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Portfolio-aspects in real options management

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Abstract

Real options theory applies techniques known from finance theory to the valuation of capital investments. The present paper investigates further into this analogy, considering the case of a portfolio of real options. An implementation of real option models in practice will mostly be concerned with a portfolio of real options, so the analysis of portfolio aspects is of both academic and practical interest. Is a portfolio of real options special? In order to shed some light on this question, the present paper will outline the relevant features of a portfolio of real options. It will show that the analogy to financial options remains great if compound option models are applied. As a result, a portfolio of real options, and therefore the firm as such, generally is to be understood as one single compound, real option.

Portfolio-aspects in real options management

1 Introduction

In the last two decades finance theory has received important insights from the growing real options literature. The idea behind the real options approach is simple and straightforward. Simple, because "real options [...] are opportunities to purchase real assets on possibly favorable terms".[1] Real options arise from a decision maker's degrees of freedom in choosing what actions to take, contingent on future events. Straightforward, because the theory of valuing the optionality in financial contracts, "option pricing theory", supplies powerful tools that can be turned to good use for valuing real options. All in all, the appeal of real options consists in quantifying the value created by the flexibility inherent in the management of investment projects, thereby providing a correct basis for making strategic investment decisions.

The body of real options literature has grown ample, modelling various and most complex settings of real options.[2] Yet the attribute "real" indicates that real options must finally aim at giving practitioners instruments which they can apply to their "real" problems. A practitioner normally is confronted with a vast opportunity set. In other words, if she wants to employ the real options theory, she has not only one real option to assess, but rather a myriad of real options, a *portfolio of real options*.

At first sight, one would expect value additivity for real options, just like it is the case for financial options. This intuition is wrong. Financial options only define distribution rights of a given number of titles whose value itself is generally not affected by the existence of the options. On the contrary, real options can have an impact on the underlying assets. Ownership of the underlying asset may be a prerequisite for owning the option, and exercising the option may modify the underlying asset. An intuitive example is an option to shut down a plant. When exercised, all other options die, i.e. become worthless. It is obvious that value additivity does not hold, since the options interact.

Few authors have addressed this issue. If so, their objective was mainly to solve a concrete problem where they had to consider interactions, not explicitly tackling the interaction phenomenon.[3] Outstanding are *Trigeorgis*[4] and *Kulatilaka*[5], who have focused

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1 Myers (1977), p. 163.
2 For a detailed literature review, see Lander/Pinches (1998).
3 See e.g. Brennan/Schwartz (1985) and Kulatilaka (1993).

on options interactions, and demonstrated that multiple real options in one project may not be valued separately. *Kester*[6] examines the case of sequenced product introductions (growth options) which lead to synergies and learning effects. *Childs/Ott/Triantis*[7] discuss two projects that can be developed in parallel, but of which only one can be implemented. Whereas these papers do consider portfolios of real options, there is none which explicitly addresses the valuation problem of a portfolio of real options in general.

The present paper borrows from the existing literature and expands the idea of options interactions towards the general real options setting most corporate decision makers are confronted with: multiple underlyings with multiple real options.[8] It shows in how far a stand-alone analysis is different from a portfolio-analysis in the context of real options and develops the associated portfolio problem by structuring the arising effects. Via a simple numerical example, it demonstrates the consequences for the valuation of a portfolio of real options.

In other words, the questions to be answered are: In which regards a portfolio of real options is special? What kind of interactions may be encountered? Which significance must be attributed to diversification, which to budgets? What are the consequences for the valuation of a portfolio of real options?

The paper is organised as follows. Section 2 develops the relevant portfolio-aspects of a portfolio of real options on an abstract basis. Section 3 applies the obtained results to a numerical example and will show how the general case of a portfolio of real options needs to be priced. Section 4 summarizes the results and concludes.

2 Portfolio-aspects in a real options-world

2.1 Assumptions and definitions

In order to build a sound basis for the following analyses throughout this paper, some assumptions are in place. First, it is assumed that all considered investment projects are at least in part irreversible, i.e. there are sunk costs. This assumption is necessary for interpreting these projects as options, otherwise most options would be worthless since

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4 See Trigeorgis (1993).
5 See Kulatilaka (1995).
6 See Kester (1993).
7 See Childs/Ott/Triantis (1998).
8 As an example, Faiz from Texaco Inc. stresses the need of considering portfolio aspects in the management of real options. See Faiz (1999).

current actions could be unwound later without costs. Second, the relevant markets are assumed to be frictionless and complete, which implies that there is a spanning portfolio for all underlying real assets. In fact, the complete markets assumption is crucial for every real options framework and is treated differently in the literature;[9] by supposing complete markets, this paper leaves it up to the reader how to treat this problem.

A set of additional assumptions will conveniently focus the following analyses. Nonetheless it is stressed that their relaxation is possible and very often of utmost interest. All options are european-type, there are no dividends during the life of the options, the risk-free rate of interest is known, constant and identical for all maturities. All real options are proprietary, i.e. the option holder has an exclusive right to exercise the option and does not have to consider the competitors' behaviour. There are no agency-conflicts, so that the optimal option exercise policy will be implemented and therefore the theoretical option value will be realized and translated into market value of the firm. There is no capital rationing.[10] Finally, the decision maker can specify all existing real options. Their origin and creation are not considered.

