delta(m-1-1/\epsilon)}\right) \\
&= O\left(\sum_{m=\kappa+2}^{\lfloor \Delta/2 \rfloor} M^{-\delta(m-1-1/\epsilon)}\right) + O\left(M^{-\delta(\underbrace{\kappa+1}_{= \lfloor 1+1/\epsilon \rfloor + 1} > 1+1/\epsilon)}\right) \\
&= O\left(\sum_{i=1}^{\lfloor \Delta/2 \rfloor - (\kappa+1)} M^{-\delta i}\right) + o(1) \\
&= O(M^{-\delta}) + o(1) \\
&= o(1).
\end{aligned}$$

Similarly, let  $L_m$  denote the number of nodes incident with  $m$  temporal single-loops, then

$$\mathbb{E}[L_m] = O\left(\sum_{1 \leq i \leq n} \sum_{1 \leq t_1, \dots, t_m \leq T} \frac{\binom{2m}{2, \dots, 2} \binom{d_i}{2m}}{M^m T^m}\right) = O\left(\frac{M_{2m}}{M^m}\right) = O\left(\frac{\Delta^{2m-1}}{M^{m-1}}\right)$$

which if  $\Delta^{2+\epsilon} = O(M)$  implies that the probability of at least one node incident with more than  $\lambda = \lfloor 1 + 1/\epsilon \rfloor$  temporal single-loops is  $o(1)$  by the same argument as above. Finally, let  $H_m$  denote the number of ordinary multi-edges of multiplicity  $m$ , then

$$\mathbb{E}[H_m] = O\left(\sum_{1 \leq i, j \leq n} \frac{m! \binom{d_i}{m} \binom{d_j}{m}}{M^m}\right) = O\left(\frac{M_m^2}{M^m}\right) = O\left(\frac{\Delta^{2m-2}}{M^{m-2}}\right)$$

which if  $\Delta^{2+\epsilon} = O(M)$  implies that the probability of at least one ordinary multi-edge of multiplicity higher than  $\eta = \lfloor 2 + 2/\epsilon \rfloor$  is  $o(1)$  by the same argument as above.  $\blacktriangleleft$

### 3 Algorithm T-Gen

T-GEN takes a realizable tuple  $\mathbf{D}$  as input and outputs a uniform random sample  $G \in \mathcal{G}(\mathbf{D})$  from the set of matching simple temporal graphs. The algorithm starts by sampling a random temporal multigraph  $G \in \mathcal{M}(\mathbf{D})$  via the temporal configuration model (see section 2). It then checks if  $G$  satisfies initial conditions on the numbers and multiplicities of non-simple edges (see subsection 3.1). In particular, the graph  $G$  is not allowed to contain temporal triple-edges, or temporal double-loops (or any higher multiplicities). If  $G$  satisfies these conditions, then the algorithm proceeds to removing all temporal single-loops and temporal double-edges during two stages. Stage 1 (subsection 3.2) removes all temporal single-loops in the graph. For this purpose three kinds of switching operations are used. The main kind of switching removes a temporal single-loop at a specified node and with a specified timestamp. After performing this kind of switching we always perform one of two auxiliary switchings. The purpose of these switchings is to equalize the probabilities between graphs which contain

ordinary multi-edges of high multiplicity and graphs which do not. Stage 2 (subsection 3.3) removes all temporal double-edges, i.e. double-edges which share the same timestamp. Doing this efficiently requires five kinds of switchings, two of which remove a temporal double-edge between two specified nodes and with a specified timestamp, and three of which are auxiliary switchings. Once all non-simple edges have been removed, the resulting simple temporal graph is output.

### 3.1 Initial Conditions

The initial conditions for the random multigraph  $G$  are as follows. Define

$$B_L = \frac{M_2}{M}, \quad B_D = \frac{M_2^2}{M^2 T},$$

let  $L = \sum_{i,t} \mathbf{W}_{i,i,t}(G)$  and  $D = \sum_{i \neq j,t} \mathbf{W}_{i,j,t}(G)$  denote the sums of the multiplicities of loops and temporal multi-edges of  $G$ , respectively, and choose three constants

$$\lambda \geq 1 + 1/\epsilon, \quad \kappa \geq 1 + 1/\epsilon, \quad \mu \geq 3 + 2/\epsilon$$

where  $\epsilon > 0$  is a constant such that  $\Delta^{2+\epsilon} = O(M)$  (or set  $\lambda = \kappa = \mu = \Delta$  if no such constant exists). Then,  $G$  satisfies the initial conditions if  $L \leq B_L$ ,  $D/2 \leq B_D$ , there are no temporal multi-loops of multiplicity  $w \geq 2$  or temporal multi-edges of multiplicity  $w \geq 3$ , and no node is incident with more than  $\lambda$  temporal single-loops or  $\kappa$  temporal double-edges.

Observe that the lower bounds on the constants  $\lambda$  and  $\kappa$  are due to Lemma 5. The additional constant  $\mu$  is due to the following result (proof in subsection 4.2).

► **Lemma 6.** *If the input tuple  $\mathbf{D} = (\mathbf{d}, T)$  satisfies  $\Delta^{2+\epsilon} = O(M)$  for a constant  $\epsilon > 0$  and  $T - \Delta = \Omega(T)$ , then with high probability none of the graphs visited during a given run of T-GEN contain an edge of multiplicity higher than  $\mu = \lfloor 3 + 2/\epsilon \rfloor$ .*

As a simple temporal graph is allowed to contain ordinary multi-edges, the constant  $\mu$  cannot be enforced by rejecting violating initial graphs. Instead, we equalize the probabilities between graphs which contain such edges and graphs which do not via the auxiliary switchings mentioned above. We describe this approach in detail in subsection 3.2 and subsection 3.3.

Finally, there are two special cases. First, if the initial multigraph  $G$  is simple, then this graph can be output without checking the preconditions or going through any of the stages. Second, if the input tuple does not satisfy

$$M > 16\Delta^2 + 4\Delta + 2B_L + 4B_D, \quad T > \Delta - 1$$

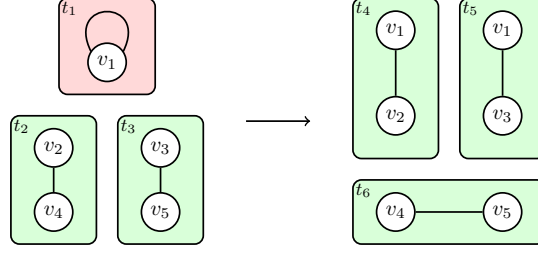
then T-GEN restarts until a simple graph is found and output. These requirements are satisfied if  $\Delta^{2+\epsilon} = O(M)$  and  $T - \Delta = \Omega(T)$  but we make them explicit here to ensure correctness on any input tuple.

In all other cases, T-GEN restarts if the graph  $G$  does not satisfy the initial conditions. Otherwise, the algorithm enters Stage 1 to remove the temporal single-loops in  $G$ .

### 3.2 Stage 1: Removal of Temporal Single-Loops

Stage 1 removes the temporal single-loops in the graph. Doing this efficiently requires multiple kinds of switching operations. In total, we use three kinds of switchings which we denote as TL,  $A_{m,n}$  and I. We formally define each kind of switching below. The switching of the main kind written as TL removes a temporal single-loop at a specified node and with





■ **Figure 1** The TL switching removes a temporal single-loop with timestamp  $t_1$  at a node  $v_1$ . A shaded region labeled with a timestamp contains all edges with this timestamp between the nodes. Red and green shades indicate non-simple and simple edges, respectively.

a specified timestamp. After performing this kind of switching, an  $A_{m,n}$  auxiliary switching is performed with a certain probability. This switching adds up to two ordinary multi-edges with multiplicities  $\max\{m, n\} \geq \mu$  to the graph to equalize the probability of producing graphs with or without these kinds of edges. In addition, we define the identity switching  $I$  which maps each graph to itself. This is done to formalize the event in which no auxiliary switching is performed. Formal definitions of the TL and  $A_{m,n}$  switchings are as follows.

► **Definition 7** (TL switching at  $v_1, t_1$ ). For a graph  $G$  such that  $(\{v_1\}, t_1)$  is a temporal single-loop, let  $(\{v_2, v_4\}, t_2)$ ,  $(\{v_3, v_5\}, t_3)$  be edges and  $t_4, t_5, t_6 \in [1, T]$  timestamps such that

- none of the edges  $(\{v_2, v_4\}, t_2)$ ,  $(\{v_3, v_5\}, t_3)$  is a loop or in a temporal double-edge,
- the nodes  $v_2, v_3, v_4, v_5$  are distinct from  $v_1$ , and  $v_4$  is distinct from  $v_5$ , and
- none of the edges  $(\{v_1, v_2\}, t_4)$ ,  $(\{v_1, v_3\}, t_5)$ ,  $(\{v_4, v_5\}, t_6)$  exist.

Then, a TL switching replaces the edges  $(\{v_1\}, t_1)$ ,  $(\{v_2, v_4\}, t_2)$ ,  $(\{v_3, v_5\}, t_3)$  with  $(\{v_1, v_2\}, t_4)$ ,  $(\{v_1, v_3\}, t_5)$ ,  $(\{v_4, v_5\}, t_6)$  (see Figure 1).

We stress that the integer subscripts of nodes are used in place of generic indices to reduce visual clutter and simplify the descriptions. Naturally, these labels may still refer to any node in the graph.

► **Definition 8** ( $A_{m,n}$  switching at  $v_1, v_2, v_3, v_4, v_5$ ). For a graph  $G$  such that  $\{v_2, v_4\}$  and  $\{v_3, v_5\}$  are non-edges, let  $(\{v_2, v_{2i+4}\}, t_i)$ ,  $(\{v_4, v_{2i+5}\}, t_{m+i})$ ,  $1 \leq i \leq m$  be incident edges at  $v_2$ ,  $v_4$ ,  $(\{v_3, v_{2m+2i+4}\}, t_{2m+i})$ ,  $(\{v_5, v_{2m+2i+5}\}, t_{2m+n+i})$ ,  $1 \leq i \leq n$  incident edges at  $v_3$ ,  $v_5$ , and  $t_{2m+2n+1}, \dots, t_{4m+4n} \in [1, T]$  timestamps such that

- none of the edges  $(\{v_2, v_{2i+4}\}, t_i)$ ,  $(\{v_4, v_{2i+5}\}, t_{m+i})$ ,  $1 \leq i \leq m$ ,  $(\{v_3, v_{2m+2i+4}\}, t_{2m+i})$  and none of  $(\{v_3, v_{2m+2i+5}\}, t_{2m+n+i})$ ,  $1 \leq i \leq n$  is a loop or in a temporal double-edge,
- the nodes  $v_1, \dots, v_{2m+2n+5}$  are all distinct, and
- none of the edges  $(\{v_2, v_4\}, t_{2m+2n+i})$ ,  $(\{v_{2i+4}, v_{2i+5}\}, t_{3m+2n+i})$ ,  $1 \leq i \leq m$  and none of  $(\{v_3, v_5\}, t_{4m+2n+i})$ ,  $(\{v_{2m+2i+4}, v_{2m+2i+5}\}, t_{4m+3n+i})$ ,  $1 \leq i \leq n$  exist.

