A Note on Forward and Backward Partial Differential Equations for Derivative Contracts with Forwards as Underlyings

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Abstract

In this note we show that the forward partial differential equation for the valuation of European options proposed by Dumas, Fleming, and Whaley (DFW) does not hold in general for a model with time and state-dependent local volatities. To clarify things we analyze the forward and backward partial differential equations for contingent claims in detail. We consider the three cases of spot options on the spot underlying, and spot and forward options on the forward. We find that the DFW forward equation is only valid in certain special settings, namely when the cost of carry for the underlying asset is equal to zero, or for the general class of homogeneous option pricing models. Furthermore, we show that in the general case of arbitrary (but deterministic) cost of carry, the spot price has to be the argument of the local volatility function in the backward equation for the forward price of a claim on the forward, not the forward price.

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1 Introduction

Both Kolmogorow's forward and backward partial differential equation (pde) are widely used in the theory of option pricing. Recently, especially the forward equation has received much attention in the context of derivative pricing under a volatility smile (see, e.g., Andersen and Brotherton-Ratcliffe (AB) [1], Dumas, Fleming, and Whaley [4], or Dupire [5]). Whereas the standard backward equation uses the underlying and calendar time as variables in the pricing function, the forward equation is based on the maturity date and the strike price of a standard European option, and it is furthermore only valid for this rather limited class of securities. The backward equation on the other hand holds for any underlying and any contingent claim and is thus far more general. Using the forward equation, however, allows the simultaneous valuation of more than one European option, which is a special advantage when finite difference methods have to be employed to solve the pde numerically.

In both theoretical analyses and empirical applications of derivative pricing models, it is sometimes convenient to use forward prices instead of spot prices. This transformation can be performed for both the underlying of a contingent claim and the claim itself. In the extreme case we could thus move from a standard spot option on the underlying to a forward option on the forward written on the underlying. This involves a proper transformation of the coordinates in a given pde, but it is not in general true that the transition from spot prices to forward prices automatically removes all terms depending on domestic and foreign interest rates (in the case of an FX contract) or interest rates and dividend yields (in the case of a contract on a dividend paying stock or stock index). As we will show below, this technical trick only yields convenient expressions for the pde when we work with the backward equation. In the case of a forward pde we cannot in general obtain a comparably simple pde based on the forward prices of the underlying asset and the option, except when the forward price is equal to the current spot price, or the pricing model has the homogeneity property.[1] Since DFW [4] do not explicitly assume either of these scenarios, a key result of our analysis is that the pde suggested in their paper is incorrect for their general model.

The remainder of the paper is organized as follows. In section 2 we will briefly summarize the theory on forward and backward pde's for contingent claim valuation. Section 3 contains the main results of the paper, namely the forward and backward equations for all four scenarios listed above, i.e. for both spot and forward options on both the underlying and the forward contract. A summary is given in section 4.

[1] See Merton [7] for a definition and an analysis of the homogeneity property of option prices.

2 Forward and Backward Equations

2.1 The General Setting

Let $(\Omega, \mathcal{F}, P)$ be a filtered probability space. The dynamics of the underlying under the risk-neutral probability measure $\hat{P}$ are given by

$$dS_t = (r - \delta)S_t\, dt + \sigma(S,t)S_t\, d\widehat{W}_t.$$

We call $r - \delta$ the *cost of carry* for S with δ representing the continuous payout rate (e.g., a dividend yield). If S is an exchange rate, we would set r equal to r_d, the domestic interest rate, and δ equal to r_f, the foreign interest rate. In the following analysis we will only refer to this case, i.e. we will consider the dynamics

$$dS_t = (r_d - r_f)S_t\, dt + \sigma(S,t)S_t\, d\widehat{W}_t \tag{1}$$

with S as the spot exchange rate. Of course all the results are also valid for the alternative interpretation of S as a dividend paying stock or stock index. Both r_d and r_f are assumed to be constant. This assumption could easily be relaxed to make r_d and r_f functions of time. However, the gain in generality is rather small compared to the cost of a much more cumbersome notation. $d\widehat{W}_t$ represents the increment of a standard one-dimensional Brownian motion, and $\sigma(S,t)$ is called the *local volatility function*.

Besides the exchange rate there is a second security called the *domestic money market account* with price B_t at time t and dynamics given by

$$dB_t = B_t\, r_d\, dt \tag{2}$$

which, given $B_0 = 1$, yields $B_t = \exp(r_d t)$.