In this paper, options to invest, options to expand and options to abandon will be treated.[11] An option to invest enables a decision maker to acquire a real asset in exchange for the investment outlay which is the strike of this call option. An option to expand is similar, allowing to expand the scope of an existing asset. An option to abandon allows to quit a project, e.g. shut down a plant, possibly for a salvage value which is the strike of this put option. Options to expand or to abandon are labelled "operating" options, since in our context they deal with an asset in place.

The term "portfolio" is understood in the most general manner: a portfolio is a set of elements, a portfolio of real options is a set of real options. "Portfolio-aspects" are regarded as the properties of the set which differ from the properties of the separate elements. In this sense, portfolio-aspects emerge by putting together real options into a portfolio.

A "project" is defined as a real asset, including all operating options that go together with it. The complete markets assumption yields that the market value of a project without any

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9 See e.g. Copeland/Koller/Murrin (1994), p.460; Luehrman (1998), p. 52; Kasanen/Trigeorgis (1994).

10 See section 2.4 for this assumption, which might astonish the reader used to capital budgets.

operating options, called "separate project", is known. The objective of this paper is to quantify the value that can be created by combining separate projects with operating options. For all analyses, the firm is assumed to maximize its market value, so value creation is always to be understood in terms of additional market value.

In the following, the portfolio aspects of a portfolio of real options are discussed, namely diversification, direct and indirect qualitative interactions, options interactions and correlation.

2.2 Portfolio-aspects and a portfolio of real assets: diversification

The assessment of a particular investment opportunity in a risky world has to take into account the stochastic correlation of this opportunity with all other opportunities. This implies that an optimal global strategy can only be found by considering all relevant alternatives simultaneously.[12]

A famous model of investment management is the model of portfolio selection by *Markowitz*.[13] He shows that, as long as there is no perfect positive correlation between the returns of the analyzed securities, risk can be reduced through diversification so that covariance is the only relevant measure in a portfolio analysis. This is a portfolio-aspect consisting solely in the diversification which results from the stochastic relationship between risky returns. As a consequence, it is purely stochastic in nature.

Leaving the finance department of the firm, the "investment program" is a set of real asset investments that a firm is planning to undertake. Of course this program is a portfolio, a set of elements, in which diversification should be relevant, since the cash-flow profiles of separate projects show correlations that in general are not perfectly positive. This implies that just like a portfolio of securities, a marginal project has to be assessed only by the covariance between its returns and the return of the rest of the portfolio. If an additional project is added to the portfolio, the aggregate risk normally will grow by less than the risk of the separate project. Furthermore, real investments normally last more than just one period, so that new projects can only be evaluated by taking into account simultaneously all other new projects and all projects in place, more specifically: the

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11 In the literature, the terminology is very homogeneous. The reader may refer to Trigeorgis (1996), pp. 9-14 for an introduction.

12 See Franke/Hax (1999), p. 306.

13 For the whole paragraph, see Markowitz (1952).

covariances of their future cash-flows. This relationship, again, is purely stochastic in nature.

We retain that generally speaking, diversification is a relevant portfolio-aspect of a portfolio of investments in real assets, in analogy to a portfolio of securities. Diversification appears as a legitimate goal for a risk-averse entrepreneur whose capital is mainly employed in his firm. Yet, if this were the case, the firm would maximize the entrepreneur's utility, not its market value. Here, a firm maximizing its market value is explicitly assumed. This firm cannot create value by diversification, because its shareholders can and will diversify their personal portfolios. Furthermore, it is assumed that market values of the separate projects exist. Consequently, on a perfect capital market the market value of a portfolio of such projects must equal the sum of the parts.[14]

It is concluded that diversification and, implicitly, the correlation of the market values of the separate projects without operating options, are not a relevant portfolio-aspect in the first place. However, considering real options, the correlation becomes relevant when the new market value that can be generated by combining separate projects with operating options is to be assessed. This will be shown in section 2.6.

Beyond these stochastic relationships, projects can affect each other on an technical or physical level. It may then become important to consider which projects are operated at the same time. In order to clearly distinguish them from the stochastic relationships encountered so far, they are defined as "qualitative interactions".

2.3 Portfolio-aspects and a portfolio of real assets: direct qualitative interactions

Developing the classification of Betge,[15] "direct qualitative interactions" are those qualitative interactions which have their origin in the investment-plan[16] as such or which result from interactions with investments already undertaken and still producing cash-flows. Direct qualitative interactions do not result from stochastic relationships, but from physical properties of the projects. This is why they cannot be avoided in any way comparable to diversification.

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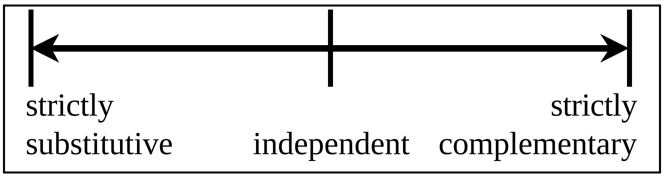
14 See Bodie/Kane/Marcus (1999), pp. 258f.

15 See Betge (1995), p.11.

16 The investment-plan comprises all investments that are to be undertaken. Together with all other plans of the firm, e.g. the financing-plan, the overall plan of the corporation is obtained. See Betge (1995), p.11 and Perridon/Steiner (1997), p.605f.

