

# The Tits alternative for non-spherical triangles of groups

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## ABSTRACT

Triangles of groups have been introduced by Gersten and Stallings. They are, roughly speaking, a generalization of the amalgamated free product of two groups and occur in the framework of Corson diagrams. First, we prove an intersection theorem for Corson diagrams. Then, we focus on triangles of groups. It has been shown by Howie and Kopteva that the colimit of a hyperbolic triangle of groups contains a non-abelian free subgroup. We give two natural conditions, each of which ensures that the colimit of a non-spherical triangle of groups either contains a non-abelian free subgroup or is virtually solvable.

## 1. Introduction

Given a commutative diagram of groups and injective homomorphisms, we may construct its colimit (in the category of groups). The colimit, or, more precisely, the colimit group, can be obtained by taking the free product of the groups and identifying the factors according to the homomorphisms. A good example is the amalgamated free product  $X *_A Y$ , which is the colimit group of the diagram  $X \leftarrow A \rightarrow Y$ .

We are interested in Corson diagrams. A Corson diagram is based on a set  $I$ . For every subset  $J \subseteq I$  with  $|J| \leq 2$ , there is a group  $G_J$  and for every two subsets  $J_1 \subset J_2 \subseteq I$  with  $|J_2| \leq 2$  there is a homomorphism  $\varphi_{J_1 J_2} : G_{J_1} \rightarrow G_{J_2}$ , see Figure 1. Note that both Artin groups and Coxeter groups have a natural interpretation as colimit groups of Corson diagrams. A triangle of groups is nothing but a Corson diagram based on a set  $I$  with  $|I| = 3$ .

Gersten and Stallings introduced the notion of curvature and proved that for non-spherical triangles of groups the natural homomorphisms  $\nu_J$  from the groups  $G_J$  to the colimit group  $\mathfrak{G}$  are injective, see [25]. A similar result holds for non-spherical Corson diagrams, see [13]. While these two results can be proved by nice arguments based on Euler’s formula for planar graphs, spherical Corson diagrams are much harder to investigate, see, for example, [2, 12].

In Section 2, we introduce the basic notions for this paper. Then, in Section 3, we give an example of a spherical triangle of groups showing that, even though the natural homomorphisms  $\nu_J : G_J \rightarrow \mathfrak{G}$  are injective, the intersections of their images may be larger than the amalgamated subgroups. But this can only happen in the spherical realm. For non-spherical triangles of groups and, more generally, non-spherical Corson diagrams, there are no large intersections, see Theorem 3.8.

At this point, it seems worth mentioning that the absence of large intersections shall not be confused with the developability of complexes of groups, which is implied by the injectivity of the natural homomorphisms  $\nu_J : G_J \rightarrow \mathfrak{G}$ , see [7, Corollary III.C.2.15].

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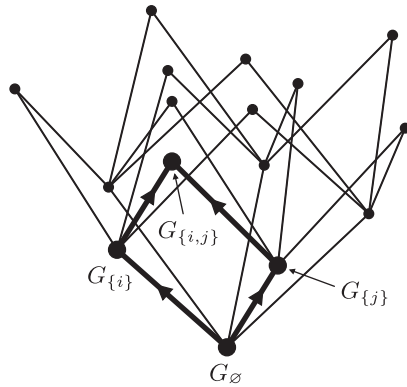


FIGURE 1. A Corson diagram based on a set  $I$  with  $|I| = 5$ . For simplicity, the homomorphisms  $\varphi_{\emptyset J} : G_{\emptyset} \rightarrow G_J$  with  $J \subseteq I$  and  $|J| = 2$  have been omitted in this figure. Note that, if  $|I| = n$ , then the graphical representation of the Corson diagram has exactly  $\frac{1}{2} \cdot (n^2 + n + 2)$  vertices.

Howie and Kopteva showed that, under mild assumptions, the colimit group of a hyperbolic triangle of groups has a non-abelian free subgroup, see [16]. In Section 4, we focus on the Euclidean case and discuss the following version of the Tits alternative: A class  $\mathcal{C}$  of groups satisfies the Tits alternative if each  $G \in \mathcal{C}$  either has a non-abelian free subgroup or is virtually solvable. The Tits alternative is named after Jacques Tits, who proved in 1972 that the class of finitely generated linear groups has this property, see [26, Corollary 1]. Since then, the Tits alternative has been proved for many other classes of groups. For a list of results and open problems, we refer the reader to [24].

As indicated above, we are interested in Euclidean triangles of groups. In the case that none of the Gersten–Stallings angles is 0, we may follow Bridson’s construction of a simplicial complex  $\mathcal{X}$ , see [6], and use billiards on a suitable triangle in the Euclidean plane to obtain geodesics in the geometric realization  $|\mathcal{X}|$ . These geodesics allow us to prove that, as soon as the simplicial complex  $\mathcal{X}$  branches, the colimit group has a non-abelian free subgroup, see Theorems 4.18 and 4.19.

The remaining cases can be analysed with quotients and amalgamated free products. In the end, we generalize the result by Howie and Kopteva mentioned above and prove that the Tits alternative holds for the class of colimit groups of non-spherical triangles of groups with the property that none of the Gersten–Stallings angles is 0 and the group  $G_{\emptyset}$  either has a non-abelian free subgroup or is virtually solvable, see Theorem 4.24, or with the property that every group  $G_J$  with  $J \subseteq I$  and  $|J| \leq 2$  either has a non-abelian free subgroup or is virtually solvable, see Theorem 4.25.

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## 2. Preliminaries

### 2.1. Corson diagrams and their colimits

Let  $I$  be an arbitrary set. Assume given for every subset  $J \subseteq I$  with  $|J| \leq 2$  a group  $G_J$  and for every two subsets  $J_1 \subset J_2 \subseteq I$  with  $|J_2| \leq 2$  an injective homomorphism  $\varphi_{J_1 J_2} : G_{J_1} \rightarrow G_{J_2}$ .

Moreover, assume the resulting diagram to be *commutative*, that is, for every sequence of subsets  $\emptyset = J_1 \subset J_2 \subset J_3 \subseteq I$  with  $|J_3| = 2$  the equation  $\varphi_{J_1 J_3} = \varphi_{J_2 J_3} \circ \varphi_{J_1 J_2}$  holds.

Since these diagrams have been introduced by Corson in [13], we refer to them as *Corson diagrams*. In the case  $|I| = 3$ , Corson diagrams are known as *triangles of groups*. Whenever we consider a triangle of groups, we may assume without loss of generality that  $I = \{1, 2, 3\}$ .

Given a Corson diagram, we will mostly be interested in its *colimit group*. The colimit group can be obtained by taking the free product of the groups  $G_J$  and identifying the factors according to the homomorphisms. Let us make this construction a little more precise. Think of each  $G_J$  as a set, and let  $R_J$  be the set of all words over the group elements and their formal inverses that represent the identity. Then, the colimit group  $\mathfrak{G}$  is given by the following presentation:

$$\mathfrak{G} = \left\langle \bigsqcup_{\substack{J \subseteq I \\ |J| \leq 2}} G_J : \bigsqcup_{\substack{J \subseteq I \\ |J| \leq 2}} R_J, \bigsqcup_{\substack{J_1 \subset J_2 \subseteq I \\ |J_2| \leq 2}} \{g = \varphi_{J_1 J_2}(g) : g \in G_{J_1}\} \right\rangle. \quad (*)$$

This presentation, though not very economic, turns out to be suitable for our purposes. For every subset  $J \subseteq I$  with  $|J| \leq 2$ , we may consider the natural homomorphism  $\nu_J : G_J \rightarrow \mathfrak{G}$  given by  $g \mapsto g$ . The colimit group, equipped with these homomorphisms, is called the *colimit*. For further reading about it, we refer the reader to [27, 1.1] and [1, Chapter III].

### 2.2. Curvature of Corson diagrams

The homomorphisms  $\nu_J : G_J \rightarrow \mathfrak{G}$  do not need to be injective. An example of a triangle of groups in which they are not has been given by Gersten and Stallings in [25, 1.4]. On the other hand, it turned out that for non-spherical triangles of groups and, more generally, for non-spherical Corson diagrams they are. Let us therefore introduce the notion of curvature. For every two distinct  $i, j \in I$ , the homomorphisms  $\varphi_{\{i\}\{i,j\}}$  and  $\varphi_{\{j\}\{i,j\}}$  uniquely determine a homomorphism  $\alpha : G_{\{i\}} *_{G_\emptyset} G_{\{j\}} \rightarrow G_{\{i,j\}}$ . If  $\alpha$  is not injective, then let  $\hat{m}$  denote the minimal length of a non-trivial element in its kernel (in the usual length function on the amalgamated free product). Recall that the homomorphisms  $\varphi_{\{i\}\{i,j\}}$  and  $\varphi_{\{j\}\{i,j\}}$  are injective, whence the minimal length  $\hat{m}$  must be even. The *Gersten–Stallings angle*  $\triangleleft_{\{i,j\}}$  is now defined by:

$$\triangleleft_{\{i,j\}} = \begin{cases} \frac{2\pi}{\hat{m}} & \text{if } \alpha \text{ is not injective,} \\ 0 & \text{if } \alpha \text{ is injective.} \end{cases}$$

Three pairwise distinct elements  $i, j, k \in I$  are called a *spherical triple* if  $\triangleleft_{\{i,j\}} + \triangleleft_{\{i,k\}} + \triangleleft_{\{j,k\}}$  is strictly larger than  $\pi$ . The Corson diagram is called *spherical* if it has a spherical triple, and *non-spherical* otherwise.

Consider a non-spherical triangle of groups. Since  $|I| = 3$ , there is only one set of three pairwise distinct elements. Depending on whether  $\triangleleft_{\{1,2\}} + \triangleleft_{\{1,3\}} + \triangleleft_{\{2,3\}}$  is strictly smaller than  $\pi$  or equal to  $\pi$ , the triangle of groups is called *hyperbolic* or *Euclidean*. This distinction is of relevance in Section 4.

### 2.3. Embedding theorems

We are now able to state the theorem about non-spherical triangles of groups that has been mentioned above.

**THEOREM 2.1** (Gersten–Stallings). *For every non-spherical triangle of groups and every subset  $J \subseteq I$  with  $|J| \leq 2$ , the natural homomorphism  $\nu_J : G_J \rightarrow \mathfrak{G}$  is injective.*

This theorem has been proved in [25]. Later, it has been generalized to non-spherical Corson diagrams in [13]. Even more has been shown in [13]. Not only the groups  $G_J$  but also the colimit groups of subdiagrams naturally embed into  $\mathfrak{G}$ . Let us clarify. Given a Corson diagram and a subset  $K \subseteq I$ , we may restrict our focus to the subdiagram spanned by the groups  $G_J$  with  $J \subseteq K$  and  $|J| \leq 2$ , see the bold vertices and arrows in Figure 1 for an example. The colimit group of such a subdiagram can be obtained by modifying (\*) as follows:

$$\mathfrak{G}_K = \left\langle \bigsqcup_{\substack{J \subseteq K \\ |J| \leq 2}} G_J : \bigsqcup_{\substack{J \subseteq K \\ |J| \leq 2}} R_J, \bigsqcup_{\substack{J_1 \subseteq J_2 \subseteq K \\ |J_2| \leq 2}} \{g = \varphi_{J_1 J_2}(g) : g \in G_{J_1}\} \right\rangle.$$

Analogously to  $\nu_J : G_J \rightarrow \mathfrak{G}$  introduced in Subsection 2.1, we may now consider the natural homomorphisms  $\tilde{\nu}_K : \mathfrak{G}_K \rightarrow \mathfrak{G}$  given by  $g \mapsto g$ .

**THEOREM 2.2 (Corson).** *For every non-spherical Corson diagram and every subset  $K \subseteq I$ , the natural homomorphism  $\tilde{\nu}_K : \mathfrak{G}_K \rightarrow \mathfrak{G}$  is injective.*

**REMARK 2.3.** It is easy to verify that for every subset  $K \subseteq I$  with  $|K| \leq 2$  there is an isomorphism  $\mu_K : G_K \rightarrow \mathfrak{G}_K$  given by  $g \mapsto g$ . Hence, the injectivity of  $\tilde{\nu}_K : \mathfrak{G}_K \rightarrow \mathfrak{G}$  implies the injectivity of  $\nu_K = \tilde{\nu}_K \circ \mu_K : G_K \rightarrow \mathfrak{G}$  and, in particular, Theorem 2.2 implies Theorem 2.1.

**REMARK 2.4.** In order to keep the notation simple, we make the following convention: Whenever we know that the homomorphisms  $\tilde{\nu}_K : \mathfrak{G}_K \rightarrow \mathfrak{G}$  are injective, for example, in case of a non-spherical Corson diagram, we do not need to mention them any more and may tacitly interpret  $\mathfrak{G}_K$  as a subgroup of  $\mathfrak{G}$ . In this case, the symbol  $\mathfrak{G}_K$  refers to the subgroup of  $\mathfrak{G}$  that is generated by the elements of the groups  $G_J$  with  $J \subseteq K$  and  $|J| \leq 2$ . Now, we can easily observe that  $K_1 \subseteq K_2$  implies  $\mathfrak{G}_{K_1} \subseteq \mathfrak{G}_{K_2}$ .

#### 2.4. Standing assumption on the Gersten–Stallings angles

We will have to make one more assumption, which has already been indicated by Gersten and Stallings in [25, p. 493, ll. 4–6] and Corson in [13, p. 567, l. 15], even though Theorems 2.1 and 2.2 hold without it, also cf. [18, p. 58, ll. 12–16; 23, p. 210, ll. 18–21].

**STANDING ASSUMPTION.** We shall always assume, without stating explicitly, that none of the Gersten–Stallings angles is equal to  $\pi$ , or, equivalently, that for every two distinct  $i, j \in I$  the equation  $\varphi_{\{i\}\{i,j\}}(G_{\{i\}}) \cap \varphi_{\{j\}\{i,j\}}(G_{\{j\}}) = \varphi_{\emptyset\{i,j\}}(G_{\emptyset})$  holds.

### 3. Intersection theorem

Assume given a Corson diagram with the property that the homomorphisms  $\tilde{\nu}_K : \mathfrak{G}_K \rightarrow \mathfrak{G}$  are injective. One question we are interested in is whether two subgroups  $\mathfrak{G}_{K_1}$  and  $\mathfrak{G}_{K_2}$  intersect only along the obvious subgroup  $\mathfrak{G}_{K_1 \cap K_2}$  or along some larger subgroup of  $\mathfrak{G}$ . In Subsection 3.1, we give an example of a spherical Corson diagram in which the homomorphisms  $\tilde{\nu}_K : \mathfrak{G}_K \rightarrow \mathfrak{G}$  are injective but there are  $K_1, K_2 \subseteq I$  such that  $\mathfrak{G}_{K_1} \cap \mathfrak{G}_{K_2} \neq \mathfrak{G}_{K_1 \cap K_2}$ . Then, we recall the notion of *disc pictures* and use it to prove an *intersection theorem* showing that this can only happen in the spherical realm.

#### 3.1. Example

Let us consider the following Corson diagram:  $I = \{1, 2, 3\}$ ,  $G_{\emptyset} = \{1\}$ ,  $G_{\{1\}} = \langle a : - \rangle$ ,  $G_{\{2\}} = \langle b : - \rangle$ ,  $G_{\{3\}} = \langle c : - \rangle$ ,  $G_{\{1,2\}} = \langle a, b : b^{-1}ab = a^2 \rangle$ ,  $G_{\{1,3\}} = \langle a, c : c^{-1}ac = a^2 \rangle$ , and

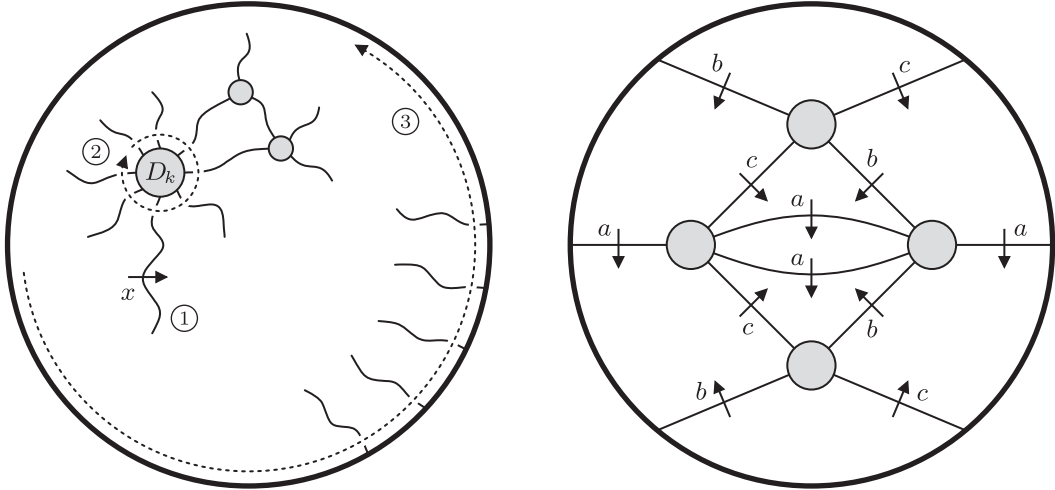


FIGURE 2. Some elements of a disc picture (left) and an example showing that  $bab^{-1} = cac^{-1}$  holds in  $G = \langle a, b, c : b^{-1}ab = a^2, c^{-1}ac = a^2, bc = cb \rangle$  (right).

$G_{\{2,3\}} = \langle b, c : bc = cb \rangle$ . Here, the homomorphisms  $\varphi_{J_1 J_2} : G_{J_1} \rightarrow G_{J_2}$  are implicitly given by  $a \mapsto a$ ,  $b \mapsto b$ , and  $c \mapsto c$ . Since  $G_\emptyset$  is trivial, the resulting diagram is commutative. Britton's Lemma [9, Principal Lemma] shows that the homomorphisms  $\varphi_{J_1 J_2} : G_{J_1} \rightarrow G_{J_2}$  are injective and the Gersten–Stallings angles amount to  $\pi/2$  each. So, it is a spherical Corson diagram in the sense of Subsection 2.2.

PROPOSITION 3.1. *The natural homomorphisms  $\tilde{\nu}_K : \mathfrak{G}_K \rightarrow \mathfrak{G}$  are injective.*

*Proof.* Since the homomorphism  $\tilde{\nu}_{\{1,2,3\}} : \mathfrak{G}_{\{1,2,3\}} \rightarrow \mathfrak{G}$  is obviously injective, it suffices to verify the injectivity of the homomorphisms  $\tilde{\nu}_K : \mathfrak{G}_K \rightarrow \mathfrak{G}$  with  $K \subseteq \{1, 2, 3\}$  and  $|K| \leq 2$ . But then, we already know from Remark 2.3 that there are isomorphisms  $\mu_K : G_K \rightarrow \mathfrak{G}_K$  with  $\nu_K = \tilde{\nu}_K \circ \mu_K$ . So, it suffices to verify the injectivity of the homomorphisms  $\nu_K : G_K \rightarrow \mathfrak{G}$ .