Then, an  $A_{m,n}$  switching replaces the edges  $(\{v_2, v_{2i+4}\}, t_i)$ ,  $(\{v_4, v_{2i+5}\}, t_{m+i})$ ,  $1 \leq i \leq m$ ,  $(\{v_3, v_{2m+2i+4}\}, t_{2m+i})$ ,  $(\{v_3, v_{2m+2i+5}\}, t_{2m+n+i})$ ,  $1 \leq i \leq n$  with  $(\{v_2, v_4\}, t_{2m+2n+i})$ ,  $(\{v_{2i+4}, v_{2i+5}\}, t_{3m+2n+i})$ ,  $1 \leq i \leq m$ ,  $(\{v_3, v_5\}, t_{4m+2n+i})$ ,  $(\{v_{2m+2i+4}, v_{2m+2i+5}\}, t_{4m+3n+i})$ ,  $1 \leq i \leq n$ .

In other words, the TL switching chooses two edges and then rewires the specified loop and the two edges such that exactly the specified loop is removed and no other non-simple edges are created or removed. Likewise, the  $A_{m,n}$  switching chooses  $m$  incident edges at two nodes  $v_2, v_4$  each and  $n$  incident edges at two nodes  $v_3, v_5$  each and then rewires the edges such

that exactly  $m$  simple edges between the nodes  $v_2, v_4$  and exactly  $n$  simple edges between the nodes  $v_3, v_5$  are created and no non-simple edges are created or removed.

After each TL switching, we perform an  $A_{m,n}$  auxiliary switching or the identity switching. To decide which switching to perform we define a probability distribution over the different types of switchings which ensures uniformity. In total, the set of  $A_{m,n}$  auxiliary switchings is

$$\Theta_A = \bigcup_{\substack{0 \leq m, n < \Delta \\ \mu \leq \max\{m, n\}}} \{A_{m,n}\}.$$

The switching to be performed is then sampled from the distribution  $(\Theta_A \cup \{I\}, P_A)$  where

$$p_A(A_{m,n}) = p_A(I) \frac{\bar{f}_{A_{m,n}}(\mathbf{W}')}{b_{A_{m,n}}(\mathbf{W}')}, \quad p_A(I) = 1 - \sum_{\theta \in \Theta_A} p_A(\theta)$$

for quantities  $\bar{f}_{A_{m,n}}(\mathbf{W}')$  and  $b_{A_{m,n}}(\mathbf{W}')$  given further below.

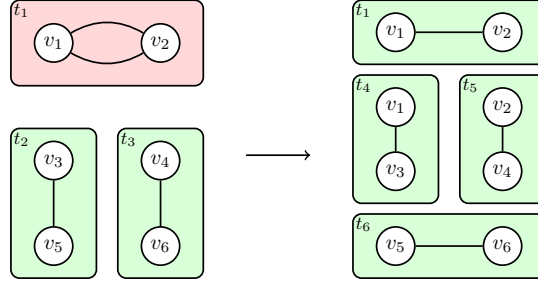
On a high level, Stage 1 runs in a loop until a rejection occurs or all temporal single-loops have been removed from  $G$ . To this end, let  $\pi$  denote a permutation of the entries in  $\mathbf{W}(G)$  such that  $\mathbf{W}_{i,i,t} = 1$ . Then, Stage 1 iterates through the temporal single-loops in the order given by  $\pi$  and performs the following steps for each temporal single-loop.

1. Let  $G$  denote the current graph,  $\mathbf{W} = \mathbf{W}(G)$  and  $(\{v_1\}, t_1)$  the loop.
2. Pick a uniform random TL switching  $S$  which removes  $(\{v_1\}, t_1)$  from  $G$ .
3. Restart (**f-reject**) with probability  $1 - \frac{f_{\text{TL}}(G)}{f_{\text{TL}}(\mathbf{W})}$ .
4. Rewire the edges according to  $S$ , let  $G'$  denote the resulting graph and  $\mathbf{W}' = \mathbf{W}(G')$ .
5. Let  $(\{v_2, v_4\}, t_2)$ ,  $(\{v_3, v_5\}, t_3)$  denote the edges removed by  $S$ .
6. Restart if  $m_{2,4} \geq \mu$  or  $m_{3,5} \geq \mu$ .
7. Restart (**b-reject**) with probability  $1 - \frac{b_{\text{TL}}(\mathbf{W}'; 2)}{b_{\text{TL}}(G', v_1 v_2 v_3 v_4 v_5; 2)}$ .
8. Choose a switching type  $\theta \sim (\Theta_A \cup \{I\}, P_A)$ .
9. If  $\theta = A_{m,n}$  for some  $A_{m,n} \in \Theta_A$ :
  - a. Restart if  $m_{2,4} \geq 1$  or  $m_{3,5} \geq 1$ .
  - b. Pick a uniform random  $A_{m,n}$  switching  $S'$  which adds an edge with node set  $\{v_2, v_4\}$  and multiplicity  $m$  and an edge with node set  $\{v_3, v_5\}$  and multiplicity  $n$  to  $G'$ .
  - c. Restart (**f-reject**) with probability  $1 - \frac{f_{A_{m,n}}(G')}{\bar{f}_{A_{m,n}}(\mathbf{W}')}$ .
  - d. Rewire the edges according to  $S'$  and let  $G''$  denote the resulting graph.
  - e. Restart (**b-reject**) with probability  $1 - \frac{b_{A_{m,n}}(\mathbf{W}')}{b_{A_{m,n}}(G'', v_1 v_2 v_3 v_4 v_5)}$ .
  - f. Set  $G' \leftarrow G''$ .
10. Restart (**b-reject**) with probability  $1 - \frac{b_{\text{TL}}(\mathbf{W}'; 0) b_{\text{TL}}(\mathbf{W}'; 1)}{b_{\text{TL}}(G', v_1; 0) b_{\text{TL}}(G', v_1 v_2 v_3; 1)}$ .
11. Set  $G \leftarrow G'$ .

To fully specify Stage 1, it remains to define the quantities used for the f- and b-rejection steps. For the f-rejection in step 3, define  $f_{\text{TL}}(G)$  as the number of TL switchings which can be performed on the graph  $G$ . The corresponding upper bound is

$$\bar{f}_{\text{TL}}(\mathbf{W}) = M^2 T^3.$$

For the b-rejections in steps 7 and 10, define  $b_{\text{TL}}(G', v_1 v_2 v_3 v_4 v_5; 2)$  as the number of timestamps  $t_2, t_3 \in [1, T]$  such that the edges  $(\{v_2, v_4\}, t_2)$ ,  $(\{v_3, v_5\}, t_3)$  do not exist in  $G'$ ,  $b_{\text{TL}}(G', v_1 v_2 v_3; 1)$  as the number of simple temporal edges  $(\{v_4, v_5\}, t_6)$  such that  $v_4, v_5$  are



■ **Figure 2** The  $\text{TD}_1$  switching removes a temporal double-edge with timestamp  $t_1$  between two nodes  $v_1, v_2$  while leaving a single-edge between the nodes.

distinct from  $v_1, v_2, v_3$ , and  $b_{\text{TL}}(G', v_1; 0)$  as the number of distinct simple temporal edges  $(\{v_1, v_2\}, t_4), (\{v_1, v_3\}, t_5)$  incident at  $v_1$ . The lower bounds on these quantities are

$$\underline{b}_{\text{TL}}(\mathbf{W}'; 2) = (T - (\mu - 1))^2, \quad \underline{b}_{\text{TL}}(\mathbf{W}'; 1) = M - 2B_L - 4B_D - 4\Delta, \quad \underline{b}_{\text{TL}}(\mathbf{W}'; 0) = k_1(k_1 - 1)$$

where  $k_i = d_i - \sum_{1 \leq t \leq T} (2\mathbf{W}'_{i,i,t} + \sum_{1 \leq j \leq n: j \neq i} \mathbf{W}'_{i,j,t})$ .

For the f-rejection in step 9c, define  $f_{\mathbf{A}_{m,n}}(G')$  as the number of  $\mathbf{A}_{m,n}$  switchings which can be performed on the graph  $G'$ . The upper bound is

$$\bar{f}_{\mathbf{A}_{m,n}}(\mathbf{W}') = \Delta^{2(m+n)} T^{2(m+n)}.$$

For the b-rejection in step 9e, define  $b_{\mathbf{A}_{m,n}}(G'', v_1 v_2 v_3 v_4 v_5)$  as the number of  $\mathbf{A}_{m,n}$  switchings which can produce the graph  $G''$ . The corresponding lower bound is

$$\underline{b}_{\mathbf{A}_{m,n}}(\mathbf{W}') = (M - 2B_L - 4B_D - 4(m+n+3)\Delta)^{m+n} (T - (\Delta - 1))^{2(m+n)}.$$

The correctness of Stage 1 is implied by the following result, the proof of which can be found in subsection 4.1.

► **Lemma 9.** *The graph  $G'$  at the end of an iteration of Stage 1 is uniformly distributed in  $\mathcal{S}(\mathbf{W}')$  given that the graph  $G$  at the start of the iteration is uniformly distributed in  $\mathcal{S}(\mathbf{W})$ .*

Showing that Stage 1 is efficient requires showing that both the probability of restarting and the run time of all iterations is small. To this end, we show the following in subsection 4.2.

► **Lemma 10.** *The probability of not restarting in Stage 1 is  $\exp(-O(\Delta^2/M)) - O(\Delta/T)$ .*

► **Lemma 11.** *The expected run time of Stage 1 is  $O(\Delta^2)$ .*

Stage 1 ends if all temporal single-loops have been removed. The algorithm then moves on to Stage 2 to remove the remaining temporal double-edges.