Before going into the details we want to point out that there are two slightly different meanings of the terms 'backward equation' and 'forward equation'. From the mathematical point of view backward and forward equations are known as the Kolmogorow equations for the transition densities. From the economic point of view there also exists a so called *backward (pricing) equation* for an almost arbitrary contingent claim $g(S,t)$ on the underlying asset S. This equation can be derived either by using duplication strategies or by integrating the transition density function. For the special case of plain vanilla options there also exists a so called *forward (pricing) equation*, which makes use of the special payoff structure for these contracts. With the Kolmogorow forward equation for the transition density one can easily derive the forward equation for a European call in the variables strike K and maturity T. This equation is of great empirical interest because data of option prices vary in K and T rather than in S and t.

2.2 Backward Equation

Let $C(S,t;K,T)$ denote the value at time t of a standard European call option with strike price equal to K and maturity date T, when the current exchange rate is equal to S.

Using standard no-arbitrage arguments, it can be shown that, given (1) and (2), the value of the option $C \equiv C(S,t;K,T)$ satisfies the fundamental Black-Scholes pde

$$r_d C = C_t + (r_d - r_f) S C_S + \frac{1}{2} S^2 \sigma^2(S,t) C_{SS} \tag{3}$$

where subscripts denote partial derivatives. This equation has to be solved under the boundary condition

$$C(S,T;K,T) = g(S) \equiv \max(S-K,0).$$

Note that in (3) K and T are held fixed, and the fundamental pde is the backward equation in the so called *backward variables* S and t. This equation holds in general for any claim with an (almost) arbitrary boundary condition.

An analogous backward equation holds for the transition density. Let $p \equiv p(S,t;y,T)$ denote the risk-neutral transition density for the exchange rate, i.e. $p(S,t;.,T)$ is the density of the process at time T when started at S at time t. Then this density satisfies the pde

$$p_t + (r_d - r_f) S p_S + \frac{1}{2} S^2 \sigma^2(S,t) p_{SS} = 0 \tag{4}$$

subject to the boundary condition

$$p(S,T;y,T) = \delta(S-y),$$

where $\delta(x)$ represents the Dirac delta function (see Shreve [8], p. 180).

2.3 Forward Equation

Under suitable regularity conditions on the local volatility function σ, there also exists a pde in the so called *forward variables* y,T of the density $p = p(S,t;y,T)$, which is given by

$$p_T + (r_d - r_f) \frac{\partial}{\partial y}[yp] - \frac{1}{2} \frac{\partial^2}{\partial y^2}[\sigma^2(y,t) y^2 p] = 0 \tag{5}$$

with boundary condition

$$p(S,t;y,t) = \delta(S-y).$$

A heuristic derivation of this result can be found in Shreve [8], pp. 306-08.

From the forward pde for the density we can derive a forward pde for plain vanilla options (see AB [1]). Using the fact that

$$\frac{\partial^2 C(S,t;K,T)}{\partial K^2} = e^{-r_d(T-t)} p(S,t;K,T),$$

and the forward pde for the density, we obtain after two integrations with respect to K the pde

$$C_T + (r_d - r_f)KC_K + r_f C - \frac{1}{2}\sigma^2(K,T)K^2 C_{KK} = 0, \tag{6}$$

which has to be solved under the boundary condition

$$C(S,t;K,t) = \max(S-K,0).$$

Note that the forward equation is only valid for a standard call, in contrast to the backward equation that holds for arbitrary claims.

3 Forward-Based Derivation of Backward and Forward Partial Differential Equations

3.1 Options on Forwards

We now consider options on forwards. Let T_1 and T_2 be the maturity dates of the call and the underlying forward F, respectively. We denote the price of the call at time t with strike K on the forward by $\bar{C}(F,t,T_2;K,T_1)$. Assuming constant domestic and foreign interest rates, the price of a forward with maturity T_2 on the underlying exchange rate is given by $F(S,t,T_2) = Se^{(r_d-r_f)(T_2-t)}$. In the special case, when the maturities of the option and the forward contract are the same, i.e. $T_1 = T_2 = T$, the price of a call on the underlying is equal to the price of a call on the forward. At time T, both options have the same payoff, since at maturity the forward price converges to the spot, and both options do not have any cash flows between t and T. So it must be true that

$$C(S,t;K,T) = \bar{C}(F,t,T;K,T) \equiv \bar{C}(F,t;K,T),$$

which means we can interpret the transition from C to $\bar{C}$ as a simple transformation of coordinates from S to F. For the general case, prices are given in Proposition 1 in Appendix A.