Recall the presentation (\*) of the colimit group  $\mathfrak{G}$  and note that, in our situation, it can be simplified by deleting superficial generators and relators so that we finally obtain the presentation  $\mathfrak{G} = \langle a, b, c : b^{-1}ab = a^2, c^{-1}ac = a^2, bc = cb \rangle$ . If  $K = \{1, 2\}$  or  $K = \{1, 3\}$ , then let  $\mathfrak{N} := \langle\langle bc^{-1} \rangle\rangle \trianglelefteq \mathfrak{G}$  and let  $\pi : \mathfrak{G} \rightarrow \mathfrak{G}/\mathfrak{N}$  be the canonical projection. It is easy to see that both  $\pi \circ \nu_{\{1,2\}}$  and  $\pi \circ \nu_{\{1,3\}}$  are isomorphisms and, hence, both  $\nu_{\{1,2\}}$  and  $\nu_{\{1,3\}}$  are injective. If  $K = \{2, 3\}$ , then let  $\mathfrak{N} := \langle\langle a \rangle\rangle \trianglelefteq \mathfrak{G}$  instead and proceed analogously. Finally, if  $|K| \leq 1$ , then  $K$  is contained in some  $\tilde{K} \subseteq \{1, 2, 3\}$  with  $|\tilde{K}| = 2$ . By construction of the colimit, we have  $\nu_K = \nu_{\tilde{K}} \circ \varphi_{K\tilde{K}}$ . Since both  $\nu_{\tilde{K}}$  and  $\varphi_{K\tilde{K}}$  are injective, their composition is injective, too.  $\square$

PROPOSITION 3.2. *The equation  $\mathfrak{G}_{\{1,2\}} \cap \mathfrak{G}_{\{1,3\}} = \mathfrak{G}_{\{1\}}$  does not hold.*

*Proof.* Use the isomorphism  $\mu_{\{1,2\}} : G_{\{1,2\}} \rightarrow \mathfrak{G}_{\{1,2\}}$  and Britton's Lemma to show that the word  $bab^{-1}$  represents an element in  $\mathfrak{G}_{\{1,2\}}$  that is not in  $\mathfrak{G}_{\{1\}}$ . On the other hand, in the colimit group  $\mathfrak{G}$ , the equations  $bab^{-1} = bca^2c^{-1}b^{-1} = cba^2b^{-1}c^{-1} = cac^{-1}$  hold. So, the words  $bab^{-1}$  and  $cac^{-1}$  represent the same element of the colimit group  $\mathfrak{G}$ , which is in  $\mathfrak{G}_{\{1,2\}} \cap \mathfrak{G}_{\{1,3\}}$  but not in  $\mathfrak{G}_{\{1\}}$ . This calculation is also illustrated in Figure 2, in terms of disc pictures.  $\square$

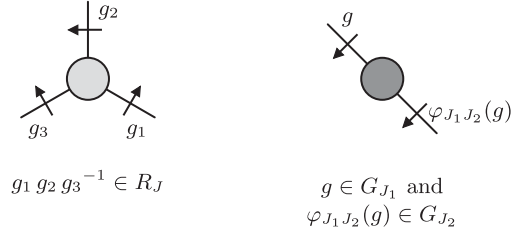


FIGURE 3. A local vertex (bright) and a joining vertex (dark).

### 3.2. Preliminaries about disc pictures

The proof of the intersection theorem involves *disc pictures*. Let us therefore follow Corson's preliminary section, see [13], and recall some basic notions.

Consider a group  $G$  and a presentation  $G = \langle X : R \rangle$ . A disc picture  $\mathcal{P}$  over this presentation consists of the disjoint union of closed discs  $D_1, D_2, \dots, D_n$  in the interior of a closed disc  $D$  and a compact 1-manifold  $M$  properly embedded into  $D \setminus \text{int}(D_1 \cup D_2 \cup \dots \cup D_n)$ . The closed discs  $D_1, D_2, \dots, D_n$  are called *vertices*, the components of  $M$  are called *arcs*. Moreover, the components of  $\text{int}(D) \setminus (D_1 \cup D_2 \cup \dots \cup D_n \cup M)$  are called *regions*. Every arc has a transversal orientation and is labelled by a generator, see ① in Figure 2. Every vertex  $D_k$  has the property that one can read off a relator *along* its boundary  $\partial D_k$ , that is, by starting at some point on  $\partial D_k \setminus M$  and going once around  $\partial D_k$  in some orientation, see ② in Figure 2. Every word that can be read off along the outer boundary  $\partial D$  is called a *boundary word* of the disc picture, see ③ in Figure 2. It is well known, and easy to verify, that a word over the generators and their formal inverses represents the identity of the group  $G$  if and only if it is a boundary word of some disc picture over the presentation  $G = \langle X : R \rangle$ . Disc pictures are, roughly speaking, duals of van Kampen diagrams. For further reading about them, we refer the reader to [4]. In addition to the above, the following notions will be of relevance for us.

**DEFINITION 3.3** ('subpicture'). Consider a closed disc  $D_{\mathcal{Q}}$  in  $D$ . If the parts of the disc picture  $\mathcal{P}$  that are contained in  $D_{\mathcal{Q}}$  assemble to a disc picture  $\mathcal{Q}$ , then we call  $\mathcal{Q}$  a *subpicture* of  $\mathcal{P}$ . Note that every boundary word of a subpicture does necessarily represent the identity of the group  $G$ . A simple kind of subpicture is a *spider*. It consists of exactly one vertex  $D_k$  and some arcs, each of which connects  $D_k$  to the outer boundary  $\partial D_{\mathcal{Q}}$  of the subpicture  $\mathcal{Q}$ .

Since we are interested in Corson diagrams and their colimit groups, we will focus on disc pictures over  $(*)$ . Here, it makes sense to distinguish between local and joining vertices.

**DEFINITION 3.4** ('local and joining vertices'). A vertex  $D_k$  is called *local* if one can read off a relator of the form  $g_1^{\varepsilon_1} g_2^{\varepsilon_2} \dots g_m^{\varepsilon_m} \in R_J$  along its boundary. Otherwise, it is called *joining*, in which case one can read off a relator of the form  $g = \varphi_{J_1 J_2}(g)$  with  $g \in G_{J_1}$  and  $\varphi_{J_1 J_2}(g) \in G_{J_2}$ , see Figure 3.

**DEFINITION 3.5** ('bridge'). Let  $\mathcal{B}$  be the union of the compact 1-manifold  $M$  and the joining vertices. The components of  $\mathcal{B}$  are called *bridges*. Every simply connected bridge has two distinct endpoints, each of which lies either on the boundary of some local vertex or on the outer boundary. Two local vertices, say  $D_k$  and  $D_l$ , are called *neighbours* if there is a bridge that connects  $D_k$  and  $D_l$ , that is, a bridge with one endpoint on the boundary  $\partial D_k$  and the other endpoint on the boundary  $\partial D_l$ .

DEFINITION 3.6 ('inner and outer'). A bridge is called *inner* if it connects two local vertices. Similarly, a region is called *inner* if its closure does not meet the outer boundary  $\partial D$ . A bridge or a region that is not inner is called *outer*.

Let us consider a non-spherical Corson diagram. As stated in Remark 2.4, we may interpret  $\mathfrak{G}_K$  as a subgroup of  $\mathfrak{G}$ . The following lemma uses this interpretation to describe the labels of the arcs of a bridge.

LEMMA 3.7. Consider a bridge with  $m$  arcs that are labelled by generators  $b_1 \in G_{J_1}$ ,  $b_2 \in G_{J_2}, \dots, b_m \in G_{J_m}$ . Then, all these generators represent the same element of the colimit group  $\mathfrak{G}$ . This element, say  $b \in \mathfrak{G}$ , is called the value of the bridge. It is contained in the subgroup  $\mathfrak{G}_{J_1 \cap J_2 \cap \dots \cap J_m}$ .

*Proof.* The first assertion is immediate. So, we only need to verify that the value of the bridge is actually contained in  $\mathfrak{G}_{J_1 \cap J_2 \cap \dots \cap J_m}$ . Let us make two observations. First, if one of the sets  $J_1, J_2, \dots, J_m$  is empty, say  $J_k = \emptyset$ , then the value of the bridge can be represented by  $b_k \in G_\emptyset$ . So,  $b \in \mathfrak{G}_\emptyset$ . This, of course, can be written as  $b \in \mathfrak{G}_{J_1 \cap J_2 \cap \dots \cap J_m}$ , whence we are done. Therefore, we may assume without loss of generality that none of the sets  $J_1, J_2, \dots, J_m$  is empty, in which case they must alternately have cardinality 1 and 2. Second, if  $m = 1$ , then there is nothing to show. So, we may assume without loss of generality that  $m \geq 2$ . But then, there must be at least one set of cardinality 1 among  $J_1, J_2, \dots, J_m$ .

(i) If all the sets of cardinality 1 are equal, say equal to  $\{i\}$ , then  $b \in \mathfrak{G}_{\{i\}}$ . But, in this case, all sets of cardinality 2 must contain  $i$ , which implies that  $J_1 \cap J_2 \cap \dots \cap J_m = \{i\}$ . Therefore,  $b \in \mathfrak{G}_{\{i\}}$  can be written as  $b \in \mathfrak{G}_{J_1 \cap J_2 \cap \dots \cap J_m}$ .

(ii) If there are two distinct sets of cardinality 1 among  $J_1, J_2, \dots, J_m$ , say  $\{i\}$  and  $\{j\}$ , then  $b \in \mathfrak{G}_{\{i\}} \cap \mathfrak{G}_{\{j\}}$ . We claim that  $\mathfrak{G}_{\{i\}} \cap \mathfrak{G}_{\{j\}} = \mathfrak{G}_\emptyset$ . Once this has been shown, we know that  $b \in \mathfrak{G}_\emptyset$ . Since  $J_1 \cap J_2 \cap \dots \cap J_m = \emptyset$ , this can, again, be written as  $b \in \mathfrak{G}_{J_1 \cap J_2 \cap \dots \cap J_m}$ . It remains to show that  $\mathfrak{G}_{\{i\}} \cap \mathfrak{G}_{\{j\}} = \mathfrak{G}_\emptyset$ . By our standing assumption, the equation  $\varphi_{\{i\}\{i,j\}}(G_{\{i\}}) \cap \varphi_{\{j\}\{i,j\}}(G_{\{j\}}) = \varphi_{\emptyset\{i,j\}}(G_\emptyset)$  holds. In order to transport this equation to the colimit group  $\mathfrak{G}$ , we apply the injective homomorphism  $\nu_{\{i,j\}} : G_{\{i,j\}} \rightarrow \mathfrak{G}$ . This yields:

$$\nu_{\{i,j\}}(\varphi_{\{i\}\{i,j\}}(G_{\{i\}})) \cap \nu_{\{i,j\}}(\varphi_{\{j\}\{i,j\}}(G_{\{j\}})) = \nu_{\{i,j\}}(\varphi_{\emptyset\{i,j\}}(G_\emptyset)).$$

Using the equations  $\nu_{\{i,j\}}(\varphi_{K\{i,j\}}(G_K)) = \nu_K(G_K) = \tilde{\nu}_K(\mu_K(G_K)) = \tilde{\nu}_K(\mathfrak{G}_K)$  for the subsets  $K \subset \{i, j\}$ , we finally obtain  $\tilde{\nu}_{\{i\}}(\mathfrak{G}_{\{i\}}) \cap \tilde{\nu}_{\{j\}}(\mathfrak{G}_{\{j\}}) = \tilde{\nu}_\emptyset(\mathfrak{G}_\emptyset)$ , which reads as  $\mathfrak{G}_{\{i\}} \cap \mathfrak{G}_{\{j\}} = \mathfrak{G}_\emptyset$  in the shorthand notation of Remark 2.4.  $\square$

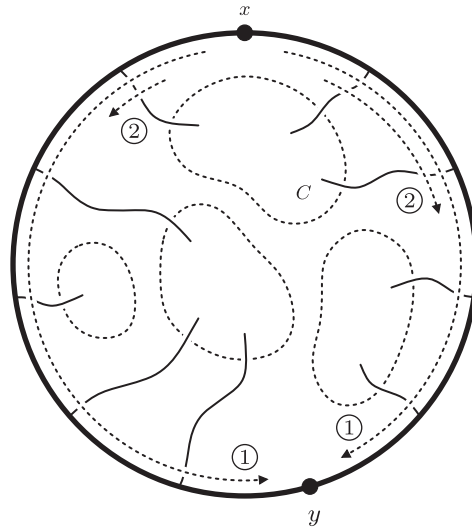
### 3.3. Statement and proof of the intersection theorem

We are now ready to discuss the intersection theorem. The proof is based on ideas and techniques that go back to Gersten and Stallings in [25] and Corson in [13].

THEOREM 3.8. For every non-spherical Corson diagram and every two subsets  $K_1, K_2 \subseteq I$ , the equation  $\mathfrak{G}_{K_1} \cap \mathfrak{G}_{K_2} = \mathfrak{G}_{K_1 \cap K_2}$  holds.

*Proof.* The inclusion ' $\supseteq$ ' is a consequence of Remark 2.4. So, we only need to verify the inclusion ' $\subseteq$ '. Suppose that there were a non-spherical Corson diagram and two subsets  $K_1, K_2 \subseteq I$  with  $\mathfrak{G}_{K_1} \cap \mathfrak{G}_{K_2} \not\subseteq \mathfrak{G}_{K_1 \cap K_2}$ . Then, we can find an element  $g \in \mathfrak{G}$  with  $g \in \mathfrak{G}_{K_1} \cap \mathfrak{G}_{K_2}$  but  $g \notin \mathfrak{G}_{K_1 \cap K_2}$ . Being contained in  $\mathfrak{G}_{K_1}$ , it can be represented by a word  $w_1$  over the generators from the groups  $G_J$  with  $J \subseteq K_1$  and  $|J| \leq 2$ , and their formal inverses. On the other hand, being contained in  $\mathfrak{G}_{K_2}$ , it can also be represented by a word  $w_2$  over the generators from the groups  $G_J$  with  $J \subseteq K_2$  and  $|J| \leq 2$ , and their formal inverses.



FIGURE 4. The disc picture  $\mathcal{P}$  is connected.

Since  $w_1$  and  $w_2$  represent the same element of the colimit group  $\mathfrak{G}$ , there is a disc picture  $\mathcal{P}$  over  $(*)$  with boundary word  $w_1 w_2^{-1}$ . By construction,  $g \notin \mathfrak{G}_{K_1 \cap K_2}$ . So, it cannot be the identity of the colimit group  $\mathfrak{G}$ . Therefore, the words  $w_1$  and  $w_2$  cannot be empty and there are at least two arcs, or one arc twice, incident with the outer boundary  $\partial D$ .

We may assume without loss of generality that the element  $g$ , the words  $w_1$  and  $w_2$ , and the disc picture  $\mathcal{P}$  are chosen in such a way that the *complexity* of the disc picture is minimal, that is, the number of local vertices is minimal and, among all disc pictures with this minimal number of local vertices, the number of bridges is minimal. This assumption has many consequences on the structure of the disc picture.

(1) *The disc picture  $\mathcal{P}$  is connected. In particular, since there are arcs incident with the outer boundary  $\partial D$ , every local vertex is incident with at least one arc. Moreover, all bridges and regions are simply connected.* We claim that if  $\mathcal{P}$  was not connected, then we could remove at least one component and, hence, obtain a disc picture with strictly fewer local vertices or with the same number of local vertices but strictly fewer bridges. In other words, we could obtain a disc picture of lower complexity.

First, note that there are two distinct points  $x, y \in \partial D \setminus M$  such that one can read off the words  $w_1$  and  $w_2$  when going from  $x$  to  $y$  along the respective side of  $\partial D$ , see ① in Figure 4. If there is a component of  $\mathcal{P}$  that is incident with at most one side of  $\partial D$ , then we can remove it. In this case, the boundary words of the disc picture may change. But the new disc picture gives rise to new words  $\tilde{w}_1$  and  $\tilde{w}_2$ . Since the removed component has been incident with at most one side of  $\partial D$ , at least one of the words  $\tilde{w}_i$  remains equal to  $w_i$ . So, both  $\tilde{w}_1$  and  $\tilde{w}_2$ , which represent the same element of the colimit group  $\mathfrak{G}$ , still represent  $g$ . *In the following steps, as here, we may obtain new words  $\tilde{w}_1$  and  $\tilde{w}_2$ , and sometimes even a new element  $\tilde{g} \in \mathfrak{G}$ . But, in each step, it is easy to see that these data could have been chosen right at the beginning.*

By the above, we may assume without loss of generality that every component of  $\mathcal{P}$  is incident with both sides of  $\partial D$ . Suppose that there is more than one such component and let  $C$  be the first one traversed when going from  $x$  to  $y$  along  $\partial D$ . For a moment, let us focus on  $C$  and ignore all the other components. Now, one can read off new words  $\tilde{w}_1$  and  $\tilde{w}_2$  along the respective sides of  $\partial D$  that represent a new element  $\tilde{g} \in \mathfrak{G}$ , see ② in Figure 4. By construction,  $\tilde{g} \in \mathfrak{G}_{K_1} \cap \mathfrak{G}_{K_2}$ . If  $\tilde{g} \notin \mathfrak{G}_{K_1 \cap K_2}$ , then the component  $C$  is already a suitable disc picture and



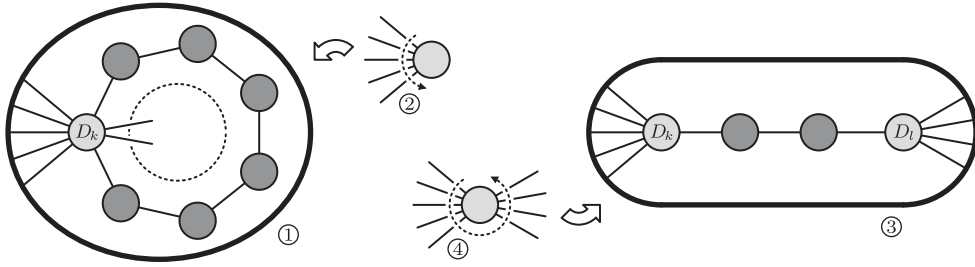


FIGURE 5. Replace the subpictures  $\mathcal{Q}$  by spiders.

we can remove the other components completely. On the other hand, if  $\tilde{g} \in \mathfrak{G}_{K_1 \cap K_2}$ , then we can keep the other components and remove  $C$ . The words that one can now read off along the respective sides of  $\partial D$  represent the element  $\tilde{g}^{-1}g \in \mathfrak{G}$ , which is the product of an element in  $\mathfrak{G}_{K_1 \cap K_2}$  and an element not in  $\mathfrak{G}_{K_1 \cap K_2}$ . Therefore,  $\tilde{g}^{-1}g \notin \mathfrak{G}_{K_1 \cap K_2}$  and, again, we end up with a suitable disc picture.

DEFINITION 3.9 (‘type of a local vertex’). Since every local vertex is incident with at least one arc, we can associate a type to every local vertex. More precisely, for every local vertex  $D_k$  there is a unique subset  $J \subseteq I$  with  $|J| \leq 2$  such that all arcs incident with  $D_k$  are labelled by generators from  $G_J$ . (Recall that the set of generators is the disjoint union of all  $G_J$  with  $J \subseteq I$  and  $|J| \leq 2$ .) In this case, we say that  $D_k$  is of type  $J$ .

(2) *The disc picture  $\mathcal{P}$  has at least one local vertex.* Since  $w_1$  and  $w_2$  cannot be empty, there is at least one arc incident with each side of  $\partial D$ . Therefore, if  $\mathcal{P}$  had no local vertex at all, then it would have to be a single bridge  $B$  connecting the two sides of  $\partial D$ . Depending on the transversal orientation of its arcs, the value of  $B$  is either  $g$  or  $g^{-1}$ . The extremal arcs of  $B$  are labelled by generators, say  $b_1 \in G_{J_1}$  and  $b_m \in G_{J_m}$  with  $J_1 \subseteq K_1$  and  $J_m \subseteq K_2$ . Using Lemma 3.7 and Remark 2.4, we can now observe that  $g \in \mathfrak{G}_{J_1 \cap J_2 \cap \dots \cap J_m} \subseteq \mathfrak{G}_{J_1 \cap J_m} \subseteq \mathfrak{G}_{K_1 \cap K_2}$ , in contradiction to  $g \notin \mathfrak{G}_{K_1 \cap K_2}$ .