### 3.3 Stage 2: Removal of Temporal Double-Edges

Stage 2 uses five kinds of switchings which we denote as  $\text{TD}_1, \text{TD}_0, \mathbf{B}_{m,n}, \mathbf{C}_{m,n,o,p}$  and  $\mathbf{I}$ . The two main switchings  $\text{TD}_1$  and  $\text{TD}_0$  remove a temporal double-edge between two specific nodes and with a specific timestamp. The difference is that the  $\text{TD}_1$  switching only removes one occurrence of the edge while the  $\text{TD}_0$  switching erases both occurrences. This is done to equalize the probability between graphs in which the removed temporal double-edge is a non-edge, or single edge. After performing a  $\text{TD}_1$  switching, the  $\mathbf{B}_{m,n}$

auxiliary switching may be performed, and after performing a  $\text{TD}_0$  switching, the  $\mathcal{C}_{m,n,o,p}$  switching may be performed. These auxiliary switchings add ordinary multi-edges with multiplicity  $\max\{m, n\} \geq \mu$  or  $\max\{m, n, o, p\} \geq \mu$  to the graph to equalize the probabilities between graphs with or without these edges. We define the  $\text{TD}_1$  switching as follows.

► **Definition 12** ( $\text{TD}_1$  switching at  $(\{v_1, v_2\}, t_1)$ ). For a graph  $G$  such that  $(\{v_1, v_2\}, t_1)$  is contained in a temporal double-edge, let  $(\{v_3, v_5\}, t_2)$ ,  $(\{v_4, v_6\}, t_3)$  be edges and  $t_4, t_5, t_6 \in [1, T]$  timestamps such that

- none of the edges  $(\{v_3, v_5\}, t_2)$ ,  $(\{v_4, v_6\}, t_3)$  is contained in a temporal double-edge,
- the nodes  $v_3, v_4, v_5, v_6$  are distinct from  $v_1$  and  $v_2$ , and  $v_5$  is distinct from  $v_6$ , and
- none of the edges  $(\{v_1, v_3\}, t_4)$ ,  $(\{v_2, v_4\}, t_5)$ ,  $(\{v_5, v_6\}, t_6)$  exist.

Then, a  $\text{TD}_1$  switching replaces the edges  $(\{v_1, v_2\}, t_1)$ ,  $(\{v_3, v_5\}, t_2)$ ,  $(\{v_4, v_6\}, t_3)$  with  $(\{v_1, v_3\}, t_4)$ ,  $(\{v_2, v_4\}, t_5)$ ,  $(\{v_5, v_6\}, t_6)$  (see Figure 2).

The  $\text{TD}_0$  switching can be defined analogously by using four edges in place of two to remove both edges contained in the temporal double-edge. To choose a  $\text{TD}_1$  or  $\text{TD}_0$  switching when removing a temporal double-edge, specify the probability distribution  $(\{\text{TD}_1, \text{TD}_0\}, P)$  where

$$p(\text{TD}_1) = p(\text{TD}_0) \frac{p_{\mathcal{C}}(\mathbb{1}) \bar{f}_{\text{TD}_1}(\mathbf{W}) \underline{b}_{\text{TD}_0}(\mathbf{W}')}{p_{\mathcal{B}}(\mathbb{1}) \bar{f}_{\text{TD}_0}(\mathbf{W}) \underline{b}_{\text{TD}_1}(\mathbf{W}')}, \quad p(\text{TD}_0) = 1 - p(\text{TD}_1)$$

for quantities  $\bar{f}_{\text{TD}_1}(\mathbf{W})$ ,  $\bar{f}_{\text{TD}_0}(\mathbf{W})$  and  $\underline{b}_{\text{TD}_1}(\mathbf{W}') = \underline{b}_{\text{TD}_1}(\mathbf{W}'; 0) \underline{b}_{\text{TD}_1}(\mathbf{W}'; 1) \underline{b}_{\text{TD}_1}(\mathbf{W}'; 2)$ ,  $\underline{b}_{\text{TD}_0}(\mathbf{W}') = \underline{b}_{\text{TD}_0}(\mathbf{W}'; 0) \underline{b}_{\text{TD}_0}(\mathbf{W}'; 1) \underline{b}_{\text{TD}_0}(\mathbf{W}'; 2)$  defined below and where  $p_{\mathcal{B}}(\mathbb{1})$  and  $p_{\mathcal{C}}(\mathbb{1})$  are the probabilities of not performing a  $\mathcal{B}_{m,n}$  and  $\mathcal{C}_{m,n,o,p}$  auxiliary switching, respectively.

Continuing with the auxiliary switchings, we now define the  $\mathcal{B}_{m,n}$  switching. The  $\mathcal{C}_{m,n,o,p}$  switching can be defined analogously by expanding the number of ordinary multi-edges created from the two edges removed by the  $\text{TD}_1$  switching to the four edges removed by the  $\text{TD}_0$  switching.

► **Definition 13** ( $\mathcal{B}_{m,n}$  switching at  $v_1, v_2, v_3, v_4, v_5, v_6$ ). For a graph  $G$  such that  $\{v_3, v_5\}$  and  $\{v_4, v_6\}$  are non-edges, let  $(\{v_3, v_{2i+5}\}, t_i)$ ,  $(\{v_5, v_{2i+6}\}, t_{m+i})$ ,  $1 \leq i \leq m$  be incident edges at  $v_3, v_5$ ,  $(\{v_4, v_{2m+2i+5}\}, t_{2m+i})$ ,  $(\{v_6, v_{2m+2i+6}\}, t_{2m+n+i})$ ,  $1 \leq i \leq n$  incident edges at  $v_4, v_6$ , and  $t_{2m+2n+1}, \dots, t_{4m+4n} \in [1, T]$  timestamps such that

- none of the edges  $(\{v_3, v_{2i+5}\}, t_i)$ ,  $(\{v_5, v_{2i+6}\}, t_{m+i})$ ,  $1 \leq i \leq m$ ,  $(\{v_4, v_{2m+2i+5}\}, t_{2m+i})$ ,  $(\{v_6, v_{2m+2i+6}\}, t_{2m+n+i})$ ,  $1 \leq i \leq n$  is contained in a temporal double-edge,
- the nodes  $v_1, \dots, v_{2m+2n+6}$  are all distinct, and
- none of the edges  $(\{v_3, v_5\}, t_{2m+2n+i})$ ,  $(\{v_{2i+5}, v_{2i+6}\}, t_{3m+2n+i})$ ,  $1 \leq i \leq m$  and none of the edges  $(\{v_4, v_6\}, t_{4m+2n+i})$ ,  $(\{v_{2m+2i+5}, v_{2m+2i+6}\}, t_{4m+3n+i})$ ,  $1 \leq i \leq n$  exist.

Then, a  $\mathcal{B}_{m,n}$  switching replaces the edges  $(\{v_3, v_{2i+5}\}, t_i)$ ,  $(\{v_5, v_{2i+6}\}, t_{m+i})$ ,  $1 \leq i \leq m$ ,  $(\{v_4, v_{2m+2i+5}\}, t_{2m+i})$ ,  $(\{v_6, v_{2m+2i+6}\}, t_{2m+n+i})$ ,  $1 \leq i \leq n$  with  $(\{v_3, v_5\}, t_{2m+2n+i})$ ,  $(\{v_{2i+5}, v_{2i+6}\}, t_{3m+2n+i})$ ,  $1 \leq i \leq m$ ,  $(\{v_4, v_6\}, t_{4m+2n+i})$ ,  $(\{v_{2m+2i+5}, v_{2m+2i+6}\}, t_{4m+3n+i})$ ,  $1 \leq i \leq n$ .

The sets of  $\mathcal{B}_{m,n}$  and  $\mathcal{C}_{m,n,o,p}$  switchings are

$$\Theta_{\mathcal{B}} = \bigcup_{\substack{0 \leq m, n < \Delta \\ \mu \leq \max\{m, n\}}} \{\mathcal{B}_{m,n}\} \quad \Theta_{\mathcal{C}} = \bigcup_{\substack{0 \leq m, n, o, p < \Delta \\ \mu \leq \max\{m, n, o, p\}}} \{\mathcal{C}_{m,n,o,p}\}$$

and the associated type distributions are  $(\Theta_B \cup \{I\}, P_B)$  and  $(\Theta_C \cup \{I\}, P_C)$  where

$$p_B(\mathbb{B}_{m,n}) = p_B(I) \frac{\bar{f}_{\mathbb{B}_{m,n}}(\mathbf{W}')}{\bar{b}_{\mathbb{B}_{m,n}}(\mathbf{W}')}, \quad p_B(I) = 1 - \sum_{\theta \in \Theta_B} p_B(\theta),$$

$$p_C(\mathbb{C}_{m,n,o,p}) = p_C(I) \frac{\bar{f}_{\mathbb{C}_{m,n,o,p}}(\mathbf{W}')}{\bar{b}_{\mathbb{C}_{m,n,o,p}}(\mathbf{W}')}, \quad p_C(I) = 1 - \sum_{\theta \in \Theta_C} p_C(\theta).$$

The main loop of Stage 2 is as follows. Let  $\pi$  denote a permutation of the entries in  $\mathbf{W}(G)$  such that  $\mathbf{W}_{i,j,t} = 2$  and  $i \neq j$ . Then, Stage 2 iterates through the temporal double-edges in the order given by  $\pi$  and performs the following steps.

