Partial symmetries of solutions to nonlinear elliptic and parabolic problems in bounded radial domains

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“Natur!
aus dem simpelsten Stoff zu den größten Kontrasten; ohne Schein der Anstrengung zu der größten Vollendung - zur genausten Bestimmtheit, immer mit etwas Weichem überzogen. Jedes ihrer Werke hat ein eigenes Wesen, jede ihrer Erscheinungen den isoliertesten Begriff, und doch macht alles eins aus.”

aus: J.W.v. Goethe, Werke, Hamburger Ausgabe in 14 Bänden, dtv, 1998, Band 13, Naturwissenschaftliche Schriften I, S. 45ff.

“Everything should be as simple as it can be, but not simpler”

Albert Einstein.

Contents

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Alberto Saldaña De Fuentes,
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Introduction

Symmetry is one of nature's greatest mysteries. It appears in a wide variety of forms and circumstances, where usually the optimization of some variable or factor is being carried out. A typical example is the spherical shape of a soap bubble, which minimizes surface tension. In most cases, however, the identification of *what* is being optimized and *how*, is a hard task and requires a deep understanding of the mechanisms behind each phenomenon.

One way to improve our understanding of symmetry is through the study of the mathematical models that describe natural phenomena. Many of these models are formulated with the aid of partial differential equations (PDE's), which capture the essence of *change*, either physical, chemical, biological, social, or anything else. In this manner, we can find models for bubbles (minimal surfaces, Young-Laplace equation [24]), crystals (curvature elasticity theory [59]), population dynamics (diffusive Lotka-Volterra models [11,34]), elementary particles (Gross-Pitaevskii equation [53]), galaxy formation (collisionless Boltzmann equation [5]), cell motility (Chemotaxis models [62]), wound healing (Chemical species model [15]), crime dynamics (crime hotspot model [7]), black holes (Einstein field equations, Kerr metric [31]), etc. The literature is very vast, and these references are only intended as a glimpse of the huge universe of mathematical models that are available in the PDE setting. The symmetry that these phenomena may have, should also be present —to some extent— in the mathematical models.

In this regard, the present thesis is devoted to the investigation of symmetry properties of solutions to boundary value problems for nonlinear partial differential equations. The study of the symmetries of a solution to a PDE became an important issue with the development of the rigorous theories of solvability starting in the last decades of the 19th century. Before 1870, the study of PDE's was centered mainly in heuristic methods for finding explicit solutions, but this could only be applied to a very restricted set of problems. It was until the rigorization program for analysis led by Weierstraß that a new horizon was opened and —with further contributions from Fredholm, Hilbert, C.G. Neumann, Poincaré, Riemann, K.H.A. Schwarz, and many others— the foundations of the modern study of PDE's were established. For an excellent recapitulation of the history of PDE's we refer the reader to the survey paper [9] and the references therein.

Since only few relevant problems in PDE admit explicit solutions, the study of qualitative properties is essential, and it is closely related to existence results (finding a solution is easier when we restrict the search to the set of symmetric functions), nonexistence results (e.g. [30]), and a wide variety of other aspects as, e.g., asymptotic convergence (see [48]).

This thesis focuses on boundary value problems in a radial bounded domain $B \subset \mathbb{R}^N$ with radial symmetric data. We are particularly interested in the following question:

Under which assumptions and to what extent is the symmetry of the data inherited to a solution?

For linear PDE's a fairly good understanding of this question has been achieved, principally due to a robust existence and uniqueness theory. Nonlinear problems, however, pose a major challenge, since even the simplest nonlinear equations have a wide variety of "symmetry breaking" phenomena, which are not present in the linear case (see the survey [42] for references in this regard). Notice that even solutions of linear boundary value problems in B can already have a very complicated shape. An easy example is given by the Dirichlet eigenvalue problem

$$\begin{cases} -\Delta u = \lambda\, u & \text{in } B, \\ u = 0 & \text{on } \partial B. \end{cases}$$

While the (up to a constant factor unique) eigenfunctions corresponding to the first eigenvalue λ_1 are radially symmetric, the eigenfunctions corresponding to the second eigenvalue λ_2 are sign changing nonradial functions of the form $u(x) = j(|x|)\frac{x}{|x|}\cdot p$ for some $p \in \mathbb{S}^{N-1}$ with some positive function j, where $\mathbb{S}^{N-1}$ denotes the unit sphere in $\mathbb{R}^N$. Moreover, the shape of the eigenfunctions continues to get more and more complex as the eigenvalue increases.

This reveals two things: first, that radial symmetry can not always be expected in this generality; and second, that extra assumptions are needed in order to obtain particular symmetry properties of a solution. Usually, extra assumptions are imposed on the solution itself. For example, we can find in the literature hypothesis on positivity [3, 26, 27, 48, 49], on stability properties [21, 44], or on being a critical point of some functional with variational structure [4, 58, 60, 61]. See also [36] for a comprehensible collection of symmetry and symmetry breaking results using different approaches.

Here we present a new kind of assumption: a simple geometric condition in terms of a reflectional inequality. With this hypothesis we characterize the symmetry of any solution of a variety of problems of the elliptic and parabolic type. Particularly, we consider scalar equations and systems of equations with two different boundary conditions. In general, the solutions to these problems exhibit drastically different qualitative properties (see the survey [42] for an overview in this regard) and this is reflected in the variety of techniques that we use in our proofs.

We focus our attention in a partial symmetry known as *foliated Schwarz symmetry*. A continuous function $u : \overline{B} \to \mathbb{R}$ is called *foliated Schwarz symmetric with respect to some unit vector* $p \in \mathbb{S}^{N-1}$ if u is axially symmetric with respect to the axis $\mathbb{R}p$ and nonincreasing in the polar angle $\theta := \arccos(\frac{x}{|x|} \cdot p) \in [0, \pi]$. The name originated in [57] to describe the symmetry of some elliptic variational problems.

We now illustrate our results with a paradigmatic example. Let us consider a classical solution (u_1, u_2) of the following well-known system arising from a model in population dynamics: the (nonautonomous) Lotka-Volterra system for the competition of two species. This system has the form

$$\begin{aligned} (u_1)_t - \mu_1 \Delta u_1 &= a_1(t)u_1 - b_1(t)u_1^2 - \alpha_1(t)u_1u_2 && \text{in } B \times (0,\infty), \\ (u_2)_t - \mu_2 \Delta u_2 &= a_2(t)u_2 - b_2(t)u_2^2 - \alpha_2(t)u_1u_2 && \text{in } B \times (0,\infty), \\ \frac{\partial u_i}{\partial \nu} &= 0 && \text{on } \partial B \times (0,\infty), \\ u_i(x,0) &= u_{0,i}(x) \geq 0 && \text{for } x \in B, \end{aligned} \tag{1}$$

where B is a ball or an annulus in $\mathbb{R}^N$, ν denotes the outward pointing normal vector, $u_{0,1}, u_{0,2} \in C(\overline{B})$, μ_1 and μ_2 are positive constants, and

$$\begin{gathered} a_i, b_i, \alpha_i \in L^\infty((0,\infty)) \ \text{ satisfy that} \\ a_i(t), b_i(t) \geq 0 \text{ for } t > 0 \quad \text{ and } \quad \inf_{t>0} \alpha_i(t) > 0 \quad \text{ for } i = 1, 2. \end{gathered} \tag{2}$$

This system is commonly used to model the competition between two different species. The coefficients μ_i, a_i, b_i, and α_i represent diffusion, birth, saturation, and competition rates respectively (see [11, 34]); and the functions u_1 and u_2 represent the population density of each species. In the literature, the system is mostly considered with constant coefficients for matters of simplicity, whereas it is more natural to assume time-dependence as in [11,37,41] in order to model the effect of different time periods (e.g. seasons) on the birth rates, the movement, or the aggressiveness of the species.

The requirement $\partial_\nu u_i = 0$ on $\partial B \times (0,\infty)$ is called *Neumann boundary conditions*, and this is a *no flux* assumption. In the model, this can be interpreted as if the species were in a closed compound, without the possibility of getting out or others coming in.

In order to explain our main symmetry result for (1) we need to introduce, for $e \in \mathbb{S}^{N-1}$, the half domain $B(e) := \{x \in B \;:\; x \cdot e > 0\}$ and the reflection $\sigma_e(x) := x - 2(x\cdot e)e$ with respect to the hyperplane perpendicular to e.

In Theorem 4.1 in Chapter 4 we show that the following simple reflection inequality on the initial profiles already results in asymptotic symmetry properties of the solutions:

$$\begin{aligned} &\text{There is } e \in \mathbb{S}^{N-1} \;\text{ such that } u_{0,1} \not\equiv u_{0,1}\circ\sigma_e,\; u_{0,2} \not\equiv u_{0,2}\circ\sigma_e \text{ and} \\ &\qquad u_{0,1} \geq u_{0,1}\circ\sigma_e,\; u_{0,2} \leq u_{0,2}\circ\sigma_e \text{ in } B(e). \end{aligned} \tag{3}$$

The first part of this assumption states that the hyperplane perpendicular to e is not a symmetry hyperplane for the initial profiles $u_{0,1}$ or $u_{0,2}$, and the second part assumes, in a weak sense, that the competing species are "slightly separated " from each other at time zero. Loosely speaking, we then observe the following asymptotic symmetry properties: *if (3) holds, then, as time advances, the population densities of the species become increasingly symmetric; in particular, they tend to be foliated Schwarz symmetric functions with a common symmetry axis but with respect to antipodal points on* $\mathbb{S}^{N-1}$ *as the time variable goes to infinity.*

This asymptotic symmetry is made precise using the concept of the *omega limit set* $\omega(u_1,u_2)$ of a solution (u_1,u_2) of (1), which is given by

$$\begin{aligned} \omega(u_1,u_2) :=\{(z_1,z_2) \in C(\overline{B}) \times C(\overline{B}) :\; &\text{there is } t_n \to \infty \text{ such that} \\ &\|u_1(\cdot,t_n) - z_1\|_{L^\infty(B)} + \|u_2(\cdot,t_n) - z_2\|_{L^\infty(B)} \to 0\}. \end{aligned}$$

For global solutions which are uniformly bounded and have equicontinuous semiorbits $\{u_i(\cdot,t) \,:\, t \geq 1\}$, the set $\omega(u_1,u_2)$ is nonempty, compact and connected. The equicontinuity can be obtained under mild boundedness and regularity assumptions on the equation and using boundary and interior Hölder estimates (see Remark 4.9 in Chapter 4). Our main symmetry result for (1) (see Theorem 4.1 in Chapter 4) then states that, *if (3) holds, then all limit profiles* $(z_1,z_2) \in \omega(u_1,u_2)$ *are foliated Schwarz symmetric with respect to a common symmetry axis and antipodal points.*

As far as we know, this is the first symmetry result regarding the Lotka-Volterra problem with competition, even in the stationary case with constant coefficients, i.e., the elliptic version of problem (1). We remark that the dynamics of this system have a very rich structure and depend strongly on the relationships between the coefficients, see for example [11, 14, 18, 19, 37, 40, 41]. In particular, the omega limit set of a solution could have a rather complicated structure due to the time dependence of the coefficients.

For a related class of Dirichlet problems for elliptic competing systems with a variational structure, Tavares and Weth proved in [60] that the ground state solutions are foliated Schwarz symmetric with respect to antipodal points. We will not assume for any of our results any variational structure that could lead to symmetry information.

More is known in the case of Dirichlet problems for *cooperative* systems. For a class of parabolic cooperative systems, Földes and Poláčik [50] proved in particular that, if the underlying domain is a ball, then all positive solutions are asymptotically radially symmetric and radially nonincreasing. On the other hand, for *elliptic* cooperative systems with variational structure and under some convexity assumptions on the data, Damascelli and Pacella [16] proved foliated Schwarz symmetry of solutions having Morse index less or equal to the dimension of the domain.

Although hypothesis (3) does not seem very strong, it is a key element in order to obtain the symmetry result. In fact, for general positive initial data, foliated Schwarz symmetry cannot be expected, as one may see already by looking at equilibria (i.e., stationary solutions) in special cases (see Theorem 4.3 in Chapter 4).

To prove our symmetry results, we develop, inspired by the techniques in [43, 49], a "parabolic rotating plane method". In the following, we briefly outline this method by focusing on the nonlinear boundary value problem

$$\begin{aligned} u_t - \Delta u &= f(t, |x|, u) && \text{in } B \times (0, \infty), \\ u &= 0 && \text{on } \partial B \times (0, \infty), \\ u(x, 0) &= u_0(x) && \text{for } x \in B. \end{aligned} \tag{4}$$

Here, as before, B is a ball or an annulus in $\mathbb{R}^N$, the initial condition u_0 is a continuous function and f satisfies some regularity assumptions. The method then consists of three steps:

1. Linearization and initialization of the method.

2. Perturbation.

3. Limit process and symmetry characterization.

In the **first step** we consider, for given $e \in \mathbb{S}^{N-1}$, the *difference function*

$$u^e : \overline{B(e)} \times [0, \infty) \to \mathbb{R}, \qquad u^e(x, t) := u(x, t) - u(\sigma_e(x), t).$$

This function satisfies an associated linearized boundary value problem with e-dependent coefficients. To start the method we need to guarantee the existence of a direction $e \in \mathbb{S}^{N-1}$ such that u^e is positive in $B(e) \times [0, \infty)$. This will be guaranteed by the parabolic maximum principle and the following extra assumption:

$$\textit{there is } e \in \mathbb{S}^{N-1} \textit{ such that } u_0 \geq u_0 \circ \sigma_e \textit{ and } u_0 \not\equiv u_0 \circ \sigma_e \textit{ in } B(e).$$

For the elliptic counterpart of (4) we assume that the reflectional inequality is directly satisfied by the solution (see Corollary 2.2). In the case of systems (for example (1)), we make a similar assumption on each of the initial profiles $u_{0,i}$; however, in this case, the reflectional inequality changes depending on the nature of the system. If it is competitive, as in (1), then (3) is assumed, whereas a different inequality assumption is required for cooperative systems (see hypothesis (h5)' in Theorem 4.5). The definition of the difference functions also changes for each case.

The **second step**, i.e., the perturbation, is actually the heart of the method and the most difficult to perform. The goal of this step is to show that the asymptotic nonnegativity of the difference function in $B(e)$ remains stable under small rotations of $e \in \mathbb{S}^{N-1}$. This is usually achieved through several lemmas and estimates for linear equations, but the statements and the proofs are completely different for each problem. Let us comment briefly on some of the elements involved in each of the cases we study.

If the equation satisfies Dirichlet boundary conditions, then the linearized problem also satisfies Dirichlet boundary conditions and the use of maximum principles for small domains is a key ingredient in the proofs, since they allow to control negative values close to the boundary. For elliptic equations, the standard maximum principle for small domains (see [6]) suffices. For the parabolic problem, more involved estimates due to Poláčik play a prominent rôle (see Theorem 2.6); loosely speaking, they guarantee that if a (possibly sign changing) solution of a linear parabolic problem is positive in a big enough subcylinder and the supremum norm of the negative part is (relatively) small, then the solution is asymptotically nonnegative.

If the problem satisfies Neumann boundary conditions, then the linearized problem has mixed boundary conditions and this prevents the use of standard maximum principles for small domains. For this case we use a different strategy: we developed a new tool which can be seen as a generalization of the Harnack inequality and the Hopf lemma. This result gives quantitative information on lower bounds for the gradient of the solution (see Theorem 3.4), and we use this to achieve the perturbation result.

In the study of systems a new factor comes into play: the *interaction* between the components of the solution. Here it is crucial to understand how the symmetry of the components is entangled by a "synchronization of symmetry" mechanism. The arguments and estimates of the scalar counterparts are relevant and can be extended in some sense, but they are insufficient to complete the perturbation argument. The main complications are related to the appearance of the so-called "semi-trivial limit profiles", that is, to elements of the omega limit set where one of the components vanishes. This makes the linearized system asymptotically very sensitive with respect to small changes in the direction e. To circumvent this problem we devise a new normalization argument which, together with the other estimates, yields the perturbation result also in this case. We emphasize that the normalization argument varies for the Dirichlet and for the Neumann case.

Lastly, in the **third step** we rotate the direction e as much as possible and as long as the asymptotic positivity of the difference function u^e is maintained. This implies local monotonicity with respect to every (cylindrical) angle. Then, using a symmetry characterization, we translate this local monotonicity into foliated Schwarz symmetry (see Chapter 1).

In contrast to the usual moving plane method on bounded domains in the form developed in [26], for elliptic problems, and in [49], for parabolic problems, in the rotating plane method the symmetry axis is not fixed a priori by our assumption on the initial profile. Nonetheless, our results imply that all the limit profiles share the same axis. In Chapter 3 Section 3.4 we discuss briefly a problem with "altered symmetry" which gives rise to a varying axis of asymptotic symmetry.

The *parabolic rotating plane method* described above is a variant of the well-known *moving plane method*. This technique has been successfully used to investigate the symmetry properties in solutions of nonlinear problems. The method has its roots in the work of Alexandrov [2], who studied minimal surfaces with constant mean curvature; then Serrin [56] elaborated it in order to analyze overdetermined boundary value problems associated with elliptic PDE's. In a seminal paper [26] from 1979, Gidas, Ni, and Nirenberg developed a powerful variant to derive, in particular, radial symmetry of positive solutions to some elliptic problems in balls. Since then, the method has widely been used and generalized in the elliptic setting; we refer the reader to the articles [6, 8] for a survey.

The use of moving plane arguments to analyze the asymptotic symmetry of solutions to parabolic problems is more recent. Consider e.g. the initial value problem (4) with B being a ball, u_0 a *positive* continuous function, and the nonlinearity f satisfying some mild regularity assumptions. We note that, in general, the solution u will not be radially symmetric for any finite time unless the initial profile u_0 is radially symmetric. This follows from the backwards-in-time uniqueness of solutions (see for example [1, Lemma A.16] or [23, Theorem 11 Section 2.3]). Nevertheless, under a suitable monotonicity assumption on the nonlinearity, u is asymptotically radially symmetric by a result of Poláčik [49, Corollary 2.6], i.e., all elements of the corresponding omega limit set are radial functions. The result relies on a parabolic version of the moving plane method related to positive solutions, and it extends to a general setting of fully nonlinear parabolic equations (see [49]). Earlier variants of moving plane techniques for parabolic problems were developed by Babin, Dancer, Hess, and Poláčik, see [3, 17, 33], starting with the study of periodic solutions and semilinear autonomous equations. We refer the reader to the survey paper [48] and the references therein for an overview of the development of the moving plane method in the parabolic case.

We point out that, both in the elliptic and in the parabolic setting, the moving plane method for radial bounded domains relies strongly on the following hypothesis:

(C) *Convexity of the domain, i.e., the underlying domain is a ball.*

(P) *Positivity of the solution.*

(M) *Monotonicity of the nonlinearity, i.e., f is nonincreasing in $|x|$.*

If we remove any of these assumptions, we cannot expect results on radial symmetry in general. See e.g. [13, 42, 46, 47] and the references therein for results on the existence of nonradial solutions in the case where (C) or (M) is violated. However, the parabolic rotating plane method explained above does not depend on (C), (P), and/or (M). In the elliptic setting, the *rotating plane method* has already been used, e.g., in [43] Pacella proves foliated Schwarz symmetry of (possibly nodal) solutions of elliptic problems in bounded domains under some extra stability and convexity assumptions. This result was later extended and generalized in [30, 44]. We refer to [45] for a survey on the rotating plane method in elliptic problems.

Let us mention that there are also techniques independent of moving plane method arguments to study the symmetry properties of solutions to PDE's. For example, the use of polarizations, symmetrizations, and variational and topological methods have been very fruitful lines of research in this regard. We refer the reader to the survey paper [61] and the references therein for an overview.

In this context, in Chapter 5, we study *radial symmetry* of semi-trivial limit profiles of some parabolic systems using stability properties of parabolic problems. To be precise, in Theorem 5.2, inspired by [32], we exploit the stability of autonomous problems with convex nonlinearities; we then complement this information with other recent estimates for positive solutions to linear equations obtained in [35]. With these tools we prove that any nonzero component of a semi-trivial limit profile must be the unique positive solution of an associated elliptic problem, and this solution is radially symmetric. Moreover, in Theorem 5.5 we consider a setting where the zero limit profile is stable, and we combine this information with results in [25] on scalar problems with asymptotically symmetric data to obtain the asymptotic radial symmetry of semitrivial limit profiles. These results are independent of rotating plane arguments and therefore they do not require any reflectional assumption on the initial profiles.

The thesis is organized as follows. The first chapter presents the definition and characterizations of foliated Schwarz symmetric functions. The second chapter is dedicated to the study of nonlinear equations under Dirichlet boundary conditions. In particular, we study problem (4), but we also consider a fully nonlinear version. Chapter 3 is devoted to semilinear equations under Neumann boundary conditions. Chapters 4 and 5 handle nonlinear systems under Neumann and Dirichlet boundary conditions respectively. In particular, Chapter 4 contains our study of the Lotka-Volterra problem given in (1).

Notation and conventions

Sets and functions

We always consider $\mathbb{R}^N$ with $N \geq 2$. Let $\mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ denote the unit sphere, and $B_r(x) := \{y \in \mathbb{R}^N : |x - y| < r\}$ denote the open ball centered at $x \in \mathbb{R}^N$ with radius $r > 0$. Moreover $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$. We denote the euclidean scalar product between two vectors $p, q \in \mathbb{R}^N$ by $p \cdot q$, and $e_i \in \mathbb{S}^{N-1}$ denotes the unit vector with the i-th coordinate equal to 1.

For two sets $U, V \subset \mathbb{R}^N$ we write $V \subset\subset U$ if V is *compactly contained* in U, that is, $\overline{V}$ is a compact set and $\overline{V} \subset U$. We put

$$\operatorname{dist}(U, V) := \inf\{|x - y| \ : \ x \in U, \, y \in V\}$$

and, if $U = \{x\}$ for some $x \in \mathbb{R}^N$, we simply write $\operatorname{dist}(x, V)$ in place of $\operatorname{dist}(\{x\}, V)$.

Let $Q \subset \mathbb{R}^{N+1}$. We denote the *parabolic boundary of* Q by

$$\partial_P Q := \overline{\{(x,t) \in \partial Q : t < M_Q\}},$$

where $M_Q := \sup\{t \in \mathbb{R} : (x,t) \in Q \text{ for some } x \in \mathbb{R}^N\}$.

Let $\Omega \subset \mathbb{R}^N$. Then Ω° denotes the interior of Ω and

$$\operatorname{inrad}(\Omega) := \sup\{r > 0 : B_r(x) \subset \Omega \text{ for some } x \in \Omega\}$$

denotes the *inner radius* of Ω. Moreover, for a real valued function $v : \Omega \to \mathbb{R}$, let $v^+ := \max\{v, 0\}$ and $v^- := -\min\{v, 0\}$ denote the positive and negative parts of v, and let $\operatorname{supp}(v) := \overline{\{v \in \Omega : v \neq 0\}}$ denote the *support* of v.

For a function $u \in L^\infty(\Omega \times (0, \infty))$, we define the *omega limit set* of u by

$$\omega(u) := \{z \in C(\overline{\Omega}) : \|u(\cdot, t_n) - z\|_{L^\infty(\Omega)} \to 0 \text{ for some } t_n \to \infty\}.$$

Analogously, if $u_i \in L^\infty(\Omega \times (0,\infty))$ for $i = 1, \dots, n$, with $n \in \mathbb{N}$, then the *omega limit set* of $u = (u_1, \dots, u_n)$ is defined by

$$\omega(u) := \omega(u_1, \dots, u_n) := \Big\{ (z_1, \dots, z_n) : z_i \in C(\overline{\Omega}),\ i = 1, \dots, n, \text{ and } \sum_{i=1}^n \|u_i(\cdot, t_n) - z_i\|_{L^\infty(\Omega)} \to 0 \text{ for some } t_n \to \infty \Big\}.$$

Function spaces

Let $|x|$ stand for the euclidean norm of $x \in \mathbb{R}^N$ and let $|E|$ be the Lebesgue measure of a measurable set $E \subset \mathbb{R}^N$. For a bounded measurable set $Q \subset \mathbb{R}^{N+1}$, a bounded continuous function $Q \to \mathbb{R}$; $(x,t) \mapsto v(x,t)$, and $p \in (0,\infty]$, we define the norms

$$[v]_{p,Q} := \left(\frac{1}{|Q|} \int_Q |v(x,t)|^p \, d(x,t) \right)^{\frac{1}{p}} \quad \text{for } p \in (0,\infty),$$
$$[v]_{\infty,Q} := \|v\|_{L^\infty(Q)} = \sup_Q |v|.$$

The gradient of v is always considered with respect to the *space* variable $x \in \mathbb{R}^N$, that is, $\nabla v = (\partial_{x_1} v, \dots, \partial_{x_N} v)$.

Let $\alpha \in (0,1]$ and $\Omega \subset \mathbb{R}^N$ be a domain. Put $Q_T := \Omega \times (\tau, T)$ for $0 \le \tau < T$. We define the parabolic Hölder spaces of bounded uniformly continuous functions

$$\begin{aligned} C^{\alpha,\alpha/2}(\overline{Q_T}) &:= \{u \in C(\overline{Q_T}) : \|u\|_{C^{\alpha,\alpha/2}(\overline{Q_T})} < \infty\}, \\ C^{1+\alpha,(1+\alpha)/2}(\overline{Q_T}) &:= \{u \in C(\overline{Q_T}) : \|u\|_{C^{1+\alpha,(1+\alpha)/2}(\overline{Q_T})} < \infty\}, \\ C^{2+\alpha,2+\alpha/2}(\overline{Q_T}) &:= \{u \in C(\overline{Q_T}) : \|u\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q_T})} < \infty\}, \end{aligned} \tag{5}$$

where

$$|v|_{\alpha;Q_T} := \sup \left\{ \frac{|v(x,t) - v(y,s)|}{|x-y|^\alpha + |t-s|^{\frac{\alpha}{2}}} \ : \ (x,t),(y,s) \in \overline{Q_T},\ (x,t) \ne (y,s) \right\},$$
$$\|u\|_{C^{\alpha,\alpha/2}(\overline{Q_T})} := \|u\|_{L^\infty(Q_T)} + |u|_{\alpha;Q_T},$$
$$\|u\|_{C^{1+\alpha,(1+\alpha)/2}(\overline{Q_T})} := \|u\|_{C^{\alpha,\alpha/2}(\overline{Q_T})} + \|\nabla u\|_{L^\infty(Q_T)} + |\nabla u|_{\alpha;Q_T},$$
$$\|u\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q_T})} := \|u\|_{C^{1+\alpha,(1+\alpha)/2}(\overline{Q_T})} + \|D^2 u\|_{L^\infty(Q_T)} + |D^2 u|_{\alpha;Q_T}.$$

For $p \in [1,\infty)$ let

$$W_p^{2,1}(Q_T) := \{u \in L^p(Q_t) \ : \ D^2 u\,,\, \nabla u\,,\, \partial_t u \in L^p(Q_T)\}$$

be the t-anisotropic Sobolev Space endowed with the norm

$$\|u\|_{W_p^{2,1}(Q_T)} := \left(\int_{Q_T} \sum_{i,j=1}^{N} |\partial_{x_i x_j} u|^p + \sum_{i=1}^{N} |\partial_{x_i} u|^p + |\partial_t u|^p \; d(x,t) \right)^{\frac{1}{p}},$$

Here the derivatives are understood in the usual weak sense and D^2u stands for the Hessian matrix of u with respect to the space variables, that is, $D^2u = (\partial_{x_i x_j} u)_{i,j=1}^N$

The space $W_{p,loc}^{2,1}(Q_T)$ is defined analogously using the space $L_{loc}^p(Q_T)$ instead of $L^p(Q_T)$.

Moreover, $W^{2,\infty}(\Omega)$ and $H^2(\Omega) = W^{2,2}(\Omega)$ are the usual Sobolev spaces and $H_0^1(\Omega)$ denotes the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$, where $C_c^\infty(\Omega)$ is the space of smooth functions with compact support in Ω.

Finally, $C_0(\Omega) := \{u \in C(\overline{\Omega}) : u(x) = 0 \text{ for all } x \in \partial\Omega\}$ is endowed with the supremum norm $\|\cdot\|_{L^\infty(\Omega)}$.

Rotating plane method notation

Let $e \in \mathbb{S}^{N-1}$ and let $B \subset \mathbb{R}^N$ be an open ball or an open annulus centered at the origin. Then $H(e) := \{x \in \mathbb{R}^N : x \cdot e = 0\}$ denotes a hyperplane perpendicular to e, $\sigma_e : \overline{B} \to \overline{B}$ denotes the reflection with respect to $H(e)$, that is, $\sigma_e(x) := x - 2(x \cdot e)e$, and $B(e) := \{x \in B : x \cdot e > 0\}$ denotes the open domain. Further let

$$\Sigma_1(e) := \{x \in \partial B(e) : x \cdot e = 0\} \quad \text{and} \quad \Sigma_2(e) := \{x \in \partial B(e) : x \cdot e > 0\}.$$

Chapter 1

Foliated Schwarz symmetry and its characterizations

In this chapter we introduce the notion of partial symmetry that is used in our symmetry results and prove some of the characterizations that will be needed in the proofs. For this whole chapter, B denotes a radial (with respect to the origin) subdomain of $\mathbb{R}^N$ with $N \geq 2$.

Definition 1.1. We say that a function $u \in C(B)$ is *foliated Schwarz symmetric with respect to some unit vector* $p \in \mathbb{S}^{N-1}$ if u is axially symmetric with respect to the axis $\mathbb{R}p$ and nonincreasing in the polar angle $\theta := \arccos(\frac{x}{|x|} \cdot p) \in [0, \pi]$.

For the characterizations of foliated Schwarz symmetry we need two auxiliary lemmas. The following lemma is standard, but we include the proof for the convenience of the reader.

Lemma 1.2. *Let $v \in C(\mathbb{R})$ be an even and 2π-periodic function, and let $\mathcal{R}$ denote the points of reflectional symmetry of v. If v is not constant, then $\mathcal{R} = \{\frac{n\pi}{k} : n \in \mathbb{Z}\}$ with some positive integer k.*

Proof. Note that if $r \in \mathcal{R}$ then v is $2r-$periodic, that is,

$$v(\theta) = v(\theta + 2mr) \qquad \text{for all } \theta \in \mathbb{R},\ m \in \mathbb{Z}. \tag{1.1}$$

We show first that $\mathcal{R} \cap [0, \pi]$ is a finite set. By contrapositive, we show that if $\mathcal{R} \cap [0, \pi]$ is infinite, then v must be constant. Indeed, if $\mathcal{R} \cap [0, \pi]$ is infinite, then for each $m \in \mathbb{N}$ there exist $r_1(m), r_2(m) \in \mathcal{R} \cap [0, \pi]$ such that $r^m := r_1(m) - r_2(m) \in (0, \frac{1}{m})$. Note that $r^m \in \mathcal{R}$ since

$$v(r_1(m) - r_2(m) + \theta) = v(r_1(m) + r_2(m) - \theta) = v(r_1(m) - r_2(m) - \theta)$$

for all $\theta \in \mathbb{R}$ and $m \in \mathbb{N}$, where we have used that $r_1(m) \in \mathcal{R}$ and that v is $2r_2(m)$-periodic by (1.1).

Now, fix $x \in \mathbb{R}$ and let $n_m \in \mathbb{Z}$ be such that $x \in (2n_m r^m\,,\, 2(n_m+1)r^m]$. In particular $0 \le x - 2n_m r^m \le r^m < \frac{1}{m}$. Then, by (1.1),

$$v(0) - v(x) = \lim_{m\to\infty} v(2n_m r^m) - v(x) = 0$$

because $v \in C(\mathbb{R})$. Therefore v must be constant. Since we have, by assumption, that v is not constant, we get that $\mathcal{R} \cap [0,\pi]$ is a finite set.

Since v is even and $2\pi-$periodic we have that $\pi \in \mathcal{R}$. Then

$$\bar{r} := \min\{r \in (0\,,\,\pi]\ :\ r \in \mathcal{R}\} \le \pi$$

and $\bar{r} \in \mathcal{R}$.

Now, let $r \in \mathcal{R}$ be any reflectional symmetry point of v and let $n_r \in \mathbb{Z}$ be such that $r \in [n_r\bar{r}, (n_r+1)\bar{r})$. Then $r - n_r\bar{r} \in \mathcal{R} \cap [0,\bar{r}) = \{0\}$ which implies that $r = n_r\bar{r}$. In particular, since $\pi \in \mathcal{R}$, $\bar{r} = \frac{\pi}{n_\pi}$. Clearly $p\bar{r} \in \mathcal{R}$ for all $p \in \mathbb{Z}$. Then we have that $\mathcal{R} = \{\frac{p\pi}{n_\pi}\ :\ p \in \mathbb{Z}\}$ as claimed. □

Lemma 1.3. *Let $v \in C(\mathbb{R})$ be an even and 2π-periodic function, and let $\mathcal{R}$ denote the points of reflectional symmetry of v. If, for some $\eta \in \mathbb{R}$,*

$$\begin{aligned} v(\eta+\phi) &\ge v(\eta-\phi) \quad \textit{for all } \phi \in [0,\pi] \textit{ and} \\ v(\eta+\phi_0) &> v(\eta-\phi_0) \quad \textit{for some } \phi_0 \in (0,\pi). \end{aligned} \tag{1.2}$$

then we have $\mathcal{R} = \{n\pi\ :\ n \in \mathbb{Z}\}$.

Proof. Since v is continuous, even, 2π-periodic, and not constant by (1.2), it follows from Lemma 1.2 that $\mathcal{R} = \{\frac{n\pi}{k}\ :\ n \in \mathbb{Z}\}$ for some positive integer k. We suppose by contradiction that $k \ge 2$. Then v is $\frac{2\pi}{k}-$periodic. Let $L \in \mathbb{Z}$ be such that $\eta \in (L\frac{\pi}{k}, (L+1)\frac{\pi}{k}]$ and M be such that $\phi_0 - L\frac{\pi}{k} \in (M\frac{2\pi}{k}, (M+1)\frac{2\pi}{k}]$. Set

$$\begin{aligned} \widetilde{\eta} &:= \eta - L\frac{\pi}{k} \ \in \ \Big(0\,,\,\frac{\pi}{k}\Big], \\ \widetilde{\phi}_0 &:= \phi_0 - M\frac{2\pi}{k} - L\frac{\pi}{k} \ \in \ \Big(0\,,\,\frac{2\pi}{k}\Big]. \end{aligned}$$

Let $w \in C(\mathbb{R})$ be given by $w(\phi) := v(\phi + L\frac{\pi}{k})$. Then the following properties are easy consequences of (1.2), the $\frac{2\pi}{k}$-periodicity of w, and the fact that $\frac{\pi}{k}$ is a point of reflectional symmetry of w.

$$\begin{aligned} w(\phi) &= w(-\phi) && \text{for all } \phi \in \mathbb{R}, \\ w\Big(\pm\frac{\pi}{k} + \phi\Big) &= w\Big(\pm\frac{\pi}{k} - \phi\Big) && \text{for all } \phi \in \mathbb{R}, \\ w(\widetilde{\eta} + \phi) &\ge w(\widetilde{\eta} - \phi) && \text{for all } \phi \in (0,\pi), \\ w(\widetilde{\eta} + \widetilde{\phi}_0) &> w(\widetilde{\eta} - \widetilde{\phi}_0). \end{aligned}$$

Since $0 < \frac{2\pi}{k} - \widetilde{\phi}_0 < \pi$, it follows that

$$\begin{aligned} w(\widetilde{\eta} + \widetilde{\phi}_0) > w(\widetilde{\eta} - \widetilde{\phi}_0) &= w\bigg(-\frac{2\pi}{k} - (\widetilde{\eta} - \widetilde{\phi}_0)\bigg) = w\bigg(\widetilde{\eta} + \frac{2\pi}{k} - \widetilde{\phi}_0\bigg) \\ &\geq w\bigg(\widetilde{\eta} - \frac{2\pi}{k} + \widetilde{\phi}_0\bigg) = w\bigg(\frac{2\pi}{k} - \widetilde{\eta} - \widetilde{\phi}_0\bigg) = w(\widetilde{\eta} + \widetilde{\phi}_0), \end{aligned}$$

which yields a contradiction. Hence $k = 1$, and thus the claim follows. □

We now recall some notation. Let $e \in \mathbb{S}^{N-1}$. Then $H(e) := \{x \in \mathbb{R}^N : x \cdot e = 0\}$ denotes a hyperplane perpendicular to e, $\sigma_e : \overline{B} \to \overline{B}$ denotes the reflection with respect to $H(e)$, that is, $\sigma_e(x) := x - 2(x \cdot e)e$, and $B(e) := \{x \in B : x \cdot e > 0\}$ denotes the half domain.

Our first proposition generalizes a result due to Brock ([10], Lemma 4.2) to characterize sets of foliated Schwarz symmetric functions with respect to a common vector.

Proposition 1.4. *Let $\mathcal{U} \subset C(\overline{B})$ and define*

$$\mathcal{M} := \{e \in \mathbb{S}^{N-1} \mid u(x) \geq u(\sigma_e(x)) \text{ for all } x \in B(e) \text{ and } u \in \mathcal{U}\}. \tag{1.3}$$

If

$$\mathbb{S}^{N-1} = \mathcal{M} \cup -\mathcal{M}, \tag{1.4}$$

i.e., if for all $e \in \mathbb{S}^{N-1}$ we have

$$u \geq u \circ \sigma_e \text{ in } B(e) \text{ for all } u \in \mathcal{U} \qquad \text{or} \qquad u \leq u \circ \sigma_e \text{ in } B(e) \text{ for all } u \in \mathcal{U},$$

then there is $p \in \mathbb{S}^{N-1}$ such that every $u \in \mathcal{U}$ is foliated Schwarz symmetric with respect to p.

Proof. We start by constructing orthogonal unit vectors $e_1, \ldots, e_{N-1}$ such that

$$u \equiv u \circ \sigma_{e_i} \qquad \text{for } i = 1, \ldots, N-1 \text{ and every } u \in \mathcal{U}. \tag{1.5}$$

For this we first consider the set

$$\mathcal{A}_1 := \{e \in \mathbb{S}^{N-1} \,:\, u(x) > u(\sigma_e(x)) \text{ for some } u \in \mathcal{U} \text{ and some } x \in B(e)\}.$$

By (1.4) we have $\mathcal{A}_1 \subset \mathcal{M}$, and $\mathcal{A}_1$ does not contain antipodal points. Moreover, $\mathcal{A}_1$ is a relatively open subset of $\mathbb{S}^{N-1}$. If $\mathcal{A}_1$ is empty, then $u \equiv u \circ \sigma_e$ for any $u \in \mathcal{U}$ and $e \in \mathbb{S}^{N-1}$, so any choice of orthonormal vectors $e_1, \ldots, e_{N-1}$ satisfies (1.5). Hence we may assume that $\mathcal{A}_1 \neq \varnothing$. Then also the relative boundary $\partial\mathcal{A}_1$ of $\mathcal{A}_1$ in $\mathbb{S}^{N-1}$ is nonempty. Let $e_1 \in \partial\mathcal{A}_1$; then any $u \in \mathcal{U}$ satisfies $u \equiv u \circ \sigma_{e_1}$. Next we consider

$$\mathcal{A}_2 := \{e \in \mathbb{S}^{N-1} \cap H(e_1) : u(x) > u(\sigma_e(x)) \text{ for some } u \in \mathcal{U} \text{ and } x \in B(e)\}.$$

If $\mathcal{A}_2$ is empty, then we may complement e_1 with any choice of orthonormal vectors $e_2, \dots, e_{N-1}$ in $\mathbb{S}^{N-1} \cap H(e_1)$ to obtain (1.5). If $\mathcal{A}_2$ is nonempty, then – by the same argument as above – also the relative boundary $\partial\mathcal{A}_2$ of $\mathcal{A}_2$ in $\mathbb{S}^{N-1} \cap H(e_1)$ is nonempty, and every vector $e_2 \in \partial\mathcal{A}_2$ satisfies $u \equiv u \circ \sigma_{e_2}$ for every $u \in \mathcal{U}$. Successively we find orthogonal vectors $e_1, \dots, e_{N-1} \in \mathbb{S}^{N-1}$ such that (1.5) holds (then the process stops since $\mathbb{S}^{N-1} \cap H(e_1) \cap H(e_2) \cap \dots \cap H(e_{N-1})$ consists merely of two antipodal points).

Without loss of generality, we may assume that the vectors $e_1, \dots, e_{N-1}$ satisfying (1.5) are the first $N-1$ coordinate vectors. Next we show that every hyperplane containing the x_N-axis is a symmetry hyperplane for every $u \in \mathcal{U}$. For this let $q = (q_1, \dots, q_N) \in \mathbb{S}^{N-1}$ be such that $\mathbb{R}e_N \subset H(q)$. By (1.4) we can assume that $q \in \mathcal{M}$ (otherwise we replace q by $-q$). Since $q_N = 0$, for $x \in B(q)$ we have that $[\sigma_{e_1} \circ \ldots \circ \sigma_{e_{N-1}}](x) = -\sigma_{e_N}(x) \notin B(q)$, and from (1.5) we deduce that

$$u(x) = u(-\sigma_{e_N}(x)) \le u(\sigma_q(-\sigma_{e_N}(x))) = u(-\sigma_{e_N}(\sigma_q(x))) = u(\sigma_q(x)) \le u(x)$$

for every $u \in \mathcal{U}$. Hence $u \equiv u \circ \sigma_q$ for every $u \in \mathcal{U}$, as claimed. We conclude that every $u \in \mathcal{U}$ is axially symmetric with respect to the axis $\mathbb{R}e_N$.