(3) *A local vertex cannot be a neighbour of itself.* If there was such a local vertex  $D_k$ , say of type  $J$ , then we could consider the subpicture  $\mathcal{Q}$  consisting of the local vertex  $D_k$ , one of the bridges that connect  $D_k$  with itself, everything that is enclosed by this bridge, and the extremal parts of the remaining arcs incident with  $D_k$ , see ① in Figure 5. Every boundary word  $w$  of the subpicture  $\mathcal{Q}$  is a word over generators from  $G_J$  and their formal inverses that represents the identity of the colimit group  $\mathfrak{G}$ . Since the natural homomorphism  $\nu_J : G_J \rightarrow \mathfrak{G}$  is injective, the word  $w$  does not only represent the identity of the colimit group  $\mathfrak{G}$  but also the identity of the group  $G_J$ . Therefore, we can remove the subpicture  $\mathcal{Q}$  and replace it by a single spider with boundary word  $w$ , see ② in Figure 5. After this modification, we obtain a disc picture with at most as many local vertices and strictly fewer bridges, and, hence, of lower complexity.

(4) *Two local vertices of the same type cannot be neighbours.* If there were two such local vertices  $D_k$  and  $D_l$ , without loss of generality  $D_k \neq D_l$ , then we could consider the subpicture  $\mathcal{Q}$  consisting of the local vertices  $D_k$  and  $D_l$ , one of the bridges that connect  $D_k$  and  $D_l$ , and the extremal parts of the remaining arcs incident with  $D_k$  and  $D_l$ , see ③ in Figure 5. By the same arguments as in (3), we can remove the subpicture  $\mathcal{Q}$  and replace it by a single spider with the same boundary word, see ④ in Figure 5. Again, we obtain a disc picture of lower complexity.

(5) *Every bridge has at least two arcs.* If there was a bridge  $B$  with only one arc, then we could distinguish between three cases. First, if  $B$  is connecting two local vertices, say of types  $J_1$  and  $J_2$ , then  $J_1 = J_2$ , in contradiction to (4). Second, if  $B$  is connecting the outer boundary  $\partial D$  with itself, then, by (1),  $B$  is already the whole disc picture, in contradiction to (2). So, we

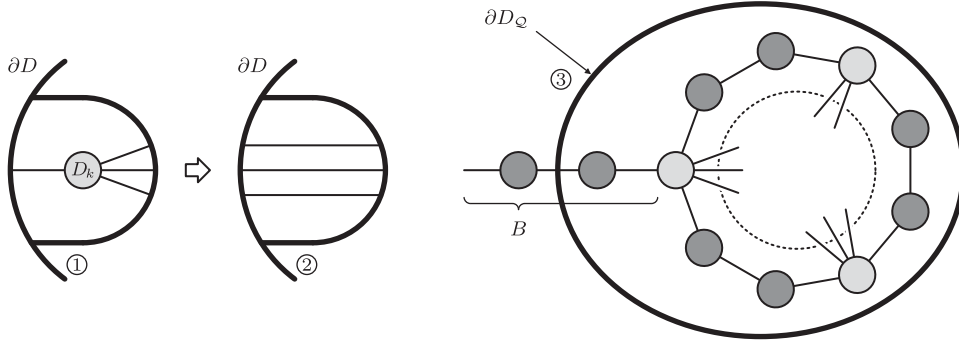


FIGURE 6. Replace the subpicture  $\mathcal{Q}$  by some arcs (left) and remove the bridge  $B$  (right).

may assume without loss of generality that  $B$  is connecting a local vertex  $D_k$ , say of type  $J$ , and the outer boundary  $\partial D$ , say at the side of  $\partial D$  along which one can read off the word  $w_1$ .

By the former,  $B$  is labelled by some generator from  $G_J$  and, by the latter,  $J \subseteq K_1$ . Now, consider the subpicture  $\mathcal{Q}$  consisting of the local vertex  $D_k$ , the bridge  $B$ , and the extremal parts of the remaining arcs incident with  $D_k$ , see ① in Figure 6. Replace it by a subpicture in which the arcs traversing  $\partial D_{\mathcal{Q}}$ , which are all labelled by generators from  $G_J$ , are extended to the outer boundary  $\partial D$ , see ② in Figure 6. This gives a disc picture with one fewer local vertex and, hence, of lower complexity.

(6) *The two regions on either side of a bridge cannot be the same.* Suppose that there was a bridge  $B$  having the same region  $R$  on either side. Then, we can find a subpicture  $\mathcal{Q}$  whose boundary  $\partial D_{\mathcal{Q}}$  is contained in  $R$ , except for one point where it crosses  $B$ , see ③ in Figure 6. Therefore, the value of  $B$  is the identity of the colimit group  $\mathfrak{G}$ . Since the natural homomorphisms  $\nu_J : G_J \rightarrow \mathfrak{G}$  are injective, the labels of the arcs of  $B$  must also be the identities of the respective groups  $G_J$ . So, we can remove the bridge  $B$  and obtain a disc picture of lower complexity.

(7) *The value of a bridge cannot be an element of  $\mathfrak{G}_{\emptyset}$ .* Suppose that there was a bridge  $B$  with value  $b \in \mathfrak{G}_{\emptyset}$ . Since  $b \in \mathfrak{G}_{\emptyset}$ , there is a generator  $b_{\emptyset} \in G_{\emptyset}$  that represents it. In fact, for every  $J \subseteq I$  with  $1 \leq |J| \leq 2$  there is a generator  $b_J \in G_J$  that represents it, namely  $b_J := \varphi_{\emptyset J}(b_{\emptyset}) \in G_J$ .

By (6), the two regions on either side of  $B$  cannot be the same. Among these two distinct regions choose the region  $R$  with the property that the arcs of  $B$  are heading away from  $R$ . Now, remove  $B$  and relabel all the remaining arcs in the boundary  $\partial R$  as follows: If an arc is labelled by a generator  $a \in G_J$  and is heading towards  $R$ , then relabel it by  $a \cdot b_J \in G_J$ . If it is heading away from  $R$ , then relabel it by  $b_J^{-1} \cdot a \in G_J$ . This guarantees that one can still read off relators along the boundaries of the remaining vertices. Here, we leave the details to the reader, see [13, Appendix] for another description and ①–③ in Figure 7 for some examples. Note that, in ②, the generator  $\varphi_{J_1 J_2}(a) \cdot b_{J_2} \in G_{J_2}$  satisfies the following equation:

$$\begin{aligned}
 \varphi_{J_1 J_2}(a) \cdot b_{J_2} &= \varphi_{J_1 J_2}(a) \cdot \varphi_{\emptyset J_2}(b_{\emptyset}) \\
 &= \begin{cases} \varphi_{J_1 J_2}(a) \cdot \varphi_{J_1 J_2}(b_{J_1}) & \text{if } J_1 = \emptyset, \\ \varphi_{J_1 J_2}(a) \cdot \varphi_{J_1 J_2}(\varphi_{\emptyset J_1}(b_{\emptyset})) & \text{otherwise} \end{cases} \\
 &= \varphi_{J_1 J_2}(a) \cdot \varphi_{J_1 J_2}(b_{J_1}) \\
 &= \varphi_{J_1 J_2}(a \cdot b_{J_1}).
 \end{aligned}$$

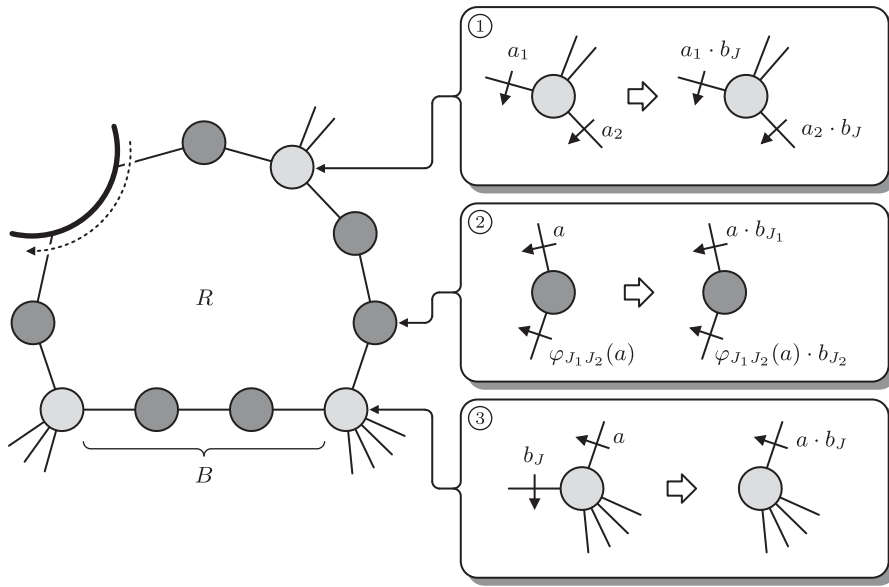


FIGURE 7. Relabel the remaining arcs in the boundary  $\partial R$ .

Therefore, one can actually read off the relator  $a \cdot b_{J_1} = \varphi_{J_1 J_2}(a \cdot b_{J_1})$  along the boundary of the respective joining vertex. So, we obtain a disc picture with the same number of local vertices and strictly fewer bridges and, hence, of lower complexity.

(8) For every bridge, there is a unique element  $i \in I$  such that the value of the bridge is in  $\mathfrak{G}_{\{i\}} \setminus \mathfrak{G}_{\emptyset}$ . Let  $B$  be a bridge with  $m$  arcs that are labelled by generators  $b_1 \in G_{J_1}$ ,  $b_2 \in G_{J_2}, \dots, b_m \in G_{J_m}$ . By (7), none of the sets  $J_1, J_2, \dots, J_m$  is empty, which implies that they must alternately have cardinality 1 and 2. By (5), every bridge has at least two arcs, that is,  $m \geq 2$ . Therefore, there must be at least one set  $J_k$  of cardinality 1, say  $J_k = \{i\}$ . So, the value of  $B$  is an element of  $\mathfrak{G}_{\{i\}}$  and, again by (7), cannot be an element of  $\mathfrak{G}_{\emptyset}$ . The element  $i \in I$  is unique; in the proof of Lemma 3.7, we have seen that for any two distinct  $i, j \in I$  the equation  $\mathfrak{G}_{\{i\}} \cap \mathfrak{G}_{\{j\}} = \mathfrak{G}_{\emptyset}$  holds, whence  $B$  cannot have a value that is simultaneously in  $\mathfrak{G}_{\{i\}} \setminus \mathfrak{G}_{\emptyset}$  and  $\mathfrak{G}_{\{j\}} \setminus \mathfrak{G}_{\emptyset}$ .

DEFINITION 3.10 (‘type of a bridge’). In this case, we say that the bridge  $B$  is of type  $i$ .

(9) If a bridge of type  $i$  is incident with a local vertex of type  $J$ , then  $i \in J$ . Similarly, if it is incident with one side of the outer boundary  $\partial D$ , then  $i \in K_1$  or  $i \in K_2$ , respectively. We give a proof of the first assertion, the proof of the second one is essentially the same. Let  $B$  be a bridge of type  $i$  that is incident with a local vertex  $D_k$  of type  $J$ . As we have seen in (8), one of the arcs of  $B$  is labelled by a generator from  $G_{\{i\}}$ . Moreover, the extremal arc of  $B$  that is incident with  $D_k$  is labelled by a generator from  $G_J$ . By Lemma 3.7 and Remark 2.4, we can conclude that  $B$  has a value in  $\mathfrak{G}_{\{i\} \cap J}$ . By (7), this value is not in  $\mathfrak{G}_{\emptyset}$ . Therefore,  $\{i\} \cap J \neq \emptyset$ , whence  $i \in J$ .

(10) There are no local vertices of type  $\emptyset$ . By (1), every local vertex is incident with at least one arc. So, if there was a local vertex of type  $\emptyset$ , then it would have to be incident with an arc that is labelled by some generator  $a \in G_{\emptyset}$ . But this arc is part of a bridge with a value in  $\mathfrak{G}_{\emptyset}$ , in contradiction to (7).

(11) There are no local vertices of type  $\{i\}$  with  $i \in I$ . First, observe that if there was such a local vertex  $D_k$ , then it would have to be a neighbour of some other local vertex  $D_l$ . Suppose it was not. Then, all bridges that are incident with  $D_k$  must either connect it to itself, which is

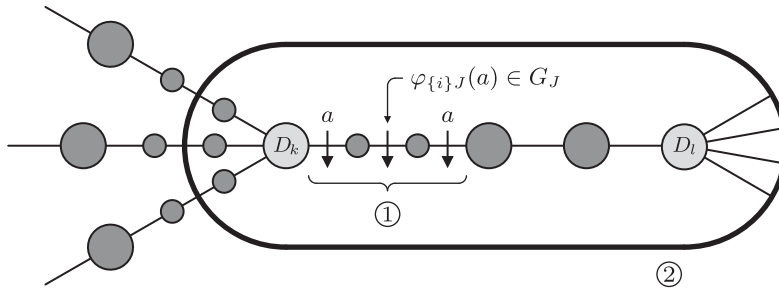


FIGURE 8. Replace each arc by a sequence of three arcs.

not possible by (3), or to the outer boundary  $\partial D$ . But, by (1), the disc picture  $\mathcal{P}$  is connected. So, this is already the whole disc picture. In particular, all bridges are incident with  $D_k$ , which is a local vertex of type  $\{i\}$ . Therefore, by (9), all bridges are of type  $i$ . But this means that each letter of  $w_1$  and  $w_2$  represents an element in  $\mathfrak{G}_{\{i\}}$ , whence  $g \in \mathfrak{G}_{\{i\}}$ . On the other hand, since both  $w_1$  and  $w_2$  are not empty, there is at least one bridge connecting  $D_k$  to either side of  $\partial D$ . Again, by (9), this implies both  $i \in K_1$  and  $i \in K_2$ . But since  $\{i\} \subseteq K_1 \cap K_2$ , we can use Remark 2.4 to conclude that  $g \in \mathfrak{G}_{K_1 \cap K_2}$ , in contradiction to  $g \notin \mathfrak{G}_{K_1 \cap K_2}$ .

So, we may assume without loss of generality that  $D_k$  is a neighbour of some other local vertex  $D_l$ . Consider a bridge that connects  $D_k$  and  $D_l$ . Again, by (9), this bridge is of type  $i$  and the local vertex  $D_l$  is of some type  $J$  with  $i \in J$ , that is,  $\{i\} \subseteq J$ . By (4), two local vertices of the same type cannot be neighbours. So, we actually obtain that  $\{i\} \subset J$ . Now, replace every arc that is incident with  $D_k$ , say labelled by some generator  $a \in G_{\{i\}}$ , by a sequence of three arcs with the same transversal orientation. The first and the third are labelled by  $a \in G_{\{i\}}$ , the second by  $\varphi_{\{i\}J}(a) \in G_J$ , see ① in Figure 8. Then, consider the subpicture  $\mathcal{Q}$  indicated in ② in Figure 8. By the same arguments as in (3) and (4), we can remove the subpicture  $\mathcal{Q}$  and replace it by a single spider with the same boundary word. Again, we obtain a disc picture of lower complexity.

**DEFINITION 3.11** ('angle and swap'). By (10) and (11), we know that every local vertex  $D_k$  is of some type  $\{i, j\}$  with distinct  $i, j \in I$ . By (9), such a local vertex is incident with bridges each of which is either of type  $i$  or of type  $j$ . Consider the connected components of  $\partial D_k \setminus M$ . They are called *angles*. An angle is called a *swap* if one of the two bridges enclosing it is of type  $i$  and the other one is of type  $j$ .

(12) *Every local vertex has at least one swap in its boundary.* Suppose that there was a local vertex  $D_k$ , say as above of type  $\{i, j\}$ , without any swap in its boundary. Then, all bridges that are incident with  $D_k$  are of the same type, say of type  $i$ .

Since  $D_k$  is a local vertex of type  $\{i, j\}$ , the arcs that are incident with  $D_k$  are labelled by generators  $a_1, a_2, \dots, a_m \in G_{\{i, j\}}$ . The respective bridges are all of type  $i$ . So, each of these generators represents an element in  $\mathfrak{G}_{\{i\}} \setminus \mathfrak{G}_{\emptyset}$ , whence we can even find generators  $\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_m \in G_{\{i\}}$  representing the same elements, that is, satisfying the equations  $\nu_{\{i, j\}}(a_s) = \nu_{\{i\}}(\tilde{a}_s) = \nu_{\{i, j\}}(\varphi_{\{i\}\{i, j\}}(\tilde{a}_s))$ . Now, we can use the injectivity of the homomorphism  $\nu_{\{i, j\}} : G_{\{i, j\}} \rightarrow \mathfrak{G}$  to conclude that  $a_s = \varphi_{\{i\}\{i, j\}}(\tilde{a}_s)$ . Similarly to the modification described above in (11), we replace every arc that is incident with  $D_k$  by a sequence of two arcs with the same transversal orientation. If the arc has been labelled by  $a_s \in G_{\{i, j\}}$ , then the new arc that is incident with  $D_k$  is labelled by  $\tilde{a}_s \in G_{\{i\}}$  whereas the other one is labelled by  $a_s \in G_{\{i, j\}}$ , see ① in Figure 9. Now, let  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m \in \{-1, 1\}$  such that the word  $a_1^{\varepsilon_1} a_2^{\varepsilon_2} \dots a_m^{\varepsilon_m}$  has originally been a boundary word of  $D_k$ . After this modification, one can

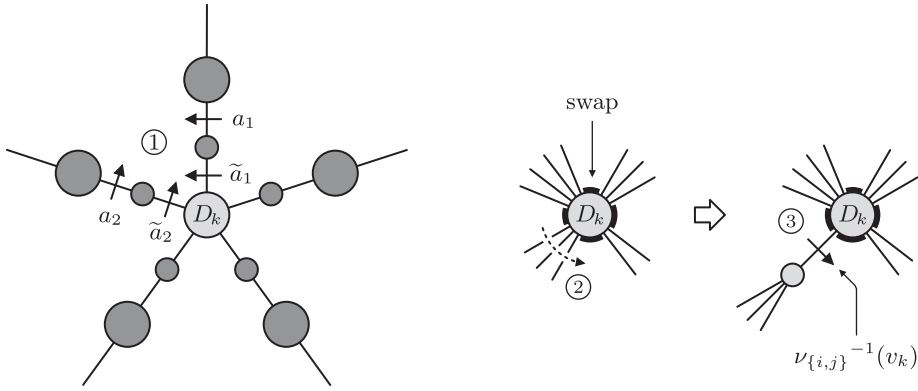


FIGURE 9. Replace each arc by a sequence of two arcs (left) and create a local vertex (right).

read off the word  $\tilde{a}_1^{\varepsilon_1} \tilde{a}_2^{\varepsilon_2} \dots \tilde{a}_m^{\varepsilon_m}$  along  $\partial D_k$ . But:

$$\varphi_{\{i\}\{i,j\}}(\tilde{a}_1^{\varepsilon_1} \tilde{a}_2^{\varepsilon_2} \dots \tilde{a}_m^{\varepsilon_m}) = a_1^{\varepsilon_1} a_2^{\varepsilon_2} \dots a_m^{\varepsilon_m} = 1 \quad \text{in } G_{\{i,j\}}$$

The injectivity of  $\varphi_{\{i\}\{i,j\}} : G_{\{i\}} \rightarrow G_{\{i,j\}}$  implies that  $D_k$  has become a local vertex of type  $\{i\}$  and can be removed as in (11). We obtain a disc picture of lower complexity.