1. Let  $G$  denote the current graph,  $\mathbf{W} = \mathbf{W}(G)$  and  $(\{v_1, v_2\}, t_1)$  the temporal double-edge.
2. Choose a switching type  $\theta \sim (\{\text{TD}_0, \text{TD}_1\}, P)$ .
3. Pick a uniform random  $\theta$  switching  $S$  which removes  $(\{v_1, v_2\}, t_1)$  from  $G$ .
4. Restart (**f-reject**) with probability  $1 - \frac{f_\theta(G)}{\bar{f}_\theta(\mathbf{W})}$ .
5. Rewire the edges according to  $S$ , let  $G'$  denote the resulting graph and  $\mathbf{W}' = \mathbf{W}(G')$ .
6. If  $\theta = \text{TD}_1$ :
  - a. Let  $(\{v_3, v_5\}, t_2), (\{v_4, v_6\}, t_3)$  denote the edges removed by  $S$ .
  - b. Restart if  $m_{3,5} \geq \mu$  or  $m_{4,6} \geq \mu$ .
  - c. Restart (**b-reject**) with probability  $1 - \frac{b_{\text{TD}_0}(\mathbf{W}'; 2)}{b_{\text{TD}_0}(G', v_1 \dots v_6; 2)}$ .
  - d. Choose a switching type  $\theta_B \sim (\Theta_B \cup \{I\}, P_B)$ .
  - e. If  $\theta = \mathbb{B}_{m,n}$  for some  $\mathbb{B}_{m,n} \in \Theta_B$ :
    - i. Restart if  $m_{3,5} \geq 1$  or  $m_{4,6} \geq 1$ .
    - ii. Pick a uniform random  $\mathbb{B}_{m,n}$  switching  $S'$  which adds an edge with node set  $\{v_3, v_5\}$  and multiplicity  $m$  and an edge with node set  $\{v_4, v_6\}$  and multiplicity  $n$  to  $G'$ .
    - iii. Restart (**f-reject**) with probability  $1 - \frac{f_{\mathbb{B}_{m,n}}(G')}{\bar{f}_{\mathbb{B}_{m,n}}(\mathbf{W}')}$ .
    - iv. Rewire the edges according to  $S'$  and let  $G''$  denote the resulting graph.
    - v. Restart (**b-reject**) with probability  $1 - \frac{b_{\mathbb{B}_{m,n}}(\mathbf{W}')}{b_{\mathbb{B}_{m,n}}(G'', v_1 \dots v_6)}$ .
    - vi. Set  $G' \leftarrow G''$ .
  - f. Restart (**b-reject**) with probability  $1 - \frac{b_{\text{TD}_1}(\mathbf{W}'; 0) b_{\text{TD}_1}(\mathbf{W}'; 1)}{b_{\text{TD}_1}(G', v_1 v_2; 0) b_{\text{TD}_1}(G', v_1 v_2 v_3 v_4; 1)}$ .
7. Else if  $\theta = \text{TD}_0$ :
  - a. Let  $(\{v_3, v_7\}, t_2), (\{v_4, v_8\}, t_3), (\{v_5, v_9\}, t_4), (\{v_6, v_{10}\}, t_5)$  denote the edges removed by  $S$ .
  - b. Restart if  $m_{3,7} \geq \mu, m_{4,8} \geq \mu, m_{5,9} \geq \mu$  or  $m_{6,10} \geq \mu$ .
  - c. Restart (**b-reject**) with probability  $1 - \frac{b_{\text{TD}_0}(\mathbf{W}'; 2)}{b_{\text{TD}_0}(G', v_1 \dots v_{10}; 2)}$ .
  - d. Choose a switching type  $\theta_C \sim (\Theta_C \cup \{I\}, P_C)$ .
  - e. If  $\theta = \mathbb{C}_{m,n,o,p}$  for some  $\mathbb{C}_{m,n,o,p} \in \Theta_C$ :
    - i. Restart if  $m_{3,7} \geq 1, m_{4,8} \geq 1, m_{5,9} \geq 1$  or  $m_{6,10} \geq 1$ .
    - ii. Pick a uniform random  $\mathbb{C}_{m,n,o,p}$  switching  $S'$  which adds an edge with node set  $\{v_3, v_7\}$  and multiplicity  $m$ , an edge with node set  $\{v_4, v_8\}$  and multiplicity  $n$ , an edge with node set  $\{v_5, v_9\}$  and multiplicity  $o$ , and an edge with node set  $\{v_6, v_{10}\}$  and multiplicity  $p$  to  $G'$ .
    - iii. Restart (**f-reject**) with probability  $1 - \frac{f_{\mathbb{C}_{m,n,o,p}}(G')}{\bar{f}_{\mathbb{C}_{m,n,o,p}}(\mathbf{W}')}$ .
    - iv. Rewire the edges according to  $S'$  and let  $G''$  denote the resulting graph.
    - v. Restart (**b-reject**) with probability  $1 - \frac{b_{\mathbb{C}_{m,n,o,p}}(\mathbf{W}')}{b_{\mathbb{C}_{m,n,o,p}}(G'', v_1 \dots v_{10})}$ .

- vi. Set  $G' \leftarrow G''$ .
- f. Restart (**b-reject**) with probability  $1 - \frac{\underline{b}_{\text{TD}_0}(\mathbf{W}'; 0) \underline{b}_{\text{TD}_0}(\mathbf{W}'; 1)}{b_{\text{TD}_0}(G', v_1 v_2; 0) b_{\text{TD}_0}(G', v_1 \dots v_6; 1)}$ .
8. Set  $G \leftarrow G'$ .

It remains to define the quantities required for the f- and b-rejection steps. For the f-rejection in step 4, define  $f_{\text{TD}_1}(G)$  and  $f_{\text{TD}_0}(G)$  as the number of  $\text{TD}_1$  and  $\text{TD}_0$  switchings which can be performed on the graph  $G$ , respectively. The corresponding upper bounds are

$$\bar{f}_{\text{TD}_1}(\mathbf{W}) = M^2 T^3, \quad \bar{f}_{\text{TD}_0}(\mathbf{W}) = M^4 T^6.$$

For the b-rejections in steps 6c and 7c, define  $b_{\text{TD}_1}(G', v_1 \dots v_6; 2)$  as the number of timestamps  $t_2, t_3 \in [1, T]$  such that  $(\{v_3, v_5\}, t_2)$ ,  $(\{v_4, v_6\}, t_3)$  do not exist in  $G'$  and  $b_{\text{TD}_0}(G', v_1 \dots v_{10}; 2)$  as the number of timestamps  $t_2, t_3, t_4, t_5 \in [1, T]$  such that  $(\{v_3, v_7\}, t_2)$ ,  $(\{v_4, v_8\}, t_3)$ ,  $(\{v_5, v_9\}, t_4)$ ,  $(\{v_6, v_{10}\}, t_5)$  do not exist in  $G'$ . The lower bounds are

$$\underline{b}_{\text{TD}_1}(\mathbf{W}'; 2) = (T - (\mu - 1))^2, \quad \underline{b}_{\text{TD}_0}(\mathbf{W}'; 2) = (T - (\mu - 1))^4.$$

For the b-rejections in steps 6f and 7f, define  $b_{\text{TD}_1}(G', v_1 \dots v_4; 1)$  as the number of simple edges  $(\{v_5, v_6\}, t_6)$  in  $G'$  such that  $v_5, v_6$  are distinct from  $v_1, v_2, v_3, v_4$  and  $b_{\text{TD}_0}(G', v_1 \dots v_6; 1)$  as the number of distinct simple edges  $(\{v_7, v_8\}, t_{10})$ ,  $(\{v_9, v_{10}\}, t_{11})$  in  $G'$  such that  $v_7, v_8, v_9, v_{10}$  are distinct from  $v_1, v_2, v_3, v_4, v_5, v_6$ . Then, define  $b_{\text{TD}_1}(G', v_1 v_2; 0)$  as the number of simple edges  $(\{v_1, v_3\}, t_4)$ ,  $(\{v_2, v_4\}, t_5)$  incident at  $v_1$  and  $v_2$  in  $G'$  and  $b_{\text{TD}_0}(G', v_1 v_2; 0)$  as the number of distinct simple edges  $(\{v_1, v_3\}, t_4)$ ,  $(\{v_2, v_4\}, t_5)$ ,  $(\{v_1, v_5\}, t_6)$ ,  $(\{v_2, v_6\}, t_7)$  incident at  $v_1$  and  $v_2$  in  $G'$ . The lower bounds are

$$\begin{aligned} \underline{b}_{\text{TD}_1}(\mathbf{W}'; 1) &= M - 4B_D - 4\Delta, & \underline{b}_{\text{TD}_0}(\mathbf{W}'; 1) &= (M - 4B_D - 4\Delta)^2, \\ \underline{b}_{\text{TD}_1}(\mathbf{W}'; 0) &= k_1 k_2, & \underline{b}_{\text{TD}_0}(\mathbf{W}'; 0) &= k_1(k_1 - 1)k_2(k_2 - 1). \end{aligned}$$

For the f-rejections in steps 6eiii and 7eiii, define  $f_{\mathcal{B}_{m,n}}(G')$ ,  $f_{\mathcal{C}_{m,n,o,p}}(G')$  as the number of  $\mathcal{B}_{m,n}$ ,  $\mathcal{C}_{m,n,o,p}$  switchings which can be performed on  $G'$ , respectively. In addition, define

$$\bar{f}_{\mathcal{B}_{m,n}}(\mathbf{W}') = \Delta^{2(m+n)} T^{2(m+n)}, \quad \bar{f}_{\mathcal{C}_{m,n,o,p}}(\mathbf{W}') = \Delta^{2(m+n+o+p)} T^{2(m+n+o+p)}.$$

For the b-rejections in steps 6ev and 7ev, define  $b_{\mathcal{B}_{m,n}}(G'', v_1 \dots v_6)$  and  $b_{\mathcal{C}_{m,n,o,p}}(G'', v_1 \dots v_{10})$  as the number of  $\mathcal{B}_{m,n}$  and  $\mathcal{C}_{m,n,o,p}$  switchings which can produce the graph  $G''$ , respectively. The corresponding lower bounds are

$$\begin{aligned} \underline{b}_{\mathcal{B}_{m,n}}(\mathbf{W}') &= (M - 4B_D - 4(m+n+3)\Delta)^{m+n} (T - (\Delta - 1))^{2(m+n)}, \\ \underline{b}_{\mathcal{C}_{m,n,o,p}}(\mathbf{W}') &= (M - 4B_D - 4(m+n+o+p+5)\Delta)^{m+n+o+p} (T - (\Delta - 1))^{2(m+n+o+p)}. \end{aligned}$$

We show the following in subsection 4.1 and subsection 4.2.

► **Lemma 14.** *The graph  $G'$  at the end of an iteration of Stage 2 is uniformly distributed in  $\mathcal{S}(\mathbf{W}')$  given that the graph  $G$  at the start of the iteration is uniformly distributed in  $\mathcal{S}(\mathbf{W})$ .*

► **Lemma 15.** *The probability of not restarting in Stage 2 is  $\exp(-O(\Delta^3/MT) - O(\Delta^2/T^2))$ .*

► **Lemma 16.** *The expected run time of Stage 2 is  $O(\Delta^3/T)$ .*

Once Stage 2 ends, the final graph is simple and can be output.

## 4 Proof of Theorem 2

It remains to show Theorem 2. We start with the uniformity of the output distribution.

### 4.1 Uniformity of T-Gen

To show Lemma 9, we first show that the upper and lower bounds specified for Stage 1 hold. To this end, let  $\mathcal{M}_0(\mathbf{D}) \subseteq \mathcal{M}(\mathbf{D})$  denote the set of temporal multigraphs which satisfy the initial conditions specified in subsection 3.1. Then, we obtain the following results.