On the one hand, projects can exclude each other, e.g. when they have an identical technical function. In this case they are strictly substitutive. On the other hand, projects may require that other projects exist at the same time, as e.g. in a production line. They are strictly complementary. These two relationships can also be gradual in the sense that the cash-flow profile of one project can be positively or negatively affected by the existence of other projects. An example for a positive gradual interaction is a plant whose productivity rises because of synergies with another plant. Furthermore, the interaction can always be mutual or just one-way. For example, it is mutual when both of two plants benefit from synergies, or one-way when one of the two plants is completely unaffected by the existence of the other. Other combinations are of course conceivable, such as two plants where one is positively and one is negatively affected by each other. In short, interaction can affect the profitability of projects or even their feasibility as such. If there is no interaction, the projects are independent. All theses cases together can be visualized as a continuum:

Exhibit 1: Continuum of direct qualitative interaction



Hax states that the analysis of interacting projects may be focused on strictly complementary projects without loss of generality.[17] This is consistent with the concepts that we have developed so far, since the interaction can take place in different ways and degrees, yet the principle of interaction remains the same. This is why section 3.4 will consider the case of a positive gradual interaction, representative for the whole continuum.

2.4 Portfolio-aspects and a portfolio of real assets: indirect qualitative interactions

The term "indirect qualitative interaction" covers those qualitative interactions which result from incongruities between plans. If plans are not perfectly harmonised, one plan may become a bottleneck, resulting in projects competing for the limited resources and entering into interaction via this competition. This interaction is beyond the investment-plan, like for example binding capital restrictions, or budgets. Due to the shortage of capital, not all profitable investments can be supported.

This kind of interaction is qualitative, because it is not inferred by simple stochastic relationships. It is indirect, because it results from general conditions and restrictions that do not necessarily go hand in hand with the investment projects and therefore could be avoided, e.g. by finding additional financing resources.

In the current context, shortage of financial resources is equivalent to all shortages in other plans than the investment-plan, because for example a lack of qualified personnel is as binding a constraint as a lack of money. Indirect interactions will not be allowed for in the present paper for two reasons. Firstly, if restrictive plans were assumed, the dynamics ("interactions") between current plans, investment choice and future plans would have to be modelled, a task that is beyond the scope of this paper. Secondly, the indirect interactions result from frictions in the relevant markets or from a voluntarily chosen organisational design of the corporation. These frictions can hardly be explained in the otherwise perfect world that is assumed: either they lead to suboptimal solutions, or alternatively they are not binding and thus irrelevant.

As a result, the relevance of portfolio-aspects following from indirect qualitative interactions is ambiguous. Henceforth, these portfolio-aspects are excluded from the analysis.

2.5 Portfolio-aspects and a portfolio of real options: options interactions

So far different kinds of interaction on the level of real assets have been developed. Now the analysis will focus on a portfolio of real options, so that the real assets are the underlying assets of the real options. This way, the previously developed interactions are implicit to the further analysis. In the following sections, interactions on the level of real options will be analyzed.[18]

Financial options exhibit value additivity. This is due to the fact that they generally do not affect the underlying asset: they merely define a distribution of a given number of titles whose value itself is generally not affected by the existence of the options. On the contrary, real options are often inseparably interconnected with the underlying asset in the sense that ownership of the underlying asset is a prerequisite for owning the option, and that exercising the option has an impact on the asset and its value. This is by definition the case of operating real options. Since these affect the asset, they interact.

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17 See Hax (1985), p. 39.
18 For this whole section, see Trigeorgis (1996), pp. 234-237.

For a comprehensive understanding of the interaction of options, consider two european-style, operating options on the same underlying asset, but with different maturities. The option maturing first is the "first" option, the other one the "second". A valuation of these two options has to take place simultaneously, yet the effects on each option can be showed separately, in a static way.

On the one hand, if the second option is added, the first option is affected because the value of its underlying is not only the value of the real asset, but the value of the asset plus the second option. Since an option value can never be negative, the value of the underlying asset will be unchanged or higher with the second option. If the first option is a call (put), its value increases (decreases) with the arrival of the second option. As an example, the value of an option to expand will increase if the second option is an option to abandon. This is quite plausible: the second option serves as an insurance against negative outcomes in the future, so that the decision maker, ceteris paribus, is more willing to exercise the option to expand. As a consequence, the first option can only be priced when the second one is taken into account.

On the other hand, exercising the first option might modify the underlying asset and its value, which is also the underlying of the second option. Because of this modification, it is possible that after exercising the option to expand, it is less probable that the option to abandon will be exercised so that the value of the second option decreases. This shows that the second option cannot be priced without considering the first one.

Since both options affect each other, they have to be priced simultaneously: the value of the first call increases because of the value of the second put, but to find the value of the call we need the value of the put and vice versa. The extreme case of negative interaction can be that the value of the portfolio equals the value of the most valuable option in isolation. This is the lower limit of value, because in the worst case all other options simply would not be exercised. The other extreme is super-additivity: the value of the portfolio is higher than the sum of the option values in isolation. Finally, it is conceivable that there is no interaction at all. Considering these cases together results in all possible options interactions being outlined. If more than two options on the same underlying asset exist, the problem becomes much more complex, but the principle of interaction remains the same: all options have to be priced simultaneously.