To complete the proof of foliated Schwarz symmetry, we may now restrict to any two-dimensional subspace of $\mathbb{R}^N$ containing the axis $\mathbb{R}e_N$, hence we may assume that $N = 2$ from now on. Let $u \in \mathcal{U}$ be a non radial function. Then there are $e_* \in \mathbb{S}^{N-1}$ and $x \in B(e_*)$ such that $e_* \cdot e_2 > 0$ and

$$u(x) > u(\sigma_{e_*}(x)) \text{ or} \tag{1.6}$$

$$u(x) < u(\sigma_{e_*}(x)). \tag{1.7}$$

Assume (1.6) first. Writing $u = u(r, \phi)$ in (permuted) polar coordinates with $x_1 = r\sin\phi$ and $x_2 = r\cos\phi$, we get that u is even in ϕ, and that there are $r > 0$ and $\eta \in (-\pi, 0)$ such that (1.2) holds for the function $\mathbb{R} \to \mathbb{R}$, $\phi \mapsto u(r, \phi)$. Hence by Lemma 1.3 there are no other points of reflectional symmetry of this function in $(-\pi, 0)$ except the origin, and by (1.4) this implies that for every $e \in \mathbb{S}^{N-1}$ with $e \cdot e_2 > 0$ we have $u \ge u \circ \sigma_e$ and $u \not\equiv u \circ \sigma_e$ in $B(e)$. Then again by (1.4) we have that

$$u \ge u \circ \sigma_e \text{ in } B(e) \text{ for all } u \in \mathcal{U} \text{ and all } e \in S \text{ with } e \cdot e_2 \ge 0,$$

and this readily implies that every $u \in \mathcal{U}$ is foliated Schwarz symmetric with respect to the unit vector e_2.

A similar argument shows that, if we assume (1.7) then every $u \in \mathcal{U}$ is foliated Schwarz symmetric with respect to the unit vector $-e_2$. The proof is finished. □

The following proposition and corollary characterize foliated Schwarz symmetry by properties related to the method of rotating planes.

Proposition 1.5. *Let $\mathcal{U} \subset C(\overline{B})$, $\mathcal{M}$ be defined as in (1.3), and let $\tilde{e} \in \mathcal{M}$. If for all two dimensional subspaces $P \subseteq \mathbb{R}^N$ containing $\tilde{e}$ there are two different points $p_1,\ p_2$ in the same connected component of $\mathcal{M} \cap P$ such that $u \equiv u \circ \sigma_{p_1}$ and $u \equiv u \circ \sigma_{p_2}$ for every $u \in \mathcal{U}$, then there is $p \in \mathbb{S}^{N-1}$ such that every $u \in \mathcal{U}$ is foliated Schwarz symmetric with respect to p.*

Proof. Let P be a two dimensional subspace with $\tilde{e} \in P$. By hypothesis there is some connected component K_P of $\mathcal{M} \cap P$ and $p_1, p_2 \in K_P$ such that $u \equiv u \circ \sigma_{p_1}$ and $u \equiv u \circ \sigma_{p_2}$ for every $u \in \mathcal{U}$. We first show that

$$K_P \text{ contains a closed halfcircle,} \tag{1.8}$$

i.e., $\{e \in \mathbb{S}^{N-1} \cap P \ : \ e \cdot e' \geq 0\} \subseteq K_P$ for some $e' \in \mathbb{S}^{N-1}$. We assume without loss of generality that

$$p_1 = (1, 0, \dots, 0), \qquad p_2 = (\cos\psi, \sin\psi, 0, \dots, 0) \quad \text{for some } \psi \in (0, 2\pi]$$

and

$$(\cos\phi, \sin\phi, 0, \dots, 0) =: p_\phi \in \mathcal{M} \qquad \text{for all } \phi \in [0, \psi]$$

(because p_1 and p_2 are in the same connected component of $\mathcal{M} \cap P$). Let $u \in \mathcal{U}$. Using polar coordinates, we define

$$\tilde{v}(r, \phi, x') := u(r\cos\phi, r\sin\phi, x') = u(x)$$

with $x \in B,\ x' = (x_3, \dots, x_N) \in \mathbb{R}^{N-2},\ \phi \in \mathbb{R}$, and $r = |x|$. If, independently of the choice of $u \in \mathcal{U}$, $\tilde{v}$ does not depend on ϕ, then $\mathcal{M} \cap P = \mathbb{S}^{N-1} \cap P$ and so (1.8) holds trivially. So, we may suppose that $u \in \mathcal{U}$ was chosen such that the function

$$v : \mathbb{R} \to \mathbb{R}, \qquad v(\phi) := \tilde{v}(r, \phi, x')$$

is non-constant for some fixed $r > 0$ and $x' \in \mathbb{R}^{N-2}$. By assumption, we then have

$$\begin{aligned} v(\phi) &= v(-\phi), && \phi \in \mathbb{R}, \\ v(\psi + \phi) &= v(\psi - \phi), && \phi \in \mathbb{R}, \\ v(\eta + \phi) &\geq v(\eta - \phi), && \eta \in (0, \psi), \phi \in (0, \pi), \end{aligned} \tag{1.9}$$

i.e., v has two points of reflectional symmetry, one at zero, and one at ψ, and the points in between satisfy the defining property of $\mathcal{M}$. Since the function is non-constant, the inequality in (1.9) must be strict for some $\eta \in (0, \psi)$ and $\phi \in (0, \pi)$. Then, by Lemma 1.3, we get that $u \not\equiv u \circ \sigma_{p_\phi}$ for $\phi \in (0, \pi)$. By assumption, we then conclude that $p_2 \neq p_\phi$ for $\phi \in (0, \pi)$, and therefore $\psi \geq \pi$. Hence (1.8) holds, as claimed.

Now since (1.8) holds independently of P, we conclude that, for all $e \in \mathbb{S}^{N-1}$ we have $e \in \mathcal{M}$ or $-e \in \mathcal{M}$, so that (1.4) holds. Hence, the assertion follows from Proposition 1.4. □

Corollary 1.6. *Let $\mathcal{U} \subset C(\overline{B})$ and suppose that the set $\mathcal{M}$ defined in (1.3) contains a nonempty subset $\mathcal{N}$ with the following properties*

(i) $\mathcal{N}$ is relatively open in $\mathbb{S}^{N-1}$;

(ii) for every $e \in \partial\mathcal{N}$ and $u \in \mathcal{U}$ we have $u \leq u \circ \sigma_e$ in $B(e)$. Here $\partial\mathcal{N}$ denotes the relative boundary of $\mathcal{N}$ in $\mathbb{S}^{N-1}$.

Then there is $p \in \mathbb{S}^{N-1}$ such that every $u \in \mathcal{U}$ is foliated Schwarz symmetric with respect to p.

Proof. By assumption, there exists $\tilde{e} \in \mathcal{N} \subset M$. Let $P \subseteq \mathbb{R}^N$ be a two-dimensional subspace containing $\tilde{e}$, and let L_P denote the connected component of $\overline{\mathcal{N} \cap P}$ containing $\tilde{e}$. Since $\mathcal{M}$ is closed, L_P is a subset of the connected component of $\mathcal{M} \cap P$ containing $\tilde{e}$. By Proposition 1.5, it suffices to show that there are different points $p_1, p_2 \in L_P$ such that $u \equiv u \circ \sigma_{p_1}$ and $u \equiv u \circ \sigma_{p_2}$ for every $u \in \mathcal{U}$.

We distinguish two cases. If $L_P = \mathbb{S}^{N-1} \cap P$, then we have $u \equiv u \circ \sigma_p$ in B for every $p \in L_P$, $u \in \mathcal{U}$ by the definition of $\mathcal{M}$ and since $L_P \subset \mathcal{M}$.

If $L_P \neq \mathbb{S}^{N-1} \cap P$, then there exists two different points p_1, p_2 in the relative boundary of L_P in $\mathbb{S}^{N-1} \cap P$. Since $\mathcal{N}$ is relatively open in $\mathbb{S}^{N-1}$, these points are contained in $\partial\mathcal{N} \subset \mathcal{M}$. Then, by assumption and the definition of $\mathcal{M}$, we have $u \equiv u \circ \sigma_{p_1}$ and $u \equiv u \circ \sigma_{p_2}$ in B for every $u \in \mathcal{U}$, as required. □

Chapter 2

Parabolic and elliptic equations with Dirichlet boundary conditions

Some of the results in this chapter were published in [54]. For the remainder of the thesis, B always denotes a ball or an annulus centered at zero in $\mathbb{R}^N$ with $N \geq 2$ and $I_B := \{|x| \ : \ x \in \overline{B}\}$. Our aim in this chapter is to prove the following

Theorem 2.1. *Let $u \in C^{2,1}(\overline{B} \times (0,\infty)) \cap C(\overline{B} \times [0,\infty))$ be a classical solution of*

$$\begin{aligned} u_t - \Delta u &= f(t,|x|,u), && x \in B,\ t > 0,\\ u(x,t) &= 0, && x \in \partial B,\ t > 0,\\ u(x,0) &= u_0(x), && x \in B, \end{aligned} \tag{2.1}$$

where the following assumptions hold.

(f1) The nonlinearity $f : [0,\infty) \times I_B \times \mathbb{R} \to \mathbb{R}$, $(t,r,u) \mapsto f(t,r,u)$ is continuous in t, r and locally Lipschitz in u uniformly with respect to t and r, i.e., for every $K > 0$ there is $L = L(K) > 0$ such that

$$|f(t,r,u_1) - f(t,r,u_2)| \leq L|u_1 - u_2|$$

for all $(t,r) \in [0,\infty) \times I_B$ and $u_1, u_2 \in [-K,K]$.

(f2) The function $f(\cdot,\cdot,0)$ is bounded on $[0,\infty) \times I_B$.

(U1) There is $e \in \mathbb{S}^{N-1}$ such that $u_0 \geq u_0 \circ \sigma_e$ and $u_0 \not\equiv u_0 \circ \sigma_e$ in $B(e)$.

(U2) The solution is uniformly bounded, i.e., $\|u\|_{L^\infty(B\times(0,\infty))} < \infty$.

Then u is asymptotically foliated Schwarz symmetric with respect to some $p \in \mathbb{S}^{N-1}$, i.e., all elements of

$$\omega(u) = \{z \in C(\overline{B}) : \|u(\cdot, t_n) - z\|_{L^\infty(B)} \to 0 \text{ for some } t_n \to \infty\}$$

are foliated Schwarz symmetric with respect to p.

This theorem characterizes the asymptotic symmetry of any solution that satisfies the assumptions, but it does not give conditions for the existence. In this regard, it is important to remark that if the nonlinearity f is continuous and locally Lipschitz in u uniformly with respect to t and $|x|$, it follows from standard semigroup theory that, for every $u_0 \in C(\overline{B})$, the corresponding local (in time) problem admits a unique solution $u \in C(\overline{B} \times [0, T(u_0)))$ for some time $T(u_0) > 0$. Further, it has been studied extensively in recent years under which assumptions on the nonlinearity f and the initial condition u_0 this unique solution exists globally in time and the corresponding orbit $\{u(\cdot, t) : t > 0\}$ is relatively compact in $C(\overline{B})$. In this case the set $\omega(u)$ is a nonempty connected compact set. We refer the reader to [12, 49, 52] and the references therein, where many specific examples are discussed which give rise to this behavior.

All the symmetry results in this thesis have direct implications for the elliptic and periodic parabolic versions of the respective problem that is being studied. For example, an immediate consequence of Theorem 2.1 is the following

Corollary 2.2. *(i) Let $f : I_B \times \mathbb{R} \to \mathbb{R}$, $(r, u) \mapsto f(r, u)$ be continuous in $r \in I_B$ and locally Lipschitz in u uniformly with respect to r. Moreover, let $u \in C^2(\overline{B})$ be a classical solution of the elliptic problem*

$$\begin{aligned} -\Delta u &= f(|x|, u) && \text{in } B, \\ u(x) &= 0 && \text{on } \partial B, \end{aligned} \tag{2.2}$$

such that (U1) holds for u in place of u_0. Then u is foliated Schwarz symmetric with respect to some $p \in \mathbb{S}^{N-1}$.

(ii) Suppose that $f : [0, \infty) \times I_B \times \mathbb{R} \to \mathbb{R}$ satisfies (f1) and is periodic in t, i.e., there is $T > 0$ such that $f(t + T, r, u) = f(t, r, u)$ for all t, r, u. Suppose furthermore that $u \in C^{2,1}(\overline{B} \times (0, \infty)) \cap C(\overline{B} \times [0, \infty))$ is a T-periodic solution of (2.1), i.e., $u(x, t + T) = u(x, t)$ for all $x \in B, t \in [0, \infty)$, and such that (U1) holds. Then $u(\cdot, t)$ is foliated Schwarz symmetric with respect to some $p \in \mathbb{S}^{N-1}$ for all times $t \in [0, \infty)$.

As mentioned in the introduction of the thesis, an easy example giving rise to a (sign changing) nonradial but foliated Schwarz symmetric solution of (2.2) – and thus also of (2.1) – is given by $f(|x|, u) = \lambda_2 u$, where λ_2 is the second Dirichlet eigenvalue of the Laplacian.

Under additional spectral assumptions on the solution, statements similar to part (i) of Corollary 2.2 have been derived in [30,44] as an intermediate step in the proof of symmetry results for solutions of (2.2) with low Morse index. We note that results on radial symmetry of nonnegative time periodic solutions had been obtained by Dancer and Hess [17] in the setting where B is a ball in $\mathbb{R}^N$ and f is nonincreasing in $|x|$.

We derive Theorem 2.1 from a more general theorem in Section 2.1 below, dealing with a class of fully nonlinear problems, as in [49]. To prove that every solution u is asymptotically foliated Schwarz symmetric, we use a parabolic rotating plane method as detailed in the introduction. Note however that assumption (U1) does not imply that the functions in $\omega(u)$ are strictly decreasing in the polar angle with respect to the symmetry axis. For instance, in case B is a ball, f is decreasing in $|x|$, and $u_0 \in C(\overline{B})$ is a nonnegative function satisfying (U1), then, as explained in the introduction, a result due to Poláčik [49, Corollary 2.6] implies that $\omega(u)$ only consists of radial functions.

This chapter is organized as follows. We introduce a fully nonlinear version of problem (2.1) in Section 2.1, then in Section 2.2 we present a family of linear parabolic problems associated with the fully nonlinear problem. The estimates required to study these linear problems are contained in Section 2.3. Section 2.4 contains a standard regularity result that will be used in all the chapters and finally, in Section 2.5 we use a parabolic rotating plane argument to prove the main results in this chapter.

To close this introduction, we would like to remark that, although Corollary 2.2 is an immediate consequence of Theorem 2.1, it can also be derived independently by a somewhat simpler argument not relying on the deep estimates in [49]. We leave the details to the reader.

2.1 A fully nonlinear version of the model problem

In this section we set up a more general framework for our symmetry result. The setting is strongly inspired by [49]. We consider the fully nonlinear parabolic problem

$$\begin{aligned} u_t(x,t) &= F(t,x,u,\nabla u, D^2u), && (x,t) \in B \times (0,\infty), \\ u(x,t) &= 0, && (x,t) \in \partial B \times (0,\infty), \\ u(x,0) &= u_0(x), && x \in B, \end{aligned} \tag{2.3}$$

where, as before, B is a bounded radial domain in $\mathbb{R}^N$, $N \geq 2$, and $D^2u = (u_{x_ix_j})_{i,j=1}^N \in \mathbb{R}^{N\times N}$ is the Hessian of u. As for the right hand side of (2.3), we consider the following assumptions.

(F1) *Reflection invariance:* We have $F : [0,\infty) \times \overline{B} \times \mathcal{B} \to \mathbb{R}$, where $\mathcal{B}$ is an open convex set in $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N\times N}$ such that $B \times \mathcal{B}$ is invariant under the transformations

$$(x,u,p,q) \mapsto (Rx,u,Rp,RqR), \text{ for every hyperplane reflection } R \in \mathbb{R}^{N\times N}.$$

Moreover, $F(t,Rx,u,Rp,RqR) = F(t,x,u,p,q)$ for every hyperplane reflection $R \in \mathbb{R}^{N\times N}$ and $(t,x,u,p,q) \in (0,\infty) \times B \times \mathcal{B}$.

(F2) *Regularity:* F is continuous on $[0,\infty)\times\overline{B}\times\mathcal{B}$ and Lipschitz in (u,p,q), uniformly with respect to x and t, i.e., there is $L > 0$ such that

$$\sup_{x\in B, t\geq 0} |F(t,x,u,p,q) - F(t,x,\tilde{u},\tilde{p},\tilde{q})| \leq L|(u,p,q) - (\tilde{u},\tilde{p},\tilde{q})|$$

for all $(u,p,q), (\tilde{u},\tilde{p},\tilde{q}) \in \mathcal{B}$. Moreover, F is differentiable with respect to q on $[0,\infty) \times \overline{B} \times \mathcal{B}$.

(F3) *Ellipticity:* There is a constant $\alpha_0 > 0$ such that

$$\partial_{q_{ij}} F(t,x,u,p,q)\xi_i\xi_j \geq \alpha_0|\xi|^2$$

for all $(t,x,u,p,q) \in [0,\infty) \times B \times \mathcal{B}$ and $\xi \in \mathbb{R}^N$. Here and below, we use the summation convention (summation over repeated indices).

We point out that these hypothesis are closely related to the ones in [49, Section 2]. However, in contrast to [49], we make no monotonicity assumptions on the nonlinearity and it may also include terms depending on the radial derivative of u. So this allows us to also consider equations like

$$u_t = g(t,|x|,u,|\nabla u|,\Delta u) + d(|x|)\nabla u \cdot x, \quad (x,t) \in B \times [0,\infty),$$

where

$$\begin{aligned} &I_B \to \mathbb{R}; && r \mapsto d(r) \\ &\mathbb{R}^5 \to \mathbb{R}; && (t,r,u,\eta,\xi) \mapsto g(t,r,u,\eta,\xi) \end{aligned}$$

are continuous functions, g is Lipschitz continuous in (u, η, ξ) uniformly with respect to (t, r), the partial derivative g_ξ exist everywhere, and $g_\xi \geq \alpha_0$ for some positive constant α_0.

The symmetry result which we want to prove in this general setting relies also on assumptions $(U1)$ and $(U2)$ for a fixed solution of (2.3), which were stated in Theorem 2.1.

Theorem 2.3. *Assume* $(F1)-(F3)$, *and let* $u \in C^{2,1}(B \times (0,\infty)) \cap C(\overline{B} \times [0,\infty))$ *be a classical solution of the problem* (2.3) *satisfying assumptions (U1),(U2) and such that the functions* $(x,t) \mapsto u(x, t+s)$ *for* $s \geq 1$ *are equicontinuous on* $B \times [0,1]$, *that is,*

$$\lim_{h\to 0} \sup_{\substack{x,\bar{x}\in\overline{B}, t,\bar{t}\in[t_0,t_0+1],\\ |x-\bar{x}|+|t-\bar{t}|<h,\\ t_0\geq 1}} |u(x,t) - u(\bar{x},\bar{t})| = 0. \tag{2.4}$$

Then there is $p \in \mathbb{S}^{N-1}$ *such that, for all* $z \in \omega(u)$, z *is foliated Schwarz symmetric with respect to* p, *i.e.,* u *is asymptotically foliated Schwarz symmetric with respect to* p.

Remark 2.4. Note that if u satisfies (2.4) and assumption (U2) from Theorem 2.1, then it follows from the Arzelà-Ascoli Theorem that $\{u(\cdot,t) : t \geq 1\}$ is a precompact set in $C_0(B)$. Therefore $\omega(u) \subset C_0(B)$ and

$$\lim_{t\to\infty} \inf_{z\in\omega(v)} \|v(\cdot,t) - z\|_{L^\infty(B)} = 0.$$

2.2 Linearization of fully nonlinear problems

To use the rotating plane method in the parabolic setting, the crucial step is to consider the linear problem satisfied by the difference between a solution of (2.3) and its reflection at a hyperplane. In this section we give the details of this linearization in the fully nonlinear setting. As before, let B denote a ball or an annulus in $\mathbb{R}^N$, $N \geq 2$. We introduce some notation first. Let $e \in \mathbb{S}^{N-1}$, then $H(e) := \{x \in \mathbb{R}^N : x \cdot e = 0\}$ denotes a hyperplane perpendicular to e, $\sigma_e : \overline{B} \to \overline{B}$ denotes the reflection with respect to $H(e)$, that is, $\sigma_e(x) := x - 2(x \cdot e)e$, and $B(e) := \{x \in B : x \cdot e > 0\}$ denotes the half domain. Moreover, we let u denote a solution of (2.3) and assume the hypothesis of Theorem 2.3.

Define $u^e : \overline{B} \times [0,\infty) \to \mathbb{R}$ by

$$u^e(x,t) := u(x,t) - u(\sigma_e(x),t) \qquad \text{for } x \in B(e),\ t > 0.$$

Then u^e is a solution of the problem

$$\begin{aligned} u_t^e &= a_{ij}^e u_{x_i x_j}^e + b_i^e u_{x_i}^e + c^e u^e && \text{in } B(e)\times(0,\infty),\\ u^e &= 0 && \text{on } \partial B(e)\times(0,\infty),\\ u^e(x,0) &= u_0(x) - u_0(\sigma_e(x)) && \text{for all } x\in B(e), \end{aligned} \tag{2.5}$$

where the coefficients $a_{ij}^e, b_i^e, c^e \in L^\infty(B\times(0,\infty))$ for $i,j=1,\dots,N$, are obtained, as in [49], via the Hadamard formulas. To make this precise, define $B\times(0,\infty)\to\mathbb{R}$; $(x,t)\mapsto u_e(x,t):=u(\sigma_e(x),t)$ and consider

$$c^e(x,t) := \begin{cases} \int_0^1 F_u(t,|x|,su+(1-s)u_e,Du,D^2u))ds, & \text{if } u^e(x,t)\neq 0,\\ 0, & \text{if } u^e(x,t)=0, \end{cases}$$

$$b_i^e(x,t) := \begin{cases} \int_0^1 F_{p_i}(t,|x|,u_e,\dots,(u_e)_{x_{i-1}},su_{x_i} \\ \qquad +(1-s)(u_e)_{x_i},u_{x_{i+1}},\dots,D^2u))ds, & \text{if } (u^e)_{x_i}(x,t)\neq 0,\\ 0, & \text{if } (u^e)_{x_i}(x,t)=0, \end{cases}$$

$$\begin{aligned} a_{ij}^e(x,t) := \int_0^1 & F_{q_{ij}}(t,|x|,u_e,Du_e,\dots,(u_e)_{x_{i^-}x_{j^-}},su_{x_ix_j}\\ &+(1-s)(u_e)_{x_ix_j},u_{x_{i^+}x_{j^+}},\dots,u_{x_Nx_N}))ds \end{aligned}$$

for $e\in\mathbb{S}^{N-1}$, $x\in B$, and $t>0$, where $(i^-,j^-),(i^+,j^+)$ stand for the pairs of indices preceding, respectively, following, (i,j) within a fixed identification of $\mathbb{R}^{N\times N}$ with $\mathbb{R}^{N^2}$.

By (F1) and (F2) the integrals make sense and the right hand side of (2.5) is equal to the difference of $F(t,|x|,u,Du,D^2u)$ and $F(t,|x|,u_e,Du_e,D^2u_e)$.

As a consequence of (F2) and (F3), there is $\beta_0>0$ such that

$$|c^e(x,t)|,|b_i^e(x,t)|,|a_{ij}^e(x,t)|<\beta_0 \qquad \text{and} \qquad a_{ij}^e(x,t)\xi_i\xi_j\geq\alpha_0|\xi|^2 \tag{2.6}$$

for all $x\in B(e)$, $t>0$, $i,j\in\{1,\dots,N\}$, $\xi\in\mathbb{R}^N$, and $e\in\mathbb{S}^{N-1}$ with $\alpha_0>0$ as in (F3).

2.3 Estimates for linear problems

In this section we quote some estimates due to Poláčik [49]. Consider the following general linear parabolic equation.

$$v_t = a_{ij}(x,t)v_{x_ix_j}+b_i(x,t)v_{x_i}+c(x,t)v, \qquad (x,t)\in U\times(\tau,T), \tag{2.7}$$

$$v=0, \qquad (x,t)\in\partial U\times(\tau,T), \tag{2.8}$$

where U is an open subset of some fixed bounded domain $\Omega \subset \mathbb{R}^N$, $0 \le \tau < T \le \infty$, the coefficients a_{ij}, b_i, c are defined on $U \times (\tau, T)$, are measurable, and satisfy that

$$\begin{aligned} &\sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \ge \alpha_0|\xi|^2, && x \in U,\ t \in [\tau, T),\ \xi \in \mathbb{R}^N, \\ &|a_{ij}(x,t)|, |b_i(x,t)|, |c(x,t)| < \beta_0, && x \in U,\ t \in [\tau, T), \end{aligned} \tag{2.9}$$

for all $i, j \in \{1, \dots, N\}$ and for some constants $\alpha_0 > 0$ and $\beta_0 \ge 1$.

When referring to a solution (resp. supersolution) of equation (2.7), we mean a function v in the Sobolev space $W^{2,1}_{N+1,loc}(U \times (\tau, T))$ such that (2.7) (resp. with "$=$ "replaced by "$\ge$ ") is satisfied almost everywhere. A solution (resp. supersolution) of the boundary value problem (2.7),(2.8) is in addition supposed to be continuous on $\overline{U} \times [\tau, T)$ and to satisfy (2.8) (resp. with "$=$ "replaced by "$\ge$ ") in the pointwise sense. We introduce now some notation. For a bounded set $Q \subset \mathbb{R}^{N+1}$, a bounded continuous function $v : Q \to \mathbb{R}$, and $p \in (0, \infty]$, let

$$\begin{aligned} [v]_{p,Q} &:= \left(\frac{1}{|Q|} \int_Q |v(x,t)|^p \, d(x,t) \right)^{\frac{1}{p}} \quad \text{if } p < \infty \quad \text{and} \\ [v]_{\infty,Q} &:= \sup_Q |v|. \end{aligned}$$

Lemma 2.5. *([49, Lemma 3.4]) Given $\varepsilon > 0$, $d > 0$, $\theta > 0$, there are positive constants κ, p determined only by N, $\operatorname{diam}(\Omega)$, α_0, β_0, d, ε, and θ with the following property. If D, U are domains in Ω with $D \subset\subset U$, $\operatorname{dist}(\overline{D}, \partial U) \ge d$, $|D| > \varepsilon$, and $v \in C(\overline{U} \times [\tau, \tau + 4\theta])$ is a supersolution of equation (2.7) with coefficients satisfying (2.9) for some $\tau \in \mathbb{R}$ and $T = \tau + 4\theta$, then*

$$\inf_{D \times (\tau+3\theta, \tau+4\theta)} v \ge \kappa [v^+]_{p, D \times (\tau+\theta, \tau+2\theta)} - e^{4m\theta} \sup_{\partial_P (U \times (\tau, \tau+4\theta))} v^-,$$

where $m = \sup_{U \times (\tau, \tau+4\theta)} c$.

If v is a solution of (2.7), then the conclusion holds with $p = \infty$ and κ is independent of ε.

Theorem 2.6. *(Special case of [49, Theorem 3.7])*
Fix $\rho \in (0, \frac{\operatorname{diam}(\Omega)}{2})$. Then there is

$$\delta = \delta(N, \operatorname{diam}(\Omega), \alpha_0, \beta_0, \rho) > 0$$

and, for every $d, \theta > 0$,

$$\mu = \mu(N, \operatorname{diam}(\Omega), \alpha_0, \beta_0, d, \theta, \rho) \ \in \ (0, 1]$$

with the following properties: If $D \subset U$ are subdomains of Ω satisfying

$$\operatorname{inrad}(D) > \rho, \qquad |U \backslash \overline{D}| < \delta, \qquad \operatorname{dist}(\overline{D}, \partial U) > d,$$

if $v \in C(\overline{U} \times [\tau, \infty))$ is a solution of problem (2.7),(2.8) *with coefficients satisfying* (2.9) *for some $\tau \in \mathbb{R}$, $T = \infty$, and if*

$$\begin{aligned} &v > 0 \qquad \text{in } \overline{D} \times [\tau, \tau + 8\theta), \\ &\|v^-(\cdot, \tau)\|_{L^\infty(U \backslash D)} \le \mu \|v\|_{L^\infty(D \times (\tau+\theta, \tau+2\theta))}, \end{aligned}$$

then the following statements hold true:

(S1) $v(x,t) > 0$ for all $(x,t) \in \overline{D} \times [\tau, \infty)$.

(S2) $\|v^-(x,t)\|_{L^\infty(U)} \to 0$ as $t \to \infty$.

Lemma 2.7. *([49, Lemma 3.5]) Given $\varepsilon > 0$, $d > 0$, $\theta > 0$, $0 < \tau_1 < \tau_2 < \tau_3 < \tau_4$, there are positive constants κ, κ_1, and p determined only by N, $\operatorname{diam}(\Omega)$, α_0, β_0, d, ε, $\tau_2 - \tau_1$, $\tau_3 - \tau_2$, $\tau_3 - \tau_4$, and θ with the following property. If D, U are domains in Ω with $D \subset\subset U$, $\operatorname{dist}(\overline{D}, \partial U) \ge d$, $|D| > \varepsilon$, and $v \in C(\overline{U} \times [\tau, T])$ is a supersolution of*

$$v_t = a_{ij}(x,t) v_{x_i x_j} + b_i(x,t) v_{x_i} + c(x,t) v + g(x,t), \quad (x,t) \in U \times (\tau, T),$$

where $\tau_1 - 2\theta \le \tau \le \tau_1 - \theta$, $T \ge \tau_4$, the coefficients satisfy (2.9)*, and $g \in L^{N+1}(U \times (\tau, \tau_4))$, then*

$$\inf_{D \times (\tau_3, \tau_4)} v \ge \kappa [v^+]_{p, D \times (\tau_1, \tau_2)} - \kappa_1 \|g^-\|_{L^{N+1}(U \times (\tau, \tau_4))} - e^m \sup_{\partial_P (U \times (\tau, \tau+4\theta))} v^-,$$

where $m = \sup_{U \times (\tau, \tau+4\theta)} c$.

If v is a solution of (2.7)*, then the conclusion holds with $p = \infty$ and κ, κ_1 are independent of ε.*

2.4 Regularity of solutions

To show that Theorem 2.3 implies Theorem 2.1 we need to show that under the assumptions of Theorem 2.1 the condition (2.4) holds. In this regard there is the following result due to Poláčik.

Proposition 2.8 (Particular case of Proposition 2.7 in [49])**.** *Let $B \subset \mathbb{R}^N$ be a ball or an annulus and let (F2) and (F3) hold. Further, assume that $(0,0,0) \in \mathcal{B}$ and that the function $F(\cdot, \cdot, 0, 0, 0)$ is bounded on $[0, \infty) \times B$.*

Then for every solution $u \in C^{2,1}(B \times (0,\infty)) \cap C(\overline{B} \times [0,\infty))$ *of (2.3) satisfying (U2), we have that*

$$\sup_{\substack{x,\bar{x}\in\overline{B},t,\bar{t}\in[s,s+1],\\ x\neq\bar{x},t\neq\bar{t},\\ s\geq 1}} \frac{|u(x,t)-u(\bar{x},\bar{t})|}{|x-\bar{x}|^{\alpha}+|t-\bar{t}|^{\frac{\alpha}{2}}} < \infty \qquad \textit{for some } \alpha > 0.$$

Indeed, being a ball or an annulus, B is smoothly bounded and therefore satisfies assumption (A) of [49, Proposition 2.7].

Although the previous Proposition suffices for our purposes in this chapter, we use instead the following lemma, which is more precise with respect to the dependence of the constants and it also includes the case of problems with Neumann boundary conditions. This will be needed in the following chapters. The proof follows standard techniques as used in [49, Proposition 2.7]. Recall the definition of the Hölder spaces for uniformly continuous functions given in (5). In the following ν denotes the outward normal vector

Lemma 2.9. *Let* $\Omega \subset \mathbb{R}^N$ *be a smooth bounded domain,* $I \subset \mathbb{R}$ *open,* $\mu \in C^1(\Omega \times I)$*,* $g \in L^{\infty}(\Omega \times I)$*, and let* $v \in C^{2,1}(\overline{\Omega} \times I) \cap C(\overline{\Omega \times I})$ *be a classical solution of*

$$v_t - \mu(x,t)\Delta v = g(x,t) \qquad \textit{in } \Omega \times I, \tag{2.10}$$

satisfying

$$v = 0 \quad \textit{on } \partial\Omega \times I \qquad \textit{or} \qquad \partial_{\nu} v = 0 \quad \textit{on } \partial\Omega \times I.$$

Suppose moreover that

$$\inf_{\Omega\times I} \mu(x,t) > \frac{1}{K},$$
$$\|v\|_{L^{\infty}(\Omega\times I)} + \|\mu\|_{C^1(\Omega\times I)} + \|g\|_{L^{\infty}(\Omega\times I)} < K.$$

for some $K > 0$*. Let* $\mathcal{I} \subset I$ *with* $\operatorname{dist}(\mathcal{I}, \partial I) \geq 1$*. Then there are constants* $C > 0$ *and* $\gamma \in (0,1)$*, depending only on* Ω *and* K *such that*

$$\|v\|_{C^{1+\gamma,(1+\gamma)/2}(\overline{\Omega}\times[s,s+1])} \leq C \qquad \textit{for all } s \in \mathcal{I}.$$

In particular, if $I = (0,\infty)$*, then for* $h \in \{v, v_{x_j} : j \in \{1,\ldots,N\}\}$*, we have that*

$$\sup_{\substack{x,\bar{x}\in\overline{\Omega},\, t,\bar{t}\in[s,s+1],\\ x\neq\bar{x},\, t\neq\bar{t},\, s\geq 1}} \frac{|h(x,t)-h(\bar{x},\bar{t})|}{|x-\bar{x}|^{\gamma}+|t-\bar{t}|^{\frac{\gamma}{2}}} < C, \tag{2.11}$$

and the semiorbit $\{v(\cdot,t) \,:\, t \geq 1\}$ *is relatively compact in* $C(\overline{\Omega})$*. Therefore* $\omega(v)$ *is a nonempty connected compact subset of* $C(\overline{\Omega})$ *satisfying*

$$\lim_{t\to\infty} \inf_{z\in\omega(v)} \|v(\cdot,t) - z\|_{L^\infty(\Omega)} = 0.$$

Proof. Fix $s \in \mathcal{I}$ and set

$$Q := \overline{\Omega} \times [s, s+1] \;\subset\; \overline{\Omega \times I}.$$

Then $v \in W^{2,1}_{N+3}(Q)$ because $v \in C^{2,1}(Q)$. Further, since $\|\mu\|_{C^1(\Omega\times I)} < K$ we have that

$$|\mu(x,t) - \mu(y,s)| \leq K|(x,t) - (y,s)| \quad \text{for all } (x,t),(y,s) \in Q.$$

Then all the assumptions of [38, Theorem 7.30, p.181] are satisfied if $v = 0$ on $\partial B \times (0,\infty)$, and all the assumptions of [38, Theorem 7.35, p.185] are satisfied if $\partial_\nu v = 0$ on $\partial B \times (0,\infty)$. In both cases, the respective theorem implies the existence of a constant $C_1(\Omega, K) = C_1 > 0$ such that

$$\|D^2 v\|_{L^{N+3}(Q)} + \|v_t\|_{L^{N+3}(Q)} \leq C_1(\|g\|_{L^{N+3}(Q)} + \|u\|_{L^{N+3}(Q)}) \leq 2C_1|\Omega|K.$$

Then, by a standard interpolation argument,

$$\|v\|_{W^{2,1}_{N+3}(Q)} \leq C_2$$

for some constant $C_2(\Omega, K) = C_2 > 0$.

By Sobolev embeddings (see, for example, [52, embedding (1.2)] or [63, Corollary 1.4.1] and the references therein), we then have that

$$v \in C^{1+\gamma,(1+\gamma)/2}(Q) \qquad \text{for } \gamma = 1 - \frac{N+2}{N+3} \;\in\; (0,1),$$

and there is a constant $C_3(\Omega) = C_3 > 0$ such that

$$\|v\|_{C^{1+\gamma,(1+\gamma)/2}(Q)} \leq C_3\|v\|_{W^{2,1}_{N+3}(Q)} \leq C_3 C_2.$$

This implies (2.11) if $I = (0,\infty)$, and the last claim follows from the Arzelà-Ascoli Theorem (see [52, Proposition 53.3]). □

2.5 Main result for scalar Dirichlet problems

Let u be a solution of (2.3) satisfying the assumptions of Theorem 2.3. We use the definitions introduced in Section 2.2. Moreover, for every $z \in \omega(u)$, let $z^e : B(e) \to \mathbb{R}$ be given by

$$z^e(x) := z(x) - z(\sigma_e(x)).$$

Set

$$\mathcal{M} := \{e \in \mathbb{S}^{N-1} \mid z^e(x) \geq 0 \text{ for all } x \in B(e) \text{ and } z \in \omega(u)\}.$$

The parabolic rotating plane method relies on the following two Lemmas.

Lemma 2.10. *Let $e \in \mathbb{S}^{N-1}$ be as in (U1). There is some $\varepsilon > 0$ such that $e' \in \mathcal{M}$ for all $e' \in \mathbb{S}^{N-1}$ with $|e' - e| < \varepsilon$.*

Proof. If $e \in \mathbb{S}^{N-1}$ is as in (U1), then it follows from (2.5) and the parabolic strong maximum principle (see for example [51]) that

$$u^e(x,t) > 0 \quad \text{ in } B(e) \times (0, \infty), \tag{2.12}$$

and therefore $e \in \mathcal{M}$. Let $\delta > 0$ be chosen as in Theorem 2.6 corresponding to $\Omega = B$, $\rho := \frac{1}{4}\operatorname{inrad}(B)$, and α_0, β_0 as in (2.6). Moreover, let $D \subset\subset B(e)$ be a subdomain such that $|B(e)\backslash D| < \delta$ and $\operatorname{inrad}(D) > \rho$. Put $d := \frac{1}{2}\operatorname{dist}(\overline{D}, \partial B(e))$, $\theta := 1$, and let $\mu \in (0,1]$ be as in Theorem 2.6 corresponding to these choices of Ω, α_0, β_0, d, θ, and ρ. By (2.12) there exists some $\eta > 0$ such that

$$u^e > \eta \qquad \text{in } \overline{D} \times [1, 9].$$

Moreover, there is some $\varepsilon > 0$ such that, for all $e' \in \mathbb{S}^{N-1}$ with $|e - e'| < \varepsilon$, $D \subset\subset B(e')$, $|B(e')\backslash D| < \delta$, $\operatorname{dist}(\overline{D}, \partial B(e')) > d$, $u^{e'} > \eta$ in $\overline{D} \times [1, 9]$, and

$$\|(u^{e'})^-(\cdot, 1)\|_{L^\infty(B(e'))} \leq \mu\eta \leq \mu\|u^{e'}\|_{L^\infty(D\times[2,3])},$$

by the continuity of u and (2.12).

Hence, for these $e' \in \mathbb{S}^{N-1}$, the hypothesis of Theorem 2.6 are satisfied with $U = B(e')$, $\tau = 1$, $\theta = 1$, and D as above. We therefore get that

$$\|(u^{e'})^-(\cdot, t)\|_{L^\infty(B(e'))} \to 0, \text{ as } t \to \infty.$$

This shows $e' \in \mathcal{M}$ for $e' \in \mathbb{S}^{N-1}$ with $|e - e'| < \varepsilon$, as claimed. □

Lemma 2.11. *Let $e \in \mathcal{M}$. If there is some $\tilde{z} \in \omega(u)$ such that $\tilde{z}^e \not\equiv 0$, then there is some $\varepsilon > 0$ such that $e' \in \mathcal{M}$ for all $e' \in \mathbb{S}^{N-1}$ with $|e - e'| < \varepsilon$.*

Proof. Since $\tilde{z}^e \not\equiv 0$ there is some $\alpha > 0$ and $x_0 \in B(e)$ such that $\tilde{z}^e(x_0) \geq 2\alpha > 0$. Let $\delta > 0$ be chosen as in Theorem 2.6 corresponding to $\Omega = B$, $\rho := \frac{1}{4}\operatorname{inrad}(B)$, and α_0, β_0 as in (2.6). Moreover, let $D \subset\subset B(e)$ be a subdomain such that $|B(e)\backslash D| < \delta$, $\operatorname{inrad}(D) > \rho$ and $x_0 \in D$. Put $d := \frac{1}{2}\operatorname{dist}(\overline{D}, \partial B(e))$, $\theta := \frac{1}{8}$, and let $\mu \in (0,1]$ be as in Theorem 2.6 (corresponding to these choices of $\Omega, \alpha_0, \beta_0, d, \theta$ and ρ). Since $z^e \geq 0$ in

$B(e)$ for all $z \in \omega(u)$ and $\operatorname{dist}(u(\cdot,t),\omega(u)) \to 0$ in $C_0(B)$ as $t \to \infty$ by Remark 2.4, there is some $T_0 > 0$ such that

$$\|(u^e)^-(\cdot,t)\|_{L^\infty(B(e))} < \frac{\mu\kappa\alpha}{8}e^{-4\beta_0} \qquad \text{for } t \geq T_0, \tag{2.13}$$

where $\kappa > 0$ is the constant given by Lemma 2.5 for $\Omega, \alpha_0, \beta_0, d$ as above and $\theta = 1$. Next, we may take $T_1 \geq T_0 + 1$ such that $\|u^e(\cdot,T_1) - \tilde{z}^e\|_{L^\infty(B(e))} < \alpha$ and therefore $u^e(x_0,T_1) > \alpha$. We then apply Lemma 2.5 to $U = B(e)$, $\tau := T_1 + 2$ and $\theta = 1$ in order to get

$$\begin{aligned}\inf_{D\times(\tau,\tau+1)} u^e &\geq \kappa\|(u^e)^+\|_{L^\infty(D\times(\tau-2,\tau-1))} - e^{4\beta_0} \sup_{\partial_P(B(e)\times(\tau-3,\tau+1))} (u^e)^- \\ &\geq \kappa\alpha - \frac{\mu\kappa\alpha}{8} \geq \frac{\kappa\alpha}{2} =: \eta > 0.\end{aligned}$$

Moreover, by continuity, there is some $\varepsilon > 0$ such that

$$D \subset\subset B(e'), \qquad |B(e')\backslash D| < \delta, \qquad \operatorname{dist}(\overline{D}, \partial B(e')) > d$$

and

$$\inf_{D\times(\tau,\tau+1)} u^{e'} \geq \frac{\eta}{2}$$
$$\|(u^{e'})^-(\cdot,\tau)\|_{L^\infty(B(e'))} \leq \|(u^e)^-(\cdot,\tau)\|_{L^\infty(B(e))} + \frac{\eta\mu}{4}.$$

for all $e' \in \mathbb{S}^{N-1}$ with $|e - e'| < \varepsilon$.