3.2 Forwards on Options on Forwards

The ultimate goal is to find the forward price of a call on a forward contract (written on the underlying).

Having obtained the function $\bar{C}$ as shown above, the forward price for delivery at time T_0 ($T_0 \leq T_1 \leq T_2$) of this option is given by

$$D(\bar{C}, t, T_0) \equiv D(F, t, T_2; K, T_1, T_0) = e^{r_d(T_0-t)}\bar{C}(F, t, T_2, K, T_1).$$

In what follows we will only consider the special case $T_0 = T_1 = T_2 = T$:

$$\begin{aligned} D(F,t,T;K,T;T) &\equiv D(F,t;K,T) \\ &= e^{r_d(T-t)}\bar{C}(F,t;K,T) \\ &= e^{r_d(T-t)}C(S,t;K,T), \end{aligned} \tag{7}$$

i.e., the forward price of the option is simply equal to the price of the option on the exchange rate times a factor reflecting the cost of carry for C.

3.3 The Backward Equation for Forward Contingent Claims on Forwards

From the Black-Scholes backward pde (3) for the claim $C(S,t;K,T)$ in the backward variables S, t with K, T fixed we obtain the following forward-based backward pde for $D \equiv D(F,t;K,T)$

$$D_t + \frac{1}{2}\sigma^2(Fe^{-(r_d-r_f)(T-t)}, t)F^2 D_{FF} = 0 \tag{8}$$

using the change of variables

$$F(S,t,T) = Se^{(r_d-r_f)(T-t)} \tag{9}$$

$$C(S,t;K,T) = D(Se^{(r_d-r_f)(T-t)}, t; K, T)\, e^{-r_d(T-t)} = D(F,t;K,T)e^{-r_d(T-t)}. \tag{10}$$

To see that (8) actually holds, compute the partial derivatives

$$\begin{aligned} C_t &= \frac{\partial}{\partial t}\left[D(Se^{(r_d-r_f)(T-t)}, t; K, T)e^{-r_d(T-t)}\right] = e^{-r_d(T-t)}\left(r_d D - (r_d - r_f)FD_F + D_t\right) \\ C_S &= e^{-r_d(T-t)}\frac{\partial}{\partial F}D(F,t;K,T)\frac{\partial}{\partial S}F(S,t,T) = e^{-r_f(T-t)}D_F \\ C_{SS} &= e^{r_d(T-t)}e^{-2r_f(T-t)}D_{FF} \end{aligned}$$

and plug the results into the standard backward pde (3) to obtain (8). Note that the first argument of the local volatility function in (8) is the *discounted* forward price (i.e. the current exchange rate), and not the forward price itself as indicated in the backward pde in DFW [4] (equation (4), p. 2087).[2]

[2] One could obtain the backward pde in the form given by DFW starting from the dynamics of the forward price under the risk-neutral measure, $dF_t = \sigma(F,t)F_t d\widehat{W}_t$. Then, however, the local volatility function of the stochastic process for the exchange rate, $S = e^{-(r_d-r_f)(T-t)}F_t$, would also be given by $\sigma(Se^{(r_d-r_f)(T-t)}, t)$. The key point thus is that the two local volatility functions for the spot and the forward price have to be identical.

So when implementing their approach on a finite difference grid, one has to make sure to specify the local volatility function properly.

3.4 The Forward Equation for Forward Calls on Forwards

We will now derive the forward pde for a forward call on a forward. DFW [4] (equation (5), p. 2088) state the following forward pde for the forward price of a standard European option:

$$D_T - \frac{1}{2}\sigma^2(K,T)K^2 D_{KK} = 0. \qquad (11)$$

As we will now show, this equation does not hold for the general cases of arbitrary combinations of r_d and r_f, or arbitrary state-dependent local volatility functions. It is only true when $r_d = r_f$ or when we consider the general class of homogeneous option pricing models. Thus, the transition from spot prices to forward prices does not automatically remove all terms depending on domestic and foreign interest rates.