This result resembles a compound option, an option on an option where the underlying of the first option is the second option. It is obvious that these two options have to be priced simultaneously: the strike is a necessary pricing input, and here the strike equals the unknown value of another option. Yet the analogy between the demonstrated real options feature and an option that is compound in the strict sense is limited, because the described real options do not only come into existence by exercising another option. For a clear distinction, the following terms are introduced. The general interaction effect between real options is defined as "time compoundness", given that they refer to the same underlying asset over time. In exchange, the special case of a compound option in the strict sense is described as "causal compoundness", because the first option, if exercised, gives birth to the second.

Both forms of compoundness are relevant portfolio-aspects of a portfolio of real options and will be discussed in what follows.

2.6 Portfolio-aspects and a portfolio of real options: correlation

As we have seen in section 2.2, the correlation of market values of the separate projects is irrelevant in the sense that a possible diversification would not create additional value. These market values are exogenous factors in the given setting. Here we seek to determine the new market value which is reached by combining separate projects, just like they can be acquired on the relevant markets, with operating options. The difference between the new and the old market value is the value created by the firm.

We have seen that a project, i.e. separate project plus operating options, has to be priced as a compound option in a real options framework. Now consider more than just one project. If theses projects are completely independent of each other, the correlation of project values is of course irrelevant. On the contrary, if these projects exhibit any kind of interaction and thus have to be priced simultaneously, correlations are of importance. When pricing many projects, as many stochastic processes for the value of the respective underlyings must be modelled, including their correlation. The valuation has to take into account how these processes move together; this is analogous to pricing standard financial compound options. Even for only two stochastic processes, and taking a model as simple as the binomial model, the calculus is very demanding. The binomial tree becomes three-dimensional, with one time- and two underlying-dimensions. Contingent risk-neutral probabilities are required, which have to be determined by an iterative numerical algorithm.[19]

Generally speaking, the correlation can be constant or changing over time. It might be affected by the exercise of real options. This can be a side effect, but it is also conceivable that there is an option whose only aim is to influence the correlation. This can be the case if by exercising the option, the sensitivity of project value towards the value of the underlying is changed, so that this option in fact serves to react to price movements. A well-known example for this is a dual-fuel steam boiler.[20]

The present paper aims at showing how a portfolio of real options is to be priced. With regards to complexity and in order to focus on the main issues of this paper, all further analyses will suppose perfect positive correlation for the case of two projects. This is legitimate, because this simplified setting leads on its own to a new form of compoundness which embraces projects and therefore is called "inter-project compoundness". If more complex correlations are permitted, the result of this paper is even more prominent, because the correlation alone would demand a simultaneous, "compound" pricing.

The correlation of the stochastic processes of the values of the underlying assets is a relevant portfolio-aspect of a portfolio of real options. For simplicity and without loss of generality, it will not be treated explicitly in the further analyses.

3 Pricing a portfolio of real options

3.1 Overview

The goal of this section is to price a portfolio of real options which incorporates all relevant portfolio aspects. For this purpose, a simple numerical example will successively be developed towards the general portfolio case. This will be carried out in three stages. Stage 1 will show a basic real options configuration: one project with one operating option. Complexity will then increase in stage 2, where a second operative option is added. Finally, stage 3 investigates the general portfolio case, asking what happens if two interacting projects with two operative options are to be modelled.

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19 In particular, see Boyle (1988).
20 See Kulatilaka, N. (1993).

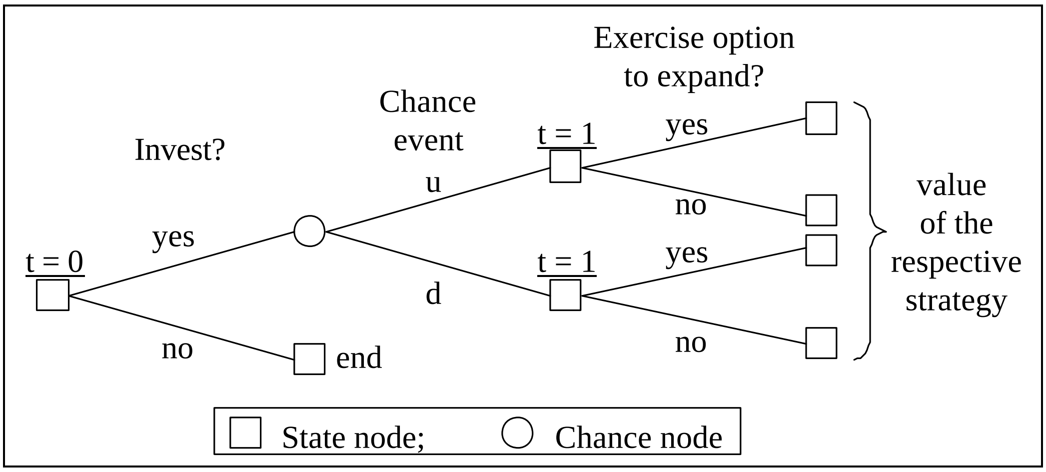
3.2 Stage 1: One project with one operating option

The most simple composition of a portfolio of real options is an investment opportunity which gives one operating option once the initial investment has been undertaken.[21] This serves as the starting point and benchmark for the later stages.