Combining this with (2.13), we find that

$$\begin{aligned}\|(u^{e'})^-(\cdot,\tau)\|_{L^\infty(B(e'))} &\leq \frac{\eta\mu}{4} + \frac{\mu\kappa\alpha}{8}e^{-4\beta_0} \\ &\leq \frac{\mu\eta}{2} \leq \mu\|(u^{e'})^+\|_{L^\infty(D\times(\tau+\frac{1}{8},\tau+\frac{1}{4}))}\end{aligned}$$

for every $e' \in \mathbb{S}^{N-1}$ with $|e - e'| < \varepsilon$. In particular, for these $e' \in \mathbb{S}^{N-1}$ the hypothesis of Theorem 2.6 are satisfied with $U = B(e')$ and $\theta = \frac{1}{8}$. Therefore

$$\|(u^{e'})^-(x,t)\|_{L^\infty(B(e'))} \to 0 \quad \text{as } t \to \infty,$$

which yields $e' \in \mathcal{M}$ for all $e' \in \mathbb{S}^{N-1}$ with $|e - e'| < \varepsilon$, as claimed. □

We are now ready to prove the main symmetry result of this chapter. We give two proofs, the first one is shorter, but the second one might provide a better insight on the rotating plane method since it is slightly more constructive.

Proof of Theorem 2.3. By Lemma 2.10 we have that $\mathcal{N} := \mathcal{M}^\circ \neq \emptyset$, where $\mathcal{M}^\circ$ denotes the relative interior of $\mathcal{M}$ in $\mathbb{S}^{N-1}$. Furthermore, Lemma 2.11 implies that $z^e \equiv 0$ in $B(e)$ for all $z \in \omega(u)$ and $e \in \partial\mathcal{N}$, where $\partial\mathcal{N}$ denotes the relative boundary of $\mathcal{N}$ in $\mathbb{S}^{N-1}$. The result follows from Corollary 1.6 applied to $\mathcal{U} = \omega(u)$. □

Alternative proof of Theorem 2.3. Let $e \in \mathbb{S}^{N-1}$ be as in (U1). Then, by Lemma 2.10, there is $\varepsilon > 0$ such that

$$e' \in \mathcal{M} \qquad \text{for all } e' \in \mathbb{S}^{N-1} \text{ with } |e' - e| < \varepsilon. \tag{2.14}$$

Let P be any two dimensional subspace of $\mathbb{R}^N$ containing e. Without loss of generality, we may assume that $e = (1, 0, \dots, 0)$ and

$$P = \{x = (x_1, 0, ..., 0, x_N) \mid x_1, x_N \in \mathbb{R}\}.$$

Define

$$e_\theta := (\cos\theta\,,\, 0\,,\, \dots\,,\, 0\,,\, \sin\theta\,) \quad \text{and} \quad z_\theta := z_{e_\theta} \in C_0(B(e_\theta))$$

for $\theta \in \mathbb{R}$ and

$$\Theta_1 := \sup\{\theta > 0 \;:\; e_\phi \in \mathcal{M} \text{ for all } 0 \le \phi \le \theta\},$$
$$\Theta_2 := \inf\{\theta < 0 \;:\; e_\phi \in \mathcal{M} \text{ for all } \theta \le \phi \le 0\}.$$

We note that $\Theta_2 < 0 < \Theta_1$ by (2.14). If $\Theta_1 - \Theta_2 \ge 2\pi$ (and in particular if $\Theta_1 = \infty$ or $\Theta_2 = -\infty$), it immediately follows from the definition of $\mathcal{M}$ that every $H(e_\theta)$, $\theta \in \mathbb{R}$, is a symmetry hyperplane for all elements of $\omega(u)$. If both Θ_1 and Θ_2 are finite and $\Theta_1 - \Theta_2 < 2\pi$, we have $z_{\Theta_1} \equiv z_{\Theta_2} \equiv 0$ for all $z \in \omega(u)$ as a consequence of Lemma 2.11, so that $H(e_{\Theta_1})$ and $H(e_{\Theta_2})$ are symmetry hyperplanes for all elements of $\omega(u)$. Moreover, $e_{\Theta_1} \neq e_{\Theta_2}$ and $e_\varphi \in \mathcal{M}$ for all $\varphi \in (\Theta_2, \Theta_1)$. Since this can be done for all two dimensional subspaces P of $\mathbb{R}^N$ containing e, we can use Proposition 1.5, applied to $\mathcal{U} = \omega(u)$, to obtain the existence of $p \in \mathbb{S}^{N-1}$ such that every $z \in \omega(u)$ is foliated Schwarz symmetric with respect to p, as claimed. □

Proof of Theorem 2.1. Let $u \in C^{2,1}(\overline{B} \times (0, \infty)) \cap C(\overline{B} \times [0, \infty))$ be a solution of (2.1) such that all the assumptions in Theorem 2.1 hold. We show how Theorem 2.3 implies Theorem 2.1.

Let $K := \|u\|_{L^\infty(B\times(0,\infty))}$. Note that u satisfies (2.10) with $\mu \equiv 1$ and

$$g(x,t) := f(t, x, u(x,t)) \le L\,\|u_1\|_{L^\infty(B\times(0,\infty))} + \|f(\cdot,\cdot,0)\|_{L^\infty(B\times(0,\infty))} < \infty$$

for all $x \in B$ and $t > 0$ by (f1), (f2), and (U2), where $L(K) = L$ is the constant given in (f1). Then, by Lemma 2.9, we have that u satisfies (2.11) and therefore (2.4).

On the other hand, we may consider (2.1) as a special case of (2.3) with $\mathcal{B} = (-K-1, K+1) \times \mathbb{R}^N \times \mathbb{R}^{N^2}$ and

$$F : [0,\infty) \times \overline{B} \times \mathcal{B} \to \mathbb{R}, \qquad F(t,x,u,p,q) = \operatorname{trace}(q) + f(t,|x|,u).$$

With this definition, assumptions (F1) and (F3) are obviously satisfied and (F2) follows from assumption (f1). Hence the assumptions of Theorem 2.1 imply those of Theorem 2.3 and this ends the proof. $\square$

Chapter 3

Parabolic equations with Neumann boundary conditions

In this chapter we turn our attention to nonlinear parabolic equations with Neumann boundary conditions. The goal of this Chapter is to prove the following

Theorem 3.1. *Let* $u \in C^{2,1}(\overline{B}\times(0,\infty))\cap(\overline{B}\times[0,\infty))$ *be a classical solution of*

$$\begin{aligned} u_t - \mu(|x|,t)\Delta u &= f(t,|x|,u), && x \in B,\ t>0, \\ \partial_\nu u &= 0, && x \in \partial B,\ t>0, \\ u(x,0) &= u_0(x), && x \in B, \end{aligned} \tag{3.1}$$

where

- *(μ) the diffusion coefficient* $\mu \in C^1(I_B \times (0,\infty))$ *is such that there are constants* $\mu^* \geq \mu_* > 0$ *with* $\|\mu_i\|_{C^1(I_B\times(0,\infty))} \leq \mu^*$ *and* $\mu_i(r,t) \geq \mu_*$ *for all* $r \in I_B$ *and* $t > 0$.

Further, assume that the hypothesis (f1), (f2), (U1), and (U2) from Theorem 2.1 are satisfied. Then there is some $p \in \mathbb{S}^{N-1}$ *such that all elements of* $\omega(u)$ *are foliated Schwarz symmetric with respect to* p.

We stress that this result holds for positive and for nodal solutions of (3.1). Analogously as in Corollary 2.2, Theorem 3.1 yields immediate corollaries for the elliptic and for the time periodic versions of (3.1).

Although many (existence) results for (2.1) have a similar version for (3.1) (with $\mu \equiv 1$), it is clear that the Neumann boundary conditions are much less restrictive than the Dirichlet counterpart, and the symmetries of the solution are coerced to a lesser degree by the influence of the boundary. To have an idea of the symmetry properties of solutions of (3.1), let us consider the following elliptic equation.

$$\begin{aligned} -\varepsilon^2 \Delta u = u - u^p \quad & \text{in } \Omega, \\ u > 0 \quad & \text{in } \Omega, \\ \partial_\nu u = 0 \quad & \text{on } \partial\Omega. \end{aligned} \tag{3.2}$$

Here ε is a small positive constant, $1 < p < \frac{N+1}{N+2}$ ($= \infty$ if $N = 2$), and $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. Solutions to (3.2) can be regarded as stationary solutions of (3.1) with $\mu \equiv \varepsilon^2$ in the case where Ω is a ball.

If Ω is the unit ball, the first interesting remark regarding symmetry properties of (3.2) is that the result of Gidas, Ni, and Nirenberg [26] does not hold for (3.2) and therefore the usual moving plane method can not be applied. Indeed, for small values of ε many nonradial positive solutions have been constructed, in particular single-peak solutions, multipeak solutions, and also solutions concentrating on multidimensional sets —lower dimensional inner spheres for example. These results can be found in [42] and the references therein together with an overview of the qualitative properties of solutions to elliptic —and some parabolic—problems, with a special focus on the differences between Neumann and Dirichlet boundary conditions.

We just mention one more revealing fact in this regard: for a general smooth domain Ω, the least energy solutions of (3.2) are known to be single-peak solutions. The location of the concentrating points is determined by the geometry of the underlying domain *and* the boundary conditions. For instance, under Neumann boundary conditions, the maximum point is located near a maximum of the mean curvature of $\partial\Omega$, whereas, for the same problem under Dirichlet boundary conditions, the concentrating point is near the most distant point to $\partial\Omega$. We refer to [42] and the references therein for the formal statements. This suggest that solutions of (3.1) tend to concentrate near the boundary rather than near the center, and therefore foliated Schwarz symmetry is a natural partial symmetry to be expected in this setting. As a matter of fact, the least-energy solution of (3.2) in a ball is foliated Schwarz symmetric, and this was proved in [39] using a version of the elliptic rotating plane method.

As in the previous chapter, the strategy to prove Theorem 3.1 is to use a parabolic rotating plane argument. However, as the above discussion suggests, the arguments used for the Dirichlet problem can not be easily extended to the present case. Indeed, Theorem 2.1 relies strongly on maximum principles for small domains (and its consequences, e.g. Theorem 2.6), which are only available under Dirichlet boundary conditions, and therefore a different approach is needed for problem (3.1). In this chapter, we develop new tools to study the symmetries of (3.1) exploiting the nature of the Neumann boundary conditions. In particular, we use the fact that it is possible to extend solutions to second order Neumann problems on B to a larger domain via inversion at the boundary. The extended solution satisfies an equation in the strong sense, and therefore it is possible to use Harnack inequalities and maximum principles to obtain information up to the boundary. Specifically, we prove a quantitative Harnack-Hopf type principle, Lemma 3.4 below, which yields information up to the non-smooth part of the boundary of cylinders over half balls and half annuli. With Lemma 3.4 we are able to show a stability property of reflection inequalities with respect to small perturbations of a hyperplane, see Lemma 3.5 below.

We give a brief outline of this chapter. We start with the details of the extension argument in Section 3.1. Next, in Section 3.2, we derive a quantitative Harnack-Hopf type lemma for equations in cylinders over half balls and half annuli under mixed boundary conditions and then use it to prove our main perturbation result in this setting, Lemma 3.5. Section 3.3 contains the proof of Theorem 3.1. Finally, in Section 3.7 we include a brief discussion on equations with angular derivative terms and the possibility of a varying axis of asymptotic symmetry.

Some of the results presented in this chapter were submitted for publication in [55].

3.1 Extension of solutions in radial domains

We now detail the extension procedure mentioned in the introduction of the chapter. First we fix some notation. Let B be a ball or an annulus in $\mathbb{R}^N$ centered at zero, and fix $0 \leq A_1 < A_2 < \infty$ such that

$$B := \begin{cases} \{x \in \mathbb{R}^N : A_1 < |x| < A_2\}, & \text{if } A_1 > 0, \\ \{x \in \mathbb{R}^N : |x| < A_2\}, & \text{if } A_1 = 0. \end{cases} \tag{3.3}$$

Note that $I_B = \{|x| \, : \, x \in \overline{B}\} = [A_1, A_2]$. Define

$$\widetilde{B} := \begin{cases} \{x \in \mathbb{R}^N : \frac{A_1^2}{A_2} < |x| < \frac{A_2^2}{A_1}\}, & \text{if } A_1 > 0, \\ \{x \in \mathbb{R}^N : |x| < 2A_2\}, & \text{if } A_1 = 0, \end{cases} \tag{3.4}$$

and for $x \in \widetilde{B} \setminus B$ we put

$$\hat{x} := \begin{cases} \dfrac{A_2^2}{|x|^2} x, & \text{if } |x| \geq A_2, \\ \dfrac{A_1^2}{|x|^2} x, & \text{if } |x| \leq A_1. \end{cases}$$

Lemma 3.2. *Let $I \subset \mathbb{R}$ be an open interval, let $\mu, g : B \times I \to \mathbb{R}$ be given functions, and let $u \in C^{2,1}(\overline{B} \times I) \cap C(\overline{B} \times \overline{I})$ be a solution of*

$$\begin{cases} u_t - \mu(x,t)\Delta u = g(x,t) & \text{in } B \times I, \\ \partial_\nu u = 0 & \text{on } \partial B \times I. \end{cases}$$

Then the function

$$\tilde{u} : \widetilde{B} \to \mathbb{R}, \qquad \tilde{u}(x,t) := \begin{cases} u(x,t), & x \in \overline{B},\ t \in I, \\ u(\hat{x},t), & x \in \widetilde{B} \setminus \overline{B},\ t \in I, \end{cases} \tag{3.5}$$

satisfies that $\tilde{u} \in W^{2,1}_{p,loc}(\widetilde{B} \times I) \cap C^{1,0}(\widetilde{B} \times I)$ for any $p \geq 1$ and it is a strong solution of the equation

$$\tilde{u}_t - \tilde{\mu}(x,t)\Delta\tilde{u} - \tilde{b}(x,t)\partial_r \tilde{u} = \tilde{g}(x,t) \qquad \text{in } \widetilde{B} \times I, \tag{3.6}$$

where $\partial_r = \frac{1}{|x|}\sum_{j=1}^{N} x_j \partial_j$ is the radial derivative and

$$\tilde{\mu}(x,t) := \begin{cases} \mu(x,t), & x \in B,\ t \in I, \\ \frac{|x|^2}{|\hat{x}|^2}\mu(\hat{x},t), & x \in \widetilde{B} \setminus B,\ t \in I, \end{cases}$$

$$\tilde{b}(x,t) := \begin{cases} 0, & x \in B,\ t \in I \\ \frac{(4-2N)|x|}{|\hat{x}|^2}\mu(\hat{x},t), & x \in \widetilde{B} \setminus B,\ t \in I, \end{cases}$$

$$\tilde{g}(x,t) := \begin{cases} g(x,t), & x \in B,\ t \in I, \\ g(\hat{x},t), & x \in \widetilde{B} \setminus B,\ t \in I. \end{cases}$$

Proof. The proof is by direct calculation. Let $t \in I$ and $x \in \widetilde{B}$ with $|x| \geq A_2$. To simplify the presentation we assume $A_1 = 0$ and we will often omit the argument $(\hat{x}, t)$. Then,

$$\begin{aligned} \nabla\tilde{u}(x,t) &= (\nabla u)\left(\frac{\delta_{ij}|x|^2 - 2x_i x_j}{|x|^4}\right)_{i,j=1}^{N} A_2^2 \\ &= \left(\frac{\partial_i u}{|x|^2} - \frac{2x_i}{|x|^4}\sum_{k=1}^{N} x_k \partial_k u\right)_{i=1}^{N} A_2^2. \end{aligned} \tag{3.7}$$

It follows, by the Neumann boundary conditions, that

$$\nabla\tilde{u}(x,t) = \nabla u(x,t) \text{ for all } x \in \overline{B}, t > 0. \tag{3.8}$$

Indeed, for $x \in B$ this follows from the definition of $\tilde{u}$, and if $x \in \partial B$, then $|x| = A_2$ and the outward normal vector is $\nu(x) = x/A_2$, thus by (3.7) we have that

$$\nabla\tilde{u}(x,t) = \nabla u(x,t)(\delta_{ij} - \nu_i\nu_j)_{i,j=1}^N = \nabla u(x,t) - \partial_\nu u(x,t)\nu = \nabla u(x,t).$$

Now we calculate the second derivatives.

$$\begin{aligned}(\partial_j(\partial_i u(\hat{x},t)))_{i,j=1}^N &= (\partial_{ji}u)_{i,j=1}^N \left(\frac{\delta_{ij}|x|^2 - 2x_ix_j}{|x|^4}\right)_{i,j=1}^N A_2^2 \\ &= \left(\frac{\partial_{ji}u}{|x|^2} - \frac{2x_j}{|x|^4}\sum_{k=1}^N x_k\partial_{ki}u\right)_{i,j=1}^N A_2^2.\end{aligned}$$

$$D^2\tilde{u}(x,t) = (\partial_{ji}(u(\hat{x},t)))_{i,j=1}^N = \left(\partial_j\left(\frac{\partial_i u}{|x|^2} - \frac{2x_i}{|x|^4}\sum_{k=1}^N x_k\partial_k u\right)\right)_{i,j=1}^N A_2^2$$

$$= A_2^2\left(\frac{\partial_j(\partial_i u)|x|^2 - \partial_i u(2x_j)}{|x|^4}\right.$$

$$\left. - 2\frac{\delta_{ij}\sum_{k=1}^N x_k\partial_k u + x_i\sum_{k=1}^N[\delta_{kj}\partial_k u + x_k\partial_j(\partial_k u)]}{|x|^4} + 8\frac{x_ix_j\sum_{k=1}^N x_k\partial_k u}{|x|^6}\right)_{i,j=1}^N$$

$$= A_2^2\left(\left(\frac{\partial_{ji}u}{|x|^4} - \frac{2x_j}{|x|^6}\sum_{k=1}^N x_k\partial_{ki}u\right)A_2^2 - \frac{2x_j\partial_i u}{|x|^4} + \frac{-2\delta_{ij}\sum_{k=1}^N x_k\partial_k u}{|x^4|} + \frac{-2x_i\partial_j u}{|x|^4}\right.$$

$$\left. + \left(\frac{-2x_i}{|x|^6}\sum_{k=1}^N x_k\partial_{jk}u + \frac{4x_jx_i}{|x|^8}\sum_{k=1}^N\sum_{l=1}^N x_kx_l\partial_{lk}u\right)A_2^2 + \frac{8x_ix_j\sum_{k=1}^N x_k\partial_k u}{|x|^6}\right)_{i,j=1}^N.$$

Then, the formula for the Laplacian is the following.

$$\Delta\tilde{u}(x,t) = \sum_{i=1}^{N} \partial_{ii}\tilde{u}(x,t) =$$
$$= \left(\frac{\Delta u}{|x|^4} - \frac{2\sum_{i=1}^{N}\sum_{k=1}^{N} x_i x_k \partial_{ki} u}{|x|^6} - \frac{2\sum_{i=1}^{N}\sum_{k=1}^{N} x_i x_k \partial_{ik} u}{|x|^6} + \frac{4\sum_{k=1}^{N}\sum_{l=1}^{N} x_k x_l \partial_{lk} u}{|x|^6} \right) A_2^4$$
$$+ \left(\frac{-2\sum_{i=1}^{N} x_i \partial_i u}{|x|^4} + \frac{-2N\sum_{k=1}^{N} x_k \partial_k u}{|x^4|} + \frac{-2\sum_{i=1}^{N} x_i \partial_i u}{|x|^4} + \frac{8\sum_{k=1}^{N} x_k \partial_k u}{|x|^4} \right) A_2^2$$
$$= \frac{A_2^4}{|x|^4} \left(\Delta u + \frac{4-2N}{A_2^2} (\nabla u \cdot x) \right).$$

Also, note that

$$\frac{4-2N}{|\hat{x}|^2} (\nabla(\tilde{u}(x,t)) \cdot x) = \frac{4-2N}{A_2^4} |x|^2 \sum_{i=1}^{N} x_i \left(\frac{\partial_i u}{|x|^2} - \frac{2x_i}{|x|^4} \sum_{k=1}^{N} x_k \partial_k u \right) A_2^2$$
$$= \frac{2N-4}{A_2^2} (\nabla u \cdot x),$$

which motivates the definition of the coefficient $\tilde{b}$ in the statement of the Lemma. If $A_1 > 0$, one can make similar calculations for $x \in \widetilde{B}$ with $|x| < A_1$. Then, it is easy to check that (3.6) holds for almost every $x \in \widetilde{B}$ and $t \in I$.

By (3.8) we have $\tilde{u} \in C^{1,0}(\widetilde{B} \times I)$. Now, fix $p \geq 1$. By assumption, $\|u\|_{W_p^{2,1}(B\times J)} < \infty$ for any subinterval $J \subset\subset I$. As calculated above, the map $x \mapsto \hat{x}$ has uniformly bounded first and second derivatives in $\tilde{B} \setminus B$, and therefore $\|\tilde{u}\|_{W_p^{2,1}(\tilde{B}\times J)} < \infty$. Since (3.6) holds almost everywhere in $\widetilde{B} \times I$ we find that $\tilde{u} \in W_{p,loc}^{2,1}(\widetilde{B} \times I)$ is a strong solution of (3.6). □

Remark 3.3. A similar extension property is valid in half balls and half annuli under mixed boundary conditions. More precisely, let $B_+ := \{x \in \overline{B} : x_N > 0\}$, $I \subset \mathbb{R}$ be an open interval, let $\mu, g : B_+ \times I \to R$ given functions and let $u \in C^{2,1}(\overline{B_+} \times I) \cap C(\overline{B_+ \times I})$ be a solution of

$$u_t - \mu(x,t)\Delta u = g(x,t) \qquad \text{in } B_+ \times I,$$

satisfying $u = 0$ on $\Sigma_1 \times I$ and $\partial_\nu u = 0$ on $\Sigma_2 \times I$, where

$$\Sigma_1 := \{x \in \partial B_+ : x_N = 0\}, \qquad \Sigma_2 := \{x \in \partial B_+ : x_N > 0\}. \tag{3.9}$$

Let $\widetilde{B_+} := \{x \in \widetilde{B} \,:\, x_N > 0\}$ and define $\tilde{u} : \widetilde{B_+} \to \mathbb{R}$ by (3.5) for $x \in \widetilde{B_+}$. Then $\tilde{u} \in W^{2,1}_{p,loc}(\widetilde{B_+} \times I) \cap C^{1,0}(\widetilde{B_+} \times \overline{I})$ for any $p > N+2$ and it is a strong solution of (3.6) in $\widetilde{B_+}$ with coefficients defined analogously as in Lemma 3.2.

3.2 A Harnack-Hopf type lemma and related estimates

The first result of this section is an estimate related to a linear parabolic boundary value problem on a (parabolic) half cylinder. The estimate can be seen as an extension of both the Harnack inequality and the Hopf lemma since it also gives information on a "tangential"derivative at corner points. A somewhat related (but significantly weaker) result for supersolutions of the Laplace equation was given in [29, Lemma A.1].

Lemma 3.4. *Let $a, b \in \mathbb{R}$, $a < b$, $I := (a,b)$, $B_+ := \{x \in \overline{B} \,:\, x_N > 0\}$. Suppose that $v \in C^{2,1}(\overline{B_+} \times I) \cap C(\overline{B_+ \times I})$ satisfies*

$$\begin{aligned} v_t - \mu\Delta v - cv &\geq 0 && \text{in } B_+^\circ \times I,\\ \frac{\partial v}{\partial \nu} &= 0 && \text{on } \Sigma_2 \times I,\\ v &= 0 && \text{on } \Sigma_1 \times I,\\ v(x,a) &\geq 0 && \text{for } x \in B_+, \end{aligned}$$

where the sets Σ_i are given in (3.9) and the coefficients satisfy

$$\frac{1}{M} \leq \mu(x,t) \leq M \quad \text{and} \quad |c(x,t)| \leq M \qquad \text{for } (x,t) \in B_+ \times I$$

with some positive constant $M > 0$. Then $v \geq 0$ in $B_+ \times (a,b)$. Moreover, if $v(\cdot, a) \not\equiv 0$ in B_+, then

$$v > 0 \text{ in } B_+ \times I \qquad \text{and} \qquad \frac{\partial v}{\partial e_N} > 0 \text{ on } \Sigma_1 \times I. \tag{3.10}$$

Furthermore, for every $\delta_1 > 0$, $\delta_2 \in (0, \frac{b-a}{4}]$, there exist $\kappa > 0$ and $p > 0$ depending only on δ_1, δ_2, B, and M such that

$$v(x,t) \geq x_N\, \kappa \left(\int_{Q(\delta_1,\delta_2)} v^p \; d(x,t) \right)^{\frac{1}{p}} \tag{3.11}$$

for all $x \in B_+$ and $t \in [a+3\delta_2\,,\, a+4\delta_2]$. where

$$Q(\delta_1,\delta_2) := \{(x,t) \,:\, x \in B_+,\, x_N \geq \delta_1,\, a+\delta_2 \leq t \leq a+2\delta_2\}.$$

Proof. We begin by showing that $v \geq 0$ in $B_+ \times I$. Let $\varepsilon > 0$ and define

$$\overline{B_+ \times I} \to \mathbb{R}; \quad (x,t) \mapsto \varphi(x,t) := e^{-Mt} v(x,t) + \varepsilon.$$

Then

$$\begin{aligned} \varphi_t - \mu\Delta\varphi - (c-M)\varphi = -\varepsilon(c-M) &\geq 0, && x \in B_+^\circ,\ t \in I,\\ \frac{\partial\varphi}{\partial\nu}(x,t) &= 0, && x \in \Sigma_2,\ t \in I,\\ \varphi(x,t) &= \varepsilon, && x \in \Sigma_1,\ t \in I,\\ \varphi(x,t) \geq \frac{\varepsilon}{2} &> 0, && x \in B_+,\ t \in [a, a+s) \end{aligned}$$

for some $s \in (0, b-a)$, since the function $(x,t) \mapsto e^{-Mt}v(x,t)$ is continuous in $\overline{B_+} \times [a,b)$. By the maximum principle (see for example [38, Theorem 2.7] or [51, Theorem 5, Chapter 3]) the minimum of u is attained at the parabolic boundary $\partial_P(B_+ \times I) = (\Sigma_1 \times \overline{I}) \cup (\Sigma_2 \times \overline{I}) \cup (B_+ \times \{a\})$. By the boundary point lemma (see for example [38, Lemma 2.8]) the minimum of φ can not be achieved in $\Sigma_2 \times [a+s, b]$ because of the Neumann boundary conditions. Since $\varphi \geq 0$ in the rest of the parabolic boundary we get that $\varphi \geq 0$ in $B_+ \times I$, that is $v(x,t) \geq -\varepsilon e^{Mt}$ in $B_+ \times I$. Letting $\varepsilon \to 0$ we have that $v \geq 0$ in $B_+ \times I$. Moreover, if $v(\cdot, a) \not\equiv 0$ in B_+, then the first claim in (3.10) follows similarly by the strong maximum principle and the boundary point lemma.

Next we note that the second claim in (3.10) is a consequence of the first claim and the inequality (3.11) (for suitably chosen δ_1, δ_2). It thus remains to prove (3.11). Let $\delta_1 > 0$, $\delta_2 \in (0, \frac{b-a}{4}]$ and consider $\widetilde{B}, \widetilde{B_+}$ as defined in (3.4) and Remark 3.3. Without loss, we may assume that

$$\delta_1 < \min\Big\{\frac{\delta_2}{2}, \frac{\operatorname{dist}(B, \partial\widetilde{B})}{3}\Big\}. \tag{3.12}$$

By Remark 3.3, there exists an extension $\tilde{v} \in W^{2,1}_{N+1,loc}(\widetilde{B_+} \times I)$ of v which satisfies $\mathcal{L}(x,t)\tilde{v} \geq 0$ in $\widetilde{B_+} \times I$ in the strong sense. Here the linear differential operator $\mathcal{L}$ is given by

$$\mathcal{L}(x,t)w := w_t - \tilde{\mu}(x,t)\Delta w - \tilde{b}(x,t)\partial_r w - \tilde{c}(x,t)w$$

with $\tilde{\mu}$, $\tilde{b}$ given as in Lemma 3.2 and

$$\tilde{c}(x,t) := \begin{cases} c(x,t), & x \in B_+,\ t \in I,\\ c(\hat{x},t), & x \in \widetilde{B}_+ \setminus B_+,\ t \in I. \end{cases}$$

Moreover, there is a positive constant β_0 which only depends on B and M such that $\tilde{\mu}$, $\tilde{b}$, and $\tilde{c}$ are uniformly bounded by β_0, and $\tilde{\mu}$ is bounded below by β_0^{-1}. Next, we define the compact sets

$$K_{\delta_1} := \{x \in B_+ \,:\, x_N \geq \frac{\delta_1}{2}\} \quad \text{and}$$
$$\widetilde{K}_{\delta_1} := \{x \in \widetilde{B}_+ \,:\, x_N \geq \frac{\delta_1}{2},\ \operatorname{dist}(x, \partial\widetilde{B}) \geq \delta_1\}$$

By Lemma 2.7 there exist $\kappa_1 > 0$ and $p > 0$, depending only on δ_1, δ_2, B, and M such that

$$\inf_{\substack{x \in \widetilde{K}_{\delta_1} \\ t \in [a+\frac{5}{2}\delta_2, a+4\delta_2]}} \tilde{v}(x,t) \geq \kappa_1 \Big(\int_{\widetilde{K}_{\delta_1} \times [a+\delta_2, a+2\delta_2]} (\tilde{v})^p \, d(x,t) \Big)^{\frac{1}{p}}$$
$$\geq \kappa_1 \Big(\int_{Q(\delta_1,\delta_2)} v^p \, d(x,t) \Big)^{\frac{1}{p}}. \tag{3.13}$$

Here we used in the last step that

$$Q(\delta_1, \delta_2) \ \subset\ K_{\delta_1} \times [a+\delta_2, a+2\delta_2] \ \subset\ \widetilde{K}_{\delta_1} \times [a+\delta_2, a+2\delta_2].$$

Next, we define

$$D := \{(x,t) \,:\, t < 0,\ x_N < \frac{\delta_1}{2},\ |x - \delta_1 e_N|^2 + t^2 < \delta_1^2\};$$
$$\Gamma_1 := \{(x,t) \,:\, t \leq 0,\ x_N < \frac{\delta_1}{2},\ |x - \delta_1 e_N|^2 + t^2 = \delta_1^2\};$$
$$\Gamma_2 := \{(x,t) \,:\, t \leq 0,\ |x - \delta_1 e_N|^2 + t^2 \leq \delta_1^2,\ x_N = \frac{\delta_1}{2}\}.$$

Note that $\Gamma_1 \cup \Gamma_2$ equals $\partial_P D$, the parabolic boundary of D. Let $x_0 \in \Sigma_1$ and $t_0 \in [a+3\delta_2, a+4\delta_2]$. By construction and (3.12), we then have

$$\{(x_0 + x, t_0 + t) \,:\, (x,t) \in D\} \subset \widetilde{B_+}^{\circ} \times [a + \frac{5}{2}\delta_2, a+4\delta_2]$$

and

$$\{(x_0 + x, t_0 + t) \,:\, (x,t) \in \Gamma_2\} \subset \widetilde{K}_{\delta_1} \times [a + \frac{5}{2}\delta_2, a+4\delta_2]. \tag{3.14}$$

Next we fix $k > 0$ such that

$$k \geq \frac{2\beta_0[\delta_1 + \beta_0 N(1+\delta_1)]}{{\delta_1}^2}.$$

Moreover, we define the function

$$z : \overline{D} \to \mathbb{R}, \qquad z(x,t) := \Big(e^{-k(|x-\delta_1 e_N|^2 + t^2)} - e^{-k{\delta_1}^2}\Big) e^{-\beta_0 t}.$$

Put also

$$\varepsilon := \frac{\min\limits_{(x,t)\in\Gamma_2} \tilde{v}(x_0+x,t_0+t)}{\max\limits_{(x,t)\in\Gamma_2} z(x,t)} > 0$$

and consider

$$w : \overline{D} \to \mathbb{R}, \qquad w(x,t) := \tilde{v}(x_0+x,t_0+t) - \varepsilon z(x,t)$$

Then $w \geq 0$ on Γ_2 and also $w \geq 0$ on Γ_1, since $z \equiv 0$ on Γ_1. Moreover, for $(x,t) \in D$ we have

$$\begin{aligned}
&\mathcal{L}(t_0+t,x_0+x)z(x,t)\\
=&[-\beta_0 - \tilde{c}(t+t_0,x_0+x)]z(x,t)\\
&+ 2k\, e^{-k(|x-\delta_1 e_N|^2+t^2)-\beta_0 t}\Big[\tilde{\mu}(t_0+t,x_0+x)(N - 2k|x-\delta_1 e_N|^2)\\
&- t - \tilde{b}(t_0+t,x_0+x)\frac{x_0+x}{|x_0+x|}\cdot(x-\delta_1 e_N)\Big]\\
\leq& 2k\, e^{-k(|x-\delta_1 e_N|^2+t^2)-\beta_0 t}\Big[\delta_1 - \frac{2k}{\beta_0}\Big(\frac{\delta_1}{2}\Big)^2 + \beta_0 N(1+\delta_1)\Big] \leq 0,
\end{aligned}$$

by the definition of k. Therefore we have

$$\mathcal{L}(t_0+t,x_0+x)w(x,t) \geq 0 \text{ for } (x,t) \in D \quad \text{and} \quad w \geq 0 \text{ on } \partial_P D = \Gamma_1 \cup \Gamma_2.$$

By the maximum principle for strong solutions, we conclude that $w \geq 0$ in $\overline{D}$ and thus in particular

$$\tilde{v}(x_0+se_N,t_0) \geq \varepsilon z(se_N,0) \geq \varepsilon_1 s \min_{(x,t)\in\Gamma_2} \tilde{v}(x_0+x,t_0+t) \qquad \text{for } s \in (0,\frac{\delta_1}{2})$$

with a constant $\varepsilon_1 \in (0, \operatorname{diam}(B)^{-1})$ depending only on the function z and B. By (3.14) and since $x_0 \in \Sigma_1$, $t_0 \in [a+3\delta_2, a+4\delta_2]$ were chosen arbitrarily, we conclude that

$$v(x,t) \;\geq\; \varepsilon_1 x_N \inf_{\substack{y\in\widetilde{K}_{\delta_1}\\ \tau\in[a+\frac{5}{2}\delta_2,a+4\delta_2]}} \tilde{v}(y,\tau) \qquad \begin{cases} \text{for } x \in B_+ \text{ with } x_N < \dfrac{\delta_1}{2}\\ \text{and } t \in [a+3\delta_2, a+4\delta_2]. \end{cases}$$

By definition of K_{δ_1} and since $0 \leq \varepsilon_1 x_N \leq 1$ for $x \in B_+$, the latter estimate holds also without the restriction $x_N < \frac{\delta_1}{2}$. Combining this fact with (3.13), we obtain that

$$v(x,t) \geq \kappa_1\varepsilon_1 x_N \bigg(\int_{Q(\delta_1,\delta_2)} v^p \; d(x,t)\bigg)^{\frac{1}{p}}$$

for all $x \in B_+$ and $t \in [a+3\delta_2\, ,\, a+4\delta_2]$, so that (3.11) holds with $\kappa := \kappa_1\varepsilon_1$. □

For the last lemma of this section, we first recall for convenience some notation. For $e \in \mathbb{S}^{N-1}$, let $H(e) := \{x \in \mathbb{R}^N : x \cdot e = 0\}$, $\sigma_e : \overline{B} \to \overline{B}$ be given by $\sigma_e(x) := x - 2(x \cdot e)e$, and $B(e) := \{x \in B : x \cdot e > 0\}$. Moreover, for a function $v : \overline{B} \times I \to \mathbb{R}$ we let $v^e : \overline{B} \times [0, \infty) \to \mathbb{R}$ be given by

$$v^e(x,t) := v(x,t) - v(\sigma_e(x), t) \qquad \text{for } x \in B(e),\ t > 0.$$

We also put

$$\begin{aligned} \Sigma_1(e) &:= \{x \in \partial B(e) : x \cdot e = 0\}, \\ \Sigma_2(e) &:= \{x \in \partial B(e) : x \cdot e > 0\}. \end{aligned} \tag{3.15}$$

To implement the rotating plane technique for the boundary value problems considered in our main results, we need to analyze under which conditions positivity of $v^e(\cdot, t)$ in $B(e)$ at some time $t \in I$ induces positivity of $v^{e'}(\cdot, t')$ in $B(e')$ for a slightly perturbed direction e' at a later time $t' > t$. The following perturbation lemma is sufficient for our purposes.

Lemma 3.5. *Let $d, k, M > 0$ be given constants and let $v \in C^{2,1}(\overline{B} \times [0,1])$ be a function satisfying the following:*

($E\chi$) *There is a function*

$$\chi : [0, \sqrt{1 + \operatorname{diam}(B)^2}\,] \to [0, \infty) \quad \textit{with} \quad \lim_{\vartheta \to 0} \chi(\vartheta) = 0$$

such that

$$|v(x,t) - v(y,s)| + |\nabla v(x,t) - \nabla v(y,s)| \le \chi(|(x,t) - (y,s)|)$$

for all $(x,t), (y,s) \in \overline{B} \times [0,1]$.

There is $e \in \mathbb{S}^{N-1}$ such that

(i) *the function v^e satisfies*

$$\begin{aligned} v^e_t - \mu \Delta v^e - c v^e &\ge 0 && \textit{in } B(e) \times (0,1), \\ \frac{\partial v^e}{\partial \nu} &= 0 && \textit{on } \Sigma_2(e) \times (0,1), \\ v^e &= 0 && \textit{on } \Sigma_1(e) \times (0,1), \\ v^e(x,0) &\ge 0 && \textit{for all } x \in B(e), \end{aligned}$$

where the coefficient functions μ and c are in $L^\infty(B(e) \times (0,1))$ and satisfy that

$$\frac{1}{M} \le \mu \le M \quad \textit{and} \quad |c| \le M \qquad \textit{in } B(e) \times (0,1),$$

(ii) $\sup\{v^e(x, \frac{1}{4}) \,:\, x \in B(e),\ x \cdot e \geq d\} \geq k,$

then there is $\rho > 0$*, depending only on* B*,* d*,* k*,* M*, and the function* χ *with*

$$v^{e'}(\cdot, 1) > 0 \quad \text{in } B(e') \qquad \text{for all } e' \in \mathbb{S}^{N-1} \text{ with } |e - e'| < \rho.$$

Remark 3.6. The result obviously remains true if v^e is replaced by $-v^e$. We will use this fact in the next chapter.