DEFINITION 3.12 ('sufficiently many swaps'). Assume given a local vertex  $D_k$  of some type  $\{i, j\}$  with distinct  $i, j \in I$ . Then,  $D_k$  has *sufficiently many swaps* in its boundary if the Gersten–Stallings angle  $\angle_{\{i,j\}} \neq 0$  and the number of swaps  $m \geq 2\pi/\angle_{\{i,j\}}$ .

(13) *Every local vertex has sufficiently many swaps in its boundary.* Suppose that there was a local vertex  $D_k$  without sufficiently many swaps in its boundary. By (13), there is at least one swap. If we start at some swap and go from swap to swap once around  $\partial D_k$ , then we can read off words  $v_1, v_2, \dots, v_m$ , see ② in Figure 9. Each of these words represents an element in  $\mathfrak{G}$ , so it makes sense to write  $v_1, v_2, \dots, v_m \in \mathfrak{G}$ . Their product  $v_1 \cdot v_2 \cdot \dots \cdot v_m$  is the identity element. We may assume without loss of generality that  $v_1, v_3, \dots, v_{m-1} \in \mathfrak{G}_{\{i\}}$  and  $v_2, v_4, \dots, v_m \in \mathfrak{G}_{\{j\}}$ . In order to show that at least one of these elements is contained in  $\mathfrak{G}_\emptyset$ , we construct their preimages under the injective homomorphisms  $\nu_K : G_K \rightarrow \mathfrak{G}$ :

$$\begin{aligned} \nu_{\{i\}}^{-1}(v_1), \nu_{\{i\}}^{-1}(v_3), \dots, \nu_{\{i\}}^{-1}(v_{m-1}) &\in G_{\{i\}}, \\ \nu_{\{j\}}^{-1}(v_2), \nu_{\{j\}}^{-1}(v_4), \dots, \nu_{\{j\}}^{-1}(v_m) &\in G_{\{j\}}. \end{aligned}$$

These preimages assemble to an element  $x := \nu_{\{i\}}^{-1}(v_1) \cdot \nu_{\{j\}}^{-1}(v_2) \cdot \dots \cdot \nu_{\{j\}}^{-1}(v_m)$  of the amalgamated free product  $G_{\{i\}} *_{G_\emptyset} G_{\{j\}}$ . Now, recall the definition of the Gersten–Stallings angle. The homomorphism  $\alpha : G_{\{i\}} *_{G_\emptyset} G_{\{j\}} \rightarrow G_{\{i,j\}}$  introduced in Subsection 2.2 satisfies:

$$\begin{aligned} \alpha(x) &= \varphi_{\{i\}\{i,j\}}(\nu_{\{i\}}^{-1}(v_1)) \cdot \varphi_{\{j\}\{i,j\}}(\nu_{\{j\}}^{-1}(v_2)) \cdot \dots \cdot \varphi_{\{j\}\{i,j\}}(\nu_{\{j\}}^{-1}(v_m)) \\ &= \nu_{\{i,j\}}^{-1}(v_1) \cdot \nu_{\{i,j\}}^{-1}(v_2) \cdot \dots \cdot \nu_{\{i,j\}}^{-1}(v_m) \\ &= \nu_{\{i,j\}}^{-1}(v_1 \cdot v_2 \cdot \dots \cdot v_m) \\ &= \nu_{\{i,j\}}^{-1}(1) \\ &= 1. \end{aligned}$$

So,  $x \in \ker(\alpha)$ . Since  $D_k$  does not have sufficiently many swaps in its boundary, we know that either the Gersten–Stallings angle  $\angle_{\{i,j\}} = 0$  or the length of  $x$ , which is at most  $m$ , is strictly smaller than  $2\pi/\angle_{\{i,j\}}$ , which is nothing but the minimal length of a non-trivial element in  $\ker(\alpha)$ . In either case,  $x$  must be trivial in  $G_{\{i\}} *_{G_\emptyset} G_{\{j\}}$ .

It is a consequence of the normal form theorem, see [20, Lemma 1], that there is an index  $k \in \{1, 2, \dots, m\}$  such that  $\nu_{\{i\}}^{-1}(v_k) \in \varphi_{\emptyset\{i\}}(G_\emptyset)$  or  $\nu_{\{j\}}^{-1}(v_k) \in \varphi_{\emptyset\{j\}}(G_\emptyset)$ , depending on the parity of  $k$ . But then:

$$v_k \in \left\{ \begin{array}{ll} \nu_{\{i\}}(\varphi_{\emptyset\{i\}}(G_\emptyset)) & \text{if } k \text{ is odd,} \\ \nu_{\{j\}}(\varphi_{\emptyset\{j\}}(G_\emptyset)) & \text{if } k \text{ is even} \end{array} \right\} = \nu_\emptyset(G_\emptyset) = \mathfrak{G}_\emptyset.$$

In either case,  $v_k \in \mathfrak{G}_\emptyset$ . Now, we add a new local vertex and a new bridge to the disc picture as illustrated in ③ in Figure 9. The arcs that had been traversed when reading off the word  $v_k$  end up at the new local vertex, which is connected to  $D_k$  by a single arc labelled by  $\nu_{\{i,j\}}^{-1}(v_k)$ . This increases both the number of local vertices and the number of bridges by 1. But still, some of the properties we have discussed so far, in particular (1), hold true and the bridge connecting the new local vertex and  $D_k$ , which has a value in  $\mathfrak{G}_\emptyset$ , can be removed as in (6) or (7). Next, we want to get rid of the new local vertex. If the removal of the bridge has made the disc picture  $\mathcal{P}$  disconnected, then we can remove one of the two components as in (1). Otherwise, there is still a path from the new local vertex to  $D_k$ , which implies that the new local vertex is a neighbour of some other local vertex. Once this is clear, we can remove it as in (13) and in the final step of (11). In either case, in particular in the latter, we do not only remove the new local vertex but also at least one more bridge. So, in either case, we obtain a disc picture of lower complexity.

After all these observations, we can give an easy combinatorial argument that yields a contradiction. The principal idea is to distribute weights over certain parts of the disc picture. Every local vertex gets the weight  $2\pi$ , every inner bridge gets the weight  $-2\pi$ , and every inner region gets the weight  $2\pi$ . For the notion of *inner*, see Definition 3.6. The weighted parts of the disc picture correspond to vertices, edges, and bounded regions of a planar graph, which is non-empty, finite, and connected. So, we may use Euler’s formula for planar graphs to calculate the total weight:

$$2\pi \cdot \# \text{ local vertices} - 2\pi \cdot \# \text{ inner bridges} + 2\pi \cdot \# \text{ inner regions} = 2\pi.$$

Let us count again. But, this time, we reallocate the weights to the regions. Every inner bridge distributes its weight  $-2\pi$  equally to the two regions on either side and every local vertex distributes its weight  $2\pi$  equally to the swaps in its boundary, each of which lets it traverse to the adjacent region. The new total weight of a region  $R$  is denoted by  $\text{wt}(R)$  and can be estimated from above using (14)–(16):

(14) *The weight traversing each swap is bounded by the Gersten–Stallings angle associated to the type of the local vertex. In particular, it is at most  $\pi/2$ .* Let  $D_k$  be a local vertex of type  $J = \{i, j\}$ . By (13),  $D_k$  has sufficiently many swaps in its boundary. So, the Gersten–Stallings angle  $\angle_{\{i,j\}} \neq 0$  and the number of swaps is at least  $2\pi/\angle_{\{i,j\}}$ . Since  $D_k$  distributes its weight  $2\pi$  equally to the swaps, the weight traversing each swap is at most  $\angle_{\{i,j\}}$ .

(15) *There are no inner regions of positive weight.* Let  $R$  be an inner region. By (1),  $R$  is an open disc. By (6), the boundary  $\partial R$  contains some number of inner bridges, say  $m$ , and the same number of angles, some of which may be swaps. By (3),  $m \geq 2$ .

By (6), each of the  $m$  inner bridges contributes  $-\pi$  to  $\text{wt}(R)$  and, by (14), each of the at most  $m$  swaps contributes at most  $\pi/2$ . Therefore, we can estimate  $\text{wt}(R)$  as follows: If  $m \geq 4$ , then  $\text{wt}(R) \leq 1 \cdot 2\pi - m \cdot \pi + m \cdot \pi/2 \leq 0$ . If  $m = 3$  and there are at most two swaps, then  $\text{wt}(R) \leq 1 \cdot 2\pi - 3 \cdot \pi + 2 \cdot \pi/2 = 0$ . If  $m = 3$  and there are exactly three swaps, then there are three pairwise distinct  $i, j, k \in I$  such that the local vertices in the boundary  $\partial R$  are of types  $\{i, j\}$ ,  $\{i, k\}$ ,  $\{j, k\}$ . Since we consider a non-spherical Corson diagram,

there are no spherical triples. In particular,  $\angle_{\{i,j\}} + \angle_{\{i,k\}} + \angle_{\{j,k\}}$  is at most  $\pi$ , whence  $\text{wt}(R) \leq 1 \cdot 2\pi - 3 \cdot \pi + \angle_{\{i,j\}} + \angle_{\{i,k\}} + \angle_{\{j,k\}} \leq 0$ . What remains is the case that  $m = 2$ . But if there were two distinct  $i, j \in I$  such that one of the inner bridges is of type  $i$  and the other is of type  $j$ , then both local vertices must be of type  $\{i, j\}$ , in contradiction to (4). So, in this case, there cannot be any swap, whence  $\text{wt}(R) = 1 \cdot 2\pi - 2 \cdot \pi + 0 = 0$ .

(16) *There are at most two outer regions of positive weight, and each of them has at most weight  $\pi/2$ .* Let  $R$  be an outer region. By (1),  $R$  is an open disc. The boundary  $\partial R$  contains some number of bridges, say  $m$ , and some number of angles. By (1), (2), and (6), exactly two of the bridges are outer, which implies that  $m \geq 2$ . Moreover, it contains exactly  $m - 1$  angles, some of which may be swaps.

Again, if  $m \geq 3$ , then  $\text{wt}(R) \leq 0 \cdot 2\pi - (m - 2) \cdot \pi + (m - 1) \cdot \pi/2 = (-m + 3) \cdot \pi/2 \leq 0$ . If  $m = 2$  and there is no swap, then  $\text{wt}(R) = 0 \cdot 2\pi - 0 \cdot \pi + 0 = 0$ . What remains is the case that  $m = 2$  and there is a swap. Then,  $\text{wt}(R)$  might well be positive, but  $\text{wt}(R) \leq 0 \cdot 2\pi - 0 \cdot \pi + 1 \cdot \pi/2 = \pi/2$ .

Next, we show that this can happen at most twice. Since there is a swap, we know that there are two distinct  $i, j \in I$  such that one of the outer bridges is of type  $i$  and the other one is of type  $j$ . If both bridges end up at the same side of  $\partial D$ , say at the side of  $\partial D$  along which one can read off the word  $w_1$ , then, by (9), both  $i \in K_1$  and  $j \in K_1$ . This allows us to remove the respective local vertex  $D_k$  of type  $\{i, j\} \subseteq K_1$  as in the final step of (5) and to obtain a disc picture of lower complexity. So, we may assume without loss of generality that the two bridges end up at different sides of  $\partial D$ . But, by (1), we know that the disc picture  $\mathcal{P}$  is connected. Hence, this can happen at most twice, namely when the boundary  $\partial R$  contains one of the two points  $x$  and  $y$  that have been chosen in (1).

By (15) and (16), the total weight given to the disc picture is at most  $\pi$ . This is a contradiction to the above observation that the total weight amounts to  $2\pi$ , which completes the proof. □

### 3.4. Interpretation

In case of a non-spherical triangle of groups, the intersection theorem says that the groups  $\mathfrak{G}_K$  with  $K \subseteq \{1, 2, 3\}$  and  $|K| \leq 2$  intersect exactly as sketched in Figure 10. One particularly nice way of reading the intersection theorem is to start with such a setting. Let  $M$  be the union of three groups with the property that each two of them intersect along a common subgroup, by which we implicitly mean that the two multiplications agree on the subgroup. The set  $M$  is equipped with a partial multiplication and the question arises whether it can be homomorphically embedded into a group, that is, whether there exists an injective map into a group such that the restriction to each of the three groups is a homomorphism. In order to give a partial answer to this question, we may interpret our three groups and their intersections, equipped with the inclusion maps, as a triangle of groups. By construction of the colimit, the natural homomorphisms  $\nu_J : G_J \rightarrow \mathfrak{G}$  agree on the intersections and, hence, yield a map  $\nu : M \rightarrow \mathfrak{G}$ . This map is injective if and only if the natural homomorphisms  $\nu_J : G_J \rightarrow \mathfrak{G}$  are injective and the equations  $\mathfrak{G}_{K_1} \cap \mathfrak{G}_{K_2} = \mathfrak{G}_{K_1 \cap K_2}$  hold.

So, if the triangle of groups is non-spherical, then the answer is affirmative. On the other hand, it is a consequence of the universal property, see [1, Chapter III], that if the map  $\nu : M \rightarrow \mathfrak{G}$  is not injective, then  $M$  cannot be homomorphically embedded into any group and the answer is negative.

## 4. Billiards theorem for triangles of groups

In the previous section, we used a combinatorial argument based on Euler’s formula for planar graphs to prove the intersection theorem. This kind of argument, be it in the language



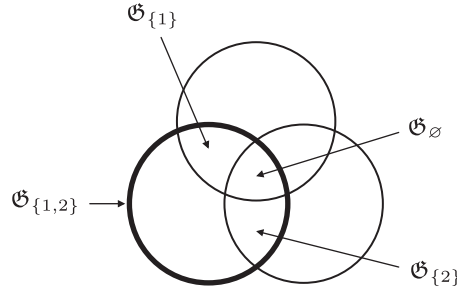


FIGURE 10. The group  $\mathfrak{G}_{\{1,2\}}$  intersecting the groups  $\mathfrak{G}_{\{1,3\}}$  and  $\mathfrak{G}_{\{2,3\}}$ .

of homotopies, see, for example, [25], in the language of van Kampen diagrams, see, for example, [15, 16], or in the language of disc pictures, see, for example, [13], turned out to be very powerful in our context, and we highlight the following two results from [15, 16]. In each of them, one considers a non-spherical triangle of groups and assumes that for every  $a \in \{1, 2, 3\}$  there is an element  $g_a \in \mathfrak{G}_{\{a\}} \setminus \mathfrak{G}_{\emptyset}$ .

**THEOREM 4.1** (Edjvet *et al.*). *The product  $g_1 g_2 g_3 \in \mathfrak{G}$  has infinite order.*

**THEOREM 4.2** (Howie–Kopteva). *If the triangle of groups is hyperbolic, then there is an  $n \in \mathbb{N}$  such that the elements  $(g_1 g_2 g_3)^n \in \mathfrak{G}$  and  $(g_1 g_3 g_2)^n \in \mathfrak{G}$  generate a non-abelian free subgroup.*

**REMARK 4.3.** Both, in Theorems 4.1 and 4.2, the authors had the case  $G_{\emptyset} = \{1\}$  in mind and, therefore, used  $\alpha : G_{\{i\}} * G_{\{j\}} \rightarrow G_{\{i,j\}}$  instead of  $\alpha : G_{\{i\}} *_{G_{\emptyset}} G_{\{j\}} \rightarrow G_{\{i,j\}}$  to define the angle  $\angle_{\{i,j\}}$ . Note that their notion of a group-theoretic angle coincides with Pride’s property- $W_k$ , which has been introduced in [22] and generalized in [23]. In fact, following their notion, the angle  $\angle_{\{i,j\}}$  is strictly smaller than  $\pi/k$  if and only if the group  $G_{\{i,j\}}$  has Pride’s property- $W_k$ . Our notion is certainly reminiscent of Pride’s property- $W_k$ , but using the amalgamated free product prevents us from measuring the intersection along the group  $G_{\emptyset}$ . One can check that Theorems 4.1 and 4.2 do also hold in our setting with arbitrary groups  $G_{\emptyset}$  and Gersten–Stallings angles as defined in Subsection 2.2, see [14] for details.

In the light of Theorem 4.2, one may wonder about the Euclidean case. Let us therefore assume that the triangle of groups is Euclidean and ask under which conditions the colimit group  $\mathfrak{G}$  has a non-abelian free subgroup. A first class of examples to look at are *Euclidean triangle groups*:

$$\Delta(k, l, m) = \langle a, b, c : a^2, b^2, c^2, (ab)^k, (ac)^l, (bc)^m \rangle$$

with  $(k, l, m) \in \{(3, 3, 3), (2, 4, 4), (2, 3, 6)\}$ .

Each of these groups happens to be the colimit group of the Euclidean triangle of groups based on the following data:

$$\begin{aligned} G_{\emptyset} &= \{1\}, & G_{\{1\}} &= \langle a : a^2 = 1 \rangle, & G_{\{2\}} &= \langle b : b^2 = 1 \rangle, & G_{\{3\}} &= \langle c : c^2 = 1 \rangle, \\ G_{\{1,2\}} &= \langle a, b : a^2 = b^2 = (ab)^k = 1 \rangle, & G_{\{1,3\}} &= \langle a, c : a^2 = c^2 = (ac)^l = 1 \rangle, \\ G_{\{2,3\}} &= \langle b, c : b^2 = c^2 = (bc)^m = 1 \rangle. \end{aligned}$$

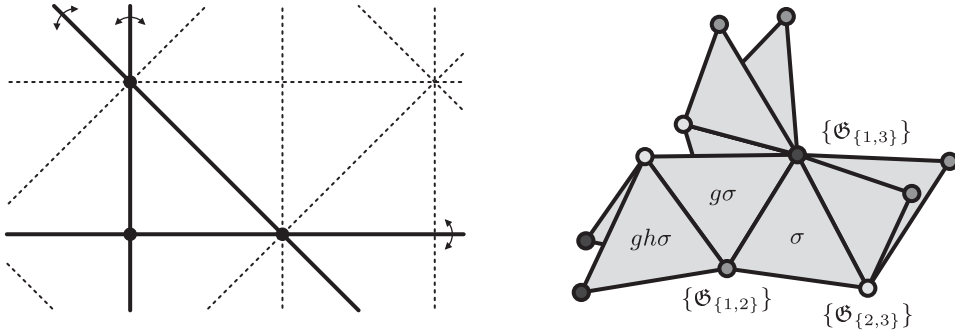


FIGURE 11. The Euclidean triangle group  $\Delta(2, 4, 4)$  realized as  $S \leq \text{isom}(\mathbb{E}^2)$  (left) and a part of the simplicial complex  $\mathcal{X}$  including translates of  $\sigma$  by group elements  $g$  and  $gh$  with  $g \in \mathfrak{G}_{\{1,3\}}$  and  $h \in \mathfrak{G}_{\{2,3\}}$  (right).

Here, as in Subsection 3.1, the homomorphisms  $\varphi_{J_1 J_2} : G_{J_1} \rightarrow G_{J_2}$  are implicitly given by  $a \mapsto a$ ,  $b \mapsto b$ , and  $c \mapsto c$ . It turns out that the Gersten–Stallings angles amount to  $\sphericalangle_{\{1,2\}} = \pi/k$ ,  $\sphericalangle_{\{1,3\}} = \pi/l$ , and  $\sphericalangle_{\{2,3\}} = \pi/m$ , whence the triangle of groups is actually Euclidean.