Example 1:
Suppose a firm has the opportunity to acquire a plant whose market value is stochastic and perfectly positively correlated with the oil price. The oil price and therefore the value of the plant are assumed to follow a two-period binomial process with the jump parameters $u = 1.3$ and $d = 0.7$. The initial market value of the plant is 100[22]. The investment decision is now-or-never, and the plant can be run immediately after investing. If the plant is run, the firm has the option to expand, increasing the capacity and market value of the plant by 60% for a strike of 50.

Generally speaking, an option can be modelled and priced via dynamic programming, because it implies the pricing of future rights to choose. For this, dynamic programming determines contingent partial plans. In the current context, these plans are sequences of exercise decisions. This is why the present valuation problem could be visualized by a decision tree and solved by an algorithm of dynamic programming such as the "roll-back algorithm".[23] Exhibit 2 depicts the decision tree for example 1:

Exhibit 2: Decision tree on stage 1



[21] This portfolio contains two options: an option to invest, i.e. to make the initial investment, and an operating option. In the following the option to invest is trivial, since market values are known and a decision has to be made immediately. The goal of the following sections is to price the operating option(s).

[22] To focus the analysis, no currency nor unit is specified throughout the paper.

[23] See Dixit/Pindyck (1994), pp. 120-124. For dynamic programming and decision trees, see Magee (1964) and Laux (1971).

Qualitatively, pricing the project with dynamic programming is equivalent to pricing it with an option pricing model. Throughout this paper, the discrete binomial model by *Cox/Ross/Rubinstein* will be used.[24] Assuming a risk-free rate of interest of 6%, the value of the project, denominated as I, is depicted as follows:[25]

Exhibit 3: Value of project I

$$I_0 \begin{cases} \xrightarrow{p} & I_1^1 = \max(130; 130 \cdot 1.6 - 50) \\ \xrightarrow{1-p} & I_1^0 = \max(70; 70 \cdot 1.6 - 50) \end{cases}$$

The risk-neutral probabilities p and (1-p) are:

(1) $p = (1 + r - d)/(u - d) = (1.06 - 0.7)/(1.3 - 0.7) = 0.6$ and $1 - p = 0.4$ [26]

Application of the risk-neutral probabilities yields I_0:

(2) $I_0 = \{0.6 \cdot \max(130;130 \cdot 1.6 - 50) + 0.4 \cdot \max(70;70 \cdot 1.6 - 50)\}/1.06 = 115.85$.

The value of the project in $t = 0$ is 115.85, the option to expand is only exercised after one up-jump. As the market value of the separate project is 100, the additional value created by the option to expand is 15.85.

Example 2:

All else equal, instead of the option to expand the plant is combined with an european-style option to abandon, permitting to sell the plant in $t = 2$ for a strike of 100, independent of the state of the oil price. In other words, the plant is suitable for an alternative use whose value is non-stochastic. The binomial tree comprises two periods now, but can still be solved with a simple backward induction.

The project is denominated as J. Starting with $J_2^2 = \max(169;100)$, $J_2^1 = \max(91;100)$ and $J_2^0 = \max(49;100)$, the induction yields $J_1^1 = 133.4$ and $J_1^0 = 94.3$, and finally $J_0 = 111.11$.

The value created by the option to abandon is thus 111.11 -100 = 11.11. The "naive" sum of the two option values, put and call, is 15.85 (call) + 11.11 (put) = 26.96. This value

24 See Cox/Ross/Rubinstein (1979).

25 The notation is as follows: the figure in subscript counts the periods, the figure in superscript counts the number of up-jumps.

26 The risk-neutral probabilities will be the same in all examples, because the jump coefficients and the risk-free interest rate are the same.

would be obtained if the portfolio of the firm consisted of these two options and if there were no interactions.

3.3 Stage 2: One project with two operating options

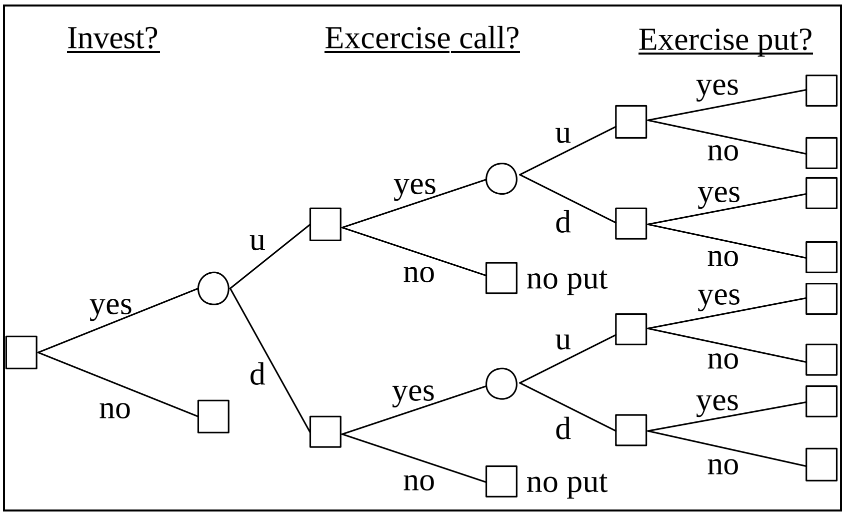
3.3.1 Causal Compoundness

Up to now, the project could be priced by the help of a simple backward induction, because there was only one option. Now two operating options will be analysed, and it will be found that the algorithm needs to be modified.