► **Lemma 17.** *For all  $G \in \mathcal{S}(\mathbf{W}) \subseteq \mathcal{M}_0(\mathbf{D})$  we have*

$$f_{\text{TL}}(G) \leq \bar{f}_{\text{TL}}(\mathbf{W}).$$

**Proof.** By Definition 7, a TL switching involves two edges  $(\{v_2, v_4\}, t_2)$ ,  $(\{v_3, v_5\}, t_3)$  and three timestamps  $t_4, t_5, t_6 \in [1, T]$ . The total number of choices for the edges and timestamps constitutes an upper bound on the number of switchings which can be performed, and there are at most  $M^2$  choices the (oriented) edges and at most  $T^3$  choices the timestamps. ◀

► **Lemma 18.** *For all  $G' \in \mathcal{S}(\mathbf{W}') \subseteq \mathcal{M}_0(\mathbf{D})$  where  $m_{2,4} < \mu$  and  $m_{3,5} < \mu$  we have*

$$\underline{b}_{\text{TL}}(\mathbf{W}'; 2) \leq b_{\text{TL}}(G', v_1 v_2 v_3 v_4 v_5; 2) \leq T^2$$

and for all  $G' \in \mathcal{S}(\mathbf{W}') \subseteq \mathcal{M}_0(\mathbf{D})$  we have

$$\underline{b}_{\text{TL}}(\mathbf{W}'; 1) \leq b_{\text{TL}}(G', v_1 v_2 v_3; 1) \leq M,$$

$$\underline{b}_{\text{TL}}(\mathbf{W}'; 0) = b_{\text{TL}}(G', v_1; 0).$$

**Proof.** The first set of inequalities follows since there are at most  $T$  and at least  $T - (\mu - 1)$  available timestamps for each edge with multiplicity at most  $\mu - 1$ . For the second set of inequalities, there are at most  $M$  choices for an edge, at most  $2B_L + 4B_D$  choices such that the edge is not simple, and at most  $4\Delta$  choices such that some of the nodes are not distinct. For the equality, observe that  $\mathbf{W}'$  (in addition to  $\mathbf{D}$ ) determines the number of incident simple temporal edges at all nodes. ◀

► **Lemma 19.** *For all  $G' \in \mathcal{S}(\mathbf{W}') \subseteq \mathcal{M}_0(\mathbf{D})$  we have*

$$f_{A_{m,n}}(G') \leq \bar{f}_{A_{m,n}}(\mathbf{W}').$$

**Proof.** The number of edges and timestamps needed for the  $A_{m,n}$  switching is  $2(m + n)$  and there are at most  $\Delta$  choices for each incident edge, and at most  $T$  choices for each timestamp. ◀

► **Lemma 20.** *For all  $G'' \in \mathcal{S}(\mathbf{W}') \subseteq \mathcal{M}_0(\mathbf{D})$  we have*

$$\underline{b}_{A_{m,n}}(\mathbf{W}') \leq b_{A_{m,n}}(G'', v_1 \dots v_5) \leq M^{m+n} T^{2(m+n)}.$$

**Proof.** The number of switchings which can produce a given graph corresponds to the number of choices for edges and timestamps needed to reverse the switching. Reversing the  $A_{m,n}$  switching requires  $m + n$  edges and  $2(m + n)$  timestamps. For each edge, there are at most  $M$  choices, at most  $2B_L + 4B_D$  choices such that the edge is not simple, and at most  $2\Delta$  choices for each node already chosen such that the nodes are not distinct. For each timestamp, there are at most  $T$  choices, and at least  $T - (\Delta - 1)$  choices such that the edge does not exist in  $G''$ . ◀

**Proof of Lemma 9.** Let  $c_{i,j} = \sum_{1 \leq t \leq T} \mathbf{1}_{w_{i,j,t} > 0}$  denote the number of distinct temporal edges between two given nodes  $v_i, v_j$  and  $N(v_i)$  the (multi-)set of incident temporal edges at a given node  $v_i$ . Then by Lemma 17, after the f-rejection in step 3 a given graph  $G' \in \mathcal{S}(\mathbf{W}')$  is produced with probability

$$\sum_{\substack{(\{v_1, v_2\}, t_4) \neq (\{v_1, v_3\}, t_5) \in N(v_1) \\ v_1 \neq v_2, w_{1,2,4}=1 \\ v_1 \neq v_3, w_{1,3,5}=1}} \sum_{\substack{(\{v_4, v_5\}, t_6) \in E \\ v_4 \neq v_5, w_{4,5,6}=1 \\ v_4 \notin \{v_1, v_2\} \\ v_5 \notin \{v_1, v_3\}}} \frac{(T - c_{2,4})(T - c_{3,5})}{\bar{f}_{\text{TL}}(\mathbf{W}')|\mathcal{S}(\mathbf{W}')|}$$

where  $v_1$  is the node at which the temporal single-loop was removed. Thus, in particular,  $G'$  is produced via switchings where we fix the three created edges  $(\{v_1, v_2\}, t_4)$ ,  $(\{v_1, v_3\}, t_5)$ ,  $(\{v_4, v_5\}, t_6)$  to three given edges which satisfy the conditions with probability

$$\frac{(T - c_{2,4})(T - c_{3,5})}{\bar{f}_{\text{TL}}(\mathbf{W}')|\mathcal{S}(\mathbf{W}')|} \propto b_{\text{TL}}(G', v_1 v_2 v_3 v_4 v_5; 2)$$

and by Lemma 18 after steps 6 and 7 with probability proportional to  $\underline{b}_{\text{TL}}(\mathbf{W}'; 2) = T - (\mu - 1)$  if  $m_{2,4}, m_{3,5} < \mu$  (which implies  $c_{2,4}, c_{3,5} < \mu$ ) and probability 0 if any of  $m_{2,4} \geq \mu, m_{3,5} \geq \mu$ . Now, if  $m_{2,4} = m_{3,5} = 0$ , we perform a type  $A_{m,n}$  switching with probability  $p_A(A_{m,n})$ , or restart the algorithm with this probability if  $0 < m_{2,4}, m_{3,5} < \mu$ . Thus, if  $0 \leq m_{2,4}, m_{3,5} < \mu$ , the probability of producing  $G'$  via switchings which create the three fixed edges is

$$p_A(l) \frac{\underline{b}_{\text{TL}}(\mathbf{W}'; 2)}{\bar{f}_{\text{TL}}(\mathbf{W}')|\mathcal{S}(\mathbf{W}')|}.$$

If instead  $\mu \leq m_{2,4} < \min\{d_2, d_4\}$  or  $\mu \leq m_{3,5} < \min\{d_3, d_5\}$ , then  $G'$  is produced only via an  $A_{m_{2,4}, m_{3,5}}$  switching on a graph with  $m_{2,4} = m_{3,5} = 0$  and by Lemma 19 and Lemma 20, after the f- and b-rejections in steps 9c and 9e, the probability of producing  $G'$  in this way is

$$p_A(A_{m_{2,4}, m_{3,5}}) \frac{\underline{b}_{A_{m_{2,4}, m_{3,5}}}(\mathbf{W}')}{\bar{f}_{A_{m_{2,4}, m_{3,5}}}(\mathbf{W}')} \frac{\underline{b}_{\text{TL}}(\mathbf{W}'; 2)}{\bar{f}_{\text{TL}}(\mathbf{W}')|\mathcal{S}(\mathbf{W}')|}.$$

It is now straightforward to verify that the probabilities  $p_A(l)$  and  $p_A(A_{m,n})$  as specified for Stage 1 equalize the expressions given above. Thus, after step 9, the probability of producing a given graph  $G'$  via switchings which create the three fixed edges no longer depends on  $c_{2,4}$  or  $c_{3,5}$ . It only remains to show that step 10 equalizes the probabilities over all choices of the three edges  $(\{v_1, v_2\}, t_4)$ ,  $(\{v_1, v_3\}, t_5)$  and  $(\{v_4, v_5\}, t_6)$ . To this end, observe that the probability of producing  $G'$  via switchings where we only fix the choices of  $(\{v_1, v_2\}, t_4)$  and  $(\{v_1, v_3\}, t_5)$  is

$$\sum_{\substack{(\{v_4, v_5\}, t_6) \in E \\ v_4 \neq v_5, w_{4,5,6}=1 \\ v_4 \notin \{v_1, v_2\} \\ v_5 \notin \{v_1, v_3\}}} p_A(l) \frac{\underline{b}_{\text{TL}}(\mathbf{W}'; 2)}{\bar{f}_{\text{TL}}(\mathbf{W}')|\mathcal{S}(\mathbf{W}')|} \propto b_{\text{TL}}(G', v_1 v_2 v_3; 1)$$

and by Lemma 18 after the second b-rejection in step 10, this probability is proportional to  $\underline{b}_{\text{TL}}(\mathbf{W}'; 1)$ . Finally, to show that the probability is equal over all choices of  $(\{v_1, v_2\}, t_4)$ ,  $(\{v_1, v_3\}, t_5)$ , observe that by Lemma 18, we have

$$\sum_{\substack{(\{v_1, v_2\}, t_4) \neq (\{v_1, v_3\}, t_5) \in N(v_1) \\ v_1 \neq v_2, w_{1,2,4}=1 \\ v_1 \neq v_3, w_{1,3,5}=1}} p_A(l) \frac{\underline{b}_{\text{TL}}(\mathbf{W}'; 1) \underline{b}_{\text{TL}}(\mathbf{W}'; 2)}{\bar{f}_{\text{TL}}(\mathbf{W}')|\mathcal{S}(\mathbf{W}')|} \propto b_{\text{TL}}(G', v_1; 0) = \underline{b}_{\text{TL}}(\mathbf{W}'; 0)$$



and thus  $G'$  is produced with probability

$$p_A(1) \frac{\underline{b}_{\text{TL}}(\mathbf{W}'; 0) \underline{b}_{\text{TL}}(\mathbf{W}'; 1) \underline{b}_{\text{TL}}(\mathbf{W}'; 2)}{\bar{f}_{\text{TL}}(\mathbf{W}) |\mathcal{S}(\mathbf{W})|}$$

which only depends on  $\mathbf{W}$ ,  $\mathbf{W}'$  and  $\mathbf{D}$ .  $\blacktriangleleft$

The proof of Lemma 14 requires showing that the upper and lower bounds for Stage 2 are correct. To this end, let  $\mathcal{M}_1(\mathbf{D}) \subseteq \mathcal{M}_0(\mathbf{D})$  denote set of temporal multigraphs which satisfy the initial conditions and which are output by Stage 1, i.e. which contain no temporal single-loops. Then, we obtain the following results.