We begin our derivation with the standard forward pde (6) for a plain vanilla call. From (7), (9), and (10) we get

$$\begin{aligned}
C_K &= e^{-r_d(T-t)} D_K \\
C_{KK} &= e^{-r_d(T-t)} D_{KK} \\
C_T &= \frac{\partial}{\partial T}\left[D(Se^{(r_d-r_f)(T-t)}, t; K, T)\, e^{-r_d(T-t)}\right] = e^{-r_d(T-t)}\left(D_T + D_F(r_d - r_f)F - r_d D\right).
\end{aligned}$$

From this we deduce the pde

$$\frac{1}{2}\sigma^2(K,T)K^2 D_{KK} = (r_d - r_f)KD_K - (r_d - r_f)D + D_T + (r_d - r_f)FD_F. \qquad (12)$$

The same result can be derived using the transition densities and performing the change of variables there (see Appendix B).

Comparing (12) to (11) shows that the two equations are equivalent if and only if

$$(r_d - r_f)[FD_F + KD_K - D] = 0 \qquad (13)$$

It is easy to see that one special case for which (13) holds is when the domestic interest rate r_d is equal to the foreign interest rate r_f. However, there is a not so obvious scenario under which (13) is true even in the case of a non-zero cost of carry, i.e. for $r_d \neq r_f$. Equation (13) is then equivalent to

$$FD_F + KD_K = D.$$

Using the relationships between the partial derivatives of D and C shown above, the spot-based version of this equation is given by

$$SC_S + KC_K = C. \qquad (14)$$

In the case of a homogeneous option pricing model, it is true for $\alpha \in \mathbb{R}$ that

$$C(\alpha S, t, \alpha K, T) = \alpha C(S, t, K, T).$$

Differentiating both sides with respect to α and evaluating the result at $\alpha = 1$ leads to equation (14).

If a given option pricing model is in fact homogeneous, the delta of the call is equal to $e^{-r_f(T-t)}\tilde{P}(S_T > K)$, where the second term represents the probability that the option will finish in the money under the equivalent martingale measure using the value of the underlying as the numeraire.

Choosing, for example, the local volatility function $\sigma(S,t) = \frac{\sigma}{S}$ yields a model in which the delta is no longer equal to the expression given above, so that this model is also not homogeneous.

4 Summary

This paper deals with pdes in the theory of contingent claim pricing. Starting from standard forward and backward equations we derive forward and backward pdes for the forward prices of claims written on the forward price of the underlying.

In the course of our analysis we find that there are two errors in the pdes proposed by DFW [4]. First, in their backward equation the first argument of the local volatility function is the forward price, whereas it should be the discounted forward price. Furthermore, their forward equation does not hold in the general case of arbitrary domestic and foreign interest rates or for non-homogeneous option pricing models.

References

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Appendix

A Calls on Forwards

Proposition 1 (Calls on Forwards) *The price of a call with strike K and maturity T_1 on a forward contract with maturity $T_2 > T_1$ is given by*

$$\bar{C}(F,t,T_2;K,T_1) = Fe^{-r_d(T_1-t)}\tilde{P}_t(F(T_1,T_2) > K) - Ke^{-r_d(T_1-t)}\hat{P}_t(F(T_1,T_2) > K)$$

where $\tilde{P}$ and $\hat{P}$ denote the martingale measures obtained by using the modified exchange rate $Se^{-r_f(T-t)}$ and the domestic money market account B as numeraires, respectively.

In the special case of constant or only time-dependent volatility this pricing equation is known as the Black formula (Black [2]).

Note that for $T_1 = T_2 = T$ this formula is nothing but the pricing formula for a call on an underlying with a continuous dividend yield equal to r_f:

$$\begin{aligned}\bar{C}(F,t;K,T) &= Fe^{-r_d(T-t)}\tilde{P}(F(T,T) > K) - Ke^{-r_d(T-t)}\hat{P}(F(T,T) > K) \\ &= Se^{-r_f(T-t)}\tilde{P}(S_T > K) - Ke^{-r_d(T-t)}\hat{P}(S_T > K) \\ &= C(S,t;K,T)\end{aligned}$$

Proof. Remember that we can value a European call on an underlying S with continuous dividend yield r_f by valuing a European call on an underlying with initial price $Se^{-r_f(T-t)}$ at time t and no dividend payments. The payoff of the call on the forward at time T_1 is

$$\begin{aligned}\max(F(T_1,T_2) - K; 0) &= F(T_1,T_2)I_{\{F(T_1,T_2)>K\}} - KI_{\{F(T_1,T_2)>K\}} \\ &:= X_{T_1} - Y_{T_1}\end{aligned}$$

where I_A denotes the indicator function for event A, $X_{T_1} = F(T_1,T_2)I_{\{F(T_1,T_2)>K\}}$, $Y_{T_1} = KI_{\{F(T_1,T_2)>K\}}$, and $F(t,T_2)$ is short for $F(S_t,t,T_2)$.