Proof. Let $e \in \mathbb{S}^{N-1}$ be such that (i) and (ii) are satisfied, and let $\kappa > 0$ and $p > 0$ be the constants given by Lemma 3.4 applied to $a = 0$, $b = 1$, $\delta_1 = d$ and $\delta_2 = \frac{1}{4}$. We first note that condition $(E\chi)$ and hypothesis (ii) imply that there exists $C_1 > 0$, depending only on B, d, k, M, and χ, such that

$$\kappa \left(\int_{Q^e} (v^e)^p \, dx \, dt \right)^{\frac{1}{p}} \geq C_1,$$

where $Q^e := \{(x, t) \,:\, x \in B(e),\ x \cdot e \geq d,\ \frac{1}{4} < t < \frac{1}{2}\}$. Then, by Lemma 3.4, it follows that

$$\nabla v^e(x, 1) \cdot e \geq C_1 \qquad \text{for all } x \in \Sigma_1(e). \tag{3.16}$$

Note that

$$\nabla v^{e'}(x, 1) = (\nabla v^{e'}(x, 1) \cdot e')e' \qquad \text{for all } e' \in \mathbb{S}^{N-1} \text{ and } x \in H(e').$$

since $v^{e'} \equiv 0$ in $H(e')$. In particular $|\nabla v^{e'}(\cdot, 1)| = |\nabla v^{e'}(\cdot, 1) \cdot e'|$ in $H(e')$. Then

$$\begin{aligned} |\nabla v^{e'}(x, 1) \cdot e'| &= |\nabla v^{e'}(x, 1)| - |\nabla v^e(x, 1)| + |\nabla v^e(x, 1) \cdot e| \\ &\geq -|\nabla v^{e'}(x, 1) - \nabla v^e(x, 1)| + \nabla v^e(x, 1) \cdot e \end{aligned}$$

for all $e' \in \mathbb{S}^{N-1}$ and $x \in H(e')$. Then, by $(E\chi)$ and (3.16), there is some $\rho_0 > 0$, depending only on B, d, k, M, and χ, such that

$$\nabla v^{e'}(x, 1) \cdot e' \geq \frac{C_1}{2} \quad \begin{cases} \text{for all } e' \in \mathbb{S}^{N-1} \text{ and } x \in \overline{B} \\ \text{with } |e - e'| < \rho_0 \text{ and } |x \cdot e'| \leq \rho_0. \end{cases} \tag{3.17}$$

By Lemma 3.4, there is some $\eta_1 > 0$ which only depends on B, d, k, M, and χ, such that

$$v^e(x, 1) \geq \eta_1 \qquad \text{for } x \in \overline{B(e)} \text{ with } x \cdot e \geq \frac{\rho_0}{2}.$$

Again by condition $(E\chi)$, we may fix $\rho \in (0, \rho_0)$, depending only on B, d, k, M and χ, such that for all $e' \in \mathbb{S}^{N-1}$ with $|e - e'| < \rho$,

$$v^{e'}(x, 1) \geq \frac{\eta_1}{2} \qquad \text{for } x \in \overline{B(e')} \text{ with } x \cdot e' \geq \frac{\rho_0}{2}. \tag{3.18}$$

For fixed $e' \in \mathbb{S}^{N-1}$ with $|e-e'|<\rho$, (3.17) ensures that

$$v^{e'}(x,1) = v(x,1) - v(\sigma_{e'}(x),1) > 0 \qquad \text{for } x \in B(e') \text{ with } x\cdot e' \le \frac{\rho_0}{2}.$$

Combining this with (3.18), we find that

$$v^{e'}(x,1) > 0 \qquad \text{for } x \in B(e'),$$

as claimed. □

3.3 Main result for scalar Neumann problems

This section is devoted to the proof of Theorem 3.1. Let B denote a ball or an annulus in $\mathbb{R}^N$, and let $u \in C^{2,1}(\overline{B}\times(0,\infty)) \cap C(\overline{B}\times[0,\infty))$ be a classical (possibly sign changing) solution of (3.1) such that the hypothesis (f1), (f2), (μ), (U1), and (U2) of Theorem 3.1 are fulfilled. We first note that

$$\begin{aligned} u_t - \mu(|x|,t)\Delta u - c(x,t)u &= f(t,|x|,0) \qquad && \text{in } B\times(0,\infty),\\ \partial_\nu u &= 0 && \text{on } \partial B\times(0,\infty) \end{aligned}$$

with

$$c(x,t) := \begin{cases} \dfrac{f(t,|x|,u(x,t)) - f(t,|x|,0)}{u(x,t)}, & \text{if} \quad u(x,t)\neq 0,\\ 0, & \text{if} \quad u(x,t) = 0 \end{cases}$$

for $x\in\overline{B}$, $t>0$. By (f1) and (U2) we have $c \in L^\infty(B\times(0,\infty))$, and thus (f2) and Lemma 2.9 imply that the functions

$$\overline{B}\times[0,1]\to\mathbb{R}; \qquad (x,t)\mapsto u(x,\tau+t), \qquad \tau\ge 1, \tag{3.19}$$

and $\overline{B}\times[0,1]\to\mathbb{R}^N$; $(x,t)\mapsto \nabla u(x,\tau+t)$, $\tau\ge 1$, are uniformly equicontinuous. Hence there exists a function

$$\chi : [0, \sqrt{1+\operatorname{diam}(B)^2}\,] \to [0,\infty) \quad \text{with} \quad \lim_{\vartheta\to 0}\chi(\vartheta) = 0$$

such that all of the functions in (3.19) satisfy condition $(E\chi)$ of Lemma 3.5. Next, we set

$$u^e(x,t) := u(x,t) - u(\sigma_e(x),t) \qquad \text{for } x\in\overline{B},\ t>0, \text{ and } e\in\mathbb{S}^{N-1}.$$

We wish to apply Corollary 1.6 to the sets $\mathcal{U} := \omega(u)$ and

$$\mathcal{N} := \{e\in\mathbb{S}^{N-1} \mid \exists\ T>0 \text{ such that } u^e(x,t)>0 \text{ for all } x\in B(e), t>T\}$$

With $\mathcal{M}$ defined as in (1.3), it is obvious that $\mathcal{N} \subset \mathcal{M}$. We note that the function u^e satisfies

$$\begin{aligned} u_t^e - \mu(|x|,t)\Delta u^e &= c^e(x,t)u^e && \text{in } B(e) \times (0,\infty), \\ \frac{\partial u^e}{\partial \nu} &= 0 && \text{on } \Sigma_2(e) \times (0,\infty), \\ u_i^e &= 0 && \text{on } \Sigma_1(e) \times (0,\infty), \end{aligned} \tag{3.20}$$

with $\Sigma_i(e)$ as defined in (3.15) and

$$c^e(x,t) := \begin{cases} \dfrac{f(t,|x|,u(x,t)) - f(t,|x|,u(\sigma_e(x),t))}{u^e(x,t)}, & \text{if} \quad u^e(x,t) \neq 0, \\ 0, & \text{if} \quad u^e(x,t) = 0 \end{cases}$$

for $x \in B$, $t > 0$.

By (f1) and (μ) there is $M > 0$ such that

$$\|c^e\|_{L^\infty(B\times(0,\infty))} \leq M \qquad \text{for all } e \in \mathbb{S}^{N-1}.$$

and

$$\frac{1}{M} \leq \mu(|x|,t) \leq M \qquad \text{for all } x \in B,\ t > 0.$$

By (U1), there exists $\tilde{e} \in \mathbb{S}^{N-1}$ such that $u^{\tilde{e}}(\cdot,0) \geq 0$, $u^{\tilde{e}}(\cdot,0) \not\equiv 0$ on $B(\tilde{e})$ and thus $u^{\tilde{e}}(x,t) > 0$ in $B(\tilde{e}) \times (0,\infty)$ by Lemma 3.4, so that $\tilde{e} \in \mathcal{N}$. Moreover, it easily follows from Lemmas 3.4 and 3.5 that $\mathcal{N}$ is a relatively open subset of $\mathbb{S}^{N-1}$. By Corollary 1.6, it therefore only remains to prove that $z \leq z \circ \sigma_e$ in $B(e)$ for every $z \in \omega(u)$ and $e \in \partial\mathcal{N}$.

We argue by contradiction, assume there exists $\hat{e} \in \partial\mathcal{N}$ and $z \in \omega(u)$ such that $z \not\leq z \circ \sigma_{\hat{e}}$ in $B(\hat{e})$. Define

$$z^e : \overline{B} \to \mathbb{R} \quad \text{by} \quad z^e(x) := z(x) - z(\sigma_e(x))$$

for $e \in \mathbb{S}^{N-1}$. Then there exist constants $d, k > 0$ such that

$$\sup\{z^{\hat{e}}(x) \,:\, x \in B,\ x \cdot \hat{e} \geq d\} > k$$

We now let $\rho > 0$ be given by Lemma 3.5 corresponding to the choices of d, k, M and χ made above. By continuity and since $\hat{e} \in \partial\mathcal{N}$, there exists $e \in \mathcal{N}$ such that

$$|e - \hat{e}| < \rho \tag{3.21}$$

and

$$\sup\{z^e(x) \,:\, x \in B,\ x \cdot e \geq d\} > k. \tag{3.22}$$

Let $(t_n)_{n\in\mathbb{N}} \subset (0,\infty)$ be a sequence with $t_n \to \infty$ and $u(t_n,\cdot) \to z$ in $L^\infty(\overline{B})$. By (3.22), there exists $n_0 \in \mathbb{N}$ such that

$$\sup\{u^e(t_n,x) \,:\, x \in B,\ x\cdot e \geq d\} > k \qquad \text{for all } n \geq n_0.$$

Moreover, by the definition of $\mathcal{N}$, there exists $T > 0$ such that $u^e(\cdot,t) > 0$ in $B(e)$ for $t \geq T$. Next, fixing $n \in \mathbb{N}$ such that $t_n \geq \max\{T + \frac{1}{4}, t_{n_0}\}$ and applying Lemma 3.5 to the function

$$\overline{B} \times [0,1] \to \mathbb{R}, \qquad (x,t) \mapsto u(x, t_n - \frac{1}{4} + t),$$

we find, using (3.21), that $u^{\hat{e}}(x, t_n + \frac{3}{4}) > 0$ for all $x \in B(\hat{e})$. Hence $\hat{e} \in \mathcal{N}$ by Lemma 3.4. Since $\mathcal{N}$ is relatively open in $\mathbb{S}^{N-1}$, this contradicts the fact that $\hat{e} \in \partial\mathcal{N}$. The proof of Theorem 3.1 is thus finished.

3.4 Equations with angular derivative terms

Note that both Theorem 2.3 and Theorem 3.1 imply that *all* the limit profiles $z \in \omega(u)$ share the *same* symmetry axis $\mathbb{R}p$. This is somewhat surprising, since the symmetry axis is not fixed by any of the assumptions in the Theorems. The direction of symmetry $p \in \mathbb{S}^{N-1}$ can not be determined without extra hypothesis; for $e \in \mathbb{S}^{N-1}$ as in (U1), we can only deduce that $p \cdot e > 0$.

However, if the symmetry of the equation is "altered" by an *angular* derivative term, then foliated Schwarz symmetric limit profiles with respect to *different* vectors are indeed a possibility, i.e., u has a varying axis of asymptotic symmetry. We have the following result in this regard.

Theorem 3.7. *Let B be a disc or annulus in $\mathbb{R}^2$ and $u \in C^{2,1}(\overline{B}\times(0,\infty)) \cap C(\overline{B}\times[0,\infty))$ be a classical solution of*

$$\begin{aligned} u_t - \Delta u + d(t)u_\theta &= f(t,|x|,u) &&\text{in } B\times(0,\infty),\\ u(x,0) &= u_0(x) &&\text{for } x \in B, \end{aligned} \tag{3.23}$$

with

$$\partial_\nu u = 0 \text{ on } \partial B \times (0,\infty) \quad \text{or} \quad u = 0 \text{ on } \partial B\times(0,\infty),$$

where $u_\theta = \partial_\theta u$ denotes the angular derivative, $d \in C([0,\infty))$, and hypothesis (f1), (f2), (U1), (U2) from Theorem 2.1 are satisfied. Then for all $z \in \omega(u)$, there is some $p_z \in \mathbb{S}^{N-1}$ such that z is foliated Schwarz symmetric with respect to p_z.

We take B a planar domain for simplicity, but from the proof it is clear that the result can be easily generalized to higher dimensions and more general equations. Notice that angular and radial derivatives prevent a direct application of the standard moving plane method, even when B is a ball and the coefficients and nonlinearity are time- and space-independent.

Proof of Theorem 3.7. The idea of the proof is simply to transform (3.23) into an equation with no angular derivative term. In the following we shall use the same symbol for a function in Cartesian coordinates and its polar coordinates version. Let $v(r, \theta, t) := u(r, \theta + D(t), t)$, where

$$D(t) := \int_0^t d(s)ds,$$

which exists for all $t > 0$ since $d \in C([0, \infty))$. Then

$$\begin{aligned} \partial_t v(r, \theta, t) &= \partial_\theta u(r, \theta + D(t), t)\partial_t D(t) + \partial_t u(r, \theta + D(t), t) \\ &= d(t)u_\theta(r, \theta + D(t), t) + u_t(r, \theta + D(t), t) \end{aligned}$$

and so, by (3.23),

$$\begin{aligned} v_t - \Delta v &= f(t, |x|, v) && \text{in } B \times (0, \infty), \\ v(x, 0) &= u_0(x) && \text{in } B, \end{aligned}$$

with

$$\partial_\nu v = 0 \text{ on } \partial B \times (0, \infty) \quad \text{or} \quad v = 0 \text{ on } \partial B \times (0, \infty).$$

Since v is just a (time dependent) continuous rotation (in space) of u, the claim follows from Theorem 2.3 applied to v if Dirichlet boundary conditions are satisfied, or from Theorem 3.20 if v satisfies Neumann boundary conditions. □

Observe that under the general assumptions of Theorem 3.7, $\omega(u)$ might only have one element, and therefore we can not guarantee that there exist different axis of symmetry for different limit profiles. But from the proof it is clear that, if $d \equiv 1$ for example, then u has a "varying axis of asymptotic symmetry" which is continuously rotating counterclockwise as the time variable increases.

Chapter 4

Parabolic systems with Neumann boundary conditions

In this chapter we begin our study of the asymptotic shape of solutions to parabolic systems. To begin with, we investigate the asymptotic symmetries of positive solutions of *competitive parabolic systems* on bounded radial domains with Neumann boundary conditions.

Recall that B denotes a ball or an annulus in $\mathbb{R}^N$ with $N \geq 2$ and that $I_B := \{|x| \,:\, x \in \overline{B}\}$. We have the following result.

Theorem 4.1. *Let $u_1, u_2 \in C^{2,1}(\overline{B} \times (0,\infty)) \cap C(\overline{B} \times [0,\infty))$ be nonnegative functions such that $u = (u_1, u_2)$ solves*

$$\begin{aligned} (u_1)_t - \mu_1(|x|,t)\Delta u_1 &= f_1(t,|x|,u_i) - \alpha_1(|x|,t)u_1u_2 && x \in B, t > 0,\\ (u_2)_t - \mu_2(|x|,t)\Delta u_2 &= f_2(t,|x|,u_i) - \alpha_2(|x|,t)u_1u_2 && x \in B, t > 0,\\ \partial_\nu u_1(x,t) = \partial_\nu u_2(x,t) &= 0 && x \in \partial B, t > 0,\\ u_1(x,0) = u_{0,1}(x)\,,\ u_2(x,0) &= u_{0,2}(x) && \text{for all } x \in B, \end{aligned} \tag{4.1}$$

where ν stands for the outward normal vector and the following holds.

(h1) For $i = 1,2$, the nonlinearity $f_i : [0,\infty) \times I_B \times [0,\infty) \to R$ is continuous and for any compact subset $K \subset [0,\infty)$ we have that

$$\sup_{\substack{r \in I_B, t>0,\\ v,\bar{v} \in K, v \neq \bar{v}}} \frac{|f_i(t,r,v) - f_i(t,r,\bar{v})|}{|v - \bar{v}|} < \infty,$$

that is, f_i is Lipschitz continuous in v uniformly with respect to t and r. Moreover $f_i(t,r,0) = 0$ for all $r \in I_B$, $t > 0$, and $i \in \{1,2\}$.

(h2) There are constants $\mu^* \geq \mu_* > 0$ *such that* $\|\mu_i\|_{C^{2,1}(I_B \times (0,\infty))} \leq \mu^*$ *and* $\mu_i(r,t) \geq \mu_*$ *for all* $r \in I_B$, $t > 0$, *and* $i \in \{1,2\}$.

(h3) There exist constants $\alpha^* \geq \alpha_* > 0$ *such that* $\alpha_* \leq \alpha_i(r,t) \leq \alpha^*$ *for all* $r \in I_B$, $t > 0$, *and* $i \in \{1,2\}$.

(h4) The solution is uniformly bounded, that is, $\max_{i=1,2} \|u_i\|_{L^\infty(B\times(0,\infty))} < \infty$.

(h5) There is $e \in \mathbb{S}^{N-1}$ *such that* $u_{0,1} \not\equiv u_{0,1} \circ \sigma_e$, $u_{0,2} \not\equiv u_{0,2} \circ \sigma_e$, *and* $u_{0,1} \geq u_{0,1} \circ \sigma_e$, $u_{0,1} \leq u_{0,2} \circ \sigma_e$ *in* $B(e)$.

Then there is some $p \in \mathbb{S}^{N-1}$ *such that* u *is asymptotically foliated Schwarz symmetric with respect to antipodal points, i.e., all elements of* $\omega(u_1)$ *are foliated Schwarz symmetric with respect to* p, *and all elements of* $\omega(u_2)$ *are foliated Schwarz symmetric with respect to* $-p$.

A direct consequence of Theorem 4.1 is the following result for the nonautonomous Lotka-Volterra model discussed in the introduction of the thesis.

Theorem 4.2. *Let* $u_1, u_2 \in C^{2,1}(\overline{B}\times(0,\infty)) \cap C(\overline{B}\times[0,\infty))$ *be nonnegative functions such that* $u = (u_1, u_2)$ *solves* (1), *where* (2) *and assumptions (h4) and (h5) from Theorem 4.1 hold. Then there is some* $p \in \mathbb{S}^{N-1}$ *such that all elements of* $\omega(u_1)$ *are foliated Schwarz symmetric with respect to* p, *and all elements of* $\omega(u_2)$ *are foliated Schwarz symmetric with respect to* $-p$.

As mentioned in the introduction of the thesis, without the assumption (h5), we can construct solutions of a stationary problem which are not foliated Schwarz symmetric. To be precise, we have the following existence result for a class of elliptic Lotka-Volterra systems.

Theorem 4.3. *Suppose that* $B := \{x \in \mathbb{R}^2 : \frac{1}{2} < |x| < 1\}$ *and let* $k \in \mathbb{N}$. *For every* $a \geq 16k^2$ *there exists* $\alpha_{k,a} > 0$ *such that the system*

$$\begin{aligned} -\Delta u_1 &= au_1 - u_1^2 - \alpha u_1 u_2 && \text{in } B, \\ -\Delta u_2 &= au_2 - u_2^2 - \alpha u_1 u_2 && \text{in } B, \\ \partial_\nu u_1 &= \partial_\nu u_2 = 0 && \text{on } \partial B, \end{aligned} \tag{4.2}$$

admits, for every $\alpha \geq \alpha_{k,a}$, *a positive classical solution* $u = (u_1, u_2)$ *such that the angular derivatives* $\frac{\partial u_i}{\partial \theta}$ *of the components change sign at least* k *times on every circle contained in* $\overline{B}$.

To prove our main result, Theorem 4.1, we use some of the tools of Chapter 3 (the extension in Lemma 3.2 and the Harnack-Hopf Lemma 3.4). However, the rotating plane method faces new complications when applied to systems. The main difficulties with systems appear when dealing with the so-called semi-trivial limit profiles, that is, elements of $\omega(u_1, u_2)$ of the form $(z, 0)$ and/or $(0, z)$. In these cases the perturbation argument within the rotating plane method cannot be carried out as in the previous Chapters. To overcome this obstacle, we apply a new normalization procedure and distinguish different cases for the asymptotics of the normalized profile. We also mention that the existence and nature of semi-trivial limit profiles are a subject of current research, and it is often linked with the concept of "permanence", see e.g. [11, 18, 19, 37, 40, 41].

It is natural to ask whether similar symmetry properties are available for other kinds of systems. Indeed, the proof of Theorem 4.1 can be adjusted to deal with a variety of problems. We have the following three results.

The first one characterizes the asymptotic symmetry of a different kind of competitive parabolic system known as *cubic systems.*

Theorem 4.4. *Let $u_1, u_2 \in C^{2,1}(\overline{B} \times (0,\infty)) \cap C(\overline{B} \times [0,\infty))$ be nonnegative functions such that $u = (u_1, u_2)$ solves*

$$\begin{aligned}
(u_1)_t - \Delta u_1 &= \lambda_1 u_1 + \gamma_1 u_1^3 - \alpha_1 u_1 u_2^2 && \text{in } B \times (0,\infty),\\
(u_2)_t - \Delta u_2 &= \lambda_2 u_2 + \gamma_2 u_2^3 - \alpha_2 u_1^2 u_2 && \text{in } B \times (0,\infty),\\
\partial_\nu u_1 &= \partial_\nu u_2 = 0 && \text{on } \partial B \times (0,\infty),\\
u_1(x,0) &= u_{0,1}(x),\ \ u_2(x,0) = u_{0,2}(x) && \text{for } x \in B,
\end{aligned} \tag{4.3}$$

where λ_i, γ_i, and α_i are positive constants for $i = 1, 2$, and assumptions (h4) and (h5) from Theorem 4.1 hold. Then there is some $p \in \mathbb{S}^{N-1}$ such that all elements of $\omega(u_1)$ are foliated Schwarz symmetric with respect to p, and all elements of $\omega(u_2)$ are foliated Schwarz symmetric with respect to $-p$.

The elliptic counterpart of this system is being studied extensively due to its relevance in the study of binary mixtures of Bose-Einstein condensates, see [22].

Now we turn our attention to *cooperative parabolic systems.* The following result studies the cooperative counterpart of (4.1).

Theorem 4.5. *Let $u_1, u_2 \in C^{2,1}(\overline{B} \times (0,\infty)) \cap C(\overline{B} \times [0,\infty))$ be nonnegative functions such that $u = (u_1, u_2)$ solves*

$$\begin{aligned}
(u_i)_t - \mu_i(|x|,t)\Delta u_i &= f_i(t,|x|,u_i) + \alpha_i(|x|,t)u_1 u_2, && x \in B,\ t > 0\\
\partial_\nu u_i &= 0, && x \in \partial B,\ t > 0,\\
u_i(x,0) &= u_{0,i}(x), && x \in B,\ i = 1, 2.
\end{aligned} \tag{4.4}$$

Suppose furthermore that assumptions $(h1)$-$(h4)$ from Theorem 4.1 hold and that *(h5)' there is $e \in \mathbb{S}^{N-1}$ such that $u_{0,i} \not\equiv u_{0,i} \circ \sigma_e$ and $u_{0,i} \geq u_{0,i} \circ \sigma_e$ in $B(e)$ for $i = 1, 2$.*

Then there is some $p \in \mathbb{S}^{N-1}$ such that all elements of $\omega(u_1) \cup \omega(u_2)$ are foliated Schwarz symmetric with respect to p.

If a stronger assumption regarding the interaction between the components of the solution is assumed (see (A3) below), then we can also consider *sign-changing* solutions and cooperative systems with more than two equations. We have the following

Theorem 4.6. *Let $J := \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ and let $u = (u_1, \ldots, u_n)$ with $u_i \in C^{2,1}(\overline{B} \times (0, \infty)) \cap C(\overline{B} \times [0, \infty))$ be a solution of*

$$\begin{aligned} (u_i)_t &= \Delta u_i + F_i(t, |x|, u) && \text{in } B \times (0, \infty), \\ \partial_\nu u_i &= 0 && \text{on } \partial B \times (0, \infty), \\ u_i(x, 0) &= u_{0,i}(x) && \text{for all } x \in B,\ i \in J, \end{aligned} \tag{4.5}$$

where

$$\max_{i \in J} \|u_i\|_{L^\infty(B \times (0,\infty))} < \infty \tag{4.6}$$

and the following holds:

(A1) For each $i \in J$ the function $F_i : [0, \infty) \times I_B \times \mathbb{R}^n \to \mathbb{R}$; $(t, r, v) \mapsto F_i(t, r, v)$ is locally Lipschitz in v uniformly with respect to r and t, that is, for any compact subset $K \subset \mathbb{R}^n$ there is some $C(K) = C > 0$ such that

$$\sup_{\substack{r \in I_B,\ t > 0, \\ v, \bar{v} \in K,\, v \neq \bar{v}}} \frac{|F_i(t, r, v) - F_i(t, r, \bar{v})|}{|v - \bar{v}|} < C.$$

Moreover, $\max_{i \in J} \sup_{r \in I_B, t > 0} |F_i(t, r, 0)| < \infty$.

(A2) For every $i, j \in J$, $i \neq j$, one has that $\partial F_i(t, r, u)/\partial u_j \geq 0$ for all $t \in [0, \infty)$, $r \in I_B$, and $u \in \mathbb{R}^n$ such that the derivative exists.

(A3) For each M there is a constant $\sigma = \sigma(M) > 0$ such that the following holds: for every nonempty subsets $I_1, I_2 \subset J$ with $I_1 \cap I_2 = \emptyset$ and $I_1 \bigcup I_2 = J$, there are $i \in I_1$ and $j \in I_2$ such that $\partial F_i(t, r, u)/\partial u_j \geq \sigma$ for all $r \in I_B$, $t \in [0, \infty)$, and $u \in \mathbb{R}^n$ with $|u| \leq M$, such that the derivative exists.

(A4) There is $e \in \mathbb{S}^{N-1}$ *such that*

$$u_{0,i} \not\equiv u_{0,i} \circ \sigma_e \qquad \text{and} \qquad u_{0,i}(x) \geq u_{0,i}(\sigma_e(x))$$

for all $x \in B(e)$ *and* $i \in J$.

Then there is some $p \in \mathbb{S}^{N-1}$ *such that all elements of* $\bigcup_{i=1}^{n} \omega(u_i)$ *are foliated Schwarz symmetric with respect to* p.

Hypothesis (A2) is a *cooperativity* assumption and (A3) is often referred to as *irreducibility.* (A3) was used for some symmetry results in [50].

The chapter is organized as follows. Section 4.1 contains an extension of Lemma 3.4 to systems and an estimate for quotients of solutions. In Section 4.2 we detail the linearization procedure for competitive parabolic systems and the proof of Theorem 4.1. Section 4.3 is devoted to cooperative systems and contains the proof of Theorems 4.5 and 4.6. In Section 4.4 we detail our study of cubic parabolic systems and prove Theorem 4.4. Finally, in Section 4.5, we prove Theorem 4.3.

Some of the results presented in this chapter were submitted for publication in [55].

4.1 Harnack-Hopf type lemma and quotient estimates

For this whole chapter we assume that B has the form given in (3.3).

We start by extending Lemma 3.4 to the case of weakly coupled systems with mixed boundary conditions.

Lemma 4.7. *Let* $a, b \in \mathbb{R}$, $a < b$, $I := (a,b)$, $B_+ := \{x \in \overline{B} \,:\, x_N > 0\}$, $J := \{1, 2, \ldots, n\}$ *for some* $n \in \mathbb{N}$, *and* $w = (w_1, w_2, \ldots, w_n)$ *with* $w_i \in C^{2,1}(\overline{B_+} \times I) \cap C(\overline{B \times I})$ *be a classical solution of*

$$\begin{aligned} (w_i)_t - \mu_i \Delta w_i &= \sum_{j \in J} c_{ij} w_j && \text{in } B_+^\circ \times I, \\ \frac{\partial w_i}{\partial \nu} &= 0 && \text{on } \Sigma_2 \times I, \\ w_i &= 0 && \text{on } \Sigma_1 \times I, \\ w_i(x, \tau) &\geq 0 && \text{for all } x \in B_+, \; i \in J, \end{aligned}$$

where $\mu_i, c_{ij} \in L^\infty(B_+ \times I)$ *satisfy that* $\inf_{B_+ \times I} \mu_i > 0$ *and that*

$$c_{ij} \geq 0 \qquad \text{on } B_+ \times I \text{ for } i \neq j. \tag{4.7}$$

Then

$$w_i \geq 0 \qquad \text{in } B_+ \times I \text{ for } i \in J. \tag{4.8}$$

Moreover, if $w_i(x,\tau) \not\equiv 0$ *for some* $i \in J$, *then*

$$w_i > 0 \text{ in } B_+ \times I \qquad \text{and} \qquad \frac{\partial w_i}{\partial e_N} > 0 \text{ on } \Sigma_1 \times I.$$

Furthermore, for every $\delta_1 > 0$, $\delta_2 \in (0, \frac{b-a}{4}]$, *there exist* $\kappa > 0$ *and* $p > 0$ *depending only on* δ_1, δ_2, B, *and* M *such that*

$$w_i(x,t) \geq x_N\, \kappa \bigg(\int_{Q(\delta_1,\delta_2)} w_i^p \; d(x,t) \bigg)^{\frac{1}{p}}$$

for all $x \in B_+$, $t \in [a + 3\delta_2, a + 4\delta_2]$, $i \in J$, *with* $Q(\delta_1, \delta_2)$ *as defined in Theorem 3.4.*

Proof. To prove (4.8), we fix $\lambda > \max_{i\in J} \sum_{j\in J} \|c_{ij}\|_{L^\infty(B_+\times I)}$ and let $\varepsilon > 0$. We define

$$v_i(x,t) := e^{-\lambda t} w_i(x,t) + \varepsilon, \qquad \text{for } x \in \overline{B_+}, t \in \overline{I} \text{ and } i \in J.$$

Then

$$\begin{aligned} (v_i)_t - \mu_i \Delta v_i - (c_{ii} - \lambda) v_i &> \sum_{j\in J\setminus\{i\}} c_{ij} v_j && \text{in } {B_+}^\circ \times I, \\ \frac{\partial v_i}{\partial \nu} &\equiv 0 && \text{on } \Sigma_2 \times I, \quad \text{and} \\ v_i &\geq \varepsilon > 0 && \text{on } \Sigma_1 \times I \,\cup\, B_+ \times \{a\}. \end{aligned}$$

We show that $v_i > 0$ in $B_+ \times I$ for all $i \in J$. We argue as in [51, Section 8]. Assume this is not the case and let

$$\bar{t} := \sup\{t \in [a,b) \,:\, v_i > 0 \text{ in } B_+ \times [a,t) \text{ for all } i \in J\}.$$

By continuity, we have $\bar{t} > a$, $v_i(\cdot, \bar{t}) \geq 0$ in B_+ for all $i \in J$ and $v_j(\bar{x}, \bar{t}) = 0$ for some $\bar{x} \in B_+$ and some $j \in J$. Since the domain is a half cylinder, the Neumann boundary conditions on Σ_2 imply that $\bar{x} \in {B_+}^\circ$ (see for example [38, Lemma 2.8]). But then

$$0 \geq (v_j)_t(\bar{x},\bar{t}) - \mu_i \Delta v_i(\bar{x},\bar{t}) - (c_{ii} - \lambda) v_i(\bar{x},\bar{t}) > \sum_{j\in J\setminus\{i\}} c_{ij} v_j(\bar{x},\bar{t}) \geq 0,$$

a contradiction. Therefore $v_i > 0$ in $B_+ \times I$ for all $i \in J$. Since $\varepsilon > 0$ was chosen arbitrarily, we conclude that (4.8) holds. Consequently, (4.7) implies that

$$(w_i)_t - \mu_i(x,t)\Delta w_i - c_{ii}(x,t) w_i = \sum_{j\in J\setminus\{i\}} c_{ij} w_j \geq 0 \qquad \text{in } B_+ \times I \text{ for } i \in J.$$

The result now follows from Lemma 3.4. □

A key ingredient of the proof of Theorem 4.1 is the following quotient estimate which compares the values of the components of u at different times. Similar estimates were obtained by Húska, Poláčik, and Safonov in [35, Corollary 3.10] (see Lemma 5.11 in Chapter 5) for positive solutions of scalar parabolic Dirichlet problems. However, the Neumann boundary conditions on ∂B allow to obtain a stronger result with a simpler proof. In the following, for matters of simplicity, we sometimes omit the argument (x, t).

Lemma 4.8. *Let B be a ball or an annulus and let a function $u \in C^{2,1}(\overline{B} \times (0, \infty)) \cap C(\overline{B} \times [0, \infty))$ be a classical solution of*

$$\begin{aligned} u_t - \mu \Delta u - c\, u &= 0 \qquad \text{in } B \times (0, \infty), \\ u &> 0 \qquad \text{in } B \times (0, \infty), \\ \partial_\nu u &= 0 \qquad \text{on } \partial B \times (0, \infty), \end{aligned} \tag{4.9}$$

where the coefficients μ and c are in $L^\infty(B \times (0, \infty))$ and satisfy that

$$\inf_{x \in B,\, t > 0} \mu(x, t) \geq \beta_0^{-1},$$

$$\|\mu\|_{L^\infty(B \times (0,\infty))} + \|c\|_{L^\infty(B \times (0,\infty))} < \beta_0$$

for some $\beta_0 > 0$. Then there is a constant $\eta > 1$ depending only on β_0 and B such that

$$\frac{1}{\eta} \leq \frac{u(x, t)}{\|u(\cdot, \tau)\|_{L^\infty(B)}} \leq \eta \quad \text{for all } x \in B,\ t \in [\tau - 3, \tau + 3],\ \text{and } \tau \geq 5.$$

Proof. Let $\tilde{u}$ denote the extension of u to $\widetilde{B}$ as defined in (3.5). Then, by Lemma 3.2, $\tilde{u}$ is a strong solution of

$$(\tilde{u})_t - \tilde{\mu}\, \Delta \tilde{u} - \tilde{b}\, \partial_r \tilde{u} = \tilde{c}\, \tilde{u} \qquad \text{in } \widetilde{B} \times (0, \infty).$$

Here $\tilde{\mu}, \tilde{b} \in L^\infty(\widetilde{B} \times (0, \infty))$ are defined as in Lemma 3.2 and $\tilde{c} \in L^\infty(\widetilde{B} \times (0, \infty))$ is defined by

$$\tilde{c}(x, t) := \begin{cases} c(x, t), & x \in B,\ t \in (0, \infty), \\ c(\hat{x}, t), & x \in \widetilde{B} \setminus B,\ t \in (0, \infty). \end{cases}$$

Also note that $\inf\limits_{B \times (0,\infty)} \tilde{\mu} \geq \mu_*$ by our assumptions on μ and the definition of $\tilde{\mu}$.

Now, fix $\tau \geq 5$. We apply Lemma 2.7 with $U = \widetilde{B}$, $D = B$, $p = \infty$, and $v = \tilde{u}$. The application yields $\kappa_1 > 0$ such that

$$\inf_{B \times (\tau - 3, \tau + 3)} u \geq \kappa_1 \|u(\cdot, \tau - 4)\|_{L^\infty(B)}, \tag{4.10}$$

since $\tilde{u}$ coincides with u on $B\times(0,\infty)$. Moreover, by the maximum principle (see for example [38, Lemma 7.1]) and the uniform bounds on the coefficients, there exists $\kappa_2 > \kappa_1$ independent of τ such that

$$\|u(\cdot,s)\|_{L^\infty(B)} \leq \kappa_2 \|u(\cdot,\tau-4)\|_{L^\infty(B)} \qquad \text{for all } s\in[\tau-3,\tau+3]. \tag{4.11}$$

For $x\in B$ and $t\in(\tau-3,\tau+3)$ we have that

$$\frac{u(x,t)}{\|u(\cdot,\tau)\|_{L^\infty(B)}} \geq \frac{\kappa_1\|u(\cdot,\tau-4)\|_{L^\infty(B)}}{\|u(\cdot,\tau)\|_{L^\infty(B)}} \geq \frac{\kappa_1}{\kappa_2},$$

by (4.11) and (4.10); on the other hand,

$$\frac{u(x,t)}{\|u(\cdot,\tau)\|_{L^\infty(B)}} \leq \frac{\kappa_2\|u(\cdot,\tau-4)\|_{L^\infty(B)}}{\|u(\cdot,\tau)\|_{L^\infty(B)}} \leq \frac{\kappa_2}{\kappa_1}.$$

Therefore the claim follows with $\eta = \frac{\kappa_2}{\kappa_1}$. □

4.2 Neumann parabolic systems with competition

For this whole section, let $u_1, u_2 \in C^{2,1}(\overline{B}\times(0,\infty))\cap C(\overline{B}\times[0,\infty))$ be functions such that $u=(u_1,u_2)$ solves (4.1) and such that assumptions (h1)-(h5) from Theorem 4.1 are fulfilled.

We begin with two remarks.

Remark 4.9. Note that u_i satisfies

$$(u_i)_t - \mu_i(|x|,t)\Delta u_i = g_i(x,t), \quad x\in B,\ t>0,$$

with $g_i : B\times(0,\infty)\to\mathbb{R}$ given by

$$g_i(x,t) = f_i(t,|x|,u_i(x,t)) - \alpha_i(|x|,t)u_1(x,t)u_2(x,t) \qquad \text{for } i=1,2.$$

Moreover, by (h1)-(h4), we have that $\inf\limits_{B\times(0,\infty)} \mu_i \geq K^{-1}$ and

$$\|u_i\|_{L^\infty(B\times(0,\infty))} + \|\mu_i\|_{C^1(B\times(0,\infty))} + \|g_i\|_{L^\infty(B\times(0,\infty))} < K \qquad \text{for } i=1,2$$

and for some constant $K>0$. Then Lemma 2.9 implies that

$$\|u_i\|_{C^{a,a/2}(\overline{B}\times[s,s+1])} < C \quad \text{for all } s\in[1,\infty),\ i=1,2 \tag{4.12}$$

for some $a\in(1,2)$ and $C>0$. In particular u_i and ∇u_i satisfy (2.11), the semiorbits $\{u_i(\cdot,t): t\geq 1\}$ are precompact in $C(\overline{B})$ for $i=1,2$, and

$$\lim_{t\to\infty} \inf_{(z_1,z_2)\in\omega(u)} \|u_1(\cdot,t)-z_1\|_{L^\infty(B)} + \|u_2(\cdot,t)-z_2\|_{L^\infty(B)} = 0,$$

where

$$\omega(u) := \{(z_1, z_2) \in C(\overline{B}) \times C(\overline{B}) \; : \; \text{there is } t_n \to \infty \text{ as } n \to \infty \; \text{ such that} \\ \lim_{n\to\infty} \|u_1(\cdot, t_n) - z_1\|_{L^\infty(B)} + \|u_2(\cdot, t_n) - z_2\|_{L^\infty(B)} = 0\}.$$

Moreover, the omega limit sets of $u = (u_1, u_2)$ and its components are related as follows:

$$\omega(u_i) = \{z_i \; : \; z = (z_1, z_2) \in \omega(u)\} \qquad \text{for } i = 1, 2. \tag{4.13}$$

Indeed, let $i \in \{1, 2\}$ and $t_n \to \infty$ be a sequence such that $u_i(\cdot, t_n) \to z_i$ in $L^\infty(B)$ as $n \to \infty$ with $z_i \in \omega(u_i)$. Then, by compactness, we can pass to a subsequence such that $u_j(\cdot, t_n) \to z_j$ in $L^\infty(B)$ as $n \to \infty$ for some $z_j \in \omega(u_j)$ with $j \in \{1, 2\}\backslash\{i\}$. This implies the inclusion "$\subseteq$ " in (4.13). Since the inclusion "$\supseteq$ " is obvious, the claim follows.

Remark 4.10. Note that u_i satisfies (4.9) with $\mu = \mu_i$ and

$$c(x,t) := \alpha_i(t, |x|)u_2(x,t) + \begin{cases} \dfrac{f_i(t, |x|, u_i(x,t))}{u_i(x,t)}, & \text{if} \quad u_i(x,t) \neq 0, \\ 0, & \text{if} \quad u_i(x,t) = 0, \end{cases}$$

where $c \in L^\infty(B \times (0, \infty))$ by (h1), (h3), and (h4). Then, by (h2) all the assumptions of Lemma 4.8 are satisfied.

4.2.1 Linearization of competitive systems

Next, we slightly change some notation used in the previous Chapters in order to deal with competitive systems of two equations. For $e \in \mathbb{S}^{N-1}$, a radial domain $B \subset \mathbb{R}^N$, $I \subset \mathbb{R}$, and a pair $v = (v_1, v_2)$ of functions $v_i : \overline{B} \times I \to \mathbb{R}$, $i = 1, 2$, we set

$$\begin{aligned} v_1^e(x,t) &:= v_1(x,t) - v_1(\sigma_e(x), t) \\ v_2^e(x,t) &:= v_2(\sigma_e(x), t) - v_2(x,t) \end{aligned} \qquad x \in \overline{B}, \; t > 0. \tag{4.14}$$

The same notation is used if the functions do not depend on time. More precisely, for a pair $z = (z_1, z_2)$ of functions $z_i : \overline{B} \to \mathbb{R}$, $i = 1, 2$, we set

$$\begin{aligned} z_1^e(x) &:= z_1(x) - z_1(\sigma_e(x)) \\ z_2^e(x) &:= z_2(\sigma_e(x)) - z_2(x) \end{aligned} \qquad x \in \overline{B}, \; t > 0. \tag{4.15}$$

Since $u = (u_1, u_2)$ solves (4.1), for fixed $e \in \mathbb{S}^{N-1}$ we have

$$\begin{aligned} (u_1^e)_t - \mu_1 \Delta u_1^e - \hat{c}_1^e(x,t)u_1^e &= \alpha_1[\hat{u}_1\hat{u}_2 - u_1u_2] = \alpha_1[u_1u_2^e - \hat{u}_2u_1^e], \\ (u_2^e)_t - \mu_2 \Delta u_2^e - \hat{c}_2^e(x,t)u_2^e &= \alpha_2[u_1u_2 - \hat{u}_1\hat{u}_2] = \alpha_2[u_2u_1^e - \hat{u}_1u_2^e] \end{aligned}$$

in $B \times (0,\infty)$ with $\hat{u}_i(x,t) := u_i(\sigma_e(x),t))$ and

$$\hat{c}_i^e(x,t) := \begin{cases} \dfrac{f_i(t,|x|,u_i(x,t)) - f_i(t,|x|,u_i(\sigma_e(x),t))}{u_i(x,t) - u_i(\sigma_e(x),t)}, & \text{if} \quad u_i^e(x,t) \neq 0, \\ 0, & \text{if} \quad u_i^e(x,t) = 0 \end{cases}$$

for $x \in B$, $t > 0$, $i = 1, 2$. Setting

$$\begin{aligned} c_1^e(x,t) &:= \hat{c}_1^e(x,t)) - \alpha_1(|x|,t)u_2(\sigma_e(x),t) \\ c_2^e(x,t) &:= \hat{c}_2^e(x,t)) - \alpha_2(|x|,t)u_1(\sigma_e(x),t) \end{aligned} \qquad x \in B,\ t > 0,$$

we thus obtain the system

$$\begin{aligned} (u_1^e)_t - \mu_1 \Delta u_1^e - c_1^e u_1^e &= \alpha_1 u_1 u_2^e \\ (u_2^e)_t - \mu_2 \Delta u_2^e - c_2^e u_2^e &= \alpha_2 u_2 u_1^e \end{aligned} \qquad \text{in } B(e) \times (0,\infty) \tag{4.16}$$

together with the boundary conditions

$$\frac{\partial u_i^e}{\partial \nu} = 0 \quad \text{on } \Sigma_2(e) \times (0,\infty), \qquad u_i^e = 0 \quad \text{on } \Sigma_1(e) \times (0,\infty), \tag{4.17}$$

where the sets $\Sigma_i(e)$ are given as in (3.15) for $i = 1, 2$. As a consequence of (h1),(h3), and (h4), we have

$$\max_{i=1,2} \|c_i^e\|_{L^\infty(B\times(0,\infty))} \leq M \qquad \text{for all } e \in \mathbb{S}^{N-1} \tag{4.18}$$

for some constant $M > 0$. Moreover, by making M larger if necessary and using (h2), we may also assume that

$$\frac{1}{M} \leq \mu_i(|x|,t) \leq M \qquad \text{for } x \in B,\ t > 0 \text{ and } i = 1, 2. \tag{4.19}$$

We note that, by (h3) and since $u_1, u_2 \geq 0$ in $B \times (0,\infty)$, system (4.16) is a weakly coupled linear parabolic cooperative system. For these systems a variety of estimates are available (see for example [51] and [50]). In particular, Lemma 4.7 can be applied to study the boundary value problem (4.16), (4.17).