► **Lemma 21.** *For all  $G \in \mathcal{S}(\mathbf{W}) \subseteq \mathcal{M}_1(\mathbf{D})$  we have*

$$\begin{aligned} f_{\text{TD}_1}(G) &\leq \bar{f}_{\text{TD}_1}(\mathbf{W}), \\ f_{\text{TD}_0}(G) &\leq \bar{f}_{\text{TD}_0}(\mathbf{W}). \end{aligned}$$

**Proof.** There are at most  $M^2$  choices for the (oriented) edges  $(\{v_3, v_5\}, t_2)$ ,  $(\{v_4, v_6\}, t_3)$  and at most  $T^3$  choices for the timestamps  $t_4, t_5, t_6 \in [1, T]$  needed to perform a  $\text{TD}_1$  switching. The  $\text{TD}_0$  switching uses twice as many edges and timestamps.  $\blacktriangleleft$

► **Lemma 22.** *For all  $G' \in \mathcal{S}(\mathbf{W}') \subseteq \mathcal{M}_1(\mathbf{D})$  where  $m_{3,5} < \mu$  and  $m_{4,6} < \mu$  we have*

$$\underline{b}_{\text{TD}_1}(\mathbf{W}'; 2) \leq b_{\text{TD}_1}(G', v_1 \dots v_6; 2) \leq T^2,$$

for all  $G' \in \mathcal{S}(\mathbf{W}') \subseteq \mathcal{M}_1(\mathbf{D})$  where  $m_{3,7}, m_{4,8}, m_{5,9}, m_{6,10} < \mu$  we have

$$\underline{b}_{\text{TD}_0}(\mathbf{W}'; 2) \leq b_{\text{TD}_0}(G', v_1 \dots v_{10}; 2) \leq T^4$$

and for all  $G' \in \mathcal{S}(\mathbf{W}') \subseteq \mathcal{M}_1(\mathbf{D})$  we have

$$\begin{aligned} \underline{b}_{\text{TD}_1}(\mathbf{W}'; 1) &\leq b_{\text{TD}_1}(G', v_1 \dots v_4; 1) \leq M, \\ \underline{b}_{\text{TD}_0}(\mathbf{W}'; 1) &\leq b_{\text{TD}_0}(G', v_1 \dots v_6; 1) \leq M^2, \\ \underline{b}_{\text{TD}_1}(\mathbf{W}'; 0) &= b_{\text{TD}_1}(G', v_1 v_2; 0), \\ \underline{b}_{\text{TD}_0}(\mathbf{W}'; 0) &= b_{\text{TD}_0}(G', v_1 v_2; 0). \end{aligned}$$

**Proof.** The first two sets of inequalities follow since there are at most  $T$  and at least  $T - (\mu - 1)$  available timestamps for each edge. For the third and fourth sets of inequalities, observe that there are at most  $M$  choices for an edge, at most  $4B_D$  choices such that the edge is not simple, and at most  $4\Delta$  choices such that the nodes are not distinct. The equalities follow from the observation that  $\mathbf{W}'$  (in addition to  $\mathbf{D}$ ) determines the number of incident simple temporal edges at each node.  $\blacktriangleleft$

► **Lemma 23.** *For all  $G' \in \mathcal{S}(\mathbf{W}') \subseteq \mathcal{M}_1(\mathbf{D})$  we have*

$$\begin{aligned} f_{\text{B}_{m,n}}(G') &\leq \bar{f}_{\text{B}_{m,n}}(\mathbf{W}'), \\ f_{\text{C}_{m,n,o,p}}(G') &\leq \bar{f}_{\text{C}_{m,n,o,p}}(\mathbf{W}'). \end{aligned}$$

**Proof.** The number of edges and timestamps needed for the  $\text{B}_{m,n}$  switching is  $2(m+n)$ , the number of edges and timestamps needed for the  $\text{C}_{m,n,o,p}$  switching is  $2(m+n+o+p)$ , and there are at most  $\Delta$  choices for each incident edge, and at most  $T$  choices for each timestamp.  $\blacktriangleleft$

► **Lemma 24.** For all  $G'' \in \mathcal{S}(\mathbf{W}') \subseteq \mathcal{M}_1(\mathbf{D})$  we have

$$\begin{aligned} \underline{b}_{\mathbf{B}_{m,n}}(\mathbf{W}') &\leq b_{\mathbf{B}_{m,n}}(G'', v_1 \dots v_6) \leq M^{m+n} T^{2(m+n)}, \\ \underline{b}_{\mathbf{C}_{m,n,o,p}}(\mathbf{W}') &\leq b_{\mathbf{C}_{m,n,o,p}}(G'', v_1 \dots v_{10}) \leq M^{m+n+o+p} T^{2(m+n+o+p)}. \end{aligned}$$

**Proof.** The number of edges and timestamps needed to reverse the  $\mathbf{B}_{m,n}$  switching is  $m+n$  and  $2(m+n)$ , the number of edges and timestamps needed to reverse the  $\mathbf{C}_{m,n,o,p}$  switching is  $m+n+o+p$  and  $2(m+n+o+p)$ . There are at most  $M$  choices for each edge, at most  $4B_D$  choices such that the edge is not simple, and at most  $2\Delta$  choices for each node already chosen such that the nodes are not distinct. Finally, there are at most  $T$  and at least  $T - (\Delta - 1)$  choices for each timestamp such that the edge does not exist in  $G''$ . ◀

**Proof of Lemma 14.** Using Lemma 21, Lemma 22, Lemma 23, and Lemma 24 in a similar style argument as in the proof of Lemma 9, after removing a temporal double-edge between two nodes  $v_1$  and  $v_2$  in step 5, a given graph  $G' \in \mathcal{S}(\mathbf{W}')$  is produced with probability

$$p(\text{TD}_1) p_{\mathbf{B}}(l) \frac{\underline{b}_{\text{TD}_1}(\mathbf{W}'; 0) \underline{b}_{\text{TD}_1}(\mathbf{W}'; 1) \underline{b}_{\text{TD}_1}(\mathbf{W}'; 2)}{\bar{f}_{\text{TD}_1}(\mathbf{W}) |\mathcal{S}(\mathbf{W})|}$$

if  $m_{1,2}(G') = 1$  and with probability

$$p(\text{TD}_0) p_{\mathbf{C}}(l) \frac{\underline{b}_{\text{TD}_0}(\mathbf{W}'; 0) \underline{b}_{\text{TD}_0}(\mathbf{W}'; 1) \underline{b}_{\text{TD}_0}(\mathbf{W}'; 2)}{\bar{f}_{\text{TD}_0}(\mathbf{W}) |\mathcal{S}(\mathbf{W})|}$$

if  $m_{1,2}(G') = 0$ . Specifying the probabilities  $p(\text{TD}_1)$  and  $p(\text{TD}_0)$  as done for Stage 2 then suffices to equalize the probabilities over all graphs in  $\mathcal{S}(\mathbf{W}')$ . ◀

We are now able to show the following.

► **Lemma 25.** Given a tuple  $\mathbf{D} = (\mathbf{d}, T)$  as input,  $T\text{-GEN}$  outputs a uniform random sample  $G \in \mathcal{G}(\mathbf{D})$ .

**Proof.** If the initial graph  $G$  is simple, then the claim follows by Theorem 3. Otherwise the initial graph  $G$  is uniformly distributed in the set  $\mathcal{S}(\mathbf{W}(G)) \subseteq \mathcal{M}(\mathbf{D})$  for some  $\mathbf{W}(G) \neq \mathbf{0}^{n \times n \times T}$ . If  $G$  satisfies the initial conditions, then all entries  $\mathbf{W}(G)_{i,i,t}$  are either 0 or 1 and all entries  $\mathbf{W}(G)_{i,j,t}$  such that  $i \neq j$  are either 0 or 2. Now, it is straightforward to check that each iteration of Stage 1 corresponds to a map  $\mathcal{S}(\mathbf{W}) \rightarrow \mathcal{S}(\mathbf{W}')$  where  $\mathbf{W}'$  is the matrix obtained from  $\mathbf{W} = \mathbf{W}(G)$  by setting exactly one entry  $\mathbf{W}_{i,i,t} = 1$  to 0, and Stage 1 ends once all such entries have been set to 0. Similarly, each iteration of Stage 2 corresponds to a map  $\mathcal{S}(\mathbf{W}) \rightarrow \mathcal{S}(\mathbf{W}')$  where  $\mathbf{W}'$  is the matrix obtained from  $\mathbf{W}$  by setting exactly two entries  $\mathbf{W}_{i,j,t} = \mathbf{W}_{j,i,t} = 2$  where  $i \neq j$  to 0, and Stage 2 ends once all such entries have been set to 0. Thus, after Stage 1 and Stage 2 end, the final graph  $G$  is a simple temporal graph with  $\mathbf{W}(G) = \mathbf{0}^{n \times n \times T}$ , and by Lemma 9 and Lemma 14, this graph is uniformly distributed in  $\mathcal{S}(\mathbf{0}^{n \times n \times T}) = \mathcal{G}(\mathbf{D})$  as claimed. ◀

We move on to the run time proof.

## 4.2 Runtime of T-Gen

The following additional results are needed for the proofs of Lemma 6, Lemma 10, Lemma 15, Lemma 11, and Lemma 16.