The algebraic structure of the colimit group  $\mathfrak{G}$  can be revealed by geometry. Consider three lines in the Euclidean plane  $\mathbb{E}^2$  enclosing a triangle with angles  $\pi/k$ ,  $\pi/l$ , and  $\pi/m$ , see Figure 11. The reflections along these lines generate a subgroup  $S \leq \text{isom}(\mathbb{E}^2)$  that is isomorphic to  $\Delta(k, l, m)$ . Since  $\text{isom}(\mathbb{E}^2)$  is solvable,  $S \leq \text{isom}(\mathbb{E}^2)$  must be solvable, too. Therefore,  $\Delta(k, l, m)$  is solvable and cannot have a non-abelian free subgroup, see, for example, [14, 17] for more detailed descriptions of the geometry.

In this section, we generalize the geometric approach. More precisely, we use a construction introduced by Bridson in [6] to study all *non-degenerate* Euclidean triangles of groups, that is, all Euclidean triangles of groups with the property that each Gersten–Stallings angle is strictly between 0 and  $\pi$ . Since the strict upper bound  $\pi$  is our standing assumption, see Subsection 2.4, being non-degenerate actually means that none of the Gersten–Stallings angles is 0. (But even if there was no such standing assumption, then it is clear that a Euclidean triangle of groups is non-degenerate if and only if none of the Gersten–Stallings angles is 0.) Given such a triangle of groups, we will construct a simplicial complex  $\mathcal{X}$ . The action of  $\mathfrak{G}$  on  $\mathcal{X}$  will give us new insight into the structure of  $\mathfrak{G}$ . Actually, it turns out that if  $\mathcal{X}$  *branches*, that is, if  $|\mathcal{X}|$  is not a topological manifold any more, then the colimit group  $\mathfrak{G}$  has a non-abelian free subgroup, see Theorem 4.19. This allows us to give an answer to a problem mentioned by Kopteva and Williams in [18, p. 58, l. 24], who wondered if the class of colimit groups of non-spherical triangles of groups satisfies the Tits alternative.

As already mentioned, the construction and the basic properties of  $\mathcal{X}$  have been introduced by Bridson in [6]. In Subsection 4.1, we summarize what is relevant for our work, and apply it from Subsection 4.2 onwards. Our proofs are based on ideas and techniques that go back to two Diplomarbeiten under supervision of Bieri, namely by Lorenz in [19] and Brendel in [5]. Both Lorenz and Brendel use altitudes in triangles to detect non-abelian free subgroups, but under additional assumptions on the Gersten–Stallings angles. We use the language of billiards instead, which gives us the flexibility we need.

#### 4.1. Bridson’s simplicial complex

Given a non-degenerate Euclidean triangle of groups, we define an abstract simplicial complex  $\mathcal{X}$  as follows (the same definition can be given for any triangle of groups, but we

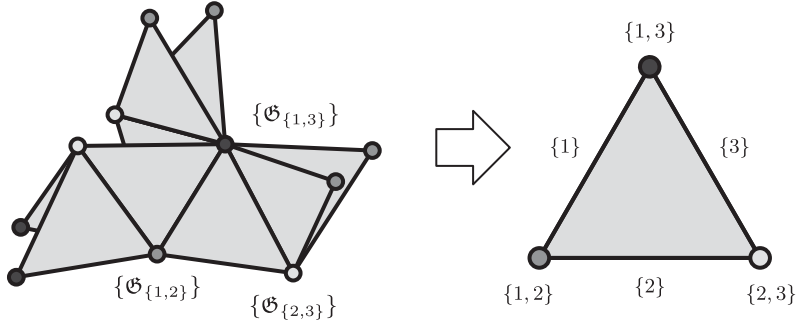


FIGURE 12. Labelling the triangle  $\Delta$  and constructing  $h : |\mathcal{X}| \rightarrow \Delta$ .

need it only in the non-degenerate Euclidean case):

$$\begin{aligned}
 0\text{-simplices} &:= \{ \{g\mathfrak{G}_{\{1,2\}}\} : g \in \mathfrak{G} \} \\
 &\quad \sqcup \{ \{g\mathfrak{G}_{\{1,3\}}\} : g \in \mathfrak{G} \} \\
 &\quad \sqcup \{ \{g\mathfrak{G}_{\{2,3\}}\} : g \in \mathfrak{G} \}, \\
 1\text{-simplices} &:= \{ \{g\mathfrak{G}_{\{1,2\}}, g\mathfrak{G}_{\{1,3\}}\} : g \in \mathfrak{G} \} \\
 &\quad \sqcup \{ \{g\mathfrak{G}_{\{1,2\}}, g\mathfrak{G}_{\{2,3\}}\} : g \in \mathfrak{G} \} \\
 &\quad \sqcup \{ \{g\mathfrak{G}_{\{1,3\}}, g\mathfrak{G}_{\{2,3\}}\} : g \in \mathfrak{G} \}, \\
 2\text{-simplices} &:= \{ \{g\mathfrak{G}_{\{1,2\}}, g\mathfrak{G}_{\{1,3\}}, g\mathfrak{G}_{\{2,3\}}\} : g \in \mathfrak{G} \}.
 \end{aligned}$$

4.1.1. *Group action and stabilizers.* We will use the letter  $\sigma$  to denote the 2-simplex that is represented by the identity, that is,  $\sigma := \{\mathfrak{G}_{\{1,2\}}, \mathfrak{G}_{\{1,3\}}, \mathfrak{G}_{\{2,3\}}\}$ . There is a natural action of  $\mathfrak{G}$  on  $\mathcal{X}$  given by left-multiplication of each coset. A fundamental domain for this action consists of the 2-simplex  $\sigma$  and its faces, see Figure 11.

Bridson mentioned in [6, p. 431, ll. 8–9] that ‘the pattern of stabiliser in this fundamental domain is precisely the original triangle of groups’. Since we distinguish between the diagram and its image in the colimit group, we would replace ‘original triangle of groups’ by ‘pattern of subgroups  $\mathfrak{G}_K$  with  $K \subseteq \{1, 2, 3\}$  and  $|K| \leq 2$ ’. Anyway, note that there is a very easy way to prove this observation using the intersection theorem because we can observe that for all pairwise distinct elements  $a, b, c \in \{1, 2, 3\}$ :

(1)

$$\text{stab}_{\mathfrak{G}}(\{\mathfrak{G}_{\{a,b\}}\}) = \{g \in \mathfrak{G} : g\mathfrak{G}_{\{a,b\}} = \mathfrak{G}_{\{a,b\}}\} = \mathfrak{G}_{\{a,b\}};$$

(2)

$$\begin{aligned}
 \text{stab}_{\mathfrak{G}}(\{\mathfrak{G}_{\{a,b\}}, \mathfrak{G}_{\{a,c\}}\}) &= \{g \in \mathfrak{G} : g\mathfrak{G}_{\{a,b\}} = \mathfrak{G}_{\{a,b\}}, g\mathfrak{G}_{\{a,c\}} = \mathfrak{G}_{\{a,c\}}\} \\
 &= \mathfrak{G}_{\{a,b\}} \cap \mathfrak{G}_{\{a,c\}} = \mathfrak{G}_{\{a\}};
 \end{aligned}$$

(3)

$$\begin{aligned}
 \text{stab}_{\mathfrak{G}}(\{\mathfrak{G}_{\{a,b\}}, \mathfrak{G}_{\{a,c\}}, \mathfrak{G}_{\{b,c\}}\}) \\
 &= \{g \in \mathfrak{G} : g\mathfrak{G}_{\{a,b\}} = \mathfrak{G}_{\{a,b\}}, g\mathfrak{G}_{\{a,c\}} = \mathfrak{G}_{\{a,c\}}, g\mathfrak{G}_{\{b,c\}} = \mathfrak{G}_{\{b,c\}}\} \\
 &= \mathfrak{G}_{\{a,b\}} \cap \mathfrak{G}_{\{a,c\}} \cap \mathfrak{G}_{\{b,c\}} = \mathfrak{G}_{\emptyset}.
 \end{aligned}$$

Hence, by (1)–(3), the stabilizers of the 0-simplices are the groups  $\mathfrak{G}_K$  with  $K \subseteq \{1, 2, 3\}$  and  $|K| = 2$ . The stabilizers of the 1-simplices and the 2-simplex are their pairwise and triple intersections, which are precisely the groups  $\mathfrak{G}_K$  with  $K \subseteq \{1, 2, 3\}$  and  $|K| = 1$  and  $|K| = 0$ .

4.1.2. *Simple connectedness.* Let us now consider the geometric realization  $|\mathcal{X}|$ . As usual, it is equipped with the *weak topology*. For details about abstract simplicial complexes and their geometric realizations, we refer the reader to [21, Sections 1.1–1.3].

LEMMA 4.4 (Behr). *The geometric realization  $|\mathcal{X}|$  is simply connected.*

Behr has proved a slightly more general version of this lemma in 1975, see [3, Satz 1.2]. Roughly speaking, he translates edge paths in  $|\mathcal{X}|$  to products  $h_1 h_2 \cdots h_n$  of elements  $h_i \in \mathfrak{G}_{K_i}$  with  $K_i \subseteq \{1, 2, 3\}$  and  $|K_i| = 2$ , and vice versa.

REMARK 4.5. In order to keep the notation simple, we use the same symbols to refer to simplices in  $\mathcal{X}$  and to their geometric realizations in  $|\mathcal{X}|$ . Moreover, whenever we talk about a simplex in  $|\mathcal{X}|$  without further specification, we mean the closed simplex.

4.1.3. *Metric structure.* The geometric realization  $|\mathcal{X}|$  can be equipped with a piecewise Euclidean metric structure. The triangle of groups is non-degenerate and Euclidean. We may therefore pick a closed triangle  $\Delta$  in the Euclidean plane  $\mathbb{E}^2$  with the property that its angles agree with the Gersten–Stallings angles. For every subset  $K \subseteq \{1, 2, 3\}$  with  $|K| = 2$  we label the corresponding vertex of  $\Delta$  with the angle  $\sphericalangle_K$  by  $K$ . For later purposes, let us label the edges of  $\Delta$ , too. The edge between the two vertices that are labelled by  $K_1$  and  $K_2$  is labelled by their intersection  $K_1 \cap K_2$ , see Figure 12.

Next, we construct a continuous map  $h : |\mathcal{X}| \rightarrow \Delta$ . Every 0-simplex is of the form  $\{g\mathfrak{G}_K\}$  for some  $K \subseteq \{1, 2, 3\}$  with  $|K| = 2$ . Map it to the vertex that is labelled by  $K$ . In order to continue this map to the higher dimensional simplices, map the 1-simplices homeomorphically to the *corresponding* edges, that is, if a 1-simplex is of the form  $\{g\mathfrak{G}_{\{a,b\}}, g\mathfrak{G}_{\{a,c\}}\}$ , then map it to the edge that is labelled by  $\{a, b\} \cap \{a, c\} = \{a\}$ . For every 2-simplex  $\tau$ , use Schoenflies’ Theorem to continue the homeomorphism  $h|_{\partial\tau} : \partial\tau \rightarrow \partial\Delta$  to  $h|_\tau : \tau \rightarrow \Delta$ . The latter ones assemble to the desired continuous map  $h : |\mathcal{X}| \rightarrow \Delta$ . Given two points  $x, y \in \tau$ , we can now measure their *local distance*:

$$d_\tau(x, y) := \|h|_\tau(x) - h|_\tau(y)\|.$$

The local distance  $d_\tau : \tau \times \tau \rightarrow \mathbb{R}$  is a metric on the single 2-simplex  $\tau$ . The geometric realization  $|\mathcal{X}|$  equipped with the local distances is called an  *$\mathbb{E}$ -simplicial complex*. For a formal definition, see [6, Section 1.1]. In order to extend the local distances to a metric on  $|\mathcal{X}|$ , we follow Bridson’s work.

DEFINITION 4.6 (*‘ $m$ -chain’*). Let  $x, y \in |\mathcal{X}|$ . An  *$m$ -chain* from  $x$  to  $y$  is a finite sequence  $\mathcal{C} = (x_0, x_1, \dots, x_m)$  of points in  $|\mathcal{X}|$  with the property that  $x_0 = x$  and  $x_m = y$ , and that for every  $1 \leq i \leq m$  both  $x_{i-1}$  and  $x_i$  are contained in some common 2-simplex  $\tau_i$ .

Let  $\mathcal{C}$  be an  $m$ -chain as above. Then, the *length* of  $\mathcal{C}$  is defined by:

$$\text{length}(\mathcal{C}) := \sum_{i=1}^m d_{\tau_i}(x_{i-1}, x_i).$$

For every  $1 \leq i \leq m$ , there is a unique geodesic from  $x_{i-1}$  to  $x_i$  in  $\tau_i$ . We call it the *segment* from  $x_{i-1}$  to  $x_i$ . The concatenation of all segments is called the *path* induced by  $\mathcal{C}$ . It is denoted by  $[\mathcal{C}]$ . Note that neither  $\text{length}(\mathcal{C})$  nor  $[\mathcal{C}]$  depends on the choice of  $\tau_i$ , that is, if there are two 2-simplices  $\tau_i$  and  $\tilde{\tau}_i$  such that  $x_{i-1}, x_i \in \tau_i \cap \tilde{\tau}_i$ , then the local distances  $d_{\tau_i}(x_{i-1}, x_i)$  and  $d_{\tilde{\tau}_i}(x_{i-1}, x_i)$  agree and the segment  $[[x_{i-1}, x_i]]$  is well defined.

Let  $x, y \in |\mathcal{X}|$ . Since  $|\mathcal{X}|$  is path connected, there is a path from  $x$  to  $y$ . By the construction given in Behr’s proof or by a direct argument, we can even find a path from  $x$  to  $y$  that is

induced by some  $m$ -chain  $\mathcal{C}$ . Hence, there is a function  $d : |\mathcal{X}| \times |\mathcal{X}| \rightarrow \mathbb{R}$  given by:

$$d(x, y) := \inf \{ \text{length}(\mathcal{C}) : \exists m \text{ such that } \mathcal{C} \text{ is an } m\text{-chain from } x \text{ to } y \}.$$

It is straightforward to see that  $d : |\mathcal{X}| \times |\mathcal{X}| \rightarrow \mathbb{R}$  is a pseudometric. On the other hand, *distinguishability*, that is,  $d(x, y) = 0$  implies  $x = y$ , is an issue. Here, a lemma from [6, Section 1.2] comes into play. It uses that  $|\mathcal{X}|$  is an  $\mathbb{E}$ -simplicial complex with a finite set of shapes, that is, with a finite set of isometry classes of simplices.

LEMMA 4.7 (Bridson). *For every  $x \in |\mathcal{X}|$ , there is an  $\varepsilon(x) > 0$  with the following property: For every  $y \in |\mathcal{X}|$  with  $d(x, y) < \varepsilon(x)$ , there is a common 2-simplex  $\tau$  containing both  $x$  and  $y$  such that the distances  $d_\tau(x, y)$  and  $d(x, y)$  agree.*

This lemma implies distinguishability, whence the pseudometric  $d : |\mathcal{X}| \times |\mathcal{X}| \rightarrow \mathbb{R}$  is actually a metric. Even more, it makes  $|\mathcal{X}|$  a complete geodesic metric space, see [6, Theorem 1.1]. As mentioned in [6, p.381, ll.25–31], the topology induced by the metric  $d : |\mathcal{X}| \times |\mathcal{X}| \rightarrow \mathbb{R}$  is coarser, and may even be strictly coarser, than the weak topology. However,  $|\mathcal{X}|$  remains simply connected as a metric space.

REMARK 4.8. Another important application of Lemma 4.7 concerns the arc length of paths. Given an  $m$ -chain  $\mathcal{C} = (x_0, x_1, \dots, x_m)$ , the metric  $d : |\mathcal{X}| \times |\mathcal{X}| \rightarrow \mathbb{R}$  allows us to determine the arc length of the path  $[[\mathcal{C}]]$ , see, for example, [7, Definition I.1.18]. In the light of Lemma 4.7, it is easy to verify that the arc length of the path  $[[\mathcal{C}]]$  agrees with  $\text{length}(\mathcal{C})$ .

4.1.4. *CAT(0) property.* From now on, we will consider  $|\mathcal{X}|$  as a metric space. A crucial observation is that  $|\mathcal{X}|$  has the CAT(0) property. In order to prove this, we will verify the *link condition*.

DEFINITION 4.9 (‘geometric link’). Let  $x \in |\mathcal{X}|$ . The (geometric) closed star  $\text{St}(x)$  is the union of the geometric realizations of all simplices that contain  $x$ . If  $y \in \text{St}(x) \setminus \{x\}$ , then there is at least one 2-simplex that contains both  $x$  and  $y$ . So, we may consider the segment  $[[x, y]]$ . Two such segments  $[[x, y]]$  and  $[[x, y']]$  are called equivalent if one of them is contained in the other. We call the set of equivalence classes the *geometric link* of  $x$ . It is denoted by  $\text{Lk}(x, |\mathcal{X}|)$ .

The geometric link  $\text{Lk}(x, |\mathcal{X}|)$  can be equipped with a metric structure. First, we consider certain subsets of  $\text{Lk}(x, |\mathcal{X}|)$ . For every 2-simplex  $\tau$  in  $\text{St}(x)$ , let  $\text{Lk}(x, \tau)$  be the subset of all elements of  $\text{Lk}(x, |\mathcal{X}|)$  that are represented by segments in  $\tau$ . Note that, as soon as one representative has this property, all representatives do. The subset  $\text{Lk}(x, \tau) \subseteq \text{Lk}(x, |\mathcal{X}|)$  has a natural metric  $d_{\text{Lk}(x, \tau)} : \text{Lk}(x, \tau) \times \text{Lk}(x, \tau) \rightarrow \mathbb{R}$  given by the Euclidean angle:

$$d_{\text{Lk}(x, \tau)} ([[x, y]]_\sim, [[x, y']]_\sim) := \angle_{h|_\tau(x)} (h|_\tau(y), h|_\tau(y')) \in [0, \pi].$$

If  $x$  is in the 1-skeleton of  $|\mathcal{X}|$ , then every  $(\text{Lk}(x, \tau), d_{\text{Lk}(x, \tau)})$  is isometrically isomorphic to a closed interval of length  $\sphericalangle_{\{1,2\}}$ ,  $\sphericalangle_{\{1,3\}}$ ,  $\sphericalangle_{\{2,3\}}$ , or  $\pi$ , see ① and ② in Figure 13. In particular, we may interpret the subsets  $\text{Lk}(x, \tau) \subseteq \text{Lk}(x, |\mathcal{X}|)$  as 1-simplices and, at least after a barycentric subdivision of each simplex, the whole geometric link  $\text{Lk}(x, |\mathcal{X}|)$  as a simplicial complex.

Even more, it is an  $\mathbb{E}$ -simplicial complex with a finite set of shapes. As in Paragraph 4.1.3, the connected components of  $\text{Lk}(x, |\mathcal{X}|)$  can be equipped with a pseudometric. This pseudometric turns out to be a metric which makes every connected component a complete geodesic metric space. If we set the distance of elements from distinct connected components to  $\infty$ , then we obtain an extended metric  $d_{\text{Lk}(x, |\mathcal{X}|)} : \text{Lk}(x, |\mathcal{X}|) \times \text{Lk}(x, |\mathcal{X}|) \rightarrow \mathbb{R} \cup \{\infty\}$ .

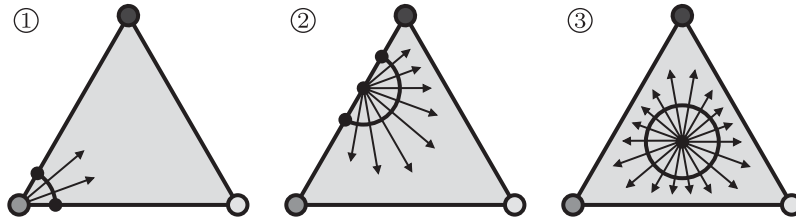


FIGURE 13. The sets  $\text{Lk}(x, \tau)$  for different points  $x \in \tau$ .