See also Remark 4.17 iv) below for a method to linearize more general competitive systems.

4.2.2 Main result for Neumann competitive systems

Let

$$\mathcal{N} := \{e \in \mathbb{S}^{N-1} : u_i^e > 0 \ \text{ in } \ B(e) \times [T,\infty) \ \text{ for } i = 1, 2 \text{ and some } T > 0\}.$$

We have the following two Lemmas which we prove at the end of this subsection.

Lemma 4.11. *The set $\mathcal{N}$ is relatively open in $\mathbb{S}^{N-1}$.*

Lemma 4.12. *For every $e \in \partial\mathcal{N}$ and every $z \in \omega(u)$ we have $z_1^e \equiv z_2^e \equiv 0$ in $B(e)$.*

Then we can proceed to the

Proof of Theorem 4.1. Define

$$\mathcal{U} := \omega(u_1) \cup -\omega(u_2) = \{z_1, -z_2 \;:\; z \in \omega(u)\} \tag{4.20}$$

and

$$\mathcal{M} := \{e \in \mathbb{S}^{N-1} \;:\; z^e \geq 0 \quad \text{in } B(e) \text{ for all } z \in \mathcal{U}\}.$$

Note that the last equality in (4.20) is a consequence of (4.13). Then we have that $\mathcal{N} \subset \mathcal{M}$. Moreover, for $e \in \mathbb{S}^{N-1}$ as in (h5), we have

$$u_i^e(\cdot, 0) \geq 0,\; u_i^e(\cdot, 0) \not\equiv 0 \quad \text{in } B(e) \quad \text{for } i = 1, 2.$$

Lemma 4.7 then implies that $u_i^e > 0$ on $B(e) \times (0, \infty)$ for $i = 1, 2$, so that $e \in \mathcal{N}$ and thus $\mathcal{N}$ is nonempty. Moreover $\mathcal{N}$ is a relatively open subset of $\mathbb{S}^{N-1}$ by Lemma 4.11 and, by Lemma 4.12, $z \equiv z \circ \sigma_e$ for all $z \in \mathcal{U}$ and $e \in \partial\mathcal{N}$. The result now follows from Corollary 1.6. □

Proof of Lemma 4.11. Let $e \in \mathcal{N}$. Then (u_1^e, u_2^e) is a solution of (4.16), and there is $T > 0$ such that u_1^e and u_2^e are positive in $B(e) \times (T, \infty)$. Thus

$$\begin{aligned} (u_1^e)_t - \mu_1 \Delta u_1^e - c_1^e u_1^e &= \alpha_1 u_1 u_2^e \geq 0 \\ (u_2^e)_t - \mu_2 \Delta u_2^e - c_2^e u_2^e &= \alpha_2 u_2 u_1^e \geq 0 \end{aligned} \qquad \text{in } B(e) \times [T, \infty),$$

since α_1 and α_2 are nonnegative by hypothesis (h3). By (4.12), the functions

$$\overline{B} \times [0, 1] \to \mathbb{R}, \qquad (x, t) \mapsto u_i(x, T + t), \qquad i = 1, 2,$$

satisfy the assumptions of Lemma 3.5. Therefore, by Remark 3.6, we find that there exists $\rho > 0$ such that $u_i^{e'}(\cdot, T+1) > 0$ in $B(e')$ for $e' \in \mathbb{S}^{N-1}$ with $|e' - e| < \rho$. Hence, by Lemma 4.7, $e' \in \mathcal{N}$ for $e' \in \mathbb{S}^{N-1}$ with $|e' - e| < \rho$, and thus $\mathcal{N}$ is open. □

Proof of Lemma 4.12. Let $z = (z_1, z_2) \in \omega(u)$, and consider an increasing sequence $t_n \to \infty$ with $t_1 > 6$ and such that $u_i(\cdot, t_n) \to z_i$ uniformly in $\overline{B}$ for $i = 1, 2$. We will only show that $z_2^e \equiv 0$ in $B(e)$ for all $e \in \partial\mathcal{N}$, since the same argument shows that $z_1^e \equiv 0$ in $B(e)$ for all $e \in \partial\mathcal{N}$. By (4.12) there exists a function $\chi : [0, \sqrt{1+\mathrm{diam(B)}^2}\,] \to [0,\infty)$ with $\lim_{\vartheta\to 0}\chi(\vartheta) = 0$ such that all the functions

$$\overline{B} \times [0,1] \to \mathbb{R}; \quad (x,t) \mapsto u_2(x,\tau+t), \quad \tau \geq 1, \tag{4.21}$$

satisfy the equicontinuity condition $(E\chi)$ of Lemma 3.5. Arguing by contradiction, we now assume that $z_2^{\hat{e}} \not\equiv 0$ in $B(\hat{e})$ for some $\hat{e} \in \partial\mathcal{N}$. By the equicontinuity of the functions in (4.21), there are $\zeta \in (0, \frac{1}{4})$, a nonempty open subset $\Omega \subset\subset B(\hat{e})$, and $k_1 > 0$ such that, after passing to a subsequence,

$$u_2^{\hat{e}} \geq k_1 \quad \text{on } \Omega \times [t_n-\zeta, t_n+\zeta] \quad \text{for all } n \in \mathbb{N}. \tag{4.22}$$

We now apply a normalization procedure for u_1, since we cannot exclude the possibility that $u_1(\cdot,t_n) \to 0$ as $n \to \infty$. Define, for $n \in \mathbb{N}$,

$$I_n := [t_n-2, t_n+2] \subset \mathbb{R}, \qquad \beta_n := \|u_1(\cdot,t_n)\|_{L^\infty(B)}$$

and the functions

$$v_n : \overline{B} \times I_n \to \mathbb{R}, \qquad v_n(x,t) = \frac{u_1(x,t)}{\beta_n}.$$

By Lemma 4.8 and Remark 4.10, there exists $\eta > 1$ such that

$$\frac{1}{\eta} \leq v_n \leq \eta \quad \text{on } B \times I_n \qquad \text{for all } n \in \mathbb{N}. \tag{4.23}$$

Moreover, we have that

$$\|v_n\|_{C^{1+\gamma,(1+\gamma)/2}(\overline{B}\times[s,s+1])} < L \quad \text{for all } s \in [t_n-1, t_n+1],\ n \in \mathbb{N}$$

and for some $\gamma \in (0,1)$ and $L > 0$; that is,

$$\sup_{\substack{x,\bar{x}\in\overline{B},\, t,\bar{t}\in[s,s+1],\\ x\neq\bar{x},\, t\neq\bar{t},\, s\in[-1,1]}} \frac{|v_n(x,t_n+t) - v_n(\bar{x},t_n+\bar{t})|}{|x-\bar{x}|^\gamma + |t-\bar{t}|^{\frac{\gamma}{2}}} < L, \tag{4.24}$$

and

$$\sup_{\substack{x,\bar{x}\in\overline{B},\, t,\bar{t}\in[s,s+1],\\ x\neq\bar{x},\, t\neq\bar{t},\, s\in[-1,1]}} \frac{|\nabla v_n(x,t_n+t) - \nabla v_n(\bar{x},t_n+\bar{t})|}{|x-\bar{x}|^\gamma + |t-\bar{t}|^{\frac{\gamma}{2}}} < L$$

for all $n \in \mathbb{N}$. This follows from Lemma 2.9 and the fact that v_n satisfies

$$\begin{aligned} (v_n)_t - \mu_1\Delta v_n &= (c-\alpha_1 u_2)v_n && \text{in } B \times I_n, \\ \partial_\nu v_n &= 0 && \text{in } \partial B \times I_n \end{aligned}$$

with $c \in L^\infty(B \times (0,\infty))$ given by

$$c(x,t) := \begin{cases} \dfrac{f_1(t,|x|,u_1(x,t))}{u_1(x,t)}, & \text{if} \quad u_1(x,t) \neq 0, \\ 0, & \text{if} \quad u_1(x,t) = 0, \end{cases}$$

for $x \in B$ and $t > 0$.

Then, by (4.24) and by adjusting the function χ above, we may also assume that all of the functions

$$\overline{B} \times [0,1] \to \mathbb{R}; \quad (x,t) \mapsto v_n(x,\tau+t), \quad \tau \in [t_n - 1, t_n + 1],\ n \in \mathbb{N},$$

satisfy the equicontinuity condition $(E\chi)$ of Lemma 3.5.

For $e \in \mathbb{S}^{N-1}$ and $n \in \mathbb{N}$ we also consider

$$v_n^e : \overline{B} \times I_n \to \mathbb{R}, \qquad v_n^e(x,t) := v_n(x,t) - v_n(\sigma_e(x),t),$$

and we note that

$$\begin{aligned} (v_n^e)_t - \mu_1 \Delta v_n^e - c_1^e v_n^e &= \alpha_1 v_n u_2^e && \text{in } B(e) \times I_n, \\ (u_2^e)_t - \mu_2 \Delta u_2^e - c_2^e u_2^e &= \alpha_2 \beta_n u_2 v_n^e && \text{in } B(e) \times I_n, \\ \partial_\nu v_n^e = \partial_\nu u_2^e &= 0 && \text{on } \Sigma_2(e) \times I_n, \\ v_n^e(x,t) = u_2^e(x,t) &= 0 && \text{on } \Sigma_1(e) \times I_n \end{aligned} \tag{4.25}$$

with $\Sigma_i(e)$ as defined in (3.15).

Set

$$Q_n := B(\hat{e}) \times [t_n - \zeta\,,\, t_n + \zeta] \qquad \text{for } n \in \mathbb{N},$$

with ζ as in (4.22). We now distinguish two cases.

$$\underline{\text{Case 1:}} \qquad \limsup_{n\to\infty} \|v_n^{\hat{e}}\|_{L^\infty(Q_n)} > 0. \tag{4.26}$$

In this case, by (4.24), there are $d \in (0,1)$, $k_2 > 0$, and $t^* \in [-\zeta,\zeta]$ such that, after passing to a subsequence,

$$\sup\{v_n^{\hat{e}}(x, t_n + t^*) : x \in B(\hat{e}),\ x \cdot \hat{e} \geq d\} \geq k_2 \qquad \text{for } n \in \mathbb{N}.$$

Without loss, we may assume that $d < \min\{x \cdot \hat{e} \,:\, x \in \Omega\}$, so that also

$$\sup\{u_2^{\hat{e}}(x, t_n + t^*) : x \in B(\hat{e}),\ x \cdot \hat{e} \geq d\} \geq k_1 \qquad \text{for } n \in \mathbb{N}$$

by (4.22). Let M be as in (4.18), (4.19), $k := \frac{1}{2}\min\{k_1, k_2\}$, and let $\rho > 0$ be the constant given by Lemma 3.5 for the choices of M, d, k, and χ made in this proof. Since $\hat{e} \in \partial\mathcal{N}$, there exists $e \in \mathcal{N}$ such that $|e - \hat{e}| < \frac{\rho}{2}$, $x_1, x_2 \in B(e)$ and, by equicontinuity,

$$\sup\{v_n^e(x, t_n + t^*) : x \in B(e),\ x \cdot e \geq d\} \geq k,$$
$$\sup\{u_2^e(x, t_n + t^*) : x \in B(e),\ x \cdot e \geq d\} \geq k$$

for all $n \in \mathbb{N}$. Since $e \in \mathcal{N}$ we can fix $n \in \mathbb{N}$ such that

$$v_n^e(x, t_n + t^* - \frac{1}{4}) \geq 0, \quad u_2^e(x, t_n + t^* - \frac{1}{4}) \geq 0, \qquad \text{for all } x \in B(e).$$

Then, applying Lemma 3.5 to the functions

$$\overline{B} \times [0,1] \to \mathbb{R};\ \ (x,t) \mapsto u_2(x, t_n + t^* - \frac{1}{4} + t),\ \ (x,t) \mapsto v_n(x, t_n + t^* - \frac{1}{4} + t),$$

we conclude that

$$u_2^{\bar{e}}(\cdot, t_n + t^* + \frac{3}{4}) > 0 \quad \text{and} \quad v_n^{\bar{e}}(\cdot, t_n + t^* + \frac{3}{4}) > 0 \qquad \text{in } B(\bar{e})$$

for all $\bar{e} \in \mathbb{S}^{N-1}$ with $|\bar{e} - e| < \rho$, and thus in particular for $\bar{e} = \hat{e}$. But this implies $u_i^{\hat{e}}(\cdot, t_n + t^* + \frac{3}{4}) > 0$ in $B(\hat{e})$ for $i = 1, 2$, and thus $\hat{e} \in \mathcal{N}$ by Lemma 4.7. This contradicts the hypothesis that $\hat{e} \in \partial\mathcal{N}$, since $\mathcal{N}$ is open by Lemma 4.11.

$$\underline{\text{Case 2:}} \qquad \lim_{n\to\infty} \|v_n^{\hat{e}}\|_{L^\infty(Q_n)} = 0. \tag{4.27}$$

In this case let

$$Q := B(\hat{e}) \times (-\zeta, \zeta)$$

and fix a nonnegative function $\varphi \in C_c^\infty(Q)$ with $\varphi \equiv 1$ on $\Omega \times (-\frac{\zeta}{2}, \frac{\zeta}{2})$. Moreover, let

$$\varphi_n \in C_c^\infty(Q_n) \quad \text{be given by} \quad \varphi_n(x,t) := \varphi(x, t_n + t), \quad n \in \mathbb{N}.$$

Setting $(u_2^{\hat{e}})^+ := \max\{u_2^{\hat{e}}, 0\}$ and $(u_2^{\hat{e}})^- := -\min\{u_2^{\hat{e}}, 0\}$, we find by (h3), (4.22), and (4.23) that, for $n \in \mathbb{N}$,

$$\begin{aligned} A_n &:= \int_{Q_n} \alpha_1 v_n u_2^{\hat{e}} \varphi_n d(x,t) = \int_{Q_n} \alpha_1 v_n [(u_2^{\hat{e}})^+ - (u_2^{\hat{e}})^-] \varphi_n d(x,t) \\ &\geq \frac{\alpha_*}{\eta} \int_{Q_n} (u_2^{\hat{e}})^+ \varphi_n d(x,t) - \alpha^* \eta \, \|(u_2^{\hat{e}})^-\|_{L^\infty(Q_n)} \, \|\varphi\|_{L^1(Q)}, \\ &\geq \frac{\alpha_*}{\eta} k_1 |\Omega| \zeta \ - \ \alpha^* \eta \, \|(u_2^{\hat{e}})^-\|_{L^\infty(Q_n)} \, \|\varphi\|_{L^1(Q)}, \end{aligned} \tag{4.28}$$

where the last term on the right hand side goes to zero as $n \to \infty$ because $\hat{e} \in \partial\mathcal{N}$. Hence we have

$$\liminf_{n\to\infty} A_n > 0.$$

On the other hand, integrating by parts, we have by (4.25) that

$$\begin{aligned} A_n &= \int_{Q_n} [(v_n^{\hat{e}})_t - \mu_1 \Delta v_n^{\hat{e}} - c_1^{\hat{e}} v_n^{\hat{e}}]\varphi_n d(x,t) \\ &= -\int_{Q_n} [v_n^{\hat{e}}(\varphi_n)_t + v_n^{\hat{e}}\Delta(\mu_1\varphi_n) + c_1^{\hat{e}} v_n^{\hat{e}}\varphi_n] d(x,t) \\ &\le \|v_n^{\hat{e}}\|_{L^\infty(Q_n)} \int_Q \Big(|(\varphi)_t| + |\Delta(\mu_1(x,t_n+t)\varphi)| + M\varphi\Big) d(x,t) \qquad (4.29) \end{aligned}$$

for $n \in \mathbb{N}$. Invoking (h2) and (4.27), we conclude that

$$\limsup_{n\to\infty} A_n \le 0.$$

So we have obtained a contradiction again, and thus the claim follows. □

Remark 4.13. At first glance, the estimates (4.28),(4.29) might appear as purely technical elements in the proof. They have however a deeper meaning: these estimates imply that the symmetry of each component u_i is *asymptotically entangled* or that the symmetry of the components is *synchronizing* as time goes forward. We will see this entanglement effect in more detail in the following section (see Remark 4.17 below), but as a first remark, note that if the competition coefficients α_i were zero, then we have a *decoupled system* and each of the components of the solution is still asymptotically foliated Schwarz symmetric (by Theorem 3.1), but not necessarily with respect to antipodal points.

4.3 Cooperative parabolic systems

Proof of Theorem 4.5. Let $u_1, u_2 \in C^{2,1}(\overline{B} \times (0,\infty)) \cap C(\overline{B} \times [0,\infty))$ be functions such that $u = (u_1, u_2)$ solves (4.4), and suppose that $(h1)$-$(h4)$ and $(h5)'$ in the statement of the theorem are satisfied. The proof is almost exactly the same as the one of Theorem 4.1 with only two changes. The first change we need to make concerns the definitions of v_2^e and z_2^e in (4.14) and (4.15). More precisely, we now set $v_i^e(x,t) = v_i(x,t) - v_i(\sigma_e(x),t)$ and $z_i^e(x) = z_i(x) - z_i(\sigma_e(x))$ for $i = 1,2$. With this change, we again arrive at the linearized system (4.16). Considering now the sets

$$\mathcal{U} := \omega(u_1) \cup \omega(u_2) = \{z_1, z_2 \,:\, z \in \omega(u)\}$$

in place of (4.20) and

$$\mathcal{N} := \{e \in \mathbb{S}^{N-1} \ : \ \exists\, T > 0 \text{ s.t. } u_i^e > 0 \text{ in } B(e) \times [T, \infty) \text{ for } i = 1, 2\},$$

we may now validate the assumptions of Corollary 1.6 in exactly the same way as in Subsection 4.2.2. Hence the proof is complete. □

Remark 4.14. (i) Note that both Theorem 4.1 and Theorem 4.5 assume that the components u_i are nonnegative, and this assumption is essential for the cooperativity of the linearized system (4.16). Without the sign restriction, systems (4.1) and (4.4) arise from each other by replacing u_i by $-u_i$ for $i = 1, 2$ and adjusting f accordingly.

(ii) Our method breaks down if the coupling term has different signs in the components, as e.g. in a predator-prey type system

$$\begin{aligned} (u_1)_t - \mu_1 \Delta u_1 &= f_1(t, |x|, u_1) + \alpha_1 u_1 u_2 \\ (u_2)_t - \mu_2 \Delta u_2 &= f_2(t, |x|, u_2) - \alpha_2 u_1 u_2 \end{aligned} \qquad \text{in } B \times (0, \infty). \tag{4.30}$$

In this case, there seems to be no way to derive a cooperative linearized system of the type (4.16) for difference functions related to hyperplane reflections. The asymptotic shape of solutions for this system (satisfying Dirichlet or Neumann boundary conditions) remains an interesting open problem. In Chapter 5 we show a partial result in this direction regarding semi-trivial limit profiles (see Theorems 5.2 and 5.4).

4.3.1 Irreducible cooperative systems of n-equations

In this subsection we prove Theorem 4.6 following the techniques used in the proof of Theorem 4.1. Let us set a framework for the parabolic rotating plane method in this setting.

Let $g : B \times (0, \infty) \to \mathbb{R}$ be given by $g(x, t) := F(t, |x|, u(x, t))$. Note that, by assumption (A1) and (4.6), we have that $\|g\|_{L^\infty(B \times (0,\infty))} < \infty$. Then, by Lemma 2.9,

$$\|u_i\|_{C^{a,a/2}(\overline{B} \times [s,s+1])} \leq C \quad \text{for all } s \geq 1, \ j \in J, \tag{4.31}$$

and for some constants $a \in (1, 2)$ and $C > 0$. Moreover

$$\lim_{t \to \infty} \inf_{z \in \omega(u)} \sum_{i=1}^{n} \|u_i(\cdot, t) - z_i\|_{L^\infty(B)} = 0. \tag{4.32}$$

For $e \in \mathbb{S}^{N-1}$, recall the sets $\Sigma_1(e)$ and $\Sigma_2(e)$ defined in (3.15) and let $u_i^e(x, t) := u_i(x, t) - u_i(\sigma_e(x), t)$, for all $x \in B(e)$, $t > 0$, and $i \in J$. Then we can use the Hadamard formulas (as in [50]) to get the following weakly coupled linear system

$$\begin{aligned} (u_i^e)_t - \Delta u_i^e &= \sum_{j=1}^{N} c_{ij}^e(x,t) u_j^e && \text{in } B(e) \times (0,\infty), \\ \frac{\partial u_i^e}{\partial \nu} &= 0 && \text{on } \Sigma_2(e) \times (0,\infty), \\ u_i^e &= 0 && \text{on } \Sigma_1(e) \times (0,\infty), \end{aligned} \tag{4.33}$$

for all $i \in J$, where

$$\begin{aligned} c_{ij}^e(x,t) := \int_0^1 & \partial_{u_j} F_i(t, |x|, u_1(x,t), \dots, u_{j-1}(x,t), s u_j(x,t) \\ &+ (1-s) u_j(\sigma_e(x), t), u_{j+1}(\sigma_e(x), t), \dots, u_n(\sigma_e(x), t)) ds, \end{aligned} \tag{4.34}$$

if $u^e(x,t) \neq 0$ and $c_{ij}^e(x,t) := 0$ if $u^e(x,t) \neq 0$, for $x \in B(e)$ and $t > 0$.

Note that by (A1)-(A2) these coefficients are defined almost everywhere and

$$c_{ij}^e \geq 0 \qquad \text{for all} \quad i,j \in J \quad \text{with} \quad i \neq j. \tag{4.35}$$

Further, by (A1) and (4.6), there is $M \geq 1$ such that

$$\|c_{ij}^e\|_{L^\infty(B \times (0,\infty))} < M \qquad \text{for all } i,j \in J. \tag{4.36}$$

Now, for this section let

$$\mathcal{N} := \{e \in \mathbb{S}^{N-1} \;:\; u_i^e > 0 \text{ in } B(e) \times [T,\infty) \text{ for } i \in J \text{ and some } T > 0\}. \tag{4.37}$$

We now prove the analog of Lemmas 4.11 and 4.12.

Lemma 4.15. *The set $\mathcal{N}$ is relatively open in $\mathbb{S}^{N-1}$.*

Proof of Lemma 4.11. Let $e \in \mathcal{N}$. Then, by (4.33), (4.35), and (4.37) there is $T > 0$ such that

$$\begin{aligned} (u_i^e)_t - \Delta u_i^e - c_{ii}^e u_i^e &\geq 0 && \text{in } B(e) \times [T,\infty), \\ \frac{\partial u_i^e}{\partial \nu} &= 0 && \text{on } \Sigma_2(e) \times [T,\infty), \\ u_i^e &= 0 && \text{on } \Sigma_1(e) \times [T,\infty). \end{aligned}$$

By (4.31) the functions

$$\overline{B} \times [0,1] \to \mathbb{R}; \quad (x,t) \mapsto u_i(x, T+t), \quad i = 1,2$$

satisfy the equicontinuity condition $(E\chi)$ from Lemma 3.5 and then, by Lemma 3.5, there exists $\rho > 0$ such that $u_i^{e'}(\cdot, T+1) > 0$ in $B(e')$ for all $e' \in \mathbb{S}^{N-1}$ with $|e' - e| < \rho$ and $i \in J$. Hence, by Lemma 4.7, $e' \in \mathcal{N}$ for $e' \in \mathbb{S}^{N-1}$ with $|e' - e| < \rho$, and thus $\mathcal{N}$ is open. $\square$

Lemma 4.16. *For every $e \in \partial\mathcal{N}$ and every $z \in \omega(u)$ we have $z_i^e \equiv 0$ in $B(e)$ for $i \in J$.*

Proof. Let $e \in \partial\mathcal{N}$, $z \in \omega(u)$, and $t_n \to \infty$ a sequence such that $u_i^e(\cdot, t_n) \to z_i^e$ uniformly as $n \to \infty$ for all $i \in J$. We argue by contradiction and consider two cases.

Case 1: Assume that $z_i^e \not\equiv 0$ in $B(e)$ for all $i \in J$. In this case, by (4.31) and (4.32), there are $k > 0$, $d > 0$, and $\varepsilon > 0$ such that, passing to a subsequence,

$$\sup\{u_i^{e'}(x, t_n) : x \in B(e'),\ x \cdot e' \geq d\} \geq k \qquad \text{for all } n \in \mathbb{N}$$

and for all $e' \in \mathbb{S}^{N-1}$ with $|e - e'| < \varepsilon$.

By (4.31) there exists a function

$$\chi : [0, \sqrt{1 + \mathrm{diam(B)}^2}\,] \to [0, \infty) \quad \text{with } \lim_{\vartheta \to 0} \chi(\vartheta) = 0$$

such that all of the functions

$$\overline{B} \times [0,1] \to \mathbb{R}; \quad (x,t) \mapsto u_i(x, t_n + t - \frac{1}{4}) \quad \text{for } \ n \in \mathbb{N}, \ \ i \in J,$$

satisfy the equicontinuity condition $(E\chi)$ of Lemma 3.5. Let $M > 0$ be as in (4.36). Let $\rho > 0$ be the constant given by Lemma 3.5 for these choices of d, k, M, and χ. Take $\hat{e} \in \mathcal{N}$ such that $|\hat{e} - e| < \frac{1}{2}\min\{\rho, \varepsilon\}$. Since $\hat{e} \in \mathcal{N}$, we may find $\bar{n} \in \mathbb{N}$ such that

$$u_i^{\hat{e}} > 0 \qquad \text{in } B(\hat{e}) \times [t_{\bar{n}} - \frac{1}{4}, \infty) \quad \text{for all } i \in J.$$

Then applying Lemma 3.5 to the functions

$$\overline{B} \times [0,1] \to \mathbb{R}; \quad (x,t) \mapsto u_i^{\hat{e}}(x, t_{\bar{n}} + t - \frac{1}{4}) \quad \text{for } i \in J,$$

we conclude that

$$u_i^{\bar{e}}(\cdot, t_{\bar{n}} + \frac{3}{4}) > 0 \ \text{ in } B(\bar{e}) \quad \text{for all } i \in J \text{ and all } \bar{e} \in \mathbb{S}^{N-1} \text{ with } |\bar{e} - e'| < \rho.$$

Thus, in particular this holds for $\bar{e} = e$. But this implies $u_i^e(\cdot, T + \frac{3}{4}) > 0$ in $B(e)$ for all $i \in J$. Therefore $e \in \mathcal{N}$ by Lemma 4.7. This contradicts the hypothesis that $e \in \partial\mathcal{N}$, since $\mathcal{N}$ is open by Lemma 4.15.

Case 2: Assume that

$$I_1 := \{i \in J : z_i^e \equiv 0\} \neq \emptyset,$$
$$I_2 := \{i \in J : z_i^e \not\equiv 0\} \neq \emptyset.$$

In this case, by (4.6) and hypothesis (A3), we can assume without loss of generality that $z_1^e \equiv 0$, $z_2^e \not\equiv 0$, and that

$$\inf_{B(e)\times(0,\infty)} c_{12}^e \geq \sigma \qquad \text{for some } \sigma > 0. \tag{4.38}$$

Then there is $t_n \to \infty$ such that $u_i^e(\cdot, t_n) \to z_i^e$ for all $i \in J$ and by equicontinuity there are $k, \zeta > 0$ and a nonempty open subset $\Omega \subset\subset B(e)$ such that, passing to a subsequence,

$$u_2^e(x,t) > k \quad \text{in } \Omega \times [t_n - 2\zeta, t_n] \qquad \text{for all } n \in \mathbb{N}. \tag{4.39}$$

Now, by [50, Lemma 3.6] applied with $D = \Omega$ and $U = B(e)$, there are positive constants κ, κ_1, and p such that

$$\inf_{x\in\Omega} u_1^e(x,t_n) \geq \kappa[(u_1^e)^+]_{p,\Omega\times(t_n-2\zeta,t_n-\zeta)} - \kappa_1 \max_{j\in J} \sup_{\partial_P(B(e)\times(t_n-3\zeta,t_n+\zeta))} (u_j^e)^-$$

for all $n \in \mathbb{N}$. Then, since $e \in \partial\mathcal{N}$, this implies that

$$\liminf_{n\to\infty} [u_1^e]_{p,\Omega\times(t_n-2\zeta,t_n-\zeta)} \leq \kappa^{-1} \lim_{n\to\infty} \inf_{x\in\Omega} u_1^e(x,t_n) = 0,$$

since $z_1^e \equiv 0$ by assumption. But this implies, by (4.31), that

$$\liminf_{n\to\infty} \|u_1^e\|_{L^\infty(\Omega\times(t_n-2\zeta\,,\,t_n-\zeta))} = 0. \tag{4.40}$$

Let $\varphi \in C_c^\infty(B(e)\times(0,\zeta))$ be a nonnegative function such that $\varphi \not\equiv 0$ and

$$\operatorname{supp}(\varphi) \subset Q := \Omega \times \left(\frac{1}{3}\zeta\,,\, \frac{2}{3}\zeta\right).$$

In particular

$$C_1 := \|\varphi\|_{L^1(B(e)\times(0,\zeta))} = \|\varphi\|_{L^1(Q)} > 0. \tag{4.41}$$

Let

$$Q_n := \Omega \times (t_n - 2\zeta, t_n - \zeta) \qquad \text{for } n \in \mathbb{N}$$

and $\varphi_n \in C_c^\infty(Q_n)$ given by $\varphi_n(x,t) := \varphi(x, t - t_n + 2\zeta)$ for $n \in \mathbb{N}$. Then, by (4.36), (4.38), (4.39), and (4.41)

$$A_n := \int_{Q_n} \sum_{j=2}^{n} c_{1j}^e(x,t) u_j^e \varphi_n d(x,t) = \int_{Q_n} \sum_{j=2}^{n} c_{1j}^e(x,t)[(u_j^e)^+ - (u_j^e)^-]\varphi_n d(x,t)$$

$$\geq \sigma k C_1 - M C_1 \sum_{j=2}^{n} \|(u_j^e)^-\|_{L^\infty(Q_n)} \tag{4.42}$$

for all $n \in \mathbb{N}$.

On the other hand, using (4.33) and integration by parts, we get that

$$A_n = \int_{Q_n} u_1^e[-(\varphi_n)_t - \Delta\varphi_n - c_{11}^e \varphi_n] d(x,t)$$

$$\leq \int_Q |\varphi_t| + |\Delta\varphi| + M|\varphi| d(x,t) \|u_1^e\|_{L^\infty(Q_n)} = C_2 \|u_1^e\|_{L^\infty(Q_n)} \tag{4.43}$$

for all $n \in \mathbb{N}$, where $C_2 := \int_Q |\varphi_t| + |\Delta\varphi| + M|\varphi| d(x,t) > 0$.

Since $e \in \partial\mathcal{N}$,

$$\lim_{n\to\infty} \|(u_j^e)^-\|_{L^\infty(Q_n)} = 0 \qquad \text{for all } j \in J,$$

and thus

$$0 < \frac{\sigma k C_1}{C_2} \leq \liminf_{n\to\infty} \|u_1^e\|_{L^\infty(\Omega_n)} = 0$$

by (4.42), (4.43), and (4.40). This is again a contradiction. Therefore we must have that $z_i \equiv 0$ for all $i \in J$. Since $z \in \omega(u)$ and $e \in \partial\mathcal{N}$ were chosen arbitrarily, this concludes the proof of the Lemma. □

Proof of Theorem 4.6. Let $\mathcal{N}$ be as in (4.37) and let $\mathcal{M}$ be as in (1.3) with $\mathcal{U} = \{z_i : z \in \omega(u)\} = \bigcup_{i=1}^n \omega(u_i)$, where this last equality follows by compactness as in (4.13). We obviously have $\mathcal{N} \subset \mathcal{M}$. Moreover, for $e \in \mathbb{S}^{N-1}$ as in (A4), we have

$$u_i^e(\cdot, 0) \geq 0,\ u_i^e(\cdot, 0) \not\equiv 0 \qquad \text{on } B(e) \text{ for all } i \in J,$$

Lemma 4.7 then implies that $u_i^e > 0$ on $B(e) \times (0, \infty)$ for $i = 1, 2$, so that $e \in \mathcal{N}$ and thus $\mathcal{N}$ is nonempty. Moreover $\mathcal{N}$ is a relatively open subset of $\mathbb{S}^{N-1}$ by Lemma 4.15 and, by Lemma 4.16, $z \equiv z \circ \sigma_e$ for all $z \in \mathcal{U}$ and $e \in \partial\mathcal{N}$. The result now follows from Corollary 1.6. □

Remark 4.17. (i) The irreducibility condition (A3) makes the proof simpler since no normalization argument is necessary. However, the allowance of more than two equations requires a Harnack inequality for systems [50, Lemma 3.6].

(ii) The cooperative system (4.4) is *not* irreducible.

(iii) The "symmetry entanglement" or "symmetry synchronization" discussed in Remark 4.13 is easier to understand under the irreducibility assumption (A3). For instance, let $\mathcal{M}$ as in the proof of Theorem 4.6 and let t_n be a sequence of times such that $t_n \to \infty$. Then, for every $e \in \mathcal{M}$ we can infer from the proof of Lemma 4.16 the equivalence

$$\lim_{n\to\infty} \|u_i^e(\cdot,t_n)\|_{L^\infty(B(e))} = 0 \quad \text{for some } i \in J \qquad \Longleftrightarrow \qquad \lim_{n\to\infty} \|u_j^e(\cdot,t_n)\|_{L^\infty(B(e))} = 0. \quad \text{for all } j \in J$$

Actually, this statement holds for all $e \in \mathbb{S}^{N-1}$, since $\mathbb{S}^{N-1} = \mathcal{M} \cup -\mathcal{M}$ by Corollary 1.4 and Theorem 4.6. Recall that a semitrivial limit profile is an element $z \in \omega(u)$ where $z_i \equiv 0$ for some $i \in \{1,\dots,n\}$. Then, the equivalence from above implies an interesting further symmetry characterization of semitrivial limit profiles: the nonzero components must be radial functions. We will prove in the next chapter (see Theorems 5.2, 5.4, and 5.5) that this is also the case for other parabolic systems with Dirichlet boundary conditions without any irreducibility assumption.

(iv) Consider $J = \{1,2\}$ and assume that the system (4.5) satisfies (A1) and that instead of (A2) we have that

(A2)' For every $i,j \in J$, $i \neq j$, it follows that $\partial F_i(t,r,u)/\partial u_j \leq 0$ for all $t \in [0,\infty)$, $r \in I_B$, and $u \in \mathbb{R}^n$ such that the derivative exists.

Then (4.5) is a *competitive* parabolic system. In this case we can also use the Hadamard formulas (4.34) to obtain a weakly coupled system as in (4.33) for the difference function (u_1^e, u_2^e) as defined in (4.14) for $e \in \mathbb{S}^{N-1}$. Therefore we can argue as in Theorem 4.6 if we also assume that

(A3)' For each M there is a constant $\sigma = \sigma(M) > 0$ such that

$$\partial F_i(t,r,u)/\partial u_j \leq -\sigma \text{ for all } i,j = 1,2,\ i \neq j,\ r \in I_B,\ t \in [0,\infty),$$

and $u \in \mathbb{R}^n$ with $|u| \leq M$ such that the derivative exists.

(A4)' There is some $e \in \mathbb{S}^{N-1}$ such that $u_{0,i} \not\equiv u_{0,i} \circ \sigma_e$ for $i = 1,2$ and $u_{0,1}(x) \geq u_{0,1}(\sigma_e(x))$, $u_{0,2}(x) \leq u_{0,2}(\sigma_e(x))$ for all $x \in B(e)$.

However if the system does not satisfy (A3)', as for example (4.1) or (4.3), then a normalization argument as in Lemma 4.12 is needed, and further assumptions have to be made to successfully apply the normalization.

4.4 Cubic parabolic systems

We now proceed to the proof of Theorem 4.4. For this section, let $u_1, u_2 \in C^{2,1}(\overline{B} \times (0,\infty)) \cap C(\overline{B} \times [0,\infty))$ be nonnegative functions such that $u = (u_1, u_2)$ solves (4.3) where λ_i, γ_i, and α_i are positive constants for $i = 1, 2$. Further assume that (h4) and (h5) from Theorem 4.1 hold.

The proof of Theorem 4.4 follows closely the proof of Theorem 4.1 except in two steps: the estimate (4.28) in that proof and the linearization procedure.

4.4.1 Linearization of cubic parabolic systems

For the linearization, let $e \in \mathbb{S}^{N-1}$ and let $u_i^e : \overline{B(e)} \times (0,\infty) \to \mathbb{R}$ for $i = 1, 2$ be given by

$$\begin{aligned} u_1^e(x,t) &:= u_1(x,t) - u_1(\sigma_e(x),t) \\ u_2^e(x,t) &:= u_2(\sigma_e(x),t) - u_2(x,t) \end{aligned} \qquad x \in B,\ t > 0.$$

To simplify the notation we write $\hat{u}_i$ as a shortcut for the function

$$B \times (0,\infty) \to \mathbb{R}\ ; \quad (x,t) \mapsto u_i(\sigma_e(x),t) \qquad \text{for } i = 1, 2,$$

and omit the arguments (x,t). Then we have that u_1^e satisfies that

$$\begin{aligned} &(u_1^e)_t - \Delta u_1^e = \lambda_1 u_1 + \gamma_1 u_1^3 - \alpha_1 u_1 u_2^2 - (\lambda_1 \hat{u}_1 + \gamma_1 \hat{u}_1^3 - \alpha_1 \hat{u}_1 \hat{u}_2^2) \\ &= \lambda_1 u_1^e + \gamma_1 (u_1^2 + u_1 \hat{u}_1 + \hat{u}_1^2) u_1^e - \alpha_1 u_1 u_2^2 + \alpha_1 \hat{u}_1 \hat{u}_2^2 + \alpha_1 u_1 \hat{u}_2^2 - \alpha_1 u_1 \hat{u}_2^2 \\ &= (\lambda_1 + \gamma_1 (u_1^2 + u_1 \hat{u}_1 + \hat{u}_1^2) - \alpha_1 \hat{u}_2^2) u_1^e + \alpha_1 u_1 (\hat{u}_2^2 - u_2^2) \\ &= (\lambda_1 + \gamma_1 (u_1^2 + u_1 \hat{u}_1 + \hat{u}_1^2) - \alpha_1 \hat{u}_2^2) u_1^e + \alpha_1 u_1 (\hat{u}_2 + u_2) u_2^e \end{aligned}$$

in $B(e) \times (0,\infty)$, and u_2^e satisfies a similar equation. Therefore, we have that

$$\begin{aligned} (u_1^e)_t - \mu_1 \Delta u_1^e - c_1^e u_1^e &= a_1^e u_2^e && \text{in } B(e) \times (0,\infty), \\ (u_2^e)_t - \mu_2 \Delta u_2^e - c_2^e u_2^e &= a_2^e u_1^e && \text{in } B(e) \times (0,\infty), \\ \frac{\partial u_1^e}{\partial \nu} = \frac{\partial u_2^e}{\partial \nu} &= 0 && \text{on } \Sigma_2(e) \times (0,\infty), \\ u_1^e = u_2^e &= 0 && \text{on } \Sigma_1(e) \times (0,\infty), \end{aligned} \tag{4.44}$$

where the sets $\Sigma_i(e)$ are given as in (3.15) and the coefficients $c_i^e, a_i \in L^\infty(B \times (0,\infty))$ are given by

$$\begin{aligned} c_i^e(x,t) &:= \lambda_i + \gamma_i [u_i^2 + u_i u_i(\sigma_e(x),t) + u_i^2(\sigma_e(x),t)] - \alpha_i u_j^2(\sigma_e(x),t), \\ a_i^e(x,t) &:= \alpha_i u_i [u_j + u_j(\sigma_e(x),t)] \geq 0, \end{aligned}$$

for all $i, j = 1, 2$, $i \neq j$.

Arguing as in Remark 4.9 using

$$g_i(x,t) := \lambda_i u_i(x,t) + \gamma_i (u_i(x,t))^3 - \alpha_i u_i(x,t)(u_j(x,t))^2$$

for $i = 1, 2$, $i \neq j$, we have that

$$\lim_{t\to\infty} \inf_{(z_1,z_2)\in\omega(u)} \|u_1(\cdot,t) - z_1\|_{L^\infty(B)} + \|u_2(\cdot,t) - z_2\|_{L^\infty(B)} = 0.$$

and that

$$\|u_i\|_{C^{a,a/2}(\overline{B}\times[s,s+1])} < C \quad \text{for all } s \in [1,\infty),\ i = 1,2 \tag{4.45}$$

for some $a \in (1, 2)$ and $C > 0$.