► **Lemma 26.** *Given that  $\Delta^{2+\epsilon} = O(M)$  for a constant  $\epsilon > 0$  and  $T - \Delta = \Omega(T)$ , we have  $p_A(l) = 1 - o(\Delta^{-1})$ .*

**Proof.** Let  $k = m + n$ . Then, the probability of choosing a type  $A_{m,n}$  switching is at most

$$p(A_{m,n}) = p_A(l) \frac{\bar{f}_{A_{m,n}}(\mathbf{W}')}{\underline{b}_{A_{m,n}}(\mathbf{W}')} < \frac{\Delta^{2k} T^{2k}}{(M - B_L - B_D - 4(k+3)\Delta)^k (T - (\Delta - 1))^{2k}}.$$

Now, if  $\Delta^{2+\epsilon} = O(M)$  and  $T - \Delta = \Omega(T)$ , then

$$\frac{\Delta^{2k} T^{2k}}{(M - B_L - B_D - 4(k+3)\Delta)^k (T - (\Delta - 1))^{2k}} = O(\Delta^{-\epsilon k}) = o(\Delta^{-k/\mu})$$

by  $B_L + B_D + 4(k+3)\Delta = O(\Delta^2) = o(M)$  and  $\epsilon > \frac{1}{\mu}$ . Thus, the type l switching is chosen with probability at least

$$p_A(l) = 1 - \sum_{\substack{0 \leq m, n < \Delta \\ \mu \leq \max\{m, n\}}} p_A(A_{m,n}) > 1 - \sum_{\mu \leq k < 2\Delta} \binom{k+1}{1} o(\Delta^{-k/\mu}) = 1 - o(\Delta^{-1})$$

as claimed. ◀

► **Lemma 27.** *Given that  $\Delta^{2+\epsilon} = O(M)$  for a constant  $\epsilon > 0$  and  $T - \Delta = \Omega(T)$ , we have  $p_B(l) = 1 - o(\Delta^{-1})$  and  $p_C(l) = 1 - o(\Delta^{-1})$ .*

**Proof.** By a similar argument as in the proof of Lemma 26. ◀

**Proof of Lemma 6.** By Lemma 5, the highest multiplicity of any ordinary multi-edge in the initial graph is at most  $\eta = \lfloor 2 + 2/\epsilon \rfloor$  with high probability. To complete the proof, we show that with high probability no two edges created due to switchings share the same node set, which in turn implies that the highest multiplicity of any ordinary multi-edge is  $\eta + 1 = \lfloor 3 + 2/\epsilon \rfloor := \mu$ .

First, observe that Lemma 26 and Lemma 27 imply that the probability of performing at least one auxiliary switching in either of Stage 1 or Stage 2 is at most  $(B_L + B_D)o(\Delta^{-1}) = o(1)$ . With the remaining probability, only TL, TD<sub>1</sub>, and TD<sub>0</sub> switchings are performed. The TL switching creates three edges with node sets  $\{v_1, v_2\}$ ,  $\{v_1, v_3\}$  and  $\{v_4, v_5\}$  where  $v_1$  is incident with a temporal single-loop, and where  $v_2$  and  $v_4, v_3$  and  $v_5$  are determined by choosing the switching uniformly at random which implies that these nodes are incident with edges  $(\{v_2, v_4\}, t_2)$ ,  $(\{v_3, v_5\}, t_3)$  chosen uniformly at random among all edges which satisfy the conditions. Likewise, the TD<sub>1</sub> switching creates three edges with node sets  $\{v_1, v_3\}$ ,  $\{v_2, v_4\}$  and  $\{v_5, v_6\}$  where  $v_1$  and  $v_2$  are incident with a temporal double-edge,  $v_3$  and  $v_5, v_4$  and  $v_6$  are incident with edges satisfying the conditions chosen uniformly at random, and the TD<sub>0</sub> switching adds two such sets of edges. Now, observe that when performing a TL switching at some node  $v_1$ , there are at least  $M - 2B_L - 4B_D - 2\Delta$  choices for the edge  $(\{v_2, v_4\}, t_2)$  and at least  $M - 2B_L - 4B_D - 3\Delta$  choices for the second edge  $(\{v_3, v_5\}, t_3)$ . Thus, the probability that a given node  $v$  with degree  $d$  takes on the role of one of  $v_2, v_3, v_4, v_5$  in such a switching chosen uniformly at random is at most

$$\frac{2d}{M - 2B_L - 4B_D - 2\Delta} + \frac{2d}{M - 2B_L - 4B_D - 3\Delta} = O\left(\frac{\Delta}{M}\right)$$

due to  $d \leq \Delta$  and  $O(B_L + B_D + \Delta) = o(M) \implies M - O(B_L - B_D - \Delta) = \Omega(M)$ . Similar calculations give the same asymptotic probability of a given node being involved in a TD<sub>1</sub> or TD<sub>0</sub> switching.

This leads to the following bounds on the probability of creating two edges with the same node set in terms of the number of iterations of a stage performed. Fix any stage, and any node  $v$ , and let  $i$  denote the number of iterations performed overall, and  $j$  the number of iterations performed such that the non-simple edge removed is incident at  $v$ . Then, the number of edges created overall is  $O(i)$  and the expected number of edges created which are incident at  $v$  is  $O(j + i\Delta/M)$ . Thus, the probability that the next switching performed at  $v$  creates an edge which shares the same node set as any edges created prior is  $O(i\Delta^2/M^2)$  for edges not incident at  $v$ , and  $O(j\Delta/M + i\Delta^2/M^2)$  for edges incident at  $v$ .

Now, recall that by the initial conditions, there are at most  $\lambda = O(1)$  incident temporal single-loops and at most  $\kappa = O(1)$  incident temporal double-edges at any node, and at most  $B_L = M_2/M = O(\Delta)$  temporal single-loops and at most  $B_D = M_2^2/MT = O(\Delta^2/T)$  temporal double-edges overall. Then, starting with Stage 1, the probability of creating two edges with share the same node set due to any of the TL switchings is at most

$$\begin{aligned} \sum_{1 \leq i \leq B_L} \sum_{1 \leq j \leq \lambda} \left( O\left(i \frac{\Delta^2}{M^2}\right) + O\left(j \frac{\Delta}{M} + i \frac{\Delta^2}{M^2}\right) \right) &< B_L \lambda \left( O\left(B_L \frac{\Delta^2}{M^2}\right) + O\left(\lambda \frac{\Delta}{M}\right) \right) \\ &= O\left(\frac{\Delta^4}{M^2} + \frac{\Delta^2}{M}\right) \\ &= o(1) \end{aligned}$$

by  $B_L = O(\Delta)$ ,  $\lambda = O(1)$ , and  $\Delta^{2+\epsilon} = O(M)$ . Likewise, for Stage 2, the probability of creating two edges with share the same node set due to any of the TD<sub>1</sub> or TD<sub>0</sub> switchings is at most

$$\begin{aligned} \sum_{1 \leq i \leq B_D} \sum_{1 \leq j \leq \kappa} \left( O\left((B_L + i) \frac{\Delta^2}{M^2}\right) + O\left((\lambda + j) \frac{\Delta}{M} + (B_L + i) \frac{\Delta^2}{M^2}\right) \right) \\ < B_D \kappa \left( O\left((B_L + B_D) \frac{\Delta^2}{M^2}\right) + O\left((\lambda + \kappa) \frac{\Delta}{M}\right) \right) \\ = O\left(\frac{\Delta^6}{T^2 M^2} + \frac{\Delta^5}{T M^2} + \frac{\Delta^3}{T M}\right) \\ = o(1) \end{aligned}$$

by  $B_D = O(\Delta^2/T)$ ,  $\kappa = O(1)$ ,  $\Delta = O(T)$  and  $\Delta^{2+\epsilon} = O(M)$ . ◀

**Proof of Lemma 10.** By Lemma 26 the probability that Stage 1 restarts in steps 9a – f is smaller than  $B_L o(\Delta^{-1}) = o(1)$  and by Lemma 6 we can assume that the highest multiplicity of an edge in  $G$  is at most  $\mu$ . Hence, the probability of not restarting is at most the probability of not f- or b-rejecting in steps 3, 7 or 10 under this assumption.

The probability of an f-rejection in step 3 equals the probability of choosing edges  $(\{v_2, v_4\}, t_2)$ ,  $(\{v_3, v_5\}, t_3)$  and timestamps  $t_4, t_5, t_6 \in [1, T]$  which do not fulfill the conditions defined for the TL switching. For each edge, this probability is at most  $(2B_L + 4B_D + 3\Delta)/M$  as there are at least  $M$  choices for each (oriented) edge, at most  $2B_L$  choices for a loop, at most  $4B_D$  choices for an edge contained in a temporal double-edge and at most  $3\Delta$  choices for an edge such that  $v_2, v_3, v_4, v_5$  is not distinct from  $v_1$  or  $v_4$  is not distinct from  $v_5$ . For the timestamps, we can assume that  $m_{1,2}, m_{1,3}, m_{4,5} \leq \mu$  so for each timestamp the probability of a rejection is at most  $(\mu - 1)/T$ . Then, the probability of not f-rejecting in a given iteration is at least

$$\frac{(M - 2B_L - 4B_D - 3\Delta)^2 (T - (\mu - 1))^3}{M^2 T^3} = \left(1 - O\left(\frac{\Delta}{M}\right)\right)^2 \left(1 - O\left(\frac{\mu}{T}\right)\right)^3.$$

In addition, by Lemma 18, the probability of a b-rejection in step 7 is at most  $(\mu - 1)/T$  for each timestamp, and the probability of a b-rejection in step 10 is at most  $(2B_L + 4B_D + 4\Delta)/M$ . Thus, the probability of not b-rejecting in a given iteration is at least

$$\frac{(M - 2B_L - 4B_D - 4\Delta)(T - (\mu - 1))^2}{MT^2} = \left(1 - O\left(\frac{\Delta}{M}\right)\right) \left(1 - O\left(\frac{\mu}{T}\right)\right)^2.$$

Finally, by the initial conditions there are at most  $B_L = M_2/M < \Delta$  iterations of Stage 1, and by using  $\mu = O(1)$ , the probability that the algorithm does not restart in Stage 1 is at least

$$\left(\left(1 - O\left(\frac{\Delta}{M}\right)\right)^2 \left(1 - O\left(\frac{\mu}{T}\right)\right)^3\right)^{B_L} = \exp\left(-O\left(\frac{\Delta^2}{M}\right) - O\left(\frac{\Delta}{T}\right)\right). \quad \blacktriangleleft$$

**Proof of Lemma 11.** By the initial conditions, there are at most  $B_L = M_2/M < \Delta$  iterations of Stage 1. Thus, the claim follows if an iteration runs in expected time  $O(\Delta)$ . The steps of an iteration which require attention are the f-rejection in step 3, the b-rejections in step 7 and 10, choosing an auxiliary switching in steps 8 – 9, the f-rejection in step 9c, and the b-rejection in step 9f.

First, note that as observed by [1], there is a simple trick to implement an f-rejection step at little additional cost. In the case of step 3, it suffices to choose the two edges and three timestamps required for the TL switching uniformly at random and restart if those choices do not satisfy the conditions given in Definition 7. Then, since there are  $\bar{f}_{\text{TL}}(\mathbf{W}) = M^2 T^3$  ways to choose two (oriented) edges and three timestamps and  $f_{\text{TL}}(G)$  choices yield a switching which can be performed on the current graph  $G$ , we restart with the desired probability of  $f_{\text{TL}}(G)/\bar{f}_{\text{TL}}(\mathbf{W})$ .