We now calculate the price of this option at time t, using martingale-based valuation techniques. Let X_t and Y_t denote the values at time t of the random cash flows X_{T_1} and Y_{T_1}, respectively. For the first term we use the martingale measure $\tilde{P}$, obtained by choosing the modified exchange rate $Se^{-r_f(T-t)}$ as the numeraire, for the second term the martingale measure $\hat{P}$ is obtained using the domestic money market account as the normalizing asset. Since

$$F(T_1, T_2) = S_{T_1} e^{(r_d - r_f)(T_2 - T_1)}$$

we obtain

$$\begin{aligned}
\frac{X_t}{S_t e^{-r_f(T_2-t)}} &= \mathbb{E}_t^{\tilde{P}} \left(\frac{X_{T_1}}{S_{T_1} e^{-r_f(T_2-T_1)}} \right) \\
&= e^{r_d(T_2-T_1)} \tilde{P}_t(F(T_1, T_2) > K) \\
\Rightarrow X_t &= F(t, T_2) e^{-r_d(T_1-t)} \tilde{P}_t(F(T_1, T_2) > K) \qquad (15)
\end{aligned}$$

and

$$\begin{aligned}
\frac{Y_t}{e^{r_d t}} &= \mathbb{E}_t^{\hat{P}} \left(\frac{Y_{T_1}}{e^{r_d T_1}} \right) \\
&= K e^{-r_d T_1} \hat{P}_t(F(T_1, T_2) > K) \\
\Rightarrow Y_t &= K e^{-r_d(T_1-t)} \hat{P}_t(F(T_1, T_2) > K) \qquad (16)
\end{aligned}$$

Subtracting (16) from (15) finally yields the proposition.

B Transformed Forward and Backward Transition Densities

Now we derive the backward and forward pdes for the modified densities. We will see that this leads to the same backward and forward pdes for the forward call, as derived in section 3.

Let $t, T \in [0, \tau]$, $t \leq T$. Define

$$\tilde{p}(\tilde{x}, t, y, T) := p(e^{-(r_d - r_f)(T-t)} \tilde{x}, t; y, T) = p(x, t; y, T)$$

as the modified transition density, where $x = e^{-(r_d - r_f)(T-t)} \tilde{x}$, and $p(x, t; y, T)$ satisfies both the standard backward pde (4) and the standard forward pde (5). It is obviously true that

$$\int \tilde{p}(\tilde{x}, t; y, T)\, dy = 1.$$

We want to transform the pdes for $p(x, t; y, T)$ into analogous pdes for $\tilde{p}(\tilde{x}, t, y, T)$. To do so, we need the following partial derivatives:

$$\frac{\partial \tilde{p}(\tilde{x}, t; y, T)}{\partial t} = \frac{\partial p(x, t; y, T)}{\partial t} + (r_d - r_f) x \frac{\partial p(x, t; y, T)}{\partial x}$$

$$\begin{aligned}
\frac{\partial \tilde{p}(\tilde{x},t;y,T)}{\partial \tilde{x}} &= e^{-(r_d-r_f)(T-t)}\frac{\partial p(x,t;y,T)}{\partial x} \\
\frac{\partial^2 \tilde{p}(\tilde{x},t;y,T)}{\partial \tilde{x}^2} &= e^{-2(r_d-r_f)(T-t)}\frac{\partial^2 p(x,t;y,T)}{\partial x^2}
\end{aligned}$$

From these and (4) we get

$$\begin{aligned}
\frac{\partial \tilde{p}(\tilde{x},t;y,T)}{\partial t} &= \frac{\partial p(x,t;y,T)}{\partial t} + (r_d-r_f)x\frac{\partial p(x,t;y,T)}{\partial x} \\
&= -\frac{1}{2}\sigma^2(x,t)x^2\frac{\partial^2 p(x,t;y,T)}{\partial x^2} \\
&= -\frac{1}{2}\sigma^2(e^{-(r_d-r_f)(T-t)}\tilde{x},t)\tilde{x}^2\frac{\partial^2 \tilde{p}(x,t;y,T)}{\partial \tilde{x}^2}
\end{aligned}$$