4.4.2 Main result for cubic systems

In this subsection we use the same notation employed in Subsection 4.2.2 to denote the *analog* set with respect to a solution (u_1, u_2) of (4.3). This should cause no confusion. Let

$$\mathcal{N} := \{e \in \mathbb{S}^{N-1} : u_i^e > 0 \text{ in } B(e) \times [T,\infty) \text{ for } i = 1,2 \text{ and some } T > 0\}. \tag{4.46}$$

Lemma 4.18. *The set $\mathcal{N}$ is relatively open in $\mathbb{S}^{N-1}$.*

Proof. The result follows in exactly the same way as in the proof of Lemma 4.11. □

Let $e \in \mathbb{S}^{N-1}$. We define the functions $z_i^e : \overline{B(e)} \to \mathbb{R}$ for $i = 1, 2$ by

$$\begin{aligned} z_1^e(x) &:= z_1(x) - z_1(\sigma_e(x)) \\ z_2^e(x) &:= z_2(\sigma_e(x)) - z_2(x) \end{aligned} \qquad x \in \overline{B},\ t > 0.$$

Then we have the following

Lemma 4.19. *For every $e \in \partial\mathcal{N}$ and every $z \in \omega(u)$ we have $z_1^e \equiv z_2^e \equiv 0$ in $B(e)$.*

Proof. Let $z = (z_1, z_2) \in \omega(u)$, and consider an increasing sequence $t_n \to \infty$ with $t_1 > 6$ and such that $u_i(\cdot, t_n) \to z_i$ uniformly in $\overline{B}$ for $i = 1, 2$. We only show that $z_2^e \equiv 0$ in $B(e)$ for all $e \in \partial\mathcal{N}$, since an analogous argument shows that $z_1^e \equiv 0$ in $B(e)$ for all $e \in \partial\mathcal{N}$. By (4.45) there exists a function

$\chi : [0, \sqrt{1 + \mathrm{diam(B)}^2}\,] \to [0, \infty)$ with $\lim_{\vartheta \to 0} \chi(\vartheta) = 0$ and such that all of the functions

$$\overline{B} \times [0,1] \to \mathbb{R}; \qquad (x,t) \mapsto u_2(x, \tau + t), \quad \tau \geq 1, \tag{4.47}$$

satisfy the equicontinuity condition $(E\chi)$ of Lemma 3.5. Arguing by contradiction, we now assume that $z_2^{\hat{e}} \not\equiv 0$ in $B(\hat{e})$ for some $\hat{e} \in \partial\mathcal{N}$. By the equicontinuity of the functions in (4.47), there exists $\zeta \in (0, \frac{1}{4})$, a nonempty open subset $\Omega \subset\subset B(\hat{e})$ and $k_1 > 0$ such that, after passing to a subsequence,

$$u_2^{\hat{e}} \geq k_1 \quad \text{on } \Omega \times [t_n - \zeta, t_n + \zeta] \quad \text{for all } n \in \mathbb{N}. \tag{4.48}$$

We now apply a normalization procedure for u_1. Define, for $n \in \mathbb{N}$,

$$I_n := [t_n - 2, t_n + 2] \subset \mathbb{R}, \qquad \beta_n := \|u_1(\cdot, t_n)\|_{L^\infty(B)}$$

and the functions

$$v_n : \overline{B} \times I_n \to \mathbb{R}, \qquad v_n(x,t) := \frac{u_1(x,t)}{\beta_n}.$$

Note that u_1 satisfies (4.9) with $\mu \equiv \mu_1$ and

$$c := \lambda_1 + \gamma_1 u_1^2 - \alpha_1 u_2^2 \;\in\; L^\infty(B \times (0, \infty)). \tag{4.49}$$

Therefore, by Lemma 4.8, there is $\eta > 1$ such that

$$\frac{1}{\eta} \;\leq\; v_n \;\leq\; \eta \quad \text{in } B \times I_n \qquad \text{for all } n \in \mathbb{N}. \tag{4.50}$$

By (4.49), (4.50), and the fact that v_n satisfies

$$\begin{aligned} (v_n)_t - \mu_1 \Delta v_n &= c\, v_n \qquad && \text{in } B \times I_n, \\ \partial_\nu v_n &= 0 && \text{on } \partial B \times I_n \end{aligned}$$

for all $n \in \mathbb{N}$, we have that Lemma 2.9 implies

$$\|v_n\|_{C^{1+\gamma,(1+\gamma)/2}(\overline{B} \times [s,s+1])} < L \quad \text{for all } s \in [t_n - 1, t_n + 1],\; n \in \mathbb{N},$$

and for some $\gamma \in (0,1)$ and $L > 0$.

As a consequence, by adjusting the function χ above, we may also assume that all of the functions

$$\overline{B} \times [0,1] \to \mathbb{R}; \quad (x,t) \mapsto v_n(x, s + t) \quad \text{for } s \in [t_n - 1, t_n + 1],\; n \in \mathbb{N},$$

satisfy the equicontinuity condition $(E\chi)$ of Lemma 3.5.

For $e \in \mathbb{S}^{N-1}$ and $n \in \mathbb{N}$ we also consider

$$v_n^e : \overline{B} \times I_n \to \mathbb{R}, \qquad v_n^e(x,t) := v_n(x,t) - v_n(\sigma_e(x),t),$$

and we note that

$$\begin{aligned} (v_n^e)_t - \mu_1 \Delta v_n^e - c_1^e v_n^e &= \beta_n^{-1} a_1^e u_2^e && \text{in } B(e) \times (0,\infty), \\ (u_2^e)_t - \mu_2 \Delta u_2^e - c_2^e u_2^e &= \beta_n a_2^e v_n^e && \text{in } B(e) \times (0,\infty), \\ \partial_\nu v_n^e = \partial_\nu u_2^e &= 0 && \text{on } \Sigma_2(e) \times I_n, \\ v_n^e(x,t) = u_2^e(x,t) &= 0 && \text{on } \Sigma_1(e) \times I_n \end{aligned} \tag{4.51}$$

with $\Sigma_i(e)$ as defined in (3.15), where

$$\beta_n^{-1} a_1^e(x,t) = \alpha_1 v_n [u_2 + u_2(\sigma_e(x),t)] \ \in \ L^\infty(B(e) \times I_n).$$

Let

$$Q_n := B(\hat{e}) \times [t_n - \zeta, t_n + \zeta]) \quad \text{ for all } n \in \mathbb{N}$$

and ζ as in (4.48). We now distinguish two cases.

$$\underline{\text{Case 1:}} \qquad \limsup_{n\to\infty} \|v_n^{\hat{e}}\|_{L^\infty(Q_n)} > 0.$$

This case leads to a contradiction in exactly the same way as in Case 1 in the proof of Lemma 4.12, see (4.26).

$$\underline{\text{Case 2:}} \qquad \lim_{n\to\infty} \|v_n^{\hat{e}}\|_{L^\infty(Q_n)} = 0. \tag{4.52}$$

In this case let

$$Q := B(\hat{e}) \times (-\zeta, \zeta)$$

and fix a nonnegative function $\varphi \in C_c^\infty(Q)$ with $\varphi \equiv 1$ on $\Omega \times (-\frac{\zeta}{2}, \frac{\zeta}{2})$. Moreover, let $\varphi_n \in C_c^\infty(Q_n)$ be given by $\varphi_n(x,t) := \varphi(x, t_n + t)$. We have that

$$\begin{aligned} A_n &:= \int_{Q_n} \frac{a_1^{\hat{e}}(x,t)}{\beta_n} u_2^{\hat{e}} \varphi_n d(x,t) = \int_{Q_n} \alpha_1 [u_2(x,t) + u_2(\sigma_{\hat{e}}(x),t)] v_n u_2^{\hat{e}} \varphi_n d(x,t) \\ &= \int_{Q_n} \alpha_1 [2u_2(x,t) + u_2^{\hat{e}}(x,t)][(u_2^{\hat{e}})^+ - (u_2^{\hat{e}})^-] \varphi_n v_n d(x,t) \\ &\geq \alpha_1 \eta^{-1} k_1^2 |\Omega| \zeta - 2\eta\alpha_1 \|u_2\|_{L^\infty(B\times(0,\infty))} \int_{Q_n} (u_2^{\hat{e}})^- \varphi_n d(x,t), \end{aligned}$$

by (4.50) and (4.48). Since the last term on the right hand side goes to zero as $n \to \infty$ because $\hat{e} \in \partial\mathcal{N}$, we have that $\liminf_{n\to\infty} A_n > 0$. On the other hand, integrating by parts, we have by (4.51) that

$$\begin{aligned} A_n &= \int_{Q_n} [(v_n^{\hat{e}})_t - \mu_1 \Delta v_n^{\hat{e}} - c_1^{\hat{e}} v_n^{\hat{e}}]\varphi_n d(x,t) \\ &= -\int_{Q_n} [v_n^{\hat{e}}(\varphi_n)_t + v_n^{\hat{e}}\Delta(\mu_1\varphi_n) + c_1^{\hat{e}} v_n^{\hat{e}}\varphi_n] d(x,t) \\ &\le \|v_n^{\hat{e}}\|_{L^\infty(\Omega_n)} \int_Q |\varphi_t| + \mu_1|\Delta\varphi| + M\varphi \; d(x,t). \end{aligned}$$

Then (4.52) implies that $\limsup_{n\to\infty} A_n \le 0$, and we also obtain a contradiction in this case. This concludes the proof. □

We now proceed to the

Proof of Theorem 4.4. Let $\mathcal{N}$ be as in (4.46) and define

$$\begin{aligned} \mathcal{U} &:= \omega(u_1) \cup -\omega(u_2) = \{z_1, -z_2 \;:\; z \in \omega(u)\}, \\ \mathcal{M} &:= \{e \in \mathbb{S}^{N-1} \;:\; z \ge z \circ \sigma_e \quad \text{in } B(e) \;\; \text{for all } z \in \mathcal{U}\}. \end{aligned}$$

We obviously have $\mathcal{N} \subset \mathcal{M}$. Moreover, for $e \in \mathbb{S}^{N-1}$ as in (h5), we have

$$u_i^e(\cdot,0) \ge 0, \; u_i^e(\cdot,0) \not\equiv 0 \qquad \text{on } B(e) \text{ for } i = 1,2.$$

Lemma 4.7 then implies that $u_i^e > 0$ on $B(e) \times (0,\infty)$ for $i = 1,2$, so that $e \in \mathcal{N}$ and thus $\mathcal{N}$ is nonempty. Moreover $\mathcal{N}$ is a relatively open subset of $\mathbb{S}^{N-1}$ by Lemma 4.18 and, by Lemma 4.19, $z \equiv z \circ \sigma_e$ for all $z \in \mathcal{U}$ and $e \in \partial\mathcal{N}$. The result now follows from Corollary 1.6. □

Remark 4.20. (i) We believe that this method can also be used to handle a wider variety of competing systems, whenever the difference function u_i^e satisfies a weakly coupled system like (4.44) (see Remark 4.17 (iv) in this regard) *and* whenever the estimate (4.28) can be executed in a similar way.

(ii) We have considered problem (4.3) with constant coefficients for simplicity, but from the proof it is clear that the result can be easily generalized to the case where μ_1, μ_2 are functions satisfying (h2), α_1, α_2, functions satisfying (h3) and $\lambda_i, \gamma_i \in L^\infty(I_B \times (0,\infty))$ for $i = 1,2$.

4.5 Existence of stationary solutions without foliated Schwarz symmetry

We now proceed to the proof of Theorem 4.3.

Proof of Theorem 4.3. We find (u_1, u_2) via the method of sub- and supersolutions. For this we fix $a \geq 16k^2$ and we define the set

$$\Omega_0 := \Big\{(r\cos\theta, r\sin\theta)\ :\ \frac{1}{2} < r < 1,\ 0 < \theta < \frac{2\pi}{k}\Big\} \subset B$$

and the auxiliary function $\varphi \in C^1(\overline{\Omega_0})$ in polar coordinates by

$$\varphi(r,\theta) := \begin{cases} 1 + \cos(2k\theta), & \text{if } \theta \in (\frac{\pi}{2k}, \frac{3\pi}{2k}), \\ 0, & \text{if } \theta \in [0, \frac{\pi}{2k}] \cup [\frac{3\pi}{2k}, \frac{2\pi}{k}]. \end{cases}$$

Note that $\varphi \in W^{2,\infty}(\Omega_0)$. Moreover, since $a \geq 16k^2$, we have that

$$\begin{aligned} [-\Delta\varphi - a\varphi + \varphi^2](r,\theta) &= \Big[\frac{(2k)^2}{r^2} - a\Big]\cos(2k\theta) - a + \varphi^2(r,\theta) \\ &\leq (a - \frac{(2k)^2}{r^2}) - a + 4 \ \leq\ -4k^2 + 4 \ \leq\ 0 \end{aligned}$$

if $(r,\theta) \in (\frac{1}{2}, 1) \times (\frac{\pi}{2k}, \frac{3\pi}{2k})$ and $[-\Delta\varphi - a\varphi + \varphi^2](r,\theta) = 0$ elsewhere. Next we define the rotated domains

$$\Omega_n := \{(r\cos\theta, r\sin\theta)\ :\ \frac{1}{2} < r < 1,\ \frac{2n\pi}{k} < \theta < \frac{2(n+1)\pi}{k}\}$$

for $n = 0, \dots, k-1$ and the functions $\psi_1, \psi_2 \in W^{2,\infty}(B) \cap C^1(\overline{B})$ in polar coordinates by

$$\begin{aligned} \psi_1(r,\theta) &:= \varphi\Big(\theta - \frac{2n\pi}{k}\Big) && \text{if } (r,\theta) \in \Omega_n \text{ for some } n \in \{0, \dots, k-1\}, \\ \psi_2(r,\theta) &:= \varphi\Big(\theta + \frac{(1-2n)\pi}{k}\Big) && \text{if } (r,\theta) \in \Omega_n \text{ for some } n \in \{0, \dots, k-1\}. \end{aligned}$$

Since $\psi_1\psi_2 \equiv 0$ in B, it follows from the above computations on φ and the boundary conditions $\partial_\nu \psi_i = 0$ on ∂B for $i = 1, 2$, that (ψ_1, ψ_2) is a subsolution of (4.2) for arbitrary $\alpha > 0$. Now, for the supersolution, fix an arbitrary $\varepsilon \in (0,1)$ and define $\psi_i^\varepsilon := \psi_i + \varepsilon \in W^{2,\infty}(B) \cap C^1(\overline{B})$ for $i = 1, 2$. Moreover, put $\alpha_{k,a} := \frac{1}{\varepsilon^2}\big[\max_{i=1,2} \|\Delta\psi_i\|_{L^\infty(B)} + 3a\big]$. Then for $\alpha \geq \alpha_{k,a}$ and $i = 1, 2$, we have

$$-\Delta\psi_i^\varepsilon - a\psi_i^\varepsilon + (\psi_i^\varepsilon)^2 + \alpha\psi_1^\varepsilon\psi_2^\varepsilon \geq -\Delta\psi_i - 3a + \alpha\varepsilon^2 \geq 0 \quad \text{in } B$$

Hence $(\psi_1^\varepsilon, \psi_2^\varepsilon)$ is a supersolution of (4.2) for $\alpha \geq \alpha_{k,a}$. Now, for $\alpha \geq \alpha_{k,a}$, a variant of the standard method of sub- and supersolutions (see e.g. [11, 20, 23]) yields the existence of a classical solution (u_1, u_2) of (4.2) such that $\psi_i \leq u_i \leq \psi_i^\varepsilon$ in B for $i = 1, 2$. Moreover, since $\varepsilon \in (0, 1)$, it is easy to see that the angular derivatives $\frac{\partial u_i}{\partial \theta}$, $i = 1, 2$ change sign at least k times on every circle contained in $\overline{B}$. □

Chapter 5

Parabolic systems with Dirichlet boundary conditions

The main objective of this chapter is to establish the following analog of Theorem 4.1 for competitive parabolic systems with *Dirichlet boundary conditions.* For simplicity we just consider the case $\mu_1 \equiv \mu_2 \equiv 1$.

Theorem 5.1. *Let $B \subset \mathbb{R}^N$ be a ball or an annulus and let $u_1, u_2 \in C^{2,1}(\overline{B} \times (0,\infty)) \cap C(\overline{B} \times [0,\infty))$ be nonnegative functions such that $u = (u_1, u_2)$ is a solution of the system*

$$\begin{aligned} (u_i)_t - \Delta u_i &= f_i(t, |x|, u_i) - \alpha_i(|x|, t) u_1 u_2, && x \in B,\ t > 0, \\ u_i(x,t) &= 0, && x \in \partial B,\ t > 0, \\ u_i(x,0) &= u_{0,i}(x), && x \in B \end{aligned} \tag{5.1}$$

for $i = 1, 2$, where the following holds.

$(h1)_D$ *For $i = 1,2$, the nonlinearity $f_i : [0,\infty) \times I_B \times [0,\infty) \to \mathbb{R}$, $(t,r,v) \mapsto f_i(t,r,v)$ is continuously differentiable in v. Further, the functions f_i and $\partial_v f_i$ are Hölder continuous in t and r for all $v \in [0,\infty)$, and locally Lipschitz continuous in v uniformly with respect to t and r. In other words, there is $\gamma > 0$ such that, for every $h \in \{f_i, \partial_v f_i \,:\, i = 1,2\}$,*

$$h(\cdot,\cdot,v) \in C^{\gamma,\gamma/2}(B \times (0,\infty)) \qquad \text{for all } v \in [0,\infty)$$

and

$$\sup_{\substack{r \in I_B, t>0,\\ v,\bar{v} \in K, v \neq \bar{v}}} \frac{|h(t,r,v) - h(t,r,\bar{v})|}{|v - \bar{v}|} < \infty,$$

for any compact subset $K \subset [0,\infty)$*. Moreover* $f_i(t,r,0) = 0$ *for all* $r \in I_B$, $t > 0$*, and* $i \in \{1,2\}$*.*

$(h2)_D$ *There are positive constants* α^*, α_**, and* β *such that* $\alpha_i \in C^{\beta,\beta/2}(I_B \times (0,\infty))$ *and* $\alpha_* \leq \alpha_i(r,t) \leq \alpha^*$ *for all* $r \in I_B$, $t > 0$*, and* $i \in \{1,2\}$*.*

If $u_{0,i}$ *satisfies hypothesis* $(h5)$ *from Theorem 4.1, then there is some* $p \in \mathbb{S}^{N-1}$ *such that all elements of* $\omega(u_1)$ *are foliated Schwarz symmetric with respect to* p*, and all elements of* $\omega(u_2)$ *are foliated Schwarz symmetric with respect to* $-p$*.*

The proof relies on a parabolic version of Serrin's boundary point lemma [56, Lemma 1] to achieve an analog of Lemma 3.5 to use a similar rotating plane argument as in Theorem 4.1. Moreover, the normalization procedure now relies on sharp estimates due to Húska, Polàčik, and Safonov [35] (see Corollary 5.12 below).

The second objective of this chapter is to study more thoroughly the symmetry properties of the *semi-trivial limit profiles*, that is, elements of $\omega(u)$ where one of the components is zero. As explained in the introduction of the thesis, systems of the type (5.1) are commonly used to model population dynamics. In this setting, the existence of a semi-trivial limit profile can be interpreted as the asymptotic extinction of one of the species. In such a situation, one may guess that the remaining species, in the (asymptotic) absence of a competing (or symbiotic) species, is likely to become asymptotically radially symmetric in B. We show that this is indeed the case if additional assumptions are made.

Our first result in this regard assumes a particular structure for the nonlinearity. For the rest of the Chapter, let $\lambda_1 > 0$ denote the first Dirichlet eigenvalue of the Laplacian in B.

Theorem 5.2. *Let* B *be a ball or an annulus and let* $u_1, u_2 \in C^{2,1}(\overline{B} \times (0,\infty)) \cap C(\overline{B} \times [0,\infty)) \cap L^\infty(B \times (0,\infty))$ *be nonnegative functions such that* $u = (u_1, u_2)$ *is a classical solution of*

$$\begin{aligned} (u_i)_t - \Delta u_i &= f_i(u_i) - \alpha_i(x,t) u_1 u_2 && \text{in } B \times (0,\infty), \\ u_i &= 0 && \text{on } \partial B \times (0,\infty), \\ u_i(x,0) &= u_{0,i}(x) && \text{for all } x \in B, i = 1,2, \end{aligned} \tag{5.2}$$

where $u_{0,i} \in C_0(B)$ *is not identically zero and* $\alpha_i \in L^\infty(B \times (0,\infty))$ *for* $i = 1,2$*. Further, assume that*

(E1) $f_i \in C^1([0,\infty))$ *is strictly concave in* $[0,\infty)$, $f_i(0) = 0$*, and*

$$f_i(s) \to -\infty \quad \text{as } s \to \infty \qquad \text{for } i = 1,2.$$

(E2) $f_i'(0) > \lambda_1$ *for* $i = 1, 2$.

Then all the semi-trivial limit profiles of u *are radially symmetric. In particular, if there is* $(z, 0) \in \omega(u)$, *then* z *is radially symmetric and it is the unique positive (weak) solution of*

$$\begin{aligned} -\Delta z = f_1(z) \quad &in \;\; B, \\ z \in C_0(B) \cap H_0^1(B)&. \end{aligned} \tag{5.3}$$

The analogous claim holds if there is $(0, z) \in \omega(u)$ *with* $z \not\equiv 0$.

Hypothesis (E1) might look a bit restrictive, but it is satisfied by many interesting models. In particular, the Lotka-Volterra model for two competing species satisfies this assumption. Theorem 5.2 then directly implies the following

Corollary 5.3. *Let* B *be a ball or an annulus and let* $u_1, u_2 \in C^{2,1}(\overline{B} \times (0, \infty)) \cap C(\overline{B} \times [0, \infty)) \cap L^\infty(B \times (0, \infty))$ *be nonnegative functions such that* $u = (u_1, u_2)$ *is a classical solution of*

$$\begin{aligned} (u_i)_t - \Delta u_i = a_i u_i - b_i u_i^2 - \alpha_i(x,t) u_1 u_2 \qquad & in \; B \times (0, \infty), \\ u_i = 0 \qquad & on \; \partial B \times (0, \infty), \\ u_i(x, 0) = u_{0,i}(x) \qquad & for \; all \; x \in B, \;\; i = 1, 2. \end{aligned}$$

where $u_{0,i} \in C_0(B)$ *is not identically zero,* $a_i > \lambda_1$, $b_i > 0$, *and* $\alpha_i \in L^\infty(B \times (0, \infty))$ *for* $i = 1, 2$. *If* $(z, 0) \in \omega(u)$ *then* $z \in C_0(B) \cap H_0^1(B)$ *is a positive radially symmetric function and it is the unique weak solution of* $-\Delta z = a_1 z - b_1 z^2$ *in* B. *The analogous claim holds for* $(0, z) \in \omega(u)$.

The Lotka-Volterra model assumes that the coefficients α_i are positive for $i = 1, 2$, but this assumption is not needed for the claim in Corollary 5.3. Indeed, Theorem 5.2 makes *no* assumption on the sign of α_1 and α_2, and therefore the result also applies to semitrivial limit profiles of cooperative, competitive, and predator-prey type systems (see (4.30)) with Dirichlet boundary conditions and nonlinearities satisfying (E1) and (E2). However, the standard predator-prey model does not satisfy (E2), see (5.4) below. Nonetheless, the symmetry of semi-trivial limit profiles can still be characterized using a slightly different approach. We have the following

Theorem 5.4. *Let* B *be a ball or an annulus and let* $u_1, u_2 \in C^{2,1}(\overline{B} \times (0, \infty)) \cap C(\overline{B} \times [0, \infty)) \cap L^\infty(B \times (0, \infty))$ *be nonnegative functions such that*

$u = (u_1, u_2)$ is a classical solution of

$$\begin{aligned} (u_1)_t - \Delta u_1 &= a_1 u_1 - b_1 u_1^2 - \alpha_1(x,t) u_1 u_2 && \text{in } B \times (0,\infty), \\ (u_2)_t - \Delta u_2 &= -a_2 u_2 - b_2 u_2^2 + \alpha_2(x,t) u_1 u_2 && \text{in } B \times (0,\infty), \\ u_i &= 0 && \text{on } \partial B \times (0,\infty), \\ u_i(x,0) &= u_{0,i}(x) && \text{for all } x \in B \end{aligned} \tag{5.4}$$

and $i = 1, 2$, where $u_{0,1}$, $u_{0,2} \in C_0(B)$ are not identically zero, $a_1 > \lambda_1$, $a_2 > 0$, $b_1 > 0$, $b_2 > 0$, and $\alpha_1, \alpha_2 \in L^\infty(B \times (0,\infty))$ are nonnegative functions. Then $0 \notin \omega(u_1)$ and if $(z, 0) \in \omega(u)$ then $z \in C_0(B) \cap H_0^1(B)$ is a positive radially symmetric function and it is the unique weak solution of $-\Delta z = a_1 z - b_1 z^2$ in B.

Here u_1 is considered to be "the prey" whereas u_2 is "the predator". The fact that the coefficient a_2 appears with a minus in (5.4) is interpreted in the model as the effect of diseases and other hazards for the predator. This minus sign is the reason why system (5.4) does not satisfy assumption (E2) in Theorem 5.2. In this case, however, the extinction of the prey is impossible, because if u_1 is close to zero then u_2 also tends to zero (the predator needs the prey to survive), and then the birth rate $a_1 > \lambda_1$ (where as before $\lambda_1 > 0$ denotes the first Dirichlet eigenvalue of the Laplacian in B) is big enough to force a recovery of u_1. We use this fact to characterize the symmetry of the semitrivial limit profiles $(z, 0)$.

In another context, we can consider systems of equations where it is possible to ensure that the zero possesses some stability, in the sense that if $0 \in \omega(u_i)$ for some $i = 1, 2$, then $\omega(u_i) = \{0\}$. In this case the asymptotic symmetry of the solution can be characterized using results due to Földes and Polàčik [25] for asymptotically symmetric equations. This results however are only available for convex domains and therefore B must be a ball. For simplicity we just develop these ideas for a very restricted class of systems, but we remark that the results in [25] are stated in a quite general framework similar to that of Theorem 2.3. We have the following

Theorem 5.5. *Let B be a ball and let $u_1, u_2 \in C^{2,1}(\overline{B} \times (0,\infty)) \cap C(\overline{B} \times [0,\infty)) \cap L^\infty(B \times (0,\infty))$ be nonnegative functions such that $u = (u_1, u_2)$ is a classical solution of*

$$\begin{aligned} (u_i)_t - \Delta u_i &= a_i(t) u_i^2 - b_i(t) u_i - \alpha_i(x,t) u_1 u_2 && \text{in } B \times (0,\infty), \\ u_i &= 0 && \text{on } \partial B \times (0,\infty), \\ u_i(x,0) &= u_{0,i}(x) && \text{for all } x \in B \end{aligned} \tag{5.5}$$

and $i = 1, 2$, *where* $u_{0,i} \in C_0(B)$ *is not identically zero,* $\alpha_i \in C(\overline{B} \times (0, \infty)) \cap L^\infty(B \times (0, \infty))$ *is nonnegative,* $a_i, b_i \in C((0, \infty)) \cap L^\infty((0, \infty))$, *and there is* $b_* > 0$ *such that* $\inf_{x \in B, t > 0} b_i(t) > b_*$. *If there is* $(z, 0) \in \omega(u)$ *such that* $z \not\equiv 0$, *then* $\omega(u_2) = \{0\}$ *and* z *is radially symmetric and strictly decreasing in the radial variable. The analogous claim holds if there is* $(0, z) \in \omega(u)$ *such that* $z \not\equiv 0$.

We now give a brief outline of this chapter. Section 5.1 is devoted to the extension of Serrin's boundary point lemma to parabolic equations and to the proof of the main perturbation lemma in this Chapter. Section 5.2 contains the proof of Theorem 5.1 and finally Section 5.3 contains the proofs of Theorems 5.2 to 5.5.

To close this introduction, let us mention that, analogously as in Sections 4.3 and 4.4, the proof of Theorem 5.1 can be adjusted to study the symmetry properties of cooperative parabolic systems and other parabolic competitive systems. Since the arguments for these adjustments are very similar to those already presented in Sections 4.3 and 4.4 we do not give further details.

5.1 A parabolic version of Serrin's boundary point lemma

First we fix some notation. Let B be a ball or an annulus in $\mathbb{R}^N$ centered at zero and fix $0 \leq A_1 < A_2 < \infty$ such that

$$B := \begin{cases} \{x \in \mathbb{R}^N : A_1 < |x| < A_2\}, & \text{if } A_1 > 0, \\ \{x \in \mathbb{R}^N : |x| < A_2\}, & \text{if } A_1 = 0. \end{cases} \tag{5.6}$$

For a subset $A \subset \overline{B}$ and $\delta > 0$ we define

$$[A]_\delta := \{x \in \overline{B} \ : \ \text{dist}(x, A) \leq \delta\}. \tag{5.7}$$

For $\mathcal{I} \subset \mathbb{R}$, $e \in \mathbb{S}^{N-1}$, and a function $v : \overline{B} \times \mathcal{I} \to \mathbb{R}$ we define

$$v^e : \overline{B} \times \mathcal{I} \to \mathbb{R} \qquad \text{by} \qquad v^e(x, t) := v(x, t) - v(\sigma_e(x), t).$$

Fix $I := [0, 1]$ and $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N$. Let $v \in C^{2,1}(\overline{B} \times I)$ be a function such that v^{e_1} satisfies

$$\begin{aligned} v_t^{e_1} - \Delta v^{e_1} - c v^{e_1} &\geq 0 && \text{in } B(e_1) \times I, \\ v^{e_1} &= 0 && \text{on } \partial B(e_1) \times I, \\ v^{e_1} &\geq 0 && \text{in } B(e_1) \times I, \end{aligned} \tag{5.8}$$

and the following holds.

(H_{α,β_0}) There are $\alpha \in (0,1)$ and $\beta_0 > 0$ such that

$$\|v\|_{C^{2+\alpha,1+\alpha/2}(B\times I)} + \|c\|_{L^\infty(B(e_1)\times I)} \leq \beta_0.$$

(H_k) There is $k > 0$ such that $\|v^{e_1}\|_{L^\infty(B(e)\times(\frac{1}{7},\frac{4}{7}))} \geq k$.

Remark 5.6. Let $v \in C^{2,1}(\overline{B}\times I)$ be a function satisfying (H_{α,β_0}) and (H_k) for some $\alpha \in (0,1)$, $\beta_0 > 0$, and $k > 0$. If $v^{e_1} \equiv 0$ on $\partial B(e) \times I$, then there is $\delta \in (0, \frac{A_1-A_2}{2})$, depending only on α, β_0, k, and B such that $|v^{e_1}| < k$ in $[\partial B(e)]_\delta \times I$. Therefore we have that

$$\|v^{e_1}\|_{L^\infty((B(e)\setminus[\partial B(e)]_\delta)\times(\frac{1}{7},\frac{4}{7}))} \geq k. \tag{5.9}$$

Our first Lemma focuses on points of the boundary $\partial B(e_1)$ which are *not* corner points.

Lemma 5.7. *Let $v \in C^{2,1}(\overline{B} \times I)$ be a function such that v^{e_1} satisfies* (5.8) *and such that assumptions (H_{α,β_0}) and (H_k) hold for some $\alpha \in (0,1)$, $\beta_0 > 0$, and $k > 0$. Then given $\delta \in (0, \frac{A_1-A_2}{2})$, there are positive constants ε and μ depending only on δ, α, β_0, k, and B such that*

$$v^{e_1}(x,t) \geq \mu x_1 \qquad \text{for all } x \in B(e_1)\setminus[\partial B]_\delta,\ t \in [\,\frac{6}{7}\,,1\,] \tag{5.10}$$

and

$$\frac{\partial v^{e_1}}{\partial \nu} > \varepsilon \qquad \text{in} \quad (\partial B(e_1)\setminus[\partial B \cap H(e_1)]_\delta) \times [\,\frac{6}{7}\,,1\,]. \tag{5.11}$$

Here ν is the inwards unit normal vector field on $\partial B(e_1)$.

Proof. By Remark 5.6 there is $\delta_1 \in (0,\delta)$ depending only on α, β_0, k, δ, and B, such that

$$\|v^{e_1}\|_{L^\infty((B(e)\setminus[\partial B(e)]_{\delta_1})\times(\frac{1}{7},\frac{4}{7}))} \geq k. \tag{5.12}$$

Let $r < \min\{\delta_1, \frac{1}{7}\}$ be such that for any $t^* \in [\frac{6}{7}, 1]$ and $x^* \in \partial B(e_1)$ with $\operatorname{dist}(x^*, H(e_1) \cap \partial B) > \delta_1$ we have that $x^* + r\nu(x^*) =: y \in B(e_1)$ and $\partial B_r(y,t^*) \cap (\partial B(e_1) \times [0,t^*]) = \{(x^*,t^*)\}$. Fix such a pair (x^*,t^*) and define the sets

$$\begin{aligned}
D &:= B_r(y,t^*) \cap B_{\frac{r}{2}}(x^*,t^*) \cap (B(e_1) \times [0,t^*]),\\
\Gamma_1 &:= \partial B_r(y,t^*) \cap \overline{B_{\frac{r}{2}}(x^*,t^*)} \cap (\overline{B(e_1)} \times [0,t^*]),\\
\Gamma_2 &:= \overline{B_r(y,t^*)} \cap \partial B_{\frac{r}{2}}(x^*,t^*) \cap (\overline{B(e_1)} \times [0,t^*]),
\end{aligned}$$

and the function $z : D \to [0,1]$ given by

$$z(x,t) := (e^{-\gamma(|x-y|^2+(t-t^*)^2)} - e^{-\gamma r^2})e^{-\beta_0(t-t^*)} \quad \text{with } \gamma = \frac{2(N+1)}{r^2}$$

An easy calculation shows that

$$\begin{aligned} & z_t - \Delta z - cz \\ &= e^{-\beta_0(t-t^*)}e^{-\gamma(|x-y|^2+(t-t^*)^2)}2\gamma(-2\gamma|x-y|^2 + N - (t-t^*)) + (-\beta_0 - c)z \\ &\le e^{-\beta_0(t-t^*)}e^{-\gamma(|x-y|^2+(t-t^*)^2)}\gamma(-\gamma r^2 + 2(N+1)) = 0 \qquad \text{in } D. \end{aligned} \tag{5.13}$$

Let $0 < d < \min\{\operatorname{dist}(\Gamma_2, \partial B(e_1) \times [\frac{5}{7}, 1])\ ,\ \delta_1\}$ and define

$$K := \{x \in B(e_1)\ :\ \operatorname{dist}(x, \partial B(e_1)) \ge d\}.$$

Note that $\Gamma_2 \subset K \times [\frac{5}{7}, 1]$ and that K only depends on α, β_0, k, δ, and B. Then by (H_{α,β_0}), (5.12), and Lemma 2.7, there is $\mu_1 = \mu_1(\alpha, \beta_0, \delta, k, B) > 0$ such that $v^{e_1} \ge \mu_1$ in $K \times [\frac{5}{7}, 1]$. In particular, we have that $v^{e_1} \ge \mu_1$ in Γ_2. Let $w : D \to \mathbb{R}$ be given by $w := v^{e_1} - \mu_1 z$. Since $z \equiv 0$ in Γ_1 and $v^{e_1} \ge 0$ in $B(e_1)$ we have that $w \ge 0$ in Γ_1 and, since $z \le 1$ in Γ_2, it follows that $w \ge 0$ on $\partial_P D = \Gamma_1 \cup \Gamma_2$. Further, by (5.8) and (5.13) we get that $w_t - \Delta w - cw \ge 0$ in D. Thus the maximum principle implies that $w \ge 0$ in D. Since $w(x^*, t^*) = 0$ we have that $\frac{\partial w(x^*,t^*)}{\partial \nu} \ge 0$ and thus

$$\frac{\partial v^{e_1}}{\partial \nu}(x^*, t^*) \ge \mu_1 \frac{\partial z}{\partial \nu}(x^*, t^*) = 2e^{-\gamma r^2}\mu_1 \gamma r =: \varepsilon > 0.$$

Since $\delta > \delta_1$, this proves (5.11). Moreover, since $\frac{\partial v^{e_1}}{\partial e_1} \ge \varepsilon$ in $(H(e_1)\backslash[\partial B]_{\delta_1}) \times [\frac{6}{7}, 1]$, there is, by (H_{α,β_0}), some $\delta_2 = \delta_2(\alpha, \beta_0, \delta, k, B) \in (0, \delta_1)$ such that

$$\frac{\partial v^{e_1}}{\partial e_1} \ge \frac{\varepsilon}{2} \qquad \text{in} \quad ([H(e_1)\backslash[\partial B]_{\delta_1}]_{\delta_2}) \times [\frac{6}{7}, 1]. \tag{5.14}$$

On the other hand, $B(e_1)\backslash[\partial B(e_1)]_{\delta_1} \subset B(e_1)\backslash[\partial B(e_1)]_{\delta_2}$ since $\delta_2 < \delta_1$. Thus, by (H_{α,β_0}), (5.12), and Lemma 2.7, there is $\mu_2 > 0$ depending only on α, β_0, δ, k, and B such that $v^{e_1} \ge \mu_2$ in $(B(e_1)\backslash[\partial B(e_1)]_{\delta_2}) \times [\frac{5}{7}, 1]$. This, together with (5.14) easily implies (5.10) for some $\mu > 0$ depending only on α, β_0, δ, k, and B. □

We now turn our attention to the *corner points* on the boundary $\partial B(e_1)$.

Lemma 5.8. *Let $v \in C^{2,1}(\overline{B} \times I)$ be a function such that v^{e_1} satisfies* (5.8) *and such that assumptions (H_{α,β_0}) and (H_k) hold for some $\alpha \in (0,1)$,*

$\beta_0 > 0$, and $k > 0$. Then there is $\varepsilon > 0$ depending only on α, β_0, k, and B such that

$$\frac{\partial^2 v^{e_1}(x,1)}{\partial s^2} > \varepsilon \qquad \text{and} \qquad \frac{\partial^2 v^{e_1}(x,1)}{\partial \tilde{s}^2} < -\varepsilon \tag{5.15}$$

for all $x \in \partial B \cap H(e_1)$, where $s = \frac{1}{\sqrt{2}}(\nu + e_1) \in \mathbb{S}^{N-1}$, $\tilde{s} = \frac{1}{\sqrt{2}}(-\nu + e_1) \in \mathbb{S}^{N-1}$, and ν is the inwards unit normal vector field on ∂B.

Proof. Let $x^* \in \partial B \cap H(e_1)$. There is $r \in (0, \frac{1}{7})$ depending only on B such that $x^* + r\nu(x^*) =: y \in H(e_1) \cap B$ and $\partial B_r(y,1) \cap (\partial B \times I) = \{(x^*,1)\}$. Define

$$\begin{aligned} U &:= B_r(y,1) \cap B_{\frac{r}{2}}(x^*,1) \cap (B(e_1) \times I), \\ \Lambda_1 &:= \partial B_r(y,1) \cap \overline{B_{\frac{r}{2}}(x^*,1)} \cap (\overline{B(e_1)} \times I), \\ \Lambda_2 &:= \overline{B_r(y,1)} \cap \partial B_{\frac{r}{2}}(x^*,1) \cap (\overline{B(e_1)} \times I), \\ \Lambda_3 &:= \overline{B_r(y,1)} \cap \overline{B_{\frac{r}{2}}(x^*,1)} \cap (H(e_1) \times I), \end{aligned}$$

and the function $\varphi : \overline{B} \times I \to [-\operatorname{diam}(B), \operatorname{diam}(B)]$ given by

$$\varphi(x,t) := x_1(e^{-\theta(|x-y|^2+(t-1)^2)} - e^{-\theta r^2})e^{-\beta_0(t-1)} \quad \text{with } \theta = \frac{2(N+3)}{r^2}.$$

A direct calculation shows that

$$\begin{aligned} &\varphi_t - \Delta\varphi - c\varphi \\ &= e^{-\theta(|(x,t)-(y,1)|^2)-\beta_0(t-1)} 2\theta x_1(-2\theta|x-y|^2 + N + 2 - t) - (\beta_0 + c)\varphi \\ &\le e^{-\theta(|(x,t)-(y,1)|^2)-\beta_0(t-1)} 2\theta x_1(-2\theta(\frac{r}{2})^2 + N + 3) = 0 \qquad \text{in } U. \end{aligned} \tag{5.16}$$

Since $\operatorname{dist}(\Lambda_2, \partial B \times I) > 0$, there is $\delta_1 > 0$ depending only on B such that $\Lambda_2 \subset (\overline{B(e_1)} \backslash [\partial B]_{\delta_1}) \times [\frac{6}{7}, 1]$. Then, by Lemma 5.7 with $\delta = \delta_1$, there is $\mu > 0$ depending only on α, β_0, k, and B such that $v^{e_1}(x,t) \ge \mu x_1$ for $x \in (\overline{B(e_1)} \backslash [\partial B]_{\delta_1})$ and $t \in [\frac{6}{7}, 1]$. In particular $v^{e_1}(x,t) \ge \mu x_1$ for $(x,t) \in \Lambda_2$. Define the function $\psi : \overline{B} \times I \to \mathbb{R}$ by $\psi(x,t) := v^{e_1}(x,t) - \mu\varphi(x,t)$. Then $\psi \ge 0$ on Λ_2, because $\varphi(x,t) \le x_1$ for $(x,t) \in \Lambda_2$. Moreover, since $v^{e_1} \ge 0$ in $B(e_1) \times I$ and $\varphi \equiv 0$ in $\Lambda_1 \cup \Lambda_3$ by definition, we get that $\psi \ge 0$ on $\partial_P U = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$. Further, by (5.8) and (5.16), $\psi_t - \Delta\psi - c\psi \ge 0$ in U. Thus the maximum principle implies that $\psi \ge 0$ in U. Now, remember that $s(x^*) = \frac{1}{\sqrt{2}}(\nu(x^*) + e_1)$. By direct calculation $\frac{\partial\varphi}{\partial s}(x^*,1) = 0$ and, since $v^{e_1} \equiv 0$ on $\partial B(e_1) \times I$, we have that

$$\frac{\partial v^{e_1}}{\partial s}(x^*,1) = \frac{1}{\sqrt{2}}(\frac{\partial v^{e_1}}{\partial e_1}(x^*,1) + \frac{\partial v^{e_1}}{\partial \nu}(x^*,1)) = 0.$$

This implies that $\frac{\partial\psi}{\partial s}(x^*, 1) = 0 = \psi(x^*, 1)$. Since $\psi \geq 0$ in U, it follows that $\frac{\partial^2\psi}{\partial s^2}(x^*, 1) \geq 0$, and therefore

$$\frac{\partial^2 v^{e_1}}{\partial s^2}(x^*, 1) \geq \mu\frac{\partial^2\varphi}{\partial s^2}(x^*, 1) = \mu(-4e^{-\theta r^2}\theta\frac{1}{\sqrt{2}}(x - y)\cdot s) = 2\mu e^{-\theta r^2}\theta r, \tag{5.17}$$

which yields the first inequality in (5.15) with $\varepsilon := 2\mu e^{-\theta r^2}\theta r > 0$. For the second inequality, note that the function v^{e_1} is antisymmetric in x with respect to $H(e_1)$, and therefore

$$\frac{\partial^2 v^{e_1}}{\partial \tilde{s}^2}(x^*, 1) = \frac{\partial^2 v^{e_1}}{\partial(-\tilde{s})^2}(x^*, 1) = -\frac{\partial^2 v^{e_1}}{\partial s^2}(x^*, 1) \leq -\varepsilon,$$

where $\tilde{s}(x^*) = \frac{1}{\sqrt{2}}(-\nu(x^*) + e_1)$. The proof is finished. □

The next lemma will be helpful to guarantee positivity near corner points.