If  $x$  is not in the 1-skeleton of  $|\mathcal{X}|$ , then it must be in the interior of some 2-simplex  $\tau$ . In this case,  $(\text{Lk}(x, \tau), d_{\text{Lk}(x, \tau)})$  is isometrically isomorphic to the standard 1-sphere  $\mathbb{S}^1$ , see ③ in Figure 13. Since  $\text{Lk}(x, \tau) = \text{Lk}(x, |\mathcal{X}|)$ , the metric  $d_{\text{Lk}(x, \tau)} : \text{Lk}(x, \tau) \times \text{Lk}(x, \tau) \rightarrow \mathbb{R}$  is already a metric  $d_{\text{Lk}(x, |\mathcal{X}|)} : \text{Lk}(x, |\mathcal{X}|) \times \text{Lk}(x, |\mathcal{X}|) \rightarrow \mathbb{R}$  on the geometric link. For more details, in particular for a remark about equivalent definitions, we refer the reader to [7, Sections 1.7.14–1.7.15]. Note that Bridson and Haefliger consider the open star instead of the closed one. But, in the end, this does not make a difference.

DEFINITION 4.10 (‘link condition’). The geometric realization  $|\mathcal{X}|$  satisfies the *link condition* if for every  $x \in |\mathcal{X}|$  and every pair of points  $a, b \in \text{Lk}(x, |\mathcal{X}|)$  with  $d_{\text{Lk}(x, |\mathcal{X}|)}(a, b) < \pi$  there is a unique geodesic from  $a$  to  $b$ , or, equivalently, if every injective loop  $\lambda : \mathbb{S}^1 \rightarrow \text{Lk}(x, |\mathcal{X}|)$  has arc length at least  $2\pi$ .

The equivalence relies on the fact that  $|\mathcal{X}|$  is a two-dimensional  $\mathbb{E}$ -simplicial complex with a finite set of shapes. One can either prove it directly or apply [7, Theorem I.7.55, (3)  $\Leftrightarrow$  (1), Theorem II.5.5, (3)  $\Leftrightarrow$  (2), Lemma II.5.6].

LEMMA 4.11 (Bridson and Gersten–Stallings). *The geometric realization  $|\mathcal{X}|$  satisfies the link condition.*

A proof of this lemma has been given by Bridson in [6, p. 431, ll. 19–25] and by Gersten and Stallings in [25, p. 499, ll. 19–28]. The argument is both simple and important, so let us outline the main ideas. The most interesting case occurs when  $x$  is a 0-simplex, without loss of generality  $x \in \{\{g\mathfrak{G}_{\{1,2\}}\} : g \in \mathfrak{G}\}$ . Recall the definition of  $\sigma$  from Paragraph 4.1.1, it allows us to describe the set of 2-simplices of  $|\mathcal{X}|$  as the orbit  $\{g\sigma : g \in \mathfrak{G}\}$ . Every injective loop  $\lambda : \mathbb{S}^1 \rightarrow \text{Lk}(x, |\mathcal{X}|)$  traverses some finite number of 1-simplices, say  $m$ . If the first one is  $\text{Lk}(x, g\sigma)$ , then the next ones are  $\text{Lk}(x, gh_1h_2 \cdots h_i\sigma)$  with elements  $h_i$  alternately in  $\mathfrak{G}_{\{1\}} \setminus \mathfrak{G}_{\emptyset}$  and  $\mathfrak{G}_{\{2\}} \setminus \mathfrak{G}_{\emptyset}$ . At the end,  $\lambda$  traverses  $\text{Lk}(x, gh_1h_2 \cdots h_{m-1}\sigma)$ . Since  $\lambda$  is a loop, we know that there is an  $h_m$  such that  $\text{Lk}(x, gh_1h_2 \cdots h_m\sigma) = \text{Lk}(x, g\sigma)$ , which is equivalent to  $h_1h_2 \cdots h_m \in \mathfrak{G}_{\emptyset}$ . Let us write  $h := h_1h_2 \cdots h_m$ . We may assume without loss of generality that  $h_1, h_3, \dots, h_{m-1} \in \mathfrak{G}_{\{1\}} \setminus \mathfrak{G}_{\emptyset}$  and  $h_2, h_4, \dots, h_m \in \mathfrak{G}_{\{2\}} \setminus \mathfrak{G}_{\emptyset}$ . Since  $h \in \mathfrak{G}_{\emptyset}$ , also  $h_mh^{-1} \in \mathfrak{G}_{\{2\}} \setminus \mathfrak{G}_{\emptyset}$ . Now, we proceed similarly to (13) in the proof of the intersection theorem. We construct the preimages under the injective homomorphisms  $\nu_K : G_K \rightarrow \mathfrak{G}$ :

$$\begin{aligned} \nu_{\{1\}}^{-1}(h_1), \nu_{\{1\}}^{-1}(h_3), \dots, \nu_{\{1\}}^{-1}(h_{m-1}) &\in G_{\{1\}} \setminus \varphi_{\emptyset\{1\}}(G_{\emptyset}), \\ \nu_{\{2\}}^{-1}(h_2), \nu_{\{2\}}^{-1}(h_4), \dots, \nu_{\{2\}}^{-1}(h_mh^{-1}) &\in G_{\{2\}} \setminus \varphi_{\emptyset\{2\}}(G_{\emptyset}). \end{aligned}$$

Again, these preimages assemble to an element  $\nu_{\{1\}}^{-1}(h_1) \cdot \nu_{\{2\}}^{-1}(h_2) \cdots \nu_{\{2\}}^{-1}(h_mh^{-1})$  of the amalgamated free product  $G_{\{1\}} *_{G_{\emptyset}} G_{\{2\}}$  that is contained in the kernel of the homomorphism  $\alpha : G_{\{1\}} *_{G_{\emptyset}} G_{\{2\}} \rightarrow G_{\{1,2\}}$  introduced in Subsection 2.2. But, by the normal form theorem, see [20, Lemma 1], this element is non-trivial. Therefore,  $m \geq 2\pi/\angle_{\{1,2\}}$ . Now,

Remark 4.8, which does also hold for  $m$ -chains in  $\text{Lk}(x, |\mathcal{X}|)$ , implies that the arc length of  $\lambda$  is equal to  $m \cdot \sphericalangle_{\{1,2\}}$ , which can be estimated from below by  $2\pi / \sphericalangle_{\{1,2\}} \cdot \sphericalangle_{\{1,2\}} = 2\pi$ , whence we are done. The link condition for the other cases, that is, if  $x$  is in the interior of a 1-simplex or in the interior of a 2-simplex, is almost immediate.

Once we have convinced ourselves that the geometric realization  $|\mathcal{X}|$  satisfies the link condition, we may apply Bridson's main theorem [6, Section 2, Main Theorem, (11)  $\Rightarrow$  (2)].

**THEOREM 4.12 (Bridson).** *Since  $|\mathcal{X}|$  is a simply connected  $\mathbb{E}$ -simplicial complex with a finite set of shapes that satisfies the link condition, it has the CAT(0) property.*

4.1.5. *Geodesics.* Let  $[\mathcal{C}]$  be the path induced by an  $m$ -chain  $\mathcal{C} = (x_0, x_1, \dots, x_m)$ . There is a necessary and, as we will see in Lemma 4.15, sufficient condition for  $[\mathcal{C}]$  to be a geodesic, namely that  $\mathcal{C}$  is *straight*, that is, that there are no obvious shortcuts at the points  $x_1, x_2, \dots, x_{m-1}$ . Let us make the notion of straightness a little more precise.

**DEFINITION 4.13 ('straight  $m$ -chain').** An  $m$ -chain  $\mathcal{C} = (x_0, x_1, \dots, x_m)$  is called *straight* if for every  $1 \leq i \leq m-1$  the distance between  $[[x_i, x_{i-1}]]_{\sim}$  and  $[[x_i, x_{i+1}]]_{\sim}$  in  $\text{Lk}(x_i, |\mathcal{X}|)$  is at least  $\pi$ .

**REMARK 4.14.** In general, it is not easy to determine the distance between  $[[x_i, x_{i-1}]]_{\sim}$  and  $[[x_i, x_{i+1}]]_{\sim}$  in  $\text{Lk}(x_i, |\mathcal{X}|)$ . But the link condition ensures that, once we are able to connect them by an injective path of arc length  $\pi$ , the distance between them is actually equal to  $\pi$ .

Now, the CAT(0) property comes into play. It allows us to conclude from the local property 'straight  $m$ -chain' to the global property 'geodesic'.

**LEMMA 4.15 (Bridson).** *If  $\mathcal{C}$  is a straight  $m$ -chain, then  $[\mathcal{C}]$  is a geodesic.*

This lemma can be proved by showing that every straight  $m$ -chain induces a local geodesic, which is an easy consequence of [6, Section 2, Main Theorem, (2)  $\Rightarrow$  (5)]. And once we know this, [7, Proposition II.1.4(2)] tells us that every local geodesic is a geodesic.

**REMARK 4.16.** By Lemma 4.15, we are now able to construct geodesics easily; and these geodesics are unique, see [6, Section 2, Main Theorem, (2)  $\Rightarrow$  (1)], which will be of relevance in the proof of the billiards theorem.

## 4.2. Billiards theorem

In this section, we still assume given a non-degenerate Euclidean triangle of groups and consider *billiard shots* and *billiard sequences* on the triangle  $\Delta$ .

4.2.1. *Billiard shots and billiard sequences.* We choose some point  $y_0$  in the interior of  $\Delta$  and some direction. Then, we consider the path that starts at  $y_0$  and goes in a straight line into the chosen direction. Eventually, this path leaves  $\Delta$ . Let  $y_1 \in \partial\Delta$  be its last point in  $\Delta$ . If this point is a vertex ('the ball is in the pocket'), then we withdraw the path. Otherwise, it is in the interior of an edge ('the ball hits the cushion'), which allows us to reflect the path according to the rule that the *angle of incidence* is equal to the *angle of reflection*. Now, we can go on. Whenever the path leaves  $\Delta$  at some point in the interior of an edge, we reflect it again. After some finite number of reflections, say at the points  $y_1, y_2, \dots, y_{m-1} \in \partial\Delta$ , we stop at some point  $y_m$  in the interior of  $\Delta$ . The sequence  $\mathcal{B} = (y_0, y_1, \dots, y_m)$  is called a *billiard sequence*, the induced path  $[\mathcal{B}] = [[y_0, y_1, \dots, y_m]]$  is called a *billiard shot*.



4.2.2. *Statement and proof of the billiards theorem.* The notion of billiard shots and billiard sequences allows us to prove that certain elements of the colimit group  $\mathfrak{G}$  are non-trivial.

DEFINITION 4.17 (‘adapted’). Given a billiard sequence  $\mathcal{B} = (y_0, y_1, \dots, y_m)$ , we call an element  $g \in \mathfrak{G}$  *adapted* to  $\mathcal{B}$ , if it is a product  $g_1 g_2 \cdots g_{m-1}$  such that each  $g_i \in \mathfrak{G}_{\{a_i\}} \setminus \mathfrak{G}_\emptyset$ , where  $\{a_i\}$  is the label of the edge whose interior contains  $y_i$ .

THEOREM 4.18. *Assume given a non-degenerate Euclidean triangle of groups and a closed triangle  $\Delta$  in the Euclidean plane  $\mathbb{E}^2$  as constructed in Paragraph 4.1.3. If an element  $g \in \mathfrak{G}$  is adapted to a billiard sequence  $\mathcal{B} = (y_0, y_1, \dots, y_m)$  on  $\Delta$  with at least one reflection, that is, with  $m \geq 2$ , then  $g$  is non-trivial.*

*Proof.* The idea is to lift the billiard shot  $[\mathcal{B}]$  to the geometric realization  $|\mathcal{X}|$ . For every  $1 \leq i \leq m$ , we use  $h|_{g_1 g_2 \cdots g_{i-1} \sigma} : g_1 g_2 \cdots g_{i-1} \sigma \rightarrow \Delta$  to lift the segment  $[[y_{i-1}, y_i]]$ . Let us make some observations.

(1) *These lifts assemble to a path in  $|\mathcal{X}|$ .* For every  $1 \leq i \leq m - 1$ , the first segment  $[[y_{i-1}, y_i]]$  is lifted by  $h|_{g_1 g_2 \cdots g_{i-1} \sigma}$ , the second segment  $[[y_i, y_{i+1}]]$  is lifted by  $h|_{g_1 g_2 \cdots g_i \sigma}$ . To show that these two lifts actually fit together, we convince ourselves that in either case  $y_i$  is lifted to the same point. If the edge whose interior contains  $y_i$  is labelled by  $\{a\}$ , then the two adjacent vertices are labelled by  $\{a, b\}$  and  $\{a, c\}$ , where  $b$  and  $c$  are the remaining two elements of  $\{1, 2, 3\}$ . We can therefore observe:

$$\begin{aligned} (h|_{g_1 g_2 \cdots g_{i-1} \sigma})^{-1}(y_i) &\in \{g_1 g_2 \cdots g_{i-1} \mathfrak{G}_{\{a,b\}}, g_1 g_2 \cdots g_{i-1} \mathfrak{G}_{\{a,c\}}\}, \\ (h|_{g_1 g_2 \cdots g_i \sigma})^{-1}(y_i) &\in \{g_1 g_2 \cdots g_i \mathfrak{G}_{\{a,b\}}, g_1 g_2 \cdots g_i \mathfrak{G}_{\{a,c\}}\}. \end{aligned}$$

Since  $g_i \in \mathfrak{G}_{\{a\}}$ , the two 1-simplices agree. Call them  $\tau$  and observe:

$$(h|_{g_1 g_2 \cdots g_{i-1} \sigma})^{-1}(y_i) = (h|_\tau)^{-1}(y_i) = (h|_{g_1 g_2 \cdots g_i \sigma})^{-1}(y_i).$$

So, the lift of  $y_i$  is well defined. Let us denote it by  $x_i$ . For the lifts of the extremal points  $y_0$  and  $y_m$ , we define analogously  $x_0 := (h|_\sigma)^{-1}(y_0)$  and  $x_m := (h|_{g\sigma})^{-1}(y_m)$ , whence the lifts of each two segments  $[[y_{i-1}, y_i]]$  and  $[[y_i, y_{i+1}]]$  assemble to the path  $[[x_{i-1}, x_i, x_{i+1}]]$  and, more general, the lifts of all segments  $[[y_0, y_1]], [[y_1, y_2]], \dots, [[y_{m-1}, y_m]]$  assemble to the path  $[\mathcal{C}]$  induced by the  $m$ -chain  $\mathcal{C} := (x_0, x_1, \dots, x_m)$ .

(2) *The  $m$ -chain  $\mathcal{C}$  is straight.* Let  $1 \leq i \leq m - 1$ . We construct a 2-chain  $\mathcal{L}$  from  $[[x_i, x_{i-1}]]_\sim$  to  $[[x_i, x_{i+1}]]_\sim$  in  $\text{Lk}(x_i, |\mathcal{X}|)$  such that  $\text{length}(\mathcal{L}) = \pi$  and the path  $[\mathcal{L}]$  is injective. Then, by Remark 4.8, the path  $[\mathcal{L}]$  is of arc length  $\pi$  and, by Remark 4.14, the distance between  $[[x_i, x_{i-1}]]_\sim$  and  $[[x_i, x_{i+1}]]_\sim$  in  $\text{Lk}(x_i, |\mathcal{X}|)$  is equal to  $\pi$ . First, consider the edge whose interior contains  $y_i$  and choose another point  $\tilde{y}_i$  in the interior of this edge. Let  $\tilde{x}_i := (h|_{g_1 g_2 \cdots g_{i-1} \sigma})^{-1}(\tilde{y}_i) = (h|_{g_1 g_2 \cdots g_i \sigma})^{-1}(\tilde{y}_i)$  be its lift. Then, move to the geometric link  $\text{Lk}(x_i, |\mathcal{X}|)$  and construct the 2-chain  $\mathcal{L} := ([[x_i, x_{i-1}]]_\sim, [[x_i, \tilde{x}_i]]_\sim, [[x_i, x_{i+1}]]_\sim)$ , see Figure 14.

Observe that the path  $[\mathcal{L}]$  traverses the interval  $\text{Lk}(x_i, g_1 g_2 \cdots g_{i-1} \sigma)$  until it reaches its endpoint  $[[x_i, \tilde{x}_i]]_\sim$ . Then, it traverses the interval  $\text{Lk}(x_i, g_1 g_2 \cdots g_i \sigma)$ . Therefore:

$$\begin{aligned} \text{length}(\mathcal{L}) &= d_{\text{Lk}(x_i, g_1 g_2 \cdots g_{i-1} \sigma)} ([[x_i, x_{i-1}]]_\sim, [[x_i, \tilde{x}_i]]_\sim) \\ &\quad + d_{\text{Lk}(x_i, g_1 g_2 \cdots g_i \sigma)} ([[x_i, \tilde{x}_i]]_\sim, [[x_i, x_{i+1}]]_\sim) \\ &= \angle_{y_i}(y_{i-1}, \tilde{y}_i) + \angle_{y_i}(\tilde{y}_i, y_{i+1}) = \pi. \end{aligned}$$

Since  $g_i \notin \mathfrak{G}_\emptyset$ , the two intervals traversed by  $[\mathcal{L}]$  are actually not the same. So,  $[\mathcal{L}]$  must be injective.

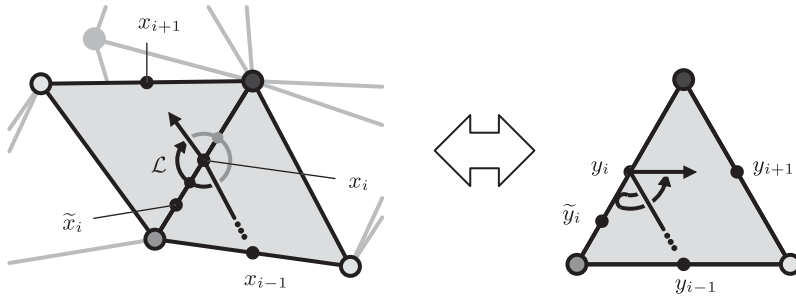


FIGURE 14. Lift the billiard shot to  $g_1g_2 \cdots g_{i-1}\sigma$  and  $g_1g_2 \cdots g_i\sigma$ .

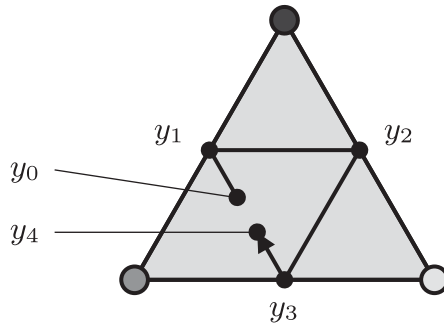


FIGURE 15. A first billiard shot.