Example 3:

The plant still gives the specified option to expand. If and only if this call is exercised, the previously specified option to abandon is obtained. As defined in section 2.5, there is a causal compoundness. An example for this setting is a plant which can only be turned to a different use after its modification, which is the consequence of exercising the call. Again, a decision tree allows us to demonstrate the features of the project. If the call is not exercised and therefore there is no put, the value of the project equals the market value of the separate project, as there is no further flexibility. The decision tree is given in exhibit 4:

Exhibit 4: Decision tree on stage 2



This project, called K, cannot be priced by the simple backward induction employed so far. A modified recursive procedure is needed. First, suppose the call is exercised in t = 1, so that there is a put. This yields:

(3) $K_2^2(\text{ex.}) = \max(169 \cdot 1.6; 100) = 270.4$;

(4) $K_2^1(\text{ex.}) = \max(91 \cdot 1.6; 100) = 145.6$, and

(5) $K_2^0(\text{ex.}) = \max(49 \cdot 1{,}6; 100) = 100$.

The put is only exercised after two down-jumps. Applying the given risk-neutral probabilities, one obtains $K_1^1(\text{ex.}) = 208$ and $K_1^0(\text{ex.}) = 120.2$. Second, suppose the call has not been exercised, so there is no further option. As the plant is not modified in this case, its value equals the known market value: $K_1^1(\text{n.ex.}) = 130$ and $K_1^0(\text{n.ex.}) = 70$.

Third, it is to be determined if exercising the call is advantageous:

(6) $K_1^1 = \max(K_1^1(\text{ex.}) - 50; K_1^1(\text{n.ex.})) = 158$, and

(7) $K_1^0 = \max(K_1^0(\text{ex.}) - 50; K_1^0(\text{n.ex.})) = 70.2$.

The call is always exercised. Finally, application of the risk-neutral probabilities yields $K_0 = 115.91$.

Deduction of the market value of the separate project yields a combined option value of 115.91-100 = 15.91. It is remarkable that the value of the two options, precisely: of the compound option, only slightly exceeds the value of the call alone, which is 15.85. This is because the put only activates after exercising the call, which boosts the value of the underlying asset for the put. The probability that the put will be exercised, contingent on exercise of the call, is very small.

This section has demonstrated that decision trees can be employed for modelling a compound option-problem. Putting the trivial case of independent decisions aside, one can generalize that a decision tree always models a compound-problem. This is because the tree contains all possible combinations of actions and realizations of the stochastic variables. Therefore one can qualify the decision tree as an exhaustive form of the binomial tree: it explicitly incorporates the iterative feature of the binomial tree, because the decision tree distinguishes the cases of exercising and not exercising the call as separate branches. In addition, it has been shown that causal compoundness can lead to significant deviations from value additivity.

3.3.2 Time Compoundness

The compoundness analysed so far is no particular feature of real options and rather well-known from financial compound options. Consequently, this section will treat the case of time compoundness, showing that it demands the iterative recursion known from causal compoundness.

Example 4:
Now the plant gives the two previously specified options to expand and to abandon. The option to abandon (maturity $t = 2$) always exists, regardless of whether the option to expand (maturity $t = 1$) has been exercised or not. One can imagine the option to abandon again as an alternative use of the plant, but now this use is not affected by exercising the other option. The decision tree is almost identical to that of Example 3, except that the put exists in both state nodes of period two.

The same iterative recursive procedure as in example 3 has to be applied, because one still has to take into account the existence of the put when deciding whether the call is to be exercised. Since the decision tree changes only in part, similarly only a part of the calculations must be repeated. The project is denominated as L.

If the call is exercised, the resulting problem becomes identical to Example 3, due to the existence of the put. Therefore, $L_1^1(\text{ex.}) = K_1^1(\text{ex.}) = 208$ and $L_1^0(\text{ex.}) = K_1^0(\text{ex.}) = 120.2$. The values in the case when the call is not exercised have to be computed anew, because the put still exists in this case. Yet since the value of the underlying asset is not modified, the problem is identical to Example 2. Therefore, $L_1^1(\text{n.ex.}) = J_1^1 = 133.4$ and $L_1^0(\text{n.ex.}) = J_1^0 = 94.3$. These two intermediate results now must be put together as in Example 3, checking whether or not exercise of the call is advantageous. Thus the problem is solved by:

(8) $$L_0 = \{0.6 \cdot \max(208 - 50; 133.4) + 0.4 \cdot \max(120.2 - 50;\ 94.3)\}/1.06 = 125.03.$$

The option value in the case of time compoundness is 125.03-100 = 25.03, clearly exceeding the value of 15.91 in the case of causal compoundness. This makes sense, since the put protects against negative outcomes when the call is not exercised. In that sense the two options complement each other, the call allowing to profit from positive outcomes and the put protecting against negative ones. Nevertheless, there is still sub-additivity, the option value differing by 1.92 from the naive sum of the parts of 26.96, which is approximately 7%.