Lemma 5.9. *Let $w \in C^2(\overline{B(e_1)})$ be a function such that the following holds.*

(i) $w \equiv 0$ on $\partial B(e_1)$.

(ii) There is $\varepsilon > 0$ such that

$$\frac{\partial^2 w(x)}{\partial s(x)^2} > \varepsilon \qquad \text{and} \qquad \frac{\partial^2 w(x)}{\partial \tilde{s}(x)^2} < -\varepsilon \qquad \text{for all } x \in H(e_1)\cap\partial B,$$

where $s(x) = \frac{1}{\sqrt{2}}(\nu(x) + e_1)$, $\tilde{s}(x) = \frac{1}{\sqrt{2}}(-\nu(x) + e_1)$, and ν is the inwards unit normal vector field on ∂B.

(iii) There is a function $\chi : [0, \mathrm{diam}(B)\,] \to [0,\infty)$ with $\lim\limits_{\vartheta\to 0}\chi(\vartheta) = 0$ such that $|D^2w(x) - D^2w(y)| \leq \chi(|x - y|)$ for all $x, y \in \overline{B(e_1)}$.

Then there exists $\delta > 0$ depending only on ε, χ, and B such that $w \geq 0$ in $[H(e_1)\cap\partial B]_\delta \cap B(e_1)$.

Proof. By hypothesis *(iii)* there is $\delta_1 = \delta_1(\varepsilon, \chi) > 0$ such that, for every $x^* \in H(e_1)\cap\partial B$ we have that

$$\frac{\partial^2 w(x)}{\partial s(x^*)^2} > \varepsilon \quad \text{and} \quad \frac{\partial^2 w(x)}{\partial \tilde{s}(x^*)^2} < -\varepsilon \quad \text{for all } x \in \overline{B_{\delta_1}(x^*)\cap B(e_1)}. \tag{5.18}$$

If $A_1 > 0$ —where A_1 is as in (5.6)—we also assume that

$$\delta_1 < \left(\frac{1}{\sin(\frac{\pi}{4})} - 1\right)A_1. \tag{5.19}$$

We show that the claim holds for any $\delta \in (0, \frac{\delta_1}{2}]$. Without loss of generality we may assume for the rest of the proof that the domain B is two-dimensional, since otherwise we can repeat the following argument to w restricted to $B(e_1) \cap P$ where P is any plane containing $\mathbb{R}e_1$ to yield the result. By contradiction, assume there are $x^* \in H(e_1) \cap \partial B$ and $y \in B(e_1) \cap B_\delta(x^*)$ such that $w(y) < 0$. Then there are $y_1 \in H(e_1) \cap B_{\delta_1}(x^*)$ and $\lambda_1 > 0$ such that $y = y_1 + \lambda_1 \tilde{s}(x^*)$. Let $L_1 := \{\lambda \geq 0 \,:\, y_1 + \lambda \tilde{s}(x^*) \in \overline{B(e_1) \cap B_{\delta_1}(x^*)}\}$ and $A := \{y_1 + \lambda \tilde{s}(x^*) \,:\, \lambda \in L_1\}$. Note that $A \cap \partial B \neq \emptyset$, by our choice of δ and (5.19). In particular, we may find $\lambda_B > 0$ such that $y_1 + \lambda_B \tilde{s}(x^*) \in \partial B$. Moreover, due to the fact that B is assumed to be two-dimensional, there is $\lambda_2 > 0$ such that $x^* + \lambda_2 s(x^*) =: y_2 \in A$. Let $L_2 := \{\lambda \geq 0 \,:\, x^* + \lambda s(x^*) \in \overline{B(e_1) \cap B_{\delta_1}(x^*)}\}$ and define the functions $f_1 : L_1 \to \mathbb{R}$ and $f_2 : L_2 \to \mathbb{R}$ by

$$f_1(\lambda) := w(y_1 + \lambda \tilde{s}(x^*)), \qquad f_2(\lambda) := w(x^* + \lambda s(x^*)).$$

By (5.18), we have that $f_2'' > \varepsilon$ in L_2. By Assumption (i) it follows that

$$f_2'(0) = \frac{\partial w(x^*)}{\partial s(x^*)} = \frac{1}{\sqrt{2}}\left(\frac{\partial w(x^*)}{\partial \nu(x^*)} + \frac{\partial w(x^*)}{\partial e_1}\right) = 0 = w(x^*) = f_2(0).$$

Therefore $f_2(\lambda) > 0$ for $\lambda \in L_2 \cap (0, \infty)$. Since $y_2 \in A$, there is $\lambda_3 > 0$ such that $f_1(\lambda_3) = w(y_2) = f_2(\lambda_2) > 0$. But then $f_1(0) = 0$, $f_1(\lambda_1) = w(y) < 0$, $f_1(\lambda_3) > 0$, and $f_1(\lambda_B) = 0$. This contradicts the fact that $f_1'' < -\varepsilon$ in L_1 by (5.18). Therefore $w \geq 0$ in $[H(e_1) \cap \partial B]_\delta \cap B(e_1)$ as claimed. □

We are ready to prove a perturbation result for supersolutions of scalar equations.

Lemma 5.10. *Let $v \in C^{2,1}(\overline{B} \times I)$ be a function such that v^{e_1} (resp. $-v^{e_1}$) satisfies* (5.8) *and such that assumptions (H_{α,β_0}) and (H_k) hold for some $\alpha \in (0,1)$, $\beta_0 > 0$, and $k > 0$. Then there exists $\rho > 0$ depending only on α, β_0, k, and B such that $v^{e'}(\cdot, 1) \geq 0$ (resp. $-v^{e'}(\cdot, 1) \geq 0$) in $B(e')$ for all $e' \in \mathbb{S}^{N-1}$ with $|e_1 - e'| < \rho$.*

Proof. Assume that v^{e_1} satisfies (5.8) and that (H_{α,β_0}) and (H_k) hold. By Lemma 5.8 and assumption (H_{α,β_0}), there are $\varepsilon_1 > 0$ and $\rho_1 > 0$ such that

$$\frac{\partial^2 v^{e'}}{\partial s(x)^2}(x, 1) > \varepsilon_1 \quad \text{and} \quad \frac{\partial^2 v^{e'}}{\partial \tilde{s}(x)^2}(x, 1) < -\varepsilon_1, \quad x \in \partial B \cap H(e')$$

for all $e' \in \mathbb{S}^{N-1}$ with $|e' - e_1| < \rho_1$. By Lemma 5.9 there is $\delta_1 > 0$ such that

$$v^{e'}(\cdot, 1) \geq 0 \qquad \text{in } [\partial B \cap H(e')]_{\delta_1} \cap B(e') \tag{5.20}$$

for all $e' \in \mathbb{S}^{N-1}$ with $|e'-e_1| < \rho_1$. Further, by Lemma 5.7 with $\delta = \delta_1$ there is $\varepsilon_2 > 0$ with $\frac{\partial v^{e_1}}{\partial \nu}(\cdot, 1) > \varepsilon_2$ on $\partial B(e_1)\backslash[\partial B \cap H(e_1)]_{\delta_1}$. Then, by (H_{α,β_0}), there are $\delta_2 \in (0, \delta_1)$ and $\rho_2 \in (0, \rho_1)$ with the property $\frac{\partial v^{e'}}{\partial \nu}(\cdot, 1) > \varepsilon_2$ in $[\partial B(e')\backslash[\partial B \cap H(e')]_{\delta_1}]_{\delta_2}$ for all $e' \in \mathbb{S}^{N-1}$ with $|e' - e_1| < \rho_2$. Since $v^{e'}(\cdot, 1) \equiv 0$ on $\partial B(e')$ we have

$$v^{e'}(\cdot, 1) > 0 \qquad \text{in } [\partial B(e')\backslash[\partial B \cap H(e')]_{\delta_1}]_{\delta_2} \cap B(e') \tag{5.21}$$

for all $e' \in \mathbb{S}^{N-1}$ with $|e' - e_1| < \rho_2$. Moreover, by Lemma 5.7 with $\delta = \delta_2$ there is $\mu > 0$ with $v^{e_1}(x, 1) \geq \mu x_1 > \frac{\mu\delta_2}{2}$ for all $x \in B(e_1)\backslash[\partial B(e_1)]_{\delta_2}$. Then, by (H_{α,β_0}), there is $\rho_3 \in (0, \rho_2)$ such that

$$v^{e'}(x, 1) > \frac{\mu\delta_2}{2} \qquad \text{for all } x \in B(e')\backslash[\partial B(e')]_{\delta_2} \tag{5.22}$$

for all $e' \in \mathbb{S}^{N-1}$ with $|e' - e_1| < \rho_3$. Therefore (5.20), (5.21), and (5.22) imply that $v^{e'}(\cdot, 1) \geq 0$ in $B(e')$ for all $e' \in \mathbb{S}^{N-1}$ with $|e' - e_1| < \rho_3$. It is easy to check that the constant ρ_3 depends only on α, β_0, k, and B so this yields the claim for v^{e_1}.

From the proof of Lemmas 5.7 and 5.8 it is clear that their claims also hold if we write $-v^{e_1}$ instead of v^{e_1}. Therefore we can argue exactly the same if we now assume that $-v^{e_1}$ satisfies (5.8). This ends the proof. □

5.1.1 Homogeneous linear equations

We quote some recent estimates from [35].

Lemma 5.11 (Particular cases of Lemma 3.9 and Corollary 3.10 in [35]). *Let* $v \in C^{2,1}(B \times (0, \infty)) \cap C(\overline{B} \times [0, \infty))$ *be a positive solution of*

$$\begin{aligned} v_t - \Delta v - cv &= 0 \qquad \textit{in } B \times (0, \infty), \\ v &= 0 \qquad \textit{on } \partial B \times (0, \infty), \end{aligned}$$

where $\|c\|_{L^\infty(B\times(0,\infty))} < \beta_0$ *for some* $\beta_0 > 0$. *Then, there are positive constants* C_1, C_2, *and* ϑ *depending only on* B *and* β_0 *such that*

$$\frac{v(x, \tau)}{\|v(\cdot, \tau)\|_{L^\infty(B)}} \geq C_1(\text{dist}(x, \partial B))^\vartheta, \qquad x \in B,\ \tau \in [1, \infty), \tag{5.23}$$

and

$$\frac{\|v(\cdot, \tau + t)\|_{L^\infty(B)}}{\|v(\cdot, \tau)\|_{L^\infty(B)}} \in \left[C_2\, ,\, \frac{1}{C_2}\right], \qquad \tau \in (1, \infty),\ t \in [0, 1], \tag{5.24}$$

where $\text{dist}(x, \partial B) := \inf\{|x - y| \ :\ y \in \partial B\}$.

By rescaling and combining the two estimates in the previous Lemma we get the following

Corollary 5.12. *Assume the hypothesis of Lemma 5.11. For any $k > 0$ there are positive constants C and ϑ depending only on B, β_0, and k such that, for any $\tau \geq 2k$,*

$$C \geq \frac{v(x, \tau + t)}{\|v(\cdot, \tau)\|_{L^\infty(B)}} \geq C^{-1}(\operatorname{dist}(x, \partial B))^\vartheta, \qquad x \in B,\ t \in [-k, k].$$

Proof. Let $\tilde{v} : B \times (0, \infty) \to \mathbb{R}$ be given by $\tilde{v}(x, t) := v(x, kt)$ for $x \in B$, $t > 0$. Clearly $\tilde{v}$ satisfies the assumptions of Lemma 5.11. Let C_1, C_2, and ϑ be the constants (depending only on B, β_0, and k) given by Lemma 5.11 for $\tilde{v}$. Note that (5.24) implies that

$$\frac{\|\tilde{v}(\cdot, \tau)\|_{L^\infty(B)}}{\|\tilde{v}(\cdot, \tau + t)\|_{L^\infty(B)}} \in \left[C_2\,, \frac{1}{C_2}\right], \qquad \tau \in (1, \infty),\ t \in [0, 1], \tag{5.25}$$

Then, by (5.23), (5.24), and (5.25),

$$\begin{aligned} C_2^{-1}\|\tilde{v}(\cdot, \tau)\|_{L^\infty(B)} \geq \tilde{v}(x, t + \tau) &\geq C_1(\operatorname{dist}(x, \partial B))^\vartheta \|\tilde{v}(\cdot, t + \tau)\|_{L^\infty(B)} \\ &\geq C_1 C_2 (\operatorname{dist}(x, \partial B))^\vartheta \|\tilde{v}(\cdot, \tau)\|_{L^\infty(B)} \end{aligned}$$

for all $x \in B$, $\tau \geq 2$, and $t \in (-1, 1)$. The result follows. □

5.2 Dirichlet parabolic systems with competition

For the remainder of the section we assume the hypothesis of Theorem 5.1, that is, let $u_1, u_2 \in C^{2,1}(\overline{B} \times (0, \infty)) \cap C(\overline{B} \times [0, \infty))$ be nonnegative functions such that $u = (u_1, u_2)$ is a classical solution of (5.1) and such that assumptions $(h1)_D$ and $(h2)_D$ from Theorem 5.1 and assumption $(h5)$ from Theorem 4.1 are fulfilled. We begin with a regularity remark.

Remark 5.13. Arguing exactly as in Remark 4.9 we have, by Lemma 2.9, that

$$\|u_i\|_{C^{a,a/2}(\overline{B} \times [s, s+1])} < C \quad \text{ for all } s \in [1, \infty),\ i = 1, 2$$

for some $a \in (1, 2)$ and $C > 0$. Then $\omega(u)$ is a nonempty compact subset of $C_0(B) \times C_0(B)$ satisfying

$$\lim_{t \to \infty} \inf_{z \in \omega(u)} \|u_1(\cdot, t) - z_1\|_{L^\infty(B)} + \|u_2(\cdot, t) - z_2\|_{L^\infty(B)} = 0.$$

Moreover, since the functions $(x,t) \mapsto f_i(t,|x|,u_i(x,t))$ and α_i are Hölder continuous in $B \times [1,\infty)$ for $i = 1,2$, a standard regularity argument (see for example [52, Remark 48.3 and Remark 47.4 (iii)] or [3, Theorem 2.2]) implies that

$$\|u_i\|_{C^{2+\gamma,1+\gamma/2}(\overline{B}\times[s,s+1])} < C \quad \text{for all } s \in [2,\infty),\ i = 1,2 \tag{5.26}$$

for some $\gamma \in (0,1)$ and $C > 0$. We briefly outline this argument. Let $I := [0,2]$, $Q := B \times I$, and let $v \in C^{2,1}(\overline{B} \times I)$ be a solution of

$$\begin{aligned} v_t(x,t) - \Delta v(x,t) &= f(t,x), && x \in B,\ t \in I, \\ v(x,t) &= 0, && x \in \partial B,\ t \in I, \end{aligned}$$

where

$$\|v\|_{C^{1+\gamma,(1+\gamma)/2}(\overline{B}\times I)} + \|f\|_{C^{\gamma,\gamma/2}(\overline{B}\times I)} < C \tag{5.27}$$

for some $\gamma \in (0,1)$ and $C > 0$. Let

$$w : B \times I \to \mathbb{R} \quad \text{be given by} \quad w(x,t) := tv(x,t).$$

Then

$$\begin{aligned} w_t(x,t) - \Delta w(x,t) &= tf(t,x) - v(x,t), && (x,t) \in B \times I, \\ w(x,t) &= 0, && (x,t) \in \partial_P(B \times I). \end{aligned}$$

Then, by [52, Theorem 48.2] (see also [38, Theorem 4.28] and [11, Section 1.6.6]), there is a constant $C_1 > 0$ depending only on B and γ such that

$$\|w\|_{C^{2+\gamma,1+\gamma/2}(\overline{B}\times I)} \le C_1(\|w\|_{L^\infty(B\times I)} + \|tf - v\|_{C^{\gamma,\gamma/2}(\overline{B}\times I)}).$$

Then there is some constant $C_2 > 0$ depending only on C, B, and γ such that

$$\|w\|_{C^{2+\gamma,1+\gamma/2}(\overline{B}\times I)} \le C_2$$

by (5.27). But this implies that

$$\|v\|_{C^{2+\gamma,1+\gamma/2}(\overline{B}\times[1,2])} \le C_3$$

for some $C_3 > 0$ depending only on C, B, and γ. This implies (5.26) by taking suitable translations of u_i.

5.2.1 Linearization of semilinear Dirichlet systems

The linearization procedure is similar to the one given in Subsection 4.2.1. We repeat it for completeness. For $e \in \mathbb{S}^{N-1}$ we set

$$\begin{aligned} u_1^e(x,t) &:= u_1(x,t) - u_1(\sigma_e(x),t) \\ u_2^e(x,t) &:= u_2(\sigma_e(x),t) - u_2(x,t) \end{aligned} \qquad x \in B,\ t > 0.$$

The same notation is used if the functions do not depend on time. More precisely, for a pair $z = (z_1, z_2)$ of functions $z_i : \overline{B} \to \mathbb{R}$, $i = 1, 2$, we set

$$\begin{aligned} z_1^e(x) &:= z_1(x) - z_1(\sigma_e(x)) \\ z_2^e(x) &:= z_2(\sigma_e(x)) - z_2(x) \end{aligned} \qquad x \in B,\ t > 0.$$

Since $u = (u_1, u_2)$ solves (5.1), for fixed $e \in \mathbb{S}^{N-1}$ we have

$$\begin{aligned} (u_1^e)_t - \Delta u_1^e - c_1^e u_1^e &= \alpha_1 u_1 u_2^e && \text{in } B(e) \times (0,\infty), \\ (u_2^e)_t - \Delta u_2^e - c_2^e u_2^e &= \alpha_2 u_2 u_1^e && \text{in } B(e) \times (0,\infty), \\ u_i^e &= 0 && \text{on } \partial B(e) \times (0,\infty), \end{aligned} \tag{5.28}$$

where $c_i^e \in L^\infty(B \times (0,\infty))$, $i = 1, 2$, is given by

$$\begin{aligned} c_1^e(x,t) &:= \hat{c}_1^e(x,t) - \alpha_1(|x|,t) u_2(\sigma_e(x),t) \\ c_2^e(x,t) &:= \hat{c}_2^e(x,t) - \alpha_2(|x|,t) u_1(\sigma_e(x),t) \end{aligned} \qquad x \in B,\ t > 0,$$

with

$$\hat{c}_i^e(x,t) := \begin{cases} \dfrac{f_i(t,|x|,u_i(x,t)) - f_i(t,|x|,u_i(\sigma_e(x),t))}{u_i(x,t) - u_i(\sigma_e(x),t)}, & \text{if} \quad u_i^e(x,t) \neq 0, \\ 0, & \text{if} \quad u_i^e(x,t) = 0 \end{cases}$$

for $x \in B$, $t > 0$, and $i = 1, 2$.

Moreover, by hypothesis $(h1)_D$ and $(h2)_D$ from Theorem 5.1, there is $M \geq 1$ such that

$$\|\alpha_i\|_{L^\infty(B\times(0,\infty))} + \|c_i^e\|_{L^\infty(B\times(0,\infty))} \leq M \tag{5.29}$$

for all $i = 1, 2$ and $e \in \mathbb{S}^{N-1}$. Further, by (5.26), me may also assume that

$$\|u_i\|_{C^{2+\gamma,1+\gamma/2}(\overline{B}\times[s,s+1])} < M \quad \text{for all } s \in [2,\infty),\ i = 1, 2 \tag{5.30}$$

for some $\gamma \in (0,1)$, by making M larger.

5.2.2 Proof of Theorem 5.1

Let

$$\mathcal{N} := \{e \in \mathbb{S}^{N-1} : u_i^e > 0 \text{ in } B(e) \times [T, \infty) \text{ for } i = 1, 2 \text{ and some } T > 0\}.$$

We have the following two Lemmas which we prove at the end of this subsection.

Lemma 5.14. *The set $\mathcal{N}$ is relatively open in $\mathbb{S}^{N-1}$.*

Lemma 5.15. *For every $e \in \partial\mathcal{N}$ and every $z \in \omega(u)$ we have $z_1^e \equiv z_2^e \equiv 0$ in $B(e)$.*

Then we can proceed to the

Proof of Theorem 5.1. Define

$$\mathcal{U} := \omega(u_1) \cup -\omega(u_2) = \{z_1, -z_2 \,:\, z \in \omega(u)\} \tag{5.31}$$

and

$$\mathcal{M} := \{e \in \mathbb{S}^{N-1} \,:\, z^e \geq 0 \quad \text{in } B(e) \text{ for all } z \in \mathcal{U}\}.$$

Note that the last equality in (5.31) is a consequence of (4.13). Then we have that $\mathcal{N} \subset \mathcal{M}$. Moreover, for $e \in \mathbb{S}^{N-1}$ as in assumption (h5), we have

$$u_i^e(\cdot, 0) \geq 0,\ u_i^e(\cdot, 0) \not\equiv 0 \quad \text{in } B(e) \quad \text{for } i = 1, 2.$$

The maximum principle then implies that $u_i^e > 0$ on $B(e) \times (0, \infty)$ for $i = 1, 2$, so that $e \in \mathcal{N}$ and thus $\mathcal{N}$ is nonempty. Moreover $\mathcal{N}$ is a relatively open subset of $\mathbb{S}^{N-1}$ by Lemma 5.14 and, by Lemma 5.15, $z \equiv z \circ \sigma_e$ for all $z \in \mathcal{U}$ and $e \in \partial\mathcal{N}$. The result now follows from Corollary 1.6. □

Proof of Lemma 5.14. Let $e \in \mathcal{N}$. Then (u_1^e, u_2^e) is a solution of (5.28), and there is $T > 0$ such that u_1^e and u_2^e are positive in $B(e) \times (T, \infty)$. Thus

$$\begin{aligned} (u_1^e)_t - \Delta u_1^e - c_1^e u_1^e &= \alpha_1 u_1 u_2^e \geq 0 \\ (u_2^e)_t - \Delta u_2^e - c_2^e u_2^e &= \alpha_2 u_2 u_1^e \geq 0 \end{aligned} \qquad \text{in } B(e) \times [T, \infty),$$

since α_1 and α_2 are nonnegative by hypothesis $(h2)_D$. Without loss of generality we may assume that $e = e_1$. Then, by (5.30), the functions

$$\overline{B \times I} \to \mathbb{R}, \qquad (x, t) \mapsto u_i(x, T + t), \qquad i = 1, 2,$$

satisfy the assumptions of Lemma 5.10. Therefore we find that there exists $\rho > 0$ such that $u_i^{e'}(\cdot, T + 1) \geq 0$ in $B(e')$ for $e' \in \mathbb{S}^{N-1}$ with $|e' - e| < \rho$. Hence, by the maximum principle for systems, $e' \in \mathcal{N}$ for $e' \in \mathbb{S}^{N-1}$ with $|e' - e| < \rho$, and thus $\mathcal{N}$ is open. □

Proof of Lemma 5.15. Let $z = (z_1, z_2) \in \omega(u)$, and consider an increasing sequence $t_n \to \infty$ with $t_1 > 6$ and such that $u_i(\cdot, t_n) \to z_i$ uniformly in $\overline{B}$ for $i = 1, 2$. We will only show that $z_2^e \equiv 0$ in $B(e)$ for all $e \in \partial\mathcal{N}$, since the same argument shows that $z_1^e \equiv 0$ in $B(e)$ for all $e \in \partial\mathcal{N}$. Let $M > 0$ be as in (5.29), (5.30) and $\gamma \in (0,1)$ as in (5.30). Then all the functions

$$\overline{B} \times [0,1] \to \mathbb{R}; \quad (x,t) \mapsto u_2(x, \tau + t), \quad \tau \geq 2, \tag{5.32}$$

satisfy condition (H_{α,β_0}) of Lemma 5.10 with $\alpha = \gamma$ and $\beta_0 = M$. In particular this implies that the functions in (5.32) are equicontinuous. Arguing by contradiction, we now assume that $z_2^{\hat{e}} \not\equiv 0$ in $B(\hat{e})$ for some $\hat{e} \in \partial\mathcal{N}$. By the equicontinuity of the functions in (5.32), there are $\zeta \in (0, \frac{1}{7})$, a nonempty open subset $\Omega \subset\subset B(\hat{e})$, and $k_1 > 0$ such that, after passing to a subsequence,

$$u_2^{\hat{e}} \geq k_1 \quad \text{on } \Omega \times [t_n - \zeta, t_n + \zeta] \quad \text{for all } n \in \mathbb{N}. \tag{5.33}$$

We now apply a normalization procedure for u_1. Define, for $n \in \mathbb{N}$,

$$I_n := [t_n - 3, t_n + 3] \subset \mathbb{R}, \qquad \beta_n := \|u_1(\cdot, t_n)\|_{L^\infty(B)}$$

and the functions

$$v_n : \overline{B} \times I_n \to \mathbb{R}, \qquad v_n(x,t) := \frac{u_1(x,t)}{\beta_n}.$$

By Corollary 5.12 there exists $\eta > 1$ and $\vartheta > 0$ such that

$$\eta \geq v_n \geq \frac{1}{\eta} \operatorname{dist}(x, \partial B)^\vartheta \quad \text{on } B \times I_n \qquad \text{for all } n \in \mathbb{N}. \tag{5.34}$$

Moreover, we have that

$$\|v_n\|_{C^{1+\tilde{\gamma},(1+\tilde{\gamma})/2}(\overline{B} \times [s,s+2])} < \tilde{C} \quad \text{for all } s \in [t_n - 2, t_n + 2],\ n \in \mathbb{N}$$

and for some $\tilde{\gamma} \in (0,1)$ and $\tilde{C} > 0$. This follows from Lemma 2.9 and the fact that v_n satisfies

$$\begin{aligned} (v_n)_t - \Delta v_n &= (c - \alpha_1 u_2) v_n \qquad && \text{in } B \times I_n, \\ v_n &= 0 && \text{in } \partial B \times I_n \end{aligned}$$

where

$$c(x,t) := \int_0^1 \partial_v f_1(t, |x|, s u_1(x,t)) ds$$

for $x \in B$ and $t > 0$. Here $\partial_v f_1$ denotes the derivative of f_1 with respect to the third variable. Note that $c \in C^{\beta,\beta/2}(B \times [1,\infty))$ for some $\beta > 0$ by assumption $(h1)_D$ and (5.26). Thus, arguing as in Remark 5.13, we have that

$$\|v_n\|_{C^{2+\gamma,1+\gamma/2}(\overline{B}\times[s,s+1])} < M \quad \text{for all } s\in[t_n-1,t_n+1],\ n\in\mathbb{N} \tag{5.35}$$

by making the constant γ smaller and M larger if necessary.

Then, we may also assume that all of the functions

$$\overline{B}\times[0,1]\to\mathbb{R};\quad (x,t)\mapsto v_n(x,\tau+t),\quad \tau\in[t_n-1,t_n+1],\ n\in\mathbb{N},$$

satisfy the equicontinuity condition (H_{α,β_0}) of Lemma 5.10 with $\alpha=\gamma$ and $\beta_0=M$.

For $e\in\mathbb{S}^{N-1}$ and $n\in\mathbb{N}$ we also consider

$$v_n^e:\overline{B}\times I_n\to\mathbb{R},\qquad v_n^e(x,t):=v_n(x,t)-v_n(\sigma_e(x),t),$$

and we note that

$$\begin{aligned} (v_n^e)_t-\Delta v_n^e-c_1^e v_n^e &= \alpha_1 v_n u_2^e && \text{in } B(e)\times I_n,\\ (u_2^e)_t-\Delta u_2^e-c_2^e u_2^e &= \alpha_2\beta_n u_2 v_n^e && \text{in } B(e)\times I_n,\\ v_n^e = u_2^e &= 0 && \text{on } \partial B(e)\times I_n. \end{aligned} \tag{5.36}$$

Set

$$Q_n := B(\hat{e})\times[t_n-\zeta\,,\,t_n+\zeta]\qquad \text{for } n\in\mathbb{N},$$

with ζ as in (5.33). We now distinguish two cases.

<u>Case 1:</u> $\limsup_{n\to\infty}\|v_n^{\hat{e}}\|_{L^\infty(Q_n)}>0.$

In this case, by (5.35), there are $d\in(0,1)$, $k_2>0$, and $t^*\in[-\zeta,\zeta]$ such that, after passing to a subsequence,

$$\sup\{v_n^{\hat{e}}(x,t_n+t^*): x\in B(\hat{e}),\ \operatorname{dist}(x,\partial B(\hat{e}))\ge d\}\ge k_2 \qquad \text{for } n\in\mathbb{N}.$$

Without loss, we may assume that

$$\sup\{u_2^{\hat{e}}(x,t_n+t^*): x\in B(\hat{e}),\ \operatorname{dist}(x,\partial B(\hat{e}))\ge d\}\ge k_1 \qquad \text{for } n\in\mathbb{N}$$

by (5.33). Next, let $\rho>0$ be the constant given by Lemma 5.10 for $\delta=d$, $k:=\frac{1}{2}\min\{k_1,k_2\}$, $\beta_0=M$, $\alpha=\gamma$, with M and γ as above. Since $\hat{e}\in\partial\mathcal{N}$, there exists $e\in\mathcal{N}$ such that $|e-\hat{e}|<\frac{\rho}{2}$, $x_1,x_2\in B(e)$ and, by equicontinuity,

$$\sup\{v_n^e(x,t_n+t^*): x\in B(e),\ \operatorname{dist}(x,\partial B(e))\ge d\}\ge k,$$
$$\sup\{u_2^e(x,t_n+t^*): x\in B(e),\ \operatorname{dist}(x,\partial B(e))\ge d\}\ge k$$

for all $n \in \mathbb{N}$. Since $e \in \mathcal{N}$ we can fix $n \in \mathbb{N}$ such that

$$v_n^e(x, t_n + t^* - \frac{1}{7}) \geq 0, \quad u_2^e(x, t_n + t^* - \frac{1}{7}) \geq 0, \qquad \text{for all } x \in B(e).$$

Without loss of generality we may assume that $e = e_1$. Then, applying Lemma 5.10 to the functions

$$\overline{B} \times [0,1] \to \mathbb{R}; \;\; (x,t) \mapsto u_2(x, t_n + t^* - \frac{1}{7} + t), \;\; (x,t) \mapsto v_n(x, t_n + t^* - \frac{1}{7} + t),$$

we conclude that

$$u_2^{\bar{e}}(\cdot, t_n + t^* + \frac{6}{7}) \geq 0 \quad \text{and} \quad v_n^{\bar{e}}(\cdot, t_n + t^* + \frac{6}{7}) \geq 0 \qquad \text{in } B(\bar{e})$$

for all $\bar{e} \in \mathbb{S}^{N-1}$ with $|\bar{e} - e| < \rho$, and thus in particular for $\bar{e} = \hat{e}$. But this implies $u_i^{\hat{e}}(\cdot, t_n + t^* + \frac{6}{7}) \geq 0$ in $B(\hat{e})$ for $i = 1, 2$, and thus $\hat{e} \in \mathcal{N}$ by the maximum principle for systems. This contradicts the hypothesis that $\hat{e} \in \partial\mathcal{N}$, since $\mathcal{N}$ is open by Lemma 5.14.

$$\underline{\text{Case 2:}} \qquad \lim_{n\to\infty} \|v_n^{\hat{e}}\|_{L^\infty(Q_n)} = 0. \tag{5.37}$$

In this case let

$$Q := B(\hat{e}) \times (-\zeta, \zeta)$$

and fix a nonnegative function $\varphi \in C_c^\infty(Q)$ with $\varphi \equiv 1$ on $\Omega \times (-\frac{\zeta}{2}, \frac{\zeta}{2})$. Moreover, let

$$\varphi_n \in C_c^\infty(Q_n) \quad \text{be given by} \quad \varphi_n(x,t) := \varphi(x, t_n + t), \quad n \in \mathbb{N}.$$

Setting $(u_2^{\hat{e}})^+ := \max\{u_2^{\hat{e}}, 0\}$ and $(u_2^{\hat{e}})^- := -\min\{u_2^{\hat{e}}, 0\}$, we find by (h3), (5.33), and (5.34) that, for $n \in \mathbb{N}$,

$$\begin{aligned} A_n &:= \int_{Q_n} \alpha_1 v_n u_2^{\hat{e}} \varphi_n d(x,t) = \int_{Q_n} \alpha_1 v_n [(u_2^{\hat{e}})^+ - (u_2^{\hat{e}})^-] \varphi_n d(x,t) \\ &\geq \frac{\alpha_*}{\eta} \int_{Q_n} \operatorname{dist}(x, \partial B)^\vartheta (u_2^{\hat{e}})^+ \varphi_n d(x,t) - \alpha^* \eta \, \|(u_2^{\hat{e}})^-\|_{L^\infty(Q_n)} \, \|\varphi\|_{L^1(Q)}, \\ &\geq \frac{\alpha_*}{\eta} k_1 |\Omega| \zeta \inf_\Omega \operatorname{dist}(x, \partial B)^\vartheta \; - \; \alpha^* \eta \, \|(u_2^{\hat{e}})^-\|_{L^\infty(Q_n)} \, \|\varphi\|_{L^1(Q)}, \end{aligned}$$

where the last term on the right hand side goes to zero as $n \to \infty$ because $\hat{e} \in \partial\mathcal{N}$. Hence we have

$$\liminf_{n\to\infty} A_n > 0.$$

On the other hand, integrating by parts, we have by (5.36) that

$$\begin{aligned}A_n &= \int_{Q_n} [(v_n^{\hat{e}})_t - \Delta v_n^{\hat{e}} - c_1^{\hat{e}} v_n^{\hat{e}}]\varphi_n d(x,t)\\ &= -\int_{Q_n} [v_n^{\hat{e}}(\varphi_n)_t + v_n^{\hat{e}}\Delta\varphi_n + c_1^{\hat{e}} v_n^{\hat{e}}\varphi_n] d(x,t)\\ &\le \|v_n^{\hat{e}}\|_{L^\infty(Q_n)} \int_Q \Big(|(\varphi)_t| + |\Delta\varphi)| + M\varphi\Big) d(x,t)\end{aligned}$$

for $n \in \mathbb{N}$. Invoking (5.37) we conclude that

$$\limsup_{n\to\infty} A_n \le 0.$$

So we have obtained a contradiction again and thus the claim follows. □

5.3 Radial symmetry of semi-trivial limit profiles

5.3.1 Systems with concave nonlinearities

For this subsection we assume that B is a ball or an annulus and that $u_1, u_2 \in C^{2,1}(\overline{B} \times (0,\infty)) \cap C(\overline{B} \times [0,\infty)) \cap L^\infty(B \times (0,\infty))$ are nonnegative functions such that $u = (u_1, u_2)$ is a classical solution of (5.2) where, for $i = 1, 2$, the coefficient and nonlinearity satisfy that $\alpha_i \in L^\infty(I_B \times (0,\infty))$, $f_i \in C^1(\mathbb{R})$ is strictly concave in $[0,\infty)$, $f_i(0) = 0$, $f(s) \to -\infty$ as $s \to \infty$, and the initial profile $u_{0,i} \in C_0(B)$ is not identically zero.

Note that

$$\begin{aligned}(u_i)_t - \Delta u_i - c_i\, u_i = f_i(0) &= 0 && \text{in } B \times (0,\infty),\\ u_i &= 0 && \text{on } \partial B \times (0,\infty),\\ u_i(x,0) &= u_{i,0}(x) && \text{for all } x \in \overline{B},\ i = 1,2,\end{aligned}$$

where $c_i \in C(\overline{B} \times (0,\infty)) \cap L^\infty(B \times (0,\infty))$ is given by

$$c_i(x,t) := \int_0^1 f'(su_i(x,t))\, ds - \alpha_i(x,t) u_j(x,t)$$

for $i \neq j$, $i, j \in \{1,2\}$. And since $u_{0,i} \ge 0$, and $u_{0,i} \not\equiv 0$ in B, $i = 1,2$, the parabolic maximum principle implies that

$$u_1 > 0 \quad \text{and} \quad u_2 > 0 \qquad \text{in } B \times (0,\infty). \tag{5.38}$$

Moreover by Lemma 2.9,

$$\sup_{\substack{x,\bar{x}\in\overline{B},\, t,\bar{t}\in[s,s+1],\\ x\neq\bar{x},\, t\neq\bar{t},\, s\ge 1}} \frac{|u_i(x,t) - u_i(\bar{x},\bar{t})|}{|x-\bar{x}|^\gamma + |t-\bar{t}|^{\frac{\gamma}{2}}} < L, \tag{5.39}$$

for some positive constants L and γ, and the semiorbits $\{u_i(\cdot,t) \,:\, t \geq 1\}$ are relatively compact in $C_0(B)$ for $i = 1, 2$. Therefore

$$\begin{aligned}&\omega(u) \text{ is a nonempty compact subset of } C_0(B) \times C_0(B) \text{ satisfying} \\ &\qquad \lim_{t\to\infty} \inf_{z\in\omega(u)} \|u_1(\cdot,t) - z_1\|_{L^\infty(B)} + \|u_2(\cdot,t) - z_2\|_{L^\infty(B)} = 0.\end{aligned} \tag{5.40}$$

The results in this section use some well-known techniques for nonlinear PDE's with concave nonlinearities as used in [32] (see also [12, Chapters 9 & 10]). The following result is the key element in the proof of Theorems 5.2 and 5.4.

Theorem 5.16. *Let $u_1, u_2 \in C^{2,1}(\overline{B} \times (0,\infty)) \cap C(\overline{B} \times [0,\infty)) \cap L^\infty(B \times (0,\infty))$ be nonnegative functions such that $u = (u_1, u_2)$ is a solution of* (5.2) *satisfying (E1) from Theorem 5.2 and with $\alpha_i \in L^\infty(I_B \times (0,\infty))$, $i = 1, 2$. Further assume that*

$$(0,0) \notin \omega(u_1, u_2). \tag{5.41}$$

If $f_1'(0) > \lambda_1$ and there is $(z, 0) \in \omega(u_1, u_2)$, then z is radially symmetric and it is the unique positive solution of (5.3). *Analogously, if $f_2'(0) > \lambda_1$ and there is $(0, z) \in \omega(u_1, u_2)$, then z is radially symmetric and it is the unique positive solution of* (5.3).

Proof. Assume (5.41), that $f_1'(0) > \lambda_1$, and that there is $(\tilde{z}, 0) \in \omega(u_1, u_2)$. It is well-known (see for example [32, Proposition 2.1]) that there is a unique positive weak solution φ of (5.3). We will prove that $\tilde{z} = \varphi$ and the radial symmetry of z follows by the uniqueness of φ.

Let $t_n \to \infty$ be an increasing sequence such that

$$\begin{aligned}\lim_{n\to\infty} \|u_1(\cdot,t_n) - \tilde{z}\|_{L^\infty(B)} &= 0, \\ \lim_{n\to\infty} \|u_2(\cdot,t_n)\|_{L^\infty(B)} &= 0.\end{aligned}$$

We proceed by contradiction. Assume that, passing to a subsequence, there is $\gamma > 0$ such that

$$\|u_1(\cdot,t_n) - \varphi\|^2_{L^2(B)} > \gamma \tag{5.42}$$

for all $n \in \mathbb{N}$, where we have also used (5.39) to estimate the L^2-norm with the L^∞-norm.