With these two observations in mind, the final conclusion that  $g$  is non-trivial in  $\mathfrak{G}$  is almost immediate. By Lemma 4.15,  $[[C]]$  is a geodesic from  $x_0 \in \sigma$  to  $x_m \in g\sigma$ . Before going on, observe that any two points in  $\sigma$  can be connected by a 1-chain, which is, of course, straight and therefore induces a geodesic. But, as mentioned in Remark 4.16, geodesics are unique. Hence, the unique geodesic between any two points in  $\sigma$  is completely contained in  $\sigma$ . Let us now go back to our situation. We assume that  $m \geq 2$ . So, the geodesic  $[[C]]$  leaves  $\sigma$  eventually and, therefore, does not end in  $\sigma$ , that is,  $x_m \notin \sigma$ , which implies that  $g\sigma \neq \sigma$  and, finally,  $g \neq 1$ .  $\square$

4.2.3. *A first example.* We conclude Subsection 4.2 with an example that illustrates the application of the billiards theorem. Assume given a triangle of groups with Gersten–Stallings angles  $\angle_{\{1,2\}} = \angle_{\{1,3\}} = \angle_{\{2,3\}} = \pi/3$ . Since none of them is equal to 0, it is easy to see that for every  $a \in \{1, 2, 3\}$  there is an element  $g_a \in \mathfrak{G}_{\{a\}} \setminus \mathfrak{G}_{\emptyset}$ . Indeed, if there was no element  $g_a \in \mathfrak{G}_{\{a\}} \setminus \mathfrak{G}_{\emptyset}$ , then  $\mathfrak{G}_{\{a\}} = \mathfrak{G}_{\emptyset}$ , which means  $\tilde{\nu}_{\{a\}}(\mathfrak{G}_{\{a\}}) = \tilde{\nu}_{\emptyset}(\mathfrak{G}_{\emptyset})$ . Now, observe that  $\nu_{\{a\}}(G_{\{a\}}) = \tilde{\nu}_{\{a\}}(\mu_{\{a\}}(G_{\{a\}})) = \tilde{\nu}_{\{a\}}(\mathfrak{G}_{\{a\}}) = \tilde{\nu}_{\emptyset}(\mathfrak{G}_{\emptyset}) = \tilde{\nu}_{\emptyset}(\mu_{\emptyset}(G_{\emptyset})) = \nu_{\emptyset}(G_{\emptyset}) = \nu_{\{a\}}(\varphi_{\emptyset\{a\}}(G_{\emptyset}))$ . Since the homomorphism  $\nu_{\{a\}} : G_{\{a\}} \rightarrow \mathfrak{G}$  is injective, we obtain  $G_{\{a\}} = \varphi_{\emptyset\{a\}}(G_{\emptyset})$ , whence  $G_{\{a\}} *_{G_{\emptyset}} G_{\{b\}}$  is generated by  $G_{\{b\}}$ . So, the homomorphism  $\alpha : G_{\{a\}} *_{G_{\emptyset}} G_{\{b\}} \rightarrow G_{\{a,b\}}$  is injective and the Gersten–Stallings angle  $\angle_{\{a,b\}} = 0$ . The product  $g := g_1g_2g_3 \in \mathfrak{G}$ , which has been considered in Theorem 4.1, is adapted to the billiard sequence  $\mathcal{B} = (y_0, y_1, y_2, y_3, y_4)$  drawn in Figure 15 and is therefore non-trivial. Even more, we may continue the billiard shot. This yields billiard sequences of the form  $(y_0, y_1, y_2, y_3, y_1, y_2, y_3, \dots, y_1, y_2, y_3, y_4)$ . Every power of  $g$ , that is, every element  $g^n \in \mathfrak{G}$  with

$n \in \mathbb{N}$ , is adapted to such a billiard sequence and is therefore non-trivial. Hence,  $g$  has infinite order.

### 4.3. Constructing non-abelian free subgroups

Again, assume given a non-degenerate Euclidean triangle of groups. So, for every  $a \in \{1, 2, 3\}$  there is an element  $g_a \in \mathfrak{G}_{\{a\}} \setminus \mathfrak{G}_\emptyset$ . Let us recall the notion of branching. We say that the simplicial complex  $\mathcal{X}$  branches if the geometric realization  $|\mathcal{X}|$  is not a topological manifold any more. It is easy to see that  $\mathcal{X}$  branches if and only if there is an  $a \in \{1, 2, 3\}$  such that the index of  $\mathfrak{G}_\emptyset$  in  $\mathfrak{G}_{\{a\}}$  is at least 3 or there are two distinct  $a, b \in \{1, 2, 3\}$  such that  $\mathfrak{G}_{\{a,b\}}$  is not generated by  $\mathfrak{G}_{\{a\}}$  and  $\mathfrak{G}_{\{b\}}$ . The following theorem says that branching already implies the existence of non-abelian free subgroups in  $\mathfrak{G}$ .

**THEOREM 4.19.** *Assume given a non-degenerate Euclidean triangle of groups. If the simplicial complex  $\mathcal{X}$  branches, then the colimit group  $\mathfrak{G}$  has a non-abelian free subgroup.*

**REMARK 4.20.** Note that Theorem 4.19 is the two-dimensional analogue of a well-known fact. Consider an amalgamated free product  $X *_A Y$  with the property that the image of  $A$  in  $X$  and the image of  $A$  in  $Y$  have index at least 2. The associated Bass–Serre tree  $\mathcal{T}$  branches if and only if one of the indices is at least 3. In this case, the amalgamated free product  $X *_A Y$  has a non-abelian free subgroup.

*Proof of Theorem 4.19.* We may assume without loss of generality that  $\triangleleft_{\{1,2\}} \geq \triangleleft_{\{1,3\}} \geq \triangleleft_{\{2,3\}}$ . So, there are exactly three possibilities for the Gersten–Stallings angles, each of which is considered in a separate column in Figure 16.

First, if there is an  $a \in \{1, 2, 3\}$  such that the index of  $\mathfrak{G}_\emptyset$  in  $\mathfrak{G}_{\{a\}}$  is at least 3, consider the element  $h \in \mathfrak{G}$  that is given in the respective entry in Figure 16. It is constructed in such a way that both  $h$  and  $h^{-1}$  are adapted to a billiard sequence  $\mathcal{B}_1$  with the following property: The billiard shot  $[[\mathcal{B}_1]]$  starts at some point in the interior of  $\Delta$  and goes orthogonally away from the edge labelled by  $\{a\}$ . After a couple of reflections, it comes back to the starting point, but in the opposite direction, see ① in Figure 16.

Note that, given an element  $g \in \mathfrak{G}$  with a decomposition into factors that are alternately from  $\{h, h^{-1}\}$  and  $\mathfrak{G}_{\{a\}} \setminus \mathfrak{G}_\emptyset$ , we may concatenate the billiard shot  $[[\mathcal{B}_1]]$  and the orthogonal reflection at the edge labelled by  $\{a\}$ , see ② in Figure 16, accordingly. This yields a billiard sequence  $\mathcal{B}_2$ , which  $g$  is adapted to. Hence, we know: *If there is at least one factor in the decomposition of  $g$ , then there is at least one reflection in the billiard sequence  $\mathcal{B}_2$  and, by the billiards theorem,  $g$  is non-trivial.*

Since the index of  $\mathfrak{G}_\emptyset$  in  $\mathfrak{G}_{\{a\}}$  is at least 3, we can find an element  $\tilde{g}_a \in \mathfrak{G}_{\{a\}}$  that is neither in  $\mathfrak{G}_\emptyset$  nor in  $g_a \mathfrak{G}_\emptyset$ . In particular, neither  $\tilde{g}_a^{-1} g_a$  nor  $g_a^{-1} \tilde{g}_a$  is in  $\mathfrak{G}_\emptyset$ . Define  $x := g_a h \tilde{g}_a^{-1} \in \mathfrak{G}$  and  $y := h g_a h \tilde{g}_a^{-1} h^{-1} \in \mathfrak{G}$ . We claim that  $x$  and  $y$  generate a non-abelian free subgroup of  $\mathfrak{G}$ . Consider a non-empty freely reduced word over the letters  $x$  and  $y$  and their formal inverses. The element  $g \in \mathfrak{G}$  that is represented by this word has a natural decomposition into factors from  $\{h^{\pm 1}, g_a^{\pm 1}, \tilde{g}_a^{\pm 1}\}$ . Cancel each  $h^{-1}h$  and subsume each  $\tilde{g}_a^{-1} g_a$  and each  $g_a^{-1} \tilde{g}_a$  to a single element in  $\mathfrak{G}_{\{a\}} \setminus \mathfrak{G}_\emptyset$ . This yields a new decomposition of  $g$  into factors that are alternately from  $\{h, h^{-1}\}$  and  $\mathfrak{G}_{\{a\}} \setminus \mathfrak{G}_\emptyset$ . It is easy to see that, despite of the cancellation of each  $h^{-1}h$ , there is at least one factor left in the new decomposition of  $g$ . So, by our preliminary discussion,  $g$  is non-trivial, which completes the proof that  $x$  and  $y$  generate a non-abelian free subgroup of  $\mathfrak{G}$ .

Second, consider the case that there are two distinct  $a, b \in \{1, 2, 3\}$  such that  $\mathfrak{G}_{\{a,b\}}$  is not generated by  $\mathfrak{G}_{\{a\}}$  and  $\mathfrak{G}_{\{b\}}$ . In this case, let  $X := \mathfrak{G}_{\{a,b\}}$ ,  $A := \langle \mathfrak{G}_{\{a\}}, \mathfrak{G}_{\{b\}} \rangle \leq \mathfrak{G}$ , and  $Y := \langle \mathfrak{G}_{\{a,c\}}, \mathfrak{G}_{\{b,c\}} \rangle \leq \mathfrak{G}$ , where  $c$  is the remaining element of  $\{1, 2, 3\}$ .

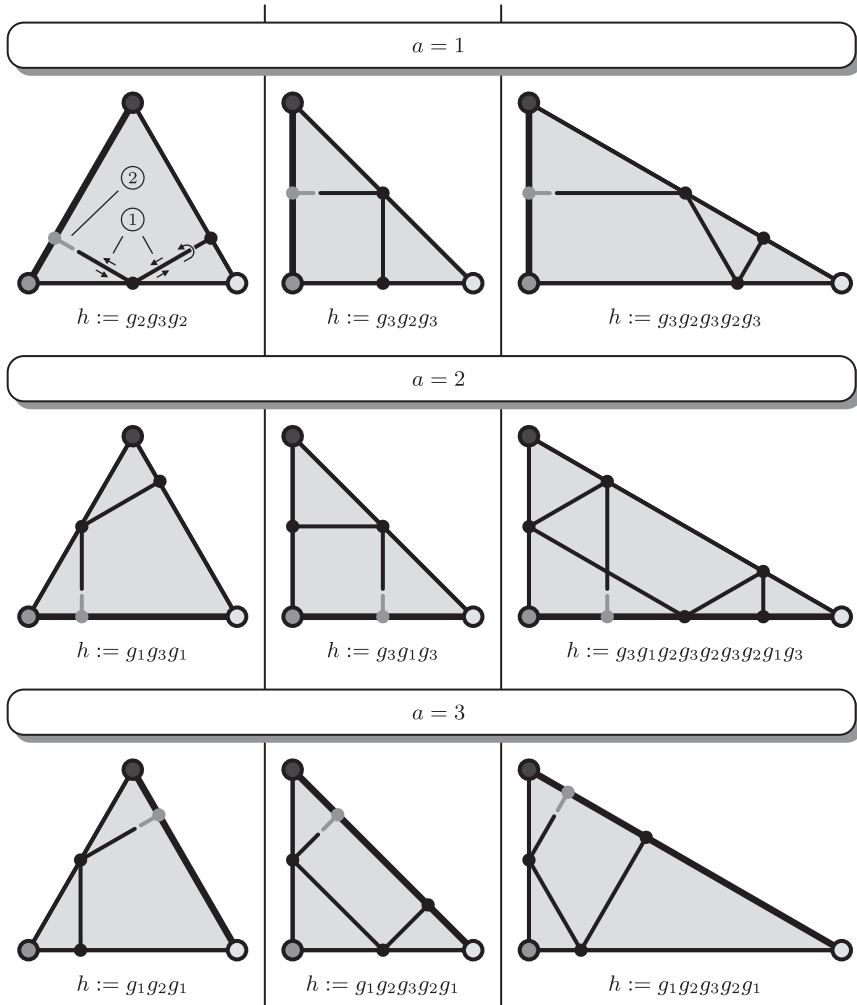


FIGURE 16. The first column refers to Gersten–Stallings angles  $\angle_{\{1,2\}} = \angle_{\{1,3\}} = \angle_{\{2,3\}} = \pi/3$ , the second to  $\angle_{\{1,2\}} = \pi/2$ ,  $\angle_{\{1,3\}} = \angle_{\{2,3\}} = \pi/4$ , the third to  $\angle_{\{1,2\}} = \pi/2$ ,  $\angle_{\{1,3\}} = \pi/3$ ,  $\angle_{\{2,3\}} = \pi/6$ .

Using the presentation (\*), one can show that  $\mathfrak{G} \cong X *_A Y$ . Here, the homomorphisms are the ones induced by the inclusions. By assumption,  $|X : A| \geq 2$ . On the other hand, we know that there is an element  $g_c \in \mathfrak{G}_{\{c\}} \setminus \mathfrak{G}_\emptyset$ . By the intersection theorem,  $\mathfrak{G}_{\{c\}} \cap \mathfrak{G}_{\{a,b\}} = \mathfrak{G}_\emptyset$ . So,  $g_c \in \mathfrak{G}_{\{c\}} \setminus \mathfrak{G}_{\{a,b\}} \subseteq Y \setminus A$  and  $|Y : A| \geq 2$ . If  $|Y : A| = 2$ , then  $A$  is a normal subgroup of  $Y$ . Again, by the intersection theorem:

$$g_c^{-1} g_a g_c \in A \cap \mathfrak{G}_{\{a,c\}} \subseteq \mathfrak{G}_{\{a,b\}} \cap \mathfrak{G}_{\{a,c\}} = \mathfrak{G}_{\{a\}},$$

$$g_c^{-1} g_b g_c \in A \cap \mathfrak{G}_{\{b,c\}} \subseteq \mathfrak{G}_{\{a,b\}} \cap \mathfrak{G}_{\{a,b\}} = \mathfrak{G}_{\{b\}}.$$

This implies that  $\angle_{\{a,c\}} = \angle_{\{b,c\}} = \pi/2$ . Hence  $\angle_{\{a,b\}} = 0$ , which is not possible since we assume the triangle of groups to be non-degenerate. So,  $|Y : A| \geq 3$  and, by Remark 4.20,  $\mathfrak{G} \cong X *_A Y$  has a non-abelian free subgroup.  $\square$

REMARK 4.21. The idea of Theorems 4.18 and 4.19 is certainly ping-pong-ish. In the above proof, we construct products  $g = g_1 g_2 \cdots g_m$  whose factors  $g_i$  are alternately from  $\{h, h^{-1}\}$  and

$\mathfrak{G}_{\{a\}} \setminus \mathfrak{G}_{\emptyset}$ . Therefore, the sequence  $\sigma \mapsto g_m \sigma \mapsto g_{m-1} g_m \sigma \mapsto \cdots \mapsto g_1 \cdots g_{m-1} g_m \sigma$  moves the fundamental domain  $\sigma$  back and forth through  $|\mathcal{X}|$ . But, instead of defining ping-pong sets, we construct geodesics to ensure that the final position of  $\sigma$  is actually different from the initial one. The language of billiards helps us to see these geodesics without getting unnecessarily confused by the surrounding complex.

#### 4.4. Tits alternative

In this section, we ask about the cases that are not covered by Theorems 4.2 and 4.19, and discuss the following version of the Tits alternative.

**DEFINITION 4.22** (‘Tits alternative’). A class  $\mathcal{C}$  of groups satisfies the *Tits alternative* if each  $G \in \mathcal{C}$  either has a non-abelian free subgroup or is virtually solvable.

**REMARK 4.23.** There are groups that neither have a non-abelian free subgroup nor are virtually solvable. For example, take Thompson’s group  $F$ . It has been shown by Brin and Squier in [8] that  $F \leq \text{PLF}(\mathbb{R})$  does not have a non-abelian free subgroup. And if  $F$  was virtually solvable, then  $[F, F]$  would have to be virtually solvable, too. But  $[F, F]$  is infinite and simple, see [11, Section 4], which implies that  $[F, F]$  cannot be virtually solvable.

We may use Thompson’s group  $F$  to prove that the Tits alternative does not hold for the class of colimit groups of non-spherical triangles of groups. For example, let  $\Gamma_1$  be the triangle of groups with the property that the groups  $G_J$  are all equal to  $F$  and the injective homomorphisms  $\varphi_{J_1 J_2} : G_{J_1} \rightarrow G_{J_2}$  are all identities. The Gersten–Stallings angles amount to  $\angle_{\{1,2\}} = \angle_{\{1,3\}} = \angle_{\{2,3\}} = 0$ , whence  $\Gamma_1$  is a degenerate hyperbolic triangle of groups. But the colimit group is isomorphic to  $F$  and, therefore, neither has a non-abelian free subgroup nor is virtually solvable.

Note that there are non-degenerate examples, too. Pick one of the three triangles of groups given in the introduction to Section 4 and replace every group  $G_J$  by  $F \times G_J$  and every injective homomorphism  $\varphi_{J_1 J_2} : G_{J_1} \rightarrow G_{J_2}$  by  $\text{id}_F \times \varphi_{J_1 J_2} : F \times G_{J_1} \rightarrow F \times G_{J_2}$ . The Gersten–Stallings angles remain the same, whence the new triangle of groups, call it  $\Gamma_2$ , is non-degenerate and Euclidean. But the colimit group is isomorphic to  $F \times \Delta(k, l, m)$  and, therefore, neither has a non-abelian free subgroup nor is virtually solvable. The latter is a consequence of the following two results: Let  $G$  be a group and let  $N \trianglelefteq G$  be a normal subgroup. The group  $G$  has a non-abelian free subgroup if and only if  $N$  or  $G/N$  does. So,  $F \times \Delta(k, l, m)$  does not have a non-abelian free subgroup. On the other hand, if  $G$  is virtually solvable, then every subgroup of  $G$  is virtually solvable, too. So,  $F \times \Delta(k, l, m)$  is not virtually solvable.

Let us now assume that  $G_{\emptyset}$  either has a non-abelian free subgroup or is virtually solvable, for example,  $G_{\emptyset} = \{1\}$  as in Remark 4.3. In the non-degenerate case, this assumption already implies the Tits alternative.

**THEOREM 4.24.** *The Tits alternative holds for the class of colimit groups of non-degenerate non-spherical triangles of groups with the property that the group  $G_{\emptyset}$  either has a non-abelian free subgroup or is virtually solvable.*

Interestingly, in the degenerate case, it does not. Just consider the triangle of groups  $\Gamma_3$  given by the following data:

$$\begin{aligned} G_\emptyset &= \{1\}, & G_{\{1\}} &= F, & G_{\{2\}} &= \langle a : a^2 = 1 \rangle, & G_{\{3\}} &= \langle b : b^2 = 1 \rangle, \\ G_{\{1,2\}} &= F \times \langle a : a^2 = 1 \rangle, & G_{\{1,3\}} &= F \times \langle b : b^2 = 1 \rangle, \\ G_{\{2,3\}} &= \langle a, b : a^2 = b^2 = 1 \rangle. \end{aligned}$$

Here, the homomorphisms  $\varphi_{J_1 J_2} : G_{J_1} \rightarrow G_{J_2}$  are given by  $\forall f \in F : f \mapsto (f, 1)$ , by  $a \mapsto (1, a)$  and  $a \mapsto a$ , and by  $b \mapsto (1, b)$  and  $b \mapsto b$ . The Gersten–Stallings angles amount to  $\angle_{\{1,2\}} = \pi/2$ ,  $\angle_{\{1,3\}} = \pi/2$ , and  $\angle_{\{2,3\}} = 0$ , whence  $\Gamma_3$  is a degenerate Euclidean triangle of groups. Its colimit group is isomorphic to  $F \times (\mathbb{Z}_2 * \mathbb{Z}_2)$  and, therefore, neither has a non-abelian free subgroup nor is virtually solvable. The following theorem is an analogue of Theorem 4.24. It includes the degenerate case.