The b-rejections in step 7 and 10 require computing the quantities  $b_{\text{TL}}(G', v_1 v_2 v_3 v_4 v_5; 2)$ , and  $b_{\text{TL}}(G', v_1 v_2 v_3; 1)$ ,  $b_{\text{TL}}(G', v_1; 0)$ . Computing  $b_{\text{TL}}(G', v_1 v_2 v_3 v_4 v_5; 2)$  only requires look ups of the multiplicities of  $\{v_2, v_4\}$  and  $\{v_3, v_5\}$  which take time  $O(1)$  if the implementation maintains a data structure to store the multiplicity of the edges in the graph. To compute  $b_{\text{TL}}(G', v_1 v_2 v_3; 1)$ , it suffices to iterate through the lists of incident edges at nodes  $v_1, v_2, v_3$  and subtract the number of simple edges which collide with those nodes from the total number of simple edges, which takes time  $O(\Delta)$ . Additionally, computing  $b_{\text{TL}}(G', v_1; 0)$  can be implemented in time  $O(1)$  by maintaining the number of simple edges incident at each node.

Computing the type distribution for steps 8 – 9 naively would take time  $\Theta(\Delta^2)$ , but there is a simple trick to speed-up the computation. Re-purposing the proof of Lemma 26, we see that

$$\sum_{\substack{0 \leq m, n < \Delta \\ \mu \leq \max\{m, n\} \\ m+n=k}} p_{\mathbf{A}}(\mathbf{A}_{m,n}) < \binom{k+1}{1} p_{\mathbf{A}}(\mathbf{A}_{k,0}) < \binom{k+1}{1} \frac{\bar{f}_{\mathbf{A}_{k,0}}(\mathbf{W})}{\underline{b}_{\mathbf{A}_{k,0}}(\mathbf{W}')} =: \bar{p}_{\mathbf{A}}(\mathbf{A}_k).$$

Thus, in time  $O(\Delta)$ , we can compute

$$\underline{p}_{\mathbf{A}}(\mathbf{l}) := 1 - \sum_{\mu \leq k < 2\Delta - 1} \bar{p}_{\mathbf{A}}(\mathbf{A}_k)$$

as a lower bound on the probability of performing the identity switching, and with this probability step 9 can be skipped. Otherwise, if one of the auxiliary switchings  $\mathbf{A}_{m,n}$  where  $m + n = k$  is chosen, then we compute the exact probabilities of these at most  $k + 1 = O(\Delta)$

switchings, and either choose a switching in accordance with the exact probabilities or restart the algorithm with probability

$$1 - \sum_{\substack{0 \leq m, n < \Delta \\ \mu \leq \max\{m, n\} \\ m+n=k}} \frac{p_{\mathbf{A}}(\mathbf{A}_{m,n})}{\bar{p}_{\mathbf{A}}(\mathbf{A}_k)}$$

to correct for the overestimate.

Finally, by Lemma 26 the probability that steps 9a – f are executed is at most  $o(\Delta^{-1})$ , so it suffices if these steps can be implemented in time  $O(\Delta^2)$ . For the f-rejection in step 9c, we first pick the incident edges of  $v_2$  and  $v_4$  and the timestamps uniformly at random, and then restart if these choices do not satisfy the conditions in step Definition 8. This results in a restart probability of  $f_{\mathbf{A}_{m,n}}(G') / (d_2^m d_4^n T^{2(m+n)})$ , and to reach the desired probability of  $f_{\mathbf{A}_{m,n}}(G') / \bar{f}_{\mathbf{A}_{m,n}}(\mathbf{W}')$ , it suffices to restart with probability  $1 - (d_2^m d_4^n T^{2(m+n)}) / (\Delta T)^{2(m+n)}$  afterwards. To implement the b-rejection in step 9f, we need to compute  $b_{\mathbf{A}_{m,n}}(G'', v_1 v_2 v_3 v_4 v_5)$  which is the number of  $\mathbf{A}_{m,n}$  switchings which can produce the graph  $G''$ . As this is equal to the number of ways to reverse an  $\mathbf{A}_{m,n}$  switching on the graph  $G''$ , we can compute this quantity as the number of choices for  $m$  and  $n$  edges between  $v_2, v_4$  and  $v_3, v_5$ , respectively,  $m+n$  additional edges, and  $2(m+n)$  timestamps which satisfy the conditions. Computing the number of choices for the edges between  $v_2, v_4$  and  $v_3, v_5$  can be done in time  $O(1)$  by looking up the number of simple edges between these nodes. Choosing an additional edge, and checking if it satisfies the conditions, e.g. if it is simple, and does not share nodes with the other edges, takes time  $O(1)$  for the first edges chosen and time  $O(\Delta)$  for the last edges, so for all  $m+n < 2\Delta - 1$  edges this takes at most time  $O(\Delta^2)$ . Finally, computing the number of available timestamps again only requires looking up the multiplicities of  $2(m+n) = O(\Delta)$  edges and takes time  $O(\Delta)$ . ◀

**Proof of Lemma 15.** By Lemma 27 the probability that Stage 2 restarts in steps 6ei – vi or 7ei – vi is smaller than  $B_D o(\Delta^{-1}) = o(1)$  and by Lemma 6 we can assume that the highest multiplicity of an edge in  $G$  is at most  $\mu$ . Hence, the probability of not restarting is at most the probability of not f- or b-rejecting in steps 4, 6c, 6f, 7c or 7f under this assumption. Furthermore, it is straightforward to check that the probability of f- or b-rejecting in a given iteration is larger if  $\theta = \text{TD}_0$  so we focus on this case.

Using a similar argument as for Lemma 10, the probability of not f-rejecting in step 3 of a given iteration is at least

$$\frac{(M - 4B_D - 5\Delta)^4 (T - (\mu - 1))^6}{M^4 T^6} = \left(1 - O\left(\frac{\Delta}{M}\right)\right)^4 \left(1 - O\left(\frac{\mu}{T}\right)\right)^6.$$

In addition, by Lemma 22 and Lemma 23, the probability of a b-rejection in step 6c or 7c is at most  $(\mu - 1)/T$  for each timestamp, and the probability of a b-rejection in step 6f or 7f is at most  $(4B_D + 4\Delta)/M$ . Then, the probability of not b-rejecting in a given iteration is at least

$$\frac{(M - 4B_D - 4\Delta)^2 (T - (\mu - 1))^4}{M^2 T^4} = \left(1 - O\left(\frac{\Delta}{M}\right)\right)^2 \left(1 - O\left(\frac{\mu}{T}\right)\right)^4.$$

Finally, by the initial conditions there are at most  $B_D = M_2^2 / MT < \Delta^2 / T$  iterations of Stage 2, and thus the probability that the algorithm does not restart in Stage 2 is at least

$$\left( \left(1 - O\left(\frac{\Delta}{M}\right)\right)^4 \left(1 - O\left(\frac{\mu}{T}\right)\right)^6 \right)^{B_D} = \exp\left(-O\left(\frac{\Delta^3}{MT}\right) - O\left(\frac{\Delta^2}{T^2}\right)\right). \quad \blacktriangleleft$$

**Proof of Lemma 16.** By the initial conditions, there are at most  $B_D = M_2^2/MT < \Delta^2/T$  iterations of Stage 2, so the claim follows if an iteration runs in expected time  $O(\Delta)$ . We focus on the f-rejection in step 4, the b-rejections in steps 6c, 6f and 7c, 7f, choosing an auxiliary switching in steps 6d – e and 7d – e, the f-rejections in step 6eiii and 7eiii, and the b-rejections in step 6ev and 7ev.

To implement the f-rejection in step 4, it again suffices to choose the two (or four) edges and three (or six) timestamps needed for the  $\text{TD}_1$  or  $\text{TD}_0$  switching uniformly at random and restart if those choices do not satisfy the conditions given in Definition 12. By similar arguments as in the proof of Lemma 11, the quantities needed for the b-rejections in step 6c, 6f and 7c, 7f can be computed in time  $O(\Delta)$ .

To choose an auxiliary switching in steps 6d – e the method described in the proof of Lemma 11 can be re-used. For steps 7d – e, define

$$\sum_{\substack{0 \leq m, n, o, p \leq \Delta \\ \mu < \max\{m, n, o, p\} \\ m+n+o+p=k}} p_{\mathcal{C}}(\mathcal{C}_{m, n, o, p}) < \binom{k+1}{3} \frac{\bar{f}_{\mathcal{C}_{k, 0, 0, 0}}(\mathbf{W})}{\underline{b}_{\mathcal{C}_{k, 0, 0, 0}}(\mathbf{W}')} =: \bar{p}_{\mathcal{C}}(\mathcal{C}_k).$$

Then, we can compute

$$\underline{p}_{\mathcal{C}}(\mathbb{I}) := 1 - \sum_{\mu \leq k < 4\Delta - 3} \bar{p}_{\mathcal{C}}(\mathcal{C}_k)$$

in time  $O(\Delta)$ . To efficiently choose one of the switchings where  $m + n + o + p = k$ , observe that  $p_{\mathcal{C}}(\mathcal{C}_{m, n, o, p})$  only depends on  $k$  and not the individual choices of  $m, n, o, p$ , thus, it suffices to compute the exact probability for one of the switchings, and the exact number of switchings, and then follow the steps described in the proof of Lemma 11.

By Lemma 27 the probability that steps 6ev or 7ev are executed is at most  $o(\Delta^{-1})$ , so it suffices if these steps can be implemented in time  $O(\Delta^2)$ . This is possible by following the steps described towards the end of the proof of Lemma 11 with the necessary modifications. ◀

The following shows the efficiency of T-GEN.

► **Lemma 28.** *T-GEN runs in expected time  $O(M)$  for a tuple  $\mathbf{D} = (\mathbf{d}, T)$  which satisfies  $\Delta^{2+\epsilon} = O(M)$  for a constant  $\epsilon > 0$  and  $T - \Delta = \Omega(T)$ .*

**Proof.** By Lemma 4 and Lemma 5 the algorithm restarts at most  $O(1)$  times before finding a multigraph which satisfies the initial conditions. In addition, if the initial multigraph satisfies the conditions, then by Lemma 10 and Lemma 15, the algorithm restarts at most  $O(1)$  times during Stage 1 and Stage 2. The run time of the temporal configuration model is  $O(M)$ , and if the initial multigraph satisfies the conditions, then by Lemma 11 and Lemma 16, the combined runtime of both Stage 1 and Stage 2 is  $O(\Delta^3/T) + O(\Delta^2) = O(M)$ . ◀

### 4.3 Proof of Theorem 2

**Proof of Theorem 2.** The claim follows by Lemma 25 and Lemma 28. ◀

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