Integrating this expression leads to the backward pde (8) for the forward call D:

$$\frac{\partial D(\tilde{x},t;K,T)}{\partial t} = -\frac{1}{2}\sigma^2(e^{-(r_d-r_f)(T-t)}\tilde{x},t)\tilde{x}^2\frac{\partial^2 D(\tilde{x},t;K,T)}{\partial \tilde{x}^2}$$

where

$$D(\tilde{x},t;K,T) = \hat{\mathbb{E}}_t\left(\max(F_T-K;0)\right) = \int_K^\infty (y-K)\tilde{p}(\tilde{x},t;y,T)dy$$

For the forward pde we additionally need the following derivatives:

$$\begin{aligned}
\frac{\partial \tilde{p}(\tilde{x},t;y,T)}{\partial T} &= \frac{\partial p(x,t;y,T)}{\partial T} - (r_d-r_f)x\frac{\partial p(x,t;y,T)}{\partial x} \\
\frac{\partial \tilde{p}(\tilde{x},t;y,T)}{\partial y} &= \frac{\partial p(x,t;y,T)}{\partial y} \\
\frac{\partial^2 \tilde{p}(\tilde{x},t;y,T)}{\partial y^2} &= \frac{\partial^2 p(x,t;y,T)}{\partial y^2}
\end{aligned}$$

From these and (5) we get

$$\begin{aligned}
\frac{\partial \tilde{p}(\tilde{x},t;y,T)}{\partial T} &= \frac{\partial p(x,t;y,T)}{\partial T} - (r_d-r_f)x\frac{\partial p(x,t;y,T)}{\partial x} \\
&= \frac{1}{2}\frac{\partial^2}{\partial y^2}[\sigma^2(y,T)y^2p(x,t;y,T)] - (r_d-r_f)\frac{\partial}{\partial y}[yp(x,t;y,T)] \\
&\quad -(r_d-r_f)x\frac{\partial p(x,t;y,T)}{\partial x} \\
&= \frac{1}{2}\frac{\partial^2}{\partial y^2}[\sigma^2(y,T)y^2\tilde{p}(\tilde{x},t;y,T)] - (r_d-r_f)\frac{\partial}{\partial y}[y\tilde{p}(\tilde{x},t;y,T)] \\
&\quad -(r_d-r_f)\tilde{x}\frac{\partial \tilde{p}(\tilde{x},t;y,T)}{\partial \tilde{x}} \qquad (17)
\end{aligned}$$

To derive the forward pde for a forward call D we follow the lines of AB [1].

$$D(\tilde{x},t;K,T) = \hat{\mathbb{E}}_t\left(\max(F_T-K;0)\right) = \int_K^\infty (y-K)\tilde{p}(\tilde{x},t;y,T)dy$$

From this it follows that

$$\frac{\partial^2 D(\tilde{x},t;K,T)}{\partial K^2} = \tilde{p}(\tilde{x},t;K,T)$$

Plugging this into (17) we get

$$\begin{aligned}
\frac{\partial \tilde{p}(\tilde{x},t;y,T)}{\partial T} &= \frac{\partial^2}{\partial K^2}\frac{\partial D(\tilde{x},t;K,T)}{\partial T} \\
&= -(r_d - r_f)\frac{\partial}{\partial K}\left(K\frac{\partial^2 D(\tilde{x},t;K,T)}{\partial K^2}\right) \\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial K^2}\left(K^2\sigma^2(K,T)\frac{\partial^2 D(\tilde{x},t;K,T)}{\partial K^2}\right) \\
&\quad -(r_d - r_f)\tilde{x}\frac{\partial^2}{\partial K^2}\left(\frac{\partial D(\tilde{x},t;K,T)}{\partial \tilde{x}}\right)
\end{aligned}$$

Integrating this twice with respect to K leads to

$$\begin{aligned}
\frac{\partial D(\tilde{x},t;K,T)}{\partial T} &= \frac{1}{2}\sigma^2(K,T)K^2\frac{\partial^2 D(\tilde{x},t;K,T)}{\partial K^2} - (r_d - r_f)K\frac{\partial D(\tilde{x},t;K,T)}{\partial K} \\
&\quad + (r_d - r_f)D(\tilde{x},t,K,T) - \frac{\partial D(\tilde{x},t;K,T)}{\partial \tilde{x}}(r_d - r_f)\tilde{x}
\end{aligned}$$

This is equation (12) with $\tilde{x} = F$.