**THEOREM 4.25.** *The Tits alternative holds for the class of colimit groups of non-spherical triangles of groups with the property that every group  $G_J$  with  $J \subseteq \{1, 2, 3\}$  and  $|J| \leq 2$  either has a non-abelian free subgroup or is virtually solvable.*

Given Theorem 4.19, the proofs of Theorems 4.24 and 4.25 are elementary. But we need an auxiliary result. It had been asked by Button and was answered independently by several authors thereafter, see [10, Problem 3] for details.

**LEMMA 4.26** (Linnell, Minasyan, Klyachko, ...). *Let  $G$  be a group and let  $N \trianglelefteq G$  be a normal subgroup. If both  $N$  and  $G/N$  are virtually solvable, then  $G$  is virtually solvable, too.*

*Proof of Theorem 4.24.* Consider a non-degenerate non-spherical triangle of groups with the property that the group  $G_\emptyset$  either has a non-abelian free subgroup or is virtually solvable. If  $G_\emptyset$ , and hence  $\mathfrak{G}_\emptyset$ , has a non-abelian free subgroup, then the colimit group  $\mathfrak{G}$  has so, too. So, we may assume without loss of generality that  $G_\emptyset$ , and hence  $\mathfrak{G}_\emptyset$ , are virtually solvable. The triangle of groups is non-degenerate. So, for every  $a \in \{1, 2, 3\}$  there is an element  $g_a \in \mathfrak{G}_{\{a\}} \setminus \mathfrak{G}_\emptyset$ . If the triangle of groups is hyperbolic, then, by Theorem 4.2, the colimit group  $\mathfrak{G}$  has a non-abelian free subgroup. So, we may assume without loss of generality that the triangle of groups is Euclidean. Now, let  $a \in \{1, 2, 3\}$ . Since  $g_a \in \mathfrak{G}_{\{a\}} \setminus \mathfrak{G}_\emptyset$ , the index of  $\mathfrak{G}_\emptyset$  in  $\mathfrak{G}_{\{a\}}$  is at least 2. If it is strictly larger than 2, then the simplicial complex  $\mathcal{X}$  branches and, by Theorem 4.19,  $\mathfrak{G}$  has a non-abelian free subgroup. So, we may assume without loss of generality that it is equal to 2. In particular,  $\mathfrak{G}_\emptyset$  is normal in  $\mathfrak{G}_{\{a\}}$ . By the same argument, we may assume without loss of generality that for every two distinct  $a, b \in \{1, 2, 3\}$  the group  $\mathfrak{G}_{\{a,b\}}$  is generated by  $\mathfrak{G}_{\{a\}}$  and  $\mathfrak{G}_{\{b\}}$ . Therefore,  $\mathfrak{G}_\emptyset$  is normal in  $\mathfrak{G}_{\{1\}}$ ,  $\mathfrak{G}_{\{2\}}$ ,  $\mathfrak{G}_{\{3\}}$ ,  $\mathfrak{G}_{\{1,2\}}$ ,  $\mathfrak{G}_{\{1,3\}}$ ,  $\mathfrak{G}_{\{2,3\}}$ , and, finally, in  $\mathfrak{G}$ .

Note that this property also holds in the triangle of groups itself, that is, for every  $J \subseteq \{1, 2, 3\}$  with  $1 \leq |J| \leq 2$  the image  $\varphi_{\emptyset J}(G_\emptyset)$  is normal in  $G_J$ . For a formal proof, apply the natural homomorphism  $\nu_J : G_J \rightarrow \mathfrak{G}$ , which is injective, and observe:

$$\begin{aligned} \nu_J \circ \varphi_{\emptyset J}(G_\emptyset) &= \nu_\emptyset(G_\emptyset) = \tilde{\nu}_\emptyset \circ \mu_\emptyset(G_\emptyset) = \tilde{\nu}_\emptyset(\mathfrak{G}_\emptyset) \\ &\leq \tilde{\nu}_J(\mathfrak{G}_J) = \tilde{\nu}_J \circ \mu_J(G_J) = \nu_J(G_J). \end{aligned}$$

We may therefore construct the *quotient triangle of groups*, which is obtained by replacing the group  $G_\emptyset$  by  $G_\emptyset/G_\emptyset \cong \{1\}$  and for every  $J \subseteq \{1, 2, 3\}$  with  $1 \leq |J| \leq 2$  the group  $G_J$  by  $G_J/\varphi_{\emptyset J}(G_\emptyset)$ . Here, one needs to verify that every injective homomorphism  $\varphi_{J_1 J_2} : G_{J_1} \rightarrow G_{J_2}$  induces an injective homomorphism between the quotients and that the Gersten–Stallings angles remain the same. We leave this work to the reader. However, the resulting diagram is a non-degenerate Euclidean triangle of groups. Moreover, using the presentation (\*) one can

show that its colimit group is isomorphic to  $\mathfrak{G}/\mathfrak{G}_\emptyset$ . We will now study the quotient triangle of groups in some more detail. For every  $a \in \{1, 2, 3\}$ , the index of  $\varphi_{\emptyset\{a\}}(G_\emptyset)$  in  $G_{\{a\}}$  is equal to 2, that is, the quotient  $G_{\{a\}}/\varphi_{\emptyset\{a\}}(G_\emptyset)$  has two elements. Therefore, the quotient triangle of groups must be isomorphic to one of the three triangles of groups given in the introduction to Section 4 and, in particular, its colimit group  $\mathfrak{G}/\mathfrak{G}_\emptyset$  is (virtually) solvable. Moreover, by assumption,  $G_\emptyset$ , and hence  $\mathfrak{G}_\emptyset$ , is virtually solvable, too. So, by Lemma 4.26, we may conclude that  $\mathfrak{G}$  is virtually solvable.  $\square$

*Proof of Theorem 4.25.* Consider a non-spherical triangle of groups with the property that every group  $G_J$  with  $J \subseteq \{1, 2, 3\}$  and  $|J| \leq 2$  either has a non-abelian free subgroup or is virtually solvable. Again, we may assume without loss of generality that the groups  $G_J$ , and hence their images  $\mathfrak{G}_J$ , are virtually solvable. Moreover, if the triangle of groups is non-degenerate, then we know by Theorem 4.24 that  $\mathfrak{G}$  either has a non-abelian free subgroup or is virtually solvable. So, we may assume without loss of generality that the Gersten–Stallings angle  $\angle_{\{2,3\}} = 0$ , which means that the homomorphism  $\alpha : G_{\{2\}} *_{G_\emptyset} G_{\{3\}} \rightarrow G_{\{2,3\}}$  induced by  $\varphi_{\{2\}\{2,3\}}$  and  $\varphi_{\{3\}\{2,3\}}$  is injective. As mentioned in the proof of Theorem 4.19, one can always show that  $\mathfrak{G} \cong X *_A Y$  with  $X := \mathfrak{G}_{\{2,3\}}$ ,  $A := \langle \mathfrak{G}_{\{2\}}, \mathfrak{G}_{\{3\}} \rangle \leq \mathfrak{G}$ ,  $Y := \langle \mathfrak{G}_{\{1,2\}}, \mathfrak{G}_{\{1,3\}} \rangle \leq \mathfrak{G}$ . Depending on  $|X : A|$  and  $|Y : A|$ , we distinguish between four cases.

(1) If  $|X : A| = 1$ , then  $\mathfrak{G}_{\{2,3\}}$  is generated by  $\mathfrak{G}_{\{2\}}$  and  $\mathfrak{G}_{\{3\}}$  or, equivalently,  $G_{\{2,3\}}$  is generated by  $\varphi_{\{2\}\{2,3\}}(G_{\{2\}})$  and  $\varphi_{\{3\}\{2,3\}}(G_{\{3\}})$ . So,  $\alpha$  is not only injective but also surjective, whence  $G_{\{2\}} *_{G_\emptyset} G_{\{3\}} \cong G_{\{2,3\}}$ . This allows us to simplify the original presentation (\*) of the colimit group  $\mathfrak{G}$  by deleting superficial generators and relators:

$$\begin{aligned} \mathfrak{G} = \langle & G_{\{1\}}, G_{\{1,2\}}, G_{\{1,3\}} : R_{\{1\}}, R_{\{1,2\}}, R_{\{1,3\}}, \\ & \{g = \varphi_{\{1\}\{1,2\}}(g) : g \in G_{\{1\}}\}, \\ & \{g = \varphi_{\{1\}\{1,3\}}(g) : g \in G_{\{1\}}\} \rangle. \end{aligned}$$

So,  $\mathfrak{G} \cong G_{\{1,2\}} *_{G_{\{1\}}} G_{\{1,3\}}$  or, equivalently,  $\mathfrak{G} \cong \mathfrak{G}_{\{1,2\}} *_{\mathfrak{G}_{\{1\}}} \mathfrak{G}_{\{1,3\}}$ . Now, we may, again, distinguish between four cases. (a) If  $|\mathfrak{G}_{\{1,2\}} : \mathfrak{G}_{\{1\}}| = 1$ , then  $\mathfrak{G} \cong \mathfrak{G}_{\{1,3\}}$ , which is virtually solvable. (b) If  $|\mathfrak{G}_{\{1,3\}} : \mathfrak{G}_{\{1\}}| = 1$ , then  $\mathfrak{G} \cong \mathfrak{G}_{\{1,2\}}$ , which is virtually solvable. (c) If  $|\mathfrak{G}_{\{1,2\}} : \mathfrak{G}_{\{1\}}| = |\mathfrak{G}_{\{1,3\}} : \mathfrak{G}_{\{1\}}| = 2$ , then  $\mathfrak{G}_{\{1\}}$  is normal in  $\mathfrak{G}_{\{1,2\}}$ ,  $\mathfrak{G}_{\{1,3\}}$ , and  $\mathfrak{G}$ . The quotient  $\mathfrak{G}/\mathfrak{G}_{\{1\}} \cong \mathbb{Z}_2 * \mathbb{Z}_2$ , which is (virtually) solvable. On the other hand,  $\mathfrak{G}_{\{1\}}$  itself is virtually solvable. So, by Lemma 4.26, the colimit group  $\mathfrak{G}$  is virtually solvable. (d) Otherwise,  $|\mathfrak{G}_{\{1,2\}} : \mathfrak{G}_{\{1\}}| \geq 2$  and  $|\mathfrak{G}_{\{1,3\}} : \mathfrak{G}_{\{1\}}| \geq 2$  and not both equal to 2. But then, by Remark 4.20,  $\mathfrak{G} \cong \mathfrak{G}_{\{1,2\}} *_{\mathfrak{G}_{\{1\}}} \mathfrak{G}_{\{1,3\}}$  has a non-abelian free subgroup.

(2) If  $|Y : A| = 1$ , then  $\mathfrak{G} \cong X = \mathfrak{G}_{\{2,3\}}$ , which is virtually solvable.

(3) If  $|X : A| = |Y : A| = 2$ , then  $A$  is normal in  $X$ ,  $Y$ , and  $\mathfrak{G}$ . The quotient  $\mathfrak{G}/A \cong \mathbb{Z}_2 * \mathbb{Z}_2$ , which is (virtually) solvable. Let us now study the normal subgroup  $A$  in some more detail. Since the homomorphism  $\alpha : G_{\{2\}} *_{G_\emptyset} G_{\{3\}} \rightarrow G_{\{2,3\}}$  is injective, its image  $\langle \varphi_{\{2\}\{2,3\}}(G_{\{2\}}), \varphi_{\{3\}\{2,3\}}(G_{\{3\}}) \rangle \leq G_{\{2,3\}}$ , and hence  $A = \langle \mathfrak{G}_{\{2\}}, \mathfrak{G}_{\{3\}} \rangle \leq \mathfrak{G}_{\{2,3\}} \leq \mathfrak{G}$ , is isomorphic to  $G_{\{2\}} *_{G_\emptyset} G_{\{3\}}$ . By a case analysis analogue to the one in (1), we can see that  $A$  is either virtually solvable, namely in cases (a)–(c), or has a non-abelian free subgroup, namely in case (d). In the former cases, by Lemma 4.26, the colimit group  $\mathfrak{G}$  is virtually solvable. In the latter case, the colimit group  $\mathfrak{G}$ , which contains  $A$  as a subgroup, has a non-abelian free subgroup.

(4) Otherwise,  $|X : A| \geq 2$  and  $|Y : A| \geq 2$  and not both equal to 2. But then, by Remark 4.20,  $\mathfrak{G} \cong X *_A Y$  has a non-abelian free subgroup.  $\square$

REMARK 4.27. Kopteva and Williams have proved that the Tits alternative holds for the class of non-spherical Pride groups that are based on graphs with at least four vertices, see [18, Theorem 1]. One way to read Theorem 4.24 is the following: The Tits alternative does not hold



for the class of non-spherical Pride groups that are based on graphs with three vertices. But once we assume that each edge is *genuine*, that is, that none of the Gersten–Stallings angles is equal to 0, it does.

### Appendix. Further applications

In Subsections 4.1 and 4.2, we have been working with non-degenerate Euclidean triangles of groups. But the construction can be extended to all non-degenerate non-spherical triangles of groups. In the hyperbolic case, one can either pick a triangle  $\Delta$  in the hyperbolic plane  $\mathbb{H}^2$ , as suggested by Bridson in [6, p. 431, ll. 13–16], or a triangle  $\Delta$  in the Euclidean plane  $\mathbb{E}^2$  ‘whose angles are perhaps a little bit larger than the group-theoretic angles’, as suggested by Gersten and Stallings in [25, p. 499, ll. 7–9]. Let us sketch an application for each of the two alternatives.

#### A.1. Normal forms

In the billiards theorem, we assume given an element  $g \in \mathfrak{G}$  that is adapted to a billiard sequence, that is, equipped with a suitable decomposition into factors from  $\mathfrak{G}_{\{1\}} \setminus \mathfrak{G}_{\emptyset}$ ,  $\mathfrak{G}_{\{2\}} \setminus \mathfrak{G}_{\emptyset}$ ,  $\mathfrak{G}_{\{3\}} \setminus \mathfrak{G}_{\emptyset}$ . But we could also go the other way and use the simplicial complex  $\mathcal{X}$  to construct decompositions. More precisely, given a non-degenerate non-spherical triangle of groups, pick a triangle  $\Delta$  either in the Euclidean plane  $\mathbb{E}^2$  or in the hyperbolic plane  $\mathbb{H}^2$  whose angles agree with the Gersten–Stallings angles and construct the simplicial complex  $\mathcal{X}$ . Note that all the results from Subsections 4.1 and 4.2 still hold true. Given an arbitrary element  $g \in \mathfrak{G}$ , consider the unique geodesic in  $|\mathcal{X}|$ , see Remark 4.16, from the barycentre of  $\sigma$  to the barycentre of  $g\sigma$ . As soon as  $g \notin \mathfrak{G}_{\emptyset}$ , the geodesic traverses several 2-simplices. First, it traverses  $\sigma$ . Then, depending on whether it leaves  $\sigma$  crossing a 0-simplex or the interior of a 1-simplex, there is an element  $g_1 \in \mathfrak{G}_K$  with  $K \subseteq \{1, 2, 3\}$  and  $|K| = 2$  or  $|K| = 1$  such that the next 2-simplex it traverses is  $g_1\sigma$ . This procedure goes on. At the end, it traverses  $g_1g_2 \cdots g_m\sigma = g\sigma$ , which yields a decomposition of  $g$  into factors from the groups  $\mathfrak{G}_K$  with  $K \subseteq \{1, 2, 3\}$  and  $1 \leq |K| \leq 2$  and one final factor from  $\mathfrak{G}_{\emptyset}$ .

Note that this decomposition is not well defined, even in the case  $\mathfrak{G}_{\emptyset} = \{1\}$ . Just imagine the geodesic running along some 1-simplex. Then, there are many possibilities to choose the respective 2-simplex  $g_1g_2 \cdots g_i\sigma$ . On the other hand, if we fix a set of coset representatives for each pair of subgroups  $\mathfrak{G}_{K_1} \leq \mathfrak{G}_{K_2}$  with  $K_1 \subset K_2 \subseteq \{1, 2, 3\}$  and  $|K_2| \leq 2$ , then there is a well-defined decomposition in terms of these coset representatives and one final factor from  $\mathfrak{G}_{\emptyset}$ . Even though it seems to be inconvenient to work with, we may call it a normal form.

#### A.2. Euclidean domination

The second alternative has the advantage that there are only three different triangles  $\Delta$ . More precisely, given a non-degenerate non-spherical triangle of groups, the Gersten–Stallings angles are always of the form  $2\pi/\hat{m}$ , where  $\hat{m}$  is even. Let us think of them as  $\pi/k$ ,  $\pi/l$ ,  $\pi/m$  with  $k, l, m \in \mathbb{N}$  and  $k \leq l \leq m$ . It is easy to see that either  $(\pi/3, \pi/3, \pi/3)$  or  $(\pi/2, \pi/4, \pi/4)$  or  $(\pi/2, \pi/3, \pi/6)$  dominates  $(\pi/k, \pi/l, \pi/m)$ , that is, coordinatewise at least as large as  $(\pi/k, \pi/l, \pi/m)$ . So, if we take the dominating triple instead of the original Gersten–Stallings angles, then, again, all the results from Subsections 4.1 and 4.2, in particular the link condition and the billiards theorem, hold true. Therefore, the proof of Theorem 4.19 extends to all non-degenerate non-spherical triangles of groups.

REMARK A.1. Our methods almost yield an alternative proof of Theorem 4.2; we cannot say anything about the generators  $(g_1g_2g_3)^n$  and  $(g_1g_3g_2)^n$  but we can prove the existence

of another non-abelian free subgroup. If the hyperbolic triangle of groups is non-degenerate and the simplicial complex  $\mathcal{X}$  branches, then Theorem 4.19 does the job. The remaining cases are elementary. If it is non-degenerate and the simplicial complex  $\mathcal{X}$  does not branch, then the quotient  $\mathfrak{G}/\mathfrak{G}_\emptyset$  is a hyperbolic triangle group, has a non-abelian free subgroup, and so has  $\mathfrak{G}$ . Finally, if it is degenerate, say with  $\angle_{\{2,3\}} = 0$ , then  $\mathfrak{G}$  contains  $\mathfrak{G}_{\{1,2\}} *_{\mathfrak{G}_{\{1\}}} \mathfrak{G}_{\{1,3\}}$  as a subgroup. Since there are elements  $g_2 \in \mathfrak{G}_{\{2\}} \setminus \mathfrak{G}_\emptyset$  and  $g_3 \in \mathfrak{G}_{\{3\}} \setminus \mathfrak{G}_\emptyset$ , the indices  $|\mathfrak{G}_{\{1,2\}} : \mathfrak{G}_{\{1\}}|$  and  $|\mathfrak{G}_{\{1,3\}} : \mathfrak{G}_{\{1\}}|$  are both at least 2. If one of these indices is equal to 2, then the respective Gersten–Stallings angle must be equal to  $\pi/2$ . But the triangle of groups is hyperbolic, so either  $\angle_{\{1,2\}}$  or  $\angle_{\{1,3\}}$  is smaller than  $\pi/2$ , which implies that either  $|\mathfrak{G}_{\{1,2\}} : \mathfrak{G}_{\{1\}}|$  or  $|\mathfrak{G}_{\{1,3\}} : \mathfrak{G}_{\{1\}}|$  is larger than 2. So, by Remark 4.20,  $\mathfrak{G}$  has a non-abelian free subgroup.

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