We have by hypothesis that

$$(0,0) \notin \omega(u). \tag{5.43}$$

Note that $\{z \in C_0(B) : (z,0) \in \omega(u)\}$ is compact, since $\omega(u)$ is compact by (5.40). In the following we fix a nonempty subdomain $U \subset\subset B$. Then, by (5.43) and Corollary 5.12 with $k=1$, there are $\widetilde{C} > 0$ and $\eta > 0$ such that

$$\inf_{\substack{x \in U, \\ (z,0) \in \omega(u)}} z(x) \geq \widetilde{C} \inf_{(z,0)\in\omega(u)} \|z\|_{L^\infty(B)} > \eta. \tag{5.44}$$

Since φ is bounded, there is some $\varepsilon = \varepsilon(\eta, \varphi) \in (0,1)$ such that

$$\|\varphi\|_{L^\infty(B)} \leq \frac{\eta}{\varepsilon}. \tag{5.45}$$

Let $c \in L^\infty(B)$ be given by

$$c(x) := \begin{cases} \frac{f_1(\varphi) - f_1(\varepsilon\varphi)}{(1-\varepsilon)\varphi}, & \text{if } x \in U, \\ \frac{f_1(\varphi)}{\varphi}, & \text{if } x \in B \backslash U. \end{cases}$$

Note that the strict concavity of f_1 and the fact that $f_1(0) = 0$ imply that

$$\begin{aligned} \frac{f_1(\varphi(x))}{\varphi(x)} - c(x) &\geq 0 \qquad \text{for all } x \in B, \\ \frac{f_1(\varphi(x))}{\varphi(x)} - c(x) &> 0 \qquad \text{for all } x \in U. \end{aligned} \tag{5.46}$$

Let

$$\delta := \inf \left\{ \int_B |\nabla w|^2 - c(x) w^2 dx \ : \ \int_B w^2 dx = 1 \, , \, w \in H_0^1(B) \right\}. \tag{5.47}$$

Since $c \in L^\infty(B)$, standard arguments (see, for example, [28, Theorem 8.38]) imply that there is a positive function $v \in H_0^1(B)$ such that

$$\int_B \nabla v \nabla w - c(x) vw \; dx = \delta \int_B vw \; dx \qquad \text{for all } w \in H_0^1(B).$$

In particular, for $w = \varphi$ we get by integration by parts that

$$\delta \int_B v\varphi \; dx = \int_B \left(\frac{f_1(\varphi)}{\varphi} - c(x) \right) \varphi v \; dx > 0,$$

by (5.46), and therefore $\delta > 0$.

Now, let

$$K := \max\{ |B| \, \|\alpha_1 u_1 (u_1 - \varphi)\|_{L^\infty(B\times(0,\infty))} \, , \, |B| \, \|u_1 - \varphi\|^2_{L^\infty(B\times(0,\infty))} \}$$

and let $k > 0$ be such that

$$K e^{-\delta k} < \gamma. \tag{5.48}$$

By Corollary 5.12 for this choice of k, there is $C > 0$ such that

$$\lim_{n\to\infty} \|u_2\|_{L^\infty(B\times[t_n-t_0,t_n])} \le C \lim_{n\to\infty} \|u_2(\cdot,t_n)\|_{L^\infty(B)} = 0,$$

for all $t_0 \in [0,k]$. Then, by (5.44) and (5.40) there is $n^* \in \mathbb{N}$ such that

$$\|u_2\|_{L^\infty(B\times[t_{n^*}-k,t_{n^*}])} < \frac{\delta\gamma}{K}, \tag{5.49}$$

$$\inf_{U\times[t_{n^*}-k,t_{n^*}]} u_1 > \eta. \tag{5.50}$$

Using (5.45), (5.50), and the strict concavity of f_1 we have that

$$\frac{f_1(u_1(x,t)) - f_1(\varphi(x))}{u_1(x,t)-\varphi(x)} \le c(x) \qquad \text{for all } x \in B,\ t\in[t_{n^*}-k, t_{n^*}].$$

By (5.49), (5.47), we have integrating by parts that

$$\begin{aligned}
&\frac{d}{dt}\|u_1(\cdot,t)-\varphi\|^2_{L^2(B)} = 2\int_B (u_1-\varphi)(u_1)_t dx\\
&= 2\int_B (u_1-\varphi)(\Delta u_1 + f_1(u_1) - \alpha_1 u_1 u_2)dx\\
&= 2\int_B -|\nabla(u_1-\varphi)|^2 + \frac{f_1(u_1)-f_1(\varphi)}{u_1-\varphi}(u_1-\varphi)^2 - \alpha_1 u_1 u_2 (u_1-\varphi)dx\\
&\le 2\int_B -|\nabla(u_1-\varphi)|^2 + c(x)(u_1-\varphi)^2 + \frac{K}{|B|}\|u_2\|_{L^\infty(B\times(t_n-k,t_n))}dx\\
&\le -2\int_B \Big[|\nabla(u_1-\varphi)|^2 - c(x)(u_1-\varphi)^2\Big]dx + K\|u_2\|_{L^\infty(B\times(t_n-k,t_n))}\\
&\le \delta\Big(-2\|u_1(\cdot,t)-\varphi\|^2_{L^2(B)} + \gamma\Big)
\end{aligned} \tag{5.51}$$

for $t \in (t_{n^*}-k, t_{n^*})$.

Let

$$\tau := \inf\Big\{t \in (t_{n^*}-k\,,\,t_{n^*}) \;:\; \|u_1(\cdot,s)-\varphi\|^2_{L^2(B)} > \gamma \text{ for all } s\in(t\,,\,t_{n^*})\Big\}.$$

Then by (5.51)

$$\frac{d}{dt}\|u_1(\cdot,t)-\varphi\|^2_{L^2(B)} \le -\delta\|u_1(\cdot,t)-\varphi\|^2_{L^2(B)} \tag{5.52}$$

for $t\in[\tau,t_{n^*}]$. Thus, (5.42), (5.52), and Gronwall's inequality yield

$$\gamma < \|u_1(\cdot,t_{n^*})-\varphi\|^2_{L^2(B)} \le e^{-\delta(t_{n^*}-\tau)}\|u_1(\cdot,\tau)-\varphi\|^2_{L^2(B)}. \tag{5.53}$$

If $\tau > t_{n^*} - k$, then by continuity $\|u_1(\cdot,\tau) - \varphi\|^2_{L^2(B)} = \gamma$ which yields a contradiction with (5.53). Then $\tau = t_{n^*} - k$, but this implies

$$\begin{aligned}\gamma < \|u_1(\cdot,t_{n^*}) - \varphi\|^2_{L^2(B)} &\le e^{-\delta k}\|u_1(\cdot,t_{n^*} - k) - \varphi\|^2_{L^2(B)}\\ &\le e^{-\delta k}|B|\ \|u_1 - \varphi\|^2_{L^\infty(B\times(0,\infty))} < \gamma,\end{aligned}$$

by (5.53) and (5.48), again a contradiction.

Therefore $\tilde z \equiv \varphi$. Since we can argue similarly for $(0,\tilde z) \in \omega(u_1,u_2)$ with $\tilde z \not\equiv 0$ the proof is finished. □

Proof of Theorem 5.2. The result follows from Theorem 5.16 once we prove that $(0,0) \notin \omega(u_1,u_2)$. We argue by contradiction. Let $t_n \to \infty$ be an increasing sequence such that $(u_1(\cdot,t_n), u_2(\cdot,t_n)) \to (0,0)$ in $C_0(B)\times C_0(B)$ as $n\to\infty$ and let $\psi \in C^2(\overline{B})\cap C_0(B)$ be the solution of

$$\begin{aligned} -\Delta\psi &= \lambda_1\psi && \text{in } B,\\ \psi &= 0 && \text{on } \partial B,\\ \psi &> 0 && \text{in } \partial B,\end{aligned} \tag{5.54}$$

with $\|\psi\|_{L^\infty(B)} = 1$.

By assumption (E2) there is $c > 0$ such that $f_i'(0) > c + \lambda_1$ for $i = 1,2$. Then, using that f_i' is decreasing (since f_i is concave) and that $f_i(0) = 0$, $i = 1,2$, we get by integration by parts that

$$\begin{aligned}\partial_t \int_B u_i(x,t)\psi(x)dx &= \int_B \Delta u_i\psi(x) + f_i(u_i)\psi(x) - \alpha_i(x,t)u_1u_2\psi(x)dx\\ &\ge \int_B [(-\lambda_1 + f_i'(u_i))u_i - \alpha_i(x,t)u_1u_2]\psi(x)dx\\ &\ge \Psi(t)\int_B u_i\psi(x)dx,\end{aligned} \tag{5.55}$$

for $t > 0$, where $\Psi : [0,\infty) \to \mathbb{R}$ is a continuous function given by

$$\Psi(t) := \inf_{\substack{x\in\overline{B}\\ i,j\in\{1,2\},\ i\neq j}} (-\lambda_1 + f_i'(u_i(x,t)) - Mu_j(x,t)),$$

with

$$M := \max_{i=1,2}\|\alpha_i\|_{L^\infty(B\times(0,\infty))}.$$

Then $\lim_{n\to\infty}\Psi(t_n) > c$. Therefore, passing to a subsequence,

$$\Psi(t_n) > c \qquad \text{for all } n\in\mathbb{N}.$$

Set

$$\tau_n := \inf \left\{ t \in (0, t_n) \; : \; \Psi(s) > c \text{ for all } s \in (t, t_n) \right\}$$

for $n \in \mathbb{N}$.

Then by (5.55), (5.38), (5.54),

$$0 \leq \int_B u_i(x, \tau_n)\psi(x)dx \leq \int_B u_i(x, t_n)\psi(x)dx \qquad \text{for } n \in \mathbb{N},\ i = 1, 2.$$

Thus, by (5.39),

$$\lim_{n\to\infty} \|u_i(\cdot, \tau_n)\|_{L^\infty(B)} = 0 \qquad \text{for } i = 1, 2. \tag{5.56}$$

Since $u_i(\cdot, 0) = u_{0,i} \not\equiv 0$ by hypothesis, we can assume that $\tau_n > 0$ for all $n \in \mathbb{N}$. Then by continuity $\Psi(\tau_n) = c$ for all $n \in \mathbb{N}$. On the other hand, (5.56) and the definition of Ψ imply that $\lim\limits_{n\to\infty} \Psi(\tau_n) > c$, and we have reached a contradiction. Thus $(0, 0) \notin \omega(u)$ and the result follows by Theorem 5.16. □

Proof of Theorem 5.4. We prove first that $0 \notin \omega(u_1)$. We argue by contradiction. Let $t_n \to \infty$ be an increasing sequence such that $t_1 > 1$ and $u_1(\cdot, t_n) \to 0$ in $C_0(B)$ as $n \to \infty$. By assumption there is $\mu > 0$ such that $a_1 > \mu + \lambda_1$. Let $M > 0$ be such that

$$\max_{i=1,2} \|u_i\|_{L^\infty(B\times(0,\infty))} + \|\alpha_i\|_{L^\infty(B\times(0,\infty))} < M,$$

and $k > 0$ be such that

$$e^{-\frac{a_2}{2}k} < \frac{\mu}{2M^2}. \tag{5.57}$$

Note that u_1 satisfies

$$\begin{aligned} (u_1)_t - \Delta u_1 - cu_1 &= 0 \qquad \text{in } B \times (0, \infty), \\ u_1 &= 0 \qquad \text{on } \partial B \times (0, \infty), \end{aligned}$$

with

$$c := a_1 - b_1 u_1 - \alpha_1 u_2 \;\in\; L^\infty(B \times (0, \infty)).$$

Then, by Corollary 5.12 with this choice of k, there is $C > 0$ such that

$$\|u_1\|_{L^\infty(B\times[t-k,t])} \leq C\|u_1(\cdot, t)\|_{L^\infty(B)} \qquad \text{for all } t > 2k. \tag{5.58}$$

In particular, $\lim_{n\to\infty} \|u_1\|_{L^\infty(B\times[t_n-k,t_n])} = 0$. Then, passing to a subsequence, we have that

$$\|u_1\|_{L^\infty(B\times[t_n-k,t_n])} \leq \frac{a_2}{2M} \qquad \text{for all } n \in \mathbb{N}.$$

But then

$$\begin{aligned} (u_2)_t - \Delta u_2 &= (-a_2 - b_2u_2 + \alpha_2 u_1)u_2 \leq -\frac{a_2}{2}u_2 && \text{in } B \times [t_n - k, t_n], \\ u_2 &= 0 && \text{on } \partial B \times (0, \infty), \end{aligned}$$

for all $n \in \mathbb{N}$. Then the parabolic maximum principle yields that

$$\|u_2(\cdot, t_n)\|_{L^\infty(B)} \leq e^{-\frac{a_2}{2}k}\|u_2(\cdot, t_n - k)\|_{L^\infty(B)} < e^{-\frac{a_2}{2}k}M < \frac{\mu}{2M} \tag{5.59}$$

for all $n \in \mathbb{N}$, by (5.57).

Let $\psi \in C^2(\overline{B}) \cap C_0(B)$ be the solution of (5.54) with $\|\psi\|_{L^\infty(B)} = 1$. Then, for $t > 0$, we get by integration by parts that

$$\begin{aligned} \partial_t \int_B u_1(x,t)\psi(x)dx &= \int_B \Delta u_1\psi(x) + (a_1 - b_1u_1 - \alpha_1(x,t)u_2)u_1\psi(x)dx \\ &\geq \int_B (-\lambda_1 + a_1 - b_1u_1 - \alpha_1(x,t)u_2)u_1\psi(x)dx \\ &> \Psi(t)\int_B u_1(x,t)\psi(x)dx, \end{aligned} \tag{5.60}$$

where $\Psi : [0,\infty) \to \mathbb{R}$ is a continuous function given by

$$\Psi(t) := \inf_{x\in B}(\mu - b_1u_1(x,t) - Mu_2(x,t)).$$

By (5.59),

$$\Psi(t_n) > \frac{\mu}{2} - b_1\|u_1(\cdot, t_n)\|_{L^\infty(B)},$$

for all $n \in \mathbb{N}$. Then $\liminf_{n\to\infty} \Psi(t_n) \geq \frac{\mu}{2}$. After passing to a subsequence, we may assume that $\Psi(t_n) \geq \frac{\mu}{3}$ for all $n \in \mathbb{N}$. Let

$$\tau_n := \inf\left\{t \in (0, t_n) \ : \ \Psi(s) \geq \frac{\mu}{3} \text{ for all } s \in (t, t_n)\right\}$$

for $n \in \mathbb{N}$. Then, by (5.60) and the nonnegativity of u_1 and ψ,

$$0 \leq \int_B u_1(x,\tau_n)\psi(x)dx \leq \int_B u_1(x,t_n)\psi(x)dx \qquad \text{for all } n \in \mathbb{N}.$$

Then

$$\lim_{n\to\infty} \|u_1(\cdot, \tau_n)\|_{L^\infty(B)} = 0. \tag{5.61}$$

Since $u_1 > 0$ in $B\times(0,\infty)$ we can assume that, passing to a subsequence, $\tau_n > 2k$ for all $n \in \mathbb{N}$. Then, by (5.58) and (5.61) we get, again up to a subsequence, that

$$\|u_1\|_{L^\infty(B\times[\tau_n-k,\tau_n])} \leq \frac{a_2}{2M} \qquad \text{for all } n \in \mathbb{N}.$$

Arguing as before this implies that $\|u_2(\cdot,\tau_n)\|_{L^\infty(B)} < \frac{\mu}{2M}$ and therefore $\liminf_{n\to\infty} \Psi(\tau_n) \geq \frac{\mu}{2}$. On the other hand, by the continuity of Ψ and the definition of τ_n, we have that $\Psi(\tau_n) = \mu/3$ for all $n \in \mathbb{N}$. We have reached a contradiction. Therefore $0 \notin \omega(u_1)$ and in particular $(0,0) \notin \omega(u)$. Then the claim follows by Theorem 5.16. $\square$

5.3.2 Systems as asymptotically symmetric equations

In this subsection $B \subset R^N$ denotes a ball.

Proof of Theorem 5.5. Note that

$$\begin{aligned} (u_1)_t &= \Delta u_1 + a_1(t)u_1^2 - b_1(t)u_1 + g(x,t), && \text{in } B\times(0,\infty),\\ u_1 &= 0 && \text{on } \partial B\times(0,\infty) \end{aligned}$$

where $g(x,t) := -\alpha_1(|x|,t)u_1(x,t)u_2(x,t)$. By our assumptions, $g \equiv 0$ on $\partial B\times(0,\infty)$.

Let $(z,0) \in \omega(u)$ with $z \not\equiv 0$. The radial symmetry of z will follow from [25, Theorem 2.2] once we have proved that

i) The functions $u_1(\cdot,t) : \overline{B} \to \mathbb{R}$ with $t \geq 1$ are equicontinuous.

ii) $\lim_{t\to\infty} \|g\|_{L^{N+1}(B\times(t,t+1))} = 0$.

iii) $\liminf_{t\to\infty} u(x,t) > 0$ for all $x \in B$.

The claim i) follows easily from our assumptions and Lemma 2.9. For the claims i) and ii), suppose for the moment that

$$\text{if } 0 \in \omega(u_i) \text{ for some } i = 1,2, \text{ then } \omega(u_i) = \{0\}. \tag{5.62}$$

Then since $(z,0) \in \omega(u)$ we have by (5.62) that $u_2(\cdot,t) \to 0$ as $t \to \infty$, and claim ii) follows since α_1 and u_1 are uniformly bounded.

Moreover, since $z \not\equiv 0$, (5.62) implies that

$$0 \notin \omega(u_1). \tag{5.63}$$

Now, note that u_i satisfies

$$\begin{aligned}(u_i)_t - \Delta u_i - c_i u_i &= 0 \qquad \text{in } B \times (0,\infty),\\ u_i &= 0 \qquad \text{on } \partial B \times (0,\infty)\end{aligned}$$

with $c_i \in L^\infty(B \times (0,\infty))$ given by

$$c_i(x,t) := a_i(t)u_i(x,t) - b_i(t) - \alpha_i(|x|,t)u_j(x,t)$$

for $i,j = 1,2$, with $i \neq j$.

Then, by Corollary 5.12 with $k = 1$, there are $C \geq 1$ and $\vartheta > 0$ such that

$$C \geq \frac{u_i(x,\tau+t)}{\|u_i(\cdot,\tau)\|_{L^\infty(B)}} \geq C^{-1}\operatorname{dist}(x,\partial B)^\vartheta \tag{5.64}$$

for all $x \in B$, $t \in [-1,1]$, $\tau \geq 2$, and $i = 1,2$.

Therefore, by (5.64) and (5.63), we have that

$$\liminf_{t\to\infty} u_1(x,t) \geq C^{-1}\operatorname{dist}(x,\partial B)^\vartheta \liminf_{t\to\infty} \|u_1(\cdot,t)\|_{L^\infty(B)} > 0$$

for all $x \in B$. Therefore the iii) follows.

Thus, it only remains to prove that (5.62) holds. We show this claim for $i = 2$ and the same arguments hold for $i = 1$. Indeed, let $t_n \to \infty$ as $n \to \infty$ be an increasing sequence such that $t_1 > 1$ and $(u_1(\cdot,t_n), u_2(\cdot,t_n)) \to (z,0)$ in $C_0(B) \times C_0(B)$ as $n \to \infty$.

By making C bigger, we may assume that

$$\|a_2\|_{L^\infty(0,\infty)} \leq C. \tag{5.65}$$

Since $u_2(\cdot,t_n) \to 0$ in $C_0(B)$ as $n \to \infty$, there is $\bar{n} \in \mathbb{N}$ such that

$$\|u_2(\cdot,t_{\bar{n}})\|_{L^\infty(B)} \leq \frac{b_*}{2C^2}. \tag{5.66}$$

Then, by (5.64), we have that

$$\|u_2\|_{L^\infty(B\times[t_{\bar{n}},t_{\bar{n}}+1)]} \leq C\|u_2(\cdot,t_{\bar{n}})\|_{L^\infty(B))} \leq \frac{b_*}{2C}. \tag{5.67}$$

Moreover, since $\alpha_2 u_1 \geq 0$ in $B \times (0,\infty)$, we have by (5.65), (5.66), and (5.67) that

$$(u_2)_t - \Delta u_2 \leq (C\|u_2\|_{L^\infty(B\times[t_{\bar{n}},t_{\bar{n}}+1)]} - b_*)u_2 \leq -\frac{b_*}{2}u_2$$

in $B \times [t_{\bar{n}}, t_{\bar{n}} + 1]$. Then the parabolic maximum principle yields that

$$\|u_2(\cdot, t_{\bar{n}} + 1)\|_{L^\infty(B)} \leq e^{-\frac{b_*}{2}} \|u_2(\cdot, t_{\bar{n}})\|_{L^\infty(B))} < \frac{b_*}{2C^2}.$$

Therefore, by (5.64),

$$\|u_2\|_{L^\infty(B \times [t_{\bar{n}}+1, t_{\bar{n}}+2])} \leq C \|u_2(\cdot, t_{\bar{n}} + 1)\|_{L^\infty(B))} < \frac{b_*}{2C}.$$

Arguing as before we get that

$$(u_2)_t - \Delta u_2 \leq -\frac{b_*}{2} u_2 \qquad \text{in } B \times [t_{\bar{n}} + 1, t_{\bar{n}} + 2].$$

Thus

$$\|u_2(\cdot, t_{\bar{n}} + 2)\|_{L^\infty(B)} \leq e^{-\frac{b_*}{2}} \|u_2(\cdot, t_{\bar{n}} + 1)\|_{L^\infty(B))} < \frac{b_*}{2C^2}.$$

Repeating this process indefinitely we obtain that

$$(u_2)_t - \Delta u_2 \leq -\frac{b_*}{2} u_2 \qquad \text{in } B \times [t_{\bar{n}}, \infty).$$

Therefore

$$\lim_{t \to \infty} \|u_2(\cdot, t)\|_{L^\infty(B)} \leq \lim_{t \to \infty} e^{-\frac{b_*}{2}(t - t_{\bar{n}})} \|u_2(\cdot, t_{\bar{n}})\|_{L^\infty(B))} = 0.$$

Then $\omega(u_2) = \{0\}$ and this ends the proof. □

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Zusammenfassung

In dieser Arbeit studieren wir die Symmetrieeigenschaften von Lösungen von verschiedenen elliptischen und parabolischen Gleichungen und Systemen. Von nun an sei B eine Kugel oder ein Annulus in $\mathbb{R}^N$, $N \geq 2$. Wir arbeiten mit einer besonderen Art von Symmetrie, welche *geblätterte Schwarz-Symmetrie* genannt wird. Wir sagen, dass eine Funktion $u \in C(B)$ *geblättert Schwarz-symmetrisch bezüglich eines Vektors* $p \in \mathbb{S}^{N-1}$ ist, wenn u axialsymmetrisch bezüglich der Achse $\mathbb{R}p$ und nichtwachsend in dem Polarwinkel $\theta := \arccos(\frac{x}{|x|} \cdot p) \in [0, \pi]$ ist.

Sei

$$I_B := \{|x| \; : \; x \in \overline{B}\}, \qquad B(e) := \{x \in B \; : \; x \cdot e > 0\},$$
$$\sigma_e : B \to B; \quad x \mapsto \sigma_e(x) := x - 2(x \cdot e)e.$$

Das folgende Resultat ist ein hinreichendes Kriterium für asymptotische Symmetrie von Lösungen von Reaktionsdiffusions-Gleichungen unter Dirichlet-Randbedingungen.

Satz 1. Sei $u \in C^{2,1}(\overline{B} \times (0,\infty)) \cap C(\overline{B} \times [0,\infty))$ eine klassische Lösung von

$$\begin{aligned} &u_t = \Delta u + f(t,|x|,u), && x \in B,\; t > 0,\\ &u(x,t) = 0, && x \in \partial B,\; t > 0,\\ &u(x,0) = u_0(x), && x \in B, \end{aligned} \tag{5.68}$$

wobei

(f1) $f : [0,\infty) \times I_B \times \mathbb{R} \to \mathbb{R}$, $(t,r,u) \mapsto f(t,r,u)$ ist stetig in t, r und lokal Lipschitz in u, gleichmäßig bezüglich t und r, d.h. für alle $K > 0$ gibt es $L = L(K) > 0$ derart, dass

$$|f(t,r,u_1) - f(t,r,u_2)| \leq L|u_1 - u_2|$$

für alle $(t,r) \in [0,\infty) \times I_B$ und $u_1, u_2 \in [-K, K]$.

(f2) $f(\cdot,\cdot,0)$ ist beschränkt auf $[0,\infty) \times I_B$.

(U1) Es gibt $e \in \mathbb{S}^{N-1}$ mit $u_0 \geq u_0 \circ \sigma_e$ und $u_0 \not\equiv u_0 \circ \sigma_e$ in $B(e)$.

(U2) $\|u\|_{L^\infty(B\times(0,\infty))} < \infty$.

Dann gibt es ein $p \in \mathbb{S}^{N-1}$ derart, dass u asymptotisch geblättert Schwarz-symmetrisch bezüglich p ist, d.h. alle Elemente in

$$\omega(u) = \{z \in C(\overline{B}) : \|u(\cdot, t_n) - z\|_{L^\infty(B)} \to 0 \text{ für eine Folge } t_n \to \infty\}$$

sind geblättert Schwarz-symmetrisch bezüglich $p \in \mathbb{S}^{N-1}$.

Unmittelbar folgt für das elliptische und das parabolische zeitperiodische Problem das folgende

Korollar 2. (i) Sei $f : I_B \times \mathbb{R} \to \mathbb{R}$, $(r, u) \mapsto f(r, u)$ stetig in $r \in I_B$ und lokal Lipschitz in u, gleichmäßig bezüglich r. Ferner sei $u \in C^2(\overline{B})$ eine klassische Lösung des elliptischen Problems

$$\begin{aligned} -\Delta u &= f(|x|, u) && \text{in } B, \\ u(x) &= 0 && \text{auf } \partial B \end{aligned}$$

derart, dass (U1) für u anstelle von u_0 gilt. Dann ist u geblättert Schwarz-symmetrisch bzgl. eines Vektors $p \in \mathbb{S}^{N-1}$.

(ii) Angenommen, $f : [0, \infty) \times I_B \times \mathbb{R} \to \mathbb{R}$ erfülle (f1) und sei periodisch in t, d.h. es gibt $T > 0$ mit $f(t + T, r, u) = f(t, r, u)$ für alle $t \geq 0$, $r \in I_B$ und $u \in \mathbb{R}$. Ferner sei $u \in C^{2,1}(\overline{B} \times (0, \infty)) \cap C(\overline{B} \times [0, \infty))$ eine T-periodische Lösung von (5.68), d.h., $u(x, t + T) = u(x, t)$ für alle $x \in B$ und $t \geq 0$, für die (U1) gilt. Dann ist $u(\cdot, t)$ geblättert Schwarz-symmetrisch bzgl. eines Vektors $p \in \mathbb{S}^{N-1}$ für alle Zeiten $t \in [0, \infty)$.

Alle nachstehenden Sätze in dieser Zusammenfassung haben ähnliche Korollare für das entsprechende elliptische und zeitperiodische parabolische Problem.

Unser nächstes Ergebnis behandelt nichtlineare parabolische Probleme mit Neumann Randbedingungen.

Satz 3. Sei $u \in C^{2,1}(\overline{B} \times (0, \infty)) \cap (\overline{B} \times [0, \infty))$ eine klassische Lösung von

$$\begin{aligned} u_t - \mu(|x|, t)\Delta u &= f(t, |x|, u), && x \in B,\ t > 0, \\ \partial_\nu u &= 0, && x \in \partial B,\ t > 0, \\ u(x, 0) &= u_0(x), && x \in B, \end{aligned}$$

wobei

(μ) $\mu \in C^1(I_B \times (0,\infty))$ und es gibt Konstanten $\mu^* \geq \mu_* > 0$ derart, dass $\|\mu_i\|_{C^1(I_B\times(0,\infty))} \leq \mu^*$ und $\mu_i(r,t) \geq \mu_*$ für alle $r \in I_B$ und $t > 0$.

Ferner seien die Voraussetzungen (f1), (f2), (U1) und (U2) aus Satz 1 erfüllt, dann gibt es ein $p \in \mathbb{S}^{N-1}$ derart, dass jedes Element $z \in \omega(u)$ geblättert Schwarz-symmetrisch bzgl. $p \in \mathbb{S}^{N-1}$ ist.

Wir betonen, dass dieses Ergebnis, wie auch Satz 1 nicht nur für positive Lösungen gilt. Die folgenden Resultate sind parabolischen Systemen gewidmet. Wir betrachten zunächst ein kompetitives System unter Neumann-Randbedingungen. Hier ν ist der außere Normalenvektor.

Satz 4. Sei $u_1, u_2 \in C^{2,1}(\overline{B} \times (0,\infty)) \cap C(\overline{B} \times [0,\infty))$ nicht-negative Funktionen derart, dass $u = (u_1, u_2)$ eine Lösung von

$$\begin{aligned} (u_i)_t - \mu_i(|x|,t)\Delta u_i &= f_i(t,|x|,u_i) - \alpha_i(|x|,t)u_1u_2, && x \in B,\ t > 0, \\ \partial_\nu u_i &= 0, && x \in \partial B,\ t > 0, \\ u_i(x,0) &= u_0^i(x), && x \in B, \end{aligned}$$

ist, wobei

(h1) für $i = 1, 2$, ist die Funktion $f_i : [0,\infty) \times I_B \times [0,\infty)$ lokal Lipschitz stetig in u_i gleichmäßich bzgl. r und t, d.h.,

$$\sup_{\substack{r\in I_B,\, t>0,\\ v,\bar v \in K,\, v\neq \bar v}} \frac{|f_i(t,r,v) - f_i(t,r,\bar v)|}{|v-\bar v|} < \infty$$

für jede kompakte Teilmenge $K \subset [0,\infty)$. Ferner gilt $f_i(t,r,0) = 0$ für alle $r \in I_B$ und $t > 0$.

(h2) $\mu_i \in C^{2,1}(I_B \times (0,\infty))$ und es gibt Konstanten $\mu^* \geq \mu_* > 0$ derart, dass $\|\mu_i\|_{C^{2,1}(I_B\times(0,\infty))} \leq \mu_*$ und $\mu_i(r,t) \geq \mu_*$ für alle $r \in I_B, t > 0$ und $i = 1, 2$.

(h3) $\alpha_i \in L^\infty(I_B \times (0,\infty))$ und es gibt Konstanten $\alpha^* \geq \alpha^* > 0$ derart, dass $\alpha_* \leq \alpha_i(r,t) \leq \alpha^*$ für alle $r \in I_B,\ t > 0$ und $i = 1, 2$.

(h4) $\|u_i\|_{L^\infty(B\times(0,\infty))} < \infty$ für $i = 1, 2$,

(h5) $u_0^1 \geq u_0^1 \circ \sigma_e$, $u_0^2 \leq u_0^2 \circ \sigma_e$ in $B(e)$ für einen $e \in \mathbb{S}^{N-1}$ mit $u_0^i \not\equiv u_0^i \circ \sigma_e$ für $i = 1, 2$.

Dann gibt es $p \in \mathbb{S}^{N-1}$ derart, dass für alle $(z_1, z_2) \in \omega(u)$, z_1 geblättert Schwarz-symmetrisch bzgl. p ist, und z_2 geblättert Schwarz-symmetrisch bzgl. $-p$ ist.

Das nächste Ergebnis charakterisiert die asymptotische Gestalt von Lösungen von kooperativen Systemen von n Gleichungen.

Satz 5. Sei $J := \{1, 2, \dots, n\}$ für ein $n \in \mathbb{N}$ und für jedes $i \in J$ sei $u_i \in C^{2,1}(\overline{B} \times (0,\infty)) \cap C(\overline{B} \times [0,\infty))$ derart, dass $u = (u_1, \dots, u_n)$ eine Lösung von

$$\begin{aligned} (u_i)_t &= \Delta u_i + F_i(t, |x|, u), && x \in B,\ t > 0, \\ \partial_\nu u_i &= 0, && x \in \partial B,\ t > 0, \\ u_i(x,0) &= u_0^i(x), && x \in B, \end{aligned}$$

ist, wobei

(A1) Für jedes $i \in J$, ist die Funktion $F_i : [0,\infty) \times I_B \times \mathbb{R}^n \to \mathbb{R}$ lokal Lipschitz in u, gleichmäßig bzgl. r und t, d.h., für jede kompakte Teilmenge $K \subset \mathbb{R}^n$ gibt es ein $C(K) = C > 0$ mit

$$\sup_{\substack{r \in I_B,\ t>0, \\ v,\bar{v} \in K,\ v \neq \bar{v}}} \frac{|F_i(t,r,v) - F_i(t,r,\bar{v})|}{|v - \bar{v}|} < C.$$

Ferner gilt $\max_{i \in J} \sup_{r \in I_B, t>0} |F_i(t,r,0)| < \infty$.

(A2) Für alle $i, j \in J$, $i \neq j$ gilt $\partial F_i(t,r,u)/\partial u_j \geq 0$ für alle $t \in [0,\infty)$, $r \in I_B$ und $u \in \mathbb{R}^n$, für welche die Ableitung existiert.

(A3) Für jedes $M > 0$ gibt es eine Konstante $\sigma = \sigma(M) > 0$ derart, dass für jede Wahl von nichtleeren Teilmengen $I_1, I_2 \subset J$, mit $I_1 \cap I_2 = \emptyset$ und $I_1 \bigcup I_2 = J$, es $i \in I_1$ und $j \in I_2$ gibt derart, dass $\partial F_i(t,r,u)/\partial u_j \geq \sigma$ für alle $r \in I_B$, $t \in [0,\infty)$, $u \in \mathbb{R}^n$, für die $|u| \leq M$ und die Ableitung existiert.

(A4) Es gibt ein $e \in \mathbb{S}^{N-1}$ derart, dass $u_0^i \not\equiv u_0^i \circ \sigma_e$ und $u_0^i(x) \geq u_0^i(\sigma_e(x))$ für alle $x \in B(e)$ und $i \in J$.

Ferner gelte

$$\max_{i \in J} \|u_i\|_{L^\infty(B \times (0,\infty))} < \infty.$$

Dann gibt es ein $p \in \mathbb{S}^{N-1}$ derart, dass z geblättert Schwarz symmetrisch bzgl. $p \in \mathbb{S}^{N-1}$ für alle $z \in \bigcup_{i=1}^{n} \omega(u_i)$ ist.

Das folgende Resultat ist einer Klasse von kubischen kompetitiven Systemen gewidmet.

Satz 6. Seien λ_i, γ_i und α_i positive Konstanten und $u_1, u_2 \in C^{2,1}(\overline{B} \times (0,\infty)) \cap C(\overline{B} \times [0,\infty))$ nichtnegative Funktionen derart, dass $u = (u_1, u_2)$ eine Lösung von

$$\begin{aligned}
(u_1)_t - \Delta u_1 &= \lambda_1 u_1 + \gamma_1 u_1^3 - \alpha_1 u_1 u_2^2 && \text{in } B \times (0,\infty),\\
(u_2)_t - \Delta u_2 &= \lambda_2 u_2 + \gamma_2 u_2^3 - \alpha_2 u_1^2 u_2 && \text{in } B \times (0,\infty),\\
\partial_\nu u_1 &= \partial_\nu u_2 = 0 && \text{auf } \partial B \times (0,\infty),\\
u_i(x,0) &= u_0^i(x) \geq 0 && \text{für } x \in B,\ i = 1,2,
\end{aligned}$$

ist, und (h4) und (h5) aus Satz 4 erfüllt. Dann gibt es $p \in \mathbb{S}^{N-1}$ derart, dass für alle $(z_1, z_2) \in \omega(u_1, u_2)$, z_1 geblättert Schwarz-symmetrisch bzgl. p und z_2 geblättert Schwarz-symmetrisch bzgl. $-p$ ist.

Wir beweisen ähnliche Sätze auch für Dirichlet-Randbedingungen. Insbesondere haben wir folgendes

Satz 7. Seien $u_1, u_2 \in C^{2,1}(\overline{B} \times (0,\infty)) \cap C(\overline{B} \times [0,\infty))$ nichtnegative Funktionen derart, dass $u = (u_1, u_2)$ eine Lösung von

$$\begin{aligned}
(u_i)_t - \mu_i(|x|,t)\Delta u_i &= f_i(t,|x|,u_i) - \alpha_i(|x|,t) u_1 u_2, && x \in B,\ t > 0,\\
u_i(x,t) &= 0, && x \in \partial B,\ t > 0,\\
u_i(x,0) &= u_0^i(x), && x \in B,
\end{aligned}$$

für $i = 1,2$ ist, wobei

$(h1)_D$ für $i = 1,2$, ist die Funktion $f_i : [0,\infty) \times I_B \times [0,\infty) \to \mathbb{R}$, $(t,r,v) \mapsto f_i(t,r,v)$ stetig differenzierbar in v. Ferner sind f_i und $\partial_v f_i$ Hölder stetig in t und r für alle $v \in [0,\infty)$, und lokal Lipschitz stetig in v gleichmäßig bzgl. t und r. D.h., es gibt $\gamma > 0$ derart, dass für jede $h \in \{f_i, \partial_v f_i \ : \ i = 1,2\}$,

$$h(\cdot,\cdot,v) \in C^{\gamma,\gamma/2}(B \times (0,\infty)) \qquad \text{für alle } v \in [0,\infty)$$

und

$$\sup_{\substack{r \in I_B, t>0,\\ v,\bar{v} \in K, v \neq \bar{v}}} \frac{|h(t,r,v) - h(t,r,\bar{v})|}{|v - \bar{v}|} < \infty,$$

für jede kompakte Teilmenge $K \subset [0,\infty)$. Ferner gilt $f_i(t,r,0) = 0$ für alle $r \in I_B$, $t > 0$, und $i \in \{1,2\}$,

$(h2)_D$ Es gibt α^*, α_*, und β derart, dass $\alpha_i \in C^{\beta,\beta/2}(I_B \times (0,\infty))$ und $\alpha_* \leq \alpha_i(r,t) \leq \alpha^*$ für alle $r \in I_B$, $t > 0$, und $i \in \{1,2\}$, und die Annahme (h5) aus Satz 4 erfüllt ist. Dann gibt es $p \in \mathbb{S}^{N-1}$ derart, dass für alle $(z_1, z_2) \in \omega(u_1, u_2)$, z_1 geblättert Schwarz-symmetrisch bzgl. p und z_2 geblättert Schwarz-symmetrisch bzgl. $-p$ ist.

Für die Beweise dieser Ergebnisse wird eine neue Variante der Moving-Plane bzw. Rotating-Plane Methode entwickelt, welche die Ideen früherer Varianten verbindet und weiterentwickelt.

Für autonome Systeme können wir asymptotische radiale Symmetrie in Spezialfällen beweisen.

Satz 8. Seien $u_1, u_2 \in C^{2,1}(\overline{B} \times (0,\infty)) \cap C(\overline{B} \times [0,\infty)) \cap L^\infty(B \times (0,\infty))$ nichtnegative Funktionen derart, dass $u = (u_1, u_2)$ eine Lösung von

$$\begin{aligned} (u_i)_t - \Delta u_i &= f_i(u_i) - \alpha_i(|x|, t) u_1 u_2 && \text{in } B \times (0,\infty), \\ u_i &= 0 && \text{auf } \partial B \times (0,\infty), \\ u_i(x, 0) &= u_0^i(x) \geq 0 && \text{für alle } x \in B, i = 1, 2, \end{aligned}$$

ist, wobei $u_{0,i} \in C_0(B)$, $u_{0,i} \not\equiv 0$, $\alpha_i \in L^\infty(I_B \times (0,\infty))$,

(E1) $f_i \in C^1([0,\infty))$ strikt konkav in $[0,\infty)$, $f_i(0) = 0$ und $f_i(s) \to -\infty$ für $s \to \infty$, für $i = 1, 2$.

(E2) $f_i'(0) > \lambda$, wobei $\lambda > 0$ der erste Eigenwert des Laplace-Operators in B mit Dirichlet-Randbedingungen ist.

Sei $(z_1, z_2) \in \omega(u_1, u_2)$ derart, dass $z_i \not\equiv 0$ und $z_j \equiv 0$ für $i, j \in J$. Dann ist z_i radialsymmetrisch. Ferner ist $z_i \in H_0^1(B) \cap C(\overline{B})$ die eindeutige positive schwache Lösung des elliptischen Problems $-\Delta\varphi = f_i(\varphi)$ in B.

Der Beweis von Satz 8 benutzt Stabilitätseigenschaften von autonomen Problemen mit konvexen Nichtlinearitäten und einige Abschätzungen für Quotienten von Lösungen von linearen Problemen.

Der Satz 8 benötigt keine Annahme bzgl. des Vorzeichens von α_1 und α_2. Deswegen kann man auch kooperative, kompetitive und Räuber - Beute Modelle mit Dirichlet-Randbedingungen betrachten, wie z.B.

$$\begin{aligned} (u_1)_t - \Delta u_1 &= a_1 u_1 - b_1 u_1^2 - \alpha_1(x,t) u_1 u_2 && \text{in } B \times (0,\infty), \\ (u_2)_t - \Delta u_2 &= a_2 u_2 - b_2 u_2^2 + \alpha_2(x,t) u_1 u_2 && \text{in } B \times (0,\infty), \\ u_i &= 0 && \text{auf } \partial B \times (0,\infty), \\ u_i(x,0) &= u_{0,i}(x) && \text{für alle } x \in B,\ i = 1, 2. \end{aligned}$$

wobei $u_{0,1}$, $u_{0,2} \in C_0(B)$ nicht identisch Null sind, $a_1 > \lambda_1$, $a_2 > \lambda$ $b_1 > 0$, $b_2 > 0$ und $\alpha_1, \alpha_2 \in L^\infty(B \times (0,\infty))$ nichtnegative Funktionen sind.