As a result, time compoundness has to be priced just like causal compoundness. This explains why it should be considered as a compound problem despite the qualitative differences. The numerical example shows that a naive pricing approach can lead to an incorrect project valuation.

3.4 Stage 3: Two projects with two operating option

3.4.1 Analogy to decision trees

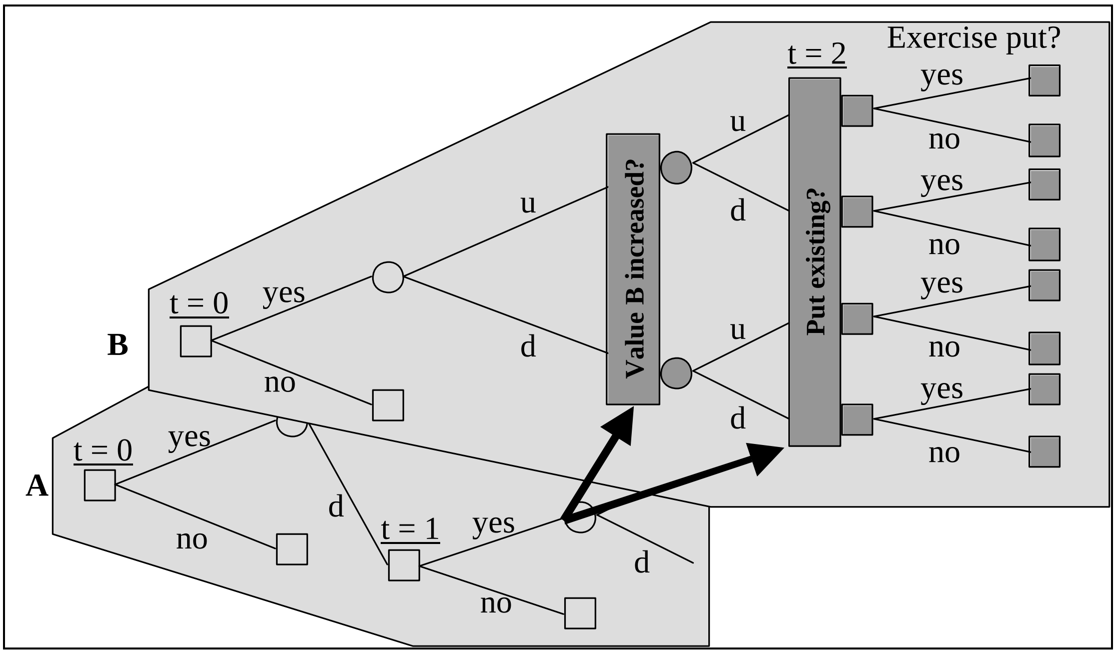
The idea of compoundness will now be expanded towards the case of two projects with direct qualitative interactions. A setting is chosen which will unite all portfolio-aspects identified in this paper and therefore is to be considered as the general case of a portfolio of real options.

Example 5:

The firm has to decide at t= 0 whether to realize two projects A and B which do not exclude each other. Both separate project values depend on the oil price as in example 1 and therefore are perfectly positively correlated. To facilitate the calculation, both separate projects have an initial market value of 100. Project A is the known plant with the option to expand at t = 1. Project B is another plant, giving the known option to abandon at t = 2, if and only if the call on A is exercised. This is a causal compoundness which is incorporated in many projects. That is why it is defined as "interproject compoundness". Furthermore, there is a direct qualitative interaction of the form that if and only if the call on A is exercised, the market value of plant B immediately rises by 20%. Consequently, each project as such is taken from stage 1, because in a stand-alone form A is identical to example 1 and B contains no options. Because of the interactions, the ensemble constitutes stage 3. As argued in section 2.3, this setting is representative for all direct qualitative interactions. One can imagine this setting as follows: Modification of A brings with it a new technology which can also be put to good use in B, for example via a better productivity which leads to a higher market value. Only with this new technology an alternative use for B is possible.

Of course each project can be modelled by a separate decision tree. Yet these trees have to be assessed simultaneously, which can be visualized by a decision tree on two levels. Level A consists of project A with the call at t = 1. Level B is project B as in example 1, with the difference that there is no call. The newly introduced feature is the link between these levels, visualized by arrows: the decision on the call on level A determines on level B whether the market value goes up, and whether there is a put. The two-level decision tree can be found in exhibit 5:

Exhibit 5: Decision Tree on stage 3



On a first step, the arrows make it impossible to apply a simple induction. Nevertheless, a new exhaustive tree on one level could easily be obtained, modelling each project combination as a separate branch: "A only", "B only", "A and B", "no investment". The branches of this new tree would be as in the examples described before, with the exception of "A and B": after the branch "exercise call on A", there is a new decision node for the put on B, taking into account the increased value of the underlying asset. This new tree could easily be solved by a simple backward induction.

The new decision tree obviously brings together interacting decisions. As a decision tree generally is a way of modelling a compound problem, it is demonstrated on an intuitive level that the possibility of investing in the two interacting projects A and B is a compound option. Therefore it should be possible to price example 5 by the help of the developed iterative recursion. This will be demonstrated in the following section.

3.4.2 Pricing the option on stage 3

It directly follows from the argumentation in the preceding section that there is an additional iteration which consists of comparing {"A", "B", "A & B", "no investment"}. "A" is project I with an option value of 15.85, "B" alone has no option and therefore no option value, and the same is of course true for "no investment". Consequently, only "A & B" is left to be assessed.

The project is denominated as M. First, consider the case when the call on A is exercised. The value of A increases by 60%, at the same time the value of B increases by 20% and there is a put on B in t = 2. This yields:

(9) $M_2^2(\text{ex.}) = 169 \cdot 1.6 + \max(169 \cdot 1.2;100) = 473.2$ (put not exercised),

(10) $M_2^1(\text{ex.}) = 91 \cdot 1.6 + \max(91 \cdot 1.2;100) = 254.8$ (put not exercised),

(11) $M_2^0(\text{ex.}) = 49 \cdot 1.6 + \max(49 \cdot 1.2;100) = 178.4$ (put exercised).

Using the binomial model one obtains $M_1^1(\text{ex.}) = (0.6 \cdot M_2^2 + 0.4 \cdot M_2^1)/1.06 = 364$ and analogously $M_1^0(\text{ex.}) = 211.55$.

If the call is not exercised, there is no interaction anymore. Therefore, the value of the portfolio of the two projects equals the sum of the parts: $M_1^1(\text{n.ex.}) = 2 \cdot 130 = 260$ and $M_1^0(\text{n.ex.}) = 140$. Comparing the values for exercising and not exercising the call, yields $M_1^1 = \max(M_1^1(\text{ex.}) - 50; M_1^1(\text{n.ex.}) = 314$ (exercise call) and analogously $M_1^0 = 161.55$ (exercise call). Finally, $M_0 = 238.70$ is obtained.

After deduction of the market values of the separate projects, the option value when implementing both projects is 38,7. Comparing this with the other values gives max(38.7; 15.85; 0) = 38.7. Project A and B are to be undertaken simultaneously.

A comparative-static analysis may start with the option value of the call as in example 1, which is 15.85. If the exercise of the call only caused an increase in the value of B without activating the put, an analogous application of the algorithm would yield 32.83. The increase in value of 16.98 is the consequence of the interproject direct qualitative interaction. The marginal value contribution of the put is then only about half of the value of the put in isolation (38.7-15.85-16.98 = 5.87; 5.87/11.11=52,8%). So, taking into account the interproject causal compoundness of the put, one finds a sub-additivity of 5.24/(5.24+38.7)=11.9%. The results are resumed as follows:

Table 1: Comparative-static analysis of value interaction

1. Option value call alone	15.85
2. Value contribution direct qualitative interaction	16.98
3. Option value put alone	11.11
4. Sub-additivity put	-5.24

Overall value (=1+2+3+4)	38.70

It can be concluded that the valuation of stage 3 required the algorithm of a compound option valuation. This demonstrates in a straightforward way that it is necessary to qualify stage 3 as a compound option. An analysis of the different value effects has proved that the compoundness leads to considerable deviations from value additivity.

4 Conclusion

This paper has analyzed the portfolio aspects of a portfolio of real options. At the beginning, the possible portfolio-aspects were structured in an argumentative way, providing a sound basis for the following analyses. It found that the relevant portfolio-aspects are direct qualitative interactions, options interactions (time and causal compoundness) and correlation. Then, a simple numerical example illustrated the developed portfolio aspects and demonstrated the uniqueness of a portfolio of real options. For this, the example was developed towards the most general and complex setting in three successive stages. Generally speaking, the elements of a portfolio of real options can come from all three stages. A portfolio can consist of several projects (or only one) from stage 1 among which no interactions exist. In this case value additivity would be obtained, but this is not the general case. Stage 2 demonstrates that due to time compoundness, a compound option pricing model is required for the valuation of a portfolio of real options. A portfolio consisting of several projects from stage 2, without further interaction between projects, would allow us to price each project as a compound option and then add up their values. This again is only a special case.

Stage 3 showed a new form of compoundness, namely interproject compoundness. It incorporated all relevant portfolio aspects of a portfolio of real options. Correlation effects have not explicitly been modelled in the numerical example but could be integrated in the proposed algorithm with some effort. Since even the presented case of perfectly positively correlated assets leads to an interproject compoundness, this effect is even more prominent when correlations are less strong or change over time.

Whereas the arguments of section 2 are of general nature, the examples in section 3 do not pretend generality. Nevertheless, it is a proof by existence: since one example with value interaction on stages 2 and 3 can be found, the possibility of interaction is proved. A portfolio of real options generally does not exhibit value additivity.

In conclusion, stage 3 is the general portfolio problem when managing real options. If there are any of the described interactions, a firm must consider its portfolio of real options as one single compound option. Managing this option consists in pricing it repeatedly over time and implementing the implicitly generated optimal exercise policy.

The absence of interaction between projects cannot be supposed a priori, at least some kind of interaction seems to be a realistic hypothesis for most real cases. If a corporation uses capital budgets, no assumption is required, because there is automatically interaction. Since we know that interaction converts a real option-problem into a compound real option-problem, the finding of this paper, in answer to the last of the questions raised in the introduction, can be generalized as follows: *a firm is a compound option*. This has to be taken into account when assessing single projects or when valuing the whole firm. It appears as an interesting future field of research to test this finding empirically.

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